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Franco Flandoli
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Stochastic Partial Differential Equations in Fluid Mechanics

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Preface

These notes originated from a series of lectures given at Waseda University in April–May 2021, supported by Top Global University Project of Waseda University. The first author expresses warm gratitude to the organizers for this opportunity, in particular to Prof. Tadahisa Funaki and Prof. Yoshihiro Shibata. The lectures and the subsequent refinements by both authors have been occasions to review classical ideas and techniques (mostly Chaps. 1 and 2) and present a new direction that has emerged in the last few years (Chaps. 3, 4, and 5).

In spite of the existence of many texts and references devoted to stochastic fluid mechanics (among others, see [3, 46, 94, 112, 113, 187, 192, 193, 256]), we have been motivated to write an additional one by a recent idea, which emerged from the paper by Lucio Galeati [147], that small-scale noise suitably introduced into the equations in transport form may lead to an enhanced dissipation. This mathematical result calls the old idea, often traced back to Joseph Valentin Boussinesq in 1877 [43] (see a discussion in [239]), sometimes called the *turbulent viscosity hypothesis*, namely that small-scale turbulence may produce an additional viscosity term in the equations.

The analogy with the result of [147] motivates us to ask ourselves several side questions, including about the origin and form of the noise in fluid mechanics. Where does the noise come from and which forms does it take in the equations? Noise has been introduced in fluid dynamic equations for a long time, starting from the foundational book of Landau and Lifshitz [197], to the literature on numerical computation of turbulence (e.g. [254]), to investigations of theoretical physics (e.g. [195]); see also P.L. Chow [71], Vishik and Fursikov [256] and very many other works, some of them quoted in other parts of this book. Generically, it is meant to describe fluctuations, random external perturbations, and the proposal to include it into the equations is generally accepted also thanks to the fact that real observations of turbulent fluids show some degree of randomness. But a precise description of its mechanical origin is missing.

In Chap. 5 of these notes, we describe a heuristic path from small-scale perturbations and turbulence to additional viscosity, through the intermediate step of transport noise. The chapter is full of open questions. Part of the mathematics that can be used to formalize such a heuristic path is given in Chaps. 1–4. Chapter 1 is limited to additive noise treated in a fully deterministic way, motivated for instance by perturbations arising near boundaries due to irregularity of the boundary profile, as described in Sect. 5.5; for a deterministic audience, this chapter may be a useful introduction to the subject. In Chap. 2, we move to truly stochastic analysis and treat stochastic Navier–Stokes equations with state-dependent noise, a motivation for velocity-dependent noise being described at the end of Sect. 5.5.

Chapters 3 and 4 are devoted to transport noise. Its introduction into fluid equations may be motivated in different ways, quoted in Chap. 3, among which we stress the geometric approach of Darryl Holm [177]. Our viewpoint is that it describes the action of small space scales on large ones; see Chap. 5, Sect. 5.3. Additive noise at small scales lifts to transport noise acting on large scales, as described in Sect. 5.1.1. Accepted transport noise as an interesting random input, Chaps. 3 and 4 develop the mathematical theory of well-posedness and the link with deterministic equations with enhanced diffusion, precisely with enhanced dissipation in the case of heat-type equations (Chap. 3) and enhanced viscosity in the case of the Navier–Stokes equations (Chap. 4).

More than a conclusive book on a fully developed theory, this is for us a starting point for a better understanding of stochasticity related to turbulence and its effects following the ideas of the turbulent viscosity hypothesis. This book will be successful if in a few years a more advanced one will be written with a more mature view of these topics.

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Chapter 1
The Navier–Stokes Equations with Deterministic Rough Force

1.1 The Deterministic Navier–Stokes Equations

1.1.1 The Newtonian Equations

These notes are based on the following mathematical model, called the incompressible Navier–Stokes equations (see [250] for more details on the physics of fluids, just sketched here). We assume that D is a regular bounded connected open domain, but for the purpose of this introductory subsection it can be more general. In D we have a fluid described by means of its velocity $u = u(t, x)$ (a vector field) and pressure $p = p(t, x)$ (a scalar field). The equations are

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f \tag{1.1}$$
$$\operatorname{div} u = 0$$

supplemented by boundary and initial conditions

$$u|_{\partial D} = 0$$
$$u|_{t=0} = u_0.$$

The density field is assumed to be constant and, up to a normalization, equal to 1, hence it does not explicitly appear in the equations. Constant density is the consequence of two assumptions: incompressibility, imposed by the equation $\operatorname{div} u = 0$, and the assumption that the density is constant at time zero, hence it remains constant. The fluid is assumed to be viscous, namely we assume

$$\nu > 0$$

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and this fact has, as a consequence, the no-slip boundary condition $u|_{\partial D} = 0$, because viscous fluids must be at rest on solid boundaries. The function f is a body force, like gravitation. The differential equation in (1.1) is a system, u being a vector field. The meaning of such an equation is the second Newton law: consider a very small portion of fluid, identified by a point $x(t)$, which moves in time. Recall that we assume a mass density equal to one. The acceleration $x''(t)$ is equal to the sum of the forces. But the velocity $x'(t)$ is equal to $u(t, x(t))$, by definition of u. Hence

$$\frac{d}{dt} u(t, x(t)) = \text{forces.}$$

This reads

$$\partial_t u + u \cdot \nabla u = \text{forces}$$

along the trajectory $x(t)$, which is the first system of differential equations in (1.1). The forces are due to pressure, viscosity and the external inputs.

We stress that the no-slip condition $u|_{\partial D} = 0$ provokes large stress near the boundary, if u is large nearby and this stress, when the viscosity is small enough, may lead to instabilities and generate vortices. This is the so–called phenomenon of the emergence of a boundary layer: close to the boundary the fluid presents a turbulent behavior for $\nu \to 0$. The thickness of the boundary layer and some control on the behavior of the fluid in this region are very challenging and mostly open questions, see [16] for a review on the topic.

Basic is the energy balance. Assuming enough regularity to perform computations, the time derivative of the global kinetic energy is given by

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_D |u(t, x)|^2 dx &= \int_D u(t, x) \cdot \partial_t u(t, x)\, dx \\ &= - \int_D u \cdot (u \cdot \nabla u)\, dx - \int_D u \cdot \nabla p dx \\ &\quad + \nu \int_D u \cdot \Delta u dx + \int_D u \cdot f dx. \end{aligned}$$

Now

$$\int_D u \cdot (u \cdot \nabla u)\, dx = \frac{1}{2} \int_D u \cdot \nabla |u|^2 dx = -\frac{1}{2} \int_D \operatorname{div} u \cdot |u|^2 dx = 0$$

(we have used also $u|_{\partial D} = 0$); similarly,

$$\int_D u \cdot \nabla p dx = - \int_D p \operatorname{div} u dx = 0$$

and

$$\int_D u \cdot \Delta u dx = -\int_D |\nabla u|^2 dx.$$

Therefore we get

$$\frac{d}{dt}\frac{1}{2}\int_D |u(t,x)|^2 dx + \nu \int_D |\nabla u|^2 dx = \int_D u \cdot f dx.$$

The interpretation is that the variation of kinetic energy is given by the dissipation into heat plus the work done by the external forces. This equation is not only very informative from the physical viewpoint but represents one of the main tools in the mathematical investigation (in dimension 3, when dealing with weak solutions, it must be replaced by an inequality).

1.1.2 A Rigorous Deterministic Theorem in d = 2

Let us recall a rigorous result about Eq. (1.1). More details on functional analytic aspects traced here can be found for instance in [146, 148, 199, 200, 205, 209, 247, 248].

Assume D is a regular bounded connected open domain. Denote by $H^k\left(D, \mathbb{R}^2\right)$, $k = 1, 2, \ldots$, the classical Sobolev spaces or vector fields and by $H_0^k\left(D, \mathbb{R}^2\right)$ the subspace of those which are zero at the boundary. Denote by H (resp. V, $D(A)$) the closure in $L^2\left(D; \mathbb{R}^2\right)$ (resp. $H^1\left(D, \mathbb{R}^2\right)$, $H^2\left(D, \mathbb{R}^2\right)$) of smooth compact support fields $v \in C_c^\infty\left(D; \mathbb{R}^2\right)$ such that $\operatorname{div} v = 0$.

It turns out that H is the space of $L^2\left(D; \mathbb{R}^2\right)$-vector fields v, divergence free, such that $v \cdot n|_{\partial D} = 0$ where n is the normal to ∂D (one can prove that $v \cdot n|_{\partial D}$ is well–defined, for divergence free L^2 vector fields). Denote by P the projection of $L^2\left(D; \mathbb{R}^2\right)$ on H. Moreover, V (resp. $D(A)$) is the space of all $v \in H_0^1\left(D, \mathbb{R}^2\right)$ (resp. $v \in H^2\left(D, \mathbb{R}^2\right) \cap H_0^1\left(D, \mathbb{R}^2\right)$) such that $\operatorname{div} v = 0$.

Define the unbounded linear operator $A : D(A) \subset H \to H$ by the identity

$$\langle Av, w \rangle = \nu \langle \Delta v, w \rangle$$

for all $v \in D(A)$ and $w \in H$, or as

$$Av = \nu P \Delta v.$$

Denote by $\mathbb{L}^4$ the space $L^4\left(D,\mathbb{R}^2\right)\cap H$, with the usual topology of $L^4\left(D,\mathbb{R}^2\right)$. Define the trilinear form $b:\mathbb{L}^4\times V\times\mathbb{L}^4\to\mathbb{R}$ as

$$b(u,v,w)=\sum_{i,j=1}^{2}\int_D u_i(x)\,\partial_i v_j(x)\,w_j(x)\,dx=\int_D(u\cdot\nabla v)\cdot w\,dx$$

(it is well–defined and continuous on $\mathbb{L}^4\times V\times\mathbb{L}^4$ by the Hölder inequality). Notice that

$$V\subset\mathbb{L}^4$$

by the Sobolev embedding theorem, hence b is also defined and continuous on $V\times V\times V$. Moreover, the following interpolation inequality (sometimes known as Ladyzhenskaya's inequality [196]) holds true: for some constant $C>0$

$$\|f\|_{L^4(D)}^2\le C\|f\|_{L^2(D)}\|f\|_{H^1(D)} \tag{1.2}$$

for all $f\in H^1(D)$. It follows that

$$\int_0^T\|u(t)\|_{\mathbb{L}^4}^4dt\le C\sup_{t\in[0,T]}\|u(t)\|_H^2\int_0^T\|u(t)\|_V^2dt. \tag{1.3}$$

This implies in particular that the integral

$$\int_0^t b(u(s),\phi,u(s))\,ds$$

in the definition below is well-defined, under the regularity of u and ϕ specified there. Note that

$$b(u,v,w)=-b(u,w,v)$$

if $u\in\mathbb{L}^4$, $v,w\in V$.

Sometimes we shall also use the operator

$$B:\mathbb{L}^4\times\mathbb{L}^4\to V'$$

defined by the identity

$$\langle B(u,v),\phi\rangle=-b(u,\phi,v)=-\int_D(u\cdot\nabla\phi)\cdot v\,dx$$

for all $\phi \in V$. Thanks to the embedding $D((-A)^{1+\epsilon}) \hookrightarrow W^{1,\infty}(D, \mathbb{R}^2)$, $\epsilon > 0$, the operator B can be extended to a continuous bilinear operator between $H \times H$ and $D((-A)^{-1-\epsilon})$. When $v \in V$, we may also write

$$\langle B(u, v), \phi \rangle = b(u, v, \phi).$$

Moreover, when $u \cdot \nabla v \in L^2\left(D; \mathbb{R}^2\right)$, it is explicitly given by

$$B(u, v) = P(u \cdot \nabla v).$$

This representation extends to several classes of pairs (u, v) at the price of suitable extensions of the projection P, that we do not discuss here (see the references mentioned above). In the sequel we shall only use the rules explicitly given above.

For smooth divergence free fields, equal to zero at the boundary, we have

$$\langle B(u, v), \phi \rangle = \int_D (u \cdot \nabla v) \cdot \phi dx = -\int_D (u \cdot \nabla \phi) \cdot v dx = -b(u, \phi, v).$$

In the sequel we denote by V' the dual of V. We may identify H with H' and thus write $D(A) \subset V \subset H \subset V'$ with continuous dense embeddings. The scalar product $\langle \cdot, \cdot \rangle$ in H "extends" to the dual pairing between V and V', which will be denoted by the same notation.

Definition 1.1 Given $u_0 \in H$ and $f \in L^2\left(0, T; V'\right)$, we say that

$$u \in C([0, T]; H) \cap L^2(0, T; V)$$

is a weak solution of Eq. (1.1) if

$$\begin{aligned} &\langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ &\quad = \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle\, ds + \int_0^t \langle f(s), \phi \rangle\, ds \end{aligned}$$

for every $\phi \in D(A)$.

The previous definition is a natural reformulation of Eq. (1.1). Indeed,

$$\int_D \phi \cdot (u \cdot \nabla u)\, dx = -\int_D u \cdot (u \cdot \nabla \phi)\, dx = -b(u, \phi, u)$$

(using also $u|_{\partial D} = 0$) and similarly,

$$\int_D \phi \cdot \Delta u dx = \int_D u \cdot \Delta \phi dx.$$

In fact we could avoid the integration by parts in the first case, and a single integration by parts is sufficient in the second case, but in this way we anticipate the poor regular case investigated later on. The following result is classical, see for instance [199, 200, 247, 248].

Theorem 1.2 *For every $u_0 \in H$ and $f \in L^2\left(0, T; V'\right)$ there exists a unique weak solution of Eq. (1.1). It satisfies*

$$\|u(t)\|_H^2 + 2\nu \int_0^t \|\nabla u(s)\|_{L^2}^2 ds = \|u_0\|_H^2 + 2\int_0^t \langle u(s), f(s)\rangle\, ds.$$

If $\left(u_0^n\right)_{n\in\mathbb{N}}$ is a sequence in H converging to $u_0 \in H$ and $(f^n)_{n\in\mathbb{N}}$ is a sequence in $L^2\left(0, T; V'\right)$ converging to $f \in L^2\left(0, T; V'\right)$, then the corresponding unique solutions $(u^n)_{n\in\mathbb{N}}$ converge to the corresponding solution u in $C\left([0, T]; H\right)$ and in $L^2(0, T; V)$.

We do not provide a proof but, when we give a proof for the stochastic case in Chap. 2, the reader may easily reconstruct one for this theorem. Since measurability is a consequence of continuity, we have the following result.

Corollary 1.3 *If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\omega \mapsto (u_0(\omega), f(\omega))$ is a measurable map from $(\Omega, \mathcal{F})$ to $H \times L^2\left(0, T; V'\right)$ (endowed with the Borel σ-algebra) then, denoting by $u(\omega)$ the weak solution corresponding to $(u_0(\omega), f(\omega))$, we have that $\omega \mapsto u(\omega)$ is measurable from $(\Omega, \mathcal{F})$ to $C([0, T]; H) \cap L^2(0, T; V)$.*

1.2 Well–Posedness of the Model with Rough Force

Consider the equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \partial_t W$$
$$\operatorname{div} u = 0$$

when W is a function of space and time, not differentiable in time, with

$$u|_{\partial D} = 0$$
$$u|_{t=0} = u_0.$$

It is a generalization of the model of the previous section, with $f = \partial_t W$, where we stress the irregularity in time of the forcing term. A justification to the introduction of this model is given in Chap. 5, see in particular Sect. 5.5. Let us stress that, although the notation W clearly alludes to a Wiener process and will do so in the subsequent chapters, here it is just a single function; even discontinuous, a priori, as in the starting examples of Sect. 5.5.

The aim of this section is to give a rigorous definition of the solution and prove, in 2D, existence and uniqueness.

The approach we follow here may look strange at first sight but (although old) is quite modern in style. We could learn the proof of the deterministic case and adapt it to the stochastic one (Galerkin approximations, compactness etc.). This has been done with great success in the literature and, indeed, we will investigate some models with such an approach in Chaps. 2 and 4. However, a different approach which became more and more successful recently with singular SPDEs, consists of two steps: a probabilistic kernel, often linear, Gaussian, followed by a nonlinear deterministic step. We do the same here: we solve the linear case, the so–called Stokes equation, with ad hoc tools, then we apply Theorem 1.2. In this chapter, thanks to the fact that the force is additive and not depending on the state of the system, we also solve the linear problem by means of deterministic tools, but in the next one we use probability.

The methodology developed here, as we said, is very classical but was proved to be useful for several different purposes. In [25] there was a sort of primitive but very instructive use of this idea, where the auxiliary variable was not the solution of the Stokes problem but just the Brownian motion itself. The approach was used several times for purposes of existence, uniqueness regularity, approximation, random dynamical system studies, even investigation of very complex regimes, both in dimensions 2 and 3 and sometimes for state-dependent noise of very simple form which allows us to apply another kind of transformation. See for instance [3, 79, 80, 104, 110, 136–140], but this list is highly incomplete and the method is used also today in order to reduce the stochastic case to a random one and use advanced deterministic results of maximal regularity, e.g. [40].

1.2.1 The Stokes Problem

Let us consider first the Stokes problem:

$$\partial_t z + \nabla q = \nu \Delta z + \partial_t W$$
$$\operatorname{div} z = 0$$

Let us argue heuristically in order to identify the solution, then we formalize the concept of the solution and the result. Thanks to the linearity of the problem, we may use semigroups to get an explicit formula:

$$z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} \partial_s W(s)\, ds.$$

Here we have denoted by e^{tA} the analytic semigroup generated by A (cf. [225] for general facts about analytic semigroups and [248] for the Stokes case). But at this level we still have the same problem of the meaning of $\partial_s W$. However, if we integrate by parts, we get

$$\begin{aligned} z(t) &= e^{tA} z_0 + \left[e^{(t-s)A} W(s) \right]_{s=0}^{s=t} - \int_0^t \frac{d}{ds} e^{(t-s)A} W(s)\, ds \\ &= e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t A e^{(t-s)A} W(s)\, ds, \end{aligned}$$

which is an expression with only W. The problem now is that $Ae^{(t-s)A}W(s)$ should be well–defined and integrable, in spite of the fact that A is an unbounded operator. The semigroup e^{tA}, being analytic, takes values in $D(A)$ for every $t > 0$ but with a singularity for $t = 0$, measured by the property

$$\|Ae^{tA}h\|_H \le \frac{C}{t}\|h\|_H.$$

The singularity $\frac{C}{t}$ is not integrable, hence we need some property of W in order to have that $Ae^{(t-s)A}W(s)$ is integrable on $[0, T]$.

We solve the previous problem in the simplest possible way by assuming that

$$W \in L^\infty(0, T; D(A)).$$

In the examples of Sect. 5.5 this is guaranteed by $\sigma_k \in D(A)$. Under this assumption we may write

$$\int_0^t Ae^{(t-s)A} W(s)\, ds = \int_0^t e^{(t-s)A} AW(s)\, ds$$

and the integral is obviously well-defined. In the two remarks below we explain two other solutions under less regularity of W.

Remark 1.4 If

$$W \in L^\infty\left(0, T; D\left((-A)^\epsilon\right)\right)$$

for some $\epsilon > 0$, then we can write

$$\int_0^t Ae^{(t-s)A} W(s)\, ds = -\int_0^t (-A)^{1-\epsilon}\, e^{(t-s)A}\, (-A)^\epsilon\, W(s)\, ds$$

and use the inequality

$$\|(-A)^{1-\epsilon}\, e^{tA}h\|_H \leq \frac{C}{t^{1-\epsilon}}\|h\|_H$$

to get the well–posedness of $z(t)$.

Remark 1.5 If

$$W \in C^{\epsilon}\left([0,T];H\right)$$

for some $\epsilon > 0$, then we can write

$$\begin{aligned}\int_0^t Ae^{(t-s)A}W(s)\,ds &= \int_0^t Ae^{(t-s)A}\left(W(s)-W(t)\right)ds + \int_0^t Ae^{(t-s)A}W(t)\,ds\\ &= \int_0^t Ae^{(t-s)A}\left(W(s)-W(t)\right)ds - W(t) + e^{tA}W(t)\end{aligned}$$

and now

$$\|Ae^{(t-s)A}\left(W(s)-W(t)\right)\|_H \leq \frac{C}{t-s}|t-s|^{\epsilon},$$

which is integrable. Therefore $z(t)$ is well-defined.

We can thus give the following definition and prove the following theorem. As just remarked, with some effort it can be extended to

$$W \in L^{\infty}\left(0,T;D\left((-A)^{\epsilon}\right)\right) \cup C^{\epsilon}\left([0,T];H\right)$$

for some $\epsilon > 0$.

Definition 1.6 Given $z_0 \in H$ and $W \in L^{\infty}(0,T;D(A))$, we say that z is a weak solution of the Stokes problem if

$$z \in L^{\infty}(0,T;H)$$

and

$$\langle z(t),\phi\rangle = \langle z_0,\phi\rangle + \int_0^t \langle z(s), A\phi\rangle\,ds + \langle W(t),\phi\rangle - \langle W(0),\phi\rangle$$

for every $\phi \in D(A)$.

Theorem 1.7 *If* $z_0 \in H$ *and* $W \in L^{\infty}(0, T; D(A))$*, then there exists one and only one weak solution of the Stokes problem; it is given by*

$$z(t) = e^{tA} z_0 + W(t) - e^{tA} W(0) + \int_0^t e^{(t-s)A} A W(s)\, ds. \tag{1.4}$$

Proof

Step 1 (uniqueness and explicit formula) Let z be a solution. Let

$$\phi \in C^1([0, T]; H) \cap C([0, T]; D(A))$$

be given. Let $0 = t_0 < \ldots < t_n = T$ be a partition of $[0, T]$, partition also denoted by π. Then, using the identities

$$\langle z(t_{i+1}), \phi(t_{i+1})\rangle - \langle z(t_{i+1}), \phi(t_i)\rangle = \int_{t_i}^{t_{i+1}} \langle z(t_{i+1}), \partial_s \phi(s)\rangle\, ds$$

$$\langle W(t_{i+1}), \phi(t_{i+1})\rangle - \langle W(t_{i+1}), \phi(t_i)\rangle = \int_{t_i}^{t_{i+1}} \langle W(t_{i+1}), \partial_s \phi(s)\rangle\, ds$$

we get

$$\begin{aligned}\langle z(t_{i+1}), \phi(t_{i+1})\rangle = \langle z(t_i), \phi(t_i)\rangle &+ \int_{t_i}^{t_{i+1}} \langle z(t_{i+1}), \partial_s \phi(s)\rangle\, ds \\ &+ \int_{t_i}^{t_{i+1}} \langle z(s), A\phi(t_i)\rangle\, ds \\ &+ \langle W(t_{i+1}), \phi(t_{i+1})\rangle - \langle W(t_i), \phi(t_i)\rangle \\ &- \int_{t_i}^{t_{i+1}} \langle W(t_{i+1}), \partial_s \phi(s)\rangle\, ds.\end{aligned}$$

It implies

$$\begin{aligned}\langle z(T), \phi(T)\rangle = \langle z_0, \phi(0)\rangle + \int_0^T \left\langle z\left(s_\pi^+\right), \partial_s \phi(s)\right\rangle ds + \int_0^T \left\langle z(s), A\phi\left(s_\pi^-\right)\right\rangle ds \\ + \langle W(T), \phi(T)\rangle - \langle W(0), \phi(0)\rangle - \int_0^T \left\langle W\left(s_\pi^+\right), \partial_s \phi(s)\right\rangle ds,\end{aligned}$$

where $s_\pi^- = t_i$, $s_\pi^+ = t_{i+1}$, if $s \in [t_i, t_{i+1}]$. Taking the limit over a sequence of partitions π_N with size going to zero, we get

$$\langle z(T), \phi(T)\rangle = \langle z_0, \phi(0)\rangle + \int_0^T \langle z(s), \partial_s \phi(s)\rangle\, ds + \int_0^T \langle z(s), A\phi(s)\rangle\, ds$$
$$+ \langle W(T), \phi(T)\rangle - \langle W(0), \phi(0)\rangle - \int_0^T \langle W(s), \partial_s \phi(s)\rangle\, ds$$

(thanks to the regularity of z, ϕ and Lebesgue dominated convergence theorem). The argument applies to every intermediate time t in place of T, hence we have

$$\langle z(t), \phi(t)\rangle = \langle z_0, \phi(0)\rangle + \int_0^t \langle z(s), \partial_s \phi(s)\rangle\, ds + \int_0^t \langle z(s), A\phi(s)\rangle\, ds$$
$$+ \langle W(t), \phi(t)\rangle - \langle W(0), \phi(0)\rangle - \int_0^t \langle W(s), \partial_s \phi(s)\rangle\, ds.$$

For such a value of t, take the function

$$\phi_t(s) := e^{(t-s)A}\psi$$

with $\psi \in D(A)$. This function is of class

$$\phi_t(\cdot) \in C^1([0,t]; H) \cap C([0,t]; D(A))$$

hence, from the previous identity,

$$\langle z(t), \psi\rangle = \left\langle z_0, e^{tA}\psi\right\rangle - \int_0^t \left\langle z(s), Ae^{(t-s)A}\psi\right\rangle ds + \int_0^t \left\langle z(s), Ae^{(t-s)A}\psi\right\rangle ds$$
$$+ \langle W(t), \psi\rangle - \left\langle W(0), e^{tA}\psi\right\rangle + \int_0^t \left\langle W(s), Ae^{(t-s)A}\psi\right\rangle ds.$$

Using the fact that A is selfadjoint and $W(s) \in D(A)$ we get

$$\langle z(t), \psi\rangle = \left\langle e^{tA} z_0, \psi\right\rangle + \langle W(t), \psi\rangle - \left\langle e^{tA} W(0), \psi\right\rangle + \int_0^t \left\langle e^{(t-s)A} AW(s), \psi\right\rangle ds$$

and finally, by the arbitrary nature of ψ, we find that z is given by the explicit formula (1.4). This also implies uniqueness.

Step 2 (existence) Formula (1.4) defines a function of class $L^\infty(0,T;H)$. The function $z(t) - W(t)$ is given by

$$z(t) - W(t) = e^{tA}(z_0 - W(0)) + \int_0^t e^{(t-s)A} AW(s)\, ds$$

and therefore, by classical results on analytic semigroups, it is differentiable for $t > 0$ and satisfies

$$\frac{d}{dt}\left(z\left(t\right)-W\left(t\right)\right)=Az(t).$$

Then it is sufficient to integrate this identity in time, taking the scalar product with $\phi \in D\left(A\right)$ and using the fact that A is selfadjoint.

■

When we have given the definition of the trilinear form b we have seen the role of the space $\mathbb{L}^4$. We need to upgrade the regularity of z in order to cope with the nonlinearity later on. Since it is sufficient for us, we restrict to $z_0 = 0$. As usual, we state and prove the result under the abundant regularity $W \in L^\infty\left(0, T; D\left(A\right)\right)$, but the result is true, in this case, also when

$$W \in L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\epsilon}\right)\right) \cup C^{\frac{1}{4}+\epsilon}\left([0, T]; H\right)$$

for some $\epsilon > 0$.

Theorem 1.8 *Let $z_0 = 0$. If $W \in L^\infty\left(0, T; D\left(A\right)\right)$, then the weak solution of the Stokes problem satisfies $z \in L^\infty\left(0, T; \mathbb{L}^4\right)$ and the map from W to z is linear continuous between these spaces. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\omega \mapsto W\left(\omega\right)$ is a measurable map from $(\Omega, \mathcal{F})$ to $L^\infty\left(0, T; D\left(A\right)\right)$ (endowed with the Borel σ-algebra) then, denoting by $z\left(\omega\right)$ the weak solution corresponding to $W\left(\omega\right)$, we have that $\omega \mapsto z\left(\omega\right)$ is measurable from $(\Omega, \mathcal{F})$ to $C\left([0, T]; H\right) \cap L^\infty\left(0, T; \mathbb{L}^4\right)$.*

Proof Without optimizing the argument, let us remark that $V \subset \mathbb{L}^4$ by the Sobolev embedding theorem and

$$\|z\left(t\right)\|_V \leq \|W\left(t\right) - e^{tA}W\left(0\right)\|_V + \int_0^t \|e^{(t-s)A}AW\left(s\right)\|_V ds.$$

Now $D\left(A\right) \subset V$, hence $\|W\left(t\right) - e^{tA}W\left(0\right)\|_V$ is bounded. And a well–known inequality for analytic semigroups gives us, for some constant $C > 0$

$$\|e^{tA}w\|_V \leq \frac{C}{\sqrt{t}}\|w\|_H$$

for all $w \in V$ and $t \in (0, T]$. Hence we deduce $z \in L^\infty\left(0, T; V\right) \subset L^\infty\left(0, T; \mathbb{L}^4\right)$. The measurability follows from the continuity, which is a consequence of linearity and boundedness. ■

As we said above, the theorem extends to more general data. The following one is an example.

Theorem 1.9 *If* $z_0 \in D\left((-A)^{\frac{1}{4}+\epsilon}\right)$ *and*

$$W \in L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\epsilon}\right)\right)$$

for some $\epsilon > 0$ *then*

$$z \in L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\frac{\epsilon}{2}}\right)\right) \subset L^\infty(0, T; \mathbb{L}^4).$$

Proof The term $e^{tA}z_0 + W(t) - e^{tA}W(0)$ in the explicit expression for $z(t)$ is of class $L^\infty\left(0, T; D\left((-A)^{\frac{1}{4}+\epsilon}\right)\right)$, directly from the assumptions. For the integral term $\int_0^t Ae^{(t-s)A}W(s)\,ds$ we have

$$(-A)^{\frac{1}{4}+\frac{\epsilon}{2}} \int_0^t Ae^{(t-s)A}W(s)\,ds = -\int_0^t (-A)^{1-\frac{\epsilon}{2}} e^{(t-s)A} (-A)^{\frac{1}{4}+\epsilon} W(s)\,ds.$$

Now $(-A)^{\frac{1}{4}+\epsilon} W \in L^\infty(0, T; H)$ and

$$\|(-A)^{1-\frac{\epsilon}{2}} e^{(t-s)A}\|_{\mathcal{L}(H,H)} \leq \frac{C}{(t-s)^{1-\epsilon/2}}.$$

These two facts imply the result. ■

Since our main goal is to consider a stochastic forcing term which is a Brownian motion, let us explain a bit better the case when W is only H valued, namely $W \in C^\alpha([0, T]; H)$, $\forall \alpha < \frac{1}{2}$, $W(0) = 0$. Arguing as in Remark 1.5, we can write

$$z(t) = e^{tA}z_0 + e^{tA}W(t) + \int_0^t Ae^{(t-s)A}(W(s) - W(t))\,ds. \tag{1.5}$$

Thus, as explained in Remark 1.5, $z(t) \in L^\infty(0, T; H)$. A result completely analogous to Theorem 1.7 can be stated also in this framework:

Theorem 1.10 *If* $z_0 \in H$ *and* $W \in C^\alpha([0, T]; H)$, $\forall \alpha < \frac{1}{2}$, $W(0) = 0$, *there exists one and only one weak solution for the Stokes problem and it is given by the formula*

$$z(t) = e^{tA}z_0 + e^{tA}W(t) + \int_0^t Ae^{(t-s)A}(W(s) - W(t))\,ds.$$

Proof

Step 1 (uniqueness and explicit formula) Let z be a solution. Let

$$\phi \in C^1([0, T]; H) \cap C([0, T]; D(A)).$$

Arguing exactly as in the proof of Theorem 1.7 we get that $z(t)$ satisfies

$$\langle z(t), \phi(t)\rangle = \langle z_0, \phi(0)\rangle + \int_0^t \langle z(s), \partial_s \phi(s)\rangle\, ds + \int_0^t \langle z(s), A\phi(s)\rangle\, ds$$
$$+ \langle W(t), \phi(t)\rangle - \langle W(0), \phi(0)\rangle - \int_0^t \langle W(s), \partial_s \phi(s)\rangle\, ds.$$

For such a value of t, take the function

$$\phi_t(s) := e^{(t-s)A}\psi$$

with $\psi \in D(A)$. This function is of class

$$\phi_t(\cdot) \in C^1([0,t]; H) \cap C([0,t]; D(A)).$$

Using ϕ_t defined above in the weak formulation satisfied by $z(t)$ we get

$$\langle z(t), \psi\rangle = \left\langle z_0, e^{tA}\psi\right\rangle + \langle W(t), \psi\rangle + \int_0^t \left\langle W(s), Ae^{(t-s)A}\psi\right\rangle ds.$$

We add and subtract $\int_0^t \left\langle W(t), Ae^{(t-s)A}\psi\right\rangle ds$ in the relation above, then, exploiting the regularity of ϕ, the semigroup and its infinitesimal generator commute. Thus, thanks to the fact that A is selfadjoint, we arrive at the following relation:

$$\langle z(t), \psi\rangle = \left\langle z_0, e^{tA}\psi\right\rangle + \langle W(t), \psi\rangle + \int_0^t \left\langle Ae^{(t-s)A}(W(s) - W(t)), \psi\right\rangle ds$$
$$+ \int_0^t \left\langle W(t), Ae^{(t-s)A}\psi\right\rangle ds$$
$$= \left\langle z_0, e^{tA}\psi\right\rangle + \left\langle e^{tA}W(t), \psi\right\rangle + \left\langle \int_0^t Ae^{(t-s)A}(W(s) - W(t))ds, \psi\right\rangle.$$

From the density of $D(A)$ in H, the first statement follows.

Step 2 (existence) As in the proof of Theorem 1.7, formula (1.5) defines a function in $L^\infty(0,T;H)$ and the relation $z(t) - e^{tA}W(t) - \int_0^t Ae^{(t-s)A}(W(s) - W(t))\ ds = e^{tA}z_0$ holds. Therefore, by classical results on analytic semigroups, we can differentiate this relation for $t > 0$, obtaining

$$\frac{d}{dt}\left(z(t) - e^{tA}W(t) - \int_0^t Ae^{(t-s)A}\,(W(s) - W(t))\ ds\right)$$
$$= A\left(z(t) - e^{tA}W(t) - \int_0^t Ae^{(t-s)A}\,(W(s) - W(t))\ ds\right).$$

Integrating this identity in time, taking the scalar product with $\phi \in D\left(A^2\right)$ we arrive at the relation below:

$$\left\langle z(t) - z(0) - e^{tA}W(t) - \int_0^t Ae^{(t-s)A}\left(W(s) - W(t)\right)\, ds, \phi\right\rangle$$
$$= \left\langle \int_0^t \left(z(s) - e^{sA}W(s) - \int_0^s Ae^{(s-r)A}\left(W(r) - W(s)\right)\, dr\right)\, ds, A\phi\right\rangle.$$

Our goal is to try to rewrite better the quantity

$$\left\langle e^{tA}W(t) + \int_0^t Ae^{(t-s)A}\left(W(s) - W(t)\right), \phi\right\rangle$$
$$- \left\langle \int_0^t \left(e^{sA}W(s) + \int_0^s Ae^{(s-r)A}\left(W(r) - W(s)\right)\, dr\right) ds, A\phi\right\rangle.$$

We first concentrate on the double integral. Thanks to the fact that A is selfadjoint and the regularity of the process $e^{(s-r)A}\left(W(r) - W(s)\right)$, it can be rewritten as

$$- \left\langle \int_0^t ds \int_0^s e^{(s-r)A}\left(W(r) - W(s)\right)\, dr, A^2\phi\right\rangle$$
$$= -\left\langle \int_0^t ds \int_0^s e^{(s-r)A}W(r)\, dr - \int_0^t ds \int_0^s e^{(s-r)A}W(s)\, dr, A^2\phi\right\rangle$$
$$= -\left\langle \int_0^t dr \int_r^t e^{(s-r)A}W(r)\, ds - \int_0^t ds \int_0^s e^{(s-r)A}W(s)\, dr, A^2\phi\right\rangle$$
$$= \left\langle \int_0^t e^{sA}W(s) - e^{(t-s)A}W(s)\, ds, A\phi\right\rangle.$$

The exchange of the order of integration is allowed thanks to the continuity of the integrand functions and the compactness of the integration set. Exploiting this relation we arrive at

$$\left\langle e^{tA}W(t) + \int_0^t Ae^{(t-s)A}\left(W(s) - W(t)\right), \phi\right\rangle$$
$$- \left\langle \int_0^t \left(e^{sA}W(s) + \int_0^s Ae^{(s-r)A}\left(W(r) - W(s)\right)\, dr\right) ds, A\phi\right\rangle$$
$$= \left\langle e^{tA}W(t) - \int_0^t Ae^{(t-s)A}W(t), \phi\right\rangle$$
$$= \langle W(t), \phi\rangle.$$

Thus the thesis follows from the density of $D(A^2)$ in $D(A)$.

■

For the remainder of the chapter, in order to treat the nonlinearity, we will need $z(t) \in L^4(0,T;\mathbb{L}^4)$ and the map $\omega \mapsto z(\omega)$, which is measurable from $(\Omega, \mathcal{F})$ to $C([0,T];H) \cap L^4\left(0,T;\mathbb{L}^4\right)$. Recall our definition of a mild solution

$$z(t) = e^{tA} z_0 + e^{tA} W(t) + \int_0^t A e^{(t-s)A} (W(s) - W(t))\, ds.$$

We will show separately that $e^{tA} z_0$ and $e^{tA} W(t) + \int_0^t A e^{(t-s)A} (W(s) - W(t))\, ds$ have the required regularity. For the first term we will use a trick which will be presented in Chap. 3 in a more difficult case, so we refer to Sect. 3.2.2 for more details. For the second one, we will exploit, again, the Hölder regularity of the Brownian motion. As discussed previously,

$$\begin{aligned}
\int_0^T \|e^{tA} z_0\|_{\mathbb{L}^4}^4 dt &\le C \int_0^T \|e^{tA} z_0\|_H^2 \|e^{tA} z_0\|_V^2 dt \\
&\le C \sup_{t\in[0,T]} \|e^{tA} z_0\|_H^2 \int_0^T \|e^{tA} z_0\|_V^2 dt \\
&\le C \|z_0\|_H^2 \int_0^T \langle -A e^{tA} z_0, e^{tA} z_0 \rangle dt \\
&= -C \|z_0\|_H^2 \int_0^T \frac{d\|e^{tA} z_0\|_H^2}{2dt} dt \\
&\le \frac{C\|z_0\|_H^2}{2} \left(\|z_0\|_H^2 - \|e^{tA} z_0\|_H^2 \right) \le \frac{C}{2} \|z_0\|_H^4.
\end{aligned}$$

For the second one, taking $\epsilon > 0$ small enough, we have

$$\begin{aligned}
&\|e^{tA} W(t) + \int_0^t A e^{(t-s)A} (W(s) - W(t))\, ds\|_{\mathbb{L}^4}^4 \\
&\quad \le C \|e^{tA} W(t)\|_H^2 \|e^{tA} W(t)\|_V^2 + C \left(\int_0^t \|A e^{(t-s)A} (W(s) - W(t))\|_{\mathbb{L}^4} ds \right)^4 \\
&\quad \le \frac{C}{t} \|W(t)\|_H^4 \\
&\qquad + C \left(\int_0^t \|A e^{(t-s)A} (W(s) - W(t))\|_H^{1/2} \|A e^{(t-s)A} (W(s) - W(t))\|_V^{1/2} ds \right)^4 \\
&\quad \le C \|W\|^4_{C^{\frac{1}{2}-\epsilon}(0,T;H)} t^{1-4\epsilon} + C \left(\int_0^t \frac{\|W\|_{C^{\frac{1}{2}-\epsilon}(0,T;H)} (t-s)^{\frac{1}{2}-\epsilon}}{(t-s)^{\frac{5}{4}}} ds \right)^4 \\
&\quad \le C(T, \epsilon, D) \|W\|^4_{C^{\frac{1}{2}-\epsilon}(0,T;H)}.
\end{aligned}$$

Thus, up to introducing another different constant C depending only from T, ϵ and the domain D, we have that

$$\|e^{\cdot A}W(\cdot)+\int_0^{\cdot} Ae^{(\cdot-s)A}(W(s)-W(\cdot))\,ds\|_{L^4(0,T;\mathbb{L}^4)} \leq C(T,\epsilon,D)\|W\|_{C^{\frac{1}{2}-\epsilon}(0,T;H)}.$$

Thus $z \in L^4\left(0, T; \mathbb{L}^4\right)$.

The measurability of the map follows for the same reasons as the case treated in Theorem 1.8.

Remark 1.11 When W is a Brownian motion in a suitable Hilbert space, or also in the case of other special stochastic processes, using rules of stochastic calculus one can improve a bit on the previous results, which therefore are sub-optimal. In fact, when W takes values in H only, it is not possible to prove that almost surely $z \in L^2(0, T; V)$ with previous deterministic tricks. However, this result is true and follows from the argument described in Sect. 3.2.2. The theory extends also to the case of Banach-valued processes. Several results can be found for instance in [90] and [179, 180], see also [1, 49, 253].

1.2.2 Auxiliary Navier–Stokes Type Equations

Let us explain first the heuristics. Having solved the Stokes problem we introduce the auxiliary variable

$$v\,(t) = u\,(t) - z(t),$$

which satisfies

$$\partial_t v + (v+z)\cdot\nabla\,(v+z) + \nabla\,(p-q) = \nu\Delta v$$
$$\operatorname{div} v = 0.$$

This equation has the form

$$\partial_t v + v\cdot\nabla v + \nabla\pi = \nu\Delta v - L\,(v,z)$$
$$\operatorname{div} v = 0$$

with the affine function

$$L\,(v,z) = v\cdot\nabla z + z\cdot\nabla v + z\cdot\nabla z.$$

Therefore the Navier–Stokes structure is preserved, for the variable v, up to a remainder which is affine. It is then not surprising that the auxiliary equation for

v is solvable similarly to the classical Navier–Stokes equations. The strategy then is solving the auxiliary equation and then deducing the solution of the Navier–Stokes equations with rough force.

To avoid confusion with the heuristics above, let us formulate the problem from scratch. Consider the modified Navier–Stokes equation

$$\partial_t v + (v+z) \cdot \nabla (v+z) + \nabla \pi = \nu \Delta v + f \tag{1.6}$$
$$\operatorname{div} v = 0$$

with

$$v|_{\partial D} = 0$$
$$v|_{t=0} = v_0.$$

Definition 1.12 Given $v_0 \in H$, $f \in L^2\left(0, T; V'\right)$ and $z \in L^4\left(0, T; \mathbb{L}^4\right)$, we say that

$$v \in C\left([0, T]; H\right) \cap L^2\left(0, T; V\right)$$

is a weak solution of Eq. (1.6) if

$$\langle v(t), \phi \rangle - \int_0^t b\left(v(s) + z(s), \phi, v(s) + z(s)\right) ds$$
$$= \langle v_0, \phi \rangle + \int_0^t \langle v(s), A\phi \rangle\, ds + \int_0^t \langle f(s), \phi \rangle\, ds$$

for every $\phi \in D(A)$.

Theorem 1.13 *For every $v_0 \in H$, $f \in L^2\left(0, T; V'\right)$ and $z \in L^4\left(0, T; \mathbb{L}^4\right)$, there exists a unique weak solution of Eq. (1.6). It satisfies*

$$\|v(t)\|_H^2 + 2\nu \int_0^t \|\nabla v(s)\|_{L^2}^2 ds = \|v_0\|_H^2 + 2\int_0^t \langle f(s), v(s)\rangle\, ds$$
$$+ 2\int_0^t \left(b(v, v, z) + b(z, v, z)\right)(s)\, ds.$$

Finally, a continuity and a measurability statement completely analogous to those of Theorem 1.2 and Corollary 1.3 hold here too.

Proof

Step 1 (uniqueness) Let $v^{(i)}$ be two solutions. The function $w = v^{(1)} - v^{(2)}$ satisfies

$$\langle w(t), \phi\rangle - \int_0^t \left(b\left(v^{(1)} + z, \phi, v^{(1)} + z\right) - b\left(v^{(2)} + z, \phi, v^{(2)} + z\right)\right) ds$$
$$= \int_0^t \langle w(s), A\phi\rangle \, ds$$

for every $\phi \in D(A)$. A simple manipulation gives us

$$b\left(v^{(1)} + z, \phi, v^{(1)} + z\right) - b\left(v^{(2)} + z, \phi, v^{(2)} + z\right) - b(w, \phi, w)$$
$$= b\left(v^{(2)} + z, \phi, w\right) + b\left(w, \phi, v^{(2)} + z\right)$$

hence

$$\langle w(t), \phi\rangle - \int_0^t b(w(s), \phi, w(s)) \, ds$$
$$= \int_0^t \langle w(s), A\phi\rangle \, ds + \int_0^t \left\langle \widetilde{f}(s), \phi\right\rangle ds$$

where

$$\widetilde{f} = -B\left(v^{(2)} + z, w\right) - B\left(w, v^{(2)} + z\right).$$

By Lemma 1.14 below, $\widetilde{f} \in L^2\left(0, T; V'\right)$. Then, by Theorem 1.2,

$$\|w(t)\|_H^2 + 2\nu \int_0^t \|\nabla w(s)\|_{L^2}^2 ds = 2\int_0^t b\left(w, w, v^{(2)} + z\right)(s) \, ds.$$

Again by Lemma 1.14, we have

$$|b\left(w, w, v^{(2)} + z\right)| \le |b\left(w, w, v^{(2)}\right)| + |b(w, w, z)|$$
$$\le \epsilon\|w\|_V^2 + \epsilon\|w\|_V^2 + \frac{C}{\epsilon^3}\|w\|_H^2 \|v^{(2)}\|_{L^4}^4$$
$$+ \epsilon\|w\|_V^2 + \epsilon\|w\|_V^2 + \frac{C}{\epsilon^3}\|w\|_H^2 \|z\|_{L^4}^4$$
$$= 4\epsilon\|w\|_V^2 + \frac{C}{\epsilon^3}\|w\|_H^2 \left(\|v^{(2)}\|_{L^4}^4 + \|z\|_{L^4}^4\right)$$

Summarizing, with $4\epsilon = \nu$, using the fact that $\|w\|_V^2 = \|\nabla w\|_{L^2}^2 + \|w\|_H^2$, renaming the constant C,

$$\|w(t)\|_H^2 + \nu \int_0^t \|\nabla w(s)\|_{L^2}^2 ds$$
$$= C \int_0^t \|w(s)\|_H^2 \left(1 + \|v^{(2)}(s)\|_{L^4}^4 + \|z(s)\|_{L^4}^4\right) ds.$$

We conclude $w = 0$ by the Gronwall lemma, using the assumption on z and inequality (1.3) for $v^{(2)}$.

Step 2 (existence) Define the sequence (v^n) by setting $v^0 = 0$ and for every $n \geq 0$, given $v^n \in C([0,T]; H) \cap L^2(0,T;V)$, let v^{n+1} be the solution of Eq. (1.1) with initial condition v_0 and with

$$f - B\left(v^n, z\right) - B\left(z, v^n\right) - B(z, z)$$

in place of f. In particular

$$\left\langle v^{n+1}(t), \phi\right\rangle - \int_0^t b\left(v^{n+1}(s), \phi, v^{n+1}(s)\right) ds$$
$$= \langle v_0, \phi\rangle + \int_0^t \left\langle v^{n+1}(s), A\phi\right\rangle ds + \int_0^t \langle f(s), \phi\rangle ds$$
$$- \int_0^t \left\langle\left(B\left(v^n, z\right) + B\left(z, v^n\right) + B(z, z)\right)(s), \phi\right\rangle ds$$

for every $\phi \in D(A)$. In order to claim that this definition is well done, we notice that

$$B\left(v^n, z\right), B\left(z, v^n\right), B(z, z) \in L^2\left(0, T; V'\right)$$

by Lemma 1.14 below.

Then let us investigate the convergence of (v^n). First, let us prove a bound. From the previous identity and Theorem 1.2 we get

$$\|v^{n+1}(t)\|_H^2 + 2\nu \int_0^t \|\nabla v^{n+1}(s)\|_{L^2}^2 ds$$
$$= \|v_0\|_H^2 + 2\int_0^t \left\langle f(s), v^{n+1}(s)\right\rangle ds$$
$$+ 2\int_0^t \left(b\left(v^n, v^{n+1}, z\right) + b\left(z, v^{n+1}, v^n\right) + b\left(z, v^{n+1}, z\right)\right)(s)\, ds.$$

It gives us (using Lemma 1.14 below)

$$\|v^{n+1}(t)\|_H^2 + \nu \int_0^t \|\nabla v^{n+1}(s)\|_{L^2}^2 ds$$
$$= \|v_0\|_H^2 + C \int_0^t \|f(s)\|_{V'}^2 ds + \epsilon \int_0^t \|v^n(s)\|_V^2 ds$$
$$+ C_\epsilon \int_0^t \|v^n(s)\|_H^2 \left(1 + \|z(s)\|_{L^4}^4\right) ds + C_\epsilon \int_0^t \|z(s)\|_{L^4}^4 ds.$$

By using the Gronwall lemma and a small constant ϵ, one can find $R > \|v_0\|_H^2$ and T small enough such that if

$$\sup_{t\in[0,T]} \|v^n(t)\|_H^2 \le R, \qquad \int_0^T \|v^n(s)\|_V^2 ds \le R \tag{1.7}$$

then the same inequalities hold for v^{n+1}.

Set $w_n = v^n - v^{n-1}$, for $n \ge 1$. From the identity above,

$$\langle w_{n+1}(t), \phi\rangle - \int_0^t \left(b\left(v^{n+1}, \phi, v^{n+1}\right) - b\left(v^n, \phi, v^n\right)\right)(s)\, ds$$
$$= \int_0^t \langle w_{n+1}(s), A\phi\rangle\, ds - \int_0^t \left\langle\left(B\left(v^n, z\right) - B\left(v^{n-1}, z\right)\right)(s), \phi\right\rangle ds$$
$$- \int_0^t \left\langle\left(B\left(z, v^n\right) - B\left(z, v^{n-1}\right)\right)(s), \phi\right\rangle ds.$$

Again as above, since

$$b\left(v^{n+1}, \phi, v^{n+1}\right) - b\left(v^n, \phi, v^n\right) - b\left(w_{n+1}, \phi, w_{n+1}\right)$$
$$= b\left(v^n, \phi, w_{n+1}\right) + b\left(w_{n+1}, \phi, v^n\right)$$

we may rewrite it as

$$\langle w_{n+1}(t), \phi\rangle - \int_0^t b\left(w_{n+1}(s), \phi, w_{n+1}(s)\right) ds$$
$$= \int_0^t \langle w_{n+1}(s), A\phi\rangle\, ds - \int_0^t \langle (B(w_n, z) + B(z, w_n))(s), \phi\rangle\, ds$$
$$+ \int_0^t \left(b\left(v^n, \phi, w_{n+1}\right) + b\left(w_{n+1}, \phi, v^n\right)\right)(s)\, ds.$$

One can check as above the applicability of Theorem 1.2 and get

$$\begin{aligned}\|w_{n+1}(t)\|_H^2 &+ 2\nu \int_0^t \|\nabla w_{n+1}(s)\|_{L^2}^2 ds \\ &= 2\int_0^t \left(b\left(w_n, w_{n+1}, z\right) + b\left(z, w_{n+1}, w_n\right)\right)(s)\, ds \\ &\quad + 2\int_0^t b\left(w_{n+1}, w_{n+1}, v^n\right)(s)\, ds.\end{aligned}$$

As above we deduce

$$\left|b\left(w_{n+1}, w_{n+1}, v^n\right)\right| \leq \frac{\nu}{2}\|w_{n+1}\|_V^2 + C\|w_{n+1}\|_H^2 \|v^n\|_{L^4}^4.$$

But

$$\left|b\left(w_n, w_{n+1}, z\right) + b\left(z, w_{n+1}, w_n\right)\right| \leq \frac{\nu}{2}\|w_{n+1}\|_V^2 + \frac{1}{4}\|w_n\|_V^2 + C\|w_n\|_H^2 \|z\|_{L^4}^4.$$

Hence

$$\begin{aligned}\|w_{n+1}(t)\|_H^2 &+ \nu \int_0^t \|\nabla w_{n+1}(s)\|_{L^2}^2 ds \\ &\leq C\int_0^t \|w_{n+1}(s)\|_H^2 \left(1 + \|v^n(s)\|_{L^4}^4\right) ds \\ &\quad + \frac{1}{4}\int_0^t \|w_n(s)\|_V^2 ds + C\int_0^t \|w_n(s)\|_H^2 \|z(s)\|_{L^4}^4 ds.\end{aligned}$$

Now we work under the bounds (1.7) and deduce, using the Gronwall lemma, for T possibly smaller than the previous one,

$$\begin{aligned}\sup_{t\in[0,T]} \|w_{n+1}(t)\|_H^2 &+ \int_0^T \|w_{n+1}(s)\|_V^2 ds \\ &\leq \frac{1}{2}\left(\sup_{t\in[0,T]} \|w_n(t)\|_H^2 + \int_0^T \|w_n(s)\|_V^2 ds\right).\end{aligned}$$

It implies that the sequence (v^n) is Cauchy in $C([0,T];H) \cap L^2(0,T;V)$. The limit v has the right regularity to be a weak solution and satisfies the weak formulation; in the identity above for v^{n+1} and v^n we may prove that

$$\int_0^t b\left(v^{n+1}(s), \phi, v^{n+1}(s)\right) ds \to \int_0^t b\left(v(s), \phi, v(s)\right) ds$$
$$\int_0^t b\left(v^n(s), \phi, z(s)\right) ds \to \int_0^t b\left(v(s), \phi, z(s)\right) ds$$
$$\int_0^t b\left(z(s), \phi, v^n(s)\right) ds \to \int_0^t b\left(z(s), \phi, v(s)\right) ds.$$

All these convergences can be proved easily by recalling the definition of b. Similarly, we can pass to the limit in the energy identity. ■

Lemma 1.14 *If $u, v \in L^4\left(0,T;\mathbb{L}^4\right)$ then*

$$B(u,v) \in L^2\left(0,T;V'\right). \tag{1.8}$$

Moreover,

$$|b(u,v,w)| \le \epsilon\|v\|_V^2 + \epsilon'\|u\|_V^2 + \frac{C}{\epsilon^2\epsilon'}\|u\|_H^2\|w\|_{L^4}^4 \tag{1.9}$$

$$|b(u,v,w)| \le \epsilon\|v\|_V^2 + \epsilon'\|w\|_V^2 + \frac{C}{\epsilon^2\epsilon'}\|w\|_H^2\|u\|_{L^4}^4, \tag{1.10}$$

where C is a constant independent of ϵ and ϵ'.

Proof Indeed,

$$|\langle B(u,v), \phi\rangle| = |b(u,\phi,v)| \le \|\phi\|_V\|u\|_{L^4}\|v\|_{L^4}$$
$$\|B(u,v)\|_{V'} \le \|u\|_{L^4}\|v\|_{L^4}$$

and thus

$$\int_0^T \|B(u(t),v(t))\|_{V'}^2 dt \le \left(\int_0^T \|u(t)\|_{L^4}^4 dt\right)^{1/2}\left(\int_0^T \|v(t)\|_{L^4}^4 dt\right)^{1/2}.$$

Moreover,

$$|b(u,v,w)| \le \|v\|_V\|u\|_{L^4}\|w\|_{L^4} \le \epsilon\|v\|_V^2 + \frac{1}{4\epsilon}\|u\|_{L^4}^2\|w\|_{L^4}^2,$$

hence the proof of (1.9) and (1.10) is the same. Let us prove the first one. From the interpolation inequality (1.2),

$$
\begin{aligned}
|b(u, v, w)| &\le \epsilon \|v\|_V^2 + \frac{C}{\epsilon} \|u\|_V \|u\|_H \|w\|_{L^4}^2 \\
&\le \epsilon \|v\|_V^2 + \epsilon' \|u\|_V^2 + \frac{C}{\epsilon^2 \epsilon'} \|u\|_H^2 \|w\|_{L^4}^4 .
\end{aligned}
$$

■

1.2.3 Final Main Result on the Equation with Rough Force

Finally, we may define the concept of the solution and prove the well–posedness for the Navier–Stokes equations with rough force

$$
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta v + f + \partial_t W \qquad (1.11)\\
\operatorname{div} u &= 0
\end{aligned}
$$

with

$$
\begin{aligned}
u|_{\partial D} &= 0 \\
u|_{t=0} &= u_0 .
\end{aligned}
$$

Definition 1.15 Given $u_0 \in H$, $f \in L^2\left(0, T; V'\right)$ and $W \in L^\infty(0, T; D(A))$, we say that

$$
\begin{aligned}
u \in\, & C([0, T]; H) \cap L^\infty\left(0, T; \mathbb{L}^4\right) \\
&+ C([0, T]; H) \cap L^2(0, T; V)
\end{aligned}
$$

is a weak solution of Eq. (1.11) if

$$
u - z \in C([0, T]; H) \cap L^2(0, T; V),
$$

where z is defined above with $z_0 = 0$ and

$$
\begin{aligned}
&\langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\
&\quad = \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle\, ds + \int_0^t \langle f(s), \phi \rangle\, ds + \langle W(t), \phi \rangle - \langle W(0), \phi \rangle
\end{aligned}
$$

for every $\phi \in D(A)$.

Theorem 1.16 *Assume* $u_0 \in H$, $f \in L^2\left(0, T; V'\right)$ *and* $W \in L^\infty(0, T; D(A))$. *Then the Navier–Stokes equation (1.11) has a unique weak solution, given by the sum of the solution of the Stokes problem and the solution of the auxiliary problem, which satisfies the energy identity of Theorem 1.13. Finally, a continuity and a measurability statement completely analogous to those of Theorem 1.2 and Corollary 1.3 hold here too.*

Proof

Step 1 (uniqueness) Let $u^{(i)}$ be two solutions. Let $v^{(i)} = u^{(i)} - z$; they are solutions of the auxiliary problem, hence they coincide, hence also $u^{(i)}$ coincide.
Step 2 (existence) Let v be a solution of the auxiliary problem. Set $u = v + z$: then u is a solution of Eq. (1.11).
Step 3 (measurability) Again, $u(\omega)$ is given by

$$u(\omega) = v(\omega) + z(\omega),$$

hence it inherits the measurability properties of $v(\omega)$ and $z(\omega)$ given by Theorems 1.13 and 1.8, respectively.

■

1.3 Summary

The main technique illustrated in this chapter is the reduction of the PDE with rough input to the classical PDE, by means of the solution of Stokes problem with rough input. Even if this approach is less flexible that the one based on probabilistic methods that will be described in Chap. 2, see the discussion in Sect. 2.1, it is quite useful and successful in a number of cases, especially in the theory of Random Dynamical Systems [78, 79] but also when the continuity of solution with respect to the input force may help, as in Large Deviation Theory, to apply the contraction theorem. Indeed, for these kinds of applications the pathwise approach leads to a final structure which is more powerful than the one based on probabilistic methods. Therefore, the topics described in this chapter have not only a pedagogical motivation, in order to introduce a little at a time to researchers with a deterministic background the issues related to SPDE, but also to present a methodology, not optimal for studying the well–posedness of stochastic partial differential equations with additive noise, but useful for treating other issues related to such problems.

Fluid dynamic equations with rough inputs (time derivative of non-differentiable stochastic processes) have been studied by many authors. Nowadays, thanks to the interest of many researchers in the singular SPDE, the regularity of the rough inputs is being pushed to the limit. However, a main question, still widely open, is a precise justification of such rough inputs. In Chap. 5 we consider this problem and suggest a research direction related to the complexity of real irregular boundaries and the input they have on the fluid.

Chapter 2
Stochastic Navier–Stokes Equations and State-Dependent Noise

2.1 Introduction

In the previous chapter the force $\partial_t W$ was a single deterministic function (more precisely a distribution). The theory applies to the case when W is a stochastic process, just by treating each realization separately, and caring about measurability a posteriori.

We may thus ask ourselves whether the theory of Chap. 1 can be extended to rough inputs which depend on the solution, in particular of the simple form

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sigma(u)\, \partial_t W$$
$$\operatorname{div} u = 0,$$

where the distributional derivative $\partial_t W$ is multiplied by a function $\sigma(u)$ of the solution. In Sect. 5.5 we motivate this generality by examples.

Application of the ideas of Chap. 1 to this case meets trouble. The problem is not just the fact that the Stokes problem

$$\partial_t z + \nabla q = \nu \Delta z + \sigma(u)\, \partial_t W$$
$$\operatorname{div} z = 0$$

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depends on u: this problem in principle could be solved by an iteration. The problem is that we cannot apply the trick of integration by parts in the mild formula for z:

$$\begin{aligned} z(t) &= e^{tA}z_0 + \int_0^t e^{(t-s)A}\sigma(u(s))\,\partial_s W(s)\,ds \\ &= e^{tA}z_0 + \left[e^{(t-s)A}\sigma(u(s))\,W(s)\right]_{s=0}^{s=t} - \int_0^t \frac{d}{ds}\left(e^{(t-s)A}\sigma(u(s))\right)W(s)\,ds \\ &= e^{tA}z_0 + \sigma(u(t))\,W(t) - e^{tA}\sigma(u(0))\,W(0) \\ &+ \int_0^t Ae^{(t-s)A}\sigma(u(s))\,W(s)\,ds + \int_0^t e^{(t-s)A}\frac{d}{ds}\sigma(u(s))\,W(s)\,ds \end{aligned}$$

and

$$\frac{d}{ds}\sigma(u(s)) = \langle D\sigma(u(s)),\partial_s u(s)\rangle$$

brings again into play the term $\partial_s W(s)$.

One way to escape this problem is to use the theory of rough paths, which however is quite elaborated for our purposes. The most classical way is, when W is related to Brownian motions, to use stochastic calculus. The purpose of this chapter is to illustrate the technique to study the Stochastic Navier–Stokes equations by stochastic calculus.

Remark 2.1 The reader has certainly noticed that we have introduced, in parallel to $\sigma(u)\,\partial_t W$, also a term $F(u)$. This is not for generality, which clearly is not our purpose in these notes. The reason is deep: if we introduce a term $\sigma(u)\,\partial_t W$, we also need to introduce a compensator $F(u)$, otherwise the physics is wrong. This is the Wong–Zakai principle: see Sects. 5.5 and 5.6.

2.1.1 Filtered Probability Space

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. A filtration indexed by $t \geq 0$ is a family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras such that $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}$ for every $t_1 \leq t_2$. We call $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\right)$ a filtered probability space, and we abbreviate it to $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. A stochastic process $(X_t)_{t\geq 0}$ on $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$, taking values in a measurable space, is adapted if X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$. It is progressively measurable if the map $(s,\omega) \mapsto X_s(\omega)$ is measurable on $([0,t]\times\Omega,\mathcal{B}(0,t)\otimes\mathcal{F}_t)$ for every $t \geq 0$ ($\mathcal{B}(0,t)$ being the Borel σ-algebra on $[0,t]$). When the target space is metric with the Borel σ-algebra, and the process is continuous, the concepts of adapted and progressively measurable are equivalent. When we deal with processes that, with respect to the time variable, are equivalence classes (with respect to zero sets for the Lebesgue measure on the time interval), like $L^2(0, T; V)$, we cannot use the concept of an adapted process since X_t (given t) is not well-defined. In this case we always use the concept of progressively measurable: for every t, the restriction on $[0, t]$ is a well-defined equivalence class and the definition applies to it.

Denote by $L^2_{\mathcal{F}_t}(\Omega, H)$ the space of random variables (in fact equivalence classes) $X : \Omega \to H$ that are $\mathcal{F}_t$-measurable and square integrable. We denote by $C_{\mathcal{F}}([0, T]; H)$ the space of continuous adapted processes $(X_t)_{t\in[0,T]}$ with values in H such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^2\right] < \infty$$

and by $L^2_{\mathcal{F}}(0, T; V)$ the space of progressively measurable processes $(X_t)_{t\in[0,T]}$ with values in V such that

$$\mathbb{E}\left[\int_0^T \|X_t\|_V^2 dt\right] < \infty.$$

Of course we may use similar notations also with different spaces in place of H and V; this is just the most common case in the sequel.

A (real-valued) Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a continuous adapted process $(W_t)_{t\geq 0}$ such that $\mathbb{P}(W_t = 0) = 1$, $W_t - W_s$ is independent of $\mathcal{F}_s$ for every $t \geq s \geq 0$, and $W_t - W_s$ is a centered Gaussian random variable with variance $t - s$ (we write $W_t - W_s \sim N(0, t-s)$). With probability one, paths are not only continuous but also locally Hölder continuous with any Hölder exponent $\alpha < \frac{1}{2}$.

A noise often used in these notes is

$$W(t, x) := \sum_{k\in K} \sqrt{\lambda_k}\sigma_k(x)\, W_t^k, \tag{2.1}$$

where K is a finite set, $\sigma_k \in D(A)$, $\left(W_t^k\right)_{t\geq 0}$ are independent Brownian motions on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. With probability one, the path $t \mapsto W(t, \cdot)$ is of class $C([0, T]; D(A))$ (also $C^\alpha([0, T]; D(A))$ for every $\alpha < \frac{1}{2}$).

The machinery introduced here and below is strongly based on the general theory on Stochastic Partial Differential Equations (SPDEs), of which wide and fundamental accounts can be found for instance in the paper of Bensoussan and Temam [25], the theses of E. Pardoux [224] and M. Viot [255], the work of Krylov and Rozovski [191], in the books of Vishik-Fursikov [256], Metivier [215], Da Prato and Zabczyk [90], [91], [92] and Prevôt and Roeckner [226], among others.

2.2 Additive Noise Under the View of Stochastic Calculus

First, let us elaborate the result of Chap. 1 under the view of stochastic calculus. Consider the Itô-type equation, in $d = 2$,

$$du + (u \cdot \nabla u + \nabla p)\, dt = \nu \Delta u dt + \sum_{k \in K} \sqrt{\lambda_k} \sigma_k dW_t^k \tag{2.2}$$

$$\operatorname{div} u = 0$$

with

$$u|_{\partial D} = 0$$

$$u(0) = u_0.$$

Definition 2.2 Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and the noise $W(t, x)$ as in (2.1), given $u_0 : \Omega \to H$, $\mathcal{F}_0$-measurable, we say that a process u is a solution of equation (2.2), if its paths are of class

$$u \in C([0, T]; H) \cap L^2(0, T; V)$$

with probability one, it is adapted as a process in H, progressively measurable in V, and

$$\langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds$$
$$= \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle\, ds + \sum_{k \in K} \sqrt{\lambda_k} \langle \sigma_k, \phi \rangle W_t^k$$

for every $\phi \in D(A)$.

Theorem 2.3 *There exists a unique solution.*

Proof Given two solutions, with probability one their paths are two solutions in the sense of the theorem of the previous chapter, hence they coincide. Path by path the existence of $u(\omega)$ is given by that theorem; since W is measurable, also u is measurable. But the measurability result can be applied on any subinterval $[0, t]$, the process u being always the same (namely the restriction to $[0, t]$ of the process on $[0, T]$), hence we have progressive measurability, which gives also adaptedness in H due to continuity. □

We want now to apply the Itô formula to compute

$$d\|u(t)\|_H^2.$$

Let us recall, for comparison, that when X_t is a process in $\mathbb{R}^d$ satisfying the equation

$$dX_t^i = b_t^i dt + \sum_{k\in K} \sigma_t^{ik} dW_t^k$$

and f is a function of class $C^2\left(\mathbb{R}^d\right)$, then

$$df\left(X_t\right) = \sum_{i=1}^{d} \partial_i f\left(X_t\right) dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k\in K} \partial_i \partial_j f\left(X_t\right) \sigma_t^{ik}\sigma_t^{jk} dt,$$

where we have to replace dX_t^i by the equation. Rigorously, all these identities have to be interpreted in integral form and the stochastic processes $X_t^i, b_t^i, \sigma_t^{ik}$ are assumed progressively measurable. In order to apply these facts we need a progressively measurable process (and this is provided by the previous theorem) and a finite-dimensional reduction.

Theorem 2.4 *If* $\mathbb{E}\left[\|u_0\|_H^2\right] < \infty$ *then*

$$u \in C_{\mathcal{F}}\left([0,T]; H\right) \cap L^2_{\mathcal{F}}\left(0,T; V\right)$$

and

$$\mathbb{E}\left[\|u(t)\|_H^2\right] + 2\nu \int_0^t \mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right] ds = \mathbb{E}\left[\|u_0\|_H^2\right] + t\sum_{k\in K} \lambda_k \|\sigma_k\|_H^2$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_H^2\right] \le \mathbb{E}\left[\|u_0\|_H^2\right] + T\sum_{k\in K} \lambda_k \|\sigma_k\|_H^2 + C\sqrt{\sum_{k\in K} \lambda_k \mathbb{E}\left[\int_0^T \langle u(s), \sigma_k\rangle^2 ds\right]}.$$

Proof Taking a complete orthonormal system in H, (e_i), made of eigenvectors of A, with eigenvalues $(-\lambda_i)$, called H_n the finite-dimensional space generated by $e_1, .., e_n$ and π_n the projection onto H_n, called $u_n(t) = \pi_n u(t)$, called finally

$$b_n(u(s)) := \sum_{i=1}^{n} b(u(s), u(s), e_i) e_i$$

we have (from the weak formulation applied to each e_i)

$$u_n(t) + \int_0^t b_n(u(s))\, ds = \pi_n u_0 + \int_0^t A u_n(s)\, ds + \pi_n W(t).$$

Taken the function $f_n(x) = \sum_{i=1}^{n} \langle x, e_i \rangle^2$, which has the properties $\partial_i f_n(x) = 2\langle x, e_i \rangle$, $\partial_i \partial_j f_n(x) = 2\delta_{ij}$, using the fact that, with $\sigma_t^{ik} = \sqrt{\lambda_k} \langle \sigma_k, e_i \rangle$, one has $\sum_{i=1}^{\infty} \left(\sigma_t^{ik}\right)^2 = \lambda_k \|\sigma_k\|_H^2$, the classical Itô formula gives us

$$\begin{aligned} d\|u_n(t)\|_H^2 &= 2\langle u_n(t), du_n(t) \rangle + \sum_{k\in K} \lambda_k \|\pi_n \sigma_k\|_H^2 dt \\ &= -2\nu \|\nabla u_n(t)\|_{L^2}^2 dt + \sum_{k\in K} \lambda_k \|\pi_n \sigma_k\|_H^2 dt \\ &\quad + 2\sum_{k\in K} \sqrt{\lambda_k} \langle u_n(t), \pi_n \sigma_k \rangle dW_t^k + b(u(s), u(s), u_n(s))\, dt, \end{aligned}$$

where we have used

$$\langle u_n(s), b_n(u(s)) \rangle = b(u(s), u(s), u_n(s)).$$

This identity has to be interpreted in integral form. Using the convergence properties of π_n and the regularity of u, it is not difficult to pass to the limit and obtain

$$\begin{aligned} \|u(t)\|_H^2 + 2\nu \int_0^t \|\nabla u(s)\|_{L^2}^2 ds &= \|u_0\|_H^2 + t\sum_{k\in K} \lambda_k \|\sigma_k\|_H^2 \\ &\quad + 2\sum_{k\in K} \sqrt{\lambda_k} \int_0^t \langle u(s), \sigma_k \rangle dW_s^k, \end{aligned} \tag{2.3}$$

where the last term is an Itô-integral. In order to take expected values we have to use a localization argument that we explain here, namely we omit the repetition below when it is used several times. For sake of simplicity of notation assume that u is a solution defined for all $t \geq 0$ (we can do this, T is arbitrary). For every $R > 0$, let τ_R be the stopping time defined as

$$\tau_R = \inf\{t > 0 : \|u(t)\|_H > R\}$$

or equal to $+\infty$ if the set is empty. Compute the previous identity at time $s \wedge \tau_R$ (it helps the fact that the process u is continuous in H):

$$\begin{aligned} \|u(s\wedge\tau_R)\|_H^2 + 2\nu \int_0^s 1_{r\leq\tau_R} \|\nabla u(r)\|_{L^2}^2 dr &= \|u_0\|_H^2 + (s\wedge\tau_R)\sum_{k\in K} \lambda_k \|\sigma_k\|_H^2 \\ &\quad + 2\sum_{k\in K} \sqrt{\lambda_k} \int_0^s 1_{r\leq\tau_R} \langle u(s), \sigma_k \rangle dW_r^k. \end{aligned}$$

Now $\mathbb{E}\left[\int_0^T 1_{r\le\tau_R}\langle u(r),\sigma_k\rangle^2\,dr\right]<\infty$ hence the Itô integrals of this identity are true martingales and we can apply Doob's inequality. Therefore

$$\mathbb{E}\left[\sup_{s\in[0,t]}\|u(s\wedge\tau_R)\|_H^2\right]\le\mathbb{E}\left[\|u_0\|_H^2\right]+T\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2$$
$$+C\sqrt{\sum_{k\in K}\lambda_k\mathbb{E}\int_0^t\langle u(r),\sigma_k\rangle^2\,1_{r\le\tau_R}dr}.$$

Now we observe that $\mathbb{E}\left[\sup_{s\in[0,t]}\|u(s\wedge\tau_R)\|_H^2\right]=\mathbb{E}\left[\sup_{s\in[0,t]}\|u(s)\|_H^2 1_{s\le\tau_R}\right]$. Thus, exploiting Young's inequality, the expression above can be rewritten as

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\|u(s)\|_H^2 1_{s\le\tau_R}\right]&\le\mathbb{E}\left[\|u_0\|_H^2\right]+T\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2\\&\quad+C\sqrt{\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2}\sqrt{\int_0^t\mathbb{E}\left[\|u(r)\|_H^2 1_{r\le\tau_R}\right]\,dr}\\&\le\mathbb{E}\left[\|u_0\|_H^2\right]+2T\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2\\&\quad+C\int_0^t\mathbb{E}\left[\sup_{s\in[0,r]}\|u(s)\|_H^2 1_{s\le\tau_R}\right]\,dr.\end{aligned}$$

By the Gronwall lemma it follows that

$$\mathbb{E}\left[\sup_{s\in[0,T]}\|u(s)\|_H^2 1_{s\le\tau_R}\right]\le C,$$

where C is a constant independent of R. Letting $R\to+\infty$, by the monotone convergence theorem we get

$$\mathbb{E}\left[\sup_{s\in[0,T]}\|u(s)\|_H^2\right]\le C.$$

This means, in particular, that $u\in C_{\mathcal{F}}([0,T];H)$ and $\mathbb{E}\left[\int_0^T\langle u(r),\sigma_k\rangle^2\,dr\right]<\infty$. Therefore $\sum_{k\in K}\sqrt{\lambda_k}\int_0^t\langle u(s),\sigma_k\rangle\,dW_s^k$ is a true martingale and its expected value is equal to zero. Starting again from relation (2.3), thanks to previous computations, the right-hand side and $\|u(t)\|_H^2$ have finite expected value, hence the same is true for the other term on the left-hand side. We get, in particular, the energy relation

$$\mathbb{E}\left[\|u(t)\|_H^2\right] + 2\nu\mathbb{E}\left[\int_0^t \|\nabla u(s)\|_{L^2}^2 ds\right] = \mathbb{E}\left[\|u_0\|_H^2\right] + t\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2.$$

From this result, which is already part of the thesis, we deduce $u \in L^2_{\mathcal{F}}(0, T; V)$. The last energy relation can be obtained starting again from relation (2.3) and exploiting Doob's inequality similarly to what we have done before for proving that $\mathbb{E}\left[\sup_{s\in[0,T]}\|u(s)\|_H^2\right] \leq C$. We omit the easy details. □

2.2.1 Consequences

The message we get from this theorem is manifold:

- The solution has integrability properties in ω reflecting analogous properties assumed on the data.
- In the modeling of emergence of vortices developed in the previous section we have made a mistake: creating vortices from nothing we introduce energy into the system. Therefore we have to include an extra dissipation mechanism. There is a loss of energy due to the impact of the flow with the obstacle (which, let us remember, is not included into the boundary conditions); part of this energy is given back in the form of emerging vortices. We do not have a sufficiently good solution to this mistake, which then we leave as an open problem. A possible proposal is adding a friction term $-\lambda(x)u$

$$du + (u\cdot\nabla u + \nabla p)\,dt = (\nu\Delta u - \lambda(x)u)\,dt + \sum_{k\in K}\sqrt{\lambda_k}\sigma_k dW_t^k$$

  with a friction coefficient possibly depending on x and localized near the boundary: in this way the physical idea is that energy of large scales is subtracted near the boundary and re-injected through the vortices σ_k. The energy balance is now

$$\mathbb{E}\left[\|u(t)\|_H^2\right] + 2\nu\int_0^t \mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right] ds + 2\mathbb{E}\left[\int_0^t\int_D \lambda(x)\,|u(s,x)|^2 dx ds\right]$$
$$= \mathbb{E}\left[\|u_0\|_H^2\right] + t\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2.$$

But we should be able to choose $\lambda(x)$ in such a way that

$$2\mathbb{E}\left[\int_0^t \int_D \lambda(x)\,|u(s,x)|^2 dx ds\right] \sim t \sum_{k\in K} \lambda_k \|\sigma_k\|_H^2.$$

We do not know how to reach this target.

- Assume $u(t)$ is a statistically stationary solution; this implies that $\mathbb{E}\left[\|u(t)\|_H^2\right] = \mathbb{E}\left[\|u_0\|_H^2\right]$ and $\mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right]$ is independent of s, which then we denote by $\mathbb{E}\left[\|\nabla u\|_{L^2}^2\right]$. Then, stressing the dependence of u on ν,

$$\epsilon := \nu\mathbb{E}\left[\|\nabla u_\nu\|_{L^2}^2\right] = \frac{1}{2}\sum_{k\in K} \lambda_k \|\sigma_k\|_H^2.$$

  The dissipation ϵ of energy due to viscosity remains constant in the inviscid limit $\epsilon \to 0$ (it is a statement of K41 theory), if the energy injection is constant. Relations like this one may be useful for investigations about turbulence and go in the direction opposite to the famous Kato's criterion, e.g. [22], [26], [42], [120], [184], [195], [203].
- We may use a small variant of the previous result to study state-dependent noise by iterations, see below.

We complete this section by listing several references, which however are just a minority of the existing ones (and excluding those already quoted elsewhere in the notes, especially in the Preface). The stochastic Navier Stokes equations with additive noise, or more generally with state-dependent noise (the case of transport noise is a particular one treated in the next chapter), has been a sort of paradigm of SPDEs and thus it has been investigated by many authors. To some extent it is possible to make a classification (but often a single work approaches several topics):

- existence, uniqueness and other foundational properties, including Markov selections, Kolmogorov equations and the difficult case of space-time white noise: [4], [14], [24], [33], [36], [47], [57], [87], [93], [106], [107], [119], [144], [158], [165], [208], [214], [229], [232], [237], [246], [261], [263], [262];
- invariant measures, stationary solutions and ergodicity: [20], [32], [48], [58], [102], [103], [129], [136], [156], [161], [162], [163], [164], [169], [170], [212], [219], [222], [231], [234];
- special properties, including large deviations, inviscid limits, existence of densities, numerical methods and optimal control: [9], [23], [30], [159], [160], [155], [157], [31], [34], [45], [55], [54], [56], [66], [73], [96], [105], [108], [154], [173], [174], [203], [211], [221], [228], [233], [243], [258].

2.3 2D Stochastic Navier–Stokes Equations

Consider now the equations

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \sigma_k(u)\, \partial_t W_t^k \tag{2.4}$$

$$\operatorname{div} u = 0$$

with

$$u|_{\partial D} = 0$$
$$u(0) = u_0.$$

Assume

$$F \in Lip(H, H)$$
$$\sigma_k \in Lip(H, H) \cap C(H, D(A)), \text{ bounded in } H, \qquad k \in K.$$

With some additional elements of stochastic analysis (Itô formula for $\|u(t)\|_H^p$ and Burkholder–Davis–Gundy inequality) one can drop the assumption that σ_k are bounded, so it is made here only for simplicity of exposition. The assumption $C(H, D(A))$ is also made just for simplicity, but it is clear from the estimates below that it is absolutely unessential.

Definition 2.5 Given $u_0 \in H$ and $f \in L^2(0, T; V')$, we say that

$$u \in C_{\mathcal{F}}([0, T]; H) \cap L^2_{\mathcal{F}}(0, T; V)$$

is a weak solution of Eq. (2.4) if

$$\begin{aligned} &\langle u(t), \phi\rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ &= \langle u_0, \phi\rangle + \int_0^t \langle u(s), A\phi\rangle\, ds + \int_0^t \langle f(s), \phi\rangle\, ds \\ &+ \int_0^t \langle F(u(s)), \phi\rangle\, ds + \sum_{k\in K} \int_0^t \langle \sigma_k(u(s)), \phi\rangle\, dW_s^k \end{aligned}$$

for every $\phi \in D(A)$.

Theorem 2.6 *For every $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ and $f \in L^2_{\mathcal{F}}(0, T; V')$ such that*

$$\mathbb{E}\left[\|u_0\|_H^r\right] + \mathbb{E}\left[\int_0^T \|f(s)\|_{V'}^r\, ds\right] < +\infty$$

for some $r > 4$, *there exists a unique weak solution of Eq. (2.4). It satisfies*

$$\mathbb{E}\left[\|u(t)\|_H^2\right] + 2\nu\mathbb{E}\left[\int_0^t \|\nabla u(s)\|_H^2 ds\right]$$
$$= \mathbb{E}\left[\|u_0\|_H^2\right] + 2\mathbb{E}\left[\int_0^t \langle u(s), f(s) + F(u(s))\rangle ds\right]$$
$$+ \sum_{k\in K} \mathbb{E}\left[\int_0^t \|\sigma_k(u(s))\|_H^2 ds\right].$$

Remark 2.7 In the language of stochastic differential equations [183], the notion of uniqueness used here corresponds to the so-called pathwise uniqueness; the notion of existence to the so-called strong existence. We use the name weak solutions not in the probabilistic sense of stochastic equations but in the analytical sense, being the formulation made against test functions.

Remark 2.8 If we take a deterministic initial condition u_0 and forcing term f, then Theorem 2.6 holds under the more natural assumptions $u_0 \in H$, $f \in L^2(0, T; V')$. Indeed, uniqueness holds under the natural assumption $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ and $f \in L^2_{\mathcal{F}}(0, T; V')$. The additional integrability assumptions are required to prove some additional bounds in order to get existence, see Step 3 in Sect. 2.4.4 below. In the case of deterministic data such estimates hold without requiring this kind of integrability, see Remark 2.25 below. We decide to consider stochastic data in the statement of Theorem 2.6 for the purpose of giving the reader a complete understanding of the main tools and difficulties in order to prove the well-posedness of a nonlinear stochastic system with state-dependent noise.

The previous theorem is, today, a sort of paradigm of the compactness method and its proof has served as a basis for several generalizations, not only in SPDE theory but also, for instance, in the framework of interacting particle systems and their macroscopic limit (see for instance [132] and references therein); this is why we give several details below, although classical, and we even extend pedagogically the discussion. We have decided also to present from scratch the technical but quite classical argument related to the realization on an auxiliary space based on the Skorohod theorem. Although great, we have the feeling that an alternative should be found, similarly to what the Gyongy–Krylov theorem does with respect to proving first weak existence and then applying the Yamada–Watanabe theorem. When the limit equation is deterministic, or in the case of additive noise, an alternative proof to Skorohod embedding exists and is described below for the 3D Navier–Stokes equations with additive noise. However, in general, we have not found a similar solution.

The proof of the previous theorem owes a lot to several investigations. Among many others, let us mention only [255], [256], [215], [118], [238], [61], [46].

2.3.1 Proof of Uniqueness

Let $u^{(i)}$ be two solutions. Then $w = u^{(1)} - u^{(2)}$ satisfies

$$\begin{aligned}
&\langle w(t), \phi \rangle - \int_0^t \left(b\left(u^{(1)}, \phi, u^{(1)}\right) - b\left(u^{(2)}, \phi, u^{(2)}\right)\right)(s)\, ds \\
&= \int_0^t \langle w(s), A\phi \rangle\, ds + \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), \phi \right\rangle ds \\
&+ \sum_{k \in K} \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), \phi \right\rangle dW_s^k
\end{aligned}$$

and since

$$\begin{aligned}
&b\left(u^{(1)}, \phi, u^{(1)}\right) - b\left(u^{(2)}, \phi, u^{(2)}\right) - b(w, \phi, w) \\
&= b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right)
\end{aligned}$$

we get

$$\begin{aligned}
&\langle w(t), \phi \rangle - \int_0^t (b(w(s), \phi, w(s)))\, ds \\
&= \int_0^t \langle w(s), A\phi \rangle\, ds + \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), \phi \right\rangle ds \\
&+ \sum_{k} \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), \phi \right\rangle dW_s^k \\
&+ \int_0^t \left(b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right)\right)(s)\, ds.
\end{aligned}$$

We need the Itô formula to continue; it can be proved similarly to Theorem 2.4. It gives us

$$\begin{aligned}
\|w(t)\|_H^2 + 2\nu \int_0^t \|\nabla w(s)\|_{L^2}^2 ds &= 2 \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), w(s) \right\rangle ds \\
&+ 2 \int_0^t \left(b\left(u^{(2)}, w, w\right) + b\left(w, w, u^{(2)}\right)\right)(s)\, ds \\
&+ \sum_{k \in K} \int_0^t \|\sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right)\|_H^2 ds \\
&+ M_t
\end{aligned}$$

where

$$M_t := \sum_k \int_0^t \left\langle \sigma_k \left(u^{(1)}(s)\right) - \sigma_k \left(u^{(2)}(s)\right), w(s)\right\rangle dW_s^k.$$

Therefore, if L_F and L_k are the Lipschitz constants of F and σ_k respectively, using estimates of Chap. 1 we get

$$\begin{aligned}\|w(t)\|_H^2 + \nu \int_0^t \|\nabla w(s)\|_{L^2}^2 ds \leq \left(2L_F + \sum_{k\in K} L_k^2\right) \int_0^t \|w(s)\|_H^2 ds \\ + C \int_0^t \|w(s)\|_H^2 \left(1 + \|u^{(2)}(s)\|_{\mathbb{L}^4}^2\right) ds \\ + M_t.\end{aligned}$$

We need now a very interesting trick that we have learned from Bjorn Schmalfuss [238]: introducing

$$\rho_t = \exp\left(-C \int_0^t \left(1 + \|u^{(2)}(s)\|_{\mathbb{L}^4}^2\right) ds\right)$$

we have, from the Itô formula again,

$$\|w(t)\|_H^2 \rho_t + \nu \int_0^t \|\nabla w(s)\|_{L^2}^2 \rho_s ds \leq \left(2L_F + \sum_{k\in K} L_k^2\right) \int_0^t \|w(s)\|_H^2 \rho_s ds + \widetilde{M}_t$$

where

$$\widetilde{M}_t := \sum_{k\in K} \int_0^t \left\langle \sigma_k \left(u^{(1)}(s)\right) - \sigma_k \left(u^{(2)}(s)\right), w(s)\right\rangle \rho_s dW_s^k.$$

Omitting the necessary localization argument entirely similar to the one used in Theorem 2.4, we get

$$\begin{aligned}&\mathbb{E}\left[\|w(t)\|_H^2 \rho_t\right] + \nu \mathbb{E}\left[\int_0^t \|\nabla w(s)\|_{L^2}^2 \rho_s ds\right] \\ &\leq \left(2L_F + \sum_{k\in K} L_k^2\right) \int_0^t \mathbb{E}\left[\|w(s)\|_H^2 \rho_s\right] ds,\end{aligned}$$

which leads to $\mathbb{E}\left[\|w(t)\|_H^2 \rho_t\right] = 0$ by the Gronwall lemma. But, thanks to the regularity of $u^{(2)}$, $\mathbb{P}(\rho_t > 0) = 1$. Hence $\mathbb{P}(w(t) = 0) = 1$. Since this is true for all t, the processes $u^{(1)}$ and $u^{(2)}$ are modifications; but they are continuous, hence they are indistinguishable.

2.4 Proof of Existence

2.4.1 Introduction

Existence for differential equations is a wide subject with many ideas. More or less, all methods consist in the construction of a sequence, based on some approximation or iteration method which allows us to define the sequence by means of easier equations than the one object of investigation. Then one has to prove convergence in a topology which allows one to pass to the limit in the approximate equations. Linear terms pass to the limit under very weak convergences, hence the demanding parts for the limit step are the nonlinear terms. When they have suitable monotonicity properties, again weak convergence is sufficient, but the Navier–Stokes nonlinearity does not have such properties. Strong convergence in a topology like H is needed. Weak convergence does not suffice to take the limit in a quadratic expression; the weak limit of the square is not the square of the weak limit, in general.

We have insisted on this classification of ideas because the existence of weakly convergent subsequences of an approximating scheme is an excellent property also in the stochastic case, it applies for instance to spaces like $L^2(\Omega, B)$ with a Banach space B. But the existence of *strongly convergent subsequences* of an approximating scheme is very demanding, in the stochastic case. And for the Navier–Stokes equations we are faced with this demanding problem.

Essentially there are two ways to get strong convergence: one is related to contraction principle arguments and consists in the proof of the Cauchy property of the sequence, in the strong topology, usually in expected value. This kind of argument is not easy to be implemented for the Navier–Stokes equations. In the deterministic setting we have seen an example of this technique in Chap. 1: for the auxiliary Navier–Stokes equations we have constructed a sequence (v_n) and proved it was Cauchy. In the stochastic case, performing similar proofs is very difficult because of the problem of *closure of moments*: we have to take expected values but the nonlinearity increases the order of the moment. Inspection into the proof of Chap. 1 reveals we have used uniform bounds on the iterates to close a certain inequality in the proof of the Cauchy property; in the deterministic case such bounds are deterministic; in the stochastic case they are in expected value and thus not easily applicable.

The alternative strategy to have strong convergence of subsequences is by compactness theorems. However, here there is a structural problem: compactness in spaces like $L^2(\Omega, B)$ is essentially impossible to prove (except for criteria based on Malliavin calculus, which however did not prove to be competitive, until now). Thus one goes to compactness of the laws, because compactness in spaces of measures is very well characterized.

But then the problem becomes that we have only subsequences of laws, which converge in strong topologies. Namely, it is not strong convergence of the original stochastic processes, only of their laws. How can we identify a limit stochastic process and pass to the limit in the equations?

Here there are several strategies, each one with advantages depending on a certain feature of the problem; or, if not advantages, it is the only one we can use.

- When we can prove the so-called pathwise uniqueness, as above in the 2D case, there is a brilliant criterion of Gyongy and Krylov which proves the convergence in probability of the approximating sequence of stochastic processes, hence upgrading the pure convergence in law. We shall explain this below.
- An alternative to this method, when pathwise uniqueness is known, is proving weak convergence of the laws; constructing a solution on an auxiliary probability space and then using a theorem of Yamada–Watanabe type (which requires pathwise uniqueness) to prove that a solution on the original probability space exists. This strategy looks longer than the previous one, hence we prefer to describe the Gyongy–Krylov approach.
- When pathwise uniqueness is not known or it is false, there is no way to upgrade the weak convergence of laws to some kind of stronger convergence of the processes. In this case the Skorohod representation theorem allows one to reformulate the approximating sequence on a new, auxiliary probability space, where it converges also almost surely, not only in law. Then one can pass to the limit. But the limit process lives in an auxiliary probability space. This is the same strategy used in the previous item, but not followed by a Yamada–Watanabe step. Hence the final result is just existence on an auxiliary space.
- For special noise, like the additive one, when pathwise uniqueness is not known, there is a trick to pass to the limit in the equation using just the weak convergence of the laws, without performing the Skorohod representation theorem step. The limit law is a solution of the equation, in a suitable sense. We shall describe this procedure below. It applies for instance to the 3D Navier–Stokes equations with additive noise.

One may add several comments to the previous list, related for instance to the concept of martingale solutions, but we limit ourselves to the previous discussion and show some of the computations for the first and the last item.

2.4.2 Gyongy–Krylov Convergence Criterion

The following result is taken from [168]. We give the details for completeness.

If (E, d) is a metric space we denote by $\left(E^2, d^2\right)$ the product space with the metric $d^2\left((x, y), \left(x', y'\right)\right) = d(x, y) + d\left(x', y'\right)$; we understand that on every one of these metric spaces the σ-field is the Borel one; and we denote by D the diagonal:

$$D = \left\{(x, x) \in E^2; x \in E\right\}.$$

Lemma 2.9 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a complete separable metric space (E, d). Assume that, for* *every pair of subsequences* $((n_1(k), n_2(k)))_{k\in\mathbb{N}}$, *with* $n_1(k) \geq n_2(k)$ *for every* $k \in \mathbb{N}$, *there is a subsequence* $(k(h))_{h\in\mathbb{N}}$ *such that the random variables* $\left(X_{n_1(k(h))}, X_{n_2(k(h))}\right)_{h\in\mathbb{N}}$ *from* $(\Omega, \mathcal{F}, \mathbb{P})$ *to* $\left(E^2, d^2\right)$ *converge in law to a measure* μ *on* E^2 *such that* $\mu(D) = 1$. *Then there exists a random variable* X *from* $(\Omega, \mathcal{F}, \mathbb{P})$ *to* (E, d) *such that* X_n *converges to* X *in probability.*

Proof It is sufficient to prove that $(X_n)_{n\in\mathbb{N}}$ is Cauchy in probability: given $\epsilon > 0$ we have to find n_0 such that for all $n, m > n_0$ one has

$$\mathbb{P}(d(X_n, X_m) \geq \epsilon) < \epsilon.$$

Let us prove this by contradiction: we assume that there exists $\epsilon_0 > 0$ such that for every k there are $n_1(k) \geq n_2(k) > k$ such that

$$\mathbb{P}\left(d\left(X_{n_1(k)}, X_{n_2(k)}\right) \geq \epsilon_0\right) \geq \epsilon_0.$$

We may perfect the construction in order to have that $n_1(k), n_2(k)$ are strictly increasing, hence they are subsequences. But by assumption there exists a subsequence $k(h)$ such that $\left(X_{n_1(k(h))}, X_{n_2(k(h))}\right)$ converges in law to μ, hence its probability of taking values in a closed set is upper semicontinuous:

$$\mu((x, y) : d(x, y) \geq \epsilon_0) \geq \limsup \mathbb{P}\left(d\left(X_{n_1(k(h))}, X_{n_2(k(h))}\right) \geq \epsilon_0\right) \geq \epsilon_0.$$

This inequality is incompatible with $\mu(D^c) = 0$, hence we have reached a contradiction. □

2.4.3 *Compactness Criteria*

Deterministic Ascoli–Arzelà Theorem

Given two Banach spaces $X \subset Y$, we say that the embedding $X \subset Y$ is compact if bounded sets of X are relatively compact in Y.

A version of the Ascoli–Arzelà theorem claims that, given two Banach spaces $X \overset{\text{compact}}{\subset} Y$, a family $F \subset C([0, T]; Y)$ with the following two properties is relatively compact in $C([0, T]; Y)$:

(i) $\{f(t); f \in F\}$ is bounded in X;
(ii) F is uniformly equicontinuous in $C([0, T]; Y)$; namely, for every $\epsilon > 0$ there is $\delta > 0$ such that $\|f(t) - f(s)\|_Y \leq \epsilon$ for every $f \in F$ and $t, s \in [0, T]$ such that $|t - s| \leq \delta$.

In particular:

Proposition 2.10 *If $p > 1$ and $X \overset{compact}{\subset} Y$, then*

$$W^{1,p}(0,T;X) \overset{compact}{\subset} C([0,T];Y).$$

Indeed, if $F \subset W^{1,p}(0,T;X)$ is bounded, and $t \in \left[\frac{T}{2},T\right]$ (similarly for $t \in \left[0,\frac{T}{2}\right]$)

$$f(t) - f(s) = \int_s^t f'(r)\,dr,$$

$$f(t) = \frac{2}{T}\int_0^{T/2} f(s)\,ds + \frac{2}{T}\int_0^{T/2}\int_s^t f'(r)\,drds,$$

$$\begin{aligned}\|f(t)\|_X &\le \frac{2}{T}\int_0^{T/2}\|f(s)\|_X ds + \frac{2}{T}\int_0^{T/2}\int_s^t \|f'(r)\|_X drds\\ &\le \frac{2}{T}\|f\|_{L^1(0,T;X)} + \|f'\|_{L^1(0,T;X)} \le C\end{aligned}$$

and

$$\|f(t) - f(s)\|_X \le \int_s^t \|f'(r)\|_X dr \le \|f'\|_{L^p(0,T;X)}|t-s|^{1/q} \le C|t-s|^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and the constant C is independent of $f \in F$. So F satisfies the assumptions of Ascoli–Arzelà theorem.

Deterministic Aubin–Lions Type Theorems

Theorem 2.11 *Let $X \subset Y \subset Z$ be three Banach spaces, with continuous dense embeddings. Assume that the embedding $X \subset Y$ is compact. Let $p \in [1,\infty)$ be given. Then the embedding*

$$L^p(0,T;X) \cap W^{1,1}(0,T;Z) \subset L^p(0,T;Y)$$

is compact.

Remark 2.12 The previous theorem, when applied to functions spaces $X \subset Y \subset Z$, treats the problem of compactness of functions of space-time. Heuristically, one needs a condition of compactness for the space variable and one for the time variable and, a priori, one could expect the need for some sort of joint compactness in the two variables. By the Ascoli–Arzelà theorem, the space or real-valued functions $W^{1,2}(0,T;\mathbb{R})$ is compactly embedded into $L^2(0,T;\mathbb{R})$. The remarkable feature of the previous theorem is that the compactness in the time variable does not require a simultaneous compactness in the space variable: the space Z can be much larger than Y. Said differently, the two compactness requirements, in space and time, are quite decoupled.

Remark 2.13 The consequence in examples is that the only key assumption turns out to be $L^p(0,T;X)$, the other being a technical consequence based on the differential equation.

Remark 2.14 Assume $p>1$ and also assume the bound is in $W^{1,r}(0,T;Z)$ with $r>1$. The previous result means that, if we have a sequence of functions (u_n) (usually solutions of an approximate equation) such that

$$\int_0^T \|u_n(t)\|_X^p dt + \int_0^T \|\frac{du_n(t)}{dt}\|_Z^r dt \le C$$

then there exists a subsequence (u_{n_k}) and a function $u \in L^p(0,T;Y)$ such that

$$\lim_{n\to\infty}\int_0^T \|u_{n_k}(t)-u(t)\|_Y^p dt = 0.$$

Moreover, $u \in L^p(0,T;X)\cap W^{1,r}(0,T;Z)$ and (u_{n_k}) can be chosen so that it converges weakly to u in $L^p(0,T;X)$ and in $W^{1,r}(0,T;Z)$ (it is here that we use $p,r>1$). The weak convergence in these topologies is a consequence of the general theory of reflexive Banach spaces; that it can be done for a unique subsequence is easy; that the limit in the strong topology of $L^p(0,T;Y)$ and weak topologies of $L^p(0,T;X)$ and $W^{1,r}(0,T;Z)$ is the same function u requires some arguments that we omit (for instance: weak convergence in $L^p(0,T;X)$ implies weak convergence in $L^p(0,T;Y)$, hence the weak limit in these topologies is the same as the strong limit in $L^p(0,T;Y)$, by uniqueness between weak and strong limit in $L^p(0,T;Y)$). Moreover, in most examples one proves also a bound of the form

$$\sup_{t\in[0,T]}\|u_n(t)\|_Y \le C.$$

By the same arguments, one may have that (u_{n_k}) converges also weak-star to u in $L^\infty(0,T;Y)$. Finally, If $Y \overset{\text{compact}}{\subset} Z$, by Proposition 2.10 we may also add strong convergence of (u_{n_k}) to u in $C([0,T];Z)$.

Essential for the stochastic case is the following generalization (see Simon [240], Corollary 5):

Theorem 2.15 *If* $\alpha r > 1 - \frac{r}{p}$ $(p, r \geq 1)$ *then*

$$L^p(0, T; X) \cap W^{\alpha,r}(0, T; Z) \overset{compact}{\subset} L^p(0, T; Y).$$

Here $\alpha \in (0, 1)$ and $W^{\alpha,r}(0, T; Z)$ is the space of functions $f \in L^r(0, T; Z)$ such that

$$\int_0^T \int_0^T \frac{\|f(t) - f(s)\|_Z^r}{|t - s|^{1+\alpha r}} dsdt < \infty.$$

Recall also that $W^{\alpha,r}(0, T; Z) \subset C([0, T]; Z)$ if $\alpha r > 1$. The reason for asking this generalization is that we do not have true time derivatives in the stochastic case, but we have fractional time regularity.

The property of continuity in time in Y of solutions sometimes follows a posteriori, from the (S)PDE. Alternatively, we may try to prove convergence of the approximating scheme in the uniform topology. Obviously, the Ascoli–Arzelà theorem provides uniform convergence but the assumptions are too difficult to be checked in (S)PDEs like those of fluid mechanics (let us remark, however, that the Ascoli–Arzelà theorem is at the foundation of most proofs of the compactness results illustrated here). To this purpose we may use the following results [240], Corollary 9, [118], Theorem 2.2:

Theorem 2.16 *Assume in addition* $(\theta \in (0, 1))$

$$\|v\|_Y \leq C\|v\|_X^{1-\theta}\|v\|_Z^{\theta} \qquad \theta \in (0, 1)$$

$$\alpha r > 1 \text{ and } p > \frac{1-\theta}{\theta}\frac{r}{\alpha r - 1} \quad (p, r \geq 1).$$

Then

$$L^p(0, T; X) \cap W^{\alpha,r}(0, T; Z) \overset{compact}{\subset} C([0, T]; Y).$$

Theorem 2.17 *If* $\alpha \in (0, 1)$, $p > 1$ *satisfy*

$$\alpha p > 1.$$

Then

$$W^{\alpha,p}(0, T; X) \overset{compact}{\subset} C([0, T]; Y).$$

Stochastic Theory

Consider now a differential equation where the solution depends also on a random parameter,

$$u = u(\omega, t, x).$$

The principle that compactness can be investigated separately in the three arguments, in principle, could still hold. However, the obstacle is that compactness in the random parameter ω is not an easy matter. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is always infinite dimensional in our examples and compactness criteria in $L^p(\Omega)$ are not natural (although something can be done by means of Malliavin calculus, see for instance [89]).

The natural approach is to consider the laws of the random objects and apply compactness arguments to these laws. It is easier due to the following basic theorem. Let (X, d) be a complete metric space and $\mathcal{B}$ the Borel σ-field. Recall we say that a family $\mathcal{G}$ of probability measures on $(X, \mathcal{B})$ is *tight* if for every $\epsilon > 0$ there is a compact set $K \subset X$ such that

$$\mu(K) \geq 1 - \epsilon$$

for all $\mu \in \mathcal{G}$.

Theorem 2.18 (Prohorov) *A family $\mathcal{G}$ of probability measures on $(X, \mathcal{B})$ is tight if and only if it is relatively compact.*

Corollary 2.19 *Assume (u_N) is a sequence of random functions from $(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(0, T; Y)$. Assume $\alpha r > 1 - \frac{r}{p}$ and that for every $\epsilon > 0$ there are $R_1, R_2 > 0$ such that*

$$\mathbb{P}\left(\|u_N\|_{L^p(0,T;X)} \geq R_1\right) \leq \epsilon,$$

$$\mathbb{P}\left(\|u_N\|_{W^{\alpha,r}(0,T;Z)} \geq R_2\right) \leq \epsilon$$

for all $N \in \mathbb{N}$. Then there exists a subsequence $\left(u_{N_k}\right)$ which converges in law, in the strong topology of $L^p(0, T; Y)$, to a random function $\widetilde{u}$ from a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ to $L^p(0, T; Y)$. Moreover, if $p, r > 1$, we may choose $\left(u_{N_k}\right)$ so that $\widetilde{u}$ takes also values in $L^p(0, T; X)$ and $W^{\alpha,r}(0, T; Z)$.

Corollary 2.20 *Assume (u_N) is a sequence of random functions from $(\Omega, \mathcal{F}, \mathbb{P})$ to $C([0, T]; Y)$. Assume $\alpha r > 1$ and that for every $\epsilon > 0$ there are $R_1 > 0$ such that*

$$\mathbb{P}\left(\|u_N\|_{W^{\alpha,r}(0,T;X)} \geq R_1\right) \leq \epsilon$$

for all $N \in \mathbb{N}$. *Then there exists a subsequence* $\left(u_{N_k}\right)$ *which converges in law, in the strong topology of* $C([0,T];Y)$*, to a random function* $\widetilde{u}$ *from a probability space* $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ *to* $C([0,T];Y)$.

Recall that the convergence in law stated in Corollary 2.19 means

$$\lim_{k\to\infty} \mathbb{E}\left[\Phi\left(u_{N_k}\right)\right] = \widetilde{\mathbb{E}}\left[\Phi\left(\widetilde{u}\right)\right]$$

for every bounded continuous function $\Phi : L^p(0,T;Y) \to \mathbb{R}$. Here $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ are the expected values on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ respectively. The convergence in law stated in Corollary 2.20 has an analogous meaning replacing $L^p(0,T;Y)$ with $C([0,T];Y)$.

Remark 2.21 Sufficient conditions for the applicability of Corollary 2.19 are uniform in N estimates of the form

$$\mathbb{E}\left[\|u_N\|_{L^p(0,T;X)}\right] \le C,$$
$$\mathbb{E}\left[\|u_N\|_{W^{\alpha,r}(0,T;Z)}\right] \le C.$$

Indeed, by the Markov inequality,

$$\mathbb{P}\left(\|u_N\|_{L^p(0,T;X)} \ge R_1\right) \le \frac{C}{R_1}$$

and similarly for the second inequality, hence given $\epsilon > 0$ we can find $R_1, R_2 > 0$ with the required properties. Similar sufficient conditions can be stated for Corollary 2.20.

Remark 2.22 The consequence of the peculiar feature of the previous corollaries that the process $\widetilde{u}$ may be defined on a new probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ is the emergence of the concept of a "weak solution in the probabilistic sense". This means that the probability space over which we find a solution is not necessarily prescribed a priori. If we are only interested in statistical properties, this is not bad, but sometimes for special investigation it is very restrictive.

2.4.4 Application to Galerkin Approximations: 2D Case

Estimates and Compactness

Step 1 (Preparation) Let us use the definitions introduced in the proof of Theorem 2.4: (e_i) is a complete orthonormal system in H made of eigenvectors of

A, with eigenvalues $(-\lambda_i)$, H_n and π_n are consequently defined, and we introduce the bilinear operator $B_n : H_n \times H_n \to H_n$ defined as

$$B_n(u, v) = \pi_n P(u \cdot \nabla v)$$

(we omit the verification that $u, v \in H_n$ imply $u \cdot \nabla v \in L^2(D; \mathbb{R}^2)$, so that P is well-defined on $u \cdot \nabla v$). Then we consider the finite-dimensional equation

$$du_n = Au_n dt - B_n(u_n, u_n) dt + f_n + F_n(u_n) + \sum_k \sigma_k^n(u_n) dW_t^k,$$

where $f_n = \pi_n f$, $F_n(u) = \pi_n F(u)$, $\sigma_k^n(u_n) = \pi_n \sigma_k(u_n)$; with initial condition $u_0^n = \pi_n u_0$. It is easy to check that

$$\langle B_n(u_n, u_n), u_n \rangle = 0.$$

Step 2 (Estimates in Square Norms) Therefore, from the Itô formula (in finite dimensions) we get

$$\begin{aligned} \|u_n(t)\|_H^2 + 2\nu \int_0^t \|\nabla u_n(s)\|_{L^2}^2 ds &= 2\int_0^t \langle f_n(s) + F_n(u_n(s)), u_n(s) \rangle ds \\ &\quad + \sum_{k \in K} \int_0^t \|\sigma_k^n(u_n(s))\|_H^2 ds + M_t^n \end{aligned} \tag{2.5}$$

where

$$M_t^n = 2\sum_{k \in K} \int_0^t \left\langle \sigma_k^n(u_n(s)), u_n(s) \right\rangle dW_t^k.$$

After having seen above various proofs, it is a simple exercise to deduce (see also Step 3 below)

$$\mathbb{E}\left[\int_0^T \|u_n(s)\|_V^2 ds\right] \le C, \tag{2.6}$$

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u_n(t)\|_H^2\right] \le C.$$

Then we investigate the $W^{\alpha,r}\left(0,T;V'\right)$ norm of u_n. In a sense, this is the most technical part but the reader will recognize that the key properties are (2.6), the rest of the proof is technicalities. For $s<t$

$$\begin{aligned}\|u_n(t)-u_n(s)\|_{V'} \le& \int_s^t \|Au_n(r)\|_{V'}dr + \int_s^t \|B_n(u_n,u_n)(r)\|_{V'}dr\\ &+\int_s^t \|f_n(r)+F_n(u_n)(r)\|_{V'}dr\\ &+\|\sum_k \int_s^t \sigma_k^n(u_n(r))\,dW_r^k\|_{V'}.\end{aligned}$$

We have

$$\mathbb{E}\left[\int_s^t \|Au_n(r)\|_{V'}dr\right] \le \sqrt{t-s}\left(\mathbb{E}\left[\int_s^t \|Au_n(r)\|_{V'}^2 dr\right]\right)^{1/2} \le C\sqrt{t-s}$$

by (2.6), and similarly,

$$\mathbb{E}\left[\int_s^t \|f_n(r)+F_n(u_n)(r)\|_{V'}dr\right] \le C\sqrt{t-s}.$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[\|\sum_k \int_s^t \sigma_k^n(u_n(r))\,dW_r^k\|_{V'}\right] &\le \left(\mathbb{E}\left[\|\sum_k \int_s^t \sigma_k^n(u_n(r))\,dW_r^k\|_{V'}^2\right]\right)^{1/2}\\ &= \left(\mathbb{E}\left[\sum_k \int_s^t \|\sigma_k^n(u_n(r))\|_{V'}^2 dr\right]\right)^{1/2}\\ &\le C\sqrt{t-s}\end{aligned}$$

because we assume σ_k^n bounded. Finally, from the usual inequalities,

$$\begin{aligned}\int_s^t \|B_n(u_n,u_n)(r)\|_{V'}dr &\le C\int_s^t \|u_n(r)\|_H\|u_n(r)\|_V dr\\ &\le C\sup_{r\in[0,T]}\|u_n(r)\|_H\int_s^t \|u_n(r)\|_V dr,\end{aligned}$$

hence

$$\mathbb{E}\left[\int_s^t \|B_n(u_n,u_n)(r)\|_{V'}dr\right] \le C\mathbb{E}\left[\sup_{r\in[0,T]}\|u_n(r)\|_H^2\right]^{1/2}$$
$$\mathbb{E}\left[\left(\int_s^t \|u_n(r)\|_V dr\right)^2\right]^{1/2}$$
$$\le C\sqrt{t-s}.$$

Putting together all these pieces,

$$\mathbb{E}\left[\|u_n(t)-u_n(s)\|_{V'}\right] \le C\sqrt{t-s},$$

which implies

$$\mathbb{E}\left[\int_0^T\int_0^T \frac{\|u_n(t)-u_n(s)\|_{V'}}{|t-s|^{1+\alpha}}dsdt\right] \le \int_0^T\int_0^T \frac{C}{|t-s|^{\frac{1}{2}+\alpha}}dsdt =: C < \infty$$

if $\alpha \in \left(0,\frac{1}{2}\right)$. The condition $\alpha r > 1-\frac{r}{p}$ of Theorem 2.15 is fulfilled for $1-\frac{1}{p} < \frac{1}{2}$, namely for $p<2$. This result is not so good for the sequel: when passing to the limit in the nonlinear term we have

$$\int_0^t \langle B_n(u_n(s),u_n(s)),\phi\rangle ds = -\int_0^t b(u_n(s),\pi_n\phi,u_n(s))ds$$

so, for $\epsilon>0$, taking $\phi\in D\left((-A)^{1+\epsilon}\right)\subset C_b^1(D)$, it is sufficient to have strong convergence of u_n in $L^2(0,T;H)$, but not in $L^p(0,T;H)$ with $p<2$. Perhaps there are arguments to overcome this difficulty thanks to the uniform in time bound of estimate (2.6), but it is interesting to show how to upgrade the integrability of solutions and thus let us develop this in the next step. Note that we required that the test function $\phi\in D(A)$ in the definition of a weak solution. We can move from test functions in $D((-A)^{1+\epsilon})$ to test functions in $D(A)$ by density of $D((-A)^{1+\epsilon})$ in $D(A)$ and exploiting the regularity of u.

Step 3 (Estimates in L^r) Take $r>2$. Assume

$$\mathbb{E}\left[\|u_0\|_H^r\right]<\infty, \qquad \mathbb{E}\left[\int_0^t \|f(s)\|_{V'}^r ds\right]<\infty.$$

Consider the function

$$f(x)=\|x\|^r$$

for $x \in \mathbb{R}^n$. We have, for $x \neq 0$,

$$\partial_i f(x) = r\|x\|^{r-1}\frac{x_i}{\|x\|} = r\|x\|^{r-2}x_i,$$

$$\begin{aligned}\partial_j\partial_i f(x) &= r x_i \partial_j \|x\|^{r-2} + r\|x\|^{r-2}\delta_{ij}\\ &= r(r-2)\|x\|^{r-4}x_i x_j + r\|x\|^{r-2}\delta_{ij},\end{aligned}$$

and we may include $x = 0$ for $r \geq 4$. Treating rigorously the case $r \in (2,4)$, unnecessary for the following, requires some more details that we omit. Then from the Itô formula we have

$$\begin{aligned}d\|u_n(t)\|_H^r &= r\|u_n(t)\|_H^{r-2}\langle u_n(t), du_n(t)\rangle\\ &+ \frac{1}{2}r(r-2)\sum_{k\in K}\|u_n(t)\|_H^{r-4}\left\langle u_n(t), \sigma_k^n(u_n(t))\right\rangle^2 dt\\ &+ \frac{1}{2}r\|u_n(t)\|_H^{r-2}\sum_{k\in K}\|\sigma_k^n(u_n(t))\|_H^2 dt,\end{aligned}$$

hence

$$\begin{aligned}&d\|u_n(t)\|_H^r + r\nu\|u_n(t)\|_H^{r-2}\|\nabla u_n(t)\|_{L^2}^2\\ &\leq r\|u_n(t)\|_H^{r-2}\langle u_n(t), f_n + F_n(u_n)\rangle dt + dM_t^{n,r}\\ &+ \frac{1}{2}r(r-1)\|u_n(t)\|_H^{r-2}\sum_k\|\sigma_k^n(u_n(t))\|_H^2 dt,\end{aligned}$$

where

$$M_t^{n,r} = r\sum_{k\in K}\int_0^t \|u_n(s)\|_H^{r-2}\left\langle\sigma_k^n(u_n(s)), u_n(s)\right\rangle dW_t^k.$$

From the usual localization argument,

$$\begin{aligned}&\mathbb{E}\left[\|u_n(t)\|_H^r\right] + r\nu\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2}\|\nabla u_n(s)\|_{L^2}^2 ds\right]\\ &\leq C_r\mathbb{E}\left[\int_0^t\left(\|u_n(s)\|_H^r + 1\right)ds\right] + C_r\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2}\|f(s)\|_{V'}^2 ds\right]\\ &+ \nu\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2}\|u_n(s)\|_V^2 ds\right] + \mathbb{E}\left[\|u_0\|_H^r\right].\end{aligned}$$

We need, from $ab \leq c_r \left(a^{\frac{r}{r-2}} + b^{\frac{r}{2}} \right) \left(\frac{r-2}{r} + \frac{2}{r} = 1 \right)$

$$\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2} \|f(s)\|_{V'}^2 ds \right] \leq \mathbb{E}\left[\int_0^t \|u_n(s)\|_H^r ds \right] + \mathbb{E}\left[\int_0^t \|f(s)\|_{V'}^r ds \right]$$

hence the additional assumption on u_0 and f. From the Gronwall lemma,

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|u_n(t)\|_H^r \right] \leq C.$$

Using this preliminary estimate and Burkholder–Davis–Gundy inequality (we omit the details) we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^r \right] \leq C. \tag{2.7}$$

Raising relation (2.5) to the power $\frac{r}{2}$ and exploiting the preliminary estimates above and the properties of the stochastic integral,

$$\mathbb{E}\left[\left(\int_0^T \|\nabla u_n(t)\|_{L^2}^2 dt \right)^{r/2} \right] \leq C. \tag{2.8}$$

Repeating the arguments above, one can check that under previous integrability assumptions on u_0 and f, thanks to relations (2.7), (2.8),

$$\mathbb{E}\left[\|u_n(t) - u_n(s)\|_{V'}^{r/2} \right] \leq C(t-s)^{r/4}.$$

It follows that

$$\mathbb{E}\left[\int_0^T \int_0^T \frac{\|u_n(t) - u_n(s)\|_{V'}^{r/2}}{|t-s|^{1+\alpha r/2}} dsdt \right] \leq \int_0^T \int_0^T \frac{C}{|t-s|^{\frac{2-r/2}{2}+\alpha r/2}} dsdt$$
$$=: C_r < \infty$$

if $\alpha r < \frac{r}{2}$. The condition $\alpha \frac{r}{2} > 1 - \frac{r}{2p}$ of Theorem 2.15 is fulfilled for $p = 2$ if $\alpha r > 2 - \frac{r}{2}$. Thus if

$$2 - \frac{r}{2} < \alpha r < \frac{r}{2}$$

both conditions are satisfied. For $r = 2$ this is impossible, as seen in the previous step, but for every $r > 2$ there exists $\alpha \in \left(0, \frac{1}{2}\right)$ with such a property. Similarly, the condition $\alpha \frac{r}{2} > 1$ of Theorem 2.17 is fulfilled if

$$2 < \alpha r < \frac{r}{2}.$$

For every $r > 4$ there exists $\alpha \in \left(0, \frac{1}{2}\right)$ with such a property. This is exactly our integrability assumption on the initial condition and the forcing term.

The conclusion is:

Theorem 2.23 *There exist* (α, r) *with* $\alpha r > 1$ *and* $C > 0$ *such that*

$$\mathbb{E}\left[\|u_n\|_{W^{\alpha,r}(0,T;V')}\right] \leq C.$$

From the previous results and the embedding $D((-A)^{\alpha}) \overset{c}{\hookrightarrow} D((-A)^{\beta})$ if $\alpha > \beta$, the corollary below follows immediately.

Corollary 2.24 *The family of laws of* u_n *is tight in* $L^2(0, T; H) \cap C([0, T]; D\left((-A)^{-\beta}\right))$ *for each* $\beta > \frac{1}{2}$.

Remark 2.25 From the proof above, it is completely clear that the additional integrability assumptions on u_0 and f are needed only to complete Step 3. Here we want to explain how to change the proof above in order to obtain similar estimates in the case of deterministic data $u_0 \in H,\ f \in L^2(0, T; V')$. We restart from the last relation obtained without considering the additional integrability assumptions, namely

$$\mathbb{E}\left[\|u_n(t)\|_H^r\right] + r\nu\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2}\|\nabla u_n(s)\|_{L^2}^2 ds\right]$$
$$\leq C_r\mathbb{E}\left[\int_0^t \left(\|u_n(s)\|_H^r + 1\right) ds\right] + C_r\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2}\|f(s)\|_{V'}^2 ds\right]$$
$$+ \nu\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2}\|u_n(s)\|_V^2 ds\right] + \|u_0\|_H^r.$$

We apply in a way different from before Young's inequality to $\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2} \|f(s)\|_{V'}^2 ds\right]$,

$$\mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2} \|f(s)\|_{V'}^2 ds\right] = \mathbb{E}\left[\int_0^t \|u_n(s)\|_H^{r-2} \|f(s)\|_{V'}^{\frac{2(r-2)}{r}} \|f(s)\|_{V'}^{\frac{4}{r}} ds\right]$$

$$\leq C_r \left(\int_0^t \mathbb{E}\left[\|u_n(s)\|_H^r\right] \|f(s)\|_{V'}^2 ds + \int_0^t \|f(s)\|_{V'}^2 ds\right).$$

From these relations, by the Gronwall Lemma

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|u_n(t)\|_H^r\right] \leq C.$$

This is the only change in the proof of Step 3 in the case of deterministic data with minor integrability assumptions, then the proof goes on exactly as above without any changes.

Application of Gyongy–Krylov Criterion and Conclusion of the Proof of Existence

Let u_n be the Galerkin sequence. Assume we have a subsequence u_{n_k} and a process u with the following properties:

1. u has the regularity prescribed by the theorem;
2. u_{n_k} converges to u in probability in $L^2(0, T; H)$;
3. u_{n_k} converges weakly to u in $L^2_{\mathcal{F}}(0, T; V)$ and weak star in $C_{\mathcal{F}}([0, T]; H)$.

Then with some work we can pass to the limit in the weak formulation of the equations; property 2 is needed to pass to the limit in the quadratic term. The existence of a subsequence with properties 1–3 comes from (2.6) (and a variant of the argument of Remark 2.14 to identify the limit as the same function). From this subsequence, from the bounds of the previous section and the compactness theorem, we may also extract another one such that u_{n_k} converges in law, in the strong topology of $L^2(0, T; H) \cap C\left([0, T]; D\left((-A)^{-\beta}\right)\right)$ for each $\beta > \frac{1}{2}$, to the law of u (again we identify the limit by a variant of the argument of Remark 2.14). The convergence in law implies convergence in probability, in the strong topology of $L^2(0, T; H) \cap C\left([0, T]; D\left((-A)^{-\beta}\right)\right)$, by the Gyongy–Krylov criterion, which is applicable as shown below in this section. The tightness in $C\left([0, T]; D\left((-A)^{-\beta}\right)\right)$, never mentioned before, is a technical requirement in order to apply the Gyongy–Krylov criterion.

Hence we have to show that the Gyongy–Krylov criterion applies. We fix $\beta > 1$, take any pair of subsequences $(n_1(k), n_2(k))$ and consider the sequence of pairs $\left(u_{n_1(k)}, u_{n_2(k)}\right)$. Since (u_n) is tight in $L^2(0,T;H) \cap C([0,T]; D\left((-A)^{-\beta}\right))$, it is very easy to check that also $\left(u_{n_1(k)}, u_{n_2(k)}\right)$ is tight in $\left(L^2(0,T;H) \cap C\left([0,T]; D\left((-A)^{-\beta}\right)\right)\right)^2$. Let $k(h)$ be a subsequence such that $\left(u_{n_1(k(h))}, u_{n_2(k(h))}\right)$ converges in law to some μ. We only need to prove that $\mu(D) = 1$. This is the final aim of this section. The proof of this fact will be split into several steps. Before this we recall a classical characterization for Wiener processes used several times below. We refer to [90] Chapter 4 for a detailed discussion on Wiener processes taking values in separable Hilbert spaces.

Theorem 2.26 *Let U_0 be a separable Hilbert space and $M(t)$ a square integrable continuous martingale with values in U_0 such that $M(0) = 0$. $M(t)$ is a Wiener process with covariance Q adapted to the filtration $\mathcal{F}_t$ and increments $M(t) - M(s)$ independent of $\mathcal{F}_s$ if and only if $\langle\langle M\rangle\rangle_t = tQ,\ t \geq 0$.*

Step 1 (Notations) To shorten the notations, let us denote the subsequences $u_{n_1(k(h))}, u_{n_2(k(h))}$ simply by $u_{n(h)}, u_{m(h)}$. We denote by W the cylindrical Wiener process on H; it is a continuous stochastic process on a larger set U_0, such that the embedding J of H in U_0 is Hilbert–Schmidt. Under these assumptions W is a well-defined continuous process with values in U_0 and covariance $Q_1 = JJ^*$; see [90] for more details on this topic. After introducing this notation, the diffusion term $\sum_{k\in K} \sigma_k(u(t))dW_t^k$ can be rewritten as $G(u)dW_t$ where $G \in Lip\,(H, L(H,H)) \cap C\,(H, L(H, D(A)))$ is the operator defined by

$$G(h) = \sum_{k\in K} \langle e_k, \cdot\rangle \sigma_k(h) \ \ \forall h \in H.$$

Thanks to this notation, applying the projector π_n on the coefficients σ_k is equivalent to considering the operator

$$G^n(h) = \sum_{k\in K} \langle e_k, \cdot\rangle \pi_n \sigma_k(h) \ \ \forall h \in H.$$

Consider the quintuple $\left(u_{n(h)}, u_{m(h)}, u_0, f, W\right)$ and call Q_h its law. We fix $\beta > 1$. Due to Corollary 2.24, this quintuple converges weakly to a measure Q on

$$\left(L^2(0,T;H) \cap C([0,T]; D\left((-A)^{-\beta}\right)\right)^2 \times H \times L^2(0,T;V') \times C([0,T];U_0).$$

By the Skorohod representation theorem there exist a new probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$, random variables $\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)$ with laws Q_h and a random variable $\left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right)$ with law Q, such that

$$\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right) \to \left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right) \ \ \widetilde{\mathbb{P}} - a.s.$$

in $\left(L^2\left(0,T;H\right)\cap C([0,T];D\left((-A)^{-\beta}\right)\right)^2 \times H \times L^2(0,T;V') \times C\left([0,T];U_0\right)$. Moreover, by the results of [252], interesting side information, not used below, is that the random variables $\left(\widetilde{u}_{n(h)},\widetilde{u}_{m(h)},\widetilde{u}_{0,h},\widetilde{f}_h,\widetilde{W}_h\right)$ can be chosen of the form

$$\left(\widetilde{u}_{n(h)},\widetilde{u}_{m(h)},\widetilde{u}_{0,h},\widetilde{f}_h,\widetilde{W}_h\right) = \left(u_{n(h)}\circ\phi_h, u_{m(h)}\circ\phi_h, u_0\circ\phi_h, f\circ\phi_h, W\circ\phi_h\right),$$

where ϕ_h are perfect maps between Ω and $\widetilde{\Omega}$. Let us consider the filtrations

$$\left(\widetilde{\mathcal{F}}_h\right)_t = \sigma\left(\widetilde{u}_{n(h)}(s),\widetilde{u}_{m(h)}(s),\widetilde{u}_{0,h},\widetilde{f}_h(s),\widetilde{W}_h(s),\ s\le t\right),$$

$$\widetilde{\mathcal{F}}_t = \sigma\left(\widetilde{u}^{(1)}(s),\widetilde{u}^{(2)}(s),\widetilde{u}_0,\widetilde{f}(s),\widetilde{W}(s),\ s\le t\right),$$

the stochastic processes

$$\begin{aligned}\widetilde{M}_h^n(t) &= (-A)^{-\beta}\left(\widetilde{u}_{n(h)}(t)-\pi_{n(h)}\widetilde{u}_{0,h}\right) + (-A)^{-\beta}\int_0^t B_{n(h)}(\widetilde{u}_{n(h)}(s),\widetilde{u}_{n(h)}(s))ds\\ &\quad - (-A)^{-\beta}\int_0^t \pi_{n(h)}\widetilde{f}_h(s)ds - (-A)^{-\beta}\int_0^t \pi_{n(h)}F\left(\widetilde{u}_{n(h)}(s)\right)ds\\ &\quad - (-A)^{-\beta}\int_0^t A\widetilde{u}_{n(h)}(s)\,ds,\end{aligned}$$

$$\widetilde{I}_h^n(t) = (-A)^{-\beta}\int_0^t G^{n(h)}\left(\widetilde{u}_{n(h)}(s)\right)d\widetilde{W}_{h,s}$$

and similarly for $m(h)$. By preservation of laws and the definition of the filtration, we have that also $\widetilde{W}_h$ is a Q_1 Wiener process in U_0, adapted to the filtration $\left(\widetilde{\mathcal{F}}_h\right)_t$. Thus the stochastic integrals are well-defined and $\widetilde{W}_h$ is a square integrable, continuous martingale with values in U_0. Let us denote, for each $s\in[0,T]$, by

$$X_s = \left(L^2\left(0,s;H\right)\cap C\left([0,s];D\left((-A)^{-\beta}\right)\right)\right)^2 \times H \times L^2(0,s;V') \times C\left([0,s];U_0\right).$$

Step 2 (Characterization of $\widetilde{W}(t)$) The first thing we want to show is that $\widetilde{W}(t)$ is a Q_1 Wiener process on U_0 with respect to the filtration $\widetilde{\mathcal{F}}_t$. Let us consider arbitrary $u, v \in U_0$ and $\psi : X_s \to \mathbb{R}$ continuous and bounded. By the integrability properties of a Q_1 Wiener process it follows immediately that the families $\left\{\langle \widetilde{W}_h(t) - \widetilde{W}_h(s), u\rangle_{U_0}\right\}_{h\in\mathbb{N}}$, $\left\{\|\widetilde{W}_h(t)\|^2_{U_0}\right\}_{h\in\mathbb{N}}$ are uniformly integrable. Thus, due to the fact that $\widetilde{W}_h(t)$ is a martingale, the following relations hold:

$$tTr(Q_1) = \widetilde{\mathbb{E}}\left[\|\widetilde{W}_h(t)\|^2_{U_0}\right] \to \widetilde{\mathbb{E}}\left[\|\widetilde{W}(t)\|^2_{U_0}\right],$$

$$\begin{aligned} 0 &= \widetilde{\mathbb{E}}\left[\langle \widetilde{W}_h(t) - \widetilde{W}_h(s), u\rangle_{U_0} \psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\right] \\ &\to \widetilde{\mathbb{E}}\left[\langle \widetilde{W}(t) - \widetilde{W}(s), u\rangle_{U_0} \psi\left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right)\right], \end{aligned}$$

$$\begin{aligned} 0 = \widetilde{\mathbb{E}}&\Big[\big(\langle \widetilde{W}_h(t), u\rangle_{U_0}\langle \widetilde{W}_h(t), v\rangle_{U_0} - \langle \widetilde{W}_h(s), u\rangle_{U_0}\langle \widetilde{W}_h(s), v\rangle_{U_0} \\ &-(t-s)\langle Q_1 u, v\rangle_{U_0}\big)\, \psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\Big] \\ &\to \widetilde{\mathbb{E}}\Big[\big(\langle \widetilde{W}(t), u\rangle_{U_0}\langle \widetilde{W}(t), v\rangle_{U_0} - \langle \widetilde{W}(s), u\rangle_{U_0}\langle \widetilde{W}(s), v\rangle_{U_0} - (t-s)\langle Q_1 u, v\rangle_{U_0}\big) \\ &\psi\left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right)\Big]. \end{aligned}$$

$\widetilde{W}(0) = 0\ \widetilde{\mathbb{P}} - a.s.$ due to the fact that $\widetilde{W}_h(0) = 0\ \widetilde{\mathbb{P}} - a.s.$ and $\widetilde{W}_h \to \widetilde{W}\ \widetilde{\mathbb{P}} - a.s.$ Therefore, due to the characterization of Wiener processes recalled before, it follows that $\widetilde{W}$ is a Q_1 Wiener process with values in U_0, adapted to $\widetilde{\mathcal{F}}_t$.

Step 3 (Identification of $\widetilde{M}^n_h(t)$, $\widetilde{M}^m_h(t)$, $\widetilde{I}^n_h(t)$, $\widetilde{I}^m_h(t)$) Now we wish to study $\widetilde{M}^n_h(t)$, $\widetilde{M}^m_h(t)$, $\widetilde{I}^n_h(t)$, $\widetilde{I}^m_h(t)$. The fact that $\widetilde{I}^n_h(t)$, $\widetilde{I}^m_h(t)$ are H valued, square integrable, continuous martingales follows immediately by the definition of the stochastic integrals and the regularity of the operator G. Moreover, by the properties of the stochastic integral, we have

$$\langle\langle \widetilde{I}^n_h\rangle\rangle_t = \int_0^t (-A)^{-\beta} G^{n(h)}\left(\widetilde{u}_{n(h)}(s)\right)\left(G^{n(h)}\left(\widetilde{u}_{n(h)}(s)\right)\right)^* (-A)^{-\beta}\, ds,$$

$$\langle\langle \widetilde{I}^m_h\rangle\rangle_t = \int_0^t (-A)^{-\beta} G^{m(h)}\left(\widetilde{u}_{m(h)}(s)\right)\left(G^{m(h)}\left(\widetilde{u}_{m(h)}(s)\right)\right)^* (-A)^{-\beta}\, ds.$$

In order to identify $\widetilde{M}^n_h(t)$ with $\widetilde{I}^n_h(t)$ (analogously for $\widetilde{M}^m_h(t)$ and $\widetilde{I}^m_h(t)$) we need to show that also $\widetilde{M}^n_h(t)$ is a square integrable continuous martingale, with values in

H and $\langle\langle \widetilde{M}_h^n - \widetilde{I}_h^n \rangle\rangle_t = 0$, $\forall t \in [0, T]$. To obtain this result we introduce another sequence of stochastic processes

$$
\begin{aligned}
M_h^n(t) &= (-A)^{-\beta} \left(u_{n(h)}(t) - \pi_{n(h)} u_0\right) + (-A)^{-\beta} \int_0^t B_{n(h)}(u_{n(h)}(s), u_{n(h)}(s)) ds \\
&- (-A)^{-\beta} \int_0^t \pi_{n(h)} f(s) ds - (-A)^{-\beta} \int_0^t \pi_{n(h)} F\left(u_{n(h)}(s)\right) ds \\
&- (-A)^{-\beta} \int_0^t A u_{n(h)}(s)\, ds \\
&= (-A)^{-\beta} \int_0^t G^{n(h)}\left(u_{n(h)}(s)\right) dW_s = I_h^n(t).
\end{aligned}
$$

The second equality follows from the weak formulation satisfied by $u_{n(h)}$. From the properties of the stochastic integral and the regularity of G, it follows immediately that M_h^n is a square integrable continuous martingale with values in H for each $h \in \mathbb{N}$ and its quadratic variation is

$$
\langle\langle M_h^n \rangle\rangle_t = \int_0^t (-A)^{-\beta} G^{n(h)}\left(u_{n(h)}(s)\right) \left(G^{n(h)}\left(u_{n(h)}(s)\right)\right)^* (-A)^{-\beta}\, ds.
$$

Let us consider arbitrary $u,\ v \in H$ and $\psi : X_s \to \mathbb{R}$ continuous and bounded. From the fact that

$$
\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right) \stackrel{\mathcal{L}}{=} \left(u_{n(h)}, u_{m(h)}, u_0, f, W\right),
$$

the regularity of the coefficients σ_k and the properties of the stochastic integral, see Theorem 4.36 in [90], we have the following relations

$$
\sup_{h\in\mathbb{N}} \widetilde{\mathbb{E}}\left[\|\widetilde{M}_h^n(t)\|_H^p\right] = \sup_{h\in\mathbb{N}} \mathbb{E}\left[\|M_h^n(t)\|_H^p\right] = C_p < +\infty \quad p \geq 2, \tag{2.9}
$$

$$
\begin{aligned}
0 &= \mathbb{E}\left[\langle M_h^n(t) - M_h^n(s), u\rangle \psi\left(u_{n(h)}, u_{m(h)}, u_0, f, W\right)\right] \\
&= \widetilde{\mathbb{E}}\left[\langle \widetilde{M}_h^n(t) - \widetilde{M}_h^n(s), u\rangle \psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
0 = \mathbb{E}\Bigg[\Bigg(&\langle M_h^n(t), u\rangle \langle M_h^n(t), v\rangle - \langle M_h^n(s), u\rangle \langle M_h^n(s), v\rangle \\
&- \left\langle \int_s^t (-A)^{-\beta} G^{n(h)}\left(u_{n(h)}(r)\right) \left(G^{n(h)}\left(u_{n(h)}(r)\right)\right)^* (-A)^{-\beta} u\, ds, v \right\rangle_H \Bigg) \\
&\psi\left(u_{n(h)}, u_{m(h)}, u_0, f, W\right)\Bigg]
\end{aligned}
$$

$$= \widetilde{\mathbb{E}}\Bigg[\bigg(\langle \widetilde{M}_h^n(t), u\rangle\langle \widetilde{M}_h^n(t), v\rangle - \langle \widetilde{M}_h^n(s), u\rangle\langle \widetilde{M}_h^n(s), v\rangle$$

$$- \left\langle \int_s^t (-A)^{-\beta} G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right)\left(G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right)\right)^* (-A)^{-\beta} u\, ds, v\right\rangle_H \bigg)$$

$$\psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\Bigg],$$

$$0 = \mathbb{E}\Bigg[\bigg(\langle M_h^n(t), u\rangle\langle I_h^n(t), v\rangle - \langle M_h^n(s), u\rangle\langle I_h^n(s), v\rangle$$

$$- \left\langle \int_s^t (-A)^{-\beta} G^{n(h)}\left(u_{n(h)}(r)\right)\left(G^{n(h)}\left(u_{n(h)}(r)\right)\right)^* (-A)^{-\beta} u\, dr, v\right\rangle_H \bigg)$$

$$\psi\left(u_{n(h)}, u_{m(h)}, u_0, f, W\right)\Bigg]$$

$$= \widetilde{\mathbb{E}}\Bigg[\bigg(\langle \widetilde{M}_h^n(t), u\rangle\langle \widetilde{I}_h^n(t), v\rangle - \langle \widetilde{M}_h^n(s), u\rangle\langle \widetilde{I}_h^n(s), v\rangle$$

$$- \left\langle \int_s^t (-A)^{-\beta} G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right)\left(G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right)\right)^* (-A)^{-\beta} u\, dr, v\right\rangle_H \bigg)$$

$$\psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\Bigg].$$

Therefore $\widetilde{M}_h^n$ is a square integrable continuous martingale with values in H,

$$\langle\langle \widetilde{M}_h^n \rangle\rangle_t = \langle\langle \widetilde{M}_h^n, \widetilde{I}_h^n \rangle\rangle_t$$
$$= \int_0^t (-A)^{-\beta} G^{n(h)}\left(u_{n(h)}(s)\right)\left(G^{n(h)}\left(u_{n(h)}(s)\right)\right)^* (-A)^{-\beta}\, ds$$

and

$$\langle\langle \widetilde{M}_h^n - \widetilde{I}_h^n \rangle\rangle_t = \langle\langle \widetilde{M}_h^n \rangle\rangle_t + \langle\langle \widetilde{I}_h^n \rangle\rangle_t - 2\langle\langle \widetilde{M}_h^n, \widetilde{I}_h^n \rangle\rangle_t = 0.$$

Thus we have the required identification. Relation (2.9) for $p > 2$ implies that the families $\|\widetilde{M}_h^n\|_H^2$ and $\|\widetilde{M}_h^m\|_H^2$ are uniformly integrable. This fact will be crucial in the next steps.

Step 4 (Limit Processes, Preparation) Since we applied Corollary 2.19 with both $p, r > 1$, it follows, in particular, that $\widetilde{u}^{(1)}$, $\widetilde{u}^{(2)}$ have paths $\widetilde{\mathbb{P}} - a.s.$ in $L^\infty(0, T; H) \cap L^2(0, T; V)$. From this and the regularity in $C([0, T]; D((-A)^{-\beta}))$, it follows that their paths have also regularity $C_w([0, T]; H)$. This kind of regularity is enough to prove the pathwise uniqueness as in Sect. 2.3.1. Thus, in order to be able to apply the Gyongy–Krylov criterion and obtaining the existence of a weak solution, it is enough to show that both $\widetilde{u}^{(1)}$ and $\widetilde{u}^{(2)}$ satisfy a weak formulation with respect to the same Wiener process $\widetilde{W}$. In fact this will imply that the measure Q restricted on $\left(L^2(0,T;H) \cap C\left([0,T];D((-A)^{-\beta})\right)\right)^2$ is concentrated on the diagonal. Thus for $j \in \{1,2\}$ we introduce the stochastic processes

$$\widetilde{M}^{(j)}(t) = (-A)^{-\beta}\left(\widetilde{u}^{(j)}(t) - \widetilde{u}_0\right) + (-A)^{-\beta}\int_0^t B(\widetilde{u}^{(j)}(s), \widetilde{u}^{(j)}(s))ds$$
$$- (-A)^{-\beta}\int_0^t \widetilde{f}(s)ds - (-A)^{-\beta}\int_0^t F\left(\widetilde{u}^{(j)}(s)\right)ds$$
$$- (-A)^{-\beta}\int_0^t A\widetilde{u}^{(j)}(s)\,ds,$$

$$\widetilde{I}^{(j)}(t) = (-A)^{-\beta}\int_0^t G\left(\widetilde{u}^{(j)}(s)\right)d\widetilde{W}_s.$$

As before, $\widetilde{I}^{(j)}$ are H valued, square integrable, continuous martingales from the properties of the stochastic integral and the regularity of the operator G. Moreover, by the properties of the stochastic integral, we have

$$\langle\langle \widetilde{I}^{(j)}\rangle\rangle_t = \int_0^t (-A)^{-\beta} G\left(\widetilde{u}^{(j)}(s)\right)\left(G\left(\widetilde{u}^{(j)}(s)\right)\right)^* (-A)^{-\beta}\,ds.$$

Step 5 (Limit Processes, Analysis of $\widetilde{M}^{(j)}$) In order to complete the proof we have to identify $\widetilde{M}^{(j)}$ with $\widetilde{I}^{(j)}$. Thus, as before, we need to show that also $\widetilde{M}^{(j)}$ is a square integrable continuous martingale, with values in H and $\langle\langle \widetilde{M}^{(j)} - \widetilde{I}^{(j)}\rangle\rangle_t = 0$, $\forall t \in [0,T]$. Note that, if $\widetilde{u}^{(j)}$ were weak solutions of the stochastic Navier Stokes equations they would satisfy $\widetilde{M}^{(j)} = \widetilde{I}^{(j)}$ for $\beta = 1$. We can move from our condition on β to the case $\beta = 1$ a posteriori via a density argument due to the regularity of $\widetilde{u}^{(j)}$. We will skip the easy details related to this point. For what concerns the analysis of $\widetilde{M}^{(j)}$, we do the computations only for $j = 1$, but the same can be done analogously for $j = 2$. Obviously, $\tilde{u}_0 \stackrel{\mathcal{L}}{=} u_0$, $\tilde{f} \stackrel{\mathcal{L}}{=} f$. Easily, we get the convergence in H $\widetilde{\mathbb{P}}$-a.s. of

$$(-A)^{-\beta}\left(\widetilde{u}_{n(h)}(t) - \pi_{n(h)}\widetilde{u}_{0,h}\right) - (-A)^{-\beta}\int_0^t A\widetilde{u}_{n(h)}(s)\,ds$$
$$- (-A)^{-\beta}\int_0^t \pi_{n(h)}\widetilde{f}_h(s)ds - (-A)^{-\beta}\int_0^t \pi_{n(h)}F\left(\widetilde{u}_{n(h)}(s)\right)ds$$

to the corresponding terms in $\widetilde{M}^{(1)}(t)$ due to convergence properties of $(\widetilde{u}_{n(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h)$, the Lipschitzianity of the operator F and the properties of the orthogonal projector $\pi_{n(h)}$. We just show the convergence of one term as an example, the others being simpler.

$$\begin{aligned}
&\left\| \int_0^t \pi_{n(h)} F\left(\widetilde{u}_{n(h)}(s)\right) ds - \int_0^t F\left(\widetilde{u}^{(1)}(s)\right) ds \right\|_H \\
&\leq \int_0^t \left\| \pi_{n(h)} \left(F\left(\widetilde{u}_{n(h)}(s)\right) ds - F\left(\widetilde{u}^{(1)}(s)\right)\right) \right\|_H ds \\
&+ \int_0^t \left\| (I - \pi_{n(h)}) F\left(\widetilde{u}^{(1)}(s)\right) \right\|_H ds \\
&\leq L_F \left(\int_0^t \|\widetilde{u}_{n(h)}(s) - \widetilde{u}^{(1)}(s)\|_H ds + \int_0^t \|(I - \pi_{n(h)})\widetilde{u}^{(1)}(s)\|_H ds \right) \\
&+ T \|(I - \pi_{n(h)}) F(0)\|_H \\
&\to 0 \ \widetilde{\mathbb{P}} - a.s.
\end{aligned}$$

exploiting the convergence of $\widetilde{u}_{n(h)}$ in $L^2(0, T; H)$ for the first term, the properties of $\pi_{n(h)}$ and dominated convergence theorem for the others. In order to obtain the almost sure convergence in H of $\widetilde{M}_{n(h)}(t)$ to $\widetilde{M}^{(1)}(t)$ it remains to show that

$$\left\| (-A)^{-\beta} \int_0^t B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)) \, ds \right\|_H \to 0 \ \widetilde{\mathbb{P}} - a.s.$$

The convergence of the nonlinear term is, in general, the most involved part of the proof. For this reason several approaches have been introduced which fit well to different situations. We start with the simplest one for this case, but at the end of the proof we will present other possibilities. Note that all the computations we did so far can be performed under the assumption $\beta = 1$; this is the unique step where we use the assumption $\beta > 1$.

$$\begin{aligned}
&\left\| (-A)^{-\beta} \int_0^t B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)) \, ds \right\|_H \\
&\leq \int_0^t \|B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B_{n(h)}(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s))\|_{D((-A)^{-\beta})} \, ds \\
&+ \int_0^t \|(I - \pi_{n(h)}) B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)))\|_{(D(-A)^{-\beta})} \, ds \\
&\leq C \int_0^t \|\tilde{u}_{n(h)}(s)\|_H \|\tilde{u}_{n(h)}(s) - \tilde{u}^{(1)}(s)\|_H ds \\
&+ C \int_0^t \|\tilde{u}_{n(h)}(s) - \tilde{u}^{(1)}(s)\|_H \|\tilde{u}^{(1)}(s)\|_H \\
&+ \int_0^t \|(I - \pi_{n(h)}) B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)))\|_{(D(-A)^{-\beta})} \, ds. \qquad (2.10)
\end{aligned}$$

The last inequality follows from the fact that $D((-A)^{\beta}) \hookrightarrow W^{1,\infty}(D)$ if $\beta > 1$. Therefore for each $u, v \in H,\ w \in D((-A)^{\beta})$,

$$\left|\int_D u(x) \cdot \nabla w(x) v(x) dx\right| \leq C\|u\|_H \|v\|_H \|w\|_{D((-A)^{\beta})}.$$

The required convergence follows from relation (2.10) due to the almost sure convergence of $\widetilde{u}_{n(h)}$ to $\widetilde{u}^{(1)}$ in $L^2(0, T; H)$ for the first two terms and properties of the projector for the remaining one.

Thus the family $\|\widetilde{M}_h^n(t)\|_H^2$ is uniformly integrable and $\|\widetilde{M}_h^n(t)\|_H^2 \to \|\widetilde{M}^{(1)}(t)\|_H^2$ $\widetilde{\mathbb{P}} - a.s.$ Therefore

$$\widetilde{\mathbb{E}}\left[\|\widetilde{M}^{(1)}(t)\|_H^2\right] = \lim_{h\to+\infty} \widetilde{\mathbb{E}}\left[\|\widetilde{M}_h^n(t)\|_H^2\right] = \lim_{h\to+\infty} \widetilde{\mathbb{E}}\left[\|\widetilde{I}_h^n(t)\|_H^2\right] < +\infty.$$

From the computations above it follows that $\widetilde{M}^{(1)}$ is a square integrable process with values in H. Now we want to show that $\widetilde{M}^{(1)}(t)$ is a martingale with quadratic variation equal to $\langle\langle \widetilde{I}^{(1)}\rangle\rangle_t$. The proof of this fact is similar to what we have done for the Wiener process. In fact due to the $\widetilde{\mathbb{P}}$-a.s. convergence of $\|\widetilde{M}_h^n(t)\|_H^2$, the fact that such random variables are uniformly integrable and $\widetilde{M}_h^n(t)$ is a square integrable martingale with quadratic variation $\langle\langle \widetilde{M}_h^n\rangle\rangle_t$, the following chain of equalities hold for each $u,\ v \in H,\ \psi : X_s \to \mathbb{R}$ continuous and bounded:

$$\begin{aligned}
0 &= \widetilde{\mathbb{E}}\left[\langle \widetilde{M}_h^n(t) - \widetilde{M}_h^n(s), u\rangle \psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\right] \\
&\to \widetilde{\mathbb{E}}\left[\langle \widetilde{M}^{(1)}(t) - \widetilde{M}^{(1)}(s), u\rangle \psi\left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right)\right],
\end{aligned}$$

$$\begin{aligned}
0 = \widetilde{\mathbb{E}}\Bigg[&\Bigg(\langle \widetilde{M}_h^n(t), u\rangle\langle \widetilde{M}_h^n(t), v\rangle - \langle \widetilde{M}_h^n(s), u\rangle\langle \widetilde{M}_h^n(s), v\rangle \\
&- \int_s^t \langle (-A)^{-\beta} G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right)\left(G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right)\right)^* (-A)^{-\beta} u, v\rangle\, dr\Bigg) \\
&\psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\Bigg] \\
\to \widetilde{\mathbb{E}}\Bigg[&\Bigg(\langle \widetilde{M}^{(1)}(t), u\rangle\langle \widetilde{M}^{(1)}(t), v\rangle - \langle \widetilde{M}^{(1)}(s), u\rangle\langle \widetilde{M}^{(1)}(s), v\rangle \\
&- \int_s^t \langle (-A)^{-\beta} G\left(u^{(1)}(r)\right)\left(G\left(u^{(1)}(r)\right)\right)^* (-A)^{-\beta} u, v\rangle_H\, dr\Bigg) \\
&\psi\left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right)\Bigg].
\end{aligned}$$

Thus $\widetilde{M}^{(1)}$ is a continuous square integrable martingale with values in H such that

$$\langle\langle \widetilde{M}^{(1)}\rangle\rangle_t = \langle\langle \widetilde{I}^{(1)}\rangle\rangle_t = \int_0^t \langle (-A)^{-\beta} G\left(\widetilde{u}^{(1)}(s)\right)\left(G\left(\widetilde{u}^{(1)}(s)\right)\right)^* (-A)^{-\beta}\, ds.$$

Step 6 (Limit Processes, Identification) We need to show that $\langle\langle \widetilde{M}^{(1)} - \widetilde{I}^{(1)}\rangle\rangle_t = 0$ for all $t \in [0, T]$ to conclude the proof. This claim is true, indeed

$$\begin{aligned}\langle\langle \widetilde{I}^{(1)} - \widetilde{M}^{(1)}\rangle\rangle_t &= \langle\langle \widetilde{M}^{(1)}\rangle\rangle_t + \langle\langle \widetilde{I}^{(1)}\rangle\rangle_t - 2\langle\langle \widetilde{I}^{(1)}, \widetilde{M}^{(1)}\rangle\rangle_t \\ &= 2\int_0^t \langle (-A)^{-\beta} G\left(\widetilde{u}^{(1)}(s)\right)\left(G\left(\widetilde{u}^{(1)}(s)\right)\right)^* (-A)^{-\beta}\, ds \\ &\quad - 2\int_0^t (-A)^{-\beta} G\left(\widetilde{u}^{(1)}(s)\right) J^{-1} d\langle\langle \widetilde{W}, \widetilde{M}^{(1)}\rangle\rangle_s. \qquad (2.11)\end{aligned}$$

Thus it remains to compute $\langle\langle \widetilde{W}, \widetilde{M}^{(1)}\rangle\rangle_t$, but this can be done thanks to the converging properties of $\widetilde{W}_h$ to $\widetilde{W}$ and the fact that they are Wiener processes with values in U_0, therefore uniformly integrable. In conclusion, if $u \in U_0,\ v \in H,\ \psi : X_s \to \mathbb{R}$ continuous and bounded, then

$$\begin{aligned}0 &= \widetilde{\mathbb{E}}\Bigg[\Bigg(\langle \widetilde{W}_{h,t}, u\rangle_{U_0}\langle \widetilde{M}_h^n(t), v\rangle - \langle \widetilde{W}_{h,s}, u\rangle_{U_0}\langle \widetilde{M}_h^n(s), v\rangle \\ &\qquad - \int_s^t \langle (-A)^{-\beta} G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right) J^{-1} Q_1 u, v\rangle\, dr\Bigg) \\ &\qquad \psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\Bigg] \\ &= \widetilde{\mathbb{E}}\Bigg[\Bigg(\langle \widetilde{W}_{h,t}, u\rangle_{U_0}\langle \widetilde{M}_h^n(t), v\rangle - \langle \widetilde{W}_{h,s}, u\rangle_{U_0}\langle \widetilde{M}_h^n(s), v\rangle \\ &\qquad - \int_s^t \langle (-A)^{-\beta} G^{n(h)}\left(\widetilde{u}_{n(h)}(r)\right) J^* u, v\rangle\, dr\Bigg) \\ &\qquad \psi\left(\widetilde{u}_{n(h)}, \widetilde{u}_{m(h)}, \widetilde{u}_{0,h}, \widetilde{f}_h, \widetilde{W}_h\right)\Bigg] \\ &\to \widetilde{\mathbb{E}}\Bigg[\Bigg(\langle \widetilde{W}_t, u\rangle_{U_0}\langle \widetilde{M}^{(1)}(t), v\rangle - \langle \widetilde{W}_s, u\rangle_{U_0}\langle \widetilde{M}^{(1)}(s), v\rangle \\ &\qquad - \int_s^t \langle (-A)^{-\beta} G\left(\widetilde{u}^{(1)}(r)\right) J^* u, v\rangle\, dr\Bigg)\psi\left(\widetilde{u}^{(1)}, \widetilde{u}^{(2)}, \widetilde{u}_0, \widetilde{f}, \widetilde{W}\right)\Bigg].\end{aligned}$$

Therefore

$$\langle\langle \widetilde{M}^{(1)}, \widetilde{W}\rangle\rangle_t = \int_0^t (-A)^{-\beta} G(\widetilde{u}^{(1)})(s) J^* \, ds,$$

thus

$$\langle\langle \widetilde{W}, \widetilde{M}^{(1)}\rangle\rangle_t = \int_0^t J \left(G(\widetilde{u}^{(1)})(s) \right)^* (-A)^{-\beta} \, ds.$$

Inserting this expression in relation (2.11) we can identify $\widetilde{M}^{(1)}$ and $\widetilde{I}^{(1)}$. Arguing analogously for $\widetilde{M}^{(2)}$ and $\widetilde{I}^{(2)}$, it follows that both $\widetilde{u}^{(1)}$ and $\widetilde{u}^{(2)}$ satisfy a weak formulation with respect to the same Wiener process $\widetilde{W}$ and this implies the thesis.

We have adapted the previous proof from several ideas on the existence of martingale solutions for stochastic partial differential equations, see for example [46], [90], [98], [118].

As anticipated in Step 5, here we extend pedagogically the discussion about the limiting behavior $\|\widetilde{M}_h^n(t)\|_H^2$, $\|\widetilde{M}_h^m(t)\|_H^2$ in order to describe other techniques to show the required convergences.

First we want show a different approach to obtain the almost sure convergence of the non linear term

$$\left\| (-A)^{-\beta} \int_0^t B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)) \, ds \right\|_H \to 0 \;\; \widetilde{\mathbb{P}} - a.s.$$

The convergence above holds for β large enough such that

$$\sum_{k=1}^{\infty} \|\nabla (-A)^{-\beta} e_k\|_{L^\infty}^2 < +\infty,$$

where e_k is an orthonormal basis of H made by eigenvectors of A. In order to prove that under this assumption the convergence of the nonlinear term holds, we note first that if $n(h) \geq k$, then

$$\left\langle B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)), (-A)^{-\beta} e_k \right\rangle = \left\langle B(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)), (-A)^{-\beta} e_k \right\rangle.$$

Therefore, if $n(h) \geq k$,

$$\begin{aligned}
&\left| \int_0^t \left\langle B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)), (-A)^{-\beta} e_k \right\rangle ds \right| \\
&\leq \int_0^t \left| \left\langle B(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)), (-A)^{-\beta} e_k \right\rangle \right| ds \\
&= \int_0^t \left| \left\langle B(\widetilde{u}_{n(h)}(s), (-A)^{-\beta} e_k), \widetilde{u}_{n(h)}(s) \right\rangle - \left\langle B(\widetilde{u}^{(1)}(s), (-A)^{-\beta} e_k), \widetilde{u}^{(1)}(s) \right\rangle \right| ds
\end{aligned}$$

$$\leq \int_0^t |\left\langle B(\widetilde{u}_{n(h)}(s), (-A)^{-\beta} e_k), \widetilde{u}_{n(h)}(s) - \widetilde{u}^{(1)}(s) \right\rangle| \, ds$$

$$+ \int_0^t |\left\langle B(\widetilde{u}_{n(h)}(s) - \widetilde{u}^{(1)}(s), (-A)^{-\beta} e_k), \widetilde{u}^{(1)}(s) \right\rangle| \, ds$$

$$\leq \|\nabla (-A)^{-\beta} e_k\|_{L^\infty} \int_0^t \|\widetilde{u}_{n(h)}(s) - \widetilde{u}^{(1)}(s)\|_H \left(\|\widetilde{u}_{n(h)}(s)\|_H + \|\widetilde{u}^{(1)}(s)\|_H \right) \, ds.$$

On the opposite side, if $n(h) < k$,

$$\left\langle B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)), (-A)^{-\beta} e_k \right\rangle = 0$$

therefore

$$|\int_0^t \left\langle B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)), (-A)^{-\beta} e_k \right\rangle \, ds|$$

$$= |\int_0^t \left\langle B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)), (-A)^{-\beta} e_k \right\rangle \, ds|$$

$$\leq \|\nabla (-A)^{-\beta} e_k\|_{L^\infty} \int_0^t \|\widetilde{u}^{(1)}(s)\|_H^2 \, ds.$$

Coming back to the convergence of the nonlinear term, we have

$$\|(-A)^{-\beta} \int_0^t B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)) \, ds\|_H^2$$

$$= \sum_{k=1}^\infty \left\langle \int_0^t B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)) \, ds, (-A)^{-\beta} e_k \right\rangle^2$$

$$= \sum_{k=1}^\infty \left(\int_0^t \left\langle B_{n(h)}(\widetilde{u}_{n(h)}(s), \widetilde{u}_{n(h)}(s)) - B(\widetilde{u}^{(1)}(s), \widetilde{u}^{(1)}(s)), (-A)^{-\beta} e_k \right\rangle \, ds \right)^2$$

$$\leq \left(\sum_{k=1}^{n(h)} \|\nabla (-A)^{-\beta} e_k\|_{L^\infty}^2 \right)$$

$$\left(\int_0^t \|\widetilde{u}_{n(h)}(s) - \widetilde{u}^{(1)}(s)\|_H \left(\|\widetilde{u}_{n(h)}(s)\|_H + \|\widetilde{u}^{(1)}(s)\|_H \right) \, ds \right)^2$$

$$+ \left(\sum_{k=n(h)}^{\infty} \|\nabla (-A)^{-\beta} e_k\|_{L^\infty}^2 \right) \left(\int_0^t \|\widetilde{u}^{(1)}(s)\|_H^2 \, ds \right)^2$$

$$\leq \left(\sum_{k=1}^{\infty} \|\nabla (-A)^{-\beta} e_k\|_{L^\infty}^2 \right)$$

$$\left(\int_0^t \|\widetilde{u}_{n(h)}(s) - \widetilde{u}^{(1)}(s)\|_H \left(\|\widetilde{u}_{n(h)}(s)\|_H + \|\widetilde{u}^{(1)}(s)\|_H\right) ds\right)^2$$
$$+ \left(\sum_{k=n(h)}^{\infty} \|\nabla(-A)^{-\beta} e_k\|_{L^\infty}^2\right) \left(\int_0^t \|\widetilde{u}^{(1)}(s)\|_H^2 \, ds\right)^2 .$$

The second factor of the first addend converges to zero a.s. and the first factor of the first addend is finite by assumption, hence the first addend goes to zero; the second one also for similar but easier reasons.

Note that in the case of the torus with periodic boundary conditions, we have

$$\sum_k \|\nabla(-A)^{-\beta} e_k\|_{L^\infty}^2 = \sum_{k \in \mathbb{Z}_0^2} \frac{1}{|k|^{4\beta-2}} < +\infty \iff \beta > 1.$$

Therefore we do not expect that the condition on β presented in this remark allows us to avoid the requirement of considering $\beta > 1$. Indeed, the condition

$$\sum_{k=1}^{\infty} \|\nabla(-A)^{-\beta} e_k\|_{L^\infty}^2 < +\infty$$

holds for $\beta > \frac{3}{2}$ in a general 2D domain. To get this bound on β, we recall that if D is a d-dimensional smooth, bounded domain, then the eigenvalues of the Stokes operator with no-slip boundary conditions satisfy the asymptotic relation

$$\lambda_k \sim C_d k^{2/d}$$

where C_d is a constant depending only from d and the volume of D, see for example [181]. Thanks to this relation and exploiting Sobolev embedding theorem we get the result easily. Indeed the following relations hold true:

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\nabla(-A)^{-\beta} e_k\|_{L^\infty}^2 &= \sum_{k=1}^{\infty} \|(-A)^{-\beta} e_k\|_{W^{1,\infty}}^2 \\
&\leq C_d \sum_{k=1}^{\infty} \|(-A)^{-\beta} e_k\|_{D((-A)^{1+\epsilon})}^2 \\
&= C_d \sum_{k=1}^{\infty} \|(-A)^{1+\epsilon-\beta} e_k\|^2 \\
&= \sum_{k=1}^{+\infty} \frac{C_d}{\lambda_k^{2(\beta-1-\epsilon)}}.
\end{aligned}$$

The last series converges, in dimension two, if and only if $2(\beta - 1 - \epsilon) > 1$. Therefore we arrive at the relation $\beta > \frac{3}{2}$.

The argument presented in the main proof can be refined in order to obtain the final result without passing for the condition $\beta > 1$. We just sketch this argument. Before we showed that, if we take $\beta = 1$ and $\epsilon > 0$ such that $\beta + \epsilon > 1$ then $\widetilde{M}_h^n(t) \to \widetilde{M}^{(1)}(t)$ a.s. in $D((-A)^{-\epsilon})$ and, thanks to the fact that the random variables $\|\widetilde{M}_h^n(t)\|^2_{D((-A)^{-\epsilon})}$ are uniformly integrable, also $\mathbb{E}\left[\|\widetilde{M}_h^n(t)\|^2_{D((-A)^{-\epsilon})}\right] \to \mathbb{E}\left[\|\widetilde{M}^{(1)}(t)\|^2_{D((-A)^{-\epsilon})}\right]$. Let us introduce the functions

$$f(x) = \|x\|_H^2, \quad f_N(x) = \|\pi_N x\|_H^2 \wedge N.$$

The first one is continuous on H, the others are continuous and bounded on $D((-A)^{\alpha})$, $\forall \alpha \in \mathbb{R}$. Obviously, it holds that

$$0 \le f_N(x) \nearrow f(x) \le +\infty, \quad \forall x \in D((-A)^{\alpha}).$$

Thanks to above computations, we have

$$\begin{aligned}\widetilde{\mathbb{E}}\left[\|\pi_N \widetilde{M}^{(1)}(t)\|_H^2 \wedge N\right] &= \lim_{h\to+\infty} \widetilde{\mathbb{E}}\left[\|\pi_N \widetilde{M}_h^n(t)\|_H^2 \wedge N\right] \\ &= \lim_{h\to+\infty} \widetilde{\mathbb{E}}\left[\pi_N \|\widetilde{I}_h^n(t)\|_H^2 \wedge N\right] \le C < +\infty,\end{aligned}$$

where C is a constant independent of N. By monotone convergence it follows immediately that $\widetilde{M}^{(1)}(t)$ is a square integrable random variable with values in H. Since with this approach we did not show that $\widetilde{M}_h^n(t) \to \widetilde{M}^{(1)}(t)$ almost surely in H we cannot take $u, v \in H$ in order to study the martingale properties of the limit processes, but we can only take $u, v \in D((-A)^{-\epsilon})$. This regularity of u and v is enough to prove the required properties and conclude the proof.

2.4.5 3D Navier–Stokes Equations with Additive Noise

Let us add a few remarks on the 3D Navier–Stokes equations in a domain D, just with additive noise, which we write briefly in abstract form

$$du = Audt + B\,(u,u)\,dt + f + F\,(u) + \sum_k \sigma_k dW_t^k. \tag{2.12}$$

Writing the theory of 3D Navier–Stokes equations in the same detail as above is not consistent with the format of these notes. Therefore we shall limit ourselves to an outline of ideas. For more elements on the deterministic theory see for instance [247], [248], [200]. For a result on weak solutions not limited to additive noise see for instance [118] and for other results on additive noise, including a theory about the Kolmogorov equation and Markov selections, see [88] and [137]. For a path-by-path approach (which provides solutions on an a priori given probability space, but is not proved to be progressively measurable) see for instance [139].

The definition of a weak solution is similar to the 2D case. However, two new elements are present. The first one is that we just require *weak continuity in* H, namely continuity in the weak topology of H:

$$u \in C\left([0, T]; H_w\right) \cap L^{\infty}(0, T; H) \cap L^2(0, T; V). \tag{2.13}$$

For every test function $\phi \in H$, the function $t \mapsto \langle u(t), \phi \rangle$ is continuous. Since we assume $u \in L^{\infty}(0, T; H)$, a property like

$$u \in C\left([0, T]; D(A)'\right)$$

implies $u \in C([0, T]; H_w)$.

The second detail is that now we cannot prove the energy identity; and if u is a weak solution (in the sense of weak regularity plus the weak formulation of the equation), we cannot even prove an energy inequality. We have to include it in the definition, if we want to use it; and the existence of weak solutions satisfying the energy inequality can be established. Sometimes the weak solutions which have an energy inequality are called Leray solutions.

The other aspect which drastically changes is the interpolation inequalities. The property (b, B, P etc. are defined as in the 2D case)

$$b(u, v, w) \le \|v\|_V \|u\|_{\mathbb{L}^4} \|w\|_{\mathbb{L}^4}$$

is always true, being given by the Hölder inequality. But the Ladyzhenskaya inequality is crucially different:

$$\|f\|_{L^4} \overset{d=2}{\le} \|f\|_{W^{\frac{1}{2},2}} \le \|f\|_{L^2}^{1/2} \|f\|_{W^{1,2}}^{1/2},$$

$$\|f\|_{L^4} \overset{d=3}{\le} \|f\|_{W^{\frac{3}{4},2}} \le \|f\|_{L^2}^{1/4} \|f\|_{W^{1,2}}^{3/4}.$$

This is due to Sobolev embedding theorem in dimension d: $W^{\alpha,p}(D) \subset L^q(D)$ if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. This increase of the power of $\|f\|_{W^{1,2}}$ has tremendous consequences. In particular, from the regularity

$$u \in L^{\infty}(0, T; H) \cap L^2(0, T; V)$$

we cannot deduce anymore $u \in L^4\left(0, T; \mathbb{L}^4\right)$, property that we have used in essential way in $d = 2$. Now we only have $u \in L^{8/3}\left(0, T; \mathbb{L}^4\right)$:

$$\int_0^T \|u(t)\|_{\mathbb{L}^4}^{8/3} dt \le C \int_0^T \|u(t)\|_H^{2/3} \|u(t)\|_V^2 dt \le C \sup_{t\in[0,T]} \|u(t)\|_H^{2/3} \int_0^T \|u(t)\|_V^2 dt.$$

The Problem of Uniqueness

Let us illustrate the problem in the particular case $F = 0$, $\sigma_k = 0$. If $u^{(i)}$ are two solutions and we set $w = u^{(1)} - u^{(2)}$, we have

$$\langle w(t), \phi\rangle - \int_0^t \left(b\left(u^{(1)}, \phi, u^{(1)}\right) - b\left(u^{(2)}, \phi, u^{(2)}\right)\right)(s)\, ds = \int_0^t \langle w(s), A\phi\rangle\, ds$$

and since

$$\begin{aligned} & b\left(u^{(1)}, \phi, u^{(1)}\right) - b\left(u^{(2)}, \phi, u^{(2)}\right) - b(w, \phi, w) \\ &= b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right) \end{aligned}$$

we get

$$\begin{aligned} & \langle w(t), \phi\rangle - \int_0^t (b(w(s), \phi, w(s)))\, ds \\ &= \int_0^t \langle w(s), A\phi\rangle\, ds + \int_0^t \left(b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right)\right)(s)\, ds. \end{aligned}$$

Up to details (in particular the next fact requires Leray solutions), we have

$$\begin{aligned} \|w(t)\|_H^2 + 2\nu \int_0^t \|\nabla w(s)\|_H^2 ds &\le 2 \int_0^t \left(b\left(u^{(2)}, w, w\right) + b\left(w, w, u^{(2)}\right)\right)(s)\, ds \\ &= 2 \int_0^t b\left(w, w, u^{(2)}\right)(s)\, ds. \end{aligned}$$

But now

$$\begin{aligned} \left|b\left(w, w, u^{(2)}\right)\right| &\le C \|w\|_V \|w\|_{\mathbb{L}^4} \|u^{(2)}\|_{\mathbb{L}^4} \\ &\le C \|w\|_V^{7/4} \|w\|_H^{1/4} \|u^{(2)}\|_{\mathbb{L}^4}. \end{aligned}$$

We may use Young's inequality $ab \leq \nu a^{8/7} + C_\nu b^8$:

$$|b\left(w, w, u^{(2)}\right)| \leq \nu \|w\|_V^2 + C_\nu \|w\|_H^2 \|u^{(2)}\|_{\mathbb{L}^4}^8$$

so that

$$\|w(t)\|_H^2 + \nu \int_0^t \|\nabla w(s)\|_H^2 ds \leq C_\nu \int_0^t \|w(s)\|_H^2 \left(\|u^{(2)}(s)\|_{\mathbb{L}^4}^8 + 1\right) ds.$$

The Gronwall lemma this time does not apply because we do not know that $u^{(2)}$ is of class $L^8\left(0, T; \mathbb{L}^4\right)$; we only know $u \in L^{8/3}\left(0, T; \mathbb{L}^4\right)$.

Estimates on Galerkin and Tightness

The definition of Galerkin approximations is the same as in 2D and the first energy inequalities are proved in the same way. We get the same bounds (2.6)–(2.7). With due work we deduce that laws of u_n are tight in $L^2(0, T; H)$. A little additional work gives tightness in

$$L^2(0, T; H) \cap C\left([0, T]; D(A)'\right).$$

Moreover, we have weak convergence in the topologies of (2.6), hence any limit measure of subsequences is supported on the regularity space of the definition of a weak solution. It remains to prove that such limit measures (which exist) correspond to solutions of the 3D Navier–Stokes equations.

Definition of Solution and Convergence

Until now a solution has been a stochastic process. However, the previous construction provides only a probability measure on certain function spaces. One can always introduce a stochastic process with such a measure as a law, but it is just an artefact, it is not defined on the original probability space where the problem was formulated. Therefore we give the following definition, which is called weak in a double sense: weak probabilistically and weak analytically.

Definition 2.27 Let $u_0 \in H$ be given. A weak solution of the 3D Navier–Stokes equations (2.12) with initial condition u_0 is a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, a family of independent Brownian motions W_t^k, $k \in K$, over such space, and a stochastic process u, with paths of class (2.13), progressively measurable (adapted in H, being weakly continuous), which satisfies

$$\begin{aligned}&\langle u(t),\phi\rangle-\int_0^t b(u(s),\phi,u(s))\,ds\\&=\langle u_0,\phi\rangle+\int_0^t\langle u(s),A\phi\rangle\,ds+\sum_{k\in K}\sqrt{\lambda_k}\,\langle\sigma_k,\phi\rangle\,W_t^k\\&+\int_0^t\langle f(s)+F(u(s)),\phi\rangle ds\end{aligned}$$

for every $\phi\in D(A)$. We also require

$$\begin{aligned}\mathbb{E}\left[\|u(t)\|_H^2\right]+2\nu\int_0^t\mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right]ds\le\|u_0\|_H^2+t\sum_{k\in K}\lambda_k\|\sigma_k\|_H^2\\+\int_0^t\mathbb{E}\left[\langle f(s)+F(u(s)),u(s)\rangle\right]\,ds.\end{aligned}$$

Notice that assuming u_0 random provokes a problem: a probability space should be defined in advance; this is not compatible with the construction. An alternative then is to prescribe the law of u_0 on H.

Let us sketch the proof of existence of such solutions. In order to simplify the notation, we will neglect the dependence of the time of the processes and the functions appearing in the integrals. Let u_n be the Galerkin approximations defined above. In fact, consider for each n the pair

$$(u_n,W_n)$$

where $W_n(t):=\sum_k\sigma_k^n W_t^k$, which is a random variable with values in

$$L^2(0,T;H)\times C([0,T];H). \tag{2.14}$$

Call Q_n its law. The family $(Q_n)_{n\in\mathbb{N}}$ is tight in this space (the tightness of the second component follows from its convergence to $W(t):=\sum_k\sigma_k W_t^k$). Let us extract a subsequence $\left(Q_{n_k}\right)$ which weakly converges to a probability measure Q. Then, for every smooth compact support divergence free test vector field $\phi(t,x)$, consider the functional

$$\begin{aligned}J_\phi(u,w)&:=1\wedge\\&\left|\int_0^T\langle u,(\partial_s+A)\phi\rangle\,ds+\int_0^T b(u,\phi,u)\,ds+\int_0^T\langle f+F(u),\phi\rangle\right.\\&\left.-\int_0^T\langle w,\partial_s\phi\rangle\,ds\right|.\end{aligned}$$

Notice that, if a sequence of functions $(u_n) \subset L^2(0,T;H)$ converges strongly to u, and ϕ is bounded, then $b(u_n,\phi,u_n)$ converges to $b(u,\phi,u)$. Thus the functional J_ϕ is continuous on the product space (2.14), and bounded. Hence

$$\lim_{k\to\infty}\int J_\phi(u,w)\,Q_{n_k}(du,dw)=\int J_\phi(u,w)\,Q(du,dw).$$

But

$$\int J(u,w)\,Q_{n_k}(du,dw)$$
$$=\mathbb{E}\Bigg[1\wedge\bigg|\int_0^T\langle u_{n_k},(\partial_s+A)\phi\rangle ds+\int_0^T b\left(u_{n_k},\phi,u_{n_k}\right)ds$$
$$+\int_0^T\langle f+F\left(u_{n_k}\right),\phi\rangle-\int_0^T\langle W_{n_k},\partial_s\phi\rangle ds\bigg|\Bigg].$$

The equation satisfied by u_{n_k} may be rewritten for time-dependent test functions ϕ as we did in Chap. 1 when dealing with the Stokes problem:

$$\int_0^T\langle u_{n_k},(\partial_s+A)\phi\rangle ds+\int_0^T b\left(u_{n_k},\pi_{n_k}\phi,u_{n_k}\right)ds+\int_0^T\langle f+F\left(u_{n_k}\right),\pi_{n_k}\phi\rangle ds$$
$$-\int_0^T\langle W_{n_k},\partial_s\phi\rangle ds=0.$$

Hence

$$\int J(u,w)\,Q_{n_k}(du,dw)$$
$$=\mathbb{E}\left[1\wedge|\int_0^T b\left(u_{n_k},\phi-\pi_{n_k}\phi,u_{n_k}\right)ds+\int_0^T\langle f+F\left(u_{n_k}\right),\phi-\pi_{n_k}\phi\rangle|\right].$$

Let us prove it goes to zero:

$$\mathbb{E}\left[|\int_0^T b\left(u_{n_k},\phi-\pi_{n_k}\phi,u_{n_k}\right)ds|\right]\le\|\phi-\pi_{n_k}\phi\|_{D(A)}\mathbb{E}\left[\int_0^T\|u_{n_k}\|_H\|u_{n_k}\|_V ds\right]$$
$$\le\|\phi-\pi_{n_k}\phi\|_{D(A)}\mathbb{E}\left[\sup_{t\in[0,T]}\|u_{n_k}(t)\|_H\int_0^T\|u_{n_k}\|_V ds\right]$$

and $\|\phi-\pi_{n_k}\phi\|_{D(A)}\to 0$ (using $\phi\in D(A)$ and the commutativity of π_{n_k} with A),

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_{n_k}(t)\|_H\int_0^T\|u_{n_k}\|_V ds\right]\le C$$

by the bounds (2.6); and

$$\mathbb{E}\left[\left| \int_0^T \left\langle f + F\left(u_{n_k}\right), \phi - \pi_{n_k}\phi \right\rangle ds \right| \right]$$
$$\leq \|\phi - \pi_{n_k}\phi\|_V \left(\mathbb{E}\left[\int_0^T \|f\|_{V'} ds \right] + C\mathbb{E}\left[\int_0^T \left(1 + \|u_{n_k}\|_H\right) ds \right] \right)$$

and the argument is similar and easier.

It follows that Q satisfies

$$\int J_\phi (u, w)\, Q\,(du, dw) = 0$$

for every ϕ. Realize Q as law of $\left(\widetilde{u}, \widetilde{W}\right)$. The second marginal of Q is the law of $W := \sum_k \sigma_k dW_t^k$, being the weak limit of the second marginal of Q_{n_k}, which is the law of W_n which converges a.s. to W; hence $\widetilde{W}$ has the same law of W. Working a little bit with Gaussianity, we may check that $\widetilde{W}$ is represented as $\sum_k \sigma_k d\widetilde{W}_t^k$ where $\widetilde{W}_t^k$ are independent Brownian motions.

We have

$$\widetilde{\mathbb{E}}\Bigg[1 \wedge \Bigg| \int_0^T \langle \widetilde{u}, (\partial_s + A)\,\phi \rangle\, ds + \int_0^T b\,(\widetilde{u}, \phi, \widetilde{u})\, ds$$
$$+ \int_0^T \langle f + F\,(\widetilde{u})\,, \phi \rangle - \int_0^T \left\langle \widetilde{W}, \partial_s \phi \right\rangle ds \Bigg| \Bigg] = 0,$$

hence $\widetilde{\mathbb{P}}$-a.s.

$$\int_0^T \langle \widetilde{u}, (\partial_s + A)\,\phi \rangle\, ds + \int_0^T b\,(\widetilde{u}, \phi, \widetilde{u})\, ds + \int_0^T \langle f + F\,(\widetilde{u})\,, \phi \rangle$$
$$- \int_0^T \left\langle \widetilde{W}, \partial_s \phi \right\rangle ds = 0$$

for every given ϕ (the negligible set where this may not hold depends on ϕ). Taking first a dense countable set of ϕ's, so that we can invert the quantifiers and then a convergence argument based on pathwise regularity, we deduce that, $\widetilde{\mathbb{P}}$-a.s., we have

$$\int_0^T \langle \widetilde{u}, (\partial_s + A)\,\phi \rangle\, ds + \int_0^T b\,(\widetilde{u}, \phi, \widetilde{u})\, ds + \int_0^T \langle f + F\,(\widetilde{u})\,, \phi \rangle$$
$$- \int_0^T \left\langle \widetilde{W}, \partial_s \phi \right\rangle ds = 0$$

for all ϕ, which is the definition of a weak solution.

We have adapted the previous proof from the case of convergence to a deterministic equation (see for instance [185], Chapter 4, for an example in the framework of particle systems). It does not extend, however, to state-dependent noise since the stochastic integral is not continuous in the noise.

2.5 Summary

The main techniques illustrated in this chapter are the use of the Itô formula, an interesting idea for uniqueness, its consequence through a criterion of Gyongy and Krylov, and especially the method of compactness, quite universal and useful in many fields.

Similarly to the remarks in the summary of Chap. 1, a main open problem related to this chapter is the link between a real irregular boundary (or other mechanisms responsible for noise terms) and stochastic models of fluids; here the problem is enriched by the dependence on the flow intensity, a very realistic feature, which poses a new technical issue, namely the presence of the Wong–Zakai corrector in the limit equation, as discussed in Sects. 5.5 and 5.6. Also the case, not treated here for simplicity (but see [118]), of noise depending on the gradient of the solution is relevant, since vortex production due to instability is related to shear. We discuss noise depending on the gradient of the solution in the next two chapters but the physical origin is different and the mathematical dependence is linear, while in the case of shear dependence it may also be nonlinear. This issue should be investigated much better.

We have also seen that noise introduces energy, on average, hence the model should be corrected by an energy loss.

Chapter 3
Transport Noise in the Heat Equation

This chapter and the following one present miscellaneous topics around the concept of transport noise, which came recently to the attention of researchers as an additional term of the Navier–Stokes equations, although relevant older works existed [52, 53, 145, 216, 217, 259]; its effects on passive scalars are on the contrary a classical subject nowadays in mathematical physics, see for instance [65, 69, 143, 149, 198, 206, 242]. We distinguish these two directions, mathematically related but physically very different:

1. the case when the transport noise affects a passive scalar (let us call this the exogenous case);
2. the case when the transport noise affects the fluid equation itself (endogenous case).

We devote Chap. 3 to the exogenous case and Chap. 4 to the endogenous one. The second level of subdivision is:

(a) action of transport noise on scalars;
(b) action on vector fields.

Essentially, our present understanding is limited to case (a). We devote to it Sects. 3.2, 3.3, 4.1, and 4.2 considering both exogenous and endogenous actions.

Finally, we discuss case (b) in Sects. 3.4 and 4.4 where we stress the limitations of our understanding. In spite of these, we hope it will be possible in the future to throw light, on this difficult subject. Due to the strong connections between this chapter and the following one we avoid adding a summary section at the end of this chapter. For this scope we suggest to read Sect. 4.5 and Chap. 5, which summarize not only the topics of Chaps. 3 and 4 but, in a sense, the meaning of these lecture notes as a journey from perturbations introduced by boundary roughness to regularization effects related to turbulence.

Concerning the investigation of transport–type noise in Navier–Stokes, Euler and related equations, the so–called endogenous case, apart from the pioneering

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works mentioned above, a great impetus has been given by two sources. One of them has been the variational approach given by D.D. Holm [177], see also [5, 6, 68, 77, 81, 82, 100, 101, 141, 150, 178] among several other works in this direction. Another one has been the realization that transport noise may have special regularizing properties, like restoring uniqueness to a PDE which has multiple solutions, see for instance [8, 20, 121] among many others for the case of scalar transport equations, [130, 131] for the transport of vector fields, and [10–13, 35, 122, 152] for nonlinear models, including the so–called dyadic model of turbulence. Some further results are summarized in [36, 113]. A new mechanism of regularization by noise was discovered more recently and will be presented below in a particular case in Sect. 4.4. Among the works in this direction let us quote [19, 97, 115, 117, 123, 126, 210].

Besides the works already quoted, several others contributed to the development of our present understanding of transport terms in fluid dynamics. Some of them are quoted below in the specific sections; let us mention in addition [15, 60, 62, 74–76, 81, 83, 84, 194, 201, 202, 220, 230, 244] among others.

3.1 Introduction: Stochastic Heat Transport

Let us oversimplify the fluid dynamics near the boundary. The following view is highly phenomenological and should be subject to much deeper research (see some progress in [135]).

We assume that the fluid, in a region near the boundary, may be approximately described by the equations

$$\partial_t u + \nabla p = \nu \Delta u - \frac{1}{\epsilon} u + \frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k,$$

$$\operatorname{div} u = 0,$$

$$u|_{\partial D} = 0.$$

This is the Stokes model, strongly incorrect in itself for turbulent fluids, but complemented by the creation of eddies/vortices (the term $\frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k$) and an extra-dissipation term of friction type ($-\frac{1}{\epsilon} u$) to compensate the extra input of energy (in the average) due to the noise.

We have intentionally parametrized the problem by $\epsilon > 0$, in the very precise way written above, because we want to explore here a special scaling limit. Physical motivations for this special rescaling can be found for instance in [133, 134, 207]. Let us also, from now on, denote u by u^ϵ. The abstract semigroup formulation of this problem, with A given by the operator $\nu P \Delta$ as in the previous chapters, is

$$u^\epsilon (t) = e^{t\left(A - \frac{1}{\epsilon}\right)} u_0 + \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{(t-s)\left(A - \frac{1}{\epsilon}\right)} \sigma_k d W_s^k.$$

In Chap. 1, in order to avoid Itô integrals and cover rough noise sources of very different type, we integrated by parts and used the following formulation:

$$u^{\epsilon}(t) = e^{t\left(A-\frac{1}{\epsilon}\right)}u_0 + \frac{1}{\epsilon}\sum_{k\in K}\sigma_k W_t^k + \frac{1}{\epsilon}\sum_{k\in K}\int_0^t e^{(t-s)\left(A-\frac{1}{\epsilon}\right)}\left(A-\frac{1}{\epsilon}\right)\sigma_k W_s^k ds.$$

When W_s^k are independent Brownian motions, both formulations are meaningful and they are equivalent. In the following lines we shall apply a Fubini–type theorem to the stochastic integral: one way to justify it rigorously is precisely to use the last formulation which involves only Lebesgue integrals.

Let us introduce two notations:

$$W^{\epsilon}(t,x) = \int_0^t u^{\epsilon}(s,x)\,ds,$$
$$W(t,x) = \sum_{k\in K}\sigma_k(x)\,W_t^k.$$

Then

$$\begin{aligned}
W^{\epsilon}(t) &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\int_0^s e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k ds \\
&= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\int_r^t e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k ds dW_r^k \\
&= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\left(A-\frac{1}{\epsilon}\right)^{-1}\left[e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}-1\right]\sigma_k dW_r^k \\
&= \frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1}\sum_{k\in K}\int_0^t e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k - \frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1}W(t).
\end{aligned}$$

Now we use the fact (well–known in the framework of Yosida approximations of semigroup theory, see [225]) that

$$\lim_{\lambda\to\infty}\lambda(\lambda-A)^{-1}h = h$$

for all $h\in H$; A^{-1} being compact in our example, we can easily verify this property using the spectral decomposition. With minor additional arguments that we leave as exercise, it follows that:

Lemma 3.1

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\|W^{\epsilon}(t)-W(t)\|_H^2\right] = 0.$$

The result is also uniform in time, with supremum inside the expected value. The message of this lemma is that u converges in distribution to a white noise, the time derivative of the space-dependent Brownian motion W.

Why is this an interesting regime? Let us investigate this issue in the case of the evolution of an auxiliary quantity: heat. Assume the fluid has a variable temperature and is not strongly influenced by temperature, hence we do not change its equation of motion. But temperature, next indicated by $\theta(t, x)$, evolves according to the diffusion-transport equation

$$\partial_t \theta + u \cdot \nabla\theta = \kappa\Delta\theta + q,$$

where $\kappa > 0$, typically small, is the heat diffusion constant and $u \cdot \nabla\theta$ is the transport due to the fluid motion; q is a heat source. If we take the limit $\epsilon \to 0$ in the model of fluid above and we apply the heuristics of the Wong–Zakai result, we find the model

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla\theta) \circ \partial_t W^k = \kappa\Delta\theta + q,$$

where the symbol $\circ$ stands for the Stratonovich operation. In Chap. 5 we explain why the correct Itô interpretation of this equation is

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla\theta)\, \partial_t W^k = (\kappa\Delta + \mathcal{L})\,\theta + q, \tag{3.1}$$

where the stochastic term is now understood in the classical Itô sense and $\mathcal{L}$ is the linear differential operator

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sigma_k(x) \cdot \nabla(\sigma_k(x) \cdot \nabla\theta(x)).$$

The result of this modeling step is that we end-up with model (3.1) for the heat diffusion under a turbulent velocity field. Taking (heuristically at this stage) the expectation of each term and introducing the mean temperature profile

$$\Theta(t, x) = \mathbb{E}[\theta(t, x)]$$

we get

$$\partial_t \Theta = (\kappa\Delta + \mathcal{L})\,\Theta + q.$$

If the noise has suitable properties, the elliptic operator $\mathcal{L}$ strongly increases the dissipation of the term $\kappa\Delta$. Moreover, we shall prove that the random field $\theta(t, x)$ is close to its average $\Theta(t, x)$ under suitable assumptions. This will lead to the statement that *turbulent transport increases the original diffusion*, a fact that is observed in experiments (it corresponds, in our daily life, to the fact that when we stir coffee the temperature rapidly decreases). This model has the power to explain a well–known experimental phenomenon, the so–called *eddy diffusion*.

The results outlined in this introductory section will be developed below in some detail but additional information can be found in the paper that initiated this research [147] and in subsequent references like [116, 223]; a different scaling can be seen in [125].

3.1.1 Divergence Form of the Operator

Let us discuss the additional term $\mathcal{L}\theta$ appearing in Eq. (3.1). Componentwise we can write

$$(\mathcal{L}\theta)(x) = \frac{1}{2}\sum_{k\in K}\sum_{i,j=1}^{d} \sigma_k^i(x)\,\partial_i\left(\sigma_k^j(x)\,\partial_j\theta(x)\right).$$

Since $\sum_{i=1}^{d}\partial_i\sigma_k^i(x) = 0$, we deduce also

$$(\mathcal{L}\theta)(x) = \frac{1}{2}\sum_{k\in K}\sum_{i,j=1}^{d} \partial_i\left(\sigma_k^i(x)\,\sigma_k^j(x)\,\partial_j\theta(x)\right).$$

Let us now introduce for the first time (but this doesn't mean it is a secondary concept) the covariance function of the noise, covariance with respect to the space variable. It is defined as

$$Q(x,y) = \mathbb{E}\left[W(t,x)\otimes W(t,y)\right] \qquad x,y\in D$$

and it is easily found to be

$$Q(x,y) = \sum_{k\in K}\sigma_k(x)\otimes\sigma_k(y).$$

Therefore we have found

$$(\mathcal{L}\theta)(x) = \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\left(Q_{ij}(x,x)\,\partial_j\theta(x)\right).$$

This is an elliptic operator in divergence form. Ellipticity comes from the property

$$\sum_{i,j=1}^{d} Q_{ij}(x,x)\,\xi_i\xi_j = \mathbb{E}\left[|W(t,x)\cdot\xi|^2\right] \geq 0$$

for all $\xi = (\xi_1,\ldots,\xi_d)\in\mathbb{R}^d$.

3.2 Existence and Uniqueness for the Heat Equation with Transport Noise

In this section we want to prove an existence and uniqueness result for the equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta)\, \partial_t W^k = (\kappa \Delta + \mathcal{L})\, \theta + q$$

in a bounded regular domain $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Other domains and boundary conditions can be studied as well.

We know two very efficient methods:

1. variational;
2. semigroups.

3.2.1 Variational Method

This method has been developed by Pardoux [224] and Krylov–Rozovskii [191], in the more general context of SPDEs with monotone operators. We limit ourselves to the ideas.

- One has to introduce a sequence of approximating problems which have a unique solution by known results. We skip this step.
- On these approximations, one has to prove estimates independent of the approximating parameter.
- *We perform such a step on the true equation, in the style of a priori estimates: we assume that we have a smooth solution and see which estimates hold.*
- Such estimates imply the existence of weakly convergent subsequences, sufficient to pass to the limit, the equation being linear. We skip the details of this step.

A Priori Estimates Using Stratonovich Formulation

If we use the Stratonovich formulation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k = \kappa \Delta \theta + q$$

and we accept that the rules of calculus (being the limit of smooth noise) are the classical ones, we get (recall $\operatorname{div}\sigma_k = 0$)

$$\frac{d}{dt}\|\theta(t)\|_{L^2}^2 = -2\left\langle \theta, \sum_{k\in K} (\sigma_k \cdot \nabla\theta) \circ \partial_t W^k \right\rangle + 2\langle \theta, \kappa\Delta\theta\rangle + 2\langle \theta, q\rangle$$
$$= -2\kappa\|\nabla\theta(t)\|_{L^2}^2 + 2\langle\theta, q\rangle$$

because

$$2\int_D \langle\theta, \sigma_k \cdot \nabla\theta\rangle = 2\int_D \theta(x)\,\sigma_k(x)\cdot\nabla\theta(x)\,dx$$
$$= \int_D \sigma_k(x)\cdot\nabla\theta^2(x)\,dx = -\int_D \operatorname{div}\sigma_k(x)\,\theta^2(x)\,dx = 0.$$

Therefore

$$\frac{d}{dt}\|\theta(t)\|_{L^2}^2 + 2\kappa\|\nabla\theta(t)\|_{L^2}^2 = 2\langle\theta(t), q(t)\rangle$$

leading to the a.s. (deterministic!) estimates. By easy classical steps one gets

$$\sup_{t\in[0,T]} \|\theta(t)\|_{L^2}^2 \le C$$
$$\int_0^T \|\nabla\theta(s)\|_{L^2}^2 ds \le C$$

with C depending only on κ, $\|\theta_0\|_{L^2}$, $\int_0^T \|q(s)\|_{L^2}^2 ds$.

A Priori Estimates Using Itô Formulation

Obviously the final result will be the same, but let us see the computation when the equation contains the Ito–Stratonovich corrector; and the Itô formula is used to perform computations, with its correcting term. We use the Itô formulation

$$\partial_t\theta + \sum_{k\in K} (\sigma_k \cdot \nabla\theta)\,\partial_t W^k = (\kappa\Delta + \mathcal{L})\,\theta + q$$

and we apply the Itô formula, to get

$$
\begin{aligned}
d\|\theta(t)\|_{L^2}^2 &= -2\sum_{k\in K}\langle\theta,(\sigma_k\cdot\nabla\theta)\rangle\, dW^k + 2\langle\theta,(\kappa\Delta+\mathcal{L})\theta+q\rangle\, dt \\
&\quad + \sum_{k\in K}\|\sigma_k\cdot\nabla\theta\|_{L^2}^2 dt \\
&= -2\kappa\|\nabla\theta(t)\|_{L^2}^2 + 2\langle\theta,q\rangle - 2\frac{1}{2}\int_D\sum_{ij} Q(x,x)\,\partial_i\theta\partial_j\theta dx dt \\
&\quad + \sum_{k\in K}\int_D\sum_{ij}\sigma_k^i(x)\,\partial_i\theta\sigma_k^j(x)\,\partial_j\theta dx dt.
\end{aligned}
$$

We obtain the same result as above. *At the level of energy estimates, the Itô term and the corrector completely balance each other.*

Maximum Principle a Priori Estimates

Let us also describe a side estimate of some interest. Consider the Kolmogorov equation

$$
\partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta + q,
$$
$$
\theta|_{t=0} = \theta_0
$$

on a time interval $[0,T]$. Introducing $\theta_T(t) = \theta(T-t)$, $u_T(t) = u(T-t)$, $q_T(t) = q(T-t)$, we get

$$
\partial_t\theta_T - u_T\cdot\nabla\theta_T + \kappa\Delta\theta_T + q_T = 0,
$$
$$
\theta_T|_{t=T} = \theta_0.
$$

Denoting by $\varphi_{s,t}(x)$ the flow associated to the equation

$$
\begin{aligned}
d\varphi_{s,t}(x) &= -u_T\left(t,\varphi_{s,t}(x)\right)dt + \sqrt{2\kappa}dB_t \qquad t\in[s,T], \\
\varphi_{s,s}(x) &= x,
\end{aligned}
$$

where B_t is an auxiliary Brownian motion, we have

$$
\begin{aligned}
d\theta_T\left(t,\varphi_{s,t}(x)\right) &= \partial_t\theta_T dt + \nabla\theta_T\cdot d\varphi_{s,t} + \kappa\Delta\theta_T dt \\
&= u_T\cdot\nabla\theta_T dt - \kappa\Delta\theta_T dt - q_T dt \\
&\quad - \nabla\theta_T\cdot u_T dt + \nabla\theta_T\cdot\sqrt{2\kappa}dB_t + \kappa\Delta\theta_T dt \\
&= -q_T dt + \nabla\theta_T\cdot\sqrt{2\kappa}dB_t
\end{aligned}
$$

and therefore

$$\mathbb{E}\left[\theta_0\left(\varphi_{s,T}(x)\right)\right]-\theta_T(s,x)=-\int_s^T \mathbb{E}\left[q_T\left(t,\varphi_{s,t}(x)\right)\right]dt.$$

Going back to the original variables we have

$$\mathbb{E}\left[\theta_0\left(\varphi_{s,T}(x)\right)\right]-\theta(T-s,x)=-\int_s^T \mathbb{E}\left[q\left(T-t,\varphi_{s,t}(x)\right)\right]dt,$$

namely,

$$\theta(t,x)=\mathbb{E}\left[\theta_0\left(\varphi_{T-t,T}(x)\right)\right]+\int_{T-t}^T \mathbb{E}\left[q\left(T-r,\varphi_{T-t,r}(x)\right)\right]dr.$$

We deduce in particular

$$\|\theta(t)\|_\infty \le \|\theta_0\|_\infty+\int_0^T \|q(r)\|_\infty dr. \tag{3.2}$$

The previous computation, performed here heuristically, can be made rigorous by convolution under very general assumptions. With due effort based on the theory of stochastic flows, it works also for the equation

$$\partial_t\theta+\sum_{k\in K}(\sigma_k\cdot\nabla\theta)\circ\partial_t W^k=\kappa\Delta\theta+q$$

in Stratonovich form, being the limit of equations with regular coefficients. The final result is the same, a deterministic (a.s.) inequality in the supremum norm, a kind of maximum principle estimate.

3.2.2 Semigroup Method

Opposite to the previous subsections which contain only an outline of the variational approach, here we give all the details of the semigroup approach. Initially, it was more difficult to understand how to apply semigroups to this kind of equations, since the regularity issues about the stochastic term are "at the limit", so to speak. The breakthrough came with the papers by Da Prato [85, 86], developed further in the book [90]. The theory was later assessed by a series of works, see the book [111]. Recently, this theory has been much extended by Agresti and Veraar [1, 2], Hytönen et al. [179, 180].

Consider the equation

$$\partial_t \theta + \sum_{k\in K} (\sigma_k \cdot \nabla \theta)\, \partial_t W^k = (\kappa \Delta + \mathcal{L})\, \theta + q. \tag{3.3}$$

Let: $H = L^2(D)$, $V = W_0^{1,2}(D)$, $D(A) = W^{2,2}(D) \cap V$, $A : D(A) \subset H \to H$,

$$A\theta = (\kappa \Delta + \mathcal{L})\, \theta.$$

e^{tA}, $t \geq 0$, the analytic semigroup generated by A (under minimal regularity assumptions on $Q(x, x)$, see [225, Chapter 7]). Then

$$\theta(t) = e^{tA}\theta_0 - \sum_{k\in K} \int_0^t e^{(t-s)A} (\sigma_k \cdot \nabla \theta(s))\, dW_s^k + \int_0^t e^{(t-s)A} q(s)\, ds.$$

We want to solve this equation by iterations. These equations are not trivial because there is a gradient of θ on the right–hand side and thus iteration requires that also the left–hand side accepts a gradient.

Notions of Solution and Main Result

Even if the definitions of $H,\ V,\ A,\ D(A)$ changed with respect to the previous chapters we keep the same notations. In particular, in the sequel we denote by V' the dual of V. We may identify H with H' and thus write $D(A) \subset V \subset H \subset V'$ with continuous dense embeddings. The scalar product $\langle \cdot, \cdot \rangle$ in H "extends" to the dual pairing between V and V', which will be denoted by the same notation. As already done in a previous chapter, let us denote by $L^2_{\mathcal{F}}(0, T; V)$ the space of progressively measurable process with values in V and by $C_{\mathcal{F}}([0, T]; H)$ the space of continuous adapted square integrable processes. Assume σ_k smooth enough, $\theta_0 \in H$, $q \in L^2(0, T; H)$. A stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is assumed to be given (thus we deal with strong solutions).

Definition 3.2 A stochastic process

$$\theta \in C_{\mathcal{F}}([0, T]; H) \cap L^2_{\mathcal{F}}(0, T; V)$$

is a weak solution if, for every $\phi \in D(A)$, we have

$$\langle \theta(t), \phi \rangle = \langle \theta_0, \phi \rangle + \int_0^t \langle \theta(s), (\kappa \Delta + \mathcal{L})\, \phi \rangle\, ds$$
$$+ \int_0^t \langle q(s), \phi \rangle\, ds + \sum_{k\in K} \int_0^t \langle \theta(s), \sigma_k \cdot \nabla \phi \rangle\, dW_t^k$$

for every $t \in [0, T]$, $\mathbb{P}$-a.s.

Notice that the stochastic integrals are well–defined since $\sigma_k \cdot \nabla\phi \in H$, hence the integrand is a continuous adapted process; the deterministic integral is obviously well–defined, since $s \mapsto \langle \theta(s), (\kappa\Delta + \mathcal{L})\phi \rangle$ is $\mathbb{P}$-a.s. continuous.

In the following alternative definition we use the heat semigroup e^{tA}.

Definition 3.3 A stochastic process

$$\theta \in C_{\mathcal{F}}([0,T]; H) \cap L^2_{\mathcal{F}}(0,T; V)$$

is a mild solution if the following identity holds:

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds - \sum_{k\in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla\theta(s)\, dW_s^k$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s.

Proposition 3.4 *The two notions of solution coincide.*

The proof is not difficult and similar to one shown in Chap. 1 for the Stokes problem. However, it can be found in [127]. The main result proved below is:

Theorem 3.5 *For every $\theta_0 \in H$ and $q \in L^2(0,T;H)$, there exists one and only one (weak or mild) solution.*

General Parabolic Equations with Itô–Type Transport Noise

In order to fully appreciate certain aspects of the previous result, consider the more general problem: the equation

$$\partial_t\theta + \sum_{k\in K} (\sigma_k \cdot \nabla\theta)\, \partial_t W^k = \sum_{i,j=1}^d \partial_j \left(a_{ij}(x)\, \partial_i\theta\right) + q, \tag{3.4}$$

where $a_{i,j}$ is strongly elliptic and sufficiently regular so that the operator

$$A\theta = \sum_{i,j=1}^d \partial_j \left(a_{i,j}(x)\, \partial_i\theta\right)$$

generates an analytic semigroup. The notions of solutions are the same.

Theorem 3.6 *Assume the exists $\eta < 1$ such that*

$$\frac{1}{2}\sum_{k\in K} (\sigma_k(x) \cdot \xi)^2 \leq \eta \sum_{i,j=1}^d a_{ij}(x)\, \xi_i \xi_j \tag{3.5}$$

for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$*. Then, for every* $\theta_0 \in H$*, there exists one and only one (weak or mild) solution.*

Auxiliary Variables and End of the Proof

In order to study the equation

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds - \sum_{k\in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla\theta(s)\, dW_s^k$$

let us consider the auxiliary system

$$v_h(t) = \sigma_h \cdot \nabla e^{tA}\theta_0 + \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} q(s)\, ds$$
$$- \sum_{k\in K} \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} v_k(s)\, dW_s^k$$

for $h \in K$. These two problems are equivalent (specifying correctly the function spaces): if $\theta(t)$ is a solution of the first one then

$$v_k(t) := \sigma_k \cdot \nabla\theta(t),$$
$$v(t) := (v_k(t))_{k\in K}$$

is a solution of the second one; and if $v(t) := (v_k(t))_{k\in K}$ is a solution of the second one, then $\theta(t)$ defined by

$$\theta(t) = e^{tA}\theta_0 + \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} q(s)\, ds - \sum_{k\in K} \int_0^t e^{(t-s)A} v_k(s)\, dW_s^k$$

is a solution of the first one. Up to details related to continuity properties of stochastic convolutions, the key lemma to prove the theorem for the first equation is the following result for the second one.

Consider the space X_T of vectors $(v_k(\cdot))_{k\in K}$ such that $v_k \in L^2_{\mathcal{F}}(0, T; H)$ and, in the case when K is countable,

$$\|v\|_T^2 := \sum_{h\in K} \mathbb{E} \int_0^T \|v_h(t)\|_H^2 dt < \infty.$$

It is a Hilbert space and $\|v\|_T$ is the induced norm.

Proposition 3.7 *There exists a unique solution* $(v_k(\cdot))_{k\in K} \in X_T$.

Proof

Step 1 (preparation) Notice that, by assumption (3.5),

$$\sum_{k\in K}\|\sigma_k\cdot\nabla f\|_{L^2}^2=\int_D\sum_{k\in K}(\sigma_k(x)\cdot\nabla f(x))^2\,dx$$
$$\leq 2\eta\int_D\sum_{i,j=1}^d a_{ij}(x)\,\partial_i f(x)\,\partial_j f(x)\,dx$$
$$=-2\eta\int_D(Af)(x)\,f(x)\,dx=-2\eta\,\langle Af,f\rangle$$

for every $f\in D(A)$. We use this fact in the inequalities below.
Moreover, we use the following fact:

$$-2\int_0^T\left\langle Ae^{tA}\theta_0,e^{tA}\theta_0\right\rangle dt=-\int_0^T\frac{d}{dt}\left\langle e^{tA}\theta_0,e^{tA}\theta_0\right\rangle dt$$
$$=-\left(\|e^{TA}\theta_0\|_H^2-\|\theta_0\|_H^2\right)\leq\|\theta_0\|_H^2.$$

Similarly, one has

$$-2\int_0^T\int_s^T\left\langle Ae^{(t-s)A}v_k(s),e^{(t-s)A}v_k(s)\right\rangle dtds$$
$$=-\int_0^T\int_s^T\frac{d}{dt}\left\langle e^{(t-s)A}v_k(s),e^{(t-s)A}v_k(s)\right\rangle dtds$$
$$=-\int_0^T\left(\|e^{(T-s)A}v_k(s)\|_{L^2}^2-\|v_k(s)\|_{L^2}^2\right)ds$$
$$\leq\int_0^T\|v_k(s)\|_{L^2}^2ds.$$

Step 2 (fixed point) Consider the map Γ defined on X_T as

$$(\Gamma v)_h(t):=w_h(t)+\sum_{k\in K}\int_0^t\sigma_h\cdot\nabla e^{(t-s)A}v_k(s)\,dW_s^k$$

$h\in K$, where we have set

$$w_h(t):=\sigma_h\cdot\nabla e^{tA}\theta_0+\int_0^t\sigma_h\cdot\nabla e^{(t-s)A}q(s)\,ds.$$

We prove it takes values in X_T and it is a contraction; thus it has a unique fixed point. Notice that, opposite to many other applications of contraction mapping principle, we do not need to take T small.

Using a result of the first step and similar estimates for the convolution integral, we get

$$\sum_{h\in K}\int_0^T \mathbb{E}\left[\|w_h(t)\|_{L^2}^2\right]dt \le C_1 < \infty.$$

Moreover, from the isometry formula and the Fubini theorem,

$$\begin{aligned}
&\sum_{h\in K}\int_0^T \mathbb{E}\left[\|\sum_{k\in K}\int_0^t \sigma_h\cdot\nabla e^{(t-s)A}v_k(s)\,dW_s^k\|_{L^2}^2\right]dt \\
&=\sum_{h\in K}\int_0^T\int_s^T \mathbb{E}\left[\sum_{h\in K}\|\sigma_h\cdot\nabla e^{(t-s)A}v_k(s)\|_{L^2}^2\right]dtds \\
&\le 2\eta\sum_{k\in K}\int_0^T\int_s^T\left\langle Ae^{(t-s)A}v_k(s),e^{(t-s)A}v_k(s)\right\rangle dtds \\
&\le \eta\|v\|_T^2,
\end{aligned}$$

having used the two facts proved in Step 1. Therefore $\Gamma v\in X_T$. By the same computation we have

$$\|\Gamma v'-\Gamma v''\|_T^2\le\eta\|v'-v''\|_T^2$$

and $\eta<1$, hence Γ is a contraction. ■

Super-Parabolicity Condition and Stratonovich Formulation

We have solved the general parabolic equation (3.4) under assumption (3.5), sometimes called the super–parabolicity condition, very famous in the theory of nonlinear filtering and Zakai equations (cf. [191, 224, 235, 236]). The parabolic equation

$$\partial_t\theta=\sum_{i,j=1}^d\partial_j\left(a_{ij}(x)\,\partial_i\theta\right)$$

is well–posed when a_{ij} is strongly parabolic, namely when there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\xi_j \geq \nu\|\xi\|^2$$

for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. The condition of the stochastic case is therefore much more restrictive. However, when the problem (3.4) comes from a Stratonovich equation of the form (3.3), we have

$$a_{ij}(x) = \kappa\delta_{ij} + \frac{1}{2}Q_{ij}(x, x)$$

with

$$Q_{ij}(x, x) = \sum_{k\in K} \sigma_k^i(x)\,\sigma_k^j(x).$$

The super-parabolicity condition in this case requires us to find $\eta \in (0, 1)$ such that

$$\begin{aligned}\frac{1}{2}\sum_{k\in K}(\sigma_k(x)\cdot\xi)^2 &\leq \eta\sum_{i,j=1}^{d}\left(\kappa\delta_{ij} + \frac{1}{2}\sum_{k\in K}\sigma_k^i(x)\,\sigma_k^j(x)\right)\xi_i\xi_j \\ &= \eta\kappa\|\xi\|^2 + \frac{\eta}{2}\sum_{k\in K}(\sigma_k(x)\cdot\xi)^2,\end{aligned}$$

namely such that

$$\sum_{k\in K}(\sigma_k(x)\cdot\xi)^2 \leq \frac{2\eta\kappa}{1-\eta}\|\xi\|^2.$$

Under the summability conditions which guarantee to have $Q(x, y)$ well–defined and bounded, such an η exists, sufficiently close to 1. Therefore the Stratonovich equation is always well–posed.

3.2.3 The Equation for the Average

We have immediately a result if we take the average, called as above

$$\Theta(t, x) := \mathbb{E}[\theta(t, x)].$$

We assume here that $\theta_0 \in H$ is deterministic.

Proposition 3.8 *If* $\theta(t,x)$ *is the solution given by Theorem 3.5, then* $\Theta(t,x)$ *is a (weak or mild) solution of the deterministic equation*

$$\partial_t \Theta = (\kappa\Delta + \mathcal{L})\Theta + q,$$
$$\Theta|_{t=0} = \theta_0.$$

Proof We take $q = 0$ for brevity. Take for instance the weak formulation, for $\phi \in D(A)$:

$$\langle \theta(t), \phi \rangle = \langle \theta_0, \phi \rangle + \int_0^t \langle \theta(s), (\kappa\Delta + \mathcal{L})\phi \rangle \, ds + \sum_{k \in K} \int_0^t \langle \theta(s), \sigma_k \cdot \nabla \phi \rangle \, dW_t^k.$$

The stochastic integral $\int_0^t \langle \theta(s), \sigma_k \cdot \nabla\phi \rangle \, dW_t^k$ is a martingale because $\theta \in L^2_{\mathcal{F}}(0,T;H)$ (it is much more than this). Therefore

$$\langle \Theta(t), \phi \rangle = \langle \theta_0, \phi \rangle + \int_0^t \langle \Theta(s), (\kappa\Delta + \mathcal{L})\phi \rangle \, ds.$$

Moreover,

$$\Theta \in C([0,T];H) \cap L^2(0,T;V)$$

as a consequence of the property

$$\theta \in C_{\mathcal{F}}([0,T];H) \cap L^2_{\mathcal{F}}(0,T;V).$$

Therefore it is a weak solution. The proof that it is a mild solution is similar, or it follows from the equivalence between the two concepts, under our regularity, in the deterministic case. ∎

3.3 When θ Is Close to Θ

In the previous section we have shown than the average Θ satisfies an equation with enhanced dissipation; this fact is well–known, see for instance [206, Chapter 4]. The behavior of the stochastic process θ may be, however, very different, a priori. In this section we show conditions under which θ is close to Θ, hence producing the dissipative properties of Θ, in a suitable sense. When so, we may speak of *eddy dissipation*: thanks to the noise, the passive scalar has dissipative properties similar to those of the solution of a deterministic equation with enhanced dissipation. Starting from the idea of [147] (see also [125]), several results in this direction have been proved, [114, 115, 115–117, 201, 202].

This research line intersects with the study of mixing properties and enhanced dissipation due to deterministic and stochastic vector fields. The deterministic literature on this subject is already too diverse for easy references; in the stochastic case let us mention [18, 21, 22, 99, 117, 153].

3.3.1 Main Assumption and Result

Define $\varepsilon_{Q,\kappa} \geq 0$ as the smallest number such that

$$\begin{aligned}&\int\int v(x)^T Q(x,y)\, v(y)\, dxdy \\ &\leq \varepsilon_{Q,\kappa} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x)\, v(x) \right) dx\end{aligned} \tag{3.6}$$

for all $v \in L^2\left(D, \mathbb{R}^d\right)$. When $v(x) = f(x)\nabla w(x)$, it gives us

$$\begin{aligned}&\int\int v(x)^T Q(x,y)\, v(y)\, dxdy \\ &\leq \varepsilon_{Q,\kappa} \int |f(x)|^2 \left(\kappa |\nabla w(x)|^2 + \frac{1}{2} \nabla w(x)^T Q(x,x)\, \nabla w(x) \right) dx \\ &\leq -\varepsilon_{Q,\kappa} \|f\|_\infty^2 \langle Aw, w\rangle.\end{aligned}$$

In the next theorem we assume $\theta_0 \in L^\infty(D)$, $q \in L^\infty([0,T]\times D)$. Call $C_\infty(T, \theta_0, q) > 0$ a constant such that

$$\sup_{s\in[0,T]} \mathbb{E}\left[\|\theta(s)\|_\infty^2\right] \leq C_\infty(T, \theta_0, q).$$

In Sect. 3.2.1 above we have outlined one method to prove a bound of this form, in that case even an a.s. bound:

$$\|\theta(t)\|_\infty \leq \|\theta_0\|_\infty + T\|q\|_\infty.$$

However, there are other bounds available, on the average, using regularity theory for $\theta(t)$, see [127], which improve the dependence on T.

Theorem 3.9 *For every $\phi \in L^2(D)$,*

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] \leq \varepsilon_{Q,\kappa} \|\phi\|_{L^2}^2 C_\infty(T, \theta_0, q).$$

Proof Recall the identity

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\,ds - \sum_{k\in K}\int_0^t e^{(t-s)A}\sigma_k \cdot \nabla\theta(s)\,dW_s^k.$$

Here $e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\,ds$ is precisely $\Theta(t)$, hence

$$\theta(t) - \Theta(t) = -\sum_{k\in K}\int_0^t e^{(t-s)A}\sigma_k \cdot \nabla\theta(s)\,dW_s^k.$$

If $\phi \in H$,

$$\langle \theta(t) - \Theta(t), \phi\rangle = \sum_{k\in K}\int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla\theta e^{(t-s)A}\phi\right\rangle dW_s^k.$$

Then (here we take advantage of the cancellations of Itô integrals)

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] = \sum_{k\in K}\mathbb{E}\left[\int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla e^{(t-s)A}\phi\right\rangle^2 ds\right].$$

Write $\phi_{t,s} := e^{(t-s)A}\phi$. Then

$$\begin{aligned}
&\sum_{k\in K}\left\langle \theta(s), \sigma_k \cdot \nabla\phi_{t,s}\right\rangle^2 \\
&\quad = \sum_{k\in K}\int\int \theta(s,x)\,\theta(s,y)\,\sigma_k(x)\cdot\nabla\phi_{t,s}(x)\,\sigma_k(y)\cdot\nabla\phi_{t,s}(y)\,dxdy \\
&\quad = \int\int \theta(s,y)\,\nabla\phi_{t,s}(y)^T\, Q(x,y)\,\nabla\phi_{t,s}(x)\,\theta(s,x)\,dxdy \\
&\quad \le -\varepsilon_{Q,\kappa}\|\theta(s)\|_\infty^2 \left\langle Ae^{(t-s)A}\phi, e^{(t-s)A}\phi\right\rangle.
\end{aligned}$$

Therefore, with the notation $C_\infty(T, \theta_0, q)$,

$$\begin{aligned}
&\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] \\
&\quad \le \varepsilon_{Q,\kappa} C_\infty(T, \theta_0, q)\int_0^t \left\langle (-A)\,e^{(t-s)A}\phi, e^{(t-s)A}\phi\right\rangle ds \\
&\quad = \varepsilon_{Q,\kappa} C_\infty(T, \theta_0, q)\int_0^t \frac{d}{ds}\|e^{(t-s)A}\phi\|_{L^2}^2 ds \\
&\quad \le \varepsilon_{Q,\kappa} C_\infty(T, \theta_0, q)\,\|\phi\|_{L^2}^2
\end{aligned}$$

after a computation already done above for $\int_0^t \frac{d}{ds}\|e^{(t-s)A}\phi\|_{L^2}^2 ds$. ∎

3.3.2 When $\varepsilon_{Q,\kappa}$ Is Small (and $\mathcal{L}$ Is Not Small)

Inequality (3.6) is not immediately transparent. Let us discuss it in two cases, which, however, do not exhaust all opportunities.

The Case When $Q(x,x)$ Is Degenerate

The first one neglects the second term on the right–hand side, the term with $Q(x,x)$, because in very relevant cases it is degenerate. This happens precisely in the case considered everywhere in these notes, namely the case of a viscous fluid in a bounded domain D, satisfying the no-slip boundary condition $u|_{\partial D} = 0$. In this case $Q(x,x) = 0$ for $x \in \partial D$. We do not exclude that, in spite of this degeneracy, $Q(x,x)$ may help on the right–hand side of (3.6). But a priori it is difficult to use it.

In this case we look for the smallest constant $\epsilon_Q \geq 0$ such that

$$\int\int v(x)^T Q(x,y)\, v(y)\, dxdy \leq \epsilon_Q \int |v(x)|^2 dx \tag{3.7}$$

for all $v \in L^2\left(D, \mathbb{R}^d\right)$. Then

$$\varepsilon_{Q,\kappa} \leq \frac{\epsilon_Q}{\kappa}$$

because, if (3.7) holds, being

$$\epsilon_Q \int |v(x)|^2 dx \leq \frac{\epsilon_Q}{\kappa} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x)\, v(x)\right) dx$$

we have that $\frac{\epsilon_Q}{\kappa}$ is a constant fulfilling (3.6), hence the smallest one is less or equal to $\frac{\epsilon_Q}{\kappa}$. We thus have:

Corollary 3.10

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2\right] \leq \frac{\epsilon_Q}{\kappa} \|\phi\|_{L^2}^2 (\|\theta_0\|_\infty + T\|q\|_\infty)^2.$$

Therefore, one way to have $\theta(t)$ close to $\Theta(t)$ is to have a very small ϵ_Q. However, any small noise realizes this target but then also the additional operator $\mathcal{L}$ is small. Thus the true question is: are there noises such that ϵ_Q is small and the operator $\mathcal{L}$ is *substantial*?

The name "substantial" may refer to different properties. We have in mind two of them:

- improvement of the decay rate κ (eddy diffusion);

- production of a significantly modified profile (turbulent boundary layer heat profile).

In [116] we have constructed a noise, made of vortex structures, in simple 2D domains, with the following properties: given $\epsilon, \delta > 0$ (small) and $\sigma^2 > 0$ (large) we have

$$\begin{aligned} &\epsilon_Q \leq \epsilon \\ &Q(x,x) \geq \sigma^2 I \qquad \text{for all } x \in D \text{ such that } d(x, \partial D) \geq \delta. \end{aligned}$$

The first condition guarantees that the profile of $\theta(t)$ (smoothed by the scalar product $\langle \theta(t), \phi \rangle$) is close to the profile of $\Theta(t)$. The second condition implies that the deterministic equation of $\Theta(t)$ has an enhanced diffusion, still effective in spite of the vanishing-diffusion boundary layer. In [116] we have proved the following dissipativity property:

Theorem 3.11 *Assume $D = B(0,1) \subset \mathbb{R}^d$. Call $\lambda_{D,\kappa,Q}$ the first eigenvalue of $-A$ (it measures the rate of decay of $\Theta(t)$). Then there exists a constant $C_{D,d} > 0$ such that*

$$\lambda_{D,\kappa,Q} \geq C_{D,d} \min\left(\sigma^2, \frac{\kappa}{\delta}\right).$$

asymptotically as $\delta \to 0$ one can take $C_{D,d} = d/2$ and one also has $\lambda_{D,\kappa,Q} \geq \frac{\kappa d}{\kappa + \delta\sigma^2}\sigma^2$.

This result corresponds to the improvement of the decay rate κ (eddy diffusion) mentioned above. Considering the other sentence, namely producing a significantly modified profile (diffusion boundary layer), we have the following result, in a modified geometry with respect to the one of these lectures (see [127] for more details and other results in this direction). The domain now is the infinite channel

$$D = \mathbb{R} \times [-1, 1]$$

with Dirichlet boundary condition for both temperature and fluid at the upper and bottom boundaries:

$$\theta(x_1, \pm 1) = \sigma_k(x_1, \pm 1) = 0 \qquad \text{for every } x_1 \in \mathbb{R}, k \in K.$$

The theoretical results are similar to those above. In addition, let us consider the stationary deterministic profile for a given $q = q(x)$, element of H: we have to solve

$$A\Theta_{st} + q = 0,$$

namely

$$\Theta_{st} = -A^{-1}q.$$

In practice, assume that in a region $x \in [-L, L] \times [-1, 1]$ the function $q(x)$ is equal to a constant q, and both the stationary solution $\Theta_{st}(x)$ and $Q(x, x)$ depend only on the vertical direction $z \in [-1, 1]$ and they are symmetric with respect to $z = 0$. The equation

$$\operatorname{div}\left(\left(\kappa I + \frac{1}{2}Q(x, x)\right)\nabla\Theta_{st}(x)\right) = -q(x)$$

becomes

$$\partial_z\left(\left(\kappa + Q_{22}(z)\right)\partial_z\Theta_{st}(z)\right) = -q.$$

It gives us

$$\left(\kappa + Q_{22}(z)\right)\partial_z\Theta_{st}(z) = -qz$$

without constants, since both sides of the identity should vanish at $z = 0$ (the function Θ_{st} is symmetric with respect to $z = 0$ and smooth, hence $\partial_z\Theta_{st}(0) = 0$). Therefore we have to solve

$$\partial_z\Theta_{st}(z) = -\frac{qz}{\kappa + Q_{22}(z)}$$
$$\Theta_{st}(1) = 0.$$

The solution of the previous equation is

$$\Theta_{st}(z) = -\int_{-1}^{z}\frac{qs}{\kappa + Q_{22}(s)}ds.$$

Without noise the solution is

$$\Theta_{st}^{Q=0}(z) = \frac{q}{\kappa}\frac{1 - z^2}{2} = \frac{q}{2\kappa} - \frac{q}{2\kappa}z^2$$

so the curvature $\frac{q}{\kappa}$ is large (for κ small) and also the maximum is large:

$$\max\Theta_{st}^{Q=0} = \frac{q}{2\kappa}.$$

Assume

$$c_2\sigma^2 1_{[-1+\delta, 1-\delta]} \leq Q_{22}(z) \leq c_2\sigma^2$$

with large σ^2 and small δ. Then

$$\frac{q}{\kappa + c_2\sigma^2}\frac{1-z^2}{2} \leq \Theta_{st}(z)(z) \leq -\int_{-1}^{z}\frac{qs}{\kappa + c_1\sigma^2 1_{[-1+\delta,1-\delta]}(s)}ds.$$

If $z \in [-1, -1+\delta]$ we have

$$\Theta_{st}(z)(z) \leq \frac{q}{\kappa}\frac{1-z^2}{2}$$

like in the case without noise but, for $z \in [-1+\delta, 0]$ we have

$$\Theta_{st}(z)(z) \leq \frac{q}{\kappa}\frac{1-(1-\delta)^2}{2} + \frac{q}{\kappa + c_1\sigma^2}\frac{(1-\delta)^2 - z^2}{2}$$
$$= C\left(\kappa, q, \delta, \sigma^2\right) - \frac{q}{\kappa + c_1\sigma^2}\frac{z^2}{2}.$$

The curvature $\frac{q}{\kappa+c_1\sigma^2}$ is much smaller than $\frac{q}{\kappa}$ and the maximum

$$\max \Theta_{st}(z) = C\left(\kappa, q, \delta, \sigma^2\right) \geq \frac{q}{\kappa + c_1\sigma^2}\frac{(1-\delta)^2}{2}$$

is very small for large σ^2 and small δ.

Figure 3.1 illustrates the modification of the profile, from the standard parabolic one of free diffusion in a steady medium, to the case of turbulent decay. The reduction in heat content can be dramatic, due to turbulence, creating a fundamental engineering problem.

The Case When $Q(x, x)$ Is Non-degenerate

In bounded domains with no-slip boundary conditions for the fluid, $Q(x, x)$ is always degenerate. However, in other geometries, like the torus or the full space, we may have non-degenerate $Q(x, x)$.

Assume, for some $\sigma^2 > 0$ (large), we have

$$Q(x, x) \geq \sigma^2 I \qquad \text{for all } x \in D.$$

Then

$$\int \left(\kappa |v(x)|^2 + \frac{1}{2}v(x)^T Q(x, x) v(x)\right) dx$$
$$\geq \left(\kappa + \frac{\sigma^2}{2}\right)\int |v(x)|^2 dx.$$

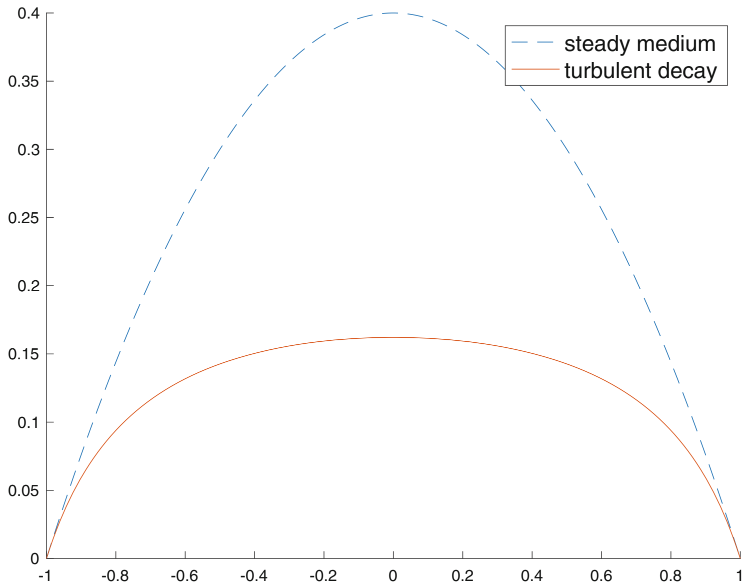

Fig. 3.1 The dashed profile is the classical parabolic profile with $Q = 0$. The solid-line profile is the one obtained by a large σ^2 and small δ

If (3.7) holds, being

$$\epsilon_Q \int |v(x)|^2 dx \leq \frac{\epsilon_Q}{\kappa + \frac{\sigma^2}{2}} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x) v(x) \right) dx$$

we deduce (as above)

$$\varepsilon_{Q,\kappa} \leq \frac{\epsilon_Q}{\kappa + \frac{\sigma^2}{2}}.$$

We thus have:

Corollary 3.12

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] \leq \frac{\epsilon_Q}{\kappa + \frac{\sigma^2}{2}} \|\phi\|_{L^2}^2 (\|\theta_0\|_\infty + T\|q\|_\infty)^2.$$

Therefore, another way to have $\theta(t)$ close to $\Theta(t)$, different from ϵ_Q small (or concurring with it) is to have σ^2 large.

Assume we are in full space $\mathbb{R}^d$. A famous noise satisfying the previous conditions (for suitable values of its parameters) is R. Kraichnan noise, [188, 189]. It is space-homogeneous, $Q(x, y) = Q(x - y)$, with the form

$$Q(z) = \sigma^2 k_0^\zeta \int_{k_0 \le |k| < k_1} \frac{1}{|k|^{d+\zeta}} e^{ik\cdot z} \left(I - \frac{k \otimes k}{|k|^2} \right) dk.$$

This model has a meaning and an interest for both positive and negative ζ. Assume $\zeta > 0$ (the so–called Kolmogorov 41 case is $\zeta = 4/3$). In this case, take $k_1 = +\infty$. Assume

$$k_0 = k_0^N$$

and take $k_0^N \to \infty$. Then

$$\begin{aligned}
Q(x, x) = Q(0) &= \sigma^2 k_0^\zeta \int_{k_0 \le |k| < \infty} \frac{1}{|k|^{d+\zeta}} \left(I - \frac{k \otimes k}{|k|^2} \right) dk \\
&\overset{k'=k/k_0}{=} \sigma^2 k_0^\zeta \int_{1 \le |k'| < \infty} \frac{1}{k_0^{d+\zeta} |k'|^{d+\zeta}} \left(I - \frac{k' \otimes k'}{|k'|^2} \right) k_0^d dk' \\
&= \sigma^2 \int_{1 \le |k| < \infty} \frac{1}{|k|^{d+\zeta}} \left(I - \frac{k \otimes k}{|k|^2} \right) dk
\end{aligned}$$

is independent of k_0 and therefore of N. This is the matrix appearing in the limit parabolic equation. But, concerning ϵ_Q, we have

$$\begin{aligned}
&\int\int v(x)^T Q(x, y) v(y) \, dx dy \\
&\le \sigma^2 k_0^\varsigma \int_{k_0 \le |k| < \infty} \frac{1}{|k|^{d+\varsigma}} |\widehat{v}(k)|^2 dk \\
&\le \sigma^2 k_0^{-d} \int_{k_0 \le |k| < \infty} |\widehat{v}(k)|^2 dk \le \sigma^2 k_0^{-d} \|v\|_{L^2}^2 .
\end{aligned}$$

Thus ϵ_Q is small if $\sigma^2 k_0^{-d}$ is small, hence if $k_0^N \to \infty$.

Remark 3.13 If $-d \le \zeta \le 0$, $k_0 = 1$, σ^2 small, and k_1 is so large that $\sigma^2 \int_{1 \le k \le k_1} \frac{1}{k^{\zeta+1}} dk$ is large, then $Q(x, x)$ is large and ϵ_Q is small. This regime is further investigated in [124].

Remark 3.14 We have seen that, in order to fulfill our conditions, the noise has to activate very small scales (large k) with high energy.

3.3.3 The Result for Long Times

The last result we want discuss in this section is an easy consequence of the techniques developed in this section and the estimate below proved in [127] in the case of an infinite channel. In this framework we can link the evolution of θ, solution of problem (3.1), to Θ_{st}, solution of the corresponding stationary problem.

Proposition 3.15 *If* $\theta_0 \in L^2(\mathcal{F}_0; D(A)),\ q(t) \equiv q \in D(A)$*, then*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|\theta(t)\|_\infty^2\right] \leq C(\|q\|_{D(A)}^2 + \|\theta_0\|_{D(A)}^2)$$

for some C *independent of* T.

Anyway, the argument of [127] can be extended also to regular two–dimensional or three–dimensional domains such that Poincaré inequality holds. Letting $C_\infty(\theta_0, q) > 0$ denote the right–hand side of previous proposition, namely

$$\sup_{t\geq 0} \mathbb{E}\left[\|\theta\,(t)\,\|_\infty^2\right] \leq C_\infty(\theta_0, q),$$

the result above holds.

Theorem 3.16 *For every* $\phi \in H$,

$$\limsup_{t\to\infty} \mathbb{E}\left[\langle \theta\,(t) - \Theta_{st}, \phi\rangle^2\right] \leq \frac{\epsilon_Q}{\kappa} \|\phi\|^2 C_\infty(\theta_0, q).$$

Proof Recall the identity

$$\theta\,(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q\,(s)\,ds - \sum_{k\in K} \int_0^t e^{(t-s)A}\sigma_k \cdot \nabla\theta\,(s)\,dW_s^k.$$

Set

$$\Theta\,(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q\,(s)\,ds.$$

Then

$$\theta\,(t) - \Theta\,(t) = -\sum_{k\in K} \int_0^t e^{(t-s)A}\sigma_k \cdot \nabla\theta\,(s)\,dW_s^k.$$

If $\phi \in H$,

$$\langle \theta\,(t) - \Theta\,(t)\,, \phi\rangle = \sum_{k\in K} \int_0^t \left\langle \theta\,(s)\,, \sigma_k \cdot \nabla\theta e^{(t-s)A}\phi \right\rangle dW_s^k.$$

Then (here we take advantage of the cancellations of Itô integrals)

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] = \sum_{k\in K} \mathbb{E}\int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla e^{(t-s)A}\phi\right\rangle^2 ds.$$

Write $\phi_{t,s} := e^{(t-s)A}\phi$. Then

$$\begin{aligned}
&\sum_{k\in K} \left\langle \theta(s), \sigma_k \cdot \nabla\phi_{t,s}\right\rangle^2 \\
&\quad = \sum_{k\in K} \int\int \theta(s,x)\,\theta(s,y)\,\sigma_k(x)\cdot\nabla\phi_{t,s}(x)\,\sigma_k(y)\cdot\nabla\phi_{t,s}(y)\,dxdy \\
&\quad = \int\int \theta(s,y)\,\nabla\phi_{t,s}(y)^T\,Q(x,y)\,\nabla\phi_{t,s}(x)\,\theta(s,x)\,dxdy \\
&\quad \le -\frac{\epsilon_Q}{\kappa}\|\theta(s)\|_\infty^2 \left\langle Ae^{(t-s)A}\phi, e^{(t-s)A}\phi\right\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] &\le \frac{\epsilon_Q}{\kappa} C_\infty(\theta_0, q) \int_0^t \left\langle (-A)\, e^{(t-s)A}\phi, e^{(t-s)A}\phi\right\rangle ds \\
&= \frac{\epsilon_Q}{\kappa} C_\infty(\theta_0, q) \int_0^t \frac{d}{ds}\|e^{(t-s)A}\phi\|^2 ds \\
&\le \frac{\epsilon_Q}{\kappa} C_\infty(\theta_0, q)\, \|\phi\|^2.
\end{aligned}$$

Now we use the fact that

$$\lim_{t\to\infty} \langle \Theta(t) - \Theta_{st}, \phi\rangle = 0.$$

Indeed,

$$\Theta(t) - \Theta_{st} = e^{tA}\left(\theta_0 + A^{-1}q\right).$$

For every $\epsilon > 0$, from the inequality $(a+b)^2 \le (1+\epsilon)\,a^2 + \left(1 + \frac{4}{\epsilon}\right) b^2$ we have

$$\begin{aligned}
&\mathbb{E}\left[\langle \theta(t) - \Theta_{st}, \phi\rangle^2\right] \\
&\quad \le (1+\epsilon)\,\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi\rangle^2\right] + \left(1 + \frac{4}{\epsilon}\right)\mathbb{E}\left[\langle \Theta(t) - \Theta_{st}, \phi\rangle^2\right].
\end{aligned}$$

This implies the result of the theorem. ■

In order to be of interest for applications, this theorem requires two conditions:

- that ϵ_Q is small;
- that Θ_{st} is significantly affected by the noise.

These two conditions have already been discussed deeply in Sect. 3.3.2, here we just refer to [127, 128] for some numerical experiments of the fact that Θ_{st} is significantly affected by the noise.

3.4 The Action of Transport Noise on Vector Fields

Our understanding of the action on vector fields is completely different with respect to the case of the action on scalar fields. The reason stays in the stretching term which formalizes the fact that vectors are (possibly) elongated by the deformation tensor of the underlying Lagrangian dynamics.

In this section we will focus our attention to the linear case (passive vector fields). The nonlinear one (vorticity formulation of the Navier–Stokes equations) will be the object of Sect. 4.4. Thus we start with the equation of a passive vector field, typically a magnetic field in applications. This investigation is related to the research on the so–called dynamo effect, see for instance [190, 260].

3.4.1 Passive Magnetic Field

The equations for a magnetic field M in a fluid u are

$$\partial_t M + u \cdot \nabla M = \eta \Delta M + M \cdot \nabla u.$$

Similarly to the scalar case, we model u by a white noise, with the Stratonovich interpretation:

$$dM + \sum_{k \in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt + \sum_{k \in K} M \cdot \nabla \sigma_k \circ dW_t^k.$$

The equation can be written as

$$dM = (\eta \Delta + \mathcal{L}) M dt + \text{Itô terms}$$

for a suitable second order differential operator $\mathcal{L}$. And $\overline{M}(t) := \mathbb{E}[M]$ satisfies

$$\partial_t \overline{M} = (\eta \Delta + \mathcal{L}) \overline{M}.$$

Thus, as above, the question arises whether $\mathbb{E}\left[\left\langle M(t)-\overline{M}(t), \phi\right\rangle^2\right]$ is small.

This question is open. We shall see below that in the case of special noise (space-homogeneous and mirror symmetric) the operator $\mathcal{L}$ is the same as the one of the scalar case. In this situation there exists the following conjecture from Krause and Rädler [190, page 12]: *"homogeneous isotropic mirror symmetric turbulence only influences the decay rate of the mean magnetic fields, which is enhanced in almost all cases of physical interest."*

The Corrector

If we define

$$B_k M = M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M$$

then the corrector is $\frac{1}{2}\sum_{k\in K} B_k B_k M$. Thus let us compute $B_k B_k M$. We have

$$\begin{aligned} B_k B_k M &= (B_k M)\cdot \nabla\sigma_k - \sigma_k \cdot \nabla (B_k M) \\ &= (M\cdot\nabla\sigma_k - \sigma_k\cdot\nabla M)\cdot\nabla\sigma_k - \sigma_k\cdot\nabla(M\cdot\nabla\sigma_k - \sigma_k\cdot\nabla M) \\ &= (M\cdot\nabla\sigma_k)\cdot\nabla\sigma_k - (\sigma_k\cdot\nabla M)\cdot\nabla\sigma_k \\ &\quad - \sigma_k\cdot\nabla(M\cdot\nabla\sigma_k) + \sigma_k\cdot\nabla(\sigma_k\cdot\nabla M). \end{aligned}$$

Lemma 3.17

$$\begin{aligned} \frac{1}{2}\sum_{k\in K} B_k B_k M = \mathcal{L}M - \sum_{k\in K}\sum_{i,j}\sigma_k^i \partial_i M_j \partial_j \sigma_k \\ + \frac{1}{2}\sum_{k\in K}\sum_{i,j}\left(\partial_j\sigma_k^i\partial_i\sigma_k - \sigma_k^i\partial_i\partial_j\sigma_k\right)M_j. \end{aligned}$$

Proof The term

$$\frac{1}{2}\sum_{k\in K}\sigma_k\cdot\nabla(\sigma_k\cdot\nabla M)$$

is equal to $\mathcal{L}M$, as in the previous sections. The term $\sigma_k\cdot\nabla(M\cdot\nabla\sigma_k)$ is equal to

$$(\sigma_k\cdot\nabla M)\cdot\nabla\sigma_k + \sum_{i,j}\left(\sigma_k^i\partial_i\partial_j\sigma_k\right)M_j$$

hence its first addendum, $(\sigma_k \cdot \nabla M) \cdot \nabla \sigma_k$, adds to another equal term in the total sum; they form the term

$$-\sum_{k \in K} \sum_{i,j} \sigma_k^i \partial_i M_j \partial_j \sigma_k$$

in the final result. The zero order term is thus the remainder of this computation. ■

Lemma 3.18 *Assume the noise is space-homogeneous:*

$$Q(x, y) = Q(x - y)$$

and $Q(x, x) = Q(0)$, a constant matrix. Then

$$\frac{1}{2} \sum_{k \in K} \sum_{i,j} \left(\partial_j \sigma_k^i \partial_i \sigma_k - \sigma_k^i \partial_i \partial_j \sigma_k \right) M_j = 0.$$

Proof

Step 1 The sum $\sum_{k \in K} \sigma_k^i(x) \sigma_k^\alpha(x)$ is constant, equal to $Q_{i,\alpha}(0)$, for every $i, \alpha = 1, 2, 3$. Thus their derivatives are equal to zero. It follows that

$$\sum_{k \in K} \left(\partial_j \sigma_k^i \right)(x) \sigma_k^\alpha(x) = -\sum_{k \in K} \sigma_k^i(x) \left(\partial_j \sigma_k^\alpha \right)(x).$$

Moreover, it follows also

$$\sum_i \partial_i \sum_{k \in K} \sigma_k^i(x) \sigma_k^\alpha(x) = 0$$

which implies

$$\sum_{k \in K} \sum_i \sigma_k^i(x) \partial_i \sigma_k^\alpha(x) = 0$$

because $\operatorname{div} \sigma_k = 0$.

Step 2 Not only the sum $\sum_{k \in K} \sigma_k^i(x) \sigma_k^\alpha(x)$ is constant, but also $\sum_{k \in K} \left(\partial_j \sigma_k^i \right)(x) \sigma_k^\alpha(x)$. Indeed, we have

$$\begin{aligned}\sum_{k \in K} \left(\partial_j \sigma_k^i \right)(x) \sigma_k^\alpha(y) &= \partial_{x_j} \sum_{k \in K} \sigma_k^i(x) \sigma_k^\alpha(y) \\ &= \partial_{x_j} Q_{i,\alpha}(x - y) = \left(\partial_j Q_{i,\alpha} \right)(x - y),\end{aligned}$$

which implies

$$\sum_{k \in K} \left(\partial_j \sigma_k^i\right)(x)\, \sigma_k^\alpha(x) = \left(\partial_j Q_{i,\alpha}\right)(0).$$

This implies

$$\partial_i \sum_{k \in K} \left(\partial_j \sigma_k^i\right)(x)\, \sigma_k^\alpha(x) = 0.$$

Step 3 Now, first the two terms we have to investigate are opposite one to the other:

$$\begin{aligned}\sum_{k \in K} \sum_i \partial_j \sigma_k^i \partial_i \sigma_k &= \partial_j \sum_{k \in K} \sum_i \sigma_k^i \partial_i \sigma_k - \sum_{k \in K} \sum_i \sigma_k^i \partial_i \partial_j \sigma_k \\ &= -\sum_{k \in K} \sum_i \sigma_k^i \partial_i \partial_j \sigma_k,\end{aligned}$$

where we have used the fact that $\sum_{k \in K} \sum_i \sigma_k^i \partial_i \sigma_k$ is equal to zero (Step 1). Therefore it is sufficient to prove that

$$\sum_{k \in K} \sum_i \partial_j \sigma_k^i \partial_i \sigma_k = 0.$$

But this term can be written as

$$\sum_i \partial_i \sum_{k \in K} \partial_j \sigma_k^i \sigma_k,$$

which is zero, because of Step 2. The identity between the previous two terms is due to the fact that $\sum_i \partial_i \partial_j \sigma_k^i = 0$, being $\operatorname{div} \sigma_k = 0$.

■

Corollary 3.19 *If the noise is space-homogeneous, then*

$$\frac{1}{2} \sum_{k \in K} B_k B_k M = \mathcal{L} M - \sum_j \partial_j Q(0) \cdot \nabla M_j$$

where $\partial_j Q(0)$ *is the matrix with entries* $\left(\partial_j Q_{\alpha,i}\right)(0)$. *In the particular case when*

$$Q(-x) = Q(x)$$

(mirror symmetry) then $\partial_j Q(0) = 0$ *and thus*

$$\frac{1}{2}\sum_{k\in K} B_k B_k M = \mathcal{L}M.$$

Proof For the first identity it remains to show that

$$\sum_{k\in K}\sum_{i,j}\sigma_k^i \partial_i M_j \partial_j \sigma_k^\alpha = \sum_{i,j}\left(\sum_{k\in K}\sigma_k^i \partial_j \sigma_k^\alpha\right)\partial_i M_j$$
$$= \sum_j \left(\partial_j Q_{\alpha,i}\right)(0)\,\partial_i M_j,$$

where we have used an identity proved in Step 2 of the previous proof.

Under mirror symmetry, $Q_{\alpha,i}(x)$ is a smooth even function, hence its derivatives at zero are equal to zero. ■

The Difficulty

We have shown that in the particular case of space-homogeneous noise with mirror symmetry the Itô form of the equation is

$$dM + \sum_{k\in K}\sigma_k\cdot\nabla M dW_t^k = (\eta\Delta + \mathcal{L})\,M dt + \sum_{k\in K} M\cdot\nabla\sigma_k dW_t^k,$$

similarly to the passive scalar case. Without mirror symmetry we would have an additional first-order differential operator, related to the so–called α–effect in the dynamo theory.

Notice first, as a secondary detail, that we have not used the assumption of isotropy in the derivation of the previous subsection. If the sentence quoted above from [190] concerns only the mean magnetic field, then it is true and without isotropy. We have proved:

Theorem 3.20 *If the noise is homogeneous and mirror symmetric, then the mean magnetic field* $\overline{M}(t) := \mathbb{E}[M]$ *satisfies the parabolic equation*

$$\partial_t \overline{M} = (\eta\Delta + \mathcal{L})\,\overline{M},$$

where

$$\mathcal{L}M = \sum_{i,j} Q_{ij}(0)\,\partial_j\partial_j M.$$

And we have shown in Sect. 3.3.2 that Kraichnan noise gives us $Q(0)$ equal to a large multiple of the identity, under some conditions on the parameters.

The problem arises if we interpret the sentence of [190] for the true magnetic field M instead of its average. We are not able anymore to prove that M is close to $\overline{M}$. The reason stands in the estimates on M. We do not have anymore the energy conservation estimate, because

$$\langle \sigma_k \cdot \nabla M, M \rangle = 0$$

hence

$$d\|M(t)\|_{L^2}^2 + 2\eta \|\nabla M(t)\|_{L^2}^2 dt = 2\sum_{k\in K} \langle M \cdot \nabla \sigma_k, M \rangle \circ dW_t^k$$

but $\langle M \cdot \nabla \sigma_k, M \rangle$ is not zero and contributes a lot, at least a priori.

Similarly, the Lagrangian property should be reformulated here as

$$M(t,x) = D\varphi_{-t}(x)\, M_0(\varphi_{-t}(x))$$

and the Lagrangian deformation tensor $D\varphi_{-t}(x)$ may have, a priori, an enormous effect of stretching on $M_0(\varphi_{-t}(x))$. Thus, even if we may start the computation as in the scalar case

$$\begin{aligned}\langle M(t), \phi \rangle - \left\langle \overline{M}(t), \phi \right\rangle &= +\sum_{k\in K} \int_0^t \left\langle M(s), e^{(t-s)A} \sigma_k \cdot \nabla \phi \right\rangle dW_t^k \\ &\quad + \sum_{k\in K} \int_0^t \left\langle M(s) \nabla \sigma_k, e^{(t-s)A} \phi \right\rangle dW_t^k,\end{aligned}$$

we do not have good estimates on $M(s)$ to control in mean square the stochastic terms.

The Purely Transport Case

If we consider the ideal model

$$dM + \sum_{k\in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt$$

where the noise acts only on the transport term, we get the equation

$$dM + \sum_{k\in K} \sigma_k \cdot \nabla M dW_t^k = (\eta\Delta + \mathcal{L})\, M dt$$

which satisfies the estimates

$$\|M(t)\|_{L^2}^2 + 2\eta \int_0^t \|\nabla M(s)\|_{L^2}^2 ds = \|M_0\|_{L^2}^2$$

$$\|M(t)\|_\infty \leq \|M_0\|_\infty .$$

Therefore we may control the difference

$$\langle M(t), \phi\rangle - \left\langle \overline{M}(t), \phi\right\rangle = \sum_{k\in K} \int_0^t \left\langle M(s), e^{(t-s)A}\sigma_k \cdot \nabla\phi\right\rangle dW_t^k$$

exactly as in the scalar scase.

From the physical viewpoint the stretching term $\sum_{k\in K} M \cdot \nabla\sigma_k \circ dW_t^k$ cannot be neglected. However, it is possible that there are regimes where its effect is small.

Remark 3.21 In this model we should not assume $\operatorname{div} M = 0$, otherwise the model is incorrect, because $\sigma_k \cdot \nabla M$ is not divergence free in general, while the other terms of the equation would be divergence free ($\sigma_k \cdot \nabla M - M \cdot \nabla\sigma_k$ is divergence free, on the contrary). If we want the additional property that M is divergence free, then we have to consider the more difficult model

$$dM + \sum_{k\in K} P(\sigma_k \cdot \nabla M) \circ dW_t^k = \eta\Delta M dt,$$

where P is the projector introduced in the previous chapters. The Itô–Stratonovich corrector now is much more complex. This difficulty is necessary in the case below of the Navier–Stokes equations, where the role of M is taken by the vorticity ω, which is divergence free. Hence the simple ideas described in this subsection are more complex, for the 3D Navier–Stokes equations, in two respects: the problem is nonlinear, hence it is not sufficient to control $\langle M(t), \phi\rangle - \left\langle \overline{M}(t), \phi\right\rangle$, and the corrector is non-local, since it contains P.

Chapter 4
Transport Noise in the Navier–Stokes Equations

Stochastic transport of passive scalars (the topic described in the previous chapter) is a well-known subject in the literature (see for instance [206]). On the contrary, this chapter introduces an analogous idea for the *internal modeling of a fluid*, which is less common and still debated. In some cases, however, it leads to results observed in the real world, hence it deserves to be investigated.

Prior to the concepts described in this chapter is the concept of vorticity, mentioned several times in these notes but never used explicitly, also because a rigorous use of vorticity in bounded domains leads to troubles (the value of the vorticity at the boundary is not known and thus a proper initial-boundary value problem for the vorticity equation cannot be settled).

Vorticity is defined as

$$\omega = \operatorname{curl} u$$

and in $d = 2$ it is a vector perpendicular to the plane of motion, hence it can be described by a scalar given by the third component of $\operatorname{curl} u$, namely

$$\omega \overset{d=2}{=} \partial_1 u_2 - \partial_2 u_1 .$$

From the Navier–Stokes equations, using some vector identities, we find the equation

$$\partial_t \omega + u \cdot \nabla \omega + \omega \cdot \nabla u = \nu \Delta \omega + \operatorname{curl} f$$

which has the advantage that the pressure has disappeared; but the term $\omega \cdot \nabla u$, called the *vortex stretching* term, provokes several troubles (it is responsible for the increase of intensity of the vorticity, which otherwise, for $\operatorname{curl} f = 0$, would be just transported by $u \cdot \nabla \omega$ and diffused by $\nu \Delta \omega$).

F. Flandoli, E. Luongo, *Stochastic Partial Differential Equations in Fluid Mechanics*, Lecture Notes in Mathematics 2328,
https://doi.org/10.1007/978-981-99-0385-6_4

In $d = 2$ one can see that $\omega \cdot \nabla u = 0$ (indeed u lives in the plane of motion, hence also ∇u, but ω is perpendicular to such plane) and therefore the equation simplifies into the diffusion-transport equation

$$\partial_t \omega + u \cdot \nabla \omega \overset{d=2}{=} \nu \Delta \omega + \operatorname{curl} f$$

which is very useful in domains "without" boundary, like the torus or the full space. It leads to additional invariants and a priori estimates with great success. See [70, 205, 209] for outstanding accounts of scientific and mathematical understandings based on vorticity. In the case $d = 2$ the velocity and the vorticity are linked by the relation

$$u = -\nabla^{\perp} (-\Delta)^{-1} \omega,$$

where $\nabla^{\perp}$ is the differential operator $\nabla^{\perp} \psi = \begin{bmatrix} \partial_2 \psi \\ -\partial_1 \psi \end{bmatrix}$ and $(-\Delta)^{-1}$ is the solution map of the boundary value problem $-\Delta \psi = \omega$ with proper boundary conditions, $\psi = 0$ where we work in a bounded domain and the velocity satisfies the no-slip boundary conditions, ψ periodic where we work in the torus. In the case $d = 3$ we can reconstruct u by ω, but the explicit linear relation is less simple, see [209]. In this case, we simply write $u = K\omega$.

After introducing the concept of vorticity, we can try to generalize the topic described in the previous chapter, stochastic transport of passive scalars, to the endogenous case. Before this we want to explain how transport noise appears in the system. Fluids, in their complex regimes that we loosely name turbulent, show the activation of several scales: we observe large-scale motions and small-scale ones at the same time, with several intermediate scales; very small vortices, larger and larger ones, up to motion at the scale of the full domain. Oversimplifying this multiscale picture, let us think that we want to split the fluid velocity into two components

$$\omega(t, x) = \overline{\omega}(t, x) + \omega'(t, x),$$

the first one containing most of the large scales, the second one mostly related to the small scales. We will return to this topic in Chap. 5 for some deeper motivations to this decomposition and some concrete possibilities to perform it. A precise subdivision is impossible, due to the multiscale nature of the problem. However, in some regime, a considerable degree of separation occurs [218].

Then we can consider the Navier–Stokes-type system

$$\begin{aligned}
\partial_t \overline{\omega} + \left(\overline{u} + u'\right) \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \left(\nabla \overline{u} + \nabla u'\right) &= \nu \Delta \overline{u} + \overline{\operatorname{curl} f}, \\
\partial_t \omega' + \left(\overline{u} + u'\right) \cdot \nabla \omega' + \omega' \cdot \left(\nabla \overline{u} + \nabla u'\right) &= \nu \Delta \omega' + \operatorname{curl} f', \\
\overline{u} = K\bar{\omega}, &\qquad u' = K\omega', \\
\overline{\omega}(0) = \overline{\omega}_0, &\qquad \omega'(0) = \omega_0'.
\end{aligned}$$

This system is equivalent to the original equation

$$\partial_t \omega + u \cdot \nabla \omega + \omega \cdot \nabla u = \nu \Delta \omega + \mathrm{curl}\, f,$$
$$u = K\omega, \qquad \omega(0) = \omega_0,$$

when

$$\mathrm{curl}\, f = \overline{\mathrm{curl}\, f} + \mathrm{curl}\, f',$$
$$\omega_0 = \overline{\omega}_0 + \omega_0'.$$

Indeed, if $(\overline{\omega}; \omega')$ is a solution of the system, then $\omega = \overline{\omega} + \omega'$ is a solution of the equations; vice versa, if ω is a solution of the equations and $\overline{\omega}$ is a solution of

$$\partial_t \overline{\omega} + u \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla u = \nu \Delta \overline{\omega} + \overline{\mathrm{curl}\, f},$$

then $\omega' = \omega - \overline{\omega}$ is a solution of

$$\partial_t \omega' + \left(\overline{u} + u'\right) \cdot \nabla \omega' + \omega' \cdot \left(\nabla \overline{u} + \nabla u'\right) = \nu \Delta \omega' + \mathrm{curl}\, f',$$
$$u' = K\omega'.$$

We may reverse the roles of $\overline{\omega}$ and ω' in the latter argument.

In the system we impose the small–large-scale subdivision only on data: on the initial condition and on the forcing term. At least for a short time, this subdivision is expected to be maintained, approximately. How much it is maintained for longer times is a very difficult issue; certainly $\overline{\omega}$, for longer times is corrupted by small scales and ω' by large scales; the open problem is how much.

Now let us come to stochastic modeling: looking at real situations with a boundary and the vortices produced near it, we suspect that the small scales are quite concentrated in a region near the boundary, the large scales are active everywhere.

Thus we replace the system above with the model

$$\partial_t \overline{\omega} + \left(\overline{u} + u'\right) \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \left(\nabla \overline{u} + \nabla u'\right) = \nu \Delta \overline{\omega} + \overline{\mathrm{curl}\, f},$$
$$\partial_t \omega' = \nu \Delta \omega' - \frac{1}{\epsilon} \omega' + \frac{1}{\epsilon} \sum_k \mathrm{curl}\, \sigma_k \partial_t W^k,$$
$$\overline{u} = K\overline{\omega}, \qquad u' = K\omega',$$
$$\overline{\omega}(0) = \overline{\omega}_0, \qquad \omega'(0) = \omega_0',$$

where both equations are considered in the full domain D but the second one is mostly active near the boundary thanks to the fact that the vector fields σ_k have small support near the boundary. A more complete model is treated in [135].

Let us look only at the equation of large scales

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u} = \nu \Delta \overline{\omega} + \overline{\mathrm{curl}\, f} - u' \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla u',$$
$$\overline{u} = K \overline{\omega}.$$

If we take the limit $\epsilon \to 0$ and argue as in the linear case of temperature diffusion, we get the equation

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} = (\nu \Delta + \mathcal{L})\, \overline{\omega} + \overline{\mathrm{curl}\, f} - \sum_{k \in K} (\sigma_k \cdot \nabla \overline{\omega})\, \partial_t W^k - \sum_{k \in K} (\nabla \sigma_k \cdot \overline{\omega})\, \partial_t W^k,$$
$$\overline{u} = K \overline{\omega}.$$

This is a *closed* model of large scales, influenced by turbulent small scales.

Is it useful and realistic? This difficult question is under investigation. Let us only mention one positive fact. Consider the associated deterministic equation

$$\partial_t \Omega + U \cdot \nabla \Omega + \Omega \cdot \nabla U = (\nu \Delta + \mathcal{L})\, \Omega + \overline{\mathrm{curl}\, f},$$
$$U = K \Omega, \qquad \Omega(0) = \overline{\omega}_0$$

(if $\overline{\omega}_0$ and $\overline{\mathrm{curl}\, f}$ are deterministic, otherwise take their expectations). This equation has, for suitable $\mathcal{L}$, stronger dissipativity properties that the original one with just $\nu \Delta$. If we can prove that $\overline{\omega}$ is close to Ω, then we get that the large-scale motion $\overline{\omega}$ reveals a stronger dissipativity, due to the presence of turbulent small scales. This is the observed phenomenon of *eddy viscosity*: turbulence improves the viscous properties. Mathematically, we can prove that $\overline{\omega}$ is close to Ω only in $d = 2$; in $d = 3$ there are essential obstructions. But at least for $d = 2$ we see that this model leads to realistic results. We will discuss some issues related to $d = 3$ in Sect. 4.4. In 2D, the procedure above leads to the simpler stochastic equation (let us write it here in Stratonovich form for simplicity of notation)

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} \overset{d=2}{=} \nu \Delta \overline{\omega} - \sum_{k \in K} \sigma_k \cdot \nabla \overline{\omega} \circ \partial_t W^k + \overline{\mathrm{curl}\, f},$$
$$\overline{u} \overset{d=2}{=} -\nabla^{\perp} (-\Delta)^{-1} \overline{\omega}.$$

This is an excellent equation, similar to the one of temperature diffusion and transport. In particular, one can discuss when $\overline{\omega}$ is close to the deterministic solution of an equation with increased dissipation of the form

$$\partial_t \Omega + U \cdot \nabla \Omega \overset{d=2}{=} (\nu \Delta + \mathcal{L})\, \Omega + \overline{\mathrm{curl}\, f},$$
$$U \overset{d=2}{=} -\nabla^{\perp} (-\Delta)^{-1} \Omega.$$

In this case, as declared before, some convergence results can be stated. In Sect. 4.1 we will prove the well-posedness of the vorticity equation with transport noise; the phenomenon of eddy viscosity, namely the convergence of $\bar{\omega}$ to Ω, will be the main object of Sect. 4.2, showing some results analogous to the ones explained in Chap. 3. We refer to [114, 117] for a more complete treatment of the convergence of $\overline{\omega}$ to Ω. Some of the results outlined in this introductory section would require a chapter in themselves and will not be developed in this book. The reader may see some of the existing results in the following references: [133–135].

Lastly, we want point out that we may perform this argument at the level of velocity, instead of vorticity. They are not equivalent, and which one is better for the physics is still debated.

4.1 Well-Posedness for the Vorticity Formulation

In this section we want to show an existence and uniqueness result for the equation

$$\partial_t \omega + \sum_{k \in K} (\sigma_k \cdot \nabla \omega)\, \partial_t W^k = (\nu \Delta + \mathcal{L})\, \omega - u \cdot \nabla \omega + q \tag{4.1}$$

in the two-dimensional torus $\mathbb{T}^2 = [0, \pi]^2$, where $u = -\nabla^{\perp}(-\Delta)^{-1}\omega$, σ_k are divergence free, smooth, vector fields and

$$\mathcal{L}\omega = \sum_{k \in K} \sigma_k \cdot \nabla(\sigma_k \cdot \nabla \omega).$$

These are the Navier–Stokes equations in vorticity formulation with transport noise. Contrary, for example, to the additive noise case, considering the equations in vorticity formulation or in velocity formulation leads us to different results. We postpone to Sect. 4.3 the discussion about the differences between vorticity formulation and velocity formulation in the transport noise framework. In this section and in the next one we are only interested to show results analogous to the ones of Sects. 3.2–3.3 in the endogenous case.

Contrary to Sect. 3.2, here we present only the variational method that we think is more suitable to treat the Navier–Stokes nonlinearity. The proof of the well-posedness of the systems of Sects. 4.1 and 4.3 are strongly inspired by the results of [44], which we suggest reading also for an alternative approach to the analysis of the system with state-dependent noise described in Chap. 2.

4.1.1 *Variational Method: Plan of Work*

We recall the plan of work for the variational approach already described in Chap. 3.

According to the results of Pardoux and Krylov–Rozovskii in the more general context of SPDEs with monotone operators [191, 224] or in a less abstract context [44], we will perform the following steps:

- One has to introduce a sequence of approximating problems which have a unique solution by known results. We will skip the details about the local existence that follows from the classical theory on stochastic differential equations with locally Lipshitz coefficients, see for example [183, 241].
- On these approximations, one has to prove estimates independent of the approximating parameter.
- Such estimates imply the existence of weakly convergent subsequences which are, indeed, global solutions of the approximating problems. Contrary to the linear case, this condition is not enough to pass to the limit in the equation, due to the nonlinear term. We will need to obtain a stronger result about the convergence of the approximations in order to pass to the limit in the nonlinear term.

4.1.2 Functional Setting and Assumptions

Let: $H = L^2_0\left(\mathbb{T}^2\right)$, $V = W^{1,2}\left(\mathbb{T}^2\right) \cap H$, $D(A) = W^{2,2}\left(\mathbb{T}^2\right) \cap V$, $A : D(A) \subset H \to H$

$$A\omega = \Delta\omega,$$

where $L^2_0(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2)$ made by zero mean functions. It is well-known that A is the infinitesimal generator of analytic semigroup of negative type and moreover V can be identified with $D((-A)^{1/2})$. Something more can be said on the fractional powers of the operator $-A$. Indeed, for each $\alpha \in \mathbb{R}$

$$D((-A)^{\alpha}) = D((-\Delta)^{\alpha}) = \{q \in W^{2\alpha,2}(\mathbb{T}^2) : \langle q, 1\rangle_{W^{2\alpha,2},W^{-2\alpha,2}} = 0\},$$

see [225] Chapter 7 for more details.

According to these notations, it follows immediately that

$$-\nabla^{\perp}(-\Delta)^{-1} \in L\left(D((-A)^{\alpha}), W^{2\alpha+1,2}(\mathbb{T}^2; \mathbb{R}^2)\right).$$

The transport term, $v \cdot \nabla\xi$ where ξ is a real-valued function and v is a divergence free vector field, is defined analogously to what we have done in Sect. 1.1.2 and satisfies similar skewness properties, namely

$$\int_{\mathbb{T}^2} \xi_1(x)v(x) \cdot \nabla\xi_2(x)dx = -\int_{\mathbb{T}^2} \xi_2(x)v(x) \cdot \nabla\xi_1(x)dx$$

every time the integrals above are well-defined. Therefore if ξ is zero mean, then also $v \cdot \nabla \xi$ is zero mean.

As already done in a previous chapter, we consider the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, thus we deal with strong solutions. Let us denote by $L^p_{\mathcal{F}}(0, T; V)$ the space of p integrable, progressively measurable processes with values in V and by $C_{\mathcal{F}}([0, T]; H)$ the space of continuous adapted square integrable processes. Assume σ_k smooth enough (just for simplicity we assume $\sigma_k \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$, but less can be required), $\omega_0 \in L^4_{\mathcal{F}_0}(\Omega, H)$, $q \in L^4_{\mathcal{F}}(0, T; H)$.

Definition 4.1 A stochastic process

$$\omega \in C_{\mathcal{F}}([0, T]; H) \cap L^2_{\mathcal{F}}(0, T; V)$$

is a weak solution of Eq. (4.1) if, for every $\phi \in D(A)$, we have

$$\langle \omega(t), \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega(s), (\nu\Delta + \mathcal{L})\phi \rangle \, ds + \int_0^t \langle \omega(s), u(s) \cdot \nabla \phi \rangle ds$$
$$+ \int_0^t \langle q(s), \phi \rangle \, ds + \sum_{k\in K} \int_0^t \langle \omega(s), \sigma_k \cdot \nabla \phi \rangle \, dW^k_s$$

for every $t \in [0, T]$, $\mathbb{P}$-a.s.

The main result proved below is:

Theorem 4.2 *For every $\omega_0 \in L^4_{\mathcal{F}_0}(\Omega, H)$ and $q \in L^4_{\mathcal{F}}(0, T; H)$, there exists one and only one weak solution of Eq. (4.1).*

Remark 4.3 The result stated here is a bit superabundant for our scope. In Sect. 4.2, we will consider deterministic initial conditions and forcing terms. We prefer to state Theorem 4.2 in full generality in order to explain several tricks for the variational method. As in Sect. 2.3 the extra integrability conditions are needed in order to get existence, but uniqueness follows under the more natural assumptions $\omega_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$, $q \in L^2_{\mathcal{F}}(0, T; H)$. Moreover, the extra integrability in time of q is not needed in the case of deterministic forcing term.

4.1.3 Galerkin Approximation and Limit Equations

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H made by eigenvectors of $-\Delta$ and λ_i the corresponding eigenvalues, λ_i are positive and nondecreasing. Let $H^N = \operatorname{span}\{e_1, \ldots, e_N\} \subseteq H$, $P^N : H \to H$ the orthogonal projector of H on H^N.

We start looking for a finite-dimensional approximation of the solution of Eq. (4.1). We define

$$\omega^N(t) = \sum_{i=1}^{N} c_{i,N}(t) e_i(x).$$

The $c_{i,N}$ have been chosen in order to satisfy $\forall e_i,\ \ 1 \le i \le N,\ t \in [0, T]$,

$$\begin{aligned} \left\langle \omega^N(t), e_i \right\rangle &= \left\langle \omega_0^N, e_i \right\rangle + \int_0^t \left\langle \omega^N(s), \left(\nu\Delta + \mathcal{L}^N\right) e_i \right\rangle ds \\ &\quad + \int_0^t \langle \omega^N(s), u^N(s) \cdot \nabla e_i \rangle ds + \int_0^t \langle q(s), e_i \rangle\, ds \\ &\quad + \sum_{k \in K} \int_0^t \left\langle \omega^N(s), \sigma_k \cdot \nabla e_i \right\rangle dW_s^k,\ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{4.2}$$

where $\omega_0^N = P^N \omega_0,\ u^N(t) = -\nabla^\perp(-\Delta)^{-1}\omega^N(t)$ and

$$\mathcal{L}^N \phi = \frac{1}{2} \sum_{k \in K} P^N \left(\sigma_k \cdot \nabla P^N(\sigma_k \cdot \nabla \phi)\right)\ \forall \phi \in H^N.$$

As stated in the plan of work, local existence and uniqueness for the solution of this system of ordinary stochastic differential equations follows from the classical theory for stochastic differential equations with locally Lipshitz coefficients. For what concerns the global existence, it follows from the a priori estimates below.

Lemma 4.4 *The Itô formula below holds:*

$$d\|\omega^N\|_H^2 + 2\nu\|\nabla\omega^N\|_{L^2}^2 dt = 2\langle q, \omega^N \rangle dt \tag{4.3}$$

and the following energy estimates are satisfied:

$$\|\omega^N(t)\|_H^2 \le \int_0^t e^{-\nu(t-s)} \frac{\|q(s)\|_H^2}{\nu} ds + e^{-\nu t}\|\omega_0\|_H^2, \tag{4.4}$$

$$\nu \int_0^t \|\nabla\omega^N(s)\|_{L^2}^2 ds \le \int_0^t \frac{\|q(s)\|_H^2}{\nu} ds + \|\omega_0\|_H^2, \tag{4.5}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega^N(t)\|_H^2\right] \le \mathbb{E}\left[\int_0^T \frac{\|q(s)\|_H^2}{\nu} ds + \|\omega_0\|_H^2\right], \tag{4.6}$$

$$\nu\mathbb{E}\left[\int_0^T \|\nabla\omega^N(t)\|_{L^2}^2\, ds\right] \le \mathbb{E}\left[\int_0^T \frac{\|q(s)\|_H^2}{\nu} ds + \|\omega_0\|_H^2\right], \tag{4.7}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega^N(t)\|_H^4\right]+\mathbb{E}\left[\int_0^T\|\omega^N(s)\|_H^2\|\nabla\omega^N(s)\|_{L^2}^2\,ds\right]\leq C, \tag{4.8}$$

$$\mathbb{E}\left[\left(\int_0^T\|\nabla\omega^N(s)\|_{L^2}^2\,ds\right)^2\right]\leq C, \tag{4.9}$$

where C is a constant possibly changing its value line by line, but independent of N.

Proof The Itô formula follows immediately from the finite dimensional Itô formula, in fact

$$d\|\omega^N\|_H^2=\sum_{i=1}^N dc_{i,N}^2=2\sum_{i=1}^N c_{i,N}dc_{i,N}+\sum_{i=1}^N\langle dc_{i,N},dc_{i,N}\rangle.$$

Thus, exploiting the weak formulation satisfied by ω^N, we have

$$d\|\omega^N\|_H^2=2\left(\langle\omega^N,\nu\Delta\omega^N\rangle+\langle q,\omega^N\rangle+\langle\omega^N,u^N\cdot\nabla\omega^N\rangle+\langle\omega^N,\mathcal{L}^N\omega^N\rangle\right)dt$$
$$+\sum_{k\in K}\langle\omega^N,\sigma_k\cdot\nabla\omega^N\rangle dW_t^k+\sum_{i=1}^N\sum_{k\in K}\langle\omega^N,\sigma_k\cdot\nabla e_i\rangle^2dt.$$

From the fact that $u^N,\ \sigma_k$ are divergence free, it follows immediately that $\langle\omega^N,u^N\cdot\nabla\omega^N\rangle=\langle\omega^N,\sigma_k\cdot\nabla\omega^N\rangle=0$. Moreover, we can notice, integrating by parts and exploiting $\operatorname{div}\sigma_k=0$, that

$$\begin{aligned}\sum_{i=1}^N\sum_{k\in K}\langle\omega^N,\sigma_k\cdot\nabla e_i\rangle^2+2\langle\omega^N,\mathcal{L}^N\omega^N\rangle&=\sum_{i=1}^N\sum_{k\in K}\langle\sigma_k\cdot\nabla\omega^N,e_i\rangle^2\\&\quad+\sum_{k\in K}\langle P^N\left(\sigma_k\cdot\nabla P^N(\sigma_k\cdot\nabla\omega^N)\right),\omega^N\rangle\\&=\sum_{k\in K}\langle\sigma_k\cdot\nabla\omega^N,P^N(\sigma_k\cdot\nabla\omega^N)\rangle\\&\quad+\sum_{k\in K}\langle\sigma_k\cdot\nabla P^N(\sigma_k\cdot\nabla\omega^N),\omega^N\rangle=0.\end{aligned}$$

Thus we arrive at the Itô formula in the statement. Starting from the Itô formula and applying the Poincaré inequality for zero mean functions in the torus, Cauchy–Schwarz, Young's inequality properly we get

$$d\|\omega^N\|_H^2+2\nu\|\nabla\omega^N\|_{L^2}^2dt\leq\left(\nu\|\omega^N\|_H^2+\frac{\|q\|_H^2}{\nu}\right)dt\leq\left(\nu\|\nabla\omega^N\|_{L^2}^2+\frac{\|q\|_H^2}{\nu}\right)dt.$$

Thus by the Gronwall lemma, we have the following relation:

$$\|\omega^N(t)\|_H^2 \leq \int_0^t e^{-\nu(t-s)} \frac{\|q(s)\|_H^2}{\nu} ds + e^{-\nu t} \|\omega_0\|_H^2,$$

$$\nu \int_0^t \|\nabla \omega^N(s)\|_{L^2}^2 ds \leq \|\omega_0\|_H^2 + \int_0^t \frac{\|q(s)\|_H^2}{\nu} ds.$$

These inequalities imply the first four energy relations stated and the last one. For what concerns the remaining one, we use the Itô formula satisfied by $\|\omega^N(t)\|_H^2$ and we apply the classical Itô formula for the function $f(x) = x^2$. Arguing as above, we get

$$\begin{aligned} d\|\omega^N\|^4 + 4\nu \|\omega^N\|_H^2 \|\nabla \omega^N\|_{L^2}^2 &= 4\|\omega^N\|_H^2 \langle q, \omega^N \rangle dt \\ &\leq \left(2\nu \|\omega^N\|_H^2 \|\nabla \omega^N\|_{L^2}^2 + \frac{2\|q\|_H^2 \|\omega^N\|_H^2}{\nu} \right) dt \\ &\leq \left(2\nu \|\omega^N\|_H^2 \|\nabla \omega^N\|_{L^2}^2 + \frac{\|\omega^N\|_H^4}{\nu^2} + \frac{\|q\|_H^4}{\nu^2} \right) dt \end{aligned}$$

and this relation implies the thesis by the Gronwall lemma. □

Remark 4.5 Only relations (4.8) and (4.9) use the further integrability assumptions on the initial conditions and the forcing term.

From the energy estimates on ω^N, there exists a subsequence, which we will denote again for simplicity by ω^N, which converges to a stochastic process ω in the way described below:

$$\omega^N \overset{*}{\rightharpoonup} \omega \; L^4(\Omega; L^\infty(0, T; H)),$$

$$\omega^N \rightharpoonup \omega \; L^4(\Omega; L^2(0, T; V))$$

and an unknown process B^* such that

$$u^N \cdot \nabla \omega^N \rightharpoonup B^* \; L^2(\Omega; L^2(0, T; V^*)).$$

Moreover, thanks to the converging properties of the projector P^N for $N \to +\infty$, the processes ω and B^* satisfies $\mathbb{P}$-a.s. for each $i \in \mathbb{N}$ and $t \in [0, T]$

$$\begin{aligned} \langle \omega(t), \phi \rangle + \int_0^t \langle B^*(s), e_i \rangle_{V^*,V} ds &= \langle \omega_0, e_i \rangle + \int_0^t \langle \omega(s), (\nu \Delta + \mathcal{L}) e_i \rangle ds \\ &+ \int_0^t \langle q(s), e_i \rangle ds \\ &+ \sum_{k \in K} \int_0^t \langle \omega(s), \sigma_k \cdot \nabla e_i \rangle dW_s^k. \end{aligned} \tag{4.10}$$

Let us explain a bit better the part related to the convergence of the term $\int_0^t \langle \omega^N(s), \mathcal{L}^N e_i \rangle ds$. We know that for each $\alpha \geq 0$, $x \in D((-A)^\alpha)$, $\|P^N x - x\|_{D((-A)^\alpha)} \to 0$. Thus, since for each $k \in K$, $\beta \geq 1/2$, the operator $\sigma_k \cdot (\nabla \cdot) \in L(D((-A)^\beta), D((-A)^{\beta-1/2}))$, then if $\phi \in D((-A)^\beta)$, $\|P^N(\sigma_k \cdot \nabla\phi) - \sigma_k \cdot \nabla\phi\|_{D((-A)^{\beta-1/2})} \to 0$. Starting from these observations it is easy to show that for each $\phi \in D(A)$, $\|\mathcal{L}^N e_i - \mathcal{L} e_i\| \to 0$. Then, thanks to the weak convergence of ω^N to ω, we have the required convergence of $\int_0^t \langle \omega^N(s), \mathcal{L}^N e_i \rangle ds$.

For what concerns the continuity in H of the process ω we can argue in the following way via Itô formula and Kolmogorov continuity theorem. From the weak formulation above we get the weak continuity in H of ω applying the Kolmogorov continuity theorem for the SDE satisfied by $\langle \omega(t), e_i \rangle$. Applying the Itô formula to $\|\omega(t)\|_H^2$ we get, arguing as in the proof of Lemma 4.4,

$$d\|\omega\|_H^2 = -2\nu\|\nabla\omega\|_{L^2}^2 dt - 2\langle B^*, \omega \rangle_{V^*,V} dt + 2\langle q, \omega \rangle dt.$$

From this, we get the continuity of $\|\omega\|_H^2$ thanks to the integrability properties of ω. Weak continuity and continuity of the norm implies strong continuity, thus we have the strong continuity of ω as a process taking values in H. Alternatively, the strong continuity in H of ω follows from the results in [224].

Remark 4.6 Without the additional energy estimates it is not possible to gain a weak convergent subsequence for the nonlinearity. In fact we have for each $\phi \in V$

$$|\langle u^N(s) \cdot \nabla\phi, \omega^N(s) \rangle| \leq C\|\nabla\phi\|_{L^2}\|\nabla\omega^N(s)\|_{L^2}\|\omega^N(s)\|_H$$

thus $\|u^N(s) \cdot \nabla\omega^N(s)\|_{V^*} \leq C\|\nabla\omega^N(s)\|_{L^2}\|\omega^N(s)\|_H$ and

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|u^N(s) \cdot \nabla\omega^N(s)\|_{V^*}^2 \, ds\right] \\ &\leq C \sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|\nabla\omega^N(s)\|_{L^2}^2 \|\omega^N(s)\|_H^2 \, ds\right] \leq C \end{aligned}$$

thanks to relation (4.8).

4.1.4 Existence, Uniqueness and Further Results

To prove the existence of the solutions of Eq. (4.1) we need the following lemma. This way of proceeding is classical in stochastic analysis, see for example [44, 227].

Lemma 4.7 *Let*

$$\tau_M = \inf\{t \in [0,T] : \|\omega(t)\|_H^2 \geq M\} \wedge \inf\{t \in [0,T] : \int_0^t \|\nabla\omega(s)\|_{L^2}^2 \, ds \geq M\} \wedge T$$

then

$$1_{[0,\tau_M]}\left(\omega^N - \omega\right) \to 0, \; in\; L^2(\Omega, L^2(0,T;H)).$$

Proof We have to show that

$$\mathbb{E}\left[\int_0^T 1_{[0,\tau_M]}(s)\|\omega^N(s) - \omega(s)\|_H^2\, ds\right] \tag{4.11}$$

converges to zero in N. Let $\tilde{\omega}^N = P^N\omega,\; \tilde{u}^N = -\nabla^{\perp}(-\Delta)^{-1}\tilde{\omega}^N$. Then, by the triangular inequality

$$\begin{aligned}(4.11) \leq\, & 2\mathbb{E}\left[\int_0^T 1_{[0,\tau_M]}(s)\|\tilde{\omega}^N(s) - \omega(s)\|_H^2\, ds\right]\\ & + 2\mathbb{E}\left[\int_0^T 1_{[0,\tau_M]}(s)\|\tilde{\omega}^N(s) - \omega^N(s)\|_H^2\, ds\right].\end{aligned}$$

Thanks to the properties of the projector P^N and dominated convergence theorem, it follows that $\tilde{\omega}^N \to \omega$ in $L^2(\Omega, L^2(0,T;V)) \cap L^2(\Omega, C(0,T;H))$, and also in weaker topologies. Therefore, we are left to show the convergence of

$$\mathbb{E}\left[\int_0^{\tau_M} \|\tilde{\omega}^N(s) - \omega^N(s)\|_H^2\, ds\right]. \tag{4.12}$$

Let $B^N(s) = u^N(s)\cdot\nabla\omega^N(s)$, then for each $i \leq N$ the following relation holds true:

$$\begin{aligned}\langle(\omega - \omega^N)(t), e_i\rangle + \int_0^t \langle B^*(s) - B^N(s), e_i\rangle_{V^*,V} ds\\ = \int_0^t \nu\langle(\omega - \omega^N)(s), \Delta e_i\rangle + \langle\omega(s), \mathcal{L}e_i\rangle ds\\ - \int_0^t \langle\omega^N(s), \mathcal{L}^N e_i\rangle\, ds\\ + \sum_{k\in K}\int_0^t \langle\omega(s) - \omega^N(s), \sigma_k\cdot\nabla e_i\rangle dW_s^k.\end{aligned}$$

Thanks to the previous relation we can compute $\frac{1}{2}d\|\tilde{\omega}^N - \omega^N\|_H^2$ via the Itô formula:

$$\begin{aligned}\frac{1}{2}d\|\tilde{\omega}^N - \omega^N\|_H^2 + \nu\|\nabla(\tilde{\omega}^N - \omega^N)\|_{L^2}^2\, dt = \langle\omega, \mathcal{L}(\tilde{\omega}^N - \omega^N)\rangle dt\\ - \langle\omega^N, \mathcal{L}^N(\tilde{\omega}^N - \omega^N)\rangle dt\end{aligned}$$

$$
\begin{aligned}
&- \langle B^* - B^N, \tilde{\omega}^N - \omega^N \rangle_{V^*,V} dt \\
&+ \sum_{k \in K} \langle \omega - \omega^N, \sigma_k \cdot \nabla(\tilde{\omega}^N - \omega^N) \rangle dW_t^k \\
&+ \frac{1}{2} \sum_{k \in K} \sum_{i=1}^{N} \langle \omega - \omega^N, \sigma_k \cdot \nabla e_i \rangle^2 dt.
\end{aligned}
\tag{4.13}
$$

Next, to better understand the behavior of the terms

$$
\langle \omega, \mathcal{L}(\tilde{\omega}^N - \omega^N) \rangle - \langle \omega^N, \mathcal{L}^N(\tilde{\omega}^N - \omega^N) \rangle + \frac{1}{2} \sum_{k \in K} \sum_{i=1}^{N} \langle \omega - \omega^N, \sigma_k \cdot \nabla e_i \rangle^2,
$$

we will first write them in an equivalent form:

$$
\begin{aligned}
2\langle \omega, \mathcal{L}(\tilde{\omega}^N - \omega^N) \rangle &- 2\langle \omega^N, \mathcal{L}^N(\tilde{\omega}^N - \omega^N) \rangle \\
&= \sum_{k \in K} \langle \omega, \sigma_k \cdot \nabla \left(\sigma_k \cdot \nabla(\tilde{\omega}^N - \omega^N) \right) \rangle \\
&- \sum_{k \in K} \langle \omega^N, \sigma_k \cdot \nabla P^N \left(\sigma_k \cdot \nabla(\tilde{\omega}^N - \omega^N) \right) \rangle \\
&= - \sum_{k \in K} \langle \sigma_k \cdot \nabla(\tilde{\omega}^N - \omega^N), \sigma_k \cdot \nabla \omega \rangle \\
&+ \sum_{k \in K} \langle P^N \left(\sigma_k \cdot \nabla(\tilde{\omega}^N - \omega^N) \right), \sigma_k \cdot \nabla \omega^N \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k \in K} \sum_{i=1}^{N} \langle \omega - \omega^N, \sigma_k \cdot \nabla e_i \rangle^2 \\
&= \sum_{k \in K} \langle \sigma_k \cdot \nabla \omega - P^N(\sigma_k \cdot \nabla \omega^N), P^N(\sigma_k \cdot \nabla \omega) - P^N(\sigma_k \cdot \nabla \omega^N) \rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2\langle \omega, \mathcal{L}(\tilde{\omega}^N - \omega^N) \rangle - 2\langle \omega^N, \mathcal{L}^N(\tilde{\omega}^N - \omega^N) \rangle + \sum_{k \in K} \sum_{i=1}^{N} \langle \omega - \omega^N, \sigma_k \cdot \nabla e_i \rangle^2 \\
&= \sum_{k \in K} \langle \sigma_k \cdot \nabla \omega - P^N(\sigma_k \cdot \nabla \omega^N), P^N(\sigma_k \cdot \nabla \omega) \\
&- P^N(\sigma_k \cdot \nabla \omega^N) - \sigma_k \cdot \nabla \tilde{\omega}^N + \sigma_k \cdot \nabla \omega^N \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{k\in K}\langle \sigma_k\cdot\nabla\omega - P^N(\sigma_k\cdot\nabla\omega^N), (I-P^N)(\sigma_k\cdot\nabla\omega)\rangle\\
&+ \sum_{k\in K}\langle \sigma_k\cdot\nabla\omega - P^N(\sigma_k\cdot\nabla\omega^N), \sigma_k\cdot\nabla(\omega-\tilde{\omega}^N)\rangle\\
&+ \sum_{k\in K}\langle \sigma_k\cdot\nabla\omega - P^N(\sigma_k\cdot\nabla\omega^N), (I-P^N)(\sigma_k\cdot\nabla\omega^N)\rangle\\
&\le \sum_{k\in K}\|\sigma_k\cdot\nabla\omega - P^N(\sigma_k\cdot\nabla\omega^N)\|_H\|(I-P^N)(\sigma_k\cdot\nabla\omega)\|_H\\
&+ \sum_{k\in K}C\|\sigma_k\cdot\nabla\omega - P^N(\sigma_k\cdot\nabla\omega^N)\|_H\|\nabla(\omega-\tilde{\omega}^N)\|_{L^2}\\
&+ \sum_{k\in K}C\|(I-P^N)(\sigma_k\cdot\nabla\omega)\|_H\|\nabla\omega^N\|_{L^2}.
\end{aligned}
$$

We now move on to treating the nonlinear term

$$
\begin{aligned}
\langle u^N\cdot\nabla\omega^N, \tilde{\omega}^N-\omega^N\rangle_{V^*,V} &= \langle(\tilde{u}^N-u^N)\cdot\nabla(\tilde{\omega}^N-\omega^N), \tilde{\omega}^N\rangle_{V^*,V}\\
&+ \langle\tilde{u}^N\cdot\nabla\tilde{\omega}^N, \tilde{\omega}^N-\omega^N\rangle_{V^*,V}.
\end{aligned}
$$

Therefore, by Ladyzhenskaya's and Young's inequalities,

$$
\begin{aligned}
|\langle(\tilde{u}^N-u^N)\cdot\nabla(\tilde{\omega}^N-\omega^N), \tilde{\omega}^N\rangle_{V^*,V}| &\le \|\nabla\tilde{\omega}^N\|_{L^2}\|\tilde{\omega}^N-\omega^N\|_{L^4}\|\tilde{u}^N-u^N\|_{L^4}\\
&\le C\|\nabla\omega\|_{L^2}\|\tilde{\omega}^N-\omega^N\|_{L^4}^2\\
&\le C\|\nabla\omega\|_{L^2}\|\tilde{\omega}^N\\
&-\omega^N\|_H\|\nabla(\tilde{\omega}^N-\omega^N)\|_{L^2}\\
&\le C\|\nabla\omega\|_{L^2}^2\|\tilde{\omega}^N-\omega^N\|_H^2\\
&+\frac{\nu}{2}\|\nabla(\tilde{\omega}^N-\omega^N)\|_{L^2}^2.
\end{aligned}
$$

To remove some positive terms which corrupt our estimates we use, again, the trick we learnt by Bjorn Schmalfuss [238], introduced in Chap. 2 and we apply the classical Itô formula to $\frac{1}{2}\mathcal{R}(t)\|\tilde{\omega}^N(t)-\omega^N(t)\|_H^2$, where $\mathcal{R}(t) = exp(-\eta_1 t - \eta_2\int_0^t\|\nabla\omega(s)\|_{L^2}^2\,ds)$. Taking the expected value for $t=\tau_M$ and exploiting previous estimates, we arrive at

$$
\begin{aligned}
&\mathbb{E}\left[\frac{1}{2}\mathcal{R}(\tau_M)\|\tilde{\omega}^N(\tau_M)-\omega^N(\tau_M)\|_H^2\right]\\
&+\nu\mathbb{E}\left[\int_0^{\tau_M} ds\mathcal{R}(s)\|\nabla(\tilde{\omega}^N(s)-\omega^N(s))\|_{L^2}^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\frac{\eta_1}{2}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\tilde{\omega}^N(s) - \omega^N(s)\|_H^2\, ds\right] \\
&- \frac{\eta_2}{2}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\nabla\omega(s)\|_{L^2}^2\|\tilde{\omega}^N(s) - \omega^N(s)\|_H^2\, ds\right] \\
&+ \left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B^*(s) - \tilde{u}^N(s)\cdot\nabla\tilde{\omega}^N(s), \tilde{\omega}^N(s) - \omega^N(s)\rangle_{V^*,V}\ ds\right]\right| \\
&+ C\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\nabla\omega(s)\|_{L^2}^2\|\tilde{\omega}^N(s) - \omega^N(s)\|_H^2\, ds\right] \\
&+ \frac{\nu}{2}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\nabla\tilde{\omega}^N(s) - \nabla\omega^N(s)\|_{L^2}^2\, ds\right] \\
&+ \sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\sigma_k\cdot\nabla\omega(s)\right. \\
&\left. - P^N(\sigma_k\cdot\nabla\omega^N(s))\|_H\|(I - P^N)(\sigma_k\cdot\nabla\omega(s))\|_H\, ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\ ds\ \mathcal{R}(s)\|\sigma_k\cdot\nabla\omega(s)\right. \\
&\left. - P^N(\sigma_k\cdot\nabla\omega^N(s))\|_H\|\nabla(\omega(s) - \tilde{\omega}^N(s))\|_{L^2}\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\ ds\ \mathcal{R}(s)\|(I - P^N)(\sigma_k\cdot\nabla\omega(s))\|_H\|\nabla\omega^N(s)\|_{L^2}\, ds\right].
\end{aligned}
\tag{4.14}
$$

If we choose $\eta_1,\ \eta_2$ large enough, we can remove some terms in the right-hand side. Let us consider the remaining terms, recalling that from the weak convergence of ω^N it follows that $\mathbb{E}\left[\int_0^T \|\nabla\omega\|_{L^2}^2\, ds\right] \leq C$. Applying Cauchy–Schwarz inequality where it is needed, we get

$$
\begin{aligned}
&\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\sigma_k\cdot\nabla\omega(s)\right. \\
&\left. - P^N(\sigma_k\cdot\nabla\omega^N(s))\|_H\|(I - P^N)(\sigma_k\cdot\nabla\omega(s))\|_H\, ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\ ds\ \mathcal{R}(s)\|\sigma_k\cdot\nabla\omega(s)\right. \\
&\left. - P^N(\sigma_k\cdot\nabla\omega^N(s))\|_H\|\nabla(\omega(s) - \tilde{\omega}^N(s))\|_{L^2}\right]
\end{aligned}
$$

$$+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M} ds\ \mathcal{R}(s)\|(I-P^N)(\sigma_k\cdot\nabla\omega(s))\|_H\|\nabla\omega^N(s)\|_{L^2}\,ds\right]$$

$$\leq C\sum_{k\in K}\mathbb{E}\left[\int_0^{T}\|(I-P^N)(\sigma_k\cdot\nabla\omega(s))\|_H^2\,ds\right]^{1/2}$$

$$+ C\mathbb{E}\left[\int_0^{T}\|\nabla(\omega(s)-\tilde{\omega}^N(s))\|_{L^2}^2\,ds\right]^{1/2}\to 0.$$

Lastly, we have to treat $\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle B^*(s)-\tilde{u}^N(s)\cdot\nabla\tilde{\omega}^N(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\right.\right.$ $\left.\left. ds\right]\right|$.

$$\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle B^*(s)-\tilde{u}^N(s)\cdot\nabla\tilde{\omega}^N(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\ \ ds\right]\right|$$

$$\leq\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle B^*(s)-u(s)\cdot\nabla\omega(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\ \ ds\right]\right|$$

$$+\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle u(s)\cdot\nabla\omega(s)-\tilde{u}^N(s)\cdot\nabla\tilde{\omega}^N(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\ \ ds\right]\right|.$$

Thanks to $\tilde{\omega}^N-\omega^N\rightharpoonup 0$ in $L^2(\Omega;L^2(0,T;V))$ the first term converges to 0. For what concerns the second one

$$\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle u(s)\cdot\nabla\omega(s)-\tilde{u}^N(s)\cdot\nabla\tilde{\omega}^N(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\,ds\right.\right.$$

$$\left.\left.\pm\int_0^{\tau_M}\mathcal{R}(s)\langle u(s)\cdot\nabla\tilde{\omega}^N(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\,ds\right]\right|$$

$$\leq\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle u(s)\cdot(\nabla\omega(s)-\nabla\tilde{\omega}^N(s)),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\,ds\right]\right|$$

$$+\left|\mathbb{E}\left[\int_0^{\tau_M}\mathcal{R}(s)\langle(u(s)-\tilde{u}^N(s))\cdot\nabla\tilde{\omega}^N(s),\tilde{\omega}^N(s)-\omega^N(s)\rangle_{V^*,V}\,ds\right]\right|$$

$$\leq\mathbb{E}\left[\int_0^{\tau_M}\|u(s)\|_{L^4}\|\omega(s)-\tilde{\omega}^N(s)\|_H^{1/2}\|\nabla(\omega(s)\right.$$

$$\left.-\tilde{\omega}^N(s))\|_{L^2}^{1/2}\|\nabla(\tilde{\omega}^N(s)-\omega^N(s))\|_{L^2}\,ds\right]$$

$$+\mathbb{E}\left[\int_0^{\tau_M}\|u(s)-\tilde{u}^N(s)\|_{L^4}\|\tilde{\omega}^N(s)\|_H^{1/2}\|\nabla\tilde{\omega}^N(s)\|_{L^2}^{1/2}\|\nabla(\tilde{\omega}^N(s)\right.$$

$$\left.-\omega^N(s))\|_{L^2}\,ds\right]$$

$$
\leq C\mathbb{E}\Bigg[\|\omega-\tilde{\omega}^N\|^2_{L^2(0,T;H)}\Bigg]^{1/4}\mathbb{E}\Bigg[\|\omega-\tilde{\omega}^N\|^2_{L^2(0,T;V)}\Bigg]^{1/4}
$$

$$
\mathbb{E}\Bigg[\|\omega^N-\tilde{\omega}^N\|^2_{L^2(0,T;H)}\Bigg]^{1/2}
$$

$$
+C\mathbb{E}\Bigg[\|\omega-\tilde{\omega}^N\|^2_{L^2(0,T;H)}\Bigg]^{1/4}\mathbb{E}\Bigg[\int_0^{\tau_M}\|\nabla\tilde{\omega}^N(s)\|^2_{L^2}\,ds\Bigg]^{1/4}
$$

$$
\mathbb{E}\Bigg[\|\omega^N-\tilde{\omega}^N\|^2_{L^2(0,T;V)}\Bigg]^{1/2}\to 0.
$$

In the last inequalities we use strongly the fact that $\|\omega(s)\|^2_H \leq M$ on $[0,\tau_M]$. In conclusion, in (4.14) all the terms on the right-hand side converge to zero as $N\to\infty$, namely we have the following relation:

$$
\mathbb{E}\left[\frac{1}{2}\mathcal{R}(\tau_M)\|\tilde{\omega}^N(\tau_M)-\omega^N(\tau_M)\|^2_H\right]
+\frac{\nu}{2}\mathbb{E}\left[\int_0^{\tau_M} ds\mathcal{R}(s)\|\nabla(\tilde{\omega}^N(s)-\omega^N(s))\|^2_{L^2}\right]\to 0. \tag{4.15}
$$

From relation (4.15), $\mathcal{R}(t)\geq C_M>0\ \forall t\leq\tau_M$ and the properties of P^N, via triangular inequality the thesis follows. □

Lemma 4.8 *$B^*=u\cdot\nabla\omega$ in $L^2(\Omega,L^2(0,T;V^*))$.*

Proof Thanks to the estimates (4.8) and (4.9) we get easily that $u\cdot\nabla\omega^N$ and $u^N\cdot\nabla\omega$ converge to $u\cdot\nabla\omega$ weakly in $L^2(\Omega;L^2(0,T;V^*))$. We do the explicit computations just for one of the two, the other one being analogous.

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T\|u(s)\cdot\nabla\omega^N(s)\|^2_{V^*}\,ds\right] &\leq C\mathbb{E}\left[\int_0^T\|\nabla u(s)\|^2_{L^2}\|\nabla\omega^N(s)\|^2_{L^2}\,ds\right]\\
&\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega(t)\|^2_H\int_0^T\|\nabla\omega^N(s)\|^2_{L^2}\,ds\right]\\
&\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega(t)\|^4_H\right]\\
&+C\mathbb{E}\left[\left(\int_0^T\|\nabla\omega^N(s)\|^2_{L^2}\,ds\right)^2\right]\\
&\leq C.
\end{aligned}
$$

Let now $\phi \in L^\infty(\Omega; L^\infty(0,T;V))$, then $u\cdot\nabla\phi \in L^2(\Omega; L^2(0,T;H))$. Thus, from the convergence properties of ω^N, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \langle u(s)\cdot\nabla\omega^N(s),\phi\rangle_{V^*,V}ds\right] &= -\mathbb{E}\left[\int_0^T \langle u(s)\cdot\nabla\phi,\omega^N(s)\rangle ds\right]\\ &\to -\mathbb{E}\left[\int_0^T \langle u(s)\cdot\nabla\phi,\omega(s)\rangle ds\right]\\ &= \mathbb{E}\left[\int_0^T \langle u(s)\cdot\nabla\omega(s),\phi\rangle_{V^*,V}ds\right].\end{aligned}$$

From the density of $L^\infty(\Omega; L^\infty(0,T;V))$ in $L^2(\Omega; L^2(0,T;V))$ and the uniform boundedness of $u\cdot\nabla\omega^N$ in $L^2(\Omega; L^2(0,T;V^*))$ we have the required claim. For what concerns the convergence of the nonlinear term, first note that, arguing as above, the sequence $\{u^N\cdot\nabla\omega^N\}_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0,T;V^*))$. Moreover, we have

$$\begin{aligned}u\cdot\nabla\omega - u^N\cdot\nabla\omega^N &= u\cdot\nabla(\omega-\omega^N) + u\cdot\nabla\omega^N\\ &\quad + u^N\cdot\nabla(\omega-\omega^N) - u^N\cdot\nabla\omega =: I_1+I_2+I_3+I_4.\end{aligned}$$

Thanks to the previous observations $I_1+I_2+I_4$ converges weakly to 0 in $L^2(\Omega; L^2(0,T;V^*))$. For what concerns I_3, let us take $\phi \in L^\infty(\Omega; L^\infty(0,T; D(A)))$ and τ_M defined as in Lemma 4.7, then we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\tau_M} \langle u^N(s)\cdot\nabla(\omega(s)-\omega^N(s)),\phi\rangle_{V^*,V}\,ds\right]\\ &\le C\mathbb{E}\left[\int_0^{\tau_M} \|\omega^N(s)\|_H\|\omega(s)-\omega^N(s)\|_H\,ds\right]\\ &\to 0\end{aligned}$$

thanks to Holder's inequality and Lemma 4.7. Since it holds that $\tau_M \nearrow T$ a.s., the thesis follows, thanks to Lemma 4.4. Indeed,

$$\begin{aligned}&\left|\mathbb{E}\left[\int_0^T \langle u^N(s)\cdot\nabla(\omega(s)-\omega^N(s)),\phi\rangle_{V^*,V}\,ds\right]\right|\\ &\le \left|\mathbb{E}\left[\int_0^{\tau_M} \langle u^N(s)\cdot\nabla(\omega(s)-\omega^N(s)),\phi\rangle_{V^*,V}\,ds\right]\right|\\ &+\left|\mathbb{E}\left[\int_{\tau_M}^T \langle u^N(s)\cdot\nabla(\omega(s)-\omega^N(s)),\phi\rangle_{V^*,V}\,ds\right]\right|\end{aligned}$$

$$\leq \left| \mathbb{E}\left[\int_0^{\tau_M} \langle u^N(s) \cdot \nabla(\omega(s) - \omega^N(s)), \phi \rangle_{V^*,V} \, ds \right] \right|$$
$$+ C\mathbb{E}\left[\int_{\tau_M}^T \|\omega(s)\|_H^2 + \left\| \omega^N(s) \right\|_H^2 \, ds \right]$$
$$\leq \left| \mathbb{E}\left[\int_0^{\tau_M} \langle u^N(s) \cdot \nabla(\omega(s) - \omega^N(s)), \phi \rangle_{V^*,V} \, ds \right] \right|$$
$$+ C\mathbb{E}\left[\int_{\tau_M}^T \left(\int_0^T \|B^*(r)\|_{V^*}^2 + \|q(r)\|_H^2 \, dr \right) ds \right] + C\mathbb{E}\left[\int_{\tau_M}^T \|\omega_0\|_H^2 \, ds \right].$$

Thus, if we fix $\epsilon > 0$ and $M > 0$ such that

$$C\mathbb{E}\left[\int_{\tau_M}^T \left(\int_0^T \|B^*(r)\|_{V^*}^2 + \|q(r)\|_H^2 \, dr + \|\omega_0\|_H^2 \right) ds \right] \leq \epsilon,$$

then

$$\text{limusup}_{N\to+\infty} |\mathbb{E}\left[\int_0^T \langle u^N(s) \cdot \nabla(\omega(s) - \omega^N(s)), \phi \rangle_{V^*,V} \, ds \right]| \leq \epsilon.$$

The thesis follows by the density of $L^\infty(\Omega; L^\infty(0, T; D(A)))$ in $L^2(\Omega; L^2(0, T; V))$ and the uniform boundedness of $u^N \cdot \nabla\omega^N$ in $L^2(\Omega; L^2(0, T; V^*))$. □

Theorem 4.9 *There is at most one weak solution of problem (4.1) in the sense of Definition 4.1.*

Proof Let ω, $\tilde{\omega}$ be two solutions and v be their difference. Let $u = -\nabla^\perp(-\Delta)^{-1}\omega$ $\tilde{u} = -\nabla^\perp(-\Delta)^{-1}\tilde{\omega}$ be the corresponding velocities and χ their difference. Thus, v and χ satisfies $\mathbb{P}$-a.s. for each $t \in [0, T]$ and $\phi \in D(A)$

$$\langle v(t), \phi \rangle = \int_0^t \langle v(s), (\nu\Delta + \mathcal{L})\phi \rangle ds + \int_0^t \langle u(s) \cdot \nabla\phi, \omega(s) \rangle ds$$
$$- \int_0^t \langle \tilde{u}(s) \cdot \nabla\phi, \tilde{\omega}(s) \rangle ds$$
$$+ \sum_{k\in K} \int_0^t \langle \sigma_k \cdot \nabla\phi, v(s) \rangle dW_s^k.$$

Arguing as in the proof of Proposition 4.10 below, v and χ satisfy the Itô formula

$$\frac{d\|v\|_H^2}{2} = \left(-\nu\|\nabla v\|_{L^2}^2 + \langle u \cdot \nabla v, \omega \rangle_{V^*,V} - \langle \tilde{u} \cdot \nabla v, \tilde{\omega} \rangle_{V^*,V} \right) dt$$
$$\pm \langle \tilde{u} \cdot \nabla v, \omega \rangle_{V^*,V} dt$$
$$= -\nu\|\nabla v\|_{L^2}^2 dt + \langle \chi \cdot \nabla v, \omega \rangle_{V^*,V} dt.$$

We apply the Itô formula to the process $\frac{1}{2}\mathcal{R}(t)\|v(t)\|_H^2$, where $\mathcal{R}(t) = exp(-\eta \int_0^t \|\nabla\omega\|_{L^2}^2\, ds)$

$$\begin{aligned}\frac{d\left(\mathcal{R}\|v\|_H^2\right)}{2} &= \mathcal{R}(-\nu\|\nabla v\|_{L^2}^2 + \langle \chi\cdot\nabla v, \omega\rangle_{V^*,V})dt - \frac{\eta\mathcal{R}}{2}\|v\|_H^2\|\nabla\omega\|_{L^2}^2 dt \\ &\le -\nu\mathcal{R}\|\nabla v\|_{L^2}^2 dt - \frac{\eta\mathcal{R}}{2}\|v\|_H^2\|\nabla\omega\|_{L^2}^2 dt \\ &+ C\mathcal{R}\|\nabla\omega\|_{L^2}\|v\|_H\|\nabla v\|_{L^2}dt \\ &\le -\nu\mathcal{R}\|\nabla v\|_{L^2}^2 dt - \frac{\eta\mathcal{R}}{2}\|v\|_H^2\|\nabla\omega\|_{L^2}^2 dt \\ &+ \frac{\nu\mathcal{R}\|\nabla v\|_{L^2}^2}{2}dt + C\mathcal{R}\|v\|_H^2\|\nabla\omega\|_{L^2}^2 dt.\end{aligned}$$

Thus, taking η large enough, we have

$$\frac{d\left(\mathcal{R}\|v\|_H^2\right)}{2} + \frac{\nu\mathcal{R}\|\nabla v\|_{L^2}^2}{2}dt \le 0$$

and the thesis follows immediately by the Gronwall lemma. □

Lemmas 4.7, 4.8 identify the nonlinear term and together with Theorem 4.9 conclude the proof of Theorem 4.2. Actually, thanks to some abstract results on stochastic processes something more can be shown, namely that the full sequence ω^N converges to ω in $L^2(\Omega; L^2(0,T;V))$ and, for each $t \in [0,T]$, $\omega^N(t)$ converges to $\omega(t)$ in $L^2(\Omega; H)$. We skip the details related to this kind of convergence, which are not necessary for the next sections, in order to keep this chapter self-contained. Some details about this kind of result and, more in general, the application of the variational method to other fluid dynamical models with transport noise can be found in [64, 204].

Lastly, we want to show that the Itô formula stated in Lemma 4.4 continues to hold for ω, the solution of problem (4.1). For what concerns the energy estimates, they continue to hold immediately due to the weak convergence of ω^N to ω, but they can proved independently starting from the Itô formula and repeating the same steps as in Lemma 4.4.

Proposition 4.10 *The Itô formula below holds:*

$$d\|\omega\|_H^2 + \nu\|\nabla\omega\|_{L^2}^2 dt = 2\langle q, \omega\rangle dt.$$

Proof Let $\tilde{\omega}^N$ be defined as in Lemma 4.7. We already know by the properties of the projector P^N that $\tilde{\omega}^N \to \omega \in L^2(0,T;V)\cap C(0,T;H)$ $\mathbb{P}$-a.s. Exploiting the weak formulation satisfied by ω with test functions e_i we get

$$\langle \omega(t), e_i \rangle = \langle \omega_0, e_i \rangle + \int_0^t \langle \omega(s), (\nu\Delta + \mathcal{L}) e_i \rangle \, ds + \int_0^t \langle \omega(s), u(s) \cdot \nabla e_i \rangle ds + \int_0^t \langle q(s), e_i \rangle \, ds + \sum_{k \in K} \int_0^t \langle \omega(s), \sigma_k \cdot \nabla e_i \rangle \, dW_s^k \ \mathbb{P}\text{-a.s.}$$

Multiplying each equation by e_i and summing up, we get

$$\begin{aligned} d\tilde{\omega}^N &= \nu\Delta\tilde{\omega}^N dt + \sum_{i=1}^N \langle \omega, u \cdot \nabla e_i \rangle e_i \, dt \\ &+ \sum_{i=1}^N \langle q, e_i \rangle e_i \, dt + \sum_{i=1}^N \langle \omega, \mathcal{L} e_i \rangle e_i \, dt \\ &+ \sum_{k \in K} \sum_{i=1}^N \langle \omega, \sigma_k \cdot \nabla e_i \rangle e_i \, dW_t^k. \end{aligned}$$

Now we can apply the Itô formula to the process $\frac{\|\tilde{\omega}^N(t)\|_H}{2}^2$ obtaining

$$\begin{aligned} \|\tilde{\omega}^N(t)\|_H^2 + 2\nu \int_0^t \|\nabla\tilde{\omega}^N(s)\|_{L^2}^2 \, ds = \|\omega_0^N\|^2 &+ 2\int_0^t \langle \omega(s), u(s) \cdot \nabla\tilde{\omega}^N(s) \rangle \, ds \\ &+ 2\int_0^t \langle q(s), \tilde{\omega}^N(s) \rangle \, ds \\ &+ 2\int_0^t \langle \omega(s), \mathcal{L}\tilde{\omega}^N(s) \rangle \, ds \\ &+ \sum_{k \in K} \sum_{i=1}^N \int_0^t \langle \omega(s), \sigma_k \cdot \nabla e_i \rangle^2 ds \\ &+ 2\sum_{k \in K} \int_0^t \langle \omega(s), \sigma_k \cdot \nabla\tilde{\omega}^N \rangle dW_s^k. \end{aligned}$$

Thanks to the properties of the projector P^N we get the Itô formula easily. The only thing we need to prove is that

$$\sum_{i=1}^N \langle \omega(s), \sigma_k \cdot \nabla e_i \rangle^2 + \langle \omega(s), \sigma_k \cdot \nabla \left(\sigma_k \cdot \nabla\tilde{\omega}^N \right) \rangle \to 0.$$

The last relation is true, in fact

$$\sum_{i=1}^{N}\langle\omega(s),\sigma_k\cdot\nabla e_i\rangle^2+\langle\omega(s),\sigma_k\cdot\nabla\left(\sigma_k\cdot\nabla\tilde{\omega}^N(s)\right)\rangle$$
$$=-\langle\omega(s),\sigma_k\cdot\nabla\left(P^N(\sigma_k\cdot\nabla\omega(s))\right)\rangle+\langle\omega(s),\sigma_k\cdot\nabla\left(\sigma_k\cdot\nabla\tilde{\omega}^N(s)\right)\rangle$$
$$=\langle\sigma_k\cdot\nabla\omega(s),P^N\left(\sigma_k\cdot\nabla\omega(s)\right)\rangle-\langle\sigma_k\cdot\nabla\omega(s),\sigma_k\cdot\nabla\tilde{\omega}^N(s)\rangle\to 0.$$

□

4.2 Eddy Viscosity for the Vorticity Equation

Let ω be the solution of the problem of the previous section. Due the presence of a non-linear term and contrary to Sect. 3.3, it is not true anymore that $\mathbb{E}\left[\omega(t)\right]$ solves the system

$$\partial_t\tilde{\omega}=(\nu\Delta+\mathcal{L})\,\tilde{\omega}-\tilde{u}\cdot\tilde{\omega}+q. \tag{4.16}$$

Therefore, a fortiori, the behavior of the stochastic process ω can be very different from the one of $\tilde{\omega}$. In this section we show conditions under which ω is close to $\tilde{\omega}$, hence producing the dissipative properties of $\tilde{\omega}$, in a suitable sense. When so, we may speak, similarly to Sect. 3.3, of *eddy viscosity*: thanks to the noise, the fluid has dissipative properties similar to those of the solution of a deterministic equation with enhanced dissipation. We will show in this section a result in this direction, but plenty more general results can be found in [64, 114, 115, 117] for several fluid dynamics models.

We now give explicit representations of the coefficients $\sigma_k,\ k\in\mathbb{Z}_0^2$,

$$\sigma_k(x)=\sqrt{2\kappa}a_k e^{ik\cdot x}=\begin{cases}\sqrt{2\kappa}\theta_k e^{i\kappa\cdot x}\frac{k^\perp}{|k|}\ if k\in\mathbb{Z}_+^2\\ \sqrt{2\kappa}\theta_k e^{i\kappa\cdot x}\frac{-k^\perp}{|k|}\ if k\in\mathbb{Z}_-^2,\end{cases}$$

where $\mathbb{Z}_+^2,\ \mathbb{Z}_-^2$ is a partition of $\mathbb{Z}_0^2=\mathbb{Z}^2\setminus\{(0,0)\}$ with $\mathbb{Z}_+^2=-\mathbb{Z}_-^2$, θ_k satisfies:

1. $\sum_{k\in\mathbb{Z}_0^2}\theta_k^2=1$.
2. $\theta_k=0$ if $|k|$ is large enough. We will denote by K the finite set of k where $\theta_k\neq 0$.
3. $\theta_k=\theta_k$ if $|k|=|l|$.

Lastly, we take an infinite sequence of complex standard Brownian motions such that $\overline{W^k} = W^{-k}$. At the end, our noise is parameterized by the coefficients κ, θ_k and the set K. Under this setting, Eq. (4.1) can be rewritten as

$$d\omega + \sum_{k\in K} (\sigma_k \cdot \nabla\omega)\, dW_t^k = ((\nu+\kappa)\,\Delta\omega - u\cdot\nabla\omega + q)\, dt. \tag{4.17}$$

The corresponding deterministic system is

$$\partial_t \tilde{\omega} = (\nu+\kappa)\,\Delta\tilde{\omega} - \tilde{u}\cdot\nabla\tilde{\omega} + q. \tag{4.18}$$

Due to the results of Sect. 4.1 and classical results on two-dimensional Navier–Stokes equations, see for example [200, 247, 248], under the assumptions $\omega_0 \in H$, $q \in L^4(0,T;H)$ there exists a unique weak solution ω (resp. $\tilde{\omega}$) of problem (4.17) (resp. (4.18)).

Let us introduce a notation used in this section in order to improve the readability of the results: if a, b are two positive numbers, then we write $a \lesssim b$ if there exists a positive constant C such that $a \le Cb$ and $a \lesssim_\alpha b$ when we want to highlight the dependence of the constant C on a parameter α.

Now we can state the main result of this section.

Theorem 4.11 *Let ω and $\tilde{\omega}$ be weak solutions to (4.17) and (4.18) respectively. Then for any $\alpha \in (0,1)$, there exists $C = C(\alpha)$ such that for any $\epsilon \in (0,\alpha]$ one has*

$$\mathbb{E}\left[\|\omega-\tilde{\omega}\|^p_{C(0,T;H^{-\alpha})}\right]^{1/p}$$
$$\lesssim_{\alpha,p,T} \sqrt{\kappa^\epsilon}\,\|\theta\|^{\alpha-\epsilon}_{\ell^\infty} R_T exp\left(\frac{C}{(\kappa+\nu)^2}\left((T(\kappa+\nu)+1)\,R_T^2\right)\right),$$

$$\mathbb{E}\left[\|\omega-\tilde{\omega}\|^p_{C(0,T;H^{-\alpha})}\right]^{1/p} \lesssim_{\alpha,p,T} \sqrt{\kappa^\epsilon}\,\|\theta\|^{\alpha-\epsilon}_{\ell^\infty} R_T exp\left(\frac{CR_T^2}{\nu^2}\right),$$

where the constant R_T, independent of the noise, is defined below.

This result was originally proven in [117] and we refer to it for a more detailed discussion on the convergence rate of some fluid dynamical models with transport noise to the corresponding deterministic systems.

4.2.1 Some Analytical Lemmas

For the remainder of the section we need to recall some classical tools on the transport term, products of functions in Sobolev spaces and the heat semigroup, see for example [117, 225] for the proof of these statements and more details on the topics.

Lemma 4.12 *Given a divergence free vector field $V \in L^2(\mathbb{T}^2;\mathbb{R}^2)$ the following bounds hold true.*

1. *If $V \in L^\infty\left(\mathbb{T}^2;\mathbb{R}^2\right)$, $f \in L^2(\mathbb{T}^2)$, then we have*

$$\|V\cdot\nabla f\|_{H^{-1}} \lesssim \|V\|_{L^\infty}\|f\|_{L^2}.$$

2. *Let $\alpha \in (1,2],\ \beta \in (0,\alpha-1),\ V \in H^\alpha(\mathbb{T}^2;\mathbb{R}^2),\ f \in H^{-\beta}(\mathbb{T}^2)$, we have*

$$\|V\cdot\nabla f\|_{H^{-1-\beta}} \lesssim_{\alpha,\beta} \|V\|_{H^\alpha}\|f\|_{H^{-\beta}}.$$

3. *Let $\beta \in (0,1)$, then for any $f \in H^\beta(\mathbb{T}^2),\ g \in H^{1-\beta}(\mathbb{T}^2)$ it holds that*

$$\|fg\|_{L^2} \lesssim_\beta \|f\|_{H^\beta}\|g\|_{H^{1-\beta}}.$$

4. *Let $\beta \in (0,1),\ V \in H^{1-\beta}(\mathbb{T}^2;\mathbb{R}^2),\ f \in L^2(\mathbb{T}^2)$, then one has*

$$\|V\cdot\nabla f\|_{H^{-1-\beta}} \lesssim_\beta \|V\|_{H^{1-\beta}}\|f\|_{L^2}.$$

The second lemma provides classical estimates on the semigroup generated by Δ:

Lemma 4.13 *Let $q \in D((-\Delta)^{\alpha/2}),\ \alpha \in \mathbb{R}$. Then:*

1. *for any $\rho \geq 0$, it holds that $\|e^{t\Delta}q\|_{H^{\alpha+\rho}} \leq C_\rho t^{-\rho/2}\|q\|_{H^\alpha}$ for some constant increasing in ρ;*
2. *for any $\rho \in [0,2]$, it holds that $\|\left(I-e^{t\Delta}\right)q\|_{H^{\alpha-\rho}} \lesssim_\rho t^{\rho/2}\|q\|_{H^\alpha}$.*

The semigroup $e^{\delta(t-s)\Delta}$ has also regularizing effects as stated in the following:

Lemma 4.14 *For any $\delta > 0,\ \alpha \in \mathbb{R},\ q \in L^2(0,T;D((-\Delta)^{\alpha/2}))$, it holds that*

$$\left\|\int_0^t e^{\delta(t-s)\Delta}q(s)\,ds\right\|^2_{H^{\alpha+1}} \lesssim \frac{1}{\delta}\int_0^t \|q(s)\|^2_{H^\alpha}\,ds\ \ \forall t \in [0,T].$$

4.2.2 The Stochastic Convolution

Similarly to Sect. 3.3, we will prove Theorem 4.2 introducing a mild formulation satisfied by ω. Thus we will need some preliminaries about the stochastic convolution in this framework.

Under our assumptions, we know by Lemma 4.10 that

$$\sup_{t\in[0,T]}\|\omega(t)\|_H^2 \le \int_0^T \frac{\|q(s)\|_H^2}{\nu} ds + \|\omega_0\|_H^2 =: R_T^2 < +\infty \ \mathbb{P}\text{-a.s.}$$

This quantity will play a crucial role in this section. Given a positive parameter $\delta > 0$, denote the stochastic integral and stochastic convolution as

$$M(t) := \sqrt{2\kappa}\sum_{k\in K}\int_0^t a_k e^{ik\cdot x}\cdot\nabla\omega dW_s^k, \tag{4.19}$$

$$Z(t) := \sqrt{2\kappa}\sum_{k\in K}\int_0^t e^{\delta(t-s)\Delta}\left(a_k e^{ik\cdot x}\cdot\nabla\omega\right) dW_s^k. \tag{4.20}$$

Lemma 4.15 *The processes M(t) and Z(t) defined above satisfy:*

(i) $M(t)$ *is a continuous martingale with values in* V^**. Moreover, it holds that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|M(t)\|_{V^*}^2\right] \lesssim \kappa R_T^2 T.$$

(ii) For each $\epsilon \in (0, 1/2)$, $p \ge 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|Z(t)\|_{H^{-\epsilon}}^p\right]^{1/p} \lesssim_{\epsilon,p,T} \sqrt{\kappa\delta^{\epsilon-1}} R_T, \tag{4.21}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|Z(t)\|_{H^{-1-\epsilon}}^p\right]^{1/p} \lesssim_{\epsilon,p,T} \sqrt{\kappa\delta^{\epsilon-1}}\|\theta\|_{\ell^\infty} R_T. \tag{4.22}$$

(iii) For $\beta \in (0, 1]$ *and* $\epsilon \in (0, \beta]$, $p \ge 1$ *it holds that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|Z(t)\|_{H^{-\beta}}^p\right]^{1/p} \lesssim_{\epsilon,p,T} \sqrt{\kappa\delta^{\epsilon-1}}\|\theta\|_{\ell^\infty}^{\beta-\epsilon} R_T. \tag{4.23}$$

Proof Up to some technicalities, the proofs of these statements are similar to some computations we did several times before. Therefore we simply provide the proof of the first statement. For a fully detailed and complete proof of all the estimates we suggest of reading [117].

The first point follows immediately by the Burkholder–Davis–Gundy inequality, Lemma 4.12.1. and the obvious relation $\frac{|k^\perp|}{|k|}\|e^{ik\cdot x}\|_{L^\infty} = 1$. Indeed,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\sum_{k\in K}\int_0^t \sigma_k\cdot\nabla\omega(s)dW_s^k\|_{V^*}^2\right]$$

$$\lesssim \kappa \mathbb{E}\left[\sum_{k\in K}\int_0^T \|a_k e^{ik\cdot x}\cdot\nabla\omega(s)\|_{V^*}^2 ds\right]$$
$$\lesssim \kappa \sum_{k\in K}\int_0^T \theta_k^2 \mathbb{E}\left[\|\omega(s)\|_H^2\right] ds \lesssim \kappa R_T^2 T.$$

Now we can state the result about the mild formulation satisfied by ω.

Theorem 4.16 *Let ω be the weak solution of problem (4.17), then ω satisfies the following integral relation $\mathbb{P}$-a.s.:*

$$\omega(t) = e^{(\kappa+\nu)t\Delta}\omega_0 - \int_0^t e^{(\kappa+\nu)(t-s)\Delta}\left(u(s)\cdot\nabla\omega(s)\right) ds - Z(t). \tag{4.24}$$

The proof is not difficult and is similar to one made in Chap. 1 for the Stokes problem. However, it can be found in [117].

4.2.3 Proof of Theorem 4.11

First we prove the following lemma about the behavior of the nonlinearity.

Lemma 4.17 *If $\omega \in H$, $\tilde{\omega} \in V$, $u = -\nabla^\perp(-\Delta)^{-1}\omega$, $\tilde{u} = -\nabla^\perp(-\Delta)^{-1}\tilde{\omega}$, then for each $\alpha \in (0,1)$ it holds that*

$$\|u\cdot\nabla\omega - \tilde{u}\cdot\nabla\tilde{\omega}\|_{H^{-\alpha-1}} \lesssim_\alpha \|\omega-\tilde{\omega}\|_{H^{-\alpha}}(\|\omega\|_H + \|\tilde{\omega}\|_V).$$

Proof

$$\begin{aligned}\|u\cdot\nabla\omega - \tilde{u}\cdot\nabla\tilde{\omega}\|_{H^{-\alpha-1}} &\le \|(u-\tilde{u})\cdot\nabla\omega\|_{H^{-\alpha-1}} + \|\tilde{u}\cdot\nabla(\omega-\tilde{\omega})\|_{H^{-\alpha-1}}\\ &=: I_1 + I_2.\end{aligned}$$

By Lemma 4.12.4. with $\beta = \alpha$ we have

$$I_1 \lesssim_\alpha \|u-\tilde{u}\|_{H^{1-\alpha}}\|\omega\|_H \lesssim \|\omega-\tilde{\omega}\|_{H^{-\alpha}}\|\omega\|,$$

Again by Lemma 4.12.2. we have

$$I_2 \lesssim \|\tilde{u}\|_{D(A)}\|\omega-\tilde{\omega}\|_{H^{-\alpha}} \lesssim \|\tilde{\omega}\|_V\|\omega-\tilde{\omega}\|_{H^{-\alpha}}.$$

Combining these two relations the thesis follows. □

Proof of Theorem 4.11 From the energy estimates we know that

$$\|\omega(t)\|_H^2 + \nu \int_0^t \|\nabla\omega(s)\|_{L^2}^2 \, ds \le \|\omega_0\|_H^2 + \int_0^t \frac{\|q(s)\|_H^2}{\nu} ds \ \mathbb{P}\text{-a.s.},$$

$$\|\tilde{\omega}(t)\|_H^2 + (\nu+\kappa) \int_0^t \|\nabla\tilde{\omega}(s)\|_{L^2}^2 \, ds \le \|\omega_0\|_H^2 + \int_0^T \frac{\|q(s)\|_H^2}{(\nu+\kappa)} ds.$$

Moreover, by the results of Sect. 4.2.2 we know that both ω and $\tilde{\omega}$ satisfy a mild formulation. Thus letting $\xi = \omega - \tilde{\omega}$ we have

$$\xi(t) = -\int_0^t e^{(\kappa+\nu)(t-s)\Delta} \left(u(s)\cdot\nabla\omega(s) - \tilde{u}(s)\cdot\nabla\tilde{\omega}(s)\right) ds - Z_t.$$

By Lemmas 4.14 and 4.17 it follows that

$$\begin{aligned}\|\xi(t)\|_{H^{-\alpha}}^2 &\lesssim_\alpha \|Z(t)\|_{H^{-\alpha}}^2 + \frac{1}{\kappa+\nu}\int_0^t \| \, u(s)\cdot\nabla\omega(s) - \tilde{u}(s)\cdot\nabla\tilde{\omega}(s)\|_{H^{-\alpha-1}}^2 \, ds \\ &\lesssim_\alpha \|Z(t)\|_{H^{-\alpha}}^2 + \frac{1}{\kappa+\nu}\int_0^t \|\xi(s)\|_{H^{-\alpha}}^2 (\|\omega(s)\|^2 + \|\tilde{\omega}(s)\|_V^2) \, ds.\end{aligned}$$

Thus by the Gronwall lemma, there exists $C = C(\alpha)$ such that

$$\|\xi(t)\|_{H^{-\alpha}}^2 \lesssim_\alpha \left(\sup_{t\in[0,T]}\|Z(t)\|_{H^{-\alpha}}^2\right) exp\left(\frac{C}{\nu+\kappa}\int_0^T \|\omega(s)\|_H^2 + \|\tilde{\omega}(s)\|_V^2 \, ds\right). \tag{4.25}$$

Taking the expectation and exploiting relation (4.23) we arrive at the first relation in the statement. Exploiting the other estimate

$$\int_0^T \|\omega(s)\|_H^2 ds \le \int_0^T \|\omega(s)\|_V^2 ds \le \frac{1}{\nu}\left(\|\omega_0\|_H^2 + \int_0^T \frac{\|q(s)\|_H^2}{\nu} ds\right)$$

and the obvious fact that $\frac{1}{\nu} + \frac{1}{\kappa+\nu} \le \frac{2}{\nu}$, starting from (4.25) we can obtain the second estimate and the proof is complete. □

4.2.4 The Result for Long Times

The last result we want to discuss in this section is an easy corollary of Theorem 4.11 and the convergence, for high viscosity, of the solution of the deterministic Navier–Stokes equations to the corresponding stationary solution in the case of $q(t) \equiv q \in H$. Obviously, we are in the framework for having existence and uniqueness of the weak solution of problem (4.17). In this section we have to consider ω a weak solution of problem (4.17), $\tilde{\omega}$ a solution of problem (4.18) and $\bar{\omega}$ a weak solution of the following stationary problem with periodic boundary conditions:

$$(\nu+\kappa)\Delta\bar{\omega} - \bar{u}\cdot\nabla\bar{\omega} + q = 0, \tag{4.26}$$

where $\bar{u} = -\nabla^{\perp}(-\Delta)^{-1}\bar{\omega}$. By a weak solution of problem (4.26) we mean a function $\bar{\omega} \in V$ such that

$$(\kappa+\nu)\langle\nabla\bar{\omega}, \nabla\phi\rangle + \langle\bar{u}\cdot\nabla\bar{\omega}, \phi\rangle = \langle q, \phi\rangle \;\; \forall\phi \in V.$$

For the sake of completeness we state and prove the classical deterministic result below, see for example [247, 248] for a detailed discussion on this topic.

Theorem 4.18 *For $\kappa+\nu$ large enough there exists a unique $\bar{\omega}$ weak solution of problem (4.26), moreover $\|\tilde{\omega}(t) - \bar{\omega}\| \to 0$ exponentially fast as $t \to +\infty$.*

Proof of Theorem 4.11 Existence and uniqueness for $\kappa+\nu$ large enough follows by the contraction mapping theorem. Indeed, let us define the map $T : V \to V$ such that to each $v \in V$ is associated $T(v)$ which is the unique weak solution of the stationary problem below:

$$(\nu+\kappa)\Delta T(v) - \chi\cdot\nabla T(v) + q = 0, \tag{4.27}$$

where $\chi = -\nabla^{\perp}(-\Delta)^{-1}v$. The existence and uniqueness of a weak solution of problems (4.27) follows immediately by the Lax–Milgram lemma due the fact that $\langle\chi\cdot\nabla z, z\rangle = 0 \; \forall z \in V$, thus the bilinear form $a(z_1, z_2) = (\kappa+\nu)\langle\nabla z_1, \nabla z_2\rangle + \langle\chi\cdot\nabla z_1, z_2\rangle$ is continuous and coercive for all $z_1,\; z_2 \in V$. Moreover, again by the Lax–Milgram lemma, the following a priori estimate holds true:

$$\|T(v)\|_V \leq \frac{\|q\|_H}{\kappa+\nu}.$$

For what concerns contractivity, let $v_1,\; v_2 \in V$, $u_1 = -\nabla^{\perp}(-\Delta)^{-1}v_1$, $u_2 = -\nabla^{\perp}(-\Delta)^{-1}v_2$ the corresponding velocity. Then $T(v_1)$ and $T(v_2)$ satisfy for all $\phi \in V$

$$\begin{aligned}(\kappa+\nu)\langle\nabla\left(T(v_1)-T(v_2)\right), \nabla\phi\rangle &= \langle u_2\cdot\nabla T(v_2), \phi\rangle - \langle u_1\cdot\nabla T(v_1), \phi\rangle \\ &= \langle u_2\cdot\nabla\left(T(v_2)-T(v_1)\right), \phi\rangle \\ &\quad - \langle(u_1-u_2)\cdot\nabla T(v_1), \phi\rangle.\end{aligned}$$

If we take $\phi = T(v_1) - T(v_2)$, we get via the Holder inequality and Sobolev embedding theorem

$$(\kappa+\nu)\|T(v_1)-T(v_2)\|_V^2 \leq C\|v_1-v_2\|_V\|T(v_1)-T(v_2)\|_V\|T(v_1)\|_V.$$

We restrict the map T to the closed ball in V centered in 0 and with radius $\frac{\|q\|_H}{\kappa+\nu} =: M$ which we will denote by B_M. Thanks to the a priori estimate $T : B_M \to B_M$. Moreover, by previous computations, if $\frac{CM}{\kappa+\nu} = \frac{C\|q\|_H}{(\kappa+\nu)^2} < 1$ the map is a contraction and we have existence and uniqueness of the solution.

For what concerns the convergence of $\tilde{\omega}$ to $\bar{\omega}$, if we denote by $\xi = \tilde{\omega} - \bar{\omega}$ and $\chi = \tilde{u} - \bar{u}$, then the following relation holds true:

$$\begin{aligned}\frac{d}{dt}\|\xi\|_H^2 + 2(\kappa+\nu)\|\nabla\xi\|_{L^2}^2 &= -2\langle \tilde{u}\cdot\nabla\tilde{\omega}, \xi\rangle + 2\langle \bar{u}\cdot\nabla\bar{\omega}, \xi\rangle \\ &= -2\langle \tilde{u}\cdot\nabla\tilde{\omega}, \xi\rangle + 2\langle \bar{u}\cdot\nabla\bar{\omega}, \xi\rangle \pm 2\langle \tilde{u}\cdot\nabla\bar{\omega}, \xi\rangle \\ &= -2\langle \chi\cdot\nabla\bar{\omega}, \xi\rangle \le C\|\nabla\bar{\omega}\|_{L^2}\|\nabla\xi\|_{L^2}\|\xi\|_H \\ &\le C\|\nabla\bar{\omega}\|_{L^2}\|\nabla\xi\|_{L^2}^2.\end{aligned}$$

The constant appearing in the chain of inequalities is due to the Sobolev embedding theorem and the boundedness of the operator $\nabla^\perp \in L(D(-\Delta)^\alpha; H^{2\alpha-1}(\mathbb{T}^2;\mathbb{R}^2))$ for each $\alpha \ge 1/2$. Thus, if $\kappa+\nu$ is large enough, letting $\alpha := 2(\kappa+\nu) - C\|\nabla\bar{\omega}\|_{L^2} \ge 2(\kappa+\nu) - \frac{C\|q\|_H}{\kappa+\nu} > 0$ we have

$$0 \ge \frac{d}{dt}\|\xi\|_H^2 + \alpha\|\nabla\xi\|_{L^2}^2 \ge \frac{d}{dt}\|\xi\|_H^2 + \alpha\|\xi\|_H^2.$$

Therefore by the Gronwall lemma

$$\|\bar{\omega} - \tilde{\omega}(t)\|_H^2 \le \|\omega_0 - \bar{\omega}\|_H^2 e^{-\alpha t}.$$

□

In this framework we can have $\kappa + \nu$ large enough without any unrealistic assumption on the viscosity ν. This is a particular property of the transport noise, which moreover guarantees a suitable convergence of ω to $\bar{\omega}$ for long times for a suitable scaling of the parameters $\{\theta_k\}_{k\in K}$. This kind of convergence is described by the theorem below.

Theorem 4.19 *For $\kappa+\nu$ large enough, for each $\delta > 0$ and $\alpha \in (0,1)$, there exists $\bar{T} = \bar{T}(\delta)$ and a sequence $\{\theta_k\}_{k\in K}$ depending on δ, $\bar{T}$ and α such that*

$$\mathbb{E}\left[\sup_{t\in[T,2\bar{T}]}\|\omega(t) - \bar{\omega}\|_{H^{-\alpha}}^2\right] \le \delta.$$

Proof of Theorem 4.11 Let $\tilde{\omega}$ be the weak solution of problem (4.18). First we fix $\delta > 0$, $\alpha \in (0,1)$. If $(\kappa+\nu)$ is large enough, by Theorem 4.18, we can find $\bar{T} = \bar{T}(\delta)$ such that

$$\|\bar{\omega} - \tilde{\omega}(t)\|_H^2 \le \delta/4, \ \forall t \ge \bar{T}.$$

Now we use the results of Theorem 4.11 for $\epsilon = \alpha/2$, thus we have

$$\mathbb{E}\left[\|\omega - \tilde{\omega}\|^2_{C(0,2\bar{T};H^{-\alpha})}\right] \lesssim_{\alpha,2\bar{T}} \kappa^{\alpha/2} \|\theta\|^{\alpha}_{\ell^\infty} \left(\frac{2\bar{T}\|q\|^2_H}{\nu} + \|\omega_0\|^2_H\right)$$
$$exp\left(\frac{C}{\nu^2}\left(\|\omega_0\|^2_H + \frac{2\bar{T}\|q\|^2_H}{2\nu}\right)\right).$$

If we take θ such that the right-hand side of the previous inequality can be bounded by $\delta/4$ then the thesis follows immediately. For example, some possible choices of θ can be found in [117, Example 1.3] □

4.3 Velocity Formulation

Contrary to the results of Sect. 4.1, here we want to discuss the well-posedness of the stochastic Navier–Stokes equations with transport noise, in the perhaps, more natural formulation: the velocity formulation. As discussed in the introduction to this chapter, so far, it is not clear which formulation of the Navier–Stokes equations with transport noise is better for the physics. We have chosen to explain all the computations in detail at the vorticity level because the portrait of the results is more clear and readable, due to the gain of regularity. For the matter of completeness, we want to explain what changes considering the well-posedness of velocity equations. We will follow again the variational method and the plan described in Sect. 4.1.1.

We will consider the stochastic problem, already written in Itô form,

$$du + \sum_{k\in K} P(\sigma_k \cdot \nabla u)dW^k_t = ((\nu A + \mathcal{L})u - P(u\cdot\nabla u) + q)\,dt \tag{4.28}$$

in the two-dimensional torus $\mathbb{T}^2 = [0,\pi]^2$, where σ_k are smooth, divergence free vector fields, P and A are the projector and the Stokes operator as described in Chap. 1, but now with periodic boundary conditions,

$$\mathcal{L}u = \sum_{k\in K} P\left(\sigma_k \cdot \nabla\left(P(\sigma_k\cdot\nabla u)\right)\right).$$

However, the argument presented in the following pages continues to be valid considering no-slip boundary conditions in a bounded domain. We have chosen to consider the case of periodic boundary conditions in order to be consistent with previous sections.

4.3.1 Functional Setting and Assumptions

Let: $H = L^2_{0,div}\left(\mathbb{T}^2;\mathbb{R}^2\right)$, $V = W^{1,2}\left(\mathbb{T}^2;\mathbb{R}^2\right) \cap H$, $D(A) = W^{2,2}\left(\mathbb{T}^2;\mathbb{R}^2\right) \cap V$, $A : D(A) \subset H \to H$

$$Au = P\Delta u,$$

where $L^2_{0,div}(\mathbb{T}^2;\mathbb{R}^2)$ is the subspace of $L^2(\mathbb{T}^2;\mathbb{R}^2)$ made by zero mean, divergence free vector fields. It is well-known that A is the infinitesimal generator of an analytic semigroup of negative type and moreover V can be identified with $D((-A)^{1/2})$, see [247, 248]. Obviously the nonlinearity of the system has the same properties described in Sect. 1.1.2.

Let us consider the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, thus we deal with strong solutions. As already done in a previous chapter, let us denote by $L^p_{\mathcal{F}}(0,T;V)$ the space of p integrable, progressively measurable processes with values in V and by $C_{\mathcal{F}}([0,T];H)$ the space of continuous adapted square integrable processes. Assume σ_k smooth enough (just for simplicity we assume $\sigma_k \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$, but less can be required), $u_0 \in L^4_{\mathcal{F}_0}(\Omega, H)$, $q \in L^4_{\mathcal{F}}(0,T;H)$.

Definition 4.20 A stochastic process

$$u \in C_{\mathcal{F}}([0,T];H) \cap L^2_{\mathcal{F}}(0,T;V)$$

is a weak solution of Eq. (4.28) if, for every $\phi \in D(A)$, we have

$$\langle u(t), \phi\rangle = \langle u_0, \phi\rangle + \int_0^t \langle u(s), (\nu A + \mathcal{L})\phi\rangle\, ds + \int_0^t b(u(s), \phi, u(s))\, ds$$
$$+ \int_0^t \langle q(s), \phi\rangle\, ds + \sum_{k\in K}\int_0^t b(\sigma_k, \phi, u(s))\, dW^k_s$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s.

The main result proved below is:

Theorem 4.21 *For every $u_0 \in L^4_{\mathcal{F}_0}(\Omega, H)$ and $q \in L^4_{\mathcal{F}}(0,T;H)$, there exists one and only one weak solution of Eq. (4.28).*

Remark 4.22 One can notice that the definitions given and the assumptions required in this section and the ones given in Sect. 4.1 are completely dual. For this reason, it is not surprising that the computations needed in order to get Theorem 4.21 and the ones needed to get Theorem 4.2 are similar. Thus, in the following, we will refer to Sect. 4.1 for the complete explanations of some arguments. In a certain sense, this section can be seen by the reader as a long exercise on the variational method for fluid dynamic models with transport noise. For pedagogical reasons we add almost all the proofs of the statements below, but we suggest that you try to prove them by yourself. Again, as in Sect. 2.3 the extra integrability conditions are needed in order to get existence, but uniqueness follows under the more natural assumptions $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H),\ q \in L^2_{\mathcal{F}}(0, T; H)$.

Remark 4.23 The difference between the two models is clear, noting that, at a formal level, taking the curl of Eq. (4.28) we do not arrive Eq. (4.1). A different, perhaps more transparent, possibility to see this is considering the evolution of the quantity $\|(-A)^{1/2}u\|^2_H$ by Itô formula, at least at a formal level. If the two models were equivalent, then we would have $d\|(-A)^{1/2}u\|^2_H = d\|\omega\|^2_{L^2}$. In fact in the case of periodic boundary conditions, integrating by parts, we have the following chain of equalities:

$$\begin{aligned}\int_{\mathbb{T}^2} |\omega(x)|^2 dx &= \int_{\mathbb{T}^2} (\partial_1 u_2(x))^2 + (\partial_2 u_1(x))^2 - 2\partial_1 u_2(x)\partial_2 u_1(x) dx \\ &= \int_{\mathbb{T}^2} (\partial_1 u_2(x))^2 + (\partial_2 u_1(x))^2 + \partial_{12} u_2(x) u_1(x) + \partial_{12} u_1(x) u_2(x) dx \\ &= \int_{\mathbb{T}^2} (\partial_1 u_2(x))^2 + (\partial_2 u_1(x))^2 - \partial_{11} u_1(x) u_1(x) - \partial_{22} u_2(x) u_2(x) dx \\ &= \int_{\mathbb{T}^2} (\partial_1 u_2(x))^2 + (\partial_2 u_1(x))^2 + (\partial_1 u_1(x))^2 + (\partial_2 u_2(x))^2\, dx.\end{aligned}$$

We already know by Lemma 4.10 that $d\|\omega\|^2_{L^2}$ has only a drift term. Now we compute $d\|(-A)^{1/2}u\|^2_H = d\|\nabla u\|^2_{L^2}$.

$$\begin{aligned}d\|(-A)^{1/2}u\|^2_H =& -\nu\|Au\|^2_H dt - \langle \mathcal{L}u, Au\rangle dt + b\,(u, u, Au)\,dt \\ & - \langle q, Au\rangle dt + \sum_{k\in K} b(\sigma_k, u, Au)^2 dt + \sum_{k\in K} b(\sigma_k, u, Au) dW^k_t.\end{aligned}$$

While $b(u, u, Au) = 0$ in the torus, see for example [192], there is no reason that $b(\sigma_k, u, Au) = 0$, thus $d\|\nabla u\|^2_{L^2}$ has also a diffusion term.

4.3.2 Galerkin Approximation and Limit Equations

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H made by eigenvectors of $-A$, and the corresponding eigenvalues, λ_i, are positive and nondecreasing thanks to our choice of removing the constant vector fields. Let $H^N = \text{span}\{e_1, \ldots, e_N\} \subseteq H$, $P^N : H \to H$ the orthogonal projector of H on H^N. We start looking for a finite-dimensional approximation of the solution of Eq. (4.28). We define

$$u^N(t) = \sum_{i=1}^N c_{i,N}(t) e_i(x).$$

The $c_{i,N}$ have been chosen in order to satisfy $\forall e_i, \quad 1 \le i \le N, \ t \in [0, T]$

$$\begin{aligned}\left\langle u^N(t), e_i \right\rangle = \left\langle u_0^N, e_i \right\rangle + \int_0^t \left\langle u^N(s), \left(\nu A + \mathcal{L}^N\right) e_i \right\rangle ds \\ + \int_0^t b\left(u^N(s), e_i, u^N(s)\right) ds \\ + \int_0^t \langle q(s), e_i \rangle\, ds + \sum_{k \in K} \int_0^t b\left(\sigma_k, e_i, u^N(s)\right) dW_s^k, \ \mathbb{P}\text{-a.s.,}\end{aligned} \tag{4.29}$$

where

$$\mathcal{L}^N \phi = \frac{1}{2} \sum_{k \in K} P^N P\left(\sigma_k \cdot \nabla P^N P(\sigma_k \cdot \nabla \phi)\right) \ \forall \phi \in H^N.$$

Local existence and uniqueness of the solution of this system of ordinary stochastic differential equations follows from the classical theory for stochastic differential equations with locally Lipshitz coefficients. For what concerns the global existence, it follows from the a priori estimates below.

Lemma 4.24 *The Itô formula below holds:*

$$d\|u^N\|_H^2 + 2\nu \|\nabla u^N\|_{L^2}^2 dt = 2\langle q, u^N \rangle dt \tag{4.30}$$

and the following energy estimates are satisfied:

$$\|u^N(t)\|_H^2 \le \int_0^t e^{-\nu(t-s)} \frac{\|q(s)\|_H^2}{\nu} ds + e^{-\nu t} \|u_0\|_H^2, \tag{4.31}$$

$$\nu \int_0^t \|\nabla u^N(s)\|_{L^2}^2 ds \le \int_0^t \frac{\|q(s)\|_H^2}{\nu} ds + \|u_0\|_H^2, \tag{4.32}$$

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u^N(t)\|_H^2\right] \le \mathbb{E}\left[\int_0^T \frac{\|q(s)\|_H^2}{\nu} ds + \|u_0\|_H^2\right], \tag{4.33}$$

$$\nu \mathbb{E}\left[\int_0^T \|\nabla u^N(t)\|_{L^2}^2\, ds\right] \le \mathbb{E}\left[\int_0^T \frac{\|q(s)\|_H^2}{\nu} ds + \|u_0\|_H^2\right], \tag{4.34}$$

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u^N(t)\|_H^4\right] + \mathbb{E}\left[\int_0^T \|u^N(s)\|_H^2 \|\nabla u^N(s)\|_{L^2}^2\, ds\right] \le C, \tag{4.35}$$

$$\mathbb{E}\left[\left(\int_0^T \|\nabla u^N(s)\|_{L^2}^2\, ds\right)^2\right] \le C, \tag{4.36}$$

where C is a constant possibly changing its value line by line, but independent of N.

Proof of Theorem 4.11 The proof of this statement is completely analogous to that of Lemma 4.4. We prefer to add it only for the matter of completeness, in order to make clear to the reader the little changes due to the presence of the Leray projector P.

The Itô formula follows immediately from the finite dimensional Itô formula. In fact

$$d\|u^N\|_H^2 = \sum_{i=1}^N dc_{i,N}^2 = 2\sum_{i=1}^N c_{i,N} dc_{i,N} + \sum_{i=1}^N \langle dc_{i,N}, dc_{i,N}\rangle.$$

Thus, exploiting the weak formulation satisfied by u^N, we have

$$\begin{aligned} d\|u^N\|_H^2 &= 2\left(\langle u^N, \nu A u^N\rangle + \langle q, u^N\rangle + b\left(u^N, u^N, u^N\right) + \langle u^N, \mathcal{L}^N u^N\rangle\right) dt \\ &\quad + \sum_{k\in K} b(\sigma_k, u^N, u^N) dW_t^k + \sum_{i=1}^N \sum_{k\in K} b\left(\sigma_k, e_i, u^N\right)^2 dt. \end{aligned}$$

From the fact that u^N, σ_k are divergence free, it follows immediately that $b\left(u^N, u^N, u^N\right) = b(\sigma_k, u^N, u^N) = 0$. Moreover, we can notice, integrating by parts and exploiting $\operatorname{div}\sigma_k = 0$, that

$$\begin{aligned} &\sum_{i=1}^N \sum_{k\in K} b\left(\sigma_k, e_i, u^N\right)^2 + 2\langle u^N, \mathcal{L}^N u^N\rangle \\ &\quad = \sum_{i=1}^N \sum_{k\in K} \left\langle \sigma_k \cdot \nabla u^N, e_i\right\rangle^2 + \sum_{k\in K} \langle P^N P\left(\sigma_k \cdot \nabla P^N P(\sigma_k \cdot \nabla u^N)\right), u^N\rangle \\ &\quad = \sum_{k\in K} \langle \sigma_k \cdot \nabla u^N, P^N P(\sigma_k \cdot \nabla u^N)\rangle \\ &\quad + \sum_{k\in K} \langle \sigma_k \cdot \nabla P^N P(\sigma_k \cdot \nabla u^N), u^N\rangle = 0. \end{aligned}$$

Thus we arrive at the Itô formula in the statement. Starting from the Itô formula and applying the Poincaré inequality for zero mean vector fields in the torus, the Cauchy–Schwarz and Young's inequality properly, we get

$$d\|u^N\|_H^2 + 2\nu\|\nabla u^N\|_{L^2}^2 dt \le \left(\nu\|u^N\|_H^2 + \frac{\|q\|_H^2}{\nu}\right) dt \le \left(\nu\|\nabla u^N\|_{L^2}^2 + \frac{\|q\|_H^2}{\nu}\right) dt.$$

Thus we obtain the following relations by the Gronwall lemma:

$$\|u^N(t)\|_H^2 \le \int_0^t e^{-\nu(t-s)} \frac{\|q(s)\|_H^2}{\nu} ds + e^{-\nu t} \|u_0\|_H^2,$$

$$\nu \int_0^t \|\nabla u^N(s)\|_{L^2}^2 ds \le \|u_0\|_H^2 + \int_0^t \frac{\|q(s)\|_H^2}{\nu} ds.$$

These inequalities imply the first four energy relations stated and the last one. For what concerns the remaining one, we use the Itô formula satisfied by $\|u^N(t)\|_H^2$ and we apply the Itô formula for the function $f(x) = x^2$. Arguing as above, we get

$$\begin{aligned} d\|u^N\|_H^4 + 4\nu \|u^N\|_H^2 \|\nabla u^N\|_{L^2}^2 &= 4\|u^N\|_H^2 \langle q, u^N \rangle dt \\ &\le \left(2\nu \|u^N\|_H^2 \|\nabla u^N\|_{L^2}^2 + \frac{2\|q\|_H^2 \|u^N\|_H^2}{\nu} \right) dt \\ &\le \left(2\nu \|u^N\|_H^2 \|\nabla u^N\|_{L^2}^2 + \frac{\|u^N\|^4}{\nu^2} + \frac{\|q\|^4}{\nu^2} \right) dt \end{aligned}$$

and this implies the thesis by the Gronwall lemma. □

Remark 4.25 Only relations (4.35) and (4.36) use the further integrability assumptions on the initial conditions and the forcing term.

From the energy estimates on u^N, there exists a subsequence, which we will denote again for simplicity by u^N, which converges to a stochastic process u in the way described below:

$$u^N \overset{*}{\rightharpoonup} u \; L^4(\Omega; L^\infty(0, T; H))$$

$$u^N \rightharpoonup u \; L^4(\Omega; L^2(0, T; V))$$

and an unknown process B^* such that

$$B(u^N, u^N) \rightharpoonup B^* \; L^2(\Omega; L^2(0, T; V^*)).$$

Moreover, thanks to the converging properties of the projector P^N for $N \to +\infty$, the processes u and B^* satisfy $\mathbb{P}$-a.s. for each $i \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} \langle u(t), \phi \rangle + \int_0^t \langle B^*(s), e_i \rangle_{V^*,V} ds = \langle u_0, e_i \rangle + \int_0^t \langle u(s), (\nu A + \mathcal{L}) e_i \rangle ds \\ + \int_0^t \langle q(s), e_i \rangle ds \\ + \sum_{k \in K} \int_0^t b(\sigma_k, e_i, u(s)) \, dW_s^k. \end{aligned} \tag{4.37}$$

For what concerns the convergence of $\int_0^t \langle u^N(s), \mathcal{L}^N e_i \rangle ds$ and the continuity of u as a process taking values in H, one can argue as in the vorticity framework.

Remark 4.26 Without the additional energy estimates it is not possible to gain a weak convergent subsequence for the nonlinearity. In fact, we have, thanks to Ladyzhenskaya's inequality, for each $\phi \in V$

$$|b(u^N, \phi, u^N)| \le C \|\nabla\phi\|_{L^2} \|\nabla u^N\|_{L^2} \|u^N\|_H,$$

thus $\|B(u^N, u^N)\|_{V^*} \le C \|\nabla u^N\|_{L^2} \|u^N\|_{L^2}$ and

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|B(u^N(s), u^N(s))\|_{V^*}^2 \, ds\right] \\ &\le C \sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|\nabla u^N(s)\|_{L^2}^2 \|u^N(s)\|_H^2 \, ds\right] \le C, \end{aligned}$$

thanks to relation (4.35).

4.3.3 Existence, Uniqueness and Further Results

To prove the existence of the solutions of Eq. (4.28) we need the following lemma. As described before, this way of proceeding is classical in stochastic analysis, see for example [44, 227].

Lemma 4.27 *Let*

$$\tau_M = \inf\{t \in [0,T] : \|u(t)\|_H^2 \ge M\} \wedge \inf\{t \in [0,T] : \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \ge M\} \wedge T$$

then

$$1_{[0,\tau_M]}(u^N - u) \to 0, \text{ in } L^2(\Omega, L^2(0,T;H)).$$

Proof of Theorem 4.11 We have to show that

$$\mathbb{E}\left[\int_0^T 1_{[0,\tau_M]}(s) \|u^N(s) - u(s)\|_H^2 \, ds\right] \tag{4.38}$$

converges to zero in N. Let $\tilde{u}^N = P^N u$. Then, by the triangular inequality

$$\begin{aligned} (4.38) \le\ & 2\mathbb{E}\left[\int_0^T 1_{[0,\tau_M]}(s) \|\tilde{u}^N(s) - u(s)\|_H^2 \, ds\right] \\ &+ 2\mathbb{E}\left[\int_0^T 1_{[0,\tau_M]}(s) \|\tilde{u}^N(s) - u^N(s)\|_H^2 \, ds\right]. \end{aligned}$$

Thanks to the properties of the projector P^N and dominated convergence theorem, it follows that $\tilde{u}^N \to u$ in $L^2(\Omega, L^2(0,T;V)) \cap L^2(\Omega, C(0,T;H))$, and also in weaker topologies. Therefore, we are left to show the convergence of

$$\mathbb{E}\left[\int_0^{\tau_M} \|\tilde{u}^N(s) - u^N(s)\|_H^2 \, ds\right]. \tag{4.39}$$

Calling $B^N = B(u^N, u^N)$, then for each $i \leq N$ the following relation holds true:

$$\begin{aligned}\langle (u-u^N)(t), e_i\rangle &+ \int_0^t \langle B^*(s) - B^N(s), e_i\rangle_{V^*,V} ds \\ &= \int_0^t \nu\langle u(s) - u^N(s), Ae_i\rangle ds \\ &+ \int_0^t \langle u(s), \mathcal{L}e_i\rangle - \langle u^N(s), \mathcal{L}^N e_i\rangle ds \\ &+ \sum_{k\in K}\int_0^t b\left(\sigma_k, e_i, u(s) - u^N(s)\right) dW_s^k.\end{aligned}$$

Thanks to the previous relation we can compute $\frac{1}{2}d\|u^N - \tilde{u}^N\|_H^2$ via the Itô formula:

$$\begin{aligned}\frac{1}{2}d\|\tilde{u}^N - u^N\|_H^2 + \nu\|\nabla(\tilde{u}^N - u^N)\|_{L^2}^2\, dt &= \langle u, \mathcal{L}(\tilde{u}^N - u^N)\rangle dt \\ &- \langle u^N, \mathcal{L}^N(\tilde{u}^N - u^N)\rangle dt \\ &- \langle B^* - B^N, \tilde{u}^N - u^N\rangle_{V^*,V} dt \\ &+ \sum_{k\in K} b\left(\sigma_k, \tilde{u}^N - u^N, u - u^N\right) dW_t^k \\ &+ \frac{1}{2}\sum_{k\in K}\sum_{i=1}^N b\left(\sigma_k, e_i, u - u^N\right)^2 dt.\end{aligned} \tag{4.40}$$

Next, to better understand the behavior of the terms

$$\langle u, \mathcal{L}(\tilde{u}^N - u^N)\rangle - \langle u^N, \mathcal{L}^N(\tilde{u}^N - u^N)\rangle + \frac{1}{2}\sum_{k\in K}\sum_{i=1}^N b\left(\sigma_k, e_i, u - u^N\right)^2$$

we will first write them in an equivalent form:

$$
\begin{aligned}
2\langle u, \mathcal{L}(\tilde{u}^N - u^N)\rangle &- 2\langle u^N, \mathcal{L}^N(\tilde{u}^N - u^N)\rangle \\
&= \sum_{k\in K}\langle u, \sigma_k \cdot \nabla \left(P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right)\right)\rangle \\
&- \sum_{k\in K}\langle u^N, \sigma_k \cdot \nabla \left(P^N P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right)\right)\rangle \\
&= -\sum_{k\in K} b\left(\sigma_k, u, P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right)\right) \\
&+ \sum_{k\in K} b\left(\sigma_k, u^N, P^N P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right)\right) \\
&= \sum_{k\in K}\langle P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right), P^N P(\sigma_k \cdot \nabla u^N)\rangle \\
&- \sum_{k\in K}\langle P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right), P\left(\sigma_k \cdot \nabla u\right)\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k\in K}\sum_{i=1}^{N} b\left(\sigma_k, e_i, u - u^N\right)^2 &= \sum_{k\in K}\langle P\left(\sigma_k \cdot \nabla u\right), P^N P\left(\sigma_k \cdot \nabla(u - u^N)\right)\rangle \\
&- \sum_{k\in K}\langle P^N P\left(\sigma_k \cdot \nabla u^N\right), P^N P\left(\sigma_k \cdot \nabla(u - u^N)\right)\rangle.
\end{aligned}
$$

Thus, by the Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
&2\langle u, \mathcal{L}(\tilde{u}^N - u^N)\rangle - 2\langle u^N, \mathcal{L}^N(\tilde{u}^N - u^N)\rangle + \sum_{k\in K}\sum_{i=1}^{N} b\left(\sigma_k, e_i, u - u^N\right)^2 \\
&= \sum_{k\in K}\langle P^N P(\sigma_k \cdot \nabla u^N) - P\left(\sigma_k \cdot \nabla u\right), P\left(\sigma_k \cdot \nabla(\tilde{u}^N - u^N)\right) \\
&- P^N P\left(\sigma_k \cdot \nabla(u - u^N)\right)\rangle \\
&= -\sum_{k\in K}\langle P\left(\sigma_k \cdot \nabla u\right) - P^N P(\sigma_k \cdot \nabla u^N), (I - P^N) P(\sigma_k \cdot \nabla u)\rangle \\
&+ \sum_{k\in K}\langle P\left(\sigma_k \cdot \nabla u\right) - P^N P(\sigma_k \cdot \nabla u^N), P\left(\sigma_k \cdot \nabla(u - \tilde{u}^N)\right)\rangle \\
&+ \sum_{k\in K}\langle P\left(\sigma_k \cdot \nabla u\right) - P^N P(\sigma_k \cdot \nabla u^N), (I - P^N) P(\sigma_k \cdot \nabla u^N)\rangle
\end{aligned}
$$

$$\leq \sum_{k\in K} \|P\,(\sigma_k \cdot \nabla u) - P^N P\left(\sigma_k \cdot \nabla u^N\right)\|_H \|(I - P^N)P(\sigma_k \cdot \nabla u)\|_H$$

$$+ \sum_{k\in K} C\|P\,(\sigma_k \cdot \nabla u) - P^N P\left(\sigma_k \cdot \nabla u^N\right)\|_H \|\nabla(u - \tilde{u}^N)\|_{L^2}$$

$$+ \sum_{k\in K} C\|(I - P^N)P(\sigma_k \cdot \nabla u)\|_H \|\nabla u^N\|_{L^2}.$$

For what concerns the nonlinear term we have

$$\langle B^N, \tilde{u}^N - u^N\rangle_{V^*,V} = \langle B(\tilde{u}^N - u^N, \tilde{u}^N - u^N), \tilde{u}^N\rangle_{V^*,V} + \langle B(\tilde{u}^N, \tilde{u}^N), \tilde{u}^N - u^N\rangle_{V^*,V}$$

and, by Ladyzhenskaya's and Young's inequalities,

$$\begin{aligned} |\langle B(\tilde{u}^N - u^N, \tilde{u}^N - u^N), \tilde{u}^N\rangle_{V^*,V}| &\leq \|\nabla \tilde{u}^N\|_{L^2} \|\tilde{u}^N - u^N\|_{L^4}^2 \\ &\leq C\|\nabla u\|_{L^2} \|\tilde{u}^N - u^N\|_H \|\nabla(\tilde{u}^N - u^N)\|_{L^2} \\ &\leq C\|\nabla u\|_{L^2}^2 \|\tilde{u}^N - u^N\|_H^2 \\ &+ \frac{\nu}{2}\|\nabla(\tilde{u}^N - u^N)\|_{L^2}^2. \end{aligned}$$

To remove some positive terms which corrupt our estimates we use, again, the trick we learnt from Bjorn Schmalfuss [238], introduced in Chap. 2 and we apply the Itô formula to the process $\frac{1}{2}\mathcal{R}(t)\|\tilde{u}^N(t) - u^N(t)\|_H^2$, where $\mathcal{R}(t) = exp(-\eta_1 t - \eta_2 \int_0^t \|\nabla u(s)\|_{L^2}^2\, ds)$. We take the expected value for $t = \tau_M$ and exploit previous estimates:

$$\mathbb{E}\left[\frac{1}{2}\mathcal{R}(\tau_M)\|\tilde{u}^N(\tau_M) - u^N(\tau_M)\|_H^2\right] + \nu\mathbb{E}\left[\int_0^{\tau_M} ds\mathcal{R}(s)\|\nabla(\tilde{u}^N(s) - u^N(s))\|_{L^2}^2\right]$$

$$\leq -\frac{\eta_1}{2}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\tilde{u}^N(s) - u^N(s)\|_H^2\, ds\right]$$

$$-\frac{\eta_2}{2}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\nabla u(s)\|_{L^2}^2 \|\tilde{u}^N(s) - u^N(s)\|_H^2\, ds\right]$$

$$+\left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B^*(s) - B(\tilde{u}^N(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V}\, ds\right]\right|$$

$$+ C\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\nabla u(s)\|_{L^2}^2 \|\tilde{u}^N(s) - u^N(s)\|_H^2\, ds\right]$$

$$+ \frac{\nu}{2}\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\|\nabla \tilde{u}^N(s) - \nabla u^N(s)\|_{L^2}^2\, ds\right]$$

$$
\begin{aligned}
&+ \sum_{k\in K} \mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s) \| P\,(\sigma_k \cdot \nabla u(s))\right. \\
&\left. - P^N P\left(\sigma_k \cdot \nabla u^N(s)\right)\|_H \|(I-P^N)P(\sigma_k \cdot \nabla u(s))\|_H\, ds\right] \\
&+ C \sum_{k\in K} \mathbb{E}\left[\int_0^{\tau_M} ds\, \mathcal{R}(s) \| P\,(\sigma_k \cdot \nabla u)\right. \\
&\left. - P^N P\left(\sigma_k \cdot \nabla u^N(s)\right)\|_H \|\nabla(u(s) - \tilde{u}^N(s))\|_{L^2}\right] \\
&+ C \sum_{k\in K} \mathbb{E}\left[\int_0^{\tau_M} ds\, \mathcal{R}(s) \|(I-P^N)P(\sigma_k \cdot \nabla u(s))\|_H \|\nabla u^N(s)\|_{L^2}\, ds\right].
\end{aligned}
$$

Taking $\eta_1,\ \eta_2$ large enough we can remove some terms in the right-hand side. Let us consider the remaining terms, recalling that the weak convergence of u^N implies $\mathbb{E}\left[\int_0^T \|\nabla u(s)\|_{L^2}^2\, ds\right] \leq C$. Applying the Cauchy–Schwarz inequality where it is needed, we get

$$
\begin{aligned}
&\sum_{k\in K} \mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s) \| P\,(\sigma_k \cdot \nabla u(s))\right. \\
&\left. - P^N P\left(\sigma_k \cdot \nabla u^N(s)\right)\|_H \|(I-P^N)P(\sigma_k \cdot \nabla u(s))\|_H\, ds\right] \\
&+ C \sum_{k\in K} \mathbb{E}\left[\int_0^{\tau_M} ds\, \mathcal{R}(s) \| P\,(\sigma_k \cdot \nabla u(s))\right. \\
&\left. - P^N P\left(\sigma_k \cdot \nabla u^N(s)\right)\|_H \|\nabla(u(s) - \tilde{u}^N(s))\|_{L^2}\right] \\
&+ C \sum_{k\in K} \mathbb{E}\left[\int_0^{\tau_M} ds\, \mathcal{R}(s) \|(I-P^N)P(\sigma_k \cdot \nabla u(s))\|_H \|\nabla u^N(s)\|_{L^2}\, ds\right] \\
&\leq C \sum_{k\in K} \mathbb{E}\left[\int_0^{T} \|(I-P^N)P(\sigma_k \cdot \nabla u(s))\|_H^2\, ds\right]^{1/2} \\
&+ C\mathbb{E}\left[\int_0^{T} \|\nabla(u(s) - \tilde{u}^N(s))\|_{L^2}^2\, ds\right]^{1/2} \\
&\to 0.
\end{aligned}
$$

Lastly, we have to treat $\left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B^*(s) - B(\tilde{u}^N(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \, ds\right]\right|$.

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B^*(s) - B(\tilde{u}^N(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \;\; ds\right]\right| \\
&\leq \left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B^*(s) - B(u(s), u(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \;\; ds\right]\right| \\
&+ \left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B(u(s), u(s)) - B(\tilde{u}^N(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \;\; ds\right]\right|.
\end{aligned}$$

Thanks to $\tilde{u}^N - u^N \rightharpoonup 0$ in $L^2(\Omega; L^2(0, T; V))$ the first term converges to 0. For what concerns the second one:

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B(u(s), u(s)) - B(\tilde{u}^N(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} ds\right.\right. \\
&\left.\left.\pm \int_0^{\tau_M} \mathcal{R}(s)\langle B(u(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \; ds\right]\right| \\
&\leq \left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B(u(s), u(s) - \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \; ds\right]\right| \\
&+ \left|\mathbb{E}\left[\int_0^{\tau_M} \mathcal{R}(s)\langle B(u(s) - \tilde{u}^N(s), \tilde{u}^N(s)), \tilde{u}^N(s) - u^N(s)\rangle_{V^*,V} \; ds\right]\right| \\
&\leq \mathbb{E}\left[\int_0^{\tau_M} \|u(s)\|_{L^4} \|u(s) - \tilde{u}(s)^N\|_H^{1/2} \|\nabla(u(s) - \tilde{u}^N(s))\|_{L^2}^{1/2} \|\nabla(\tilde{u}^N(s)\right. \\
&\left. - u^N(s))\|_{L^2} \; ds\right] \\
&+ \mathbb{E}\left[\int_0^{\tau_M} \|u(s) - \tilde{u}^N(s)\|_{L^4} \|u^N(s) - \tilde{u}^N(s)\|_{L^4} \|\nabla u(s)\|_{L^2} \; ds\right]
\end{aligned}$$

$$\begin{aligned}
&\leq C\mathbb{E}\left[\int_0^{\tau_M} \|\nabla u(s)\|_{L^2}^{1/2} \|\nabla(\tilde{u}^N(s) - u^N(s))\|_{L^2} \|\nabla(\tilde{u}^N(s) - u(s))\|_{L^2}^{1/2} \|\tilde{u}^N(s)\right. \\
&\left. - u(s)\|_H^{1/2} \; ds\right] \\
&+ C\mathbb{E}\left[\left(\int_0^T \|u(s) - \tilde{u}^N(s)\|_{L^4}^4 \; ds\right)^{1/4} \left(\int_0^T \|u^N(s) - \tilde{u}(s)^N\|_{L^4}^4 \; ds\right)^{1/4}\right] \\
&\leq C\mathbb{E}\left[\|\tilde{u}^N - u\|_{L^\infty(0,T;H)}^{1/2} \|u\|_{L^2(0,\tau_M,V)}^{1/2} \|\tilde{u}^N - u\|_{L^2(0,T,V)}^{1/2} \|\tilde{u}^N - u^N\|_{L^2(0,T,V)}\right]
\end{aligned}$$

$$+ C\mathbb{E}\Bigg[\|u-\tilde{u}^N\|^{1/2}_{L^2(0,T;V)}\|u^N-\tilde{u}^N\|^{1/2}_{L^2(0,T;V)}\|u-\tilde{u}^N\|^{1/2}_{L^\infty(0,T;H)}\|u^N$$

$$-\tilde{u}^N\|^{1/2}_{L^\infty(0,T;H)}\Bigg]$$

$$\leq C\mathbb{E}\Bigg[\|\tilde{u}^N-u\|^2_{L^\infty(0,T;H)}\Bigg]^{1/4}\mathbb{E}\Bigg[\|\tilde{u}^N-u\|^2_{L^2(0,T;V)}\Bigg]^{1/4}$$

$$\Bigg(\mathbb{E}\Bigg[\|\tilde{u}^N-u^N\|^2_{L^2(0,T;V)}\Bigg]^{1/2}+\mathbb{E}\Bigg[\|\tilde{u}^N-u^N\|^2_{L^\infty(0,T;H)}\Bigg]^{1/4}$$

$$\mathbb{E}\Bigg[\|\tilde{u}^N-u^N\|^2_{L^2(0,T;V)}\Bigg]^{1/4}\Bigg)$$

$$\to 0.$$

In several steps we use the fact that $\|u(t)\|^2_H \leq M$ on $[0,\tau_M]$ and $\int_0^{\tau_M}\|\nabla u(s)\|^2_{L^2} \leq M$. At the end, we find the following relation:

$$\mathbb{E}\left[\frac{1}{2}\mathcal{R}(\tau_M)\|\tilde{u}^N(\tau_M)-u^N(\tau_M)\|^2_H\right]$$
$$+\frac{\nu}{2}\mathbb{E}\left[\int_0^{\tau_M} ds\mathcal{R}(s)\|\nabla(\tilde{u}^N(s)-u^N(s))\|^2_{L^2}\right]\to 0. \tag{4.41}$$

From relation (4.41), $\mathcal{R}(t) \geq C_M > 0\ \forall t \leq \tau_M$ and the properties of P^N, via triangular inequality the thesis follows. □

Remark 4.28 Until the final estimates on the nonlinear term, the proof of Lemma 4.27 is analogous to the proof of Lemma 4.7 up to some technicalities due to the presence of the Leray projection P. The final estimates of the nonlinear term are a bit different due to the lack of regularity of some terms. Thus we prefer to show this proof in all its details.

Lemma 4.29 *$B^* = B(u,u)$ in $L^2(\Omega, L^2(0,T;V^*))$.*

Proof of Theorem 4.11 Due to the experience gained on the vorticity formulation, we hope at this point that it is clear that Lemma 4.27 is the crucial result in order to identify the nonlinear term. We add some computations, similar to what we have done for Lemma 4.8 above. The only changes are due to the lack of regularity of the process u with respect to the previous section.

Thanks to the estimates (4.35), (4.36) and the further results about the weak (resp. weak$*$) convergence of u^N to u in $L^4(\Omega;L^2(0,T;V))$ (resp. $L^4(\Omega;L^\infty(0,T;H))$) we get that $B(u,u^N)$ and $B(u^N,u)$ converge weakly to $B(u,u)$ in $L^2(\Omega;L^2(0,T;V^*))$ easily. We do the explicit computations just for one of the two, the other one is analogous:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \|B(u(s), u^N(s))\|_{V^*}^2 \, ds\right] \\
&\le \mathbb{E}\left[\int_0^T \|u(s)\|_{L^4}^2 \|\nabla u^N(s)\|_{L^4}^2 \, ds\right] \\
&\le C\mathbb{E}\left[\|u\|_{L^\infty(0,T;H)} \|u^N\|_{L^\infty(0,T;H)} \|u\|_{L^2(0,T;V)} \|u^N\|_{L^2(0,T;V)}\right] \\
&\le C\mathbb{E}\left[\|u\|_{L^\infty(0,T;H)}^4\right] + C\mathbb{E}\left[\|u^N\|_{L^\infty(0,T;H)}^4\right] \\
&+ C\mathbb{E}\left[\left(\int_0^T \|\nabla u^N(s)\|_{L^2}^2 \, ds\right)^2\right] + C\mathbb{E}\left[\left(\int_0^T \|\nabla u(s)\|_{L^2}^2 \, ds\right)^2\right] \\
&\le C.
\end{aligned}$$

Now let $\phi \in L^\infty(\Omega; L^\infty(0, T; V))$, then $B(u, \phi) \in L^\infty(\Omega; L^\infty(0, T; V^*))$, thus from the convergence properties of u^N we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \langle B(u(s), u^N(s), \phi\rangle_{V^*,V} ds\right] &= -\mathbb{E}\left[\int_0^T \langle u(s) \cdot \nabla\phi^N, u(s)\rangle ds\right] \\
&\to -\mathbb{E}\left[\int_0^T \langle u(s) \cdot \nabla\phi, u(s)\rangle ds\right] \\
&= \mathbb{E}\left[\int_0^T \langle B(u(s), u(s)), \phi\rangle_{V^*,V} ds\right].
\end{aligned}$$

From the density of $L^\infty(\Omega; L^\infty(0, T; V))$ in $L^2(\Omega; L^2(0, T; V))$ and the uniform boundedness of $B(u, u^N)$ in $L^2(\Omega; L^2(0, T; V^*))$ we have the required claim. For what concerns the convergence of the nonlinear term, first note that, arguing as above, the sequence $\{B(u^N, u^N)\}_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0, T; V^*))$. Moreover, we have

$$\begin{aligned}
B(u, u) - B(u^N, u^N) &= B(u, u - u^N) + B(u, u^N) \\
&\quad + B(u^N, u - u^N) - B(u^N, u) =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Thanks to the previous observations, $I_1 + I_2 + I_4$ converges weakly to 0 in $L^2(\Omega; L^2(0, T; V^*))$. For what concerns I_3, let us take $\phi \in L^\infty(\Omega; L^\infty(0, T; D(A)))$ and τ_M defined as in Lemma 4.27, then we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_M} \langle B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right] \\
&= -\mathbb{E}\left[\int_0^{\tau_M} \langle B(u^N(s), \phi), u(s) - u^N(s)\rangle_{V^*,V} \, ds\right]
\end{aligned}$$

$$\leq C\mathbb{E}\left[\int_0^{\tau_M} \|\nabla u^N(s)\|_{L^2} \|u(s) - u^N(s)\|_H \, ds\right] \to 0$$

thanks to Holder's inequality and Lemma 4.27. Since it holds that $\tau_M \nearrow T$ a.s., we have, thanks to Lemma 4.24,

$$\left|\mathbb{E}\left[\int_0^T B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right|$$
$$\leq \left|\mathbb{E}\left[\int_0^{\tau_M} \langle B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right|$$
$$+ \left|\mathbb{E}\left[\int_{\tau_M}^T \langle B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right|$$
$$\leq \left|\mathbb{E}\left[\int_0^{\tau_M} \langle B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right|$$
$$+ C\mathbb{E}\left[\int_{\tau_M}^T \|u^N(s)\|_{L^4} \|u(s) - u^N(s)\|_{L^4} \, ds\right]$$
$$\leq \left|\mathbb{E}\left[\int_0^{\tau_M} \langle B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right|$$
$$+ C\mathbb{E}\left[\int_{\tau_M}^T \|u^N(s)\|_H^2 \, ds\right]^{1/4} \mathbb{E}\left[\int_{\tau_M}^T \|u(s) - u^N(s)\|_H^2 \, ds\right]^{1/4}$$
$$\leq \left|\mathbb{E}\left[\int_0^{\tau_M} \langle B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right|$$
$$+ C\mathbb{E}\left[\int_{\tau_M}^T \left(\int_0^T \|q(r)\|_H^2 + \|B^*(r)\|_{V^*}^2 \, dr + \|u_0\|_H^2\right) ds\right]^{1/2}.$$

Therefore, if we fix $\epsilon > 0$ and $M > 0$ such that

$$C\mathbb{E}\left[\int_{\tau_M}^T \left(\int_0^T \|q(r)\|_H^2 + \|B^*(r)\|_{V^*}^2 \, dr + \|u_0\|_H^2\right) ds\right]^{1/2} \leq \epsilon,$$

then

$$\text{limusup}_{N\to+\infty} \left|\mathbb{E}\left[\int_0^T B(u^N(s), u(s) - u^N(s)), \phi\rangle_{V^*,V} \, ds\right]\right| \leq \epsilon.$$

The thesis follows by the density of $L^\infty(\Omega; L^\infty(0, T; D(A)))$ in $L^2(\Omega; L^2(0, T; V))$ and the uniform boundedness of $B(u^N, u^N)$ in $L^2(\Omega; L^2(0, T; V^*))$. □

Theorem 4.30 *There is at most one weak solution of problem (4.28) in the sense of Definition 4.20.*

Proof of Theorem 4.11 Even if this proof is analogous to the one presented for Theorem 4.9, we prefer to add it in order to make clear to the reader the simplifications of the argument described in Sect. 2.3.1 thanks to the presence of the transport noise.

Let $u_1, \ u_2$ be two solutions and v be their difference. Thus, v satisfies $\mathbb{P}$-a.s. for each $t \in [0, T]$ and $\phi \in D(A)$,

$$\begin{aligned}\langle v(t), \phi \rangle &= \int_0^t \langle v(s), (\nu A + \mathcal{L})\phi \rangle ds + \int_0^t b\left(u_1(s), \phi, u_1(s)\right) ds \\ &- \int_0^t b\left(u_2(s), \phi, u_2(s)\right) ds \\ &+ \sum_{k \in K} \int_0^t b\left(\sigma_k, \phi, v(s)\right) dW_s^k.\end{aligned}$$

Arguing as in the proof of Lemma 4.10 and 4.24, v satisfies the Itô formula below:

$$\begin{aligned}\frac{d\|v\|_H^2}{2} &= \Big(-\nu\|\nabla v\|_{L^2}^2 + b(u_1, v, u_1) - b(u_2, v, u_2) \pm b(u_2, v, u_1)\Big)\, dt \\ &= -\nu\|\nabla v\|_{L^2}^2 dt + b(v, v, u_1) dt.\end{aligned}$$

We apply the Itô formula to the process $\frac{1}{2}\mathcal{R}(t)\|v(t)\|_H^2$, where $\mathcal{R}(t) = exp(-\eta \int_0^t \|\nabla u_1(s)\|_{L^2}^2 \, ds)$

$$\begin{aligned}\frac{d\left(\mathcal{R}\|v\|_H^2\right)}{2} &= \mathcal{R}\Big(-\nu\|\nabla v\|_{L^2}^2 + b(v, v, u_1)\Big)\, dt - \frac{\eta\mathcal{R}}{2}\|v\|_H^2\|\nabla u_1\|_{L^2}^2 dt \\ &\le -\nu\mathcal{R}\|\nabla v\|_{L^2}^2 dt - \frac{\eta\mathcal{R}}{2}\|v\|_H^2\|\nabla u_1\|_{L^2}^2 dt \\ &+ C\mathcal{R}\|\nabla u_1\|_{L^2}\|v\|_H\|\nabla v\|_{L^2} dt \\ &\le -\nu\mathcal{R}\|\nabla v\|_{L^2}^2 dt - \frac{\eta\mathcal{R}}{2}\|v\|_H^2\|\nabla u_1\|_{L^2}^2 dt \\ &+ \frac{\nu\mathcal{R}\|\nabla v\|_{L^2}^2}{2} dt + C\mathcal{R}\|v\|_H^2\|\nabla u_1\|_{L^2}^2 dt.\end{aligned}$$

Thus, taking η large enough, we have

$$\frac{d\left(\mathcal{R}\|v\|_H^2\right)}{2} + \frac{\nu\mathcal{R}\|\nabla v\|_{L^2}^2}{2} dt \le 0$$

and the thesis follows immediately by the Gronwall lemma. □

Lemmas 4.27 and 4.29 identify the nonlinear term and together with Theorem 4.30 conclude the proof of Theorem 4.21. Again, invoking some abstract results on stochastic processes, it can be shown also that the full sequence u^N converges to u in $L^2(\Omega; L^2(0, T; V))$ and for each $t \in [0, T]$ $u^N(t)$ converges to $u(t)$ in $L^2(\Omega; H)$, see [44] for some details. Arguing as in Sect. 4.1.4 one can prove that the Itô formula and the estimates stated in Lemma 4.24 for the approximating sequence u^N continue to hold for the limit point u. We omit the easy details.

4.4 The 3D Navier–Stokes Equations with Transport Noise

The final topic we want to discuss is the vorticity formulation for the 3D Navier–Stokes equation with transport noise. In this case we should find

$$\partial_t\overline{\omega} + (\overline{u} \cdot \nabla\overline{\omega} - \overline{\omega} \cdot \nabla\overline{u}) \overset{d=3}{=} \nu\Delta\overline{\omega} + \overline{\text{curl } f}$$
$$- \sum_{k\in K} (\sigma_k \cdot \nabla\overline{\omega} - \overline{\omega} \cdot \nabla\sigma_k) \circ \partial_t W^k.$$

Indeed, in the original vorticity equation there are two quadratic terms

$$u \cdot \nabla\omega - \omega \cdot \nabla u$$

and in both of them we have to replace u by $\left(\overline{u} + u'\right)$, and then u' by noise. The previous stochastic equation has been investigated, at the level of local-in-time existence and uniqueness of smooth solutions (see in particular [81] dealing with the more difficult case of $\nu = 0$), but the link with an equation of the form

$$\partial_t\Omega + U \cdot \nabla\Omega \overset{d=3}{=} (\nu\Delta + \mathcal{L})\,\Omega + \Omega \cdot \nabla U + \overline{\text{curl } f} \tag{4.42}$$

has not been understood until now. Maybe there are fluid regimes where there is a link (see the discussion in [126]), but this is still an open problem.

On the contrary, if we investigate the model, in 3D, with just transport noise,

$$\partial_t\overline{\omega} + (\overline{u} \cdot \nabla\overline{\omega} - \overline{\omega} \cdot \nabla\overline{u}) \overset{d=3}{=} \nu\Delta\overline{\omega} + \overline{\text{curl } f}$$
$$- \sum_{k\in K} P\,(\sigma_k \cdot \nabla\overline{\omega}) \circ \partial_t W^k,$$

it is possible to prove a rigorous link with (4.42). Notice that we have introduced the projection $P : L^2 \to H$ in this equation: in general the term $\sigma_k \cdot \nabla\overline{\omega}$ is not divergence free, while the sum of all other terms is divergence free, hence without the projection there would be no solution in general. Moreover, notice that the previous model has been investigated only on the 3D torus, to avoid the problem of the boundary conditions for the vorticity.

One can prove that the solution $\overline{\omega}$ of the stochastic Navier–Stokes equations is close (in a suitable topology) to the solution Ω of the deterministic Navier–Stokes equations (4.42) with increased dissipation. This fact has a very important consequence: that *well-posedness is improved by noise*. In the deterministic case, the larger the viscosity, the longer the time interval of existence and uniqueness of smooth solutions; this interval becomes even infinite when the sizes of the initial condition and the viscosity (and the forcing term if it is not zero) satisfy a certain relation. Since the noise has the effect of introducing an extra dissipation, it has the effect of increasing the length of the time interval of existence and uniqueness of smooth solutions of the stochastic equation, length that again becomes infinite under certain conditions.

This is the first known regularization by noise result for 3D Navier–Stokes equations; it has been proved in [126]. See also the "deterministic" variant obtained by a Wong–Zakai approximation based on rough paths [123]. These works leave open the very difficult question of whether the same result holds when the noise affects also the stretching term. Results for regularization by noise along similar lines, but for other equations, have been developed in [115].

4.4.1 The Result in the Case of Only Transport

Consider, on the 3D torus, the vorticity equation with noise only in the transport component:

$$\partial_t \omega + u \cdot \nabla \omega + P\left(u' \cdot \nabla \omega\right) = \Delta \omega + \omega \cdot \nabla u,$$

with noise u' of the form

$$u'(t, x) = \sum_k \sigma_k(x) \circ \partial_t W_t^k.$$

The projection in the term $P\left(u' \circ \nabla \omega\right)$, necessary for compatibility, is the source of great technical difficulties (the Itô–Stratonovich corrector is a nonlocal differential operator).

Call ω the unique local solution, for $\omega_0 \in H$ (the space L^2 with usual conditions).

Theorem 4.31 *Given $T, R_0, \epsilon > 0$ there exists $(\sigma_k)_{k \in K}$ with the following property: for every initial condition $\omega_0 \in H$ with $\|\omega_0\|_H \leq R_0$, the 3D Navier–Stokes equations with transport noise (and viscosity = 1) has a global unique solution on $[0, T]$, up to probability ϵ.*

The full proof requires too many details, see [126]. Let us mention only one fact. The norm $\|\omega(t)\|_H^2$ can be controlled *locally* from

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega,$$

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_H^2 + \|\nabla \omega(t)\|_{L^2}^2 = \langle \omega \cdot \nabla u, \omega \rangle.$$

The term $\langle \omega \cdot \nabla u, \omega \rangle$ describes the *stretching* of vorticity ω produced by the deformation tensor ∇u. This is the potential source of unboundedness of $\|\omega(t)\|_H^2$.

Sobolev and interpolation inequalities:

$$\langle \omega \cdot \nabla u, \omega \rangle \leq \|\omega\|_{L^3}^3 \leq \|\omega\|_{W^{1/2,2}}^3 \leq \|\omega\|_{L^2}^{3/2} \|\omega\|_{W^{1,2}}^{3/2} \leq \|\omega\|_{W^{1,2}}^2 + \|\omega\|_{L^2}^6$$

lead to

$$\frac{d}{dt}\|\omega(t)\|_H^2 \leq C\|\omega\|_H^6,$$

which provides only a local control.

However, the interval of existence depends on the *viscosity coefficient* ν:

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \nu \Delta \omega$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_H^2 + \nu\|\nabla \omega(t)\|_H^2 &= \langle \omega \cdot \nabla u, \omega \rangle \\ &\leq \|\omega\|_{L^2}^{3/2}\|\omega\|_{W^{1,2}}^{3/2} \\ &\leq \nu\|\nabla \omega(t)\|_H^2 + \frac{C}{\nu^3}\|\omega\|_H^6 \end{aligned}$$

$$\frac{d}{dt}\|\omega(t)\|_H^2 \leq \frac{C}{\nu^3}\|\omega\|_H^6.$$

The explosion is delayed for large ν. Not only that: beyond a threshold the solution is global.

This is the key for a regularization by noise: transport noise improves dissipation, hence it delays blow-up.

4.5 Summary

In this chapter and in the previous one we have discussed a second class of noise: the one of transport type. There is a third class, variant of the second one, namely noise of transport-stretching type in 3D, which is only mentioned but should receive due attention.

Noise of transport type in the equations for auxiliary quantities, like heat, has been investigated by several authors. In Chap. 5 we will see that it can be introduced as a Wong–Zakai limit, in order to emphasize the presence of a correcting term. In the case of heat transport our investigation culminates in the proof of a property of eddy dissipation.

But similar ideas may be applied to the internal structure of the fluid itself when we introduce the subdivision in large and small scales. Here the noise is used to summarize the dynamics of small scales and affects the closed equation for the large scales. This is the motivation for considering stochastic Navier–Stokes equations with transport-type noise (and, as mentioned above, also with transport-stretching noise in 3D). The 2D case starts to be well understood and, in particular, similarly to the case of heat transfer, we proved a result of eddy viscosity: turbulence enhances the viscosity of the fluid itself. This fact, clearly observed in real situations, is perhaps the main confirmation that the heuristic discussion made here about stochastic modeling of small scales and consequent transport noise in the large ones may have a deep physical meaning, in spite of poor justification at the level of continuum mechanics that we can provide at present.

Moving these ideas to the 3D case but with the limitation of a transport-type noise, we may show that noise improves the theory of 3D Navier–Stokes equations. This was a long-standing project in the case of additive noise, frustrated however by several technical difficulties. The case of transport noise proved to be more promising. However, for future research, understanding the case of transport-stretching noise must be considered the most important open problem.

Let us also add the following very heuristic remark. In these notes we started from additive perturbations. The introduction of such noise will be motivated in Chap. 5 by the roughness of boundaries. Additive noise, as just mentioned, has not been shown to improve so much the theory of 3D Navier–Stokes equations. But additive noise in the small scales, as shown in the present chapter, may lead to a multiplicative transport noise in the large scales. And transport noise has a better regularizing power. In the end it seems, then, that *it is the additive noise at small scales which regularizes*! Presumably the long-standing conjecture that additive noise regularizes could be correct but the path to reveal its power is very complex. Until now the efforts to prove that additive noise regularizes were based on the similarity with the finite-dimensional case, where additive noise is so successful. But this is probably too abstract a viewpoint for the Navier–Stokes equations. The deep reason of regularization stands inside the links between scales, a fact true for fluid dynamics and not for general evolution equations.

Chapter 5
From Small-Scale Turbulence to Eddy Viscosity and Dissipation

5.1 Introduction: The Global Heuristic Scheme

The previous chapters have been intentionally restricted to purely mathematical results and techniques. However, the choice made above of subjects and their order is motivated by certain intuitions, related to turbulence, that we aim to describe in this final chapter. For a general and wide introduction to turbulence, see the books of A. Chorin [70] and U. Frisch [142].

Usually below we refer to the 3D Navier–Stokes equations in vorticity form:

$$\partial_t \omega + u \cdot \nabla \omega + \omega \cdot \nabla u = \nu \Delta \omega + f,$$

$$\omega|_{t=0} = \omega_0,$$

unless differently specified, because some arguments are closer to our intuition when formulated for the vorticity.

Said in a nutshell, the global heuristic scheme we aim to illustrate here starts from the decomposition in large and small space scales introduced in Chap. 4 (see Sect. 5.1.1 for a critical discussion of this decomposition)

$$\partial_t \overline{\omega} + u \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla u = \nu \Delta \overline{\omega} + \overline{f},$$

$$\partial_t \omega' + u \cdot \nabla \omega' + \omega' \cdot \nabla u = \nu \Delta \omega' + f',$$

$$u = \overline{u} + u',$$

then concentrate the attention on the large scales

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u} - \nu \Delta \overline{\omega} - \overline{f}$$

$$= -u' \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla u',$$

F. Flandoli, E. Luongo, *Stochastic Partial Differential Equations in Fluid Mechanics*, Lecture Notes in Mathematics 2328,
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and find a suitable model of the small scales u' so that the large scales satisfy a closed equation, with small scales acting as a given input.

In order to construct a model of small scales u' we argue that, in certain turbulent regimes, it is reasonable to perturb the Navier–Stokes equations by a small-scale additive noise (see Sects. 5.2 and 5.5)

$$d\omega + (u \cdot \nabla\omega + \omega \cdot \nabla u - \nu\Delta\omega)\, dt = \overline{f} + \underbrace{d\,(\mathrm{curl}\, W)}_{\text{small scale}},$$

i.e. $f' = d\,(\mathrm{curl}\, W)$, so that such noise appears in the small-scale component of the previous decomposition

$$d\omega' + \left(u \cdot \nabla\omega' + \omega' \cdot \nabla u\right) dt = \nu\Delta\omega' dt + d\,(\mathrm{curl}\, W).$$

It leads (see Sect. 5.3), in a suitable scaling limit based on modeling assumptions which include a form of scale separation, to the choice

$$u' = \frac{dW}{dt}$$

and thus to the Stratonovich-type equation with transport noise

$$d\overline{\omega} + (\overline{u} \cdot \nabla\overline{\omega} + \overline{\omega} \cdot \nabla\overline{u})\, dt + dW \circ \nabla\overline{\omega} + \overline{\omega} \circ \nabla dW = \left(\nu\Delta\overline{\omega} + \overline{f}\right) dt$$

for the large scales.

Finally (see Sect. 5.4), in a further suitable scaling limit, the equation approximates a deterministic equation with enhanced viscosity

$$\partial_t\overline{\omega} + \overline{u} \cdot \nabla\overline{\omega} + \overline{\omega} \cdot \nabla\overline{u} = (\nu + K)\,\Delta\overline{\omega} + \overline{f}.$$

The full procedure becomes a justification of the well-known claim (Boussinesq 1877 [43]) that small scale-turbulence acts as an eddy viscosity.

We shall stress in the next section that the intermediate decomposition step is reasonable only locally in time. However, the final model with enhanced viscosity is reasonable without time restrictions.

Our hope is to understand more closely this procedure in order to clarify its range of validity and possibly its modifications depending on specific flows. See for instance [151, 182, 239] for criticisms about simplistic eddy viscosity models, asking for a better understanding of the underlying ideas and therefore the subsequent modifications.

5.1.1 Large and Small Space Scales

The first heuristic ingredient of the intuition described in Sect. 5.1 is the duality between *small and large space scales*. There are rigorous definitions, see the remark below, but they do not fit so well with the heuristic idea we have in mind, hence we mention them mostly for comparison.

Remark 5.1 Let $\omega(t, x)$ be the vorticity field. If, given t, it is a square integrable function, we may define (at least) two large-scale projections:

$$\pi_N \omega(t) = \sum_{k \in K_N} \omega_k(t) e_k,$$

$$\Pi_\epsilon \omega(t) = \theta_\epsilon * \omega(t),$$

where (e_k) is a complete orthonormal system in the Hilbert space where $\omega(t)$ lives, K_N is a set of "first" modes, θ_ϵ is a smooth mollifier and $*$ denotes convolution. These projections depend on the choice of N and ϵ.

The large-scale filters described in the Remark are very useful and rigorous. However, they do not correspond precisely to our intuition of large-scale vortex structures. Anyway, the operator Π_ϵ could be a surrogate, if nothing better is available, to single out large vortices.

Unfortunately, making rigorous the intuition of vortex structures is very difficult. However, let us stress that such intuition may be very strong after we had the chance, nowadays, to see the results of numerical simulations where these vortex structures are clearly visible: in 2D we may observe small vortices merging into larger ones, see [41, 218]; and in 3D we have seen large vortex tubes producing small filament like vortices by instabilities, see [37, 38, 70, 186, 254]. In nature, vortices are also visible many times, see the figures below in this chapter.

Given an initial condition $\omega_0 = \omega_0(x)$, we assume (at least by using Π_ϵ above) that we are able to decompose it into a large-scale component and a small-scale one:

$$\omega_0 = \overline{\omega}_0 + \omega_0'.$$

Similarly, if a force $f = f(t, x)$ acts on the fluid, we assume we can make an analogous splitting:

$$f = \overline{f} + f'.$$

Splitting the data is feasible. The difficulty comes when we want a splitting of the solution $\omega(t, x)$. There are two strategies, that we call *explicit and implicit*. The *explicit strategy* consists in choosing a projection P (like those of the Remark above) and defining

$$\overline{\omega} = P\omega,$$
$$\omega' = \omega - \overline{\omega}.$$

Then the pair $\left(\overline{\omega}, \omega'\right)$ satisfies a coupled system of equations (of course if one does this, it is natural to define the splitting of ω_0 and f above using P itself). The advantage of this approach is that we are sure, by definition, that $\left(\overline{\omega}, \omega'\right)$ is a splitting in large and small scales. The drawback is that the coupled system may be complicated. For instance, using as P the convolution Π_ϵ with a smooth mollifier, we get for $\overline{\omega}$ the equation

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u} = \nu \Delta \overline{\omega} + \overline{f} + R,$$

where the reminder R is given by

$$R = \overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u} - \Pi_\epsilon \left(u \cdot \nabla \omega + \omega \cdot \nabla u\right).$$

The analysis or simplification by modeling of this remainder is quite difficult. When $\overline{u}, \overline{\omega}$ are averages, R corresponds to the Reynold stress term in the vorticity formulation, whose modeling was widely investigated, but it remains a subject of great debate, with features depending on specific flows.

The *implicit strategy* consists in the study of the system

$$\partial_t \overline{\omega} + u \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla u = \nu \Delta \overline{\omega} + \overline{f},$$
$$\partial_t \omega' + u \cdot \nabla \omega' + \omega' \cdot \nabla u = \nu \Delta \omega' + f',$$

with initial conditions

$$\overline{\omega}|_{t=0} = \overline{\omega}_0, \qquad \omega'|_{t=0} = \omega'_0,$$

where u is the result of the Biot–Savart law on the full vorticity $\omega = \overline{\omega} + \omega'$, hence also decomposable in two parts

$$u = \overline{u} + u'.$$

If a pair $\left(\overline{\omega}, \omega'\right)$ is a solution to this system (e.g. in distributional sense), then $\omega = \overline{\omega} + \omega'$ is a solution (in distributional sense) of the full equation. The system for $\left(\overline{\omega}, \omega'\right)$ contains all information to solve the true Navier–Stokes equations. This was the approach that led us to the Navier–Stokes equations with transport noise described in Chap. 4. Let us discuss the pros and cons of this approach.

In this approach the choice of the splitting of ω_0 and f is free and may better correspond to our intuition of large and small vortex structures. This is an advantage, together with the most important one that the system of equations is relatively easy compared to those obtained by the explicit strategy described above. Indeed, the equation for large scales reads

$$\begin{aligned}&\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u} - \nu \Delta \overline{\omega} - \overline{f} \\ &= -u' \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla u'.\end{aligned}$$

It opens the door to choose a priori a model of small scales u', e.g. a suitable stochastic process, and get a closed equation for large scales (it is related to what we do below, in a sense, although we try to be as strictly as possible in the choice of u').

The drawback is that the two components of the solution $\left(\overline{\omega}, \omega'\right)$ may lose the property of representing only large and small scales, as time goes on. Initially, they are a correct large- and small-scale decomposition, by definition of the splitting $\omega_0 = \overline{\omega}_0 + \omega_0'$. But for how long should we expect that $\left(\overline{\omega}, \omega'\right)$ is a reasonable decomposition in large and small structures?

Let us briefly discuss this extremely difficult and open issue, distinguishing 2D from 3D. As we have remarked above, we all know that in 2D small vortex structures merge into larger ones (inverse cascade), see for example [41, 109]. Choose an initial condition of the form $\omega_0 = \omega_0'$, made only of small scales and assume for simplicity $f = 0$. The solution for the system (which can be proved to be unique) is simply $\left(0, \omega'\right)$, namely ω' is the full vorticity field. But we know it develops large scales by inverse cascade, hence it is not a small-scale field anymore, after a relatively short time. And the large scales which are created do not appear in the "large-scale component" $\overline{\omega}$, which remains equal to zero. How could we modify the equations in such a way that large-scale structures created by inverse cascade are shifted from the ω' to the $\overline{\omega}$ component? At present we do not know.

In terms of energy and enstrophy, it is commonly accepted that, in 2D, energy moves from small to large scales and enstrophy from large to small, see the outstanding work of Kraichnan [188] or the more recent reviews [41, 245]. For the equations in vorticity form,

$$\begin{aligned}\partial_t \overline{\omega} + u \cdot \nabla \overline{\omega} &= \nu \Delta \overline{\omega} + \overline{f}, \\ \partial_t \omega' + u \cdot \nabla \omega' &= \nu \Delta \omega' + f',\end{aligned}$$

here rewritten in 2D without the stretching terms $\overline{\omega} \cdot \nabla u$ and $\omega' \cdot \nabla u$, it is not easy to see the flux of energy. But enstrophy is certainly preserved by both terms $u \cdot \nabla \overline{\omega}$ and $u \cdot \nabla \omega'$, hence no flux of enstrophy is accepted by this *implicit* decomposition. This is certainly another drawback of the method.

In 3D, small scales are produced by instabilities of large ones (direct cascade). We could reverse the arguments above and identify drawbacks of the implicit decomposition, symmetric with respect to those in 2D. If it is already an open problem to detect correcting terms in 2D or 3D separately, we cannot even be hopeful of finding a general decomposition model which works well in both cases.

Keeping these objections in mind, we develop some of our heuristic arguments below under the hope that for a relatively short time interval $t \in [0, T]$ the decomposition $\omega = \overline{\omega} + \omega'$ provided by the above system is not bad. We hope that future research may improve this approximation.

Remark 5.2 The theory developed by Darryl Holm, Dan Crisan, Mémin and collaborators in a series of works (see for instance [67, 81, 171, 177, 213]) starts from different viewpoints and cannot be easily compared to the approach described here, but it has some similarities, for instance in the structure of the noise. The point we want to stress here is that, in order to stay close to data, the stochastic modeling of that approach is applied locally in time, on short time intervals, with a suitable restarting procedure at every step. This reminds us of the constraint mentioned above of locality in time of the implicit decomposition.

5.2 Small-Scale Turbulence and Additive Noise

Assume the fluid is turbulent. This is not a unique and well-defined concept. For instance, think of a fast fluid along a solid boundary, developing a turbulent boundary layer, or a shear flow developing a turbulent region by instability. Large-scale motion and structures, specific to the geometry of the flow, coexist with small-scale ones, maybe more universal. For instance, in the case of the turbulent boundary layer, we observe a mean flow and possibly other large-scale elements like large scale-vortex structures, superimposed on an extremely complex small-scale motion made of small hairpin vortices arising at the boundary, others apparently detaching from the boundary and traveling in the interior, others arising from the previous ones by further instabilities and so on. The small-scale turbulence mentioned in the title of this section refers to this complex motion. See for instance [172] for a review on the complexity of the so-called turbulence coherent structures. In [172], the authors describe some possible mechanisms behind turbulent boundary layer flow and their mechanism of generation. From a mathematical viewpoint, a not completely exhaustive view, is that, under suitable assumptions, the solution of the Navier–Stokes equations with no-slip boundary conditions can be split in two parts: a regular part far from the boundary of the domain which is the solution of the Euler equations and a rougher part in the boundary layer of the domain which is the solution of the so-called Prandtl equations. Without entering into the details of the assumptions about the validity of previous result and the meaning of the Prandtl equations, which are out of the heuristic scope of this chapter, let us simply point out that, so far, the validity of this decomposition under the natural assumptions of Theorem 1.2 is an open problem. It would have a deep impact on our knowledge about turbulence, because it would imply that some conjectures raised by Kolmogorov in the last century on the behavior of a turbulent fluid for $\nu \sim 0$ were false. We refer to [16] for a recent review on this topic.

Recall the decomposition in large and small scales of the previous section and the desire to have an a priori model of small scales. We concentrate on the case when small scales are turbulent. A key part of our heuristic proposal outlined in Sect. 5.1 is the claim that some turbulent regimes can be described by a stochastic equation of the form

$$d\omega + \left(u \cdot \nabla\omega + \omega \cdot \nabla u - \nu\Delta\omega - \overline{f}\right) dt = d\,(\text{curl}\, W),$$
$$\omega|_{t=0} = \omega_0,$$

with a space-dependent Brownian motion

$$W = W\,(t, x)$$

mostly made of *small-scale structures*. In Sect. 5.5 below we add details to this proposal in the particular case of vortices created at boundaries by small obstacles.

Having in mind the large-small-scale decomposition, we move immediately to the system

$$\partial_t\overline{\omega} + u \cdot \nabla\overline{\omega} + \overline{\omega} \cdot \nabla u = \nu\Delta\overline{\omega} + \overline{f}$$
$$d\omega' + \left(u \cdot \nabla\omega' + \omega' \cdot \nabla u - \nu\Delta\omega'\right) dt = d\,(\text{curl}\, W)$$

which is equivalent to the previous one.

The next step of the story is, unfortunately, some kind of simplification of the second equation of the previous system. The most extreme simplification would be

$$d\omega' = -\epsilon^{-1}\omega' dt + \epsilon^{-1} d\,(\text{curl}\, W).$$

In several papers, see [7, 95, 133–135], has been investigated the possibility to relax this extreme simplification in the direction of the true equation

$$d\omega' + \left(u \cdot \nabla\omega' + \omega' \cdot \nabla u - \nu\Delta\omega'\right) dt = d\,(\text{curl}\, W),$$

but a full solution is still obscure. One of the most critical aspects of the simplification is justifying the addition of the term $-\epsilon^{-1}\omega' dt$, while several developments have been made to remove the deletion of other terms. One vague argument in favour of $-\epsilon^{-1}\omega' dt$ is that, ω' being made of small scales, there is a vague correspondence between $\nu\Delta\omega'$ and $-\epsilon^{-1}\omega'$ (think of the Fourier representation of $\Delta\omega'$). Another vague argument is that, when we have inserted the small-scale force $d\,(\text{curl}\, W)$ already a few lines before the simplification, we have arbitrarily introduced energy into the system (one can prove this) and thus we have to compensate it by some dissipation and $-\epsilon^{-1}\omega' dt$ looks like a candidate, although very phenomenological.

Clearly, the issues just discussed require more investigation. For the time being, let us accept the proposal that the following system:

$$\partial_t \overline{\omega} + u \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla u = \nu \Delta \overline{\omega} + \overline{f}$$
$$d\omega' = -\epsilon^{-1} \omega' dt + \epsilon^{-1} d\,(\text{curl}\, W),$$
$$u = \overline{u} + u',$$

is a very simplified model of small-scale turbulence coupled with large-scale motion (we could add some terms to the second equation but we stay at this level for simplicity of exposition).

5.3 Action of Small-Scale Turbulence on Large-Scales: Transport Noise Under Scale Separation

The first equation of the previous system, the equation for the large scales, reads

$$\begin{aligned} &\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u} \\ &+ u' \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla u' \\ &= \nu \Delta \overline{\omega} + \overline{f}. \end{aligned}$$

When ϵ is very small, more precisely in the limit as $\epsilon \to 0$, it can be proved, see [7, 95, 133–135], that $\overline{\omega}$ solves the stochastic equation

$$\begin{aligned} &d\overline{\omega} + (\overline{u} \cdot \nabla \overline{\omega} + \overline{\omega} \cdot \nabla \overline{u})\, dt \\ &+ dW \circ \nabla \overline{\omega} + \overline{\omega} \circ \nabla dW \\ &= \left(\nu \Delta \overline{\omega} + \overline{f}\right) dt. \end{aligned}$$

The Stratonovich operation $\circ$ in the stochastic terms naturally arises in accordance with the general Wong–Zakai principle. Since it is a key ingredient of the next approximation, the one leading to eddy viscosity, we give a heuristic presentation in Sect. 5.6 below.

The proofs of the result above are not trivial but the intuition is clear: when ϵ^{-1} is very large, the balance of terms in the equation

$$d\omega' = -\epsilon^{-1} \omega' dt + \epsilon^{-1} d\,(\text{curl}\, W)$$

leads to the approximation

$$\omega' dt \sim d\,(\text{curl}\, W),$$

which means (assuming W divergence free)

$$u' \sim \frac{dW}{dt}.$$

Taking ϵ very small corresponds to an assumption of *time-scale separation*. The splitting $\omega = \overline{\omega} + \omega'$ until now was relatively generic, except for the simplifications then made in the equation of ω'. But now, assuming that ϵ is very small, we assume that parameters in the equation of small scales are extremized compared to those of the equation for $\overline{\omega}$. Analogously to the problems of averaging with two time scales, a very large parameter ϵ^{-1} in the second equation means that the typical time over which ω' varies is much shorter than the one over which $\overline{\omega}$ varies. This time-scale separation is an assumption, we do not have justifications.

The time-scale separation heuristically corresponds to a *space-scale separation*. If the typical velocities observed into a fluid flow are of order U, vortex structures with average velocity of rotation U have a ratio of space and time scales (radius of the vortex times 2π over period of revolution) of order U. Hence smaller space scales, those of ω', correspond to smaller time scales, hence the assumption of small ϵ is in the right direction. But very small ϵ means very small space scales too, hence we need to assume that the fluid is composed of large vortex structures plus very small ones.

Proving scale separation or proving that existing intermediate scales do not spoil the arguments remain open problems.

5.4 Eddy Viscosity and Eddy Diffusion

Since the nineteenth century, scientists like Boussinesq started recognizing that turbulence may be responsible for an increase of viscosity and diffusion: for instance, if the fluid traveling through a pipe is turbulent, it slows down and exchanges more heat through the boundary. This idea is also at the foundation of Large Eddy Simulation (LES) theory.

Finding mathematical proofs of this fact is always a challenging question; see [27, 239] for a review. There are rigorous theories that investigate the problem but it is relevant to find new ideas, also because the precise regimes under which these facts are true and the precise form of the extra-viscous or dissipative terms is not always known (think to the variety of models in LES theory including Smagorinsky one [28, 29]).

The ideas developed above plus the results of Chaps. 3 and 4 are an attempt to provide a new justification. The way turbulence is inserted into the equations is by the term $d\,(\text{curl}\,W)$, or more precisely by the equation

$$d\omega' = -\epsilon^{-1}\omega' dt + \epsilon^{-1} d\,(\text{curl}\,W)$$

or its modifications, which should incorporate realistic features of the region affected by small-scale turbulence. Given that model, under the assumption of scale separation we deduce a stochastic model of large scales $\overline{\omega}$ affected by transport-type noise. When the noise has the features described in Chaps. 3 and 4 we deduce deterministic equations, precisely for passive quantities in Chap. 3, hence subject to an eddy dissipation, and for the fluid field $\overline{\omega}$ itself in Chap. 4, hence subject to an eddy viscosity. We hope to develop this topic further in order to contribute to a better understanding of different dissipative terms corresponding to different fluid regimes. Indeed in LES theory, the additional dissipative term is not linear, thus it cannot be modeled via the transport noise described in Chaps. 3 and 4.

The ultimate conjecture arising from such arguments is that turbulence could even produce a depletion of emerging singularities, thanks to its eddy viscosity effect; however, the 3D structure of this question remains poorly understood.

5.5 More on Additive Noise at Small Scales: Vortex Production at Boundaries

This section is a complement to Sect. 5.2: we try to explain in more detail, still quite heuristically, how additive noise may arise in turbulent fluids.

5.5.1 Generation of Vortices Near Obstacles

Vortices are produced by instability even on a flat boundary. This fact, however, is already incorporated in a mathematical model based on deterministic Navier–Stokes equations in a domain with smooth boundary; thus it does not require the artificial introduction of a noise.

Different is the case of vortices produced by irregularities of the boundary or by several small or complicated obstacles in the middle of the fluid domain. In principle, if we describe precisely such irregularities in the mathematical model, then the deterministic model should be sufficient. But this is never done in practice, the irregularities being too detailed for a mathematical description. However, some attempts in this direction can be found in [17, 63]. It is here that it is meaningful to introduce noise: as a phenomenological replacement of a realistic element which is discarded by the deterministic part of the mathematical model. In the case of irregularities of a boundary this is important, since the consequences in the fluid motion of such irregularities are relevant, visible, macroscopic.

The precise physical description of the generation of vortices is a difficult topic in itself. Here we take a phenomenological viewpoint: emergence of vortices near obstacles is commonly observed and we content ourselves with an ad hoc inclusion of this fact into the equations. Deep research is mandatory on this issue.

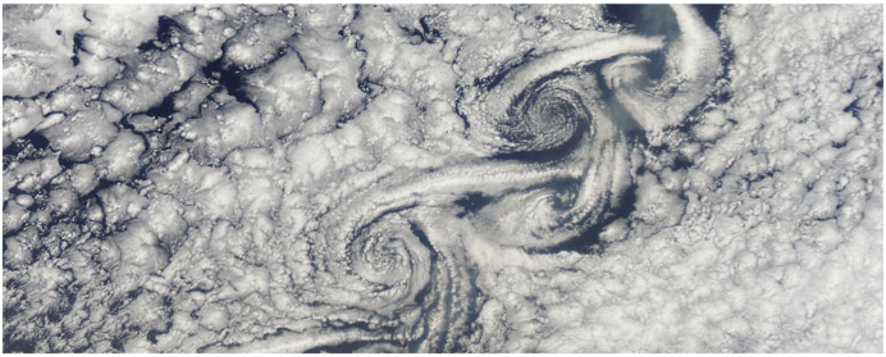

Fig. 5.1 Cloud vortices off Madeira and Canary Islands. Images by the MODIS Rapid Response team, NASA

Fig. 5.2 Ice floes, Kamchatka Coast, Russia. Image courtesy of the Earth Science and Remote Sensing Unit, NASA Johnson Space Center, eol.jsc.nasa.gov. NASA photo ID: ISS030-E-162344

Assume that the velocity field at time t is $u(t, x)$. Assume that, as a consequence of an obstacle in the domain (Fig. 5.1[1]) or at the boundary (Figs. 5.2[2] and 5.3[3]), a modification occurs and in a very short time we have a field $u(t + \Delta t, x)$ which is not just equal to the smooth evolution of $u(t, x)$. We may idealize and think that at

[1] https://visibleearth.nasa.gov/images/117121/cloud-vortices-off-madeira-and-canary-islands.

[2] https://eol.jsc.nasa.gov/SearchPhotos/photo.pl?mission=ISS030&roll=E&frame=162344.

[3] https://visibleearth.nasa.gov/images/148350/lake-erie-astir/148350f.

Fig. 5.3 Lake Erie Astir. NASA Earth Observatory images by Joshua Stevens

time t we had a jump:

$$u\left(t^{+},x\right)=u\left(t^{-},x\right)+\sigma\left(x\right),$$

where $\sigma\left(x\right)$ is presumably localized in space and corresponds to a vortex structure. Continuum mechanics does not make jumps; we idealize a fast change due to an instability as a jump, for a cleaner mathematical description.

We emphasize that vortices produced by irregularities (as well as by instabilities) appear as discrete events. Figures 5.1, 5.2 and 5.3 show wonderful instances visible in nature thanks to special events (otherwise, usually, vortices are not visible), like the perturbation of clouds due to the presence of an island and the freezing of water into ice structures of vortex type. Those of Fig. 5.1 are an example of von Karman vortices and are produced with a rather deterministic time interval, opposite to the randomized description below but the scaling limit results described in the sequel would hold also in such a case. Figures 5.1, 5.2 and 5.3 have the merit of showing very isolated and clearly visible vortices. In general, the complexity of a rough boundary profile produces a more disordered pattern of vortices, as schematically represented in Fig. 5.4. A wonderful example in nature is shown in Fig. 5.5:[4] thanks to the different colouring due to phytoplankton, we may appreciate the complexity of vortical structures close to a rough boundary.

[4] https://visibleearth.nasa.gov/images/65000/phytoplankton-bloom-off-argentina.

Fig. 5.4 Schematic representation of several vortices produced by a complex family of boundary obstacles. Picture by Claudia Flandoli

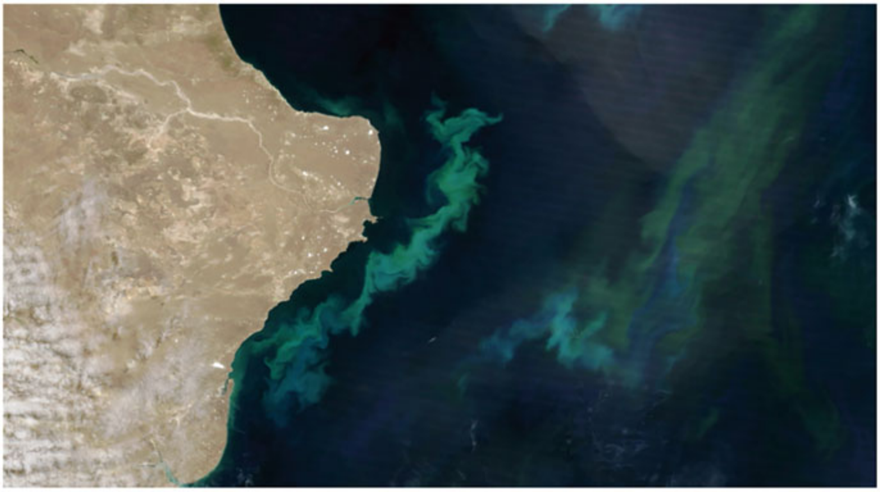

Fig. 5.5 Phytoplankton bloom off Argentina. Jacques Descloitres, MODIS Rapid Response Team, NASA/GSFC

Assume that, due to several obstacles in the boundary at certain locations x_k, $k \in K$, we may observe jumps of the form

$$u\left(t^{+}, x\right) = u\left(t^{-}, x\right) + \sigma_k\left(x\right), \tag{5.1}$$

where $\sigma_k\left(x\right)$ is a perturbation around x_k. We assume that K is finite, but one can generalize, see [124].

The way to incorporate these jumps into the Navier–Stokes equations is by means of an impulsive force:

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \sum_{k \in K} \sum_i \delta \left(t - t_i^k\right) \sigma_k.$$

Here, for each $k \in K$, we denote by $t_1^k < t_2^k < \ldots$ the sequence of jump times of class k. This way the fluid moves according to the free Navier–Stokes equations between two consecutive jumps times (reorder the full family $\left\{t_i^k; k \in K, i \in \mathbb{N}\right\}$ and consider two consecutive elements); and fulfils (5.1) at the jump times, with the correct $k \in K$. The previous one enters the framework of fluid mechanics SPDEs with kick force, see for instance [42, 72, 193].

We may assume that the jump times are random or deterministic (for the latter case, think of Karman vortices past an obstacle, as in one of the pictures above). For some later purposes it is the same, for others it is mathematically more convenient to assume them random, thus we do so. We assume that $t_{i+1}^k - t_i^k$ has exponential distribution with mean time τ^k, $\mathbb{P}\left(t_{i+1}^k - t_i^k > s\right) = e^{-s/\tau^k}$, and that all these random time intervals are independent. We may equivalently describe this by means of a family $\left\{\left(N_t^k\right)_{t \geq 0}; k \in K\right\}$ of independent standard (rate 1) Poisson processes, rescale their times as N_{t/τ^k}^k and define $t_1^k < t_2^k < \ldots$ as the random times when the Poisson process N_{t/τ^k}^k jumps (at time t_1^k it jumps from 0 to 1, at time t_2^k from 1 to 2 and so on). We have

$$\sum_{k \in K} \sum_i \delta \left(t - t_i^k\right) \sigma_k = \sum_{k \in K} \sigma_k \frac{dN_{t/\tau^k}^k}{dt},$$

where the time derivative of the jump process N_{t/τ^k}^k is understood in the sense of distributions.

It is then clear that we introduce the function

$$W(t, x) = \sum_{k \in K} \sigma_k(x) N_{t/\tau^k}^k = \sum_{k \in K} \sum_{i \in \mathbb{N}: t_i^k \leq t} \sigma_k(x)$$

and write the equation in the form

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \partial_t W. \tag{5.2}$$

This arises the mathematical question: can we study an equation of this form when $W(t)$ is not differentiable in a classical sense?

The Brownian Limit

In many examples the vortices appear in opposite pairs

$$\pm \sigma (x)$$

as in the wake after an obstacle of Fig. 5.1 above. At a boundary, usually the primary vortices always have the same sign but secondary vortices are often in pairs. With a large degree of idealization (this issue certainly requires more investigation) let us assume that each vortex σ_k appears in pairs by means of two independent Poisson processes $N^{k,1}_{t/\tau^k}$, $N^{k,2}_{t/\tau^k}$ with the same rate:

$$\frac{1}{\sqrt{2}}\left(\sigma_k (x) \frac{dN^{k,1}_{t/\tau^k}}{dt} - \sigma_k (x) \frac{dN^{k,2}_{t/\tau^k}}{dt}\right).$$

The factor $\frac{1}{\sqrt{2}}$ is just to normalize and maintain the notation τ^k for the mean time between consecutive generations, now understanding the generations of $\pm\sigma_k$ as a single process. The full process $W(t, x)$ thus has the form

$$W(t, x) = \sum_{k\in K} \frac{1}{\sqrt{2}}\sigma_k (x) \left(N^{k,1}_{t/\tau^k} - N^{k,2}_{t/\tau^k}\right). \tag{5.3}$$

Let us parametrize by n the jump times and the vortex intensities, as:

$$W_n (t, x) = \sum_{k\in K} \frac{1}{n}\sigma_k (x) \frac{N^{k,1}_{n^2 t/\tau^k} - N^{k,2}_{n^2 t/\tau^k}}{\sqrt{2}}.$$

The heuristics is that we make many more jumps but of smaller size. The precise rescaling has been chosen in order to have a non-zero finite limit. Indeed, the average of $W_n (t, x)$ is zero and the variance is equal to

$$\mathbb{E}\left[|W_n (t, x)|^2\right] = t \sum_{k\in K} \frac{|\sigma_k (x)|^2}{\tau^k},$$

which is finite and non-zero in the limit when $n \to \infty$. Let us check the previous result: since $\mathbb{E}\left[N^{k,j}_{n^2 t/\tau^k}\right] = \frac{n^2 t}{\tau^k}$, $Var\left[N^{k,j}_{n^2 t/\tau^k}\right] = \frac{n^2 t}{\tau^k}$, and $N^{k,1}_{n^2 t/\tau^k}$, $N^{k,2}_{n^2 t/\tau^k}$ are independent,

$$\mathbb{E}\left[|\frac{1}{n}\sigma_k(x)\frac{N^{k,1}_{n^2t/\tau^k} - N^{k,2}_{n^2t/\tau^k}}{\sqrt{2}}|^2\right]$$
$$= \frac{1}{2n^2}|\sigma_k(x)|^2\mathbb{E}\left[|N^{k,1}_{n^2t/\tau^k} - \frac{n^2t}{\tau^k} - N^{k,2}_{n^2t/\tau^k} + \frac{n^2t}{\tau^k}|^2\right]$$
$$= \frac{1}{2n^2}|\sigma_k(x)|^2 2Var\left[N^{k,j}_{n^2t/\tau^k}\right] = t\frac{|\sigma_k(x)|^2}{\tau^k}$$

and then a similar argument applies to the sum in k.

The Donsker invariance principle (see [39]) claims that, as $n \to \infty$,

$$\frac{1}{n}\left(N_{n^2t} - n^2t\right) \to W_t \text{ (Brownian motion)}$$

the convergence being in law and uniform on compact sets. A multidimensional version of the Donsker theorem similarly gives us that the stochastic process $W_n(t, x)$ converges in law to

$$W(t, x) := \sum_{k\in K}\frac{1}{\sqrt{\tau^k}}\sigma_k(x)\, W^k_t,$$

where $\left(W^k_t\right)_{t\geq 0}$ are independent Brownian motions. The corresponding Navier–Stokes equations, in the usual language of stochastic differential equations, have the form

$$du + (u\cdot\nabla u + \nabla p)\,dt = \nu\Delta u dt + \sum_{k\in K}\frac{1}{\sqrt{\tau^k}}\sigma_k dW^k_t.$$

Summarizing, we have at least two examples in mind of non-differentiable functions $W(t)$ which motivate the study of Eq. (5.2), non-classical because of the distributional time derivative: the case when $W(t)$ is a piecewise constant function, and the case when it is the trajectory of a process, a linear combination of Brownian motions. Recall that, with probability one, a trajectory of Brownian motion is nowhere differentiable, not of bounded variation, not Hölder of exponent $\alpha \geq \frac{1}{2}$ on any interval, but it is locally Hölder of any exponent $\alpha < \frac{1}{2}$. The analysis described in this section motivates the interest in studying Navier–Stokes equations with rough force as we have done in Chaps. 1 and 2. Some ideas described here are related to [161].

5.5.2 Scaling the Previous Example

Consider the previous system before introducing the scaling parameter n, namely Eq. (5.2) with the forcing $W(t,x)$ given by (5.3). Let us observe this system at a new space-time scale (if it may be of interest: think of observing changes minute by minute, when the vortex generation happens every few seconds). Assume $D = \mathbb{R}^2$ and the positions where the vortices are created correspond to a cluster of islands in the ocean. Let

$$u_\lambda(t,x) := \lambda^\alpha u\left(\lambda^\beta t, \lambda x\right).$$

Then

$$\begin{aligned}\partial_t u_\lambda(t,x) &= \lambda^{\alpha+\beta}(\partial_t u)\left(\lambda^\beta t, \lambda x\right),\\ \Delta u_\lambda(t,x) &= \lambda^{\alpha+2}(\Delta u)\left(\lambda^\beta t, \lambda x\right),\\ u_\lambda(t,x)\cdot\nabla u_\lambda(t,x) &= \lambda^{2\alpha+1}(u\cdot\nabla u)\left(\lambda^\beta t, \lambda x\right),\end{aligned}$$

hence we have to choose $\beta = 2$ and $\alpha = 1$ to have the same multiplier, that is λ^3, and we get

$$\begin{aligned}\partial_t u_\lambda + u_\lambda\cdot\nabla u_\lambda + \nabla p_\lambda &= \nu\Delta u_\lambda + \lambda^3(\partial_t W)\left(\lambda^2 t, \lambda x\right)\\ \operatorname{div} u_\lambda &= 0.\end{aligned}$$

But

$$\lambda^3(\partial_t W)\left(\lambda^2 t, \lambda x\right) = \partial_t W_\lambda(t,x),$$

where

$$W_\lambda(t,x) := \lambda W\left(\lambda^2 t, \lambda x\right) = \sum_{k\in K}\frac{1}{\lambda\sqrt{2}}\sigma_k^\lambda(x)\left(N^{k,1}_{\lambda^2 t/\tau^k} - N^{k,2}_{\lambda^2 t/\tau^k}\right),$$

where

$$\sigma_k^\lambda(x) = \lambda^2\sigma_k(\lambda x).$$

Assume λ is large, like the parameter n of the previous section. In the rescaled unit of time, we make very many jumps, of larger size; but also much more concentrated, since $\sigma_k^\lambda(x)$ is rescaled as classical mollifiers.

Let us observe this force by a test function ϕ (just to avoid that the pointwise observation may suffer some regularity issue)

$$\langle W_\lambda(t), \phi\rangle = \sum_{k\in K} \frac{1}{\lambda\sqrt{2}} \left(N^{k,1}_{\lambda^2 t/\tau^k} - N^{k,2}_{\lambda^2 t/\tau^k}\right) \int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx.$$

We have zero mean and (as above)

$$\begin{aligned}\mathbb{E}\left[\langle W_\lambda(t), \phi\rangle^2\right] &= \sum_{k\in K} \frac{1}{2\lambda^2} 2\frac{\lambda^2 t}{\tau^k} \left(\int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx\right)^2 \\ &= \sum_{k\in K} \frac{t}{\tau^k} \left(\int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx\right)^2.\end{aligned}$$

We get

$$\int_{\mathbb{R}^2} \sigma_k^\lambda(x)\,\phi(x)\,dx \overset{y=\lambda x}{=} \int_{\mathbb{R}^2} \sigma_k(y)\,\phi\left(\frac{y}{\lambda}\right)dy \to \phi(0)\int_{\mathbb{R}^2} \sigma_k(y)\,dy.$$

So again we see that we have a finite non-zero limit.

What may we conclude? It is difficult to get a rich conclusion, because $\sigma_k^\lambda(x)$ converge to a vector-valued space-distribution Ξ_k (a so-called current), the one such that

$$\Xi_k(\phi) = \phi(0)\int_{\mathbb{R}^2} \sigma_k(y)\,dy.$$

Thus the limit process is

$$W(t,x) := \sum_{k\in K} \frac{1}{\sqrt{\tau^k}} \Xi_k W_t^k,$$

which is distributional in space, not only non-differentiable in time. Investigating this problem seems to be a challenging mathematical task.

There is a variant which should be mentioned: if we suspend the requirement that σ_k is localized and ask that the created structures are point vortices, then

$$\sigma_k(x) = \frac{1}{\pi}\frac{(x-x_0)^\perp}{|x-x_0|^2}$$

and $\sigma_k^\lambda(x) = \sigma_k(x)$! In this case the limit process is a vector field in space (not a distribution), but with infinite energy:

$$\int_{\mathbb{R}^2} |\sigma_k(x)|^2 dx = +\infty.$$

See [124] for results on a related model.

5.5.3 *Example of State-Dependent Noise*

The examples of noise presented in Sect. 5.5.1 allow us to deal with the Stochastic Navier–Stokes equations as if they were deterministic: given a single noise realization, we solve the equation. This was the approach we developed in Chap. 1, useful in relatively few cases. Indeed, the case treated in Chap. 1 had the special feature that the random input was independent of the solution. But in real situations, as in Fig. 5.6, the noise may vary depending on the solution.

Mathematically speaking, in Chap. 1 the noise, motivated by the discussion presented in Sect. 5.5.1, entered the equation as an additive force; this was the key property which allowed us to study the linear Stokes problem first, independently of the solution of the nonlinear one. There are other cases (different from the additive case) which can be treated by similar ideas, but few.

From the discussion above, we can understand that the noise introduced in Chap. 1 has the following interpretation: *vortices emerge at a boundary due to obstacles and roughness*. However, this interpretation neglects some facts. Indeed, when the fluid is at rest, certainly no vortex is created; similarly, we do not expect frequent creations if the velocity of the flow is very small. The rate of creation of vortices hence should depend on some feature of the flow itself. This doesn't mean

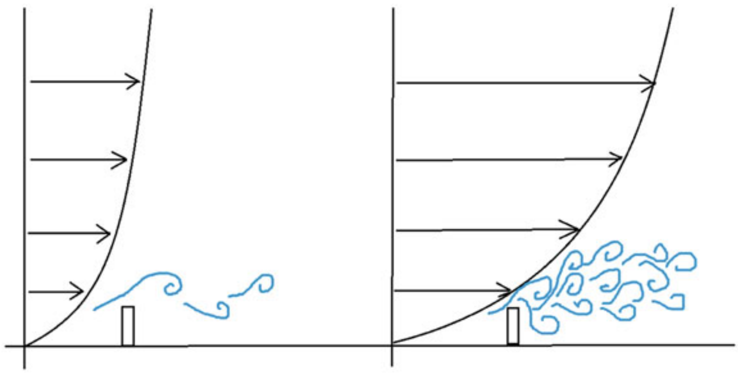

Fig. 5.6 The average wind speed influences the rate of production of vortices past an obstacle

that the model of Chap. 1 is useless: it is reasonable when the mean flow is roughly constant, and the rates τ^k should be taken appropriately with respect to the constant mean flow value.

When the state $u(t,\cdot)$ affects the rate of creation, we may use the concept of non-homogeneous Poisson process with random time-dependent rate: we introduce (corresponding to each k) an instantaneous rate $\lambda_k(u(t))$ depending on an average intensity of $u(t,\cdot)$, e.g.

$$\lambda_k(u(t)) = \chi^2\left(\frac{1}{|B(x_k,r)|}\int_{B(x_k,r)}|u(t,y)|dy\right),$$

where χ^2 is a nondecreasing non-negative function, equal to zero in zero and $r > 0$ is a length scale relevant to the problem. Then we introduce the cumulative rate

$$\Lambda_k(t) = \int_0^t \lambda_k(u(s))\,ds$$

and finally we modify the Poisson process N_t^k by this rate, namely we consider the process

$$N^k_{\Lambda_k(t)}.$$

The case previously considered was simply

$$\lambda_k(u(t)) = \lambda_k, \qquad \Lambda_k(t) = \lambda_k t, \qquad N^k_{\lambda_k t}.$$

The jump times of the noise in the equation will be the jump times of this processes, which are delayed or accelerated depending on the average intensity of $u(t)$:

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f + F(u) + \sum_{k\in K}\sigma_k\partial_t N^k_{\Lambda_k(t)} \tag{5.4}$$

or

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f + F(u) + \sum_{k\in K}\frac{1}{\sqrt{2}}\sigma_k\partial_t\left(N^{k,1}_{\Lambda_k(t)} - N^{k,2}_{\Lambda_k(t)}\right) \tag{5.5}$$

depending whether we assume that both vortices $\sigma_k(x)$ and $-\sigma_k(x)$ appear and are equally likely.

This is already a very interesting model which could deserve investigation. Otherwise, in the case of (5.5), we may rescale the noise as

$$\sum_{k\in K}\frac{1}{n\sqrt{2}}\sigma_k(x)\left(N^{k,1}_{n^2\Lambda_k(t)} - N^{k,2}_{n^2\Lambda_k(t)}\right). \tag{5.6}$$

Notice that, in order to increase the rate at time t, we have to use the instantaneous rate $n^2 \lambda_k(t)$, whence the expression $n^2 \Lambda_k(t)$ (instead of $\Lambda_k(n^2 t)$ which has a completely different and wrong meaning).

Recalling the convergence of rescaled Poisson processes to Brownian motion discussed in Sect. 5.5.1, it can be proved that the limit process of (5.6), in law, is

$$\sum_{k \in K} \sigma_k(x) B^k_{\Lambda_k(t)},$$

where B^k_t are independent Brownian motions. Then, by a deep theorem on martingales (e.g. [183]), there exists (possibly on a larger probability space) independent Brownian motions W^k_t such that, in law

$$B^k_{\Lambda_k(t)} = \int_0^t \sqrt{\lambda_k(u(s))} dW^k_s$$

(jointly in k). This result is undoubtedly advanced and not trivial even at the heuristic level, but notice at least the analogy with the coefficients $\sqrt{\lambda_k}$ in the case of constant rate: when $\lambda_k(u(s)) = \lambda_k$, $\Lambda_k(t) = \lambda_k t$, the previous identity reads

$$B^k_{\lambda_k t} = \int_0^t \sqrt{\lambda_k} dW^k_s = \sqrt{\lambda_k} W^k_t$$

and it is well-known that $\lambda_k^{-1/2} B^k_{\lambda_k t}$ is a new Brownian motion.

The final equation is

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \sigma_k \sqrt{\lambda_k(u)} \partial_t W^k_t.$$

We write it in the form

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \sigma_k(u) \partial_t W^k_t \tag{5.7}$$

by introducing the maps $\sigma_k : H \to H$ given by

$$\sigma_k(u)(x) = \sigma_k(x) \sqrt{\lambda_k(u)}.$$

This is exactly the equation treated in Chap. 2.

5.6 The Wong–Zakai Corrector and Stratonovich Integrals

In more than one place in these notes we invoke the Wong–Zakai principle and write the final equations in the Stratonovich form, or Itô form plus the corrector which plays a very key role in Chaps. 3 and 4. Without any aim to prove results here, for which we address specialized literature like [249, 251, 257], let us illustrate some of the ideas.

5.6.1 A One-Dimensional Example

Equations (5.4)–(5.5) are mathematically correct (whether they are physically relevant should be investigated more deeply). On the contrary, Eq. (5.7) requires a special choice of $F(u)$ to be the right one:

$$F(u) = \frac{1}{2} \sum_{k \in K} D\sigma_k(u)\, \sigma_k(u).$$

Here by $D\sigma_k(u)$ we mean the Frechét Jacobian of $\sigma_k(u)$, which is a linear bounded operator from H to H, under suitable assumptions, and $D\sigma_k(u)\,\sigma_k(u)$ is the application of the linear map $D\sigma_k(u)$ to the element $\sigma_k(u)$ of H. Results of Wong–Zakai type for fluid dynamic equations have been proved, see [175, 176]. They are very technical and based on methods different from those described here. Hence we limit ourselves to explaining the emergence of the term $D\sigma_k(u)\,\sigma_k(u)$ in the simple case of a one-dimensional ordinary differential equation [257].

Consider the one-dimensional equation, with $\sigma(x) \geq \nu > 0$,

$$\frac{dX_t^\epsilon}{dt} = \sigma\left(X_t^\epsilon\right) \frac{dW_t^\epsilon}{dt},$$

where W_t^ϵ is an approximation of a Brownian motion W_t. It is an equation with separated variables. Then

$$\frac{\frac{dX_t^\epsilon}{dt}}{\sigma\left(X_t^\epsilon\right)} = \frac{dW_t^\epsilon}{dt},$$

$$\int_0^T \frac{\frac{dX_t^\epsilon}{dt}}{\sigma\left(X_t^\epsilon\right)} dt = \int_0^T \frac{dW_t^\epsilon}{dt} dt,$$

$$\Phi\left(X_T^\epsilon\right) - \Phi(x_0) = W_T^\epsilon, \qquad \Phi'(x) = \frac{1}{\sigma(x)},$$

$$X_t^\epsilon = \Phi^{-1}\left(\Phi(x_0) + W_t^\epsilon\right).$$

Hence $X^{\epsilon}_{\cdot}$ converges weakly to $X_{\cdot}$ given by

$$X_t = \Phi^{-1}\left(\Phi\left(x_0\right) + W_t\right).$$

From the Itô formula, since

$$\begin{aligned} D\Phi^{-1}(x) &= \frac{1}{\Phi'\left(\Phi^{-1}(x)\right)} = \sigma\left(\Phi^{-1}(x)\right), \\ D^2\Phi^{-1}(x) &= D\left[\sigma\left(\Phi^{-1}(x)\right)\right] = \sigma'\left(\Phi^{-1}(x)\right) D\Phi^{-1}(x) \\ &= \sigma'\left(\Phi^{-1}(x)\right)\sigma\left(\Phi^{-1}(x)\right), \end{aligned}$$

$$\begin{aligned} dX_t &= \sigma\left(\Phi^{-1}\left(\Phi\left(x_0\right) + W_t\right)\right) dW_t \\ &\quad + \frac{1}{2}\sigma'\left(\Phi^{-1}\left(\Phi\left(x_0\right) + W_t\right)\right)\sigma\left(\Phi^{-1}\left(\Phi\left(x_0\right) + W_t\right)\right) dt \\ &= \sigma\left(X_t\right) dW_t + \frac{1}{2}\sigma'\left(X_t\right)\sigma\left(X_t\right) dt. \end{aligned}$$

We have found the corrector above.

Our conclusion, supported by the previous heuristic evidence, is that the right formulation of Eq. (5.7) is the stochastic equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu\Delta u + f + \frac{1}{2}\sum_{k\in K} D\sigma_k(u)\,\sigma_k(u) + \sum_{k\in K}\sigma_k(u)\,\partial_t W_t^k.$$

Remark 5.3 Using the notion of Stratonovich stochastic integral, different from the Itô one, denoted by $\int_0^t \sigma_k(u(s)) \circ dW_s^k$, one can write the equation in the more natural form

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu\Delta u + f + \sum_{k\in K}\sigma_k(u) \circ \partial_t W_t^k$$

because

$$\int_0^t \sigma_k(u(s)) \circ dW_s^k = \int_0^t \sigma_k(u(s))\, dW_s^k + \frac{1}{2}\int_0^t D\sigma_k(u(s))\,\sigma_k(u(s))\, ds$$

when u solves the equation above. Manipulations (the chain rule) with the Stratonovich formulation are similar to classical calculus, but taking expected values is not suitable, the fundamental cancellations of Itô integrals are hidden. Therefore it is very important to know the Itô formulation.

5.6.2 The Case of the Heat Equation

Key to the facts described in Chaps. 3 and 4, see in particular Sect. 3.1, is the emergence of the additional operator $\mathcal{L}$, which is a specific consequence of Stratonovich formulation; we feel we need to justify it heuristically, at least for the exogenous case. Researchers used to stochastic calculus have a tendency to accept a priori the Stratonovich formulation (it is a correct attitude!) and thus accept the presence of $\mathcal{L}$ as an obvious fact. But looking at the problem with the eyes of a more general scientist, the presence of the additional operator $\mathcal{L}$ is a revolution that requires an explanation.

The rigorous literature on Wong–Zakai-type results for SPDEs is wide, see for instance [50, 51, 59, 135, 166, 167, 249, 251]. For the purpose of this heuristic explanation, let us consider the heat transport equation

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \kappa \Delta \theta^\epsilon + q, \tag{5.8}$$

where

$$u^\epsilon (t) = \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \sigma_k dW_s^k.$$

This is a simplified model with respect to the one of Sect. 3.1 (we drop the Stokes operator A, taking $u_0 = 0$ is only to simplify notations).

Theorem 5.4 *If $\sigma_k \in D(A)$, $\phi \in C^\infty(D)$,*

$$\theta^\epsilon|_{t=0} = \theta_0 \in L^\infty(D),$$

then the weak solution θ^ϵ of Eq. (5.8) with initial condition θ_0 satisfies for every $t \geq 0$

$$\lim_{\epsilon \to 0} \left\langle \theta^\epsilon (t), \phi \right\rangle = \langle \theta (t), \phi \rangle$$

in probability, where $\theta(t)$ is the unique weak solution of equation

$$d\theta + \sum_{k \in K} \sigma_k \cdot \nabla \theta dW^k = (\kappa \Delta \theta + \mathcal{L}\theta + q)\, dt \tag{5.9}$$

with

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sigma_k (x) \cdot \nabla (\sigma_k (x) \cdot \nabla \theta (x)).$$

The unique solvability of Eq. (5.9) has been proved in Chap. 3. The unique solvability of Eq. (5.8) is classical, along with estimates of the form

$$\|\theta^{\epsilon}(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\nabla\theta^{\epsilon}(s)\|_{L^2}^2 ds = \|\theta_0\|_{L^2}^2$$

$$\|\theta^{\epsilon}(t)\|_{\infty} \le \|\theta_0\|_{\infty}. \tag{5.10}$$

Let us give only the idea of proof of Theorem 5.4, subset of the results of [223]. Recall that, with the notations

$$W^{\epsilon}(t,x) = \int_0^t u^{\epsilon}(s,x)\, ds,$$
$$W(t,x) = \sum_{k\in K} \sigma_k(x)\, W_t^k$$

in Chap. 3 we have proved that

$$\lim_{\epsilon\to 0} \mathbb{E}\left[\|W^{\epsilon}(t) - W(t)\|_H^2\right] = 0.$$

Let us introduce also some additional notations:

$$\xi_t^{k,\epsilon} = \frac{1}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-s)} dW_s^k,$$
$$W_t^{k,\epsilon} = \int_0^t \xi_s^{k,\epsilon} ds$$

so that $u^{\epsilon}(t,x) = \sum_{k\in K} \sigma_k(x)\, \xi_t^{k,\epsilon}$, $W^{\epsilon}(t,x) = \sum_{k\in K} \sigma_k(x)\, W_t^{k,\epsilon}$.

We use the weak formulation and try to pass to the limit term by term, taking great advantage of the fact that the equation is linear. In the weak formulation of Eq. (5.8), let us concentrate only on the difficult term

$$\int_0^t \left\langle u^{\epsilon}(s)\cdot\nabla\phi, \theta^{\epsilon}(s)\right\rangle ds$$

and split it on the partition π_{ϵ}:

$$\int_0^t \left\langle u^{\epsilon}(s)\cdot\nabla\phi, \theta^{\epsilon}(s)\right\rangle ds = \sum_{t_i\le t}\int_{t_i}^{t_{i+1}} \left\langle u^{\epsilon}(s)\cdot\nabla\phi, \theta^{\epsilon}(s)\right\rangle ds.$$

Just for notational convenience (at the end we go back to the general case) assume $u^{\epsilon}(t)$ is made only of a single term

$$u^{\epsilon}(t,x)=\sigma(x)\,\xi_t^{\epsilon}$$

where

$$W_t^{\epsilon}:=\int_0^t \xi^{\epsilon}(s)\,ds\to W_t.$$

Then

$$\begin{aligned}&\int_{t_i}^{t_{i+1}}\left\langle u^{\epsilon}(s)\cdot\nabla\phi,\theta^{\epsilon}(s)\right\rangle ds\\&=\int_{t_i}^{t_{i+1}}\left\langle \sigma\cdot\nabla\phi,\theta^{\epsilon}(s)\right\rangle \xi_s^{\epsilon}ds\\&=\int_{t_i}^{t_{i+1}}\left\langle \sigma\cdot\nabla\phi,\theta^{\epsilon}(t_i)\right\rangle \xi_s^{\epsilon}ds+\int_{t_i}^{t_{i+1}}\left\langle \sigma\cdot\nabla\phi,\left(\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i)\right)\right\rangle \xi_s^{\epsilon}ds\\&=\left\langle \sigma\cdot\nabla\phi,\theta^{\epsilon}(t_i)\right\rangle\left(W_{t_{i+1}}^{\epsilon}-W_{t_i}^{\epsilon}\right)+\int_{t_i}^{t_{i+1}}\left\langle \sigma\cdot\nabla\phi,\left(\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i)\right)\right\rangle \xi_s^{\epsilon}ds.\end{aligned}$$

The sum over the partition of the first term converges to the Itô integral $\int_0^t \langle\sigma\cdot\nabla\phi,\theta(s)\rangle\,dW_s$. More difficult is to understand the limit of

$$\sum_{t_i\le t}\int_{t_i}^{t_{i+1}}\left\langle \sigma\cdot\nabla\phi,\left(\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i)\right)\right\rangle \xi_s^{\epsilon}ds. \tag{5.11}$$

Notice first a potential mistake: one could think that, $\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i)$ being small for $s\in[t_i,t_{i+1}]$, this sum will converge to zero. But ξ_s^{ϵ}, being related (in the limit) to the derivative of BM, is large, and the product $(\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i))\,\xi_s^{\epsilon}$ could have a non-zero compensation. Indeed, it has: roughly speaking, $(\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i))$ behaves like $\sqrt{t_{i+1}-t_i}$ and ξ_s^{ϵ} diverges like $\frac{1}{\sqrt{t_{i+1}-t_i}}$.

The way to capture the precise asymptotics is to use Eq. (5.8) again, written here for a generic test function ψ:

$$\left\langle \psi,\theta^{\epsilon}(s)-\theta^{\epsilon}(t_i)\right\rangle-\int_{t_i}^{s}\left\langle \sigma\cdot\nabla\psi,\theta^{\epsilon}(r)\right\rangle \xi_r^{\epsilon}dr=\int_{t_i}^{s}\left\langle \kappa\Delta\psi,\theta^{\epsilon}(r)\right\rangle dr. \tag{5.12}$$

Take $\psi=\sigma\cdot\nabla\phi$ to connect with the above term (5.11) to be investigated. We have now to deal with the two terms

$$\sum_{t_i\le t}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left\langle \sigma\cdot\nabla\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}(r)\right\rangle \xi_r^{\epsilon}\xi_s^{\epsilon}drds$$

and

$$\sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{s} \left\langle \kappa \Delta \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(r \right) \right\rangle dr \right) \xi_s^{\epsilon} ds. \tag{5.13}$$

Having assumed sufficient smoothness of σ and ϕ, we may use (5.10) to bound $\theta^{\epsilon}(r)$ uniformly and find (the inequality is even a.s., with a deterministic constant $C > 0$)

$$|\int_{t_i}^{s} \left\langle \kappa \Delta \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(r \right) \right\rangle dr| \le C \left(t_{i+1} - t_i \right).$$

Since $\int_{t_i}^{t_{i+1}} |\xi_s^{\epsilon}| ds$ is infinitesimal in a suitable probabilistic sense, it is easy to show that the term (5.13) goes to zero in probability. The difficult term is

$$\sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(r \right) \right\rangle \xi_r^{\epsilon} \xi_s^{\epsilon} dr ds.$$

But we start to see an auxiliary second-order differential operator $(\sigma \cdot \nabla \sigma \cdot \nabla)$ arising here and this encourages us to continue the computation. One has to play again the same trick above: rewrite the previous expression as

$$\begin{aligned} &\sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(t_i \right) \right\rangle \xi_r^{\epsilon} \xi_s^{\epsilon} dr ds \\ &= \sum_{t_i \le t} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(t_i \right) \right\rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{\epsilon} \xi_s^{\epsilon} dr ds \end{aligned}$$

plus the remainder

$$R_{\epsilon} := \sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi \right), \theta^{\epsilon} \left(r \right) - \theta^{\epsilon} \left(t_i \right) \right\rangle \xi_r^{\epsilon} \xi_s^{\epsilon} dr ds.$$

This time, one can show that the remainder is infinitesimal. The heuristic idea comes from the fact that it contains the product of three terms, all roughly speaking of order $\sqrt{t_{i+1} - t_i}$:

$$\theta^{\epsilon}(r) - \theta^{\epsilon}(t_i), \qquad W^{\epsilon}(t_{i+1}) - W^{\epsilon}(t_i), \qquad W^{\epsilon}(t_{i+1}) - W^{\epsilon}(t_i).$$

Again (5.12) and (5.10) are useful here.

Finally, we have to understand the limit of

$$\sum_{t_i \leq t} \left\langle \sigma \cdot \nabla \left(\sigma \cdot \nabla \phi\right), \theta^{\epsilon}\left(t_i\right)\right\rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{\epsilon} \xi_s^{\epsilon} dr ds.$$

In the case of general noise with several independent Brownian motions, we have to understand the limit of

$$\sum_{t_i \leq t} \left\langle \sigma_k \cdot \nabla \left(\sigma_{k'} \cdot \nabla \phi\right), \theta^{\epsilon}\left(t_i\right)\right\rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{k,\epsilon} \xi_s^{k',\epsilon} dr ds.$$

One can prove the following property on the joint quadratic variation:

$$\lim_{\epsilon \to 0} \sum_{t_i \leq t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{k,\epsilon} \xi_s^{k',\epsilon} dr ds \to \frac{1}{2} \delta_{k,k'} t$$

uniformly in time, in probability. From properties of Riemann–Stieltjes integrals, it follows that the previous sum converges to

$$\frac{\delta_{k,k'}}{2} \int_0^t \left\langle \sigma_k \cdot \nabla \left(\sigma_{k'} \cdot \nabla \phi\right), \theta\left(s\right)\right\rangle ds.$$

The final result is that, in the weak sense,

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_0^t u^{\epsilon}\left(s\right) \cdot \nabla \theta^{\epsilon}\left(s\right) ds \\ &= \sum_{k \in K} \int_0^t \sigma_k \cdot \nabla \theta dW_s^k + \frac{1}{2} \sum_{k \in K} \int_0^t \left(\sigma_k \cdot \nabla \sigma_k \cdot \nabla\right) \theta\left(s\right) ds. \end{aligned}$$

5.7 Summary

Opposite to the previous chapters, which were mathematically rigorous, this one aims to present heuristically an ideal path from small-scale turbulence to eddy viscosity, going through models with additive noise and transport-type noise.

The topic of eddy viscosity is very important for applications and numerical computations. Although the idea is classical, a precise knowledge of the additional elliptic operator, in the large-scale dynamics, which better represents the impact of turbulent small scales, is still not fully clear, in particular in the regions close to boundaries; for instance, presumably this operator should be degenerate elliptic close to the boundary, but the kind of degeneracy may be better understood. Having a new strategy to link small turbulent scales to such an operator may give new insights; the ideal path described in this chapter seems to be a new promising link.

As stated in the preface of this book, several issues in this ideal path are still open and very difficult. Let us mention a few. The starting point is an additive noise at small scales. In this chapter we illustrated a few preliminary ideas about it, motivated by boundary irregularities. However, a more precise form and justification is needed. The second step is the transfer of this additive noise to a transport noise at large scales; the research on this topic is active, but certainly not complete, for instance because it is mostly based on what we called implicit strategy, opposite to explicit ones. The belief that a transport-type noise should appear is supported also from the comparison with other theories and approaches, but there is a chance that other terms should be added, for a more precise description.

Assuming that the previous two problems are sufficiently understood, we have in our hands stochastic equations of Navier–Stokes type for the large scales, with a kind of transport noise representing the action of small turbulent scales. The final step is proving that this stochastic model is close to a deterministic one with eddy viscosity. We have completed this last step in the idealized case of a torus and a simple noise, essentially space homogeneous. But a more interesting case is when there is a boundary, with the small-scale turbulence in the boundary layer; we do not have information on this case yet. The fluid velocity is zero at the boundary, so it is the turbulent small-scale component and this should lead to a degeneracy of the eddy viscosity near the boundary; this is an issue which will require closer investigation.

Similarly, the fluid velocity, or even better its gradient, should play a role in the intensity of the turbulent component, leading to state-dependent small-scale noise, then yielding state-dependent transport-type noise and finally a state-dependent elliptic operator for the eddy viscosity, like for instance the Smagorinsky model or other nonlinear models of large eddy simulation theory. This generalization has not been developed.

In spite of all these difficult open questions, we hope that this chapter provides some convincing motivations for investigating stochastic versions of Navier-Stokes equations, both in the more classical case of additive noise and in the intriguing one of transport-type noise, as well as emphasis on the investigation of the boundary, related to these topics.

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