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Equations of Mathematical Physics

Generalized Functions and Historical Notes

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Contents

Preface

> Presently, the notion of function is not as finally crystallized and definitely established as it seemed at the end of the 19th century; one can say that at present this notion is still in evolution, and that the dispute concerning the vibrating string is still going on only, of course, in different scientific circumstances, involving other personalities and using other terms.
>
> Luzin (1935)

It is symbolic that in that same year of 1935, S. L. Sobolev, who was 26 years old at that time, submitted to the journal *Matematicheskii Sbornik (Sbornik: Mathematics)* his famous paper Sobolev (1936), published at the same time in its brief form in *Doklady AN SSSR (Doklady Mathematics)* (Sobolev 1935). This work established the foundation of a completely new outlook on the concept of function, unexpected even for N. N. Luzin—here we speak about the generalized functions. It is also symbolic that Sobolev's work was devoted to the Cauchy problem for hyperbolic equations and, particularly, to the vibrating string problem. There is no doubt that the above Luzin's statement about the concept of a function will be relevant for a long time, and the incentive for the development of this fundamental concept of mathematics will be, as before, the equations of mathematical physics. This special role of the equations of mathematical physics (in other words, partial differential equations, which are closely related to natural phenomena) can be explained by the fact that they express the mathematical essence of fundamental laws of the natural sciences and consequently provide a source and stimulus for the development of fundamental mathematical concepts and theories. In the present book, we give fundamentals of the main ideas, concepts, and results of equations of mathematical physics in the context of various theories of generalized functions (including, of course, L. Schwartz's theory of distributions).

The crucial role in the appearance of the theory of generalized functions (in the sense the theory of distributions) was played by J. Hadamard,[1] K. O. Friedrichs, S. Bochner, and especially by L. Schwartz, who published, in 1944–1948, a series of remarkable papers on generalized functions, and in 1950–1951 the two-volume book (Schwartz 1950–1951), which immediately became classical. Being a masterpiece and oriented to a wide circle of specialists, this book attracted the attention of many people to the theory of generalized function.[2] A substantial contribution to its development was also made by such prominent mathematicians as I. M. Gelfand, M. I. Vishik, L. Hörmander, V. S. Vladimirov, V P. Maslov, and many others. As a result, the theory of generalized functions has changed all modern analysis and, in the first place, the entire theory of partial differential equations. Therefore, the foundations of the theory of distributions became necessary for general education of physicists and mathematicians.

Thus, it is not surprising that a number of excellent monographs and textbooks (see, in particular, Schwartz (1950–1951); Sobolev (2008); Gelfand et al. (1958–1966); Lions and Magenes (1968); Tréves (1980); Hörmander (1983–1985); Shubin (1987); Gilbarg and Trudinger (1983); Palamodov (1991); Kanwal (1998); Taylor (1974); Reed and Simon (1972); Vladimirov (1971); Oleinik (2007); Vekua (1962); Courant (1992); Shilov (1965); Godunov (1979); Mikhailov (1978)) are devoted to the equations of mathematical physics and distributions. However, most of them are intended for rather well-prepared readers. As for the present book, I hope it will be clear even to undergraduates majoring in physics and mathematics and will serve as a starting point for a deeper study of the above-mentioned books and papers.

[1] See the footnote on p. 148.

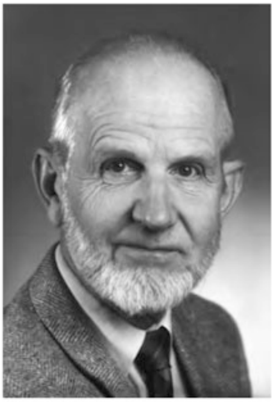
[2] As one of the referees of the present book noted, "Sobolev defined completely distributions in his paper of 1936. However, he did not get much credit for his remarkable results. Sobolev's paper was published in French, but Laurent Schwartz referred to it only in a minuscule footnote. In two papers, Kiselman (2007, 2019), possible reasons behind this neglect are briefly mentioned." The referee also points out the following phrase of Jesper Lützen, who has written a most interesting book Lützen (1982) on the prehistory of the theory of distributions: "It is not enough to do something important; you must also say that you have done something important." Schwartz's substantial contribution was application of the Fourier transform for the Sobolev spaces $W^{p,m}$ in the case $p = 2$. In this Hilbert case, the Parseval identity holds for the Fourier transform. This significantly simplified the involved proofs in the theory of general Sobolev spaces, which promoted the active development and popularization of generalized functions in linear problems and led to the introduction of other important spaces of generalized functions. In turn, general Sobolev spaces $W^{p,m}$, Sobolev–Slobodetskii spaces, and their generalizations are very important in the study of nonlinear problems, approximation problems, optimal control problems, etc. (see, in particular, Vishik (1961, 1963); Dubinskii (1968); Lions (1969); Ladyzhenskaya and Ural'tseva (1968); Ladyzhenskaya et al. (1988); Besov et al. (1978); Temam (1979); Pokhozhaev (1983); Vishik and Fursikov (1988); Babin and Vishik (1992); Fursikov (1999); Chepyzhov and Vishik (2001); Tikhomirov (1990); Arestov (1996); Besov (2013); Malykhin (2016); Vasil'eva (2020); Alimov and Tsar'kov (2021); Konyagin (2022)).

The contents of the book are reflected in the table of contents. In short, the book provides an interrelated presentation of a number of basic ideas, concepts, and results of the theory of generalized functions and equations of mathematical physics.

Chapter 1 introduces the reader to the initial elements of the language of generalized functions in the context of classical equations of mathematical physics (the Laplace equations, the heat equation, the string equation). Here we also present the basics of the theory of the Lebesgue integral and introduce the Riesz spaces of integrable functions. In the section devoted to the heat equation, the reader will be able to get acquainted with the method of dimension and similarity, which is usually not presented for mathematical students, and which is very useful at the initial stage of understanding the mathematical physics. Many useful results can be easily obtained using the concept of a δ-sequence on some or other space of test functions (see §2). In particular, in this way, we obtain the theorem on the inverse Fourier transform for functions from the Sobolev class $W^{1,n}(\mathbb{R}^n)$ (see §17).

Chapter 2 is devoted to elements of the theory of generalized functions in the sense of Schwartz (distributions in the sense of Schwartz). We believe that our presentation may be interesting also to specialists. In particular, in §16 we prove that the topology in the space $\mathcal{D}'$ of Schwartz distributions, which is usually postulated as something given from heaven, is actually uniquely determined from the following two natural requirements: first, the space $\mathcal{D}'$ must contain the space $\mathcal{D}^\flat$ of Sobolev derivatives (i.e., derivatives of finite order of locally integrable functions), and second, any linear functional on C_0^∞ with point support and which is continuous in this topology should be representable as a linear combination of the δ-function and its derivatives. (This linear combination is necessarily finite thanks to one important theorem of É. Borel, which is proved in §15.)

Sergei L'vovich Sobolev



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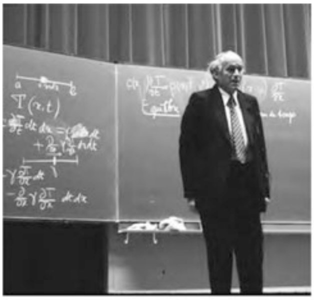

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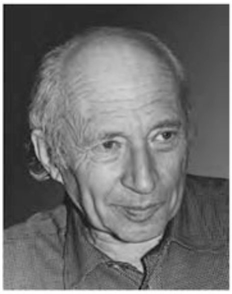

Israel Moiseevich Gelfand

Mark Iosifovich Vishik

Viktor Pavlovich Maslov

Sergei L'vovich Sobolev and Nikolai Maximovich Günther

Vasily Sergeyevich Vladimirov

Chapter 3 introduces the reader to some modern methods of studying the equations of mathematical physics that are related in one way or another related to the Fourier transform. Here we give the fundamentals of the theory of Sobolev spaces, the theory of pseudo-differential operators, and the theory of elliptic problems. We also give in §23 applications of these theories to the inverse problems of magneto-electroencephalography. In §24, we derive explicit numerically realizable formulas for solutions of elliptic (in particular, quasi-linear) equations with Cauchy data on analytic boundary. These formulas are based on Lemma 24.3 on a conformal and isometric transformation of an analytic closed curve to a circle. This lemma is also a key ingredient in construction of numerically realizable formulas for Poincaré–Steklov operators in §25. In §26, questions related to the Fourier–Hörmander integral operator and the canonical Maslov operator are discussed.

In recent years, Luzin's statement that the dispute over the concept of function is still ongoing has received a new confirmation. Once again, the incentive was mathematical physics. So, in the theory of superstrings and nanoscopic systems, the problem of construction of the theory of p-adic generalized functions was put forward (see, for example, Vladimirov (1988) and Volovich and Kozyrev (2009)). The actual problem of multiplication of generalized solutions of equations of mathematical physics was considered, in particular, in Livchak (1969); Colombeau (1985); Egorov (1990); Danilov et al. (1998); Oberguggenberger et al. (2003); Albeverio et al. (2005); Shelkovich (2008); Abreu et al. (2016); Shao and Huang (2017). One of the most important achievements here was Egorov (1990), and the main ideas and results of this paper are given in Appendix A. In Appendix B, A. B. Antonevich described the general method of construction of mnemonic function algebras.[3] In Appendix C, S. N. Samborski presents his very promising theory on the *extension* of the classical differentiation operator and nonlinear operators with partial derivatives. These three valuable appendices complete the presentation of the book.

A few remarks on the style of presentation are worth making. Part of the material is given in the "definition–theorem–proof" form, which is convenient for presentation of the results in a clear and concentrated form. However, it seems reasonable to allow the student not only to study a priori given ("from above") definitions and proofs of theorems, but also to open them when thinking about the questions that arise. A number of sections serve this purpose. In addition, some material is given in the form of exercises and problems, which are printed in petit and labelled by the letter P (with possible reference to "Parking for thinking and solving the problem"). So, sometimes the reading does require some effort. However, more difficult problems are provided with hints or references.

There are a lot of footnotes in the book, which is not too typical for mathematical literature, although there are some significant exceptions, for example, the book Anosov (2018). The importance of the footnotes can be partially explained by V. F. Dyachenko's humorous remark: "The most important thing should be written in footnotes. They are the only ones that are read." But speaking seriously, most of the footnotes are introduced so that when reading a book (which sometimes requires some effort), the reader can take a little break and feel the "faces" of familiar (and sometimes unfamiliar) names—authors of theorems, concepts, etc. (the stories of many of them are truly amazing and inspiring). And I believe that even the sophisticated reader will find something interesting and unknown here. For example,

(1) Why solutions to the Laplace equation are called harmonic functions?

(2) Why a transformation that Laplace did not even think about is named after him?

In addition, some details of the relationship between the great ones are often helpful in understanding the continuity of some ideas, and so the "picture" becomes more complete when it is clear who learned from whom.

I will also clarify the meaning of the term "generalized functions" used in the book—these are not only elements of the space $\mathcal{D}^\flat$ of Sobolev derivatives, but also elements of the space $\mathcal{D}'$ of (Schwartz) distributions or of the space $\mathcal{D}^\#(\Omega)$. By generalized functions, we will also mean Mikio Sato hyperfunctions and Antonevich

[3] Mnemonics in ancient Greek is the art of memorization. I believe that the term *mnemonic function*, which was introduced by A. B. Antonevich, reflecting the need to *remember* the method of approximation of a singularity by smooth functions, will be adopted in the scientific literature.

mnemonic function, etc. In short, this is all what develops the concept of a *function*, which N. N. Luzin prophetically wrote about.

The theorems, lemmas, definitions, exercises, etc. are numbered as $k.m$, where k is the section number and m is the number of the subsection in this section.

The present book *Equations of Mathematical Physics, Generalized Functions and Historical Notes* is a new substantially revised and significantly expanded version of the author's book *Generalized Functions in Mathematical Physics. Main Ideas and Concepts*, which was published in Russian in 2020 by Moscow Center for Continuous Mathematical Education. In turn, the 2020 edition is a reworked (extended and edited) version of the two author's books with the same title (the first edition (in Russian) was published by Moscow State University (1992) and its translation, by Nova Science Publishers, New York (2001)).

I am extremely grateful to Yu. V. Egorov, A. B. Antonevich, and S. N. Samborski, who kindly responded to my request to write their three significant appendices to the main text of the book. I deeply acknowledge the useful discussions with M. S. Agranovich, M. I. Vishik, A. I. Komech, S. V. Konyagin, V. P. Palamodov, V. M. Tikhomirov, and M. A. Shubin and their critical remarks that helped improve the text of the first and second editions (in 1972 and 2001). My special thanks go to D. E. Shcherbakov, who initiated the work on this revised edition and provided great assistance in preparing it for publication in 2020. I am also grateful to E. D. Kosov for his careful editing of the book, which made it possible to eliminate many typos and inaccuracies.

I am grateful to my good friend Alexey Alimov for the high-quality translation of this book. Just look at one masterpiece "with the directness of a Roman (!!!)," which vividly and bluntly characterizes Poincaré, who unexpectedly and bluntly credited the Laplace transform to Laplace, who, in the actual fact, did not even think about this transform. Many thanks also to Donna Chernyk, an Editor of *Mathematics* at Springer Nature, who supplied able assistance in matters pertaining to publication. The author is also grateful to Collections l'Ècole polytechnique—Palaiseau and Sofia Broström (the daugther of Lars Hörmander) for the photos provided.

All comments and questions about the book are welcome:
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Oka river near Tarusa

Tarusa, Russia A. S. Demidov
July 2022

Basic Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$—the sets of all natural, integer, and nonnegative integer numbers.

$\mathbb{R}$ and $\mathbb{C}$—the sets of all real and complex numbers $\mathbb{C} \ni z = \mathrm{Re} z + i \mathrm{Im} z$.

$\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$—the extended $\mathbb{R}$ and $\mathbb{C}$.

$X \times Y$—the Cartesian product of sets X and Y.

$X^1 = X, \ X^n = X^{n-1} \times X$.

i—the imaginary unit; $i^2 = -1$; $\mathring{\imath} = 2\pi i$—the imaginary 2π.

$x < y$, $x \le y$, $x > y$, $x \ge y$—the order relation in $\mathbb{R}$.

$a \gg 1$—a real number $a \in \mathbb{R}$ is sufficiently large.

$\{x \in X \colon P\}$—the set of elements from X satisfying condition P.

$]a, b] = \{x \in \mathbb{R} : a < x \le b\}$; $[a, b]$, $]a, b[$ and $[a, b[$ are defined similarly $]a, b]$.

$\{a_n\}$—a sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$.

$f \colon X \ni x \mapsto f(x) \in Y$—the mapping $f \colon X \to Y$ associating with an $x \in X$ its image $f(x) \in Y$. Sometimes, we will write $f(x)$ in place of f in cases where it is clear that f is meant rather than its value $f(x)$.

$A \Subset \Omega$— See Definition 3.10 on page 7.

1_A—the characteristic function of a set $A \subset X$; in other words, $1_A = 1$ in A and $1_A = 0$ outside A (i. e., in $X \setminus A$).

$\Rightarrow$—the necessity sign.

$\Longleftrightarrow$—if and only if.

$\stackrel{\mathrm{def}}{=}$—by definition.

$(x|y)$—the inner product of elements x and y in a Hilbert space.

Chapter 1
Introduction to Problems of Mathematical Physics

1 Temperature at a Point? No! In Volumes Contracting to the Point

Temperature. We know this word from our childhood. The temperature can be measured by a thermometer. This first impression concerning the temperature is, in a sense, nearer to the essence of matter than the representation of the temperature as a function of a point in space and time. Why? Because the concept of the temperature as a function of a point arose as an abstraction in connection with the conception of continuous medium. Actually, a physical parameter of the medium under consideration (for instance, its temperature) is first measured by a device in a "large" volume containing a fixed point ξ, after which it is measured using a device with better resolution in a smaller volume (containing the same point), and so on. As a result, we obtain a (finite) sequence of numbers $(a_1, \dots, a_M)$—the values of the physical parameter in the sequence of embedded volumes containing the point ξ. We next idealize the medium by assuming that the construction of such a numerical sequence is possible for an infinite nested system of volumes contracting to the point ξ. This gives us an infinite numerical sequence $\{a_m\}$. If we assume (this is the crux of the conception of the continuous medium[1]) that such a sequence exists and has a limit (which does not depend on the choice of the system of nested in domains), then this

[1] In some problems of mathematical physics, first of all in nonlinear ones, it is reasonable (see Appendices A and B) to consider a more general conception of the continuous medium in which a physical parameter (say, temperature, density, velocity,...) is characterized not by the values measured by one or another set of "devices," in other words, not by a functional of these "devices," but by a "convergent" sequence of such functionals, which define, as in nonstandard analysis (Zvonkin and Shubin 1984) (or infinitesimal analysis (Gordon et al. 2013)) a fine structure of a neighborhood of one or another point of the continuous medium. In this regard, we also note the paper Malyshev V.A. and Malyshev S.V. (2022), which develops a purely deterministic (without any stochastic) approach to derivation of macro-laws of continuum mechanics from the micro-laws of point particles. This reveals the presence of multiple scales in the transition to the continuum. For more on this subject, see also the journal "Structure of Mathematical Physics," Editor-in-Chief V.A. Malyshev (now late, former address: Moscow State University, Moscow, Russia).

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limit is considered as the value of the physical parameter (for instance, temperature) of the medium at the point ξ.

Thus, the concept of continuous medium occupying a domain[2] Ω assumes that the numerical characteristic f of a physical parameter considered in this domain Ω is a function in the usual sense: a mapping from the domain Ω into the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. Moreover, the function f is assumed to have the following property:

$$\langle f, \varphi_m^\xi \rangle = a_m, \quad m = 1, \dots, M. \tag{1.1}$$

Here a_m are the above numbers, and the left-hand side of (1.1), which is defined[3] by

$$\langle f, \varphi^\xi \rangle = \int f(x)\varphi^\xi(x)\, dx,$$

which is the "average" value of the function f measured near the point $\xi \in \Omega$ by using a "device," which we will denote by $\langle \cdot, \varphi^\xi \rangle$. The "resolving power" of the "device" depends on the "device (test) function" $\varphi^\xi : \Omega \longrightarrow \mathbb{R}$. This function is normalized as follows:

$$\int \varphi^\xi(x)\, dx = 1.$$

Let us note that more physical are "devices," in which φ^ξ has the form of a "cap" in the ρ-neighborhood of the point ξ, i.e., $\varphi^\xi(x) = \varphi(x - \xi)$ for $x \in \Omega$, where the function $\varphi \colon \mathbb{R}^n \ni x = (x_1, \dots, x_n) \mapsto \varphi(x) \in \mathbb{R}$ has the following properties:

$$\varphi \ge 0, \qquad \int \varphi = 1, \text{ and } \varphi(x) = 0 \quad \text{for } |x| \overset{\text{def}}{=} \sqrt{x_1^2 + \ldots + x_n^2} > \rho, \tag{1.2}$$

and $\rho \in]0, 1]$ is such that $\{x \in \Omega \mid |x - \xi| < \rho\} \subset \Omega$. It can be frequently assumed that the "device" measures f uniformly in the domain $\omega \in \Omega$. In this case, $\varphi = \frac{1_\omega}{|\omega|}$, where 1_ω is the characteristic function of the domain ω (i.e., $1_\omega = 1$ in ω, and $1_\omega = 0$ outside ω), and $|\omega|$ is the volume of the domain ω (i.e., $|\omega| = \int 1_\omega$). In particular, if $\Omega = \mathbb{R}^n$ and $\omega = \{x \in \mathbb{R}^n : |x| < \alpha\}$, then

$$\varphi(x) = \begin{cases} \dfrac{\alpha^{-n}}{|B_n|} & \text{for } |x| \le \alpha, \\ 0 & \text{for } |x| > \alpha, \end{cases} \tag{1.3}$$

where $|B_n|$ is the volume of the unit ball $B_n \overset{\text{def}}{=} \{x \in \mathbb{R}^n : |x| < 1\}$ in $\mathbb{R}^n$.

P 1.1 There is a popular phrase to remember the first 9 figures in π: "May, I have a large container of coffee beans." Indeed, writing down in succession the number of letters in this phrase, we get the first 9 figures in π: $\pi \approx 3.14159265$. The exact value of π is given by the well-known formula

[2] In what follows, unless otherwise stated, a domain Ω is an open connected subset of $\mathbb{R}^n$ with sufficiently smooth or piecewise smooth $(n-1)$-dimensional boundary $\partial\Omega$.

[3] Integration of a function g over a fixed (in this context) domain will be often written without indication of the domain of integration and sometimes simply as $\int g$.

$\pi = |B_2|$ (or by the formula $|B_3| = 4\pi/3$). Try to evaluate $|B_n|$ for $n > 3$ (this value will be required in what follows).

Hint It is clear that $|B_n| = \sigma_n/n$, where σ_n is the surface area of the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$, because

$$|B_n| = \int_0^1 r^{n-1}\sigma_n\, dr.$$

If the calculation of σ_n for $n > 3$ seems to be difficult or uninteresting, consider the following short and unexpectedly beautiful solution.

Solution We have

$$\left(\int_{-\infty}^{\infty} e^{-t^2}\, dt\right)^n = \int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \int_0^{\infty} e^{-r^2} r^{n-1}\sigma_n\, dr = \frac{\sigma_n}{2}\Gamma\left(\frac{n}{2}\right), \tag{1.4}$$

where Γ is the *Euler function* defined for $\operatorname{Re}\lambda > 0$ by

$$\Gamma(\lambda) = \int_0^{\infty} t^{\lambda-1} e^{-t}\, dt. \tag{1.5}$$

For $n = 2$, the right-hand side of (1.4) is π. Hence

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}. \tag{1.6}$$

Thus, $\sigma_n = 2\pi^{n/2}\Gamma^{-1}(n/2)$. Taking $n = 3$, we obtain $2\Gamma(3/2) = \sqrt{\pi}$. Using the well-known formula $\Gamma(\lambda + 1) = \lambda \cdot \Gamma(\lambda)$ (which can be derived from (1.5) by integration by parts), from which it follows that $\Gamma(n + 1) = n!$, we find that $\Gamma(1/2) = \sqrt{\pi}$. As a result,

$$\sigma_{2k} = \frac{2\pi^k}{(k-1)!}, \quad \sigma_{2k+1} = \frac{2\pi^k}{\left(k-\frac{1}{2}\right)\cdot\left(k-\frac{3}{2}\right)\cdot \ldots \cdot \frac{3}{2}\cdot\frac{1}{2}}. \tag{1.7}$$

2 The δ-Sequence and the δ-Function

In the preceding section, the idea was indicated that the definition of a function $f : \Omega \to \mathbb{R}$ (or of a function $f : \Omega \to \mathbb{C}$) as a mapping from a domain $\Omega \subset \mathbb{R}^n$ in $\mathbb{R}$ (or in $\mathbb{C}$) is equivalent to evaluation of its "average" values

$$\langle f, \varphi\rangle = \int_{\Omega} f(x)\varphi(x)\, dx, \quad \varphi \in \Phi, \tag{2.1}$$

where Φ is a sufficiently "rich" set of functions on Ω. A fairly general result in this regard is given in § 10. Here, we prove a simple but useful lemma involving a domain $\Omega \subset \mathbb{R}^n$, a point $\xi \in \Omega$, and a function $\delta_\varepsilon : \mathbb{R}^n \ni x \mapsto \delta_\varepsilon(x) \geq 0$, satisfying the condition

$$\int_{\mathbb{R}^n} \delta_\varepsilon(x)\, dx = \lim_{\sigma\to 0}\int_{|x-\xi|\leq\sigma} \delta_\varepsilon(x-\xi)\, dx = 1, \tag{2.2}$$

where $\varepsilon = \sigma$ or $\varepsilon = \frac{1}{\sigma}$. If $\varepsilon = \sigma$ (see Fig. 1.1), then as δ_ε one can take, for example, the function

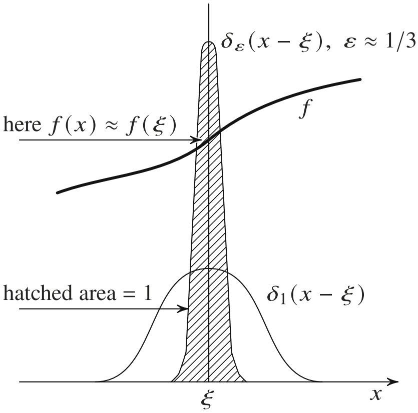

Fig. 1.1: δ-function via a δ-shaped sequence

$$x \mapsto \delta_\varepsilon(x) = \frac{\varphi(x/\varepsilon)}{\varepsilon^n},$$

where $\varphi \geq 0$, $\varphi = 0$ outside $B_n = \{|x| < 1\}$, and $\int \varphi = 1$.

Lemma 2.1 *Let* $f \in C(\Omega)$. *Then*

$$f(\xi) = \lim_{\sigma \to 0} \int_\Omega f(x)\delta_\varepsilon(x-\xi)\,dx, \quad \xi \in \Omega, \quad \varepsilon = \sigma \quad \text{or} \quad \varepsilon = 1/\sigma, \tag{2.3}$$

i.e., the value $f(\xi)$ *of a continuous function* f *can be recovered from the family of "average values"*

$$\left\{\int f(x)\delta_\varepsilon(x-\xi)\,dx\right\}_{0<\sigma\ll 1}.$$

Proof For any $\eta > 0$, there exists $\sigma > 0$ such that $|f(x) - f(\xi)| \leq \eta$ if $|x - \xi| \leq \sigma$. Hence the absolute value of the expression

$$\left(\lim_{\sigma\to 0}\int_\Omega f(x)\delta_\varepsilon(x-\xi)\,dx\right) - f(\xi) = \lim_{\sigma\to 0}\int_\Omega (f(x) - f(\xi))\delta_\varepsilon(x-\xi)\,dx$$

is majorized by η, because $\lim_{\sigma\to 0}\int_{|x-\xi|\leq\sigma}\delta_\varepsilon(x-\xi)\,dx = 1$, and

$$\lim_{\sigma\to 0}\int_{|x-\xi|\leq\sigma}|f(x)-f(\xi)|\delta_\varepsilon(x-\xi)\,dx \leq \eta \lim_{\sigma\to 0}\int_{|x-\xi|\leq\sigma}\delta_\varepsilon(x-\xi)\,dx.$$

Definition 2.2 Let Φ be a subspace in the space $C(\Omega)$, and let $\xi \in \Omega$. Also let $\nu \to \nu_0 \in \overline{\mathbb{R}}$. Consider a sequence of functions $\delta_\nu \colon x \mapsto \delta_\nu(x-\xi)$ such that

$$f(\xi) = \lim_{\nu\to\nu_0}\int_\Omega f(x)\delta_\nu(x-\xi)\,dx, \quad \xi \in \Omega, \tag{2.4}$$

for any function $f \in C(\Omega)$ (for any $f \in \Phi$). Then the mapping $\nu \mapsto \delta_\nu(x-\xi)$, where $x \in \Omega$, is called a *δ-sequence* (on the space Φ) concentrated near the point ξ. The last words are usually skipped.

In § 4, we will give example of δ-sequences on certain subspaces $\Phi \subset C(\Omega)$, from which some important results will be obtained. Some examples of such subspaces Φ will be given in § 3.

Definition 2.3 The *δ-function* in other words the *Dirac function*[4] concentrated at a point ξ is defined on $C(\Omega)$ by a (linear) space of functions is a (linear) mapping from this function space into a real line.

$$\delta_\xi \colon C(\Omega) \ni f \mapsto f(\xi) \in \mathbb{R} \quad (\text{or } \mathbb{C}), \quad \xi \in \Omega, \tag{2.5}$$

The following notation for the δ-function defined by (2.5) is also used: $\delta(x-\xi)$, and its evaluation on a function $f \in C(\Omega)$ is denoted by

$$\langle f(x), \delta(x-\xi)\rangle = f(\xi) \quad \text{or} \quad \langle \delta(x-\xi), f(x)\rangle = f(\xi),$$

or by

$$\langle f, \delta_\xi\rangle = f(\xi) \quad \text{or} \quad \langle \delta_\xi, f\rangle = f(\xi).$$

The Dirac function can be interpreted as a measuring instrument at a point (a "thermometer" measuring the "temperature" at a point) or as a point source (see, for example, Demidov et al. (2016). If $\xi = 0$, then one simply writes δ or $\delta(x)$.

Definition 2.4 Let $\Gamma = \bigcup_{k=0}^{n-1} \Gamma_k$ be a subset of $\mathbb{R}^n$, where Γ_k is either the empty set or a piecewise-smooth surface of dimension k (for $k = 0$ this is a finite number of points x_j, and for $k = 1$ this is a finite number of disjoint curves). The linear functional

$$\delta\big|_\Gamma \colon C(\mathbb{R}^n) \ni f \mapsto \int_\Gamma f(\xi)\, d\xi = \sum_{k=0}^{n-1} \int_{\Gamma_k} f(\xi)\, d\xi, \tag{2.6}$$

where $\int_{\Gamma_0} f(\xi)\, d\xi = \sum_j f(x_j)$, is the *$\delta$-function concentrated on* Γ.

This concept is used in many problems of mathematical physics (see, for example, Demidov (1975b, 1977, 1978a,b, 2000, 2002, 2004, 2006, 2010a,b); Demidov ane Badjadi (1983); Demidov and Moussaoui (2004); Demidov et al. (1996, 2012, 2013, 2016); Bezrodnykh and Demidov (2011)).

P 2.5 Let ω be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary γ of length $|\gamma|$. Let

$$f_\varepsilon(x) = \frac{1}{\varepsilon|\gamma|}\chi(x)e^{-r(x)/\varepsilon}, \quad x \in \omega, \quad 1/\varepsilon \gg 1,$$

[4] Paul Dirac (1902–1984) was the English theoretical physicist, one of the founders of quantum mechanics, laureate of the Nobel Prize in physics in 1933. In his paper "The quantum theory of the emission and absorption of radiation" (Dirac 1927), the concept of the δ-function played an important role (note that this important function was used earlier "behind the scenes"; see footnote 69 on p. 50).

where $r(x)$ is the distance from a point $x \in \omega$ to γ. Suppose that $\chi \in C^\infty(\omega)$, $\chi = 1$ near γ and $\chi = 0$ outside some large neighborhood of γ. Verify that

(1) For any $r_0 > 0, \eta > 0$ there exists $\varepsilon_0 > 0$ such that $f_\varepsilon(x) \le \eta$ for $r(x) \ge r_0$ and $1/\varepsilon \ge 1/\varepsilon_0$.

(2) $\lim_{\varepsilon \to 0} \int_\omega f_\varepsilon(x)\,dx = 1$,

and then prove that $f_\varepsilon(x) \to \delta\big|_\gamma$ as $\varepsilon \to 0$.

3 Some Spaces of Smooth Functions: Partition of Unity

The spaces of smooth functions, which will be introduced in this section, play a very important role in the analysis. In particular, they give examples of the space Φ in the "averaging" formula (2.1).

Definition 3.1 Let Ω be an open set in $\mathbb{R}^n$ and $\overline{\Omega}$ be the closure of Ω in $\mathbb{R}^n$, and let $m \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$, i.e., m is an integer nonnegative number.[5]

3.2 $C^m(\Omega)$ is the *space* of *functions* $\varphi\colon \Omega \to \mathbb{C}$ which are m times *continuously differentiable*; in other words, $\partial^\alpha \varphi$ is continuous in Ω for $|\alpha| \le m$. Here and in what follows,

$$\partial^\alpha \varphi(x) = \frac{\partial^{|\alpha|}\varphi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n, \quad \alpha_j \in \mathbb{Z}_+.$$

The vector $\alpha = (\alpha_1, \dots, \alpha_n)$ is called a *multiindex*.

By $C_b^m(\Omega)$, we denote the subspace of bounded functions in $C^m(\Omega)$.

3.3 $C^m(\overline{\Omega}) = C^m(\mathbb{R}^n)|_\Omega$, i.e.,[6] $C^m(\overline{\Omega})$ is the *restriction of the space* $C^m(\mathbb{R}^n)$ to Ω. In other words, $\varphi \in C^m(\overline{\Omega})$ means that there exists a function $\psi \in C^m(\mathbb{R}^n)$ such that $\varphi(x) = \psi(x)$ for $x \in \Omega$.

3.4 $PC^m(\Omega)$ is the *space of functions m times piecewise continuously differentiable in* Ω; this means that $\varphi \in PC^m(\Omega)$ if and only if the following two conditions are satisfied. First, $\varphi \in C^m(\Omega \setminus K_0)$ for some compact set[7] $K_0 \subset \Omega$. Second, for any compact set $K \subset \overline{\Omega}$, there exists a finite number of domains $\Omega_j \subset \Omega$, $j = 1, \dots, N$, each of which is the intersection of a finite number of domains with smooth boundary, such that $K \subset \bigcup_{j=1}^N \overline{\Omega}_j$ and $\varphi|_\omega \in C^m(\overline{\omega})$ for any connected component ω of the set

$$\Big(\bigcup_{j=1}^N \Omega_j\Big) \setminus \Big(\bigcup_{j=1}^N \partial\Omega_j\Big).$$

[5] These definitions of spaces of functions defined on a domain Ω can be obviously extended to the case when the functions are defined on a (smooth or piecewise smooth) boundary of the domain. In the notation of the spaces defined below, the index $m = 0$ is usually omitted.

[6] The space $C^m(\overline{\Omega})$ is in general different from the *space of functions m times continuously differentiable up to the boundary*. However, these spaces are equal if the boundary of the domain is sufficiently smooth.

[7] A set $K \subset \mathbb{R}^n$ is a *compact set* (or is *compact*) if and only if it is bounded and closed.

By $PC_b^m(\Omega)$ we denote the subspace of bounded functions in $PC^m(\Omega)$.

3.5 The *support* $\operatorname{supp}\varphi$ of a function $\varphi \in C(\Omega)$ is the smallest closed in Ω set outside of which the function φ is zero. In other words, $\operatorname{supp}\varphi$ is the closure in Ω of the set $\{x \in \Omega\colon\ \varphi(x) \neq 0\}$.

3.6 $C_0^m(\overline{\Omega}) = \{\varphi \in C^m(\overline{\Omega})\colon\ \operatorname{supp}\varphi \text{ is a compact set}\}$.

3.7 $C_0^m(\Omega) = \{\varphi \in C_0^m(\overline{\Omega})\colon\ \operatorname{supp}\varphi \subset \Omega\}$.

3.8 The intersection with respect to m of the above function spaces is denoted by the same symbol with m replaced by ∞. In particular, $C^\infty(\Omega) = \bigcap_m C^m(\Omega)$, $C^\infty(\overline{\Omega}) = \bigcap_m C^m(\overline{\Omega})$, $C_0^\infty(\Omega) = \bigcap_m C_0^m(\Omega)$.

3.9 If $\varphi \in C_0^m(\Omega)$ (or $\varphi \in C_0^\infty(\Omega)$) and $\operatorname{supp}\varphi \subset \omega$, where ω is a subdomain of the domain Ω, then the function φ is identified with its restriction to ω. In this case, we write $\varphi \in C_0^m(\omega)$ (or $\varphi \in C_0^\infty(\omega)$).

Definition 3.10 A set A is *compactly embedded* in Ω if its closure $\overline{A}$ is compact and $\overline{A} \subset \Omega$. In this case, we write $A \Subset \Omega$.

It is clear that $C_0^m(\Omega) = \{\varphi \in C^m(\Omega)\colon\ \operatorname{supp}\varphi \Subset \Omega\}$, and

$$C_0^m(\Omega) \subsetneq C_0^m(\overline{\Omega}) \subsetneq C^m(\overline{\Omega}) \subsetneq C^m(\Omega) \subsetneq PC^m(\Omega),$$

where the first and third inclusions become equalities if $\Omega = \mathbb{R}^n$.

Example 3.11 It is clear that the function

$$\varphi_m\colon \mathbb{R} \ni x \mapsto \varphi_m(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^{m+1} & \text{for } x > 0 \end{cases}$$

lies in $C^m(\mathbb{R})$. Next, $\lim_{x\downarrow 0} x^{m+1}\, e^{-\frac{1}{x}} = 0$, and hence the function

$$\varphi_\infty\colon \mathbb{R} \ni x \mapsto \varphi_\infty(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ e^{-\frac{1}{x}} & \text{for } x > 0 \end{cases}$$

lies in $C^\infty(\mathbb{R})$.

Example 3.12 The function

$$\varphi\colon \mathbb{R}^n \ni x \mapsto \varphi(x) = \begin{cases} \exp\Big(\dfrac{2}{(|x|^2 - 1)}\Big) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

lies in $C_0^\infty(\mathbb{R}^n)$ *qua* the product of the functions $\varphi_\infty(1 + |x|)$ and $\varphi_\infty(1 - |x|)$.

Example 3.13 Let $\varepsilon > 0$. We set

$$\varphi_\varepsilon(x) = \varphi(x/\varepsilon), \tag{3.1}$$

where φ is the function from Example 3.12. Then the function

$$\delta_\varepsilon \colon \mathbb{R}^n \ni x \mapsto \frac{\varphi_\varepsilon(x)}{\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\,dx} \tag{3.2}$$

lies in $C_0^\infty(\mathbb{R}^n)$, and besides

$$\begin{gathered}\delta_\varepsilon(x) \geq 0 \text{ for any } x \in \mathbb{R}^n,\\ \delta_\varepsilon(x) = 0 \quad \text{for } |x| > \varepsilon, \quad \text{and} \quad \int_{\mathbb{R}^n} \delta_\varepsilon(x)\,dx = 1.\end{gathered} \tag{3.3}$$

Example 3.14 Let $g(t) = \delta_\varepsilon(t+1+\varepsilon) - \delta_\varepsilon(t-1-\varepsilon)$, where δ_ε satisfies conditions (3.3). Then (see Figs. 1.2 and 1.3)

$$C_0^\infty(\mathbb{R}) \ni \varphi \colon t \mapsto \varphi(t) = \int_{-\infty}^t g(\tau)\,d\tau, \quad 0 \leq \varphi \leq 1,$$

and $\varphi(t) = 1$ for $|t| \leq 1$.

Example 3.15 Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$ be the Euclidean coordinates of a point $x \in \mathbb{R}^n$. For φ from Example 3.14, we set

$$\begin{gathered}\psi_\nu(x) = \sum_{|k|=\nu} \varphi(x_1 + 2k_1) \cdot \ldots \cdot \varphi(x_n + 2k_n),\\ k_j \in \mathbb{Z}, \quad |k| = |k_1| + \ldots + |k_n|.\end{gathered}$$

Then the family $\{\varphi_\nu\}_{\nu=0}^\infty$ of functions

$$\varphi_\nu(x) = \psi_\nu(x) \Big/ \Big(\sum_{\nu=0}^\infty \psi_\nu(x)\Big)$$

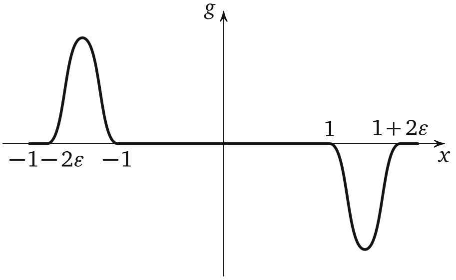

Fig. 1.2: The function $g = d\varphi/dx$, $\varphi \in C_0^\infty$

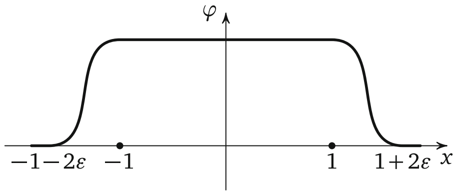

Fig. 1.3: The function $\varphi = \int g$ satisfying (3.3)

forms a *partition of unity* in $\Omega = \mathbb{R}^n$, i.e., $\varphi_\nu \in C_0^\infty(\Omega)$, and moreover,

(1) For any compact set $K \subset \Omega$, only a finite number of functions φ_ν is nonzero in K.

(2) $0 \le \varphi_\nu(x) \le 1$ and $\sum_\nu \varphi_\nu(x) = 1$ for all $x \in \Omega$.

Proposition 3.16 *For any domain $\omega \Subset \Omega$, there exists a function $\varphi \in C_0^\infty(\Omega)$ such that* $0 \le \varphi \le 1$ *and* $\varphi(x) = 1$ *for* $x \in \omega$.

Proof Let $\varepsilon > 0$ be such that 3ε is smaller than the distance from ω to $\partial\Omega = \overline{\Omega} \setminus \Omega$. Let ω_ε be the ε-neighborhood of ω. Then the function

$$x \mapsto \varphi(x) = \int_{\omega_\varepsilon} \delta_\varepsilon(x-y)\,dy, \quad x \in \Omega,$$

has the required properties, where the function δ_ε is defined by (3.2). □

Definition 3.17 Given a domain Ω, let $\Omega = \bigcup \Omega_\nu$, where $\Omega_\nu \Subset \Omega$. If each compact set $K \Subset \Omega$ has nonempty intersection only with a finite number of domains Ω_ν, then by definition the family $\{\Omega_\nu\}$ forms a *locally finite cover* of the domain Ω.

Theorem 3.18 (On Partition of Unity) *Let $\{\Omega_\nu\}$ be a locally finite cover of Ω. Then there exists a partition of unity subordinate to a locally finite cover, i.e., there exists a family of functions $\varphi_\nu \in C_0^\infty(\Omega_\nu)$ satisfying conditions* 1–2 *of Example* 3.15.

The proof is quite clear (see, for example, Vladimirov (1994, p. 19)). The *partition of unity* is a very common and convenient tool with the help of which some problems for the whole domain Ω can be reduced to problems for subdomains covering Ω (see, in particular, §11, 20, 22, and 26).

4 Examples of δ-Sequences

The examples in this section are given in the form of exercises. Exercise 4.1 will be used below in the derivation of the *Poisson formula* for the solution of the Laplace equation (see §5), and Exercise 4.2 will be used in the derivation of the *Poisson formula* for the solution of the heat equation (see §6). Exercise 4.3 will be used in the proof of the theorem on the inversion of the Fourier transform (see §17), and using Exercise 4.4, one can easily verify (see §19) the famous Weierstrass theorem on approximation of continuous functions by polynomials.

P 4.1 Verify that the sequence $\{\delta_y\}_{y\to+0}$ of functions

$$\delta_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2},$$

where $x \in \mathbb{R}$, is a δ-*sequence* on the space $C_b(\mathbb{R})$ (see §3.2) but is not a δ-sequence on $C(\mathbb{R})$.

P 4.2 Verify that the sequence $\{\delta_t\}_{t\to+0}$ of functions

$$\delta_t(x) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)},$$

where $x \in \mathbb{R}$, is not a δ-sequence on $C(\mathbb{R})$, but is a δ-sequence on the space $\Phi \subset C(\mathbb{R})$ of functions such that, for any $\varphi \in \Phi$, there exists an $a > 0$ such that $|\varphi(x)\exp(-ax^2)| \to 0$ as $|x| \to \infty$.

P 4.3 Verify that the sequence $\{\delta_\nu\}_{\nu\to\infty}$ of functions

$$\delta_\nu : \mathbb{R} \ni x \mapsto \delta_\nu(x) = \frac{\sin \nu x}{\pi x}$$

is a δ-sequence on the space of functions φ such that

$$\int_{\mathbb{R}} |\varphi(x)|\,dx < \infty, \quad \int_{\mathbb{R}} |\varphi'(x)|\,dx < \infty.$$

P 4.4 Considering the polynomials $\delta_k(x) = \frac{k}{\sqrt{\pi}}\big(1 - \frac{x^2}{k}\big)^{k^3}$, where $x \in \mathbb{R}$ and $k \in \mathbb{N}$, show that the sequence $\{\delta_k\}_{k\to\infty}$ is a δ-sequence on the space $C_0(\mathbb{R})$ but is not a δ-sequence on $C_b(\mathbb{R})$ (cf. Exercise 4.1).

Remark 4.5 When solving 4.1–4.4, it is useful to draw sketches of graphs of the corresponding functions. Exercises 4.1 and 4.2 are simple, while Exercises 4.3 and 4.4 are more difficult, because the corresponding functions are alternating. In § 13, we will prove Lemma 13.11, using which one can easily solve Exercises 4.3 and 4.4. For Exercises 4.2–4.4, one should use the well-known equalities

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy \overset{(1.6)}{=} \sqrt{\pi}, \quad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi, \quad \lim_{\nu\to\infty}(1 - a/\nu)^\nu = e^{-a}.$$

It seems that the simplest proof of the second equality here is as follows. Setting

$$\frac{1}{x} = \int_0^\infty e^{-xy}\,dy,$$

we have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \Big(\int_0^\infty \sin x e^{-xy}\,dx\Big)\,dy.$$

By twice integrating by parts, we see that the inner integral is $\frac{1}{1+y^2}$. As a result,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{1}{1+y^2}\,dy = \arctan x\Big|_0^\infty = \frac{\pi}{2}.$$

5 On the Laplace Equation

In analogy to the title "Three pearls of number theory" of the well-known book Khinchin (1998),[8] the three classical equations in partial derivatives—the Laplace equation, the heat equation, and the string equation—can be named the *three pearls of mathematical physics*. One of these pearls was found by Laplace,[9] when he was analyzing the Newton gravitation law.[10]

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[8] Alexander Yakovlevich Khinchin (1894–1959) was the corresponding member of the USSR Academy of Sciences (1939), one of the founders of modern probability theory.

[9] Pierre-Simon Laplace (1749–1827), was a French mathematician, physicist, and mechanician. In 1773, he proved that the orbits of planets are stable. For this paper, Laplace was elected as an associate member of French Academy of Sciences at age 24. In 1785, he was elected as Member of French Academy of Sciences. In the same year, 1785, in one of the exams to the military school, Laplace highly appreciates the knowledge of the 17-year-old applicant Bonaparte. Laplace and Bonaparte always maintained warm relationship. In 1799, Laplace published first two volumes of his classical "Celestial Mechanics," in which, in particular, he concluded that the Saturnian rings cannot be continuous, for otherwise it would be unstable. (On Saturnian rings, see the recent fundamental studies (Zelikin 2015, 2019)). Laplace was one of the founders of probability theory. In his "Théorie analytique des probabilités" (1812), he introduced addition and multiplication of probabilities and defined the expectation. He also used the *generating function* $z \mapsto G(z) = \sum_{n\geq 0} a_n z^n$ corresponding to the sequence $\{a_n\}_{n\geq 0}$—this concept was repeatedly used by Euler since 1741 in his works on number theory. (The use of generating functions for the Bessel function was proved instrumental in obtaining the efficient formula (21.8) and the unexpected formula (21.11) for the eigenfunction of the two-dimensional Fourier transform). A simple algebra shows that to the sequence $\{e^{-ns} a_{n+1}/s\}_{n\geq 0}$, where $s > 0$, there corresponds the generating function defined by $\int_0^\infty e^{-sx} f(x)\,dx$, $f(x) = \sum_{0\leq x<n} a_n$, i.e., the so-called Laplace transform (of which Laplace did not even think about; see Remark 18.1 below).

It should be noted that Laplace proved an important limit theorem—the so-called de Moivre–Laplace theorem (a particular case of which was presented in the first book on probability written by de Moivre (1667–1754), a student and assistant of Newton). Laplace also published papers on capillarity theory (1806), continued Newton's studies on the velocity of sound in vacuum, gave a barometric formula for evaluation of the air density as a function of altitude above the Earth surface. In addition, Laplace was engaged with problems of electrodynamics and developed mathematical foundations of potential theory. Laplace's name was included in the list of 72 greatest scientists of France, placed on the ground floor of the Eiffel Tower. For more on Laplace personality and works, see, for example, Bell (1937).

[10] This is what Albert Einstein (see Zu Isaak Newton 200. Todestage. Nord und Süd, 1927, 50, 36–40.) writes on the Newton gravitation law: "But how could Newton find the forces acting on celestial bodies? It is clear that a correct expression for these forces cannot be merely extemporize. He had no choice but to act in the reverse order and find these forces from the known motions of the planets and the Moon. Having obtained these motions, he evaluated the acceleration, and knowing them, he was able to find the forces. He did all this, being a 23-year-old young man and being in a village solitude." Newton also obtained a formula for the gravity force between celestial bodies and postulated it as a working conjecture for any bodies. This is a general physical law derived from empirical observations by what Isaac Newton called inductive reasoning. The first test of Newton's theory of gravitation between masses in the laboratory was the Cavendish experiment conducted by the British scientist Henry Cavendish in 1798 (approximately 71 years after Newton's death). Newton gave credit in his Principia to two people: Bullialdus (who wrote without proof that there was a force on the Earth towards the Sun) and Borelli (who wrote that all planets were attracted towards the Sun). In 1692, in his third letter to Bentley, he wrote "That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through

According to Godunov (1979, §1), Laplace proposed to get rid of the obvious formula for the gravity force between bodies and replace it by the differential equation $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0$ satisfied by the potential u of the gravity force $\mathbf{F} = -\nabla u$ everywhere outside these bodies.

Definition 5.1 A function $u \in C^2(\Omega)$ is called *harmonic*[11] on an open set $\Omega \subset \mathbb{R}^n$ if it satisfies in Ω the *Laplace equation* $\Delta u = 0$, where by the Greek letter Δ ("delta") we denote the *Laplace operator*[12] or the *Laplacian* defined by the formula

$$\Delta u(x) \stackrel{\text{def}}{=} \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}, \tag{5.1}$$

where $x_1, \ldots, x_n$ are the Euclidean coordinates of a point $x \in \Omega \subset \mathbb{R}^n$.

Laplace's idea mentioned by S. K Godunov turned out to be very fruitful and gave a great impetus to the development of mathematical physics and, in particular, of electrostatics and magnetism. This is why the equation $\Delta u = 0$, which holds for the potential of the gravity forces outside the points at which they are concentrated, is rightfully called after Laplace. It is also worth pointing out that already in 1757 Euler[13] in his memoir "General motion laws of liquids" derives the Laplace equation $\Delta u = 0$ for the potential u of the velocity $\mathbf{V} = \nabla u$ of vortex-free flow of inviscid liquid, i.e., $\operatorname{div} \mathbf{V} = 0$, where the operators ∇ (gradient) and div(divergence) were introduced by Hamilton (see footnote 81 on page 68). And even earlier, in 1752,

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which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it." How does this mysterious force manifest itself outside bodies? The question here was given by Laplace.

A certain disappointment here is that Laplace, who considered himself the best mathematician of France, did not refer to the brilliant and modest Lagrange (see footnote 36 on the page 31), who, among many important achievements, had also shown that the gravity force is potential.

[11] Stefan Nemirovski (a corresponding member of the Russian Academy of Science) kindly informed me about the website https://math.stackexchange.com/a/4300729, which says that the term harmonic function was first used (in the sense of Definition 5.1) by Poincaré in his memoir "Sur les équations de la physique mathématique" (1894) https://link.springer.com/content/pdf/10.1007/BF03012493.pdf. In this memoir, Poincaré calls (see pp. 87 and p. 153) *harmonic* the functions which vanish on the boundary of the domain and satisfy either the Helmholtz equation $\Delta u + k^2 u = 0$ or the Laplace equation in the domain, or the biharmonic equation). However, even earlier in the paper Dynamical Problems regarding Elastic Spheroidal Shells and Spheroids of Incompressible Liquid" (1863), William Thomson called the solutions of equation (5.1) in the three-dimensional case spherical harmonics. It is clear why spherical, but why harmonics? In this regard, I think that the answer comes from the ancient and medieval teachings about the musical and mathematical structure of the cosmos (the harmony of spheres (luminaries)). Cf. Aristotle "De Caelo" ("On the Heavens"): ". . . the theory that the movement of the stars produces a harmony, i.e., that the sounds they make are concordant, in spite of the grace and originality with which it has been stated, is nevertheless untrue" (The Complete Works of Aristotle, Princeton, New Jersey, 1984).

[12] An *operator* is a mapping $f : X \to Y$, where X and Y are function spaces.

[13] Leonhard Euler (1707–1783) was a Swiss, Prussian, and Russian mathematician and mechanician, one of the greatest mathematicians of all times.

the Laplace equation appears in the D'Alembert's studies[14] and in 1755 in Euler's studies related to the so-called Cauchy–Riemann conditions[15]

In the next section, we will see that in the absence of internal sources or sinks of heat in a body, the steady temperature in it is a harmonic function.[16] It is clear that such a steady temperature depends on the thermal regime at the boundary of the body. An important example of such regimes is given by the so-called Dirichlet conditions,[17] when a fixed temperature is maintained at the boundary of the body (possibly depending on the boundary point). The corresponding problem

$$\Delta u = 0 \quad \text{in} \;\; \Omega, \qquad u = f \quad \text{on} \;\; \partial\Omega, \tag{5.2}$$

is known as the *Dirichlet problem for the Laplace equation* (in the domain Ω). This problem, which is also called the *first boundary-value problem*, has various interpretations. Below, we shall also deal with other important boundary-value problems: the *second boundary-value problem* $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = f$, with the derivative along the normal ν to the boundary (the Neumann problem; see footnote 31 on p. 125), and the *third boundary-value problem* with the condition $au + b\frac{\partial u}{\partial \nu} = f$, where $ab \geq 0$, $a + b \neq 0$,

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[14] Jean D'Alembert (1717–1783) was a French polymath, widely known as a philosopher, mathematician, and mechanician.

[15] Augustin-Louis Cauchy (1789–1857) was a French mathematician and mechanician. He made remarkable contributions to analysis, algebra, mathematical physics, and many other areas of mathematics, one of the founders of continuum mechanics. His name is included in the list of the greatest scientists in France.
Bernhard Riemann (1826–1866) was a German mathematician, mechanicist, and physicist. During his short life, he laid the foundation for the geometric direction of the theory of analytic functions; he introduced the so-called Riemann surfaces, developed the theory of conformal maps, and in this connection gave the basic ideas of topology. Riemann found the relation between the distribution of primes and the properties of the zeta function, in particular, with the distribution of its zeros in the complex domain. The corresponding Riemann Hypothesis (one of the seven so-called Millennium Problems selected by the Clay Mathematics Institute in 2000) states that all non-trivial zeros of the zeta function have a real part equal to 1/2. In a number of works, he investigated the problem of expansion of functions into trigonometric series and, in this connection, determined necessary and sufficient conditions for integrability in the Riemannian sense, which was important for the theory of sets and functions of a real variable. The so-called Riemannian geometry he created was the precursor of the general relativity theory.

[16] If in a body Ω, there is an internal source or sink of heat $f : \Omega \ni x \mapsto f(x)$, then the steady temperature u in it satisfies the equation $\Delta u = f$, known as the *Poisson equation* (see footnote 19 on p. 16).

[17] Johann Peter Gustav Lejeune Dirichlet (1805–1859) was a German mathematician. His ancestors were natives of the Belgian town of Richelet (De Richelet, in French), which explains the origin of the unusual surname for the German language. Part of it, namely "Lejeune" (the young man (Le Jeune, in French)), has a similar origin—the grandfather was called "a young man from Richelieu." In 1855, Dirichlet became, as a successor of F. Gauss, a professor of higher mathematics at the University of Göttingen. Novel Dirichlet's works mainly deal with number theory, series theory, integral calculus, and some problems of mathematical physics

In 1831, Dirichlet married the sister of the famous composer and conductor Jakob Ludwig Felix Mendelssohn Bartholdy. Dirichlet died of a heart attack a few months after the death of his wife.

which is frequently called the Robin problem in honor of the French mathematician Victor Robin (1855–1897) and also called the Newton problem.[18]

Of particular importance is also the Cauchy problem for elliptic equations (including the Laplace equation), which appear in the study of important applied problems. The explicit numerically realizable formulas for solving these problems are presented in §24. They can serve as a test for construction of numerical algorithms for solution of ill-posed problems.

P 5.2 It can be shown that if a function f in problem (5.2) is constant and Ω is a ball or a circular cylinder, then u (and the domain itself) is completely characterized by the polar radius, i.e., $u(x) = v(\rho)$, where

$$\rho = \sqrt{x_1^2 + \ldots + x_k^2},$$

$k = 3$ in the case of a ball and $k = 2$ in the case of a circular cylinder. Verify that Δu also depends on ρ, and besides,

$$\Delta u = \frac{\partial^2 v(\rho)}{\partial \rho^2} + \frac{k-1}{\rho}\frac{\partial v(\rho)}{\partial \rho}. \tag{5.3}$$

Remark 5.3 Consideration of problem (5.2) in model domains such as a ball, a circular cylinder, or a half-space can be useful in the analysis of this problem in quite general domains.

Consider the simplest case when Ω is the half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Let us find a function $u \in C^2(\mathbb{R}^2_+)$ that satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ in $\mathbb{R}^2_+$ and the *boundary* (or, as some people put it, the *boundary-value*) *condition*

$$\lim_{y \to +0} u(x, y) = f(x), \tag{5.4}$$

where x is a point of continuity of a piecewise continuous bounded(!) function f.

The required function u is a harmonic function of two variables. Such functions are closely related to *analytic functions* of a single complex variable, i.e., to functions

[18] This boundary operator, bearing the name of V. Robin, arose much earlier in I. Newton's works (where he studied the process of heat transfer). But not without reason the French Academy of Sciences twice awarded Robin the Prix Francœur (1893 and 1897), as well as the Prix Ponsel (1895). After all, Robin, considering the problem of the distribution of the skinned (i.e., surface) current density in a conductor, reduced it to an integral equation (long before the appearance of the theory of integral equations) and, in the case of a convex conductor, found a solution to this equation by the method of successive approximations. Note also that the Dirichlet boundary condition, as well as the corresponding boundary-value problem, is named after Dirichlet since the solvability of the Laplace equation with this boundary condition was first proved by Dirichlet in his lectures, following the intuitively correct assumption made by Gauss that there is the integral of the square of the gradient of a function attains its minimum value. This assumption, known as Dirichlet's principle, was rigorously proved 30 years later by Hilbert. But even before that, the German mathematician Carl Gottfried Neumann (1832–1925) was able to prove the solvability of the Dirichlet problem, as well as a boundary-value problem with Neumann boundary operator named after him, using the representation the desired solution in the form of the so-called simple and double layer potentials.

$$w(z) = u(x, y) + iv(x, y), \quad z = x + iy \in \mathbb{C},$$

for which u and v are twice differentiable and satisfy the *Cauchy–Riemann conditions*

$$u_x - v_y = 0, \quad v_x + u_y = 0. \tag{5.5}$$

Here, the subscript denotes the derivative with respect to the corresponding variable (i.e.,, $u_x = \frac{\partial u}{\partial x}, \ldots, u_{yy} = \frac{\partial^2 u}{\partial y^2}, \ldots$). From (5.5), we get

$$u_{xx} + u_{yy} = (u_x - v_y)_x + (v_x + u_y)_y = 0, \quad v_{xx} + v_{yy} = 0.$$

So, the real and imaginary parts of an analytic function $w(z) = u(x, y) + iv(x, y)$ are harmonic functions.

According to Exercises 5.13 and 5.17 that follow, our problem has at most one *bounded* solution. To find this solution, we note that the imaginary part of the analytic function

$$\ln(x + iy) = \ln |x + iy| + i \arg(x + iy), \quad (x, y) \in \mathbb{R}^2_+,$$

coincides with $\operatorname{arccot}(x/y) \in]0, \pi[$. Therefore, this function is harmonic in $\mathbb{R}^2_+$. Besides,

$$\lim_{y\to+0} \operatorname{arccot} \frac{x}{y} = \begin{cases} \pi & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases}$$

Using these properties of $\operatorname{arccot} \frac{x}{y}$, we can construct functions harmonic in $\mathbb{R}^2_+$ and having piecewise constant boundary values. In particular, the function

$$\varphi_{a,b} : \mathbb{R}^2_+ = \mathbb{R}_x \times \mathbb{R}_+ \ni (x, y) \mapsto \frac{1}{\pi}\left[\operatorname{arccot} \frac{x - b}{y} - \operatorname{arccot} \frac{x - a}{y}\right]$$

is harmonic in $\mathbb{R}^2_+$ (this function is the "angle" subtended of the interval $[a, b] \subset \mathbb{R}_x$ at the point (x, y)). For $y \to +0$, this function becomes the characteristic function of this interval. The harmonic function

$$\mathbb{R}^2_+ \ni (x, y) \mapsto P_\varepsilon(x, y) = \frac{1}{2\pi\varepsilon}\left[\operatorname{arccot} \frac{x - \varepsilon}{y} - \operatorname{arccot} \frac{x + \varepsilon}{y}\right]$$

satisfies the boundary condition

$$\lim_{y\to+0} P_\varepsilon(x, y) = \delta_\varepsilon(x) \quad \text{for } |x| \neq \varepsilon,$$

where the function δ_ε is defined by (1.3). On the other hand, if x is a point of continuity of f, then by Lemma 2.1

$$f(x) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \delta_\varepsilon(\xi - x) f(\xi)\, d\xi.$$

This allows us to assume that the function

$$\mathbb{R}^2 \ni (x, y) \mapsto \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} P_\varepsilon(\xi - x, y) f(\xi)\, d\xi \tag{5.6}$$

is equal (at points of continuity of f) to $f(x)$ as $y \to +0$ and that this function is harmonic in $\mathbb{R}^2$ because

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\Big(\sum_{k=1}^{N} P_\varepsilon(\xi_k - x, y) f(\xi_k)(\xi_{k+1} - \xi_k)\Big) =$$

$$= \sum_{k=1}^{N} f(\xi_k)(\xi_{k+1} - \xi_k))\Big[\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) P_\varepsilon(\xi_k - x, y)\Big] = 0.$$

Making formally $\varepsilon \to 0$ in (5.6), we get the *Poisson integral*, or the *Poisson formula*[19]

$$u(x, y) = \int_{-\infty}^{\infty} f(\xi) P(x - \xi, y)\, d\xi, \quad \text{where } P(x, y) = \frac{1}{\pi}\frac{y}{x^2 + y^2}, \tag{5.7}$$

because

$$\lim_{\varepsilon \to 0} P_\varepsilon(x, y) = -\frac{1}{\pi}\frac{\partial}{\partial x}\Big(\operatorname{arccot}\frac{x}{y}\Big) = \frac{1}{\pi}\frac{y}{x^2 + y^2}.$$

Note that condition (5.4) holds in view of Exercise 4.1. We also note that the function u is bounded. Indeed,

$$|u(x, y)| = \int_{-\infty}^{\infty} |f(\xi)| P(x - \xi, y)\, d\xi \le C \int_{-\infty}^{\infty} P(x - \xi, y)\, dy = C.$$

Let us now show that the function u is harmonic in $\mathbb{R}^2_+$. Differentiating (5.7), we get

$$\frac{\partial^{j+k} u(x, y)}{\partial x^j \partial y^k} = \int_{-\infty}^{\infty} f(\xi) \frac{\partial^{j+k}}{\partial x^j \partial y^k} P(x - \xi, y)\, d\xi \tag{5.8}$$

for all $j \ge 0$ and $k \ge 0$. Here the differentiation under the integral is possible, because

$$\Big|\frac{\partial^{j+k} P(x - \xi, y)}{\partial x^j \partial y^k}\Big| \le \frac{C}{1 + |\xi|^2} \tag{5.9}$$

for $|x| < R$ and $\frac{1}{R} < y < R$, where C depends only on $j \ge 0$, $k \ge 0$, and $R > 1$. In view of (5.8), we have

$$\Delta u(x, y) = \int_{-\infty}^{\infty} f(\xi) \Delta P(x - \xi, y)\, d\xi.$$

But $\Delta P(x - \xi, y) = \Delta P(x, y)$, and $\Delta P(x, y) = 0$ in $\mathbb{R}^2_+$, inasmuch as

[19] Simón Denis Poisson (1781–1840) was a famous French mathematician, mechanicist, and physicist. With his name important mathematical objects are associated: the Poisson distribution in probability theory, the Poisson brackets in differential geometry, and the Poisson coefficient in the theory of elasticity.

$$P(x, y) = -\frac{1}{\pi}\frac{\partial}{\partial x}\left(\operatorname{arccot}\frac{x}{y}\right), \quad \Delta\left(\operatorname{arccot}\frac{x}{y}\right) = 0, \quad \Delta\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\Delta.$$

So, we have shown that the *Poisson integral* (5.7) gives a bounded solution of problem (5.2) in $\mathbb{R}^2_+$.

P 5.4 Prove estimate (5.9).

Remark 5.5 The function $P = P_{\mathbb{R}^2_+}$, as defined by (5.7), is known as the *Poisson kernel* (in $\mathbb{R}^2_+$). It can be interpreted as a solution of the problem $\Delta P = 0$ in $\mathbb{R}^2_+$, $P(x, 0) = \delta(x)$, where $\delta(x)$ is the δ-function.[20]

Remark 5.6 Note that (in the sense of the above definition), under the boundary condition (5.4), the *Poisson kernel* $P = P_{\mathbb{R}^2_+}$ is an *unbounded* (in $\mathbb{R}^2_+$) solution of problem (5.2) if $f(x) = 0$ for $x \neq 0$ and if $f(0)$ is equal, say, to 1. On the other hand, for this (piecewise continuous) boundary function f, there is a bounded solution $u(x, y) \equiv 0$.

P 5.7 Find an unbounded solution $u \in C^\infty(\mathbb{R}^2_+)$ of problem (5.2)–(5.4) for $f(x) \equiv 0$.

P 5.8 Let $k \in \mathbb{R}$ and let u be a solution of problem (5.2) in $\mathbb{R}^2_+$ given by (5.7). Find $\lim_{y\to+0}(x_0 + ky, y)$ in two cases:

(1) The function f is continuous.

(2) The function f has a discontinuity of the first kind at the point x_0.

P 5.9 Let $u\colon (x, y) \mapsto u(x, y)$ be a harmonic function in a domain $\omega \subset \mathbb{R}^2 \backsimeq \mathbb{C}$ and let

$$z\colon \Omega \ni \zeta = \xi + i\eta \mapsto z(\zeta) = x(\xi, \eta) + iy(\xi, \eta), \quad \text{where } \Omega \subset \mathbb{R}^2 \backsimeq \mathbb{C},$$

be an analytic function of a complex variable ζ with values in ω. Verify that the formula

$$U(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)), \quad (\xi, \eta) \in \Omega,$$

defines a harmonic function in Ω.

Hint Verify that $U_{\xi\xi} + U_{\eta\eta} = |z'(\zeta)|^2(u_{xx} + u_{yy})$.

[20] Formula (5.7), which gives a solution of the problem $\Delta u = 0$ in $\mathbb{R}^2_+$, $u(x, 0) = f(x)$, can be clearly interpreted as follows. The field is excited by the source function $f(x)$, which is the "sum" with respect to ξ of the point sources $f(\xi)\delta(x-\xi)$. One point source $\delta(x-\xi)$ generates the field $P(x-\xi, y)$, and hence, since the problem is linear, the "sum" of these sources will generate the field, which is the "sum" (i.e., the integral) with respect to ξ of the fields of the form $f(\xi)P(x-\xi, y)$. Physicists usually say that there is a *superposition* of fields generated by point sources. This *principle of superposition* can be traced in many formulas for solutions of *linear* problems in mathematical physics (see in this connection (5.14), (6.18), (7.17), ...). Mathematicians usually use the term "convolution" in such cases (see §19).

Theorem 5.10 (On the Mean Value) *Let $u(x)$ be a function harmonic in a domain $\Omega \subset \mathbb{R}^n$. Then the value of u at the center of any ball $B(a, R) = \{x \in \mathbb{R}^n : |x - a| < R\} \Subset \Omega$ is equal to the mean value of u on the boundary of the ball $B = B(a, R)$ and is also equal to the mean value of u over the entire ball $B = B(a, R)$, i.e.,*

$$u(a) = \frac{1}{|S|}\int_S u(s)\,ds = \frac{1}{|B|}\int_B u(x)\,dx, \tag{5.10}$$

where $|S| \overset{(1.7)}{=} \sigma_n R^{n-1}$ is the area of the $(n-1)$-dimensional sphere $S = \partial B(a, R)$ and $|B| = \int_0^R \sigma_n \rho^{n-1}\,d\rho = \frac{\sigma_n}{n} R^n$ is the volume of the ball B.

Proof For $n = 2$, Theorem 5.10 follows from the Cauchy integral formula for the analytic function $w = u + iv$ (see, for example, Lavrent'ev and Shabat (1977)). Indeed,

$$w(0) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{w(\zeta)\,d\zeta}{\zeta} \Rightarrow u\Big|_{\zeta=0} = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\tau})\,d\tau. \tag{5.11}$$

For $n = 1$, the result is clear (the graph of u is a straight line). For an arbitrary $n \geq 2$, we will use the Gauss integral formula for a harmonic function u:

$$\int_{\Gamma=\partial\Omega} \frac{\partial u}{\partial \nu} d\Gamma = 0. \tag{5.12}$$

Here ν is the normal vector to Γ. Formula (5.12), which will be proved in § 7 (see (7.7)), expresses the following remarkable fact: the total flow of the gradient of a harmonic function across the boundary of any body is zero. Let us apply (5.12) to the ball $B(a, \rho)$, where $\rho \in]0, R]$. Introducing the polar coordinates $r = |x - a|$ and $\omega = \frac{x-a}{r}$, we get

$$\begin{aligned} 0 &\overset{(5.12)}{=} \int_{\partial B(a,\rho)} \frac{\partial u}{\partial r}(s)\,ds = \rho^{n-1}\int_{|\omega|=1} \frac{\partial u}{\partial r}(a + \rho\omega)\,d\omega \\ &= \rho^{n-1}\frac{\partial}{\partial \rho}\int_{|\omega|=1} u(a + \rho\omega)\,d\omega = \rho^{n-1}\frac{\partial}{\partial \rho}\left[\rho^{1-n}\int_{\partial B(a,\rho)} u(s)\,ds\right]. \end{aligned}$$

Hence, recalling that $|\partial B(a, \rho)| = \sigma_n \rho^{n-1}$, we find that

$$R^{1-n}\int_{\partial B(a,R)} u(s)\,ds = \rho^{1-n}\int_{\partial B(a,\rho)} u(s)\,ds \overset{\rho\to 0}{\to} \sigma_n u(a),$$

which shows that $u(a) = \frac{1}{|S|}\int_{\partial B(a,R)} u(s)\,ds$. □

P 5.11 Use (5.11) to derive formulas for the solution of the Dirichlet problem in the disk and in the half-plane.[21]

[21] In what follows, we will obtain formulas for the solution of the Dirichlet problem in a ball and in a half-space on any dimension.

Hint Given a point $a = re^{i\psi}$ in the disk $\{|z| < 1\}$ and the function $u\colon z \mapsto u(z) = U(\zeta(z))$ (for the mapping $z \mapsto \zeta(z) = \frac{z-a}{1-\bar{a}z}$ of the disk $\{|z| < 1\}$ to the disk $\{|\zeta| < 1\}$), note that

$$u(a) \overset{(5.11)}{=} \frac{1}{2\pi} \int_0^{2\pi} U(e^{i\tau})\, d\tau,$$

and then verify that from the formula

$$\{|z| = 1\} \ni e^{i\theta} \mapsto e^{i\tau} = \frac{e^{i\theta} - re^{i\psi}}{1 - re^{-i\psi}e^{i\theta}} \in \{|\zeta| = 1\}, \tag{5.13}$$

which relates the boundary points of the disks, one gets the equality

$$d\tau = \frac{1 - r^2}{1 - 2r\cos(\theta - \psi) + r^2} d\theta,$$

because

$$\left\{ ie^{i\theta} \frac{1 - re^{-i(\theta-\psi)}}{[1 - re^{i(\theta-\psi)}]} \right\} = i\frac{e^{i\theta} - re^{i\psi}}{1 - re^{i(\theta-\psi)}} \overset{(5.13)}{=} ie^{i\theta} = (ie^{i\theta})'_\tau \overset{(5.13)}{=} \left(\frac{e^{i\theta} - re^{i\psi}}{1 - re^{-i\psi}e^{i\theta}} \right)'_\theta \frac{d\theta}{d\tau},$$

and

$$\left(\frac{e^{i\theta} - re^{i\psi}}{1 - re^{-i\psi}e^{i\theta}} \right)'_\theta = ie^{i\theta} \frac{1 - r^2}{[1 - re^{i(\theta-\psi)}]^2} \frac{1 - re^{-i(\theta-\psi)}}{1 - re^{-i(\theta-\psi)}}$$

$$\left\{ ie^{i\theta} \frac{1 - re^{-i(\theta-\psi)}}{[1 - re^{i(\theta-\psi)}]} \right\} \frac{1 - r^2}{\left(1 - re^{i(\theta-\psi)}\right)\left(1 - re^{-i(\theta-\psi)}\right)}.$$

Hence, in the disk $D = \{re^{i\psi} : r < R, \psi \in \mathbb{R}/(2\pi)\}$, we have

$$u(r, \psi) = \int_0^{2\pi} u(R, \theta) P_D(r, \psi - \theta)\, d\theta, \tag{5.14}$$

where

$$P_D(r, \varphi) = \frac{1}{2\pi} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos\varphi}. \tag{5.15}$$

To derive (5.7) from (5.11), one should note that if $z \mapsto \zeta(z) = \frac{z-a}{z-\bar{a}}$ is the mapping of the half-plane $\mathbb{C}_+ \ni a = x + iy$ to the unit disk $\{|\zeta| < 1\}$ and if the boundary points $t \in \mathbb{R}$ and $e^{i\tau} = \frac{t-a}{t-\bar{a}}$ are identified, then

$$u(a) \overset{(5.11)}{=} \frac{1}{2\pi} \int_0^{2\pi} U(e^{i\tau})\, d\tau$$

for the function $u(z) = U(\zeta(z))$, and $d\tau = \frac{2y\, dt}{(t-z)(t-\bar{z})} = \frac{2y\, dt}{(t-x)^2 + y^2}$.

P 5.12 Interpret function (5.15) similarly to the function $P_{\mathbb{R}^2_+}$ in Remark 5.5.

P 5.13 Using Theorem 5.14 (see below), show that problem (5.2) has a unique solution under the assumption that f is continuous.

Theorem 5.14 (Maximum Principle) *If a function u is harmonic in a domain Ω, then* $\sup_\Omega u = \sup_{\overline{\Omega}} u$.

Theorem 5.14 is a direct corollary of the following *strong maximum* (*minimum*) *principle*.

Theorem 5.15 (Strong Maximum Principle[22]) *If a function harmonic in a domain* Ω *takes the value* $\sup_\Omega u$ (*or* $\inf_\Omega u$), *at some point* $a \in \Omega$, *then* $u = \text{const}$.

Proof Let $M = \sup_\Omega u$ and let $\Omega_M = \{x \in \Omega\colon u(x) = M\}$. The set Ω_M is closed in Ω, because the function u is continuous. To prove the required equality $\Omega_M = \Omega$, it remains to show that Ω_M is open in Ω. But this is so, because the neighborhood $B(a, \rho) \subset \Omega$ of any point $a \in \Omega_M$ lies in Ω_M. Indeed, $u\big|_{B(a,\rho)} = M$, because

$$0 \le u(a) - M \overset{(5.10)}{=} \frac{1}{|B|}\int_B (u(x) - M)\,dx \le 0.$$

Theorem 5.16 *Let* u *be a continuous function in a domain* $\Omega \subset \mathbb{R}^n$. *Then* u *is harmonic if and only if the mean value property* (5.10) *is satisfied in any ball* $B \subset \Omega$.

Proof We need to check the converse of Theorem 5.10. According to Exercise 5.11, for any ball $B \subset \Omega$, there exists a function v harmonic in B such that $v = u$ on ∂B. The difference $w = u - v$ satisfies the mean value property for any ball lying in B. Therefore, the uniqueness result can be applied to w (the proof of this result depends only on the mean value property). Hence $w = 0$ in Ω, and therefore, the function u is harmonic.[23] □

P 5.17 Using Theorem 5.18 (see below), verify the uniqueness of a bounded solution of problem (5.2) in the case of a piecewise continuous function f. Cf. Exercise 5.13.

Theorem 5.18 (On the Discontinuous Majorant) *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with boundary* $\partial\Omega$ *and let* F *be a finite point set* $x_k \in \overline{\Omega}$, $k = 1, \dots, N$. *Next, let* $u(x)$ *and* $v(x)$ *be two functions harmonic in* $\Omega \setminus F$ *and continuous in* $\overline{\Omega} \setminus F$. *Suppose that there exists a constant* M *such that* $|u(x)| \le M$ *and* $|v(x)| \le M$ *for all* $x \in \overline{\Omega} \setminus F$. *If* $u(x) \le v(x)$ *for any* $x \in \partial\Omega \setminus F$, *then* $u(x) \le v(x)$ *for all* $x \in \overline{\Omega} \setminus F$.

Proof We first note that the function $\ln|x|$, where $x \in \mathbb{R}^2 \setminus \{0\}$, is harmonic. We set

$$w_\varepsilon(x) = u(x) - v(x) - \sum_{k=1}^{N} \frac{2M}{\ln(d/\varepsilon)} \ln\frac{d}{|x - x_k|}.$$

[22] This result for some second-order elliptic equations more general than the Laplace equation was first proved in 1927 by Eberhard Frederich Ferdinand Hopf (1902–1983), who was a German and American mathematician, one of the founders of the ergodic theory. See also Lemma 5.19, Eq. (11.27), Theorem 11.23, and the footnote on the Wiener–Hopf factorization on p. 131.

[23] French mathematician Jean Frédéric Delsarte (1903–1968) was one of the founders of the Bourbaki group. According to Delsart (1958), at least in $\mathbb{R}^3$, a continuous function u is harmonic if the mean value property for this function is satisfied only for any two balls with center at each point $a \in \mathbb{R}^3$.

Here $0 < \varepsilon < d$, where d is the diameter of Ω. Hence $\ln\left(\frac{d}{|x-x_k|}\right) \geq 0$. Let Ω_ε be the domain obtained from Ω by cutting disks of radius ε with centers at $x_k \in F$, $k = 1, \ldots, N$. It is clear that the function w_ε is harmonic in Ω_ε and is continuous in $\overline{\Omega}_\varepsilon$, and $w_\varepsilon(x) \leq 0$ for $x \in \partial\Omega_\varepsilon = \overline{\Omega}_\varepsilon \setminus \Omega_\varepsilon$. By the maximum principle, $w_\varepsilon(x) \leq 0$ for $x \in \Omega_\varepsilon$. It remains to make $\varepsilon \to 0$. □

From Theorem 5.15 it follows that the inequality $\partial u/\partial\nu|_{x=x_0} \geq 0$ holds at a boundary point $x_0 \in \partial\Omega$ at which a function u harmonic in Ω attains its maximum, where ν is the outer normal vector to Ω. This assertion also holds for domains with piecewise smooth boundary, and in particular, for a square (in a square, a harmonic function that attains its maximum at a corner point has the *zero* derivative in any direction). If the boundary of a domain is sufficiently smooth, then this boundary can be touched from inside by some ball. Hence the following substantial and very useful extension of Theorem 5.15 holds.

Lemma 5.19 (Giraud–Hopf–Oleinik Theorem on the Boundary Derivative[24]) *Let Γ be a smooth $(n-1)$-dimensional boundary of a domain $\Omega \Subset \mathbb{R}^n$ and let a_{ij}, b_k, and c be smooth functions in $\overline{\Omega}$, $c(x) \leq 0$, and*

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2,$$

where $|\xi|^2 = \sum_{1\leq j\leq n} \xi_j^2$ and $\alpha > 0$. Next, let $u(x)$ be continuous in $\overline{\Omega}$ function satisfying in Ω the equation[25]

$$\sum_{i,j} a_{ij}(x)u_{x_ix_j} + \sum_k b_k(x)u_{x_k} + c(x) = 0, \qquad x \in \Omega.$$

Suppose that $x_0 \in \Gamma$ and $u(x_0) > u(x)$ for any $x \in \Omega$. Then[26]

This lemma (which holds, in particular, for the Laplace equation) was proved in Giraud (1932). In the case where the continuous are only coefficients, the lemma was independently proved by Hopf (1952) for $c = 0$ and Oleinik (1952) for $c \geq 0$. However, both proofs (Hopf and Oleinik) hold only when the coefficients are bounded. For a detailed account of the problem, see Apushkinskaya and Nazarov (2022).

P 5.20 Prove Lemma 5.19 for the Laplace operator.[27] This is not so difficult.

[24] Georges Julien Giraud (1889–1943) was a French mathematician, a specialist in differential and singular integral equations. Olga Arsenievna Oleinik (1925–2001) was a prominent specialist in differential equations, mathematical physics and its applications, Academician of the USSR Academy of Sciences. On Hopf, see the footnote on p. 20.

[25] This is the same (mentioned in the footnote on p. 20) second-order linear elliptic equation, for which the 25-year-old E. Hopf proved the strengthened *maximum principle*.

[26] Since the boundary Γ is smooth, at any point $x_0 \in \Gamma$ there exists the outer normal derivative $\partial u/\partial\nu|_{x=x_0}$. The crux of the theorem is that this derivative is not zero. For the principal ideas of the proof, see Problem P 5.20.

Hint For $n > 2$, the proof is the same as for $n = 2$, but for $n = 2$ the idea is more transparent, because in this case one can draw a portion γ of the smooth boundary of the planar domain Ω. On γ, we mark a point x° and draw the annulus $K = \{R/2 \le r = |x - x^*| \le R\} \subset \Omega$, which touches γ at the point x°. One should show that $\frac{\partial u}{\partial r}\big|_{r=R} > 0$. By the strengthened maximum principle, $u(x)-u(x^\circ) < 0$ in K, excluding the point x°. Hence, by adding to $u(x)-u(x^\circ)$ the term $\varepsilon v(r)$, where $\varepsilon > 0$ is very small, and $v(r) = e^{ar^2} - e^{aR^2} > 0$ everywhere, excluding the point x°, we will get that $w(x) = u(x)-u(x^\circ)+\varepsilon v(r) \le 0$ on ∂K. Moreover, $\Delta v(r) = e^{ar^2}\left[4a^2 - 2na\right] > 0$ for $a \gg 1$. Hence $\Delta w = \varepsilon \Delta v > 0$ in the annulus K. It is easily checked that this implies the inequality $\frac{\partial w}{\partial r}\Big|_{r=R} \ge 0$. Thus, $\frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} + \varepsilon \frac{\partial v}{\partial r}\Big|_{r=R} \ge 0$, which implies $\frac{\partial u}{\partial r} + \varepsilon \frac{\partial v}{\partial r}\Big|_{r=R} \ge 0$, i.e., $\frac{\partial u}{\partial r}\Big|_{r=R} \ge -\varepsilon \frac{\partial v}{\partial r}\Big|_{r=R} > 0$.

The proof of Lemma 5.19 in the general case is similar.

6 On the Heat Equation

In order to heat a body occupying a domain $\Omega \subset \mathbb{R}^3$ from a temperature $u_0 = \text{const}$ to a temperature $u_1 = \text{const}$, it requires to supply it with energy (in the form of heat) $C \cdot (u_1 - u_0) \cdot |\Omega|$, where $|\Omega|$ is the volume of Ω and C is the (positive) coefficient known as the specific heat capacity. Let $u(x, t)$ be the temperature at a point $x = (x_1, x_2, x_3) \in \Omega$ at time t. Let us derive the differential equation satisfied by the function u. We will assume that the physical model of the real process is such that the functions considered below in connection with this process (the thermal energy, the temperature, and the heat flow) are sufficiently smooth. The variation of the thermal energy in the parallelepiped

$$\Pi = \{x \in \mathbb{R}^3 : x_k^\circ < x < x_k^\circ + h_k, k = 1, 2, 3\}$$

over time τ (starting from time t°) can be written as

$$C \cdot [u(x^\circ, t^\circ + \tau) - u(x^\circ, t^\circ)] \cdot |\Pi| + o(\tau \cdot |\Pi|)$$
$$= C \cdot [u_t(x^\circ, t^\circ) \cdot \tau + o(\tau)] \cdot |\Pi| + o(\tau \cdot |\Pi|), \quad (6.1)$$

where $|\Pi| = h_1 \cdot h_2 \cdot h_3$, and $o(A)$ is considered for $A \in \mathbb{R}$ as $A \to 0$.

[27] In the case of the two-dimensional Laplace equation, the conclusion of Lemma 5.19 holds even in the case where the boundary of the domain Ω cannot be touched from inside by a disk, but near the point x_0 the tangent to the boundary (as a function of the natural parameter) satisfies the Hölder condition. Indeed, if a function harmonic in a disk is not a constant, then at a point of its maximum its normal derivative is nonzero. But this derivative differs from $\frac{\partial u}{\partial \nu}\Big|_{x=x_0}$ only by the factor $f'(z)\big|_{z=f^{-1}(x_0)}$, which is nonzero according to one result of Kellogg (1931) on a conformal mapping $f : z \mapsto f(z)$ of the disk ontoΩ. Oliver Dimon Kellogg (1878–1932) was an American mathematician. He died on 27 August 1932 of a heart attack while climbing Doubletop Mountain near Greenville, Maine.

This change in thermal energy is associated with the presence of a heat flow across the boundary of the parallelepiped Π. According to the Fourier law, the heat flow passing through a certain area in the normal direction to this area in a unit of time is proportional to the (negative) proportionality coefficient $-k$ of the temperature derivative along this normal vector. The coefficient of proportionality $k > 0$ is called the coefficient of thermal conductivity. Thus, the amount of thermal energy entered during the time τ in the parallelepiped Π through the plate $x_1 = x_1^\circ + h_1$ is

$$k(x_1^\circ + h_1, x_2^\circ, x_3^\circ) \cdot \frac{\partial u}{\partial x_1}(x_1^\circ + h_1, x_2^\circ, x_3^\circ; t^\circ) \cdot \tau \cdot h_2 \cdot h_3 + o(\tau \cdot |\Pi|),$$

and the amount of energy released during the same time through the plate $x_1 = x_1^\circ$ is equal to

$$k(x_1^\circ, x_2^\circ, x_3^\circ) \cdot \frac{\partial u}{\partial x_1}(x_1^\circ, x_2^\circ, x_3^\circ; t^\circ) \cdot \tau \cdot h_2 \cdot h_3 + o(\tau \cdot |\Pi|).$$

Therefore, the variation of the thermal energy in Π due to the heat flow along the x_1-axis is equal to

$$\left[\frac{\partial}{\partial x_1}(k(x^\circ)\frac{\partial u}{\partial x_1}(x^\circ, t^\circ))h_1 + o(h_1)\right] \cdot \tau \cdot h_2 \cdot h_3 + o(\tau \cdot |\Pi|).$$

It is clear that the sum of the variations of the thermal energy in Π in all three directions is equal to the total change in thermal energy in Π, which is (6.1). Dividing this equality by $\tau \cdot |\Pi|$ and making τ, h_1, h_2, and h_3 to 0, we get the heat equation

$$C\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\Big(k\frac{\partial u}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(k\frac{\partial u}{\partial x_2}\Big) + \frac{\partial}{\partial x_3}\Big(k\frac{\partial u}{\partial x_3}\Big). \tag{6.2}$$

If the coefficients C and k are constant, then (6.2) can be written as

$$\frac{\partial u}{\partial t} = a\Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\Big), \quad a = \frac{k}{C} > 0.$$

Remark 6.1 If the temperature distribution is independent of time (i.e., $u_t = 0$), the temperature u satisfies the Laplace equation (if $k = \text{const}$). Thus, the *Dirichlet problem* for the Laplace equation (see §5) can be interpreted as the problem on the distribution of the steady (stationary) temperature in the body if a time-independent distribution of temperature is set on the surface of the body. This is also true for more involved problems, for example, for problems dealing with heat and mass transfer (Demidov and Yatsenko 1994).

If one is interested in the distribution of the temperature inside the body, where (for some time) the effect of boundary conditions is not very significant, then the situation is idealized and the following problem is considered:

$$C\frac{\partial u}{\partial t} = \operatorname{div}(k\nabla u), \quad (x,y,z) \in \mathbb{R}^3, \quad t > 0, \quad u|_{t=0} = f(x,y,z),$$

where f is the temperature distribution (in a body without boundary, i.e., in $\mathbb{R}^3$) at time $t = 0$. This problem is sometimes called the Cauchy problem for the *heat equation*.

Assume that f and k and, therefore, u are independent of y and z. Then u is a solution of the problem

$$C\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(k\frac{\partial u}{\partial x}\Big), \quad (x,t) \in \mathbb{R}^2_+ = \{x \in \mathbb{R}, t > 0\}, \tag{6.3}$$

$$u|_{t=0} = f(x). \tag{6.4}$$

Working Conjecture 6.2 The method by which problem (5.2) was solved in $\mathbb{R}^2_+$ suggests[28] that in the case of problem (6.3), (6.4), its solution can apparently be expressed by the formula

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)v(x-\xi,t)\,d\xi;$$

here v is the solution of (6.3) satisfying condition

$$\lim_{t\to+0} v(x,t) = \delta(x), \tag{6.5}$$

where δ is the δ-function.

Below (see Theorem 6.5) it will be shown that this conjecture is true and that the function v can be explicitly written down. This function satisfies the following conditions:

$$C\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(k\frac{\partial v}{\partial x}\Big), \quad \int_{-\infty}^{\infty} Cv\,dx = Q; \tag{6.6}$$

here Q is the total amount of heat. So, v is some function G of five independent variables, x, t, C, k, and Q, i.e.,

$$v = G(x,t,C,k,Q). \tag{6.7}$$

Remark 6.3 The method by which the function v will be found has its origins in mechanics (Sedov 1959) and is called the *method of transition to dimensionless parameters* (variables).

Let $\{p\}$ be the value of a dimensional quantity in the "original" system of units, and let $\{p^*\}$ be its value in a "different" system of units. If $[p]$ is the dimension of this quantity in the "original" system and $[p^*]$ is its dimension in the "different" system, then

$$\{p\}[p] = \{p^*\}[p^*]. \tag{6.8}$$

For example, if $[t]$ is measured in seconds and $[t^*]$ in minutes, then $\frac{\{t\}}{\{t^*\}} = 60$. So,

[28] See footnote 20 on p. 17.

$$\{p^*\} = \{p\}\frac{[p]}{[p^*]} \;\Rightarrow\; \{p^*\} = \{p\}[p] \quad \text{if} \quad [p^*] = 1. \tag{6.9}$$

Note that, in the SI system of units, v, x, t, and Q are measured in units: $[v] = \mathrm{K}$, $[x] = \mathrm{m}$, $[t] = \mathrm{s}$, $[Q] = \mathrm{J}$. In view of conditions (6.6), we have $[C][v]/[t] = [k][v]/[x]^2$ and $[C][v][x] = [Q]$. Hence the dimensions of the quantities C and k are expressed by the formulas[29]

$$[C] = \mathrm{J}/(\mathrm{m}\cdot\mathrm{K}), \quad [k] = \mathrm{J}\cdot\mathrm{m}/(\mathrm{s}\cdot\mathrm{K}).$$

Since C, k, and Q play the role of parameters of the function $v(x,t)$, it is preferable to express the units for the function v and one of its arguments, for example x, in terms of the dimensions of the quantities t, C, k, and Q, which are naturally called "base" parameters. We have

$$[x] = \sqrt{[t]\cdot[k]/[C]}, \quad [v] = [Q]/\sqrt{[t]\cdot[k]\cdot[C]}. \tag{6.10}$$

If we take as a new system of units the one in which the "base" parameters are taken as units, then in this new system of units they become dimensionless $[t^*] = 1$, $[C^*] = 1$, $[k^*] = 1$, and $[Q^*] = 1$, their numerical values t^*, C^*, k^*, and Q^* are equal to 1. Hence, in accordance with (6.8) in this new system of units, we have

$$[t] = \frac{1}{\{t\}}, \quad [C] = \frac{1}{\{C\}}, \quad [k] = \frac{1}{\{k\}}, \quad [Q] = \frac{1}{\{Q\}}, \tag{6.11}$$

in the same way as $[t] = \text{min.} = \frac{\text{hr.}}{60} = \frac{t^*}{\{t\}} = \frac{1}{\{t\}}$. In the new system of units, the other parameters are also dimensionless in view of (6.10), and hence by (6.8) they are equal to their numerical values. Hence

$$\begin{aligned} x^* &= \{x^*\} \overset{(6.9)}{=} \{x\}[x] \overset{(6.10)}{=} \{x\}\sqrt{[t]\cdot[k]/[C]} \overset{(6.11)}{=} \frac{\{x\}\sqrt{\{C\}}}{\sqrt{\{t\}\{k\}}}, \\ v^* &= \{v^*\} \overset{(6.9)}{=} \{v\}[v] \overset{(6.10)}{=} \{v\}\sqrt{[t]\cdot[k]/[C]} \overset{(6.11)}{=} \frac{\{v\}\sqrt{\{t\}\{k\}\{C\}}}{\{Q\}}. \end{aligned} \tag{6.12}$$

We now note that the dependence $v \overset{(6.7)}{=} G(x,t,C,k,Q)$ expresses the law, which does not depend on the choice of the system of units. So, $v^* = G(x^*, t^*, C^*, k^*, Q^*)$, and hence since $t^* = C^* = k^* = Q^* = 1$, we get

$$v(x,t) = \frac{Q}{\sqrt{kCt}}\, g\!\left(\sqrt{\frac{C}{kt}}\cdot x\right), \quad \text{where } g(y) = G(y,1,1,1,1). \tag{6.13}$$

[29] If we talk about a real three-dimensional physical object, the integral in (6.6) would be over a three-dimensional domain. Therefore, in the real three-dimensional physical world, $[C] = \mathrm{J}/(\mathrm{m}^3\cdot\mathrm{K})$ and $[k] = \mathrm{J}/(\mathrm{m}\cdot\mathrm{s}\cdot\mathrm{K})$.

Remark 6.4 One can get (6.13) from purely mathematical considerations. Namely, changing the variables

$$t^* = \frac{t}{\sigma_t}, \quad C^* = \frac{C}{\sigma_C}, \quad k^* = \frac{k}{\sigma_k}, \quad Q^* = \frac{Q}{\sigma_Q}, \quad x^* = \frac{x}{\sigma_x}, \quad v^* = \frac{v}{\sigma_v}$$

and requiring that v^* be equal to $G(x^*, t^*, C^*, k^*, Q^*)$, i.e., requiring that

$$C^* \cdot \frac{\partial v^*}{\partial t^*} = \frac{\partial}{\partial x^*}\Big(k^* \cdot \frac{\partial v^*}{\partial x^*}\Big), \qquad \int_{-\infty}^{\infty} C^* v^* \, dx = Q^*,$$

from (6.6), we get that $\sigma_x = \sqrt{\frac{\sigma_t \sigma_k}{\sigma_C}}$ and $\sigma_v = \frac{\sigma_Q}{\sqrt{\sigma_t \sigma_k \sigma_C}}$. Using (6.8) and taking the scale coefficients $\sigma_t = \frac{[t^*]}{[t]}$, $\sigma_C = \frac{[C^*]}{[C]}$, $\sigma_k = \frac{[k^*]}{[k]}$, and $\sigma_Q = \frac{[Q^*]}{[Q]}$, we again arrive at (6.13).

However, the use of dimension considerations can be helpful. First, this allows one to test the correctness of the occurrence of certain parameters when setting the problem: the dimensions in both parts of any equality should be consistent. And second, it allows one to find a necessary replacement of variables (not necessarily related only to scale factors). All this allows one to automatically (and therefore easily) get rid of the "extra" parameters and thereby facilitate both the analysis and calculations.[30] In addition, a transition to dimensionless parameters allows one to apply similarity considerations, which sometimes significantly facilitate the solution of both very difficult problems (see, for example, Sedov (1959)) and problems like

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[30] Consider, for example, the problem of the temperature field of an infinite plate of thickness $2S$ with initial temperature $T_0 = \text{const}$, when there is a heat exchange between the surface of the plate (with the heat transfer coefficient α) and the medium of temperature of $T_1 = \text{const}$. In other words, consider the problem

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial \xi^2}, \quad \tau > 0, \quad |\xi| < S; \quad \mp k\frac{\partial T}{\partial \xi}\Big|_{\xi=\pm S} = \alpha(T - T_1)|_{\xi=\pm S}; \quad T|_{\tau=0} = T_0.$$

Problems of this kind arise, for example, when selecting the parameters k and α which characterize the mode of quenching of some or other grade of steel (characterized by the parameter a). In this example, an infinite steel plate of thickness $2S$ can be interpreted (as is sometimes done in practice) as a localization of critical parts of an article (for example, the race of a bearing).

The function $T = f(\tau, \xi, a, S, k, \alpha, T_1, T_0)$ depends a priori on eight parameters. Tabulating the values of such a function (for which each of the parameters takes at least 10 values) is clearly impractical, since it would require analyzing a million pages. But by changing to the dimensionless parameters

$$u = (T - T_1)/(T_1 - T_0), \quad x = \xi/S, \quad t = a\tau/S^2, \quad \sigma = k/(\alpha S),$$

the problem is reduced to the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad |x| < 1; \quad \Big(u \pm \sigma\frac{\partial u}{\partial x}\Big)\Big|_{x=\pm 1} = 0; \quad u|_{t=0} = 1, \tag{6.14}$$

whose solution $u = u(t, x, \sigma)$ can be represented (and this is important in applications) in the form of a compact table (one page for each value of $\sigma \geq 0$).

the following one. *A channel of length L is drilled through the center of the ball. Find the volume of the remaining part of the ball.*[31]

Let us return to formula (6.13). In order to find the function v, we substitute its expression (6.13) in the heat Eq. (6.2). As a result, we get

$$Q\sqrt{\frac{C}{kt^3}}\Big[\frac{g(y)}{2}+y\frac{g'(y)}{2}+g''(y)\Big]=0,$$

i.e.,

$$\frac{(yg(y))'}{2}+g''(y)=0.$$

So, the function g satisfies the linear equation

$$g'(y)+\frac{yg(y)}{2}=\text{const}\,. \tag{6.15}$$

If g is odd, i.e., $g(-y)=g(y)$, then $g'(0)=0$, and hence g satisfies the homogeneous equation (6.15), whose solution is clearly given by $g(y)=Ae^{-y^2/4}$. The constant A can be found from the second condition in formula (6.6):

$$Q=\int_{-\infty}^{\infty}Cv\,dx=\frac{ACQ}{\sqrt{kCt}}\int_{-\infty}^{\infty}e^{-Cx^2/4kt}\,dx=2AQ\int_{-\infty}^{\infty}e^{-\xi^2}\,d\xi,$$

i.e., $A\overset{(1.6)}{=}1/(2\sqrt{\pi})$. Therefore,

$$v(x,t)=\frac{Q}{2\sqrt{kC\pi t}}e^{-Cx^2/(4kt)}. \tag{6.16}$$

Theorem 6.5 *Assume that, for some $\sigma\in[1,2[$, $a>0$ and $M>0$,*

$$|f(x)|\le M\exp(a|x|^{\sigma})\quad \textit{for any}\;\; x\in\mathbb{R} \tag{6.17}$$

for a function $f\in C(\mathbb{R})$. Then the function $u\colon\{(x,t)\in\mathbb{R}^2: t>0\}\to\mathbb{R}$, as defined by

$$u(x,t)=\int_{-\infty}^{\infty}f(\xi)P(x-\xi,t)\,d\xi,\quad P(x,t)=\frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)}, \tag{6.18}$$

is a solution to the heat equation

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[31] At first, it may seem that in this problem, which was posed by the American mathematician and popularizer of science Martin Gardner (1914–2010), it is required to calculate the integral that expresses the volume of the drilled channel. But since the radius of the ball is not set, it may seem that the problem is set incorrectly. However (see Rybakov 2014), the required volume V should be proportional to the cube of a given linear size, i.e., $V=kL^3$, where k is some dimensionless constant. It does not depend on the radius of the ball, and hence it is the same if the length L is equal to $2R$, where R is the radius of the ball. In this problem, $V=\frac{4}{3}\pi R^3$. Hence $k=\frac{\pi}{6}$, solving the problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{6.19}$$

in $\mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 \colon t > 0\}$. *This solution is infinitely differentiable and satisfies the initial condition* (*or, as some say, the Cauchy condition*)

$$\lim_{t\to+0} u(x,t) = f(x). \tag{6.20}$$

Besides, for any $T > 0$, *there exists a* $C(T) > 0$ *such that*

$$|u(x,t)| \le C(T)e^{(2a|x|)^\sigma} \tag{6.21}$$

for all $x \in \mathbb{R}$ *and* $t \in [0,T]$.

Proof From the construction of the function (6.16), it follows that the function $P(x,t) = (4\pi t)^{-1/2}\exp\left(-\frac{x^2}{4t}\right)$ satisfies (6.6) with $C = k = Q = 1$. Hence (6.20) follows from Exercise 4.2, while (6.19), as well as the smoothness of the function u, is secured by the well-known theorem on the differentiation of integrals with respect to a parameter (see, for example, Zorich (2016)), because the corresponding integral converges uniformly. Indeed, for all $R > 1$ and $\lambda > 0$, there exists an $N > 1$ such that

$$\int_{|\xi|>N}\left|\frac{\partial^{j+k}}{\partial x^j \partial t^k}(f(\xi)P(x-\xi,t))\right| d\xi < \lambda \tag{6.22}$$

for $x \in [-R,R]$, $t \in [1/R,R]$, because for such x and t the integrand in (6.22) is estimated in terms of $C_R|f(\xi)|P(x-\xi,t)$ if $j+k>0$.

Let us now verify estimate (6.21). Note that, for $\sigma \in [1,2[$ and some $\varepsilon > 0$ and $C_\varepsilon > 0$,

$$|\xi|^\sigma \le 2^\sigma(|x|^\sigma + |\xi - x|^\sigma), \quad |\xi - x|^\sigma \le \varepsilon(\xi - x)^2 + C_\varepsilon|\xi - x|.$$

We choose ε so as to have $1 - 4T\cdot a_\sigma \cdot \varepsilon > 0$, where $a_\sigma = a \cdot 2^\sigma$. Hence, for $t \le T$ (cf. Godunov 1979, p. 42), we have

$$\begin{aligned}
|u(x,t)| &\le \frac{M}{2\sqrt{\pi}}\int e^{a|\xi|^\sigma}\cdot e^{-(x-\xi)^2/(4t)}\frac{d\xi}{\sqrt{t}} \\
&\le M_1 e^{(2a|x|)^\sigma}\int e^{a(2|\xi-x|)^\sigma}\cdot e^{-((x-\xi)/(2\sqrt{t}))^2}\frac{d\xi}{2\sqrt{t}} \\
&\le M_1 e^{(2a|x|)^\sigma}\int e^{-(1-4Ta_\sigma\varepsilon)(x-\xi)^2/(4t)}\cdot e^{a_\sigma C_\varepsilon(|x-\xi|/(2\sqrt{t}))2\sqrt{t}}\frac{d\xi}{2\sqrt{t}}.
\end{aligned}$$

Putting $\eta = (\xi - x)(1 - 4Ta_\sigma\varepsilon)^{1/2}/(2\sqrt{t})$, we find that

$$|u(x,t)| \le C(T)e^{(2a|x|)^\sigma}\int e^{-\eta^2+\alpha|\eta|\cdot 2\sqrt{t}}\,d\eta.$$

This implies estimate (6.21), because

$$\int_0^\infty e^{-\eta^2+\alpha\eta\cdot 2\sqrt{t}}\,d\eta = e^{\alpha^2 t}\int_0^\infty e^{-(\eta-\alpha\sqrt{t})}\,d\eta \le e^{\alpha^2 t}\int_0^\infty e^{-\zeta^2}\,d\zeta.$$

Remark 6.6 In general, there exists a solution to problem (6.19), (6.20), which is different from (6.18). So, for example, the solution of problem (6.19), (6.20) for $f = 0$ is given by the series

$$u(x,t) = \sum_{m=0}^{\infty} \varphi^{(m)}(t)\cdot x^{2m}/(2m)!, \quad (x,t)\in\mathbb{R}^2_+, \tag{6.23}$$

with $\varphi \in C^\infty(\mathbb{R})$ satisfying the conditions

$$\operatorname{supp}\varphi \subset [0,1], \quad |\varphi^{(m)}(t)| \le (\gamma m)! \quad \text{for any } m\in\mathbb{Z}_+, \tag{6.24}$$

where $1 < \gamma < 2$. (The condition $\gamma < 2$ is needed for the uniform (with respect to x and t, $|x| \le R < \infty$) convergence of both the series (6.23) and its derivatives.) This simple but important fact was noticed in 1935 by A. N. Tikhonov[32] in Tychonoff (1935), who in the construction of the series (6.23) used a nontrivial result of Carleman[33] (see Carleman 1926) on the existence of a nonzero function φ with properties (6.24). It is worth pointing out that the nonzero solution (6.23) (satisfying the condition $u(x,0) = 0$) of the heat equation constructed by Tikhonov grows faster as $|x| \to \infty$ than $\exp(Cx^2)$ for any $C > 0$ (and slower than $\exp(Cx^\sigma)$, where $\sigma = 2/(2-\gamma) > 2$). On the other hand, based on the *maximum principle* for solutions of the heat equation (see, for example, Godunov (1979); Mikhailov (1978); Tychonoff (1935); Friedman (1964)), it can be shown that the solution of problem (6.19), (6.20) is unique if condition (6.21) is satisfied. The uniqueness theorem in a broader class of functions was proved in 1924 by Holmgren[34] in Holmgren (1924).

The next theorem follows from Remark 6.6.

Theorem 6.7 *If a function $f \in C(\mathbb{R})$ satisfies condition* (6.17)*, then formula* (6.18) *represents a solution of problem* (6.19)*,* (6.20)*, and this solution is unique in the class* (6.21).

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[32] Andrey Nikolaevich Tikhonov (1906–1993) was a mathematician and geophysicist, an Academician of the USSR Academy of Sciences, who proposed a method for regularizing ill-posed problems.

[33] Torsten Carleman (1892–1949) was a Swedish mathematician, an outstanding analyst. His main works were devoted to integral equations and the theory of functions.

[34] Erik Albert Holmgren (1872–1943) was a Swedish mathematician. T. Carleman, who was his pupil, exceeded his master. But Holmgren is also considered a classic—in 1901 he proved an important uniqueness theorem in the class of smooth functions (not necessarily analytic, as in the Cauchy–Kovalevskaya theorem) for the solution of the Cauchy problem for differential equations with analytic coefficients. No less significant was Holmgren's idea of the proof, in which the solvability of the conjugate problem was tested. An example of the application of his idea is presented in the proof of Theorem 11.9.

7 The Ostrogradsky–Gauss Formula: The Green Formulas and the Green Function

We start with the clear formula $\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx$, which clearly follows from the definition of the integral over an interval $]a, b[$, which is defined as the limit of the corresponding integral sums over *oriented* small intervals whose length tends to zero. The orientation of the small intervals is induced by the *orientation* of the interval (a, b), which corresponds to the direction from a to $b > a$. In the n-dimensional setting, the integral $\int_\Omega g(x_1, \ldots, x_n)\,dx_1 \ldots dx_n$ over a bounded domain $\Omega \subset \mathbb{R}^n$ is also the limit of the sum of terms of the form $g(x)dx_1 \wedge \ldots \wedge dx_n$, where $x \in \Omega$, and

$$dx_1 \wedge \ldots \wedge dx_n = \det\begin{pmatrix} dx_1 & 0 & \ldots & 0 \\ 0 & dx_2 & \ldots & 0 \\ \ldots\ldots & & & \\ 0 & 0 & \ldots & dx_n \end{pmatrix}$$

is the *oriented* volume of the parallelepiped with "infinitely small" edges dx_k. If you swap some two edges, then in the mirror one can see the former parallelepiped. So the permutation of two edges is similar to the transition from the integral $\int_a^b g(x)\,dx$ to the integral $\int_b^a g(x)\,dx$. In other words, the *oriented* volume will change the sign. Hence

$$dx_1 \wedge \ldots \wedge dx_n = (-1)^m dx_{j_1} \wedge \ldots \wedge dx_{j_n},$$

where m is the signum of a permutation $\begin{pmatrix} 1 & 2 & \ldots & n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix}$, i.e., the number of pairs of elements (not necessarily neighboring) at which the succeeding element has smaller number than the previous one. In particular, $dx_1 \wedge dx_3 \wedge dx_2 = -dx_1 \wedge dx_2 \wedge dx_3$, and $dx_3 \wedge dx_1 \wedge dx_2 = dx_1 \wedge dx_2 \wedge dx_3$.

Now we note that the Newton–Leibniz formula $\int_a^b f'(x)\,dx = f(b) - f(a)$ can be written in the form $\int_{(a,b)} f'(x)\,dx = f(b)\alpha(b) + f(a)\alpha(a)$. Here the points a and b constitute the boundary $\Gamma \overset{\text{def}}{=} \partial(a, b)$ of the interval (a, b), and $\alpha(x)$ is the cosine of the angle between the outer normal vector ν to the interval (a, b) at a point $x \in \Gamma$ and the coordinate x-axis, i.e., $\alpha(b) = 1$, $\alpha(a) = -1$.

The extension of the Newton–Leibniz formula to the case when Ω is a bonded domain in $\mathbb{R}^n$ with smooth $(n-1)$-dimensional boundary $\partial\Omega$, and $f = (f_1, \ldots, f_n)$ is a vector function with components $f_k \in C(\bar{\Omega})$ such that $\partial f_k/\partial x_k \in PC(\Omega)$ is known as the *Ostrogradsky–Gauss formula*

$$\int_\Omega \operatorname{div} f(x)\,dx = \int_{\partial\Omega} \sum_{k=1}^n f_k(x)\alpha_k\,d\Gamma, \tag{7.1}$$

where $\alpha_k = \alpha_k(x)$ is the cosine of the angle between the outward normal vector ν to $\Gamma = \partial\Omega$ at a point $x \in \Gamma$ and the kth coordinate axis, and $d\Gamma$ is the "area element" of Γ, i.e., the undirected (positive) $(n-1)$-dimensional volume of the parallelepiped $d\Gamma = ds_1 \wedge \ldots \wedge ds_{n-1}$. This means that the local system of coordinates $(s_1, \ldots, s_{n-1}, -\nu)$, where ν is the outer normal vector to Γ, has the same orientation as the system of coordinates $(x_1, \ldots, x_{n-1}, x_n)$. Formula (7.1) was first published Ostrogradsky[35] in 1831 (Note sur les intégrales définies, *Mem. l'Acad.*) and much later by Gauss[36] in a somewhat specialized form that does not contain an expression for divergence. However, in 1813 Gauss obtained this formula in a special case, and an even more special case was considered by Lagrange,[37] in 1762. Formula (7.1) is a particular case of the well-known *Stokes–Poincaré* theorem on integration of differential forms on a manifold with boundary (see, for example, Zorich (2016), which is expressed by an easy-to-remember formula:[38]

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega. \tag{7.2}$$

This formula implies (7.1) with

$$\omega = \sum_k (-1)^{k-1} f_k(x)\, dx_1 \wedge \ldots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \ldots \wedge dx_n,$$

because

$$d\omega = \sum \frac{\partial f_k(x)}{\partial x_k}\, dx, \quad \text{and} \quad \omega|_{\partial\Omega} = \sum f_k(x)\alpha_k\, d\Gamma.$$

If $f_k(x) = A_k(x)v(x)$, where $v \in PC^2(\Omega) \cap C^1(\overline{\Omega})$, then from (7.1) it follows that

$$\int_\Omega v\Big(\sum_{k=1}^n \frac{\partial A_k}{\partial x_k}\Big)\, dx = -\int_\Omega \sum_{k=1}^n A_k \frac{\partial v}{\partial x_k}\, dx + \int_{\partial\Omega} v \sum_{k=1}^n A_k \alpha_k\, d\Gamma. \tag{7.3}$$

Putting $A_k = \frac{\partial u}{\partial x_k}$, where $u \in PC^2(\Omega) \cap C^1(\overline{\Omega})$, we get the *first Green formula*[39] for the Laplace operator,

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[35] Mikhail Vasilyevich Ostrogradsky (1801–1862) was the recognized leader of mathematicians of the Russian Empire of the mid-nineteenth century.

[36] Johann Carl Friedrich Gauss (1777–1855) was a German mathematician, physicist, astronomer, and geodesist, one of the greatest mathematicians of all times.

[37] Joseph Louis Lagrange (1736–1813) was a French mathematician, physicist, and mechanician of Italian origin, one of the greatest mathematicians of the eighteenth century, and the author of the classical "Analytic mechanics." He made enormous contributions to calculus, number theory, probability theory, and numerical methods. Legendre is one of the founders of variational calculus.

[38] Formula (7.2), which for $\omega = a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3$ coincides with the classical Stokes formula, was obtained in 1889 Henri Poincaré (1854–1912) in Vol. III of his "New Methods of Celestial Mechanics." The classical Stokes formula itself first appeared as a postscript to a letter from Sir William Thomson (1824–1907), Lord Kelvin, to his equally famous colleague Sir George Stokes (1819–1903). Stokes published it in 1854 as an exam question for Cambridge University students. The form $(-1)^k dx_1 \wedge \ldots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \ldots \wedge dx_n$ is an oriented area of the $(n-1)$-dimensional parallelogram with sides $dx_1, \ldots, dx_{k-1}, dx_{k+1}, \ldots, dx_n$, with the same orientation as the standard $\mathbb{R}^n$-orientation of the element $dx = dx_1 \wedge \ldots \wedge dx_n$.

[39] George Green (1793–1841) was an English mathematician and physicist.

$$\int_\Omega v\Delta u\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\,d\Gamma - \int_\Omega \sum_{k=1}^n \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k}\,dx, \tag{7.4}$$

where Δ is the *Laplace operator* (see §5). Subtracting from (7.4) Eq. (7.4) with swapped u and v, we get the *second Green formula* for the Laplace: operator

$$\int_\Omega (v\Delta u - u\Delta v)\,dx = \int_{\partial\Omega}\Big(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big)\,d\Gamma. \tag{7.5}$$

Setting $v \equiv 1$ in (7.5), we get the following remarkable corollary:

$$\int_\Omega \Delta u\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial \nu}\,d\Gamma. \tag{7.6}$$

In particular, if a function $u \in C^1(\overline{\Omega})$ is harmonic in Ω, then[40]

$$\int_{\partial\Omega}\frac{\partial u}{\partial \nu}\,d\Gamma = 0. \tag{7.7}$$

This is the *integral Gauss formula*. We rewrite (7.5) in the form

$$\int_\Omega u(y)\Delta v(y)\,dy = \int_\Omega v(y)\Delta u(y)\,dy + \int_{\partial\Omega}\Big[u(y)\frac{\partial v}{\partial \nu}(y) - v(y)\frac{\partial u}{\partial \nu}(y)\Big]\,d\Gamma, \tag{7.8}$$

take a point $x \in \Omega$, and replace in (7.8) the function v by the function $E_\alpha(x,\cdot) \in PC^2(\Omega)$, which depends on x as a parameter and satisfies the equation

$$\Delta_y E_\alpha(x,y) \equiv \sum_{k=1}^n \frac{\partial^2}{\partial y_k^2}E_\alpha(x,y) = \delta_\alpha(x-y), \tag{7.9}$$

where δ_α is defined in (1.3) and $1/\alpha \gg 1$. Taking into account Exercise 7.1 (see below), we make α to 0. As a result, using Lemma 2.1, we get

$$u(x) = \int_\Omega E(x-y)\Delta u(y)\,dy + \int_{\partial\Omega}\Big[u(y)\frac{\partial E(x-y)}{\partial \nu} - E(x-y)\frac{\partial u(y)}{\partial \nu}\Big]\,dy, \tag{7.10}$$

where

$$E(x) = \begin{cases} \dfrac{1}{2\pi}\cdot \ln|x| & \text{for } x \neq 0, n = 2, \\ -|x|^{2-n}/((n-2)\sigma_n) & \text{for } x \neq 0, n \geq 3. \end{cases} \tag{7.11}$$

P 7.1 Taking into account Exercise 5.2 and Theorem 5.14, show that the general solution of (7.9), which depends only on $|x-y|$, can be written in the form $E_\alpha(x-y) + \text{const}$, where for $|x| \geq \alpha$ the function $E_\alpha \in C^1(\mathbb{R}^n)$ coincides with the function (7.11), and the estimate $|E_\alpha(x)| \leq |E(x)|$ holds for $|x| < \alpha$. By σ_n we denote (see Exercise 1.1) the area of the unit sphere in $\mathbb{R}^n$.

[40] Formula (7.6) expresses the important fact: the total gradient flow of a harmonic function through the boundary of any body is zero (this is quite clear from everyday experience if one takes into account that the Laplace equation is the stationary heat equation).

Hint The function $f_\alpha : \rho \mapsto f_\alpha = \frac{dE_\alpha(\rho)}{d\rho}$ is continuous and $\rho\frac{df_\alpha}{d\rho} + (n-1)f_\alpha = \begin{cases} 0 & \text{for } \rho \geq \alpha \\ \frac{\rho}{\alpha^n|B_n|} & \text{for } 0 \leq \rho \leq \alpha\,. \end{cases}$ Changing $\rho = \ln t$, $g(t) = f_\alpha(\rho(t))$, we arrive at the equation $g' + (n-1)g = \begin{cases} 0 & \text{for } \rho = e^t \geq \alpha \\ \frac{e^t}{\alpha^n|B_n|} & \text{for } 0 \leq \rho = e^t \leq \alpha\,. \end{cases}$ Since f_α is continuous, we get

$$f_\alpha(\rho) = \begin{cases} C\rho^{-(n-1)} & \text{for } \rho \geq \alpha \\ C\alpha^{-(n-1)} + \frac{\rho}{n\alpha^n|B_n|} & \text{for } 0 \leq \rho \leq \alpha \end{cases}$$

and hence

$$E_\alpha(\rho) = \begin{cases} C\int \rho^{-(n-1)}\,d\rho & \text{for } \quad \rho \geq \alpha \\ \\ E_\alpha(\alpha) + \frac{\rho}{n\alpha^n|B_n|} & \text{for } 0 \leq \rho \leq \alpha\,. \end{cases}$$

From (7.6) and (1.3), we have

$$\int_{|x|=\alpha} \frac{\partial E_\alpha}{\partial \nu}\,d\Gamma = \int_{|x|<\alpha} \Delta E_\alpha\,dx = 1. \tag{7.12}$$

Now it remains to invoke (7.12) and (7.6).

In particular, for $n = 2$, we have $E_\alpha(\rho) = C\ln\rho + D$. By the assumption $E_\alpha \in C^1$, and hence

$$\frac{dE_\alpha(\rho)}{d\rho}\bigg|_{\rho=\alpha+0} = \frac{dE_\alpha(\rho)}{d\rho}\bigg|_{\rho=\alpha-0} = \frac{C}{\alpha}\,.$$

As a result, the multiplicative constant C is $\frac{1}{2\pi}$, i.e., $E_\alpha(\rho) = \frac{1}{2\pi}\ln\rho + D$, because

$$1 = \int_{|x|<\alpha} \Delta E_\alpha\,dx = \int_{\sqrt{x_1^2+x_2^2}=\alpha} \frac{dE_\alpha}{d\rho}\bigg|_{|x|=\alpha} dx = \int_0^{2\pi} d\varphi \int_0^\alpha \frac{C}{\alpha}\,d\rho = \frac{C}{\alpha}2\pi\alpha = 2\pi C\,.$$

Let $x \in \Omega$. Consider[41] the function $g(x,\cdot)\colon \overline{\Omega} \ni y \mapsto g(x,y)$, which is a solution of the following Dirichlet problem for the homogeneous Laplace equation with the special boundary condition:

$$\Delta_y g(x,y) = 0 \quad \text{in } \Omega, \qquad g(x,y) = -E(x-y) \quad \text{for } y \in \Gamma = \partial\Omega. \tag{7.13}$$

Next, we substitute the function $g(x,\cdot)$ in (7.8) in place of v and add the resulting equality to (7.10). As a result, we get the following integral representation of the function $u \in PC^2(\Omega) \cap C^1(\overline{\Omega})$:

$$u(x) = \int_\Omega G(x,y)\Delta u(y)\,dy + \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial \nu}u(y)\,d\Gamma; \tag{7.14}$$

here

$$G(x,y) = E(x-y) + g(x,y). \tag{7.15}$$

The function (7.15) is called the *Green function* of the Dirichlet problem for the Laplace equation

[41] In § 22 (see Corollary 22.33), we give a theorem on solvability of problems much more general than problem (7.13). In § 22, we also give a theorem on smoothness of solutions.

$$\Delta u = f \quad \text{in } \Omega, \qquad u = \varphi \text{ on } \partial\Omega. \tag{7.16}$$

This name is appropriate here, because by (7.14) the solution of problem (7.16), where $f \in PC(\Omega)$, $\varphi \in C(\partial\Omega)$, can be represented via the function G in the form

$$u(x) = \int_\Omega f(y)G(x, y)\, dy + \int_{\partial\Omega} \varphi(y) \frac{\partial G(x, y)}{\partial \nu}\, d\Gamma. \tag{7.17}$$

Formula (7.17) is frequently called the *Green formula*.

P 7.2 Let $\Omega = \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, and $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Let $x^* = (x', -x_n)$ be the reflection of the point x about the hyperplane $x_n = 0$. Verify that $G(x, y) = E(x, y) - E(x^*, y)$. Check (cf. formula (5.7)) that

$$-\frac{\partial G(x, y)}{\partial y_n}\Big|_{y_n=0} = \frac{2}{\sigma_n} \frac{x_n}{[(x_1 - y_1)^2 + \ldots + (x_{n-1} - y_{n-1})^2 + x_n]^{n/2}}.$$

Remark 7.3 For sufficiently general domains, it is quite difficult to evaluate and/or analyze the solution of, say, the Dirichlet problem with the use of the Green formula (7.14). However, this difficulty can be circumvented, to some extent, using the machinery of the so-called double layer potentials (see, for example, Maz'ya (1988)).

8 The Lebesgue Integrals

With the light hand of E. B. Dynkin,[42] who was the first to read the theory of the Lebesgue integral to the 2nd year mathematics students of the Faculty of Mechanics and Mathematics of Moscow State University in the fall of 1964 as part of the Calculus mandatory course, now the theory of the Lebesgue integral is usually included in the curriculum for junior mathematics students. However, perhaps some readers are not familiar with this topic, and for them we give the necessary results in this and the next sections. At the first reading, one can quickly read the definitions and results in these sections and then proceed further. Here at least two facts are worth noting: (1) if a function f is piecewise continuous in $\Omega \Subset \mathbb{R}^n$, then it is Riemann integrable and (2) a Riemann integrable function is also Lebesgue integrable and its Riemann integral coincides with its Lebesgue integral (see Lemma 8.21). As necessary (when it comes to limit transitions under the sign of the integral, changing the order of integration, etc.), it will be advisable to return to a more careful reading of §8 and 9 and the cited textbooks.

In §§1 and 2, we outlined the idea of representation (definition) of a function by its "averages." This idea is related to the concept of the integral. According to Cauchy, with any function f continuous on $[a, b]$, one may associate the number

[42] Evgeny Borisovich Dynkin (1924–2014) was a Soviet and American mathematician. He is known for his works in group theory and Lie algebras, as well as in probability theory, member of the US National Academy of Sciences, and honorary member of the Moscow Mathematical Society (1995).

known as the integral of f over $\Omega =]a, b[$. It is denoted by $\int_a^b f(x)\,dx$ and it is defined as the limit of the so-called integral sums. Namely, for any $k = 1, \ldots, N$, for each $\xi_k \in]a_k, b_k[=]a_k, a_{k+1}[\subset]a, b[$, where $a_1 = a$, $b_N = b$, there exists the limit $\lim_{N\to\infty} \sum_{k=1}^N f(\xi_k)(b_k - a_k)$ as the diameter of the partition of $]a, b[$ into subintervals $]a_k, b_k[$ tends to zero an $N \to \infty$, i.e., $\max_{1\le k\le N}(b_k - a_k) \to 0$.

Clearly, for a continuous function f, the above Cauchy definition of the integral $\int_a^b f(x)\,dx$ is equivalent to saying that as $N \to \infty$ the so-called *lower Darboux sum*[43]

$$S_N^-(f) = \sum_{k=1}^N m_k^-(f)(b_k - a_k), \qquad m_k^-(f) = \inf_{x\in[a_k,b_k]} f(x)$$

and the *upper Darboux sum*

$$S_N^+(f) = \sum_{k=1}^N m_k^+(f)(b_k - a_k), \qquad m_k^+(f) = \sup_{x\in[\alpha_k,\beta_k]} f(x)$$

have the property that $S_N^-(f) \uparrow S^-(f)$, $S_N^+(f) \downarrow S^+(f)$ and $S^-(f) = S^+(f) = \int_a^b f(x)\,dx$.

This property is not satisfied by the famous Dirichlet function, which is discontinuous everywhere

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases} \tag{8.1}$$

P 8.1 Verify that the limits of the lower and upper Darboux sums for the Dirichlet function are different. Hence this function is not Riemann integrable. Note that the Dirichlet function is different from some other (!) continuous (which functions?) on a nullset, and this function is not almost everywhere continuous. Moreover, $D(x)$ is everywhere discontinuous.

In connection with Dirichlet and Riemann's development of the concept of a function as a pointwise map to a numerical line, the question arose about the class of functions integrable in the Cauchy sense. The answer was given by Riemann, and this is why the integral introduced by Cauchy is called the Riemann integral. Namely, Riemann proved (see Lemma 8.16) that the limit of the Cauchy integral sums exists and is finite if and only if the function f is bounded and continuous *almost everywhere*. The phrase "*almost everywhere*" (abbreviated a.e.) means that some property $P(x)$, which depends on a point $x \in \Omega \subset \mathbb{R}^n$, holds everywhere except a set $A \subset \Omega$ of zero measure $\mu(A)$. In the case $\Omega \subset \mathbb{R}$, this means that, for any $\varepsilon > 0$, there exists a union of intervals $E = \bigcup_{k=1}^\infty \Pi_k$, $\Pi_k =]a_k, b_k[$ such that (1) $A \subset E$ and (2)$\sum_{k=1}^\infty \mu(\Pi_k) < \varepsilon$, where $\mu(\Pi_k) = b_k - a_k$.

P 8.2 Give the meaning in the n-dimensional case to the phase: "the measure of a set $A \subset \Omega \subset \mathbb{R}^n$ is zero" (the n-dimensional measure of the set A is zero).

[43] Jean Gaston Darboux (1842–1917) was a French mathematician. He worked in the field of calculus and differential geometry.

Hint See the definitions that follow.

Definition 8.3 A set $E \subset \mathbb{R}^n$ is *elementary* if it is a union of a finite or countable number of parallelepipeds

$$\Pi_k = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_j \in]a_{j_k}, b_{j_k}[\,\}, \quad k \in \mathbb{N}.$$

Definition 8.4 A set $A \subset \mathbb{R}^n$ is a *nullset* or *has measure zero* (more precisely, its *n-dimensional measure is zero*) if, for any $\varepsilon > 0$, there exists an elementary set $E = \bigcup_{k=1}^{\infty} \Pi_k$ such that (1) $A \subset E$ and (2) $\sum_{k=1}^{\infty} \mu(\Pi_k) < \varepsilon$, where $\mu(\Pi_k) = \prod_{j=1}^{n}(b_{j_k} - a_{j_k})$ is the measure (the volume) of the parallelepiped Π_k.

P 8.5 Verify that $A = \bigcup_{j \geq 1} A_j$ is a nullset if and only if so is any A_j. Prove as a corollary that the set of rational numbers $\mathbb{Q}$ is a nullset in $\mathbb{R}$, and the measure of the set of irrational numbers lying in a closed interval $[a, b]$ is $b - a$.

Hint Use the equality $\sum_{j \geq 1} \frac{\varepsilon}{2^j} = \varepsilon$.

Definition 8.6 The *Cantor*[44] *set* C_0 is a subset of the interval $[0, 1]$ obtained from the interval $[0, 1]$ by sequentially throwing out at the kth iteration step $(k \geq 0)$ 2^k intervals from the middle of the 2^k closed intervals remaining in the $(k-1)$st iteration step, and the length of each interval to be thrown out is $\frac{1}{3}$ of the length of the corresponding closed interval. So, the interval $\left]\frac{1}{3}, \frac{2}{3}\right[$ is thrown out of the closed interval $[0, 1]$ at the zeroth step, two middle intervals of length $\left(\frac{1}{3}\right)^2$ are thrown out of the remaining two closed intervals at the first step, and so on.

P 8.7 Verify that set C_0 has the cardinality of a continuum, i.e., there exists a one-to-one correspondence between C_0 and $[0, 1]$, and hence between C_0 and the entire real line $\mathbb{R}$. Verify that the set C_0 is a nullset (despite the fact that C_0 is in a one-to-one correspondence with $[0, 1]$); in other words, the measure of all the removed intervals (i.e., their total length) is 1.

Hint The points of the set C_0 (the end points of the removed intervals) are in a one-to-one correspondence with the representations of numbers in the ternary number system (in which only the digits 0, 1, and 2 are used) and, therefore, with the binary representations of numbers that use only the digits 0 and 1. But such infinite binary representations are in a one-to-one correspondence with the points from the closed interval $[0, 1] \supset C_0$.

Remark 8.8 An interval $]\alpha, \beta[\subset [c, d]$ is called the middle interval if its midpoint coincides with the midpoint of $[c, d]$. If the middle interval of length $\frac{a}{2}$, where $0 < a < 1$, is thrown out of the interval $[0, 1]$, and the throw the middle intervals of length $\frac{a}{8}$ out of the remaining two closed intervals, and continue this procedure by throwing out at the nth step the middle intervals of length $\frac{a}{2^{2n+1}}$ from the remaining 2^n closed intervals, we get the set C_a by throwing out from the $[0, 1]$ a countable number of intervals of total length $a < 1$. This shows that C_a is not a nullset, but this set, as well as C_0, is nowhere dense in $[0, 1]$ and closed.

44 Georg Ferdinand Ludwig Philipp Cantor (1845–1918) was a German mathematician. He was born in Saint Petersburg, Russia. Cantor's first works were devoted to Fourier series. In these studies, he created the theory of irrational numbers, which received wide recognition. In 1874, Cantor proved that the set of real numbers is uncountable.

Of course, in the case of discontinuous functions, the difference between $m_k^-(f) = \inf_{x\in[\alpha_k,\beta_k]} f(x)$ and $m_k^+(f) = \sup_{x\in[\alpha_k,\beta_k]} f(x)$ at the boundary points of the intervals $]a_k, b_k[$ does not vanish, in general, when making $N \to \infty$. However, a set A on which difference may appear is a nullset (according to Problem 8.5). Hence the above difference between $m_k^-(f)$ and $m_k^+(f)$ has no effect on the Darboux sums of the function f satisfies

$$f = f^- = f^+ \quad \text{almost everywhere in } \Omega =]a, b[\,, \tag{8.2}$$

and[45] $f_N^- \uparrow f^-$ a.e., $f_N^+ \downarrow f^+$ a.e., where[46] $f_N^\pm(x) = \sum_{k=1}^N m_k^\pm \cdot 1_{[a_k,b_k]}(x)$. In this case[47] a function $f : \Omega \to \mathbb{R}$ is called *Riemann integrable*, and the number $S^-(f) = S^+(f)$ is called the Riemann integral of the function f and is denoted as $\int_\Omega f(x)\,dx$ (or as $\int_a^b f(x)\,dx$ if $\Omega =]a, b[\subset \mathbb{R}$).

P 8.9 Verify that the Riemann function[48]

$$R(x) = \begin{cases} 1/n & \text{if } x = m/n \text{is an irreducible fraction,} \\ 0 & \text{otherwise (i.e., at irrational points)} \end{cases} \tag{8.3}$$

(1) Is continuous a.e.; more precisely, it is discontinuous at any rational x and continuous at any irrational x

(2) Is Riemann integrable, i.e., $S^-(R) = S^+(R)$

Hint

1. Let $N \in \mathbb{N}$ and let $M_N \subset [-N, -N+1] \cup [N-1, N]$ be the set of points of the form $\frac{m}{n}$, where $n \le 2^{N+1}$. Then $R(x) < 2^{-(N+1)}$ at the remaining points.

2. Let us compose the upper Darboux sum corresponding to a partition of the real line which involves the union $O = \bigcup O_N$ of intervals O_N containing the points of the set M_N; $\mu(O_N) < 2^{-N}$. We have $\sup_{x\in\mathbb{R}\setminus O} R(x) < 2^{-N}$. Hence $S^+(R) < \frac{2}{2^N}$ (because $R \le 1$). Hence $S^+(R) = 0$, since N is arbitrary.)

The above construction of the Riemann integral can be easily carried over to the multivariate case, where $\Omega \Subset \mathbb{R}^n$ is a domain, i.e., if its boundary $\partial\Omega$ is piecewise-smooth. To this end, one should, in place of the system of vector $]a_k, b_k[$, construct a system of disjoint parallelepipeds

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[45] The notation $f_N \uparrow f$ a.e. means that the nondecreasing sequence of functions $f_N : \Omega \to \mathbb{R}$ *converges almost everywhere* (a.e.) to a function $f : \Omega \to \mathbb{R}$, i.e., for almost all $x \in \Omega$,

$$f_1(x) \le f_2(x) \le \ldots \le f_N(x) \le \ldots \qquad \text{and} \qquad \lim_{N\to\infty} f_N(x) = f(x)\,.$$

The convergence $f_N \downarrow f$ a.e. is defined similarly.

[46] Here, as everywhere, 1_A is the characteristic function of a set A, i.e., $1_A = 1$ on A and $1_A = 0$ outside A.

[47] So by a function one means any function that differs from the given one on a nullset (a set of measure zero).

[48] To imagine the graph of this function, look from the origin at an array of one-dimensional "trees" of unit height planted at the lattice points in one quadrant of a square lattice (http://users.livejournal.com/-winnie/456636.html).

$$\Pi_k = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, \text{ where } a_{j_k} < x_j < b_{j_k} \}, \quad k \in \mathbb{N},$$

such that the closure of their union is connected and the volume[49] of the domain $\Omega_\Pi = \Omega \setminus \{\bigcup_{1 \le k \le N} \Pi_k\}$ tends to zero as $N \to \infty$.

Definition 8.10 By a *step function* in a domain $\Omega \subset \mathbb{R}^n$, we mean a function $f \colon \Omega \to \mathbb{R}$ which in Ω is a finite linear combination of the characteristic functions of some parallelepipeds $\Pi_k \subset \Omega$, $k = 1, \dots, N$, where $N \in \mathbb{N}$, i.e.,

$$f(x) = \sum_{k=1}^{N} c_k \cdot 1_{\Pi_k}(x), \quad c_k \in \mathbb{R}, \quad x \in \Omega. \tag{8.4}$$

In this case the sum $\sum_{k=1}^{N} c_k \cdot \mu(\Pi_k)$ is called the *integral of a step function* (8.4). We denote this sum by $\int_\Omega f(x)\,dx$, or shortly by $\int f(x)\,dx$, or sometimes simply by $\int f$.

Definition 8.11 Let $\{\Pi\} = \{\Pi_k\}_{k=1}^N$ be a system of disjoint parallelepipeds. Assume that the closure of their union is connected and that the volume of the domain $\Omega_\Pi = \Omega \setminus \left\{\bigcup_{1 \le k \le N} \Pi_k\right\}$ does not tend to zero as $N \to \infty$.

We say that $\{\Pi\} = \{\Pi_k\}_{k=1}^N$ is a partition $\partial\{\Pi\} \stackrel{\text{def}}{=} \bigcup_{k=1}^N \partial\Pi_k$ of Ω in cells of scale $\|\Delta_\Pi\| \stackrel{\text{def}}{=} \max_{1 \le k \le N} \left(\max_{1 \le j \le n} (b_{j_k} - a_{j_k})\right)$. We say that a partition $\partial\{\Pi^2\} = \bigcup_{k=1}^{N_2} \partial\Pi_k^2$ is embedded in $\partial\{\Pi^1\} = \bigcup_{j=1}^{N_1} \partial\Pi_j^1$, i.e., $\partial\{\Pi^1\} \supseteq \partial\{\Pi^2\}$ if, for any Π_k^2, there exists a Π_j^1 such that $\Pi_j^1 \supseteq \Pi_k^2$.

P 8.12 Reformulate in terms of the elements of the set $\mathbb{R}$ the previous definition with $n = 1$ and $\Omega =]a, b[\subset \mathbb{R}$.

Definition 8.13 Let $\partial\Pi = \bigcup_{k=1}^N \partial\Pi_k$ be a partition of a set $\Omega \subset \mathbb{R}^n$ in cells, and let a function f be bounded ($|f(x)| \le \text{const}$ for any $x \in \Omega$). The numbers

$$S_\Pi^+ = \sum_{k=1}^N m_k^+\, \mu(\Pi_k) \quad \text{and} \quad S_\Pi^- = \sum_{k=1}^N m_k^-\, \mu(\Pi_k),$$

where $m_k^+ = \sup_{x \in \Pi_k} f(x)$, $m_k^- = \inf_{x \in \Pi_k} f(x)$, are called, respectively, the *upper* and *lower Darboux sums* of the function f with respect to the partition $\partial\Pi$.

P 8.14 (a) Let[50] $\partial\{\Pi^1\} \supseteq \dots \supseteq \partial\{\Pi^n\} \dots$ Verify that $\bigcup_{n \ge 1} \partial\{\Pi^n\}$ is a nullset. Show that the sequence of the corresponding lower (upper) Darboux sums of a bounded function f is monotone

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[49] Here, we use the fact that $\partial\Omega$ is piecewise-smooth. In the general case of an open set Ω, its volume (or measure) can be defined if one can find families of inscribed and circumscribed unions of parallelepiped such that the volume of their (set-theoretic) difference can be arbitrarily small; in this case one says that Ω is measurable, and its measure (volume) is defined as the limit of inscribed (and, therefore, circumscribed) families of unions of parallelepipeds.

[50] See Definition 8.11.

nondecreasing (nonincreasing), and hence the limits $\lim_{n\to\infty} S^-_{\Pi^n}$ and $\lim_{n\to\infty} S^+_{\Pi^n}$ exist. Besides, the limits

$$S^-(f) = \lim_{n\to\infty} S^-_{\Pi^n}, \quad S^+(f) = \lim_{n\to\infty} S^+_{\Pi^n}$$

of these monotone sequences $S^-_{\Pi^n}(\uparrow)$ and $S^+_{\Pi^n}(\downarrow)$ are independent of the choice of a sequence $\{\partial\Pi^n\}_{n\geq 1}$ partitions if $\lim_{n\to\infty} \|\Delta^n_\Pi\| = 0$ (i.e., if all linear sizes of the partition cells $\partial\{\Pi^n\}$ tend to zero).

(b) Verify that the upper $S^+_{\Pi^n}$ and lower $S^-_{\Pi^n}$ Darboux sums of a function f with respect to a partition $\partial\{\Pi^n\} = \bigcup_{k=1}^{N_n} \partial\Pi^n_k$ are equal, respectively, to the integrals of the following step functions:

$$f^\pm_{\Pi^n} : \Omega \ni x \mapsto f^\pm_\Pi(x) = \sum_{k=1}^{N_n} m^\pm_k \cdot 1_{\Pi^n_k}(x); \tag{8.5}$$

besides, $f^-_{\Pi^n} \uparrow$ a.e. and $f^+_{\Pi^n} \downarrow$ a.e. if $\lim_{n\to\infty} \|\Delta^n_\Pi\| = 0$.

(c) Verify that $S^-(f) = S^+(f)$ if $f^-_{\Pi^n} \uparrow f$ a.e. and $f^+_{\Pi^n} \downarrow f$ a.e. if $\lim_{n\to\infty} \|\Delta^n_\Pi\| = 0$.

The following definition is suggested by assertion (c) of Exercise 8.14.

Definition 8.15 Assume that $f^-_{\Pi^n} \uparrow f^-$ a.e. and $f^+_{\Pi^n} \downarrow f^+$ a.e. if $\lim_{n\to\infty} \|\Delta^n_\Pi\| = 0$ and

$$f = f^- = f^+ \quad \text{a.e. in } \Omega. \tag{8.6}$$

Then the function $f : \Omega \to \mathbb{R}$ is called *Riemann integrable*, and $S^-(f) = S^+(f)$ is called the (*definite*) *Riemann integral* of the function f (written $\int_\Omega f(x)\,dx$).

Lemma 8.16 *A function $f : \Omega \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and continuous almost everywhere.*

Proof If step functions (8.5) are such that $f^-_{\Pi^m}(x) \uparrow f(x)$ and $f^+_{\Pi^m}(x) \downarrow f(x)$ for a.e. x, and if x_0 is a point of continuity of the step functions $f^-_{\Pi^m}$ and $f^+_{\Pi^m}$ (note that the other points form a nullset), then this point x_0 is also a point of continuity of f, and hence $|f(x_0)| < \infty$. Conversely, if f is almost everywhere continuous, then $f^-_{\Pi^m}(x_0) \uparrow f(x_0)$ and $f^+_{\Pi^m}(x_0) \downarrow f(x_0)$ at each point x_0 of continuity of the function f. Hence $S^-(f) = S^+(f)$. □

Thus, the space of Riemann integrable functions is very large. However, this space is incomplete with respect to the convergence defined by the Riemann integral, in the same way as the set of rational numbers (unlike the real numbers) is incomplete with respect to the Euclidean distance on the real line[51]

Indeed, setting

$$f_n(x) = \begin{cases} x^{-1/2} & \text{for } x \in]1/n, 1], \\ 0 & \text{for } x \in]0, 1/n], \end{cases} \tag{8.7}$$

we note that $\int_0^1 |f_m(x) - f_n(x)|\,dx \to 0$ as m and $n \to \infty$, i.e., $\{f_k\}$ is a Cauchy sequence with respect to the convergence defined by the Riemann integral. Moreover,

[51] The adjective "complete," referring to the concept of a complete metric space (a space is complete if any Cauchy sequence in this space converges to some element of this space; see, for example, Shilov (1965)) emphasizes that this space is free from even smallest "holes" (it is completely filled, like a vessel with water). However, mechanicists talk about continuous media, rather than complete media.

for any point $x \in]0,1[$, we have $f_k(x) \uparrow f(x) = x^{-1/2}$ and $\int_0^1 f_k(x)\,dx \le 2$. From Theorem 8.26 (see below), it follows that the function $f\colon x \mapsto x^{-1/2}$ is the only limit for the Cauchy sequence $\{f_k\}$, but this function is bounded, and hence for it the upper Darboux sum is infinite (see Definition 8.13), i.e., this function f is not Riemann integrable.

Thus, the space of Riemann integrable functions is incomplete with respect to the convergence defined by the Riemann integral (see also Exercise 8.33). This and a number of other serious reasons prompted (see, for example, Tumanov (1975)) the development of the concept of the integral. A special role is played here by the *Lebesgue integral*. In 1901, 26-year-old Lebesgue[52] introduced (see Definition 8.18) the space $L(\Omega)$ of Lebesgue integrable functions defined on an open set $\Omega \subset \mathbb{R}^n$. He also introduced the integral that now bears his name. He defined this integral axiomatically as a functional $\int\colon L(\Omega) \ni f \mapsto \int f \in \mathbb{R}$, which for $\Omega =]a,b[$ is denoted in the standard way and has the following consistent (as it was proved by Lebesgue) six properties:

(1) $\displaystyle\int_a^b f(x)\,dx = \int_{a+h}^{b+h} f(x-h)\,dx$ for any a, b and h.

(2) $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx = 0$ for any a, b and c.

(3) $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ for any a and b.

(4) $\displaystyle\int_a^b f(x)\,dx \ge 0$ if $f \ge 0$ and $b > a$.

(5) $\displaystyle\int_0^1 1\cdot dx = 1$.

(6) If, for any x, $f_n(x)$ converges increasingly to $f(x)$, then the integral of f_n tends to the integral of f.

To quote from Lebesgue (1904, § XII,1): "condition (6) plays a special role. It is not as simple or as necessary as the first five conditions." Nevertheless, it was condition (6) that became the cornerstone of Lebesgue's construction of his theory of integration.

Below we give an explicit construction of the Lebesgue integral, mainly following the scheme of P. J. Daniell[53] (see also Shilov and Gurevich (2013))[54] and Shilov (2016)).

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[52] Henri Léon Lebesgue (1875–1941) was a prominent French mathematician. He is best known as the author of the theory of integration. He also worked on dimension theory, theory of functions, theory of differentiation, and much more (see Tumanov 1975).

[53] Percy John Daniell (1889–1946) was a British mathematician. The scheme of construction of the Lebesgue integral, which was proposed by him in 1918 and which is equivalent to the Lebesgue construction, has advantages in generalizing the integral to objects more involved than functions (for example, linear functionals). He also obtained important results in the theory of random processes.

[54] Georgi Evgen'evich Shilov (1917–1974) was a professor at Moscow State University, a prominent specialist in theory of functions and partial differential equations, the author of world-famous monographs and textbooks.

P 8.17 Let $f_n^k \uparrow f^k$ a.e. and $\int f_n^k \leq$ const for $k = 1$ and $k = 2$. Verify that the numbers $\int f^1 = \lim_{n\to\infty} \int f_n^1$ and $\int f^2 = \lim_{n\to\infty} \int f_n^2$ are well defined.

Definition 8.18 A function f is called *Lebesgue integrable* in a domain Ω (written $f \in L(\Omega)$) if

$$f = f^1 - f^2 \quad \text{almost everywhere in } \Omega, \tag{8.8}$$

where f^1 and f^2 are the limits of almost everywhere nondecreasing sequences of step functions with bounded integrals, i.e., $f_n^k \uparrow f^k$ a.e. and $\int f_n^k \leq$ const for $k = 1$ and $k = 2$. The number $\int_\Omega f(x)\,dx = \int f^1 - \int f^2$, where $\int f^k = \lim_{n\to\infty} \int f_n^k$, is called the *Lebesgue integral* of the function f.[55]

Remark 8.19 A function $f\colon \Omega \to \mathbb{R}$ is *measurable* if $|f| < \infty$ a.e. and if f is the limit of an almost everywhere converging sequence of step functions. It can be shown (see, for example, Shilov (2016)) that a bounded measurable function is Lebesgue integrable.

P 8.20 Let $(x, y) \in \Omega =]0, 1[\times]0, 1[$ and let

$$f^1(x, y) = \frac{y^2}{(x^2 + y^2)}, \quad f^2(x, y) = \frac{x^2}{(x^2 + y^2)}.$$

Verify that the function $f = f^1 - f^2$ is Lebesgue integrable.

Lemma 8.21 *A Riemann integrable function is also Lebesgue integrable; its Riemann integral coincides with its Lebesgue integral*

Proof By the assumption, $f_{\Pi^n}^- \uparrow f^- = f = f^+ \leq f_{\Pi^1}^+$ a.e. in Ω and $S^-(f) = S^+(f)$. We have $\int f_{\Pi^n}^- \leq \int f_{\Pi^1}^+ < \infty$, and hence, putting $f^1 = f$, $f^2 = 0$, we verify (8.8) and the equality $S^+(f) = \int f$. □

P 8.22 Prove the following results:

1. If $f \in L(\Omega)$, then the functions $\max\{f, 0\}$ and $\min\{-f, 0\}$, and hence $|f| = \max\{f, 0\} - \min\{f, 0\}$, are Lebesgue integrable.
2. A function f is Lebesgue integrable if $|f| \in L(\Omega)$.

Hint If f_n is a step function, $f_n \uparrow f$ a.e. and $\int f_n \leq$ const, then the functions $\max\{f_n, 0\}$ are step functions. Besides,

$$\max\{f_n, 0\} \uparrow \max\{f, 0\} \text{ a.e.} \quad \text{and} \quad \int \max\{f, 0\} \leq \text{const}.$$

Corollary 8.23 *Let f and $g \in L(\Omega)$. Then* $\max\{f, g\}$ *and* $\min\{f, g\}$ *are Lebesgue integrable, because*

$$\max\{f, g\} = \frac{1}{2}[(f + g) + |f - g|] \quad \textit{and} \quad \min\{f, g\} = -\max\{-f, -g\}.$$

[55] One can verify that the so-defined integral does not depend either on the choice of difference of f^1 and f^2 or on the choice of approximating sequences f_n^k.

P 8.24 Verify that the Dirichlet function (8.1) is Lebesgue integrable. Find its integral.

P 8.25 Give an example of a sequence of functions $f_n \in L(\Omega)$ for which $\int_\Omega f_n(x)\,dx \not\to 0$, but $f_n(x) \to 0$ for any $x \in \Omega$.

The next two important results on the taking the limit under the integral sign i will be helpful in proving the completeness of the space $L = L(\Omega)$. For simple proofs of these results, see, for example, Shilov (2016).

Theorem 8.26 (Beppo Levi[56]) *Let* $\{f_n\}$ *be a sequence of functions* $f_n \in L = L(\Omega)$ *and* $f_{n+1}(x) \geq f_n(x)$ *for any* $n \in \mathbb{N}$ *and each* $x \in \Omega$*. If there is a constant* C *such that* $\int f_n \leq C$ *for any* n*, then the limit* $f(x) = \lim_{n\to\infty} f_n(x)$ *exists and is finite, and besides,* $f \in L$ *and* $\lim_{n\to\infty} \int f_n = \int f$*.*

Lemma 8.27 (P. Fatou[57]) *Let* $g_n \in L$*,* $g_n \geq 0$ *and* $g_n \to g$ *a.e. If* $\int g_n \leq C < \infty$ *for any* n*, then* $g \in L$ *and* $0 \leq \int g \leq C$*.*

P 8.28 Show by examples the importance of each condition in Beppo Levi's and Fatou's lemmas.

Considering the completeness of the space $L = L(\Omega)$, the natural distance between two functions f and g in this space is as follows:

$$\|f - g\| \overset{\text{def}}{=} \int_\Omega |f(x) - g(x)|\,dx. \tag{8.9}$$

P 8.29 Verify that the functional

$$\|\cdot\| \colon L \ni f \mapsto \|f\| \overset{(8.9)}{=} \int_\Omega |f(x)|\,dx \tag{8.10}$$

is a *norm*, i.e., it has the following properties: $\|f\| > 0$ for $f \neq 0 \in L$, $\|0\| = 0$, $\|\lambda f\| = |\lambda| \cdot \|f\|$ for any $\lambda \in \mathbb{R}$, $\|f + g\| \leq \|f\| + \|g\|$.

Definition 8.30 The *norm convergence* $f_n \to f$ as $n \to \infty$ in a normed space X equipped with a norm $\|\cdot\|$ is defined as $\|f_n - f\| \to 0$.

Definition 8.31 A normed space is *complete* if, for any *Cauchy sequence* $\{f_n\}_{n\geq 1}$ (this means that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$), there exists an $f \in X$ such that $\|f_n - f\| \to 0$. A complete normed linear space is called a *Banach space*.[58]

[56] Unlike the French mathematicians Maurice Lévy (1838–1910) and Paul Pierre Lévy (1886–1971), Beppo Levi (1875–1961) was an Italian mathematician. In addition to his works on the Lebesgue integral, B. Levy made a great contribution to the theory of the resolution of singularities of algebraic surfaces.

[57] Pierre Fatou (1878–1929) was a French mathematician. He made important contributions to the theory of the Lebesgue integral and to the development of iterations of rational functions of a complex variable, which led to Mandelbrot sets.

[58] Stefan Banach (1892–1945) was a Polish mathematician. He gave the definition of a normed space and obtained fundamental results for linear operators on Banach spaces.

Theorem 8.32 (The Riesz–Fischer Completeness Theorem[59]) *The space L equipped with the norm $\|\varphi\| = \int |\varphi|$ is a Banach space.*

Proof Let $\|\varphi_n - \varphi_m\| \overset{\text{def}}{=} \int |\varphi_n - \varphi_m| \to 0$ as $n, m \to \infty$. Then there exists an increasing sequence of indexes $\{n_k\}_{k\ge 1}$ such that $\|\varphi_n - \varphi_{n_k}\| \le 2^{-k}$ for all $n > n_k$. We set

$$f_N(x) = \sum_{k=1}^{N-1} |\varphi_{n_{k+1}}(x) - \varphi_{n_k}(x)|.$$

The sequence $\{f_N\}_{N=2}^{\infty}$ is increasing and $\int f_N \le 1$. By Beppo Levi's theorem, $f = \lim_{N\to\infty} f_N \in L(\Omega)$. Therefore, the series

$$\sum_{k=1}^{\infty} |\varphi_{n_{k+1}}(x) - \varphi_{n_k}(x)|$$

converges almost everywhere. Hence the series $\sum_{k=1}^{\infty} \big(\varphi_{n_{k+1}}(x) - \varphi_{n_k}(x)\big)$ also converges almost everywhere. In other words, for almost all x, there exists the limit[60] $\varphi(x) \overset{\text{def}}{=} \lim_{m\to\infty} \varphi_{n_m}(x)$. We claim that $\varphi \in L$ and $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$. For any $\varepsilon > 0$, there exists an $N \ge 1$ such that $\int |\varphi_{n_m}(x) - \varphi_{n_k}(x)|\, dx \le \varepsilon$ for $n_m \ge N$, $n_k \ge N$. Using Fatou's lemma and setting $g_{n_m}(x) = |\varphi_{n_m}(x) - \varphi_{n_k}(x)|$, we make $n_m \to \infty$. As a result, we get $|\varphi - \varphi_{n_k}| \in L$ and $\int |\varphi(x) - \varphi_{n_k}(x)|\, dx \le \varepsilon$. By Exercise 8.22, we have $\varphi - \varphi_{n_k} \in L$. Hence $\varphi \in L$ and $\|\varphi - \varphi_{n_k}\| \to 0$ as $k \to \infty$. It follows that $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$, because

$$\|\varphi - \varphi_n\| \le \|\varphi - \varphi_{n_k}\| + \|\varphi_{n_k} - \varphi_n\|.$$

P 8.33 Construct an example of a Cauchy sequence with respect to the convergence defined by the Riemann integral, but which has no limit with respect to this convergence, and which, unlike sequence (8.7), is *bounded*.

Solution The sequence of characteristic functions of the closed intervals remaining after the nth step in the construction of the Cantor set C_a of positive measure a (see Remark 8.8) is monotone decreasing. Hence by Beppo Levy's theorem and Lemma 8.21, it converges to the characteristic function of the set C_a. But this limit function, which is Lebesgue integrable, is not Riemann integrable, because any upper Darboux sum is $a > 0$, whereas its lower Darboux sum is zero (because the set C_a is nowhere dense).

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[59] In 1907, two important theorems were published in vol. 144 of C. R. Acad. Sci. Paris. The first theorem was proved by Frigyes Riesz (1880–1956), a Hungarian mathematician, one of the founders of functional analysis. Namely, he proved that the boundedness of the norm $\|f\|_2 \overset{\text{def}}{=} \big(\int |f(x)|^2\, dx\big)^{1/2}$ of a function f (i.e., the condition $|f|^2 \in L$) is equivalent to saying that the Fourier series of f converges in the norm $\|\cdot\|_2$.

The second theorem to the effect that the space L^2 is complete was proved by the Austrian mathematician Ernst Sigismund Fischer (1875–1954), a specialist in analysis and algebra. As a corollary of his theorem on completeness of L^2, Fischer derived the above Riesz theorem.

[60] So, a *Cauchy sequence in L contains a subsequence converging almost everywhere*. We will use this fact in the proof of Corollary 9.7. (See also Lemma 10.2.)

The next theorem on taking the limit under the integral sign is one of the most important theorems of the theory of Lebesgue integral.

Theorem 8.34 (The Lebesgue Dominated Convergence Theorem) *Let $f_n \in L(\Omega)$ and let $f_n(x) \to f(x)$ a.e. in Ω. If there exists a function $g \in L(\Omega)$ (called a majorant) such that $|f_n(x)| \le g(x)$ for all $n \ge 1$, then $f \in L(\Omega)$ and $\int f = \lim_{n\to\infty} \int f_n$.*

Proof Let $L(g) = \{\varphi \in L(\Omega) \mid -g \le \varphi \le g\}$. This set is closed with respect to taking monotone limits, because (by Beppo Levi's theorem) if $\varphi_n \in L(g)$ and $\varphi_n \uparrow \varphi^+$ or $\varphi_n \downarrow \varphi^-$, then the limit functions φ^+ and φ^- lie in $L(g)$. Note that, as $k \to \infty$,

$$L(g) \ni \max\{f_n, f_{n+1}, \ldots, f_{n+k}\} \uparrow F_n^+ \stackrel{\text{def}}{=} \sup\{f_n, f_{n+1}, \ldots\}$$

and

$$L(g) \ni \min\{f_n, f_{n+1}, \ldots, f_{n+k}\} \downarrow F_n^- \stackrel{\text{def}}{=} \inf\{f_n, f_{n+1}, \ldots\}.$$

Hence $F_n^\pm \in L(g)$. It is clear that $F_n^+ \downarrow$ and $F_n^- \uparrow$. So, for almost all $x \in \Omega$ (namely, for $x \in \Omega$ at which $f_n(x) \to f(x)$), we have

$$F_n^+(x) = \sup\{f_n(x), f_{n+1}(x), \ldots\} \downarrow f(x)$$

and

$$F_n^-(x) = \inf\{f_n(x), f_{n+1}(x), \ldots\} \uparrow f(x).$$

Since $L(g)$ is closed with respect to monotone limits, we find that $f \in L(g) \subset L$. And since $F_n^-(x) \le f_n(x) \le F_n^+(x)$ for almost all $x \in \Omega$, we have $\int F_n^- \le \int f_n \le \int F_n^+$. Now $\lim_{n\to\infty} \int f_n = \int f$ since $\int F_n^\pm \to \int f$. □

Remark 8.35 It can be shown that if a function f is Lebesgue integrable, then the characteristic function $1_{\{f \le a\}}$ of the set

$$\{f \le a\} \stackrel{\text{def}}{=} \{x \in \Omega \mid f(x) \le a\}$$

is also Lebesgue integrable.

P 8.36 Verify that the functions $1_{\{a<f\le b\}}$, $1_{\{f>b\}}$, $1_{\{f=c\}}$, which are defined similarly to $1_{\{f\le a\}}$, are Lebesgue integrable.
Hint Use the equality $1_{\{a<f\le b\}} = 1_{\{f\le b\}} - 1_{\{f\le a\}}$.

Definition 8.37 If the characteristic function 1_A of a set $A \subset \Omega$ is integrable, then the set A is called *measurable*; the number $\mu(A) = \int 1_A$ is called the (*Lebesgue*) *measure* of A.

Remark 8.38 It can be shown that a bounded open or closed set is measurable. A countable intersection of measurable sets is also measurable. And if a countable union of measurable sets is bounded, then this union is also measurable.

Below we will construct a set $A \subset]0, 1[\subset \mathbb{R}$ which is not (Lebesgue) measurable (i.e., the characteristic function of this set is not Lebesgue integrable).

The following two properties of the Lebesgue measure μ will be needed in the construction:

(1) The measure is invariant under translations, i.e., $\mu(A + x) = \mu(A)$ for any point $x \in \mathbb{R}$ and any set $A \subset \mathbb{R}$ of finite measure $\mu(A)$.

(2) The measure is countably additive, i.e., $\mu(A) = \sum_{j=1}^{\infty} \mu(A_j)$ if $A = \bigcup_{j=1}^{\infty} A_j$, where $\{A_j\}_{j=1}^{\infty}$ is a pairwise disjoint family of sets A_j of finite measure.

Now to construct the required set $A \subset]0, 1[$ suffices to choose from the interval $]0, 1[$ a countable set of disjoint subsets A_j such that $\mu(A_j) = \mu(A) > 0$ and $A = \bigcup_{j=1}^{\infty} A_j$. In this case, assuming the integrability of the characteristic function of the set A, we will get the contradiction: $\mu(A) = \sum_{j=1}^{\infty} \mu(A_j) = \infty$.

We define the set A (using the so-called axiom of choice of set theory) as the set of representatives of the cosets of the half-open interval $]0, 1]$, whose points are identified if they differ by a rational number. We take one representative from each coset. For example, denoting by $\{x\} = x - [x]$ the fractional part of a number x, we can take $\sqrt{2}$ as a representative of the coset $\{\sqrt{2} + \mathbb{Q}\}$. Let $r_1, r_2, \ldots$ be the sequence of all rational numbers. Now we defined the set A_j as the set of numbers of the form $\{x_A + r_j\}$, where $x_A \in A$.

Proposition 8.39 *If a function f is Lebesgue integrable and bounded* ($m \le f(x) \le M$)*, then the Lebesgue integral $\int f(x)\,dx$ is $\lim_{\sigma\to 0} S_\sigma(f)$, where $\sigma = \max_k(y_k - y_{k-1})$, $m = y_0 < y_1 < \ldots < y_{N_\sigma} = M$, and*

$$S_\sigma(f) = y_0\,\mu\{x\colon\ f(x) = y_0\} + \sum_{k=1}^{N_\sigma} y_k \mu\{x\colon\ y_{k-1} < f(x) \le y_k\},$$

i.e.,

$$\int f(x)\,dx = \lim_{\sigma\to 0} \int \left[y_0\, 1_{\{f(x)=y_0\}} + \sum_{k=1}^{N_\sigma} y_k\, 1_{\{y_{k-1}<f(x)\le y_k\}} \right] dx. \tag{8.11}$$

Proof Assume first that $N_\sigma = 1$. By (8.8), the function

$$F_1 \stackrel{\text{def}}{=} \left[m\, 1_{\{f(x)=y_0\}} + M\, 1_{\{m<f(x)\le M\}} \right]$$

can be written as $\varphi_1 - \psi_1$, and besides, $\varphi_{k1} \uparrow \varphi_1$ a.e. and $\psi_{k1} \uparrow \psi_1$ a.e., where φ_{k1} and ψ_{k1} are step functions. We will augment the partition with points from the closed interval $[m, M]$, thereby increasing the number $N = N_\sigma$. We will first show that the function

$$F_{N_\sigma} = y_0\, 1_{\{f(x)=y_0\}} + \sum_{k=1}^{N_\sigma} y_k\, 1_{\{y_{k-1}<f(x)\le y_k\}}$$

can be written as $\varphi_N - \psi_N$, and $\varphi_{kN} \uparrow \varphi_N$ a.e. and $\psi_{kN} \uparrow \psi_N$ a.e., where φ_{kN} and ψ_{kN} are step functions. To this end, we assume (without loss of generality) that $m = y_0 \ge 0$, and we add a new partition point $\widehat{y}_k$ such that $y_{k-1} < \widehat{y}_k < y_k$. Hence $F_{(N+1)_\sigma} = \varphi_{N+1} - \psi_{N+1}$, and besides $\varphi_{k(N+1)} \uparrow \varphi_{N+1} \ge \varphi_N$ a.e. and $\psi_{k(N+1)} \uparrow \psi_{N+1} \ge \psi_N$ a.e., where $\varphi_{k(N+1)}$ and $\psi_{k(N+1)}$ are step functions, because

$$\varphi_{N+1} = \varphi_N + \widehat{y}_k \, 1_{\{\widehat{y}_k < f(x) \leq \widehat{y}_k\}},$$

and

$$\psi_{N+1} = \psi_N + \Big[\big(y_k - \widehat{y}_k\big) \, 1_{\{y_{k-1} < f(x) \leq \widehat{y}_k\}} + \widehat{y}_k \, 1_{\{\widehat{y}_k < f(x) \leq y_k\}}\big)\Big].$$

So, $\varphi_N \uparrow \varphi$ a.e., $\psi_N \uparrow \psi$ a.e. and $\int \varphi_N \leq$ const and $\int \psi_N \leq$ const. Hence by Beppo Levi's theorem, $\varphi \in L$, $\int \varphi_N \to \int \varphi$, and $\psi \in L$, $\int \psi_N \to \int \psi$. Therefore, $F_{N_\sigma} = \varphi_N - \psi_N \to \varphi - \psi \in L$. Note that $|F_{N_\sigma}| \leq 2|\varphi - \psi|$ and $y_k \, 1_{\{y_{k-1} < f(x) \leq y_k\}} \to y_k \, 1_{\{f(x) = y_k\}}$ as $\sigma \to 0$. Hence $\int F_{N_\sigma} \to \int f$ by Theorem 8.34 (the Lebesgue dominated convergence theorem). □

P 8.40 Applying Proposition 8.39 and setting $y_k = \left(\frac{k}{n}\right)^2$, calculate

$$\int_0^1 x^2 \, dx.$$

Example 8.41 Let us evaluate the integral $A = \int_{-\infty}^{\infty} e^{-x^2} \, dx$. The squared integral is

$$\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dxdy \overset{(8.11)}{=} \lim_{\sigma_n = \frac{1}{2^n} \to 0} \sum_{k=1}^{k=2^n} \frac{k}{2^n} \mu \Big\{ \frac{k}{2^n} \leq e^{-(x^2+y^2)} < \frac{k+1}{2^n} \Big\}.$$

Here $\mu\{\frac{k}{2^n} \leq e^{-(x^2+y^2)} < \frac{k+1}{2^n}\}$ is the area of the annulus $\{r_2^2 \geq x^2 + y^2 > r_1^2\}$, where $r_2^2 = -\ln k\sigma_n$, and $r_1^2 = -\ln(k+1)\sigma_n$. So,

$$\mu \Big\{ \frac{k}{2^n} \leq e^{-(x^2+y^2)} < \frac{k+1}{2^n} \Big\} = \pi \big[\ln(k+1)\sigma_n - \ln k\sigma_n\big] = \pi \ln\Big(1 + \frac{1}{k}\Big).$$

Next, see [61] and so

$$\lim_{n\to\infty} \sum_{k=1}^{k=2^n} \frac{c_k}{2^n} = C, \quad \text{where} \quad C = \lim_{k\to\infty} c_k \,. \tag{8.12}$$

As a result,

$$A^2 = \lim_{n\to\infty} \sum_{k=1}^{k=2^n} \frac{k}{2^n} \big[\pi \ln(1 + \frac{1}{k})\big] = \pi \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

[61] Here is a simple proof of equality (8.12), which was shown to me by A. L. Pyatnitsky. Since $\lim_{k\to\infty} c_k < \infty$, it follows that $|c_k| < L$ for some $L > 0$ and so for any $\varepsilon > 0$ there exists a $k_0 > 0$ such that $|c_k - C| < \varepsilon$ if $k > k_0$. Next,

$$\Big|\sum_{k=1}^{k=2^n} \frac{c_k}{2^n} - C\Big| = \Big|\sum_{k=1}^{k=2^n} \frac{c_k - C}{2^n}\Big| \leq \Big|\sum_{k=1}^{k=k_0} \frac{c_k - C}{2^n}\Big| + \Big|\sum_{k=k_0}^{k=2^n} \frac{c_k - C}{2^n}\Big| \leq \frac{L + |C|}{2^n} k_0 + \varepsilon.$$

Hence $\limsup_{n\to\infty} |\sum_{k=1}^{k=2^n} \frac{c_k}{2^n} - C| \leq \varepsilon$, and since ε is arbitrary, we arrive at the required equality $\limsup_{n\to\infty} |\sum_{k=1}^{k=2^n} \frac{c_k}{2^n} - C| = 0$.

Theorem 8.42 (Fubini[62]) *Let Ω_x be an open set in $\mathbb{R}^k$ and Ω_y be an open set in $\mathbb{R}^m$. If $f\colon \Omega \ni (x, y) \mapsto f(x, y)$ is an integrable function in the direct product $\Omega = \Omega_x \times \Omega_y$, then*

(1) *For almost all $x \in \Omega_x$ (respectively, $y \in \Omega_y$), the function*

$$f(\cdot, y)\colon \Omega_x \ni x \mapsto f(x, y)$$

(respectively, $f(x, \cdot)\colon \Omega_y \ni y \mapsto f(x, y)$) lies in the space $L(\Omega_x)$ (respectively, in $L(\Omega_y)$).

(2) $\left(\int_{\Omega_x} f(x, \cdot)\, dx\right) \in L(\Omega_y)$ *and* $\left(\int_{\Omega_y} f(\cdot, y)\, dy\right) \in L(\Omega_x)$.

(3) $\int_{\Omega} f(x, y)\, dx\, dy = \int_{\Omega_y} \left[\int_{\Omega_x} f(x, y)\, dx\right] dy = \int_{\Omega_x} \left[\int_{\Omega_y} f(x, y)\, dy\right] dx.$

For a proof, see, for example, Shilov (2016).

Remark 8.43 The existence of the two repeated integrals

$$\int_{\Omega_y} \left[\int_{\Omega_x} f(x, y)\, dx\right] dy \quad \text{and} \quad \int_{\Omega_x} \left[\int_{\Omega_y} f(x, y)\, dy\right] dx$$

does not in general imply that they are equal or that the function f is integrable in $\Omega = \Omega_x \times \Omega_y$. This follows from the example of the function

$$f\colon\,]0, 1[\times]0, 1[\ni (x, y) \mapsto f(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \tag{8.13}$$

P 8.44 Verify directly that the function (8.13), unlike the function appearing in Exercise 8.20, is not Lebesgue integrable.

The next lemma is an analogue of the well-known theorem on permutation of repeated series with nonnegative terms.

P 8.45 Verify directly that the function (8.13), unlike the function appearing in Exercise 8.20, is not Lebesgue integrable.

The next lemma is an analogue of the well-known theorem on permutation of repeated series with nonnegative terms.

Lemma 8.46 *Let f be a measurable[63] and nonnegative function on $\Omega = \Omega_x \times \Omega_y$. Assume that there exists the repeated integral*

$$\int_{\Omega_x} \left[\int_{\Omega_y} f(x, y)\, dx\right] dy = A.$$

Then $f \in L(\Omega)$, and hence property (3) of Theorem 8.42 holds.

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[62] Guido Fubini (1879–1943) was an Italian mathematician. The main topic of his research was differential geometry.

[63] See Remark 8.19 on p. 41.

Proof The function f is measurable and bounded, $f_n = \min(f, n) \in L(\Omega)$. In view of Fubini's theorem, $\int f_n = \int_{\Omega_x} \left[\int_{\Omega_y} f_n(x, y)\, dx\right] dy \le A$. Note that $f_n \uparrow f$. Hence by Beppo Levi's theorem, $f \in L(\Omega)$. □

Theorem 8.47 (see Shilov 2016) *Let* $f \in L(\mathbb{R})$, $g \in L(\mathbb{R})$ *and let*

$$F(x) = \int_0^x f(t)\, dt, \quad G(x) = \int_0^x g(t)\, dt.$$

Then

$$\int_a^b F(x)g(x)\, dx + \int_a^b f(x)G(x)\, dx = F(b)G(b) - F(a)G(a).$$

Moreover, the function F *has, for almost all* $x \in \mathbb{R}$, *the derivative*

$$F'(x) = \lim_{h\to 0} \big(F(x + h) - F(x)\big)/h$$

and

$$F'(x) = f(x).$$

9 The Riesz Spaces L^p and L^p_{loc}

Definition 9.1 Let $1 \le p < \infty$. The *space* $L^p(\Omega)$ (or simply L^p) of *integrable* (or *pth power summable*) *functions* is the complex linear space of complex-valued functions[64] f defined on Ω and such that[65] $|f|^p \in L(\Omega)$. If $f \in L^1(\Omega)$, then the integral of f is defined by

$$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

Lemma 9.2 *Let* $p \in [1, \infty)$. *Then the mapping*

$$\|\cdot\|_p \colon L^p \ni f \mapsto \|f\|_p = \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p}, \tag{9.1}$$

which will be sometimes denoted by $\|\cdot\|_{L^p}$, *is a norm.*

Proof We need to verify the triangle inequality, i.e.,

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \tag{9.2}$$

[64] More precisely, the space of classes of functions $\{f\}\colon \Omega \to \mathbb{C}$, where $g \in \{f\} \iff g = f$ almost everywhere.

[65] The space L^p is frequently called the *Riesz space*, referring to Frigyes Riesz (see p. 43), who introduced this space and established its basic properties. His younger brother Marcel Riesz (1886–1969) was also a mathematician; his works related to Fourier series, Dirichlet series, mathematical physics, and Clifford algebra.

In the case of norm (9.1), this inequality is known as the *Minkowski inequality*[66]. For $p = 1$, this inequality is clear. Let us prove it for $p > 1$ based on the well-known (see, for example, Shilov (2016) and/or Kolmogorov and Fomin (1980) *Hölder inequality*[67]

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q, \quad \text{where } 1/p + 1/q = 1, \quad p > 1. \tag{9.3}$$

We have

$$\begin{aligned}\int |f+g|^p &\leq \int |f+g|^{p-1}|f| + \int (|f+g|^{p-1}|g|) \\ &\leq \Big[\int |f+g|^{(p-1)\cdot q}\Big]^{1/q} \Big\{\Big[\int |f|^p\Big]^{1/p} + \Big[\int |g|^p\Big]^{1/p}\Big\}.\end{aligned}$$

However,

$$\Big[\int |f+g|^{(p-1)\cdot q}\Big]^{1/q} = \Big[\int |f+g|^p\Big]^{1-(1/p)}.$$

Remark 9.3 The function (9.1) is not a norm for $p < 1$. Indeed if $f(x) = 1$ for $0 < x < 1$, $f(x) = 0$ for $-1 < x < 0$, and $g(x) = 1 - f(x)$, then

$$\|f + g\|_p = 2^{\frac{1}{p}} > \|f\|_p + \|g\|_p = 2.$$

The proof of the following result is similar to that of Theorem 8.32.

Lemma 9.4 *Let* $1 \leq p < \infty$. *The space* L^p, *equipped with the norm* (9.1), *is a Banach space.*

Lemma 9.5 *The complexification of the space of step functions*[68] *is dense in* L^p, $1 \leq p < \infty$.

Proof It suffices to show that, for any $f \in L^p$, $f \geq 0$, there exists a sequence $\{h_k\}$ of step functions such that

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[66] Hermann Minkowski (1864–1909) was a German mathematician, who developed the geometric theory of numbers and the geometric four-dimensional model of the theory of relativity, which was instrumental in a deep mathematical interpretation of the properties of the electromagnetic field.

[67] Otto Ludwig Hölder (1859–1937) was a German mathematician. He worked on the convergence of Fourier series and in 1884 discovered an inequality named after him. His name appears in the Hölder continuity condition, which is important in the analysis. Hölder's contribution to group theory is also significant. In particular, he proved the theorem (known as the Jordan–Hölder theorem) on uniqueness of factor groups in a composition series. In his obituary, van der Waerden writes "... to read Hölder's papers over and over again is a profound intellectual pleasure." ("Nachruf auf Otto Hölder," Mathematische Annalen 1939. V. 116. pp. 157–165). At the same time, Hölder, like another major German mathematician Bieberbach (who solved Hilbert's 18th problem), actively supported Nazism. In 1933, he signed the oath of allegiance of German professors to Adolf Hitler and the National Socialist state.

[68] The *complexification* of a real linear space X is the complex linear space of elements of the form $f = g + ih$, where g and $h \in X$.

$$\|f - h_k\|_p \to 0 \quad \text{as } k \to \infty. \tag{9.4}$$

In the case $p = 1$, we use Definition 8.18 and take a sequence $\{h_k\}_{k=1}^{\infty}$ such that $h_k \uparrow f \in L$ and $\int h_k \to \int f$. As a result, we get (9.4). For $1 < p < \infty$, consider the set

$$E_n = \left\{x \in \Omega\colon \frac{1}{n} \le f(x) \le n\right\}$$

and define $f_n(x) = 1_{E_n}(x) \cdot f(x)$, where 1_{E_n} is the characteristic function of the set E_n. We have $f_n \uparrow f$, and hence $(f - f_n)^p \downarrow 0$. By Beppo Levi's theorem,

$$\lim_{n\to\infty} \|f - f_n\|_p^p = \int_\Omega |f(x) - f_n(x)|^p \, dx \to 0.$$

Hence, for any $\varepsilon > 0$, there exists an $n \ge 1$ such that $\|f - f_n\|_p < \frac{\varepsilon}{2}$. We fix this n. Note that

$$\int 1_{E_n} = \int 1_{E_n}^p \le \int n^p |f|^p < \infty.$$

By Hölder inequality,

$$\int f_n = \int 1_{E_n} f \le \left(\int 1_{E_n}^q\right)^{1/q} \left(\int f^p\right)^{1/p} < \infty.$$

Since $f_n \in L(\Omega)$ and $f_n(x) \in [0, n]$ for any $x \in \Omega$, in Ω there exists a sequence $\{h_k\}$ of step functions with values in $[0, n]$ such that $\lim_{k\to\infty} \int |f_n - h_k| = 0$. It follows that

$$\|f_n - h_k\|_p = \left[\int |f_n - h_k|^p\right]^{1/p} = \left[\int \left(|f_n - h_k|^{p-1} |f_n - h_k|\right)\right]^{1/p}$$
$$\le n^{1-(1/p)} \left[\int |f_n - h_k|\right]^{1/p} \to 0 \quad \text{as } k \to \infty.$$

Choosing a K such that $\|f_n - h_k\|_p < \varepsilon/2$ for $k \ge K$, we have

$$\|f - h_k\|_p \le \|f - f_n\|_p + \|f_n - h_k\|_p < \varepsilon \quad \forall k \ge K.$$

Theorem 9.6 *Let $f \in L^1(\Omega)$, let $f = 0$ almost everywhere outside some $K \Subset \Omega$, and let $\rho > 2\varepsilon > 0$ be the distance between K and $\partial\Omega$. Next, let $\delta_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\delta_\varepsilon(x) \ge 0$, $\delta_\varepsilon(x) = 0$ for $|x| > \varepsilon$ and $\int \delta_\varepsilon = 1$. Then the function*[69]

[69] By the proposal of Nikolai Maximovich Günther (1871–1941), a Corresponding Member of the USSR Academy of Sciences, the function $R_\varepsilon(f)$ is called the *Steklov regularization* of a function f. This regularization was introduced in 1907 by Vladimir Andreevich Steklov (1864–1926) in his studies on the justification of the Fourier method for solving the main problems of mathematical physics with the help of the potential theory developed by him and in his proof of the closedness of the respective systems of eigenfunctions. Thus, by applying formula (9.5) in his studies, Steklov used the concept of the *δ-function* (see footnote 4 on p. 5) 20 years before P. Dirac. Soon Steklov's idea was developed by N. M. Günther, who created the theory of functions of domains, which was

$$R_\varepsilon(f) = f_\varepsilon : \Omega \ni x \mapsto f_\varepsilon(x) = \int f(y)\delta_\varepsilon(x-y)\,dy \tag{9.5}$$

lies in the space $C_0^\infty(\Omega)$. *Moreover,*

$$\lim_{\varepsilon\to 0} \|f - f_\varepsilon\|_p = 0, \quad 1 \le p < \infty. \tag{9.6}$$

Proof It is clear that $f_\varepsilon \in C_0^\infty(\Omega)$. Let us prove equality (9.6). By Lemma 9.5 for any $\eta > 0$, there exists a function $h = h_1 + ih_2$, where h_1 and h_2 are step functions, such that $\|f - h\|_p < \eta$. We have

$$\|f - f_\varepsilon\|_p \le \|f - h\|_p + \|h - R_\varepsilon(h)\|_p + \|R_\varepsilon(f-h)\|_p.$$

Let us show that $\|R_\varepsilon(g)\|_p \le \|g\|_p$. Since $\int_\Omega \delta_\varepsilon(x-y)\,dx = 1$, for $p = 1$, we have

$$\int_\Omega\left[\int_\Omega |g(y)|\cdot\delta_\varepsilon(x-y)\,dy\right]dx = \int_\Omega\left[\int_\Omega \delta_\varepsilon(x-y)\,dx\right]|g(y)|\,dy = \|g\|_1,$$

and for $p > 1$ we get

$$\begin{aligned}
\|R_\varepsilon(g)\|_p^p &= \int_\Omega |g_\varepsilon(x)|^p\,dx \\
&= \int_\Omega\left[\int_\Omega (\delta_\varepsilon(x-y))^{(p-1)/p)}(\delta_\varepsilon(x-y)^{1/p}|g(y)|)\,dy\right]^p dx \\
&\overset{(9.3)}{\le} \int_\Omega\left[\left(\int_\Omega \delta_\varepsilon(x-y)\,dy\right)^{(p-1)/p}\cdot\left(\int_\Omega \delta_\varepsilon(x-y)|g(y)|^p\,dy\right)^{1/p}\right]^p dx \\
&= \int_\Omega\left[\int_\Omega \delta_\varepsilon(x-y)|g(y)|^p\,dy\right]dx = \int_\Omega\left[\int_\Omega \delta_\varepsilon(x-y)\,dx\right]|g(y)|^p\,dy,
\end{aligned}$$

i.e., $\|R_\varepsilon(g)\|_p^p \le \int_\Omega |g(y)|^p\,dy$. So, $\|f - f_\varepsilon\|_p \le 2\eta + \|h - R_\varepsilon(h)\|_p$. Since $h = \sum_{k=1}^N c_k \cdot 1_{\Pi_k}$, where $c_k \in \mathbb{C}$, and since $\left[1_{\Pi_k} - R_\varepsilon(1_{\Pi_k})\right] = 0$ outside the ε-neighborhood of the parallelepiped Π_k, we find that

the forerunner of generalized functions, the brainchild of S. L. Sobolev, who became the student of N. M. Günther under the following circumstances.

In his lecture for second year students, Günther made a reference to Saltykov's theorems. The student Sobolev, who was listening to the lecture, approached Günther after the lecture and expressed doubts about the validity of Saltykov's theorems, which he had just heard at the lecture. Günther did not pay much attention to what has been said. But a week later, Sobolev brought a 22-page manuscript (which became his first scientific work, published 90 years later in the appendix to the book Soboleva and Chechkin (2017)), where counterexamples to Saltykov's theorems were constructed. After that, Sobolev became a student of Günther.

V. A. Steklov made important contributions in mechanics, quadrature formulas, and asymptotic methods. He was the initiator of the creation in 1921 and the first director of the Physical and Mathematical Institute of the Russian Academy of Sciences, which in 1934 was divided into two world-famous institutes: the Lebedev Physical Institute and the Steklov Mathematical Institute.

$$\|h - R_\varepsilon(h)\|_p^p = \left|\int_\Omega \sum_{k=1}^N c_k \cdot (1_{\Pi_k} - R_\varepsilon(1_{\Pi_k}))\right|^p dx$$
$$\leq \left(\sum_{k=1}^N |c_k|\right)^p \max_k \int_\Omega \left|1_{\Pi_k} - R_\varepsilon(1_{\Pi_k})\right| dx \leq C \cdot \varepsilon.$$

For $\varepsilon < \eta^p/C$, we get $\|h - R_\varepsilon(h)\|_p < \eta$. □

From Theorem 9.6, we have the following important corollary.

Corollary 9.7 *The space $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < \infty$.*

Proof Let $g \in L^p(\Omega)$. Note that, for any $\eta > 0$, there exists a $K \Subset \Omega$ such that $\|g - g \cdot 1_K\|_p < \eta$, and by Theorem 9.6 there exists an $\varepsilon > 0$ such that $\|g \cdot 1_K - R_\varepsilon(g \cdot 1_K)\|_p < \eta$. □

P 9.8 Let $u \in C(\overline{\Omega})$, i.e., u is continuous in $\overline{\Omega}$. Verify that

$$\|u - R_\varepsilon(u)\|_C \stackrel{\text{def}}{=} \sup_{x\in\Omega} \left|(u - R_\varepsilon(u))(x)\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Definition 9.9 Let $p \in [1, \infty[$. By $L^p_{\text{loc}}(\Omega)$ (or L^p_{loc}), we denote the *space of functions $f: \Omega \to \mathbb{C}$ that are pth power locally integrable*, i.e., $f \cdot 1_K \in L^p(\Omega)$ for all $K \Subset \Omega$. The space $L^p_{\text{loc}}(\Omega)$ is equipped with the *convergence*: $f_j \to f$ in $L^p_{\text{loc}}(\Omega)$ if $\|1_K \cdot (f_j - f)\|_p \to 0$ as $j \to \infty$ for all $K \Subset \Omega$.

Definition 9.10 $L^\infty(\Omega)$ is the *space of essentially bounded functions* in Ω, i.e., the space of $f \in L^1_{\text{loc}}(\Omega)$ satisfying

$$\|f\|_\infty = \inf_{\omega\in\Omega} \sup_{x\in\omega} |f(x)| < \infty, \quad \mu(\Omega \setminus \omega) = 0. \tag{9.7}$$

Condition (9.7) means that the function f is almost everywhere bounded, i.e., there exists an $M < \infty$ such that $|f(x)| \leq M$ almost everywhere. We also set $\|f\|_\infty = \inf M$.

P 9.11 Verify that formula (9.7) defines a norm in the space $L^\infty(\Omega)$ in which this space is a Banach space.

Remark 9.12 The use of the subscript ∞ in the notation of the space and norm (9.7) is justified by the fact that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$ if $\Omega \Subset \mathbb{R}^n$.

P 9.13 Verify that, for $1 < r < s < \infty$,

$$C \subsetneq L^\infty \subsetneq L^s_{\text{loc}} \subsetneq L^r_{\text{loc}} \subsetneq L^1_{\text{loc}}.$$

Definition 9.14 Let X be a normed space with norm $\|\cdot\|$. By X' we denote the space of continuous linear functionals on X. The space X' is called the *dual* space of X. Sometimes the dual of X is also denoted by $L(X;\mathbb{R})$ and $L(X;\mathbb{C})$ (in this notation it is clear whether $'X'$ is real or complex).

P 9.15 Verify that $(\mathbb{R}^n)' = L(\mathbb{R}^n; \mathbb{R})$ is isomorphic to $\mathbb{R}^n$.
Hint Any continuous linear functional (function) on $\mathbb{R}$ acts by the formula $\mathbb{R} \ni x \mapsto ax$, where a is a real number. On the other hand, any real number a defines a continuous linear functional $\mathbb{R} \ni x \mapsto ax \in \mathbb{R}$.

P 9.16 Verify that the space X' with the norm

$$\|f\|' = \sup_{x \in X} \frac{|\langle f, x\rangle|}{\|x\|},$$

where $\langle f, x\rangle$ is the value of f on $x \in X$, is a Banach space.
Hint Here it is not necessary that the normed space X be complete. However, in the proof we need the completeness of the real line.

Theorem 9.17 (F. Riesz) *If $1 \le p < \infty$, then $(L^p)' = L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ ($q = \infty$ for $p = 1$). More precisely,*

(1) *For any $f \in L^q(\Omega)$, there exists $F \in (L^p(\Omega))'$, i.e., a continuous linear functional F on $L^p(\Omega)$, such that*

$$\langle F, \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx \quad \forall \varphi \in L^p(\Omega). \tag{9.8}$$

(2) *For any $F \in (L^p(\Omega))'$, there exists a unique element (function) $f \in L^q(\Omega)$ for which equality (9.8) holds.*

(3) *The correspondence $I\colon (L^p)' \ni F \mapsto f \in L^q$ is an isometric isomorphism, i.e., I is a linear bijection and $\|IF\|_q = \|F\|'_p$.*

Proof Assertion (1) and the estimate $\|F\|'_p \le \|f\|_q$ are clear for $p = 1$. For $p > 1$, the Hölder inequality should be applied. The proof of assertion (2) and the estimate $\|F\|'_p \ge \|f\|_q$ require more work (see, for example, Yosida (1965)). □

10 $L^1_{loc}(\Omega)$ as the Space of Linear Functionals on $C_0^\infty(\Omega)$

The idea outlined in §§1 and 2 on representation of a function by its "averages" can now be formalized in a general form as the following theorem.

Theorem 10.1 *Any function $f \in L^1_{loc}(\Omega)$ is uniquely recovered (as an element of $L^1_{loc}(\Omega)$) from the linear functional*

$$\langle f, \cdot\rangle\colon C_0^\infty(\Omega) \ni \varphi \mapsto \langle f, \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx \in \mathbb{C}, \tag{10.1}$$

i.e., from the set of numbers $\langle f, \varphi\rangle$, where $\varphi \in C_0^\infty(\Omega)$.

Proof Assume that to one functional there correspond two different functions f_1 and f_2. Then $\int (f_1 - f_2)\varphi = 0$ for all $\varphi \in C_0^\infty$. Now the next Lemma 10.2 implies that $f_1 = f_2$ almost everywhere. □

Lemma 10.2 *Let* $f \in L^1_{\text{loc}}(\Omega)$*. If* $\int_\Omega f(x)\varphi(x)\,dx = 0$ *for all* $\varphi \in C_0^\infty(\Omega)$*, then* $f = 0$ *almost everywhere.*

Proof If $\omega \Subset \Omega$, then $|f| \cdot 1_\omega = f \cdot g$, where $g(x) = 1_\omega \cdot \exp[-i \arg f(x)]$. In particular, $g(x) = \operatorname{sgn} f(x) \cdot 1_\omega(x)$ for a real function f. As noted in the footnote on p. 43, there exists a sequence of functions $\varphi_n \in C_0^\infty(\Omega)$ such that $f \cdot \varphi_n \to f \cdot g$ almost everywhere in Ω as $n \to \infty$, and besides, $|\varphi_n| \leq 1$. By Lebesgue's theorem, $\int_\Omega f \cdot g = \lim_{n\to\infty} \int_\Omega f \cdot \varphi_n$. Next, $\int_\omega |f| = \int_\Omega f \cdot g$, and so $\int_\omega |f| = 0$, because $\int_\Omega f \cdot \varphi_n = 0$. This shows that $f = 0$ almost everywhere in ω. Now $f = 0$ almost everywhere in Ω, since $\omega \Subset \Omega$ is arbitrary. □

11 Simplest Hyperbolic Equations: Generalized Sobolev Solutions

In this section, on an example of the simplest partial differential equation $u_t + u_x = 0$, which is sometimes called the transfer equation, we illustrate one of the main achievements of the theory of generalized functions. Here, we are talking about a new understanding of a solution of a differential equation, more precisely, about a new (expanded) understanding of the differential equations. With this proviso, one can consider important problems of mathematical physics, which have no solution in the usual sense. This new understanding of equations of mathematical physics and their solutions, as formulated by S. L. Sobolev[70] in 1935 (see, for example, Sobolev (2008) under the name "generalized solutions," allows one, in particular, to prove the existence and uniqueness of the generalized solution of the Cauchy problem

$$Lu \equiv u_t + u_x = 0, \quad (x,t) \in \mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 : t > 0\}, \tag{11.1}$$

$$u\big|_{t=0} = f(x), \quad x \in \mathbb{R}, \tag{11.2}$$

for Eq. (11.1) for any function $f \in PC(\mathbb{R})$ (and even for $f \in L^1_{\text{loc}}$; see Theorem 11.3). Here, we also mention (see Exercise 11.10) the theorem on continuous dependence of the solution of this problem on $f \in L^1_{\text{loc}}(\mathbb{R})$.

[70] Sergey L'vovich Sobolev (1908–1989) was one of the greatest mathematicians of the twentieth century. At the age of 24, he became a corresponding member, and at the age of 30, an Academician of the USSR Academy of Sciences. In 1945, he was involved in the work on the creation of the atomic bomb, and in 1951, for outstanding achievements in this work, he was awarded the title of Hero of Socialist Labor. S. L. Sobolev was very gentle, benevolent person. In exams, he almost always gave excellent grades, very rarely good grades. But once he gave a bad grade on an exam in mathematical physics. According to M. I. Zelikin, the following story happened. When asked by one student to allow him to answer an exam question by simply reading out the relevant pages of Sobolev's textbook, Sobolev replied briefly: "Read, please." The student began to read. Sobolev listened approvingly, but suddenly at some point expressed doubt about the validity of what the student said. The latter eagerly began to say: "Well, it's quite obvious, because it's written here." Sobolev's response was: "Unfortunately, there is a typo here. I have to give you a bad grade."

Let us explain the essence of the problem. Equation (11.1) is equivalent to the system

$$u_t + u_x \cdot \frac{dx}{dt} = 0, \quad \frac{dx}{dt} = 1.$$

Hence the equality $\frac{d}{dt}u(t + a, t) = 0$ holds along the line $x = t + a$, where a is a real parameter. From this equality, we have $u(t+a, t) = u(a, 0)$ for any t. Moreover, $f(x) = \lim_{t\to+0} u(x, t)$, and hence[71] $u(x, t) = f(x-t)$. This formula defines a solution of problem (11.1), (11.2) if f is a differentiable function. The same formula shows that problem (11.1), (11.2) has no solution (differentiable or even continuous) if f is discontinuous, for example, if $f(x) = \theta(x)$, where $\theta\colon \mathbb{R} \ni x \mapsto \theta(x) \in \mathbb{R}$ is the *Heaviside function*,[72] which is defined as

$$\theta(x) = 1 \quad \text{for } x \geq 0 \quad \text{and} \quad \theta(x) = 0 \quad \text{for } x < 0. \tag{11.3}$$

However, problem (11.1), (11.2) with the initial function (11.3) appears (at least at the formal level) in the study of propagation of plane sound waves. The corresponding process is described by the so-called system of differential *acoustic equations*

$$u_t + \frac{1}{\rho} p_x = 0, \quad p_t + \rho \cdot c^2 u_x = 0, \quad \rho > 0, \quad c > 0. \tag{11.4}$$

Here ρ is the density, c is the characteristics of the compressible medium, and $u = u(x, t)$ and $p = p(x, t)$ are, respectively, the velocity and pressure at time t at the point x. Setting

$$\alpha = u + p/(\rho \cdot c), \quad \beta = u - p/(\rho \cdot c),$$

we get the equivalent system $\alpha_t + c\alpha_x = 0$, $\beta_t - c\beta_x = 0$ of two transfer equations. So, problem (11.1), (11.2) with the initial function (11.3) can be considered as the

[71] This formula implies that for each fixed t the graph of the function $x \mapsto u(x, t)$ can be obtained by translating by t the graph of the function f to the right along the x-axis. This is why Eq. (11.1) is called the transfer equation.

[72] Oliver Heaviside (1850–1925) was an English autodidactic electrical engineer, mathematician, and physicist. In 1892, he published works on the application of the symbolic calculus (formal operations with the symbol p of the differential operator d/dt), which was popular in the middle of the nineteenth century, to solving problems on the theory of the propagation of electrical vibrations in wires. Unlike his predecessors, Heaviside defined the inverse operator unambiguously, setting $\frac{1}{p}f(t) = \int_0^t f(u)\,du$ and assuming that $f(u) = 0$ for $u < 0$. So, replacing the differentiation d/dt in the equation $\dot{x} - x = 1$ by the multiplication by p, we get after formal transformations

$$x = \frac{1}{p-1} = \frac{1}{p}\Big(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots\Big),$$

and hence, considering what has been said about the symbol $\frac{1}{p}$ and assuming that $x(t) = 0$ for $t < 0$, we get

$$x(t) = \int_0^t \Big(1 + t + \frac{t^2}{2} + \ldots\Big) = e^t - 1.$$

problem of propagation of acoustic waves with initial velocity $u(x,t) = \theta(x)$ and zero initial pressure.

P 11.1 Verify that any C^1-solution of system (11.4) can be written as

$$\begin{aligned} u(x,t) &= \frac{\varphi(x-ct)+\psi(x+ct)}{2}, \\ p(x,t) &= \frac{\varphi(x-ct)-\psi(x+ct)}{2}, \quad \text{where } \varphi \in C^1,\ \psi \in C^1. \end{aligned} \tag{11.5}$$

P 11.2 Prove the following theorem.

Theorem 11.3 *For any functions $f \in C^1(\mathbb{R})$, $F \in C(\overline{\mathbb{R}^2_+})$, the Cauchy problem*

$$u_t + u_x = F(x,t) \quad \text{in } \mathbb{R}^2_+, \quad u\big|_{t=0} = f(x), \quad x \in \mathbb{R},$$

has a unique solution $u \in C^1(\overline{\mathbb{R}^2_+})$.

As already mentioned, for $f(x) = \theta(x)$, problem (11.1), (11.2) has no regular solution (i.e., a solution in the usual sense). However, the argument leading to the formula $u(x,t) = f(x-t)$ (and the formula itself) suggests that the function $f(x-t)$ should be called a solution to problem (11.1), (11.2) for any function $f \in PC(\mathbb{R})$ (and even for $f \in L^1_{\text{loc}}(\mathbb{R})$). Moreover, the following lemma holds.

Lemma 11.4 *Let $f \in L^1_{\text{loc}}(\mathbb{R})$, and let $\{f_n\}$ be a sequence of functions $f_n \in C^1(\mathbb{R})$ such that*[73]

$$f_n \to f \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}) \quad \text{as } n \to \infty.$$

Then the function $u\colon \mathbb{R}^2_+ \ni (x,t) \mapsto u(x,t) = f(x-t)$ lies in $L^1_{\text{loc}}(\mathbb{R}^2_+)$, and besides, $u = \lim_{n\to\infty} u_n$ in $L^1_{\text{loc}}(\mathbb{R}^2_+)$, where $u(x,t) = f(x-t)$, and $u_n(x,t)$ satisfies Eq. (11.1) and the initial condition $u_n\big|_{t=0} = f_n(x)$.

Proof It suffices to consider case $f \geq 0$, because $f = f^1 + if^2$, and $f^k = f^k_+ - f^k_-$, where $f^k_\pm = \max(\pm f^k, 0)$. Putting $y = x - t$, we change the variables $(x,t) \mapsto (y,t)$. Note that $u(x,t) = f(y)$ and

$$\int_a^b \left(\int_c^d u(x,t)\,dx \right) dt \leq \int_a^b \left(\int_{c-b}^{d-a} f(y)\,dy \right) dt < \infty$$

for any a, b, c, d such that $0 < a < b$, $c < d$. Hence, by Lemma 8.46, we have $u \in L^1_{\text{loc}}(\mathbb{R}^2_+)$. Next, for the same a, b, c, d,

$$\int_a^b \int_c^d |u_n(x,t) - u(x,t)|\,dx\,dt \leq (b-a) \int_{c-b}^{d-a} |f_n(y) - f(y)|\,dy \to 0$$

as $n \to \infty$. □

[73] As the editor of the book E. D. Kosov noted, here one may take $f_n \in C_0^\infty(|x| < n)$ such that $\|f - f_n\|_{L^1(|x|<n)} < 1/n$.

Of course, one can simply define the solution of problem (11.1), (11.2) by the formula $u(x,t) = f(x-t)$ even for $f \in L^1_{\mathrm{loc}}$. However, this definition has a major drawback: using a concrete formula one can determine the solution only for a narrow class of problems. Lemma 11.4 suggests an approximative definition devoid of this drawback.

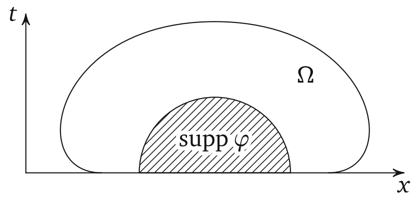

Fig. 1.4: The support supp φ of φ lies in $\overline{\Omega}$

Definition 11.5 Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. We say that a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is a *generalized solution* of problem (11.1), (11.2) if there exists a sequence of solutions $u_n \in C^1(\overline{\mathbb{R}^2_+})$ of Eq. (11.1) such that

$$u_n \to u \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^2_+) \quad \text{and} \quad u_n\big|_{t\to 0} \to f \text{ in } L^1_{\mathrm{loc}}(\mathbb{R})$$

as $n \to \infty$.

The approximative approach to the definition of a generalized solution can be applied to a wide class of problems. So, this method was used above (without explicitly mentioning), for example, when constructing a generalized solution to the equation $\Delta E(x) = \delta(x)$ (see formula (7.13)), and also the generalized solutions of the problem $\Delta P = 0$ in $\mathbb{R}^2_+$, $P(x,0) = \delta(x)$ (see Remark 5.5). However, the approximative definition, despite its technical convenience, has a significant drawback: it does not reveal the real mathematical object, the "generalized" differential equation, whose direct solution is the "generalized solution" to be defined.

It is reasonable to search for a suitable definition of the generalized solution of differential equations (and the corresponding "generalized" differential equations) by analyzing the derivation of equations of mathematical physics (within the framework of a particular concept of a continuous medium). The analysis conducted in §§1 and 2 (see Lemma 10.2) and also the *Ostrogradsky–Gauss formula* (7.3) suggest (as will be seen from Proposition 11.7) the following definition.

Definition 11.6 Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. A function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ is called a *generalized solution* of problem (11.1), (11.2) if it satisfies the integral identity

$$\int_{\mathbb{R}^2_+} (\varphi_t + \varphi_x) u(x,t)\, dx\, dt + \int_{\mathbb{R}} \varphi(x,0) f(x)\, dx = 0 \quad \forall \varphi \in C^1_0(\overline{\mathbb{R}^2_+}). \tag{11.6}$$

Proposition 11.7 *If* $u \in C^1(\overline{\mathbb{R}^2_+})$*, then condition* (11.6) *is equivalent to problem* (11.1)*,* (11.2) *in its classical sense.*

Proof Let $\varphi \in C^1_0(\overline{\mathbb{R}^2_+})$, and let Ω be a bounded domain in $\mathbb{R}^2_+$ with boundary $\Gamma = \partial\Omega$. From (7.3), it follows that

$$\int_\Omega \left[(u_t + u_x)\varphi + (\varphi_t + \varphi_x)u\right] dx\, dt = \int_{\partial\Omega} (\varphi \cdot u)[\cos(\nu, t) + \cos(\nu, x)]d\Gamma. \quad (11.7)$$

If $\operatorname{supp}\varphi \subset \overline{\Omega}$ and (see Fig. 1.4)

$$(\operatorname{supp}\varphi \cap \partial\Omega) \subset \mathbb{R}_x = \{(x, t) \in \mathbb{R}^2 : t = 0\},$$

then (11.7) can be written as

$$\int_{\mathbb{R}^2_+} (u_t + u_x)\varphi\, dx\, dt + \int_{\mathbb{R}^2_+} (\varphi_t + \varphi_x)u\, dx\, dt = -\int_{\mathbb{R}} (\varphi u)\big|_{t=0}\, dx. \quad (11.8)$$

From (11.6) and (11.8), we get the equality

$$\int_{\mathbb{R}^2_+} (u_t + u_x)\varphi\, dx\, dt = \int_{\mathbb{R}} [f(x) - u(x, 0)]\varphi(x, 0)\, dx,$$

which holds for any function $\varphi \in C^1_0(\overline{\mathbb{R}^2_+})$ and, in particular, for any function $\varphi \in C^1_0(\mathbb{R}^2_+)$. Hence

$$(11.1) \iff \int_{\mathbb{R}^2_+} (u_t + u_x)\varphi\, dx\, dt = 0 \quad \forall\varphi \in C^1_0(\mathbb{R}^2_+),$$

and therefore,

$$(11.2) \iff \int_{\mathbb{R}} f(x)\varphi(x, 0)dx = \int_{\mathbb{R}} u\big|_{t=0} \cdot \varphi(x, 0)dx \ \ \forall\varphi \in C^1_0(\overline{\mathbb{R}^2_+}).$$

Hence by Lemma 10.2 we have the implication (11.6) $\Leftrightarrow$ (11.1), (11.2). □

Proposition 11.7 shows that Definition 11.6 is consistent with the definition of the usual (differentiable, or regular) solution of problem (11.1), (11.2). The following Theorem 11.9 justifies what is new in Definition 11.6 and also shows that the integral identity (11.6) is the "generalized" differential equation discussed above.

Remark 11.8 The proof of Proposition 11.7 goes back to the Lagrangian derivation of the Euler–Lagrange equation and the transversality conditions in the calculus of variations (see Shilov 2016; Gindikin 2007).

Theorem 11.9 *For any function* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$*, problem* (11.1), (11.2) *has a generalized solution* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$*, and this solution is unique.*

Proof Let us first prove the *existence*. The function

$$u_n \colon \mathbb{R}^2_+ \ni (x,t) \mapsto u_n(x,t) = f_n(x-t)$$

is a regular solution of Eq. (11.1)) satisfying the initial condition $u_n\big|_{t=0} = f_n(x)$. Hence, by Proposition 11.7, we have

$$\int_{\mathbb{R}^2_+} (\varphi_t + \varphi_x) \cdot u_n \, dx\, dt + \int_{\mathbb{R}} f_n(x)\varphi(x,0)\, dx = 0 \quad \forall \varphi \in C^1_0(\overline{\mathbb{R}^2_+}). \tag{11.9}$$

On the other hand, by Lemma 11.4, the sequence $\{u_n\}_{n=1}^{\infty}$ converges in $L^1_{\text{loc}}(\mathbb{R}^2_+)$ to a function u such that $u(x,t) = f(x-t)$. It remains to check that this function $u \in L^1_{\text{loc}}(\mathbb{R}^2_+)$ satisfies (11.6). To this end, we note that, for any function $\varphi \in C^1_0(\overline{\mathbb{R}^2_+})$, there exist $a_\varphi > 0$ and $b_\varphi > 0$ such that

$$\operatorname{supp} \varphi \subset \{(x,t) \in \mathbb{R}^2 \colon |x| \le a_\varphi, 0 \le t \le b_\varphi\}.$$

Hence

$$\begin{aligned} &\left| \int_{\mathbb{R}^2_+} (u_n(x,t) - u(x,t))(\varphi_t + \varphi_x)\, dx\, dt \right| \\ &\qquad \le [\max_{(x,t)} |\varphi_t + \varphi_x|] \cdot \int_0^{b_\varphi} \left(\int_{-a_\varphi}^{a_\varphi} |f_n(x-t) - f(x-t)|\, dx \right) dt \\ &\qquad \le M_\varphi \cdot b_\varphi \int_{|x| \le a_\varphi + b_\varphi} |f_n(x) - f(x)|\, dx \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Now (11.6) follows from (11.9).

Let us prove the *uniqueness*. Let u_1 and u_2 be two generalized solutions of problem (11.1), (11.2). Then their difference $u = u_1 - u_2$ satisfies the relation $\int_{\mathbb{R}^2_+} (\varphi_t + \varphi_x) u\, dx\, dt = 0$ for any function $\varphi \in C^1_0(\overline{\mathbb{R}^2_+})$. Let us show that $u(x,t) = 0$ almost everywhere. By Lemma 10.2, it suffices to show that the equation

$$\varphi_t + \varphi_x = g(x,t), \quad (x,t) \in \mathbb{R}^2_+, \tag{11.10}$$

has a solution $\varphi \in C^1_0(\overline{\mathbb{R}^2_+})$ for any function $g \in C^\infty_0(\mathbb{R}^2_+)$. Let $T > 0$ be such that $g(x,t) \equiv 0$ if $t \ge T$. Setting

$$\varphi(x,t) = \int_T^t g(x - t + \tau, \tau)\, d\tau,$$

it is clear (see Fig. 1.5) that $\varphi \in C^1_0(\overline{\mathbb{R}^2_+})$ and φ is a solution of Eq. (11.10). □

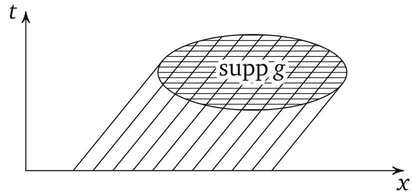

Fig. 1.5: supp φ is a shift of supp g along the characteristic of Eq. (11.10)

P 11.10 Verify that the generalized solution of problem (11.1), (11.2) depends continuously in $L^1_{\text{loc}}(\mathbb{R}^2_+)$ on the initial function $f \in L^1_{\text{loc}}(\mathbb{R})$.

P 11.11 By analyzing the proof of Theorem 11.9, show that Definition 11.6 is equivalent to Definition 11.5.

P 11.12 Using Definition 11.6, verify that the solution of problem (11.1), (11.2) is given by $u(x,t) = \theta(x-t)$ if $f(x) = \theta(x)$.

In the following exercises, $Q = \{(x,t) \in \mathbb{R}^2 : x > 0, t > 0\}$.

P 11.13 Consider the problem (see Fig. 1.6)

$$u_t + u_x = 0 \qquad \text{in } Q, \tag{11.11}$$

$$u\big|_{t=0} = f(x), \quad x > 0, \tag{11.12}$$

$$u\big|_{x=0} = h(t), \quad t > 0. \tag{11.13}$$

This problem is called *mixed*, because it simultaneously involves the *initial condition* (11.12) and the *boundary condition* (11.13). Verify that problem (11.11)–(11.13) has a unique solution $u \in C^1(\overline{Q})$ if and only if $f \in C^1(\overline{\mathbb{R}_+})$, $h \in C^1(\overline{\mathbb{R}_+})$, and $f(0) = h(0)$, $f'(0) = -h'(0)$.

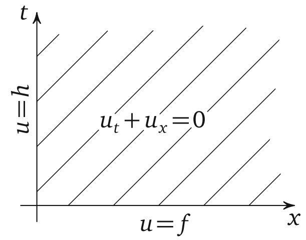

Fig. 1.6: The characteristics of the equation $u_t + u_x = 0$

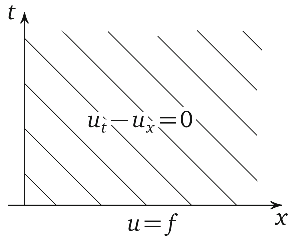

Fig. 1.7: The characteristics of the equation $u_t - u_x = 0$

P 11.14 Verify that the problem (see Fig. 1.7)

$$u_t - u_x = 0 \qquad \text{in } Q, \tag{11.14}$$

$$u\big|_{t=0} = f(x), \quad x > 0, \tag{11.15}$$

has a unique solution $u \in C^1(\bar{Q})$ if and only if $f \in C^1(\overline{\mathbb{R}_+})$. Cf. problem (11.11)–(11.13). Compare the *characteristics*, i.e., the families of lines $dx/dt = 1$ and $dx/dt = -1$ (see the figure) along which the solutions of Eqs. (11.11) and (11.14) are constant.

P 11.15 Consider the mixed problem for the system of acoustic equations

$$u_t + (1/\rho)p_x = 0, \quad p_t + \rho c^2 u_x = 0, \quad (x,t) \in Q, \tag{11.16}$$

$$u\big|_{t=0} = f(x), \quad p\big|_{t=0} = g(x), \quad x > 0, \tag{11.17}$$

$$p\big|_{x=0} = h(t), \quad t > 0, \tag{11.18}$$

where f, g, and h are functions from $C^1(\overline{\mathbb{R}_+})$.

1. Draw the level lines of the functions $u \pm \frac{1}{\rho c} p$.

2. Verify that problem (11.16)–(11.18) has a unique solution $u \in C^1(\bar{Q})$, $p \in C^1(\bar{Q})$ if and only if

$$h(0) = g(0) \quad \text{and} \quad f'(0) + \frac{1}{\rho c^2} h'(0) = 0. \tag{11.19}$$

Verify that this solution (u, p) is given by (11.5), where

$$\varphi(y) = f(y) + \frac{1}{\rho c} g(y), \quad \psi(y) = f(y) - \frac{1}{\rho c} g(y) \quad \text{if } y > 0, \tag{11.20}$$

and

$$\varphi(y) = \frac{2}{\rho c} h(-y/c) + f(-y) - \frac{1}{\rho c} g(-y) \quad \text{if } y \le 0. \tag{11.21}$$

Remark 11.16 Frequently, instead of the system (11.4) of acoustic equations, one considers the second-order equation

$$p_{tt} - c^2 p_{xx} = 0. \tag{11.22}$$

This equation, which clearly follows from system (11.4) if $p \in C^2$, $u \in C^2$, is called the *equation of the string* because the graph of the function p can be interpreted as small vibrations of a string (a one-dimensional object). The oscillations of an n-dimensional body are described by the *wave equation*

$$p_{tt} - c^2 \Delta p = 0, \quad p = p(x,t), \quad x \in \mathbb{R}^n, \ t > 0. \tag{11.23}$$

Here Δ is the *Laplace operator*. For $n = 2$, the wave equation describes the vibrations of a membrane, and for $n = 3$, the vibrations of a three-dimensional medium.

P 11.17 Following the constructions that follow from (11.5), show that the general solution of the equation of the string (11.22) has the form

$$p(x,t) = f(x - ct) + g(x + ct). \tag{11.24}$$

Express f and g in terms of the initial form and the initial velocity of the string given by $p\big|_{t=0} = \varphi(x)$ and $p\big|_{t=0} = \psi(x)$. Verify the formula

$$p(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\xi)\, d\xi, \tag{11.25}$$

which was first discovered by Euler in 1748 (but which is commonly called the d'Alembert's formula, since d'Alembert proved (11.24) in 1747).

Given $\varphi(x) = 0$, $\psi(x) = \begin{cases} |x| - 1 & \text{if } |x| < 1 \\ 0 & \text{if } 1 < |x| \end{cases}$ draw the graph of the function $x \mapsto u(t, x)$ for various t and compare with what you can see when a pebble is thrown into a calm pond.

Remark 11.18 The equation of the string was the first partial differential equation appeared in mathematics (thanks to B. Taylor).[74] This equation was the source of a lengthy but extremely fruitful discussion (see Narasimhan 1990; Luzin 1935), in which the concept was maturated (by such classics as d'Alembert, Euler, D. Bernoulli, Fourier, Riemann, and many others).

Let us return to problem (11.16)–(11.18). Consider the case $f = g = 0$, $h = 1$, in which initially (at $t = 0$) the velocity u and the pressure p are zero, and on the

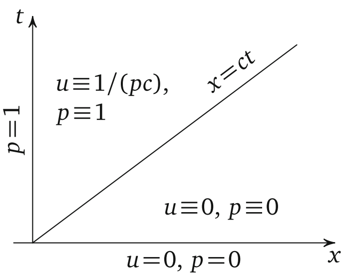

Fig. 1.8: The solution of problem (11.16)–(11.18) for $f = g = 0$ and $h = 1$

boundary $x = 0$ the pressure $p = 1$ is maintained. From (11.5), (11.20), (11.21), we get (see Fig. 1.8)

$$\begin{cases} u = 0, & p = 0 \quad \text{if } t < \dfrac{x}{c}, \\ u = \dfrac{1}{\rho c}, & p = 1 \quad \text{if } t \geq \dfrac{x}{c}. \end{cases} \tag{11.26}$$

The functions u and p are discontinuous, which is not surprising, since condition (11.19) is violated. But on the other hand, formulas (11.26) correspond well to physical processes.

P 11.19 Give an appropriate definitions of generalized solutions for the following problems:

(1) $u_t + u_x = F(x, t)$ in $Q = \{(x, t) \in \mathbb{R}^2 : x > 0, t > 0\}$.
$u\big|_{t=0} = f(x)$, $x > 0$. $u\big|_{x=0} = h(t)$, $t > 0$.

(2) $u_t + (1/\rho)p_x = F(x, t)$, $p_t + \rho \cdot c^2 u_x = G(x, t)$ in Q.
$u\big|_{t=0} = f(x)$, $p\big|_{t=0} = g(x)$, $x > 0$; $u\big|_{x=0} = h(t)$, $t > 0$.

[74] Brook Taylor (1685–1731) was an English mathematician, best known for the Taylor formula. His extended memoir "Methodus Incrementorum Directa et Inversa" (1715) ("Direct and Indirect Methods of Incrementation") in addition to his well-known formula contains the foundations of the theory of oscillations of strings. He was also the first to theoretically study the refraction of light rays in the Earth atmosphere.

(3) $p_{tt} - c^2 p_{xx} = F(x,t)$ in the half-strip $\Omega = \{(x,t) \in \mathbb{R}^2_+ : 0 < x < 1\}$.
$p\big|_{t=0} = f(x)$, $p_t\big|_{t=0} = g(x)$, $0 < x < 1$;
$p\big|_{x=0} = h_0(x)$, $p\big|_{x=1} = h_1(x)$, $t > 0$.

Under what conditions on the functions f, g, h, F, G the solutions of these problems lie, say, in the space C^1, PC^1 or $L^1_{\rm loc}$. Prove the theorems on existence, uniqueness, and continuous dependence (cf. Exercise 11.10).

As a "seed" preceding the topic of questions to which §26 and all three appendices to this book are devoted, we conclude this section with the nonlinear equation

$$u_t + (u^2/2)_x + g_x(x,t) = \varepsilon u_{xx}, \quad u = u(x,t), \tag{11.27}$$

where $\varepsilon \geq 0$, and g is a given function. This equation (with $g = 0$) was first appeared in 1948 in the paper of Burgers[75] (Adv. Appl. Mech. 1948. vol. 1, pp. 171–199). This equation is usually known as the *Burgers equation* and sometimes as the Hopf equation (although Hopf's paper appeared 2 years later; see the footnotes on p. 65 and on p. 20). To be fair, Eq. (11.27) should be called the *Burgers–Florin equation*, because in the same year 1948 Florin[76] (1948) studied more general partial differential equations with several spatial variables and, in particular, the equation

$$S_t + (S_x)^2/2 + g = \varepsilon S_{xx} \tag{11.28}$$

for a function S, which can be written as the integral

$$S(x,t) = \int_{(0,0)}^{(x,t)} P\,dx + Q\,dt, \tag{11.29}$$

where P and Q are given in terms of the solution u of Eq. (11.27):

$$P(x,t) = u(x,t), \quad Q(x,t) = -u^2(x,t)/2 + \varepsilon u_x(x,t) - g(x,t).$$

So, $P_t = u_t$, $Q_x = -u \cdot u_x + \varepsilon u_{xx} - g_x(x,t)$, which implies the condition $P_t = Q_x$ ensuring the correctness of formula (11.29). Note that $u(x,t) = S_x(x,t)$.

In hydrodynamics, the Burgers–Florin Eq. (11.27) is a model equation for $\varepsilon > 0$ for the Navier–Stokes system,[77] and for $\varepsilon = 0$, for the Euler system (see Rozhdestvenski and Yanenko 1983).

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[75] Johannes (Jan) Martinus Burgers (1895–1981) was a Dutch physicist. In addition to the Burgers equation, the Burgers vector in the dislocation theory and the Burgers material in the theory of viscoelasticity are also associated with his name.

[76] Viktor Anatol'evich Florin (1899–1960) was a Corresponding Member of the USSR Academy of Sciences, hydraulic engineer, the head of the scientific school of Soil Mechanics in Leningrad.

[77] Claude-Louis Navier (1785–1836) was a French mechanical engineer. After graduating from the Corps of Bridges and Roads (Corps des Ponts et Chaussées), he supervised the construction of bridges in the Seine department, in particular, the pedestrian bridge to the Île de la Cité in Paris. Navier is considered one of the founders of the modern theory of elasticity. But he is known primarily for the fact that, based on the model of molecular forces, he was the first to derive the Navier–Stokes equation for an incompressible fluid in 1822 in "Memoire sur les lois du mouvement des fluides" (published in 1827 in Memoires de l'Academie des sciences de l'Institut de France). The case of compressible fluid was considered by O. Cauchy in 1828, S. Poisson in 1829 and

We first consider Eq. (11.27) for $\varepsilon = 0$ and $g \equiv 0$. In this problem, a regular (C^1-smooth) solution of this equation satisfies the system $\frac{dx}{dt} = u$, $\frac{du}{dt} = 0$. So, along the curve defined by the equation $\frac{dx}{dt} = u(x, t)$ and known as the *characteristic*, the solution is constant, and hence $u(x, t) = u(x, 0)$ along this curve. Therefore, the above curve is in the actual fact the straight line $x = a + f(a)t$, where the function f defines the initial data for the solution, i.e., $u(x, 0) = f(x)$. If f is a decreasing function, for example,

$$f(x) = \begin{cases} 1 & \text{for } x \leq -1, \\ -x & \text{for } |x| \leq 1, \\ -1 & \text{for } x \geq 1, \end{cases}$$

then the characteristics form the family of straight lines

$$x(t) = \begin{cases} t + a & \text{for } a \leq -1, \\ a(1 - t) & \text{for } |a| \leq 1, \\ -t + a & \text{for } a \geq 1 \end{cases}$$

parameterized by the points a of the Ox-axis, from which they emerge. They intersect on the line $\gamma = \{x = 0,\ t \geq 1\}$. And up to this line, the desired solution u takes the value $f(a)$ along the characteristic originating from the point $(a, 0)$.

P 11.20 Let $f(x) = -\tanh(x/h)$, where $h > 0$. Verify that the solution $u = u_h$ of the problem

$$u_t + (u^2/2)_x = 0 \quad \text{in } \mathbb{R}^2_+, \quad u\big|_{t=0} = f(x), \quad x \in \mathbb{R}, \tag{11.30}$$

has a discontinuity on the line

$$\gamma = \{x = 0,\ t \geq h\}.$$

Find $\lim_{h\to 0} u_h$.

As Exercise 11.20 shows, the Cauchy problem (11.30) may fail to have a continuous solution in $\mathbb{R}^2_+$, even for analytic initial data. This effect, which is well known in hydrodynamics, is associated with the appearance of so-called *shock waves* characterized by a sudden change in density, velocity, etc. As we can see, physics suggests that the solution of problem (11.30) should be sought as a generalized solution from the class PC^1.

On the other hand, the solution of problem (11.30) may be continuous even for discontinuous initial data. For example, if

$$f(x) = \operatorname{sgn}(x),$$

then the characteristics fan out, and as one readily verifies,

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A. de Saint-Venant in 1843. However, only J. Stokes in 1845, consistently proceeding from the continuum concept, derived these equations for viscous (and, in particular, compressible) liquids in the paper "On the theories of internal friction of fluids in motion, and of the equilibrium and motion of elastic solids," which was published in 1849 in Transactions of the Cambridge Philosophical Society.

$$u(x,t) = \begin{cases} -1 & \text{for } x \le -1, \\ x/t & \text{for } |x| \le 1, \\ 1 & \text{for } x \ge 1. \end{cases}$$

This is a continuous but not differentiable solution. Its graph has a kink along the characteristics $t = \pm x > 0$. This type of solutions corresponds to a *rarefaction wave* in hydrodynamics.

P 11.21 Assume that the generalized solution u of problem (11.30) has a discontinuity along the curve

$$\gamma = \{(x,t) \in \mathbb{R}^2 \colon x = \lambda(t),\ \lambda \in C^1[\alpha,\beta]\}.$$

Prove (see the hint in Exercise 12.6) that along this line γ, which is known as the discontinuity line, the *Rankine–Hugoniot condition*[78]

$$\frac{d\lambda(t)}{dt} = \frac{u(\lambda(t)+0,t) + u(\lambda(t)-0,t)}{2}. \tag{11.31}$$

Remark 11.22 Note that the parameter q in Eq. (11.27) is usually interpreted as the flow viscosity coefficient. The viscosity smooths out the shock waves, and this is the basis of the so-called *vanishing viscosity method*. In this method, a generalized (discontinuous) solution of Eq. (11.27) with $\varepsilon = 0$ is constructed as the limit as $\varepsilon \to 0$ of the classical (smooth) solution of Eq. (11.27) with $\varepsilon > 0$.

In particular, for $g \equiv 0$ and $\varepsilon > 0$, Eq. (11.27) can, surprisingly, be reduced to the well-studied heat equation. Moreover, the following theorem holds.

Theorem 11.23 (Florin–Hopf–Cole[79]) *The solution of Eq. (11.27) is given by* $u = -2\varepsilon(\ln G)_x$, *where* G *is the solution of the linear parabolic equation*

$$G_t = \varepsilon G_{xx} + \frac{g(x,t)}{2\varepsilon} G. \tag{11.32}$$

Proof Let u be a solution of (11.27) and let the function S be given by (11.29). Putting $G = \exp\left[-\frac{S}{2\varepsilon}\right]$, we find that G is a solution of (11.32), and $u = -2\varepsilon(\ln G)_x$, because $u = S_x$. □

11.24 Let us now consider the situation when the characteristics of Eq. (11.27) intersect not on a curve, but at a point. Namely, consider the Cauchy problem for Eq. (11.27) (denoting the function u by p) for $\varepsilon = 0$, $f(x) = 1$, and $g(x) = \frac{x^2}{2}$, i.e., the problem[80]

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[78] William John Macquorn Rankine (1820–1872) was a Scottish mechanical engineer and physicist, one of the founders of engineering thermodynamics. In 1849, independently of Clausius, he obtained the general equations of thermodynamics, which express the ratio between the amount of heat and mechanical energy. In his studies of shock waves, in 1870 he was the first to obtain the condition (11.31), which was independently found in 1885 by Pierre Henri Hugoniot (1851–1887), a French professor of mechanics and ballistics, in the process of studying the expansion of gases associated with artillery gun firing.

[79] Florin V. A., some of the simplest nonlinear problems of the consolidation of a water-saturated earth medium (Florin 1948; Hopf 1950; Cole 1951).

$$p_t + (p^2/2)_x + x = 0 \quad \text{in } \mathbb{R}^2_+, \quad p\big|_{t=0} = 1, \quad x \in \mathbb{R}. \tag{11.33}$$

For this problem, the *characteristics*, which are given by the conditions

$$x_t - p = 0 \quad \text{in } \mathbb{R}^2_+, \quad x\big|_{t=0} = x^\circ, \quad x^\circ \in \mathbb{R}, \tag{11.34}$$

are no longer straight lines. What is more, the regular solution of problem (11.30) is not constant on these lines but is given by the ratio

$$\frac{d}{dt} p(x(t), t) = -x(t). \tag{11.35}$$

Hence $\ddot{x} + x = 0$, $x(0) = x^\circ$, $\dot{x}(0) = p(0) = 1$ and so

$$x(t) = x(x^\circ, t), \quad \text{where } x(x^\circ, t) = \sin t + x^\circ \cos t. \tag{11.36}$$

Formula (11.36) defines a family of characteristics that for $t \in [0, T]$ fill the set

$$\Omega = \big(\mathbb{R} \times [0, T]\big) \setminus \{|x - x_k| > 0,\ t = t_k,\ k \geq 1\}, \tag{11.37}$$

where the points $(x_k, t_k) = \big((-1)^{k-1}, \frac{\pi}{2}(2k-1)\big)$ are the points of intersection of the characteristics. Note that

$$x^\circ(x, t) \overset{(11.36)}{=} \frac{x - \sin t}{\cos t} \quad \text{for } t \neq t_k. \tag{11.38}$$

From the equation $\frac{d}{dt} p(x(t), t) \overset{(11.35)}{=} -x(t) \overset{(11.36)}{=} -\sin t - x^\circ \cos t$, it follows that

$$p(x(x^\circ, t), t) = \cos t - x^\circ \sin t \overset{(11.38)}{\Longrightarrow} p(x, t) = \frac{1 - x \sin t}{\cos t}. \tag{11.39}$$

Since $x(x^\circ, t_k) = x_k$, we find that $\lim_{t \to t_k} p(x(x^\circ, t), t) = (-1)^k x^\circ \in \mathbb{R}$, i.e., for $t = t_k$ the quantity $p(x(x^\circ, t), t)$ is not single-valued. Let's say more precisely, noting first that the graph of the function (11.39) is a ruled surface formed by straight lines of the form

$$p = \lambda_t (x - \sin t) + \cos t, \quad \lambda_t = -\tan\ t, \tag{11.40}$$

which are inclined to the Ox-axis at an angle $\theta(t) = -t$ are inclined to the Ox axis at an angle $(x(0, t), t) \mapsto \cos t$.

[80] The solution of problem (11.33) is discussed in §26 in connection with the short-wave asymptotics for a quantum mechanical oscillator defined by the equation

$$ih\psi_t + \frac{1}{2} h^2 \psi_{xx} = \frac{x^2}{2} \psi \quad \text{for } 1/h \gg 1$$

for the wave function $\psi\colon (x, t) \mapsto \psi(x, t) = \varphi(x, t) e^{\frac{i}{h} S(x,t)}$, where $S(x, 0) = x$.

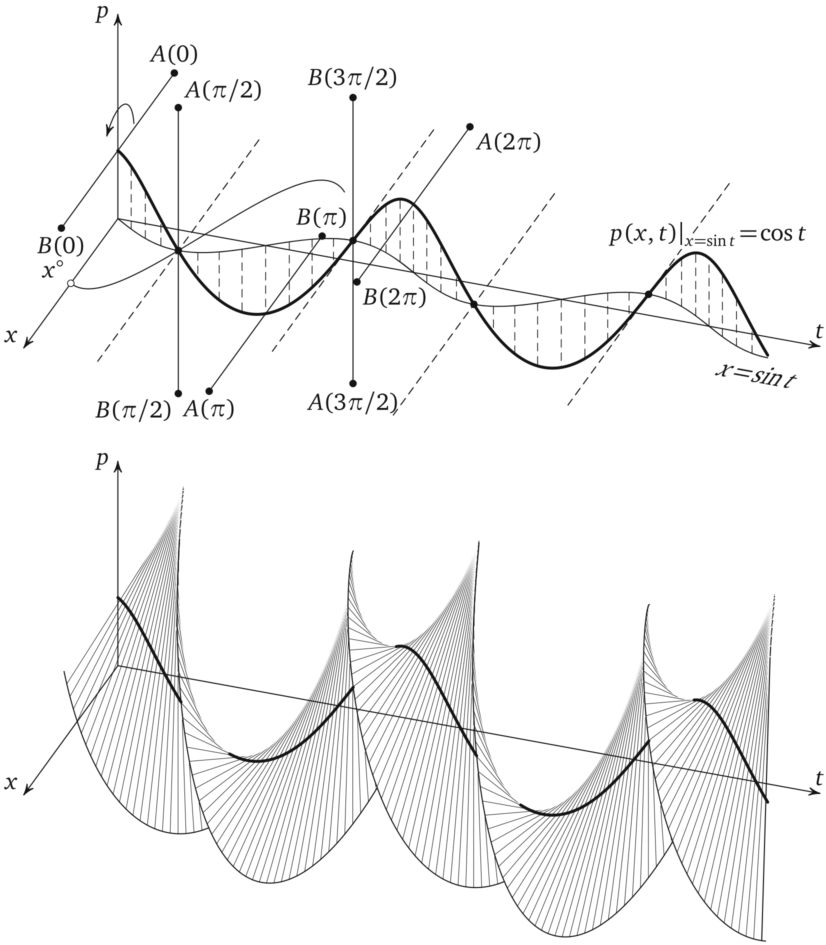

Fig. 1.9: The Lagrangian manifold (see Definition 26.22) representing the solution of problem (11.33)

Taking the set of all such lines for $t \in [0, T]$, we get the two-dimensional smooth surface $\Lambda^2_{[0,T]} \subset \mathbb{R}^3 = \mathbb{R}_x \times \mathbb{R}_t \times \mathbb{R}_p$ in the coordinate space (x, p, t) (see Fig. 1.9).

The projection of the surface $\Lambda^2_{[0,T]}$ onto the plane $\mathbb{R}_x \times \mathbb{R}_t$ is precisely the set Ω, and hence the manifold $\Lambda^2_{[0,T]}$ can be associated with the graph of the "generalized" solution $p\colon (x, t) \mapsto p(x, t)$ of problem (11.33). Hence, on Ω (for $t \neq t_k$), the differential form

$$\omega = p\, dx - H(p, x, t)\, dt, \quad \text{where } H(p, x, t) = \frac{p^2 + x^2}{2}, \tag{11.41}$$

is well defined. Note that in terms of the function H, Eqs. (11.33) and (11.34) can be written as a *canonical Hamilton system* (in the *Hamiltonian form*):[81]

$$x_t - H_p(p(x,t),x,t) = 0, \quad p_t + H_x(p(x,t),x,t) = 0. \tag{11.42}$$

In view of Eqs. (11.42), the differential

$$d\omega = -\big[p_t + H_x(p(x,t),x,t)\big]\, dx \wedge dt$$

of the form ω is zero. Hence, in the simply connected domains,

$$\Omega_k = \Omega \bigcap \Big\{t_k < t < t_{k+1} = \frac{\pi}{2}(2k+1)\Big\}, \quad k \geq 0, \tag{11.43}$$

the integral of ω does not depend on a path connecting the start and end points of the integration.

Lemma 11.25 *The function* $S\colon \Omega \ni (x,t) \mapsto S(x,t) \overset{\text{def}}{=} \int_{(0,0)}^{(x,t)} \omega$ *is continuous, vanishes at the points* (x_k, t_k) *for any* $k \geq 1$*, and such that*

$$S(x,t) = \frac{\sin 2t}{4} + \frac{1}{\cos t}(x - \sin t) - \frac{\tan t}{2}(x^2 - \sin^2 t). \tag{11.44}$$

Proof Assume first that $(x,t) \in \Omega_0$. Since the integral $\int_{(0,0)}^{(x,t)} \omega$ is independent of the choice of an integration path $\gamma \subset \Omega_0$ between $(0,0)$ and (x,t) and since $\dot{x} = p$, it can be written as the sum of two integrals

$$I_1 = \int_0^t \frac{p^2(x(0,\tau),\tau) - x^2(0,\tau)}{2} d\tau \quad \text{and} \quad I_2 = \int_{x(0,t)}^x p(\xi,t) d\xi.$$

Since $p(\xi,t) \overset{(11.39)}{=} \frac{1-\xi \sin t}{\cos t}$, $x(0,t) \overset{(11.36)}{=} \sin t$, $p(0,t) = \cos t$, we find that

$$I_1 = \frac{1}{2}\int_0^t \cos 2\tau \, d\tau, \quad I_2 = \int_{\sin t}^x \frac{1 - \xi \sin t}{\cos t}\, d\xi.$$

As a result, we arrive at (11.44) for $t < \frac{\pi}{2}$, or, what is the same,[82]

$$S(x,t) = -\frac{x^2+1}{2}\tan t + \frac{x}{\cos t}. \tag{11.45}$$

[81] William Rowan Hamilton (1805–1865) was an Irish mathematician and physicist. In his works of 1834–1835 on "Hamiltonian mechanics," the variational principle of least action was formulated, which proved to be a universal and highly effective tool in mathematical models of physics, especially in quantum mechanics and general relativity. He also proposed the system of hypercomplex numbers (the so-called quaternions). Hamilton also laid the foundations of vector analysis, including its symbolism. The main operations of vector analysis grad (gradient), div (divergence), rot (rotor), as well as the Laplace operator are expressed in terms of the operator ∇ introduced by him in 1853 (this operator is called "nabla" because of its similarity to the skeleton of the ancient Assyrian musical instrument nabla, related to the harp).

[82] The representation of $S(x,t)$ by (11.45) can be found in the book Bagrov et al. (2004, p. 126).

Despite the apparent singularity of the function S at $t = t_k$ (since $\cos t_k = 0$), the function does not actually have it. Indeed,

$$x(x^\circ, t) - \sin t \overset{(11.36)}{=} x^\circ \cos t,$$

and hence

$$\lim_{t \to t_k} S(x,t)\Big|_{x=x(x^\circ,t)} \overset{(11.44)}{=} x^\circ \Big(1 - \frac{x_k + \sin t_k}{2}\Big) \sin t_k = 0. \tag{11.46}$$

Assume now that $(x,t) \in \Omega_1$. Representing $S(x,t) = \int_{(x_1,t_1)}^{(x,t)} \omega$ as the sum of two integrals

$$\int_{t_1}^{t} \frac{p^2(x(0,\tau),\tau) - x^2(0,\tau)}{2}\, d\tau \quad \text{and} \quad \int_{x(0,t)}^{x} p(\xi,t)\, d\xi,$$

we arrive at (11.44) for $(x,t) \in \Omega_1$. The proof of the lemma, including the formula

$$S(x_k, t_k) = 0 \quad \text{for any } k \geq 1 \tag{11.47}$$

concludes by induction on $k \geq 1$. □

Remark 11.26 From formula (11.45) and the equality $p\big|_{t=0} \overset{(11.33)}{=} 1$, it follows that the function S is a solution of the problem

$$S_t + \frac{1}{2}(x^2 + S_x^2) = 0, \quad S(x,0) = x. \tag{11.48}$$

Chapter 2
The Spaces $\mathcal{D}^\flat$, $\mathcal{D}^\sharp$ and $\mathcal{D}'$: Elements of the Distribution Theory (Generalized Functions in the Sense of Schwartz)

12 The Space $\mathcal{D}^\flat$ of Sobolev Derivatives

The definition of the generalized solution $u \in L^1_{\mathrm{loc}}$ to one or another problem of mathematical physics, as given by Sobolev (1936), and, in particular, Definition 11.6 is based on Theorem 10.1 and the *Ostrogradsky–Gauss formula* (7.3). Recall that Theorem 10.1 asserts the equivalence of the two following representations of an element $u \in L^1_{\mathrm{loc}}$, i.e.,

$$\Omega \ni x \mapsto u(x) \;\Leftrightarrow\; C_0^\infty(\Omega) \ni \varphi \mapsto \int_\Omega u(x)\varphi(x)\,dx,$$

and formula (7.3) implies that, for the differential operator $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$ and any function $u \in C^{|\alpha|}(\Omega)$, we have

$$\int_\Omega \big(\partial^\alpha u(x)\big)\varphi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)\partial^\alpha\varphi(x)\,dx \quad \forall\varphi \in C_0^\infty(\Omega).$$

So, the functional

$$\partial^\alpha u\colon C_0^\infty(\Omega) \ni \varphi \mapsto \langle \partial^\alpha u, \varphi\rangle, \tag{12.1}$$

where

$$\langle \partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\int_\Omega u(x)\partial^\alpha\varphi(x)\,dx \quad \forall\varphi \in C_0^\infty(\Omega),$$

defines the function $\partial^\alpha u(x)$ if $u \in C^{|\alpha|}(\Omega)$. Functional (12.1) is also defined for $u \in L^1_{\mathrm{loc}}(\Omega)$. Hence, following Sobolev, we give the following definition.

Definition 12.1 Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex. The *derivative of order* α of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is the *functional* $\partial^\alpha u$ defined by (12.1).

Using Theorem 10.1, formula (7.3), and the identity

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$$\int_\Omega a(x)\Big(\partial^\alpha u(x)\Big)\varphi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)\partial^\alpha\Big(a(x)\varphi(x)\Big)\,dx,$$
$$u \in C^{|\alpha|}(\Omega), \quad \varphi \in C_0^\infty(\Omega),$$

which is valid for any function $a \in C^\infty(\Omega)$, we introduce the multiplication of the functional $\partial^\alpha u$, where $u \in L^1_{\mathrm{loc}}(\Omega)$, by a function $a \in C^\infty(\Omega)$ as follows:

$$a\partial^\alpha u\colon C_0^\infty(\Omega) \ni \varphi \mapsto (-1)^{|\alpha|}\int_\Omega u(x)\partial^\alpha(a(x)\varphi(x))\,dx \in \mathbb{C}. \tag{12.2}$$

Definition 12.2 The *space of Sobolev derivatives* is the space $\mathcal{D}^\flat(\Omega)$ of functionals

$$\sum_{|\alpha|<\infty} \partial^\alpha u_\alpha, \quad \text{where } u \in L^1_{\mathrm{loc}}(\Omega),$$

equipped with the operation of multiplication (12.2).

Example 12.3 Consider the function $x_+ \in L^1_{\mathrm{loc}}(\mathbb{R})$ defined by $x_+ = x$ for $x > 0$, $x_+ = 0$ for $x < 0$. Let us find its derivatives. We have

$$\langle x_+', \varphi\rangle = -\langle x_+, \varphi'\rangle = -\int_{\mathbb{R}} x_+\varphi'(x)\,dx = -\int_{\mathbb{R}_+} x\varphi'(x)\,dx$$
$$= -x\varphi(x)\Big|_0^\infty + \int_0^\infty \varphi(x)\,dx = \int_{\mathbb{R}} \theta(x)\varphi(x)\,dx = \langle\theta,\varphi\rangle,$$

i.e., $x_+' = \theta$ is the Heaviside function defined by (11.3). Let us find x_+'', i.e., θ'. We have

$$\langle\theta',\varphi\rangle = -\langle\theta,\varphi'\rangle = -\int_0^\infty \varphi'(x)\,dx = -\varphi(x)\Big|_0^\infty = \varphi(0) = \langle\delta,\varphi\rangle, \tag{12.3}$$

i.e., $\theta' = \delta(x)$ is the δ-Dirac function. Note also that

$$\langle\delta^{(k)},\varphi\rangle = (-1)^k\langle\delta,\varphi^{(k)}\rangle = (-1)^k\varphi^{(k)}(0) \tag{12.4}$$

and hence, for any $l \geq k \geq 1$ and also, for any $m \in \mathbb{N}$, we have

$$|\langle\delta^{(k-1)},\varphi\rangle| = |\langle\theta^{(k)},\varphi\rangle| \leq C\|\varphi\|_{C^l(\mathbb{R})} \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}) \tag{12.5}$$

and

$$|\langle\delta^{(m)},\varphi\rangle| \leq C\|\varphi\|_{C^0(\mathbb{R})} \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\}). \tag{12.6}$$

P 12.4 Let $\theta_\varepsilon \in C^\infty(\mathbb{R})$, $0 \leq \theta_\varepsilon(x) \leq 1$, and let $\theta_\varepsilon(x) \equiv 1$ for $x > \varepsilon$ and $\theta_\varepsilon(x) \equiv 0$ for $x < -\varepsilon$. We set $\delta_\varepsilon(x) = \theta_\varepsilon'(x)$. Verify that

$$\lim_{\varepsilon\to 0}\langle\delta_\varepsilon^{(k)},\varphi\rangle = (-1)^k\varphi^{(k)}(0) \quad \forall\varphi \in C_0^\infty(\mathbb{R}) \quad \text{and} \quad \forall k \geq 0.$$

Remark 12.5 Using (12.4) it becomes possible to extend the functional $\delta^{(k)}$ from the function space $C_0^\infty(\mathbb{R})$ to the space of functions k times continuously differentiable at the point $x = 0$ (cf. Definition 2.2). On the other hand, formula (12.3) has no sense in the space $C(\mathbb{R})$, because the functional θ is not defined on $C(\mathbb{R})$.

Consider the function $\theta_\pm : \mathbb{R}^n \ni x \mapsto \theta_\pm(x)$ defined by

$$\theta_\pm(x) = 1_{Q_\pm}(x), \quad x \in \mathbb{R}^n, \tag{12.7}$$

where $Q_\pm = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \pm x_k > 0 \ \forall k\}$. If $n = 1$, then $\theta_+ = \theta$ is the Heaviside function, and $\theta_- = 1 - \theta_+$.

P 12.6 (cf. Exercise 11.21) Let $F \in C^1(\mathbb{R})$, $\lambda \in C^1(\mathbb{R})$, $u_\pm \in C^1(\mathbb{R}^2)$, $u(x,t) = u_+(x,t)\theta_+(x - \lambda(t)) + u_-(x,t)\theta_-(x-\lambda(t))$ for $(x,t) \in \Omega \subset \mathbb{R}^2$. Find $u_t \overset{\text{def}}{=} \partial_t u(x,t)$ and $\big(F(u)\big)_x \overset{\text{def}}{=} \partial_x(F(u(x,t))$, by noting that

$$F\big(u(x,t)\big) = F\big(u_+(x,t)\big)\theta_+(x - \lambda(t)) + F\big(u_-(x,t)\big)\theta_-(x - \lambda(t)).$$

Verify that $u_t + \big(F(u)\big)_x = g$ almost everywhere in Ω if and only if, first, $u_t + \big(F(u)\big)_x \equiv g$ in $\Omega \setminus \gamma$, where $\gamma = \{(x,t) \in \mathbb{R}^2 \mid x = \lambda(t)\}$, and second, the *Rankine–Hugoniot condition*

$$\frac{d\lambda(t)}{dt} - \frac{F\big(u(\lambda(t)+0,t)\big) - F\big(u(\lambda(t)-0,t)\big)}{u(\lambda(t)+0,t) - u(\lambda(t)-0,t)} = 0 \tag{12.8}$$

holds on the line γ. The left-hand side of (12.8) is the coefficient multiplying the δ*-function* in the expression $u_t + \big(F(u)\big)_x - g$.

P 12.7 Verify that $\frac{\partial^n}{\partial x_1 \dots \partial x_n}\theta_+ = \delta(x)$.

P 12.8 Verify that the function $(x,t) \mapsto E(x,t) = \theta(t - |x|)/2$ is the *fundamental solution* of the string operator, i.e.,

$$\big(\partial^2/\partial t^2 - \partial^2/\partial x^2\big)E(x,t) = \delta(x,t).$$

Here $\delta(x,t)$ is the δ-function in $\mathbb{R}^2$: $\langle \delta(x,t), \varphi\rangle = \varphi(0,0)$ for any function $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.

P 12.9 Having noted that

$$\lim_{\varepsilon \to 0}\Big[\int_{|x|>\varepsilon} \ln|x| \cdot \varphi'(x)\,dx\Big] = \lim_{\varepsilon \to 0}\Big[\ln\varepsilon(\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx\Big]$$

for any $\varphi \in C_0^\infty(\mathbb{R})$, show that $\frac{d}{dx}\ln|x| = \text{v. p.}\frac{1}{x}$, i.e., $\big\langle \frac{d}{dx}\ln|x|, \varphi\big\rangle = \text{v. p.}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}\,dx$, where v. p. $\int_{-\infty}^{\infty} x^{-1}\varphi(x)\,dx$ is the so-called *principal value* (*valeur principale* in French) of the integral $\int_{-\infty}^{\infty} x^{-1}\varphi(x)\,dx$, which is defined by the formula

$$\text{v. p.}\int_{-\infty}^{\infty} x^{-1}\varphi(x)\,dx = \lim_{\varepsilon \to 0}\int_{|x|>\varepsilon} x^{-1}\varphi(x)\,dx. \tag{12.9}$$

P 12.10 Having noted that

$$\lim_{\varepsilon \to +0} \ln(x \pm i\varepsilon) = [\ln|x| \pm i\pi\theta(-x)],$$

prove the simplest variant of the Sochocki–Plemelj formulas[1]

[1] Formula (12.10) was first proved in the thesis of Julian Vasil'evich Sochocki (1842–1927), a professor of Mathematics at St. Petersburg University, in 1873, i.e., in the year of birth of Josip Plemelj (1873–1967), a Slovene mathematician, who reproved these formulas in (1908, Monats. Mathem. Physik, Bd. 19. 205–210).

$$\frac{1}{x \mp i0} = \text{v. p.}\frac{1}{x} \pm i\pi\delta(x), \tag{12.10}$$

which are frequently used in mathematical physics, i.e., $\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{\varphi(x)\,dx}{x\mp i\varepsilon} = \text{v. p.}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}\,dx \pm i\pi\varphi(0)$ for any function $\varphi \in C_0^\infty(\mathbb{R})$.

Remark 12.11 From (12.10) it follows that

$$\delta(x) = f(x - i0) - f(x + i0), \quad \text{where } f(x+iy) = \frac{1}{2\pi i}(x+iy)^{-1}, \tag{12.11}$$

i.e., the δ-function (*qua* an element of the space $\mathcal{D}^\flat(\mathbb{R})$) can be written as the difference of two boundary values on $\mathbb{R}$ of two functions which are analytic, respectively, in $\mathbb{C}_+$ and in $\mathbb{C}_-$, where $\mathbb{C}_\pm = \{z = x + iy \in \mathbb{C} : \pm y > 0\}$. This simple observation has deep generalizations in the theory of *hyperfunctions* (see, for example, Schapira (1970) and Chap. 9 in Hörmander (1958); Hörmander (1965)).

Remark 12.12 Any continuous function $F \in C(\mathbb{R})$ (*qua* an element of the space L^1_{loc}) has the Sobolev derivative $F' \in \mathcal{D}^\flat(\mathbb{R})$. If this derivative is locally integrable (in other words if $F(x) = \int_a^x f(y)\,dy + F(a)$, where $f \in L^1_{\text{loc}}(\mathbb{R})$), then by Theorem 8.47

$$F'(x) = \lim_{h\to 0} h^{-1}(F(x+h) - F(x)) \quad \text{for almost all } x \in \mathbb{R}. \tag{12.12}$$

Besides, formula (12.12) completely defines the Sobolev derivative F'. It is worth pointing out that the last assertion ceases to hold (even under the assumption that (12.12) holds) if $F' \notin L^1_{\text{loc}}(\mathbb{R})$. So, for example, the *Cantor function* (see Kolmogorov and Fomin (1980) or Shilov (2016)), which corresponds to the Cantor set of nonzero measure (see Remark 8.8), i.e., continuous a monotone function $K \in C[0, 1]$ which is equal to $(2k - 1) \cdot 2^{-n}$ on the kth ($k = 1, \dots, 2^{n-1}$) interval $I_n =]a_n^k, b_n^k[$ of rank n, has, for almost all $x \in [0, 1]$, the zero derivative, but its Sobolev derivative K' is nonzero. Namely,

$$K' = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}(2k-1)\cdot 2^{-n}\big(\delta(x - b_n^k) - \delta(x - a_n^k)\big). \tag{12.13}$$

P 12.13 Prove (12.13).

13 The Space $\mathcal{D}^\#$ of Generalized Functions

Elements of the space $\mathcal{D}^\flat$ were defined as finite linear combinations of the functionals $\partial^\alpha u_\alpha$, i.e., the derivatives of functions $u_\alpha \in L^1_{\text{loc}}$. If we neglect the concrete form of the functionals, i.e., consider an arbitrary linear functional

$$f : C_0^\infty(\Omega) \ni \varphi \mapsto \langle f, \varphi\rangle \in \mathbb{C}, \tag{13.1}$$

then we get an element of the space $\mathcal{D}^{\#}(\Omega)$, which we will call a generalized function (in the domain Ω). We give the precise definition.

Definition 13.1 $\mathcal{D}^{\#}(\Omega)$ is the space of all linear functionals (13.1). In this space, the operations of differentiation ∂^α and multiplication by a function $a \in C^\infty(\Omega)$ are introduced by the formulas:

$$\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle, \quad \langle af, \varphi \rangle = \langle f, a\varphi \rangle, \tag{13.2}$$

here φ is any function from $C_0^\infty(\Omega)$.

Definition 13.2 Let $f \in \mathcal{D}^{\#}(\Omega)$, $K = \overline{K} \subset \Omega$, let

$$|\langle f, \varphi \rangle| \le C \|\varphi\|_{C^l(\Omega)} \quad \forall\, \varphi \in C_0^\infty(\Omega) \tag{13.3}$$

for any $l \ge k \ge 1$ (cf. (12.5), (12.6)), and let

$$|\langle \partial^\alpha f, \varphi \rangle| \le C \|\varphi\|_{C^0(\Omega)} \quad \forall\, \varphi \in C_0^\infty(\Omega \setminus K) \tag{13.4}$$

for any α such that $|\alpha| \le m \in \mathbb{Z}_+$. Then the number k is the *order of singularity of the generalized function* f, and the number $m = |\alpha|$ is the *order of smoothness of the generalized function* $f \in \mathcal{D}^{\#}(\Omega)$ on an open set $\Omega \setminus K$.

P 13.3 Let $f = \sum_{k=0}^\infty \delta^{(k)}(x-k)$, $x \in \mathbb{R}$, i.e.,

$$\langle f, \varphi \rangle = \sum_{k=0}^\infty (-1)^k \varphi^{(k)}(k) \; \forall \varphi \in C_0^\infty(\mathbb{R}).$$

Verify that $\mathcal{D}^\flat(\Omega) \subsetneq \mathcal{D}^{\#}(\Omega)$ by checking that $f \in \mathcal{D}^{\#}(\mathbb{R})$, but $f \notin \mathcal{D}^\flat(\mathbb{R})$. Find the order of singularity of the function f.

The following result holds, even though $\mathcal{D}^\flat(\Omega) \subsetneq \mathcal{D}^{\#}(\Omega)$.

Lemma 13.4 (du Bois-Reymond[2]) *If* $f \in \mathcal{D}^{\#}(\mathbb{R})$ *and* $f' = 0$*, then* $f = \text{const}$. *Hence* $f \in \mathcal{D}^\flat(\mathbb{R})$.

Proof We have $\langle f', \varphi \rangle = \langle f, \varphi' \rangle = 0$ for any function $\varphi \in C_0^\infty(\mathbb{R})$. Consider a function $\varphi_0 \in C_0^\infty(\mathbb{R})$ such that $\int \varphi_0 = 1$. Any function $\varphi \in C_0^\infty(\mathbb{R})$ can be written in the form $\varphi = \varphi_1 + \left(\int \varphi\right)\varphi_0$, where $\varphi_1 = \varphi - \left(\int \varphi\right)\varphi_0$. Note that $\int \varphi_1 = 0$. Let $\psi(x) = \int_{-\infty}^x \varphi_1(\xi) d\xi$. We have $\psi \in C_0^\infty(\mathbb{R})$ and $\psi' = \varphi_1$. Hence $\langle f, \varphi \rangle = \langle f, \psi' \rangle + \langle f, \left(\int \varphi\right)\varphi_0 \rangle$. Now, because $\langle f, \psi' \rangle = 0$, we find that $\langle f, \varphi \rangle = C \int \varphi$, where $C = \langle f, \varphi_0 \rangle$. □

[2] Paul David Gustave du Bois-Reymond (1831–1889) was a German mathematician, whose parents were of French descent (his mother was the daughter of a French representative in Berlin). Du Bois-Reymond is known for his works on the theory of functions of a real variable. He constructed the first example of a continuous function whose Fourier series diverges at some point. Of course, du Bois-Reymond's lemma stated here was proved by him in a different formulation: if $f \in C^1(\mathbb{R})$ and $\int_{\mathbb{R}} f'(x)\varphi(x)\,dx = 0$ for any function $\varphi \in C_0^\infty(\mathbb{R})$, then $f = \text{const}$.

Generalizing the notion of a *δ-sequence* we introduce

Definition 13.5 A sequence of functionals $f_\nu \in \mathcal{D}^\#$ is said to *weakly converge* to $f \in \mathcal{D}^\#$ in the space $\Phi \supset C_0^\infty$ if $\lim_{\nu\to\infty} \langle f_\nu, \varphi\rangle = \langle f, \varphi\rangle$ for any function $\varphi \in \Phi$. If $\Phi = C_0^\infty$, then the words "in the space C_0^∞" are usually omitted.

Definition 13.6 We say that a subspace X of the space $\mathcal{D}^\#$ is *weakly sequentially complete* if, for any sequence $\{f_\nu\}_{\nu=1}^\infty$ of functionals $f_\nu \in X$ such that

$$\langle f_\nu - f_\mu, \varphi\rangle \to 0 \quad \forall \varphi \in C_0^\infty \quad \text{as } \nu, \mu \to \infty,$$

there exists an $f \in X$ such that $f_\nu \to f$ in $\mathcal{D}^\#$.

P 13.7 Verify that $\mathcal{D}^\flat$ is not weakly sequentially complete.

P 13.8 Verify that $\mathcal{D}^\#$ is weakly sequentially complete.

Lemma 13.9 *If $f_\nu \to f$ in $\mathcal{D}^\#$ on the space $\Phi \supset C_0^\infty$, then $\partial^\alpha f_\nu \to \partial^\alpha f$ in $\mathcal{D}^\#$ on the space Φ for any α.*

Proof We have $\langle \partial^\alpha f_\nu, \varphi\rangle = (-1)^{|\alpha|}\langle f_\nu, \partial^\alpha\varphi\rangle \to (-1)^{|\alpha|}\langle f, \partial^\alpha\varphi\rangle = \langle \partial^\alpha f, \varphi\rangle$. □

Example 13.10 Let $\langle f_\nu, \varphi\rangle = \int_{\mathbb{R}} \frac{\sin \nu x}{\nu}\varphi(x)\,dx$. Then $f_\nu' = \cos \nu x$, $f_\nu'' = -\nu\cdot\sin \nu x, \dots$ Hence $\langle f_\nu, \varphi\rangle \to 0$ for any function $\varphi \in C_0^\infty$ as $\nu \to \infty$. So, $\cos \nu x \to 0$ in $\mathcal{D}^\#$, $\nu \sin \nu x \to 0$ in $\mathcal{D}^\#$, and so on.

Lemma 13.11 *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, $b_k < a_k$ for any k, lie in $\Omega \subset \mathbb{R}^n$, and let*

$$\Pi = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < |x_k - a_k| < \sigma_k \ \forall k\} \subset \Omega.$$

Assume that a sequence $\{f_\nu\}_{\nu=1}^\infty$ of functions $f_\nu \in L^1_{\text{loc}}(\Omega)$ is such that the function

$$F_\nu(x) = \int_{b_1}^{x_1} \dots \int_{b_n}^{x_n} f_\nu(y)\,dy_1 \dots dy_n$$

has the following two properties:

(1) $|F_\nu(x)| \le G(x)$, $x \in \Omega$, *where* $G \in L^1_{\text{loc}}(\Omega)$.

(2) $F_\nu(x) \to \theta_+(x - a)$ *v. p. in* Ω, *where the function* θ_+ *is defined in* (12.7).

Then f_ν weakly converges to $\delta(x - a)$ in the space

$$\Phi = \left\{\varphi \in C(\Omega) : \varphi \in L^1(\Omega),\ \partial^n\varphi/\partial x_1 \dots \partial x_n \in L^1(\Omega)\right\}. \tag{13.5}$$

Proof In view of Theorems 8.34, 8.42, and 8.47, for any function $\varphi \in \Phi$, we have

$$\begin{aligned}\langle f_\nu, \varphi\rangle &= \langle \frac{\partial^n F_\nu}{\partial x_1 \dots \partial x_n}, \varphi\rangle = (-1)^n \langle F_\nu, \frac{\partial^n \varphi}{\partial x_1 \dots \partial x_n}\rangle \\ &= (-1)^n \int_\Omega F_\nu(x) \frac{\partial^n\varphi(x)}{\partial x_1 \dots \partial x_n}\,dx \to (-1)^n \int_{a_1}^\infty \dots \int_{a_n}^\infty \frac{\partial^n \varphi(x)\,dx}{\partial x_1 \dots \partial x_n} \\ &= -(-1)^n \int_{a_2}^\infty \dots \int_{a_n}^\infty \frac{\partial^{n-1}\varphi(x)\,dx}{\partial x_2 \dots \partial x_n}\,dx_2 \dots dx_n = \varphi(a).\end{aligned}$$

P 13.12 Solve Problems 4.3 and 4.4 with the help of Lemma 13.11.

Let us generalize the notion of the support of a function (see § 3). To this aim, we give an exact meaning to the phrase common in physics: "$\delta(x) = 0$ for $x \neq 0$".

Definition 13.13 Let $f \in \mathcal{D}^{\#}(\Omega)$ and ω be an open subset of Ω. We say that f *is zero* (*vanishes*) *on* ω (written $f|_{\omega} = 0$ or $f(x) = 0$ for $x \in \omega$) if $\langle f, \varphi \rangle = 0$ for any function $\varphi \in C_0^{\infty}(\omega)$.

Definition 13.14 By the *null set of a functional* $f \in \mathcal{D}^{\#}(\Omega)$ we mean the maximum open set $\Omega_0 = \Omega_0(f) \subset \Omega$ on which f vanishes, i.e., $f|_{\Omega_0} = 0$, and besides, the condition $f|_{\omega} = 0$ implies that $\omega \subset \Omega_0$.

It is clear that $\Omega_0(f)$ is the union of all $\omega \subset \Omega$ such that $f|_{\omega} = 0$.

Definition 13.15 Let $f \in \mathcal{D}^{\#}(\Omega)$. The *support* $\operatorname{supp} f$ of a functional f is the complement of the null set $\Omega_0(f)$, i.e., the set $\Omega \backslash \Omega_0(f)$.

P 13.16 Let $f \in \mathcal{D}^{\#}(\Omega)$. Verify that $x \in \operatorname{supp} f$ if and only if, for any neighborhood $\omega \subset \Omega$ of the point x, there exists a function $\varphi \in C_0^{\infty}(\omega)$ such that $\langle f, \varphi \rangle \neq 0$. Also show that Definition 13.15 is equivalent to Definition 3.5 if $f \in C(\Omega)$.

P 13.17 Find $\operatorname{supp} \delta^{(\alpha)}(x)$ and $\operatorname{supp}[(x_1 + \ldots + x_n)\delta^{(\alpha)}(x)]$.

P 13.18 Let $f \in \mathcal{D}^{\#}(\Omega)$ and let $a \in C^{\infty}$ such that $a(x) = 1$ for $x \in \operatorname{supp} f$. Is it true that $af = f$?

P 13.19 Let ω be an open set in Ω such that $\omega \supset \operatorname{supp} f$, $f \in \mathcal{D}^{\#}(\Omega)$. Verify that $af = f$ if $a(x) = 1$ for $x \in \omega$.

P 13.20 Let $f \in \mathcal{D}^{\#}(\Omega)$ be a generalized function with compact support and let $\psi \in C_0^{\infty}(\Omega)$ and $\psi \equiv 1$ on an open set $\omega \supset \operatorname{supp} f$. Verify that the formula $\langle F, \varphi \rangle = \langle f, \psi\varphi \rangle$ for any function $\varphi \in C^{\infty}(\Omega)$ defines the *extension of the functional* f to the space $C^{\infty}(\Omega)$, i.e., F is a linear functional on $C^{\infty}(\Omega)$ such that $\langle F, \varphi \rangle = \langle f, \varphi \rangle$ for any function $\varphi \in C_0^{\infty}(\Omega)$.

14 Functions Not Locally Integrable as Generalized

The idea of representability of a function $f : \Omega \to \mathbb{C}$ with the help of its "averaging" functional (10.1) was applied above only to locally integrable functions. However, in many problems of analysis, an important role is played by functions which are not locally integrable. This leads to the so-called *regularization problem* of diverging integrals:[3] let $g : \Omega \ni x \mapsto g(x)$ be a function locally integrable everywhere in Ω except a subset $N \subset \Omega$. It is required to find functionals $f \in \mathcal{D}^{\#}$ such that

$$\langle f, \varphi \rangle = \int_{\Omega} g(x)\varphi(x)\, dx \quad \forall \varphi \in C_0^{\infty}(\Omega \setminus N). \tag{14.1}$$

[3] In vol. 1 of Gelfand et al. (1958–1966) the authors write: "we assign credit for the concept of the regularization of a divergent integral to Hadamard and to M. Riesz, although Cauchy had already dealt with it (in defining the gamma function outside the region of convergence of the integral), and even Euler no doubt made use of similar considerations in his calculations."

In this case one says that the functional f regularizes the (divergent) integral $\int_\Omega g(x)\,dx$.

It is clear that functionals f satisfying (14.1) can be represented in the form

$$f = f_0 + f_1, \quad f_0 \in F_0,$$

where f_1 is a solution of the particular regularization problem (i.e., f_1 satisfies the condition (14.1)), and F_0 is the linear subspace of functionals $f_0 \in \mathcal{D}^\#(\Omega)$ such that

$$\langle f_0, \varphi\rangle = 0 \quad \forall \varphi \in C_0^\infty(\Omega\backslash N). \tag{14.2}$$

The problem of the description of the subspace F_0 is connected only with the set $N \supset \operatorname{supp} f_0$. In the case $N = x_0 \in \Omega$, this question, i.e., the problem of the general form of functionals with point support, is considered in § 15. As for the particular regularization problem, we will give one example and for problems P14.2–P14.5 in the single real variable setting.

Example 14.1 Consider the regularization of the function $1/x$. In other words, let us find a functional $f \in \mathcal{D}^\#(\mathbb{R})$ such that $xf = 1$. Note that (see (12.9))[4]

$$\left\langle \text{v. p.}\frac{1}{x}, \varphi\right\rangle = \int_{-\infty}^{\infty} \frac{1}{x}\varphi(x)\,dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}\backslash 0).$$

So, the functional v. p.$(1/x)$ regularizes the function $1/x$. Since $\langle \delta, \varphi\rangle = 0$ for any function $\varphi \in C_0^\infty(\mathbb{R}\backslash 0)$, we find that

$$\text{v. p.}\left(\frac{1}{x}\right) + C \cdot \delta(x),$$

where $C \in \mathbb{C}$, and hence (see (12.10)) the functionals $1/(x \pm i0)$ also regularize the function $1/x$.

P 14.2 Verify that

$$\left\langle \text{v. p.}\frac{1}{x}, \varphi\right\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(-x)}{2x}\,dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}).$$

P 14.3 Let $m \geq 1$ and let $a \in C_0^\infty(\mathbb{R})$. For $k \geq 2$, consider the functional

$$\text{v. p.}\left(\frac{1}{x^k}\right) \in \mathcal{D}^\#(\mathbb{R})$$

defiled by

$$\left\langle \text{v. p.}\frac{1}{x^k}, \varphi\right\rangle = \int_0^\infty \frac{1}{x^k}\left\{\varphi(x) + \varphi(-x) - 2\left[\varphi(0) + \ldots + \frac{x^{k-2}}{(k-2)!}\varphi^{(k-2)}(0)\right]\right\}dx$$

for $k = 2m$ and by

[4] The notation v. p. (*principal meaning* derived from the French "valeur principale") was introduced by Cauchy.

$$\left\langle \text{v. p.} \frac{1}{x^k}, \varphi \right\rangle = \int_0^\infty \frac{1}{x^k} \left\{ \varphi(x) + \varphi(-x) - 2\left[x\varphi'(0) + \ldots + \frac{x^{k-2}}{(k-2)!}\varphi^{(k-2)}(0) \right] \right\} dx$$

for $k = 2m + 1$.

Verify that the functional v. p.$\left(\frac{1}{x^k}\right)$ regularizes the function $\frac{1}{x^k}$.

P 14.4 Let P be a polynomial of $x \in \mathbb{R}$. Find a functional $f \in \mathcal{D}^\#(\mathbb{R})$ satisfying the equation $P(x)f = 1$. In other words, regularize the integral

$$\int_{-\infty}^{\infty} P^{-1}(x)\varphi(x)\, dx.$$

P 14.5 Let $e_+^{-1}(x) = \begin{cases} e^{-1}(x) = e^{-1/x} & x > 0 \\ e^{-1}(x) = 0 & x \le 0 \end{cases}$ where $x \in \mathbb{R}$. Find a functional $f \in \mathcal{D}^\#(\mathbb{R})$ satisfying the equation $e_+^{-1}(x)f = 1$. Compare with P 16.25. See also (Gelfand et al. 1958–1966, Vol 2), which indicates spaces of test functions for which the regularization of functions with arbitrarily strong singularities meaning.

15 Generalized Functions with Point Support: The Borel Theorem

As shown in § 14, the problem of regularization of a function locally integrable everywhere in $\Omega \subset \mathbb{R}^n$, except at a point $\xi \in \Omega$, leads to the problem of the general form of a functional $f \in \mathcal{D}^\#(\Omega)$ concentrated at the point ξ, i.e., satisfying the condition $\operatorname{supp} f = \xi$. It is clear that (see 13.17) a finite sum of a δ-function and its derivatives concentrated at the point ξ, i.e., the sum

$$\sum_{|\alpha| \le N} c_\alpha \delta^{(\alpha)}(x - \xi), \quad c_\alpha \in \mathbb{C}, \quad N \in \mathbb{N}, \tag{15.1}$$

is an example of such a functional.

But does the sum (15.1) provide the general form of a functional $f \in \mathcal{D}^\#$ *supported* at the point ξ? It can be shown (see the remark at the end of this section) that the answer to this question is negative. However, the following theorem holds.

Theorem 15.1 *If* $f \in \mathcal{D}^\#$ *and* $f = \sum_\alpha c_\alpha \delta^{(\alpha)}(x - \xi)$*, then* $c_\alpha = 0$ *for* $|\alpha| > N_f$ *for some* N_f.

Proof According to the Borel theorem that follows[5] Borel (1895) there exists a function $\varphi \in C_0^\infty(\Omega)$ such that, for any α,

[5] Félix Édouard Justin Émile Borel (1871–1956) was a French mathematician, one of the most famous mathematicians of the twentieth century. Together with his student Henri Lebesgue he was one of the founders of measure theory and its applications in probability theory. From 1934 he was president of the French Academy of Sciences. During the Second World War, he participated in the French Resistance.

$$\partial^\alpha \varphi(x)\Big|_{x=\xi} = (-1)^{|\alpha|}/c_\alpha, \quad \text{if } c_\alpha \neq 0,$$
$$\partial^\alpha \varphi(x)\Big|_{x=\xi} = 0, \qquad\qquad \text{if } c_\alpha = 0.$$

For such function φ we have $\langle \sum_\alpha c_\alpha \delta^{(\alpha)}(x-\xi), \varphi\rangle = \sum_\alpha 1$, where the sum is taken over α such that $c_\alpha \neq 0$. □

Theorem 15.2 (É. Borel) *For any set of numbers $a_\alpha \in \mathbb{C}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, and for any point $\xi \in \Omega \subset \mathbb{R}^n$, there exists a function $\varphi \in C_0^\infty(\Omega)$ such that $\partial^\alpha \varphi\big|_{x=\xi} = a_\alpha$ for any α.*

Proof Without loss of generality, we can assume that $\xi = 0 \in \Omega$. If the coefficients a_α grow “not too fast” as $|\alpha| \to \infty$ (more precisely, if there exist $M > 0$ and $\rho > 0$ such that $\sum_{|\alpha|=k} a_\alpha \leq M\rho^{-k}$ for any $k \in \mathbb{N}$), then the existence of the required function is clear. Indeed, in our case, the series $\sum_\alpha a_\alpha x^\alpha/\alpha!$, where $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$, converges in the ball $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$, and hence we can take

$$\varphi(x) = \psi(x/\rho) \sum_\alpha a_\alpha x^\alpha/\alpha! \in C_0^\infty(B_\rho) \subset C_0^\infty(\Omega)$$

as the required function, where

$$\psi \in C_0^\infty(\mathbb{R}^n), \quad \psi = 0 \text{ for } |x| > 1, \quad \psi = 1 \text{ for } |x| < \frac{1}{2}.$$

However, in the general case, the series $\sum_\alpha a_\alpha x^\alpha/\alpha!$ can diverge in B_ρ. What is the reason of the divergence? Obviously, because it is impossible to guarantee the sufficiently fast decrease of the function $a_\alpha x^\alpha/\alpha!$ as $|\alpha| \to \infty$ for all x lying in a fixed ball B_ρ. One can try to improve the situation by considering the series

$$\sum_\alpha \psi(x/\rho_\alpha) \cdot a_\alpha x^\alpha/\alpha!, \tag{15.2}$$

where ρ_α converges sufficiently fast to zero as $|\alpha| \to \infty$. If it occurs that the series (15.2) converges to a function $\varphi \in C^\infty$, then, as one can easily see, $\varphi \in C_0^\infty(\Omega)$ and $\partial^\alpha \varphi\big|_{x=0} = a_\alpha$. Indeed, setting $\gamma = (\gamma_1, \dots, \gamma_n) \leq \beta = (\beta_1, \dots, \beta_n)$ if $\gamma_k \leq \beta_k$ for any k, and $\beta - \gamma = (\beta_1 - \gamma_1, \dots, \beta_n - \gamma_n)$, we have

$$\begin{aligned}
\partial^\alpha \varphi\big|_{x=0} &= \sum_\beta \frac{a_\beta}{\beta!}\Big(\sum_{\gamma\leq\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!}\Big(\partial^{a-\gamma}\psi\Big)\Big|_{x=0}\Big(\partial^\gamma x^\beta\Big)\Big|_{x=0}\Big) \\
&= \sum_\beta \frac{a_\beta}{\beta!}\Big(\partial^\alpha x^\beta\big|_{x=0}\Big) = \sum_{\beta\neq\alpha} \frac{a_\beta}{\beta!}\Big(\partial^\alpha x^\beta\big|_{x=0}\Big) + a_\alpha = a_\alpha.
\end{aligned}$$

It remains to show that the series (15.2) converges to $\varphi \in C^\infty(\Omega)$. Note that since $\sum_\alpha = \sum_{|\alpha|\leq k} + \sum_{|\alpha|>k}$, it is sufficient to verify that there exist numbers $\rho_\alpha < 1$ such that

$$\sum_{j>k} \sum_{|\alpha|=j} \psi(x/\rho_\alpha) a_\alpha x^\alpha/\alpha! \in C^k(\Omega) \quad \forall k.$$

Let us try to find $\rho_\alpha = \rho_j$ depending only on $j = |\alpha|$. If we can show that

$$\left|\partial^\beta(\psi(x/\rho_{|\alpha|})a_\alpha x^\alpha/\alpha!\right| \le C_\alpha \rho_\alpha \tag{15.3}$$

for any β such that $|\beta| \le k$, where $C_\alpha = C_\alpha(\psi) < \infty$, then by taking $\rho_j = 2^{-j}\left(\sum_{|\alpha|=j} C_\alpha\right)^{-1}$ we obtain

$$\sum_{j>k}\sum_{|\alpha|=j}\left|\partial^\beta(\psi(x/\rho_{|\alpha|})a_\alpha x^\alpha/\alpha!\right| \le \sum_{j>k}\left[\rho_j \sum_{|\alpha|=j} C_\alpha\right] \le 1.$$

So, it remains to prove inequality (15.3). For $|\alpha| > k \ge |\beta|$ we have

$$\begin{aligned}\left|\partial_x^\beta\left(\psi\left(\frac{x}{\rho_{|\alpha|}}\right)\frac{a_\alpha x^\alpha}{\alpha!}\right)\right| &\le \frac{|a_\alpha|}{\alpha!}\sum_{\gamma\le\beta}\frac{\beta!}{\gamma!(\beta-\gamma)!}\left|\partial_x^\gamma\psi\left(\frac{x}{\rho_{|\alpha|}}\right)\right|\cdot|\partial^{\beta-\gamma}x^\alpha|\\
&\le \frac{|a_\alpha|}{\alpha!}\sum_{\gamma\le\beta}\frac{\beta!}{\gamma!(\beta-\alpha)!}\left(\frac{1}{\rho_{|\alpha|}}\right)^{|\gamma|}\cdot\left|\partial_t^\gamma\psi(t)\big|_{t=x/\rho_{|\alpha|}}\cdot x^{\alpha-\beta+\gamma}\right|\cdot\alpha!\\
&\le |a_\alpha|\sum_{\gamma\le\beta}\frac{\beta!}{\gamma!(\beta-\alpha)!}\cdot\left|\partial_t^\gamma\psi(t)\big|_{t=x/\rho_{|\alpha|}}\right|\cdot\rho_{|\alpha|},\end{aligned}$$

where the last inequality holds because $\psi(t) = 0$ for $|t| > 1$. □

Now let us return to the question on the general form of a generalized function $f \in \mathcal{D}^\#(\Omega)$ with support at a point $\xi = 0 \in \Omega$. We note, first of all (see Problem 13.20) that

$$\langle f, \varphi\rangle = \langle f, a\varphi\rangle \quad \forall\varphi \in C^\infty(\Omega)$$

for any function $a \in C_0^\infty(\Omega)$ such that $a \equiv 1$ in some neighborhood of the point $\xi = 0$. In particular, the functional f is defined on polynomials. Putting $c_\alpha = (-1)^{|\alpha|}\langle f, x^\alpha/\alpha!\rangle$, we find that

$$\langle f, \varphi\rangle = \sum_{|\alpha|<N} c_\alpha\langle\delta^{(\alpha)}, \varphi\rangle + \langle f, r_N\rangle \quad \forall N,$$

where

$$r_N(x) = a\left(\frac{x}{\varepsilon_N}\right)\left[\varphi(x) - \sum_{|\alpha|<N}\varphi^{(\alpha)}(0)x^\alpha/\alpha!\right], \quad 0 < \varepsilon_N < 1. \tag{15.4}$$

It is rather tempting to assume that

$$\langle f, r_N\rangle \to 0 \quad \text{as} \quad N \to \infty \tag{15.5}$$

for an appropriate sequence $\{\varepsilon_N\}_{N=1}^\infty$, $0 < \varepsilon_N < 1$, because in this case from Theorem 15.1 we have the following lemma.

Lemma 15.3 *If $f \in \mathcal{D}^\#(\Omega)$, $\operatorname{supp} f = 0 \in \Omega$ and if condition (15.5) is met, then there exists an $N \in \mathbb{N}$ such that $f = \sum_{|\alpha|<N} c_\alpha\delta^{(\alpha)}$.*

However, condition (15.5) may fail to hold if $f \in \mathcal{D}^\#$. The corresponding example can be constructed using the Hamel basis[6] (see, for example, Kolmogorov and Fomin (1980)).

16 The space $\mathcal{D}'$ of distributions (Schwartz generalized functions)

It seems natural to have a theory of generalized functions in which condition (15.5) is satisfied, and therefore, the conclusion of Lemma 15.3 holds. This modest wish (leading, as one can see below, to the theory of Schwartz[7] distributions) unwittingly suggests the following program:

(1) Introduce a convergence in the space $C_0^\infty(\Omega)$ such that

$$\lim_{N\to\infty} r_N = 0 \in C_0^\infty(\Omega) \tag{16.1}$$

in the sense of this convergence, where r_N is defined by (15.4).

(2) Consider only the functionals $f \in \mathcal{D}^\#(\Omega)$ that are continuous with respect to this convergence.

It is clear that one can introduce different convergences according to which $r_N \to 0$ as $N \to \infty$. Which one should be chosen? Considering this question, one should take into account that the choice of a convergence also determines the subspace of those linear functionals on C_0^∞ that are continuous with respect to this convergence. Hence it seems reasonable to augment the above conditions (1) and (2) with the following requirement:

(3) The subspace of functionals continuous with respect to this convergence should include the space $\mathcal{D}^\flat$ of Sobolev derivatives (because this space, as has been shown, plays a very important role in the problems of mathematical physics).

According to Theorem 16.1 that follows, requirement (3) uniquely determines the convergence in the space C_0^∞; moreover (in view of Lemma 16.10, see below) condition (16.1) will also be satisfied.

Theorem 16.1 *Let $\{\varphi_j\}$ be a sequence of functions $\varphi_j \in C_0^\infty(\Omega)$. Then the following two conditions are equivalent*:

(1°) *$\langle f, \varphi_j\rangle \to 0$ as $j \to \infty$ for any function $f \in \mathcal{D}^\flat$.*

(2°) *There exists a compact set $K \subset \Omega$ such that* $\operatorname{supp}\varphi_j \subset K$ *for any j, and* $\max_{x\in\Omega}|\partial^\alpha \varphi_j(x)| \to 0$ *as $j \to \infty$ for any α.*

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[6] Georg Hamel (1877–1954) was a German mathematician and mechanicist. His works on the foundations of mathematics and axiomatic in mechanics are particularly well known. In 1905, he published a paper in which he proposed a clear and detailed approach to the use of the axiom of choice in construction of a basis of integers, later called the Hamel basis, as a vector space over the rationals.

[7] Laurent Schwartz (1915–2002) was a famous French mathematician, Fields Prize winner in 1950, a member of the "Nicolas Bourbaki group" (as well as his legendary student, one of the greatest mathematicians of the twentieth century, Alexander Grothendieck (1928–2014)).

Proof The implication $2° \to 1°$ is clear. The converse assertion follows from Lemmas 16.2–16.5.

Lemma 16.2 *For any α, there exists a C_α such that*

$$\max_{x\in\Omega} |\partial^\alpha \varphi_j(x)| \le C_\alpha \quad \text{for any } j.$$

Proof For any α, consider the sequence of functionals

$$\varphi_j^{(\alpha)}\colon L^1(\Omega) \ni f \mapsto \int_\Omega f(x)\partial^\alpha\varphi_j(x)\,dx, \quad j \ge 1.$$

It is clear that these functionals are linear and continuous, because $\partial^\alpha\varphi_j \in C_0^\infty(\Omega)$. By 1°, we have $\langle\varphi_j^{(\alpha)}, f\rangle \to 0$ as $j \to \infty$ for any function $f \in L^1$. Hence by the Banach–Steinhaus theorem[8] there exists a constant C_α such that $\|\partial^\alpha\varphi_j\|_{(L^1(\omega))'} = \|\partial^\alpha\varphi_j\|_\infty \le C_\alpha$ for any j, where the equality of the norms follows from Riesz's Theorem 9.17. □

Lemma 16.3 *For any α and any $x_0 \in \Omega$,*

$$\partial^\alpha\varphi_j(x_0) \to 0 \ \text{as} \ j \to \infty.$$

Proof Since $\delta^{(\alpha)}(x - x_0) \in \mathcal{D}^\flat(\Omega)$, we have

$$\partial^\alpha\varphi_j(x_0) = \langle\delta^{(\alpha)}(x - x_0), (-1)^{|\alpha|}\varphi_j(x)\rangle \to 0.$$

Lemma 16.4 *There exists a compact set $K \subset \Omega$ such that* supp $\varphi_j \subset K$ *for any j.* □

[8] The *Banach–Steinhaus theorem* (1927) asserts the following. *Let X be a Banach space and let $\{\varphi_j\}$ be a family of continuous linear functionals on X. If, for any $x \in X$, there exists a $C_x < \infty$ such that $|\langle\varphi_j, x\rangle| \le C_x$ for any j, then there exists a constant $C < \infty$ such that $|\langle\varphi_j, x\rangle| \le C$ for $\|x\| \le 1$ for any j.*

Proof Assume the contrary. If the sequence of functionals φ_j were not bounded for $\|x\| \le 1$, then it would not be bounded on any ball $B_r(a) = \{x \in X\colon \|x - a\| \le r\}$. Consider a point $x_1 \in B_1(0)$, a functional φ_{k_1}, and a number $r_1 < 1$ such that $|\langle\varphi_{k_1}, x\rangle| > 1$ for $x \in B_{r_1}(x_1) \subset B_1(0)$. (Such r_1 and x_1 exist by the hypothesis and since the functionals φ_j are continuous.) Next, take a point $x_2 \in B_{r_1}(x_1)$, a functional φ_{k_2}, and a number $r_2 < r_1$ such that $|\langle\varphi_{k_2}, x\rangle| > 2$ for $x \in B_{r_2}(x) \subset B_{r_1}(x_1)$. Continuing this construction, we obtain a sequence of closed nested balls $B_{r_k}(x_k)$ whose radii tend to zero. In this case, $|\langle\varphi_{k_j}, x_0\rangle| > j$ for $x_0 \in \cap B_{r_k}$ (the intersection $\cap B_{r_k}$ is nonempty since X is complete). □

Hugo Steinhaus (1887–1972), one of the founders of L'vov mathematical school. In the spring of 1916, thanks to a lucky chance, he dramatically changed the life of 24-year-old Stefan Banach (see p. 42), an unemployed man who had completed two courses of the Polytechnic University before the war and earned a living as a tutor. Soon their joint paper appeared, the first for Banach (who in 1924 was elected a corresponding member of the Polish Academy of Sciences).

Proof Assume the contrary. We set $K_j = \bigcup_{k<j} \operatorname{supp}\varphi_k$. It can be assumed that the intersection $(\operatorname{supp}\varphi_j)\bigcap(\Omega\backslash K)$ is nonempty. Therefore, there exists an $x_j \in \Omega\backslash K_j$ such that $\varphi_j(x_j) \neq 0$. For each j, we choose a $\lambda_j > 0$ such that

$$\frac{|\varphi_j(x)|}{|\varphi_j(x_j)|} > \frac{1}{2} \quad \forall x \in V_j = \{|x - x_j| < \lambda_j\} \subset M_j = \operatorname{supp}\varphi_j\backslash K_j. \tag{16.2}$$

Since the intersection $V_j \bigcap V_k$ is empty for $j \neq k$, consider the function $f \in L^1_{\mathrm{loc}}(\Omega)$, which is zero outside $\bigcup_{j\geq 1} V_j$ and such that

$$f(x) = a_j|\varphi_j(x_j)|^{-1}\exp[-i\arg\varphi_j(x)] \quad \text{for } x \in V_j, \quad j \geq 1, \tag{16.3}$$

where $a_j > 0$ constants, which we will now select so that to prove the inequality[9]

$$\left|\int_\Omega f\varphi_j\,dx\right| \geq j \tag{16.4}$$

from which the lemma follows. We have $\operatorname{supp} f\varphi_j \subset (V_j \bigcup K_j)$, because $\operatorname{supp}\varphi_j \subset (M_j \bigcup K_j)$. Hence the last integral in the equality

$$\int_\Omega f\varphi_j\,dx = \int_{V_j} f\varphi_j\,dx + \int_{(\operatorname{supp} f\varphi_j)\backslash V_j} f\varphi_j\,dx$$

is estimated as

$$\left|\int_{K_j} f\varphi_j\,dx\right| \leq \max_\Omega|\varphi_j| \int_{K_j} |f|\,dx \leq A_j,$$

where $A_j = C\sum_{k<j}|a_k| \cdot \mu(V_k)$. Taking $a_j = 2(A_j + j)$, we arrive at estimate (16.4), because the inequality $\int_{V_j} f\varphi_j\,dx \geq a_j/2$ holds in view of (16.2), (16.3). □

Lemma 16.5 *For any multiindex α, any $\varepsilon > 0$, and any point $x_0 \in \Omega$, there exist $\lambda > 0$ and $\nu \geq 1$ such that $|\varphi_j^{(\alpha)}(x)| < \varepsilon$ for $|x - x_0| < \lambda$ and $j \geq \nu$.* □

Proof Assume the contrary. Then there exist α, $\varepsilon_0 > 0$, $x_0 \in \Omega$ such that, for any j, there exists an $x_j \in \{x \in \Omega\colon |x - x_0| < 1/j\}$ satisfying $|\varphi_j^{(\alpha)}(x_j)| \geq \varepsilon_0$. But on the other hand,

$$|\varphi_j^{(\alpha)}(x_j)| \leq |\varphi_j^{(\alpha)}(x_j) - \varphi_j^{(\alpha)}(x_0)| + |\varphi_j^{(\alpha)}(x_0)| \to 0,$$

and hence

$$|\varphi_j^{(\alpha)}(x_j) - \varphi_j^{(\alpha)}(x_0)| \leq C|x_j - x_0| \to 0, \text{ and } \partial^\alpha\varphi_j(x_0) \to 0$$

by Lemmas 16.2 and 16.3. □

This completes the proof of Theorem 16.1. □

[9] Inequality (16.4) contradicts the original condition 1°.

Remark 16.6 In fact, a little more has been proved than was announced in Theorem 16.1. Namely, condition 2° follows from the fact $\langle f, \varphi_j \rangle \to 0$ as $j \to \infty$ for any function $f \in L^1_{\text{loc}}(\Omega)$ and for any Sobolev derivative $f = \partial^\alpha g$, where $g \in L^1(\Omega)$.

Now we can define the spaces $\mathcal{D}$ and $\mathcal{D}'$ introduced by Schwartz (see Schwartz 1950–1951).

Definition 16.7 The space $\mathcal{D}(\Omega)$, which is sometimes called the *space of test functions* (cf. § 1), is the space $C_0^\infty(\Omega)$ equipped with the following convergence of sequence of function $\varphi_j \in C_0^\infty(\Omega)$ to a function $\varphi \in C_0^\infty(\Omega)$:

(a) There exists a compact set K such that $\operatorname{supp} \varphi_j \subset K$ for any j.

(b) For any $\beta = (\beta_1, \ldots, \beta_n)$ and for any $\sigma > 0$, there exists an $N = N(\beta, \sigma) \in \mathbb{N}$ such that

$$|\partial^\beta \varphi_j(x) - \partial^\beta \varphi(x)| < \sigma \quad \forall x \in \Omega \quad \text{for } j \geq N.$$

In this case we write $\varphi_j \xrightarrow{\mathcal{D}} \varphi$ as $j \to \infty$ (or $\lim_{j\to\infty} \varphi_j = \varphi$ in $\mathcal{D}$).

Remark 16.8 It is clear that $\mathcal{D}(\Omega) = \bigcap_{s\geq 0} \mathcal{D}_s(\Omega)$, where $\mathcal{D}_s(\Omega)$ is the function space $C_0^s(\Omega)$ equipped with the convergence, which differs from the one introduced in Definition 16.7 only by the fact that the multiindex β in condition (b) satisfies the condition $|\beta| \leq s$. It can be shown (see Exercise 16.23) that

$$\mathcal{D}^\flat(\Omega) = \bigcup_{s\geq 0} \mathcal{D}'_s(\Omega)$$

(i.e., $f \in \mathcal{D}^\flat \Leftrightarrow$ there exists an $s \geq 0$ such that $f \in \mathcal{D}'_s$), where $\mathcal{D}'_s(\Omega)$ is the space of linear functionals on $\mathcal{D}_s(\Omega)$ which are continuous with respect to this convergence in $\mathcal{D}_s(\Omega)$. The *spaces* $\mathcal{D}_s$ and $\mathcal{D}'_s$ were introduced by Sobolev (see Sobolev 1936).

Definition 16.9 The space $\mathcal{D}'(\Omega)$ of Schwartz distributions (also called the *space of Schwartz generalized functions*) is the space of continuous linear functionals on $\mathcal{D}(\Omega)$, i.e., linear functionals on $\mathcal{D}(\Omega)$ which are continuous with respect to the convergence on $\mathcal{D}(\Omega)$.

Lemma 16.10 *There exists a sequence ε_N (from which r_N is defined by (15.4)) such that*

$$\lim_{N\to\infty} r_N \xrightarrow{\mathcal{D}} 0.$$

Proof By the Taylor formula

$$r_N(x) = a\Big(\frac{x}{\varepsilon_N}\Big) \sum_{|\alpha|=N+1} \frac{N+1}{\alpha!} x^\alpha \int_0^1 (1-t)^N \varphi^{(\alpha)}(tx)\,dt.$$

Hence by the Leibniz formula we get

$$|\partial^\beta r_N(x)| \leq C_N (\varepsilon_N)^{N+1-|\beta|} \leq (1/2)^{N/2}$$

for $N \geq 2|\beta|$ if $\varepsilon_N \leq \frac{1}{2} C_N^{-\frac{2}{N+2}}$. □

The next result follows from Lemmas 15.3 and 16.10.

Theorem 16.11 (L. Schwartz) *Let* $f \in \mathcal{D}'(\Omega)$ *and let* $\operatorname{supp} f = 0 \in \Omega$. *Then there exist* $N \in \mathbb{N}$ *and* $c_\alpha \in \mathbb{C}$ *such that*

$$f = \sum_{|\alpha| \le N} c_\alpha \delta^{(\alpha)}.$$

P 16.12 Let $f_k \in \mathcal{D}'(\mathbb{R})$, where $k = 0$ or $k = 1$, and let $x \cdot f_k(x) = k$. Show (cf. Example 14.1) that $f_0(x) = C\delta(x)$, $f_1(x) = \text{v. p.}\frac{1}{x} + C\delta(x)$, where $C \in \mathbb{C}$.

The following series of Exercises 16.13–16.25 concerns the question on the structure (general form) of distributions. Some hints are given at the end of the section.

P 16.13 Verify that the following assertions are equivalent:

(a) f is a distribution with compact support, i.e., $f \in \mathcal{D}'(\Omega)$ and $\operatorname{supp} f$ is a compact set in Ω.

(b) $f \in \mathcal{E}'(\Omega)$; this means that f is a continuous linear functional on $\mathcal{E}(\Omega)$, i.e.,

$$\lim_{j\to\infty} \varphi_j = \varphi \quad \text{in } \mathcal{E} \iff \lim_{j\to\infty} a\varphi_j = a\varphi \quad \text{in } \mathcal{D} \ \ \forall a \in C_0^\infty(\Omega)$$

on the space $C^\infty(\Omega)$ with this convergence.

P 16.14 Verify that $f \in \mathcal{D}'(\Omega)$ if and only if $f \in \mathcal{D}^\#(\Omega)$ and, for any compact set $K \subset \Omega$, there exist constants $C = C(K, f) > 0$ and $N = N(K, f) \in \mathbb{N}$ such that

$$|\langle f, \varphi\rangle| \le C \cdot p_{K,N}(\varphi) \tag{16.5}$$

for any function $\varphi \in C_0^\infty(K, \Omega) = \{\psi \in C_0^\infty(\Omega) \colon \operatorname{supp}\psi \subset K\}$, where

$$p_{K,N}(\varphi) = \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \varphi(x)|. \tag{16.6}$$

P 16.15 (Cf. Exercise 16.14)

Let $\bigcup_{M\ge 1} K_M = \Omega$, where K_M are compact sets in $\mathbb{R}^n$. Verify that $f \in \mathcal{E}'(\Omega)$ (see Exercise 16.13) if and only if $f \in \mathcal{D}^\#(\Omega)$ and there exist constants $C = C(f) > 0$ and $N = N(f) \ge 1$ such that $|\langle f, \varphi\rangle| \le C \cdot p_N(\varphi)$ for any function $\varphi \in C_0^\infty(\Omega)$, where

$$p_N(\varphi) = \sum_{|\alpha|\le N} \sup_{x\in K_N} |\partial^\alpha \varphi(x)|. \tag{16.7}$$

P 16.16 (Continuation) Let $f \in \mathcal{E}'(\Omega)$, $\operatorname{supp} f \subset \omega \Subset \Omega \subset \mathbb{R}^n$. Using equality (16.7) and the inequality

$$|\psi(x)| \le \int_\Omega \Big| \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \psi(x) \Big| \, dx \quad \forall \psi \in C_0^\infty(\Omega),$$

show that there exist numbers $C > 0$ and $m \ge 1$ such that

$$|\langle f, \varphi\rangle| \le C \int_\Omega \Big| \frac{\partial^{nm}}{\partial x_1^m \ldots \partial x_n^m} \varphi(x) \Big| \, dx \quad \forall \varphi \in C_0^\infty(\omega). \tag{16.8}$$

P 16.17 (Continuation) Checking that the function $\varphi \in C_0^\infty(\omega)$ can be uniquely recovered from its derivative $\psi = \frac{\partial^{nm}\varphi}{\partial x_1^m \ldots \partial x_n^m}$, show that the linear functional $l \colon \psi \mapsto \langle f, \varphi\rangle$, which is defined on the subspace

$$Y = \left\{\psi \in C_0(\omega)\colon \psi = \frac{\partial^{nm}}{\partial x_1^m \dots \partial x_n^m}\varphi,\ \varphi \in C_0^\infty\right\}$$

of the space $L^1(\omega)$, is continuous.

P 16.18 (Continuation) Applying the *Hahn–Banach theorem* on the continuation of linear continuous functionals (see Kolmogorov and Fomin 1980, p. 169), show that there exists a function $g \in L^\infty(\omega)$ such that

$$\int_\omega g(x)\frac{\partial^{nm}}{\partial x_1^m \dots \partial x_n^m}\varphi(x)\,dx = \langle f, \varphi\rangle \quad \forall \varphi \in C_0^\infty(\omega).$$

P 16.19 (Continuation) Show that the following theorems hold.

Theorem 16.20 (On the General Form of Compactly Supported Distributions $f \in \mathcal{E}'$) *Let $f \in \mathcal{E}'(\Omega)$. Then there exist a function $F \in C_0(\Omega)$ and a number $M \geq 0$ such that $f = \partial^\alpha F$, where $\alpha = (M, \dots, M)$, i.e.,*

$$\langle f, \varphi\rangle = (-1)^{|\alpha|}\int_\Omega F(x)\partial^\alpha\varphi(x)\,dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Theorem 16.21 (On the General Form of Distributions $f \in \mathcal{D}'$) *Let $f \in \mathcal{D}'(\Omega)$. Then there exists a sequence of functions $F_\alpha \in C(\Omega)$ parameterized by $\alpha \in \mathbb{Z}_+^n$ such that $f = \sum_\alpha \partial^\alpha F_\alpha$. More precisely, $F_\alpha = \sum_{j=1}^\infty F_{\alpha_j}$, $F_{\alpha_j} \in C(\Omega)$, and*

(1) supp $F_{\alpha_j} \subset \Omega_j$, *where $\{\Omega_j\}_{j\geq 1}$ is a locally finite cover of Ω.*

(2) *for any $j \geq 1$ there exists an $M_j \geq 1$ such that $F_{\alpha_j} = 0$ for $|\alpha| > M_j$.*

P 16.22 (Peetre 1960[10]) Let $A\colon \mathcal{D}(\Omega) \to \mathcal{E}'(\Omega)$ be a continuous linear with *localization property,* i.e.,

$$\operatorname{supp} Au \subset \operatorname{supp} u \quad \forall u \in \mathcal{D}(\Omega). \tag{16.9}$$

Verify that A is a differential operator, more precisely, for any compact set $K \subset \Omega$, there exists an $N(K) < \infty$ and a family $\{c_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ of functions $c_\alpha \in C^\infty(\Omega)$ such that, for any $u \in \mathcal{D}(\Omega)$ and $x \in K$,

$$(Au)(x) = \sum_{|\alpha|\leq m(x)} c_\alpha(x)\partial^\alpha u(x),$$

where $m(x) \leq N(K)$ for any compact set $K \subset \Omega$.

P 16.23 (See Remark 16.8) Verify that $\mathcal{D}^\flat(\Omega) = \bigcup_s \mathcal{D}'_s(\Omega)$.

Remark 16.24 By Definition 13.2, a functional $f \in \mathcal{D}'$ has a finite order of singularity if there exist $k \geq 1$ and functions $f_\alpha \in L^1_{\mathrm{loc}}$, where $|\alpha| \leq k$, such that $f = \sum_{|\alpha|=k} \partial^\alpha f_\alpha$. The least k for which such a representation of f is possible, is called its order of singularity. In these terms, the space of Sobolev derivatives $\mathcal{D}^\flat$ is, according to Definition 12.2, the space of all distributions with finite order of singularity.

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[10] Jaak Peetre (1935–2019), was a Swedish mathematician of Estonian origin. In 1944, he arrived in Sweden with his family. In 1963, he was appointed professor at Lund University. Member of the Royal Swedish Academy of Sciences since 1983. Main areas of research: partial differential equations, operator interpolation, differential geometry, functional analysis.

P 16.25 Resolve the following paradox. On the one hand, the discontinuous function

$$f(x, y) = \{\begin{cases} \mathrm{Re}\big(e^{-1/z^4}\big) & \text{for } z \neq 0, z = x + iy \in \mathbb{C}, \\ 0 & \text{for } z = 0 \end{cases} \tag{16.10}$$

(being the real part of a function analytic in $\mathbb{C}\backslash 0$ with zero second derivatives with respect to x and y at the origin) is a solution of the Laplace equation on the plane. On the other hand, by Theorem 16.21 and the a priori estimate (21.7) from § 22 we have: if $f \in \mathcal{D}'(\Omega)$ and $\Delta f \equiv 0$ in Ω, then $f \in C^\infty(\Omega)$. The first proof of this fact, which was given by Schwartz in Schwartz (1950–1951) (see also the book by Gelfand[11] and Shilov (Gelfand et al. 1958–1966, Vol 2), depends on Theorem 5.10 on the arithmetic mean of harmonic functions.

P 16.26 Verify that $\mathcal{E}$ is metrizable, while $\mathcal{D}$ is not.

Remark 16.27 One can introduce in $\mathcal{D}$ (respectively, in $\mathcal{E}$) the structure of a so-called (see Kolmogorov and Fomin 1980; Robertson 1980) linear locally convex topological space[12] such that the convergence in this space coincides with the one introduced above. For example, a neighborhood of the origin in $\mathcal{D}$ can be defined with the

[11] Academician Israel Moiseevich Gelfand (1913–2009) was one of the greatest mathematicians of the twentieth century, author of many theoretical studies and applied research papers in which mathematical methods are applied in the field of physics, seismology, biology, neurophysiology, medicine. Born in Okny, Kherson Governorate, Russian Empire. After finishing only nine classes of school, a year later he came to Moscow and, working as a monitor in the Lenin library, engaged in self-education. By a lucky chance, he found himself in the field of view of A. N. Kolmogorov, who immediately appreciated his talent and was able to overcome the formal difficulties for enrolling I. M. in his graduate school. In 1938, I. M. presented, and in 1940 defended his doctoral thesis. He is the recipient of numerous national and international awards; honorary Doctor of seven foreign universities, including Harvard and Oxford; honorary foreign member of the American Academy of Arts and Sciences. A very informative and interesting article about I. M. can be found in Wikipedia.

Here is what A. A. Kirillov writes (https://www.math.upenn.edu/~kirillov/): "The *world-famous Gelfand seminar on functional analysis* was intended for high elementary school students interested in mathematics, capable students, excellent graduate students, and outstanding professors. This seminar was a kind of "window into the real world" not only for novice mathematicians but also for many established scientists. In 1964, with the help I. G. Petrovskii, the rector of Moscow State University, Gelfand founded the famous All-Union Correspondence Mathematical School. An analogue of such a school was established in the USA in 1990. I believe that the effect of this modest enterprise, existing on private donations, exceeds what has been achieved by several US presidents with multibillion dollar investments in the improvement of mathematics education. In my opinion, this is one of the greatest services of I. M. to humanity."

For the *Gelfand biological seminar* and on the typical Gelfand style of conducting seminars, see, for example, http://iitp.ru/ru/userpages/325/103.htm.

I will also cite here a little-known fact related to Gelfand's activities in the atomic project in the early 1950s. He was in charge of calculations there. In those years, computer programs were written in machine code, and machines worked only at night. An emergency happened once: the computer stopped. Formally, Gelfand was responsible for the code. They came for him urgently and brought him to sort it out. And he, who has never written this code, figured it out! According to V.M. Tikhomirov (see his book "We remember them," Izd. Popech. Soveta Mech. Math. Faculty of MSU, 2017): "This is one of the most remarkable testimonies to the genius of an outstanding man in my entire life.")

[12] A linear space X is called a *locally convex topological space* if it is a *topological* (Kolmogorov and Fomin 1980) space, the operations of addition and multiplication by a number are continuous and, moreover, any neighborhood of the origin in X contains a convex neighborhood of the origin.

help of any finite set of everywhere positive functions $\gamma_m \in C(\Omega)$ ($m = 0, 1, \dots, M$; $M \in \mathbb{Z}_+$) as the set of all functions $\varphi \in C_0^\infty(\Omega)$ such that $|\partial^\alpha \varphi| < \gamma_{|\alpha|}$ if $|\alpha| \le M$. The topology in $\mathcal{E}$ can be introduced from the distance defined by the formula in the hint to Exercise 16.26. Thus, $\mathcal{E}$ is a *Fréchet*[13] *space*, i.e., a complete metric linear locally convex topological space. The Banach–Steinhaus theorem also holds in Fréchet spaces (see, for example, Robertson (1980)): the space of continuous linear functionals on a Fréchet space (in particular, the space $\mathcal{E}'$) is weakly sequentially complete. Although $\mathcal{D}$ is not a Fréchet (see Exercise 16.26), the space $\mathcal{D}'$ is also weakly sequentially complete (for a direct proof, see, for example, Gelfand et al. (1958–1966), Vol. 1, Addendum).

Hints to Exercises P. 16.13–P. 16.26.

16.13. If (a) were not implied by (a), then there would exist a sequence of points x_k such that $x_k \to \partial\Omega$, and $f \neq 0$ in the neighborhood of x_k.

16.14. If $f \in \mathcal{D}'$, but estimate (16.5) were false, then there would exist a $K = \overline{K} \subset \Omega$ such that, for any $N \ge 1$, there would exist a function $\varphi_N \in C_0^\infty(\Omega)$ satisfying the condition $\operatorname{supp} \varphi_N \subset K$, and besides,

$$|\langle f, \varphi_N \rangle| \ge N \sum_{|\alpha| \le N} \sup_K |\varphi_N^{(\alpha)}|.$$

Hence $\psi_N = \varphi_N \cdot |\langle f, \varphi_N \rangle|^{-1} \to 0$ in $\mathcal{D}$, but $|\langle f, \psi_N \rangle| = 1$.

16.15. Since $\langle f, \varphi \rangle = \langle f, \rho\varphi \rangle$, where $\rho \in C_0^\infty$, $\rho \equiv 1$ on $\operatorname{supp} f$ one can put $K_N = \operatorname{supp} \rho$. *Warning*: in general $K_N \neq \operatorname{supp} f$. Indeed, following the book (Schwartz 1950–1951, v. 1, p. 94) consider a functional $f \in \mathcal{E}'(\mathbb{R})$ such that

$$\langle f, \varphi \rangle = \lim_{m \to \infty} \Big[\Big(\sum_{\nu \le m} \varphi\Big(\frac{1}{\nu}\Big) \Big) - m\varphi(0) - (\ln m)\varphi'(0) \Big].$$

It is clear that $\operatorname{supp} f$ is the set of points $\frac{1}{\nu}$, $\nu \ge 1$, together with their limit point $x = 0$. Consider a sequence of functions $\varphi_j \in C_0^\infty(\mathbb{R})$ such that $\varphi_j(x) = 0$ for $x \le \frac{1}{j+1}$, $\varphi_j(x) = \frac{1}{\sqrt{j}}$ for $\frac{1}{j} \le x \le 1$. Taking $K = \operatorname{supp} f$ in (16.6), we get $p_{K,N}(\varphi_j) \to 0$ as $j \to \infty$ for any $N \ge 1$, whereas $\langle f, \varphi \rangle = j/\sqrt{j} \to \infty$.

16.16. Use the inequality

$$\sup_K |\partial^\alpha \varphi(x)| \le C_j(K) \sup_K \Big| \frac{\partial}{\partial x} \partial^\alpha \varphi(x) \Big|.$$

[13] René Maurice Fréchet (1878–1973) was a French mathematician. In 1906, he introduced such fundamental concepts of analysis as compactness, completeness, and metric space; he also worked in the field of probability theory. His name is associated with such concepts as Fréchet derivative, Fréchet filter, Fréchet surface, etc.

16.17. Apply inequality (16.8).

16.18. By Riesz's theorem 9.17 we have $(L^1)' = L^\infty$.

16.20. Extend g outside ω by zero (see Exercise 16.18) and put $F(x) = (-1)^{mn}\int_{y<x} g(y)\,dy$.

16.21. Let $\sum \psi_j \equiv 1$ be a *partition of unity*. Then

$$\langle f, \varphi\rangle = \sum_j \langle \psi_j f, \varphi\rangle = \sum_j \sum_{|\alpha|\le M_j} \langle \partial^\alpha F_{\alpha_j}, \varphi\rangle = \sum_\alpha \langle \partial^\alpha \sum_j F_{\alpha_j}, \varphi\rangle.$$

16.22. Since $A\colon \mathcal{D}(\Omega) \to \mathcal{E}'(\Omega)$ and since $\operatorname{supp} Au \overset{(16.9)}{\Subset} \Omega$, for any $a \in K \Subset \Omega$ one can define the functional $A_a \in \mathcal{E}'$ such that $\langle A_a, u\rangle = (Au)(x)\big|_{x=a}$. We have $\operatorname{supp} A_a = a$, and hence by Theorem 16.11,

$$A_a : \mathcal{E} \ni u \mapsto Au\big|_{x=a} = \sum_{|\alpha|\le m(a)} \left(a_\alpha(x)\partial^\alpha u(x)\right)\big|_{x=a}.$$

We have $\sup_u |\langle A_a, u\rangle| \le C_{m(a)} < \infty \qquad \forall a \in K$. Hence by applying the Banach–Steinhaus theorem for the space $\mathcal{E}(\Omega)$, which is a Fréchet space (see Remark 16.27), we get $\sup_u |\langle A_a, u\rangle| \le C < \infty$, which implies (by Borel's Theorem 15.1) that $\sup_{a\in K} |m(a)| < \infty$ for any $K = \overline{K} \subset \Omega$. Using Exercise 13.19 and applying A to $\frac{(a-x)^\alpha}{\alpha!}$, we ind that $c_\alpha \in C^\infty(\Omega)$. 16.23. Apply Theorem 16.21.

16.25. The function (16.10) does not lie in $\mathcal{D}'$ (i.e., it cannot be regularized in $\mathcal{D}'$). The same is true for any function $f \in C^\infty(\mathbb{R}\backslash 0)$ which fails to satisfy the estimate $|f(x)| \le C|x|^{-m}$ for any $m \in \mathbb{N}$ and $C > 0$ for $0 < |x| < \varepsilon$, where $1/\varepsilon \gg 1$. The last fact can be proved, by constructing a sequence of numbers $\varepsilon_j > 0$ such that $\int_{\mathbb{R}^n} f(x)\varphi_j(x)\,dx \to \infty$ as $j \to \infty$ for the function $\varphi_j(x) = \varepsilon_j \varphi(jx)$, where $\varphi \in C_0(\mathbb{R})$, $\varphi = 0$ outside the domain $\{1 < |x| < 4\}$, $\int \varphi = 1$, but $\varphi_j \to 0$ in $\mathcal{D}$ as $j \to \infty$.

16.26. The space $\mathcal{E}$ can be endowed with the distance $\rho(\varphi, \psi) = d(\varphi - \psi)$, where $d(\varphi) = \sum_1^\infty 2^{-N} \min(p_N(\varphi), 1)$, and p_N is defined in (16.7). The space $\mathcal{D}$ is not metrizable, because the sequence $\varphi_{k,m}(x) = \varphi(x/m)/k$, where $\varphi \in \mathcal{D}(\mathbb{R})$, fails to satisfy the following metric space property: if $\varphi_{k,m} \to 0$ as $k \to \infty$, then, for any m, there exists $k(m)$ such that $\varphi_{k(m),m} \to 0$ as $m \to \infty$.

Chapter 3
Pseudo-Differential Operators and Fourier Operators

17 Fourier Series and Fourier Transform. The Spaces $\mathcal{S}$ and $\mathcal{S}'$

In 1807, Jean-Baptiste Fourier[1] had his say in the famous (going from the beginning of the XVIII century) dispute about the sounding string (Narasimhan 1990, Ch. XII). According to Luzin (1935),[2] he made a discovery that "caused the greatest perplexity and confusion among all mathematicians." It overturned all concepts and became a source of new deep ideas for the development of such concepts as function, integral, and trigonometric series. . . Fourier's discovery (strange as it may seem at first glance) consists in a formal rule for calculating the coefficients

$$a_k = \frac{1}{p}\int_{-p/2}^{p/2} u(y)e^{-\mathring{\imath}(k/p)y}\,dy, \quad \mathring{\imath} = 2\pi i, \quad i = \sqrt{-1}, \quad k \in \mathbb{Z}, \tag{17.1}$$

which are represented (for $k \neq 0$) in terms of the numbers

[1] Jean-Baptiste Joseph Fourier (1768–1830) was a French mathematician and physicist, whose name appears on the first floor of the Eiffel Tower in the list of the greatest scientists in France He, the son of a tailor, became president of the local Revolutionary Committee in 1794 but did not get along with Robespierre. It was only thanks to the Coup d'État of 9 Thermidor (July 27, 1794) that he was not executed. Favored by Napoleon, Fourier received from him the title of baron and was awarded the Legion of Honor In 1804, to fulfill the duties of the prefect of the department, Fourier moved to Grenoble (where the university now bears his name). There, he becomes the Fourier that the entire mathematical world knows: he deduced the heat equation and invented the *method of separation of variables*—one of the most common methods for solving problems in mathematical physics, which he used to find a solution to the mixed problem for the heat equation (see (17.18)). And most importantly, Fourier presented the initial function as a trigonometric series with explicitly given coefficients.

[2] Nikolai Nikolaevich Luzin (also spelled Lusin), (1883–1950) was an Academician of the USSR Academy of Sciences, founder of the Moscow Mathematical school. Among his pupils we mention A. N. Kolmogorov, M. A. Lavrentiev, L. A. Lyusternik, A. A. Lyapunov, D. E. Men'shov, P. S. Novikov, M. Ya. Suslin, P. S. Uryson, A. Ya. Khinchin, L. G. Shnirelman. For Luzitania, a loose group of young Moscow mathematicians, see Lyusternik, "The early years of the Moscow Mathematics School" (Russian Math. Surveys, 22:1 (1967), 133–157).

A. S. Demidov, *Equations of Mathematical Physics*,
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$$c_k = a_k + a_{-k}, \quad d_k = i[a_k - a_{-k}], \quad k \geq 1,$$

and which are called the *Fourier coefficients* in the "expansion"

$$u(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{\mathring{i}(k/p)x} = a_0 + \sum_{k\geq 1} \Big[c_k \cos\Big(2\pi \frac{k}{p} x\Big) + d_k \sin\Big(2\pi \frac{k}{p} x\Big) \Big] \tag{17.2}$$

of an "arbitrary" function $u \colon \Omega = \left]-\frac{p}{2}, \frac{p}{2}\right[\ni x \mapsto u(x) \in \mathbb{C}$ in harmonics

$$e^{\mathring{i}(k/p)x} = \cos\Big(2\pi \frac{k}{p} x\Big) + i \sin\Big(2\pi \frac{k}{p} x\Big), \quad k \in \mathbb{Z}. \tag{17.3}$$

The trigonometric series (17.2) is called the *Fourier series* of the function u (in the system of functions[3] (17.3)).

It should be borne in mind that for a randomly chosen continuous function (such a function, as a rule, is not differentiable everywhere), its Fourier series will almost certainly diverge at a given point. The first result concerning the convergence of the Fourier series was obtained by 24 years old L. Dirichlet (see, for example, Zorich 2016): if a function u is piecewise continuous on $\left[-\frac{p}{2}, \frac{p}{2}\right]$ and if the number of its intervals of monotonicity is finite, then the Fourier series of the function u converges to u at every point of continuity of u, and moreover, if u is continuous and $u\left(-\frac{p}{2}\right) = u\left(\frac{p}{2}\right)$, then the series (17.2) converges to u uniformly. A substantially stronger result of C. Jordan,[4] which was published in 1881, is known as the *Dirichlet–Jordan theorem.*

Theorem 17.1 (Dirichlet–Jordan) *Let $u(x)$ be a piecewise continuous function on $\left[-\frac{p}{2}, \frac{p}{2}\right]$ of bounded variation.*[5] *Then the Fourier series of this function converges to this function uniformly on each compact set not containing its discontinuity points; at each discontinuity point the Fourier series of $u(x)$ converges to the arithmetic mean of its limit values at this point.*

The *Fourier coefficients* (17.1) are defined for any function $u \in L^1$. However, the Fourier series may diverge at some points even for continuous functions (see the footnote on p. 75, and Shilov 2016, Kolmogorov and Fomin 1980, and also Exercise 17.11). Concerning the integrable functions, 19-year-old A. N. Kolmogorov[6]

[3] In this regard, see (17.19), (17.20).

[4] Marie Ennemond Camille Jordan (1838–1922), was a French mathematician who made fundamental contributions to group theory. His theorems on reducing a matrix to a Jordan normal form, on a Jordan curve, and Jordan's lemma on estimating the integral along an arc on the complex plane are widely known. The Jordan measure was the forerunner of the Lebesgue measure.

[5] Such functions $u(x)$ can be written as the difference of two monotone (nondecreasing) functions; their derivative $u'(x)$ is almost everywhere finite, and their graphs on a closed interval $[a, b]$ have finite length $\int_a^b \left(1 + [u'(x)]^2\right)^{1/2} dx$. In particular (as is easily seen), the function $[0, 1] \ni x \mapsto x^\alpha \sin \frac{1}{x}$ has a bounded variation for $\alpha > 1$, unlike in the case $\alpha \leq 1$.

[6] Andrey Nikolayevich Kolmogorov (1903–1987) was one of the most outstanding mathematicians of XX century. His book "Foundations of the Theory of Probability" (1933) had laid the foun-

constructed in 1922 (see Tikhomirov 1991) his famous example of a function $u \in L^1$ whose Fourier series diverges almost everywhere (later he also constructed an example of an integrable function whose Fourier series diverges everywhere). In 1990, the Salem Prize was awarded to Konyagin (1988) for the solution (1988) of the famous Luzin (Louzine) problem of the theory of trigonometric series. Namely, Konyagin shown that a trigonometric series cannot converge to infinity on a set of positive measure.

The following theorem on *convergence of Fourier series* in the space L^2 (see, for example, Shilov 2016) is important: for any function

$$u \in L^2(\Omega), \quad \text{where } \Omega = \left]-\frac{p}{2}, \frac{p}{2}\right[,$$

the series (17.2) converges to u in $L^2(\Omega)$, i.e.,

$$\|u - \sum_{|k|\le N} a_k e_k\|_{L^2} \to 0 \quad \text{as} \quad N \to \infty, \tag{17.4}$$

where

$$e_k : \Omega \ni x \mapsto e_k(x) = \exp\left(\mathring{\imath}\frac{k}{p}x\right). \tag{17.5}$$

This theorem reveals the transparent geometric meaning of the Fourier coefficients. Indeed, consider the complex-valued functional

$$L^2(\Omega) \times L^2(\Omega) \ni (u, v) \mapsto (u|v) = \int_{-p/2}^{p/2} u(x)\bar{v}(x)\, dx,$$

where $\bar{v}$ is the complex conjugate of the function v. It is clear that this functional defines the *inner product*.[7] The space $L^2(\Omega)$ with respect to which functions (17.5) are orthogonal (as can be easily checked), i.e.,

$$(e_k|e_m) = 0 \quad \text{for} \quad k \neq m. \tag{17.6}$$

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dation for modern probability theory, based on the theory of measure. He obtained fundamental results in topology, geometry, mathematical logic, classical mechanics, turbulence theory, algorithm complexity theory, information theory, function theory, trigonometric series theory, measure theory, approximation of functions, set theory, differential equations, dynamical systems, functional analysis, and in a number of other areas of mathematics and its applications.

[7] This means that functional $(u, v) \mapsto (u|v)$ is linear in the first argument, and in addition, $(u|u) > 0$ if $u \neq 0$, and $(u|v) = \overline{(v|u)}$, where the bar means complex conjugation. Note that the function $u \mapsto \|u\| = \sqrt{(u|u)}$ is a norm, and moreover,

$$|(u|v)| \le \|u\| \cdot \|v\|$$

(cf. formula (9.3) for $p = 2$). In his work on integral equations, David Hilbert (1862–1943) introduced the function spaces which are complete with respect to the norm generated by the inner product. Such spaces are called *Hilbert spaces* in honor of this great mathematician. In particular, $L^2(\Omega)$ is a Hilbert space.

Hence choosing $N \geq |m|$ and multiplying the function

$$\Big(u - \sum_{|k|\leq N} a_k e_k\Big)$$

by e_m (via the inner product), we obtain

$$\Big|\Big(u - \sum_{|k|\leq N} a_k e_k | e_m\Big)\Big| \leq \Big\|u - \sum_{|k|\leq N} a_k e_k\Big\|_{L^2} \|e_m\|_{L^2}.$$

Now by (17.4)

$$a_m = \frac{(u|e_m)}{(e_m|e_m)}, \quad m \in \mathbb{Z}. \tag{17.7}$$

So, the coefficient a_k is the algebraic value of the orthogonal projection of the vector $u \in L^2(\Omega)$ to the direction of the vector e_k.

Now that the geometric meaning of the Fourier coefficients has become clear, it may seem surprising that, as N. N. Luzin writes, "neither the subtle analytical mind of d'Alembert, nor the creative efforts of Euler, D. Bernoulli and Lagrange,"[8] were able to solve this most difficult question,[9] i.e., the question about formulas for the coefficients a_k in formula (17.2). However, we should not forget that the geometric transparency of the above formulas (17.7) was made possible only because the Fourier formulas (17.1) put on the agenda issues, whose solution made it possible to give the precise meaning to such words as "function," "representation of a function by a trigonometric series," and much, much more.

Remark. As for the "most difficult question" mentioned by N. N. Luzin, its appearance is related to the problem of a sounding string (see Narasimhan 1990, Luzin 1935), which is the first system with infinite number of degrees of freedom which was mathematically investigated. Already in 1753, D. Bernoulli came to the conclusion that the most general motion of a string can be obtained by summing the principal oscillations. In other words, the general solution $u = u(x, t)$ of the

[8] There is an abundant literature about the great classics Jean D'Alembert (1717–1783), Leonhard Euler (1707–1783), Daniel Bernoulli (1700–1782), and Joseph Louis Lagrange (1736–1813).

[9] Even though the expansion $\frac{\pi}{2} - \frac{x}{2} = \sum_{k\geq 1} \frac{\sin kx}{k}$ in a trigonometric series (converging for $0 < x < 2\pi$) was first put forward by Euler himself (in 1744); later he in 1752 derived the formula $\frac{x^2}{4} - \frac{\pi^2}{12} = \sum_{k\geq 1}(-1)^k \frac{\cos kx}{k^2}$, which holds for $|x| \leq \pi$ and gives (for $x = \pi$) the famous equality $\sum_{k\geq 1} \frac{1}{k^2} = \frac{\pi^2}{6}$. However, Euler (as well as D. Bernoulli and Lagrange) believed that the solution to this problem surpassed the possibilities of calculus (Yushkevich 1968). However (surprisingly!), in his paper of 1777 (published after his death in 1798) Euler derived the desired formula $a_k = \frac{2}{\pi}\int_0^\pi f(x) \cos kx\, dx$, (which, however, was found earlier in 1759 in the work of A. Clairaut, unknown to Euler) for the coefficients of the expansion

$$f(x) = \frac{a_0}{2} + \sum_{k\geq 1} a_k \cos kx$$

in cosines (and noted a similar possibility for the sine expansion). He did this by applying the now well-known technique: multiplying by $\cos mx$ (by $\sin mx$) both parts of the expansion and then integrating termwise. Only after 30 years, Fourier, without referring to Euler (apparently not knowing about his work of 1777), closed this question, by taking into account "only" the orthogonality of the sine and cosine.

differential equation of a string

$$u_{tt} - u_{xx} = 0, \quad |x| < \frac{p}{2}, \quad t > 0, \tag{17.8}$$

which satisfies, for instance, the periodicity condition

$$u\Big(-\frac{p}{2}, t\Big) - u\Big(\frac{p}{2}, t\Big) = 0, \quad u_x\Big(-\frac{p}{2}, t\Big) - u_x\Big(\frac{p}{2}, t\Big) = 0, \tag{17.9}$$

can be represented as the sum of harmonics propagating to the right and to the left (along the characteristics $x \pm t = 0$, cf. §11), more precisely,

$$u(x, t) = \sum_{k \in \mathbb{Z}} \big(a_k^+ e^{i\lambda_k(x+t)} + a_k^- e^{i\lambda_k(x-t)}\big), \tag{17.10}$$

where $a_k^\pm \in \mathbb{C}$, and $\lambda_k = 2\pi k/p$. Indeed, Eq. (17.8) and the boundary conditions (17.9) are linear and homogeneous. Hence a linear combination of functions satisfying (17.8), (17.9) also satisfies these equations. This fact suggests an idea to find the general solution of problem (17.8), (17.9) by summing (with indeterminate coefficients) the particular solutions of Eq. (17.8) satisfying the periodicity conditions (17.9). Equation (17.8) is an equation for which there exists an infinite series of particular solutions with *separated variables*—these are nonzero solutions of the form $\varphi(x)\psi(t)$. Indeed, substituting this function in (17.8), we obtain $\varphi_{xx}(x)\psi(t) = \varphi(x)\psi_{tt}(t)$. As a result,

$$\varphi_{xx}(x)/\varphi(x) = \psi_{tt}(t)/\psi(t) = \text{const}. \tag{17.11}$$

The periodicity condition (17.9) implies that $\varphi \in X$, where

$$X = \Big\{\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})\colon \varphi\Big(-\frac{p}{2}\Big) = \varphi\Big(\frac{p}{2}\Big),\ \varphi'\Big(-\frac{p}{2}\Big) = \varphi'\Big(\frac{p}{2}\Big)\Big\}. \tag{17.12}$$

Hence the function φ is necessarily (see formula (17.11)) an *eigenfunction* of the operator

$$-d^2/dx^2\colon X \to L^2(\Omega), \quad \Omega =]-p/2, p/2[. \tag{17.13}$$

This means that φ is a nonzero function of X satisfying the condition

$$-\frac{d^2\varphi}{dx^2} = \mu \cdot \varphi \tag{17.14}$$

with some constant $\mu \in \mathbb{C}$, which is known as the *eigenvalue* of operator (17.13). Since $\varphi \in X$, it follows that the number μ can be only positive (because otherwise $\varphi \equiv 0$). We denote μ by λ^2. Now condition (17.14) implies that

$$\varphi(x) = a e^{i\lambda x}, \quad \lambda \in \mathbb{R}, \quad a \in \mathbb{C}\backslash\{0\}.$$

Obviously, this formula is consistent with the condition $\varphi \in X$ if and only if $\lambda = \lambda_k = 2k\pi/p$, $k \in \mathbb{Z}$. Thus, taking into account (17.11), we obtain

$$\varphi_k(x)\psi_k(t) = a_k^+ e^{i\lambda_k x} e^{i\lambda_k t} + a_k^- e^{i\lambda_k x} e^{-i\lambda_k t}, \quad \alpha_k^\pm \in \mathbb{C},$$

thereby verifying D. Bernoulli's formula (17.10).

The Bernoulli formula brought into use the principle of composition of oscillations as well as many serious mathematical problems. One of them is connected with finding of the coefficients $a_k^\pm$ in (17.10) for any specific oscillation (cf. Exercise 11.19) that is determined by the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \tag{17.15}$$

i.e., by the initial deviation of the string from the equilibrium position and by the initial velocity of the motion of its points. In other words, D. Bernoulli's formula posed the question of finding the coefficients $a_k^{\pm}$ from the conditions

$$\sum_{k\in\mathbb{Z}}(a_k^+ + a_k^-)e^{i\lambda_k x} = f(x), \quad \sum_{k\in\mathbb{Z}} i\lambda_k(a_k^+ - a_k^-)e^{i\lambda_k x} = g(x).$$

It is curious that in 1759, i.e., 6 years after the work of D. Bernoulli, formulas (17.1), which give an answer to this question, were almost found by the 23-year-old Lagrange. All that remained for him to do in his research was to rearrange the limits in order to obtain these formulas. However, as Luzin writes in Luzin (1935), "Lagrange's thought was directed in a different way and he, almost touching the discovery, so little realized it that he flung about D. Bernoulli the remark "It is disappointing that such a witty theory is inconsistent."

As has been said, half a century later the answer to this question was given by Fourier who wrote formulas (17.1). This is the reason why the method, whose scheme was presented on the example of solution of problem (17.8), (17.9), (17.15), is called the *Fourier method* (see, for instance, Vladimirov 1971, Godunov 1979). (For obvious reasons, this method is also called the *method of separation of variables*.) This method is quite popular in mathematical physics.

P 17.2 Use the Fourier method to find the solution of the equation of the string

$$u_{tt} - u_{xx} = 0, \quad |x| < L, \quad t > 0, \tag{17.16}$$

satisfying the conditions

$$u\Big|_{|x|=L} = 0, \quad u\Big|_{t=0} = \begin{cases} 1-|x| & \text{if } |x| < 1 \\ 0 & \text{if } 1 < |x| < L \end{cases}, \quad u_t\Big|_{t=0} = 0. \tag{17.17}$$

Show that, for $t < L - 1$

$$u(t,x) = \frac{1}{2}\left[u(0, x+t) + u(0, x-t)\right].$$

Compare this result with formula (11.25). *Hint*: $2\cos(\lambda_k t)\cos(\lambda_k x) = \cos\lambda_k(x+t) + \cos\lambda_k(x-t)$.

The reader can easily find by the Fourier method the solution of the Dirichlet problem for the Laplace equation in a rectangle, by preliminary considering the special case

$$\Delta u\Big|_{(x,y)\in]0,1[^2} = 0, \quad u\Big|_{x=0} = u\Big|_{x=1} = 0, \quad u\Big|_{y=0} = f(x), \quad u\Big|_{y=1} = g(x).$$

It is also easy to obtain, by the Fourier method, the solution

$$u(x,t) = u_N(x,t) + \sum_{k>N} \frac{2\sin\lambda_k}{\lambda_k[1+\sigma\sin^2\lambda_k]} e^{-\lambda_k^2 t}\cos\lambda_k x, \quad N \ge 0, \quad u_0 \equiv 0, \tag{17.18}$$

to problem (6.14) for the heat equation. In formula (17.18), λ_k^2 is the kth ($k \in \mathbb{N}$) eigenvalue of the operator

$$-d^2/dx^2 \colon Y \to L^2(\Omega), \quad \Omega =]-1, 1[,$$

defined on the space

$$Y = \{\varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) \colon (\varphi \pm \sigma\varphi')\big|_{x=\pm 1} = 0\},$$

where $\sigma \ge 0$ is the parameter of problem (6.14), and $\lambda_k \in](k-1)\pi, (k-1/2)\pi]$ is the kth root of the equation $\cot\lambda = \sigma\lambda$. The eigenfunctions $\varphi_k(x) = \cos\lambda_k x$ of the operator $-d^2/dx^2 \colon Y \to L^2(\Omega)$ satisfy (cf. (17.6)) the orthogonality condition

$$(\varphi_k, \varphi_m) = \int_{-1}^{1} \varphi_k(x)\varphi_m(x)\,dx = 0 \quad \text{for } k \neq m. \tag{17.19}$$

Indeed, integrating by parts (applying the Ostrogradsky–Gauss formula in the multivariate case), taking into account the boundary conditions $(\varphi \pm \sigma\varphi')\big|_{x=\pm 1} = 0$, and using $-\varphi_k'' = \lambda_k^2\varphi_k$, we get

$$(\lambda_k^2 - \lambda_m^2)(\varphi_k, \varphi_m) = \int_{-1}^{1} (\varphi_k\varphi_m'' - \varphi_m\varphi_k'')\,dx$$
$$= \varphi_k\varphi_m'\big|_{-1}^{1} - \int_{-1}^{1} \varphi_k'\varphi_m' - \varphi_m\varphi_k'\big|_{-1}^{1} + \int_{-1}^{1} \varphi_k'\varphi_m' = 0.$$

It can be shown (see, for example, Vladimirov 1971) that the eigenfunctions φ_k, $k \in \mathbb{N}$ from (cf. (17.4)) a *complete system* in $L^2 = \overline{Y}$. This means that, for any $u \in L^2$ and $\varepsilon > 0$, there exist $N \geq 1$ and numbers $a_1, \ldots, a_N$ such that $\|u - \sum_{k=1}^{N} a_k\varphi_k\|_{L^2} < \varepsilon$. Hence (cf. (17.2)–(17.7)), the formal series

$$\sum_{k=1}^{\infty} \frac{(u, \varphi_k)}{(\varphi_k, \varphi_k)}\varphi_k \tag{17.20}$$

converges to u in L^2. The series (17.20) is called the *Fourier series* of the function u in the *orthogonal* (see (17.19)) *system of functions* φ_k. The reader can easily verify that $(1, \varphi_k)/(\varphi_k, \varphi_k) = 2\sin\lambda_k/(\lambda_k[1+\sigma\sin^2\lambda_k])$, as well as that series (17.18) converges uniformly together with all its derivatives for $t \geq \varepsilon$ for any $\varepsilon > 0$ and gives a smooth (except the angular points $(x, t) = (\pm 1, 0)$), and unique (see, for example, Friedman 1964) solution of problem (6.14).

Another fact is worth mentioning. The series (17.18) converges rapidly for large t. It can be shown that, for any $k \geq 1$,

$$|u(x,t) - u_N(x,t)| < 10^{-k}/N \quad \text{for } t > k/(4.3N^2). \tag{17.21}$$

However, for small t, series (17.18) converges very slowly. Hence, for small t, it is advisable to use a different representation of the solution of problem (6.14), which will be derived in §18 using the dimensionality considerations (see §6) and the so-called Laplace transform.

Substituting formally (17.1) in (17.2), we get

$$u(x) = \sum_{k=-\infty}^{\infty} \frac{1}{p} e^{\mathring{\imath}(k/p)x} \int_{-p/2}^{p/2} e^{-\mathring{\imath}(k/p)y} u(y)\,dy. \tag{17.22}$$

Making $p \to \infty$ in (17.22), we get, for an "arbitrary" function $u\colon \mathbb{R} \to \mathbb{C}$ the following (formal!) expression:

$$u(x) = \int_{-\infty}^{\infty} e^{\mathring{\imath}x\xi}\Big(\int_{-\infty}^{\infty} e^{-\mathring{\imath}y\xi} u(y)\,dy\Big)\,d\xi. \tag{17.23}$$

Let us complete the formal calculations and give a precise definition

Definition 17.3 Let $\xi \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $x\xi = \sum_{k=1}^{n} x_k\xi_k$, i.e., $x\xi = (x|\xi)$ is the inner product of x and ξ. A mapping

$$\mathbf{F}\colon L^1(\mathbb{R}^n) \ni u \mapsto \widetilde{u} = \mathbf{F}u \in \mathbb{C},$$

where[10]

$$\widetilde{u}(\xi) = \int_{\mathbb{R}^n} e^{-\mathring{\imath}x\xi} u(x)\, dx, \quad \mathring{\imath} = 2\pi i, \tag{17.24}$$

is called the *Fourier transform*, and $\widetilde{u} = \mathbf{F}u$ is called the *Fourier image* of a function $u \in L^1(\mathbb{R}^n)$.

Lemma 17.4 *If $u \in L^1(\mathbb{R}^n)$, then $\boldsymbol{F}u \in C(\mathbb{R}^n)$ and $\|\boldsymbol{F}u\|_C \le \|u\|_{L_1}$.*

Proof From Theorem 8.34 (the Lebesgue dominated convergence theorem) it follows that $\widetilde{u} = \mathbf{F}u \in C(\mathbb{R})$; moreover, $|\widetilde{u}(\xi)| \le \int_{\mathbb{R}^n} |u(x)|\, dx$. □

Example 17.5 Let $u_\pm(x) = \theta_\pm(x)e^{\mp ax}$, where $x \in \mathbb{R}$, $a > 0$, and $\theta_\pm$ is defined in (12.7). Then $\widetilde{u}_\pm(\xi) = \frac{1}{a \pm \mathring{\imath}\xi}$. Note that $\widetilde{u}_\pm \notin L^1$, even though $u_\pm \in L^1$. We also note that the function $\widetilde{u}_\pm$ extends analytically to the complex half-plane $\mathbb{C}_\mp$.

In Theorem 17.8 we will give sufficient conditions under which the formal expression (17.23) acquires the exact meaning of one of the most important formulas in analysis. We first need the following definition.

Definition 17.6 Let $p \ge 1$ and let $k \in \mathbb{Z}$. A function $u \in L^p(\Omega)$ is an element of the *Sobolev space* $W^{p,k}(\Omega)$ if all the Sobolev derivatives $\partial^\alpha u$, where $|\alpha| \le k$, lie in $L^p(\Omega)$. The convergence in the space $W^{p,k}$ is measured in the norm

$$\|u\|_{W^{p,k}} = \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^p}, \tag{17.25}$$

i.e., $u_j \to u$ in $W^{p,k}$ as $j \to \infty$ if $\|u - u_j\|_{W^{p,k}} \to 0$ as $j \to \infty$.

It is easily checked that $W^{p,k}$ is a Banach space.

Lemma 17.7 *The following embedding holds*[11] *$W^{1,n}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, i.e., for any $\{u\} \in W^{1,n}$, there exists a unique function $u \in C$ that coincides almost everywhere with any representative of $\{u\}$, and besides, $\|u\|_C \le \|\{u\}\|_{W^{1,n}}$.*

Proof From Theorem 8.42 (Fubini) and Theorem 8.47 it follows that the function u can be written as

$$u(x) = \int_{-\infty}^{x_1} \Big[\int_{-\infty}^{x_2} \dots \Big[\int_{-\infty}^{x_n} \frac{\partial^n u(y_1, \dots y_n)}{\partial y_1 \partial y_2 \dots \partial y_n}\, dy_n\Big] \dots dy_2\Big]\, dy_1,$$
$$x = (x_1, \dots, x_n),$$

which implies its continuity and the estimate $\|u\|_C \le \int \Big|\frac{\partial^n u(x)\, dx}{\partial x_1 \dots \partial x_n}\Big|$. □

[10] Formula (17.24), in contrast to other formulas, for example, given by the integral $\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx$, with i in place of $\mathring{\imath}$ in the exponential, is preferable in two respects: (1) there is no need to keep in mind the appropriate constant before the integral, (2) formula (17.24) can be extended to the case $n = \infty$.

[11] Lemma 17.7 is a simple special case of the Sobolev embedding theorem (see, for example, Sobolev 2008, Lions and Magenes 1968, Besov 2001). Note that the embedding $W^{p,k}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, which holds for $n/p < k$, is violated if $p > 1$ and $n/p = k$ (see, in particular, Exercise 20.6, where the case $p = 2$ is considered).

Theorem 17.8 *If $u \in W^{1,n}(\mathbb{R}^n)$, then, for any $x \in \mathbb{R}^n$,*

$$u(x) = \lim_{N\to\infty} u_N(x), \quad \text{where } u_N(x) = \int_{-N}^{N} \dots \int_{-N}^{N} e^{\mathring{\imath}x\xi}\widetilde{u}(\xi)d\xi_1 \dots d\xi_n; \quad (17.26)$$

here $\widetilde{u} = \boldsymbol{F}u$ is the Fourier transform of the function $u(x)$.

Proof Note that the function u is continuous by Lemma 17.7. From Fubini's theorem it follows that

$$u_N(x) = \int_{-\infty}^{\infty}\left(\dots\left(\int_{-\infty}^{\infty} u(y)\frac{\partial\theta_N(y_1 - x_1)}{\partial y_1}\,dy_1\right)\dots\right)\frac{\partial\theta_N(y_n - x_n)}{\partial y_n}\,dy_n,$$

where $\theta_N(\sigma) = \int_{-1}^{\sigma}\delta_N(s)ds$, and $\delta_N(s) = \int_{-N}^{N} e^{\mathring{\imath}s\xi}d\xi = \frac{\sin 2\pi Ns}{\pi s}$. Note that (cf. Exercises 4.3 and 13.12) $\theta_N(\sigma) \to \theta(\sigma)$, $\sigma \in \mathbb{R}$, and $|\theta_N(\sigma)| \le$ const for any $N \in \mathbb{N}$. Indeed, let $\lambda_k = \int_{k/2N}^{(k+1)/2N}\delta_N(\sigma)d\sigma$ for $k \in \mathbb{Z}_+$. Then λ_k does not depend on N and $|\lambda_k| \downarrow 0$ as $k \to \infty$, and besides, $\lambda_{2k} > -\lambda_{2k+1}$ and $2\sum_{k=0}^{\infty}\lambda_k = \int_{-\infty}^{\infty}\frac{\sin x}{\pi x}\,dx = 1$. So, $\theta_N(\sigma) \to \theta(\sigma)$ and $|\theta_N(\sigma)| \le 2\lambda_0$. Next, one should integrate by parts (as in the proof of Lemma 13.11), apply Lebesgue's theorem, and get $u_N(x) \to u(x)$. □

Remark 17.9 The above proof of Theorem 17.8, which contains, in particular, the solution to Exercises 4.3 and 13.12, shows (in view of the proof of Lemma 17.7) that the assertion of Theorem 17.8 also holds under broader assumptions: it suffices to require that both the function u and n its derivatives $\frac{\partial^k u}{\partial x_1 \partial x_2 \dots \partial x_k}$, $k = 1, \dots, n$, be integrable in $\mathbb{R}^n$.

Remark 17.10 The conclusion of Theorem 17.8 has sense, of course, only for $u \in L^1 \cap C$. The next exercise demonstrates; however, this necessary condition is not sufficient for relations (17.26) to hold.

P 17.11 Construct (cf. Kolmogorov and Fomin 1980) a sequence of functions $\varphi_N \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty}\varphi_N(y)\frac{\sin Ny}{y}\,dy \to \infty$$

(as $N \to \infty$) and $\|\varphi_N\|_{L_1} + \|\varphi_N\|_C \le 1$.
Hint Apply the Banach–Steinhaus theorem (see the footnote 8 on p. 83) to show that there exists a function $\varphi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ for which equality (17.26) violates in at least one point $x \in \mathbb{R}$.

The formal expression (17.23) and Theorem 17.8 suggest the feasibility of introducing the transformation

$$\mathbf{F}^{-1}\colon L^1(\mathbb{R}) \ni \widetilde{u} \mapsto \mathbf{F}^{-1}\widetilde{u} \in \mathbb{C},$$

which is defined by the formula

$$(\mathbf{F}^{-1}\widetilde{u})(x) = \int_{\mathbb{R}^n} e^{\mathring{\imath}x\xi}\widetilde{u}(\xi)d\xi, \quad \mathring{\imath} = 2\pi i, \quad x \in \mathbb{R}^n. \quad (17.27)$$

This formula differs from (17.24) by the sign of the exponent. The transformation $\mathbf{F}^{-1}$ is called the *inverse Fourier transform*, because $u = \mathbf{F}^{-1}\mathbf{F}u$ if $u \in W^{1,n}(\mathbb{R}^n)$, and $\mathbf{F}u \in L^1(\mathbb{R})$. Following Schwartz (1950–1951), we define the *space of rapidly converging functions* $\mathcal{S} = \mathcal{S}(\mathbb{R}^n) \subset W^{1,n}(\mathbb{R}^n)$. In the space $\mathcal{S}$ (see Theorem 17.18), the transformations $\mathbf{F}^{-1}$ and $\mathbf{F}$ are automorphisms (i.e., linear invertible maps of $\mathcal{S}$ onto itself).

Definition 17.12 The space $\mathcal{S}(\mathbb{R}^n)$ consists of the functions $u \in C^\infty(\mathbb{R}^n)$ satisfying the following condition: for any multiindexes $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, there exists a number $C_{\alpha\beta} < \infty$ such that, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$|x^\alpha \partial_x^\beta u(x)| \le C_{\alpha\beta}, \quad \text{where } \; x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \; \partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}.$$

Under this condition, one says that the sequence of functions $u_j \in \mathcal{S}$ converges in $\mathcal{S}$ to u ($u_j \to u$ in $\mathcal{S}$) as $j \to \infty$ if, for any $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that $p_m(u_j - u) \le \varepsilon$ for any $j \ge j_0$, where

$$p_m(v) = \sup_{x \in \mathbb{R}^n} \Big((1 + |x|)^m \sum_{|\alpha| \le m} |\partial^\alpha v(x)| \Big).$$

It is clear that $e^{-x^2} \in \mathcal{S}(\mathbb{R})$, but $e^{-x^2} \sin(e^{x^2}) \notin \mathcal{S}(\mathbb{R})$.

P 17.13 Verify that $\mathcal{S}$ is a Fréchet space (see Remark 16.27) in which the distance ρ can be defined by

$$\rho(u, v) = d(u - v), \quad \text{where } \; d(\varphi) = \sum_{m=1}^{\infty} 2^{-m} \inf(1, p_m(\varphi)).$$

P 17.14 Verify that $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$ (see Exercise 16.13.P). In particular, show that the convergence in $\mathcal{D}$ (in $\mathcal{S}$) implies the convergence in $\mathcal{S}$ (in $\mathcal{E}$). Verify that $\mathcal{D}$ is dense in $\mathcal{S}$, and $\mathcal{S}$ is dense in $\mathcal{E}$.

P 17.15 Integrate by parts to verify the following lemma.

Lemma 17.16 *For any multiindex α, β and any $u \in \mathcal{S}$,*

$$(-\mathring{\imath})^{|\beta|} \boldsymbol{F}[\partial_x^\alpha (x^\beta u(x))](\xi) = (\mathring{\imath})^{|\alpha|} \xi^\alpha \partial_\xi^\beta \widetilde{u}(\xi), \quad \widetilde{u} = \boldsymbol{F}u. \tag{17.28}$$

Corollary 17.17 *The following embedding holds: $\boldsymbol{F}\mathcal{S} \subset \mathcal{S}$, i.e., $\boldsymbol{F}u \in \mathcal{S}$ if $u \in \mathcal{S}$.*

Proof Since $u \in \mathcal{S}$, for any fixed $N \in \mathbb{N}$ and any $\alpha, \beta \in \mathbb{Z}_+^n$, there exists $d_{\alpha\beta} > 0$ such that $|\partial_x^\alpha (x^\beta u(x))| \le \frac{d_{\alpha\beta}}{(1+|x|)^N}$. Hence by Lemma 17.16 we have

$$|\xi^\alpha \partial_\xi^\beta \widetilde{u}(\xi)| \; \le \|\mathbf{F}[\partial_x^\alpha (x^\beta u)]\|_C \le d_{\alpha\beta} \int (1 + |x|)^{-N} \, dx, \tag{17.29}$$

which shows that $\widetilde{u} \in \mathcal{S}$. □

Theorem 17.18 *The mappings $\boldsymbol{F}\colon \mathcal{S} \to \mathcal{S}$ and $\boldsymbol{F}^{-1}\colon \mathcal{S} \to \mathcal{S}$ are continuous automorphisms of the space $\mathcal{S}$, and hence they are inverses of each other.*

Proof The mapping $\mathbf{F}$ is linear and, in addition, by Theorem 17.8, is monomorphic. Let us check that, for any $\widetilde{u} \in \mathcal{S}$, there exists a $u \in \mathcal{S}$ such that $\mathbf{F}u = \widetilde{u}$. We set $u_0 = \mathbf{F}\widetilde{u}$. Since $u_0 \in \mathcal{S}$, from Theorem 17.8 we have $\widetilde{u} = \mathbf{F}^{-1}\mathbf{F}\widetilde{u} = \mathbf{F}^{-1}u_0$. Consider the function $u(x) = u_0(-x)$. We have $\widetilde{u} = \mathbf{F}^{-1}u_0 = \mathbf{F}u$. Inequality (17.29) immediately implies that $\mathbf{F}u_j \to 0$ in $\mathcal{S}$ if $u_j \to 0$ in $\mathcal{S}$. The same arguments also hold for $\mathbf{F}^{-1}$. □

Theorem 17.19 (Plancherel[12]) *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then*

$$(\boldsymbol{F}f, \boldsymbol{F}g)_{L^2} = (f, g)_{L^2}. \tag{17.30}$$

In addition,

$$\langle \boldsymbol{F}f, g\rangle = \langle f, \boldsymbol{F}g\rangle, \quad \textit{i.e.,} \int_{\mathbb{R}^n} \widetilde{f}(\xi)g(\xi)\,d\xi = \int_{\mathbb{R}^n} f(x)\widetilde{g}(x)\,dx. \tag{17.31}$$

Proof From Fubini's theorem we have equality (17.31), because

$$\int_{\mathbb{R}^n} f(x)\widetilde{g}(x)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\xi}g(\xi)\,dx\,d\xi = \int_{\mathbb{R}^n} \widetilde{f}(\xi)g(\xi)\,d\xi.$$

Let $h = \overline{\mathbf{F}g}$. Then $g = \overline{\mathbf{F}h}$, inasmuch as

$$g(\xi) = (\mathbf{F}^{-1}\bar{h})(\xi) = \int e^{ix\xi}\bar{h}(x)\,dx = \overline{\int e^{-ix\xi}h(x)\,dx} = (\overline{\mathbf{F}h})(\xi).$$

Substituting $g(\xi) = \bar{\bar{h}}(\xi)$ and $\widetilde{g}(x) = \bar{h}(x)$ in (17.31), we arrive at (17.30). □

Note that both sides of equality (17.31) define the following linear continuous functionals on $\mathcal{S}$:

$$f\colon \mathcal{S} \ni \widetilde{g} \mapsto \int f(x)\widetilde{g}(x)\,dx, \quad \widetilde{f}\colon \mathcal{S} \ni g \mapsto \int \widetilde{f}(\xi)g(\xi)\,d\xi.$$

In this connection, (following L. Schwartz) we give two definitions.

Definition 17.20 $\mathcal{S}'(\mathbb{R}^n)$ is the *space of tempered generalized functions*, i.e., the space continuous linear functionals $f\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ equipped with the operation of differentiation $\langle \partial^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, \partial^\alpha\varphi\rangle$, where $\alpha \in \mathbb{Z}_+^n$, and the operation of multiplication $\langle af, \varphi\rangle = \langle f, a\varphi\rangle$ by any *tempered function* a, i.e., a function

[12] This theorem was proved in 1910 by the Swiss mathematician Michel Plancherel (1885–1967). The analogue of (17.30) for periodic functions, which extends the Pythagorean theorem (the sum of the squared Fourier coefficients of a function $u \in L^2(0,1)$ is equal to its squared norm) for formulated in 1799 by the French mathematician Marc-Antoine Parseval (1755–1836). Hence Theorem 17.19 is frequently called the Parseval identity.

$a \in C^\infty(\mathbb{R}^n)$ satisfying the condition: for any α, there exists a $C_\alpha < \infty$, for which there exists an $N_\alpha < \infty$ such that $|\partial^\alpha a(x)| \le C_\alpha(1+|x|)^{N_\alpha}$.

Definition 17.21 Let $f \in \mathcal{S}'$, $g \in \mathcal{S}'$. Then the formulas

$$\langle \mathbf{F}f, \varphi\rangle = \langle f, \mathbf{F}\varphi\rangle \ \forall\varphi \in \mathcal{S} \quad \text{and} \quad \langle \mathbf{F}^{-1}g, \psi\rangle = \langle g, \mathbf{F}^{-1}\psi\rangle \ \forall\psi \in \mathcal{S} \qquad (17.32)$$

define the generalized functions $\widetilde{f} = \mathbf{F}f \in \mathcal{S}'$ and $\mathbf{F}^{-1}g \in \mathcal{S}'$, which are called, respectively, the *Fourier transform of the generalized function* $f \in \mathcal{S}'$ and the *inverse Fourier transform of the generalized function* $g \in \mathcal{S}'$.

Remark 17.22 Using the embedding $L^2 \subset \mathcal{S}'$, we can speak about the Fourier transform $\widetilde{f}$ of a function $f \in L^2(\mathbb{R})$. In any way, from Theorem 17.19 it follows that $\|\widetilde{f} - \widetilde{f_N}\|_{L^2} \to 0$, where $f_N = 1_{[-N,N]}f$.

Example 17.23 It is clear that $\delta \in \mathcal{S}'$, $1 \in \mathcal{S}'$. Let us find $\mathbf{F}\delta$ and $\mathbf{F}1$. We have

$$\langle \mathbf{F}\delta, \varphi\rangle = \langle \delta, \mathbf{F}\varphi\rangle = \widetilde{\varphi}(0) = \lim_{\xi\to 0}\int e^{-\mathring{\imath}x\xi}\varphi(x)\,dx = \int \varphi(x)\,dx = \langle 1, \varphi\rangle,$$

i.e., $\mathbf{F}\delta = 1$. Similarly, $\mathbf{F}^{-1}\delta = 1$. Next, we have

$$\langle \mathbf{F}1, \varphi\rangle = \langle 1, \mathbf{F}\varphi\rangle = \langle \mathbf{F}^{-1}\delta, \mathbf{F}\varphi\rangle = \langle \delta, \mathbf{F}^{-1}\mathbf{F}\varphi\rangle,$$

i.e., $\mathbf{F}1 = \delta$. Similarly, $\mathbf{F}^{-1}1 = \delta$.

P 17.24 Verify (cf. Exercise 17.13) that $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

P 17.25 Prove (cf. Exercise 16.19 and Shilov 1965) that $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if there exists a finite sequence $\{f_\alpha\}_{|\alpha|\le N}$ of functions $f_\alpha \in C(\mathbb{R}^n)$, satisfying the condition $|f_\alpha(x)| \le C(1+|x|)^m$ and such that $f = \sum_{|\alpha|\le N} \partial^\alpha f_\alpha$. Hence $\mathcal{S}' \subset \mathcal{D}'$.

P 17.26 Verify that the mappings $\mathbf{F}\colon \mathcal{S}' \to \mathcal{S}'$ and $\mathbf{F}^{-1}\colon \mathcal{S}' \to \mathcal{S}'$ are reciprocal automorphisms of the space $\mathcal{S}'$ which are continuous relatively the weak convergence in $\mathcal{S}'$, i.e., if $\nu \to \infty$, then

$$\langle \mathbf{F}_\nu, \varphi\rangle \to \langle \mathbf{F}, \varphi\rangle \ \forall\varphi \in \mathcal{S} \iff \langle f_\nu, \varphi\rangle \to \langle f, \varphi\rangle \ \forall\varphi \in \mathcal{S}.$$

P 17.27 Putting $1_{[a,b]} = \theta(x-a) - \theta(x-b)$ and considering, for $\nu \to \infty$, the sequences $\delta_\nu(x) = 2\nu \cdot 1_{[-1/\nu,1/\nu]}(x)$ and $1_\nu(x) = 1_{[-\nu,\nu]}(x)$, evaluate $\mathbf{F}\delta$ and $\mathbf{F}1$ (cf. Example 17.23).

P 17.28 Verify that the Fourier transform of the function $\theta_\pm$, which is defined by (12.7), is given by $\widetilde{\theta}_\pm(\xi) = \pm\frac{1}{\mathring{\imath}\xi \pm 0}$.
Hint Consider $f_\nu(x) = \theta_\pm(x)e^{\mp x/\nu}$ as $\nu \to \infty$.

Remark 17.29 The space $\mathcal{S}'$ is complete with respect to the weak convergence, since $\mathcal{S}$ is a Fréchet space (see Exercise 17.13 and Remark 16.27).

P 17.30 Verify that formula (17.28) holds for any $u \in \mathcal{S}'$.

Lemma 17.31 *Let*[13] $f \in \mathcal{E}'(\mathbb{R}^n)$. *Then* $\widetilde{f} = \boldsymbol{F}$ *is a tempered function* (*see Definition* 17.20), *and besides,*

$$\widetilde{f}(\xi) = \langle f(x), e^{-\mathring{\imath}x\xi}\rangle. \qquad (17.33)$$

[13] Recall that the space $\mathcal{E}'$ is defined in Exercise 16.13.

Proof By Theorem 16.20, we have $f = \partial^\alpha g$, where $g \in C_0(\mathbb{R}^n)$. Hence

$$\begin{aligned}\langle \widetilde{f}(\xi), \varphi(\xi)\rangle &= \langle \partial_x^\alpha g(x), (\mathbf{F}\varphi)(x)\rangle = (-1)^{|\alpha|}\langle g(x), \partial_x^\alpha \widetilde{\varphi}(x)\rangle \\ &= (\mathring{\imath})^{|\alpha|}\langle g(x), \mathbf{F}[\xi^\alpha \varphi(\xi)](x)\rangle = (\mathring{\imath})^{|\alpha|}\int g(x)\Big[\int e^{-\mathring{\imath}x\xi}\xi^\alpha\varphi(\xi)d\xi\Big]\,dx.\end{aligned}$$

Since $g(x)e^{-\mathring{\imath}x\xi}\xi^\alpha\varphi(\xi) \in L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, an appeal to Fubini's theorem shows that

$$\begin{aligned}\langle \widetilde{f}(\xi), \varphi(\xi)\rangle &= \int\Big[\int (\mathring{\imath})^{|\alpha|}g(x)e^{\mathring{\imath}x\xi}\xi^\alpha\,dx\Big]\varphi(\xi)\,d\xi \\ &= \int \langle g(x), (\partial_x)^\alpha e^{-\mathring{\imath}x\xi}\rangle\varphi(\xi)d\xi = \int\langle f(x), e^{-\mathring{\imath}x\xi}\rangle\varphi(\xi)d\xi.\end{aligned}$$

Thus by Exercise 13.20 we have $\widetilde{f}(\xi) = \langle f(x), e^{-\mathring{\imath}x\xi}\rangle$. Similarly,

$$\partial^\beta \widetilde{f}(\xi) = \langle f(x), (-\mathring{\imath}x)^\beta e^{-\mathring{\imath}x\xi}\rangle. \tag{17.34}$$

Since $f \in \mathcal{E}' \subset \mathcal{S}'$, from Definition 17.12 it follows that there exists an $N \geq 1$ such that

$$|\langle f(x), \psi(x)\rangle| \leq N \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq N} (1+|x|)^N \cdot |\partial^\alpha\psi(x)| \quad \forall\psi \in \mathcal{S}.$$

Hence $|\partial^\beta \widetilde{f}(\xi)| = |\langle f(x), (-\mathring{\imath}x)^\beta e^{-\mathring{\imath}x\xi}\rangle| \leq C(1+|\xi|)^N$. □

18 The Fourier–Laplace Transform. The Paley–Wiener Theorem

Formula (17.28) from §17 (which holds for $u \in S'$, see Exercise 17.30) contains the following important property of the Fourier transform, which is often expressed in the following words: "The composition of the Fourier transform and the differentiation operator acts as the multiplication of the original function by the independent variable." More precisely, we have

$$\mathbf{F}(D_x^\alpha u(x)) = \xi^\alpha \widetilde{u}(\xi), \quad \text{where } D_x^\alpha = (\mathring{\imath})^{-|\alpha|}\partial_x^\alpha, \quad \text{and } \widetilde{u} = \mathbf{F}u, \quad u \in S'. \tag{18.1}$$

Property (18.1) allows one to reduce, in a sense, problems involving linear differential equations to algebraic ones. For example, applying the Fourier transformation to the differential equation

$$A(D_x)u(x) \equiv \sum_{|\alpha|\leq m} a_\alpha D_x^\alpha u(x) = f(x), \quad a_\alpha \in \mathbb{C}, \quad f \in \mathcal{S}', \tag{18.2}$$

we obtain an equivalent "algebraic" equation

$$A(\xi)\widetilde{u}(\xi) \equiv \Big(\sum_{|\alpha|\le m} a_\alpha \xi^\alpha\Big)\widetilde{u}(\xi) = \widetilde{f}(\xi), \quad \widetilde{f} \in \mathcal{S}'. \tag{18.3}$$

According to Hörmander (1958)[14] and Łojasiewicz (1959),[15] Eq. (18.3) always has a solution $\widetilde{u} \in \mathcal{S}'$ (see Remark 19.2). Hence the function $\mathbf{F}^{-1}\widetilde{u}$ is well defined. This function is a solution of the differential equation (18.2), because

$$f(x) = \mathbf{F}^{-1}(\widetilde{f}(\xi)) = \mathbf{F}^{-1}(A(\xi)\widetilde{u}(\xi)) \overset{(18.1),(18.3)}{=} A(D_x)\mathbf{F}^{-1}(\widetilde{u}(\xi)).$$

The above idea of constructing a solution to a linear differential equation using the Fourier transform is in many ways similar to the idea employed in operational calculus (see, for example, Gordon et al. 2013), which uses the so-called Laplace transform, which was first introduced in science by Niels Abel (see the footnote 9 on the page 11). The *Laplace transform* sends a function f, which depends on the argument $t \in \mathbb{R}_+$ and integrable "with weight" e^{-st} for any $s > 0$, to the function

$$L[f](s) = \int_0^\infty e^{-st} f(t)dt, \quad s > 0. \tag{18.4}$$

Remark 18.1 "The concept of generating functions, which was mentioned in footnote 9 on p. 11, appeared for the first time in the papers by N. Bernoulli and J. Stirling. Later this concept was used by Euler in his works on number theory, and then, 70 years later, by Laplace. According to Knuth (1997), "A similar concept, which has become known as the "Laplace transform," was actually introduced by another mathematician N. Abel[16] in the memoir *Sur les fonctions génératrices et leurs déterminantes*, Oeuvres Completes de N. H. Abel (Ed. B. Holmboe[17]) Tome Second (Christiania: Grondahl, 1839), 77–88. Indeed, Abel starts his memoir with the phrase: Soit $\varphi(x, y, z, \ldots)$ une fonction quelconque... On peut toujours trouver une fonction $f(u, v, p, \ldots)$ telle que

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[14] Lars Valter Hörmander (1931–2012) was an outstanding Swedish mathematician, one of the founders of the modern general theory of linear partial differential equations and pseudo-differential operators, who was awarded the Fields Prize (1962), the Wolf Prize (1988), and the Steele Prize (2006).

[15] Stanislaw Łojasiewicz (1926–2002) was a Polish mathematician. In the early 1950s, at the invitation of L. Schwartz, he worked in France, where he established the famous inequality that gives an upper bound for the distance from a point of an arbitrary compact set to the zero-level set of a real multivariate analytic function. This inequality has found applications in various branches of mathematics, including real algebraic geometry, analysis, and theory of differential equations.

[16] Niels Abel (1802–1829), a Norwegian genius, who was little-know during his lifetime. Abel died at the age of 26 from tuberculosis. Niels Henrik Abel "had left mathematicians such a rich legacy that they will have something to do in the next 150 years." So said Charles Hermite (1822–1901), the leader of the French mathematicians of the second half of the XIX century. The monetary size of the prestigious Abel Prize for mathematicians is comparable to that of the Nobel Prize.

[17] Bernt Michael Holmboe (1795–1850) was a mathematics teacher at the Christiania Cathedral School, where he taught Abel mathematics both at school and privately. The two became friends and remained so until Abel's early death. In the preface, Holmboe writes: "Tous les memoires contenus dans ce volume ont été écrits avant que notre auteur commencer ses voyages." This shows that this Abel work was written before 1825.

$$\varphi(x, y, z, \ldots) = \int e^{xu+yv+zp+\ldots} f(u, v, p, \ldots) dxdydz \ldots,$$

i.e., Abel introduces a fairly general integral transform including, as a particular, case transformation (18.4). Then he establishes a number of useful properties of this mapping in the context of differential equations and applies them to general examples of generating functions.

"The Development of the Laplace Transform, 1737–1937: I. Euler to Spitzer, 1737–1880" is the name of the paper Deakin (1985), which describes sometimes impolite discussion involving many participants who in the second part of the XIX century wished to give the palm to some or other mathematician for introduction of transformation (18.4). This discussion was unexpected terminated by A. Poincaré. Let me give a few comments on this. Poincaré begins his memoir "Sur les équations linéaires aux différentielles ordinaires et aux différences finies," Am. J. Math. 7 (1885), 1–56 as follows (p. 203): "Les résultats que je vais chercher à démontrer dans le présent mémoire et qui se rapportent tant à certaines équations différentielles linéaires qu' à des équations analogues, mais à différences finies, ont déjà été énoncés les uns dans un mémoire que j'ai présenté à l'Académie des Sciences pour le concours du Grand Prix des Sciences Mathématiques le 1-er Juin 1880 et qui est resté inédit, les autres dans une communication verbale faite ła Société Mathématique de France en Novembre 1882 et dans une note insérée aux Comptes Rendus de l'Académie des Sciences le 5 Mars 1883." A possible translation is as follows. "The results, that I will try to demonstrate in this memoir, and which pertain both to some linear differential equations and to similar finite-difference equations, have already been announced in my memoir presented to the French Academy of Sciences for the Grand Prix contest on 15 June 1880 (still unpublished), in my oral communication given at Société Mathématique de France in 1882, and in the note published in Proceedings of the Academy of Sciences on 5 March 1883.

Next on pp. 217–218 he writes: "Revenons maintenant aux equations différentielles. Nous avons vu dans le §1 que si l'on envisage l'intégrale générale Y de l'équation $\sum P_k \frac{d^k Y}{dx^k} = 0$, étudiée dans ce paragraphe, la dérivée logarithmique tend vers une certaine limite a, mais qu'on n'en pouvait pas conclure immédiatement que y $e^{-\alpha x}$ tend vers une limite finie et déterminée. C'est pourtant ce qui a lieu en général; mais pour le démontrer, nous serons forcés d'employer la transformation de Bessel. Voici en quoi consiste cette transformation. On pose $Y = \int v e^{zx} dz$ v étant une fonction de z qu'il reste à déterminer et l'intégrale étant prise le long d'un chemin imaginaire convenablement choisi. L'intégration par parties donne . . . .: Le chemin d'intégration devra être choisi de telle façon que le terme tout connu de cette intégration par parties soit nul, sans cependant que l'intégrale y le soit elle-même. On aura ensuite . . . " A possible translation is as follows: "Let us return now to differential equations. In §1, we saw that if one considers the general integral Y of the equation $\sum P_k \frac{d^k Y}{dx^k} = 0$, which we study in this section, then the logarithmic derivative tends to some limit A. However, one cannot directly conclude that $Y = e^{-\alpha x}$ tends to a finite limit. However, this is what usually takes place in the general case. But for a proof of this fact we are forced to invoke the Bessel transform, which is defined as follows. Let $Y = \int ve^{zx}dz$, where v is the unknown function of z, and the integral is taken over an appropriate path." (Here, as I think, Poincaré implicitly speaks about the inverse Laplace transform defined by the integral over some path in the complex plane.) "Integrating by part, we get... The integration path should be taken so that the known term of this integration by parts be zero, but the integral would not vanish. As a result, we get...

Poincaré completes his memoir on p. 258 with the directness of a Roman(!!!): "Paris, 10 novembre 1884. Noter. Dans les mémoires antérieurs, le nom de Bessel doit être remplacé partout par le nom de Laplace" ("In my previous memoirs the name of Bessel should be replaced everywhere by Laplace'). ' So, without warning, straight from the shoulder, only in his last 4th memoir Poincaré mentions (at the very end) for the first time Laplace's name, thereby completing with his authority all the discussions about the name of the transform, of which Laplace did not even think about, and which in all his four memoirs Poincaré always attributed to the name of Bessel without giving any arguments and any references), most likely for a greater effect of anticipatorily unexpected change of Bessel for the name of his compatriot (which happened against the backdrop of the defeat of France in the Franco-German war of 1870).

Let us illustrate the above idea of operational calculus by the example of problem (6.14), where we consider only the case $\sigma = 0$. In other words, consider the problem

$$u_t = u_{xx}, \quad t > 0, \quad |x| < 1; \quad u\big|_{x=\pm 1} = 0; \quad u\big|_{t=0} = 1. \tag{18.5}$$

As already noted in §17, the series (17.18), which was constructed by the Fourier method and which gives a solution to this problem, converges very slowly for small t. This, however, could have been foreseen in advance, since the Fourier series slowly converges for discontinuous functions and at the corner points of the half-strip $\{|x| < 1,\ t > 0\}$ the function $u(x,t)$ has a discontinuity. In this regard, we first consider the problem

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial \xi^2},\ \xi > 0,\ \tau > 0; \quad T\big|_{\xi=0} = T_1;\ T\big|_{\tau=0} = T_0, \tag{18.6}$$

which simulates the temperature distribution near the corner point

Changing to the dimensionless parameters in the standard way (see §6)

$$r = \xi/\sqrt{a\tau}, \quad u = (T - T_1)/(T_0 - T_1),$$

from (18.6) we get $u(\tau, \xi) = f(r)$, where the function f satisfies the conditions

$$f''(r) + \frac{r}{2}f'(r) = 0, \quad f(0) = 0, \quad f(\infty) = 1.$$

It follows that $u(\tau, \xi) = \operatorname{erf}(\xi/(2\sqrt{a\tau})) = 1 - \operatorname{erfc}(\xi/(2\sqrt{a\tau}))$, where

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-\eta^2}d\eta, \quad \operatorname{erfc}(y) = 1 - \operatorname{erf}(y).$$

These preliminary arguments suggest that for small t the solution $u(x,t)$ of problem (18.5) is well approximable by the following sum

$$1 - \left[\operatorname{erfc}\Big(\frac{1-x}{2\sqrt{t}}\Big) + \operatorname{erfc}\Big(\frac{1+x}{2\sqrt{t}}\Big)\right]. \tag{18.7}$$

This hint will allow us to obtain a representation of the solution of problem (18.5) in the form of a series rapidly converging for small t with the help of the Laplace transform. Denoting by $v(s, x)$ the function $L[u(\cdot, x)](s)$, where u is the solution of problem (18.5), and taking into account the two following obvious properties of the Laplace transform

$$L[1](s) = 1/s, \quad L[f'](s) = s \cdot L[f](s) - f(0), \tag{18.8}$$

we rewrite problem (18.5) in the ("algebraic" in the variable s) form

$$(s \cdot v(s, x) - 1) - v_{xx}(s, x) = 0, \quad v(s, x)\big|_{x=\pm 1} = 0, \quad s > 0.$$

This problem can be solved explicitly. Obviously, its solution is the function

$$v(s, x) = \frac{1}{s} - \frac{1}{s} \cdot \frac{\cosh(\sqrt{s}x)}{\cosh(\sqrt{s})}.$$

So, the solution u of problem (18.5) satisfies the relation

$$L[u(\cdot, x)](s) = \frac{1}{s} - \frac{1}{s} \cdot \frac{\cosh(\sqrt{s}x)}{\cosh(\sqrt{s})}. \tag{18.9}$$

Formulas (18.7) and (18.9) suggest that in order to obtain the representation of the solution of problem (18.5) in the form of a series rapidly converging for small t, we should

(1) Find the Laplace transform of the function $f_y(t) = \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right)$.

(2) Represent the right-hand side of formula (18.9) in the form of a series whose members are function of the form $L[f_y]$.

Below we shall show that

$$L[f_y](s) = \frac{1}{s} \exp(-y\sqrt{s}). \tag{18.10}$$

On the other hand, expressing cosh in terms of exp and representing $(1 + q)^{-1}$, where $q = \exp(-2\sqrt{s}) < 1$, as the series $1 - q + q^2 - q^3 + \ldots$, we obtain

$$\frac{\cosh(\sqrt{s}x)}{s \cdot \cosh\sqrt{s}} = \sum_{n=0}^{\infty}(-1)^n \frac{\exp[-\sqrt{s}(2n + 1 - x)] + \exp[-\sqrt{s}(2n + 1 + x)]}{s}. \tag{18.11}$$

From (18.8)–(18.11) it follows that the solution of problem (18.5) can be written as the series

$$u(x, t) = 1 + \left[\sum_{n=0}^{N}(-1)^{n+1} a_n\right] + r_n, \tag{18.12}$$

where $a_n = \operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)$, and $r_N = \sum_{n>N}(-1)^{n+1} a_n$.

P 18.2 Verify that

$$|r_N| \leq \frac{2}{N}\sqrt{\frac{t}{\pi}} \exp(-N^2/t). \tag{18.13}$$

P 18.3 By comparing estimate (18.13) with estimate (17.21), show that, for $t \leq \frac{1}{4}$, it is more convenient to use the representation of the solution of problem (18.5) in the form (18.12), and for $t \geq \frac{1}{4}$, in the form (17.18).

Let us show that (18.10) follows from the formula

$$L[(f_y)'](s) = \exp(-y\sqrt{s}) \tag{18.14}$$

and the second formula in (18.8), because $f_y(0) = 0$, $(f_y)'(t) = \frac{y}{2}\pi^{-1/2}t^{-3/2}e^{-y^2/(4t)}$. In view of Exercise 18.4 formula (18.14) it can be proved as follows:

$$L[f_y'](s) = \frac{y}{2\sqrt{\pi}}\int_0^\infty t^{-3/2}\cdot e^{-y^2/(4t)}dt = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-[\eta^2+(y^2s)/(4\eta^2)]}d\eta$$
$$= \frac{2}{\pi}e^{-y\sqrt{s}}\int_0^\infty e^{-(\eta-a/\eta)^2}d\eta = e^{-y\sqrt{s}}. \quad (\text{Replace: } \eta = \frac{y}{2\sqrt{t}},\ a = \frac{y}{2}\sqrt{s}.)$$

P 18.4 Let $F(a) = \int_0^\infty \exp[-(\eta - \frac{a}{\eta})^2]\,d\eta$, where $a > 0$. Verify that $F \equiv \frac{\sqrt{\pi}}{2}$.
Hint Use the relation $F'(a) \equiv 0$.

There is a close relation between the Fourier and Laplace transforms, which can be found by analyzing the equality

$$\partial^\beta \widetilde{f}(\xi) = \langle f(x), (-\mathring{\imath}x)^\beta e^{-\mathring{\imath}x\xi}\rangle, \quad f' \in \mathcal{E}'(\mathbb{R}^n), \quad \beta \in \mathbb{Z}_+^n,$$

which was proved in Lemma 17.31. The right-hand side of this equality is meaningful for any complex $\xi \in \mathbb{C}^n$ and any $\beta \in \mathbb{Z}_+^n$ and is a continuous function in $\mathbb{C}^n$. Thus, as is known from the theory of functions of a complex variable, the function

$$\widetilde{f}\colon \mathbb{C}^n \ni \xi \mapsto \widetilde{f}(\xi) = \langle f(x), e^{-\mathring{\imath}x\xi}\rangle \in \mathbb{C}$$

is analytic and can be treated as the *Fourier transform in the complex domain*. This function is sometimes called the *Fourier–Laplace transform*. This name can be justified by the fact that, for instance, for the function $f = \theta_+ f \in L^1(\mathbb{R})$ (cf. Example 17.5), the function

$$\mathbb{C}_- \ni \xi \mapsto \int_{-\infty}^\infty e^{-\mathring{\imath}x\xi} f(x)\,dx = \int_0^\infty e^{-\mathring{\imath}x\xi} f(x)\,dx \in \mathbb{C},$$

which is analytic in the lower half-plane $\mathbb{C}_-$, is, for real ξ (respectively, for imaginary $\xi = -\frac{is}{2\pi}$, where $s > 0$), the Fourier transform (respectively, the Laplace transform) of the function f.

The important role of the Fourier-Laplace transformation consists in the fact that, due to the so-called Paley–Wiener[18] theorems, certain properties of this analytic

[18] Norbert Wiener (1894–1964) was an American scientist, outstanding mathematician and philosopher, founder of cybernetics and the theory of artificial intelligence. his autobiographical book "I am mathematician" he writes about his co-author, the English mathematician Raymond Paley (1907–1933): "... He was the leader of the young generation of British mathematicians, and if he had not come to an untimely end he would be the mainstay of British mathematics at the present moment." And further: "My role was primarily that of suggesting problems and the broad lines on which they might be attacked, and it was generally left to Paley to draw the strings tight. He brought me a superb mastery of mathematics as a game and a vast number of tricks that added up to an armament by which almost any problem could be attacked, yet he had almost no sense of the orientation of mathematics among the other sciences... One interesting problem which we attacked together was that of the conditions restricting the Fourier transform of a function vanishing on the half line. This is a sound mathematical problem on its own merits, and Paley attacked it with vigor, but what helped me and did not help Paley was that it is essentially a problem in electrical engineering. It had

function allow one to determine whether this function is the Fourier–Laplace transform of the function f and to characterize the properties of this function f. In §22, we shall use (cf. Example 17.5) the following theorem.

Theorem 18.5 (Paley–Wiener) *Let $\widetilde{f}$ be an analytic function in $\mathbb{C}_-$ and let*

$$\sup_{\eta>0}\int_{-\infty}^{\infty}|\widetilde{f}(\xi - i\eta)|^2 d\xi < \infty.$$

Then $\widetilde{f}$ is the Fourier transform of the function $f = \theta_+ f \in L^2(\mathbb{R})$.

For a proof, see, for example, Yosida (1965).

19 Fundamental Solutions. Convolution

At the beginning of §18 it was noted that the differential equation

$$A(D_x)u(x) \equiv \sum_{|\alpha|\le m} a_\alpha \partial_x^\alpha u(x) = f(x), \quad a_\alpha \in \mathbb{C}, \quad f \in \mathcal{E}', \tag{19.1}$$

has a solution $u \in \mathcal{S}'$. In contrast to Eq. (18.2), the function f in (19.1) belongs to $\mathcal{E}' \subset \mathcal{S}'$. This fact allows us to give an "explicit" formula for the solution of Eq. (19.1), in which the role of the function f is emphasized. In this connection, we note that the formula

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} f(y)\frac{\exp(-q|x-y|)}{|x-y|}\,dy, \quad q \ge 0, \quad f \in \mathcal{E}' \cap PC_b, \tag{19.2}$$

gives (see Example 21.6 below) the solution of the equation $-\Delta u + q^2 u = f$. But this formula ceases to be meaningful for $q = 0$ if supp f is not compact (for example, if $f = 1$).

In order to deduce the desired "explicit" formula for the solution u of Eq. (19.1), we represent the function $\widetilde{u} = \mathbf{F}u$ in the form $\widetilde{u}(\xi) = \widetilde{f}(\xi)\widetilde{e}(\xi)$, where $\widetilde{e} \in \mathcal{S}'$ is the solution of the equation $A(\xi)\cdot\widetilde{e}(\xi) = 1$ (see Remark 19.2). Once this is done, it will remain to express the function $u = \mathbf{F}^{-1}(\widetilde{f}\cdot\widetilde{e})$ in terms of $f = \mathbf{F}^{-1}\widetilde{f}$ and the function $e = \mathbf{F}^{-1}\widetilde{e}$, which (in view of the relation $A(\xi)\,\widetilde{e}(\xi) \equiv 1$ and Example 17.23) satisfies the equation

been known for many years that there is a certain limitation on the sharpness with which an electric wave filter cuts a frequency band off, but the physicists and engineers had been quite unaware of the deep mathematical grounds for these limitations. In solving what was for Paley a beautiful and difficult chess problem, completely contained within itself, I showed at the same time that the limitations under which the electrical engineers were working were precisely those which prevent the future from influencing the past." With these final words, Wiener hints that the answer they are interested in about the properties of a function should have been formulated as an opportunity to analytically extend this function to some region of the complex plane.

$$A(D_x)\, e(x) = \delta(x). \tag{19.3}$$

Definition 19.1 We say that $E \in \mathcal{D}'$ is a *fundamental solution* of the operator $A(D_x)$ if[19] $A(D_x)\, E(x) = \delta(x)$.

Remark 19.2 Any differential operator with constant coefficients has (see Hörmander 1958, Łojasiewicz 1959) a fundamental solution from the class $\mathcal{S}'$. However, the appearance of the space $\mathcal{D}'$ in Definition 19.1 is justified by the fact that, for some differential operators, in $\mathcal{D}'$ it is possible (as was shown by Hörmander) to construct a fundamental solution which is locally more smooth than the fundamental solution from $\mathcal{S}'$. (Note that two fundamental solutions E_1 and E_2 of the operator $A(D_x)$ differ by the function $v = E_1 - E_2$ satisfying the equation $A(D_x)v = 0$.)

If $A(\xi) \neq 0$ for any $\xi \in \mathbb{R}^n$, then the formula $E(x) = \mathbf{F}^{-1}(1/A(\xi))$, obviously, gives a fundamental solution of the operator $A(D_x)$. In this case $E \in \mathcal{S}'$, because $1/A(\xi) \in \mathcal{S}'$. In the general case, a fundamental solution can be constructed, for instance, by regularizing the integral $\int \widetilde{\varphi}(\xi)d\xi/A(\xi)$ (cf. Exercise 14.4), which can be most simply effected for $\varphi \in \mathcal{D}$, because in this case the regularization (by virtue of analyticity of the function $\widetilde{\varphi} = \mathbf{F}\varphi$) is possible by extending (with respect to ξ) to the complex domain, where $A(\xi) \neq 0$ (see, for example, Shilov 1965).

19.3 Let us give examples of fundamental solutions. From Exercise 7.1 it follows that the function (7.11) is a fundamental solution of the Laplace operator. The function $(x, t) \mapsto E(x, t) = \theta(t - |x|)/2$ is a fundamental solution of the string operator (see Exercise 12.8). For the heat operator $\partial_t - \partial_{xx}$, the fundamental solution is given by the formula $E(x, t) = \theta(t)P(x, t)$, where the function P is defined by (6.18). Indeed, by the properties of the function P (see §6), for any function $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\begin{aligned}\langle E_t - E_{xx}, \varphi\rangle &= -\langle E, \varphi_t + \varphi_{xx}\rangle = -\lim_{\varepsilon\to+0}\int_\varepsilon^\infty\int_{\mathbb{R}}(\varphi_t + \varphi_{xx})E\, dx\, dt \\ &= \lim_{\varepsilon\to+0}\Big[\int_\varepsilon^\infty\int_{\mathbb{R}}(P_t - P_{xx})\varphi\, dx\, dt + \int_{-\infty}^\infty P(x,t)\varphi(x,t)dx\Big] = \varphi(x,t)\big|_{x=t=0}.\end{aligned}$$

Prior to representing the solution of Eq. (19.1) in terms of $f \in \mathcal{E}'$ and the fundamental solution $e \in \mathcal{S}'$ of the operator $A(D_x)$, we give the definition of the convolution of two functions (which has already implicitly appeared in the footnote on p. 17).

Assume first that φ and ψ lie in $L(\mathbb{R}^n)$. In this problem,

$$\int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n}|\varphi(y)|\, dy\Big]|\psi(z)|dz \overset{\text{Lemma 8.46}}{=} \int_{\mathbb{R}^{2n}}|\varphi(y)\psi(z)|\, dy\, dz < \infty.$$

Consider the product of the Fourier transforms of these functions, i.e.,

[19] The fundamental solution of the string operator is given in Exercise 12.8, and for the Laplace operator, is given by formula (7.11). The function defined by (6.16) for $Q = 1$ is called the fundamental solution of the Cauchy problem (6.3)–(6.4) for the heat equation, because it satisfies Eq. (6.3) and because equality (6.5) holds.

$$\int_{\mathbb{R}^n} e^{-\imath y\xi}\varphi(y)\,dy \int_{\mathbb{R}^n} e^{-\imath z\xi}\psi(z)\,dz = \int_{\mathbb{R}^{2n}} e^{-\imath(y+z)\xi}\varphi(y)\psi(z)\,dy\,dz.$$

Putting $y + z = x$, we get

$$\widetilde{\varphi}(\xi)\cdot\widetilde{\psi}(\xi) = \int_{\mathbb{R}^n} e^{-\imath\xi x}\Big[\int_{\mathbb{R}^n}\varphi(y)\psi(x-y)\,dy\Big]\,d\xi. \tag{19.4}$$

Definition 19.4 The function[20] $\mathbb{R}^n \ni x \mapsto (\varphi * \psi)(x)$, which is represented by the inner integral in (19.4), i.e., which is given by the formula

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n}\varphi(y)\psi(x-y)\,dy, \tag{19.5}$$

is called the *convolution* of the functions $\varphi \in L(\mathbb{R}^n)$ and $\psi \in L(\mathbb{R}^n)$.

P 19.5 Verify that $\varphi * \psi = \psi * \varphi$, and if $\varphi \in C_0^{|\alpha|}$, $\psi \in C_0^{|\alpha|}$, then $D^\alpha(\varphi * \psi) = (D^\alpha\varphi) * \psi = \varphi * (D^\alpha\psi)$.

Definition 19.6 Let $f = D^\alpha f_\alpha \in \mathcal{E}'$ and $g = D^\beta g_\beta \in \mathcal{E}'$, where by Theorem 16.20 the functions f_α and g_β lie in $C_0(\mathbb{R}^n)$. The *convolution* $f * g$ of the distributions $f = D^\alpha f_\alpha \in \mathcal{E}'$ and $g = D^\beta g_\beta \in \mathcal{E}'$ is the generalized function $D^{\alpha+\beta}(f_\alpha * g_\beta)$.

Note that for functions $f \in \mathcal{E}'$ and $g \in \mathcal{E}'$ the product of their Fourier transforms is well defined (by Lemma 17.31), and in addition,

$$\mathbf{F}^{-1}(\widetilde{f}\cdot\widetilde{g}) = f * g. \tag{19.6}$$

Indeed, $\mathbf{F}[D^{a+\beta}(f_a * g_\beta)] = \xi^{a+\beta}\mathbf{F}(f_a * g_\beta) = (\xi^a\widetilde{f_a})(\xi^\beta\widetilde{g}_\beta)$.

P 19.7 Let $f \in \mathcal{E}'$ and let $g \in \mathcal{S}'$. We set $g_\nu(x) = \psi\big(\frac{x}{\nu}\big)g(x)$, where $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \equiv 1$ for $|x| < 1$. Verify that $g_\nu \in \mathcal{E}'$, $g_\nu \to g$ in $\mathcal{S}'$; $\widetilde{f}\cdot\widetilde{g}_\nu \to \widetilde{f}\cdot\widetilde{g}$ in $\mathcal{S}'$, and

$$\langle f * g_\nu, \varphi\rangle \to \langle \mathbf{F}^{-1}(\widetilde{f}\cdot\widetilde{g}), \varphi\rangle \quad \forall\varphi \in \mathcal{S}.$$

The next definition, which generalizes (19.6), is based on Exercise 19.7.

Definition 19.8 The *convolution* $f*g$ of distributions $f \in \mathcal{E}'$ and $g \in \mathcal{S}'$ is a function from $\mathcal{S}'$ defined by the formula

$$f * g = \mathbf{F}^{-1}(\widetilde{f}\cdot\widetilde{g}).$$

(Note that $\widetilde{f} \in C^\infty$, $\widetilde{g} \in \mathcal{S}'$.)

P 19.9 (Cf. Exercise 19.7) Let $f \in \mathcal{E}'$, $g \in \mathcal{S}'$. Verify that $f * g = g * f$; $D^\alpha(f * g) = (D^\alpha f) * g = f * (D^\alpha g)$; $\delta * g = g$.

From the above (cf. the footnote 20 on p. 17) we have the following theorem.

[20] By Fubini's theorem, this function is defined for a.e. $x \in \mathbb{R}^n$ and belongs to $L(\mathbb{R}^n)$.

Theorem 19.10 *The desired "explicit" formula for the solution of Eq. (19.1) has the form $u = f * e$, where $e \in \mathcal{S}'$ is the fundamental solution of the operator $A(D_x)$.*

P 19.11 Prove the *Weierstrass*[21] theorem on uniform approximation of a continuous function $f \in C(K)$ by polynomials on a compact set $K \subset \mathbb{R}^n$: for any $\varepsilon > 0$ there exists a polynomial p such that $\|f(x) - p(x)\|_{C(K)} < \varepsilon$.

Let Ω be a neighborhood of K, $x = (x_1, \dots, x_n) \in \Omega$. Following the scheme of the proof of Lemma 13.11 and taking into account Exercise 4.4, we set

$$p(x) = \int_\Omega f(y)\delta_\nu(x-y)\,dy, \quad \text{where } \delta_\nu(x) = \prod_{m=1}^{n}\Big[\frac{\nu}{\sqrt{\pi}}\Big(1 - \frac{1}{\nu}x_m^2\Big)^{\nu^3}\Big].$$

However, you can do it more gracefully, assuming

$$\delta_\nu(x) \overset{\text{cf. (4.2)}}{=} \Big(\frac{\nu}{4\pi}\Big)^{n/2} e^{-\nu|x|^2/4}$$

and using the fact that an entire (i.e., analytic with respect to $x \in \mathbb{R}^n$) function is uniformly approximable by polynomials on any compact set.

Lemma 19.12 *Let $u \in L^1$, $v \in L^2$. Then $u * v \in L^2$ and*

$$\|u * v\|_{L^2} \le \|u\|_{L^1} \cdot \|v\|_{L^2}. \tag{19.7}$$

Proof We have

$$\Big|\int u(\xi - \eta)v(\eta)d\eta\Big|^2 \le \Big(\int |u(\xi - \eta)|d\eta\Big) \cdot A(\xi) = \|u\|_{L^1} \cdot A(\xi),$$

where $A(\xi) = \int |u(\xi-\eta)|\cdot|v(\eta)|^2 d\eta$. But $\int A(\xi)d\xi = \|v\|^2_{L^2} \cdot \|u\|_{L^1}$ by Lemma 8.46. □

20 On the Spaces H^s

The study of generalized solutions of equations of mathematical physics leads in a natural way to the family of Banach spaces $W^{p,m}$ introduced by Sobolev. For $p \ge 1$ and $m \in \mathbb{Z}_+$ the space $W^{p,m}(\Omega)$ is the Banach space of the functions $u \in L^p(\Omega)$ with finite norm

[21] Oddly enough, Prussian bureaucracy promoted maturation of the greatest German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897). V.I. Arnold, referring to A. Poincaré obituary of Weierstrass, writes (see Arnold 2008 and Arnold 2003): "Weierstrass began his career as a physical education teacher at school. He was particularly successful in teaching his high school students to work on parallel bars. But the Prussian rules required a gymnasium teacher to submit a written work at the end of the year proving his professional aptitude. And Weierstrass presented an essay on elliptic functions and integrals. No one in the gymnasium could understand this essay, so it was sent to the university for evaluation. And very soon he was transferred to the university, where he quickly became one of the most outstanding and famous mathematicians of the century, both in Germany and in the world."

$$\|u\|_{W^{p,m}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha u|^p \, dx \right)^{1/p} \tag{20.1}$$

(see Definition 17.6). Here $\partial^\alpha u = v$ is the generalized derivative of the function u, i.e.,

$$\int_\Omega v\,\varphi\,dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \varphi\,dx \quad \forall \varphi \in C_0^\infty(\Omega). \tag{20.2}$$

The function v satisfying conditions (20.2) was called by Sobolev the weak derivative of order α of the function u. Maybe, this is the reason, why the letter W appeared in the designation of Sobolev spaces.

For $p = 2$, the spaces $W^{p,m}$ are Hilbert spaces (see the footnote 7 on p. 93). They are denoted (apparently, in honor of Hilbert) by H^m. These spaces play a very important role in modern analysis. For their role in elliptic equations, see §22. A detailed account of the theory of these spaces can be found, for instance, in the book Besov et al. (1978) and in the paper Volevich and Paneah (1965).

We present some elements of the theory of H^s-spaces in the form of a series of definitions, problems, and observations.

P 20.1 Using formula (18.1), verify that for $m \in \mathbb{Z}_+$ the space $H^m(\mathbb{R}^n)$ is the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1 + |\xi|)^m (Fu)(\xi) \in L^2(\mathbb{R}^n)$.

Definition 20.2 Let $s \in \mathbb{R}$. The *space* $H^s = H^s(\mathbb{R}^n)$ consists of $u \in \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ with finite norm[22]

$$\|u\|_s = \|\langle \xi \rangle^s \cdot \widetilde{u}(\xi)\|_{L^2(\mathbb{R}^n)}, \quad \text{where } \langle \xi \rangle = 1 + |\xi|, \text{ and } \widetilde{u} = Fu. \tag{20.3}$$

P 20.3 Verify that $\mathcal{S} \subset H^\alpha \subset H^\beta \subset \mathcal{S}'$ if $\alpha > \beta$, the embedding operators is continuous, and the embedded spaces are dense in the enveloping spaces.

Theorem 20.4 (Sobolev Embedding Theorem) *If $s > \frac{n}{2} + m$, then $H^s(\mathbb{R}^n) \subset C_b^m(\mathbb{R}^n)$, and besides, there exists a constant $C < \infty$ such that*

$$\|u\|_{(m)} \le C \|u\|_s \quad \forall u \in H^s, \tag{20.4}$$

where $\|u\|_{(m)} \stackrel{\text{def}}{=} \sum_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} |\partial^\alpha u(x)|$.

Proof Note that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$|\varphi(x)| = \left| \int \widetilde{\varphi}(\xi) \langle \xi \rangle^s \cdot \langle \xi \rangle^{-s} e^{\mathring{\imath} x\xi} d\xi \right| \le \|\varphi\|_s \left(\int \langle \xi \rangle^{-2s} d\xi \right)^{1/2}.$$

This implies inequality (20.4). For a given $u \in H^s$, we choose $u_n \in \mathcal{S}$ such that $\|u_n - u\|_s \to 0$. In view of (20.4), $\{u_n\}$ is a Cauchy sequence in $C_b^m(\mathbb{R}^n)$. Hence there exists a function $v \in C_b^m(\mathbb{R}^n)$ such that $\|u_n - v\|_{(m)} \to 0$. As a result, $\|u_n - v\|_{L^2(\Omega)} \to 0 \ \forall \Omega \Subset \mathbb{R}^n$. Now, we have $u = v$ a. e., since $\|u_n - v\|_{L^2(\Omega)} \le \|u_n - u\|_s \to 0$. □

[22] In what follows, the norm of the space L^p will be denoted by $\|\cdot\|_{L^p}$.

P 20.5 Let $u(x) = \varphi(r)\ln(-\ln r)$, where $x \in \mathbb{R}^2$, $r = |x|$, $\varphi \in C_0^\infty(\mathbb{R}^2)$, and $\varphi(r) = 1$ for $r < \frac{1}{3}$ $\varphi(r) = 0$ for $r > \frac{2}{3}$. Verify that $u \in H^1(\mathbb{R}^2)$. This shows that $H^{n/2}(\mathbb{R}^n)$ cannot be embedded into $C(\mathbb{R}^n)$.

P 20.6 Verify that $\delta \in H^s(\mathbb{R}^n)$ for $s < -\frac{n}{2}$.

Theorem 20.7 (Sobolev Trace Theorem)
Let $s > \frac{1}{2}$. Then, for any function $u(x) \in H^s(\mathbb{R}^n)$ (which is in general discontinuous), the trace $\gamma u \in H^{s-1/2}(\mathbb{R}^{n-1})$ is defined, which (for a continuous $u(x)$) coincides with the restriction $u\big|_{x_n=0}$ of $u(x)$ to the hypersurface $x_n = 0$. Moreover, there exists a $C < \infty$ such that

$$\|\gamma u\|'_{s-1/2} \le C\|u\|_s \quad \forall u \in H^s(\mathbb{R}^n), \tag{20.5}$$

where $\|\cdot\|'_\sigma$ is the norm in the space $H^\sigma(\mathbb{R}^{n-1})$.

Proof Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$u(x', 0) = \int_{\mathbb{R}^{n-1}} e^{\mathring{\imath}x'\xi'} \left[\int_{\mathbb{R}} \widetilde{u}(\xi', \xi_n) d\xi_n \right] d\xi'.$$

Hence

$$\left(\|\gamma u\|'_{s-1/2}\right)^2 = \int_{\mathbb{R}^{n-1}} \langle \xi' \rangle^{2s-1} \left| \int_{\mathbb{R}} \widetilde{u}(\xi', \xi_n) d\xi_n \right|^2 d\xi',$$

and so

$$\left| \int \widetilde{u}(\xi', \xi_n)\, d\xi_n \right|^2 \le \int \langle \xi \rangle^{-2s} d\xi_n \int \langle \xi \rangle^{2s} |\widetilde{u}(\xi)|^2\, d\xi_n.$$

Further, $\int \langle \xi \rangle^{-2s} d\xi_n \le C_s \langle \xi' \rangle^{-s+1/2}$, and $C_s = C \int (1 + z^2)^{-s} dz < \infty$ for $s > \frac{1}{2}$; ($z = \xi_n(1 + |\xi'|^2)^{-1/2}$). Hence $\|\gamma u\|'_{s-1/2} \le C\|u\|_s$ for $u \in \mathcal{S}$. If $u \in H^s(\mathbb{R}^n)$ and $\lim_{n\to\infty} \|u_n - u\|_s = 0$, where $u_n \in \mathcal{S}$, then there exists a $w \in H^{s-1/2}(\mathbb{R}^{n-1})$ such that $\|\gamma u_n - w\|'_{s-1/2} \to 0$; moreover, w is independent of the choice of the sequence $\{u_n\}$. By definition, $\gamma u = w$, and so (20.5) holds. □

Definition 20.8 Let Ω be a domain in $\mathbb{R}^n$. The operator

$$P = P_\Omega \colon \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\Omega),$$

satisfying $\langle Pf, \varphi \rangle = \langle f, \varphi \rangle$ for any function $\varphi \in \mathcal{D}(\Omega)$ is called the *restriction operator* of distributions from $\mathcal{D}'(\mathbb{R}^n)$ to the domain Ω.

Definition 20.9 We denote by $H^s(\Omega)$ the space $P_\Omega H^s(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{s,\Omega} = \inf \|Lf\|_s, \quad f \in H^s(\Omega), \tag{20.6}$$

where the infimum is taken over all extensions $Lf \in H^s(\mathbb{R}^n)$ of the function $f \in H^s(\Omega)$ (i.e., $P_\Omega Lf = f$). If it is clear from the context that $f \in H^s(\Omega)$, then we may omit the index Ω and simply write $\|f\|_s$ in place of $\|f\|_{s,\Omega}$.

Definition 20.10 The space $H^s(\Gamma)$, where $\Gamma = \partial\Omega$ is a smooth boundary of a domain $\Omega \Subset \mathbb{R}^n$, is the completion of the space $C^\infty(\Gamma)$ in the norm

$$\|\rho\|'_{s,\Gamma} = \sum_{k=1}^{K} \|\varphi_k \rho\|'_s. \tag{20.7}$$

Here $\|\cdot\|'_s$ is the norm of the space $H^s(\mathbb{R}^{n-1})$, $\sum_{k=1}^K \varphi_k \equiv 1$ is the partition of unity (see §3) subordinate to the finite cover $\bigcup_{k=1}^K \Gamma_k = \Gamma$, where $\Gamma_k = \Omega_k \cap \Gamma$, and Ω_k is the n-dimensional domain in which the normal vectors to Γ do not intersect. Next, the function $\varphi_k\rho \in C_0^\infty(\mathbb{R}^{n-1})$ is defined by the formula

$$(\varphi_k\rho)(y') = \varphi_k(\sigma_k^{-1}(y')) \cdot \rho(\sigma_k^{-1}(y')),$$

where σ_k is a diffeomorphism of $\mathbb{R}^n$ (affine outside some ball) which "unbends" Γ_k. This means that if $x \in \Omega_k$, then the nth coordinate $y_n = y_n(x)$ of the point $y = (y', y_n) = \sigma_k(x)$ is equal to the coordinate of this point on the inward normal vector to Γ. If it is clear from the context that $\rho \in H^s(\Gamma)$, then in parallel with $\|\rho\|'_{s,\Gamma}$ we will sometimes write $\|\rho\|'_s$.

Remark 20.11 Definition 20.10 of the space $H^s(\Gamma)$ is correct, i.e., it does not depend on the choice of the cover, the partition of unity, and the diffeomorphism σ_k. In the book Shubin (1987), this fact is elegantly proved with the help of the machinery of pseudo-differential operators (which we will consider in the next section).

P 20.12 Verify that the operator $C(\overline{\Omega}) \cap H^s(\Omega) \ni u \mapsto u|_\Gamma \in C(\Gamma)$ extends to a continuous operator $\gamma\colon H^s(\Omega) \to H^{s-1/2}(\Gamma)$ if $s > \frac{1}{2}$.

Remark 20.13 The function $\gamma u \in H^{s-1/2}(\Gamma)$, where $s > \frac{1}{2}$, is called the boundary value of a function $u \in H^s(\Omega)$. One can easily show that $H^{s-1/2}(\Gamma)$, where $s > \frac{1}{2}$, is the space of boundary values of functions from $H^s(\Omega)$ (see, for example, Volevich and Paneah 1965). The condition $s > \frac{1}{2}$ is essential that follows from the example of the function $u \in H^{1/2}(\mathbb{R}_+)$ considered in Exercise 20.5.

Remark 20.14 The well-known *Arzelà's theorem*[23] (see, for example, Kolmogorov and Fomin 1980, Yosida 1965) asserts that if a family $\{f\}$ of functions $f \in C(\overline{\Omega})$, defined on $\Omega \Subset \mathbb{R}^n$, is uniformly bounded (i.e., $\|f\| = M < \infty$ for any function f) and equicontinuous (for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any function f whenever $|x - y| < \delta$), then this family $\{f\}$ contains a subsequence converging in $C(\overline{\Omega})$.

In the case $\Omega = [0, 1]$, this theorem is proved as follows (the general case is similar). Let $\varepsilon_k = M/2^{k+1}$, and let δ_k be such that $|f(x) - f(y)| < \varepsilon_k$ for any function f whenever $|x - y| < \delta_k$. Assume first that $k = 1$. Consider the columns $P_j = \{j\delta_1 \le x \le (j+1)\delta_1\}$. We have $|f| \le \text{const}$ for any function f, and hence in the column P_1 there exists infinitely many functions from the family $\{f\}$ whose graphs lie in some two neighboring rectangles of height ε_1. In the column P_2, these graphs

[23] Cesare Arzelà (1847–1912) was an Italian mathematician.

may continue only into the four neighboring rectangles of height ε_1. In addition, at least two neighboring rectangles of height ε_1 contain the graphs of an infinitely many functions from $\{f\}$. Continuing this process, we get a "trail" S_1 of width $2\varepsilon_1$ containing the graphs of an infinite subset of functions from the family $\{f\}$. We denote this subset of functions by $\{f_1\}$ and fix one of these functions f_1^*. Similarly, from the family of functions $\{f_1\}$ we single out an infinite subset of functions $\{f_2\}$ whose graphs lie in the "trail" S_2 of width $2\varepsilon_2$. Among such functions, we fix one function f_2^*. Proceeding in this way, we will construct $f_3^*, f_4^*, \dots$ By construction, the graphs of these functions f_n^* lie in the "trail" S_n of width $2\varepsilon_n$. It remains to note that

$$\max_{x\in\Omega} |f_n^*(x) - f_m^*(x)| \le 2\varepsilon_n \quad \text{for any } m > n.$$

Analogues of Arzelà's theorem also hold for the Sobolev spaces H^s and their generalizations (see, for example, Besov 2001 and the references cited there). In particular, the following theorem holds.

Theorem 20.15 (On Compactness of the Embedding) *Assume that a sequence of functions $u_n \in H^s(\Omega)$ (respectively, $u_n \in H^s(\partial\Omega)$), where $\Omega \Subset \mathbb{R}^n$, is such that $\|u_n\|_s \le 1$ (respectively, $\|u_n\|_s' \le 1$). Then this sequence contains a subsequence converging in $H^t(\Omega)$ (respectively, in $H^t(\partial\Omega)$) if $t < s$.*

21 On Pseudo-Differential Operators

The class of pseudo-differential operators (PsDO),[24] which is wider than the class of differential operators, includes the operators of the form

$$Au(x) = \int_\Omega K(x, x-y)u(y)\,dy, \quad u \in C_0^\infty(\Omega). \tag{21.1}$$

Here $K \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ and $K \in C^\infty(\Omega \times (\mathbb{R}^n \backslash 0))$. In particular, if

$$K(x, x-y) = \sum_{|\alpha|\le m} a_\alpha(x) \cdot \delta^{(\alpha)}(x-y),$$

then $Au(x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x)$. Another example of PsDO is given by singular integral operators (Mikhlin 1965). However, the exclusive place in modern mathematical physics occupied by the theory of PsDO (which took shape in the mid-1960s, see Vishik and Eskin 1964, Kohn and Nirenberg 1965,[25] Hörmander 1965) is determined not only by specific important examples. The actual fact is that PsDO provide

[24] One of the referees of the present book recommended to get rid of the abbreviation "PDO," which is widely used in Russian mathematical literature (see, for example, Arutyunov and Mishchenko 2013) in order to have no unnecessary association with Partial Differential Operators.

[25] Thanks to good graces of Kohn and Nirenberg (1965), the term "pseudo-differential operators" is now widely accepted.

a powerful and convenient tool for studying linear differential operators (primarily, elliptic ones), because the pseudo-differential operators form an algebra—one can not only add and multiply such operators (and take their compositions) but also "divide" by nonzero operators. So, solutions of some differential equations can be written down in terms of PsDO.

Before giving the corresponding definitions and results, we will briefly explain the main idea underlying the application of the PsDO by proving an important theorem on the smoothness of solutions to the elliptic differential equation with constant coefficients

$$a(D)u \equiv \sum_{|\alpha|\le m} a_\alpha D^\alpha u = f. \tag{21.2}$$

The ellipticity means that

$$a_m(\xi) \equiv \sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0 \quad \text{for } |\xi| \neq 0.$$

This is equivalent to the condition

$$|a(\xi)| \equiv \Big|\sum_{|\alpha|\le m} a_\alpha \xi^\alpha\Big| \ge C|\xi|^m \quad \text{for } |\xi| \ge M \gg 1. \tag{21.3}$$

Theorem 21.1 *Let condition* (21.3) *be satisfied and let* $f \in H^{s-m}$ *for some* s. *Then a solution* $u(x)$ *of Eq.* (21.1) *lies in* H^s *if* $u(x) \in H^{s-N}$ *for some* $N > 0$.

Proof Of course, this result can be established by constructing a fundamental solution of the operator $a(D)$ and by examining its properties (see, for example, Hörmander 1983–1985). However, instead of solving the difficult problem of regularizing the function $1/a(\xi)$, where $\xi \in \mathbb{R}^n$ (this problem will appear after application of the Fourier transform to Eq. (21.1) written in the form $\mathbf{F}^{-1}a(\xi)\mathbf{F}u = f$), it is enough to "just notice" two facts. First, using inequality (21.3), it is possible to "remove" the singularity of the function $1/a$ using a factor $\rho \in C^\infty$ such that $\rho \equiv 1$ for $|\xi| \ge M+1$, $\rho \equiv 0$ for $|\xi| \le M$. Second,

$$(\mathbf{F}^{-1}(\rho/a)\mathbf{F})(\mathbf{F}^{-1}a\mathbf{F})u = u + (\mathbf{F}^{-1}r\mathbf{F})u, \quad \text{where } r = \rho - 1. \tag{21.4}$$

Hence, in view of the obvious inequalities

$$|\rho(\xi)/a(\xi)| \le C(1+|\xi|)^{-m}, \quad |r(\xi)| \le C_N(1+|\xi|)^{-N} \quad \forall N \ge 1, \tag{21.5}$$

which imply the inequalities

$$\|(\mathbf{F}^{-1}(\rho/a)\mathbf{F})f\|_s \le C\|f\|_{s-m}, \quad \|(\mathbf{F}^{-1}r\mathbf{F})u\|_s \le C\|u\|_{s-N}, \tag{21.6}$$

we get as a result the so-called *a priori estimate*

$$\|u\|_s \le C\big(\|f\|_{s-m} + \|u\|_{s-N}\big), \quad f = a(D)u, \quad u \in H^s, \tag{21.7}$$

where C does not depend on u. From inequality (21.7) we get the above result on smoothness of solutions of the elliptic equation (21.1). □

The name "*a priori*" for estimate (21.7) of the solution of Eq. (21.1) is due to the fact that it can be established (see the above) before clarifying the solvability of Eq. (21.1), i.e., in the *a priori* form.

The simplicity of the above derivation of the a priori estimate (21.1) shows the important role of operators of the form $\mathbf{F}^{-1}a\mathbf{F}$. Such operators are called *pseudo-differential operators* constructed from the *symbol* $a = a(\xi)$. We will denote such operators also by $Op(a(\xi))$ or $a(D)$. The class of PsDO deepens of the class of symbols considered. If $a(x,\xi) = \sum a_\alpha(x)\xi^\alpha$, then

$$a(x, D)u(x) = Op(a(x,\xi))u(x) = \sum a_\alpha(x)D_x^\alpha u(x).$$

If $a(x,\xi)$ is a function which is positive homogeneous and having zero order with respect to ξ, i.e., $a(x,t\xi) = a(x,\xi)$ for $t > 0$, then $a(x,D) = Op(a(x,\xi))$ is a singular integral operator (Mikhlin 1965), namely

$$Op(a(x,\xi))u(x) = b(x)u(x) + \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{c(x, x-y)}{|x-y|^n}u(y)\,dy.$$

Here $c(x,tz) = c(x,z)$ for $t > 0$ and $\int_{|z|=1} c(x,z)\,dz = 0$. In particular, in the one-dimensional case, when $a(\xi) = a_+\theta_+(\xi) + a_-\theta_-(\xi)$, where θ_+ is the Heaviside function and $\theta_- = 1 - \theta_+$, we have

$$Op(a(x,\xi))u = \frac{a_+ + a_-}{2\pi}u(x) + \frac{i}{2\pi}\text{v.p.}\int \frac{a_+ - a_-}{x-y}u(y)\,dy,$$

which follows from Exercise 17.28 and relation (12.10).

If $n = 2$, then the x-representation of the operator $Op(a(\xi))$ can be derived from the two following propositions.

Proposition 21.2 *Let* $x_1 = r\cos 2\pi\theta$, $x_2 = r\sin 2\pi\theta$, *and let* $U(r,\theta) \stackrel{\text{def}}{=} u(x_1,x_2) = \sum_{m\in\mathbb{Z}} U_m(r)e^{\mathring{\imath}m\theta}$, $U_m(r) \in \mathbb{C}$. *Then*

$$\boldsymbol{F}_{x\to\xi}\, u(x) = \sum_{n\in\mathbb{Z}}(-i)^n e^{\mathring{\imath}\omega n}\int_0^\infty rU_n(r)J_n(2\pi|\xi|r)\,dr. \tag{21.8}$$

Here $x = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$, $\xi_1 = |\xi|\cos 2\pi\omega$, $\xi_2 = |\xi|\sin 2\pi\omega$, *and*

$$J_n\colon \mathbb{R}\ni a\mapsto J_n(a) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^\pi \cos(nt - a\sin t)\,dt$$

is the Bessel[26] *function of nth order.*

[26] Friedrich Wilhelm Bessel (1784–1846), a German astronomer, born in Westphalia, the son of a poor government employee. At the age of 15, he entered an export-import firm. During his

Proof We have $\mathbf{F}_{x\to\xi}\, u(x) = \int_0^\infty r\Big(\int_0^1 U(r,\theta)e^{-\mathring{\imath}|\xi|r\cos 2\pi(\theta-\omega)}d\theta\Big)dr$, and[27]

$$e^{-\mathring{\imath}|\xi|r\cos 2\pi(\theta-\omega)} = \sum_{n\in\mathbb{Z}} J_n(-2\pi|\xi|r)i^n e^{\mathring{\imath}n(\theta-\omega)}. \tag{21.9}$$

This implies (21.8), because

$$\int_0^1 e^{\mathring{\imath}(n-m)\theta}d\theta = \begin{cases} 0 & \text{for } m \neq n, \\ 1 & \text{for } m = n \end{cases}$$

and $J_{-n}(-a) = J_n(a)$. □

The proof of the following result is similar.

Proposition 21.3 *Let*

$$\xi_1 = |\xi|\cos 2\pi\omega, \quad \xi_2 = |\xi|\sin 2\pi\omega,$$
$$\widetilde{V}(|\xi|,\omega) \overset{\text{def}}{=} \widetilde{v}(\xi_1,\xi_2) = \sum_{m\in\mathbb{Z}} \widetilde{V}_m(|\xi|)e^{-\mathring{\imath}m\omega}, \quad \widetilde{V}_m(\rho) \in \mathbb{C}.$$

Then[28]

$$\boldsymbol{F}^{-1}_{\xi\to x}\widetilde{v}(\xi) = \sum_{n\in\mathbb{Z}} i^n e^{-\mathring{\imath}\varphi n} \int_0^\infty |\xi|\widetilde{V}_n(|\xi|)J_n(2\pi|\xi|\rho)\, d|\xi|, \tag{21.10}$$

apprenticeship, dreaming of travel, he studied languages, geography, the habits of distant peoples, and the principles of navigation, which led him to astronomy and mathematics. In 1804 he wrote a paper on Halley's Comet in which he calculated the orbit from observations made in 1607. He sent it to the German astronomer Wilhelm Olbers, who was so impressed that he arranged it its publication the same year in the important German technical journal Monatliche Correspondenz and proposed Bessel as an assistant at the Lilienthal observatory of the celebrated lunar observer J. H. Schröter. Bessel, who was liked and appreciated by his commercial firm, was obliged to choose between a position of relative affluence if he remained in it and poverty and the stars if he left it. He decided for the latter.

Bessel was a scientist whose works laid the foundations for a better determination than any previous method had allowed of the scale of the universe and the sizes of stars, galaxies, and clusters of galaxies. In addition, he made fundamental contributions to accurate positional astronomy, the exact measurement of the positions of celestial bodies; to celestial mechanics, dealing with their movements; and to geodesy, the study of Earth's size and shape. Further, he enlarged the resources of pure mathematics by his introduction and investigation of what are now known as Bessel functions (see, for example, Watson 1995, Whittaker and Watson 2020), which he used first in 1817 to investigate the very difficult problem of determining the motion of three bodies moving under mutual gravitation. Seven years later he developed Bessel functions more fully for the treatment of planetary perturbations. Much credit for the final establishment of a scale for the universe in terms of solar system and terrestrial distances, which depends vitally on accurate measurement of the distances of the nearest stars from Earth, must go to Bessel.

27 The *generating function* for $J_n(\mu)$, i.e., the formal power series $\sum_{n\in\mathbb{Z}} J_n(\mu)\, t^n$, is $e^{\frac{\mu}{2}(t-\frac{1}{t})}$ (see, for example, Lavrent'ev and Shabat 1977). Putting $t = ie^{\mathring{\imath}(\theta-\omega)}$, we get (21.9).

28 $\mathbf{F}^{-1}_{\xi\to x}\widetilde{v}(\xi) = \int_0^\infty |\xi|\big(\int_0^1 \widetilde{V}(|\xi|,\omega)e^{\mathring{\imath}\rho|\xi|\cos 2\pi(\omega-\varphi)}d\omega\big)d|\xi|$ and (cf. (21.9)) we have $e^{\mathring{\imath}\rho|\xi|\cos 2\pi(\omega-\varphi)} = \sum_{n\in\mathbb{Z}} J_n(2\pi\rho|\xi|)i^n e^{\mathring{\imath}n(\omega-\varphi)}$.

where $\xi = (\xi_1, \xi_2)$, $x = (x_1, x_2)$, $x_1 = \rho\cos 2\pi\varphi$, $x_2 = \rho\sin 2\pi\varphi$.

The following corollary is also useful.

Corollary 21.4 *If* $n = 2$, *then*

$$\boldsymbol{F}^{-1}_{\xi\to x}\frac{1}{|\xi|} = \int_0^\infty |\xi|\frac{1}{|\xi|}J_0(2\pi|x||\xi|)\,d|\xi| = \frac{1}{2\pi|x|}. \tag{21.11}$$

An important class of symbols in the theory of PsDO is given in the next definition.

Definition 21.5 A C^∞-function

$$a\colon \mathbb{R}^n\times\mathbb{R}^n \ni (x,\xi)\mapsto a(x,\xi) = a_0(x,\xi) + a_1(\xi) \tag{21.12}$$

is a *symbol* of the class $S^m = S^m(\mathbb{R}^n)$, where $m\in\mathbb{R}$ if, for any multiindexes α and β, there exist $C_{\alpha\beta}\in\mathcal{S}(\mathbb{R}^n)$ and $C_\beta\in\mathbb{R}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a_0(x,\xi)| \le C_{\alpha\beta}(x)\cdot\langle\xi\rangle^{m-|\beta|},\quad |\partial_\xi^\beta a_1(\xi)|\le C_\beta\langle\xi\rangle^{m-|\beta|}, \tag{21.13}$$

where $\langle\xi\rangle = 1+|\xi|$. In this case, we write $a\in S^m$.

Example 21.6 Let $a(\xi) = \varepsilon^2 + 1/(|\xi|^2+q^2)$, $\varepsilon\ge 0$, $q>0$. Then $a\in S^0(\mathbb{R}^n)$ for $\varepsilon>0$ and $a\in S^{-2}(\mathbb{R}^n)$ for $\varepsilon = 0$. If $n=3$, then (cf. (19.2))

$$a(D)u(x) = \varepsilon^2 u(x) + \pi\int_{\mathbb{R}^3}\frac{e^{-2\pi q|x-y|}}{|x-y|}u(y)\,dy,\quad u\in C_0^\infty. \tag{21.14}$$

Indeed, we set $f = \left(\frac{1}{|\xi|^2+q^2}\right)u$, which is equivalent to the equation $u = (|D|^2+q^2)f$, i.e., $-\Delta f + (2\pi q)^2 f = 4\pi^2 u$. In view of the estimate $\|f\|_s\le C\|u\|_{s+2}$, the solution of the last equation is unique in H^s. This solution can be written in the form $f = 4\pi^2 G * u$, where Vladimirov (1971) (cf. Exercise 7.1)

$$G(x) = \exp(-2\pi q|x|)/4\pi|x|\in H^0$$

is a fundamental solution of the operator $-\Delta + (2\pi q)^2$.

Lemma 21.7 (On Continuity) *Let* $a\in S^m$. *Then, for any* $s\in\mathbb{R}$, *there exists a constant* $C>0$ *such that*

$$\|a(x,D)u\|_{s-m}\le C\|u\|_s\quad\forall u\in C_0^\infty(\mathbb{R}^n). \tag{21.15}$$

In other words, the operator $a(x,D)$, *which is defined by the formula*

$$Op(a(x,\xi))u(x) = \int e^{ix\xi}a(x,\xi)\widetilde{u}(\xi)d\xi,\quad \widetilde{u} = Fu, \tag{21.16}$$

with symbol $a\in S^m$, *and which is clearly defined on* C_0^∞, *extends to a continuous mapping from* $H^s(\mathbb{R}^n)$ *into* $H^{s-m}(\mathbb{R}^n)$.

Proof If $a(x,\xi) = a_1(\xi)$, then estimate (21.15) is clear. Hence it suffices to establish this estimate for $a \overset{(21.12)}{=} a_0$. Setting $A_0 v = a_0(x, D)v$, we note that

$$(\widetilde{A_0 v})(\xi) = \int \Big(\int e^{i(x,\eta-\xi)} a_0(x,\eta)\, dx \Big) \widetilde{v}(\eta) d\eta.$$

Taking into account inequalities (21.13) and Lemma 17.16, we find that

$$|(\widetilde{A_0 v})(\xi)| \le C_\alpha \int \langle \eta \rangle^m \langle \xi - \eta \rangle^{-|\alpha|} |\widetilde{v}(\eta)| d\eta, \quad |\alpha| \gg 1.$$

From the Peetre inequality (Peetre 1962)[29]

$$\langle \xi \rangle^s \le C_s \langle \eta \rangle^s \langle \xi - \eta \rangle^{|s|}, \tag{21.17}$$

which follows from the triangle inequality $|\xi| \le |\eta| + |\xi - \eta|$, we get

$$\langle \xi \rangle^{s-m} |(\widetilde{A_0 v})(\xi)| \le C_{\alpha,s} \int \langle \eta \rangle^s \langle \xi - \eta \rangle^{|s-m|-|\alpha|} |\widetilde{v}(\eta)| d\eta.$$

It remains to employ inequality (19.7). □

Definition 21.8 Let $a \in S^m$. The operator $Op(a(x,\xi))$ is called *elliptic* if (cf. (21.3)) there exist $M > 0$ and $C > 0$ such that

$$|a(x,\xi)| \ge C|\xi|^m \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad |\xi| \ge M. \tag{21.18}$$

P 21.9 Following the above proof of estimate (21.7) and using Lemma 21.10 (on composition), which is given below, prove the *a priori* estimate

$$\|u\|_s \le C(\|a(x,D)\|_{s-m} + \|u\|_{s-N}) \quad \forall u \in H^s, \quad C = C(s,N), \quad N \ge 1 \tag{21.19}$$

for the elliptic operator $Op(a(x,\xi))$ with symbol $a \in S^m$

Hint Taking into account inequality (21.18) and setting $R = Op\Big(\frac{\rho(\xi)}{a(x,\xi)}\Big)$, where $\rho \in C^\infty$, $\rho = 1$ for $|\xi| \ge M + 1$, $\rho = 0$ for $|\xi| \le M$, show that

$$R \cdot Op(a(x,\xi))u = u + Op(r(x,\xi))u, \quad \text{where } r \in S^{m-N}.$$

Lemma 21.10 (On Composition) *Let $a \in S^k$, $b \in S^m$. Then, for any $N \ge 1$, the generalized Newton–Leibniz formula holds*

$$a(x,D) \cdot Op(b(x,\xi)) = \sum_{|\alpha|<N} Op\big[(\partial_\xi^\alpha a(x,\xi))(D_x^\alpha b(x,\xi))\big]/\alpha! + T_N;$$

here $\|T_N v\|_{s+N-(k+m)} \le C\|v\|_s$ for any $v \in H^s$, and C does not depend on v.

For a proof, see, for example, Vishik and Eskin (1965), Kohn and Nirenberg (1965).

[29] See Remark 10 on p. 87.

Definition 21.11 An operator $T\colon C_0^\infty \to \mathcal{S}'$ is called a *smoothing* operator if, for any $N \geq 1$ and $s \in \mathbb{R}$, there exists a $C > 0$ such that $\|Tu\|_{s+N} \leq C\|u\|_s$ for any $u \in H^s$.

P 21.12 Consider a sequence of functions $a_j \in S^{m_j}$, where $j \in \mathbb{N}$, and $m_j \downarrow -\infty$ as $j \uparrow +\infty$. Then there exists a function $a \in S^{m_1}$ such that

$$(a - \sum_{j<N} a_j) \in S^{m_N} \quad \forall N > 1. \tag{21.20}$$

Hint Following the idea of the proof of Borel's theorem (see Theorem 15.2), define

$$a(x,\xi) = \sum_{j=1}^{\infty} \varphi(\xi/t_j) a_j(x,\xi),$$

where $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(\xi) = 0$ for $|\xi| \leq \frac{1}{2}$, $\varphi(\xi) = 1$ for $|\xi| \geq 1$, and choose t_j that converge to $+\infty$ as $j \to \infty$ so rapidly that the inequality

$$\left|\partial_\xi^\alpha D_x^\beta (\varphi(\xi/t_j) a_j(x,\xi))\right| \leq 2^{-j} \langle \xi \rangle^{m_{j-1}}$$

holds for $|x| \leq 1$ and $|\alpha| + |\beta| + 1 \leq j$.

For a solution, see, for example, Shubin (1987).

Definition 21.13 The operator $A\colon C_0^\infty \to \mathcal{S}'$ is called a *pseudo-differential operator of class L* if $A = Op(a(x,\xi)) + T$, where $a \in S^m$ for some $m \in \mathbb{R}$, and T is a smoothing operator. Any function $\sigma_A \in S^m$ such that $(\sigma_A - a) \in S^{-N}$ for any N is called a *symbol* of the operator $A \in L$.

P 21.14 By applying Lemma 21.10, show that the operator $A \in L$ has (cf. Exercise 16.22) the *pseudolocality property*; in other words, if $\varphi, \psi \in C_0^\infty$, $\psi = 1$ on $\operatorname{supp}\varphi$, then $\varphi A(1-\psi)$ is a smoothing operator.

Remark 21.15 The class L is invariant not only with respect to the composition operation but also with respect to a replacement of variables. The corresponding theorem, which will be given in the next section, is quite important. Initially (in 1964), it was implicitly announced in a series of notes by M.I. Vishik and G.I. Eskin (see, for example, Vishik and Eskin 1964) on boundary-value problems for elliptic pseudo-differential equations, the study of which was associated with a replacement of variables that locally straighten the boundary of the domain. The proof of this theorem was given in their paper Vishik and Eskin (1965). What is essential in this theorem is not only the formula for transformation of the symbol of a PsDO with replacement of the variables but also the fact that it implies the possibility of an invariant definition of a pseudo-differential operator, i.e., setting it in terms that are not related to any coordinates. Such invariant definition was given in the same year 1965 by Hörmander in Hörmander (1965). We will return to this definition of Hörmander in §26.

Theorem 21.16 (On Change of Variables) *Let $a \in S^m$. Then, in the coordinate system defined by a diffeomorphism (affine outside some ball) $\sigma\colon x \mapsto y = \sigma(x)$, for any $N \geq 1$, the operator $a(x, D_x)$ can be represented in the form*

$$\sum_{|\alpha|<N} Op\left[\varphi_\alpha(y,\eta)(\partial_\xi^\alpha a(x,\xi)\big|_{\xi={}^t\sigma'(x)\eta;\; x=\sigma^{-1}(y)}\right] + T_N, \tag{21.21}$$

where ${}^t\sigma'(x)$ *is the matrix transpose to* $\sigma'(x) = \frac{\partial\sigma}{\partial x}$, *and* $\varphi_\alpha(y,\eta)$ *is a polynomial in* η *of degree at most* $|\alpha|/2$ *given by the formula*

$$\varphi_\alpha(y,\eta) = \frac{1}{\alpha!}D_z^\alpha \exp\{\mathring{\imath}(\sigma(z)-\sigma(x)-\sigma'(x)(h-x),\eta)\}\big|_{z=x,\, x=\sigma^{-1}(y)}.$$

Moreover, $\|T_N v\|_{s+[N/2+1]-m} \le C\|v\|_s$ *for any* $v \in H^s$.

22 On Elliptic Problems

In §5, we considered (for some domain Ω) the simplest elliptic problem, viz., the *Dirichlet problem* for the Laplace equation. To this problem, one can reduce another important elliptic problem—the *problem with oblique derivative* in the disk $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ for the Laplace equation:

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial \lambda} = f \quad \text{on } \Gamma = \partial\Omega, \qquad f \in C^\infty(\Gamma). \tag{22.1}$$

Here $\frac{\partial}{\partial\lambda} = \left(a\frac{\partial}{\partial x} - b\frac{\partial u}{\partial y}\right)$ is the differentiation along the direction λ (which is possibly slanted with respect to the normal vector to the boundary of Γ). This direction depends on the smooth vector field

$$\sigma : \Gamma \ni s \mapsto \sigma(s) = (a(s), b(s)) \in \mathbb{R}^2, \quad a^2(s) + b^2(s) \neq 0 \quad \forall s \in \Gamma.$$

We identify a point s of the unit circle Γ with its polar angle $\varphi \in [0, 2\pi]$. If $\sigma(\varphi) = (\cos\varphi, -\sin\varphi)$, then $\lambda = \nu$ is the (outer) normal vector to Γ; if $\sigma(\varphi) = -(\sin\varphi, \cos\varphi)$, then $\lambda = \tau$ is the tangent vector to Γ. All these and other cases are important in applications However, our interest in problem (22.1) is primarily due to the fact that it vividly illustrates the problems related to general elliptic problems

It turns out (see Exercises 22.1–22.4) that the solvability of problem (22.1) depends on the *degree of the mapping* σ with respect to the origin, which is the integer number

$$N = \{\arg[a(2\pi) + ib(2\pi)] - \arg[a(0) + ib(0)]\}/(2\pi).$$

It is clear that N the number of signed rotations made by the point $\sigma(\varphi)$ around the origin when moving along the closed curve $\sigma : \mathbb{R}/(2\pi) \ni \varphi \mapsto \sigma(\varphi) = (a(\varphi), b(\varphi))$.

If $N < 0$, then for the solvability of problem (22.1) it is necessary and sufficient that the right-hand side f be "orthogonal" to some subspace of dimension $\beta_N = 2|N| - 1$. More precisely, there exist $(2|N| - 1)$ linearly independent functions $\Phi_j \in L^2(\Gamma)$ such that problem (22.1) is solvable if and only if

$$\int_\Gamma f\Phi_j d\Gamma = 0 \quad \forall j = 1, \ldots, \beta = 2|N| - 1.$$

Note that the dimension α_N of the space of solutions of the homogeneous problem is 1.

For $N \geq 0$, the dimension β_N of the subspace to which the right-hand side f is orthogonal is zero, i.e., problem (22.1) is always solvable. But this solvability is nonunique: the dimension α_N of the space of solutions of the homogeneous problem is $2N + 2$.

So, for problem (22.1) we have $\alpha_N - \beta_N = 2N + 2$ for any N. Regarding the numbers α_N and β_N, the following series of exercises will help to understand why they are equal to the values that were announced above.

P 22.1 Setting $U = u_x$, $V = -u_y$, show that the solution u of problem (22.1) gives the solution W of the following *Riemann–Hilbert problem*:[30] find an analytic function $W = U + iV$ in Ω, continuous in $\overline{\Omega}$, and satisfying the boundary condition $aU + bV = f$ for a, b and f which are continuous on Γ. Verify that the converse of this result is also true: the solution W of this Riemann–Hilbert problem defines, up to an additive constant, the solution u of problem (22.1).

P 22.2 (Continuation.) Verify that the continuous function $g(\varphi) = \arg[a(\varphi) + ib(\varphi)] - N\varphi$ is defined on $\Gamma = \{z = \exp(i\varphi)\}$. Construct the analytic function $p + iq$ on $\Omega = \{|z| < 1\}$ from the solution of the Dirichlet problem $\Delta q = 0$ in Ω, $q = g$ on Γ, and show that the function $c(z) = z^N \cdot \exp(p(x, y) + iq(x, y))$, which is analytic on Ω, where $z = x + iy$, satisfies on Γ the condition $c = \rho(a + ib)$, where $\rho = e^p/|a + ib| > 0$.

P 22.3 (Continuation.) Let $N \geq 0$, and let $\zeta = \xi + i\eta$ be an analytic function on Ω such that

$$\operatorname{Re}\zeta = \rho f/|c|^2 \quad \text{on } \Gamma. \tag{22.2}$$

Putting $U(x, y) + iV(x, y) = c(z)\zeta(z)$, verify that

$$\rho(aU + bV) = (\operatorname{Re} c)U + (\operatorname{Im} c)V = |c|^2 \operatorname{Re}(U + iV)/\zeta = \rho f \text{ on } \Gamma,$$

i.e., $(aU + bV) = f$ on Γ. Verify that for $N \geq 0$ the general solution of the Riemann–Hilbert problem can be written in the form $c(z)[\zeta(z) + W_0(z)]$, where $W_0 = 0$ for $|z| = 1$, and the function W_0 is analytic for $0 < |z| < 1$ and has a pole for $z = 0$ of multiplicity at most N. Using Theorem 5.18, show that

$$W_0(z) = i\mu_0 + \sum_{k=-N}^{-1} [(\lambda_k + i\mu_k)z^k - (\lambda_k - i\mu_k)z^{-k}], \quad \text{where } \lambda_k \in \mathbb{R},\ \mu_k \in \mathbb{R},$$

i.e., $W_0(z)$ is a linear combination of $2N + 1$ linearly independent functions.

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[30] Riemann in his famous thesis (see Narasimhan 1990) identified two different problems, which later became called the Riemann-Hilbert problem. The assumption about the solvability of one of these Riemann problems was formulated by D. Hilbert in 1900 as the 21st problem in his list of the so-called 23 Hilbert problems. After the long efforts of many mathematicians, the final point in the study of the 21st problem was put by A. A. Bolibruch in 1989 (see, for example, http://www.mccme.ru/free-books/globus/globus1.pdf/). Another problem of Riemann was formulated by him as follows: find in a domain an analytic function $W = U + iV$ satisfying on the boundary of the domain a relation of the form $F(U, V) = 0$. However, Riemann himself expressed only general considerations about the solvability of such a problem and the method of its solution. In reality, the problem of solvability was first investigated (with some inaccuracies) by D. Hilbert in the work of 1904 in the case of a simply connected domain and $F(U, V) = aU + bV = f$ with continuous a, b and f. Later, numerous important generalizations were obtained (see, for example, Bezrodnykh and Demidov 2011, Bezrodnykh 2017, 2018, Bezrodnykh and Vlasov 2016, Bezrodnykh and Vlasov 2021, Bezrodnykh 2022a and Bezrodnykh 2022b).

P 22.4 (Continuation.) Let $N < 0$. Verify that if $U + iV$ is a solution of the Riemann–Hilbert problem, then condition (22.2) is satisfied for the function $\zeta = (U + iV)/c$. Next, writing the function $\operatorname{Re}\zeta$, which is harmonic for $|z| < 1$, as the Poisson integral (5.7) and expanding into a Fourier series $\operatorname{Re}\zeta$ for $|z| = 1$, prove that $\zeta(z) = (\lambda_0/2 + ic) + \sum_{k=1}^{\infty}(\lambda_n - i\mu_n)z^n$ for $|z| < 1$, where $c \in \mathbb{R}$, and

$$\lambda_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\frac{\rho(\varphi)\cos(n\varphi)d\varphi}{a^2(\varphi)+b^2(\varphi)}, \quad \mu_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\frac{\rho(\varphi)\sin(n\varphi)d\varphi}{a^2(\varphi)+b^2(\varphi)}.$$

Based on this result and using the fact that the function ζ has for $z = 0$ a zero of multiplicity $|N| \geq 1$, show that for $N < 0$, there do not exist more than one solution of the Riemann–Hilbert problem, and besides, show that a necessary and sufficient condition for the solvability of the Riemann–Hilbert problem for $N < 0$ is the condition $\lambda_0 = \ldots = \lambda_{|N|-1} = \mu_1 = \ldots = \mu_{|N|-1} = 0$, i.e., the “orthogonality” of the function f to the $(2|N| - 1)$-dimensional space.

Remark 22.5 For a solution of Problems 22.1–22.4, see, for example, §24 in the textbook Godunov (1979).

The particular case of problem (22.1), where $\lambda = \nu$ is the normal vector to Γ, is called the *Neumann problem*[31] for the Laplace equation. In this problem $N = -1$, because $a(\varphi) + ib(\varphi) = \exp(-i\varphi)$. The Neumann problem is solvable if and only if $\int_\Gamma f\,d\Gamma = 0$; besides, the solution is defined up to an additive constant. In the fact, if $\int_\Gamma f\,d\Gamma = 0$, then the continuous function

$$g(s) = g(s_0) + \int_{s_0}^{s} f(\varphi)d\varphi$$

is defined on Γ. From the function g, we construct the solution v of the Dirichlet problem $\Delta v = 0$ in Ω, $v = g$ on Γ. The real part of the analytic function $u + iv$, i.e., the function u (which is defined up to an additive constant), is the solution of the Neumann problem under study, because $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \tau} = f$, where $\frac{\partial}{\partial \tau}$ is the differentiation along the tangent to Γ. Conversely, if u is a solution to the Neumann problem, then

[31] It should be borne in mind that *physically it is justified not to set the normal derivative at the points of the boundary, but to set the flow through the boundary, i.e., the functional on smooth functions, which can even be a generalized function*. This remark made by Mark Iosifovich Vishik (1921–2012), a major expert on the equations of mathematical physics (see http://www.jip.ru/2012/155-157-2012.pdf), about whom academician S. P. Novikov wrote in 2011 as an outstanding mathematician and one of his teachers, was formalized in the joint paper of together Vishik and Sobolev in the Reports of the USSR Academy of Sciences (Doklady: Mathematics) (Vishik and Sobolev 1956). Their brief note (only three pages long) served as the basis for a well-known three-volume monograph (Lions and Magenes 1968) by two correspondence students of Vishik. One of them, Jacques-Louis Lions (1928–2001), the President of the French Academy of Sciences from 1997 to 1998, repeatedly (along with Laurent Schwartz, Lars Hörmander, Louis Nirenberg and many other major mathematicians, including Academicians of the Russian Academy of Sciences I. M. Gelfand, V. E. Zakharov, A. M. Il’in, S. V. Konyagin, V. P. Maslov, S. P. Novikov, and A. T. Fomenko) gave talks at the M. I. Vishik’s seminar at Moscow State University. Vishik’s second correspondence student was Enrico Magenes (1923–2010), president of the Italian Mathematical Union from 1973 to 1975. It was he who invited the author of these lines to the Free Boundary Problems seminar in Pavia for the report (Demidov 1980) who told that his villa in Pavia was built on the money received for the monograph (Lions and Magenes 1972-1973), which developed the ideas outlined in the note by Vishik and Sobolev.

the "orthogonality" condition $\int_\Gamma f\,d\Gamma = 0$ is satisfied, which can be readily seen[32] from the Gauss formula (7.6). Next, from the first Green formula (7.4) (under the additional condition $u \in C^2(\overline{\Omega})$) it follows that[33] if u_1, u_2 are two solutions of the Neumann problem, then $u = u_1 - u_2 = \text{const}$, because

$$\int_\Omega (u_x^2 + u_y^2)\,dx\,dy = 0, \quad \text{i.e.,} \quad u_x = u_y \equiv 0 \Longleftrightarrow u = \text{const}.$$

22.6 Consider now the general elliptic differential equation

$$a(x, D)u \equiv \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u = f \tag{22.3}$$

in a domain $\Omega \Subset \mathbb{R}^n$ with smooth boundary Γ. The ellipticity means that (cf. formula (21.3))

$$\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ne 0 \quad \text{for } x \in \overline{\Omega} \quad \text{and} \quad |\xi| \ne 0. \tag{22.4}$$

The example of the problem with oblique derivative shows that it makes sense to ask the following question. Find how many boundary conditions

$$b_j(x, D)u\big|_\Gamma \equiv \sum_{|\beta| \le m_j} b_{j\beta}(x) D^\beta u\big|_\Gamma = g_j \text{ on } \Gamma, \quad j = 1, \ldots, \mu, \tag{22.5}$$

should be specified (i.e., what is the number μ) and what must be the boundary operators[34] b_j to satisfy the following two conditions:

(1) the solution u of problem (22.3)–(22.5) is defined uniquely up to some finite-dimensional subspace $X_0 \subset H^s(\Omega)$;

(2) problem (22.3)–(22.5) is solvable for any right-hand side

$$h = (f, g_1, \ldots, g_\mu) \in H^{s,M} \stackrel{\text{def}}{=} H^{s-m}(\Omega) \times \prod_{j=1}^{\mu} H^{s-m_j-1/2}(\Gamma), \tag{22.6}$$

which is possibly orthogonal to some finite-dimensional subspace $Y_0 \subset H^{s,M}$? Here $H^{s,M}$ is a Banach space of functions $h = (f, g_1, \ldots, g_\mu)$ with the norm

$$\|h\|_{s,M} = \|f\|_{s-m} + \sum_{j=1}^{\mu} \|g_j\|'_{s-m_j-1/2}.$$

[32] Under the additional condition $u \in C^2(\overline{\Omega})$. This condition holds for $f \in C^\infty(\Gamma)$ by the *a priori* estimates for elliptic problems (see below).

[33] However, both results are true (see, for example, Petrovsky 1967, §28 and §35) also without the additional assumption that $u \in C^2(\overline{\Omega})$.

[34] The example of the problem $\Delta u = f$ in Ω, $\Delta u = g$ on Γ, shows that one cannot specify arbitrary boundary operators (22.5).

Definition 22.7 A *boundary-value problem* (22.3)–(22.5) is called *elliptic* if the boundary-value (in other words, boundary) conditions (22.5) for the elliptic equation (22.3) are such that conditions 1 and 2 from the previous section are satisfied.

One of the most important goals of this section is to show that Definition 22.7, which characterizes the functional properties of the operator $\mathcal{A}$, is equivalent (at least for $n \geq 3$) to the Definition 22.23 (see below) for the ellipticity of the problem (22.3)–(22.5), which is formulated in algebraic terms for the highest symbols of the operator $\mathcal{A}$.

Considering problem (22.3)–(22.5) in the form of the equation $\mathcal{A}u = h$ for the operator

$$\mathcal{A}: H^s(\Omega) \ni u \mapsto \mathcal{A}u \in H^{s,M}, \tag{22.7}$$

where $\mathcal{A}u \equiv (a(x,D)u, \gamma b_1(x,D)u, \dots, \gamma b_\mu(x,D)u)$, and γ is the boundary-value operator on Γ (see (20.5)), we will use the following standard notation. If X and Y are linear spaces and A is a linear operator from X into Y, then

$$\operatorname{Ker} A \overset{\text{def}}{=} \{x \in X: Ax = 0\}, \quad \operatorname{Coker} A \overset{\text{def}}{=} Y/\operatorname{Im} A,$$

where $\operatorname{Im} A = \{y \in Y: y = Ax,\ x \in X\}$ is the range of the operator A, and $Y/\operatorname{Im} A$ is the quotient space of Y modulo $\operatorname{Im} A$, i.e., the linear space of cosets modulo $\operatorname{Im} A$ (see Kolmogorov and Fomin 1980). Recall that the linear spaces $\operatorname{Ker} A$ and $\operatorname{Coker} A$ are called, respectively, the *kernel* and the *cokernel* of the operator A. If X and Y are Banach spaces, then by $L(X,Y)$ we denote the space of continuous linear operators from X into Y.

The next lemma has important applications.

Lemma 22.8 *Let* $A \in L(X,Y)$*, and let* $\dim \operatorname{Coker} A < \infty$*. Then the set* $\operatorname{Im} A$ *is closed in* Y*.*

Explanation Consider an example. Let A be the embedding operator of $X = C^1[0,1]$ into $Y = C[0,1]$. It is clear that $\operatorname{Im} A \neq Y = \overline{\operatorname{Im} A}$. From Lemma 22.8 it follows that $\dim \operatorname{Coker} A = \infty$. This can readily be understood directly. Indeed, let $\varphi_\alpha(t) = |t-\alpha|$, where $\alpha \in]0,1[$, $t \in [0,1]$. We have $\varphi_\alpha \notin \operatorname{Im} A$, $\varphi_\alpha - \varphi_\beta \notin \operatorname{Im} A$ for $\alpha \neq \beta$, i.e., the elements φ_α are representatives of linear independent vectors in $Y/\operatorname{Im} A$. So, $\dim(Y/\operatorname{Im} A) = \infty$.

Proof By the condition, $\dim\ Y/\operatorname{Im} A < \infty$. Hence $Y = \operatorname{Im} A \dot{+} L$ (the direct sum of linear spaces), where $\dim L < \infty$, and hence L, as equipped with the norm $\|\cdot\|_L$, is a Banach space. Consider the operator A_1 acting from the Banach space[35] $X_1 = X/\operatorname{Ker} A \times L$ into the Banach space $Y = \operatorname{Im} A \dot{+} L$ by the formula $A_1(\{x\}, l) = Ax + l$. The inverse operator of A_1 exists, because the kernel of A_1 is zero. It is easily checked that the operator A_1 is continuous. The operator A is a surjection. By the

[35] The norm in the direct product $X/\operatorname{Ker} A \times L$ is defined by $\|(\{x\}, l)\|_{X_1} = \|\{x\}\|_{X/\operatorname{Ker} A} + \|l\|_L$, where $\|\{x\}\|_{X/\operatorname{Ker} A} = \inf_{a \in \operatorname{Ker} A} \|x + a\|_X$ is the norm on the quotient space $X/\operatorname{Ker} A$ of cosets of the linear space X over $\operatorname{Ker} A$. The quotient space is complete in this norm (see, for example, Kolmogorov and Fomin 1980).

Banach theorem[36] the operator A_1^{-1} is continuous. It remains to note that $\operatorname{Im} A$ is the preimage of the closed set $X/\operatorname{Ker} A \times \{0\} \subset X_1$ under the continuous mapping $A_1^{-1}\colon Y = \operatorname{Im} A \dot{+} L \to X/\operatorname{Ker} A \times L$, and hence $\operatorname{Im} A$ is closed. □

In the case of the operator equation $\mathcal{A}u \overset{(22.7)}{=} h \in Y$, $Y = H^{s,M}$ is a Hilbert space. In this problem, the condition $\operatorname{Im} \mathcal{A} = \overline{\operatorname{Im} \mathcal{A}}$ is equivalent to saying that $\operatorname{Coker} \mathcal{A}$ is isomorphic to the orthogonal complement $\operatorname{Im} \mathcal{A}$ in $H^{s,M}$.

Definition 22.9 An operator $A \in L(X, Y)$ is called a *Fredholm operator*[37] if the dimensions of its kernel and cokernel are equal. The operator $A \in L(X, Y)$ is called a *Noetherian operator*[38] if

$$\alpha = \dim \operatorname{Ker} A < \infty, \quad \beta = \dim \operatorname{Coker} A < \infty. \tag{22.8}$$

The number $\operatorname{ind} A = \alpha - \beta \in \mathbb{Z}$ is called the *index* of the operator A. Condition (22.8) is frequently written in the short form: $\operatorname{ind} A < \infty$.

According to Definition 22.7, the operator

$$\mathcal{A}\colon H^s(\Omega) \ni u \mapsto \mathcal{A}u \in H^{s,M}$$

of the elliptic problem (22.3)–(22.5), is a Noetherian operator. The following result is one of the most important theorems in the theory of Noetherian (and hence elliptic) operators.

Theorem 22.10 (On the Stability of Index) *If a family of Noetherian operators* $A_t\colon X \to Y$ *is continuous with respect to* $t \in [0, 1]$, *i.e.,*

[36] Let X and Y be Banach spaces, $A \in L(X, Y)$. If $\operatorname{Ker} A = 0$, then $A^{-1}\colon \operatorname{Im} A \to X$ exists. However, the operator A^{-1} may fail to be continuous (for example, see the comment to Lemma 22.8.). The Banach theorem (see Kolmogorov and Fomin 1980) asserts that A^{-1} is continuous if $\operatorname{Im} A = Y$.

[37] Erik Ivar Fredholm (1866–1927) was a Swedish mathematician. His paper "Sur une classe d'équations fonctionnelles" (1903) in Acta Mathematica was one of the most important milestones in the creation of the operator theory. David Hilbert developed the concept of Hilbert space, in particular in connection with the study of Fredholm integral equations.

[38] In honor of a famous German mathematician Fritz Alexander Ernst Noether (1884–1941). Until 1933, we was a professor of mathematics in Wrocław (Breslau) University of Science and Technology. In his paper "Uber eine Klasse singularer Integralgleichungen" (Math. Ann. 1921m vol. 82, pp. 42–63) he gave an example of a linear one-dimensional singular integral operator with different (unlike Fredholm operators) dimensions of the kernel and cokernel. After Hitler came to power, Fritz and his distinguished sister Amalie Emmy Noether (1882–1935), best known for her contributions to abstract algebra and theoretical physics, emigrated: she to the United States, where she became a teacher at a women's college in Pennsylvania, while he, through the mediation of the Society for the Assistance of German Scientists, was invited to Tomsk University (USSR), where as a professor he worked as a head of the department of mathematical physics and theoretical mechanics. His articles were published in Soviet scientific journals. In September 1935, Fritz came to Moscow as an honorary guest of a special session of the Moscow Mathematical Society dedicated to the memory of his great sister, who died in the spring of the same year. But in November 1937, Fritz was accused of espionage and sentenced to 25 years in prison, and in September 1941, he was shot. Rehabilitated in 1988 by the Supreme Court of the USSR.

$$\|A_t u - A_\tau u\|_Y \le C(t,\tau)\|u\|_X,$$

where $C(t,\tau) \to 0$ *as* $t \to \tau$, *then* $\operatorname{ind} A_0 = \operatorname{ind} A_1$.

This first proof of this theorem was given by F. Atkinson[39] in Atkinson (1951) (for extensions, see also Gokhberg and Krein 1957). In the proof, which is based on Atkinson's theorem (Atkinson 1951) on the index of the product of two Noetherian operators A_1 and A_2, $\operatorname{ind} A_0 A_1 = \operatorname{ind} A_0 + \operatorname{ind} A_1$, one constructs the left and right *regularizers* of a Noetherian operator $A\colon X \to Y$, i.e., operators R_l and R_r, for which $R_l A = I_X + T$ and $A R_r = I_Y + S$, where $I_X + T\colon X \to X$ and $I_Y + S\colon Y \to Y$ are completely continuous operators, and which, therefore, have zero index by the well-known Fredholm alternative (Kolmogorov and Fomin 1980). Regularizers of this kind will be constructed below in Lemma 22.15 for the elliptic boundary-value problem (22.3)–(22.5). For this purpose, we shall also need the result of the following exercise.

P 22.11 Verify that, for any (!) $s \in \mathbb{R}$, there exists a continuous *extension operator* $\Phi\colon H^s(\Omega) \to H^s(\mathbb{R}^n)$. Verify that operator (22.7) is continuous for $s > \max_j(m_j) + 1/2$.
Hint For $\Omega = \mathbb{R}^n_+$, as Φ we can take the operator

$$\Phi f = Op(\langle\xi_-\rangle^{-s})\theta_+ Op(\langle\xi_-\rangle^{s}) Lf, \quad Op\big(a(\xi)\big) \overset{(21.16)}{=} \mathbf{F}^{-1} a(\xi)\mathbf{F}, \tag{22.9}$$

where $\langle\xi_-\rangle = \xi_- + 1$, $\xi_- = -i\xi_n + |\xi'|$, $L\colon H^s(\Omega) \to H^s(\mathbb{R}^n)$ is any extension operator, and θ_+ is the characteristic function of $\mathbb{R}^n_+$. In view of Theorem 18.5 (Paley–Wiener), the function $\theta_+ Op(\langle\xi_-\rangle^s)Lf$ does not depend on L, because the function $\langle\xi_-\rangle^s(\widetilde{L_1 f} - \widetilde{L_2 f})$ extends analytically with respect to ξ_n into $\mathbb{C}_+$. It follows that $Op(\langle\xi_-\rangle^s)(L_1 f - L_2 f) = \theta_- g \in L^2$. Hence

$$\|\Phi f\|_{s,\mathbb{R}^n} \le C \inf_L \|\theta_+ Op(\langle\xi_-\rangle^s)Lf\|_{0,\mathbb{R}^n} \le C \inf_L \|Lf\|_{s,\mathbb{R}^n} = C\|f\|_{s,\mathbb{R}^n_+}.$$

If $\overline{\Omega}$ is a compact set in $\mathbb{R}^n$, then

$$\Phi f = \varphi \cdot f + \sum_{k=0}^{K} \psi_k \cdot \Phi_k(\varphi_k \cdot f),$$

where $\sum_{k=0}^K \varphi_k \equiv 1$ in Ω, $\varphi_k \in C_0^\infty(\Omega_k)$, $\bigcup_{k=1}^K \Omega_k$ is a covering of the domain Ω such that

$$\bigcup_{k=1}^K \Omega_k \supset \Gamma = \partial\Omega; \quad \psi_k \in C_0^\infty(\Omega_k), \quad \psi_k\varphi_k = \varphi_k,$$

and Φ_k is the operator defined by (22.9) in the local coordinates, which "unbend" $\Gamma = \partial\Omega$.

[39] Frederick Valentine Atkinson (1916–2002) was educated at Oxford, and in 1939 he defended his doctoral thesis on average values of the Riemann zeta function, carried out under the supervision of Edward Titchmarsh. Then he spent three years in India, deciphering Japanese codes. He was fluent in Latin, Ancient Greek, Urdu, German, Hungarian, and Russian. He returned to Oxford in 1946, and from 1948 to 1955 was Professor of Mathematics at University College, Nigeria. During his stay in Nigeria, in September 1948, we presented his paper Atkinson (1951) to the editorial office of the Sbornik:Mathematics, which was published in 1951. In this paper, he modestly wrote that he was actually studying from an abstract point of view the known properties of singular integral equations. From 1955 until the end of his life, F. Atkinson lived in Canada. From 1989 to 1991, he was president of the Canadian Mathematical Society.

Lemma 22.12 *If* $\operatorname{ind} \mathcal{A} < \infty$*, then the so-called elliptic a priori estimate holds:*

$$\|u\|_s \le C(\|\mathcal{A}u\|_{s,M} + \|u\|_{s-1}) \quad \forall u \in H^s(\Omega); \tag{22.10}$$

here C does not depend on u.

Proof Let X_1 be the orthogonal complement of $X_0 = \operatorname{Ker} \mathcal{A}$ in H^s. We have $\mathcal{A} \in L(X_1, Y_1)$, where $Y_1 = \operatorname{Im} \mathcal{A}$, and $\mathcal{A}$ is the isomorphism of the spaces X_1 and Y_1. The space Y_1 is closed (Lemma 22.8), and hence is a Banach space. By the Banach theorem, $\mathcal{A}^{-1} \in L(Y_1, X_1)$. Let p be the orthogonal projection of X onto X_0. We have

$$\begin{aligned} \|u\|_s &\le \|pu\|_s + \|(1-p)u\|_s = \|pu\|_s + \|\mathcal{A}^{-1}\mathcal{A}(1-p)u\|_s \le \\ &\le \|pu\|_s + C_1\|\mathcal{A}(1-p)u\|_{s,M} \le \|pu\|_s + C_1\|\mathcal{A}u\|_{s,M} + C_2\|pu\|_s. \end{aligned}$$

It remains to observe that $\|pu\|_s \le C\|u\|_{s-1}$. But this is indeed so, because $pu \in X_0$, $\dim X_0 < \infty$, and hence $\|pu\|_s \le C\|pu\|_{s-1}$ (since the continuous function $\|v\|_s$ is bounded on the finite-dimensional sphere $\|v\|_{s-1} = 1$, $v \in X_0$). □

Lemma 22.13 *Inequality* (22.10) *implies that* $\dim \operatorname{Ker} \mathcal{A} < \infty$.

Proof If $\dim \operatorname{Ker} \mathcal{A} = \infty$, then $X_0 = \operatorname{Ker} \mathcal{A}$ possesses an orthonormal system of vectors $\{u_j\}_{j=1}^{\infty}$. We have $\|u_k - u_m\|_s^2 = 2$. From (22.10) we get $\|u_k - u_m\|_s = \sqrt{2} \le C\|u_k - u_m\|_{s-1}$, because $\mathcal{A}(u_k - u_m) = 0$. Hence $\|u_k - u_m\|_{s-1} \ge \sqrt{2}/C$. Therefore, the sequence $\{u_j\}$, which is bounded in $H^s(\Omega)$, does not have a subsequence converging in $H^{s-1}(\Omega)$. But this contradicts the compactness of the embedding of $H^s(\Omega)$ into $H^{s-1}(\Omega)$ (see Theorem 20.15). □

Remark 22.14 Lemmas 22.12 and 22.13 reveal the role of the *a priori* estimate (22.10), which will be proved below in Theorem 22.26. The way towards its proof is suggested by the proof of the *a priori* estimate (21.19) in $\mathbb{R}^n$ (see the hint to Exercise 21.9), and also following lemma.

Lemma 22.15 *Let* $R \in L(H^{s,M}, H^s)$*, and let*

$$R\,\mathcal{A}u = u + Tu, \qquad \|Tu\|_{s+1} \le C\|u\|_s \quad \textit{and} \tag{22.11}$$

$$\mathcal{A}\,Rh = h + Sh, \quad \|Sh\|_{s+1,M} \le C\|h\|_{s,M}. \tag{22.12}$$

Then $\operatorname{ind} \mathcal{A} < \infty$.

Proof By the assumption, $R \in L(H^{s,M}, H^s)$. Hence (22.11) implies inequality (22.10), and therefore, $\dim \operatorname{Ker} \mathcal{A} < \infty$. Next, $S\colon H^{s,M} \to H^{s,M}$ is a *compact* (or *completely continuous*) operator (Kolmogorov and Fomin 1980), i.e., S maps any bounded subset of $H^{s,M}$ into a compact set. This follows from relations (22.12) and since the embedding of $H^{s+1,M}$ into $H^{s,M}$ is compact (Theorem 20.15). Now the Fredholm theorem (see Shilov 2016, Kolmogorov and Fomin 1980) implies that $\dim \operatorname{Coker}(1 + S) < \infty$. And since $\operatorname{Im} \mathcal{A} \supset \operatorname{Im}(1 + S)$, we finally have $\dim \operatorname{Coker} \mathcal{A} < \infty$. □

An operator R satisfying conditions (22.11) and (22.12) is known as a *regularizator* of the operator $\mathcal{A}$.

Definition 22.16 Let $\Gamma = \partial\Omega$, where $\Omega \Subset \mathbb{R}^{n+1}$. A pseudo-differential operator[40] $A\colon H^s(\Gamma) \to H^{s-m}(\Gamma)$ of class L^m on a closed manifold Γ is called *elliptic* if its symbol a satisfies (cf. inequality (21.3)) the condition

$$|a(x,\xi)| \geq C|\xi|^m \quad \text{for } x \in \Gamma \quad \text{and} \quad |\xi| \gg 1. \tag{22.13}$$

P 22.17 Verify that ind $A < \infty$.
Hint Let $\sum \varphi_k \equiv 1$ be a *partition of unity* subordinate to a finite covering $\cup\Gamma_k = \Gamma$, and $\psi_k \in C_0^\infty(\Gamma_k)$, $\psi_k\varphi_k = \varphi_k$. Show (cf. the hint to Exercise 21.9) that the operator

$$Rf = \sum \psi_k Op(\rho_k(\xi)/a_k(x,\xi))\varphi_k f, \quad f \in H^{s-m}(\Gamma), \tag{22.14}$$

where $\rho \in C^\infty(\mathbb{R}^n)$, $\rho = 1$ for $|\xi| \geq M+1$ and $\rho = 0$ for $|\xi| \leq M$, is a regularizator for A.

22.18 Let us continue the study of the boundary-value problem (22.3)–(22.5). In what follows, it is assumed that the leading coefficients of the operator $a(x, D)$ are real if $\dim\Omega = 2$.

Lemma 22.19 *The principal symbol $a_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ of the operator $a(x, D)$ always admits a factorization*[41] *i.e. (see* Vishik and Eskin 1965*), the function*

$$a_m(y,\eta) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\Big|_{\xi={}^t\sigma'(x)\eta;\ x=\sigma^{-1}(y)}, \tag{22.15}$$

where σ is defined in Theorem 21.16, can be written in the form

$$a_m(y,\eta) = a_+(y,\eta)\,a_-(x,\eta), \quad \eta = (\eta', \eta_n) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{22.16}$$

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[40] Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ be the coordinate representation of a linear functional v on the tangent space to Γ at a point $p \in \Gamma$ with local coordinates $x = (x_1, \ldots, x_n)$. The functional (vector) v is called a *cotangent* functional (vector). The set of all such vectors, which is denoted by $T_p^*\Gamma$, is isomorphic to $\mathbb{R}^n$. The value of v on the tangent vector $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is given by the formula $(\xi, \nabla_x) = \xi_1\partial/\partial x_1 + \ldots + \xi_n\partial/\partial x_n$. If $y = \sigma(x)$ is a different local coordinate system of the same point $p \in \Gamma$ and if $\eta = (\eta_1, \ldots, \eta_n)$ is the corresponding coordinate representation of the cotangent vector v, then from the equality $(\xi, \nabla_x) = (\eta, \nabla_y)$ we have $\xi = {}^t\sigma'(x)\eta$, where ${}^t\sigma'(x)$ is defined in Theorem 21.16. The set $\cup_{p\in\Gamma} T_p^*\Gamma$ is equipped with the natural structure of a smooth manifold. This manifold is called the *cotangent bundle*. Let a function $a \in C^\infty(T_p^*\Gamma)$ be such that, for points from $\Gamma_k \subset \Gamma$ with local coordinates x, the function a coincides with some function $a_k \in S^m$. Let $\sum \varphi_k \equiv 1$ be a partition of unity subordinate to the cover $\cup\Gamma_k = \Gamma$, and let $\psi_k \in C_0^\infty(\Gamma_k)$, $\psi_k\varphi_k = \varphi_k$. From Theorem 21.16 on the change of variables it follows that the formula $A\colon H^s(\Gamma) \ni u \mapsto Au = \sum \varphi_k Op(a_k(x,\xi))\psi_k u \in H^{s-m}(\Gamma)$ defines uniquely, up to an operator $T \in L(H^s(\Gamma), H^{s-m+1}(\Gamma))$, a continuous linear operator, which is called the *pseudo-differential operator* of class L^m with *symbol* a.

[41] The equation in convolutions on the half-axis $\int_0^\infty K(x, x-y)u(y)\,dy = f(x)$, which was examined by N. Wiener and E. Hopf (see Wiener and Hopf 1931) in their study of problem of radiation equilibrium inside stars, was successfully solved by them using their idea of factorization of the symbol $\mathbf{F}_{z\to\xi}K(x,z)$ of the integral operator.

Here the function $a_\pm$, as well as the function $a_\pm^{-1}$, is continuous for $\eta \neq 0$, and in addition, for any $\eta' \neq 0$ it extends analytically with respect to η_n in the complex half-plane $\mathbb{C}_\mp$. Moreover,

$$a_\pm(y, t\eta) = t^\mu a_\pm(y, \eta) \quad \textit{for } \ t > 0, \quad (\eta', \eta_n) \in \mathbb{R}^{n-1} \times \mathbb{C}_\mp,$$

where the number μ is integer,[42] *and $m = 2\mu$.*

Proof If the coefficients $a_\alpha(x)$ for $|\alpha| = m$ are real, then, for $\eta' \neq 0$, the equation $a_m(y, \eta) = 0$ has only $m = 2\mu$ complex conjugate roots $\eta_n = \pm i\lambda_k(y, \eta') \in \mathbb{C}_\pm$ for η_n, and

$$a_\pm(y, \eta) = c_\pm(y) \prod_{k=1}^{\mu} (\pm\eta_n - i\lambda_k(y, \eta')), \quad c_\pm(x) \neq 0. \tag{22.17}$$

For $n \geq 3$, (22.17) always holds. Indeed, the function $a_m(u, \eta)$ is homogeneous with respect to η, and hence, for $\eta_n \neq 0$, to each root $\eta_n = \pm i\lambda(y, \eta') \in \mathbb{C}_\pm$ of the equation $a_m(u, \eta) = 0$, where $\eta = (\eta', \eta_n)$, there corresponds the root $\eta_n = \mp i\lambda(y, -\eta') \in \mathbb{C}_\mp$. It remains to note that the function $\lambda(y, \eta')$ is continuous with respect to $\eta' \neq 0$, and hence, since the sphere $|\eta'| = 1$ is connected for $n \geq 3$ and in view of the condition $\eta_n \neq 0$, we have $\mp\lambda(y, -\eta') \in \mathbb{C}_\mp$ for any $\eta' \neq 0$. □

Remark 22.20 It is clear that the symbol $|\eta|^2$ of the Laplace operator admits the factorization $|\eta|^2 = \eta_+\eta_-$, where $\eta_\pm = \pm i\eta_n + |\eta'|$. However, the symbol of the operator $(\partial/\partial y_2 + i\partial/\partial y_1)^m$ cannot be factored, since $(\eta_n + i\eta')^{-m}$ extends analytically with respect to η_n into $\mathbb{C}_+$ (into $\mathbb{C}_-$) only for $\eta' > 0$ $(\eta' < 0)$. This example explains the necessity of the constraint formulated in §22.18.

22.21 Let us formulate the *Shapiro–Lopatinskii condition*,[43] which is also known as the *complementing condition* (Agmon et al. 1959), on symbols of the boundary operators $b_j(x, D)$. We fix a point $x_0 \in \Gamma$ and single out the leading terms of the symbols of the operators $a(x_0, D)$ and $b_j(x_0, D)$, which are written in the coordinates $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, which locally "unbend" Γ. This means that near x_0 the boundary of Γ is given by the equation $y_n = 0$, where y_n is the inner normal vector to Γ. Taking into account that

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[42] We shall see below that the number μ, which is equal to the degree of homogeneity of the function $a_+(y, \eta)$ with respect to η_n, and which is called the *index of factorization* of the symbol a_m, is not accidentally indicated by the same letter as the desired number of boundary operators $b_j(x, D)$ in problem (22.3)–(22.5).

[43] In 1945/46, I. M. Gelfand at his seminar at Moscow State University formulated the problem of finding well posed boundary-value problems, for example, for linear elliptic differential equations. A couple of years later, the editorial office of the journal "Sbornik: Mathematics" received the paper Shapiro (1951) by his wife, Zorya Yakovlevna Shapiro (1914–2013), which answered this problem in algebraic terms for some elliptic linear systems in the case of the Dirichlet problem. Conditions for the coefficients of a linear elliptic system and the coefficients of the boundary-value operators, which were sufficient (and, as it later turned out, necessary) for reducing the boundary-value problem to a system of regular integral equations of the Fredholm type, were found in the papers of 1952–1953 (see Lopatinskii 1953) written by Yaroslav Borisovich Lopatinskii (1906–1981), who was elected a corresponding member in 1951, and in 1965, an academician of the Academy of Sciences of the Ukrainian SSR.

$$a_m(x_0,\eta) \overset{(22.16)}{=} a_+(x_0,\eta)a_-(x_0,\eta),$$

we consider the polynomials

$$\eta_n \mapsto \sum_{k=1}^{\mu} b_{jk}(\eta')\eta_n^k \equiv b_{m_j}(x_0,\eta) \quad \text{mod } a_+(x_0,\eta), \tag{22.18}$$

i.e., the remainders on division of the polynomials

$$\eta_n \mapsto b_{m_j}(x_0,\eta) = \sum_{|\beta|=m_j} b_{j\beta}(x_0)\xi^\beta\big|_{\xi={}^t\sigma'(x_0)\eta}, \quad j=1,\ldots,\frac{m}{2},$$

by the polynomial $\eta_n \mapsto a_+(x_0,\eta)$. The Shapiro–Lopatinskii condition means that the polynomials (22.18) are linearly independent, i.e.,

$$\det(b_{jk}(x,\eta')) \neq 0 \quad \forall x \in \Gamma, \quad \forall \eta' \neq 0. \tag{22.19}$$

In other words, the principal symbols $b_{m_j}(x,\eta)$ of the boundary operators, which are considered as polynomials of η_n, are linearly independent modulo the polynomial $\eta_n \mapsto a_+(x_0,\eta)$.

Remark 22.22 In the case of a differential operator $a(x,D)$ or a pseudo-differential operator $a(x,D)$ with rational symbol, for example, as in Example 21.6, we have

$$a_+(\eta',\eta_n) = (-1)^\mu a_-(-\eta',-\eta_n).$$

Hence the function $a_+(x_0,\eta)$ in Condition (22.19) can be replaced by $a_-(x_0,\eta)$. By the same reason, in these cases it is unessential whether y_n is the inward or outward normal vector to Γ.

According to Theorem 22.26 (see below), Definition 22.7 of ellipticity of problem (22.3)–(22.5), which was formulated in terms of functional properties of an operator $\mathcal{A}$, is equivalent to the following definition, which is given in algebraic terms and related to higher-order symbols of the operator $\mathcal{A}$.

Definition 22.23 Problem (22.3)–(22.5) and the corresponding operator $\mathcal{A}$ are called *elliptic* if Conditions (22.4) and (22.19) are satisfied.

Example 22.24 Let $a(x,D)$ be an elliptic operator of order $m=2\mu$. Let $B_j(x,D) = \partial^{j-1}/\partial\nu^{j-1} + \ldots$, $j=1,\ldots,\mu$, where ν is the normal vector to Γ, and dots denote an operator of order $j-1$. Then $\det(b_{jk}(x,\eta')) = 1$. So, in this case, problem (22.3)–(22.5) is elliptic.

P 22.25 Let λ be a smooth vector field on $\Gamma = \partial\Omega$, where $\overline{\Omega}$ is a compact set in $\mathbb{R}^n$. Verify that the *Poincaré problem*

$$a(x,D)u \equiv \sum_{|\alpha|\le 2} a_\alpha(x)D^\alpha u = f \quad \text{in } \Omega, \quad \partial u/\partial\lambda + b(x)u = g \quad \text{on } \Gamma \tag{22.20}$$

for an elliptic operator$a(x, D)$ is elliptic for $n \geq 3$ if and only if the field λ is not tangent to Γ at any point of the curve Γ . Verify also that in the case $n = 2$ problem (22.20) is elliptic (under Condition (22.13)) for any nondegenerate field λ.

Theorem 22.26 *Assume that the operator* $\mathcal{A}\colon H^s(\Omega) \to H^{s,M}(\Omega)$ *corresponding to problems* (22.3)–(22.5), *i.e., to the problem*

$$a(x, D)u \equiv \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha u = f \quad in\ \Omega \Subset \mathbb{R}^n,$$

$$b_j(x, D)u\big|_\Gamma \equiv \sum_{|\beta|\leq m_j} b_{j\beta}(x)D^\beta u\big|_\Gamma = g_j \quad on\ \Gamma, \quad j = 1, \ldots, \mu = m/2,$$

is elliptic in the sense of Definition 22.23. *Let* $s > \max_j(m_j) + 1/2$. *Then* $\operatorname{ind}\mathcal{A} < \infty$ *and*

$$\|u\|_s \leq C\Big(\|a(x, D)u\|_{s-m} + \sum_{j=1}^{\mu} \|b_j(x, D)u\big|_\Gamma\|'_{s-m_j-1/2} + \|u\|_{s-1}\Big). \qquad (22.21)$$

Proof We give only a sketch of the proof (for details, see the paper Vishik and Eskin 1965 and the books Agmon et al. 1959, Eskin 1973). Using the partition of unity (as has been suggested in hints to to Exercises 22.11 and 22.17) and taking into account Exercise 21.9, we can reduce the problem of construction of the regularizer R for the operator $\mathcal{A}$ to the case when $\Omega = \mathbb{R}^n_+$, and the symbols $a(x, \xi)$ and $b_j(x, \xi)$ are independent of x. In this case, we define the operator $R\colon H^{s,M} \to H^s$ by the formula

$$RF = P_+Op(r_+/a_+)\theta_+Op(r_-/a_-)Lf + \sum_{j=1}^{\mu} P_+Op(c_j)(g_j - f_j). \qquad (22.22)$$

Here $P_+\colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n_+)$ is the restriction operator to $\mathbb{R}^n_+$;

$$L\colon H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$$

denotes some (any) extension operator. Next, $r_\pm = \xi_\pm^\mu/\langle\xi_\pm\rangle^\mu$. These functions "remove" the singularities of the symbols $1/a_\pm$ at the point $\xi = 0$, since $\xi_\pm = \pm i\xi_n + |\xi'|$, and $\langle\xi_\pm\rangle = \xi_\pm + 1$. Note that (in contrast to the similar function ρ in (21.4)), the functions $r_\pm$ extend analytically with respect to $\xi_n \in \mathbb{C}_\mp$. Finally,

$$c_j(\xi) = \sum_{k=1}^{\mu} c_{jk}(\xi')(\xi_n^{k-1}/a_+(\xi)),$$

where $(c_{kj}(\xi'))$ is the inverse matrix (see (22.19)) of $(b_{jk}(\xi'))$, and

$$f_j = \gamma B_j(D) \cdot R_0 f, \quad \text{where } R_0 f = P_+Op(r_+/a_+)\theta_+Op(r_-/a_-)Lf.$$

Using Theorem 18.5 (Paley–Wiener), one can easily show that the function $R_0 f$ does not depend on L (cf. Exercise 22.11) and vanishes at $x_n = 0$, together with the derivatives with respect to x_n of order $j < \mu$.

Note that $Au = P_+ Op(a_-)Op(a_+)u_+$, where by $u_+ \in H^0(\mathbb{R}^n)$ we denote the extension by zero for $x_n < 0$ of the function $u \in H^s(\mathbb{R}^n)$. In view of the Paley–Wiener theorem

$$\theta_+ Op(r_-/a_-)Op(a_-)f_- = 0 \quad \forall f_- \in H^0(\mathbb{R}^n)$$

if $P_+ f_- = 0$. Hence

$$R_0 Au = P_+ Op(r_+/a_+)\theta_+ Op(r_-/a_-)Op(a_-)Op(a_+)u_+ = \\ = P_+ Op(r_+/a_+)\theta_+ Op(a_+)u_+ + T_1 u = u + T_2 u,$$

where $\|T_j u\|_{s+1} \le C\|u\|_s$. The operator R_0 is a regularizator for the operator corresponding to the Dirichlet problem with zero boundary conditions. A similar analysis shows that in this case of a half-space, operator (22.22) is a regularizator for $\mathcal{A}$. □

The following corollary follows from estimate (22.21).

Corollary 22.27 *Let* $u \in H^{s-1}(\Omega)$, $\mathcal{A}u \in H^{s,M}(\Omega)$. *Then* $u \in H^s(\Omega)$. *In particular, if* $u \in H^s(\Omega)$ *is a solution of problem* (22.3)–(22.5), *and* $f \in C^\infty(\overline{\Omega})$, $g_j \in C^\infty(\Gamma)$, *then* $u \in C^\infty(\overline{\Omega})$.

Proposition 22.28 *Under the hypotheses of Theorem* 22.26, $\operatorname{Ker}\mathcal{A}$, $\operatorname{Coker}\mathcal{A}$, *and hence* $\operatorname{ind}\mathcal{A}$ *do not depend on* s.

Proof By Corollary 22.27, we have $u \in H^t$ for any $t > s$ if $u \in H^s$ and $\mathcal{A}u = 0$. So, $\operatorname{Ker}\mathcal{A}$ does not depend on s. We further note that $H^{s,M}$ is the direct sum $\mathcal{A}(H^s)\dot{+}Q$, where Q is a finite-dimensional subspace. Besides, $H^{t,M}$ is dense in $H^{s,M}$ for $t > s$. Hence (see Lemma 2.1 in Gokhberg and Krein 1957) $Q \subset H^{t,M}$. So by Corollary 22.27 we get

$$H^{t,M} = H^{t,M} \cap H^{s,M} = H^{t,M} \cap \mathcal{A}(H^s)\dot{+}H^{t,M} \cap Q = \mathcal{A}(H^{t,M})\dot{+}Q,$$

i.e., $\operatorname{Coker}\mathcal{A}$ does not depend on s. □

Remark 22.29 Even though $\operatorname{Ker}\mathcal{A}$ and $\operatorname{Coker}\mathcal{A}$ do not depend on s, but $\dim\operatorname{Ker}\mathcal{A}$ and $\dim\operatorname{Coker}\mathcal{A}$ may vary if the operator $\mathcal{A}$ is perturbed by an operator of lower order or by an operator with arbitrarily small norm.[44] This can be easily seen even in the one-dimensional case. Nevertheless, $\operatorname{ind}\mathcal{A}$ does not depend on these perturbations by Theorem 22.10.

Remark 22.30 Theorem 22.10 gives us a convenient method for investigation of solvability of elliptic equations $\mathcal{A}u = F$. Indeed, assume that, for a family of elliptic operators

[44] The elliptic theory was constructed with the help of such operators.

$$\mathcal{A}_t = (1-t)\mathcal{A} + t\mathcal{A}_1 \colon H^s \to H^{s,M}$$

it is known that $\operatorname{ind}\mathcal{A}_1 = 0$. Then $\operatorname{ind}\mathcal{A} = 0$. If, in addition, we can establish that $\operatorname{Ker}\mathcal{A} = 0$, then the equation $\mathcal{A}u = F$ is uniquely solvable. If $\dim\operatorname{Ker}\mathcal{A} = 1$, then the equation $\mathcal{A}u = F$ is solvable for any function F orthogonal in $H^{s,M}$ to some nonzero function, and the solution itself is uniquely defined up to a one-dimensional $\operatorname{Ker}\mathcal{A}$.

Example 22.31 Let us give (following Agranovich and Vishik 1964) an example of an elliptic operator of a quite general form with zero index

$$\mathcal{A}_1 = (a(x,D), b_1(x,D)\big|_\Gamma, \dots, b_\mu(x,D)\big|_\Gamma) \colon H^s(\Omega) \to H^{s,M}(\Omega)$$

for an elliptic boundary-value problem in a domain $\Omega \Subset \mathbb{R}^n$ with smooth boundary Γ. For this operator, $\operatorname{ind}\mathcal{A}_1 = 0$. It is assumed that

$$a(x,\xi) = \sum_{|\alpha|+k\le 2\mu} a_\alpha(x)\xi^\alpha q^k, \quad b_j(x,\xi) = \sum_{|\beta|+l\le m_j} b_{j\beta}(x)\xi^\beta q^l,$$

where $q \ge 0$. It is also assumed that the $a(x,D)$ is *elliptic with parameter*, that is,

$$a_{2\mu}(x,\xi,q) = \sum_{|\alpha|+k=2\mu} a_\alpha(x)\xi^\alpha q^k \ne 0 \quad \forall(\xi,q)\ne 0,\ \forall x \in \overline{\Omega}.$$

In this problem $a_{2\mu}(x,\eta,q)$ can be factored as $a_{2\mu}(x,\eta,q) = a_+(x,\eta,q)a_-(x,\eta,q)$ (see Lemma 22.19). It is also assumed that the following analogue of the Shapiro–Lopatinskii condition is satisfied: for any $x \in \Gamma$, the principal symbols

$$b_j(x,\xi) = \sum_{|\beta|+l\le m_j} b_{j\beta}(x)\xi^\beta q^l\big|_{\xi={}^t\sigma'(x)\eta}, \quad j = 1,\dots,\mu,$$

of the boundary operators, considered as polynomials in η_n, are linearly independent modulo the function $a_+(x,\eta,q)$, considered as a polynomial with respect to η_n.

Under these assumptions. we can now repeat the proof of Theorem 22.26. We first replace $\langle\xi\rangle = 1+|\xi|$ in the definition of the norm of the space H^s by $\langle\xi\rangle = 1+q+|\xi|$. Using the obvious inequality, we have

$$\|(1+q+|\xi|)^s\widetilde{u}(\xi)\|_{L^2(\mathbb{R}^n)} \le \frac{1}{q}\|(1+q+|\xi|)^{s+1}\|_{L^2(\mathbb{R}^n)}$$

and hence the regularizator R of the operator $\mathcal{A}_1$ (see the proof of Theorem 22.26) satisfies the relations

$$R\cdot\mathcal{A}_1 u = u + Tu, \qquad \|Tu\|_s \le \frac{1}{q}\|u\|_s,$$

$$\mathcal{A}_1\cdot RF = F + T_1F, \quad \|T_1F\|_{s,M} \le \frac{1}{q}\|F\|_{s,M}.$$

Therefore, for $q \gg 1$, the operators s $1 + T$ and $1 + T_1$ are automorphisms of the corresponding spaces, and the equation $\mathcal{A}_1 u = F$ for $q \gg 1$ is uniquely solvable. This shows that $\operatorname{ind} \mathcal{A}_1 = 0$.

The following proposition easily follows from Remark 22.30 and Example 22.31.

Proposition 22.32 *Let $\mathcal{A}: H^s \to H^{s,M}$ be the operator corresponding to the problem from Example 22.24. Then $\operatorname{ind} \mathcal{A} = 0$. (In particular, this result holds for the Dirichlet problem for an elliptic operator $a(x, D)$ satisfying condition 22.18 and for the elliptic Poincaré problem considered in Exercise 22.25).*

Corollary 22.33 *The Dirichlet problem*[45]

$$\Delta u = f \in H^{s-2}(\Omega), \quad u = g \in H^{s-1/2}(\Gamma), \quad s \geq 1,$$

for the Poisson equation in a domain $\Omega \Subset \mathbb{R}^n$ with sufficiently smooth boundary Γ is uniquely solvable. Moreover,

$$\|u\|_s \leq C(\|f\|_{s-2} + \|g\|'_{s-1/2}). \tag{22.23}$$

Proof By the maximum principle (Theorem 5.14), $\operatorname{Ker} \mathcal{A} = 0$. Hence $\operatorname{Coker} \mathcal{A} = 0$, because $\operatorname{ind} \mathcal{A} = 0$. Furthermore, since $\operatorname{Ker} \mathcal{A} = 0$, from the general elliptic estimate (22.21) we have (22.23). Indeed, arguing by contradiction, consider a sequence $\{u_n\}$ such that $\|u_n\|_s = 1$, and $\|\mathcal{A} u_n\|_{s,M} \to 0$. Since the embedding of $H^s(\Omega)$ into $H^{s-1}(\Omega)$ is compact, and using (22.21) we can assume that u_n converges in H^s to $u \in H^s$. Since $\|u_n\|_s = 1$, we have $\|u\|_s = 1$. But $\|u\|_s = 0$, because $\|\mathcal{A} u\|_{s,M} = \lim \|\mathcal{A} u_n\|_{s,M} = 0$. □

Corollary 22.34 *The Neumann problem*

$$\Delta u = f \in H^{s-2}(\Omega), \quad \frac{\partial u}{\partial \nu} = g \in H^{s-3/2}(\Gamma), \quad s > 3/2, \tag{22.24}$$

in a domain $\Omega \Subset \mathbb{R}^n$ with sufficiently smooth boundary Γ is solvable if and only if

$$\int_\Omega f(x)\, dx - \int_\Gamma g(\gamma) d\Gamma = 0, \tag{22.25}$$

In this case, the solution $u(x)$ is determined up to a constant.

Proof The necessity of condition (22.25) is immediate the Gauss formula (7.7). From the first Green formula (or the Giraud–Hopf–Oleinik Lemma 5.19) it follows that $\operatorname{Ker} \mathcal{A}$ consists of constants. Hence, $\dim \operatorname{Coker} \mathcal{A} = 1$, inasmuch as $\operatorname{ind} \mathcal{A} = 0$. Hence problem (22.24) is solvable if the right-hand side $F = (f, g)$ satisfies one and (only one) orthogonality condition. So, the necessary condition (22.25) is also sufficient for solvability of problem (22.24). □

[45] This pertains, in particular, to problem (7.13).

Remark 22.35 The method of Remark 22.30 for investigating the solvability of elliptic equations can also be applied in more general cases, for example, for problems with conjugation conditions on the surfaces of discontinuity of coefficients (Demidov 1969).

Remark 22.36 The theory of elliptic boundary-value problems for differential operators considered in this section can be naturally extended to pseudo-differential operators (see Vishik and Eskin 1965, Eskin 1980).

In particular, the following theorem was proved in Demidov (1973), Demidov (1975a) (see also Eskin 1973).

Theorem 22.37 *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\Gamma = \partial\Omega$, and let $f \in H^s(\Omega)$, where $s > 3/2$. Then the integral equation*

$$\varepsilon^2 u(x) + \frac{1}{4\pi}\int_\Omega \frac{e^{-q|x-y|}}{|x-y|} u(y)\,dy = f(x), \quad q \geq 0, \quad \varepsilon \geq 0, \tag{22.26}$$

has a solution, and this solution is unique. Namely, for $\varepsilon = 0$ the solution is given by

$$u = u_0 + \rho_0\,\delta\big|_\Gamma, \quad u_0 \in H^{s-2}(\Omega), \quad \rho_0 \in H^{s-1}(\Gamma), \tag{22.27}$$

where $\delta\big|_\Gamma$ is the δ-function concentrated on Γ. More precisely, $u(x)$ lies in the function space $H^{(-1)}_{s-2}$ (which was introduced in Demidov 1973*), in which u_0 and ρ_0 are related.*[46]

If $\varepsilon > 0$, then

$$u(x) = u_0(x) + \frac{1}{\varepsilon}\rho_0(y')\varphi\, e^{-y_n/\varepsilon} + r_0(x,\varepsilon), \tag{22.28}$$

where $\|r_0\|_{L^2} \leq C\sqrt{\varepsilon}$, y_n is the distance along the normal vector from x to $y' \in \Gamma$, and $\varphi \in C^\infty(\overline{\Omega})$, $\varphi \equiv 1$ in a small neighborhood of the curve Γ, and $\varphi \equiv 0$ outside a neighborhood of slightly larger radius. Besides,

$$\frac{1}{\varepsilon}\rho_0(y')\varphi\, e^{-y_n/\varepsilon} \to \rho_0\,\delta\big|_\Gamma, \quad \textit{if } \varepsilon \to 0.$$

In next section (in §23.7) we describe a method for finding the component u_0 of the sought-for solution u. From the knowledge of this component, one can easily find the solution (22.28) of Eq. (22.26) for $\varepsilon > 0$, and hence, the density ρ_0.

Hint In order to understand why formula (22.27) gives a solution of Eq. (22.26) for $\varepsilon = 0$, we first consider a slightly different integral equation of the first kind, namely

$$\int_{\mathbb{R}^3_+} \frac{e^{-2\pi\lambda|x-y|}u(y)dy}{|x-y|} = f(x), \quad \text{where } \lambda > 0, \quad x \in \mathbb{R}^3_+. \tag{22.29}$$

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[46] This relation is reflected in formulas (22.31), (22.32) (see below).

In this equation, the kernel of the corresponding operator involves an additional factor $e^{-2\pi\lambda|x-y|}$, where $\lambda > 0$. Following, for example, (22.9), and taking into account that $f_+\big|_{\mathbb{R}^3_+} = f$ and $f_-\big|_{\mathbb{R}^3_+} = 0$, we consider the continuous extension $\Phi f = f_+ + f_-$ of the function f and note that Eq. (22.29) can be written as

$$Op\Big(\frac{1}{|\xi|^2+\lambda^2}\Big)u_+ = \Phi f, \quad u_+(y) = \begin{cases} u, & \text{if } y \in \mathbb{R}^3_+, \\ 0 & \text{otherwise.} \end{cases} \tag{22.30}$$

Indeed, (22.30) $\Leftrightarrow \big(-\Delta + (2\pi\lambda)^2\big)\Phi f = 4\pi^2 u_+$, where Δ is the *Laplace operator*. The solution Φf of the last equation can be written as the convolution $4\pi^2 G * u_+$ of the function u_+ with the fundamental solution $G(x) = \exp(-2\pi\lambda|x|)/4\pi|x|$ of the operator $-\Delta + (2\pi\lambda)^2$ (see Vladimirov 1971, cf. (7.11)). Let $|\xi'|^2_\lambda \overset{\text{def}}{=} \xi_1^2 + \xi_2^2 + (2\pi\lambda)^2$, and let $\theta(y)$ be the characteristic function of the half-space $\mathbb{R}^3_+$. Then the solution of Eq. (22.30) is given by the formula[47]

$$u_+(x) = Op\big(i\xi_3 + |\xi'|_\lambda\big)\theta(x)Op\big(-i\xi_n + |\xi'|_\lambda\big)\Phi f.$$

Next, using $Op(i\xi_3 + |\xi'|_\lambda)Op(-i\xi_3 + |\xi'|_\lambda) = \frac{1}{(2\pi)^2}\big(-\Delta + (2\pi\lambda)^2\big)$ and $Op(i\xi_3) = \frac{1}{2\pi}\frac{\partial}{\partial x_3}$, we find that $u_+(x) = u_0(x) + \rho_0(x')\delta(x')$, where

$$u_0(x) = \theta(x)\Big(-\frac{1}{(2\pi)^2}\Delta + \lambda^2\Big)f, \tag{22.31}$$

$$\rho_0(x') = -\frac{1}{4\pi^2}\frac{\partial}{\partial x_3}f(x_1,x_2,x_3)\Big|_{x_3=0} + \frac{1}{2\pi}Op(|\xi'|_\lambda)f(x_1,x_2,0). \tag{22.32}$$

The condition $\lambda > 0$ in Eqs. (22.29) and (22.30) is essential (because for $\lambda = 0$ the corresponding operators are not defined). However, if the domain Y is bounded, then using the partition of unity and applying the general elliptic theory (and, in particular, Theorem 22.10 on stability of the index of elliptic operators), it proved possible (see Demidov 1973) to show that the operator $\mathfrak{I}\colon u \mapsto \mathfrak{I}u = \int_Y \frac{u(y)\,dy}{|x-y|}$ is an isomorphism of appropriate function spaces. Moreover, in this way, formula (22.27) can also be established.

23 The Direct, Inverse, and Central Problems of Magneto-Electroencephalography

23.1 Magneto-electroencephalography is a non-invasive (not even disturbing the skin) method of brain imaging The *direct* MEEG-problem calls for evaluation of

[47] According to Theorem 18.5 (Paley–Wiener), $\theta(x)Op\big(-i\xi_n + |\xi'|_\lambda\big)f_- = 0$, and hence $u_+(x)$ is independent of f_-. Note that the condition $f \in H^s(\Omega)$ from Theorem 22.37, where $s > 3/2$, appears because $Op\big(-i\xi_n + |\xi'|_\lambda\big)\Phi f \in H^{s-1}$, and since the operator $\theta\colon H^t \ni g \mapsto \theta\cdot g \in H^t$ is bounded (see Stein 1957) if and only if $|t| < 1/2$.

the electromagnetic field from some explicit formulas (including the Biot–Savart formula) by using the volume density of electric charges and the so-called current dipoles $\mathbf{q}\colon Y \to \mathbb{R}^3$ (current dipole moments Hämäläinen et al. 1993) caused by synchronous activity of large masses of neurons in the brain, which occupies a set $Y \subset \mathbb{R}^3$. In contrast to the direct problem, the *inverse* MEEG-problem, according to the terminology accepted among biophysicists, is the search for the distribution of current dipoles based on the measurement data of electric induction $\mathcal{D} = \varepsilon\mathcal{E}$ and the magnetic induction $\mathcal{B} = \mu\mathcal{H}$ at a large number of points $\mathbf{x}_k$ on the surface X, which is the inside of the helmet covering the patient's head (Hämäläinen et al. 1993; Stroganova et al. 2011; Ichkitidze et al. 2014; Boto et al. 2018)[48] By $\mathcal{H}$ and $\mathcal{E}$ we denote the intensities of the magnetic and electric fields, which are functions of

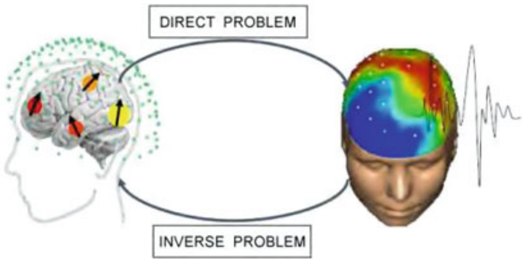

Fig. 3.1: The direct and inverse MEEG-problems

$(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_+$, where t is time, and $\mu = \mu(\mathbf{x}) > 0$ and $\varepsilon = \varepsilon(\mathbf{x}) > 0$ are, respectively, the magnetic and dielectric permittivity, which are assumed to be independent of t (Fig. 3.1).

Since the 1980s, when active studies of the inverse MEEG-problem (in the above sense) have begun, the opinion about its ill-posedness has become widespread in scientific and popular science literature (see, for example, Sheltraw and Coutsias 2003, Shestakova et al. 2012 and the literature cited there). Often an appeal was made to the authority of Helmholtz, who allegedly (see Helmholtz 1853) expressed this opinion. However, the authors of such claims either (as in the case of Shestakova et al. 2012) never read this Helmholtz's article or interpreted it too straightforwardly.

[48] Synchronous neural currents induce very weak magnetic fields that are more than 10^4 times weaker than the Earth's magnetic field. Therefore, they are measured by ultra-sensitive magnetometers. So far, the most common magnetometers are the so-called SQUID magnetometers (Superconducting Quantum Interference Device), whose sensors are accommodated in the helmet in the form of a fixed flask covering the patient's head. SQUID magnetometers are very expensive (there are only two or three dozen of them in the modern world). The fact is that quantum sensors function only at ultra-low temperatures. In addition, SQUID magnetometers are installed in special rooms shielded from external magnetic signals, including the Earth's magnetic field. Perhaps in the near future, SQUID magnetometers will be partially replaced by SERF magnetometers (Spin Exchange Relaxation Free), which have recently been studied in experiments involving portable individual 3D helmets (Boto et al. 2018).

In the actual fact, in the second part of Helmholtz (1853) in the section "Theorem von der gleichen gegenseitigen Wirkung zweier elektromotorischen Flächenelemente," Helmholtz in details (see pp. 353–359) showed that the distribution of the current inside the conductor cannot be uniquely recovered only from the direct knowledge of the electromagnetic field outside the conductor. But this is no surprise of course: supporting examples can be easily constructed in the spherically symmetric case (see, for example, Hämäläinen et al. 1993, p. 430).

Nevertheless, this does not mean that the inverse MEEG problem is ill-posed if it is understood in the sense that it pinpoints and reveals the main difficulty, and does not obscure it, namely as the search for the distribution of current dipoles (and, possibly, the volume density of electric charges) based on the solution of the main central MEEG problem, i.e., *a priori* essentially different[49] electromagnetic fields in Y are possible, which are found from experimental data of measuring the electromagnetic field at a finite set of points outside the domain Y or on just two surfaces X_1 and X_2, which are the inner parts of the helmet on the patient's head. The latter circumstance would allow us to know on X_j not only the approximate value of the electromagnetic field but also the gradients of its components.

Below it will be shown (see also Demidov 2018) that (with the knowledge of the electromagnetic field, whose recovery is discussed in §23.7), the inverse MEEG problem is absolutely correct: it has a solution (stable with respect to perturbations), which is unique, but this solution lies in a special class of functions unknown to biophysicists.[50] The solution has the form $\mathbf{q} = \mathbf{q}_0 + \mathbf{p}_0\delta\big|_{\partial Y}$, where $\mathbf{q}_0$ is a usual function defined in the domain Y occupied by the brain, and $\mathbf{p}_0\delta\big|_{\partial Y}$ (see Definition 2.4) is the *δ-function* on the boundary of the domain Y with some density $\mathbf{p}_0$. Moreover, $\mathbf{p}_0$ and $\mathbf{q}_0$ are interrelated, as pointed out in Theorem 22.37.

23.2 We will start from the *Maxwell equations*

$$\begin{aligned} &\partial_t\mathcal{B}(\mathbf{x},t) + \operatorname{rot}\mathcal{E}(\mathbf{x},t) = 0, \quad \operatorname{div}\mathcal{B}(\mathbf{x},t) = 0,\\ -&\varepsilon(\mathbf{x})\partial_t\mathcal{E}(\mathbf{x},t) + \operatorname{rot}\mathcal{H}(\mathbf{x},t) = \mathbf{J}^v(\mathbf{x}) + \mathbf{J}^p(\mathbf{x}), \quad \operatorname{div}\mathcal{D}(\mathbf{x},t) = \rho, \end{aligned} \tag{23.1}$$

where ρ is the volume density of electric charges. It is known that for biological media, the coefficient μ from the formula $\mathcal{B} = \mu\mathcal{H}$ is nearly equal to the constant μ_0, which is the magnetic permeability of the vacuum. Hence in the case of the MEEG-

[49] Two functions are called *essentially different* (cf. Demidov and Savelyev 2010) if their relative difference exceeds, say, 10÷20%, and, in addition, the neighborhood of a point of absolute maximum of one of these functions is a neighborhood of a point of absolute minimum of the other function.

[50] Biophysicists usually assume that the solution to the inverse MEEG problem is a certain set of point sources at desired points with desired coefficients. In this case, a particular multivariate linear algebra problem that appears in a huge number of studies on this topic turned out to be ill-posed, in particular, because it was initially considered in function spaces that were inadequate for the problem. Numerous attempts to correct the resulting ill-posed problems with the help of various approaches (usually via the so-called Tikhonov regularization, which was advertised by the "father of regularization of ill-posed problems" and his followers as a cure for all diseases), of course, cannot always give a reliable result, since these techniques do not reveal the true causes of ill-posedness, but only obscure them, thereby slowing down the real solution to such problems (see, for example, Leweke et al. 2022 and the references cited there).

problem, we will assume that μ = const. The extension to the case $\mu \neq$ const, which is necessary in problems related to scanning magnetic microscopes,[51] of reconstruction of the magnetization parameters of an engineering object (for example, a ship) from the measured values of the magnetic field outside its body, and some others, does not present fundamental difficulties. Hence in what follows,[52] we will assume that $\mu = 1$.

The current $\mathbf{J}^v = \sigma\mathcal{E}$ is commonly known as the volume (or ohmic) current (more precisely, its density), because it obeys the Ohm law related to the electrical conductivity coefficient $\sigma = \sigma(\mathbf{x}) \geq 0$, which is assumed to be independent of t. The volume current is the result of the action of a macroscopic electric field on the charge carriers in the conducting medium of the brain. Neural activity is caused by the so-called primary (or main) current $\mathbf{J}^p$, which occurs as a result of dielectric polarization and is a combination of the motion of charges inside or near the brain cell. The particles with these charges are parts of the molecules. They are displaced from their equilibrium positions by the action of an external electric field, but they do not leave the molecule in which they are contained.

It may be frequently assumed that the function σ is piecewise constant, $\sigma|_{Y_\pm} = \sigma_\pm$, $\sigma|_{Y_0} = \sigma_0$, and

$$\sigma_+ = 0 \quad \text{in } Y_+, \qquad \sigma_0 > 0 \quad \text{in } Y_0, \qquad \sigma_- > \sigma_0 \quad \text{in } Y_-, \tag{23.2}$$

where $Y = Y_-$ (i.e., the brain domain), Y_0 is the domain occupied by the tissues (skull, etc.) lying between $Y = Y_-$ and the domain $Y = Y_+$, which corresponds to the air around the head. These domains are shown schematically in Fig. 3.2.

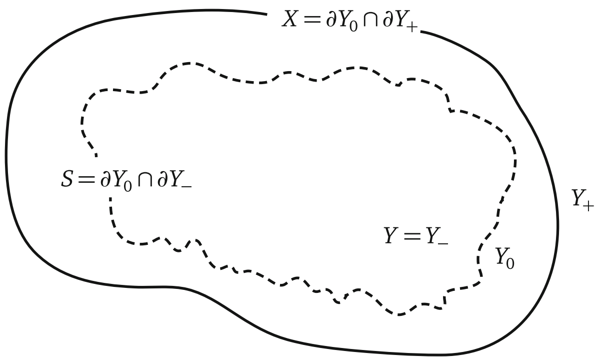

Fig. 3.2: The domains $Y = Y_-$ (brain), Y_0 (skull X and soft bio-tissues), and Y_+ (surrounding space)

[51] These tools (Martin 1987), which are capable of registering magnetic fields, for example, in integrated circuits and in Magnetotactic bacteria, are used in materials science, mineralogy, and paleomagnetic analysis (Acuna et al. 2008; Degen 2008; Weiss et al. 2007).

[52] However, some useful formulas for the case $\mu \neq$ const (including the generalized Biot–Savart formula) will be given in Remark 23.8.

In the inverse problem, it is required to find the dipole distribution of the current dipoles $\mathbf{q}\colon Y \to \mathbb{R}^3$ (first of all, the significant local maxima of the distribution components) based on the measurement data of the fields $\mathcal{B}$ and $\mathcal{E}$. Here of special importance is the fact mentioned in the fundamental paper Hämäläinen et al. (1993) and which pertains to the relation between the frequencies ω of oscillations of the electromagnetic field $\mathcal{H}(\mathbf{x}, t) = \mathbf{H}(\mathbf{x})e^{i\omega t}$, $\mathcal{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x})e^{i\omega t}$ and the frequency of electric oscillations in the brain cells. The analysis conducted in Hämäläinen et al. (1993, p. 426) shows that, for the system (23.1), a quasi-static approximation corresponding to the main term of the asymptotics as $\omega \to 0$ is justified. In Hämäläinen et al. (1993, p. 426) it is also noted that "*A current dipole $\boldsymbol{q}$„ approximating a localized primary current, is a widely used concept in neuromagnetism... In EEG and MEG applications, a current dipole is used as an equivalent source for the unidirectional primary current that may extend over several square centimeters of cortex.*"

23.3 As a result, we get the following equations:

$$\operatorname{rot}\mathbf{E} = 0, \quad \operatorname{rot}\mathbf{H} = (\sigma\mathbf{E} + \mathbf{q}), \quad \operatorname{div}\mathbf{B} = 0, \quad \operatorname{div}\mathbf{D} = \rho. \tag{23.3}$$

It is known that

$$\operatorname{rot}\mathbf{E} = 0 \;\Leftrightarrow\; \mathbf{E} = -\nabla\Phi \quad \text{and} \quad \operatorname{div}\mathbf{B} = 0 \;\Leftrightarrow\; \mathbf{B} = \operatorname{rot}\mathbf{A}. \tag{23.4}$$

We have $\operatorname{div}(\varepsilon\mathbf{E}) = \rho$, and hence

$$-\varepsilon\Delta\Phi - \nabla\varepsilon\nabla\Phi = \rho. \tag{23.5}$$

Physically, the potential Φ of the field $\mathbf{E} = -\nabla\Phi$ is constant at infinity and hence can be assumed to be zero. For similar reasons, the vector potential $\mathbf{A}$ of the field $\mathbf{B} = \operatorname{rot}\mathbf{A}$ is also chosen to be zero at infinity. We have $\operatorname{rot}(\operatorname{rot}\mathbf{A}) = \nabla\operatorname{div}\mathbf{A} - \Delta\mathbf{A}$, and hence $\Delta\mathbf{A} = -\operatorname{rot}\mathbf{B} + \nabla\operatorname{div}\mathbf{A}$. But $\operatorname{rot}\mathbf{B} \overset{(23.3)}{=} \sigma\mathbf{E} + \mathbf{q}$ (because $\mathbf{B} = \mathbf{H}$ for $\mu = 1$), and $\mathbf{E} = -\nabla\Phi$. Hence

$$\Delta\mathbf{A}(\mathbf{x}) = -\mathbf{q}(\mathbf{x}) + \nabla\big[\sigma(\mathbf{x})\Phi(\mathbf{x}) + \operatorname{div}\mathbf{A}(\mathbf{x})\big] - \Phi(\mathbf{x})\nabla\sigma(\mathbf{x}). \tag{23.6}$$

The vector potential $\mathbf{A}$ is defined up to a potential field. Indeed, $\operatorname{rot}(\mathbf{A} - \mathbf{A}^*) = 0 \overset{(23.4)}{\Leftrightarrow} \mathbf{A} - \mathbf{A}^* = \nabla\varphi$, i.e., $\mathbf{A} = \mathbf{A}^* + \nabla\varphi$, where φ is some function. Taking as φ the solution of the equation $\Delta\varphi = -\operatorname{div}\mathbf{A}^* - \sigma\Phi$, subject to the condition $\varphi\big|_\infty = 0$ (because $\mathbf{A}^*\big|_\infty = 0$, $\Phi\big|_\infty = 0$), we find that

$$\sigma(\mathbf{x})\Phi(\mathbf{x}) + \operatorname{div}\mathbf{A}(\mathbf{x}) = 0, \tag{23.7}$$

and now (23.6) implies

$$\Delta\mathbf{A}(\mathbf{x}) = -\mathbf{F}(\mathbf{x}), \quad \text{where } \mathbf{F}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) + \Phi(\mathbf{x})\nabla\sigma(\mathbf{x}). \tag{23.8}$$

It is worth pointing out that Eqs. (23.7) and (23.8) are not independent—they are equivalent to Eq. (23.3) and hence are related by the implicit relation between $\mathbf{A}$, Φ, ρ and $\mathbf{q}$ given by the Eq. (23.3).

Equation (23.8) is equivalent to the formula

$$\mathbf{q}(\mathbf{x}) = -\Delta\mathbf{A}(\mathbf{x}) - \Phi(\mathbf{x})\nabla\sigma(\mathbf{x}). \tag{23.9}$$

This formula gives the required solution of the inverse problem, but only when the potentials $\mathbf{A}$ and Φ are known in Y. However, what is known *a priori* about them (in addition to the fact that they are equal to zero at infinity) is that there are measurement data for the fields $\mathbf{B} = \operatorname{rot}\mathbf{A}$ and $\mathbf{E} = -\nabla\Phi$ at a finite set of points $\mathbf{x}_k \in X$ (see the right-hand side of the figure). Nevertheless, in §23.7 it will be shown that from these data and the results of §23.4 one can identify "essentially" different (cf. Demidov and Savelyev 2010) approximations of the electromagnetic field, and, moreover, they can be identified in the entire(!) space $\mathbb{R}^3 \supset Y$. And they correspond to the desired *a priori* possible "essentially" different solutions to the inverse problem

23.4 Let us show that along the normal vector to $S = \partial Y$ the graphs of the components of the potential $\mathbf{A}$, which obeys Eq. (23.8), have in general a corner[53] on S. Hence $\Delta\mathbf{A}$, and therefore, the sought-for solution $\mathbf{q}$ contains the *δ-function* on the boundary of the domain Y with some density $\mathbf{p}_0$. Let

$$\mathbf{A} = (a_1, a_2, a_3), \quad \text{where } \Delta a_j(\mathbf{x}) = \delta(\mathbf{x}),\ a_j(\infty) = 0,$$

i.e., $a_j(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}$. Then

$$\Delta\mathbf{A}(\mathbf{x}) \overset{(23.8)}{=} -\int_{\mathbb{R}^3} \mathbf{F}(\mathbf{y})\Delta\mathbf{a}(\mathbf{x}-\mathbf{y})\,d\mathbf{y} = \Delta\Big[-\int_{\mathbb{R}^3} \mathbf{F}(\mathbf{y})\mathbf{a}(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\Big].$$

As a result,

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathbf{F}(\mathbf{y})\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \overset{(23.8)}{=} \frac{1}{4\pi}\int_{\mathbb{R}^3} \Big(\mathbf{q}(\mathbf{y}) + \Phi(\mathbf{y})\nabla\sigma(\mathbf{y})\Big)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \tag{23.10}$$

because the *Laplace equation* has a unique solution, which vanishes at infinity (as already noted, $\mathbf{A}\big|_\infty = 0$).

If a function σ satisfying condition (23.2) is piecewise constant, then

$$\int_{\mathbb{R}^3} \frac{\Phi(\mathbf{y})\nabla\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} = (\sigma_+ - \sigma_0)\mathbf{n}_X \int_X \frac{\Phi_0(\mathbf{y}_X)\,d\mathbf{y}_X}{|\mathbf{x}-\mathbf{y}_X|} + (\sigma_0 - \sigma_-)\mathbf{n}_S \int_S \frac{\Phi_0(\mathbf{y}_S)\,d\mathbf{y}_S}{|\mathbf{x}-\mathbf{y}_S|},$$

where $\mathbf{n}_X$ and $\mathbf{n}_S$ are unit normal vectors to $X = \partial Y_0 \cap \partial Y_+ = \partial Y_+$ and $S = \partial Y_0 \cap \partial Y_- = \partial Y$.

As a result, we have the integral equation of the first kind

$$\int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in Y, \tag{23.11}$$

[53] In view of (23.5) and (23.7), the potential Φ has a similar singularity.

where $\mathbf{f}(\mathbf{x}) = 4\pi\mathbf{A}(\mathbf{x}) - \int_{\mathbb{R}^3} \frac{\Phi(\mathbf{y})\nabla\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}$. According to §22.37 and the comments given in this section, we have the following theorem.

Theorem 23.5 *Let*[54] $f \in C^\infty(\overline{Y})$. *Then Eq.* (23.11) *is uniquely solvable and its solution has the form*

$$\boldsymbol{q}(\boldsymbol{y}) = \boldsymbol{q}_0(\boldsymbol{y}) + \boldsymbol{p}_0(\boldsymbol{y}')\delta\big|_{\partial Y}, \tag{23.12}$$

where $\delta\big|_{\partial Y}$ *is the* δ*-function on* ∂Y, *and* $\boldsymbol{q}_0 \in C^\infty(\overline{Y})$ *and* $\boldsymbol{p}_0 \in C^\infty(\partial Y)$ *are related as elements of the space* $H_\infty^{(-1)}(Y)$ *defined in* Demidov (1973).

One more remark is worth making. If the electric field data are not specified, i.e., if they are *a priori* arbitrary, then the right-hand side of Eq. (23.11) is defined only up to $\nabla\varphi$. In this case, the components of the solution of Eq. (23.11) are linearly dependent, and hence, there is an infinite-dimensional ambiguity in the choice of the solution. In the case $Y = \mathbb{R}^2$, this fact was established by various methods in Baratchart et al. (2013) and Demidov et al. (2015), in which calculation formulas were put forward).

We also note (see §22.37) that there is a relation between the solution $\mathbf{q}$ of (23.11) (the integral equation of the first kind) and the solution $\mathbf{q}_\eta$ of the integral equation of the second kind

$$\eta^2\mathbf{q}_\eta(\mathbf{x}) + \int_Y \frac{\mathbf{q}_\eta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in Y, \quad \eta > 0. \tag{23.13}$$

Namely, the following theorem holds (see Demidov 1975a).

Theorem 23.6 *The solution of Eq.* (23.13) *has the form*

$$\boldsymbol{q}_\eta(\boldsymbol{x}) = \boldsymbol{q}_0(\boldsymbol{x}) + \frac{1}{\eta}\boldsymbol{p}_0(\boldsymbol{y}')\varphi\, e^{-y_n/\eta} + r_0(\boldsymbol{x}, \eta), \tag{23.14}$$

where $\|r_0\|_{L^2} \le C\sqrt{\eta}$, $\boldsymbol{y}_n$ *is the distance along the normal vector from* $\boldsymbol{x}$ *to* $\boldsymbol{y}' \in \Gamma$, *and* $\varphi \in C^\infty(\overline{Y})$, $\varphi \equiv 1$ *in the small neighborhood of* ∂Y *and* $\varphi \equiv 0$ *outside some larger neighborhood.*

23.7 Let us now discuss the above *central* problem of magneto-electroencephalography on electromagnetic field reconstruction. For simplicity, we will assume that the dielectric permeability ε is constant. In this case, Eq. (23.8) for the potential Φ has the same form of the inhomogeneous Poisson equation ($\Delta u = g$) as the equation for the scalar components of the potential $\mathbf{A}$. Besides, by specifying the function g,

[54] If we assume less regularity for the function $\mathbf{f}$, then the components $\mathbf{q}_0$ and $\mathbf{p}_0$ of the solution (23.12) are also less regular. The exact smoothness classes in terms of Sobolev spaces are given in Demidov (1973), where it is also shown that the operator

$$\mathfrak{I}\colon \mathbf{q} \mapsto \mathfrak{I}\mathbf{q} = \int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}$$

is an isomorphism of the corresponding function spaces.

we simulate the setting of the test values ρ and $\mathbf{q}$. To these values, there corresponds the solution of the *direct* MEEG-problem, which can be represented (in terms of the function u) by the potentials $\mathbf{A}_g = \mathbf{A}(\rho, \mathbf{q})$ and $\Phi_g = \Phi(\rho, \mathbf{q})$. So, the central MEEG-problem leads to the problem of minimization of the functional

$$G(\rho, \mathbf{q}) \overset{\text{def}}{=} \sum_k \Big(\|\mathbf{B}(\mathbf{x}_k) - \operatorname{rot} \mathbf{A}_g\Big|_{\mathbf{x}=\mathbf{x}_k}\|^2 + \|\mathbf{E}(\mathbf{x}_k) + \nabla\Phi_g\Big|_{\mathbf{x}=\mathbf{x}_k}\|^2\Big), \tag{23.15}$$

where $\mathbf{B}(\mathbf{x}_k)$ and $\mathbf{E}(\mathbf{x}_k)$ are the known measurement data of the electromagnetic field at the points $\mathbf{x}_k \in X$. As for the potentials $\mathbf{A}_g$ and Φ_g, which are *a priori* unknown, they can be modeled using their singularity (the corner on $S = \partial Y$ along the normal vector to S; see p. 144). At the same time, taking into account numerous works on measuring the magnetic field gradient (see, for example, Magnetic field 2002) and the possibility of measuring the field on close surfaces,[55] we assume that it is possible to measure not only the electromagnetic field at a finite set of points on the surface X but also, with some accuracy η, to find the gradient of this field, namely

$$\sum_k \Big(\|\nabla\mathbf{B}(\mathbf{x}_k) - \alpha_k\|^2 + \|\nabla\mathbf{E}(\mathbf{x}_k) - \beta_k\|^2\Big) \le \eta, \tag{23.16}$$

where α_k and β_k are data of measurement of gradients at the points $\mathbf{x}_k$.

We consider here only the spherical model. In this case, the domains $Y = Y_-$ and Y_+ appearing in (23.2) are such that Y is the ball $|x| < R = 1$, and $Y_+ = \mathbb{R}^3 \setminus Y$. So (see the right-hand side of the figure), $X = \partial Y = \partial Y_+$, and $Y_0 = \varnothing$. Consider the spherical coordinates (r, θ, φ). Let

$$Y_n^{\pm} : (\theta, \varphi) \mapsto \sum_{m=0}^{n} \big[A_{nm}^{\pm} \cos(m\varphi) + B_{nm}^{\pm} \sin(m\varphi)\big] P_n^{(m)}(\cos\theta)$$

be the so-called spherical functions[56] parameterized by the coefficients $A_{nm}^{\pm}$ and $B_{nm}^{\pm}$. Next, let

$$u(r, \theta, \varphi) = \begin{cases} \displaystyle\sum_{n\ge 0}\Big[\sum_{k\ge 2} C_k^- r^{n+k} + D_n^- r^n\Big] Y_n^-(\theta, \varphi) & \text{in } Y_-, \\ \displaystyle\sum_{n\ge 1}\Big[\sum_{k\ge 2} C_k^+ \Big(\frac{1}{r}\Big)^{n+k} + D_n^+ \Big(\frac{1}{r}\Big)^{n}\Big] Y_n^+(\theta, \varphi) & \text{in } Y_+. \end{cases} \tag{23.17}$$

Note that

[55] See https://ieeexplore.ieee.org/abstract/document/7014222/.

[56] By $P_n^{(m)}$ we denote the associated Lagrange functions, i.e.,

$$P_n^{(m)}(t) = (1 - t^2)^{\frac{m}{2}} \frac{d^m}{dt^m} P_n(t), \quad \text{and} \quad P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}\big[(t^2 - 1)^m\big].$$

$$\sum_{n\geq 0}\Big[\sum_{k\geq 2} C_k^- + D_n^-\Big] Y_n^-(\theta,\varphi) = \sum_{n\geq 1}\Big[\sum_{k\geq 2} C_k^+ + D_n^+\Big] Y_n^+(\theta,\varphi), \tag{23.18}$$

which is the continuity condition of the function u. Hence $g = \Delta u$ (as a function of (r, θ, φ)) is given by

$$g(r,\theta,\varphi) = \begin{cases} \sum\limits_{n\geq 0}\sum\limits_{k\geq 2} g_{kn}^-(r) r^{n+k-2} Y_n^-(\theta,\varphi) & \text{in } Y_-, \\ \sum\limits_{n\geq 1}\sum\limits_{k\geq 2} g_{kn}^+(r)\Big(\dfrac{1}{r}\Big)^{n+k+2} Y_n^+(\theta,\varphi) & \text{in } Y_+, \end{cases} \tag{23.19}$$

where the functions $g_{kn}^{\pm}(r) = C_k^{\pm}(n+k)(n+k+1) - n(n+1)r^2$ are related[57] by (23.18).

The problem is linear and hence the measurement data can be reduced basic spherical harmonics. For each of such harmonics, function (23.17), which depends on the family of numerical parameters $\mathcal{N} = \{C_n^{\pm}, D_n^{\pm}\}$, represents (according to the above) the potentials $\mathbf{A} = \mathbf{A}_{\mathcal{N}}$ and $\Phi = \Phi_{\mathcal{N}}$. In turn, these potentials determine the approximations $\operatorname{rot}\mathbf{A}_{\mathcal{N}}$ and $\nabla\Phi_{\mathcal{N}}$ of the quantities $\operatorname{rot}\mathbf{A}_g$ and $\nabla\Phi_g$, which enter (23.15) and which are not given *a priori*. So, functional (23.15) is approximated by the functional

$$H(\mathcal{N}) \stackrel{\text{def}}{=} \sum_k \Big(\|\mathbf{B}(\mathbf{x}_k) - \operatorname{rot}\mathbf{A}_{\mathcal{N}}\big|_{\mathbf{x}=\mathbf{x}_k}\|^2 + \|\mathbf{E}(\mathbf{x}_k) + \nabla\Phi_{\mathcal{N}}\big|_{\mathbf{x}=\mathbf{x}_k}\|^2\Big).$$

By minimizing the latter functional on the elements of N^* satisfying the condition

$$\sum_k \big(\|\nabla\mathbf{B}(\mathbf{x}_k) - \nabla\operatorname{rot}\mathbf{A}_{\mathcal{N}^*}\big|_{\mathbf{x}=\mathbf{x}_k}\|^2 + \|\nabla\mathbf{E}(\mathbf{x}_k) - \nabla\nabla\Phi_{\mathcal{N}^*}\big|_{\mathbf{x}=\mathbf{x}_k}\|^2\big) \stackrel{(23.16)}{\leq} \eta,$$

one can identify *a priori* possible essentially different *solutions of the central MEEG-problem*, because, for the potentials $\mathbf{A} = \mathbf{A}_{\mathcal{N}^*}$ and $\Phi = \Phi_{\mathcal{N}^*}$ thus obtained, using (23.9) one can find in the domain Y the component $\mathbf{q}_0$ of the required solution

$$\mathbf{q}(\mathbf{x}) \stackrel{(23.12)}{=} \mathbf{q}_0(\mathbf{x}) + \mathbf{p}_0(\mathbf{y}')\delta\big|_{\partial Y}.$$

From the knowledge of this component $\mathbf{q}_0$ one can efficiently obtain solution (23.14) of Eq. (23.13), and therefore, find $\mathbf{p}_0$.

In the general case, the spherical functions should be replaced by multiparameter set of functions corresponding to the domains Y^-, Y_0, Y_+.

As an alternative to the above method of finding the magnetic field (more precisely, essentially different fields) in the spherical case, one may probably consider the compression sensing technique (see, for example, Foucart and Rauhut 2013).

Remark 23.8 Following the footnote 52 on p. 142, we give useful formulas for situations where $\mu \neq \text{const}$.

[57] This reflects the aforementioned relation between the components of solution (23.11) of Eq. (23.12).

The first of these formulas reads as:

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int_Y \Big(\mathbf{Q}(\mathbf{y}) + \Phi(\mathbf{y})\nabla\sigma_\mu(\mathbf{y}) - \mathbf{H}(\mathbf{y}) \times \nabla\mu(\mathbf{y})\Big)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{23.20}$$

This formula extends (23.10) in the case $\sigma \neq$ const, $\mu \neq$ const. Here $\mathbf{Q} = \mu\mathbf{q}$, $\sigma_\mu = \sigma\mu$. Since $\operatorname{rot}\mathbf{F}(\mathbf{y}) = 0$ and $\nabla\frac{1}{|\mathbf{x}-\mathbf{y}|} = -\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}$, we find that

$$\operatorname{rot}\Big(\frac{1}{|\mathbf{x}-\mathbf{y}|}\mathbf{F}(\mathbf{y})\Big) = \frac{\operatorname{rot}\mathbf{F}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} + \mathbf{F}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} = \mathbf{F}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}.$$

Hence for $\mathbf{B} = \operatorname{rot}\mathbf{A}$ using (23.20) we get the generalized Biot–Savart formula,[58]

$$\int_Y \mathbf{Q}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}d\mathbf{y} = \mathbf{B}_{\sigma\mu}(\mathbf{x}), \tag{23.21}$$

where $\mathbf{B}_{\sigma\mu} = 4\pi\mathbf{B} - \int_Y \Big(\Phi(\mathbf{y})\nabla\sigma_\mu(\mathbf{y}) - \mathbf{H}(\mathbf{y}) \times \nabla\mu(\mathbf{y})\Big) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}d\mathbf{y}$.

24 The Cauchy Problem for Elliptic Equations. Explicit Formulas

24.1 The Dirichlet problem for an elliptic equation (for example, the Laplace equation $\Delta u = 0$) is well-posed in a bounded domain Ω. It is uniquely solvable and its solution u is stable under perturbations of the Dirichlet data. However, in many important problems of natural science, it is fundamentally impossible to know the Dirichlet data on the entire boundary of the Ω.

For example, some problems of geodesy, thermonuclear reaction in a tokamak, magneto-electroencephalography are reduced to the Laplace equation in a doubly connected domain Ω with Dirichlet data only on the outer boundary Γ, since there are not a priori data on the inner boundary. Therefore, to find the solution u corresponding to a real process, they try to use additional experimental data on the part of the boundary of Ω accessible to researchers. In the examples above, this is possible only on the outer boundary Γ of doubly connected domain Ω.

As such additional data in the case of second-order equations, the experimental data of measuring the gradient u on Γ are used and, consequently, the value of the normal derivative of $\frac{\partial u}{\partial \nu}$ on Γ. As a result, the Cauchy problem for the elliptic equation arises. But, as Hadamard[59] noted, the slightest high-frequency perturbations of the

[58] The formula $\mathbf{B}(\mathbf{x}) = \frac{\mathbf{q}\times(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}$ for the field $\mathbf{B}$ induced by the current dipole $\mathbf{q}$ was experimentally found in 1820 by French physicists Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841) in the process of observing the effect of the galvanic current running through a conductor on the magnetic arrow.

[59] Here we quote from the remarkable and comprehensive book Mazya and Shaposhnikova (1999): “This book presents a fascinating story of the long life and great accomplishments of Jacques

zero Cauchy data lead to an exponential deviation from the true solution. This has led J. Hadamard to his famous example

$$\begin{gathered} U_{xx} + U_{yy} = 0, \quad \text{for} \quad y > 0, \\ U(x,0) = 0, \quad U_y(x,0) = f_\varepsilon(x), \qquad \text{where} \quad f_\varepsilon(x) = e^{-\frac{1}{\varepsilon}} \sin\left(x/\varepsilon^2\right) \end{gathered} \tag{24.1}$$

of the Cauchy problem for the Laplace equation, which illustrates the available difficulties. Despite the fact that the initial data in (24.1) tend to zero uniformly together with all derivatives as $\varepsilon \to 0$, the solution of this problem, as given by the exact formula

$$U(x, y) = e^{-\frac{1}{\varepsilon}} \sin(x/\varepsilon^2) \sinh(y/\varepsilon^2),$$

tends to infinity for any $y > 0$ as $\varepsilon \to 0$.

Therefore, numerical algorithms for finding a solution must overcome the instability of the solution due to inevitable random high-frequency errors in the representation of numerical data in computer memory. Such problems are referred to ill-posed problems (see, for example, Novikoff 1938, Romanov and Kabanikhin 1991, Romanov and Kabanikhin 1994, Kabanikhin 2008, Kabanikhin 2011, Bezrodnykh and Demidov 2011). Numerical analysis of such problems calls for development of special methods and numerical algorithms. Explicit formulas in such problems provide a main tool for testing such methods and numerical algorithms. Below, we will present such explicit numerically realizable formulas in the form of converging series for Cauchy problems for elliptic equations.

Hadamard (1865–1963), who was once called “the living legend of mathematics.” As one of the last universal mathematicians, Hadamard’s contributions to mathematics are landmarks in various fields. His life is linked with world history of the twentieth century in a dramatic way. This work provides an inspiring view of the development of various branches of mathematics during the nineteenth and twentieth centuries. Part I of the book portrays Hadamard’s family, childhood and student years, scientific triumphs, and his personal life and trials during the first two world wars. The story is told of his involvement in the Dreyfus affair and his subsequent fight for justice and human rights. Also recounted are Hadamard’s worldwide travels, his famous seminar, his passion for botany, his home orchestra, where he played the violin with Einstein, and his interest in the psychology of mathematical creativity. Hadamard’s life is described in a readable and inviting way. The authors humorously weave throughout the text his jokes and the myths about him. They also movingly recount the tragic side of his life. Stories about his relatives and friends, and old letters and documents create an authentic and colorful picture. The book contains over 300 photographs and illustrations. Part II of the book includes a lucid overview of Hadamard’s enormous work, spanning over six decades. The authors do an excellent job of connecting his results to current concerns. While the book is accessible to beginners, it also provides rich information of interest to experts. Vladimir Maz’ya and Tatyana Shaposhnikova were the 2003 laureates of the Insitut de France’s Prix Alfred Verdaguer. One or more prizes are awarded each year, based on suggestions from the Academie francaise, the Academie de sciences, and the Academie de beaux-arts, for the most remarkable work in the arts, literature, and the sciences. In 2003, the award for excellence was granted in recognition of Maz’ya and Shaposhnikova’s book, Jacques Hadamard, A Universal Mathematician, which is both an historical book about a great citizen and a scientific book about a great mathematician.” It will be no wonder if the book Mazya and Shaposhnikova (1999) (the Russian translation of which appeared in 2008) will become one of the favorite books of many mathematicians and physicists.

24.2 We start with the following simple observation: if some domain $\Omega \subset \mathbb{C}$, which is mapped conformably via

$$f : \Omega \ni z \mapsto f(z) \in D$$

onto the unit disk D and if the boundary $\Gamma = \partial\Omega$ of the domain Ω is mapped isometrically onto the boundary $\mathbb{T} = \partial D$ of the disk D. Here, the isometricity[60] means that $|\frac{df(z)}{dz}| \equiv 1$ for $z \in \Gamma$. It turns out that, in this case, the domain Ω is some disk be obtained from the disk D by its translation and rotation. Since for an analytic function $w : \Omega \ni z \mapsto w(z) = f'(z)$ its modulus on Γ is identically equal, this function (by virtue of the inverse theorem boundary correspondence) is univalent. Thus, the maximum and minimum of $|w|$ are equal to unity at the boundary, and (by virtue of the maximum principle) also inside Ω. Consequently, $f(z) = e^{i\varphi}z + \text{const}$, and hence Γ is the shift and rotation of the circle $\mathbb{T}$.

However, the situation totally changes if f is assumed to be conformal only on the boundary $\Gamma = \partial\Omega$ of a given simply connected domain Ω and if this boundary is analytic (of course, of the same length 2π as $\mathbb{T} = \partial D$).

We note first that on Γ one may introduce the natural parameter $s \in \mathbb{T} \overset{\text{def}}{=} \mathbb{R}/2\pi$ equal to the length of an arc $\smile P_0P_s$ of the curve Γ in the positive direction from some fixed point P_0 with Cartesian coordinates $\big(x(0), y(0)\big)$ to a point P_s with coordinates $\big(x(s), y(s)\big)$. Without loss of generality it can be assumed that $\big(x(0), y(0)\big) = (1, 0)$. As a result,

$$\dot{x}(s) = -\sin N(s), \quad x(0) = 1; \qquad \dot{y}(s) = \cos N(s), \quad y(0) = 0, \tag{24.2}$$

where $N(s)$ is the angle measured in radians between the x-axis and the outer normal vector ν to Γ at the point $P_s \in \Gamma$. In view of (24.2), the curve Γ is closed if and only if

$$\int_0^{2\pi} \sin N(s)\, ds = \int_0^{2\pi} \cos N(s)\, ds = 0 \quad \Leftrightarrow \quad \int_0^{2\pi} e^{iN(\theta)-i\theta}\, d\theta = 0\,. \tag{24.3}$$

Since the length of the curve Γ is given, this curve is completely determined by the angle function $s \mapsto N(s)$ and, correspondingly, the 2π-periodic function

$$Q : \mathbb{T} = \mathbb{R}/2\pi \ni s \mapsto Q(s) \overset{def}{=} N(s) - s\,. \tag{24.4}$$

We denote by $\mathbf{Q}^\gamma$ the complex-valued analytic continuation of the real analytic function Q to some neighborhood V_Γ of the curve $\Gamma \subset \mathbb{R}^2 \simeq \mathbb{C} \ni z = x + iy$ that contains all points lying at distance $\gamma > 0$ from Γ. It is known (see, for example, Arnold 1989) that the coefficients L_k and M_k in the Fourier-representation

[60] Roughly speaking, isometricity can be visualized as follows: any closed curve Γ, made of inextensible and incompressible wire can be deformed to a circle. It i clear that in this case the length of Γ is 2π.

$$Q(s) = \sum_{k\geq 0}(L_k \cos ks + M_k \sin ks) \tag{24.5}$$

of this 2π-periodic analytic Q have the following exponential estimates

$$|L_k| \leq Ce^{-k\gamma}, \qquad |M_k| \leq Ce^{-k\gamma} \tag{24.6}$$

with a multiplicative constant R majorizing the modulus of the function $\mathbf{Q}^\gamma$.

Lemma 24.3 *The following results hold.*

(1) *There exists a neighborhood $V_\mathbb{T} \ni \zeta = \rho e^{i\theta}$ of the unit circle $\mathbb{T} = \{\rho = 1,\ \theta \in \mathbb{R}/2\pi\}$ in which the functions A and B, as defined by*

$$\begin{aligned} A(\rho,\theta) &= \sum_{k\geq 1}\frac{\rho^k - \rho^{-k}}{2}\{M_k \cos k\theta - L_k \sin k\theta\}, \\ B(\rho,\theta) &= \sum_{k\geq 0}\frac{\rho^k + \rho^{-k}}{2}\{L_k \cos k\theta + M_k \sin k\theta\}, \end{aligned} \tag{24.7}$$

are harmonically conjugate.

(2) *The mapping*

$$z : V_\mathbb{T} \ni \zeta = \rho e^{i\theta} \mapsto z(\zeta) = x(\rho,\theta) + iy(\rho,\theta) \in V_\Gamma, \tag{24.8}$$

with

$$z(\zeta) = 1 + \int_1^\zeta e^{A+iB}\, d\zeta \quad \Leftrightarrow \quad A + iB = \ln\frac{dz}{d\zeta} \tag{24.9}$$

is well defined.

(3) *The mapping* (24.8) *is univalent and maps isometrically the circle $\mathbb{T}$ onto the curve Γ, i.e.,*

$$\left|\frac{dz(\zeta)}{d\zeta}\right| \equiv 1 \text{ for } \zeta \in \mathbb{T}. \tag{24.10}$$

Proof From (24.6) and (24.7) it follows that the functions A and B are harmonic in $V_\mathbb{T}$. It is clear that the Cauchy–Riemann conditions

$$\frac{\partial A}{\partial\rho} = \frac{1}{\rho}\frac{\partial B}{\partial\theta}, \qquad \frac{\partial B}{\partial\rho} = -\frac{1}{\rho}\frac{\partial A}{\partial\theta}$$

are satisfied, and hence the function

$$C : V_\mathbb{T} \ni \zeta = \rho e^{i\theta} \mapsto C(\zeta) = A(\rho,\theta) + iB(\rho,\theta)$$

is analytic in $V_\mathbb{T}$. Next, in view of (24.4), (24.5) and (24.7) we have

$$e^{C(\zeta)}\Big|_{\zeta=e^{i\theta}} = e^{iN(\theta)-i\theta}$$

and hence, by (24.3), $\int_{|\zeta|=1} e^{C(\zeta)}\,d\zeta = 0$. This shows that both the integral (24.9) and the mapping (24.8) are well defined. We have $A + iB \overset{(24.9)}{=} \ln \frac{dz}{d\zeta}$ and $\left|\frac{dz}{d\zeta}\right|_{\zeta\in\mathbb{T}} = e^A\big|_{\rho=1} \overset{(24.7)}{=} 1$. Next, $1 = \left|\frac{dz}{d\zeta}\right|_{\rho=1} = \frac{ds(\theta)}{d\theta}$. It can be assumed that $s(\theta) = \theta$, i.e., $B\big|_{\rho=1} \overset{(24.5)\text{--}(24.7)}{=} Q\big(s(\theta)\big) \overset{(24.5)}{=} N(s(\theta)) - s(\theta) = N(\theta) - \theta$. Therefore, $\arg \frac{dz}{d\zeta}\big|_{\zeta=e^{i\theta}} = B(\rho,\theta)\big|_{\rho=1}$ depends continuously on $\zeta \in \mathbb{T}$. Using the equality $e^A\big|_{\rho=1} \overset{(24.7)}{=} 1$ and the geometrical interpretation of $\arg \frac{dz}{d\zeta} = \arg dz - \arg d\zeta$, we get the conclusion of the theorem: $\zeta = \rho e^{i\theta} \mapsto z(\zeta)$ is an isometric mapping of $\mathbb{T}$ onto Γ. □

Remark 24.4 Using Lemma 24.3 one can reduce the Cauchy problem in any domain with analytic boundary to the Cauchy problem in the disk with the same(!) Cauchy data and hence represent the solution of the original problem as converging series. Lemma 24.3 was applied in some or other sense in Demidov (1996), Demidov (2010a), Demidov and Platushchikhin (2010), Demidov (2020), Demidov (2021). The results obtained below in this section are based on Lemma 24.3.

24.5 We will consider here (unlike Demidov 2021) only the case of a simply connected domain Ω with analytic boundary Γ of length 2π. The functions on Γ can be considered as functions of the natural parameter s corresponding to a point $P_s \in \Gamma$. Let $F : s \mapsto F(s)$ and $G : s \mapsto G(s)$ be real functions satisfying the hypotheses of Dirichlet–Jordan's theorem on Fourier series expansion. By $V_\Gamma \subset \mathbb{R}^2$ we denote a two-sided neighborhood of the curve Γ. Our aim here is to present a construction of numerically realizable formulas for the solutions in V_Γ of the following two equations:

$$\operatorname{div}\big(\alpha(w)\nabla w\big) = 0 \quad \text{in} \quad V_\Gamma, \tag{24.11}$$

$$\operatorname{div}\big(\beta\nabla w\big) = 0 \quad \text{in} \quad V_\Gamma \tag{24.12}$$

with the Cauchy data

$$w(P_s) = F(s), \quad \frac{\partial w}{\partial \nu}(P_s) = G(s). \tag{24.13}$$

Here, $\alpha > 0$ and $\beta > 0$ are differentiable functions (α on $\mathbb{R}$, β in V_Γ), and ν is the unit normal vector to Γ.

In the following results we assume that the primitive of the function α is invertible and β is a squared harmonic function.

The change of variables $\zeta = \rho^{i\theta} \mapsto \big(x(\rho,\theta), y(\rho,\theta)\big)$, which is introduced in Lemma 24.3 via the univalent mapping $z(\zeta) = 1 + \int_1^\zeta e^{A+iB}\,d\zeta$, gives the following equivalences for $u(\rho,\theta) = w(x,y)$:

$$\operatorname{div}\big(\alpha(w)\nabla w\big) \overset{(24.11)}{=} 0 \quad \Leftrightarrow \quad \operatorname{div}\big(a(u)\nabla u\big) = 0 \quad \Leftrightarrow \quad a(u)\Delta u + a'(u)|\nabla u|^2 = 0, \tag{24.14}$$

where $a(u) = \alpha(w)$, and $\Delta = \nabla^2$ is the Laplace operator.

Following Polyanin and Zaitsev (2012) (see also §9.4.12 on p. 695), we set $U \overset{\text{def}}{=} \int a(u)\,du$. Hence $\Delta U = a(u)\big(\Delta u + \frac{a'(u)}{a(u)}|\nabla u|^2\big)$. This shows that U is harmonic in $V_{\mathbb{T}}$.

Thus, we reach

Lemma 24.6 *The solution $u = u(\rho,\theta)$ of the quasi-linear equation*

$$\operatorname{div}\big(a(u)\nabla u\big) = 0, \quad \textit{where} \quad a(u) = \frac{1}{u^{2k}},$$

in a neighborhood $V_{\mathbb{T}}$ of the circle $\mathbb{T}$ is given by the simple explicit formula

$$u(\rho,\theta) = -\left(\frac{1}{(2k-1)U(\rho,\theta)}\right)^{\frac{1}{2k-1}},$$

in terms of the harmonic function U on $V_{\mathbb{T}}$.

From this clear lemma, we almost readily obtain

Theorem 24.7 *Let the mapping $u \mapsto U(u) = \int a(u)\,du$, where $a(u) = \alpha(w)$, be invertible (as in Lemma 24.6), i.e., the function $u : U \mapsto u(U)$ is well defined. Then the following two assertions hold.*

(1) *The solution w of the original Cauchy problem for the quasi-linear equation $\operatorname{div}\big(\alpha(w)\nabla w\big) \overset{(24.11)}{=} 0$ is given by the formula*

$$w(x)\Big|_{x=x(\rho,\theta)} = u\big(U(\rho,\theta)\big), \tag{24.15}$$

where U is a harmonic function in $V_{\mathbb{T}}$.

(2) *The original Cauchy fata for the function w are transformed to the Cauchy data*

$$U\Big|_{\rho=1} = \Phi^U(\theta) \overset{\text{def}}{=} \int a(u)\,du\Big|_{u=F(\theta)}, \quad \frac{\partial U}{\partial \rho}\Big|_{\rho=1} = \Psi^U(\theta) \overset{\text{def}}{=} \frac{G(\theta)}{u'(U)\big|_{U=F(\theta)}}. \tag{24.16}$$

Proof Formula (24.15) is as straightforward as Lemma 24.6. The first of (24.16) is a reformulation of the condition $w(P_s) \overset{(24.13)}{=} F(s)$, and the second one follows from the equalities $\frac{\partial}{\partial\rho}u\big(U(\rho,\theta)\big)\big|_{\rho=1} \overset{(24.10)}{=} \frac{\partial w}{\partial \nu}(P_s) \overset{(24.13)}{=} G(s)$. □

Using as above the change of variables $\zeta = \rho^{i\theta} \mapsto \big(x(\rho,\theta), y(\rho,\theta)\big)$, as defined by the univalent mapping (24.8) and setting $b(\rho,\theta) = \beta(x,y)$, we obtain

$$\operatorname{div}\big(\beta\nabla w\big) \overset{(24.12)}{=} 0 \ \text{ in } \ V_\Gamma \quad \Leftrightarrow \quad \operatorname{div}\big(b\nabla u\big) = 0 \ \text{ in } \ V_{\mathbb{T}} \tag{24.17}$$

for $u(\rho,\theta) = w(x\rho,\theta), y\rho,\theta))$. We apply to (24.17) the Moutard transform (see Grinevich and Novikov 2019) in its simplest form, which relates a pair (b,u) to the pair (c, v), where $c = \sigma^2 b$, $v = u/\sigma$, and $\sigma > 0$ is some fixed positive solution of the equation $\operatorname{div}\left(\frac{c}{\sigma^2}\nabla w\right) \overset{(24.12)}{=} 0$.

The following result holds.

Lemma 24.8 *Let $c = 1$, i.e., $1 = \sigma^2 b$. Then both v and $Z \overset{\text{def}}{=} -1/\sigma$ are harmonic functions, and, moreover, $b = Z^2$.*

Proof We set $\partial_x = \partial_1, \partial_y = \partial_2$, and note that $\partial_j v = \sigma^2 b \partial_j v = \sigma^2 b \partial_j\left(\frac{u}{\sigma}\right) = b\sigma\partial_j u - bu\partial_j\sigma$. Hence $\partial_j(\partial_j v) = \partial_j[\sigma(b\partial_j u) - u(b\partial_j\sigma)] = \sigma\partial_j(b\partial_j u) + (\partial_j\sigma)(b\partial_j u) - u\partial_j(b\partial_j\sigma) - (\partial_j u)(b\partial_j\sigma) = \{\sigma\partial_j(b\partial_j u) - u\partial_j(b\partial_j\sigma)\} +$
$+ \{(\partial_j\sigma)(b\partial_j u) - (\partial_j u)(b\partial_j\sigma)\} = \{\sigma\partial_j(b\partial_j u) - u\partial_j(b\partial_j\sigma)\}$.

Summating over j, we get $\operatorname{div}(\nabla v) = \sigma \operatorname{div}(b\nabla u) - u \operatorname{div}(b\nabla\sigma)$. We have $\operatorname{div}(b\nabla u) = 0$ and $\operatorname{div}(b\nabla\sigma) = 0$, which gives $\Delta v = 0$. Note that $u = \sigma v$.

As direct consequence of Lemma 24.6 with $k = 1$, we see that $Z = -1/\sigma$ is a harmonic function. Finally, the equality $b = Z^2$ follows from the fact that $b = 1/\sigma^2$ and $\sigma = -1/Z$. □

Theorem 24.9 *Let $b = Z^2$, where Z is a harmonic function in $V_{\mathbb{T}} \ni (\rho, \theta)$. Then the solution u of Eq. (24.17) can be written in the form $u = \frac{v}{\sqrt{b}}$, where $\Delta v = 0$. Moreover, the original Cauchy data (24.13) for $w(x, y) = u(\rho, \theta)$ are transformed to the Cauchy data*

$$v\Big|_{\rho=1} = \Phi^v(\theta), \qquad \frac{\partial v}{\partial\rho}\Big|_{\rho=1} = \Psi^v(\theta) \tag{24.18}$$

for the function v harmonic in $V_{\mathbb{T}}$, where

$$\Phi^v(\theta) = F(\theta)Z\Big|_{\rho=1}, \quad \Psi^v(\theta) = Z\Big|_{\rho=1}\Big(G(\theta) + \frac{1}{2}F(\theta)\big(Z'_\rho/Z\big)\Big|_{\rho=1}\Big).$$

Proof In view of Lemma 24.8, one has to check that $u = -1/Z$ is a solution of the equation $\operatorname{div}\left(\beta\nabla w\right) \overset{(24.12)}{=} 0$ in $V_\Gamma \quad\Leftrightarrow\quad \operatorname{div}\left(b\nabla u\right) = 0$ in $V_{\mathbb{T}}$, which can be written as

$$-Z^2\Delta\Big(\frac{1}{Z}\Big) + 2Z^3\Big|\nabla\Big(\frac{1}{Z}\Big)\Big|^2 = 0\,.$$

We have $\nabla\left(\frac{1}{Z}\right) = -\frac{1}{Z^2}\nabla Z$ and

$$\Big|\nabla\Big(\frac{1}{Z}\Big)\Big|^2 = \frac{1}{Z^4}|\nabla Z|^2, \quad \Delta\Big(\frac{1}{Z}\Big) = \Big(-\frac{1}{Z^2}\nabla Z\Big) = \frac{2}{Z^3}|\nabla Z|^2 - \frac{1}{Z^2}\Delta Z$$

and hence

$$-Z^2\Delta\Big(\frac{1}{Z}\Big) + 2Z^3\Big|\nabla\Big(\frac{1}{Z}\Big)\Big|^2 = \Delta Z = 0\,.$$

Finally, formulas (24.18) are direct consequences of the isometry (see Lemma 24.3) between the circle and the curve Γ, which is effected by the mapping (24.8). □

It remains to note that each of the above functions U or v, being harmonic in a neighborhood of the unit circle and satisfying the Cauchy data, is represented as a series

$$\operatorname{Re}\left(\varphi_0 + \psi_0 \ln \rho + \frac{1}{2} \sum_{k \geq 1} \left\{ \left(\varphi_k + \frac{\psi_k}{k}\right) \rho^k + \left(\varphi_k - \frac{\psi_k}{k}\right) \rho^{-k} \right\} e^{ik\theta} \right).$$

25 On the Poincaré–Steklov Operators and Explicit Formulas

In 1869, Schwartz[61] Schwarz (1869) proposed the idea of alternating constructing the solution of the Dirichlet problem for the two-dimensional Laplace equation in a "complicated" domain consisting of a finite union of "simple" domains (see, for example, Courant 1992, Demidov et al. 2005), which uses solutions of the Dirichlet problem in "simple" domains Ω_k. Such algorithm is called the Schwartz alternating method.[62]

The idea of Schwarz's alternating method was developed in the so-called methods of decomposition of a "complicated" domain $\Omega = \cup_{k\geq 1}^m \omega_k$ into "simple" domains. This algorithm (see, for example, Agoshkov 2020) can be used for effective parallelization of numerical calculations when constructing a solution $u : \Omega \to \mathbb{R}$ of the boundary-value problem in the original complicated domain. Namely, the desired solution u can be constructed by "gluing" the solutions of u_k if one knows a correspondence between the boundary conditions on u_k for the adjacent domains ω_k. In some problems (see, for example, Grinberg 1948, Maergoiz 1971) these conditions have the form

$$(1-\lambda)u_k - (1+\lambda)u_j = 2f, \quad |\lambda| \leq 1, \qquad \text{and} \quad \frac{\partial u_k}{\partial \nu} = \frac{\partial u_j}{\partial \nu}, \tag{25.1}$$

here ν is the normal vector to the boundary of the adjacent domains ω_k and ω_j. In many problems such boundary operators arise in the form $a_k u_k + b_k \frac{\partial u_k}{\partial \nu} = f_k$, where $a_k b_k \geq 0$, $a_k + b_k \neq 0$ and ν is the outer normal to the boundary of the corresponding "simple" domain. In particular, we can talk about correspondences between the Dirichlet operator $\mathcal{D} : u \mapsto u\big|_{\partial\Omega}$ and the Neumann operator $\mathcal{N} : u \mapsto \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ whose alternate compositions $\mathcal{DN} : u\big|_{\partial\Omega} \mapsto \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ and $\mathcal{ND} : \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} \mapsto u\big|_{\partial\Omega}$ are called,

[61] Carl Hermann Amandus Schwartz (1843–1921) was an outstanding German mathematician, a member of the Berlin Academy of Sciences. His supervisor was Weierstrass. H. A. Schwarz is the author of remarkable results in various fields of mathematics. Many important concepts are associated with his name, viz: *the Schwartz lemma* in complex analysis; *Schwarz–Christoffel integral*; the *symmetry principle* for analytic continuation of functions; the famous *Schwarz example* of a polyhedral surface inscribed in a cylinder, which has arbitrarily large area (1890) as a counterexample to the erroneous definition of the surface area, which was given in his textbook a French academician Joseph Alfred Serret (1819–1885), one of the authors of the famous formulas Frenet–Serret; extremely useful in analysis the Cauchy–Bunyakovsky–Schwarz inequality $|(x \mid y)| \leq \|x\| \cdot \|y\|$ proved by Schwartz in 1884. This inequality appeared in 1821 for the finite-dimensional case in Cauchy's textbook on calculus, and in the case of L^2 in Bounjakowsky (1859) by Victor Bouniakowsky (1804–1889), vice-president of the Russian Academy of Sciences, specialist in number theory and probability theory.

[62] A similar idea was used in Demidov (1994) and Demidov and Yatsenko (1994) for constructing solutions of elliptic boundary-value problems with nonlinear conjugation conditions.

respectively, the *Dirichlet–Neumann operator* and the *Neumann–Dirichlet operator*. The boundary conditions (25.1) and the composition

$$\Big(\alpha_1 u + \beta_1 \frac{\partial u}{\partial \nu}\Big)\Big|_{\partial\Omega} \mapsto \Big(\alpha_2 u + \beta_2 \frac{\partial u}{\partial \nu}\Big)\Big|_{\partial\Omega}, \qquad |\alpha_k| + |\beta_k| \neq 0,$$

of the operators $\mathcal{R}_1 : u \mapsto \big(\alpha_1 u + \beta_1 \frac{\partial u}{\partial \nu}\big)\big|_{\partial\Omega}$, $\mathcal{R}_2 : u \mapsto \big(\alpha_2 u + \beta_2 \frac{\partial u}{\partial \nu}\big)\big|_{\partial\Omega}$, bearing Robin's name, are called (see, for example, Khoromskij and Wittum 2004, Novikov and Taimanov 2018) *Poincaré–Steklov operators*, following V. I. Lebedev[63] (Lebedev and Agoshkov 1983). The point is that these operators also arise when solving general boundary-value problems by decomposition methods, while Poincaré and Steklov obtained fundamental results for such problems. Recall that the search for methods for construction of solutions to boundary-value problems, which was initiated by Euler, who proposed the idea of the method of separation of variables, had produced, for a long time, solutions only in very special cases. And it was only Poincaré (1896) who, for the first time, raised the question of representing a solution of general boundary-value problems by eigenfunction series expansions (called "fundamental" by Poincaré). This Poincaré's memoir served as a starting point for V. I. Steklov (see, for example, Stekloff 1900, Stekloff 1983, in which a general method of separation of variables was rigorously justified for constructing solutions to main problems of mathematical physics of the nineteenth century.

Below we consider the question of constructing explicit numerically realizable formulas for the Poincaré–Steklov operators as applied to harmonic functions in a simply connected domain Ω. And this, by virtue of the results of §24, makes it possible to obtain explicit numerically realizable formulas for the Poincaré–Steklov operators in the case of the equations

$$\operatorname{div}\big(\alpha(w)\nabla w\big), \qquad \operatorname{div}\big(\beta\nabla w\big) = 0\,.$$

Remark 25.1 Let $\Omega_\varepsilon \subset \mathbb{R}^2$ be a bounded simply connected domain with C^1-smooth boundary $\Gamma_\varepsilon = \partial\Omega_\varepsilon$. If the domain Ω differs from Ω_ε only in that its boundary $\Gamma = \partial\Omega$ is analytic and in the differs from Γ_ε in order by at most $\varepsilon > 0$ in the C^1-Hausdorff metric, then the Poincaré–Steklov operators for Ω_ε and Ω differ in the C^1-metric by the same order.

We restrict ourselves here to the construction of numerically realizable explicit formulas for the Poincaré–Steklov operator of the form $\mathcal{DR}$. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad \Omega, \qquad u\big|_{P_s} = F(s), \quad P_s = \big(x(s), y(s)\big) \in \Gamma \tag{25.2}$$

for the Laplace equation with Dirichlet boundary condition in the case of a simply connected domain $\Omega \subset \mathbb{R}^2$ with *analytic* boundary Γ and if length $|\Gamma| = 2\pi$. Here,

[63] Vyacheslav Ivaanovich Lebedev (1930–2010) is a prominent specialist in computational mathematics, laureate of the USSR State Prize for research on nuclear reactors (1987), his supervisor in graduate school was S.L. Sobolev.

as above in §24, $s \in \mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/2\pi$ is the natural parameter on Γ defined as the length of the arc $\smile P_0P_s$ of the curve Γ counted in positive direction from the point P_0 (for definiteness, with coordinates $(x, y) = (1, 0)$) to the point P_s, and the function F is given by the Fourier series

$$F : \mathbb{T} \ni s \mapsto \sum_{k\geq 0}(a_k \cos ks + b_k \sin ks)\,. \tag{25.3}$$

Repeating the construction of §24 leading to Lemma 24.3, we use the conformal mapping

$$z : V_{\mathbb{T}} \ni \zeta = \rho e^{i\theta} \mapsto z(\zeta) = x(\rho, \theta) + iy(\rho, \theta) \in V_\Gamma\,, \tag{25.4}$$

with

$$z(\zeta) = 1 + \int_1^\zeta e^{A+iB}\, d\zeta \quad \Leftrightarrow \quad A + iB = \ln \frac{dz}{d\zeta}\,.$$

This mapping (25.4) is univalent and maps isometrically the circle $\mathbb{T}$ onto the curve Γ, i.e.,

$$\left|\frac{dz(\zeta)}{d\zeta}\right| \equiv 1 \text{ for } \zeta \in \mathbb{T}\,. \tag{25.5}$$

Consider the curve $\gamma = z(C) \subset V_\Gamma$, where $C = \{|\zeta| = r < 1\}$. Let g be the trace on γ of the solution u to problem (25.2). We put

$$f : \mathbb{T} \ni \theta \mapsto f(\theta) = g(z)\Big|_{z=z(re^{i\theta})\in\gamma}.$$

The Fourier series of this periodic function has the form

$$f : \mathbb{T} \ni \theta \mapsto \sum_{k\geq 0}(c_k \cos k\theta + d_k \sin k\theta)\,. \tag{25.6}$$

Using (25.3) and (25.6), we set $\lambda_0 = c_0$ and for $k \geq 1$ we put

$$\lambda_k = a_k - \frac{c_k r^k - a_k r^{2k}}{1 - r^{2k}}\,, \quad \mu_k = b_k - \frac{d_k r^k - b_k r^{2k}}{1 - r^{2k}}\,,$$
$$\varphi_k = \frac{c_k - a_k r^k}{1 - r^{2k}}\,, \quad \psi_k = \frac{d_k - b_k r^k}{1 - r^{2k}}\,.$$

Then the function

$$U(\rho, \theta) = \lambda_0 + \sum_{k\geq 1}\left\{\rho^k\Big[\lambda_k \cos k\theta + \mu_k \sin k\theta\Big] + \left(\frac{r}{\rho}\right)^k\Big[\varphi_k \cos k\theta + \psi_k \sin k\theta\Big]\right\}, \tag{25.7}$$

which is harmonic in the annulus $V_{\mathbb{T}} = \{r < \rho < 1;\ \theta \in \mathbb{T}\}$, satisfies the boundary conditions $U(1, \theta) = F(\theta)$ and $U(r, \theta) = f(\theta)$.

Theorem 25.2 (Demidov 2023)

$$\mathcal{DNF}(s) \stackrel{\text{def}}{=} \frac{\partial u}{\partial \nu}\Big|_{P_s=P_{s(\theta)}\in\Gamma}$$
$$\stackrel{(25.5)}{=} \frac{\partial U(\rho,\theta)}{\partial \rho}\Big|_{\rho=1} = \sum_{k\geq 1} k\Big[(\lambda_k - \varphi_k)\cos k\theta + (\mu_k - \psi_k)\sin k\theta)\Big] \quad (25.8)$$

and

$$\mathcal{DR}(u\big|_{s\in\Gamma}) = \Big(\alpha u + \beta\frac{\partial u}{\partial \nu}\Big)\Big|_{P_s\in\Gamma} \stackrel{(25.7),(25.8)}{=} \alpha U(1,s) + \beta\frac{\partial U(\rho,s)}{\partial\rho}\Big|_{\rho=1}$$

for the operator $\mathcal{DR}$.

The constructions leading to Theorem 25.2 include a formula for $u\big|_\gamma$, where u is the solution to the Dirichlet problem (25.2). Let us say a few words here about the construction of explicit numerically realizable formulas for this solution u.

It is known (see Krylov and Bogolyubov 1929 and Kantorovich and Krylov 1962 that

$$u(x,y) = \int_0^{2\pi} \mu(t)K(x(t)-x, y(t)-y)\,dt, \qquad \text{where} \quad K(\xi(t),\eta(t)) = \frac{\eta'(t)\xi - \eta\xi'}{\xi^2+\eta^2}$$

i.e., $K(\xi(t),\eta(t)) = \frac{d}{dt}\arctan\frac{\eta(t)}{\xi(t)}$, and μ is the solution of the integral equation

$$\mu(\tau) + \frac{1}{\pi}\int_0^{2\pi} \mu(t)\frac{d}{dt}\arctan\frac{y(t)-y(\tau)}{x(t)-x(\tau)}\,dt = \frac{1}{\pi}F(\tau). \qquad (25.9)$$

Approximating the integral in (25.9) by a sum (for more details, see Vlasov and Bakushinskii 1963) and taking in sequence $\tau = t_1,\dots,t_n$, we obtain a system of linear algebraic equations, from which an approximation to μ (i.e., to the solution of the original Dirichlet problem) is found.

Remark 25.3 Numerical implementation of the Dirichlet-Robin operator can be found in Demidov and Samokhin (2023).

Numerical implementation of the Robin1–Robin2 operator, i.e. operator

$$\Big(\alpha_1 u + \beta_1\frac{\partial u}{\partial\nu}\Big)\Big|_\Gamma \mapsto \Big(\alpha_2 u + \beta_2\frac{\partial u}{\partial\nu}\Big)\Big|_\Gamma, \quad |\alpha_k| + |\beta_k| \neq 0,$$

reduces to the consideration of the above Dirichlet–Robin2 problem using construction (see, for example, Zhou and Cai 2016) solution such a Robin1–Dirichlet γ-problem $\Big(\alpha_1 u + \beta_1\frac{\partial u}{\partial\nu}\Big)\Big|_\Gamma \mapsto u\Big|_\gamma$.

Remark 25.4 Another particular case of the Poincaré–Steklov operator is the Grinberg–Maergoiz operator $\mathcal{G}_\mathcal{M}$, which for $-1 \leq \lambda < 1$ is defined by the conditions

$$(1-\lambda)u(P_s^-) - (1+\lambda)u(P_s^+) = 2F(P_s), \qquad \frac{\partial u}{\partial\nu}(P_s^-) = \frac{\partial u}{\partial\nu}(P_s^+), \qquad (25.10)$$

where the value of a function at P_s^+ (at P_s^-, respectively) is defined as the limit of the values at $P_s \in \Gamma$ along the normal ν from the outer (inner, respectively) side of Γ. Explicit formulas for the traces $u\big|_\gamma$ and $u\big|_\Gamma$ of the solution of problem (25.10) are given above for the Dirichlet problem, i.e., for $\lambda = -1$, and, for $|\lambda| < 1$, Maergoiz (1971) gives analogous formulas in terms of potentials with density satisfying a second-order integral equation. This makes it possible to find $\mathcal{G}_{\mathcal{M}}(F)$ (as in Theorem 25.2).

26 On the Fourier–Hörmander Operator and the Canonical Maslov Operator

Just simple! Can this be said about the canonical operator introduced by V. P. Maslov[64] in 1965 in his Doctoral thesis (Maslov 1965)? After all, even Maslov himself wrote Maslov (2006)[65] in 2000: "I would like to write my lecture ("Quantization of Thermodynamics"—*A. D.*) in a way that will be understood by both mathematicians and physicists. This is a very challenging task. I tried to solve it once when I was writing my first book, "Perturbation Theory and Asymptotic Methods" (i. e. Maslov 1965—*A. D.*), but it turned out that neither of them understood this."[66]

26.1 Nevertheless, in this section we attempt to give a (relatively) simple presentation of the elements of the theory of the canonical operator using a purely methodological approach. In contrast to the usual presentation of the construction of this operator, as if "it is given from above," we will try to identify it in a natural way, considering the specific problems of quantum mechanics that are asymptotically close to classical mechanics problems, on which originally the method of the canonical operator was focused.

But let us start with the first topic, more precisely, with some aspects concerning the *Fourier–Hörmander*[67] integral operator. According to Remark 21.15, in Hörman-

[64] Viktor Pavlovich Maslov (born 1930) is a Russian physicist and mathematician, a specialist in mathematical physics, a member of the Russian Academy of Sciences, a bright personality, whose personal characteristics can be seen, for example, from his numerous interviews http://trv-science.ru/2010/07/06/perepletenie-traektorij-zhizni/, dedicated to the memory of V. I. Arnold, but mainly in the context of the significance of his own book Maslov (1965). In the same interview in the correspondence discussion with L. Hörmander, he said: "This study (the book Maslov (1965)—*A. D.*) was checked by such subtle and remarkable mathematicians as G. I. Eskin and O. A. Ladyzhenskaya, and I answered all their questions exhaustively in the presence of such specialists as V. P. Palamodov and S. P. Novikov."

[65] See also http://www.ega-math.narod.ru/Nquant/Demidov.htm#VPM.

[66] See, however, Remark 99 on p. 182 on a significant and immediate reaction (as an opponent of V. P. Maslov's doctoral dissertation) of V. I. Arnold, who significantly enriched the method of canonical operator with his famous article Arnold (1967), which is almost identical to his review of 1965. The widespread acceptance of Maslov's method was also promoted by Leray (1972–73), who, in particular, initiated the translation of Maslov's doctoral dissertation "Théorie des Perturbations et Méthodes Asymptotiques" (Paris: Dunod, 1972) and Arnold's paper Arnold (1967).

[67] Hörmander himself, already a recognized classic by the age of 30, modestly called the operators introduced by him the Fourier operators in Hörmander (1968)]

der (1965) an invariant definition of a pseudo-differential operator on a differentiable manifold Ω was given; in particular, in a domain $\Omega \subset \mathbb{R}^n$. This definition is a natural generalization of the following fact pointed out by Hörmander: a differential operator P of order m with smooth coefficients on a smooth manifold Ω can be defined as a continuous linear operator for which

$$e^{-\mathring{\imath}\lambda g} P(f e^{\mathring{\imath}\lambda g}) = \sum_{j=0}^{m} P_j(f, g)\lambda^j, \quad \mathring{\imath} \stackrel{\text{def}}{=} 2\pi i, \tag{26.1}$$

for any $f \in C_0^\infty(\Omega)$ and $g \in C^\infty(\Omega)$, i.e., the function $e^{-\mathring{\imath}\lambda g} P(f e^{\mathring{\imath}\lambda g})$ is a polynomial of λ of degree m. It is clear that a differential operator P of order m has this property. The proof of the converse result is based on the *Peetre theorem* (see Problem 16.22 on p. 87 and the hint to this problem on p. 90) to the effect that a continuous linear operator $P\colon \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ with *localization property*

$$\operatorname{supp}(Pu) \subset \operatorname{supp}(u) \quad \forall u \in C_0^\infty(\Omega) \tag{26.2}$$

is a differential operator. So, we need only to check that (26.1) implies condition (26.2). To this end, we first assume that the *support* of some function $u \in C_0^\infty(\Omega)$ lies in the neighborhood $\mathcal{O}(x')$ of the point x', and a function $f \in C_0^\infty(\Omega)$ is 1 in $\mathcal{O}(x')$ and $f(x'') = 0$, where the point $x'' = x''(f)$, not lying in $\operatorname{supp} u$, as well as the point x', lies in an open (not necessarily connected) set $\omega \subset \Omega$, in which a local coordinate system $x_1, \dots, x_n$ is defined. Putting $g(x) = x\xi$, where $\xi \in \mathbb{R}^n$, we get

$$e^{-\mathring{\imath}\lambda x\xi} P(f(x)e^{\mathring{\imath}\lambda x\xi}) = \sum_{j=0}^{m} p_j(x, \xi)\lambda^j, \quad p_j(x'', \xi) = 0.$$

The left-hand side of this equality is infinitely differentiable with respect to (x, ξ) and remains the same if ξ is replaced by ξ/t and λ is replaced by $t\lambda$. Hence the functions p_j (which depend parametrically on the chosen function f) are homogeneous with degree j with respect to ξ and are differentiable with respect to ξ. Therefore,

$$p_j(x, \xi) = \sum_{|\alpha|=j} p_j(x)\xi^\alpha,$$

where $p_j \in C^\infty(\omega)$ and $p_j(x'') = 0$. Next, because

$$u(x) = (fu)(x) \stackrel{(17.27)}{=} \int \widetilde{u}(\xi) f(x) e^{\mathring{\imath}x\xi}\, d\xi, \quad \widetilde{u}(\xi) \stackrel{(17.24)}{=} \mathbf{F}_{x\to\xi} u(x),$$

and since $P\colon \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is a continuous operator, we find that

$$(Pu)(x) = \int P\Big(f(x)e^{\mathring{\imath}x\xi}\Big)\widetilde{u}(\xi)d\xi =$$
$$= \sum_{j=0}^{m}\sum_{|\alpha|=j}\int e^{\mathring{\imath}x\xi}p_j(x)\xi^{\alpha}\widetilde{u}(\xi)d\xi \overset{(17.28)}{=} \sum_{j=0}^{m}\sum_{|\alpha|=j} p_j(x)\mathcal{D}^{\alpha}u(x).$$

Since $p_j(x'') = 0$, we get $(Pu)(x'') = 0$. This fact in combination with the partition of unity gives the required inclusion (26.2).

For our purposes it is sufficient to consider the case when the manifold Ω is a domain in $\mathbb{R}^n$. In this case, $g(x) = x\xi$, where $\xi \in \mathbb{R}^n$.

Definition 26.2 (See Hörmander 1965) A continuous linear operator $A\colon \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is called a *pseudo-differential operator* if, for $m_0 > m_1 > m_2 > \ldots \to -\infty$ for all $f \in C_0^{\infty}(\Omega)$ and $g(x) = x\xi$, the asymptotic expansion

$$e^{-\mathring{\imath}\lambda g}A(fe^{\mathring{\imath}\lambda g}) \asymp \sum_{j=0}^{\infty} A_j(f,g)\lambda^{m_j}, \quad \lambda \to +\infty, \tag{26.3}$$

holds (cf. (26.1)). Namely, for $\lambda \geq 1$ for each natural N the difference

$$\lambda^{-m_N}\Big(e^{-\mathring{\imath}\lambda g}A(fe^{\mathring{\imath}\lambda g}) - \sum_{j=0}^{N-1} A_j(f,g)\lambda^{m_j}\Big)$$

lies in a bounded subset of $C^{\infty}(\Omega)$.

One can check that $A_j(f,tg) = t^{m_j}A_j(f,g)$ for $t > 0$, and

$$(Af)(x) = \int e^{\mathring{\imath}x\xi}a(x,\xi)\widetilde{f}(\xi)d\xi, \tag{26.4}$$

where $a(x,\xi) = e^{-\mathring{\imath}\lambda x\xi}A(fe^{\mathring{\imath}\lambda x\xi})$. Moreover, for any $K \Subset \Omega$ there exists a constant $C_K > 0$ such that (cf. (21.20))

$$\Big|\partial_x^{\alpha}\partial_{\xi}^{\beta}\big(a(x,\xi) - \sum_{j=0}^{N-1} a_j(x,\xi)\big)\Big| \leq C_K|\xi|^{m_N-|\beta|}, \quad x \in K, \tag{26.5}$$

where

$$a_j(x,\xi) \overset{\text{def}}{=} e^{-\mathring{\imath}\lambda x\xi}A_j(fe^{\mathring{\imath}\lambda x\xi}) \in C^{\infty}\big(\Omega \times (\mathbb{R}^n \setminus \{0\})\big).$$

Operators of the form (26.4) are typical for construction of solutions to elliptic problems (cf. formula (22.14)), however, nonelliptic problems involve the operators

$$(Af)(x) = \int e^{\mathring{\imath}\sigma(x,\xi)}a(x,\xi)\widetilde{f}(\xi)d\xi, \quad a \in S^m, \tag{26.6}$$

in which the function σ may be quite general. For example, the solution of the Cauchy problem for the wave equation

$$\Delta u - u_{tt} = 0, \quad u\big|_{t=0} = 0, \quad u_t\big|_{t=0} = f \in C_0^\infty(\mathbb{R}^n) \tag{26.7}$$

is expressed for $\sigma_\pm(x,\xi;t) = x\xi \pm |\xi|t$ by the formula

$$u(x,t) = \int \frac{e^{\mathring{\imath}\sigma_+(x,\xi;t)} - e^{\mathring{\imath}\sigma_-(x,\xi;t)}}{2\mathring{\imath}|\xi|}\widetilde{f}(\xi)d\xi.$$

Formula (26.6) can be (only formally!, as $m_0 + n \geq 0$) rewritten in the form $I_\varphi(af)(x) = Af(x)$, where

$$I_\varphi(af)(x) = \iint e^{\mathring{\imath}\varphi(x,y,\xi)}a(x,y,\xi)f(y)\,dy\,d\xi, \tag{26.8}$$

and $\varphi(x,y,\xi) = \sigma(x,\xi) - y\xi$ is the so-called phase. We will assume that $\varphi\colon (x,y,\xi) \mapsto \varphi(x,y,\xi)$ is a real function which is positively homogeneous in ξ of degree 1 and is smooth for $\xi \neq 0$. In this problem, one can choose coefficients of the differential operator $M = \sum \alpha_j \frac{\partial}{\partial \xi_j} + \sum \beta_j \frac{\partial}{\partial y_j}$ such that $e^{\mathring{\imath}\varphi(x,y,\xi)} = Me^{\mathring{\imath}\varphi(x,y,\xi)}$. Hence, for $m + n < k$, the integral on the right of (26.8) can be understood as the limit

$$I_\varphi(af)(x) = \lim_{\varepsilon\to 0}\iint e^{\mathring{\imath}\varphi(x,y,\xi)}L^k\big(a(x,y,\xi)\chi(\varepsilon\xi)f(y)\big)\,dy\,d\xi, \tag{26.9}$$

which exists by the Lebesgue theorem 8.34. Here L^k is the kth power of the operator L adjoint to M, and χ is a $C_0^\infty(\mathbb{R}^n)$-function, which equals 1 in the neighborhood of the origin.

26.3 Taking into account inequality (26.5) and Remark 16.24, we note that the distribution $A\colon \mathcal{D} \ni f \mapsto I_\varphi(a(x,\xi)f$ has the singularity order $k = m + n + 1$. Moreover, $I_\varphi(af)\colon \Omega \ni x \overset{C^\infty}{\mapsto} I_\varphi(af)x$ for the points $x \in \Omega$, at which $\varphi'_\xi \overset{\text{def}}{=} \nabla_\xi\varphi(x,y,\xi) \neq 0$ for $\xi \neq 0$. The *singular support* of a distribution (denoted by sing supp) is the complement of the maximal open set in which the distribution is infinitely differentiable. Hence, setting

$$\Gamma_\varphi = \{(x,\xi)\colon \varphi'_\xi = 0\}, \quad C_\varphi = \{\,(x,\varphi'_x)\colon (x,\xi) \in \Gamma_\varphi\,\},$$

we get $\operatorname{sing\,supp} A \subset \{x\colon \nabla_\xi\varphi(x,y,\xi) = 0\}$, i.e.,

$$\operatorname{sing\,supp} A \subset \operatorname{proj}\Gamma_\varphi = \operatorname{proj} C_\varphi,$$

where proj is the projection operator onto Ω.

A phase φ is said to be *nondegenerate* if the differentials $d\frac{\partial\varphi}{\partial\xi_j}$, $j = 1,\ldots,n$, are linearly independent Γ_φ. Hence (by the implicit function theorem) Γ_φ is a manifold of dimension n. For problem (26.7), we have

$$\varphi = (x-y)\xi \pm |\xi|t, \quad \Gamma_\varphi = \{x - y = \mp t\operatorname{sgn}\xi\},$$
$$\operatorname{proj}\Gamma_\varphi = \{(x-y)^2 = t^2\}.$$

Therefore, the singularities of the solution to this problem are contained in the light cone $|x|^2 \le t^2$.

One of the goals of the theory of Fourier–Hörmander integral operators is to study where the singularities of solutions of very general equations of mathematical physics are concentrated. Questions of this kind are of considerable interest, in particular, for reconstruction of an image from the measurement of scattered waves (Krishnan and Quinto 2015), for reconstruction of singularities from bounded X-ray computed tomography data (Quinto 2017).

The next theorem is another important result of the theory of Fourier–Hörmander integral operators, which essentially generalizes formula (26.3) for pseudo-differential operators.

Theorem 26.4 (See Hörmander 1972) *Let*

$$(Af)(x) = \iint e^{\hat{\imath}\varphi(x,y,\xi)} a(x,y,\xi) f(y)\, dy\, d\xi, \quad f \in C_0^\infty(\mathbb{R}^n),$$

where $\varphi(x,\cdot,\cdot)$ is a nondegenerate phase, $a(x,\cdot,\cdot) \in S^m$ and vanishes outside the set

$$\{(y, \lambda\xi)\colon \lambda \ge 1,\ (y,\xi) \in K\}$$

for some compact set K. Let $\psi \in C^\infty(\mathbb{R}^n)$ be a real function and $\nabla\psi \ne 0$ on $\operatorname{supp} f$. Next, suppose that there is precisely one point (y_s, ξ_s) for which $y_s \in \operatorname{supp} f$, $\varphi'_\xi(y_s,\xi_s) = 0$, $\varphi'_y(y_s,\xi_s) = \psi'_y(y_s)$ and $\det Q \ne 0$, where

$$Q = \begin{pmatrix} \varphi''_{\xi\xi} & \varphi''_{\xi x} \\ \varphi''_{\xi x} & \varphi''_{xx} - \psi''_{xx} \end{pmatrix}.$$

Then

$$e^{\hat{\imath}\lambda\psi(y_s)}(Afe^{-\hat{\imath}\lambda\psi} - |\det Q|^{-1/2} e^{\frac{1}{2}\hat{\imath}\operatorname{sgn} Q} a(y_s, \lambda\xi_s) f(y_s) \in S^{m-1}.$$

This theorem has a close connection to the formulas obtained by V. P. Maslov in 1965 for the canonical operator, to which the reset of this section is devoted.

26.5 As already mentioned, we will try to identify the construction of the canonical operator by considering specific problems of quantum mechanics that are asymptotically close to classical mechanics. We shall start with the construction of an asymptotic formula for $1/h \gg 1$ of the solution[68] $\Psi\colon \mathbb{R}\times\mathbb{R}_+ \ni (x,t) \mapsto \Psi(x,t) \in \mathbb{C}$ of the Cauchy problem for the following variant of the *Schrödinger equation*:[69]

[68] In 1930, American mathematician M. Stone (1903–1989) proved a fundamental result (Stone 1932) on unitary groups $t \mapsto e^{it\mathcal{H}}$ in L^2. This fact proved instrumental (see, for example, Mackey 1963, Faddeev and Yakubovskiĭ 2009, Takhtajan 2008, Berezin and Shubin (2012), and also Reed and Simon 1972, Yosida 1965) in the derivation of the existence and uniqueness theorem of the solution $\Psi \in C^1(\mathbb{R}_+; L^2(\mathbb{R}^n))$ to fairy general problems of quantum mechanics, and, in particular, for $n = 1$ for problem (26.10). The membership of the function Ψ to the space $C^1(\mathbb{R}_+; L^2(\mathbb{R}^n))$ means that the mapping $t \mapsto \Psi(\cdot,t)$ is continuously differentiable, and $\|\Psi(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$.

[69] Erwin Rudolf Josef Alexander Schrödinger (1887–1961) was an Austrian physicist, one of the founders of quantum mechanics, winner of the Nobel Prize in Physics (1933).

$$ih\Psi_t + \frac{h^2}{2}\Psi_{xx} = \frac{x^2}{2}\Psi, \quad \Psi(x,0)\Big|_{x=x^\circ} = \Phi(x^\circ)e^{\frac{i}{h}x^\circ}. \tag{26.10}$$

Here the initial function, which has amplitude $\Phi \in C_0^\infty(\mathbb{R})$, normalized by[70]

$$\|\Phi\|_{L^2(\mathbb{R})} = 1, \tag{26.11}$$

is locally a sinusoidal wave emanating from the point $x^\circ \in \mathbb{R}$ and oscillating with frequency $1/h \gg 1$. Over time, the structure of such a "rigid" (X-ray radiation type) wave can, of course, change both frequency and amplitude. Therefore, it seems natural to look for the first approximation of the asymptotics (depending[71] *a priori* on x°) in the form[72]

$$\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)} \tag{26.12}$$

with unknown real functions S and φ. Substituting this function ψ in Eq. (26.10), we get

$$\left(\varphi\left[S_t + \frac{x^2 + S_x^2}{2}\right] - ih\left[\varphi_t + S_x\varphi_x + \frac{S_{xx}\varphi}{2}\right] + \frac{h^2}{2}\varphi_{xx}\right)e^{\frac{i}{h}S(x,t)} = 0, \tag{26.13}$$

$$e^{\frac{i}{h}S(x,0)}\varphi(x,0) = \Phi(x)e^{\frac{i}{h}x}. \tag{26.14}$$

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[70] According to V. I Arnold (Arnold 2008): "Not everyone knows how huge the role of H. Weyl was in the maturation of quantum mechanics. Schrödinger writes that for some time he was unable to get the spectra of atoms observed in the experiment by using the already known "wave–particle" duality, even though, in the class of functions he considered for the Schrödinger equation, the spectrum was found to be continuous (as in the Fourier integral), which contradicted the observations. This is because the domain on which the equation was considered was unbounded.

However, Weyl, with whom Shrödinger spoke about his difficulties, pointed out that he once faced this difficulty in elasticity theory, where he considered oscillations and waves in unbounded domains: to obtain a discrete spectrum one should pose boundary conditions at infinity—for example, requiring that the ψ-function should be square integrable. Shrödinger immediately followed this suggestion and obtained the required spectrum of hydrogen. As a result, the matrix quantum mechanics was rapidly replaced by the wave quantum mechanics."

[71] This is so, for example, if x is replaced by $\ln|x|$, i.e., in the case of the equation

$$ih\Psi_t + \frac{h^2}{2}x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\Psi\right) = \frac{\ln^2|x|}{2}\Psi.$$

[72] This is the so-called *WKB method*, named after physicists Gregor Wentzel (1898—1978), Hendrik Anthony Kramer (1894–1952), and Léon Brillouin (1889–1969), who all developed it in 1926 for problems of quantum mechanics (see, for example, Maslov and Fedoryuk 1976). This method is also called the *Debye procedure*, since it was first applied to partial differential equations by the famous Dutch physicist and physical chemist Peter Debye (1884–1966), winner of the Nobel Prize in Chemistry (1936), who moved to the United States in 1939. It should be emphasized that the WKB method allows one to construct, generally speaking, only the so-called *formal asymptotics*, i.e., it produces one function (*a priori* there may be several of them), which asymptotically satisfies only the equation but may not be asymptotically close to the solution. Usually it is not an easy task to justify the asymptotics, i.e., to prove that the formal asymptotics is asymptotically close to the solution. It also happens that some or other formal asymptotics is not asymptotically close to the solution.

Since $1/h \gg 1$, from Eqs. (26.13) and (26.14) we conclude that

$$S_t + \frac{1}{2}\left(x^2 + S_x^2\right) = 0, \quad S(x,0) = x, \tag{26.15}$$

$$\varphi_t + S_x\varphi_x + \frac{1}{2}S_{xx}\varphi = 0, \quad \varphi(x,0) = \Phi(x). \tag{26.16}$$

26.6 Note that problem (26.15) is precisely problem (11.48), whose solution was already found and given by formula (11.44). We give this formula again:

$$S(x,t) = \frac{\sin 2t}{4} + \frac{1}{\cos t}(x - \sin t) - \frac{\tan t}{2}(x^2 - \sin^2 t). \tag{26.17}$$

Here[73]

$$(x,t) \in \Omega \overset{(26.67)}{=} \big(\mathbb{R} \times [0,T]\big) \setminus \{|x - x_k| > 0,\ t = t_k,\ k \geq 1\}, \tag{26.18}$$

and $(x_k, t_k) = \big((-1)^{k-1}, \frac{\pi}{2}(2k-1)\big)$ are the points of intersection of the characteristics

$$x(x^\circ, t) \overset{(11.36)}{=} \sin t + x^\circ \cos t, \quad x^\circ \in \mathbb{R}, \quad t \in [0,T], \tag{26.19}$$

(which precisely cover the domain Ω) pertaining to the Cauchy problem for the Hamiltonian system[74]

$$\dot{p} + H_x \overset{(11.33)}{=} 0, \quad p(0) = 1, \quad \dot{x} - H_p \overset{(11.34)}{=} 0, \quad x(0) = x^\circ \tag{26.20}$$

with the Hamiltonian $H(p,x,t) = \frac{p^2+x^2}{2}$.

Let us give two more previously established formulas, which, however, follow from (26.19) and (26.20):

$$x^\circ(x,t) \overset{(11.38)}{=} \frac{x - \sin t}{\cos t} \quad \text{for } t \neq t_k \tag{26.21}$$

and $p(x(x^\circ,t),t) \overset{(11.39)}{=} \cos t - x^\circ \sin t$. Hence by (26.21) we get

$$x(p,t) = \frac{1 - p\cos t}{\sin t} \quad \text{for } t \neq k\pi. \tag{26.22}$$

26.7 Let us now consider problem (26.16). Note that

$$S_x(x,t) \overset{(26.17)}{=} \frac{1 - x\sin t}{\cos t}, \quad S_{xx}(x,t) = -\tan t. \tag{26.23}$$

[73] Despite the apparent singularity of the function $S\colon \Omega \ni (x,t) \mapsto S(x,t)$ for $t = t_k$ (because $\cos t_k = 0$), it actually does not exist. Moreover, $\lim_{t\to t_k} S(x,t)\big|_{x=x(x^\circ,t)} = 0$ according to what was said after (11.45) on p. 68.

[74] See footnote 81 on p. 68.

Hence the characteristic[75] $t \mapsto y(t) = y(x^\circ, t)$ of Eq. (26.16), which subject to the conditions $\frac{dy}{dt} = S_y(y,t)$ and $y(0) = x^\circ$, is given by the formula

$$y(x^\circ, t) = x^\circ \cos t + \sin t \overset{(26.19)}{=} x(x^\circ, t), \tag{26.24}$$

and hence, it coincides with the characteristic of Eq. (11.34). This is not surprising because

$$S_x(x,t) \overset{(11.39),\,(26.23)}{=} p(x,t). \tag{26.25}$$

In turn, the sought-for solution φ of problem (26.16) satisfies the equation

$$\frac{d\varphi}{dt} = -\frac{1}{2} S_{xx}(x,t)\varphi, \quad \text{i.e.,} \ \frac{d\varphi}{dt} \overset{(26.23)}{=} \frac{\tan t}{2}\varphi,$$

and, therefore, in view of the equality $\varphi(x^\circ, 0) = \Phi(x^\circ)$, we get[76]

$$\varphi(x(x^\circ,t),t) = \frac{\varphi(x^\circ,0)}{\sqrt{|\cos t|}} \overset{(26.24)}{=} \Phi\Big(\frac{x(x^\circ,t) - \sin t}{\cos t}\Big)\frac{1}{\sqrt{|\cos t|}}. \tag{26.26}$$

We have $\int_{-\infty}^{\infty} f(-\mu)\,d\mu = \int_{-\infty}^{\infty} f(\mu)\,d\mu$ (in particular, for $d\mu = \frac{dx}{\cos t}$), and hence

$$\|\varphi(\cdot,t)\|_{L^2(\mathbb{R})} \overset{(26.26)}{=} \|\Phi\|_{L^2(\mathbb{R})} \overset{(26.11)}{=} 1 \quad \text{for} \ t \neq \frac{(2k-1)\pi}{2}. \tag{26.27}$$

And since $\Phi \in C(\mathbb{R}) \cap L^2(\mathbb{R})$, for any $\varepsilon > 0$ there exists an $a_\varepsilon > 0$ such that $\int_{|x^\circ|>a_\varepsilon} \Phi^2(x^\circ)\,dx^\circ = \int_{\frac{|x-\sin t|}{|\cos t|}>a_\varepsilon} \varphi^2(x(x^\circ,t),t)\,dx < \varepsilon$. Hence, taking into account (26.27), we find that

$$\lim_{t \to \frac{(2k-1)\pi}{2}} \varphi^2(\cdot,t) = \delta(x - x_k), \quad \text{where} \ x_k = (-1)^{k-1}, \tag{26.28}$$

and so

$$\lim_{t \to \frac{(2k-1)\pi}{2}} \int_{\mathbb{R}} \varphi(x,t) g(x)\,dx = 0 \tag{26.29}$$

for any function $g \in C(\mathbb{R})$.

Theorem 26.8 *For $t \le T < \frac{\pi}{2}$, the function*

$$\psi\colon (x,t) \mapsto \psi(x,t) \overset{(26.12)}{=} \varphi(x,t) e^{\frac{i}{h}S(x,t)}, \tag{26.30}$$

where φ and S are defined by (26.17) and (26.26), is the asymptotics in $h \to 0$ of the solution $\Psi \in C^1(\mathbb{R}_+; L^2(\mathbb{R}))$ of problem (26.10) corresponding to the initial function

[75] *A priori*, this characteristic might be different from the characteristic (26.19). Hence their designations are different here.

[76] Bearing in mind a generalization to the multivariate setting, we note that the factor $\frac{1}{\sqrt{|\cos t|}}$ appearing in (26.26) is $\left|\det \frac{\partial x(x^\circ,t)}{\partial x^\circ}\right|^{-1/2}$.

$\Phi \in H^2(\mathbb{R})$. *Namely,*

$$\|\Psi(\cdot,t)-\psi(\cdot,t)\|_{L^2(\mathbb{R})} \le \frac{h}{2}\ln\Big|\tan\Big(\frac{t}{2}+\frac{\pi}{4}\Big)\Big|\cdot\|\Phi''\|_{L^2(\mathbb{R})}. \tag{26.31}$$

However, for any $h>0$,

$$\lim_{t\to\frac{\pi}{2}} \|\Psi(\cdot,t)-\psi(\cdot,t)\|_{L^2(\mathbb{R})} > 1, \tag{26.32}$$

i.e., ψ *is not an asymptotics for* Ψ *on the interval* $0 \le t < \frac{\pi}{2}$.

Proof Taking into account (26.13)–(26.16), we insert in Eq. (26.10) first $Z = \Psi - \psi$ and then $\overline{Z} \overset{\text{def}}{=} \operatorname{Re} Z - i \operatorname{Im} Z$. As a result, we get

$$\begin{cases} ihZ_t + \dfrac{1}{2}\big(h^2 Z_{xx} - x^2 Z\big) = -\dfrac{h^2}{2} e^{\frac{i}{h}S}\varphi_{xx}, \\ -ih\overline{Z}_t + \dfrac{1}{2}\big(h^2 \overline{Z}_{xx} - x^2 \overline{Z}\big) = -\dfrac{h^2}{2} e^{-\frac{i}{h}S}\varphi_{xx}. \end{cases} \tag{26.33}$$

Multiplying the first of these equations by $\overline{Z}$, and the second, by Z, and subtracting, this gives

$$ih\big(\overline{Z}Z_t + Z\overline{Z}_t\big) + \frac{h^2}{2}\big(\overline{Z}Z_{xx} - Z\overline{Z}_{xx}\big) = ih^2\varphi_{xx}\cdot \operatorname{Im}\big(e^{-\frac{i}{h}S}Z\big).$$

Next, we divide by ih and integrate with respect to x, taking into account that

$$\int_{\mathbb{R}} \big(\overline{Z}Z_{xx} - Z\overline{Z}_{xx}\big)\,dx = \int_{\mathbb{R}} \big(\overline{Z}Z_x - Z\overline{Z}_x\big)_x\,dx = 0.$$

Hence, denoting by $\|\cdot\|$ the norm in $L^2(\mathbb{R})$, we get[77]

$$\frac{\partial}{\partial t}\|Z\|^2 = h\int_{\mathbb{R}} \varphi_{xx}\cdot \operatorname{Im}\big(e^{-\frac{i}{h}S}Z\big)\,dx \overset{(9.3)}{\le} h\|\varphi_{xx}\|\cdot\|Z\|. \tag{26.34}$$

We have

$$\|\varphi_{xx}(\cdot,\tau)\|^2 \overset{(26.26)}{=} \frac{1}{\cos^2\tau}\int_{-\infty}^{\infty} \big(\Phi''(y)\big|_{y=\frac{x-\sin\tau}{\cos\tau}}\big)^2 \frac{dx}{|\cos\tau|} = \frac{\|\Phi''\|^2_{L^2(\mathbb{R})}}{\cos^2\tau},$$

and hence, dividing both parts of inequality (26.34) by $\|Z\|$ and taking into account that $Z\big|_{t=0} = 0$, we get $\|Z(\cdot,t)\| \le h\int_0^t \frac{d\tau}{2\cos\tau}\|\Phi''\|$, i.e., inequality (26.31) is satisfied.

Let us prove inequality (26.32). According to footnote 77, we have

$$\|\Psi(\cdot,t)\|^2 \overset{(26.10)}{=} \|\Phi(\cdot)\|^2.$$

[77] Applying the same argument to Ψ in place of $Z = \Psi - \psi$ we get the system, whose only difference from (26.33) is that its right-hand side contains 0. Hence finally we get $\frac{\partial}{\partial t}\|\Psi\|^2 = 0$.

Moreover, $\|\Phi(\cdot)\|^2 \overset{(26.11)}{=} 1$, $\|\psi(\cdot,t)\|^2 \overset{(26.12)}{=} \|\varphi(\cdot,t)\|^2 \overset{(26.27)}{=} 1$, and[78]

$$\|\Psi(\cdot,t)-\psi(\cdot,t)\|^2 = \|\Psi(\cdot,t)\|^2 + \|\psi(\cdot,t)\|^2 - 2\,\mathrm{Re}\big(\Psi(\cdot,t)\big|\psi(\cdot,t)\big).$$

Hence

$$\lim_{t\to\frac{\pi}{2}} \|\Psi(\cdot,t)-\psi(\cdot,t)\|^2 = 2,$$

because $\lim_{t\to\frac{\pi}{2}} \big(\Psi(\cdot,t)\big|\psi(\cdot,t)\big) = 0$. Indeed,

$$\Big(\Psi(\cdot,t)\Big|\psi(\cdot,t)\Big) = \Big(\Psi\Big(\cdot,\frac{\pi}{2}\Big)-\Psi(\cdot,t)\Big|\psi(\cdot,t)\Big) + \Big(\Psi\Big(\cdot,\frac{\pi}{2}\Big)\Big|\psi(\cdot,t)\Big),$$

and as $t \to \frac{\pi}{2}$ we have

$$\Big|\Big(\Psi\Big(\cdot,\frac{\pi}{2}\Big)-\Psi(\cdot,t)\Big|\psi(\cdot,t)\Big)\Big| \le \Big\|\Psi\Big(\cdot,\frac{\pi}{2}\Big)-\Psi(\cdot,t)\Big\| \to 0,$$

because $\Psi \in C^1(\mathbb{R}_+; L^2(\mathbb{R}))$ and $\big(\Psi\big(\cdot,\frac{\pi}{2}\big)\big|\psi(\cdot,t)\big) \to 0$ in view of (26.29). □

The asymptotics of Ψ in $h \to 0$ for $t \le T$ for any $T < \infty$ will be given in Theorem 26.14. Here an important role will be played by the Fourier transform, in terms of which one formulates the main axioms of quantum mechanics, which is related to the asymptotics of Ψ as $h \to 0$ we are interested in. So, in the next two subsections we recall some facts from quantum mechanics.

26.9 In 1926, almost immediately after the publication of the Schrödinger equation, M. Born[79] suggested a probabilistic interpretation of the wave function, i.e., the solution of the Schrödinger equation. Later, this interpretation had become generally accepted (though initially only Schrödinger himself did not agree with interpretation). A year later, N. Bohr[80] and W. Heisenberg,[81] during their joint work in Copenhagen, improved Born's interpretation. According to its basic principles, quantum mechanics is a statistical theory, since the measurement of the initial conditions of a micro-object changes its state, which in turn leads to the probabilistic description of the wave function, which is also called the position function Here, not the object itself is physically significant, but the square of its module, which means the probability of finding the micro-object under consideration somewhere in the space. Therefore, in quantum mechanics one seeks not the coordinates of a particle and its momentum, but the distribution of their probabilities.

So (in the one-dimensional case under consideration), we have

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[78] Recall that $(x\,|\,y)$ denotes the inner product of points x and y in a Hilbert space.

[79] Max Born (1882–1970) was a German and British physicist and mathematician, one of the founders of quantum mechanics, Born won the 1954 Nobel Prize in Physics. One of his pupils was Robert Oppenheimer, the "father of the atomic bomb."

[80] Niels Henrik David Bohr (1885–1962) was a Danish physicist, one of the creators of modern physics. In 1922 Bohr was awarded the Nobel Prize in Physics.

[81] Werner Karl Heisenberg (1901–1976) as a German theoretical physicist and one of the key pioneers of quantum mechanics, a Nobel laureate in physics (1932).

(1) the *position function* $\psi\colon (x,t) \mapsto \psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$, which obeys the Schrödinger equation (26.10) and characterizes the probability $P\{a < x < b\} = \int_a^b |\psi(x,t)|^2\, dx \le \|\psi\|^2_{L^2(\mathbb{R})} = 1$ for the quantum particle to stay in the interval $]a,b[$, and

(2) the *momentum function* $\widetilde{\psi}\colon (p,t) \mapsto \widetilde{\psi}(p,t)$, where p is the dual variable to x with respect to the *Hamiltonian system* (26.20).

At the same time, according to one of the axiomatics of quantum mechanics, the momentum function $\widetilde{\psi}$ is (up to some scale coefficients) the Fourier transform of the position function.

By shifting along the x-axis, we can assume that the expectation $M[x] \stackrel{\text{def}}{=} \int_a^b x|\psi(x,t)|^2\, dx$ of the variable x is zero. In this case, the variance $D[x] \stackrel{\text{def}}{=} M[(x - M[x])^2]$ of the random variable x (its mean square deviation) is

$$D[x] = \int_{-\infty}^{\infty} x^2 |\psi(x,t)|^2\, dx.$$

Note that a translation along the x-axis does not change $|\widetilde{\psi}(p,t)|$, because

$$\int_{-\infty}^{\infty} e^{-\imath xp}\psi(x-a)\, dx = e^{-\imath ap}\int_{-\infty}^{\infty} e^{-\imath xp}\psi(x)\, dx.$$

Hence the expectation of the momentum can also be assumed to be zero. Hence the variance of the momentum is

$$D[p] = \int_{-\infty}^{\infty} p^2 |\widetilde{\psi}(p,t)|^2\, dp.$$

The following theorem holds.

Theorem 26.10 (The Uncertainty principle[82])
The following inequality holds:

$$D[x] \cdot D[p] \ge \frac{1}{4}. \tag{26.35}$$

Proof We have $0 \le \int_{-\infty}^{\infty} \left|\mu x\psi(x,t) + \psi'(x,t)\right|^2 dx = I_0\mu^2 + I_1\mu + I_2$.
Here $I_0 = \int_{-\infty}^{\infty} x^2 |\psi(x,t|^2\, dx$, i.e., $I_0 = D[x]$. Next, $I_1 = -1$, because

$$I_1 = \int_{-\infty}^{\infty} x\big(\psi(x,t)\bar{\psi}(x,t)\big)'_x\, dx = x\big(\psi(x,t)\bar{\psi}(x,t)\big)\Big|_{-\infty}^{\infty} - \|\psi\|^2_{L^2}.$$

Finally, using (17.28) and the *Parseval identity* (17.30) we have

[82] Heisenberg established relation of type (26.35) in 1927 by analyzing the methods of measuring the coordinates and momenta of particles. Thus, the evaluation of the coordinate of a particle by means of a beam(s) of light directed at it will be the more accurate, the shorter is the wavelength of the light wave and, accordingly, the greater is the photon momentum. This also leads to an increase in the uncertainty of the momentum of the particle to which the photon transmits a part of its momentum.

$$I_2 = \int_{-\infty}^{\infty} |\psi'_x(x,t)|^2\,dx = \int_{-\infty}^{\infty} p^2|\widetilde{\psi}(p,t)|^2\,dp = D[p].$$

So, $D[x]\mu^2 - \mu + D[p] \geq 0$, which is equivalent to inequality (26.35). □

Let us now recall that $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$. Moreover,

$$S(x_k,t_k) \overset{(11.47)}{=} 0, \quad \text{and} \quad \varphi^2(x,t_k) \overset{(26.28)}{=} \delta(x - x_k)$$

holds at the points $(x_k,t_k) = \big((-1)^{k-1}, \frac{\pi}{2}(2k-1)\big)$ (and only at these points). Hence at these points (x_k,t_k) (and only at these points) the position function ψ is the *δ-function*. This means that at time points $t = t_k$ the coordinate of the quantum particle is uniquely determined (and is equal to $x = x_k$). Hence by Theorem 26.10 the momentum $p(t_k)$ is uncertain. However,

$$\lim_{t\to t_k} p(x(x^\circ,t),t) \overset{(11.39)}{=} \lim_{t\to t_k}\big(\cos t - x^\circ \sin t\big) = (-1)^k x^\circ. \tag{26.36}$$

This fact fully corresponds to the previously noted specifics of the smooth manifold $\Lambda^2_{[0,t]} \subset \mathbb{R}_x \times \mathbb{R}_p \times \mathbb{R}_t$ (see p. 67) and which is a ruled surface formed by straight lines at $\theta(t) = -t$ to the Ox-axis and sliding along the graph of the function $(x(0,t),t) \mapsto \cos t$.

We note also that, in full agreement with Theorem 26.10, for any $x \in \mathbb{R}$ the momentum p assumes, for $t = k\pi$, a strictly definite value, namely $p(x,k\pi) \overset{(11.39)}{=} (-1)^{k-1}$.

26.11 Let us now return to the question of constructing the asymptotics Ψ as $h \to 0$ for $t \leq T$ for any $T < \infty$ (and not only for the case $t \leq T < \frac{\pi}{2}$, which was considered in Theorem 26.8). The construction of asymptotics presented there in terms of functions depending on (x,t) could not be continued up to $t = \frac{\pi}{2}$, because at time $t = \frac{\pi}{2}$ all the characteristics of (26.19) intersect at one point, which is the projection of the manifold $\Lambda^2_{[0,t]}\big|_{t=\frac{\pi}{2}} \subset \mathbb{R}_x \times \mathbb{R}_p$ onto the x-axis. However, there are two circumstances that will help to overcome this difficulty. First, the manifold $\Lambda^2_{[0,t]} \subset \mathbb{R}_x \times \mathbb{R}_p \times \mathbb{R}_t$ is one-one projected onto the plane $\mathbb{R}_p \times \mathbb{R}_t$ for $t \neq k\pi$, and second, according to the previous subsection, the variables x and p are related via the Fourier transform. Hence we will construct the asymptotics both in terms of functions, depending on (x,t) for $|t - k\pi| < \varepsilon < \frac{\pi}{2}$ and in terms of functions depending on (p,t) for $\big|t - \frac{(2k-1)\pi}{2}\big| < \varepsilon$. In this way, we write down the Schrödinger equation in the corresponding variables and match these asymptotics on the common time intervals, passing the original initial data as if "on the baton."

In particular, for $\big|t - \frac{\pi}{2}\big| < \varepsilon < \frac{\pi}{2}$ we need to change to the (p,t)-variables. To this end, we write down Eq. (26.10) in Fourier images by applying the h-Fourier transform, which is given (in the n-dimensional case) by the formula

$$[\breve{\mathbf{F}}v](p) = \Big(\frac{1}{2\pi h}\Big)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i}{h}xp}\, v(x)\,dx. \tag{26.37}$$

Up to the factor $i^{n/2}$, whose role will be indicated below, the *operator* $\breve{\mathbf{F}}$[83] appearing in this formula is a modification of one of the common forms of the Fourier transform, namely $\left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} e^{-ix\xi}u(x)\,dx$ (see the footnote 10 on p. 98), where we used the equality

$$u(x) \overset{\text{cf. (17.23)}}{=} \int_{\mathbb{R}^n} e^{ix\xi}\left(\frac{1}{2\pi}\right)^n\left(\int_{\mathbb{R}^n} e^{-iy\xi}u(y)\,dy\right)d\xi,$$

which implies the formula for the inverse h-Fourier transform

$$[\breve{\mathbf{F}}^{-1}\breve{v}](x) = \left(\frac{1}{2\pi h}\right)^{n/2}\int_{\mathbb{R}^n} e^{\frac{i}{h}xp}\breve{v}(p)\,dp, \tag{26.38}$$

we made the change $\xi = \frac{p}{h}$.

Both formula (17.28) and the equalities

$$\left(-\frac{i}{h}\right)^{|\beta|}\breve{\mathbf{F}}[\partial_x^\alpha\left(x^\beta v(x)\right)](p) = \left(\frac{i}{h}\right)^{|\alpha|}p^\alpha\partial_p^\beta[\breve{\mathbf{F}}v(x)](p) \tag{26.39}$$

are clear. Application of these formulas to Eq. (26.10) again transforms it to the Schrödinger equation, but now in the variables (p,t):

$$ih\breve{\Psi}_t + \frac{1}{2}h^2\breve{\Psi}_{pp} = \frac{p^2}{2}\breve{\Psi}, \quad \breve{\Psi} = \breve{\mathbf{F}}\Psi. \tag{26.40}$$

As in the case of Eq. (26.10), we shall search the asymptotic solution of Eq. (26.40) in the form $\breve{\psi}(p,t) = \breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}(p,t)}$, from which we get the system of equations (cf. formulas (26.15)–(26.16)):

$$\breve{S}_t + \frac{1}{2}\left(p^2 + \breve{S}_p^2\right) = 0, \quad \breve{\varphi}_t + \breve{S}_p\breve{\varphi}_p + \frac{1}{2}\breve{S}_{pp}\breve{\varphi} = 0. \tag{26.41}$$

Let us write down the Cauchy data for this system. They should match the asymptotics of the solution of problem (26.10) for some $t^\circ \in]0, \frac{\pi}{2}[$, in other words, they should satisfy the relation[84]

$$\breve{\mathbf{F}}[\varphi(x,t)e^{\frac{i}{h}S(x,t)}] = \breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}(p,t)} + O(h). \tag{26.42}$$

The left-hand side of Eq. (26.42) is $\frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}}\varphi(x,t)e^{\frac{i}{h}[S(x,t)-xp]}\,dx$, which is asymptotically,[85] for $1/h \gg 1$,

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[83] Pronounced "Ef breve." Symbol ˘ (breve, Ital.) means brevity, what is in agreement with small h.

[84] Here we limit ourselves to constructing the asymptotics of the original problem only up to $O(h)$.

[85] Formula (26.43) can be proved by the method of stationary phase (Fedoryuk 1987, 1971) by reducing the problem to the analysis of the integral $I(h) = \int_0^a \chi(y)e^{\frac{i}{h}y^2}\,dy$ with an appropriate function χ. The asymptotics of $I(h)$ can be found from evaluation of the Euler integral $\int_0^\infty e^{iy^2}\,dy = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}$ (with integration along the ray spanned by the vector $1+i$ (see Lavrent'ev and Shabat 1977) and integration by parts. Next, in order to change to the integral $I(h)$, one should fix t and p, put $g(x) = S(x,t) - xp$, $f(x) = \varphi(x,t)$, and note the principal contribution in the asymptotics as

$$\left\{ \frac{\varphi(x,t)}{\sqrt{|S_{xx}(x,t)|}} e^{\frac{i}{h}[S(x,t)-xp]} e^{\frac{i}{h}\frac{\pi}{4}\operatorname{sgn} S_{xx}(x,t)} \right\}\Big|_{x=x(p,t)} + O(h). \tag{26.43}$$

Here and in what follows[86] $x(p,t)$ is the stationary point of the phase $S(x,t) - xp$, i.e., $S_x(x,t)\big|_{x=x(p,t)} = p(t)$. From (26.42)–(26.43) we have

$$\breve{S}(p,t) = \Big([S(x,t) - xp] + \frac{i\pi}{4}\operatorname{sgn} S_{xx}(x,t)\Big)\Big|_{x=x(p,t)}, \tag{26.44}$$

and

$$\breve{\varphi}(p,t) = \frac{\varphi(x,t)}{\sqrt{|S_{xx}(x,t)|}}\Big|_{x=x(p,t)}. \tag{26.45}$$

Recalling formula (26.21) for $x^\circ(x,t)$ and using the formula

$$x(p,t) \overset{(26.22)}{=} \frac{1 - p\cos t}{\sin t} \quad \text{for } t \neq k\pi, \tag{26.46}$$

which also follows from the equalities

$$S_x(x,t) \overset{(26.23)}{=} \frac{1 - x\sin t}{\cos t} \overset{(26.25)}{=} p(x,t),$$

we get

$$x^\circ(p,t) \overset{\text{def}}{=} x^\circ(x,t)\big|_{x=x(p,t)} = \frac{\cos t - p}{\sin t}. \tag{26.47}$$

Next, employing the equivalence (11.45) $\Leftrightarrow$ (26.17) for S, using (26.26) for φ, and taking into account that $S_{xx}(x,t) \overset{(26.23)}{=} -\tan t$, we get for system (26.41) the following Cauchy data for $t = t^\circ \in \left]0, \frac{\pi}{2}\right[$:

$$\breve{S}(p,t^\circ) = \frac{(1+p^2)\cos t^\circ - 2p}{2\sin t^\circ} - \frac{i\pi}{4}, \quad \breve{\varphi}(p,t^\circ) = \frac{\Phi\big(x^\circ(p,t^\circ)\big)}{\sqrt{\sin t^\circ}}. \tag{26.48}$$

The corresponding solution of system (26.41) for $t \in]0,\pi[$ is given, up to the order,[87] $O(h)$ by the formulas:[88]

$h \to 0$ comes from the localization in the neighborhood of the point $\hat{x}$, where $g'(\hat{x}) = 0$, and where, respectively, the exponent is no longer rapidly oscillating (and hence the positive and negative parts of its real part are no longer integrally annihilate each other). Of course, one may assume that $\hat{x} = 0$, $g'(\hat{x}) = 0$ and $g''(\hat{x}) > 0$. Now by changing the variable $x = \lambda(y)$ (so that $g(x) = y^2$, $\operatorname{sgn} x = \operatorname{sgn} y$), we get $I(h)$, where $\chi(y) = f(\lambda(y))\lambda'(y)$.

[86] Of course, in order not to confuse $x(p,t)$ with $x(x^\circ, t)$, we could write $x(p,t)$ in place of $x_S(p,t)$. But we will not deviate here from the notation accepted in the theory of the canonical operator.

[87] According to (26.42); see the footnote 84 on p. 171.

[88] These formulas define the solution of system (26.41), which can be easily verified directly. Another way is to reduce problem (26.41), (26.48) to the already solved problems (26.15) and (26.16). To this end, one should in (26.41) replace p by $-x$, write $t - \frac{\pi}{2}$ in place of t, and multiply $\breve{\varphi}$ by

$$\breve{S}(p,t) = \frac{p^2+1}{2}\frac{\cos t}{\sin t} - \frac{p}{\sin t} - \frac{i\pi}{4}, \quad \breve{\varphi}(p,t) = \frac{\Phi\bigl(x^\circ(p,t)\bigr)}{\sqrt{|\sin t|}}. \tag{26.49}$$

Hence from (26.46) we get

$$\frac{\partial x(p,t)}{\partial p} = -\breve{S}_{pp}(p,t). \tag{26.50}$$

Problem 26.12 Verify the next theorem following the scheme of the proof of Theorem 26.8.

Theorem 26.13 *For* $\left|t - \frac{\pi}{2}\right| < \varepsilon < \frac{\pi}{2}$*, the function*

$$\breve{\psi}\colon (p,t) \mapsto \breve{\psi}(p,t) = \breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}(p,t)}, \quad p \in \mathbb{R}, \tag{26.51}$$

where $\breve{S}$ *and* $\breve{\varphi}$ *are defined* (26.49)*, is the asymptotics for the solution* Ψ *of problem* (26.10)*. Namely, as* $h \to 0$*, we have*

$$\int_{\mathbb{R}} \left|\Psi(x)\big|_{x=x(p,t)} - \breve{\psi}(p,t)\right|^2 dp \le C_\varepsilon h^2, \quad \text{where } x(p,t) = \frac{1 - p\cos t}{\sin t}.$$

However, $\lim_{t\to\pi} \|\Psi\bigl(x_S(\cdot,t)\bigr) - \breve{\psi}(\cdot,t)\|_{L^2(\mathbb{R})} > 1$ *for any* $h > 0$*, i.e.,* $\breve{\psi}$ *is not an asymptotics for* Ψ *if* $t < \pi$*.*

As for the subsequent time interval (i.e., the interval $|t - \pi| < \varepsilon < \frac{\pi}{2}$), the construction of the asymptotics should be carried out (as mentioned above) in terms of functions that depend on variables (x,t), by applying to Eq. (26.40) some or other modification of the Fourier transform. If in this way, we consider the mapping $\breve{\mathbf{F}}^{-1}$ given by (26.38). As a result, we get $ih\Psi_t + \frac{1}{2}h^2\Psi_{xx} = \frac{x^2}{2}\Psi$. Taking again the sought-for asymptotics in the form $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$, we get the system of equations (cf. (26.15), (26.16)):

$$S_t + \frac{1}{2}(x^2 + S_x^2) = 0, \quad \varphi_t + S_x\varphi_x + \frac{1}{2}S_{xx}\varphi = 0. \tag{26.52}$$

For this system, for some $t \in \left]\frac{\pi}{2}, \pi\right[$, we find the Cauchy data matching the asymptotics (26.51), or, in other words, satisfying the relation

$$\breve{\mathbf{F}}^{-1}[\breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}(p,t)}] = \varphi(x,t)e^{\frac{i}{h}S(x,t)} + O(h). \tag{26.53}$$

The left-hand side of (26.53) is $\frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} \breve{\varphi}(p,t)e^{\frac{i}{h}[\breve{S}(p,t)+xp]}\,dx$, which, for $1/h \gg 1$ is asymptotically equal to

$$\left\{e^{\frac{i}{h}[\breve{S}(p,t)+xp]}e^{\frac{i\pi}{4}\operatorname{sgn}\breve{S}_{pp}(p,t)}\frac{\breve{\varphi}(p,t)}{\sqrt{|\breve{S}_{pp}(p,t)|}}\right\}\Bigg|_{p=p(x,t)} + O(h). \tag{26.54}$$

$e^{\frac{i\pi}{4}}$. After that, one should apply the inverse transforms to the solutions of (11.45) and (26.26) of problems (26.15) and (26.16).

Here $p(x,t)$ is the stationary point of the phase $\breve{S}(p,t)+xp$. From (26.53), (26.54) we get

$$S(x,t)=\Big(\big[\breve{S}(p,t)+xp\big]+\frac{i\pi}{4}\operatorname{sgn}\breve{S}_{pp}(p,t)\Big)\Big|_{p=p(x,t)}, \tag{26.55}$$

and

$$\varphi(x,t)=\frac{\breve{\varphi}(p,t)}{\sqrt{|\breve{S}_{pp}(p,t)|}}\Big|_{p=p(x,t)}. \tag{26.56}$$

Since $\breve{S}_p(p,t)\overset{(26.49)}{=}\frac{p\cos t-1}{\sin t}$, we find that $p(x,t)=\frac{1-x\sin t}{\cos t}$. Hence employing (26.49) for $\breve{\varphi}$ and using the equality $\breve{S}_{pp}(p,t)\overset{(26.49)}{=}\cot t$, we get the Cauchy data for $t=t^\circ\in\,]\frac{\pi}{2},\pi[$:

$$S(x,t^\circ)=-\frac{x^2+1}{2}\tan t^\circ+\frac{x}{\cos t^\circ}-2\frac{i\pi}{4},\qquad \varphi(x,t^\circ)=\frac{\Phi\Big(\frac{x-\sin t^\circ}{\cos t^\circ}\Big)}{\sqrt{|\cos t^\circ|}}$$

for problem (26.55), (26.56). Hence the solution of Eqs. (26.55) and (26.56) for $|t-\pi|<\varepsilon<\frac{\pi}{2}$ corresponding to the Cauchy data is given, up to $O(h)$, by the formulas

$$S(x,t)=-\frac{x^2+1}{2}\tan t+\frac{x}{\cos t}-2\frac{i\pi}{4},\qquad \varphi(x,t)=\frac{\Phi\Big(\frac{x-\sin t}{\cos t}\Big)}{\sqrt{|\cos t|}}.$$

Constructing the asymptotics in succession on the set of intervals

$$O_m=\{|t-m\pi|<\varepsilon\}\quad\text{and}\quad O_{m+\frac{1}{2}}=\Big\{\Big|t-\frac{(2m+1)\pi}{2}\Big|<\varepsilon\Big\},$$

where $m=0,1,\dots,M$, $M\in\mathbb{N}$, and $\varepsilon\in\,]\frac{3}{8}\pi,\frac{\pi}{2}[$, which cover the closed interval $0\le t\le T$, $T=M\pi$, we arrive at the following result.

Theorem 26.14 *Let $\Psi\in C^1(\mathbb{R}_+;L^2(\mathbb{R}))$ be a solution of problem (26.10) corresponding to the initial function $\Phi\in H^2(\mathbb{R})$, $T=M\pi$, $M\in\mathbb{N}$, and let $\mathcal{K}\colon\Phi\mapsto\mathcal{K}\Phi$ be the operator given by the formula*

$$[\mathcal{K}\Phi](x,t)=\begin{cases}\alpha_m(t)\psi_m(x,t) & \text{for } t\in O_m,\\ \alpha_{m+\frac{1}{2}}(t)\breve{\psi}_m(p,t) & \text{for } x=x(p,t),\ t\in O_{m+\frac{1}{2}}.\end{cases} \tag{26.57}$$

Here $\alpha_j\in C_0^\infty(O_j)$, $2j\in\mathbb{Z}_+$, and $\sum_{m=0}^{2M+1}\alpha_{\frac{m}{2}}(t)\equiv 1$ for $0\le t\le T$,

$$\psi_m(x,t) = \varphi(x,t)e^{\frac{i}{h}S_m(x,t)}, \quad \breve{\psi}_m(p,t) = \breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}_m(p,t)},$$
$$x(p,t) = \frac{1 - p\cos t}{\sin t},$$
$$S_m(x,t) = -\frac{x^2+1}{2}\tan t + \frac{x}{\cos t} - m\frac{i\pi}{2}, \qquad \varphi(x,t) = \frac{\Phi\left(\frac{x-\sin t}{\cos t}\right)}{\sqrt{|\cos t|}},$$
$$\breve{S}_m(p,t) = \frac{p^2+1}{2}\frac{\cos t}{\sin t} - \frac{p}{\sin t} - \left(m + \frac{1}{2}\right)\frac{i\pi}{2}, \quad \breve{\varphi}(p,t) = \frac{\Phi\left(\frac{\cos t - p}{\sin t}\right)}{\sqrt{|\sin t|}}.$$

Then $\mathcal{K}\Phi$ *is the asymptotics of the function* Ψ *as* $h \to 0$ *in the sense that, for* $t \le T < \infty$*, there exists a* $C(T) < \infty$ *such that*

$$\|\mathcal{K}\Phi(\cdot,t) - \Psi(\cdot,t)\|_{L^2(\mathbb{R})} \le C(T)h.$$

Remark 26.15 By Theorem 26.14, the principal term of the asymptotics satisfies the condition

$$[\mathcal{K}\Phi]\left(x, t + \frac{\pi}{2}\right) = e^{-i\frac{\pi}{2}}[\mathcal{K}\Phi](x,t), \tag{26.58}$$

which implies its 2π-periodicity,

$$[\mathcal{K}\Phi](x, t + 2\pi) = [\mathcal{K}\Phi](x,t)$$

and the variation of the phase by $-\frac{\pi}{2}$ at the points $(x_m, t_m) = \left((-1)^{m-1}, \frac{\pi}{2}(2m-1)\right)$, at which the *characteristics of the Hamiltonian system* (26.20) converge and at which there is a sharp increase in the radiation intensity. Facts of this kind have long been known in optics: the phase jump at the *focal* points[89] belonging to the *caustic* formed by the converging rays of a cylindrical wave (see, for example, §45 of Sommerfeld's[90] book Sommerfeld 1954), and the concentration of the intensity of light radiation on the caustics, thanks to which Archimedes burned up, according to legend, the Roman fleet.[91]

[89] From Latin *focalis*, focal; caustic, from Latin *caustica*, burning. A clear example of caustics can be seen with a regular mug. Slanting incident light rays reflecting from its interior and intersecting, highlight the caustic (its envelope) in the form of a cardioid on the surface of the liquid half poured into the mug. The brightest point is the tip of the cardioid—the point at which converge all the rays falling into the mug (Arnold 2016).

[90] Arnold Johannes Wilhelm Sommerfeld (1868–1951) was an outstanding German theoretical physicist and mathematician. He founded large Munich school of theoretical physics. Among his students are Nobel Prize winners W. Pauli, W. Heisenberg, H. Bethe (in physics) and P. Debye and L. Pauling (in chemistry).

[91] Francesca Aicardi, a student of V. I. Arnold, drew his attention (see Prasolov and Tsfasman 2004) to the fact that Aristophanes in "The Clouds" attributes to Socrates an even earlier use of caustics in business matters: he advises his client to choose a sunny place at the court session and, having bought a lens at the pharmacy, use the solar caustic to burn his debt obligation shown to the court by the opponent. And in our time, the concave configuration of some glazed buildings sometimes leads to arson of parked cars; see https://www.autonews.ru/news/5b9b428a9a7947b2d57b7899.

26.16 Let us proceed to the construction of the asymptotics of the multivariate analog of problem (26.10), namely we consider the problem:[92]

$$ih\Psi_t + \frac{1}{2}h^2\Delta\Psi = \frac{1}{2}|x|^2\Psi, \quad \Psi(x,0)\big|_{x=x^\circ} = \Phi(x^\circ)e^{\frac{i}{h}x_1^\circ}, \tag{26.59}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, Δ is the *Laplace operator*, $\frac{1}{h} \gg 1$, and $\Phi \in C_0^\infty(\mathbb{R}^2)$.

Substituting $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$ with the sought-for S and φ in (26.59), and arguing as in problem (26.10), we see that, up to an additive summand of order $O(h^2)$, we have

$$S_t + \frac{1}{2}(|\nabla S|^2 + |x|^2) = 0, \quad S(x,0) = x_1, \tag{26.60}$$

$$\varphi_t + (\nabla S|\nabla\varphi) + \frac{1}{2}\varphi\Delta S = 0, \quad \varphi(x,0) = \Phi(x). \tag{26.61}$$

As in the one-dimensional case, we shall seek S in the form

$$S(x,t) = \int_{(0,0)}^{(x,t)} \omega, \quad \text{where } \omega \overset{\text{cf. (11.41)}}{=} p\,dx - H(p,x,t)\,dt \tag{26.62}$$

with the Hamiltonian

$$H(p,x,t) = \frac{1}{2}(p_1^2 + p_2^2 + x_1^2 + x_2^2) \tag{26.63}$$

corresponding $\left(\frac{h}{i}\frac{\partial}{\partial x_k} \leftrightarrow p_k\right)$ to problem (26.59). Hence $S_t = -H$ and $\frac{\partial S}{\partial x_k} = p_k$, and therefore, $\dot p + H_x = 0$. The corresponding *Hamiltonian system* $\dot p + H_x = 0$, $\dot x - H_p = 0$ takes the form

$$\dot p_1 = -x_1, \quad \dot p_2 = -x_2, \quad \dot x_1 = p_1, \quad \dot x_2 = p_2 \tag{26.64}$$

with the initial data

$$p_1(0) = 1, \quad p_2(0) = 0, \quad x_1(0) = x_1^\circ, \quad x_2(0) = x_2^\circ \tag{26.65}$$

(here we used the relations $S(x,0) = x_1$ and $\frac{\partial S}{\partial x_k} = p_k$).

For $t \in [0,T]$, the characteristics

$$x(t) = x(x^\circ,t) \overset{(11.36)}{=} \begin{cases} x_1(x^\circ,t) = \sin t + x_1^\circ\cos t, \\ x_2(x^\circ,t) = x_2^\circ\cos t \end{cases} \tag{26.66}$$

of the Cauchy problem (26.64)–(26.65) fill in the entire set

$$\Omega = \left(\mathbb{R}^2 \times [0,T]\right) \setminus \left\{|x_1 - x_1^k| > 0,\ x_2 = x_2^k,\ t = t_k,\ k \ge 1\right\}, \tag{26.67}$$

[92] A similar analysis applies to the case of interaction of heavy and light particles, i.e., when the parameter h appears at the derivatives of only some spatial variables, for example, as in the equation $ih\Psi_t + \frac{1}{2}\left[h^2\Psi_{x_1x_1} + \Psi_{x_2x_2}\right] = \frac{1}{2}|x|^2\Psi$.

while

$$(x_1^k, x_2^k, t_k) = \Big((-1)^{k-1}, 0, \frac{\pi(2k-1)}{2}\Big) \tag{26.68}$$

are precisely the intersection points of the characteristics. Note that the initial point $x^\circ = (x_1^\circ, x_2^\circ) \in \mathbb{R}^2$ is reconstructed from (x, t) for $t \neq t_k$. Namely, in view of (26.66) we have

$$x_1^\circ(x,t) = \frac{x_1 - \sin t}{\cos t}, \quad x_2^\circ(x,t) = \frac{x_2}{\cos t}. \tag{26.69}$$

From (26.66), (26.69) and using the equation $\dot{p} + H_x = 0$, we get

$$p_1(x,t) = \frac{1 - x_1 \sin t}{\cos t}, \quad p_2(x,t) = -x_2 \tan t. \tag{26.70}$$

Hence

$$p_1(x^\circ, t) = \cos t - x_1^\circ \sin t, \quad p_2(x^\circ, t) = -x_2^\circ \sin t \tag{26.71}$$

and

$$x_1(p,t) = \frac{1 - p_1 \cos t}{\sin t}, \quad x_2(p,t) = -p_2 \cot t. \tag{26.72}$$

Formulas (26.70) (cf. (11.40)) define the mapping of the plane $\mathbb{R}_x^2 \stackrel{\text{def}}{=} \{x = (x_1, x_2)\}$ onto the plane $R_{p(t)}^2 \stackrel{\text{def}}{=} \big\{p = (p_1(x,t), p_2(x,t))\big\}$. Note that, for $t = t_k$, the projection of the plane $\mathbb{R}_{p(t)}^2$ onto the plane $\mathbb{R}_x^2$ is the point $(x_1^k, x_2^k) = \big((-1)^{k-1}, 0\big)$, while for the remaining t this projection coincides with the plane $\mathbb{R}_x^2$.

The union $\bigcup_{t\in[0,T]} \mathbb{R}_{p(t)}^2$ of the set of planes $\mathbb{R}_{p(t)}^2$ forms a smooth three-dimensional manifold[93]

$$\Lambda_{[0,T]}^3 \subset \mathbb{R}^5 = \mathbb{R}_x^2 \times \mathbb{R}_p^2 \times \mathbb{R}_t \tag{26.73}$$

in the coordinate space (x, p, t). The image of the projection of $\Lambda_{[0,T]}^3$ onto $\mathbb{R}_x^2 \times \mathbb{R}_t$ is precisely the set Ω ((26.67)), and hence the three-dimensional manifold $\Lambda_{[0,T]}^3$ (for the two-dimensional analogue of this manifold, see p. 67) can be associated with the "graph" of the solution $p = p(x,t)$ of system (26.64).

In view of (26.63), (26.64), the differential

$$d\omega = -\sum_{k=1}^{2} \big[\dot{p}_k + H_{x_k}(p(x,t), x, t)\big]\, dx_k \wedge dt.$$

of the form

$$\omega = p_1\, dx_1 + p_2\, dx_2 - H(p,x,t)\, dt \tag{26.74}$$

is zero. Hence the integral $I(x,t) = \int_{(0,0)}^{(x,t)} \omega$ is defined in the simply connected domains

$$\Omega_k = \Omega \cap \Big\{t_k < t < t_{k+1} = \frac{\pi}{2}(2k+1)\Big\}, \quad k \geq 0, \tag{26.75}$$

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[93] With the natural topology.

because it is independent of the path between the initial and the final points of integration.

Lemma 26.17 *Formula $S(x,t) = I(x,t)$ defines the solution of problem* (26.60). *This solution $S\colon \Omega \ni (x,t) \mapsto S(x,t)$ is continuous,*[94] *and moreover, $S(x,t) = 0$ for $t = t_k$, and for $t \neq t_k$ and $x = (x_1, x_2)$ we have the equality*

$$S(x,t) = \frac{\sin 2t}{4} + \frac{x_1 - \sin t}{\cos t} - \frac{\tan t}{2}(|x|^2 - \sin^2 t), \tag{26.76}$$

i.e.,

$$S(x,t) = \frac{x_1}{\cos t} - \frac{\tan t}{2}(|x|^2 + 1).$$

Proof The integral $\int_{(0,0)}^{(x,t)} p_1(x,t)\,dx_1 + p_2(x,t)\,dx_2 - H(p,x,t)\,dt$ is independent of the choice if the integration path $\gamma \subset \Omega_0$ connecting the points $(0,0) \in \mathbb{R}^2_x \times \mathbb{R}_t$ and (x,t). Hence, using (26.63) and the equality $\dot{x} = p$, it can be written as the sum of two integrals

$$I_1 = \int_0^t \frac{|p\big(x(0,\tau),\tau\big)|^2 - |x(0,\tau)|^2}{2} d\tau \quad \text{and} \quad I_2 = \int_{(\sin t,0)}^{(x_1,x_2)} \sum_{j=1}^{2} p_j(\xi,t) d\xi_j.$$

In view of (26.66) and (26.70) we have $|p(x(0,\tau),\tau)|^2 = \cos^2\tau$, $|x(0,\tau)|^2 = \sin^2\tau$, and $p_1(\xi,t) = \frac{1-\xi_1 \sin t}{\cos t}$, $p_2(\xi,t) = -\xi_2 \tan t$. Hence

$$I_1 = \frac{1}{2}\int_0^t \cos 2t\,dt, \quad I_2 = \int_{\sin t}^{x_1} \frac{1 - \xi_1 \sin t}{\cos t}\,d\xi_1 - \int_0^{x_2} \xi_2 \tan t\,d\xi_2.$$

It follows that (26.76) holds for $t < \frac{\pi}{2}$, and hence $S(x(x^\circ,t),t) \to 0$ as $t \uparrow \frac{\pi}{2}$, because

$$S(x(x^\circ,t),t) \overset{(26.66)}{=} \frac{\sin 2t}{4} + x_1^\circ - \frac{\sin t}{2}\big[x_1^\circ\big(x_1^\circ + 2\sin t\big) + (x_2^\circ)^2 \cos t\big].$$

Assume now that $(x,t) = (x_1, x_2, t) \in \Omega_k$, where $k = 1$. Writing

$$S(x,t) = \int_{(x_1^k, x_2^k, t_k)}^{(x_1,x_2,t)} \omega$$

as the sum of two integrals

$$\int_{t_k}^t \frac{|p\big(x(0,\tau),\tau\big)|^2 - |x(0,\tau)|^2}{2} d\tau \quad \text{and} \quad \int_{(x_1^k,x_2^k)}^{(x_1,x_2)} \sum_{j=1}^{2} p_j(\xi,t) d\xi_j,$$

[94] In contrast to the popular opinion that "the function S has singularities at focal points" (see, for example, on p. 105 of the book Maslov and Fedoryuk 1976). See also in this regard footnote 73 on p. 165.

we verify (26.76) for $(x,t) \in \Omega_1$. An induction in $k \geq 1$ completes the proof of the lemma, including the formula $S(x_1^k, x_2^k, t_k) = 0$ for any $k \geq 1$. □

Let us now consider problem (26.61), i.e., the problem

$$\varphi_t + (\nabla S|\nabla\varphi) + \frac{1}{2}\varphi\Delta S = 0, \quad \varphi(x,0) = \Phi(x). \tag{26.77}$$

As in the one-dimensional case, it is easily checked that the characteristics of Eq. (26.77) coincide with the characteristics of (26.66). Along these characteristics, we have

$$\frac{d\varphi\big(x(t),t\big)}{dt} = -\frac{1}{2}\varphi(x,t)\big[S_{x_1x_1}(x,t) + S_{x_2x_2}(x,t)\big]\big|_{(x,t)=(x(t),t)},$$

i.e., $\frac{d\varphi}{dt} = \frac{\sin t}{\cos t}\varphi$. Therefore,

$$\varphi(x(x^\circ,t),t) = \frac{\varphi(x^\circ,0)}{|\cos t|} \overset{(26.69)}{=} \Phi\Big(\frac{x_1 - \sin t}{\cos t}, \frac{x_2}{\cos t}\Big)\frac{1}{|\cos t|}. \tag{26.78}$$

Problem 26.18 Following the proof of Theorems 26.8, 26.13 and 26.14, verify the next theorem.

Theorem 26.19 *The solution $\Psi \in C^1(\mathbb{R}_+; L^2(\mathbb{R}^2))$ of problem (26.59) with the initial function $\Phi \in H^2(\mathbb{R}^2)$ satisfies the asymptotic estimate*

$$\|\mathcal{K}\Phi(\cdot,t) - \Psi(\cdot,t)\|_{L^2(\mathbb{R})} \leq C(T)h \quad \text{as } h \to 0,$$

where $t \leq T = M\pi$, $M \in \mathbb{N}$, and $\mathcal{K}\colon \Phi \mapsto \mathcal{K}\Phi$ is the operator defined by

$$[\mathcal{K}\Phi](x,t) = \begin{cases} \alpha_m(t)\psi_m(x,t) & \text{for } t \in O_m, \\ \alpha_{m+\frac{1}{2}}(t)\breve{\psi}_m(p,t) & \text{for } x = x(p,t), t \in O_{m+\frac{1}{2}}. \end{cases} \tag{26.79}$$

Here $\alpha_j \in C_0^\infty(O_j)$, $2j \in \mathbb{Z}_+$, and moreover, $\sum_{m=0}^{2M+1} \alpha_{\frac{m}{2}}(t) \equiv 1$ for $0 \leq t \leq T$,

$$O_m = \{|t - m\pi| < \varepsilon\}, \quad O_{m+\frac{1}{2}} = \Big\{\Big|t - \frac{(2m+1)\pi}{2}\Big| < \varepsilon\Big\}, \quad \varepsilon \in \Big]\frac{3}{8}\pi, \frac{\pi}{2}\Big[,$$

$$\psi_m(x,t) = \varphi(x,t)e^{\frac{i}{h}S_m(x,t)}, \quad \breve{\psi}_m(p,t) = \breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}_m(p,t)},$$

$$x = (x_1,x_2), \quad x_1(p,t) = \frac{1 - p_1\cos t}{\sin t}, \quad x_2(p,t) = -\frac{p_2\cos t}{\sin t}t,$$

$$S_m(x,t) = S(x,t) - \mu_m\frac{i\pi}{2}, \quad S(x,t) = \frac{x_1}{\cos t} - \frac{\tan t}{2}(|x|^2 + 1),$$

where[95] *$\mu_m = 2m$. Moreover, by Lemma 26.17 we have*

[95] In the n-dimensional case, $\mu_m = nm$. See Remark 26.20 and also §26.25 on p. 188, where a more general setting is considered.

$$S_m(x,t) \overset{(26.62)}{=} \left[\int_{(0,0)}^{(x,t)} p(x,t)\,dx - H(p(x,t),x,t)\,dt\right] - \mu_m \frac{i\pi}{2}$$

and (*see the footnote* 76 *on p.* 166)

$$\varphi(x,t) = \frac{\Phi\left(\frac{x_1-\sin t}{\cos t}, \frac{x_2}{\cos t}\right)}{|\cos t|} = \frac{\Phi\left(\frac{x_1-\sin t}{\cos t}, \frac{x_2}{\cos t}\right)}{\left|\det \frac{\partial x(x^\circ,t)}{\partial x^\circ}\right|^{1/2}}, \tag{26.80}$$

where $x(x^\circ, t)$ *is given by* (26.66),

$$\check{S}_m(p,t) = \frac{p_1^2+p_2^2+1}{2}\frac{\cos t}{\sin t} - \frac{p_1}{\sin t} - \mu_{(m+\frac{1}{2})}\frac{i\pi}{2},$$

i.e., as follows from $S_{xx} \overset{(26.76)}{=} 2\tan t$ *and the relations* (26.44)–(26.48),

$$\check{S}_m(p,t) = \left([S(x,t) - xp] - \frac{i\pi}{4}\operatorname{sgn} S_{xx}(x,t)\right)\Big|_{x=x(p,t)} - \mu_m\frac{i\pi}{2},$$

$$\check{\varphi}(p,t) = \frac{\Phi\left(\frac{\cos t-p_1}{\sin t}, -\frac{p_1}{\sin t}\right)}{|\sin t|} \overset{cf.\,(26.80)}{=} \frac{\Phi\left(\frac{\cos t-p_1}{\sin t}, -\frac{p_1}{\sin t}\right)}{\left|\det \frac{\partial p(x^\circ,t)}{\partial x^\circ}\right|^{1/2}},$$

where $p(x^\circ, t)$ *is given by* (26.71).

Remark 26.20 Following the footnote 76 on p. 166 (i.e., with further generalizations in mind), it is worth noting that the number μ_m appearing in Theorem 26.19 is the sum $\sum_{j=1}^m \operatorname{ind}_j$, where ind_j is the so-called *Morse index*[96] of the phase trajectory $]t_j', t_j''[\ni t \mapsto x(x^\circ,t)$, $t_j' < t_j < t_j''$, $|t_j''-t_j'| \ll 1$, $\det\frac{\partial x(x^\circ,t_j)}{\partial x^\circ} = 0$. In case we consider here, $\det\frac{\partial x(x^\circ,t)}{\partial x^\circ} \overset{(26.66)}{=} \cos^2 t$, and hence $t_j = (j-\frac{1}{2})\pi$. Moreover,[97]

$$\operatorname{ind}_j \overset{\text{def}}{=} \operatorname{inerdex}\frac{\partial x(p,t_j')}{\partial p} - \operatorname{inerdex}\frac{\partial x(p,t_j'')}{\partial p}, \tag{26.81}$$

where $\operatorname{inerdex}\frac{\partial x(p,t)}{\partial p}$ is the number of negative eigenvalues of the symmetric matrix $\frac{\partial x(p,t)}{\partial p}$. In view of Eq. (26.72) we have

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[96] Harold Calvin Marston Morse (1892–1977) was a famous American mathematician, widely known primarily for his outstanding results in analysis, calculus of variations, and differential topology, describing the relationship of the algebraic/topological properties of a topological space with critical points of functionals defined on it. For the Morse theory, see, for example, in Vassiliev (2014), Milnor (1963) and Chap. 3 of V. A. Sharafutdinov's lectures on Riemannian geometry; see http://math.nsc.ru/LBRT/d6/chair/study.htm.

[97] In the book Maslov and Fedoryuk (1976), the formula for ind (on p. 147) is given with the wrong sign due to a typo (t' and t'' were swapped), which arose, most likely, due to the fact that sometimes this formula is written out in terms of the matrix $\check{S}_{pp}$, for which (in contrast to p. 20 of the book Maslov and Fedoryuk 1976) the equality $\check{S}_{pp}(p,t) = -\frac{\partial x(p,t)}{\partial p}$ holds (see (26.50) and formula (6.6) on p. 135 of the book Maslov and Fedoryuk 1976).

$$\frac{\partial x(p,t)}{\partial p} = \begin{bmatrix} -\cot t & 0 \\ 0 & -\cot t \end{bmatrix},$$

which implies that inerdex $\frac{\partial x(p,t_j')}{\partial p} = 2$ and inerdex $\frac{\partial x(p,t_j'')}{\partial p} = 0$. As a result,

$$\mu_m \overset{\text{def}}{=} \sum_{j=1}^{m} \text{ind}_j = 2m \quad \text{for } n = 2.$$

We also note the following Morse's result, which will be important in the sequel: ind_j is the number degenerations (counting multiplicity) of the mapping

$$]t_j', t_j''[\ni t \mapsto \det \frac{\partial x(x^\circ, t)}{\partial x^\circ} \overset{(26.66)}{=} \cos^2 t.$$

26.21 The above method of the proof of Theorems 26.8 and 26.13 is largely based on the classical method of characteristics, which was briefly mentioned in §11. Although it is also applicable in a more general setting concerning the topic of this section, for example, in relation to problem (26.59) (see Exercise 26.18), it is, however, powerless, in particular, in constructing the short-wave asymptotics of solutions to those problems in quantum mechanics for which the potential and/or the initial phase are not explicitly specified, for example, as in this case

$$\begin{gathered} ih\Psi_t + \frac{h^2}{2}\Delta\Psi = v(x)\Psi, \\ \Psi(x,0)\big|_{x=x^\circ} = \Phi(x^\circ)e^{\frac{i}{h}S_0(x^\circ)}, \quad \Phi \in C_0^\infty(\mathbb{R}^n). \end{gathered} \tag{26.82}$$

Here the potential[98] $v \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfying the condition $v(x) \to \infty$ as $|x| \to \infty$ and the initial phase $S_0 \colon \mathbb{R}^n \ni x \overset{C^\infty}{\mapsto} S_0(x) \in \mathbb{R}$ are not given explicitly. This does not allow us to construct, in analogy with problems (26.10) and (26.59), an asymptotic approximation $\varphi(x)e^{\frac{i}{h}S(x,t)}$ to the solution of problem (26.82) in the absence of explicit formulas for S and φ.

Of course, for specific v and S_0, one can understand (for example, numerically) how intersect the characteristics $t \to x(x^\circ, t)$ that emanate for $t = 0$ from the point x°, i.e., where $\det \frac{\partial x(x^\circ,t)}{\partial x^\circ} = 0$. But what is next? Is there a phase jump on the caustic? What is it like?

Despite the fact that the method of characteristics is no longer capable of providing answers to these questions, nevertheless, as a tool for studying the Hamiltonian system, it contributed to the development of a general V. P. Maslov's concept, which allowed one not only to answer these questions but also to study various linear problems of mathematical physics with small parameter.

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[98] In this case, as pointed out in the footnote 68 on p. 163, problem (26.82) is uniquely solvable in the space $C^1(\mathbb{R}_+; L^2(\mathbb{R}^n))$.

The main objects of Maslov's theory are the so-called *Lagrangian*[99] *manifolds* (see Definition 26.22; for example, the manifold $\Lambda^2_{[0,T]} \subset \mathbb{R}_x \times \mathbb{R}_t \times \mathbb{R}_p$ given on p. 67) and the operator $\mathcal{K}$ associated with the Lagrangian manifold (which Maslov called the *canonical operator*) and the so-called *Arnold–Maslov index* (or simply the *Maslov index*). An example of the operator $\mathcal{K}$ appears in Theorem 26.19, where the corresponding Maslov index is the integer number[100] μ_m in the formulas for S_m, which characterizes the phase jump on the caustics.

Definition 26.22 Let Λ^n be an n-dimensional manifold embedded in the $2n$-dimensional *phase space* $\mathbb{R}^n_x \times \mathbb{R}^n_p$, and let

$$\omega^1 = \sum_{k=1}^{n} p_k \, dx_k$$

be the associated differential form.[101] A manifold $\Lambda = \Lambda^n$ is a *Lagrangian manifold* if on Λ the differential $d\omega^1 = \sum_{k=1}^n dp_k \wedge dx_k$ of the form ω^1 is zero. In other words,

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[99] Arnold, whom Maslov asked to be an opponent (see http://trv-science.ru/2010/07/06/perepletenie-traektorij-zhizni/), in a footnote on the second page of the paper Arnold (1967) (almost identical to the report of Arnold-the opponent, which Maslov emphasized in the above interview), notes: "The name comes from the "Lagrange brackets" in classical mechanics." Recall that the "Lagrange brackets" are defined by the formula $[u, v]_{p,x} = \sum_{k=1}^n \left(\frac{\partial x_k}{\partial u}\frac{\partial p_k}{\partial v} - \frac{\partial p_k}{\partial u}\frac{\partial x_k}{\partial v}\right)$ in the case when the coordinates $(x, p) \in \mathbb{R}^{2n}$ corresponding to the system of Hamiltonian equations $\dot p = -H_x$, $\dot x = H_p$ can be defined as functions of the variables $(u, v) \in \mathbb{R}^2$.

[100] In the case considered here, this number is the Morse index, which is defined as the number of degenerations (with due account of the multiplicity) of det $\frac{\partial r(x^\circ,\tau)}{\partial x^\circ}$ on the phase trajectory $[0, t] \ni \tau \mapsto r(x^\circ, \tau) \in \Lambda^{n+1}_{[0,t]}$ (see Remark 26.20 and §26.25). Even before Maslov Maslov (1965) in 1965 introduced the concept of the index, now known as the Maslov index, its properties were described in the symplectic theory of Sturm's theorem. This theory was constructed in the 1930–1950s by Morse (see Morse 1930, Morse 1934 and Victor Borisovich Lidskii Lidskii 1955 (1924–2008)), a honored professor at the Moscow Institute of Physics and Technology, specialist in mathematical physics, theory of differential equations and spectral theory of operators, a participant of the Second World War, a hero-scout (see Note 1943), awarded with military orders. For the development of the symplectic theory of Sturm's theorem and its relation to the Maslov index, see, in particular, Arnold (1967) and Pushkar' (1998).

[101] The differential form $p \wedge dx - H\,dt$ is denoted by ω, and the form $p \wedge dx$, by ω^1. Its differential $d\omega^1$ is a nondegenerate skew-symmetric bilinear form in $\mathbb{R}^{2n}$ of the most simple form: $dp \wedge dx$. This simplest (or *symplectic*) form (from Latin simplex, simple) takes any nondegenerate skew-symmetric bilinear form in the space $\mathbb{R}^{2n}$ in the coordinates (x, p) (known as the *symplectic* coordinates), whose existence follows from Darboux's theorem (see, for example, Arnold 1989). The *space* $\mathbb{R}^n_x \times \mathbb{R}^n_p$ with the structure defined by the differential form $d\omega^1 = \sum_{k=1}^n dp_k \wedge dx_k$ is known as the *symplectic space*. In the context of the corresponding transformation group, this term was introduced by Hermann Weyl (1885–1955), an outstanding German mathematician, theoretical physicist and philosopher, who in the footnote on p. 165 of this book Weyl (1939) writes: "The name 'complex group' formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word 'complex' in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective '*symplectic*.' "

$$\int_{\gamma=\partial\Omega} \omega^1 \overset{(7.2)}{=} \int_{\Omega} d\omega^1 = 0$$

for any closed contour γ describing a simply connected domain $\Omega \subset \Lambda$.

It is shown (see Arnold 1989) that the phase flow

$$(x(0), p(0)) \mapsto (x(t), p(t)) = g^t(x(0), p(0)), \quad g^t : \Lambda \to g^t\Lambda, \tag{26.83}$$

as given by the system of Hamiltonian equations[102] $\dot{p} = -H_x$, $\dot{x} = H_p$ under the transformation group g^t satisfying the condition $g^{t_1+t_2} = g^{t_1}g^{t_2}$ is the *canonical mapping*,[103] in other words, $\int_\gamma \omega^1 = \int_{g^t\gamma} \omega^1$ for any closed contour γ lying in a simply connected subdomain of Λ. So, a Hamiltonian flow carries one Lagrangian manifold into a different one; in our setting, Λ is mapped into $g^t\Lambda$. Moreover,

$$\int_\gamma p\,dx - H\,dt = \int_{g^t\gamma} p\,dx - H\,dt, \quad \text{because} \quad \int_{g^t\gamma} H\,dt = 0.$$

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[102] This is why this flow is called a *Hamiltonian flow*. The phase trajectory $t \mapsto (x(t), p(t))$ is frequently called a *bicharacteristics*, and its projections onto $\mathbb{R}^n_x$, i.e., the characteristic $t \mapsto x(t)$, is sometimes called a *ray*.

[103] This concept and the term itself was introduced by Carl Gustav Jacobi (1804–1851), one of the greatest mathematicians of the XIX century His contributions to complex analysis, linear algebra, dynamics, and other branches of mathematics and mechanics are enormous. Jacobi was distinguished by exceptional diligence and a complete absence of vanity and envy: when Abel published a new work that largely strengthened Jacobi's results, he limited himself to the remark: "This is above my work and above my praise." But not everything is so sweet. Jacobi was a human being. To quote from O. Ore's book "Niels Henrick Abel. Mathematician Extraordinary": "Bessel was both a friend and admirer of Jacobi, bet even so felt that he could not conceal the fact that the young mathematician was known both for his unusual ability and for his arrogance 'He is undoubtedly very talented, but here he has made almost everyone his enemy since he arrived, because he has said something unpleasant to each, and said it in a manner which they cannot forgive. However, I hope that these small stupidities soon will not be mentioned anymore; toward me he has always been a well-behaved young man.; Jacobi gradually did improve his insolent manners, as so many young men have done in the course of time. Even at this stage, in the competition which arose between Abel and Jacobi, the two rivals always expressed themselves with courtesy, respect, and admiration for the works and discoveries of the other." An extensive class of integrals was named Abelian by Jacobi's suggestion. His name is associated, in particular, with such important concepts in mathematics as the Jacobian, the Jacobi identity, and the Jacobi elliptic functions. He proved (see, for example, Arnold 1989) that when it is possible to find a canonical transformation $(x, p) \mapsto (X, P)$ such that in the coordinates (X, P) the Hamiltonian does not depend on X, then the original Hamiltonian equations can be solved in quadratures if a solution of the equation

$$\frac{\partial S}{\partial t} + H\Big(\frac{\partial S}{\partial X}, X, t\Big) = 0 \tag{26.84}$$

is found (cf. (26.60)). This is why this equation is called the Hamilton–Jacobi equation. Noting the power of this method, Arnold writes in his book Arnold (1989) that "many problems solved by Jacobi are generally not amenable to solving by other methods."

The differential form $\omega = p\,dx - H\,dt$, which appears here and also in (11.41) and (26.74), is called the *Poincaré–Cartan integral invariant*[104] (Kozlov 1998). For $t \in [0, T]$, with this form one associates the $(n+1)$-dimensional manifold

$$\Lambda^{n+1}_{[0,T]} \subset \mathbb{R}^n_x \times \mathbb{R}^n_p \times \mathbb{R}_t \tag{26.85}$$

formed by the integral curves of the Hamiltonian equations. Hence this manifold is embedded in the *extended phase space*. From the Hamiltonian equations it follows that $d\omega\big|_{\Lambda^{n+1}_{[0,T]}} = 0$ (cf. (26.74)).[105]

26.23 Let us now turn to the main purpose of this section: the presentation the basic construction of the method of the canonical Maslov operator. We will do this in relation to the asymptotics as $h \to 0$ of the solution to problem (26.82). But first, for the particular case of this problem, namely for the problem

$$L_t\Psi = 0, \quad \Psi(x, 0)\big|_{x=x^\circ} = \Phi(x^\circ)e^{\frac{i}{h}S_0(x^\circ)}, \quad \Phi \in C_0^\infty(\mathbb{R}^n), \tag{26.86}$$

where

$$L_t\Psi \overset{\text{def}}{=} ih\Psi_t + \frac{h^2}{2}\Delta\Psi - \frac{|x|^2}{2}\Psi,$$

we represent formula[106] (26.79) for the asymptotics of its solution in a more compact form. We will use the terms and notation accepted in the theory of the canonical Maslov operator.

We note first of all that the initial data in this problem define the Lagrangian manifold

$$\Lambda^n_0 = \big\{(x, p) \in \mathbb{R}^n_x \times \mathbb{R}^n_p, \text{ where } p\colon x \mapsto p(x) = \nabla S_0(x)\big\},$$

while the initial data themselves can be represented in the form

$$\Psi(x, 0) = [\mathcal{K}_{\Lambda^n_0}\Phi](x), \tag{26.87}$$

where $\Lambda^n_t \overset{(26.83)}{=} g^t\Lambda^n_0$, and the operator $\mathcal{K}_{\Lambda^n_t}$ is given for $0 \le t \le T = M\pi$, $M \in \mathbb{N}$, by the formula

[104] Élie Joseph Cartan (1869–1951) was a French a member of the French Academy of Sciences. He made a significant contribution to differential geometry (of special importance is the theory of external forms), the theory of continuous groups and their representations (mainly, Lie groups, for which he laid the foundation of the algebraic theory of Lie groups and described representations of semisimple Lie groups), and the theory of differential equations. His son, Henri Paul Cartan (1904–2008), was also an outstanding mathematician of the XX century.

[105] $\Lambda^{n+1}_{[0,T]}$ is a Lagrangian manifold in $\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}^n_p \times \mathbb{R}_{p_0}$, where $p_0 = -H(x(t), p(t))$ (see, for example, Maslov and Fedoryuk 1976, pp. 117, 155, 185).

[106] More precisely, its natural extension for an arbitrary n and arbitrary real-valued phase $S_0 \in C^\infty(\mathbb{R}^n)$.

$$[\mathcal{K}_{\Lambda_t^n}\Phi](x) = \alpha_{m+\frac{j}{2}}(t)\hat{\psi}_m(r,t) \quad \text{for } t \in O_{m+\frac{j}{2}},\ j = 0, 1. \tag{26.88}$$

Here, $m = 0, \dots, M \in \mathbb{N}$, $O_{m+\frac{j}{2}} = \left\{\left|t - \frac{(2m+j)\pi}{2}\right| < \varepsilon \in \left]\frac{3}{8}\pi, \frac{\pi}{2}\right[\right\}$ is one of the $2m+1$ intervals covering the closed interval $0 \le t \le T$. Next, $\alpha_j \in C_0^\infty(O_j)$, $2j \in \mathbb{Z}_+$ and $\sum_{m=0}^{2M+1} \alpha_{\frac{m}{2}}(t) \equiv 1$. By (r,t) we denote the point on $\Lambda_{[0,T]}^{n+1} \overset{(26.85)}{\subset} \mathbb{R}_x^n \times \mathbb{R}_t^n \times \mathbb{R}_p$ whose projection onto $\mathbb{R}_x^n \times \mathbb{R}_t$ for $t \in O_m$ is (x,t), and the projection onto $\mathbb{R}_p^n \times \mathbb{R}_t$ for $t \in O_{m+\frac{1}{2}}$ is $(p(x),t)$. Finally, $\hat{\psi}_m(r,t) = \hat{\varphi}(r,t)e^{\frac{i}{h}\hat{S}_m(r,t)}$, where[107]

$$\hat{S}_m(r,t) = \begin{cases} S(x,t) - \mu_m \dfrac{i\pi}{2}, & t \in O_m, \\ [S(x,t) - xp(x)]_{x=x(p,t)} - \mu_{(m+\frac{1}{2})}\dfrac{i\pi}{2}, & t \in O_{m+\frac{j}{2}}, \end{cases}$$

$\mu_k = nk$, and

$$\hat{\varphi}(r,t) = \frac{\Phi\big(x^\circ(r,t)\big)}{\sqrt{J(r,t)}}, \quad J(r,t) = \left|\det\frac{\partial r(x^\circ,t)}{\partial x^\circ}\right| = \frac{dr}{d\sigma(r,t)} \ne 0, \tag{26.89}$$

and $d\sigma(r,t)$, as follows from the last equality, is a volume element (i.e., the Lebesgue measure) on $\Lambda_{[0,T]}^{n+1}$.

Using the concepts and formulas from the previous section we can now give a general scheme of the method of the canonical Maslov operator in relation to the construction of the asymptotics for the solutions of problems similar to problems (26.79) and (26.86). This scheme is given by the following diagram:

$$\begin{array}{ccccc} \Psi(0) = \Phi e^{\frac{i}{h}S_0} \longrightarrow & \Lambda_0^n, \Phi & \xrightarrow{g^t} & \Lambda_t^n = g^t\Lambda_0^n, \Phi & \\ & \downarrow \mathcal{K} & & \downarrow \mathcal{K} & \\ \Psi(0) = & [\mathcal{K}_{\Lambda_0^n}\Phi] & \xrightarrow{L_t} & [\mathcal{K}_{\Lambda_t^n}\Phi]h \to 0 & \overset{h\to 0}{\asymp} \ \Psi(t). \end{array} \tag{26.90}$$

Of special importance in this diagram is the *Lagrangian manifold* Λ and the *canonical operator* $\mathcal{K}$ on Λ, which establishes a relation between the sought-for asymptotics of the function Ψ (which is defined on the *configuration space*) and the corresponding function Φ on Λ.

These two objects lay the basis for the *method of the canonical Maslov operator* in relation to numerous various problems, and in particular, to the aforementioned problem (26.82), i.e., the problem

$$ih\Psi_t + \frac{h^2}{2}\Delta\Psi = v(x)\Psi, \quad \Psi(x,0)\big|_{x=x^\circ} = \Phi(x^\circ)e^{\frac{i}{h}s(x^\circ)}, \tag{26.91}$$

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[107] We have $S(x,t) = \int_{(0,0)}^{(x,t)} p\,dx - H(p,x,t)\,dt$ (see Lemma 26.17 and equality (26.62)), and $x(p,t)$ is a stationary point of the phase $S(x,t) - xp$, i.e., $S_x(x,t)\big|_{x=x(p,t)} = p(t)$.

where $\Phi \in C_0^\infty(\mathbb{R}^n)$. Let us use this problem to illustrate the algorithm outlined in the next subsection.

26.24 Algorithm for the Construction of the Maslov Operator. Applying to problem (26.91), as in the case of problems (26.10) and (26.59), the WKB method, or, in other words, the Debye procedure (see the footnote 72 on p. 164) for searching the formal[108] asymptotics of the function Ψ in the form $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$ with sought-for real functions φ and S, we get

$$S_t + \frac{1}{2}|\nabla S|^2 + v(x) = 0, \quad S(x,0) = s(x), \tag{26.92}$$

$$\varphi_t + (\nabla S|\nabla\varphi) + \frac{1}{2}\varphi\Delta S = 0, \quad \varphi(x,0) = \Phi(x). \tag{26.93}$$

But now, in the absence of specific data on the functions v and s, it is not possible to explicitly write down the manifold $\Lambda^{n+1}_{[0,T]}$ woven, for $t \in [0,T]$, from the phase trajectories of the corresponding Hamiltonian system $\dot p + H_x = 0$, $\dot x - H_p = 0$, where $H(p,x,t) = \frac{p^2}{2} + v(x)$. Therefore, one has to reason abstractly, assuming that, for a locally finite cover by simply connected domains of the manifold $\Lambda^{n+1}_{[0,T]}$ with local coordinates

$$(r,t) = (x_{i_1}\dots x_{i_k} p_{i_{k+1}}\dots p_{i_n} t), \quad k \le n, \tag{26.94}$$

it is possible (analytically or numerically) to identify which of these domains can be projected onto $\mathbb{R}^n_x \times \mathbb{R}_t$ (such domains are called *regular*) and which (*singular*) domains can be projected only onto those $(n+1)$-dimensional planes $\mathbb{R}^n_{(x,p)} \times \mathbb{R}_t$ (such planes are also called *singular*) for which in (26.94) the strict inequality $k < n$ holds.

A point $A = (r,t) \in \Lambda^{n+1}_{[0,T]}$ is a *focal point* if its neighborhood is projected only onto *singular* $(n+1)$-dimensional planes $\mathbb{R}^n_{(x,p)} \times \mathbb{R}_t$, i.e., if $J(x,t) \overset{\text{def}}{=} \left|\det \frac{\partial x(x^\circ,t)}{\partial x^\circ}\right| = 0$, where $t \mapsto x(x^\circ,t)$ is the characteristic emanating for $t=0$ from the point $x^\circ \in \mathbb{R}^n_x$. The *caustic* is the projection of the set of all focal points onto $\mathbb{R}^n_x \times \mathbb{R}_t$. In other words, this is the set on which all the characteristics (rays) of the Hamiltonian system intersect. To the ray $t \mapsto x(x^\circ,t)$ there corresponds the phase trajectory $t \mapsto r(x^\circ,t) \in \Lambda^{n+1}_{[0,T]}$, where $r(x^\circ,0) = (x^\circ,p^\circ)$, and $p^\circ = \nabla s(x)$. One says that t_j, where $j \ge 1$, is the jth *focus* on the phase trajectory if $r(x^\circ,t_j)$ is a *focal* point. For $0 \le t < t_1$, the phase trajectory lies in the regular domain $\Omega_0 \subset \Lambda^{n+1}_{[0,t_1[}$. In this domain, $J(x,t) \overset{\text{def}}{=} \left|\det \frac{\partial x(x^\circ,t)}{\partial x^\circ}\right| \ne 0$, whence we have the dependence x° on (x,t) and the following formula (cf. (26.89)) for the formal asymptotics:

[108] In contrast to the (true) asymptotics of the solution, which is a function that differs from the solution by a value that tends to zero as $h \to 0$, the formal asymptotics only satisfies the conditions of the problem up to a value that tends to zero at $h \to 0$, but this does not guarantee proximity to the solution. A formal asymptotic will be (true) asymptotic if the operator of the problem has continuous inverse in the corresponding spaces. However, even when this is the case, it is often very difficult to verify this.

$$\Psi(x,t) \asymp \psi_0(x,t) = \frac{\Phi(x^\circ(x,t))}{\sqrt{J(x,t)}} e^{\frac{i}{h}S_0(x,t)} \quad \text{for } t \in [t_0, t_1[; \tag{26.95}$$

here $t_0 = 0$, and S_0 is the solution of problem (26.92). An obstacle occurs as the focal point is approached: $\psi_0(x,t)$ tends to infinity, because $J(x(x^\circ, t_1), t_1) = 0$. But, as mountaineers say, "a smart person will not go up the mountain, a smart person will go around the mountain." It is possible to bypass the focal point due to the fact that its presence implies the existence of the domain $\Omega_{1/2} \subset \Lambda^{n+1}_{|t-t_1|=\varepsilon\ll 1}$ which intersects Ω_0 and which contains, for $t_1 \le t < t_1 + \varepsilon$, the phase trajectory $t \mapsto r(x^\circ, t) \in \Lambda^{n+1}_{[0,T]}$ which can be diffeomorphically projected both onto $\mathbb{R}^n_x \times \mathbb{R}_t$ and onto some *singular* plane $\mathbb{R}^n_{(x,p)} \times \mathbb{R}_t$ with local coordinates

$$(r,t) = (x_{i_1}, \ldots, x_{i_k}, p_{i_{k+1}}, \ldots, p_{i_n}, t), \quad k < n. \tag{26.96}$$

Hence, as in the proof of Theorem 26.19, one can, for $|t - t_1| \ll 1$, construct an asymptotics of the form $\breve{\varphi}(r,t)e^{\frac{i}{h}\breve{S}(r,t)}$ in the local variables (26.96) by applying to Eq. (26.82) the h-modification of the Fourier transform (with respect to $x_{i_1} \ldots x_{i_k}$)

$$[\breve{\mathbf{F}}_k v](p) = \left(\frac{1}{2\pi h}\right)^{k/2} \int_{\mathbb{R}^k} e^{-\frac{i}{h}(x_{i_1}p_{i_1}+\ldots+x_{i_k}p_{i_k})} v(r)\, dx_{i_1} \ldots x_{i_k},$$

which is agreed in $\{t_0 < t < t_1\} \bigcap \{|t - t_1| \ll 1\}$ with asymptotics (26.95) via the relation

$$[\breve{\mathbf{F}}_k \psi_0(r,t)](p) = \breve{\varphi}(r,t)e^{\frac{i}{h}\breve{S}(r,t)} + O(h). \tag{26.97}$$

Thus, the initial Cauchy data are transmitted, like a "baton," to the domain $\Omega_{1/2} \subset \Lambda^{n+1}_{|t-t_1|\ll 1}$.

By using the stationary phase method (see the footnote 85 on p. 171) one can evaluate the left-hand side of Eq. (26.97), which makes it possible to find in $\Omega_{1/2}$ the functions $\breve{\varphi}$ and $\breve{S}$ that correspond to the original Cauchy data. These data can be transmitted to the functions φ_1 and S_1, which specify the asymptotics of the solution in the interval $]t_1, t_2[$ in the standard (for the WKB method) form: $\varphi_1(x,t)e^{\frac{i}{h}S_1(x,t)}$. The transmission of the initial Cauchy data (the "baton") is effected on the set $\{t_1 < t < t_2\} \cap \{|t - t_1| \ll 1\}$ via the relation

$$[\breve{\mathbf{F}}_n^{-1} \breve{\varphi}(p,t)e^{\frac{i}{h}\breve{S}(p,t)}](x) = \varphi(x,t)e^{\frac{i}{h}S(x,t)} + O(h).$$

As a result (arguing in the same way as in the proof of Theorem 26.19), we get the asymptotic formula

$$\Psi(x,t) \asymp \frac{\Phi(x^\circ(x,t))}{\sqrt{J(x,t)}} e^{\frac{i}{h}S_1(x,t) - \frac{i\pi}{2}(n-k_1)} \quad \text{for } t \in]t_1, t_2[, \tag{26.98}$$

which is similar to (26.95). The number $\mu_1 = n - k_1$ is the *Morse index* (see the footnote 100 on p. 182), which is equal to the degree of the degeneracy of the determinant of the matrix $\frac{\partial r(x^\circ, t_1)}{\partial x^\circ}$, whose rank k_1 was initially denoted by k.

The algorithm for constructing the canonical Maslov operator, which "essentially reduces the original partial differential equation on the configuration space to an ordinary differential equation along the trajectories of a Hamiltonian vector field on a Lagrangian manifold" (Nazaikinskii 2014) is completed in this case by the procedure (repeating the previous constructions) of changing from the interval $]t_{j-1}, t_j[$ to the interval $]t_j, t_{j+1}[$ and the determination of the function S_j on this interval. As a result, we get the asymptotic formula

$$\Psi(x,t) \asymp \frac{\Phi(x^\circ(x,t))}{\sqrt{J(x,t)}} e^{\frac{i}{h} S_j(x,t) - \frac{i\pi}{2}\mu_j} \quad \text{for } t \in]t_j, t_{j+1}[. \tag{26.99}$$

Here $\mu_j = \sum_{l=1}^{j} (n - k_l)$ is the Morse index of the phase trajectory $t \mapsto r(x^\circ, t)$ we consider, i.e., the number of focal points on this trajectory counting their multiplicities. This means that $m_l = (n - k_l)$ is the degree of the degeneracy of the determinant $J(x,t) \overset{\text{def}}{=} \left| \det \frac{\partial x(x^\circ,t)}{\partial x^\circ} \right|$ at the point t_l.

26.25 The Arnold–Maslov Index.[109] The matrix $\frac{\partial r(x^\circ,t_1)}{\partial x^\circ}$, whose rank is k_1, is constructed, as noted above, with the local coordinates (26.96) of the *singular* plane $\mathbb{R}^n_{(x,p)} \times \mathbb{R}_t$ onto which the domain $\Omega_{1/2} \subset \Lambda^{n+1}_{|t-t_1|\ll 1}$ containing the trajectory $t \mapsto r(x^\circ, t) \in \Lambda^{n+1}_{[0,T]}$ is diffeomorphically projected. But in general the domain $\Omega_{1/2} \subset \Lambda^{n+1}_{|t-t_1|\ll 1}$ can be projected to a different *singular* plane, and in this case the rank of the corresponding matrix $\frac{\partial r(x^\circ,t_1)}{\partial x^\circ}$ can be $\widetilde{k}_1 \neq k_1$. The same applies to all the numbers k_l. Nevertheless, the exponent $e^{-\frac{i\pi}{2}\mu_j}$ in (26.99) will not change, because the following extremely important, but by no means simply provable, although quite expected[110] fact is true, which was established by Maslov in the book Maslov (1965):

$$\text{the difference} \quad \widetilde{k}_l - k_l \quad \text{is a multiple of 4.} \tag{26.100}$$

A detailed proof of this result can be found, for example, in the book Maslov and Fedoryuk (1976) (Lemma 6.4 and Proposition 7.3). However, the essence of this fact can be easily identified by following the instructions for Problem 26.26 presented below.

As already pointed out above, $\mu_j = \sum_{l=1}^{j} m_l$, where $m_l = (n - k_l)$. Morse proved (see, for example, § 15 in the book Milnor 1963) that m_l coincides (cf.

[109] V. I. Arnold Arnold (2006) writes: "I've involuntarily repeated twice the experience of Poincaré, who attributed his result to Lorentz, when I was working on my reports on doctoral dissertations of Maslov and Gudkov on symplectic topology and real algebraic geometry.

Maslov told me that the integer I called the "Maslov index" in my report on his thesis should not be attributed to him, because only its residue modulo 4 had a physical importance in the quasi-classical theory, while my integer was useless."

[110] Indeed, the method of constructing the formula was the one from which (see Theorem 26.19) the asymptotics for the same problem was constructed in the particular special case: $v(x) = |x|^2$ and $s(x) = x_1$. Therefore, there is not much reason to doubt that formula (26.99) is correct. Hence formula (26.99) should not change (up to an additive term $O(h)$) if k_1 is replaced by $\widetilde{k}_1$, which is equivalent to condition (26.100).

formula (26.81)) with the number

$$\operatorname{ind}\gamma(r',r'') \stackrel{\text{def}}{=} \operatorname{inerdex}\frac{\partial x(r,t')}{\partial p} - \operatorname{inerdex}\frac{\partial x(r,t'')}{\partial p}, \tag{26.101}$$

which is the index[111] of the non-self-intersecting curve $\gamma(r',r'')$ parameterized by $t \in [t',t'']$. This curve, which is a part of the phase trajectory containing the focal point $r_l = r(t_l)$, connects the initial point $r' = r(t')$ and the final point $r'' = r(t'')$, which are not focal points. Moreover, inerdex $\frac{\partial x^\circ(r,t)}{\partial p}$ is the negative inertia index of the quadratic form corresponding to the symmetric matrix $\frac{\partial x^\circ(r,t)}{\partial p}$ (in other words, the number of its negative eigenvalues).

Formula (26.101) has sense, of course, for any oriented curve lying on a Lagrangian manifold (not necessarily formed by the phase trajectories of a Hamiltonian system). In this general case, (26.101) defines the so-called *Maslov index*, which was introduced by Maslov in the book Maslov (1965).

The geometric definition of the index and the related algebraic and topological aspects were given by V. I. Arnold in his review of Maslov's doctoral thesis (see the footnote 99 on p. 182) and later in his famous paper Arnold (1967), which triggered many deep studies (see, for example, Novikov 1970, Vassiliev 1981, Karasev and Maslov 1984). Arnold's definition of the Maslov index can be expressed by the formula

$$\operatorname{ind}\gamma(r',r'') \stackrel{\text{def}}{=} \nu_+ - \nu_-, \tag{26.102}$$

where $\nu_\pm$ is the number of focal points on $\gamma(r',r'')$ such that the derivative $\frac{\partial x_{i_k}}{\partial p_{i_k}}$ changes its sign from $\mp$ to $\pm$ as these points are traversed in the direction of increasing argument p_{i_k}. Here without loss of generality (see Arnold 1967 and Arnold 1989) it is assumed that the local coordinates $(x_{i_1},\dots,x_{i_k},p_{i_{k+1}}\dots,p_{i_n})$ can be chosen (and are chosen) so that they define (as functions of the variables $(p_{i_1},\dots,p_{i_k},x_{i_{k+1}}\dots,x_{i_n})$) a Lagrangian manifold in the neighborhood of the corresponding focal points at which $\frac{\partial x_{i_k}}{\partial p_{i_k}} = 0$.

Problem 26.26 Verify formula (26.100) and the equality of the right-hand sides in (26.101) and (26.102) for the problem[112]

$$ih\Psi_t + \frac{1}{2}h^2\Delta\Psi = \frac{1}{2}\sum_{m=0}^{n}\Big(\frac{x_m}{m}\Big)^2\Psi, \quad \Psi(x,0)\Big|_{x=x^\circ} = \Phi(x^\circ)e^{\frac{i}{h}x_1^\circ}, \tag{26.103}$$

i.e., problem (26.91), where $v(x) = \frac{1}{2}\sum_{m=0}^{n}\big(\frac{x_m}{m}\big)^2$, and $s(x_1,\dots,x_n) = x_1$.

Hint To begin with, verify that the Lagrangian manifold $\Lambda^{n+1}_{[0,T]}$ corresponding to this problem is formed by the phase trajectories $\big(x(x^\circ,t), p(x^\circ,t), t\big)$ of the Hamiltonian system with the Hamiltonian $H(x,p,t) = \frac{1}{2}\big(|p|^2 + \sum_{m=1}^{n}\frac{x_m^2}{m^2}\big)$ emanating from the point $x^\circ \in \mathbb{R}^n$. Moreover,

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[111] Recall that $(r,t) \stackrel{(26.96)}{=} (x_{i_1},\dots,x_{i_k},p_{i_{k+1}},\dots,p_{i_n},t)$.

[112] From the previously considered problem (26.59) this problem differs only by the factor $\frac{1}{m^2}$ multiplying x_m^2.

$$x_1(x^\circ, t) = \sin t + x_1^\circ \cos t \quad \text{and} \quad x_m(x^\circ, t) = x_m^\circ \cos \frac{t}{m} \quad \text{for } m \geq 2.$$

Next, show that the rank of the matrices $\frac{\partial r(x^\circ, t_j)}{\partial x^\circ}$ at the focal points (for $n = 2$ they appear in the focuses $t_j' = \frac{\pi}{2} + \pi(j-1)$ and $t_j'' = \pi + 2\pi(j-1)$) is $n-1$, and that the neighborhood of the focal points is projected to singular planes, of which one is $\mathbb{R}_p^n \times \mathbb{R}_t$.

Returning to formula (26.99), we note that, in general, several rays can come to the point (x, t) from different initial points $x_1^\circ, \ldots, x_N^\circ$ of the coordinate space. This is possible (for an example, see the footnote 71 on p. 164) if the Lagrangian manifold consists (in the language of Riemannian surfaces) of different "sheets," and at the point (x, t), the points $(x, p_1, t), \ldots, (x, p_N, t)$ are projected, which lie on these "sheets" and into which the corresponding phase trajectories come. In this problem, the resulting formula for the asymptotics is the sum of their contributions, of which each is given by a formula of the form (26.99).

The following problem can serve as an illustration of possible applications of the canonical operator method to various problems

26.27 The Scattering Problem. More precisely, the problem of the influence of an inhomogeneity of a medium on the propagating high-frequency wave traveling in the plane $\mathbb{R}^2 \ni x = (x_1, x_2)$ in the direction x_1 with velocity $c(x) = \sqrt{\rho(x)}^{-1}$, where the function[113] $\rho \in C^\infty(\mathbb{R}^2)$ is known to be 1 *a fortiori* only for $|x| \geq 1$. This problem was first considered by Rayleigh in 1889. However, his assumption about molecular scattering in gases was erroneous, as pointed out by L. I. Mandelstam in 1907. Modern scattering theory is related to the Schrödinger equation, as well as to the Helmholtz equation. From this point of view, the scattering problem was considered, in particular, by Vainberg in Chapters V, X and XI of his book Vainberg (1982), (for a more general setting, see his earlier papers Vainberg 1975 and Vainberg 1977.

The above wave is described by the function

$$U\colon (x, t) \mapsto U(x, t) = \Psi(x) e^{\frac{i}{h}(t - x_1)}, \qquad \frac{1}{h} \gg 1,$$

satisfying the wave equation $U_{\tau\tau} = c^2(x)\Delta U$, where $\tau = t - x_1$. It follows that Ψ is a solution of the *Helmholtz equation*[114]

[113] The quantity $\rho(x) > 0$ reflects the properties of the medium (its inhomogeneity), which has an effect on the velocity of the wave passing through it.

[114] Hermann Ludwig Ferdinand von Helmholtz (1821–1894) was an outstanding German physicist, physician, physiologist, psychologist, acoustics, who made significant contributions in several scientific fields like physiological optics and acoustics, electrodynamics and thermodynamics, the law of conservation of energy, and the principle of least action, the theory of vortices, etc. One circumstance is noteworthy, which in some measure was the reason for the extraordinary breadth of Helmholtz's scientific research. In an after-dinner speech at the celebration of his seventieth birthday (1891) he recalls: "Nun sollte ich zur Universität übergehen. Die Physik galt damals noch für eine brodlose Kunst. Meine Eltern waren zu grosser Sparsamkeit gezwungen; also erklärte mir der Vater, er wisse mir nicht anders zum Studium der Physik zu helfen, als wenn ich das der Medicin mit in den Kauf nähme. Ich war dem Studium der lebenden Natur durchaus nicht abgeneigt und ging ohne viel Schwierigkeit darauf ein" (Hermann von Helmholtz: Erinnerungen).

$$\Delta\Psi + \frac{\rho(x)}{h^2}\Psi = 0, \quad x = (x_1, x_2) \ni \mathbb{R}^2, \tag{26.104}$$

subject to Sommerfeld radiation conditions[115]

$$\Psi(x) = O(r^{1/2}), \quad \frac{\partial\Psi(x)}{\partial r} - \frac{i}{h}\Psi(x) = o(r^{-1/2}) \quad \text{as } r = |x| \to \infty. \tag{26.105}$$

Applying the *Debye procedure* (in other words, the WKB method) of searching the asymptotics of the function Ψ in the form $\psi(x) = \varphi(x)e^{\frac{i}{h}S(x)}$ with sought-for real functions S (the phase) and φ (the amplitude), we get, as in Eqs. (26.60) and (26.61), two equations

$$|\nabla S|^2 - \rho(x) = 0, \quad 2(\nabla S|\nabla\varphi) + \varphi\Delta S = 0. \tag{26.106}$$

The first of these equations[116] is the stationary *Hamilton–Jacobi equation*. To this equation there correspond the Hamiltonian equations

$$\frac{dx}{ds} = H_p, \ \frac{dp}{ds} = -H_x \ \overset{H(p,x)=|p|^2-\rho(x)}{\Longrightarrow} \ \frac{dx}{ds} = 2|p|, \ \frac{dp}{ds} = \nabla\rho(x). \tag{26.107}$$

We have $\psi(x) = e^{\frac{i}{h}x_1}$ near the straight line

$$\Lambda_0^1 = \{(x_1, x_2) = (-2, b), \ b \in \mathbb{R}\},$$

because $\rho(x) = 1$ for $x < -1$. Moreover, $S(x)\big|_{\Lambda_0^1} = -2$ and $S_{x_1}(x)\big|_{\Lambda_0^1} = 1$. Hence we also get the initial data:

$$x\big|_{s=0} = (-2, \xi) \quad \text{and} \quad p\big|_{s=0} = (1, 0). \tag{26.108}$$

From these data it follows that $H\big|_{s=0} = 0$. And because $\frac{\partial H}{\partial s} = H_p\frac{dp}{ds} + H_x\frac{dx}{ds} \overset{(26.107)}{=} 0$, we find that $H \equiv 0$, i.e., $|p|^2 = \rho(x) \le C < \infty$. So,

$$|x(s)| \le 2Cs + \max(2, |\xi|),$$

and hence for $s \le \sigma < \infty$ the solution of problem (26.107), (26.108) is bounded. An important conclusion follows: the phase trajectories of system (26.107) emanating from the Lagrangian manifold Λ_0^1 form the two-dimensional Lagrangian manifold $\Lambda^2_{[0,\sigma]}$ in the phase space $\mathbb{R}^2_x \times \mathbb{R}^2_p$.

Here ξ and s the (global) coordinates on $\Lambda^2_{[0,\sigma]}$. Let $r_{s_j} = (\xi_j, s_j)$ be the sequence $(s_1 < s_2 < \ldots)$ of *focal* points on $\Lambda^2_{[0,\sigma]}$ (these are the points at which $J \overset{\text{def}}{=} \left|\frac{\partial x(\xi,s)}{\partial(\xi,s)}\right| = 0$). The projection onto the plane $\mathbb{R}^2_x$ of all focal points is the *caustic*, i.e.,

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[115] These conditions distinguish a unique solution to the Helmholtz equation (see, for example, Vladimirov 1971). In particular, $\Psi(x) = e^{\frac{i}{h}x_1}$ for $\rho(x) \equiv 1 \ \Leftrightarrow \ c(x) \equiv 1$.

[116] It has a special name—the *eikonal equation*, from the Greek "eiko" (image). The word "icon" has the same origin.

the intersection set of the characteristics (rays) of the Hamiltonian system. At the points where $J \neq 0$, $S\big(x(\xi, s)\big)$ is equal to

$$S\big(x(\xi,0)\big) + \int_0^s 2|p|^2 d\sigma = S\big(x(\xi,0)\big) + 2\int_0^s \rho\big(x(\xi,\sigma)\big)\,d\sigma,$$

because $dS = p\,dx$ (since $S_x = p$), and $\frac{dx}{ds} = 2|p|$ and $|p|^2 = \rho(x)$.

For the second of equations (26.106) (the transport equation) a similar argument gives the initial condition: $\varphi(x)\big|_{\Lambda_0^1} = 1$. And since $2(\nabla S|\nabla\varphi) = 2(p|\nabla\varphi) = \big(\frac{dx}{ds}|\nabla\varphi\big) = \frac{d}{ds}\varphi$, the equation itself assumes the form $\frac{d}{ds}\varphi(x) + \Delta S(x)\varphi(x) = 0$. One can check that

$$\Delta S(x(\xi,s)) = \frac{1}{2}\Big(\ln\frac{J}{2}\Big)'_s,$$

and $J\big|_{s=0} = 2$. As a result, we get $\varphi(x) = \sqrt{2/J}$.

Next, one should proceed as in the proof of Theorem 26.8. With the exception of the caustic neighborhood, the asymptotics of the function Ψ is given, for $s_j < s < s_{j+1}$, in the form almost analogous to (26.99). The justification of the asymptotics obtained in this way is presented in the last two chapters of the book Vainberg (1982) (and in the papers Vainberg 1975, Vainberg 1977 for a more general situation).

In addition, we note that in the case where $\rho(x)$ depends only[117] on $|x|$, there are relatively simple explicit asymptotic formulas obtained in Dobrokhotov et al. (2013) (see also Nazaikinskii 2014), which holds even *in the neighborhood of focal points and caustics.* One of these formulas is as follows:

$$\Psi(x) = \Big(\frac{i}{2\pi h}\Big)^{1/2} a(|x|) J_0\Big(\frac{T(|x|)}{h}\Big) + O(h). \tag{26.109}$$

Here $a(|x|) = \frac{2\pi}{\rho(|x|)}\frac{T(|x|)}{|x|}$, $T(r) = \int_0^r \rho^2(\xi)\,d\xi$, and J_0 is the Bessel function. Therefore, $\Psi(x)$ for $|x| > R \gg 1$ is asymptotically, as $h \to 0$,

$$\Big(\frac{i}{2\pi h}\Big)^{1/2} a(|x|) J_0\Big(\frac{|x| + \lambda_x}{h}\Big), \quad \text{where } \lambda_x = \int_0^R \big(\rho^2(\xi) - 1\big)\,d\xi.$$

The proof of (26.109) is based on the ideas developed, in particular, in Dobrokhotov et al. (2014), Dobrokhotov et al. (2017), which proved instrumental in effectively solving many new problems.

[117] Thus, we are talking about a radial wave originating from the origin (or entering it), and therefore, the caustic in this case is only $x = 0$.

Appendix A
New Approach to the Theory of Generalized Functions (*Yu. V. Egorov*)

1 Drawbacks of the Theory of Distributions

L. Schwartz's theory of distributions was created mainly by 1950s and quickly gained popularity not only among mathematicians but also among representatives of other natural sciences. This is explained to a large extent by the fact that the theory is based on fundamental physical principles, and hence it was found to be completely natural. On the other hand, the theory of distributions proved instrumental in obtaining many wonderful mathematical results in recent years. However, it soon became clear that this theory has two significant drawbacks, which seriously hinder its application in mathematics and in other natural sciences.

The first drawback is related to the fact that for distributions it is impossible in the general case to define the operation of multiplication so that this operation would be associative. This is seen, for example, from the following Schwartz's arguments: the product $(\delta(x)\cdot x)\cdot\left(\frac{1}{x}\right)$ is defined and is equal to 0 because each distribution can be multiplied by an infinitely differentiable function ($\delta(x)$ is multiplied by x and $\left(\frac{1}{x}\right)$ by 0). On the other hand, the product $\delta(x)\cdot\left(x\cdot\left(\frac{1}{x}\right)\right)$ is also defined and is equal to $\delta(x)$.

Moreover, L. Schwartz proved the following result.

Theorem 1.1 *Let A be an associative algebra with differentiation (this is a linear operator $D\colon A\to A$ such that $D(f\cdot g)=f\cdot D(g)+D(f)\cdot g$). Assume that the space $C(\mathbb{R})$ of continuous functions on the real line is a subalgebra in A, and D coincides with the ordinary differentiation on the set of continuously differentiable functions, and the identically one function, is a unit of the algebra A. Then A cannot contain an element $\delta\neq 0$ such that $x\cdot\delta(x)=0$.*

Let us show that the product $\delta\cdot\delta$ is not defined in the space of distributions. Let $\omega(x)$ be a $C_0^\infty(\mathbb{R})$-function such that $\int\omega(x)\,dx=1$, $\omega(0)=1$. We set $\omega_\varepsilon(x)=\frac{\omega(x/\varepsilon)}{\varepsilon}$. It is natural to put

$$\delta\cdot\delta=\lim\omega_\varepsilon^2,$$

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so that $(\delta \cdot \delta, \varphi) = \lim \int \omega_\varepsilon^2(x)\varphi(x)\,dx$. However,

$$(\omega_\varepsilon^2, \omega) = \int \omega_\varepsilon^2(x)\omega(x)\,dx = \varepsilon^{-1} \int \omega^2(x)\omega(\varepsilon x)\,dx \to \infty$$

as $\varepsilon \to 0$, proving the claim. So, the theory of distributions cannot be effectively applied for solution of nonlinear problems.

Another substantial drawback of the theory of distributions stems from the fact that even linear equations with infinitely differentiable coefficients, which are "ideal" for this theory, may fail to have solutions. For example, this is so for the equation

$$\frac{\partial u}{\partial x} + \frac{ix\partial u}{\partial y} = f(x, y).$$

One can find an infinitely differentiable function f with compact support on the (x, y)-plane such that this equation would have no solutions in the class of distributions in any neighborhood of the origin. In the actual fact, the number of such functions f is relatively big—they form a set of second category in $C_0^\infty(\mathbb{R}^2)$!

2 Shock Waves

The theory of discontinuous solutions of differential equations plays an important role in gas dynamics, hydrodynamics, theory of elasticity, and other branches of mechanics. Discontinuous solutions appear naturally in the study of *shock waves*. Here we mean the phenomenon when the principal characteristics of the medium have different values on different sides of some surface (called the wave front). Even though in the actual fact these quantities vary continuously, their gradient in the neighborhood of the wave front is substantial and so they can be easily described with the help of discontinuous functions.

For example, in gas dynamics, surges in pressure, density, and other quantities take place at distances of order 10^{-10} m. The gas dynamics equations read as

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \quad \text{(the continuity equation)},\\ (\rho v)_t + (\rho v^2 + p)_x &= 0 \quad \text{(the motion equation)},\\ p &= f(\rho, T) \quad \text{(the state equation)}. \end{aligned}$$

Here ρ is the gas density, v is the velocity of gas particles, p is the pressure, and T is the temperature. The first two equations are in the divergent from, and hence generalized solutions can be obtained by integration by parts, as in the theory of distributions. Here it is usually assumed that

$$\rho = \rho_1 + \theta(x - vt)(\rho_2 - \rho_1), \quad p = p_1 + \theta(x - vt)(p_2 - p_1),$$

where θ is the Heaviside function, which equals 0 for negative values of the argument and 1 for positive arguments, and the smooth functions ρ_1, p_1, ρ_2, p_2 are values of the density and pressure, respectively, to the left and right of the wave front surface.

A significant drawback of such a description is that here only one (general) Heaviside function is used. If we replace it by a smooth function θ_ε, which changes from 0 to 1 on a small interval of length ε, then the state equation will be violated in this transition zone, which may affect the results of the calculations.

An analysis of this situation suggests the following natural way to circumvent this difficulty: to describe the functions ρ and p, one should use different functions Θ_ε. In the limit, as $\varepsilon \to 0$, these functions tend to a single (common) Heaviside function, but for $\varepsilon \neq 0$ they should be such that the state equation would be satisfied.

In fact, this situation appears in applied mathematics quite often: for a correct adequate description of a phenomenon with the help of discontinuous functions, it is necessary to memorize the method of approximation of these discontinuous functions by smooth ones. But the impossibility of such memorization, which is principal for the theory of distributions, is the main drawback of this theory, which makes it inapplicable in nonlinear problems.

Let us now describe a new theory involving the theory of distributions and at the same time free from the above disadvantage. This theory originates in the works of the French mathematician J. F. Colombeau.[1]

3 New Definition of Generalized Functions

No matter how broad is the space of generalized functions, the space of infinitely differentiable functions should be dense in it. This fairly natural assumption is generally accepted and is justified by practical applications, and we have no reason to abandon it. Hence it is natural to define the space of generalized functions as the completion of the space of infinitely differentiable functions in some topology (which effectively defines the required space). For example, the space of distributions can be defined by considering all possible sequences of infinitely differentiable functions $\{f_j\}$ in which each sequence $\int f_j(x)\varphi(x)\,dx$ has finite limit as $j \to \infty$ if $\varphi \in C_0^\infty$.

Let Ω be some domain in the space $\mathbb{R}^n$. Consider the space of sequences $\{f_j\}$ of infinitely differentiable functions in Ω. Two sequences $\{f_j\}$ and $\{g_j\}$ from this space will be called equivalent if, for each compact subset $K \subset \Omega$, there exists an $N \in \mathbb{N}$ such that $f_j(x) = g_j(x)$ for $j > N$, $x \in K$. Now a *generalized function* is defined as the set of sequences equivalent to $\{f_j\}$. The space of generalized functions thus defined will be denoted by $G(\Omega)$.

If a generalized function is such that, for some of its representative $\{f_j\}$ and each function φ from[2] $\mathcal{D}(\Omega)$, the limit

[1] See, in particular, Colombeau (1983), and also Colombeau (1984). However, close ideas (in application to multiplication of distributions) had appeared significantly earlier in the paper Livchak (1969). *Note by A. S. Demidov*.

[2] In this context, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$.

$$\lim_{j\to\infty}\int_\Omega f_j(x)\varphi(x)\,dx$$

exists, then we can define a distribution corresponding to this generalized function. Conversely, to each distribution $g \in \mathcal{D}'(\Omega)$, one can associate a generalized function defined by the representative

$$f_j = g \cdot \chi_j * \omega_\varepsilon, \tag{3.1}$$

where $\varepsilon = \frac{1}{j}$, $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(x/\varepsilon)$, and χ_j is a function from the space $C_0^\infty(\Omega)$ which is equal to 1 at the points lying at distance $\geq \frac{1}{j}$ from the boundary of the domain Ω. So, $\mathcal{D}'(\Omega) \subset G(\Omega)$.

If a generalized function is defined by a representative $\{f_j\}$, then by its *derivative of order* α we mean the generalized function defined by the representative $\{D^\alpha f_j\}$. The product of two generalized functions defined by representatives $\{f_j\}$ and $\{g_j\}$ is the generalized function corresponding to the representative $x \mapsto \{f_j(x)g_j(x)\}$.

If F is an arbitrary smooth function of k real variables, then, for any k generalized functions $f_1, \ldots, f_k$, the generalized function $F(f_1, \ldots, f_k)$ is defined.

It is worth pointing out that in contrast to the theory of distributions, where $x \cdot \delta(x) = 0$ (this fact was used in the above Schwartz's example), the product $x \cdot$ "δ"(x) is different from 0.

Generalized functions have the *locality property*. If Ω_0 is a subdomain of Ω, then, for each generalized function f, the restriction $f\big|_{\Omega_0} \in G(\Omega_0)$ is defined. Moreover, one can define the restriction to each smooth submanifold lying in Ω and even define $f(x)$ at each point x from Ω. Here one should only take into account that such restriction is a generalized function on the corresponding submanifold. In particular, the values of generalized complex-valued functions at a point have sense only as *generalized complex numbers*, which are defined as follows.

One considers the set of all sequences of complex number s$\{c_j\}$. In this set, an equivalence relation is defined so that two sequences are equivalent if they are equal for large j. The resulting classes of equivalent sequences are called *generalized complex numbers*.

A generalized function f *is equal to* 0 *in* Ω_0 if there exist $N \in \mathbb{N}$ and a representative $\{f_j\}$ such that $f_j = 0$ in Ω_0 for $j > N$. The smallest closed set outside of which $f = 0$ is called the *support* of the function f. Note, however, that this leads to paradoxes from the point of view of the theory of distributions: it may happen, for example, that the support of a function f is a singleton, but the value of f at this point is zero.

If a domain Ω is covered by a finite or countable family of domains Ω_j and if at each of these domains a generalized function f_j is defined such that $f_i - f_j = 0$ on the intersection of the domains Ω_i and Ω_j, then the generalized function f is defined uniquely such that its restriction to Ω_j coincides with f_j.

4 The Weak Equality

In analogy with the theory of distributions, one can introduce the concept of a weak equality in the theory of generalized function. Namely, generalized functions f and g are said to be *weakly equal* (written $f \sim g$) if, for some of their representatives $\{f_j\}$ and $\{g_j\}$,

$$\lim_{j\to\infty} \int_\Omega [f_j(x) - g_j(x)]\varphi(x)\,dx = 0,$$

for any function φ from C_0^∞.

In particular, two generalized complex numbers defined by sequences $\{a_j\}$ and $\{b_j\}$ are weakly equal (written $a \sim b$) if $\lim(a_j - b_j) = 0$ as $j \to \infty$. It is clear that, for distributions, the weak equality coincides with the ordinary one. If $f \sim g$, then $D^\alpha f \sim D^\alpha g$ for any α. The next result shows that the weak equality is not "too weak."

Theorem 4.1 *If $f \in G(\mathbb{R})$, $f' \sim 0$, and if the limit $\lim_{j\to\infty} \int_\mathbb{R} f_j(x)h(x)\,dx = C$ exists and is finite for some function h from $C_0^\infty(\mathbb{R})$ for which $\int h(x)\,dx = a \neq 0$, then $f \sim$ const.*

Proof By the hypothesis, $\lim_{j\to\infty} \int_\mathbb{R} f_j(x)\varphi'(x)\,dx = 0$ for any function φ from $C_0^\infty(\mathbb{R})$. Hence

$$\lim_{j\to\infty} \int_\mathbb{R} f_j(x) \cdot \left[\sigma(x) - a^{-1}h(x)\int_\mathbb{R} \sigma(y)\,dy\right] dx = 0$$

for any function σ from $C_0^\infty(\mathbb{R})$, i.e.,

$$\lim_{j\to\infty} \int f_j(x)\sigma(x)\,dx = Ca^{-1}\int \sigma(x)\,dx.$$

The above theorem implies, for example, that systems of ordinary differential equations with constant coefficients have no weak solutions, except the classical ones.

If f and g are functions continuous in a domain Ω, then their product fg is weakly equal to the product of the generalized functions corresponding to the functions f and g. The following more general *theorem* holds: *if $F \in C^\infty(\mathbb{R}^p)$ and $f_1, \ldots, f_p$ are continuous functions, then the continuous function $F(f_1, \ldots, f_p)$ is weakly equal to the generalized function $F(g_1, \ldots, g_p)$, where g_k is the generalized function weakly equal to f_k.*

Note that the concept of the weak equality can generate theorems which are paradoxical from the point of view of classical mathematics: for example, the system of equations

$$y \sim 0, \quad y^2 \sim 1, \quad 0 < x < 1$$

is solvable. Among its solutions, we mention, say, the generalized function corresponding to $f(\varepsilon, x) = \sin(x/\varepsilon)$.

Consider now the Cauchy problem

$$\partial u/\partial t = F(t, x, u, \dots, D^\alpha u, \dots) \quad \text{for } t > 0, \quad u(0, x) \sim \Phi(x).$$

Here $u = (u_1, \dots, u_N)$ is an unknown vector and F and Φ are given vectors. It can be shown that such problem always has a weak solution in the class of generalized functions without any assumptions on the type of equations. Namely, consider some representatives $\{\Phi_j\}$ and $\{F_j\}$ of the classes Φ and F and consider the Cauchy problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= F_j(t, x, v(t-\varepsilon, x), \dots, D^\alpha v(t-\varepsilon, x), \dots) && \text{for } t > 0, \\ v(t, x) &= \Phi_j(x) && \text{for } -\varepsilon \le t \le 0, \end{aligned}$$

where $\varepsilon = 1/j$. It is clear that

$$v(t, x) = \Phi_j(x) + \int_0^t F_j(s, x, \Phi_j(x), \dots, D^\alpha \Phi_j(x), \dots) ds$$

for $0 < t \le \varepsilon$. Next, the same method produces $v(t, x)$ for $\varepsilon < t \le 2\varepsilon$, and so on. So, for each $t = t_0 > 0$, we get the generalized function $\{v_j(t_0, x)\}$, which is called the weak solution.

This construction agrees closely with the classical definition of a solution and with the definition of a solution in the theory of distributions. If this generalized function lies in the class C^m, where m is the maximal order of derivatives of u on the right of the equation, then it also satisfies the equation in the ordinary sense. If the function F is linear in u, if its derivatives depend smoothly on t (so that one can consider solutions of the Cauchy problem in the class of distributions), and if the resulting generalized function is a distribution, then this function will be a solution also in the sense of the theory of distributions.

Appendix B
Algebras of Mnemonic Functions
(*A. B. Antonevich*)

1 Introduction

As was noted in the main text of the book and in Egorov's Appendix A, the impossibility of correct definition of the product of distributions is an impediment for applications to nonlinear equations, which involve such products. Similar obstacles also occur in the study of linear equations with generalized coefficients. A typical example here is the equation

$$-\Delta u + a\delta u - \lambda u = f,$$

which appears in the point interaction theory (Albeverio et al. 1988). In this equation, the coefficient are delta functions.[1]

In this regard, various approaches to the problem of multiplication of distributions were developed: V. K. Ivanov (1972, 1979, 1981), C. Christov and B. Damianov (1979), E. Rosinger (2006), V.P. Maslov (1980), S.T. Zavalishin and A.N. Sesekin (1991), etc. The greatest resonance in this direction was caused by the works of the French mathematician J.-F. Colombeau (1985) and his followers Biagioni (1988) and Oberguggenberger (1992). The basic idea here is to build algebras, more precisely, *differential algebras*[2] consisting of new objects preserving a series of properties of distributions. Such objects were called *new generalized functions*. In specific problems, for an adequate description of a number of phenomena with the help of discontinuous or generalized functions, one frequently has to *remember* the way in which they are approximated by smooth functions—this is the approach adopted in the theory of new generalized functions (as stated in Egorov's appendix). Since the word "new" loses its semantic load with time, the elements of the constructed algebras will be called *mnemonic functions*, i.e., functions with memory.

[1] The paper Demidov (1970) considers problems with "surge"-type coefficients, whose singularity is even stronger than that of the δ-function.

[2] Here, by a differential algebra, we mean a vector space, in which, for any pair of elements, the (associative and commutative) multiplication operation is defined (usually, the symbol " $\cdot$ " is omitted), the derivation is defined, and the Leibniz rule is satisfied: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

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Egorov's Appendix A describes the most common (and most simple) algebra of this type proposed by him. Below we will describe the general method of construction of such algebras (obtained on the basis of analysis of the previous structures proposed in Antonevich and Radyno (1991, 1994)), introduce the algebra of mnemonic functions on the circle, and, for this model example, following the approach of Antonevich et al. (2018), discuss a number of questions about the algebras of mnemonic functions, which have not been addressed before.

2 General Scheme of Construction of Algebras of Mnemonic Functions

Assume that we have a space $\mathcal{D}'$ of distributions in a domain Ω. As an original object for construction of the corresponding algebras, we will consider the differential algebra $\widetilde{G}(\Omega)$ consisting of all families f_ε of infinitely differentiable functions on Ω that depend on the small parameter ε. Sometimes we will assume that the small parameter takes only the values $\frac{1}{j}$, i.e., we will consider sequences $\{f_j\}$ of such functions.

Let us first discuss relations of such families to distributions.

One says that a family f_ε is *associated with a distribution* f if it converges to f in the space of distributions, i.e., if $\langle f_\varepsilon, \varphi\rangle \longrightarrow \langle f_\varepsilon, \varphi\rangle$ for all test functions φ. More detailed information about the properties of the functions f_ε can be derived by analyzing the asymptotic behavior of the quantities $\langle f_\varepsilon, \varphi\rangle$. In this analysis, one frequently encounters cases when this family of functionals admits an asymptotic expansion in $\mathcal{D}'$ in powers of ε:

$$\langle f_\varepsilon, \varphi\rangle = \sum_{k=k_0}^{\infty} \langle u_k, \varphi\rangle \varepsilon^k, \quad \text{where } u_k \in \mathcal{D}',\ u_{k_0} \neq 0.$$

In particular, for $k_0 = 0$ the family f_ε is associated with the distribution u_0, for $k_0 > 0$ it is an infinitely small quantity, and for $k_0 < 0$ it is an infinitely large quantity.

We let $G_{\mathrm{as}}(\Omega)$ denote the subspace consisting of all equivalence classes $[f_\varepsilon]$ of families f_ε associated with distributions, and by $\mathcal{N}_0$ we denote the subspace of equivalence classes of families associated with zero. On $G_{\mathrm{as}}(\Omega)$, the equivalence relation generates the map of taking the limit

$$\mathrm{Lim}\colon G_{\mathrm{as}}(\Omega) \ni [f_\varepsilon] \mapsto \mathrm{Lim}([f_\varepsilon]) \stackrel{\mathrm{def}}{=} \lim_{\varepsilon\to 0} f_\varepsilon \in D'(\Omega).$$

Here, with each distribution u, one associates the large set $\mathrm{Lim}^{-1}(u)$ consisting of the families associated with u.

Another link toward distributions comes from consideration of approximating operators. A linear operator R acting from $\mathcal{D}'$ into $\widetilde{G}(\Omega)$ is called a *method of approximation* if, for any distribution u, its range is a family of smooth functions converging to u. This is equivalent to saying that $\operatorname{Lim} R(u) = u$, i.e., the operator R is the right inverse of the map Lim and defines an embedding of the original space of distributions into $\widetilde{G}(\Omega)$.

By using each such embedding, one can define the product of distributions, which is an element of $\widetilde{G}(\Omega)$. Namely, by definition, this product is given by the formula

$$u \otimes_R v = R(u)R(v). \tag{2.1}$$

If the product $R(u)R(v)$ is associated with a distribution h, then it is natural to consider this distribution h as the product uv generated by the given method of approximation R. In the general case, the product $R(u)R(v)$ is not associated with a distribution, but information about its properties can be obtained from its asymptotic expansion.

From Schwartz's example, it follows that the multiplication given by (2.1) cannot agree with the multiplication of a distribution by a smooth function, i.e., the equality $R(uv) = R(u)R(v)$ cannot be satisfied for all smooth functions u and all distributions v. So, under this approach the multiplication operation is corrected and becomes associative. But this change is usually small: the difference $R(uv) - R(u)R(v)$ is infinitely small.

The algebra $\widetilde{G}(\Omega)$ is quite broad and so it should be naturally used to introduce an equivalence relation such that multiplication sends equivalent elements to equivalent elements. This can be archived by changing to the factor algebra by some ideal J. Under this equivalence relation, each set $\operatorname{Lim}^{-1}(u)$ should split into several equivalence classes because different elements from $\operatorname{Lim}^{-1}(u)$ behave differently under multiplication. At the same time, it is desirable that an ideal should be large enough in order to minimize the number of such classes.

Moreover, in some problems, the entire algebra $\widetilde{G}(\Omega)$ is not used, and it suffices to consider some algebras thereof. The subalgebra may have ideals wider than in the algebra $\widetilde{G}(\Omega)$. As a result, equivalence classes may become wider than when considering the entire algebra.

So, the general process of construction of algebras of mnemonic functions is as follows: one first fixes some class of methods of approximation R and then chooses a subalgebra $\widetilde{A}(\Omega) \subset \widetilde{G}(\Omega)$ containing the ranges of the operators R, after which an ideal J is chosen in $\widetilde{A}(\Omega)$. Hence the required *algebra of mnemonic functions* $A(\Omega)$ is defined as the factor algebra

$$A(\Omega) = \widetilde{A}(\Omega)/J. \tag{2.2}$$

Usually there are no difficulties in choosing the subalgebra $\widetilde{A}(\Omega)$, and the main difficulty is related to the choice of an ideal J, i.e., the choice of the form of an equivalence relation.

The first condition on the ideal is that the operator R should generate an embedding of the original space of distributions in the factor algebra. This condition is satisfied if the ideal J is contained in $\mathcal{N}_0$, and in this case this ideal is small in a sense. On the other hand, for applications it is more convenient to deal with factor algebras involving some equalities not shared by $\widetilde{A}(\Omega)$.

For example, Egorov's construction involves the ideal

$$J_E = \left\{ f_\varepsilon : \forall K \Subset \Omega \ \exists \varepsilon_K : \quad f_\varepsilon(x) = 0 \ \text{ for } \ x \in K, \varepsilon < \varepsilon_K \right\}$$

and the embeddings R defined by formula (3.1) in Appendix A. Under the action of this operator,

$$R(f') \neq R(f)', \quad \text{but} \quad R(f') - R(f)' \in J_E.$$

Hence, after changing to equivalence classes, we get the "good" embedding property: a derivative is carried to a derivative.

In the general case, certain equalities in the factor algebra are fulfilled if elements of special form lie in the ideal, which implies that the ideal J should be quite large.

Below, we will give some properties of embeddings of great value for applications; on a specific example of an algebra of mnemonic functions on the circle, we will try to find out which properties may be possessed by various embedding under a given ideal J. It will turn out that sometimes the selected properties are incompatible, and in this case more involved algebras should be constructed.

To compare the algebra of mnemonic functions and the original space of distributions, we note that $\mathcal{D}'$ is isomorphic to the factor space $G_{\mathrm{as}}(\Omega)/\mathcal{N}_0$, i.e., it can be constructed by a similar scheme.

This allows one to identify the following sources of incorrectness in the problem of multiplication of classical generalized functions:

1. The space G_{as} is not an algebra. As a result, the factor space contains no elements which could serve as candidates for the product for an arbitrary pair of elements.

2. The subspace $\mathcal{N}_0$ is not an ideal in the algebra $\widetilde{G}(\mathbb{T}^1)$. As a result, products of representatives from the same equivalence class go into different classes, and hence the product of classes (i.e., distributions) is not defined correctly.

3. The fact that $\mathcal{N}_0$ is not a subalgebra in $\widetilde{G}(\mathbb{T}^1)$ results in statements of the type $0 \times 0 \neq 0$.

Note that in conceptual terms the transition from distributions to mnemonic functions is similar to that from points of a manifold M to points of its tangent bundle. Indeed, the tangent vector at a given point of a manifold can be defined as the class of equivalent curves which approach this point in a similar fashion (i.e., in one direction), i.e., such class preserves ("stores") information only about this direction. From this point of view, elements from $\mathrm{Lim}^{-1}(u)$ can be looked upon as curves in the space of distributions passing through the point u, and the corresponding mnemonic functions are classes of equivalent curves that equally (in a sense) approach u.

3 Algebras of Mnemonic Functions on the Circle

Let us consider the space of distributions on the circle, construct the corresponding *algebra of mnemonic functions*, and using this example, discuss a series of general questions appearing in the construction of algebras of mnemonic functions.

Let us recall the necessary facts. The space $C^\infty_{2\pi}(\mathbb{R})$ consists of complex-valued infinitely differentiable 2π-periodic functions, and the topology on this space is defined by the countable family of norms

$$p_m(\varphi) = \sum_{j=0}^{m} \max_z |\varphi^{(j)}(z)|, \quad \varphi \in C^\infty(\mathbb{T}^1). \tag{3.1}$$

This space is isomorphic to the space $C^\infty(\mathbb{T}^1)$ of infinitely differentiable functions on the circle $\mathbb{T}^1 = \{z\colon |z| = 1\}$, the corresponding isomorphism being given by $z = e^{it}$. The space of generalized functions (distributions) $\mathcal{D}'(\mathbb{T}^1)$ is defined as the dual space of $C^\infty(\mathbb{T}^1)$, i.e., this space consists of continuous linear functionals on $C^\infty(\mathbb{T}^1)$.

The space $C(\mathbb{T}^1)$ is embedded into $\mathcal{D}'(\mathbb{T}^1)$. The corresponding embedding is given by

$$C(\mathbb{T}^1) \ni u \longrightarrow \langle u, \varphi\rangle = \frac{1}{2\pi}\int_{\mathbb{T}^1} u(z)\varphi(z)|dz| = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\varphi(e^{it})\,dt. \tag{3.2}$$

Here, the normalizing factor $\frac{1}{2\pi}$ is introduced so that the resulting formulas would be more manageable.

Each distribution $f \in \mathcal{D}'(\mathbb{T}^1)$ can be represented as a sum of *Fourier series*

$$f = \sum_{-\infty}^{\infty} C_k z^k, \tag{3.3}$$

where the coefficients increase not faster than some power of $|k|$.

In particular, the delta function $\langle \delta_\xi, \varphi\rangle = \varphi(\xi)$ has the expansion

$$\delta_\xi = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \xi^{-k} z^k.$$

The distribution $\mathcal{P}\left(\frac{1}{z-1}\right)$ defined by

$$\left\langle \mathcal{P}\left(\frac{1}{z-1}\right), \varphi\right\rangle = \int_{\mathbb{T}^1} \frac{\varphi(z)}{z-1}\,dz$$

is of special importance at various places in analysis and, in particular, in the theory of analytic functions and in the theory of singular integral equations. Here, the integral is understood in the sense of the Cauchy principal value. The Fourier expansion of this distribution reads as

$$\mathcal{P}\left(\frac{1}{z-1}\right) = i\pi\left[\sum_{-\infty}^{-1} z^k - \sum_{0}^{+\infty} z^k\right].$$

Suppose that, in accordance with the general scheme, $\widetilde{G}(\mathbb{T}^1)$ is a differential algebra consisting of families f_ε of smooth functions on the circle. In the choice of the subalgebra $\widetilde{A}(\mathbb{T}^1) \subset \widetilde{G}(\mathbb{T}^1)$, we will start from the fact that, for typical R, the approximating families $R(u) = f_\varepsilon$ satisfy estimates of the form

$$p_m(f_\varepsilon) \leq \frac{C}{\varepsilon^{m+\nu}}.$$

The space consisting of the families f_ε satisfying such estimates is not an algebra, and hence we consider the broader space $\widetilde{A}(\mathbb{T}^1)$ consisting of such families $\{f_\varepsilon\}$ such that there exist numbers μ and ν for which the following estimate holds:

$$p_m(f_\varepsilon) \leq \frac{C}{\varepsilon^{\mu m+\nu}}.$$

This space with natural operations is a differential algebra.

Next, consider the set

$$J(\mathbb{T}^1) = \{g_\varepsilon : \forall\, p \text{ and } m\, \exists C : \; p_m(g_\varepsilon) \leq C\varepsilon^p\}.$$

This set is contained in $\mathcal{N}_0$ and is a differential ideal in $\widetilde{A}(\mathbb{T}^1)$; moreover, in a certain sense, this ideal is the best one—it is the largest of the ideals in $\widetilde{A}(\mathbb{T}^1)$, which can be defined in terms of the growth rate of the norms of $p_m(f_\varepsilon)$.

The algebra of mnemonic functions on the circle $A(\mathbb{T}^1)$ is defined as the factor algebra

$$A(\mathbb{T}^1) = \widetilde{A}(\mathbb{T}^1)/J(\mathbb{T}^1).$$

This algebra contains the *algebra of generalized complex numbers* $\widetilde{\mathbb{C}}$ generated by families of constant f_ε (independent of $z = e^{it}$).

4 Properties of Embedding

Let us find out which properties are satisfied by embeddings of the space of distributions $\mathcal{D}'(\mathbb{T}^1)$ into the algebra of mnemonic functions $A(\mathbb{T}^1)$.

1. *Invariance under rotations.* In the case of a circle, it is natural to suppose that the embedding should be invariant under rotations. The invariance property here is essential because it is equivalent to the equality

$$R(f') = R(f)',$$

which means that the embedding commutes with the differentiation.

Since each operator that commutes with rotations of the circle is the convolution operator with some distribution, each rotation-invariant method of approximation has the form

$$R(f) = f_\varepsilon = f * \psi_\varepsilon, \tag{4.1}$$

where $*$ is the convolution operation in the space $D'(\mathbb{T}^1)$, and ψ_ε is some family of distributions. The convolution operation uses the group structure of the circle, and the point 1 is highlighted because it is a neutral element of the group. Hence $\delta_1 * \psi_\varepsilon = \psi_\varepsilon$, and as a result, ψ_ε is the family of smooth functions converging to δ_1. So, under the invariance condition, the method of approximation is uniquely determined from approximations δ_1.

Using expansion (3.3) for f and the Fourier series expansion for ψ_ε

$$\psi_\varepsilon(z) = \sum A_k(\varepsilon) z^k,$$

we get

$$R(f) = f_\varepsilon(z) = \sum A_k(\varepsilon) C_k z^k, \tag{4.2}$$

and from the convergence of ψ_ε to δ_1, it follows that $A_k(\varepsilon) \to 1$ as $\varepsilon \to 0$.

The most simple and natural method of approximation is that given by Fourier series. Each distribution f expands in a series (3.3), and hence the formula

$$R_F(f) = f_n = \sum_{-n}^{n} C_k z^k \tag{4.3}$$

defines the embedding $D'(\mathbb{T}^1)$ into $\widetilde{A}(\mathbb{T}^1)$ (in this example, by $\widetilde{A}(\mathbb{T}^1)$ we denote the algebra of mnemonic functions generated by sequences of smooth functions).

From the point of view of Fourier theories, formulas of the kind (4.2) define summation methods of such series. The issue with summation of series is related, for example, with the fact that, for a continuous function f, the sequence of partial sums (4.3) may fail to converge uniformly. But there exist summation methods of the form (4.2) which improve the convergence: for these methods, $f_\varepsilon(z)$ converges uniformly to f. Similarly, embeddings given by (4.2) may have properties that are absent in embedding (4.3).

Below, we will analyze some properties of embeddings invariant under rotation.

2. *Locality of multiplication.* We will say that an embedding R satisfies the *multiplication locality property* if $R(f)R(g) = 0$ whenever the supports of the distributions g and f are disjoint.

The supports of two δ-functions concentrated at different points are disjoint. Under a given method of approximation of the form (4.1), the product

$$R(\delta_\xi) \times R(\delta_1) = \psi_\varepsilon(z) \times \psi_\varepsilon(\xi z) \tag{4.4}$$

is a nonzero element of the algebra $\widetilde{A}(\mathbb{T}^1)$. But if the supports of the functions $\psi_\varepsilon(z)$ contract to the point 1 as $\varepsilon \to 0$, then, for sufficiently small ε, the product (4.4) is zero and lies in the ideal, and the multiplication locality property is satisfied in the factor algebra. Note that this property also holds if the values of $\psi_\varepsilon(z)$ as $z \neq 1$ rapidly converge to zero because in this case the product (4.4) also lies in the ideal $J(\mathbb{T}^1)$.

It is easily verified that if the multiplication locality property holds for products of δ-functions, then it also holds for arbitrary distributions.

Approximations usually considered in spaces of distributions on the line are constructed as follows. Let $\psi \in D(\mathbb{R})$ and $\int_{\mathbb{R}} \psi(t)\, dt = 1$. Then the family

$$\psi_\varepsilon(t) = \frac{1}{\varepsilon}\psi\Big(\frac{t}{\varepsilon}\Big) \tag{4.5}$$

converges to δ_0, and the supports of these functions converge to the point 0. The corresponding method of approximation is given by expression (4.1), where $*$ is the convolution operation in the space $D'(\mathbb{R})$. Under this method of approximation, the multiplication locality property is satisfied. Here (4.5) is a δ-like family with given *profile* ψ. Such approximations can be conveniently studied because the behavior of the approximating family f_ε is described in terms of the properties of one fixed function ψ.

In particular, the above method of approximation can be applied to periodic distributions and has the multiplication locality property. But here the functions (4.5) are not periodic and so a modification of the construction is required in order to implement the above approximation via the convolution on the circle.

Let $\psi_0 \in D(\mathbb{R})$, $\int_{\mathbb{R}} \psi_0(t)\, dt = 1$, and the support lies inside the interval $(-\pi, \pi)$. For each ε, we set $\psi_\varepsilon(t) = \frac{1}{\varepsilon}\psi\big(\frac{t}{\varepsilon}\big)$ for $-\pi \le t \le \pi$ and extend this function to $\mathbb{R}$ with period 2π, i.e., consider the function

$$\psi_\varepsilon(z) = \sum_j \frac{1}{\varepsilon}\psi_0\Big(\frac{t + 2j\pi}{\varepsilon}\Big). \tag{4.6}$$

Hence the formula

$$R(f) = f_\varepsilon = f * \psi_\varepsilon, \tag{4.7}$$

where $*$ is the convolution operation in the space $D'(\mathbb{T}^1)$, defines the same method of approximation with the multiplication locality property.

But the detailed analysis of this method of approximation on the circle is more involved compared with that in the case of the straight line. This is related, for example, to the fact that there is no simple relation between the functions ψ_ε for different ε.

5 Compatibility of the Embedding with Multiplication in $C^\infty(\mathbb{T}^1)$

The interest in the works of J. F. Colombeau stems from the fact that he solved the problem of construction of an embedding that satisfies the condition of compatibility with the multiplication in $C^\infty(\mathbb{T}^1)$. In the case under consideration, Colombeau's problem is formulated as follows:

Let R_0 be the natural embeddings of the algebra $C^\infty(\mathbb{T}^1)$ into $G(\mathbb{T}^1)$, i.e., an embedding for which with a function $f \in C^\infty(\mathbb{T}^1)$ one associates a stationary (not depending on ε) family $f_\varepsilon = f$.

It is required to construct an embedding R of the space of distributions in the differential algebra G which, for infinitely differentiable functions, coincides with the natural embedding R_0 into G.

In the case $G = A(\mathbb{T}^1)$, in the Colombeau problem, it is required to construct an embedding such that

$$R(f) - f = f_\varepsilon - f \in J(\mathbb{T}^1) \quad \text{for all } f \in C^\infty(\mathbb{T}^1). \tag{5.1}$$

If (5.1) is satisfied, then the space $C^\infty(\mathbb{T}^1)$ embeds into $G(\mathbb{T}^1)$ as an algebra, i.e.,

$$R(fg) = R(f)R(g) \quad \text{for all } f \in C^\infty(\mathbb{T}^1), g \in C^\infty(\mathbb{T}^1). \tag{5.2}$$

As already noted, the equality

$$R(fg) = R(f)R(g) \quad \text{for all } \ f \in C^\infty(\mathbb{T}^1), \ \ g \in D'(\mathbb{T}^1) \tag{5.3}$$

cannot be satisfied for any embedding. Hence in the Colombeau problem, condition (5.2) should be satisfied; this condition is a relaxation of the impracticable condition (5.3).

Here we are dealing with the case when the natural restrictions on the embedding may not be satisfied. Namely, if approximations (4.7) are constructed by convolutions with functions of the form (4.6), where ψ_0 is a compactly supported function, then the matching conditions (5.1) and (5.2) with multiplication of smooth functions are not satisfied.

This follows from the fact that the expansion of the difference $f_\varepsilon - f$ starts with ε in the power equal to the number of the first nonzero moment of the function ψ_0, where the moments are the numbers

$$M_j(\psi) = \int_{-\infty}^{+\infty} t^j \psi(t)\, dt, \quad j \in \mathbb{N}.$$

But any compactly supported function ψ_0 has nonzero moments, and therefore, $f_\varepsilon - f$ is not contained in the ideal and, moreover, does not lie in any ideal in the algebra $\widetilde{A}(\mathbb{T}^1)$.

Condition (5.1) means that, for smooth functions, the approximations under consideration should converge fast to f. There exist many such methods of approximations—the most natural among them is the one given by (4.3) via the Fourier series. From the standard properties of Fourier series, it follows that the mapping R_F is an embedding of $D'(\mathbb{T}^1)$ in the algebra of mnemonic functions $A(\mathbb{T}^1)$. For this embedding, the compatibility conditions with multiplications (i.e., equalities (5.1) and (5.2)) are satisfied. However, the multiplication locality property is not met.

6 Joint Locality and Compatibility with Multiplication of Smooth Functions

From the results of the previous section, it follows that if one considers embeddings into $G(\mathbb{T}^1)$ generated by compactly supported functions, then the multiplication locality property is met, but there is no compatibility with multiplication, and the embedding generated by partial Fourier series is compatible with multiplication, but fails to have the locality property. Let us show that there exist embeddings into $G(\mathbb{T}^1)$ satisfying both these properties.

Consider the *Schwartz space* $\mathcal{S}(\mathbb{R})$, i.e., the set of functions infinitely differentiable and decreasing at infinity faster than any power of $\frac{1}{t}$. One difference of this space from the space of compactly supported functions is that in $\mathcal{S}(\mathbb{T}^1)$ there exist functions ψ such that

$$M_0(\psi) = 1, \quad M_j(\psi) = 0 \quad \text{for } j \in \mathbb{N}.$$

We choose such a function ψ_0 and construct the following family of periodic smooth functions:

$$\psi_\varepsilon(t) = \sum_j \frac{1}{\varepsilon}\psi_0\Big(\frac{t + 2j\pi}{\varepsilon}\Big). \tag{6.1}$$

A simple algebra shows that for this family ψ_ε the embedding (4.1) satisfies both the compatibility conditions (5.1) and (5.2) and the multiplication locality property.

In his studies, J. F. Colombeau used only methods of approximation generated by compactly supported functions. Hence in order to meet condition (5.1) he had to construct more involved algebras than $A(\mathbb{T}^1)$. Let us give the most simple variant of such an algebra, which will be called the *modified Colombeau algebra*, because it is constructed by singling out the most substantial steps in the more involved Colombeau's construction.

Consider the families of infinitely differentiable functions $\{f_{q,\varepsilon}\}$ depending on two parameters ε and $q \in \mathbb{N}$. We let $\widetilde{G_C}(\mathbb{T}^1)$ denote the set consisting of all such families for each of which there exist μ and ν satisfying

$$p_m(f_{q,\varepsilon}) \leq \frac{C}{\varepsilon^{\mu m+\nu}}. \tag{6.2}$$

This set is a differential algebra in which the subset

$$J_C(\mathbb{T}^1) = \big\{g_{q,\varepsilon} : \exists\ \mu_1 \text{ and } \nu_1 : \ p_m(g_{q,\varepsilon}) \leq C\varepsilon^{q-\mu_1 m-\nu_1}\big\}$$

is a differential ideal.

The *modified Colombeau algebra* is defined as the factor space

$$G_C(\mathbb{T}^1) = \widetilde{G_C}(\mathbb{T}^1)/J_C(\mathbb{T}^1).$$

To construct an embedding into this algebra, we choose a sequence of compactly supported functions ψ_q such that their supports lie in the neighborhood of the point 0 and $M_j(\psi_q) = 0$ for $1 \leq j < q$.

The *mapping* $R_C\colon f \mapsto f_{q,\varepsilon} = f * \psi_{q,\varepsilon}$ *defines an embedding* $D'(\mathbb{T}^1)$ *into* $G_C(\mathbb{T}^1)$ *for which both the compatibility condition* (5.1) *with multiplication of smooth functions and the multiplication locality property are satisfied.*

7 Analytic Representations of Distributions

Consider another method of approximation. This method, which is frequently used in analysis, is based on the known analytic representation of a distribution. In the case of a circle, this representation is defined as follows. Consider the expansion of the distribution f in a Fourier series and write down two series

$$f^+(z) = \sum_0^\infty C_k z^k, \tag{7.1}$$

$$f^-(z) = \sum_{-\infty}^{-1} C_k z^k. \tag{7.2}$$

Here the series (7.1) converges in the disk $|z| < 1$ and its sum $f^+(z)$ is an analytic function; the series (7.2) converges for $|z| > 1$ and its sum is an analytic function. This defines the mappings $P^\pm\colon f \mapsto f^\pm$ and so the distribution f can be identified with the pair (f^+, f^-), i.e., with a piecewise analytic function on the plane. In particular, we have

$$\delta_\xi = \left(-\frac{1}{z-\xi}, \frac{1}{z-\xi}\right), \quad \mathcal{P}\left(\frac{1}{z-\xi}\right) = \left(\frac{\pi i}{z-\xi}, \frac{\pi i}{z-\xi}\right).$$

The above analytic representation of a generalized function generates an approximation of the distribution f by smooth functions via values of the analytic representation on the circle of radius $r = 1 - \varepsilon$ and of radius $\frac{1}{r} = \frac{1}{1-\varepsilon}$:

$$R_a(f) = f_\varepsilon(z) = f^+((1-\varepsilon)z) + f^-\left(\frac{z}{1-\varepsilon}\right). \tag{7.3}$$

Using the representation (4.2), we get

$$R_a(f) = f_\varepsilon(z) = \sum_{-\infty}^{\infty} C_k (1-\varepsilon)^{|k|} z^k, \tag{7.4}$$

which corresponds to the Abel method of summation of series.

For example,

$$R_a(\delta_\xi) = -\frac{1}{rz-\xi} + \frac{1}{\frac{z}{r}-\xi}, \tag{7.5}$$

$$R_a\Big(\mathcal{P}\Big(\frac{1}{x-\xi}\Big)\Big) = \frac{\pi i}{rz-\xi} + \frac{\pi i}{\dfrac{z}{r}-\xi}. \tag{7.6}$$

Formula (7.3) *defines an invariant embedding of the space $D'(\mathbb{T}^1)$ into the algebra of mnemonic functions such that equality* (5.2) *is satisfied for pairs of smooth functions of the form $f = (f^+, 0)$ and $g = (g^+, 0)$ and also for pairs of smooth functions of the form $f = (0, f^-)$ and $g = (0, g^-)$. The multiplication locality property for such embedding is not satisfied.*

8 Examples of Multiplications of Schwartz Distributions

Let us study the products of distributions under approximations (7.3). The result of multiplication of distributions $f = (f^+, f^-)$ and $g = (g^+, g^-)$ can be written in the form

$$\begin{aligned} R_a(f)R_a(g) &= \Big[f^+((1-\varepsilon)z) + f^-\Big(\frac{z}{1-\varepsilon}\Big)\Big]\Big[g_+((1-\varepsilon)z) + g^-\Big(\frac{z}{1-\varepsilon}\Big)\Big] \\ &= f^+((1-\varepsilon)z)g_+((1-\varepsilon)z) + f^-\Big(\frac{z}{1-\varepsilon}\Big)g^+((1-\varepsilon)z) \\ &\quad + f^+((1-\varepsilon)z)g^-\Big(\frac{z}{1-\varepsilon}\Big) + f^-\Big(\frac{z}{1-\varepsilon}\Big)g^-\Big(\frac{z}{1-\varepsilon}\Big). \end{aligned}$$

Here

$$f^+((1-\varepsilon)z)g^+((1-\varepsilon)z) + f^-\Big(\frac{z}{1-\varepsilon}\Big)g^-\Big(\frac{z}{1-\varepsilon}\Big) = R_a((f^+g^+, f^-g^-)),$$

i.e., the sum of the first and fourth terms is an approximation of the distribution, which has the analytic distribution (f^+g^+, f^-g^-) and, naturally, converges to this distribution.

For a fixed ε, the sum of two remaining terms

$$\gamma_\varepsilon(z) := f^-\Big(\frac{z}{1-\varepsilon}\Big)g^+((1-\varepsilon)z) + f^+((1-\varepsilon)z)g^-\Big(\frac{z}{1-\varepsilon}\Big) \tag{8.1}$$

is the function analytic in the annulus

$$K_\varepsilon = \Big\{z\colon 1-\varepsilon < |z| < \frac{z}{1-\varepsilon}\Big\}.$$

Applying the operators $P^\pm$, we get an analytic representation of this function and the formula for multiplication of distributions

$$R_a(f)R_a(g) = (h_\varepsilon^+(z), h_\varepsilon^-(z)), \tag{8.2}$$

where

$$h_\varepsilon^+(z) = f^+(z)g^+(z) + P^+[\gamma_\varepsilon(z)],$$
$$h_\varepsilon^-(z) = f^-(z)g^-(z) + P^-[\gamma_\varepsilon(z)].$$

In consideration of examples, of special interest are the cases where the product-mnemonic function (2.1) is associated with a distribution. One usually says that this is the case if the smoothness of one factor compensates the singularity of the other one. There are only few examples for which the product of distributions with common singularities is well defined. But a researcher may draw arbitrarily large numbers of such examples. So, for instance, if, for given $f = (f_+, f_-)$ and $g = (g_+, g_-)$, the condition $\gamma_\varepsilon(z) \equiv 0$ satisfies, where $\gamma_\varepsilon(z)$ is given by (8.1), then the product of distributions $f \times g$ is associated with a distribution with analytic representation (f_+g_+, f_-g_-). Note that almost all nontrivial examples considered earlier correspond to this case. For example, this condition is satisfied for the product of distributions δ_1 and $P\big(\frac{1}{z-1}\big)$. As a result,

$$\delta_1 \times P\Big(\frac{1}{z-1}\Big) = \Big(-\frac{\pi i}{(z-\xi)^2}, \frac{\pi i}{(z-\xi)^2}\Big).$$

However, the analysis of the products of distributions $f = (f_+, 0)$ and $g = (0, g_-)$ involves difficulties, because in this case it is required to describe the behavior of the images $\gamma_\varepsilon(z)$ under the action of the operators $P^\pm$. An explicit description can be given in the case of rational f^+ and g^-, because for such functions the application of the operators $P^\pm$ reduces to the known partial fraction expansion of a rational fraction. The most illustrative example is the product of the distributions $f = \big(\frac{1}{z-\xi}, 0\big)$ and $g = \big(0, \frac{1}{z-\eta}\big)$. Here, we get

$$R_a(f)R_a(g) = \frac{1}{rz-\xi}\cdot\frac{1}{z/r-\eta} = C_1(r)R_a(f) + C_2(r)R_a(g), \tag{8.3}$$

where $C_1(r) = \frac{r^2}{\xi-\eta r^2}$, $C_2(r) = -\frac{1}{\xi-\eta r^2}$. If $\xi \neq \eta$, then there exist finite limits of the coefficients $C_1(r)$ and $C_2(r)$ as $r \longrightarrow 1$, and the product under consideration is associated with a distribution with the analytic representation

$$\Big(\frac{1}{\xi-\eta}\cdot\frac{1}{z-\xi}, -\frac{1}{\xi-\eta}\cdot\frac{1}{z-\eta}\Big).$$

And if $\xi = \eta$, i.e., if the factors have singularities at the same point, then as $r = 1-\varepsilon \longrightarrow 1$ the coefficients tend to infinity and have the expansions

$$C_1(r) = \frac{1}{2\varepsilon\xi} - \frac{1}{2} + \ldots,\ C_2(r) = -\frac{1}{2\varepsilon} + \frac{1}{2} + \ldots$$

As a result, we get the asymptotic expansion of the product

$$\left(\frac{1}{z-\xi}, 0\right) \times \left(0, \frac{1}{z-\xi}\right) \sim \frac{1}{\varepsilon}\delta_\xi + \frac{1}{\pi i}\mathcal{P}\left(\frac{1}{z-\xi}\right) + \ldots,$$

in which the first term contains an infinitely large coefficient.

Similar calculations in the analysis of the product $\delta_\xi \delta_\eta$ show that the asymptotic expansion of the product of delta functions concentrated at different points is the sum of delta functions with infinitely small coefficients, and the square of a delta function is a delta function with infinitely large coefficient.

In relation to this example, we note that the multiplication locality requirement is not quite physically justified. For example, a δ-function is an idealized model of the situation when the density of the distribution of the substance in a small neighborhood of a given point is so large that the major part of the mass is concentrated in this neighborhood, and the density is small outside this neighborhood. The multiplication of two such densities corresponding to delta functions concentrated at different point results in the product of the large density by the small one, but such product may lead to a nonzero density near some points, as is the case with the product $\delta_\xi \delta_\eta$.

Note that the product of delta functions with infinitely small coefficients is also a delta function with infinitely small coefficient. In other words, the corresponding mnemonic functions form a subalgebra. Such a subalgebra was discovered earlier in the paper Maslov (1980) from different considerations.

9 Conclusions

The above embeddings of the entire space of distributions into the algebra of mnemonic function are primarily of theoretical interest, but particular problems involve particular questions.

For example, in one approach to the study of linear differential equations with generalized coefficients, one replaces their coefficients by distributions approximating their mnemonic functions. As a result, the solution of the equation is reduced to the investigation of the behavior as $\varepsilon \to 0$ of the solutions of a family of equations with smooth coefficients that depend on ε. Under this approach, different methods of approximation may be used for different coefficients.

Nonlinear equations usually do not involve generalized coefficients, but of special interest in such equations are solutions with singularities, for example, discontinuous *shock wave* type solutions, which cannot be substituted in the equation. The main approach to obtaining such solutions consists in augmenting the equation with the terms containing the small parameter, after which the limit of solutions of the new equation as $\varepsilon \to 0$ is declared as a solution.

In these applied examples, the principal issue is that the original problem is ill-posed, because the mathematical model of the process under study is too rough and the corresponding equation has no solution. The common point in these approaches is that the statement of the problem is refined by augmenting the equation with infinitely small parameters or by specifying approximations for the coefficients. Such a refinement is the introduction of additional information, which is extracted from the subject area and is not contained in the originally posed problem.

Appendix C
Extensions of First-Order Partial Differential Operators (*S. N. Samborski*)

Analysis and its applications are replete with constructions in which some functions are associated with others, and the obvious trend of many recent decades is to express these structures as mapping (operators) in suitable function spaces. However, this is not always possible, for example, for numerous generalizations of derivatives—this topic is the subject of the so-called nonsmooth analysis. This is also the case in the definition of nonclassical solutions of equations as functions satisfying some or other relations (for example, minimax solutions of game theory equations) or a pair of inequalities (as in the case of viscous solutions of Hamilton–Jacobi (and more general) equations) or as functions that are limits of solutions of different (for example, singularly perturbed) equations. It seems natural and attractive to define similar solutions as "ordinary solutions" for suitable extensions of the "corresponding" operators in "corresponding" spaces. And if this does not happen, this suggests that there are simply no suitable spaces.

At this point, it is appropriate to turn to the very concept of a "space." In the above examples, one speaks about sets of functions equipped with classical algebraic structures (without which it is impossible to even formalize equations of mathematical physics) and topological structures.

Of course, there should be a relation between such structures, and such a relation has long been formulated, and, apparently undergone very little revisions since then. Here we speak about the *continuity* of algebraic operations (topological vector spaces, K-spaces, etc.).

Below we will get rid of the requirement of *continuity*, replacing it with a less stringent requirement of *closeness* (in the sense of closed graphs). For example, for the addition operation, this means that

$$\text{if} \quad f_i \to f, \quad g_i \to g \quad \text{and} \quad f_i + g_i \to h, \quad \text{then} \;\; h = f + g.$$

Let us give two examples of such spaces. The first *space*, *denoted by* $C_{\mathrm{ae}}(X)$ (continuous almost everywhere), consists of the equivalence classes of Riemann integrable (i.e., bounded and continuous almost everywhere) functions on a compact set X. The equivalence means the coincidence almost everywhere. Unlike the case

A. S. Demidov, *Equations of Mathematical Physics*,
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of L^1, the elements of this space have "values" at points of X, can be identified with some subset of the set of set-valued functions (interval-valued, in the case of scalar functions), and feature some classical algebraic structures. The principal property of the metric introduced in C_{ae} is that it preserves the values under limits and outside the set of continuous mappings. This *value preservation property does not hold* in the framework of spaces satisfying the condition of continuity of algebraic operation. Thanks to this property, the extension by closure of the operator of classical differentiation in C_{ae} inherits the known properties of classical differentiation (for example, linearity). Moreover, its domain is stable with respect to the max and min operations, which is important in many practical applications[1] (optimization, games, etc.).

The second *space, denoted by*[2] $\mathbb{S}$, has the following remarkable property, which also *does not hold* in the framework of spaces with continuous operations. Namely, the space $\mathbb{S}$ is the completion of the set of continuous functions with respect to a metric so that this space is also a completion in the sense of the partial order. Recalling that $\mathbb{R}$ has the same properties with respect to $\mathbb{Q}$ (agreement between the Cantor and Dedekind completions), it can be said that $\mathbb{S}$ plays the same role for the set C of continuous functions (with uniform *convergence*) as $\mathbb{R}$ plays for $\mathbb{Q}$. Of course, permitting a slight "abuse of language," it can be said that these are precisely the properties that, for example, are responsible in $\mathbb{R}$ for the intermediate values property (which fails to hold in $\mathbb{Q}$) and similar properties of intermediate values in $\mathbb{S}$ for partial derivative operators (which do not hold in C).

These properties of intermediate values will be used (*in the "spaces–operator"* language) for the study of evolution equations. The simplest result of this kind is an analogue of the classical Peano's theorem on local (in time) existence of solutions of ordinary differential equations only under the continuity condition of the "right-hand side," but now for first-order partial differential equations. In the space $\mathbb{S}$, we construct such extensions of operators with first-order partial derivatives, for which the solutions of the equations $\mathcal{H}y = f$ (with a new appropriately extended domain for $\mathcal{H}$) are the limits as $\varepsilon \to 0$ of the classical solutions y_ε of the equations

$$\varepsilon \Delta y_\varepsilon + \mathcal{H} y_\varepsilon = f.$$

Regularization of solutions of the original first-order equation $\mathcal{H}y = f$ by perturbation of this equation by a smoothing operator (in our setting, by the Laplace operator Δ) with small coefficient ε tending to zero is commonly called (following the hydrodynamic analogy) the vanishing viscosity method. A regularization of this kind dates back to E. Hopf (1950), J. D. Cole (1951), and to many other later studies. These methods can be applied to define nonclassical (i.e., nondifferentiable and even

[1] So, for univariate functions, the values of the "generalized" derivative are closed intervals. For example, if $f(x) = |x|$, then $f'(0) = [-1, 1]$. So, the derivative of a function is an interval-valued function. The sum of intervals is defined (as the set of sums of their points). General interval-valued functions do not form a vector space, and hence one cannot speak about linearity of the derivative. If $g = -f$, then $(f + g)'(0) = 0 \neq f'(0) + g'(0) = [-2, 2]$. Meanwhile C_{ae} is a vector space, and the linearity problem becomes correct (in particular, for the above example).

[2] Unlike the space $\mathcal{S}$, which was introduced by L. Schwartz.

discontinuous) solutions of unperturbed equations, which, however, correspond to real (and important) physical processes, as well as of equations related to control theory, game theory, and various variational problems. Later, many attempts have been made to define such solutions without passing to the limit, which gives, for example, great advantages in numerical applications. Since the 1990s and to this day, the theory of "viscous solutions" has become the most popular. This theory was originated in Crandall et al. (1992); Crandall (1996), and in many other studies aimed at motivation of the "maximum principle." The reader familiar with these works will see how the definition of a "viscous solution" was reflected in the construction of the domain of the extension of operators in the space $\mathbb{S}$. However, our motivation is completely different and is related to expansion of the differentiation in $\mathbb{S}$.

1 Preliminaries and Notation

In what follows, X and Y will denote compact sets in $\mathbb{R}^n$ which coincide with the closure of their interior. A set $X_0 \subset X$ is said to be *nowhere dense* if its closure does not contain any nonempty open set. A *set of first category* (or a *meager set*) is a union of countably many nowhere dense sets.[3] The class of sets of second category consists of complements of sets of first category in X.

Proposition 1.1 (see Oxtoby 1980) *Let $f\colon X \to \mathbb{R}^m$ and let $C_f \subset X$ be the set of points of continuity of the mapping f. If C_f is dense in X, then C_f is a set of second category.*

Recall that a function $f\colon X \to \mathbb{R} \bigcup \pm\infty$ is called *lower* (*upper*) *semi-continuous* at a point $x \in X$ if, for any sequence $x_i \to x$, for which $\lim_i f(x_i) \in \mathbb{R} \bigcup \pm\infty$ exists, this limit is not smaller (respectively, not greater) than $f(x)$, i.e.,

$$\lim_{y\to x} \inf f(y) \geq f(x) \quad \text{(respectively } \lim_{y\to x} \sup f(y) \leq f(x)\text{).}$$

Definition 1.2 A mapping defined on a dense subset of X and assuming values in a set of (possibly infinite) intervals[4] from $\mathbb{R}$ will be called an *interval-valued mapping*.

Let F be an interval-valued mapping. Then

$$F_*(x) = \liminf \xi_i, \quad F^*(x) = \limsup \xi_i, \tag{1.1}$$

$$F_*\colon X \to \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad F^*\colon X \to \mathbb{R} \cup \{+\infty\} \tag{1.2}$$

are the (single-valued!) lower (upper) semi-continuous functions defined everywhere in X, where the limits $\liminf$ and $\limsup$ are taken over all possible sequences

[3] For $X = [0, 1]$, the set of rational points on $[0, 1]$ is a set of first category.—*A.D.*

[4] In this appendix, an interval means either open, closed, or half-open interval.

$\{x_i\}$ from the domain F for which $x_i \to x$ and over all possible sequences $\{\xi_i\}$ for which $\xi_i \in F(x_i)$. Functions (1.1) will be called the *semi-continuous envelopes* of an interval-valued function f.

Let $f\colon X \to \mathbb{R}$ be a function and let $C_f \subset X$ be the set of points of continuity of f. Next, let C_f be dense in X. Let $f\big|_{C_f}$ be the restriction of the function f to the (dense (!) in X) subset C_f. Consider its semi-continuous envelopes

$$f^- = (f\big|_{C_f})_* \quad \text{and} \quad f^+ = (f\big|_{C_f})^*. \tag{1.3}$$

These are lower (upper) semi-continuous functions defined everywhere on X. If $C_f = X$, then $f^- = f^+$.

Definition 1.3 An interval $[f^-(x), f^+(x)]$ is called the *value* at a point $x \in X$ of a bounded function f defined on a dense subset of X.

A mapping $\mathcal{F}$ from a metric space A to a metric space B is an *operator* if $\mathcal{F}\colon A \to B$ is defined on a dense subset $\mathfrak{D}$ of A. Such a mapping will be denoted by $(\mathcal{F}, \mathfrak{D})$ (this notation expresses its domain).

An operator is called *closed* if its graph is closed in $A \times B$ (in other words, if $f_i \to f \in A$ and $\mathcal{F} f_i \to F \in B$, then $f \in \mathfrak{D}$ and $\mathcal{F} f = F$). Clearly, any continuous mapping is closed (but not otherwise in general).

An operator is called *preclosed* if the closure of its graph is a graph of a closed operator (this is equivalent to saying that if $f_i \to f \in A$, $g_i \to f \in A$, $\mathcal{F} f_i \to F$, $\mathcal{F} g_i \to G$, then $F = G$). If an operator $(\mathcal{F}, \mathfrak{D})$ is preclosed, then its extension by the rule

$$\text{if} \quad f = \lim f_i \text{ and } \exists \lim \mathcal{F} f_i, \quad \text{then} \quad \mathcal{F} f \stackrel{\text{def}}{=} \lim \mathcal{F} f_i \tag{1.4}$$

is called the *extension by closure* and denoted by $(\mathcal{F}, \mathfrak{D} \uparrow)$. There can be many closed extensions (even in the same space), but the *extension by closure differs from other extensions by the smallest extension of the domain of a preclosed operator*. If, for a sequence of closed operators $(\mathcal{F}_i, \mathfrak{D}_i)$, we have the implication

$$f_i \in \mathfrak{D}_i, \quad f_i \to f, \quad \mathcal{F}_i f_i \to F \quad \Rightarrow \quad f \in \mathfrak{D} \quad \text{and} \quad \mathcal{F} f = F,$$

then we will write $(\mathcal{F}, \mathfrak{D}) \subset \lim(\mathcal{F}_i, \mathfrak{D}_i)$.

2 The Space $C_{\mathrm{ae}}(X, E)$

Definition 2.1 Let E be a finite-dimensional normed space. We let $C_{\mathrm{ae}}(X, E)$ denote the set of equivalence classes of bounded and almost everywhere continuous mappings from X into E, where the equivalence means the coincidence almost everywhere.

Let $f \in C_{\mathrm{ae}}(X, \mathbb{R})$ and let $\varphi \in f$. Since φ is a continuous function on a dense subset of X, in accordance with Definition 1.3 with this function one can associate an interval-valued function. This interval-valued function does not depend on the choice of the representative φ in the class f but depends only on the class f itself. We will continue to denote this function by f and write $f(x) = [f^-(x), f^+(x)]$. If f and g are different as elements from $C_{\rm ae}$, then the *interval-valued functions* f and g are also different and vice versa. Let $f \in C_{\rm ae}(X, \mathbb{R}^m)$, $f = (f_1, \ldots, f_m)$. We define $f(x)$ by the formula

$$f(x) = Co\{(\xi_1, \ldots, \xi_n):\ \xi_i \in f_i(x)\}, \tag{2.1}$$

where $Co\{A\}$ denotes the convex hull of a set $A \subset \mathbb{R}^m$. Recall that, for two compact sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^m$, the formula

$$\mathsf{h}(A, B) \overset{\text{def}}{=} \sup_{a \in A, b \in B} \{\mathsf{h}(a, B),\ \mathsf{h}(b, A)\}, \quad \text{where } \mathsf{h}(a, B) = \inf_{b \in B} \|a - b\|, \tag{2.2}$$

defines the Hausdorff distance between A and B.

The set of Lipschitz functions $\mathrm{Lip}_k(X, \mathbb{R})$ on X with values in $\mathbb{R}$ and with Lipschitz constant k is a conditionally complete sublattice of the *lattice*[5] of continuous functions on X. Hence, for any $f \in C_{\rm ae}(X, \mathbb{R})$ and any $n \in \mathbb{N}$,

$$f_n^- = \sup\{\varphi \in \mathrm{Lip}_n(X, \mathbb{R}):\ \varphi \le f\},$$
$$f_n^+ = \inf\{\varphi \in \mathrm{Lip}_n(X, \mathbb{R}):\ \varphi \ge f\}$$

are defined. Let $f, g \in C_{\rm ae}(X, \mathbb{R})$. Now the formula

$$\mathsf{r}(f, g) = \sup_n \{\mathsf{h}(\mathsf{gr} f_n^-, \mathsf{gr} g_n^-); \mathsf{h}(\mathsf{gr} f_n^+, \mathsf{gr} g_n^+); \|f_n^- - g_n^-\|; \|f_n^+ - g_n^+\|\}, \tag{2.3}$$

where $\|\varphi\| = \int_X |\varphi|\, d\mu$, defines a metric in the space $C_{\rm ae}(X, \mathbb{R})$). Indeed, assume that $\mathsf{r}(f, g) = 0$. If we assume that $f \ne g$, then there exist $x \in C_f \cap C_g$, numbers α, β ($\alpha < \beta$), and a neighborhood of the point x in which $f < \alpha$ and $g > \beta$ (either $g < \alpha$ or $f > \beta$). Hence there exists an $n \in \mathbb{N}$ such that $f_n^+ \ne g_n^+$ and $f_n^- \ne g_n^-$, a contradiction.

Theorem 2.2 (see Samborski 2004) *The space $C_{\rm ae}(X, \mathbb{R})$ equipped with metric (2.3) is complete, and the set of continuous functions $C(X, \mathbb{R})$ is dense in it.*

The next result is easy.

Proposition 2.3 *Let $f, g, h \in C_{\rm ae}(X, \mathbb{R})$, and let $\{f_i\}$ and $\{g_i\}$ ($i \in \mathbb{N}$) be sequences of elements from $C_{\rm ae}(X, \mathbb{R})$. Then*

(1) If $f_i \to f$, $x_i \to x$ in X, $\xi_i \in f_i(x_i)$, and $\xi_i \to \xi$ in $\mathbb{R}$, then $\xi \in f(x)$.

(2) If $f_i \to f$ and $g_i \to g$ and $X = C_f \bigcup C_g$, then $f_i + g_i \to f + g$ and $f_i \cdot g_i \to f \cdot g$.

(3) If $f_i \to f$, then the condition $\xi_i \to f(x)$ holds for any $x \in C_f$ and any $\xi_i \in f_i(x)$.

[5] A *lattice* is a partially ordered set in which each two-element subset has both least upper bound and greatest lower bound. A lattice is *conditionally complete* if each bounded subset of this lattice has east upper bound and greatest lower bound.

(4) *If the function f is continuous on X, then the convergence $f_i \to f$ is uniform, i.e., for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|\xi - f(x)| < \varepsilon$ for any $i > N$, $\xi \in f_i(x)$.*

(5) *The addition, multiplication operations, and the* min- *and* max-*operations are closed, i.e., if $f_i \to f$, $g_i \to g$, then*

$$\begin{aligned} f_i + g_i \to h &\Rightarrow h = f + g, \\ f_i \cdot g_i \to h &\Rightarrow h = f \cdot g, \\ \min(f_i, g_i) \to h &\Rightarrow h = \min(f, g), \\ \max(f_i, g_i) \to h &\Rightarrow h = \max(f, g). \end{aligned}$$

Remark 2.4 The existence of a natural isomorphism between $C_{\text{ae}}(X, \mathbb{R}^m)$ and $(C_{\text{ae}}(X, \mathbb{R}))^m$ under which $f \in C_{\text{ae}}(X, \mathbb{R}^m)$ corresponds to $(f_1, \ldots, f_m)$, where $f_i \in C_{\text{ae}}(X, \mathbb{R})$, makes it possible to endow $C_{\text{ae}}(X, \mathbb{R}^m)$ with the metric $\mathsf{r}(f, g) = \max_i \mathsf{r}(f_i, g_i)$, for which $C_{\text{ae}}(X, \mathbb{R}^m)$ is a complete space.

3 Differentiation in $C_{\text{ae}}(X, \mathbb{R}^m)$

Let $(x_1, x_2, \ldots, x_n)$ be the coordinates in X, and let D_i be the operator of (classical) partial differentiation with respect to x_i acting from $C_{\text{ae}}(X, \mathbb{R}^m)$ into $C_{\text{ae}}(X, (\mathbb{R})^* \otimes \mathbb{R}^m)$ and whose domain consists of the mappings with continuous partial differential. We let D denote the operator of (classical) *differentiation* acting from $C_{\text{ae}}(X, \mathbb{R}^m)$ into the space $C_{\text{ae}}(X, (\mathbb{R}^n)^* \otimes \mathbb{R}^m)$ and whose domain consists of continuously differentiable functions.

Theorem 3.1 *The operators D and D_i are preclosed. If $\boldsymbol{D}$ (respectively, $\boldsymbol{D}_i$) is the extension by closure of the operator D (respectively, D_i) and $\mathfrak{D}$ (respectively, $\mathfrak{D}_i$) is its domain, then*

(1) $\mathfrak{D} = \bigcap_i \mathfrak{D}_i$ *and* $\boldsymbol{D} = (\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots, \boldsymbol{D}_n)$.

(s2) $\mathfrak{D}$ *and* $\mathfrak{D}_i$ *are linear subspaces, and* $\boldsymbol{D}$ *and* $\boldsymbol{D}_i$ *are linear operators.*

Proof It suffices to consider the case $m = 1$ by putting $i = 1$. Assume on the contrary that the operator D_1 is not preclosed. Let $F_i \to F$ and $G_i \to F$, where F_i and G_i are continuously differentiable, and $D_1 F_i \to \Phi$ and $D_1 G_i \to \Psi$, but $\Phi \neq \Psi$. The last inequality implies that there exist an open set $\mathcal{U} \subset X$ and numbers α and β $(\alpha < \beta)$ and $N \in \mathbb{N}$ such that, for any $x \in \mathcal{U}$, we have $D_1 F_i(x) < \alpha$, $D_1 G_i(x) > \beta$ for $i > N$. Let $x' = (x_1', x_2, \ldots, x_n)$, $x'' = (x_1''$ and $x_2, \ldots, x_n)$ be two continuity points in $\mathcal{U}$ of the mapping F. Using the property of preservation of values in C_{ae} under limits (see assertion (1) in Proposition 2.3) and changing to subsequences, we get

$$\lim_i F_i(x') = \lim_i G_i(x') = F(x'); \;\; \lim_i F_i(x'') = \lim_i G_i(x'') = F(x'').$$

On the other hand,

$$F_i(x') - F_i(x'') < \alpha|x_1' - x_1''|\,; \quad G_i(x') - G_i(x'') > \beta|x_1' - x_1''|.$$

This contradiction shows that the operator D_1 is preclosed and hence so is the operator D. If $F, G \in \mathfrak{D}_1$, $\Phi = D_1F$, $\Psi = D_1G$, then both the Newton–Leibniz formula

$$F(x_1, y) = \int_a^{x_1} \Phi(t, y)\, dt + F(a, y)$$

for F and for G hold almost everywhere for nearly all $y = (x_2, \ldots, x_n)$ such that $\Phi(\cdot, y)$ and $\Psi(\cdot, y)$ are continuous. It follows that the operators $\mathbf{D}_1$ and $\mathbf{D}$ are linear. □

Corollary 3.2 *Let $\Theta = (\Theta_1, \ldots, \Theta_n)$ be a unit vector in $\mathbb{R}^n$, and let $F \in \mathfrak{D}$. Then, for each $x \in X$,*

$$\boldsymbol{D}F(x) \cdot \Theta = \Big(\sum_i \Theta_i\, \boldsymbol{D}_i F\Big)(x)$$

for convex sets[6] *in $\mathbb{R}^n$.*

Among generalizations of the differential in nonsmooth analysis, the most known is the Clark differential (Clarke 1983) for Lipschitz function $f\colon X \to \mathbb{R}$. This differential is defined by the formula

$$(D_{Cl} f)(x) \stackrel{\text{def}}{=} Co\, \mathrm{Cl}\{\lim_i Df(x_i)\},$$

where the limit is taken over all possible sequences x_i converging to x and consisting of the points of differentiability of the function f. (Here we use Rademacher's theorem,[7] according to which a Lipschitz function is differentiable almost everywhere.)

We say that a mapping $F \in C_{\mathrm{ae}}$ is differentiable (in C_{ae}) if $F \in \mathfrak{D}$, i.e., the differentiation operator $\mathbf{D}$ applies to the mapping F.

Proposition 3.3 *If $F \in \mathfrak{D}$, then the mapping F is Lipschitz. A Lipschitz mapping F is contained in $\mathfrak{D}$ if and only if the mapping $x \mapsto DF(x)$, which is defined at points of (classical) differentiability of the mapping F, is continuous almost everywhere. The class $\boldsymbol{D}F$ lies in $C_{\mathrm{ae}}(X, (\mathbb{R}^n)^* \otimes \mathbb{R}^m)$ and contains the mapping $x \mapsto DF(x)$. Moreover, $\boldsymbol{D}F(x) = D_{Cl}F(x)$ for each $x \in X$ for convex subsets of $\mathbb{R}^m$.*

The following example shows that a Lipschitz function may fail to lie in $\mathfrak{D}$.

Example 3.4 Let $K = [0, 1] \setminus \bigcup_i I_i$ be a *Cantor set of positive measure*, where $\{I_i\}$ is a countable system of pairwise disjoint intervals such that K is nowhere dense. We set $\Phi(x) = \int_0^x \chi_K\, d\mu$, where χ_K is the indicator function of K. Then the function Φ

[6] For a convex set $A \subset (\mathbb{R}^n)^* \otimes \mathbb{R}^m$ and a vector $b \in \mathbb{R}^n$, we denote by $A \cdot b$ the convex subset of $\mathbb{R}^m$ consisting of all vectors $\alpha \cdot b$, where $\alpha \in A$. Note that A can be uniquely recovered from the sets $\{A \cdot b\colon\ b \in \mathbb{R}^n,\ \|b\| = 1\}$.

[7] *H. Rademacher*, Über partielle und totale Differenzierbarkeit I, *Math. Ann.* **89** (1919), 340–359.

is Lipschitz but does not lie in $\mathfrak{D}$ because $D_{Cl}\Phi(x) = [0,1]$ on a set of nonzero measure.

Note that neither Corollary 3.2 nor the following proposition holds in general for the Clarke differential.

Proposition 3.5 *Let* $F, G \in C_{\mathrm{ae}}(X, \mathbb{R})$ *be differentiable. Then*

(1) *The sum* $F + \lambda G$*, where* $\lambda \in \mathbb{R}$*, is differentiable, and* $\boldsymbol{D}(F + \lambda G) = \boldsymbol{D}F + \lambda \boldsymbol{D}G$.

(2) *The product* $F \cdot G$ *is differentiable, and*

$$\boldsymbol{D}(F \cdot G) = F \cdot \boldsymbol{D} + G \cdot \boldsymbol{D}F.$$

(3) *The fraction* $1/G$ *is differentiable if* $G(x) \neq 0$ *for any* x*, and* $\boldsymbol{D}(1/G) = -\boldsymbol{D}G/G^2$.

(4) $\max(F, G)$ *and* $\min(F, G)$ *are differentiable, and* $\boldsymbol{D}(\max(F, G))$ *coincides in* C_{ae} *with the class containing the mapping*

$$x \mapsto \begin{cases} \boldsymbol{D}F(x) & \text{if } F(x) \geq G(x), \\ \boldsymbol{D}G(x) & \text{if } G(x) > F(x). \end{cases}$$

Proof Assertion (1) follows from Theorem 3.1. Assertions (2) and (3) follow from the application of the chain rule to the composition $\varphi \circ \psi$, where φ is a classically differentiable function (log, exp) and $\psi \in \mathfrak{D}$. In view of assertion (1) and the equality $\max(F, G) = \max(F - G, 0) + G$, it suffices to consider the case $G = 0$. Arguing as in the proof of Theorem 3.1, we reduce the problem to the case of a single variable $x \in X$. Let $A = \{x \in X \colon F(x) = 0,\ DF(x) \neq 0\}$. The function $x \mapsto DF(x)$ is continuous almost everywhere, and hence each point $x \in A$ has a neighborhood in which $DF(\cdot)$ preserves the sign. Hence in this neighborhood the function F is monotone. Therefore, the measure of A is zero, and $\max(F, G) \in \mathfrak{D}$. □

Among finite operations over functions that occur in the classical differential calculus, it remains to consider the composition of mappings and the chain rule for differentials. Let $F \in C_{\mathrm{ae}}(X, E)$. A representative $\varphi \in F$ is called *regular* if the set of its points of continuity coincides with the set of points of continuity of the function F. For example, if $E = \mathbb{R}$ and $F(x) = [F^-(x), F^+(x)]$, then the functions F^- and F^+ are regular representatives of the function F.

Let us define the composition of mappings. Let $F \in C_{\mathrm{ae}}(X, \mathbb{R}^m)$, $G \in C_{\mathrm{ae}}(Y, \mathbb{R}^k)$, and let $F(X) \subset Y$. A class $H \in C_{\mathrm{ae}}(X, \mathbb{R}^k)$ is called a composition of mappings F and G (written $G \circ F$) if, for any regular functions, $\varphi \in F$ and $\psi \in G$ the mapping $x \mapsto \psi(\varphi(x))$ is a representative in H.

Example 3.6 Let $X = [-1, 1]$, $Y = [-1, 1]^2$ and $F(x) = (x, ax)$. Let B be the class containing the mapping

$$(x, y) \mapsto \begin{cases} 1 & \text{if } y > 0, \\ -1 & \text{if } y < 0 \end{cases}$$

as a representative. Then the composition $B \circ F$ is defined if and only if $a \neq 0$.

Proposition 3.7 *Let $F\colon X \to Y \subset \mathbb{R}^m$ and $G\colon Y \to \mathbb{R}$ be differentiable mappings in C_{ae}. Assume that their composition $G \circ F$ is differentiable (this is always so for $m = 1$). Then its differential $\boldsymbol{D}(G \circ F)$ defines an element from $C_{\text{ae}}(X, (\mathbb{R}^n)^*)$ that coincides with $(\boldsymbol{D}G)(F(\cdot))\boldsymbol{D}F(\cdot)$.*

Example 3.8 Let us return back to Example 3.4. Let x_i be the midpoint of the interval I_i and let $2r_i$ be its length. We set $Y = [0, 1] \times [-1, 1]$ and define

$$L = Y \setminus \bigcup_i B\big((x_i, 0), r_i\big),$$

where $B(z, r)$ is the open ball of radius r with center at z. Then the function $G\colon (x, y) \mapsto \int_0^x \chi_L \, d\mu$, where χ_L is the indicator function of the set L, is differentiable, and its differential is an element from $C_{\text{ae}}(X, \mathbb{R})$. If $F(x) = (x, 0)$, then F and G are differentiable in C_{ae}, but $G \circ F$ is not.

The next theorem is useful in practical applications of the above differential calculus.

Theorem 3.9 *Let $X \subset \mathbb{R}^n$ and let a mapping $F\colon X \to \mathbb{R}^m$ be obtained from a **finite** number of real analytic functions via a **finite** number of operations $\lambda\cdot$, $+$, $\cdot$, $1/\cdot$,* max*,* min*, and their compositions ($\lambda \in \mathbb{R}$). Then F is differentiable in $C_{\text{ae}}(X, \mathbb{R}^m)$ and $\boldsymbol{D}F$ can be evaluated by the formulas given in Propositions 3.5 and 3.7 and Corollary 3.2.*

4 The Space $\mathbb{S}(X, E)$

Let E be a normed space over $\mathbb{R}$. By $\mathbb{S}^b(X, E)$, we denote the set of bounded functions on X with values in E, each of which is continuous on some dense subset of X. We will write $f \sim g$ if two functions f and g from $\mathbb{S}^b(X, E)$ coincide on the set $C_f \bigcap C_g$ of common points of continuity. It is clear that in this case $g \sim f$, and if in addition $g \sim h$, then $f \sim h$ (because by Proposition 1.1 the intersection $C_f \bigcap C_g$ is a set of second category in X for any f and g from $\mathbb{S}^b$).

Definition 4.1 $\mathbb{S}(X, E)$ is the set of classes of equivalence with respect to $\sim$, i.e.,

$$\mathbb{S}(X, E) = \mathbb{S}^b(X, E)/\!\sim .$$

This is equivalent to saying that $\mathbb{S}(X, E)$ is the set of classes of equivalent functions on X, each of which is continuous on some subset of second category in X, and the equivalence means coincidence on a set of second category.

In cases, where the concrete form of a normed space E and/or a set X plays no role, we will write $\mathbb{S}$ in place of $\mathbb{S}(X, E)$. If $f \in \mathbb{S}$, then the set

$$C_f = \bigcup_{\varphi \in f} C_\varphi,$$

which consists of the union of points of continuity for all representatives of an element f, will be called the set of points of continuity of an element $f \in \mathbb{S}$.

If $T\colon E \times E \to E$ is some binary operation in E, then it generates in $\mathbb{S}(X, E)$ the corresponding binary operation according to the usual rule: $T(f, g)$ is the equivalence class of the function $x \mapsto T(\varphi(x), \psi(x))$, where φ (respectively, ψ) is any representative of the class f (respectively, g). So, $\mathbb{S}(X, E)$ becomes a *vector space* over $\mathbb{R}$, and $\mathbb{S}(X, \mathbb{R})$ is a *commutative ring* (where the multiplication is distributively related to the commutative addition). Moreover, $\mathbb{S}(X, E)$ is a modulus over $\mathbb{S}(X, \mathbb{R})$, i.e., over this generalization of a vector of the space in which the multiplication by a scalar is replaced by that by elements of the ring $\mathbb{S}(X, \mathbb{R})$. Note that $\mathbb{S}(X, \mathbb{R})$ is a *lattice*,[8] and the inequality $f \le g$ for elements of the factor space $\mathbb{S}(X, E) = \mathbb{S}^b(X, E)/\!\sim$ means that the inequality $\varphi \le \psi$ for some (and hence for any) representatives $\varphi \in f$ and $\psi \in g$ holds on a set of second category.

Theorem 4.2 (see Samborski 2004) *The lattice $\mathbb{S}(X, \mathbb{R})$ is conditionally complete, and the sublattice $C(X, \mathbb{R})$ of continuous functions is dense in it (i.e., each element $f \in \mathbb{S}(X, \mathbb{R})$ is the supremum of some family of elements from $C(X, \mathbb{R})$ and simultaneously the infimum of a different family of elements from $C(X, \mathbb{R})$).*

Let $f \in \mathbb{S}(X, \mathbb{R})$, and $\varphi \in f$. Similarly to the case of the space $C_{\mathrm{ae}}(X, \mathbb{R})$, the *interval-valued function*

$$x \mapsto f(x) = [f^-(x), f^+(x)], \tag{4.1}$$

corresponding to φ, is independent of the choice of φ but depends only on the class f. In what follows, we will identify the class $f \in \mathbb{S}(X, \mathbb{R})$ and the corresponding interval-valued function (4.1).

Recall that to an interval-valued function $f \in \mathbb{S}(X, \mathbb{R})$ there correspond its semicontinuous envelopes

$$f^- \overset{(1.3)}{=} (f|_{C_f})_* \quad \text{and} \quad f^+ \overset{(1.3)}{=} (f|_{C_f})^*.$$

Proposition 4.3 *The interval-valued function*

$$x \mapsto [\varphi(x), \psi(x)]$$

corresponds to some element $f \in \mathbb{S}(X, \mathbb{R})$ (i.e., $\varphi = f^-$ and $\psi = f^+$) if and only if

$$\varphi^* = \psi \quad \textit{and} \quad \psi_* = \varphi. \tag{4.2}$$

Proposition 4.4 *The interval-valued functions $f \in \mathbb{S}(X, \mathbb{R})$ have the following properties:*

1. ***Separation on open subsets.*** *If f and g lie $\mathbb{S}(X, \mathbb{R})$ and if $f(x_0) \ne g(x_0)$ at some point x_0 (for definiteness, $f^+(x_0) > g^+(x_0)$), then there exist an open subset*

[8] See the remark on p. 217.

$U \subset X$ and numbers α and $\beta < \alpha$ such that the inequalities[9] *$\xi > \alpha$ and $\eta < \beta$ hold for any $x \in U$ and any $\xi \in f(x)$, $\eta \in g(x)$.*

2. ***Compactness of the graph*** $\mathrm{gr} f \overset{\text{def}}{=} \left\{\bigcup_{x\in X, \xi\in f(x)}(x, \xi)\right\} \subset X \times \mathbb{R}$.

3. ***Existence of maximum and minimum.*** *The function $f \in \mathbb{S}(X, \mathbb{R})$ attains on X both its maximum and minimum values. Namely, there exist $\widehat{x} \in X$ and $\xi = \max f \in f(\widehat{x})$ such that $\xi \geq \eta$ for any $x \in X$ and $\eta \in f(x)$. The minimum* $\min f$ *of the function f is defined similarly.*

4. ***Existence of intermediate values.*** *If $[a, b] \subset X$,*

$$\xi \in f(a) = [f^-(a), f^+(a)], \quad \eta \in f(b) = [f^-(b), f^+(b)],$$

then, for any $\theta \in [\min(\xi, \eta), \max(\xi, \eta)]$, there exists a $c \in [a, b]$ such that $\theta \in f(c)$.

5. ***Recovery on a dense subset.*** *The function $f \in \mathbb{S}(X, \mathbb{R})$ is uniquely recovered from its values on any (!) dense subset $X_0 \subset X$.*

6. ***Quasicontinuity.*** *If $f \in \mathbb{S}(X, \mathbb{R})$, then f^- and f^+ are quasicontinuous functions.*[10] *Moreover, on the set $QC(X)$ of quasicontinuous functions on X, the relation $\varphi \sim \psi$, which means that φ coincides with ψ on some dense subset of X, is an equivalence relation. The quotient space $QC(X)/\sim$ thus obtained can be naturally identified with $\mathbb{S}(X, \mathbb{R})$.*

Note that, for any $f \in \mathbb{S}(X, \mathbb{R})$ and $n \in \mathbb{N}$,

$$f_n^- = \sup\{\varphi \in \mathrm{Lip}_n(X, \mathbb{R}) \colon \varphi \leq f\},$$
$$f_n^+ = \inf\{\varphi \in \mathrm{Lip}_n(X, \mathbb{R}) \colon \varphi \geq f\}$$

are defined. Let $f, g \in \mathbb{S}(X, \mathbb{R})$. The formula

$$\mathsf{s}(f, g) \overset{\text{def}}{=} \sup_n\{\mathsf{h}(\mathrm{gr} f_n^-, \mathrm{gr} g_n^-),\ \mathsf{h}(\mathrm{gr} f_n^+, \mathrm{gr} g_n^+)\}, \tag{4.3}$$

defines a metric in the space $\mathbb{S}(X, \mathbb{R})$.

Theorem 4.5 (see Samborski 2004) *The space $\mathbb{S}(X, \mathbb{R})$ equipped with the metric (4.3) is complete, and the set $C(X, \mathbb{R})$ of continuous functions is dense in it.*

Let $f \in \mathbb{S}(X, \mathbb{R}^m)$, $f = (f_1, \ldots, f_m)$. We define $f(x)$ (cf. (2.1)) by

$$f(x) = Co\{(\xi_1, \ldots, \xi_n) \colon \xi_i \in f_i(x)\},$$

where $Co\{A\}$ is the convex hull of $A \subset \mathbb{R}^m$. The natural isomorphism between $\mathbb{S}(X, \mathbb{R}^m)$ and $(\mathbb{S}(X, \mathbb{R}))^m$, under which $f \in \mathbb{S}(X, \mathbb{R}^m)$ corresponds to $(f_1, \ldots, f_m)$, where $f_i \in \mathbb{S}(X, \mathbb{R})$, enables one to endow $\mathbb{S}(X, \mathbb{R}^m)$ with the metric

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[9] Unlike continuous functions, the open subset $U \subset X$ in general is not a neighborhood of the point x_0. Similarly, if $f^+(x_0) > g^+(x_0)$ and $f^-(x_0) < g^-(x_0)$, then there exist U and V such that $f > g$ in U and $f < g$ in V. Moreover, one can choose U and V so that x_0 would lie in the intersection of their closures.

[10] Recall that a function $\varphi \colon X \to \mathbb{R}$ is called *quasicontinuous* if, for any $x \in X$, $\varepsilon > 0$, there exists an open set U such that $x \in \mathrm{Cl}\, U$ and $y \in U \implies |f(x) - f(y)| < \varepsilon$.

$$\mathsf{s}(f, g) = \max_i \mathsf{s}(f_i, g_i),$$

with respect to which $\mathbb{S}(X, \mathbb{R}^m)$ is a complete metric space.

Proposition 4.6 *Let $f, g, h \in \mathbb{S}(X, \mathbb{R})$, and let $\{f_i\}, \{g_i\}$ be sequences of elements from $\mathbb{S}(X, \mathbb{R})$. Then*

(1) *If $f_i \to f$ in $\mathbb{S}(X, \mathbb{R})$, $x_i \to x$ in X, $\xi_i \in f_i(x_i)$ and $\xi_i \to \xi$ in $\mathbb{R}$, then $\xi \in f(x)$.*

(2) *The convergence $f_i \to f$ is uniform if the function f is continuous on X.*

(3) *The addition, multiplication operations, and the* min *and* max *operations are closed, i.e., if $f_i \to f$ and $g_i \to g$, then*

$$\begin{aligned} f_i + g_i \to h \ &\Rightarrow \ h = f + g, \\ f_i \cdot g_i \to h \ &\Rightarrow \ h = f \cdot g, \\ \min(f_i, g_i) \to h \ &\Rightarrow \ h = \min(f_i, g_i), \\ \max(f_i, g_i) \to h \ &\Rightarrow \ h = \max(f_i, g_i). \end{aligned}$$

5 Differentiation in $\mathbb{S}(X, \mathbb{R})$

We will identify $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ in the usual way with the help of the inner product $\langle\cdot,\cdot\rangle$. Let $X \subset \mathbb{R}^n$ and let $h\colon X \to \mathbb{R}$ be a lower semi-continuous function. Then, for any open set $U \subset X$, there exist $\varphi \in C^1(X)$ (the set of continuously differentiable functions on X) and $x \in U$ such that $\varphi \le h$ and $\varphi(x) = h(x)$. Indeed, it suffices to "push up" a small ball until it reaches the graph of h.

Definition 5.1 Let $f \in \mathbb{S}(X, \mathbb{R})$, $x \in X$. The subsets of $(\mathbb{R}^n)^*$ defined by

$$\begin{aligned} (D^- f)(x) &= \{D\varphi(x)\colon \varphi \in C^1,\ \varphi \le f,\ \varphi(x) = f^-(x)\}, \\ (D^+ f)(x) &= \{D\varphi(x)\colon \varphi \in C^1,\ \varphi \ge f,\ \varphi(x) = f^+(x)\} \end{aligned}$$

are called, respectively, the *subdifferential* and the *superdifferential* of a function f at a point x.

One can easily verify that, for each x, the sets $(D^- f)(x)$ and $(D^+ f)(x)$ are closed and convex (possibly, empty). Moreover, $(D^\pm f)(x)$ is nonempty if $x \in X_\pm$, where X_+ and X_- are dense in X.

We will write $\zeta \in (D^\pm f)(x)$ assuming that $x \in X_\pm$.

Proposition 5.2 *Let $X \subset \mathbb{R}$. The operator of classical differentiation D defined on the set of continuously differentiable functions is preclosed qua an operator in $\mathbb{S}(X, \mathbb{R})$. Let $\boldsymbol{D}$ be its extension by closure and let $\mathfrak{D}$ be the domain of this extension. Then*

$$f \in \mathfrak{D} \ \Leftrightarrow \ \boldsymbol{D} f = \left[\left(D^- f(\cdot)\right)_*, \left(D^+ f(\cdot)\right)^*\right] \in \mathbb{S}(X, \mathbb{R}).$$

Here, as in Proposition 4.3, the symbol "$*$" denotes the semi-continuous envelope of an interval-valued function defined on a dense subset.

Proposition 5.3 *Let ξ be a smooth vector field in $X \subset \mathbb{R}^n$. The operator of classical differentiation in the direction of the field ξ, which is defined on mappings continuously differentiable in the direction ξ, is preclosed qua an operator in $\mathbb{S}(X, \mathbb{R})$. Let $\boldsymbol{D}_\xi$ be its extension by closure, and let $\mathfrak{D}_\xi$ be the domain of this extension. Then*

$$f \in \mathfrak{D}_\xi \;\Leftrightarrow\; \boldsymbol{D}_\xi f = \left[\langle D^- f^-(\cdot), \xi(\cdot)\rangle_*, \langle D^+ f^+(\cdot), \xi(\cdot)\rangle^*\right] \in \mathbb{S}(X, \mathbb{R}).$$

The total differentiation operator D, which is defined on the set of continuously differentiable functions, is a preclosed operator from $\mathbb{S}(X, \mathbb{R})$ into $\mathbb{S}(X, \mathbb{R}^*)$. If $(x_1, \ldots, x_n)$ are coordinates in X and $\boldsymbol{D}_i$ and $\boldsymbol{D}$ are the extensions by closure of the operators of partial differentiation with respect to x_i and of the total differentiation, then $\boldsymbol{D} = (\boldsymbol{D}_1 \ldots, \boldsymbol{D}_n)$.

Proposition 5.4 *Let $F = [F^-, F^+] \in \mathbb{S}(X, \mathbb{R})$, where $X = [a, b] \subset \mathbb{R}$. Let Φ be the set of measurable functions φ on X such that*

$$F^-(x) \le \varphi(x) \le F^+(x) \quad \textit{for any} \;\; x \in X.$$

Then the following two conditions are equivalent:

(1) *$f \in \mathfrak{D}$ and $\boldsymbol{D} f = F$.*

(2) *There exists a $\varphi \in \Phi$ such that $f(x) = f(a) + \int_{[a,x]} \varphi \, d\mu \quad (a \in X)$.*

Proposition 5.4 shows that in general $\mathfrak{D}$ is not a linear subspace in $\mathbb{S}(X, \mathbb{R})$. The corresponding example can be easily constructed with the help of *Cantor sets* (of first category) of *positive measure*. Hence, without dwelling on the differential calculus in $\mathbb{S}(X, \mathbb{R})$, we only note that the complete analogue of Theorem 3.9 holds, in which one should replace C_{ae} by $\mathbb{S}$.

We complete this section with the following lemma, which will be repeatedly applied in what follows.

Lemma 5.5 *Let $u \in \mathbb{S}(X, \mathbb{R})$ and $\mathcal{U}$ be a neighborhood in X. Then there exist $x \in \mathcal{U}$, $l \in \mathbb{R}$, $\Theta \in (\mathbb{R}^n)^*$, sequences $\{x_i^-\}$ and $\{x_i^+\}$ converging to x in $\mathcal{U}$, and sequences $\{\Theta_i^-\}$ and $\{\Theta_i^+\}$ converging to Θ in $(\mathbb{R}^n)^*$ such that*

$$u^-(x_i^-) \to l, \quad u^+(x_i^+) \to l, \quad \Theta_i^- \in D^- u(x_i^-), \quad \Theta_i^+ \in D^+ u(x_i^+).$$

Proof We give only a general scheme of the proof, contenting ourselves for simplicity with the case $\dim X = 2$ (the argument in the case $\dim X = 1$ is trivial). All the open sets that appear below are assumed to be homeomorphic to an open ball.

If F has in $\mathcal{U}$ a dense set of minimum points, then by an appropriate smooth deformation φ of the domain $\mathcal{U}$ one can obtain for $u\varphi$ also a point of local maximum. Taking this point for x and using the quasi-continuity of the function $u^\pm$, we arrive at the conclusion of the lemma with $\theta = 0$ for $u\varphi$.

In what follows, we will assume that in $\mathcal{U}$ there are no extremum points of the function $u^\pm$.

We let $T_\mathcal{U}$ denote the set of connected components of the level sets of the restriction of u to $\mathcal{U}$. If u assumes the value λ on an element from $T_\mathcal{U}$, then this element will be denoted by $\tilde{\lambda}$. The set $T_\mathcal{U}$ can be endowed with the natural topology: a neighborhood of the point $\tilde{\lambda}$ consists of all $\tilde{\mu}$ falling in the neighborhood of the point $\tilde{\lambda}$ as a subset of $\mathcal{U}$ (this is an analogue of the well-known construction in the case of a continuous u and a closed $\mathcal{U}$). Assume that the points at which the level set passing through such points divides the neighborhood of this point in at least two parts are dense in $\mathcal{U}$. Then it can be easily verified that the result of this lemma is true for $\theta = 0$.

So, in what follows, we will assume that the neighborhood $\mathcal{U}$ has the following property: for any $x \in \mathcal{U}$, the component of the level set $\tilde{\lambda}$ containing x divides $\mathcal{U}$ in precisely two disjoint sets, the tree $T_{\mathcal{U}}$ is homeomorphic to an interval, and the function $\tilde{\lambda} \to \lambda$ on $T_{\mathcal{U}}$ is strictly increasing.

Let u_n^+ and u_n^- be the above Lipschitz majorant and minorant of the function u. We set

$$\theta_n^- \stackrel{def}{=} \{x \in \mathcal{U} \colon u_n^-(x) = u^-(x)\}, \quad \theta_n^+ \stackrel{def}{=} \{x \in \mathcal{U} \colon u_n^+(x) = u^+(x)\}.$$

If one of these sets (and hence, the other one too) contains an open subset, then on this set the function u is Lipschitz, proving the lemma. So, we can assume that $\theta_n^\pm$ are closed nowhere dense cycle-free subsets.

Assume there exists a point $x \in \mathcal{U}$ such that, in any of its neighborhood V, the set $V \setminus \theta_n^-$ consists of an infinite number of connected components. The same is also true for $V \setminus \theta_n^+$. One can construct sequences $\{x_i^-\}$, $\{x_i^+\}$ converging to x ($x_i^- \in V \setminus \theta_n^-$, $x_i^+ \in V \setminus \theta_n^+$), sequences $\{\theta_i^- \in D^- u_n^-(x_i^-)\}$, $\{\theta_i^+ \in D^+ u_n^+(x_i^+)\}$, and $\|\theta_i^- - \theta_i^+\| \to 0$, $i \to \infty$. Since $\|D^\pm u_i^\mp((x_i^\pm)\| \le n$, there exists the limit θ of two sequences $\{\theta_i^-\}$ and $\{\theta_i^-\}$, where $\|\theta\| \le n$.

Now let us assume that $x \in \theta_n^\pm$ for any $n \in \mathbb{N}$ and that, for any neighborhood $V \ni x$, the set $V \setminus \theta_n^\pm$ consists of a finite number of connected components. We denote by $\{\mathcal{U}_i^-\}$ the contracting system of neighborhoods

$$\mathcal{U}_i^- = \bigcup \{\tilde{\lambda} : \lambda \in]\lambda'_{i,-}, \lambda''_{i,-}[\},$$

where $\tilde{\lambda}$ is the set of level u_n^- corresponding to λ. Let

$$t_i^- = \inf\{\|x - y\|, x \in \lambda'_{i,-}, y \in \lambda''_{i,-}\},$$

and assume that, for any i,

$$(\lambda'_{i,-} - \lambda''_{i,-})/t_i^- \le K,$$

where K is a constant independent of i. Similarly, we define $\mathcal{U}_i^+$, t_i^+ with "−" replaced everywhere by "+," but with the preservation of K.

For simplicity, we assume that $\theta_n^\pm$ (for all n) is a union of intervals (in $\mathbb{R}^2$) and $D^- u_n^-$ ($D^+ u_n^+$, respectively) is constant on the interior of each interval (this assumption changes only very insignificantly the ensemble D^- and D^+ in $\mathcal{U}_i^\pm$). Let I^- be the union of intervals I from θ_n^- such that the estimate $\|D^- u_n(x)\| \le K$ holds for $x \in \operatorname{Int} I$. In a similar manner, we define I^+ by changing "−" by "+." Let α_i be the greatest angle between the intervals from $I_{n_i}^-$ and $I_{n_i}^+$ (in $\mathcal{U}_i$). If there exist an $\alpha > 0$ and an infinite sequence $\{n_i\}$ such that $\alpha_{n_i} > \alpha$, then there exist sequences $\{x_i^- \in I_{n_i}^-\}$, $\{x_i^+ \in I_{n_i}^+\}$, $\theta_i \in D^- u_{n_i}^-(x_i^-) \bigcap D^+ u_{n_i}^+(x_i^+)$ $\|\theta_i\| \le K/\sin\frac{\alpha}{2}$. In this case, the proof completes by passing to the limit.

Assume now that $\alpha_i \to 0$. In $\mathcal{U}_i$, we choose rectilinear coordinates (x, y) so that the x-coordinate (up to α_i) would be directed along the intervals $I_{n_i}^\pm$. For a fixed i and neighborhoods V^- and V^+ in $\mathcal{U}_i$, consider the functions

$$\Phi_i^-(x) = \inf_{(x,y)\in V^-} u_{n_i}^-(x, y), \quad \Phi_i^+(x) = \sup_{(x,y)\in V^+} u_{n_i}^+(x, y).$$

Now let us choose the above neighborhoods V^- and V^+ so as to satisfy the following properties:

(1) $\Phi_i^\pm$ are monotone functions with sufficiently close end points of the graphs and such that $\Phi_i^- \le \Phi_i^+$.

(2) If $|D^\pm \Phi_i^\mp(x)| \le K$, then there exist $y^\pm(x)$ such that

$$\Phi_i^\pm(x) = u_{n_i}^\pm(x, y^\pm(x)).$$

(3) $V^\pm$ be the connected part of the set $\mathcal{U}_i^\pm$ consisting of $\tilde{\lambda}$, $\lambda \in]\lambda'_{i,\pm}, \lambda''_{i,\pm}[$.

By Properties 1 and 3, there exist x^- and x^+ such that $D^- \Phi_i^-(x^-)$ is sufficiently close to $D^+ \Phi_i^+(x^+)$ and $|D^- \Phi_i^-(x^-)| \le K$. Now, from Property 2, it follows that $D^- u_i^-(x^-, y(x^-))$ and

$D^+u_i^+(x^+, y(x^+))$ are sufficiently close and their norms are bounded by the constant K. Hence the same is also true for $D^-u^(x^-, y(x^-))$ and $D^+u^+(x^+, y(x^+))$. The proof of the lemma completes by taking the limit. □

6 Equations with Partial Derivatives

Let us now give more details about the formula that specifies the domain of the extension by closure. To this end, we consider the following examples of operators with partial derivatives.

Example 1. Let $X = [0, 1]^2$ and $\mathcal{H}y = H(Dy)$, where

$$H\Big(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}\Big) = \frac{\partial y}{\partial x_1} - \frac{\partial y}{\partial x_2}.$$

It is easily shown that the domain $\mathfrak{D}_{\mathcal{H}}$ of the extension in $\mathbb{S}$ by closure of the operator $(\mathcal{H}, C^1 \uparrow)$ is as follows:

$$\mathfrak{D}_{\mathcal{H}} = \big\{f \in \mathbb{S} : \mathcal{H}f = \big((H(D^-f))_*, (H(D^+f))^*\big) \in \mathbb{S}\big\}.$$

Note that the action of the operator $\mathcal{H}$ is distinct from that of $f \to H(\mathbf{D}f)$ because the latter expression is inapplicable to discontinuous functions.

Example 2. For nonlinear equations, the desired extensions not always coincide with extensions by closure in C of the set of smooth functions. So, the simplest variational problem consisting in finding the distance from a point $x \in I = [-1, 1]$ to the nearest end point of the interval I leads to the Hamilton–Jacobi equation $\mathcal{H}y = |y'| = 1$. The solution $y = -|x| + 1$ is not smooth and is not the limit in $\mathbb{S}$ (or in C) of smooth functions y_i such that $\mathcal{H}y_i \to 1$.

It is easily shown that the domain of the closed extension $\mathcal{H}$, for which the solutions of the equation $\mathcal{H}y = F$ correspond to those of the variational problem under consideration, is the set

$$\mathfrak{D}_{\mathcal{H}} = \big\{f \in \mathbb{S} : \mathcal{H}f = \big(|D^-f|_*, |D^+f|^*\big) \in \mathbb{S}\big\}.$$

Example 3. Consider the so-called singularly perturbed ordinary differential equation

$$-\varepsilon y'' + yy' = f. \tag{6.1}$$

Our aim is to extend in $\mathbb{S}$ the operator $\mathcal{H} : y \mapsto \mathcal{H}(y) = yy'$ so that the solutions in $\mathbb{S}$ of the equation $\mathcal{H}(y) = yy'$ would correspond precisely to the limits as $\varepsilon \to 0$ of smooth solutions of Eq. (6.1). Considering the slow–fast field on the plane corresponding to Eq. (6.1), one can show that the required extension has the domain

$$\mathfrak{D}_{\mathcal{H}} = \Big\{f \in \mathbb{S} : \big((H(y^-, D^-y))_*, (H(y^+, D^+y))^*\big) \in \mathbb{S}\Big\}.$$

Note that as in Example 2 this extension is not always an extension by closure from the set of smooth functions. Solutions in $\mathbb{S}$ may be discontinuous functions and be physical (for example, the solutions $\mathfrak{D}_{\mathcal{H}}$ for the equation $yy' = 1$ describe stationary *shock waves*).

In a general setting, let

$$H\colon X \times \mathbb{R} \times (\mathbb{R}^n)^* \to \mathbb{R}$$

be a continuous function. We consider the operator from $\mathbb{S}(X, \mathbb{R})$ into $\mathbb{S}(X, \mathbb{R})$, which is defined ab initio on the set of smooth functions as the mapping $y \to \mathcal{H}y$, $(\mathcal{H}y)(x) = H(x, y(x), Dy(x))$.

Theorem 6.1

1. *The operator $\mathcal{H}$ admits the closure in the space $\mathbb{S}(X, \mathbb{R})$.*

2. *Let $(\mathcal{H}, \mathfrak{D}_{\mathcal{H}})$ be the extension of the operator $\mathcal{H}$ with domain $\mathfrak{D}_{\mathcal{H}}$ consisting of all $f \in \mathbb{S}(X, \mathbb{R})$ for which*

$$\mathcal{H}f = \left(\left(H(\cdot, f^-(\cdot), D^- f(\cdot))\right)_*, \left(H(\cdot, f^+(\cdot), D^+ f(\cdot))\right)^*\right) \in \mathbb{S}(X, \mathbb{R}).$$

Then the operator $(\mathcal{H}, \mathfrak{D}_{\mathcal{H}})$ is closed in $\mathbb{S}(X, \mathbb{R})$.

Proof It suffices to verify assertion (2), from which assertion (1) follows. Let $f_i \to f \in \mathbb{S}$, $f_i \in \mathfrak{D}_{\mathcal{H}}$, and $\mathcal{H}f_i \to F = (F^-, F^+) \in \mathbb{S}$. Note, first of all, that if $f_i \to f \in \mathbb{S}$, then, for any $\varepsilon > 0$, $\zeta \in D^- f(x)$, there exists an $N \in \mathbb{N}$ such that, for any $n \geq N$, there exist x_n and $\eta_n \in D^- f_n(x_n)$ satisfying the conditions

$$\|x - x_n\| < \varepsilon, \quad |f_n^-(x_n) - f^-(x)| < \varepsilon, \quad \|\zeta - \eta_n\| < \varepsilon.$$

(Analogous properties also hold if "−" is replaced by "+.")

From this remark, the continuity of H, and the quasi-continuity of F^-, F^+ (see property 6 in Proposition 4.4), it follows that there exists a neighborhood $\mathcal{V}$ in which

$$\left(H(\cdot, f^-(\cdot), D^- f(\cdot))\right)_* \geq F^-, \quad \left(H(\cdot, f^+(\cdot), D^+ f(\cdot))\right)^* \leq F^+. \tag{6.2}$$

Since $f \in \mathbb{S}$, from Lemma 5.5, it follows that

$$\left(H(\cdot, f^-(\cdot), D^- f(\cdot))\right)_* \leq \left(H(\cdot, f^+(\cdot), D^+ f(\cdot))\right)^*.$$

Hence form inequalities (6.2), we get the equality

$$\left(\left(H(\cdot, f^-(\cdot), D^- f(\cdot))\right)_*, \left(H(\cdot, f^+(\cdot), D^+ f(\cdot))\right)^*\right) = (F^-, F^+),$$

and therefore, $f \in \mathfrak{D}_{\mathcal{H}}$. □

The next theorem shows that the inclusion of the vanishing viscosity in the physical model described by the equation $\mathcal{H}y = f$, i.e., the addition of the term $-\varepsilon\nabla^2 y$ (where ∇^2 is the *Laplace operator*), gives as $\varepsilon \to 0$ solutions from $\mathfrak{D}_{\mathcal{H}}$. It is also worth pointing out the stability under perturbations of H.

Theorem 6.2

1. *Let* H, $H_\varepsilon\colon X\times\mathbb{R}\times(\mathbb{R}^n)^*\to\mathbb{R}$ *be continuous functions, let* $(\mathcal{H},\mathfrak{D}_{\mathcal{H}})$ *be the operator in* $\mathbb{S}(X,\mathbb{R})$ *defined in Theorem 6.1, and let* $\mathcal{H}_\varepsilon y = H_\varepsilon(\cdot, y(\cdot), Dy(\cdot))$. *Then the operator* $y\to -\varepsilon\nabla^2 y+\mathcal{H}_\varepsilon$ *in* $\mathbb{S}(X,\mathbb{R})$, *which is defined on the class* C^2 *of twice continuously differentiable functions, is preclosed.*

2. *Let* $H_\varepsilon\to H$ *as* $\varepsilon\to 0$ *uniformly on compact sets. Then*

$$\lim_{\varepsilon\to 0}(-\varepsilon\nabla^2+\mathcal{H}_\varepsilon, C^2\uparrow)\subset(\mathcal{H},\mathfrak{D}_{\mathcal{H}}).$$

Proof

1. That the operator is preclosed is proved similarly to the above.

2. Let $\mathfrak{D}_{\mathcal{H}}$ and $\mathcal{H}f=F$. Then both functions f^- and f^+ satisfy the definition of "viscosity solutions" in the sense of Crandall et al. (1992); Crandall (1996) and hence share their properties. This implies assertion (2). □

For the closed extension $(\mathcal{H},\mathfrak{D}_{\mathcal{H}})$, it is interesting to describe the *essential domain* (which is usually more simple), i.e., the set $\mathfrak{D}^1$ such that $(\mathcal{H},\mathfrak{D}_{\mathcal{H}})=(\mathcal{H},\mathfrak{D}^1\uparrow)$. For example, for the operator $\mathbf{D}_\xi$ (from Proposition 5.3) as $\mathfrak{D}^1$, one can take the set of continuously differentiable mappings, and however, this set cannot be used in this way in Example 2 (on p. 227).

Of course, such more simple essential domain $\mathfrak{D}^1$ depends on H. In many problems, in particular, in approximate calculations, it is useful to know the structure of $\mathfrak{D}^1$. To illustrate this, we give one particular result of this kind (see Samborski 2007).

Proposition 6.3 *Let* $H\colon X\times(\mathbb{R}^n)^*\to\mathbb{R}$ *be a uniformly continuous function convex in the second argument and* $\mathfrak{D}^1$ *be the set of functions of the form* $\min(\varphi_1,\dots,\varphi_k)$, *where* $k\in\mathbb{N}$ *and the functions* $\varphi_1,\dots,\varphi_k$ *are continuously differentiable. Then*

$$(\mathcal{H},\mathfrak{D}_{\mathcal{H}})=(\mathcal{H},\mathfrak{D}^1\uparrow).$$

Let us now consider the problem of existence in $\mathfrak{D}_{\mathcal{H}}$ of solutions of the equation $\mathcal{H}y=f$. The next theorem establishes the intermediate values property.

Theorem 6.4 *Let* $a,b\in\mathfrak{D}_{\mathcal{H}}$, $a\le b$ *and* $\mathcal{H}(a)\le\mathcal{H}(b)$. *Then for any* $f\in\mathbb{S}(X,\mathbb{R})$, *satisfying inequality*

$$\mathcal{H}(a)\le f\le\mathcal{H}(b),$$

there exists a $c\in\mathfrak{D}_{\mathcal{H}}$, $a\le c\le b$, *such that* $\mathcal{H}(c)=f$.

We will prove a stronger result (and, what is important, this result is more convenient for applications). Let $a,b\in\mathbb{S}(X,\mathbb{R})$. We set

$$\mathcal{H}_+a=\Big(\big(H(\cdot,a^+(\cdot),D^+a(\cdot))\Big)^*,\quad \mathcal{H}_-b=\Big(\big(H(\cdot,b^-(\cdot),D^-b(\cdot))\Big)_*.$$

Recall that these functions are defined everywhere in X and have values in $\mathbb{R}\bigcup\pm\infty$.

Theorem 6.5 *Let $a, b \in \mathbb{S}(X, \mathbb{R})$, $a \le b$. Next, let $f = (f^-, f^+) \in \mathbb{S}(X, \mathbb{R})$ be such that, for each x,*

$$\mathcal{H}_+(a)(x) \le f^+(x), \quad f^-(x) \le \mathcal{H}_-(b)(x). \tag{6.3}$$

Then there exists a $c \in \mathfrak{D}_{\mathcal{H}}$ such that $\mathcal{H}(c) = f$.

Proof In $\mathbb{S}(X, \mathbb{R})$, consider the set

$$\Phi = \left\{u \in [a, b] \colon \mathcal{H}_-(u)(x) \ge f^-(x) \ \forall x \in X\right\}.$$

consisting of all functions bounded from below by a. This set is nonempty because $b \in \Phi$. Hence, by Theorem 4.2, there exists

$$c = (c^-, c^+) = \inf \Phi \in \mathbb{S}(X, \mathbb{R}).$$

Let us verify the inequalities

$$\mathcal{H}_+(c)(x) \le f^+(x), \quad \mathcal{H}_-(c)(x) \ge f^-(x). \tag{6.4}$$

1. Let $x \in X$ be such that $D^-c(x) \ne \varnothing$ and $\xi \in D^-c(x)$. Then, for any ε, there exist $u \in \Phi$, $x \in X$, $\xi' \in D^-u(x')$ such that

$$\max\{ |x - x'|, |c^-(x) - u^-(x')|, |\xi - \xi'|\} < \varepsilon.$$

The function H is continuous and $H(x', u^-(x'), \xi') \ge f^-(x)$, and hence, making $\varepsilon \to 0$, we arrive at the second inequality in (6.4) for any $x \in X$.

2. Let $x \in X$ be such that $D^+c(x) \ne \varnothing$ and $\xi \in D^+c(x)$. Two cases are possible.

Case (a): $c^+(x) = a^+(x)$. Then, by the conditions of the theorem,

$$H(x, c^+(x), \xi) \le \mathcal{H}_+(a)(x) \le f^+(x).$$

Case (b): $c^+(x) > a^+(x)$. Assume on the contrary that

$$H(x, c^+(x), \xi) > f^+(x).$$

Let $\varphi \in C^1$ be a function such that $\varphi \ge c^+$, $\varphi(x) = c^+(x)$, and $D\varphi(x) = \xi$. For a sufficiently small $\varepsilon > 0$, the quantity $d(x) = \min(c, \varphi(x) - \varepsilon)$ has the following properties:

(i) If $x' \in X$ is such that $d(x') = \varphi(x') - \varepsilon$, then $D^-d(x') = D^-\varphi(x') = D\varphi(x') = \xi'$ and ξ' is close to ξ.

(ii) $H(x', d(x'), D^-d(x')) > f^-(x')$.

(iii) At any point x', at which $D^-d(x') \ne \varnothing$, $D^-d(x')$ coincides either with $D^-c(x')$ or with $D\varphi(x')$.

Hence $d \in \Phi$, and since $d(x) < c(x)$, we get a contradiction with the definition of c as $\inf \Phi$. So, for any $x \in X$, we have the first of (6.4).

Let us now apply Lemma 5.5 to $c = \inf \Phi$. In each neighborhood $\mathcal{U} \subset X$, there exist (by continuity of H) a point x, sequences $x_i^- \to x$, $x_i^+ \to x$, an element $\theta \in \mathbb{R}^n*$, and sequences $\theta_i^- \to \theta$, $\theta_i^+ \to \theta$, satisfying the condition

$$\lim_{i\to\infty} H(x_i^-, c^-(x_i^-), \theta_i^-) - H(x_i^+, c^+(x_i^+), \theta_i^+) = 0,$$

where $\theta_i^- \in D^-c(x_i^-)$ and $\theta_i^+ \in D^+c(x_i^+)$. By inequalities (6.4), we have

$$(\mathcal{H}_-(c), \mathcal{H}_+(c)) = (f^-, f^+) \in \mathbb{S}(X, \mathbb{R})$$

on a dense subset of X. Therefore, $c \in \mathfrak{D}_{\mathcal{H}}$. □

To apply Theorem 6.5 to boundary-value (boundary) problems, one should choose functions a and b so as to satisfy the equalities $a\big|_\Gamma = b\big|_\Gamma = \varphi$ on the boundary Γ (or on a part of it). Now the solution c thus obtained also satisfies the condition $c\big|_\Gamma = \varphi$.

Let $(t, x) \in [0, T] \times X$, $y = y(t, x)$, $D_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Consider the Cauchy problem for the equation

$$\mathcal{F}y = f, \quad \text{where } \mathcal{F}y = \frac{\partial}{\partial t}y + H(\cdot, \cdot, y(\cdot, \cdot), D_x y). \tag{6.5}$$

Let $(\mathcal{F}, \mathfrak{D}_{\mathcal{F}})$ be the closed extension in $\mathbb{S}([0, T] \times X, \mathbb{R})$ of the operator $\mathcal{F}$ described in Theorem 6.1. The sought-for solution, if it exists, lies in the set $\mathfrak{D}_{\mathcal{F}}$ in $\mathbb{S}$. Hence the "initial datum" for y at $t = 0$ can be taken *only* from the set (denoted by $\mathfrak{D}^0_{\mathcal{F}}$), which consists of the pairs (φ^-, φ^+) of two semi-continuous functions obtained by the restriction at $t = 0$ of the pairs (u^-, u^+) of functions from $\mathfrak{D}_{\mathcal{F}}$. Note that it does not follow, a priori, that (φ^-, φ^+) defines an element from $\mathbb{S}(X, \mathbb{R})$.

Theorem 6.6 (Analogue of Peano's Theorem) *Let*

$$H : [0, T] \times X \times \mathbb{R} \times (\mathbb{R}^n)^* \to \mathbb{R}$$

be a continuous function, $\varphi \in \mathfrak{D}^0_{\mathcal{F}}$, $f \in \mathbb{S}([0, T] \times X, \mathbb{R})$. *Then there exist* $T' \in (0, T]$ *and* $y \in \mathfrak{D}_{\mathcal{F}} \subset \mathbb{S}([0, T] \times X, \mathbb{R})$ *such that*

$$\mathcal{F}y = f \quad \text{on } [0, T'] \times X \quad \text{and} \quad y\big|_{t=0} = \varphi.$$

Proof Since $\varphi \in \mathfrak{D}^0_{\mathcal{F}}$, there exists a $u \in \mathfrak{D}_{\mathcal{F}}$ such that $u|_{t=0} = \varphi$. Consider $a(t, \cdot) = u(t, \cdot) - Kt$, $b(t, \cdot) = u(t, \cdot) + Kt$ ($K > 0$). Since u and f lie in $\mathbb{S}([0, T] \times X, \mathbb{R})$, they are bounded. Choosing a sufficiently large K and a sufficiently small T' and using the continuity of H, we find that

$$\mathcal{F}^+(a)(t, x) \le f^+(t, x), \quad f^-(t, x) \le \mathcal{F}^-(b)(t, x)$$

for all $t \in [0, T']$ and $x \in X$. Now an appeal to Theorem 6.5 shows that there exists $y \in \mathfrak{D}_{\mathcal{F}}$ such that $\mathcal{F}y = f$. And since $a(0, \cdot) = b(0, \cdot) = \varphi$, we find that $y|_{t=0} = \varphi$. □

By combining the trick used in the proof of Theorem 6.6 with the intermediate value property, one can obtain existence results for initial-boundary-value problems. For such problems, uniqueness is also possible. However, this question will not be addressed here. Let us illustrate the above on an example of the evolution Hamilton–Jacobi equation.

Theorem 6.7 *Let* $H: [0,T] \times X \times (\mathbb{R}^n)^* \to \mathbb{R}$ *be a continuous function,* $\varphi \in \mathfrak{D}^0_{\mathcal{F}}$, $\psi: [0,T] \times \partial X \to \mathbb{R}$. *Next, let* a, $b \in \mathfrak{D}_{\mathcal{F}}$, $a \le b$, *and*

$$a\big|_{[0,T]\times\partial X} = b\big|_{[0,T]\times\partial X} = \psi, \quad a\big|_{\{0\}\times\partial X} \le \varphi \le b\big|_{\{0\}\times\partial X}.$$

Then, for any $f \in \mathbb{S}([0,T]\times X, \mathbb{R})$ *such that* $\mathcal{F}(a) \le f \le \mathcal{F}(b)$, *there exists a solution* $y \in \mathfrak{D}_{\mathcal{F}}$ *of Eq.* (6.5) *such that*

$$y\big|_{t=0} = \varphi \quad and \quad y\big|_{[0,T]\times\partial X} = \psi.$$

Proof Consider a function $u \in \mathfrak{D}_{\mathcal{F}}$ such that $u\big|_{t=0} = \varphi$. Next, we construct the functions

$$a_1 = \max\{u - Kt, a\}, \quad b_1 = \min\{u + Kt, b\}.$$

After this, for sufficiently large K, we apply Theorem 6.5 to $\mathcal{F}$ and to the interval $[a_1, b_1]$ in $S([0,T] \times X, \mathbb{R})$. □

7 Compactness in the Space $\mathbb{S}$

The results on compactness of the kernel, Noetherian property, etc., play an important role in the theory of linear operators. The condition that an operator A has finite-dimensional kernel can be equivalently expressed as follows: for any $M \in \mathbb{R}$, the set of solutions of the equation $Ay = 0$ satisfying the inequality $\|y\| \le M$ is compact. In this form, this property can also be considered for nonlinear operators.

The following *compactness test in the space* $\mathbb{S}(X, \mathbb{R})$ is an analogue of the classical Arzelà–Ascoli theorem for the Banach space C. Recall that to each $f \in \mathbb{S}(X, \mathbb{R})$ there correspond two sequences of Lipschitz functions $\{f_n^-\}$ and $\{f_n^+\}$ converging to f (see Proposition 4.3), and, therefore, the sequence $\{\mathsf{h}(f_n^-, f_n^+)\}$ converges to zero as $n \to \infty$.

Theorem 7.1 (Compactness Test) *Let* $\mathcal{F}$ *be a family of elements from* $\mathbb{S}(X, \mathbb{R})$. *Then this family is precompact if and only if the following conditions are satisfied*:

(1) *The family* $\mathcal{F}$ *is uniformly bounded, i.e., there exists an* $M > 0$ *such that* $f(x) \in [-M, M]$ *for any* $f \in \mathcal{F}$, $x \in X$.

(2) *The sequences* $\{\mathsf{h}(f_n^-, f_n^+)\}$ *tend to zero as* $n \to \infty$ *equicontinuously with respect to* $f \in \mathcal{F}$, *i.e., for any* ε, *there exists an* $N \in \mathbb{N}$ *such that* $\mathsf{h}(f_n^-, f_n^+) < \varepsilon$ *for any* $f \in \mathcal{F}$ *and any* $n > N$.

To conclude, we give one result (of Noetherian type in the theory of linear equations). This result fully uses the language of "spaces–operators," whose introduction for nonlinear equations was the main purpose of this appendix. As before, let $\mathcal{H}y = H(x, Dy(\cdot))$ and $\mathfrak{D}_{\mathcal{H}}$ be the domain of the closed extension defined in Theorem 6.1.

Theorem 7.2 (see Samborski 2007) *Let $H\colon X \times (\mathbb{R}^n)^* \to \mathbb{R}$ be a uniformly continuous function convex in the second argument (a Hamiltonian). Assume that the following conditions are satisfied*:

(1) *There exists a $C \in \mathbb{R}$ such that, for any $p \in (\mathbb{R}^n)^*$, $(x', x'') \in X \times X$,*

$$|H(x', p) - H(x'', p)| \le C(1 + \|p\|)\|x' - x''\|.$$

(2) *For any $x \in X$, $p \in (\mathbb{R}^n)^*$, $p \neq 0$, the function $\lambda \mapsto H(x, \lambda p)$ from $\mathbb{R}$ into $\mathbb{R}$ is not constant.*

Then, for any closed bounded subset A of $\mathbb{S}(X, \mathbb{R})$, the following assertions hold:

(1) *For any $f \in \mathbb{S}(X, \mathbb{R})$, the set of all solutions from A of the equation $\mathcal{H}y = f$ is compact.*

(2) *The image $\mathcal{H}(A \bigcap \mathfrak{D}_{\mathcal{H}})$ of the set $A \bigcap \mathfrak{D}_{\mathcal{H}}$ is closed in $\mathbb{S}(X, \mathbb{R})$.*

References

Abreu E. et al. (2016). Weak asymptotic methods for scalar equations and systems. J. Math. Anal. Appl. **444**(2), 1203–1232.

Acuna M. et al. (2008). Mars crustal magnetization: a window into the past? The Martian Surface: Composition, Mineralogy and Physical Properties ed. J. F. Bell. Cambridge: Cambridge University Press, 242–262.

Agoshkov V.I. (2020). New Technique for Formulation of Domain Decomposition Algorithms. Comput. Math. and Math. Phys. **60**, 353–369.

Agranovich M.S., Vishik M.I. (1964). Elliptic problems with a parameter and parabolic problems of general type. Russian Math. Surveys **19**(3), 53–157.

Agmon S. et al. (1959). Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. in: Communications on Pure and Applied Mathematics. **12**(4), 623–727.

Albeverio S. et al. (1988). *Solvable Models in Quantum Mechanics*, Springer, New York.

Albeverio S.A. et al. (2005). Non-linear singular problems in p-adic analysis: associative algebras of p-adic distributions. Izv. Math., **69**(1), 221–263.

Alimov A.R., Tsar'kov I.G. (2021). *Geometric Approximation Theory*, Springer Monographs in Mathematics, Springer, Cham.

Anosov D.V. (2018). *Differential equations: we solve and draw*. Moscow, MTsNMO.

Antonevich A.B., Radyno Ya.V. (1991). A general method for constructing algebras of generalized functions. Dokl. Math., **43**(3), 680–684.

Antonevich A.B., Radyno Ya.V. (1994). Antonevich A.B., Radyno Ya.V. (1994). On the problem of distributions multiplication. Problems and Methods in Mathematical Physics. Stuttgart-Leipzig. (Teubner-Texte sur Mathematik; Bd. 134), 9–14.

Antonevich A.B. et al. (2018). Algebra of mnemofunctions on a circle. PFMT, no. 3(36), 55–62.

Apushkinskaya D.E., Nazarov A.I. (2022). The normal derivative lemma and surrounding issues Russian Math. Surveys, 77:2, 189–249.

Arestov V.V. (1996). Approximation of unbounded operators by bounded operators and related extremal problems, Russian Math. Surveys, 51:6, 1093–1126.

Arnold V.I. (1967). Characteristic class entering in quantization conditions. Funct. Anal. Appl., **1**(1), 1–13.

Arnold V.I. (1983). *Geometrical methods in the theory of ordinary differential equations*, Grundlehren der Mathematischen Wissenschaften, vol. 250, Springer-Verlag, New York.

Arnold V.I. (1989). *Mathematical Methods of Classical Mechanics*. Springer

Arnold V.I. (2003). *New Obscurantism and Education in Russia*. Moscow, Fazis.

Arnold V.I. (2006). Forgotten and neglected theories of Poincaré, Russian Math. Surveys **61**:1.

Arnold V.I. (2008). *What is Mathematics?* MCNMO, Moscow.

Arnold V.I. (2016). *Mathematical Understanding of Nature*. Moscow, MTsNMO (in Russian).

A. S. Demidov, *Equations of Mathematical Physics*,
https://doi.org/10.1007/978-3-031-30358-6

Arutyunov A.A., Mishchenko A.S. (2013). Reduction of the calculus of pseudodifferential operators on a noncompact manifold to the calculus on a compact manifold of doubled dimension. Math. Notes **94**, 455–469.

Atkinson F. V. (1951). The normal solubility of linear equations in normed spaces. Matem. Sb. (N.S.) **28**(1), 3–14.

Babin A.V., Vishik M.I. (1992). *Attractors of Evolution Equations*. Elsevier.

Bagrov V.G. et al. (2004). *Methods of Mathematical Physics. Asymptotical Methods*. Tomsk, TPU.

Baratchart L. et al. (2013). Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions. Inverse Problems. **29**, 1–29.

Bell E.T. (1937). *Men of Mathematics* New York: Simon & Schuster.

Berezin F.A., Shubin M.A. (2012). *The Schrödinger Equation*. Springer

Bezrodnykh S.I. (2017). Finding the coefficients in the new representation of the solution of the Riemann–Hilbert problem using the Lauricella function. Math. Notes **101**(5), 759–777.

Bezrodnykh S.I. (2018). The Lauricella hypergeometric function $F_D^{(N)}$, the Riemann–Hilbert problem, and some applications. Russian Math. Surveys, **73**(6), 941–1031.

Bezrodnykh S.I. (2022a). Analytic continuation of Lauricella's function $F(N)D$ for large in modulo variables near hyperplanes $zj = zl$, Integral Transforms and Special Functions, 33:4, 276–291.

Bezrodnykh S.I. (2022b). Analytic continuation of Lauricella's function $F_D^{(N)}$ for variables close to unit near hyperplanes $\{z_j = z_l\}$, Integral Transforms and Special Functions, 33:5, 419–433.

Bezrodnykh S.I., Demidov A.S. (2011). On the uniqueness of solution Cauchy's inverse problem for the equation $\Delta u = au + b$. J. Asymptotic Analysis. *4*, 95–121.

Bezrodnykh S.I., Vlasov V.I. (2016). On a New Representation for the Solution of the Riemann–Hilbert Problem. Math. Notes. **99**(6), 932–937.

Bezrodnykh S.I., Vlasov V.I. (2021). Asymptotics of the Riemann–Hilbert Problem for the Somov Model of Magnetic Reconnection of Long Shock Waves. Math. Notes, **110**:6, 853–871.

Besov O.V. et al. (1978). *Integral Representations of Functions and Embedding Theorems*. John Wiley and Sons, New York.

Besov O.V. (2001). On the compactness of Embeddings of weighted Sobolev spaces on a domain with irregular boundary. Proc. Steklov Inst. Math., 232, 66–87.

Besov O.V. (2013). Kolmogorov widths of Sobolev classes on an irregular domain. Proc. Steklov Inst. Math., 280, 34–45.

Biagioni H.A. (1988). *Introduction to a nonlinear theory of generalized functions*. Brazil: UNICAMP.

Borel E. (1895). Sur quelques points de la théorie des fonctions. Ann. Sci. Ecole Norm. **12**(3), 9–55.

Boto E. et al. (2018). Moving magnetoencephalography towards real-world applications with a wearable system. Nature. **555**(7698), 657–661.

Bounjakowsky W. (1859). Mémoires de l'Académie des sciences de St-Pétersbourg. 7 série, t. **1**, No. 9.

Carleman T. (1926). *Les Fonctions Quasianalitiques*. Paris.

Chepyzhov V.V., Vishik M.I. (2001). *Attractors for Equations of Mathematical Physics*. Colloquim Publications, 49, AMS, Providence.

Christov C., Damianov B. (1979). Asymptotic functions—a new class of generalized functions. Bulgar. J. Phys. **6** (1978), 543–556; **1**, 3–23; **3**, 245–256; **4**, 377–397.

Clarke F.H. (1983). *Optimisation and Nonsmooth Analysis*. NY: Wiley.

Cole J.D. (1951). On a quasi-linear parabolic equation occurring in aerodynamics, Quart. Appl. Math., vol. 9, 225–236.

Colombeau J.-F. (1983). Une multiplication générale des distributions, C. R. Acad. sci. Paris, V. 296. 357–360.

Colombeau J.-F. (1984). *New generalized functions and multiplication of distributions* / Ed. J. F. Colombeau. Amsterdam: North"=Holland (North-Holland Mathematics Studies; V. 84).

Colombeau J.-F. (1985). *Elementary introduction to new generalized functions*. Amsterdam: North Holland.

Courant R. (1992). *Partial Differential Equations*. New York.

Crandall M.G. et al. (1992). User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) **27**, 1–67.

Crandall M.G. (1996). Viscosity solutions: a Primer. Lect. Notes in Math. **1660**, Springer.

Danilov V.G. et al. (1998). Algebras of the singularities of singular solutions to first-order quasi-linear strictly hyperbolic systems. Theoret. and Math. Phys., **114**(1), 1–42.

Deakin M. (1985). Euler's Invention of Integral Transforms. Archive for History of Exact Sciences **33**(4), 307–319. https://www.jstor.org/stable/41133757

Degen C.L. (2008). Scanning magnetic field microscope with a diamond single-spin sensor. Appl. Phys. Lett. **92**.

Delsart J. (1958). Note sur une propriété nouvelle des fonctions harmoniques. Comptes Rendus de l'Académie des Sciences, Série A. **246** (9) 1358–1360.

Demidov A.S. (1969). Unique solvability of boundary value problems for a second-order elliptic equation with certain conjugation conditions on the discontinuity surfaces coefficients. Vestnik MGU. Matem. Melk. no. 3, 30–36.

Demidov A.S. (1970). Asymptotic behavior of the solutions of boundary value problems for a linear second order elliptic equation with coefficients that have a "surge." Tr. Mosk. Mat. Obs., **23**, 77–112.

Demidov A.S. (1973). Elliptic pseudodifferential boundary value problems with a small parameter in the coefficient of the leading operator. Math. USSR-Sb., **20**(3), 439–463.

Demidov A.S. (1975a). Asymptotic behavior of the solution of a boundary value problem for elliptic pseudodifferential equations with a small parameter multiplying the highest operator Tr. Mosk. Mat. Obs., **32**, 119–146.

Demidov A.S.(1975b). The form of a steady plasma subject to the skin effect in a tokamak with non-circular cross-section. Nuclear Fusion. **15**, 765–768.

Demidov A.S. (1977). Sur les problèmes paramètre dans l'opérateur elliptiques pseudo-différentiels, à petit principal. P. 108–122; Sur la perturbation «singulière dans un problème à frontiére libre. P. 123–130. (Lecture Notes in Math. Springer-Verlag, V. 594, Proceedings of the Conference Held at the Ecole Centrale de Lyon, December 8–10. 1976).

Demidov A.S. (1978a). Equilibrium form of a steady plasma. Physics of Fluids. V. 21, 902–904.

Demidov A.S. (1978b). On a problem with free boundary in the theory of a plasma in equilibrium. Trudy Sem. Petrovsk. V. 4, 65-82 (in Russian).

Demidov A.S.(1980). Configurations du plasma stationnaire équilibré. Free boundary problems. Proceedings of a Sem. held in Pavia in 1979. Roma, Vol. I, 467–486.

Demidov A.S. (1994). Solvability and algorithm for the numerical solution of elliptic boundary value problems with nonlinear conjugation conditions, Russ. Math. Surveys **49**(4), 127.

Demidov A.S. (1996). [A complete asymptotics of the solution of the Dirichlet problem for a two-dimensional Laplace equation with rapidly oscillating boundary data. (English. Russian original) Zbl 0887.35023 Dokl. Math. 53, No. 1, 81-83 (1996); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 346, No. 6, 732–734.

Demidov A.S. (2000). Some Applications of the Helmholtz-Kirchhoff Method (Equilibrium Plasma in Tokamaks, Hele-Shaw Flow, and High-frequency Asymptotics). Russ. J. Math. Phys. V. 7, No. 2, 166-186.

Demidov A.S. (2002). On the evolution of a weak perturbation of a circle in the problem of a Hele-Shaw flow. (English. Russian original) Zbl 1062.35081 Russ. Math. Surv. 57, No. 6, 1212-1214 (2002); translation from Usp. Mat. Nauk 57, No. 6, 177-178.

Demidov A.S. (2004). Evolution of the perturbation of a circle in the Stokes-Leibenson problem for a Hele-Shaw flow J. Math. Sci. (N.Y.) 123, no. 5, 4381–4403.

Demidov A.S. et al. (2005). Schwartz alternating method for elliptic boundary-value problems satisfying the maximum principle, Math. Notes **78**, 577–580.

Demidov A.S. (2006). Evolution of the perturbation of a circle in the Stokes-Leibenson problem for the Hele-Shaw flow. Part II. J. Math. Sci. (N.Y.) 139, No. 6, 7064–7078.

Demidov A.S. (2010a). Functional geometric method for solving free boundary problems for harmonic functions. Russian Math. Surveys, 65:1 1–94.

Demidov A.S. (2010b). Inverse problem for the Grad-Shafranov equation with affine right-hand side. Russian J. Math. Phys. V.17, No. 2. 145–153.

Demidov A.S. (2018). Inverse problems in magneto-electroscaning (in encephalographiy, for magnetic microscopes, etc.). J. App. Analysis and Comp. **8**(3), 915–927.

Demidov A.S. (2020). Explicit Representation of the Solution to the Cauchy Problem for the 2nd Order 2D Elliptic Equations. Advances in Mathematics Research / Ed. A.R. Baswell. V. 27. New York: Nova Science Publishers.

Demidov A.S. (2021). On Numerically Implementable Explicit Formulas for the Solutions to the 2D and 3D Equations $\mathrm{div}(\alpha(w)\nabla w) = 0$ and $\mathrm{div}(\beta\nabla w) = 0$ with Cauchy Data on an Analytic Boundary, Funct. Anal. Appl., 55:1, 52–58.

Demidov A.S. (2023). On explicit numerically realizable formulas for the Poincaré–Steklov operators (to appear).

Demidov A.S., Badjadi A. (1983). Existence, nonexistence and regularity theorems in a problem with a free boundary. Math. Sbornik, No. 1, 64-81. 1985 AMS Mathematics of the USSR-Sbornik, Vol. 50, No 1, 50–67.

Demidov A.S. et al. (1996). On inverse and direct free boundary problems in the theory of plasma equilibrium in a Tokamak. C. R. Acad. Sci., Paris, Sér. I 323, No. 4, 353–358.

Demidov A.S. et al. (2012). Boundary Control for the Plane Nonstationary Euler Hydrodynamic Problem with Free Boundary. in ebook *New Developments in Hydrodynamics Research", Nova Science Publishers Serie Physics Research and Technology*, 31–62.

Demidov A.S. et al. (2013). Cauchy problem for Stokes–Leibenson approximation in a Hele–Shaw corner cell. Comptes Rendus Mecanique, Volume 341, Issues 11–12, 755–759.

Demidov A.S. et al. (2015). On inverse problem magneto-encephalography. Quasilinear equations, inverse problems and their applications. Moscow, Conference handbook and proceedings, 22.

Demidov A.S. et al. (2016). Stokes–Leibenson problem for Hele–Shaw flow: a critical set in the space of contours. Russian J. of Math. Ph. 23(1), 35–55.

Demidov A.S., Moussaoui M. (2004). An inverse problem originating from magnetohydro dynamics. Inverse problems Vol. 20, no. 1, 137–154.

Demidov A.S., Samokhin A.S. (2023). Explicit formulas for Poincaré–Steklov operators and their numerical realization (to appear).

Demidov A.S., Platushchikhin D.A. (2010). Explicit formula for the gradient of a harmonic function from its Cauchy data on the analytic curve. Mathematical Notes, 135–137.

Demidov A.S., Savelyev V.V. (2010). Essentially different distributions of current in the inverse problem for the Grad–Shafranov equation. Russian J. of Math. Ph. 17, 56–65.

Demidov A.S., Yatsenko E.S. (1994). Investigation of heat and mass transfer in the evaporation zone of a heat pipe operating by the "inverted meniscus" principle. Int. J. HeatMass Transfer. 37(14), 2155–2163.

Dirac P.A. (1927). The Quantum Theory of the Emission and Absorption of Radiation. Proc. R. Soc. London. Ser. A. **114**.

Dobrokhotov S.Yu. et al. (2013). New formulas for Maslov's canonical operator in a neighborhood of focal points and caustics in two-dimensional semiclassical asymptotics. Theor. Math. Physicsm, **177**(3), 1579–1605.

Dobrokhotov S.Yu. et al. (2014). Fourier Integrals and a New Representation of Maslov's Canonical Operator Near Caustics. Amer. Math. Soc. Transl. **233**, 95–115.

Dobrokhotov S.Yu. et al. (2017). New integral representations of the Maslov canonical operator in singular charts. Izvestiya: Mathematics, **81**(2), 286–328.

Dubinskii Yu.A. (1968). Quasilinear elliptic and parabolic equations of arbitrary order, Russian Math. Surveys, 23:1, 45–91.

Egorov Yu.V. (1990). A contribution to the theory of generalized functions. Russian Math. Surveys, **45**(5), 1–49.

Eskin G.I. (1973). Asymptotics of solutions of elliptic pseudodifferential equations with a small parameter. Dokl. Akad. Nauk SSSR **211**(3), 547–550.

Eskin G.I. (1980). *Boundary Value Problems for Elliptic Pseudodifferential Equations*. Math. Monog.

Faddeev L.D., Yakubovskiĭ O.A. (2009). *Lectures on Quantum Mechanics for Mathematics Students.* (Vol. 47). AMS.

Fedoryuk M.V. (1971). The stationary phase method and pseudodifferential operators. Russian Math. Surveys, **6**(1), 65–115.

Fedoryuk M.V. (1987). *Asymptotic Analysis: Integrals and Series*. Moscow, Nauka.

Florin V.A. (1948). Some of the simplest non-linear problems of the consolidation of a water-saturated earth medium, Izv. AN SSSR, Otd. Tekhn. Nauk, no. 9, 1389–1402.

Foucart S., Rauhut H. (2013). *A Mathematical Introduction to Compressive Sensing*. New York: Springer Science+Business Media.

Friedman A. (1964). *Partial Differential equations of parabolic type*. Englewood-Cliffs.

Fursikov A.V. (1999). *Optimal Control of Distributed Systems. Theory and Applications.* AMS.

Gelfand I.M. et al. (1958–1966). *Generalized Functions.* Vols. 1–6. Moscow, GIFML.

Gilbarg G.D., Trudinger N. (1983). *Elliptic Partial Differential Equations of Second Order.* Springer.

Gindikin S. (2007). *Tales of Mathematicians and Physicists.* Springer Science & Business Media.

Giraud G. (1932). Généralisation des problèmes sur les opérations du type elliptique. Bull. des Sciences Math., **56**, 248–272.

Godunov S.K. (1979). Equations of Mathematical Physics. Nauka, Moscow [in Russian].

Gokhberg I.Ts., Krein M.G. (1957). Fundamental aspects of defect numbers, root numbers and indexes of linear operators. Uspekhi Mat. Nauk, **12**(2), 43–118.

Gordon E.I. et al. (2013). *Infinitesimal Analysis* (Vol. 544). Springer Science & Business Media.

Grinberg G.A. (1948). *Selected Problems in Mathematical Theory of Electric and Magnetic Phenomena*, Izd. AN SSSR, Moscow (in Russian).

Grinevich P.G., Novikov R.G. (2019). Moutard transforms for the conductivity equation, Letters in Mathematical Physics, **109**, 2209–2222.

Hämäläinen M. et al. (1993). Magnetoencephalography—theory, instrumentation, and applications to noninvasive studies of the working human brain. Reviews of Modern Physics. **65**(2), 413–497.

Helmholtz H. (1853). ´Uber einige Gesetze der Vertheilung elektrischer Str´ome in k´orperlichen Leitern mit Anwendung auf die thierisch-elektrischen Versuche. Ann. Phys. Chem. **89**, 211–233, 353–377.

Holmgren E. (1924). Sur les solutions quasianalytiques d'équations de la chaleur. Stockholmm, (Arkiv för Math, Astron. Fysik., Bd. 18.) Häfte 2, no. 9, 64–95.

Hopf E. (1950). The partial differential equation $u_t + uu_x = u_{xx}$. Commun. Pure Appl. Math. 1950, Vol. 3, 201–230.

Hopf E. (1952). A remark on linear elliptic differential equations of second order. Proc. Amer. Math. Soc. **3**, 791–793.

Hörmander L. (1958). On the division of distributions by polynomials, Ark. Mat. **3**, 555–568.

Hörmander L. (1965). Pseudo-differential operators. Comm. Pure Appl. Math. **18**, 501–517.

Hörmander L. (1968). The spectral function of an elliptic operator. Acta mathematica, **121**(1), 193–218.

Hörmander L. (1972). Integral Fourier operators. Matematika (Periodic. Collection of Translations of Foreign Articles) **16**(1), 17–61.

Hörmander L. (1983–1985). *The Analysis of Linear Partial Differential Operators.* Vols. I–IV. Springer.

Ichkitidze L.P. et al. (2014). Magnetic field sensors in medical diagnostics. Med. Tekhn. no. 6, 19–23.

Ivanov V.K. (1972). Hyperdistributions and the multiplication of Schwartz distributions. Dokl. Akad. Nauk SSSR, **204**(5), 1045–1048.

Ivanov V.K. (1979). On the algebra of elementary generalized functions. Dokl. Akad. Nauk SSSR, **246**(4), 805–806.

Ivanov V.K. (1981). Asymptotic approximation to a product of generalized functions. Soviet Math. (Izvestiya VUZ. Matem.) no. 1, 19–26.

Kabanikhin S.I. (2008). *Inverse and Ill-Posed Problems.* Sibirskoe Nauchnoye Izdatel'stvo, Novosibirsk.

Kabanikhin S.I. (2011). *Inverse and Ill-Posed Problems. Theory and Applications.* De Gruyter, Germany.

Kantorovich L.V., Krylov V.I. (1962). *Approximate Methods of Higher Analysis*, Noordhoff, 1958.

Kanwal R.P. (1998). *Generalized Functions Theory and Technique.* Birkhauser.

Karasev M.V., Maslov V.P. (1984). Asymptotic and geometric quantization. Russian Math. Surveys **39**(6), 133–205.

Kellogg O.D. (1931). On the derivatives of harmonic functions on the boundary. Trans. Amer. Math. Soc. 33, 486-510.

Kiselman C.O. (2007). *Enkonduko al distribucioj. Acta Sanmarinensia*, Vol. VI, No. 4, 105–132. Academiae Internationalis Scientiarum (AIS). Göttingen: Leins Verlag.

Kiselman C.O. (2019). Language choice in scientific writing: The case of mathematics at Uppsala University and a Nordic journal. Normat. Nordisk matematisk tidskrift 61, No. 2–4, 111–132.

Khinchin A.I. (1998). *Three pearls of number theory.* Dover Books on Mathematics, Dover Publications, Inc., Mineola, NY.

Khoromskij B.N., Wittum G. (2004). *Elliptic Poincaré–Steklov Operators*. Lecture Notes in Computational Science and Engineering, vol. 36. Springer, Berlin, Heidelberg.

Knuth D.E. (1997). *The Art of Computer Programming. Fundamental Algorithms.* Addison Wesley.

Kohn J.J., Nirenberg L. (1965). An algebra of pseudo-differential operators. Communications on Pure and Applied Mathematics, 18(1-2), 269–305.

Kolmogorov A.N., Fomin S.V. (1980). *Elementi di teoria delle funzioni e di analisi funzionale.* Editori Riuniti University Press.

Konyagin S.V. (1988). Limits of indeterminacy of trigonometric series, Math. Notes, 44:6, 910–920.

Konyagin S.V. (2022). ed. Proceedings of MIAN, 318.

Kozlov V.V. (1998). *Integral Invariants after Poincaré and Cartan.* URSS, Moscow [in Russian].

Krishnan V.P., Quinto E.T. (2015). Microlocal analysis in tomography. *Ed. O. Scherzer. Handbook of mathematical methods in imaging*. New York: Springer, Ch. 18., 847–902.

Krylov N.M., Bogolyubov N.N. (1929). Approximate solution of the Dirichlet problem, Dokl. Akad. Nauk SSSR, No. 12, 283–289 (in Russian).

Ladyzhenskaya O.A., Ural'tseva N.N. (1968). *Linear and Quasilinear Elliptic Equations*, Mathematics in Science and Engineering, vol. 46, New York and London: Academic Press.

Ladyzhenskaya O.A. et al. (1988). *Linear and Quasi-linear Equations of Parabolic Type.* American Mathematical Soc.

Lavrent'ev M.A., Shabat B.V. (1977). *Méthodes de la Théorie des Fonctions d'une Variable Complexe.* Moscow: Mir.

Lebedev V.I., Agoshkov V.I. (1983). Poincaré–Steklov Operators and Their Applications in Analysis, Otd. Vychisl. Mat. Akad. Nauk SSSR, Moscow (in Russian).

Lebesgue H.L. (1904). *Leçons sur l'intégration et la recherche des fonctions primitives.* Paris, Gauthier–Villars.

Leray J. (1972–73). *Compléments à la théorie d'Arnold de l'indice de Maslov.* Séminaire Jean Leray E.D.P. Collège de France.

Leweke S. et al. (2022). Vector-valued spline method for the spherical multiple-shell electromagnetoencephalography problem. Inverse Problems, Vol. 38, No 8.

Lidskii V.B. (1955). Oscillation theorems for canonical systems of differential equations. Dokl. Akad. Nauk SSSR, **102**(5), 877–880.

Lions J.-L. (1969). *Quelques méthodes de résolution des problémes aux limites non lineaires*. Paris, Dunod.

Lions J.L., Magenes, E. (1968). *Problèmes aux Limites non Homogènes et Applications.* Dunod, Paris

Lions J.-L., Magenes, E. (1972-1973). *Non-homogeneous boundary value problems and applications.* V. 1–3. Springer.

Livchak Ya.B. (1969). To the theory of generalized functions. Proc. Riga Algebra Seminar. Riga. Latv. Gos. Univ., 98–164.

Łojasiewicz S. (1959). Sur le problèm de division. Studia Math. 18, 87–136.

Lopatinskii Ya.B. (1953). On one method for the reduction of boundary-value problems for systems of differential equations of elliptic type to regular integral equations. Ukr. Mat. Zh., 5(2), 123–151.

Lützen J. (1982). The prehistory of the theory of distributions. Studies in the History of Mathematics and Physical Sciences, 7. Berlin; New York, NY: Springer-Verlag. viii+232.

Luzin N.N. (1935). Function In: Great Soviet Encyclopedy, 1st ed., vol. 59, 314–334; see also In: Collected Works, vol. 3, AN SSSR, Moscow, 1959.

Mackey G.W. (1963). *The Mathematical Foundations of Quantum Mechanics.* New York.

Maergoiz I.D. (1971). On the numerical solution of boundary value problems in potential theory by the method of integral equations, Siber. Math. J. **12**, 951-958 (in Russian).

Magnetic field (2002). Magnetic field gradient measurement on magnetic cards using magnetic force microscopy. AIP Conf. Proc. 2002. V. 615. P. 999.

Malykhin Yu.V. (2016). Relative widths of Sobolev classes in the uniform and integral metrics, Proc. Steklov Inst. Math., 293, 209–215.

Malyshev V.A., Malyshev S.V. (2022). Toy Model of Shock Flow, Markov Processes Relat. Fields 28, 477–488.

Martin Y. (1987). Magnetic imaging by “force microscopy” with 1000 Åresolution. Appl. Phys. Lett. **50**, 1455–1547.

Maslov V.P. (1965). *Perturbation Theory and Asymptotic Methods.* Izd. Moskov. Univ., Moscow [in Russian].

Maslov V.P. (1980). Three algebras corresponding to nonsmooth solutions of systems of quasilinear hyperbolic equations. Uspekhi Mat. Nauk **35**(2), 252–253.

Maslov V.P. (2006). *Quantization of Thermodynamics. Lectures of the Demidov Prize laureates (1993–2004)* Izd-vo Ural Univ., Ekaterinburg.

Maslov V.P., Fedoryuk M.V. (1976). *Semiclassical Approximation for the Equations of Quantum Mechanics.* Nauka, Moscow [in Russian].

Maz’ya V.G. (1988). Boundary integral equations. In:, Analysis 4, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 27, VINITI, Moscow, 1988, 131–228.

Mazya V., Shaposhnikova T. (1999). *Jacques Hadamard, A Universal Mathematician (History of Mathematics)*

Mikhailov V.P. (1978). *Partial Differential Equations.* Mir Publishers, Moscow.

Mikhlin S.G. (1965). *Multidimensional Singular Integrals and Integral Equations.* International Series of Monographs in Pure and Applied Mathematics. Vol. 83. Oxford. Pergamon Press.

Milnor J. (1963). *Morse Theory.* Princeton University Press, Princeton.

Morse M. (1930). A generalization of the Sturm theorems in n-space. Math. Ann. **103**, 52–69.

Morse M. (1934). *The Calculus of Variations in the Large.* New York AMS Coll. Publ.; V. 18), 80–106.

Narasimhan R., ed. (1990). *Riemanns Gesammelte Werke.* Teubner/Springer.

Nazaikinskii V.E. (2014). Generalizations of the Canonical Maslov Operator and Their Applications in Mathematical Physics Doctoral Thesis. Moscow.

Note (1943). Note in the newspaper “Stalinist Warrior” No. 146 of October 31, 1943, reprinted many years later by the Fiztech newspaper “Za nauku.” See also http://mipt-museum.ru/

Novikoff P. (1938). Sur le problème inverse du potentiel Dokl. Akad. Nauk SSSR 18, 165–168.

Novikov S.P. (1970). Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. applications to differential topology and the theory of characteristic classes. I II. Math. USSR-Izv., **4**(2), 257–292; Math. USSR-Izv., **4**(3), 479–505.

Novikov R.G., Taimanov I.A. (2018). Darboux–Moutard transformations and Poincaré–Steklov operators, Proc. Steklov Inst. Math. **302**, 315–324.

Oberguggenberger M. (1992). *Multiplication of distributions and application to partial differential equations.* M. Oberguggenberger. Harlow: Longman Scientific and Technical, (Pitman Research Notes in Mathematics series; V. 259).

Oberguggenberger M. et al. (2003). Local properties of Colombeau generalized functions. Math. Nachr. **256**, 88–99.

Oleinik O.A. (1952). On properties of solutions of certain boundary problems for equations of elliptic type. Mat. Sb. (N.S.), 30(72):3, 695–702.

Oleinik O.A. (2007). *Lectures on partial differential equations.* Binom, Moscow [in Russian].

Oxtoby J.C. (1980). *Measure and Category.* Springer, New York.

Palamodov V.P. (1991). *Generalized functions and harmonic analysis.* in "Commutative Harmonic Analysis 3," Probl. Mat. Fund. Napr., 72, VINITI, Moscow, 5–134.

Peetre J. (1960). Rectification à l'article "Une caractérisation abstraite des opérateurs différentiels". Math. Scand. **8**, 116–120.

Peetre J. (1962). Elliptic Partial Differential Equations of Higher Order. Univ. of Maryland, Inst. Fluid Dynamics.

Petrovsky I.G. (1967). *Partial Differential Equations* Philadelphia, Saunders.

Pokhozhaev S.I. (1983). On the solvability of quasilinear elliptic equations of arbitrary order, Sb. Math., 45:2, 257–271.

Poincaré H. (1896). La méthode de Neumann et le problème de Dirichlet, Acta Math., Vol. XX, 59–142.

Polyanin A.D., Zaitsev V.F. (2012). *Handbook of Nonlinear Partial Differential Equations*, 2nd ed., CRR Press, Taylor & Francis Group.

Prasolov V.V., Tsfasman M.A. (2004). The Globe. General Mathematical Seminar. 1. Moscow, MtsNMO [in Russian].

Pushkar' P.E. (1998). Maslov index and symplectic Sturm theorems. Funct. Anal. Appl. **32**, 172–182.

Quinto E.T. (2017). Artifacts and Visible Singularities in Limited Data X-Ray Tomography. Sensing and Imaging **18**(9).

Reed M., Simon B. (1972). *Methods of Modern Mathematical Physics.* Vols. I–IV. Academic Press.

Robertson W. (1980). *Topological Vector Spaces.* CUP Archive.

Romanov V.G., Kabanikhin S.I. (1991). *Inverse Problems of Geoelectrics*, Nauka, Moscow.

Romanov V.G., Kabanikhin S.I. (1994). *Inverse Problems for Maxwell's Equations,* Utrecht, The Netherlands, VSP.

Rosinger, E. (2006). *Distributions and Nonlinear Partial Differential Equations.* Springer.

Rozhdestvenski B.L., Yanenko N.N. (1983). Systems of Quasilinear Equations and their Applications to Gas Dynamics. Providence, R.L., AMS.

Rybakov A. (2014). From the teacher's notebook: Kvant no. 5–6, 18–23.

Samborski S.N. (2004). On metric completeness and order completeness. Proc. Steklov Inst. Math., 247, 209–216.

Samborski S.N. (2007). Extensions of first-order partial differential expressions and viscosity solutions of Hamilton–Jacoby equations. Diff. Integral Equations. **20**(1) (2007), 63–76.

Schwartz L. (1950–1951). *Theorie des distributions.* I, II. Paris.

Schwarz H.A. (1869). Uber einige Abbildungsaufgaben, Ges. Math., Abh. II., 65–83.

Sedov L.I. (1959). *Similarity and Dimensional Methods in Mechanics,* New York, Academic Press.

Shao Zh., Huang, M. (2017). Interactions of delta shock waves for the Aw-Rascle traffic model with split delta functions. J. Appl. Anal. Comput. **7**(1) 119–133.

Schapira P. (1970). *Théorie des Hyperfonctions.* Springer, Berlin.

Shapiro Z.Ya. (1951). The first boundary problem for an elliptic system of differential equations. Mat. Sb. (N.S.), **28**(70):1 55–78.

Shelkovich V.M. (2008). δ- and δ'-shock wave types of singular solutions of systems of conservation laws and transport and concentration processes Russian Math. Surveys, **63**(3) 473–546.

Sheltraw D., Coutsias E. (2003). Invertibility of current density from near-field electromagnetic data. J. of Applied Physics. **94**(8), 5307–5315.

Shestakova A.N. et al. (2012). Magnetoencephalography—the latest method of functional mapping of the human brain. Eksper. Psikh. **5**(2) 119–134.

Shilov G.E. (1965). *Mathematical Analysis (Second Special Course).* Nauka, Moscow, [in Russian].

Shilov G.Ye. (2016). *Mathematical Analysis: A Special Course.* Elsevier.

Shilov G.E., Gurevich B.L. (2013). Integral, *Measure and Derivative.* A Unified Approach. Courier Corporation.

Shubin M.A. (1987). *Pseudodifferential Operators and Spectral Theory.* Springer, Berlin.
Sobolev S.L. (1935). The Cauchy problem in the space of functionals. Dokl. AN SSSR. **3**(8) 291–294.
Sobolev S.L. (1936). Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales. Matem. Sb. **43**(1), 39–71.
Sobolev S.L. (2008). *Some Applications of Functional Analysis in Mathematical Physics* (Vol. 90). AMS.
Soboleva T.S., Chechkin G.A. (2017). *Sergey L'vovich Sobolev. Facets of Talent* (Great Mathematicians of the Twentieth Century). Moscow, KURS [in Russian].
Sommerfeld A. (1954). *Optics.* Academic Press, New York, 1954.
Stein E.M. (1957). Note on singular integrals. Proc. Amer. Math. Soc. **8**, 250–254.
Stekloff W. (1900). Les méthodes générales pour résoudre les problèmes fondamentaux de la physique mathématique. Ann. fac. sci. Toulouse. Sér. 2. **2**, 207–272.
Stekloff W. (1983). *The Main Problems of Mathematical Physics*. 2nd ed. Ed. V. S. Vladimirov. Moscow, Nauka (in Russian).
Stone M.H. (1932). On one-parameter unitary groups in Hilbert space. Ann. Math. **33**(3), 643–648.
Stroganova T.A. et al. (2011). *EEG alpha activity in the human brain during perception of an illusory kanizsa square.* Neuroscience and Behavioral Physiology **41**(2), 130–139.
Takhtajan L.A. (2008). *QuantumMechanics for Mathematicians.* Grad. Stud. in Math., Vol. 95 (AMS, Providence, Rhode Island; RKhD, Moscow–Izhevsk, 2011).
Taylor M. (1974). *Pseudo Differential Operators.* Springer.
Temam R. (1979). *Navier–Stokes Equations. Theory and Numerical Analysis.* 2nd Edition.
Tikhomirov V.M. (1990). *Approximation Theory.* Analysis II. Convex Analysis and Approximation Theory, Encyclopaedia Math. Sci., vol. 14, Springer-Verlag, Berlin 1990, 93–243.
Tikhomirov V.M., ed. (1991). *Selected Works of A.N. Kolmogorov: Vol. I: Mathematics and Mechanics* (Vol. 25). Springer Science & Business Media.
Tréves F. (1980). *Introduction to Pseudodifferential and Fourier Integral Operators.* Plenum, New York.
Tumanov I.M. (1975). *Henri Léon Lebesgue.* Nauka, Moscow [in Russian].
Tychonoff A. (1935). Théorèmes d'unicité pour l'équation de la chaleur. Matem. Sb. **42**(2), 199–216.
Vainberg B.R. (1975). On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \to \infty$ of solutions of non-stationary problems Russian Math. Surveys, **30**(2), 1–58.
Vainberg B.R. (1977). Quasiclassical approximation in stationary scattering problems. Funct. Anal. Appl., **11**(4) (1977), 247–257.
Vainberg B.R. (1982). *Asymptotic Methods in Equations of Mathematical Physics.* Izd. Moskov. Univ., Moscow (1982) [in Russian].
Vassiliev V.A. (1981). Characteristic classes of Lagrangian and Legendre manifolds dual to singularities of caustics and wave fronts. Funct. Anal. Appl. **15**(3), 164–173.
Vassiliev V.A. (2014). *Topology for Undergraduates.* Moscow, MTsNMO [in Russian].
Vasil'eva A.A. (2020). Order estimates for the Kolmogorov widths of weighted Sobolev classes with restrictions on derivatives, Eurasian Math. J., 11:4, 95–100.
Vekua I.N. (1962). *Generalized Analytic Functions.* Pergamon Press.
Vishik M.I. (1961). Boundary-value problems for quasilinear strongly elliptic systems of equations having divergence form. Dokl. Akad. Nauk SSSR, **138**:3, 518–521.
Vishik M.I. (1963). Quasi-linear strongly elliptic systems of differential equations in divergence form. Trans. Moscow Math. Soc. **12**, 125–184.
Vishik M.I., Eskin G.I. (1964). Boundary-value problems for general singular equations in a bounded domain. Dokl. Akad. Nauk SSSR. **155**(1), 24–27.
Vishik M.I., Eskin G.I. (1965). Equations in convolutions in a bounded region. Russian Math. Surveys. **20**(3), 85–151.
Vishik M.I., Fursikov A.V. (1988). *Mathematical Problems of Statistical Hydromechanics.*

Vishik, M.I., Sobolev, S.L. (1956). General formulation of some boundary value problems for elliptic partial differential equations. Dokl. Akad. Nauk SSSR **111**(3), (1956), 521–523.

Vladimirov, V.S. (1971). *Equations of Mathematical Physics.* Marcel Dekker, New York.

Vladimirov V.S. (1988). Generalized functions over the field of p-adic numbers. Russian Math. Surveys, **43**(5), 19–64.

Vladimirov V.S. (1994). *Generalized Functions in Mathematical Physics.* URSS [in Russian]

Vlasov V.K., Bakushinskii A.B. (1963). The method of potentials and the numerical solution of the Dirichlet problem for the Laplace equations. Zh. Vychisl. Matem. i Matem. Fiz., **3**(3), 574–580 (in Russian).

Volevich L.R., Paneah B.P. (1965). Certain spaces of generalized functions and embedding theorems. Russian Math. Surveys, **20**(1), 1–73.

Volovich I.V., Kozyrev S.V. (2009). p-adic mathematical physics: fundamental constructions, applications to complex and nanoscopic systems. Mathematical Physics and its Applications. Introductionary Courses, 1. Samara, Samara State Univ. [in Russian].

Watson G.N. (1995). A *Treatise on the Theory of Bessel Functions.* Cambridge University Press.

Weiss B.P. et al. (2007). Paleomagnetic analysis using SQUID microscopy. J. Geophys Res. V. 112.

Weyl H. (1939). *The Classical Groups. Their Invariants and Representations.* Princeton University Press, Princeton, N.J.

Whittaker E.T., Watson G.N. (2020). *A Course of Modern Analysis.* Courier Dover Publications.

Wiener N., Hopf E. (1931). Uber eine Klasse singularer Integralgleichungen. Sitz. Akad. Wiss. Phys.-Math. Berlin: Kl. Verlag der Akademie der Wissenschaften, 696–706.

Yosida K. (1965). *Functional Analysis.* Springer, Berlin.

Yushkevich A.P. (1968). *History of Mathematics in Russia until 1917.* Moscow, Nauka [in Russian].

Zavalishin S.T., Sesekin A.N. (1991). *Pulse Processes: Models and Application.* Moscow, Nauka.

Zelikin M.I. (2015). Fractal Theory of Saturn's Ring. Proc. Steklov Inst. Math., 291, 87–101.

Zelikin M.I. (2019). Fractal Theory of Saturn's Ring. II: Electromagnetic Phenomena. Proc. Steklov Inst. Math., 304, 153–159.

Zhou Y., Cai M.A. (2016). Numerical Solution of the Robin Problem of Laplace Equations with a Feynman–Kac Formula and Reflecting Brownian Motions. J. Sci. Comput., **69**(1), 107–121.

Zorich V.A. (2016). *Mathematical analysis, I, II.* Springer.

Zvonkin A.K., Shubin, M.A. (1984). Non-standard analysis and singular perturbations of ordinary differential equations. Russian Math. Surveys, **39**(2), 69–131.

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