HANDBOOK OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS FOR ENGINEERS AND SCIENTISTS

SECOND EDITION

Andrei D. Polyanin
Vladimir E. Nazaikinskii

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

HANDBOOK OF
LINEAR PARTIAL DIFFERENTIAL EQUATIONS FOR ENGINEERS AND SCIENTISTS

SECOND EDITION

HANDBOOK OF
LINEAR PARTIAL DIFFERENTIAL EQUATIONS FOR ENGINEERS AND SCIENTISTS

SECOND EDITION

Andrei D. Polyanin
Vladimir E. Nazaikinskii

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20151119

International Standard Book Number-13: 978-1-4665-8149-4 (eBook - PDF)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

CONTENTS

PREFACE TO THE SECOND EDITION

The **Handbook of Linear Partial Differential Equations for Engineers and Scientists**, a unique reference for scientists and engineers, contains nearly 4,000 linear partial differential equations with solutions as well as analytical, symbolic, and numerical methods for solving linear equations. First-, second-, third-, fourth-, and higher-order linear equations and systems of coupled equations are considered. Equations of parabolic, hyperbolic, elliptic, mixed, and other types are discussed. A number of new linear equations, exact solutions, transformations, and methods are described. Formulas for effective construction of solutions are given. A number of specific examples where the methods described in the book are used are considered. Boundary value problems and eigenvalue problems are described. Symbolic and numerical methods for solving PDEs with Maple, Mathematica, and MATLAB® are considered. All in all, the handbook contains many more linear partial differential equations than any other book currently available.

In selecting the material, the authors have given highest priority to the following major topics:

- Equations and problems that arise in various applications (heat and mass transfer theory, wave theory, elasticity, hydrodynamics, aerodynamics, continuum mechanics, acoustics, electrostatics, electrodynamics, electrical engineering, diffraction theory, quantum mechanics, chemical engineering sciences, control theory, etc.).
- Systems of coupled equations that arise in various fields of continuum mechanics and physics.
- Analytical and symbolic methods for solving linear equations of mathematical physics.
- Equations of general form that depend on arbitrary functions and equations that involve many free parameters; exact solutions of such equations are of major importance for testing numerical and approximate analytical methods.

The second edition has been substantially updated, revised, and expanded. More than 1,500 linear equations and systems with solutions, as well some methods and many examples, have been added, which amounts to over 700 pages of new material (including 250 new pages dealing with methods).

New to the second edition:

- Some second-, third-, fourth-, and higher-order linear PDEs with solutions.
- Systems of coupled partial differential equations with solutions.
- First-order linear PDEs with solutions.
- Some analytical methods including decomposition methods and their applications.
- Symbolic and numerical methods with Maple, Mathematica, and MATLAB.
- Some transformations, asymptotic formulas and solutions.
- Many new examples and figures included for illustrative purposes.
- Some long tables, including tables of various integral transforms.
- Extensive table of contents and detailed index.

Note that Chapters 1–12 of the book can be used as a database of test problems for numerical, approximate analytical, and symbolic methods for solving linear partial differential equations and systems of coupled equations. To satisfy the needs of a broad audience with diverse mathematical backgrounds, the authors have done their best to avoid special terminology whenever possible. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner with necessary references made to books where these methods are considered in more detail. Many sections are written so that they can be read independently from each other. This allows the reader to get to the heart of the matter quickly.

Separate sections of the book can serve as a basis for practical courses and lectures on equations of mathematical physics and linear PDEs.

We would like to express our keen gratitude to Alexei Zhurov for fruitful discussions and valuable remarks. We are very thankful to Inna Shingareva and Carlos Lizárraga-Celaya, who wrote three chapters (22–24) of the book at our request.

The authors hope that the handbook will prove helpful for a wide audience of researchers, university and college teachers, engineers, and students in various fields of applied mathematics, mechanics, physics, chemistry, economics, and engineering sciences.

Andrei D. Polyanin
Vladimir E. Nazaikinskii

PREFACE TO THE FIRST EDITION

Linear partial differential equations arise in various fields of science and numerous applications, e.g., heat and mass transfer theory, wave theory, hydrodynamics, aerodynamics, elasticity, acoustics, electrostatics, electrodynamics, electrical engineering, diffraction theory, quantum mechanics, control theory, chemical engineering sciences, and biomechanics.

This book presents brief statements and exact solutions of more than 2000 linear equations and problems of mathematical physics. Nonstationary and stationary equations with constant and variable coefficients of parabolic, hyperbolic, and elliptic types are considered. A number of new solutions to linear equations and boundary value problems are described. Special attention is paid to equations and problems of general form that depend on arbitrary functions. Formulas for the effective construction of solutions to nonhomogeneous boundary value problems of various types are given. We consider second-order and higher-order equations as well as the corresponding boundary value problems. All in all, the handbook presents more equations and problems of mathematical physics than any other book currently available.

For the reader's convenience, the introduction outlines some definitions and basic equations, problems, and methods of mathematical physics. It also gives useful formulas that enable one to express solutions to stationary and nonstationary boundary value problems of general form in terms of the Green's function.

Two supplements are given at the end of the book. Supplement A lists properties of the most common special functions (the gamma function, Bessel functions, degenerate hypergeometric functions, Mathieu functions, etc.). Supplement B describes the methods of generalized and functional separation of variables for nonlinear partial differential equations. We give specific examples and an overview application of these methods to construct exact solutions for various classes of second-, third-, fourth-, and higher-order equations (in total, about 150 nonlinear equations with solutions are described). Special attention is paid to equations of heat and mass transfer theory, wave theory, and hydrodynamics as well as to mathematical physics equations of general form that involve arbitrary functions.

The equations in all chapters are in ascending order of complexity. Many sections can be read independently, which facilitates working with the material. An extended table of contents will help the reader find the desired equations and boundary value problems. We refer to specific equations using notation like "1.8.5.2," which means "Equation 2 in Subsection 1.8.5."

To extend the range of potential readers with diverse mathematical backgrounds, the author strove to avoid the use of special terminology wherever possible. For this reason, some results are presented schematically, in a simplified manner (without details), which is, however, quite sufficient in most applications.

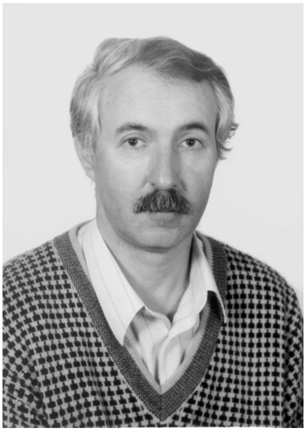
Separate sections of the book can serve as a basis for practical courses and lectures on equations of mathematical physics.

The author thanks Alexei Zhurov for useful remarks on the manuscript.

The author hopes that the handbook will be useful for a wide range of scientists, university teachers, engineers, and students in various areas of mathematics, physics, mechanics, control, and engineering sciences.

Andrei D. Polyanin

AUTHORS



Andrei D. Polyanin, D.Sc., is an internationally renowned scientist of broad interests and is active in various areas of mathematics, mechanics, and chemical engineering sciences. He is one of the most prominent authors in the field of reference literature on mathematics.

Professor Polyanin graduated with honors from the Faculty of Mechanics and Mathematics at Lomonosov Moscow State University in 1974. He received his Ph.D. in 1981 and D.Sc. in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, Professor Polyanin has been working at the Institute for Problems in Mechanics of the Russian Academy of Sciences; he is also Professor of applied mathematics at Bauman Moscow State Technical University and at National Research Nuclear University MEPhI. He is a member of the Russian National Committee on Theoretical and Applied Mechanics and the Mathematics and Mechanics Expert Council of the Higher Certification Committee of the Russian Federation.

Professor Polyanin has made important contributions to the theory of differential and integral equations, mathematical physics, engineering mathematics, theory of heat and mass transfer, and chemical hydrodynamics. He has obtained exact solutions for several thousand ordinary differential, partial differential, delay partial differential, and integral equations.

Professor Polyanin has authored more than 30 books in English, Russian, German, and Bulgarian as well as over 170 research papers and three patents. He has written a number of fundamental handbooks, including A. D. Polyanin and V. F. Zaitsev's *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, 1995 and 2003); A. D. Polyanin and A. V. Manzhirov's *Handbook of Integral Equations* (CRC Press, 1998 and 2008); A. D. Polyanin's *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC Press, 2002); A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux's *Handbook of First Order Partial Differential Equations* (Taylor & Francis, 2002); A. D. Polyanin and V. F. Zaitsev's *Handbook of Nonlinear Partial Differential Equations* (Chapman & Hall/CRC Press, 2004 and 2012); A. D. Polyanin and A. V. Manzhirov's *Handbook of Mathematics for Engineers and Scientists* (Chapman & Hall/CRC Press, 2007), and A. D. Polyanin and A. I. Chernoutsan's (Eds.) *A Concise Handbook of Mathematics, Physics, and Engineering Sciences* (Chapman & Hall/CRC Press, 2010).

Professor Polyanin is editor-in-chief of the international scientific educational website *EqWorld—The World of Mathematical Equations* and he is editor of the book series *Differential and Integral Equations and Their Applications* (Chapman & Hall/CRC Press, London/Boca Raton). Professor Polyanin is a member of the editorial board of the journals *Theoretical Foundations of Chemical Engineering*, *Mathematical Modeling and Computational Methods* (in Russian), and *Bulletin of the National Research Nuclear University MEPhI* (in Russian).

In 1991, Professor Polyanin was awarded the Chaplygin Prize of the Russian Academy of Sciences for his research in mechanics. In 2001, he received an award from the Ministry of Education of the Russian Federation.

Address: Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Ave., Bldg 1, 119526 Moscow, Russia

Home page: http://eqworld.ipmnet.ru/polyanin-ew.htm

Vladimir E. Nazaikinskii, D.Sc., is an actively working mathematician specializing in partial differential equations, mathematical physics, and noncommutative analysis. He was born in 1955 in Moscow, graduated from the Moscow Institute of Electronic Engineering in 1977, defended his Ph.D. in 1980 and D.Sc. in 2014, and worked at the Institute for Automated Control Systems, Moscow Institute of Electronic Engineering, Potsdam University, and Moscow State University. Currently he is a senior researcher at the Institute for Problems in Mechanics, Russian Academy of Sciences. He is the author of seven monographs (V. P. Maslov and V. E. Nazaikinskii, *Asymptotics of Operator and Pseudo-Differential Equations*, Consultants Bureau, New York, 1988; V. Nazaikinskii, B. Sternin, and V. Shatalov, *Contact Geometry and Linear Differential Equations*, Walter de Gruyter, Berlin-New York, 1992; V. Nazaikinskii, B. Sternin, and V. Shatalov, *Methods of Noncommutative Analysis. Theory and Applications*, Walter de Gruyter, Berlin-New York, 1996; V. Nazaikinskii, B.-W. Schulze, and B. Sternin, *Quantization Methods in Differential Equations*, Taylor and Francis, London-New York, 2002; V. Nazaikinskii, A. Savin, B.-W. Schulze, and B. Sternin, *Elliptic Differential Equations on Singular Manifolds*, CRC Press, Boca Raton, 2005; V. Nazaikinskii, A. Savin, and B. Sternin, *Elliptic Theory and Noncommutative Geometry*, Birkhäuser, Basel, 2008; V. Nazaikinskii, B.-W. Schulze, and B. Sternin, *The Localization Problem in Index Theory of Elliptic Operators*, Springer, Basel, 2014) and more than 90 papers on various aspects of noncommutative analysis, asymptotic problems, and elliptic theory.

Address: Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Ave., Bldg 1, 119526 Moscow, Russia

BASIC NOTATION AND REMARKS

Latin Characters

curl $\mathbf{u}$	curl of a vector $\mathbf{u}$, sometimes also denoted by rot $\mathbf{u}$
div $\mathbf{u}$	divergence of a vector $\mathbf{u}$; div $\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$ in the two-dimensional case $\mathbf{u} = (u_1, u_2)$
$\mathscr{E}$	fundamental solution of the Cauchy problem
$\mathscr{E}_{\rm e}$	fundamental solution corresponding to an operator (or fundamental solution of an equation)
grad a	gradient of a scalar a, also denoted by ∇a, where ∇ is the nabla vector differential operator
$\operatorname{Im}[A]$	imaginary part of a complex number A
G	Green function
$\mathbb{R}^n$	n-dimensional Euclidean space, $\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1, \dots, n\}$
$\operatorname{Re}[A]$	real part of a complex number A
$r,\ \varphi,\ z$	cylindrical coordinates, $r = \sqrt{x^2 + y^2}$ with $x = r\cos\varphi$ and $y = r\sin\varphi$
$r,\ \theta,\ \varphi$	spherical coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ with $x = r\sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and $z = r\cos\theta$
t	time ($t \geq 0$)
w	unknown function (dependent variable)
x, y, z	space (Cartesian) coordinates
$x_1, \dots, x_n$	Cartesian coordinates in n-dimensional space
$\mathbf{x}$	n-dimensional vector, $\mathbf{x} = \{x_1, \dots, x_n\}$
$\lvert\mathbf{x}\rvert$	magnitude (length) of n-dimensional vector, $\lvert\mathbf{x}\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
$\mathbf{y}$	n-dimensional vector, $\mathbf{y} = \{y_1, \dots, y_n\}$

Greek Characters

Δ	Laplace operator
Δ_2	two-dimensional Laplace operator, $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
Δ_3	three-dimensional Laplace operator, $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
Δ_n	n-dimensional Laplace operator, $\Delta_n = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$
$\Delta\Delta$	biharmonic operator; $\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ in the two-dimensional case
$\delta(x)$	Dirac delta function; $\displaystyle\int_{-a}^{a} f(y)\delta(x-y)\,dy = f(x)$, where $f(x)$ is any continuous function, $a > 0$, and $-a < x < a$
δ_{nm}	Kronecker delta, $\delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases}$
$\vartheta(x)$	Heaviside unit step function, $\vartheta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$

Brief Notation for Derivatives

Partial derivatives:

$$w_x = \partial_x w = \frac{\partial w}{\partial x}, \quad w_t = \partial_t w = \frac{\partial w}{\partial t}, \quad w_{xx} = \partial_{xx} w = \frac{\partial^2 w}{\partial x^2}, \quad w_{xt} = \partial_{tx} w = \frac{\partial^2 w}{\partial x \partial t},$$

$$w_{tt} = \partial_{tt} w = \frac{\partial^2 w}{\partial t^2}, \quad w_{xxx} = \partial_{xxx} w = \frac{\partial^3 w}{\partial x^3}, \quad w_{xxt} = \partial_{xxt} w = \frac{\partial^3 w}{\partial x^2 \partial t}, \quad \ldots$$

Ordinary derivatives for $f = f(x)$:

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f^{(n)}_x = \frac{d^n f}{dx^n} \quad \text{with} \quad n \ge 4.$$

Special Functions

$\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\bigl(\frac{1}{3}t^3 + xt\bigr)\,dt$	Airy function; $\operatorname{Ai}(x) = \frac{1}{\pi}\bigl(\frac{1}{3}x\bigr)^{1/2} K_{1/3}\bigl(\frac{2}{3}x^{3/2}\bigr)$
$\operatorname{Ce}_{2n+p}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k+p}^{2n+p} \cosh[(2k+p)x]$	even modified Mathieu functions, where $p = 0, 1$; $\operatorname{Ce}_{2n+p}(x,q) = \operatorname{ce}_{2n+p}(ix,q)$
$\operatorname{ce}_{2n}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k}^{2n} \cos 2kx$	even π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n}(q)$ are eigenvalues
$\operatorname{ce}_{2n+1}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k+1}^{2n+1} \cos[(2k+1)x]$	even 2π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n+1}(q)$ are eigenvalues
$D_\nu = D_\nu(x)$	parabolic cylinder function; it satisfies the equation $y'' + \bigl(\nu + \frac{1}{2} - \frac{1}{4}x^2\bigr)y = 0$
$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \exp\bigl(-\xi^2\bigr)\,d\xi$	error function
$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp\bigl(-\xi^2\bigr)\,d\xi$	complementary error function
$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\bigl(e^{-x^2}\bigr)$	Hermite polynomial
$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$	Hankel function of the first kind; $i^2 = -1$
$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$	Hankel function of the second kind
$F(a,b,c;x) = 1 + \sum\limits_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$	hypergeometric function, $(a)_n = a(a+1)\ldots(a+n-1)$
$I_\nu(x) = \sum\limits_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$	modified Bessel function of the first kind
$J_\nu(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$	Bessel function of the first kind
$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}$	modified Bessel function of the second kind

$L_n^s(x) = \frac{1}{n!}x^{-s}e^x\frac{d^n}{dx^n}\left(x^{n+s}e^{-x}\right)$	generalized Laguerre polynomial
$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$	Legendre polynomial
$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$	associated Legendre functions
$\mathrm{Se}_{2n+p}(x,q) = \sum_{k=0}^{\infty} B_{2k+p}^{2n+p}\sinh[(2k+p)x]$	odd modified Mathieu functions, where $p=0,1$; $\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q)$
$\mathrm{se}_{2n}(x,q) = \sum_{k=0}^{\infty} B_{2k}^{2n}\sin 2kx$	odd π-periodic Mathieu functions; these satisfy the equation $y''+(a-2q\cos 2x)y=0$, where $a=b_{2n}(q)$ are eigenvalues
$\mathrm{se}_{2n+1}(x,q) = \sum_{k=0}^{\infty} B_{2k+1}^{2n+1}\sin[(2k+1)x]$	odd 2π-periodic Mathieu functions; these satisfy the equation $y''+(a-2q\cos 2x)y=0$, where $a=b_{2n+1}(q)$ are eigenvalues
$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu)-J_{-\nu}(x)}{\sin(\pi\nu)}$	Bessel function of the second kind
$\gamma(\alpha,x) = \int_0^x e^{-\xi}\xi^{\alpha-1}\,d\xi$	incomplete gamma function
$\Gamma(\alpha) = \int_0^\infty e^{-\xi}\xi^{\alpha-1}\,d\xi$	gamma function
$\Phi(a,b;x) = 1+\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}$	degenerate hypergeometric function, $(a)_n = a(a+1)\ldots(a+n-1)$

Miscellaneous Remarks

1. The previous handbooks by Polyanin (2002) and Polyanin, Zaitsev, and Moussiaux (2002) were extensively used in compiling this book; references to these sources are often omitted.
2. The conventional abbreviations ODE and PDE stand for "ordinary differential equation" and "partial differential equation," respectively.
3. The conventional abbreviations 2D equation and 3D equation stand for "two-dimensional equation" and "three-dimensional equation," respectively.
4. Throughout the book, unless explicitly specified otherwise, all parameters occurring in the equations considered are assumed to be real numbers.
5. The term "exact solution" with regard to linear PDEs and systems of PDEs is used in the following cases:
   - the solution is expressible in terms of elementary functions;
   - the solution is expressible via special functions, in closed form via infinite function series, and/or via definite (indefinite) integrals; the solution may depend on arbitrary functions, which may occur in the equation itself or in the initial and boundary conditions.
6. If a formula or a solution contains derivatives of some functions, then the functions are assumed to be differentiable.
7. If a formula or a solution contains finite or definite integrals, then the integrals are supposed to be convergent.

8. If a formula or a solution contains an expression like $\frac{f(x)}{a-2}$, then the assumption that $a \neq 2$ is implied but often not stated explicitly.
9. Equations are numbered separately within each subsection. In Chapters 1–12, when referring to a particular equation, we use notation like 3.2.1.5, which denotes Eq. 5 in Section 3.2.1.
10. The symbol $\odot$ indicates references to literature sources whenever
    - at least one of the solutions was obtained in the cited source;
    - the cited source provides further information on the equations in question and their solutions.
11. The symbol ► marks the beginning of a small section; such sections are referred to as paragraphs.
12. The symbol $\rightrightarrows$ stands for uniform convergence.

Part I

Exact Solutions

Chapter 1

First-Order Equations with Two Independent Variables

1.1 Equations of the Form $f(x,y)\frac{\partial w}{\partial x} + g(x,y)\frac{\partial w}{\partial y} = 0$

◆ *For brevity, often only a principal integral*

$$\Xi = \Xi(x,y)$$

of an equation will be presented in Section 1.1. The general solution of the equation is given by

$$w = \Phi(\Xi),$$

where $\Phi = \Phi(\Xi)$ *is an arbitrary function.*

1.1.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x and y.**

1. $a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = 0.$

General solution: $w = \Phi(bx - ay)$, where Φ is an arbitrary function.

⊙ *Literature*: E. Kamke (1965).

2. $a\frac{\partial w}{\partial x} + (bx + c)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \frac{1}{2}bx^2 + cx - ay.$

3. $\frac{\partial w}{\partial x} + (ax + by + c)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = (abx + b^2y + a + bc)e^{-bx}.$

4. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}=0.$

For $a=b$, this is a *conoid equation*. Principal integral: $\Xi=|x|^b|y|^{-a}$.

⊙ *Literature*: E. Kamke (1965).

5. $\displaystyle ay\frac{\partial w}{\partial x}+bx\frac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=bx^2-ay^2$.

⊙ *Literature*: E. Kamke (1965).

6. $\displaystyle y\frac{\partial w}{\partial x}+(y+a)\frac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=x-y+a\ln|y+a|$.

7. $\displaystyle (ay+bx+c)\frac{\partial w}{\partial x}-(by+kx+s)\frac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=ay^2+kx^2+2(bxy+cy+sx)$.

8. $\displaystyle (a_1x+b_1y+c_1)\frac{\partial w}{\partial x}+(a_2x+b_2y+c_2)\frac{\partial w}{\partial y}=0.$

The principal integral is determined by solutions of the following auxiliary system of algebraic equations for the parameters s, λ, μ, α, β, and γ:

$$(a_1-s)(b_2-s)=a_2b_1, \tag{1}$$
$$a_1\lambda+a_2\mu=s\lambda, \qquad b_1\lambda+b_2\mu=s\mu, \tag{2}$$
$$c_1\alpha+c_2\beta-s\gamma=c_1\lambda+c_2\mu, \tag{3}$$
$$(a_1-s)\alpha+a_2\beta=\lambda s, \quad b_1\alpha+(b_2-s)\beta=\mu s. \tag{4}$$

Case 1: $(a_1-b_2)^2+4a_2b_1\neq 0$. Equation (1) has two different roots s_1 and s_2. To these roots there correspond two sets of solutions, λ_1, μ_1 and λ_2, μ_2, of system (2).

1.1. If $a_1b_2-a_2b_1\neq 0$, then $s_1\neq 0$ and $s_2\neq 0$. Hence the principal integral has the form

$$\Xi=\frac{|s_1(\lambda_1x+\mu_1y)+\lambda_1c_1+\mu_1c_2|^{s_2}}{|s_2(\lambda_2x+\mu_2y)+\lambda_2c_1+\mu_2c_2|^{s_1}}.$$

1.2. If $a_1b_2-a_2b_1=0$, then $s_1=s=a_1+b_2$ and $s_2=0$.

Principal integral for $\lambda_2c_1+\mu_2c_2\neq 0$:

$$\Xi=s\frac{\lambda_2x+\mu_2y}{\lambda_2c_1+\mu_2c_2}-\ln|s_1(\lambda_1x+\mu_1y)+\lambda_1c_1+\mu_1c_2|.$$

Principal integral for $\lambda_2c_1+\mu_2c_2=0$:

$$\Xi=\lambda_2x+\mu_2y.$$

Case 2: $(a_1-b_2)^2+4a_2b_1=0$. Equation (1) has the double root $s=\frac{1}{2}(a_1+b_2)$. System (2) gives λ and μ not equal to zero simultaneously.

2.1. If $s\neq 0$, then we find γ from (3) and take nonzero α and β that satisfy relations (4). This leads to the principal integral

$$\Xi=\ln|s(\lambda x+\mu y)+c_1\lambda+c_2\mu|-\frac{s(\alpha x+\beta y+\gamma)}{s(\lambda x+\mu y)+c_1\lambda+c_2\mu}.$$

2.2. If $s=0$, then $b_2=-a_1$. We have

$$\Xi=a_2x^2-2a_1xy-b_1y^2+2c_2x-2c_1y.$$

⊙ *Literature*: E. Kamke (1965).

► **Coefficients of equations are quadratic in x and y.**

9. $\dfrac{\partial w}{\partial x}+(ax^2+bx+c)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx-y.$

10. $\dfrac{\partial w}{\partial x}+(ay^2+by+c)\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $4ac-b^2>0$:

$$\Xi=x-\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ay+b}{\sqrt{4ac-b^2}}.$$

2°. Principal integral for $4ac-b^2<0$:

$$\Xi=x-\frac{2}{\sqrt{b^2-4ac}}\ln\left|\frac{2ay+b-\sqrt{b^2-4ac}}{2ay+b+\sqrt{b^2-4ac}}\right|.$$

11. $\dfrac{\partial w}{\partial x}+(ay+bx^2+cx)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.7.1 with $f(x)=a$ and $g(x)=bx^2+cx$.

12. $\dfrac{\partial w}{\partial x}+(axy+bx^2+cx+ky+s)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=y\exp\bigl(-\frac{1}{2}ax^2-kx\bigr)-\displaystyle\int(bx^2+cx+s)\exp\bigl(-\tfrac{1}{2}ax^2-kx\bigr)\,dx.$

13. $\dfrac{\partial w}{\partial x}+(y^2-a^2x^2+3a)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{\exp(ax^2)}{x(xy-ax^2+1)}+\displaystyle\int\exp(ax^2)\frac{dx}{x^2}.$

14. $\dfrac{\partial w}{\partial x}+(y^2-a^2x^2+a)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.1.59 with $n=1$.

15. $\dfrac{\partial w}{\partial x} + (y^2 + axy + a)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.1.60 with $n = 1$.

16. $\dfrac{\partial w}{\partial x} + (y^2 + axy - abx - b^2)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.1.61 with $n = 1$.

17. $\dfrac{\partial w}{\partial x} + k(ax + by + c)^2\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.8.6 with $f(z) = kz^2$.

18. $x\dfrac{\partial w}{\partial x} + (ay^2 + cx^2 + y)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.1.75 with $b = 1$.

19. $x\dfrac{\partial w}{\partial x} + (ay^2 + bxy + cx^2 + y)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.1.76 with $n = 1$.

20. $(ax + c)\dfrac{\partial w}{\partial x} + \bigl[\alpha(ay + bx)^2 + \beta(ay + bx) - bx + \gamma\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \ln|ax + c| - \int \frac{dv}{\alpha v^2 + \beta v + \gamma + bc/a}, \qquad v = ay + bx.$$

21. $ax^2\dfrac{\partial w}{\partial x} + by^2\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{by} - \dfrac{1}{ax}.$

22. $(ax^2 + b)\dfrac{\partial w}{\partial x} - \bigl[y^2 - 2xy + (1 - a)x^2 - b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = -\displaystyle\int \frac{dx}{ax^2 + b} + \frac{1}{y - x}.$

23. $(a_1x^2 + b_1x + c_1)\dfrac{\partial w}{\partial x} + (a_2y^2 + b_2y + c_2)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \displaystyle\int \frac{dx}{a_1x^2 + b_1x + c_1} - \int \frac{dy}{a_2y^2 + b_2y + c_2}.$

24. $(x - a)(x - b)\dfrac{\partial w}{\partial x} - \bigl[y^2 + k(y + x - a)(y + x - b)\bigr]\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $a \neq b$:

$$\Xi = \frac{y + k(y + x - a)}{y + k(y + x - b)}\left(\frac{x - a}{x - b}\right)^k, \qquad k \neq 0, \quad k \neq -1.$$

2°. Principal integral for $a = b$:

$$\Xi = \frac{(x - a) + \bigl[y + k(y + x - a)\bigr]}{\bigl[y + k(y + x - a)\bigr](x - a)}, \qquad k \neq 0, \quad k \neq -1.$$

25. $\boldsymbol{(a_1y^2 + b_1y + c_1)\frac{\partial w}{\partial x} + (a_2x^2 + b_2x + c_2)\frac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \frac{1}{3}a_1y^3 + \frac{1}{2}b_1y^2 + c_1y - \frac{1}{3}a_2x^3 - \frac{1}{2}b_2x^2 - c_2x.$

26. $\boldsymbol{y(ax+b)\frac{\partial w}{\partial x} + (ay^2 - cx)\frac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \dfrac{(ax+b)^2}{cx^2+by^2}.$

27. $\boldsymbol{(ay^2 + bx)\frac{\partial w}{\partial x} - (cx^2 + by)\frac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \frac{1}{3}ay^3 + \frac{1}{3}cx^3 + bxy.$

28. $\boldsymbol{(ay^2 + bx^2)\frac{\partial w}{\partial x} + 2bx\frac{\partial w}{\partial y} = 0.}$

This is a special case of equation 1.1.8.2 with $f(x) = bx^2$ and $g(y) = ay^2$.

29. $\boldsymbol{(ay^2 + bx^2)\frac{\partial w}{\partial x} + 2bxy\frac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \dfrac{bx^2 - ay^2}{y}.$

30. $\boldsymbol{(ay^2 + x^2)\frac{\partial w}{\partial x} + (bx^2 + c - 2xy)\frac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = ay^3 - bx^3 + 3(x^2y - cx).$

31. $\boldsymbol{(Ay^2 + Bx^2 - a^2B)\frac{\partial w}{\partial x} + (Cy^2 + 2Bxy)\frac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = (x-a)E + 2aB\int \frac{E\,dv}{v(Av^2 - Cv - B)}, \qquad v = \frac{y}{x-a},$$

where $E = \exp\left[\displaystyle\int \frac{(Av^2 + B)\,dv}{v(Av^2 - Cv - B)}\right].$

32. $\boldsymbol{(ay^2 + bx^2 + cy)\frac{\partial w}{\partial x} + 2bx\frac{\partial w}{\partial y} = 0.}$

This is a special case of equation 1.1.8.2 with $f(x) = bx^2$ and $g(y) = ay^2 + cy$.

33. $\boldsymbol{(Axy + Bx^2 + kx)\frac{\partial w}{\partial x} + (Dy^2 + Exy + Fx^2 + ky)\frac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = xV + k\int \frac{V\,dv}{(A-D)v^2 + (B-E)v - F}, \qquad v = \frac{y}{x},$$

where $V = \exp\left[\displaystyle\int \frac{(Av + B)\,dv}{(A-D)v^2 + (B-E)v - F}\right].$

34. $\boldsymbol{(Axy + Aky + Bx^2 + Bkx)\dfrac{\partial w}{\partial x} + \big[Cy^2 + Dxy + k(D-B)y\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = (x+k)E + kB\int \frac{E\,dv}{v\big[(C-A)v + D - B\big]}, \qquad v = \frac{y}{x+k},$$

where $E = \exp\left[\displaystyle\int \frac{(Av+B)\,dv}{v\big[(A-C)v + B - D\big]}\right]$.

35. $\boldsymbol{(Ay^2 + Bxy + Cx^2 + kx)\dfrac{\partial w}{\partial x} + (Dy^2 + Exy + Fx^2 + ky)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = xV + k\int \frac{V\,dv}{Av^3 + (B-D)v^2 + (C-E)v - F}, \qquad v = \frac{y}{x},$$

where $V = \exp\left[\displaystyle\int \frac{(Av^2 + Bv + C)\,dv}{Av^3 + (B-D)v^2 + (C-E)v - F}\right]$.

36. $\boldsymbol{(Ay^2 + Bxy + Cx^2)\dfrac{\partial w}{\partial x} + (Dy^2 + Exy + Fx^2)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{(Av^2 + Bv + C)\,dv}{Av^3 + (B-D)v^2 + (C-E)v - F} + \ln|x|, \qquad v = \frac{y}{x}.$$

37. $\boldsymbol{(Ay^2 + 2Bxy + Dx^2 + a)\dfrac{\partial w}{\partial x} - (By^2 + 2Dxy - Ex^2 - b)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = Ay^3 - Ex^3 + 3(Bxy^2 + Dx^2y + ay - bx)$.

38. $\boldsymbol{(y^2 - 2xy + x^2 + ay)\dfrac{\partial w}{\partial x} + ay\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \dfrac{a}{x-y} + \ln|y|$.

39. $\boldsymbol{(xf_1 - f_2)\dfrac{\partial w}{\partial x} + (yf_1 - f_3)\dfrac{\partial w}{\partial y} = 0, \qquad f_n = a_n + b_nx + c_ny.}$

Hesse's equation. The introduction of the homogeneous coordinates $x = \xi_2/\xi_1$, $y = \xi_3/\xi_1$ leads to an equation with three independent variables for $w = w(\xi_1, \xi_2, \xi_3)$:

$$g_1\frac{\partial w}{\partial \xi_1} + g_2\frac{\partial w}{\partial \xi_2} + g_3\frac{\partial w}{\partial \xi_3} = 0,$$

where $g_n = a_n\xi_1 + b_n\xi_2 + c_n\xi_3$ $(n = 1, 2, 3)$. See 2.1.1.21 for the solution of this equation.

⊙ *Literature*: E. Kamke (1965).

► **Coefficients of equations contain integer powers of x and y.**

40. $\displaystyle \frac{\partial w}{\partial x} + (y^2 + bx^2y - a^2 - abx^2)\frac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.3 with $f(x) = bx^2$.

41. $\displaystyle \frac{\partial w}{\partial x} + (ax^2y + bx^3 + c)\frac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.1 with $f(x) = ax^2$ and $g(x) = bx^3 + c$.

42. $\displaystyle \frac{\partial w}{\partial x} + (ax^2y + by^3)\frac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.2 with $k = 3$, $f(x) = ax^2$, and $g(x) = b$.

43. $\displaystyle \frac{\partial w}{\partial x} + (axy + b)y^2\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v(av^2 + bv + 1)} - \ln|x|, \qquad v = xy.$$

44. $\displaystyle \frac{\partial w}{\partial x} + A(ax + by + c)^3\frac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.8.6 with $f(z) = Az^3$.

45. $\displaystyle x\frac{\partial w}{\partial x} + \bigl[ax^4y^3 + (bx^2 - 1)y + cx\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^3 + bv + c} - \frac{x^2}{2}, \qquad v = xy.$$

46. $\displaystyle x^2\frac{\partial w}{\partial x} + (ax^2y^2 + bxy + c)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + (b+1)v + c} - \ln|x|, \qquad v = xy.$$

47. $\displaystyle (ax^2y + b)\frac{\partial w}{\partial x} - (axy^2 + c)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \frac{1}{2}ax^2y^2 + by + cx$.

48. $\displaystyle (ax + by^3)\frac{\partial w}{\partial x} - (cx^3 + ay)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = axy + \frac{1}{4}by^4 + \frac{1}{4}cx^4$.

◆ *See also equations 1.1.1.56–1.1.1.111 for integer values of exponents.*

► **Coefficients of equations contain fractional powers.**

49. $\dfrac{\partial w}{\partial x} + (a\sqrt{x}\,y + b)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y\exp\bigl(-\frac{2}{3}ax^{3/2}\bigr) - b\int \exp\bigl(-\frac{2}{3}ax^{3/2}\bigr)\,dx.$

50. $\dfrac{\partial w}{\partial x} + (a\sqrt{x}\,y + b\sqrt{y}\,)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.2 with $k = \frac{1}{2}$, $f(x) = a\sqrt{x}$, and $g(x) = b$.

51. $\dfrac{\partial w}{\partial x} + (a\sqrt{x}\,y + bx\sqrt{y}\,)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.2 with $k = \frac{1}{2}$, $f(x) = a\sqrt{x}$, and $g(x) = bx$.

52. $\dfrac{\partial w}{\partial x} + A\sqrt{ax + by + c}\,\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.8.6 with $f(z) = A\sqrt{z}$.

53. $x\dfrac{\partial w}{\partial x} + \bigl(ay + b\sqrt{y^2 + cx^2}\,\bigr)\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $a \neq 1$:

$$\Xi = \ln|x| - \int \frac{du}{(a-1)u + b\sqrt{u^2 + c}}, \qquad u = \frac{y}{x}.$$

2°. Principal integral for $a = 1$:

$$\Xi = |x|^{-b-1}\bigl(y + \sqrt{y^2 + cx^2}\,\bigr).$$

54. $(ax + b\sqrt{y}\,)\dfrac{\partial w}{\partial x} - (c\sqrt{x} + ay)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = axy + \frac{2}{3}by^{3/2} + \frac{2}{3}cx^{3/2}.$

55. $\sqrt{f(x)}\,\dfrac{\partial w}{\partial x} + \sqrt{f(y)}\,\dfrac{\partial w}{\partial y} = 0, \qquad f(t) = \sum_{\nu=0}^{4} a_\nu t^\nu.$

Principal integral: $\Xi = \left[\dfrac{\sqrt{f(x)} + \sqrt{f(y)}}{x - y}\right]^2 - a_4(x+y)^2 - a_3(x+y).$

⊙ *Literature*: E. Kamke (1965).

◆ *See also equations in 1.1.1.56–1.1.1.111 for fractional values of exponents.*

► **Coefficients of equations contain arbitrary powers of x and y.**

56. $\dfrac{\partial w}{\partial x} + (ay + bx^k)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = ye^{-ax} - b\displaystyle\int x^k e^{-ax}\,dx.$

57. $\dfrac{\partial w}{\partial x} + (ax^k y + bx^n)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y\exp\Bigl(-\dfrac{a}{k+1}x^{k+1}\Bigr) - b\displaystyle\int x^n \exp\Bigl(-\dfrac{a}{k+1}x^{k+1}\Bigr)\,dx.$

58. $\dfrac{\partial w}{\partial x} + (ay^2 + bx^n)\dfrac{\partial w}{\partial y} = 0.$

The principal integral $\Xi(x,y)$ can be found as the general solution $\Xi(x,y) = C$ of the special Riccati equation $y'_x = ay^2 + bx^n$, which is considered in the handbooks by G. M. Murphy (1960), E. Kamke (1977), and A. D. Polyanin and V. F. Zaitsev (2003).

59. $\dfrac{\partial w}{\partial x} + (y^2 + anx^{n-1} - a^2x^{2n})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n \neq -1$:

$$\Xi = \frac{E}{y - ax^n} + \int E\,dx, \qquad E = \exp\Bigl(\frac{2a}{n+1}x^{n+1}\Bigr).$$

2°. Principal integral for $n = -1$ and $a \neq -\frac{1}{2}$:

$$\Xi = \frac{xy + a + 1}{(2a+1)(xy - a)}x^{2a+1}.$$

3°. Principal integral for $n = -1$ and $a = -\frac{1}{2}$:

$$\Xi = \frac{2}{2xy + 1} + \ln|x|.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

60. $\dfrac{\partial w}{\partial x} + (y^2 + ax^n y + ax^{n-1})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n \neq -1$:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\Bigl(\frac{a}{n+1}x^{n+1}\Bigr).$$

2°. Principal integral for $n = -1$ and $a \neq 1$:

$$\Xi = \frac{xy + a}{(a-1)(xy+1)}x^{a-1}.$$

3°. Principal integral for $n = -1$ and $a = 1$:

$$\Xi = \frac{1}{xy + 1} + \ln|x|.$$

61. $\dfrac{\partial w}{\partial x}+(y^2+ax^ny-abx^n-b^2)\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $n\neq -1$:

$$\Xi=\frac{1}{y-b}\exp\left(2bx+\frac{a}{n+1}x^{n+1}\right)+\int\exp\left(2bx+\frac{a}{n+1}x^{n+1}\right)dx.$$

2°. Principal integral for $n=-1$:

$$\Xi=\frac{x^ae^{2bx}}{y-b}+\int x^ae^{2bx}\,dx.$$

62. $\dfrac{\partial w}{\partial x}+(ax^ny^2+bx^{-n-2})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\ln|x|-\int\frac{dv}{av^2+(n+1)v+b},\qquad v=x^{n+1}y.$$

63. $\dfrac{\partial w}{\partial x}+(ax^ny^2+bmx^{m-1}-ab^2x^{n+2m})\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $n+m\neq -1$:

$$\Xi=\frac{E}{y-bx^m}+a\int x^nE\,dx,\qquad E=\exp\left(\frac{2ab}{n+m+1}x^{n+m+1}\right).$$

2°. Principal integral for $n+m=-1$ and $m\neq 2ab$:

$$\Xi=\frac{x^{2ab}}{y-bx^m}+\frac{a}{2ab-m}x^{2ab-m}.$$

3°. Principal integral for $n+m=-1$ and $m=2ab$:

$$\Xi=\frac{x^m}{y-bx^m}+a\ln x.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

64. $\dfrac{\partial w}{\partial x}-\big[(n+1)x^ny^2-ax^{n+m+1}y+ax^m\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{x^{-n-1}E}{x^{n+1}y-1}-(n+1)\int x^{-n-2}E\,dx,\qquad E=\exp\left(\frac{a}{n+m+2}x^{n+m+2}\right).$$

65. $\dfrac{\partial w}{\partial x} + (ax^n y^2 + bx^m y + bcx^m - ac^2x^n)\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $m, n \neq -1$:

$$\Xi = \frac{E}{y+c} + a\int x^n E\,dx, \qquad E = \exp\biggl(\frac{b}{m+1}x^{m+1} - \frac{2ac}{n+1}x^{n+1}\biggr).$$

2°. Principal integral for $n = -1$:

$$\Xi = \frac{x^{-2ac}}{y+c}\exp\biggl(\frac{b}{m+1}x^{m+1}\biggr) + a\int x^{-2ac-1}\exp\biggl(\frac{b}{m+1}x^{m+1}\biggr)\,dx.$$

3°. Principal integral for $m = -1$:

$$\Xi = \frac{x^b}{y+c}\exp\biggl(-\frac{2ac}{n+1}x^{n+1}\biggr) + a\int x^{n+b}\exp\biggl(-\frac{2ac}{n+1}x^{n+1}\biggr)\,dx.$$

66. $\dfrac{\partial w}{\partial x} + \bigl[ax^n y^2 - ax^n(bx^m + c)y + bmx^{m-1}\bigr]\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n \neq -1$ and $m + n \neq -1$:

$$\Xi = \frac{E}{y - bx^m - c} + a\int x^n E\,dx, \qquad E = \exp\biggl(\frac{abx^{n+m+1}}{n+m+1} + \frac{acx^{n+1}}{n+1}\biggr).$$

2°. Principal integral for $n = -1$ and $m \neq 0$:

$$\Xi = \frac{x^{ac}}{y - bx^m - c}\exp\biggl(\frac{ab}{m}x^m\biggr) + a\int x^{ac-1}\exp\biggl(\frac{ab}{m}x^m\biggr)\,dx.$$

3°. Principal integral for $n \neq -1$ and $m = -1 - n$:

$$\Xi = \frac{x^{ab}}{y - bx^{-n-1} - c}\exp\biggl(\frac{ac}{n+1}x^{n+1}\biggr) + a\int x^{ab+n}\exp\biggl(\frac{ac}{n+1}x^{n+1}\biggr)dx.$$

67. $\dfrac{\partial w}{\partial x} - \bigl[anx^{n-1}y^2 - cx^m(ax^n + b) + cx^m\bigr]\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $m \neq -1$ and $m + n \neq -1$:

$$\Xi = \frac{E}{(ax^n+b)\bigl[(ax^n+b)y - 1\bigr]} - an\int \frac{x^{n-1}E}{(ax^n+b)^2}\,dx,$$
$$E = \exp\biggl(\frac{acx^{m+n+1}}{m+n+1} + \frac{bcx^{m+1}}{m+1}\biggr).$$

2°. Principal integral for $m = -1$ and $n \neq 0$:

$$\Xi = \frac{x^{bc}}{(ax^n+b)\bigl[(ax^n+b)y - 1\bigr]}\exp\biggl(\frac{ac}{n}x^n\biggr) - an\int \exp\biggl(\frac{ac}{n}x^n\biggr)\frac{x^{bc+n-1}}{(ax^n+b)^2}\,dx.$$

3°. Principal integral for $n \neq -1$ and $m = -1 - n$:

$$\Xi = \frac{x^{ac}}{(ax^n+b)\bigl[(ax^n+b)y-1\bigr]}\exp\biggl(-\frac{bc}{n}x^{-n}\biggr) - an\int \frac{x^{ac+n-1}}{(ax^n+b)^2}\exp\biggl(-\frac{bc}{n}x^{-n}\biggr)\,dx.$$

68. $\dfrac{\partial w}{\partial x} + (ax^n y^2 + bx^m y + ckx^{k-1} - bcx^{m+k} - ac^2 x^{n+2k})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $m \neq -1$ and $n+k \neq -1$:

$$\Xi = \frac{E}{y - cx^k} + a\int x^n E\,dx, \qquad E = \exp\left(\frac{2ac}{n+k+1}x^{n+k+1} + \frac{b}{m+1}x^{m+1}\right).$$

2°. Principal integral for $m = -1$ and $n+k \neq -1$:

$$\Xi = \frac{x^b E}{y - cx^k} + a\int x^{b+n} E\,dx, \qquad E = \exp\left(\frac{2ac}{n+k+1}x^{n+k+1}\right).$$

3°. Principal integral for $m \neq -1$ and $n+k = -1$:

$$\Xi = \frac{x^{2ac}}{y - cx^k}\exp\left(\frac{b}{m+1}x^{m+1}\right) + a\int x^{2ac+n}\exp\left(\frac{b}{m+1}x^{m+1}\right)dx.$$

4°. Principal integral for $m = -1$, $n+k = -1$, and $2ac + b \neq k$:

$$\Xi = \frac{ay + (ac+b-k)x^k}{(2ac+b-k)(y - cx^k)}x^{2ac+b-k}.$$

5°. Principal integral for $m = -1$, $n+k = -1$, and $2ac + b = k$:

$$\Xi = \frac{x^k}{y - cx^k} + a\ln x.$$

69. $\dfrac{\partial w}{\partial x} + (ax^{2n+1} y^3 + bx^{-n-2})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^3 + (n+1)v + b} - \ln|x|, \qquad v = x^{n+1}y.$$

70. $\dfrac{\partial w}{\partial x} + (ax^n y^3 + 3abx^{n+m} y^2 - bmx^{m-1} - 2ab^3 x^{n+3m})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n + 2m \neq -1$:

$$\Xi = \frac{E}{(y + bx^m)^2} + 2a\int x^n E\,dx, \qquad E = \exp\left(-\frac{6ab^2}{n+2m+1}x^{n+2m+1}\right).$$

2°. Principal integral for $n = -2m - 1$:

$$\Xi = \frac{x^{-6ab^2}}{(y + bx^m)^2} + \frac{a}{3ab^2 + m}x^{-2(3ab^2+m)}.$$

3°. Principal integral for $n = -2m - 1$ and $m = -3ab^2$:

$$\Xi = \frac{x^{2m}}{(y + bx^m)^2} + 2a\ln|x|.$$

71. $\dfrac{\partial w}{\partial x} + (ax^n y^3 + 3abx^{n+m} y^2 + cx^k y - 2ab^3 x^{n+3m} + bcx^{m+k} - bmx^{m-1})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $k \neq -1$ and $n + 2m \neq -1$:

$$\Xi = \frac{E}{(y + bx^m)^2} + 2a\int x^n E\,dx, \qquad E = \exp\biggl(\frac{2c}{k+1}x^{k+1} - \frac{6ab^2}{n+2m+1}x^{n+2m+1}\biggr).$$

2°. Principal integral for $k = -1$ and $n + 2m \neq -1$:

$$\Xi = \frac{x^{2c}E_2}{(y + bx^m)^2} + 2a\int x^{n+2c}E_2\,dx, \qquad E_2 = \exp\biggl(-\frac{6ab^2}{n+2m+1}x^{n+2m+1}\biggr).$$

3°. Principal integral for $k \neq -1$ and $n + 2m = -1$:

$$\Xi = \frac{x^{-6ab^2}E_1}{(y + bx^m)^2} + 2a\int x^{n-6ab^2}E_1\,dx, \qquad E_1 = \exp\biggl(\frac{2c}{k+1}x^{k+1}\biggr).$$

4°. Principal integral for $k = n + 2m = -1$ and $c \neq 3ab^2 + m$:

$$\Xi = \frac{x^{2(c-3ab^2)}}{(y + bx^m)^2} + \frac{a}{c - 3ab^2 - m}x^{2(c-3ab^2-m)}.$$

5°. Principal integral for $k = n + 2m = -1$ and $c = 3ab^2 + m$:

$$\Xi = \frac{x^{2m}}{(y + bx^m)^2} + 2a\ln|x|.$$

72. $\dfrac{\partial w}{\partial x} + \Bigl(ay^n + bx^{\frac{n}{1-n}}\Bigr)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{av^n + \frac{1}{n-1}v + b} - \ln|x|, \qquad v = yx^{\frac{1}{n-1}}.$$

73. $\dfrac{\partial w}{\partial x} + (ax^{m-n-mn}y^n + bx^m)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \ln|x| - \int\frac{dv}{av^n - (m+1)v + b}, \qquad v = yx^{-m-1}.$$

74. $\dfrac{\partial w}{\partial x} + (ax^n y^k + bx^m y)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.2 with $f(x) = bx^m$ and $g(x) = ax^n$.

75. $\boldsymbol{x\dfrac{\partial w}{\partial x} + (ay^2 + by + cx^{2b})\dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $ac > 0$:

$$\Xi = \frac{b}{\sqrt{ac}} \arctan\left(\sqrt{\frac{a}{c}}\,x^{-b}y\right) - x^b.$$

2°. Principal integral for $ac < 0$:

$$\Xi = \frac{b}{2\sqrt{-ac}} \ln \frac{ax^{-b}y - \sqrt{-ac}}{ax^{-b}y + \sqrt{-ac}} - x^b.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

76. $\boldsymbol{x\dfrac{\partial w}{\partial x} + \bigl[ay^2 + (n + bx^n)y + cx^{2n}\bigr]\dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $n \neq 0$:

$$\Xi = \int \frac{dv}{av^2 + bv + c} - \frac{1}{n}x^n, \qquad v = x^{-n}y.$$

2°. Principal integral for $n = 0$:

$$\Xi = \int \frac{dy}{ay^2 + by + c} - \ln|x|.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

77. $\boldsymbol{x\dfrac{\partial w}{\partial x} + (ax^n y^2 + by + cx^{-n})\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + (b+n)v + c} - \ln x, \qquad v = x^n y.$$

78. $\boldsymbol{x\dfrac{\partial w}{\partial x} + (ax^n y^2 + my - ab^2 x^{n+2m})\dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $m + n \neq 0$:

$$\Xi = \frac{x^m E}{y - bx^m} + a\int x^{m+n-1}E\,dx, \qquad E = \exp\left(\frac{2ab}{m+n}x^{m+n}\right).$$

2°. Principal integral for $m = -n$:

$$\Xi = \frac{x^{2ab}(y + bx^m)}{2b(y - bx^m)}.$$

79. $\boldsymbol{x\dfrac{\partial w}{\partial x} + \bigl[x^{2n}y^2 + (m-n)y + x^{2m}\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \arctan(x^{n-m}y) - \dfrac{x^{n+m}}{n+m}.$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

80. $x\dfrac{\partial w}{\partial x}+\big[ax^{2n}y^2+(bx^n-n)y+c\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=n\int\frac{dv}{av^2+bv+c}-x^n,\qquad v=x^ny.$$

81. $x\dfrac{\partial w}{\partial x}+\big[ax^{2n+m}y^2+(bx^{n+m}-n)y+cx^m\big]\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $n+m\neq0$:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\frac{x^{n+m}}{n+m},\qquad v=x^ny.$$

2°. Principal integral for $n+m=0$:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\ln x,\qquad v=x^ny.$$

82. $x\dfrac{\partial w}{\partial x}+(ay^3+3abx^ny^2-bnx^n-2ab^3x^{3n})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{(y+bx^n)^2}+2a\int x^{-1}E\,dx,\qquad E=\exp\biggl(-\frac{3ab^2}{n}x^{2n}\biggr).$$

83. $x\dfrac{\partial w}{\partial x}+\big[ax^{2n+1}y^3+(bx-n)y+cx^{1-n}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^3+bv+c}-x,\qquad v=x^ny.$$

84. $x\dfrac{\partial w}{\partial x}+\big[ax^{n+2}y^3+(bx^n-1)y+cx^{n-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^3+bv+c}-\frac{1}{n}x^n,\qquad v=xy.$$

85. $x\dfrac{\partial w}{\partial x}+(y+ax^{n-m}y^m+bx^{n-k}y^k)\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $n\neq1$:

$$\Xi=\int\frac{dv}{av^m+bv^k}-\frac{x^{n-1}}{n-1},\qquad v=\frac{y}{x}.$$

2°. Principal integral for $n=1$:

$$\Xi=\int\frac{dv}{av^m+bv^k}-\ln|x|,\qquad v=\frac{y}{x}.$$

86. $\boldsymbol{y\dfrac{\partial w}{\partial x} + \big\{x^{n-1}[(1+2n)x+an]y - nx^{2n}(x+a)\big\}\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = (x^{n+1}+ax^n-y)^{-1/n} + \int \frac{dv}{a-v^{-n}}, \qquad v = x(x^{n+1}+ax^n-y)^{-1/n}.$$

87. $\boldsymbol{y\dfrac{\partial w}{\partial x} + \big\{[a(2n+k)x^k+b]x^{n-1}y - (a^2nx^{2k}+abx^k-c)x^{2n-1}\big\}\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = x^{-k}E - ak\int \frac{E\,dv}{nv^2-bv-c}, \qquad v = x^{-n}y - ax^k,$$

where $E = \exp\biggl(-k\displaystyle\int \frac{v\,dv}{nv^2-bv-c}\biggr)$.

88. $\boldsymbol{x(2axy+b)\dfrac{\partial w}{\partial x} - \big[a(m+3)xy^2+b(m+2)y-cx^m\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = cx^{m+2}\big[cx^m - 2(m+1)y(axy+b)\big]$.

89. $\boldsymbol{x^2(2axy+b)\dfrac{\partial w}{\partial x} - (4ax^2y^2+3bxy-cx^2-k)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = (cx^2+k)^2 - 4cx^3y(axy+b)$.

90. $\boldsymbol{ax^m\dfrac{\partial w}{\partial x} + by^n\dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $m \neq 1$ and $n \neq 1$:

$$\Xi = b(n-1)x^{1-m} - a(m-1)y^{1-n}.$$

2°. Principal integral for $m = 1$ and $n \neq 1$:

$$\Xi = b\ln|x| + \frac{a}{n-1}y^{1-n}.$$

3°. Principal integral for $m \neq 1$ and $n = 1$:

$$\Xi = \frac{b}{m-1}x^{1-m} + a\ln|y|.$$

4°. Principal integral for $m = n = 1$:

$$\Xi = b\ln|x| - a\ln|y|.$$

⊙ *Literature*: E. Kamke (1965).

91. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x}+(by+cx^m)\dfrac{\partial w}{\partial y}=0.}$

1°. Principal integral for $n\neq 1$:

$$\Xi=e^{-F}y-\frac{c}{a}\int e^{-F}x^{m-n}\,dx,\qquad F=\frac{b}{a(1-n)}x^{1-n}.$$

2°. Principal integral for $n=1$ and $am\neq b$:

$$\Xi=x^{-b/a}y-\frac{c}{am-b}x^{\frac{am-b}{a}}.$$

3°. Principal integral for $n=1$ and $am=b$:

$$\Xi=x^{-b/a}y-\frac{c}{a}\ln|x|.$$

92. $\boldsymbol{ax^k\dfrac{\partial w}{\partial x}+(y^n+bx^my)\dfrac{\partial w}{\partial y}=0,\qquad n\neq 1.}$

1°. Principal integral for $m\neq k-1$:

$$\Xi=e^{-F}y^{1-n}+\frac{n-1}{a}\int e^{-F}x^{-k}\,dx,\qquad F=\frac{(1-n)b}{a(m+k-1)}x^{m-k+1}.$$

2°. Principal integral for $m=k-1$ and $(n-1)b\neq ma$:

$$\Xi=x^{\frac{(n-1)b}{a}}y^{1-n}+\frac{n-1}{(n-1)b-ma}x^{\frac{(n-1)b-ma}{a}}.$$

3°. Principal integral for $m=k-1$ and $(n-1)b=ma$:

$$\Xi=x^{\frac{(n-1)b}{a}}y^{1-n}+\frac{n-1}{a}\ln|x|.$$

93. $\boldsymbol{x(ax^k+b)\dfrac{\partial w}{\partial x}+\big[\alpha x^ny^2+(\beta-anx^k)y+\gamma x^{-n}\big]\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=x^{-k}E+ka\int\frac{E\,dv}{\alpha v^2+(\beta+bn)v+\gamma},\qquad v=x^ny,$$

where $E=\exp\left[kb\displaystyle\int\frac{dv}{\alpha v^2+(\beta+bn)v+\gamma}\right]$.

94. $\boldsymbol{(y+Ax^n+a)\dfrac{\partial w}{\partial x}-(nAx^{n-1}y+kx^m+b)\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=y^2+\dfrac{2k}{m+1}x^{m+1}+2(Ax^ny+ay+bx)$.

95. $\boldsymbol{(y+ax^{n+1}+bx^n)\dfrac{\partial w}{\partial x}+(anx^n+cx^{n-1})y\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi = x^{-1}E - a\int \frac{E\,dv}{nv^2-(bn+c)v+bc}, \qquad v = x^{-n}y+b,$$

where $E=\exp\biggl[-\displaystyle\int\frac{v\,dv}{nv^2-(bn+c)v+bc}\biggr]$.

96. $\boldsymbol{x(2ax^ny+b)\dfrac{\partial w}{\partial x}-\bigl[a(3n+m)x^ny^2+b(2n+m)y-Ax^m-Cx^{-n}\bigr]\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=(Ax^{n+m}+C)^2-2A(n+m)x^{2n+m}y(ax^ny+b)$.

97. $\boldsymbol{(ax^n+bx^2+xy)\dfrac{\partial w}{\partial x}+(cx^n+bxy+y^2)\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\frac{1}{n-2}(ay-cx)^{n-2}+\int(v+b)(av-c)^{n-3}\,dv, \qquad v=\frac{y}{x}.$$

98. $\boldsymbol{(ay^n+bx^2+cxy)\dfrac{\partial w}{\partial x}+(ky^n+bxy+cy^2)\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\frac{1}{n-2}(kx-ay)^{n-2}-\int\frac{(k-av)^{n-3}(b+cv)}{v^n}\,dv, \qquad v=\frac{y}{x}.$$

99. $\boldsymbol{(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}+(cy^2-bx^{m-1}y+ax^{n-2})\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=c\int\frac{E\,dx}{ax^n+bx^m+c}+\frac{xE}{xy+1}, \qquad E=\exp\biggl[-\int\frac{(bx^m+2c)\,dx}{x(ax^n+bx^m+c)}\biggr].$$

100. $\boldsymbol{(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}+(ax^{n-2}y^2+bx^{m-1}y+c)\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=a\int\frac{x^{n-2}E\,dx}{ax^n+bx^m+c}+\frac{E}{y-x}, \qquad E=\exp\biggl[\int\frac{(2ax^n+bx^m)\,dx}{x(ax^n+bx^m+c)}\biggr].$$

101. $\boldsymbol{(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}+(\alpha x^ky^2+\beta x^sy-\alpha\lambda^2x^k+\beta\lambda x^s)\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\alpha\int\frac{x^kE\,dx}{ax^n+bx^m+c}+\frac{E}{y+\lambda}, \qquad E=\exp\biggl(\int\frac{\beta x^s-2\alpha\lambda x^k}{ax^n+bx^m+c}\,dx\biggr).$$

102. $x(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}-\big[sx^ky^2-(ax^n+bx^m+c)y-s\lambda x^{k+2}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{y-x\sqrt{\lambda}}{y+x\sqrt{\lambda}}\exp\left(2s\sqrt{\lambda}\displaystyle\int\frac{x^k\,dx}{ax^n+bx^m+c}\right).$

103. $(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}$
$+\big[(ax^n+bx^m+c)y^2-an(n-1)x^{n-2}-bm(m-1)x^{m-2}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{1}{(ax^n+bx^m+c)\big[(ax^n+bx^m+c)y+anx^{n-1}+bmx^{m-1}\big]}+\int\frac{dx}{(ax^n+bx^m+c)^2}.$$

104. $(ax^n+by^n+x)\dfrac{\partial w}{\partial x}+(\alpha x^ky^{n-k}+\beta x^my^{n-m}+y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{1}{n-1}x^{n-1}E-\int\frac{E\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av},\qquad v=\frac{y}{x},$$

where $E=\exp\left[(1-n)\displaystyle\int\frac{(bv^n+a)\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av}\right].$

105. $(ax^n+by^n+Ax^2+Bxy)\dfrac{\partial w}{\partial x}$
$+(\alpha x^ky^{n-k}+\beta x^my^{n-m}+Axy+By^2)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{1}{n-2}x^{n-2}E-\int\frac{(Bv+A)E\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av},\qquad v=\frac{y}{x},$$

where $E=\exp\left[(2-n)\displaystyle\int\frac{(bv^n+a)\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av}\right].$

106. $(ay^m+bx^n+s)\dfrac{\partial w}{\partial x}-(\alpha x^k+bnx^{n-1}y+\beta)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=a\varphi(y)+\alpha\psi(x)+bx^ny+sy+\beta x,$$

where

$$\varphi(y)=\begin{cases}\dfrac{y^{m+1}}{m+1} & \text{if } m\neq-1,\\ \ln|y| & \text{if } m=-1,\end{cases}\qquad \psi(x)=\begin{cases}\dfrac{x^{k+1}}{k+1} & \text{if } k\neq-1,\\ \ln|x| & \text{if } k=-1.\end{cases}$$

107. $(ax^n y^m + x)\dfrac{\partial w}{\partial x} + (bx^k y^{n+m-k} + y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{n+m-1}x^{n+m-1}E - a\int \frac{E\,dv}{v^m(bv^{n-k} - av)}, \qquad v = \frac{y}{x},$$

where $E = \exp\left[a(1-n-m)\int \dfrac{dv}{bv^{n-k} - av}\right]$.

108. $x(ax^n y^m + \alpha)\dfrac{\partial w}{\partial x} - y(bx^n y^m + \beta)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{(y^a x^b)^A}{A} + \frac{(y^\alpha x^\beta)^B}{B}, \qquad \text{where} \quad A = \frac{m\beta - n\alpha}{a\beta - b\alpha}, \quad B = \frac{mb - na}{a\beta - b\alpha}.$$

109. $x(anx^k y^{n+k} + s)\dfrac{\partial w}{\partial x} - y(bmx^{m+k} y^k + s)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = aky^n + bkx^m - s(xy)^{-k}$.

110. $(ax^n y^m + Ax^2 + Bxy)\dfrac{\partial w}{\partial x} + (bx^k y^{n+m-k} + Axy + By^2)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{n+m-2}x^{n+m-2}E - a\int \frac{(Bv+A)E\,dv}{v^m(bv^{n-k} - av)}, \qquad v = \frac{y}{x},$$

where $E = \exp\left[a(2-n-m)\int \dfrac{dv}{bv^{n-k} - av}\right]$.

111. $(ax^n y^m + bxy^k)\dfrac{\partial w}{\partial x} + (\alpha y^s + \beta)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{1-n}x^{1-n}E - a\int \frac{y^m E}{\alpha y^s + \beta}\,dy, \qquad E = \exp\left[b(n-1)\int \frac{y^k\,dy}{\alpha y^s + \beta}\right].$$

1.1.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda y - ae^{\lambda x}$.

2. $\dfrac{\partial w}{\partial x} + \big(ae^{\lambda x} + b\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda(bx - y) + ae^{\lambda x}$.

3. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda y}+b\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda(bx-y)+\ln\left|b+ae^{\lambda y}\right|.$

4. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda y+\beta x}+b\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=e^{b\lambda x-\lambda y}+\dfrac{a\lambda}{\beta+b\lambda}e^{(\beta+b\lambda)x}.$

5. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda y+\beta x}+be^{\gamma x}\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=e^{-\lambda y}E+a\lambda\displaystyle\int e^{\beta x}E\,dx,$ where $E=\exp\left(\dfrac{b\lambda}{\gamma}e^{\gamma x}\right).$

6. $ae^{\lambda x}\dfrac{\partial w}{\partial x}+be^{\beta y}\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{\beta b}e^{-\beta y}-\dfrac{1}{\lambda a}e^{-\lambda x}.$

7. $\left(ae^{\lambda x}+b\right)\dfrac{\partial w}{\partial x}+\left(ce^{\beta x}+d\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=y-\displaystyle\int\dfrac{ce^{\beta x}+d}{ae^{\lambda x}+b}\,dx.$

8. $\left(ae^{\lambda x}+b\right)\dfrac{\partial w}{\partial x}+\left(ce^{\beta y}+d\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda\beta(dx-by)-d\beta\ln\left|ae^{\lambda x}+b\right|+b\lambda\ln\left|ce^{\beta y}+d\right|.$

9. $\left(ae^{\lambda y}+b\right)\dfrac{\partial w}{\partial x}+\left(ce^{\beta x}+d\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\beta\lambda(dx-by)+c\lambda e^{\beta x}-a\beta e^{\lambda y}.$

10. $\left(ae^{\lambda x}+be^{\beta y}\right)\dfrac{\partial w}{\partial x}+a\lambda e^{\lambda x}\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=ae^{\lambda x-y}-\dfrac{b}{\beta-1}e^{(\beta-1)y}.$

11. $\left(ae^{\lambda x+\beta y}+c\mu\right)\dfrac{\partial w}{\partial x}-\left(be^{\gamma x+\mu y}+c\lambda\right)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{a}{\beta-\mu}e^{(\beta-\mu)y}+\dfrac{b}{\gamma-\lambda}e^{(\gamma-\lambda)x}-ce^{-\lambda x-\mu y}.$

► **Coefficients of equations contain exponential and power-law functions.**

12. $\dfrac{\partial w}{\partial x} + (y^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int E\,dx, \qquad E = \exp\biggl(\frac{2a}{\lambda}e^{\lambda x}\biggr).$$

13. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + by + a(\lambda - b)e^{\lambda x} - a^2 e^{2\lambda x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int E\,dx, \qquad E = \exp\biggl(\frac{2a}{\lambda}e^{\lambda x} + bx\biggr).$$

14. $\dfrac{\partial w}{\partial x} + (y^2 + ae^{\lambda x}y - abe^{\lambda x} - b^2)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - b} + \int E\,dx, \qquad E = \exp\biggl(2bx + \frac{a}{\lambda}e^{\lambda x}\biggr).$$

15. $\dfrac{\partial w}{\partial x} - (y^2 - axe^{\lambda x}y + ae^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy - 1)} - \int \frac{E}{x^2}\,dx, \qquad E = \exp\biggl[\frac{a}{\lambda^2}(\lambda x - 1)e^{\lambda x}\biggr].$$

16. $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + be^{-\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + \lambda v + b} - x, \qquad v = e^{\lambda x}y.$$

17. $\dfrac{\partial w}{\partial x} + \bigl[ae^{\lambda x}y^2 + b\mu e^{\mu x} - ab^2 e^{(\lambda + 2\mu)x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\mu x}} + a\int e^{\lambda x}E\,dx, \qquad E = \exp\biggl[\frac{2ab}{\lambda + \mu}e^{(\lambda + \mu)x}\biggr].$$

18. $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + by + ce^{-\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + (b + \lambda)v + c} - x, \qquad v = e^{\lambda x}y.$$

19. $\dfrac{\partial w}{\partial x} + \bigl[ae^{\lambda x}y^2 + \mu y - ab^2e^{(\lambda+2\mu)x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\mu x}} + a\int e^{\lambda x}E\,dx, \qquad E = \exp\left[\frac{2ab}{\lambda+\mu}e^{(\lambda+\mu)x} + \mu x\right].$$

20. $\dfrac{\partial w}{\partial x} + \bigl[e^{\lambda x}y^2 + ae^{\mu x}y + a\lambda e^{(\mu-\lambda)x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + \lambda e^{-\lambda x}} + \int e^{\lambda x}E\,dx, \qquad E = \exp\left(\frac{a}{\mu}e^{\mu x} - 2\lambda x\right).$$

21. $\dfrac{\partial w}{\partial x} - \bigl[\lambda e^{\lambda x}y^2 - ae^{\mu x}y + ae^{(\mu-\lambda)x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - e^{-\lambda x}} - \lambda\int e^{\lambda x}E\,dx, \qquad E = \exp\left(\frac{a}{\mu}e^{\mu x} - 2\lambda x\right).$$

22. $\dfrac{\partial w}{\partial x} + \bigl[ae^{\lambda x}y^2 + abe^{(\lambda+\mu)x}y - b\mu e^{\mu x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + be^{\mu x}} + a\int e^{\lambda x}E\,dx, \qquad E = \exp\left[-\frac{ab}{\lambda+\mu}e^{(\lambda+\mu)x}\right].$$

23. $\dfrac{\partial w}{\partial x} + \bigl[ae^{(2\lambda+\mu)x}y^2 + (be^{(\lambda+\mu)x} - \lambda)y + ce^{\mu x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{av^2 + bv + c} - \frac{1}{\mu+\lambda}e^{(\mu+\lambda)x}, \qquad v = e^{\lambda x}y.$$

24. $\dfrac{\partial w}{\partial x} + \bigl[e^{\lambda x}(y - be^{\mu x})^2 + b\mu e^{\mu x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y - be^{\mu x}} + \dfrac{1}{\lambda}e^{\lambda x}.$

25. $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + bnx^{n-1} - ab^2e^{\lambda x}x^{2n})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^n} + a\int e^{\lambda x}E\,dx, \qquad E = \exp\left(2ab\int x^n e^{\lambda x}\,dx\right).$$

26. $\dfrac{\partial w}{\partial x}+(e^{\lambda x}y^2+ax^n y+a\lambda x^n e^{-\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+\lambda e^{-\lambda x}}+\int e^{\lambda x}E\,dx,\qquad E=\exp\biggl(\frac{a}{n+1}x^{n+1}-2\lambda x\biggr).$$

27. $\dfrac{\partial w}{\partial x}+(\lambda e^{\lambda x}y^2+ax^n e^{\lambda x}y-ax^n e^{2\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{2\lambda x}E}{y-e^{\lambda x}}+\lambda\int e^{\lambda x}E\,dx,\qquad E=\exp\biggl(a\int x^n e^{-\lambda x}\,dx\biggr).$$

28. $\dfrac{\partial w}{\partial x}+(ae^{\lambda x}y^2-abx^n e^{\lambda x}y+bnx^{n-1})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^n}+a\int e^{\lambda x}E\,dx,\qquad E=\exp\biggl(ab\int x^n e^{\lambda x}\,dx\biggr).$$

29. $\dfrac{\partial w}{\partial x}+(ax^n y^2+b\lambda e^{\lambda x}-ab^2x^n e^{2\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-be^{\lambda x}}+a\int x^n E\,dx,\qquad E=\exp\biggl(2ab\int x^n e^{\lambda x}\,dx\biggr).$$

30. $\dfrac{\partial w}{\partial x}+(ax^n y^2+\lambda y-ab^2x^n e^{2\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-be^{\lambda x}}+a\int x^n E\,dx,\qquad E=\exp\biggl(\lambda x+2ab\int x^n e^{\lambda x}\,dx\biggr).$$

31. $\dfrac{\partial w}{\partial x}+(ax^n y^2-abx^n e^{\lambda x}y+b\lambda e^{\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-be^{\lambda x}}+a\int x^n E\,dx,\qquad E=\exp\biggl(ab\int x^n e^{\lambda x}\,dx\biggr).$$

32. $\dfrac{\partial w}{\partial x}+\bigl[ax^n y^2-ax^n(be^{\lambda x}+c)y+b\lambda e^{\lambda x}\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-be^{\lambda x}-c}+a\int x^n E\,dx,$$

$$\text{where }E=\begin{cases}\exp\biggl(\dfrac{ac}{n+1}x^{n+1}+ab\displaystyle\int x^n e^{\lambda x}\,dx\biggr) & \text{if } n\neq -1,\\ x^{ac}\exp\biggl(ab\displaystyle\int \frac{e^{\lambda x}}{x}\,dx\biggr) & \text{if } n=-1.\end{cases}$$

33. $\dfrac{\partial w}{\partial x}+\big[ax^ne^{2\lambda x}y^2+(bx^ne^{\lambda x}-\lambda)y+cx^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\int x^ne^{\lambda x}\,dx,\qquad v=e^{\lambda x}y.$$

34. $\dfrac{\partial w}{\partial x}+\big[ae^{\lambda x}(y-bx^n-c)^2+bnx^{n-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{y-bx^n-c}+\dfrac{a}{\lambda}e^{\lambda x}.$

35. $\dfrac{\partial w}{\partial x}+\big(y^2+2a\lambda xe^{\lambda x^2}-a^2e^{2\lambda x^2}\big)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-ae^{\lambda x^2}}+\int E\,dx,\qquad E=\exp\biggl(2a\int e^{\lambda x^2}\,dx\biggr).$$

36. $\dfrac{\partial w}{\partial x}+\big(ae^{-\lambda x^2}y^2+\lambda xy+ab^2\big)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\Bigl[\dfrac{1}{b}y\exp\bigl(-\tfrac{1}{2}\lambda x^2\bigr)\Bigr]-ab\displaystyle\int\exp\bigl(-\tfrac{1}{2}\lambda x^2\bigr)\,dx.$

37. $\dfrac{\partial w}{\partial x}+\big(ax^ny^2+\lambda xy+ab^2x^ne^{\lambda x^2}\big)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\Bigl[\dfrac{y}{b}\exp\bigl(-\tfrac{1}{2}\lambda x^2\bigr)\Bigr]-ab\displaystyle\int x^n\exp\bigl(\tfrac{1}{2}\lambda x^2\bigr)\,dx.$

38. $\dfrac{\partial w}{\partial x}+(ae^{2\lambda x}y^3+be^{\lambda x}y^2+cy+de^{-\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^3+bv^2+(c+\lambda)v+d}-x,\qquad v=e^{\lambda x}y.$$

39. $\dfrac{\partial w}{\partial x}+(ae^{\lambda x}y^3+3abe^{\lambda x}y^2+cy-2ab^3e^{\lambda x}+bc)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{2cx}E}{(y+b)^2}+2a\int e^{(\lambda+2c)x}E\,dx,\qquad E=\exp\biggl(-\frac{6ab^2}{\lambda}e^{\lambda x}\biggr).$$

40. $x\dfrac{\partial w}{\partial x}+(ae^{\lambda x}y^2+ky+ab^2x^{2k}e^{\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\dfrac{y}{bx^k}-ab\displaystyle\int x^{k-1}e^{\lambda x}\,dx.$

41. $\displaystyle x\frac{\partial w}{\partial x} + \big[ax^{2n}e^{\lambda x}y^2 + (bx^n e^{\lambda x} - n)y + ce^{\lambda x}\big]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + bv + c} - \int x^{n-1}e^{\lambda x}\,dx, \qquad v = x^n y.$$

42. $\displaystyle y\frac{\partial w}{\partial x} + e^{\lambda x}\big[(2a\lambda x + a + b)y - e^{\lambda x}(a^2\lambda x^2 + abx - c)\big]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = xE + \int \frac{vE\,dv}{\lambda v^2 - bv - c}, \qquad v = e^{-\lambda x}y - ax,$$

where $E = \exp\left(a\displaystyle\int \frac{dv}{\lambda v^2 - bv - c}\right)$.

43. $\displaystyle ae^{\lambda x}\frac{\partial w}{\partial x} + by^m\frac{\partial w}{\partial y} = 0.$

1°. Principal integral for $m \neq 1$: $\displaystyle \Xi = \frac{1}{b(1-m)}y^{1-m} + \frac{1}{\lambda a}e^{-\lambda x}.$

2°. Principal integral for $m = 1$: $\displaystyle \Xi = \frac{1}{b}\ln y + \frac{1}{\lambda a}e^{-\lambda x}.$

44. $\displaystyle (ae^y + bx)\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0.$

1°. Principal integral for $b \neq 1$: $\displaystyle \Xi = xe^{-by} - \frac{a}{1-b}e^{(1-b)y}.$

2°. Principal integral for $b = 1$: $\Xi = xe^{-y} - ay.$

45. $\displaystyle (ax^n e^{\lambda y} + bxy^m)\frac{\partial w}{\partial x} + e^{\mu y}\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{1-n}x^{1-n}E - a\int e^{(\lambda-\mu)y}E\,dy, \qquad E = \exp\left[b(n-1)\int y^m e^{-\mu y}\,dy\right].$$

46. $\displaystyle (ax^n y^m + bxe^{\lambda y})\frac{\partial w}{\partial x} + y^k\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{n-1}x^{n-1}E + a\int y^{m-k}E\,dy, \qquad E = \exp\left[b(n-1)\int y^{-k}e^{\lambda y}\,dy\right].$$

47. $\displaystyle (ax^n y^m + bxy^k)\frac{\partial w}{\partial x} + e^{\lambda y}\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{1-n}x^{1-n}E - a\int y^m e^{-\lambda y}E\,dy, \qquad E = \exp\left[b(n-1)\int y^k e^{-\lambda y}\,dy\right].$$

1.1.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $\dfrac{\partial w}{\partial x}+a\sinh(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda y-a\cosh(\lambda x)$.

2. $\dfrac{\partial w}{\partial x}+a\sinh(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\mu x-\ln\bigl|\tanh(\tfrac{1}{2}\mu y)\bigr|$.

3. $\dfrac{\partial w}{\partial x}+\bigl[y^2-a^2+a\lambda\sinh(\lambda x)-a^2\sinh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\cosh(\lambda x)}+\int E\,dx,\qquad E=\exp\biggl[\frac{2a}{\lambda}\sinh(\lambda x)\biggr].$$

4. $\dfrac{\partial w}{\partial x}+\lambda\bigl[\sinh(\lambda x)y^2-\sinh^3(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{\exp\bigl[\frac{1}{2}\cosh(2\lambda x)\bigr]}{y-\cosh(\lambda x)}+\lambda\int\sinh(\lambda x)\exp\bigl[\tfrac{1}{2}\cosh(2\lambda x)\bigr]\,dx.$$

5. $\dfrac{\partial w}{\partial x}+\bigl\{[a\sinh^2(\lambda x)-\lambda]y^2-a\sinh^2(\lambda x)+\lambda-a\bigr\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{\sinh(\lambda x)\bigl[\sinh(\lambda x)y-\cosh(\lambda x)\bigr]}+\int\biggl[a-\frac{\lambda}{\sinh^2(\lambda x)}\biggr]E\,dx,$$
$$E=\exp\biggl[\frac{a}{2\lambda}\cosh(2\lambda x)\biggr].$$

6. $\sinh(\lambda x)\dfrac{\partial w}{\partial x}+a\sinh(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\mu\ln\bigl|\tanh(\tfrac{1}{2}\lambda x)\bigr|-\lambda\ln\bigl|\tanh(\tfrac{1}{2}\mu y)\bigr|$.

7. $\sinh(\mu y)\dfrac{\partial w}{\partial x}+a\sinh(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda\cosh(\mu y)-a\mu\cosh(\lambda x)$.

► **Coefficients of equations contain hyperbolic cosine.**

8. $\dfrac{\partial w}{\partial x}+a\cosh(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\sinh(\lambda x)-\lambda y$.

9. $\dfrac{\partial w}{\partial x} + a\cosh(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\lambda x - 2\arctan(e^{\lambda y})$.

10. $\dfrac{\partial w}{\partial x} + \big\{[a\cosh^2(\lambda x) - \lambda]y^2 - a\cosh^2(\lambda x) + \lambda + a\big\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cosh(\lambda x)\big[\cosh(\lambda x)y - \sinh(\lambda x)\big]} + \int\left[a - \frac{\lambda}{\cosh^2(\lambda x)}\right]E\,dx,$$
$$E = \exp\left[\frac{a}{2\lambda}\cosh(2\lambda x)\right].$$

11. $2\dfrac{\partial w}{\partial x} + \big\{[a - \lambda + a\cosh(\lambda x)]y^2 + a + \lambda - a\cosh(\lambda x)\big\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - \tanh\left(\frac{1}{2}\lambda x\right)} + \frac{1}{2}\int\big[a - \lambda + a\cosh(\lambda x)\big]E\,dx,$$

where

$$E = \big[\cosh(\tfrac{1}{2}\lambda x)\big]^{\frac{2(a-\lambda)}{\lambda}}\exp\left[a\int\cosh(\lambda x)\tanh\left(\tfrac{1}{2}\lambda x\right)dx\right].$$

12. $(ax^n + bx\cosh^m y)\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int y^{-k}E\,dy, \qquad E = \exp\left[b(n-1)\int y^{-k}\cosh^m y\,dy\right].$$

13. $(ax^n + bx\cosh^m y)\dfrac{\partial w}{\partial x} + \cosh^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int\frac{E\,dy}{\cosh^k(\lambda y)}, \qquad E = \exp\left[b(n-1)\int\frac{\cosh^m y\,dy}{\cosh^k(\lambda y)}\right].$$

14. $(ax^n y^m + bx)\dfrac{\partial w}{\partial x} + \cosh^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int\frac{y^m E\,dy}{\cosh^k(\lambda y)}, \qquad E = \exp\left[b(n-1)\int\frac{dy}{\cosh^k(\lambda y)}\right].$$

15. $\cosh(\mu y)\dfrac{\partial w}{\partial x} + a\cosh(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \mu a\sinh(\lambda x) - \lambda\sinh(\mu y)$.

► **Coefficients of equations contain hyperbolic tangent.**

16. $\dfrac{\partial w}{\partial x} + a\tanh(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda y - a\ln\bigl[\cosh(\lambda x)\bigr]$.

17. $\dfrac{\partial w}{\partial x} + a\tanh(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\lambda x - \ln\bigl|\sinh(\lambda y)\bigr|$.

18. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + a\lambda - a(a+\lambda)\tanh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}}{y - a\tanh(\lambda x)} + \displaystyle\int \bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}\,dx.$

19. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + 3a\lambda - \lambda^2 - a(a+\lambda)\tanh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}}{\sinh^2(\lambda x)\bigl[y - a\tanh(\lambda x) + \lambda\coth(\lambda x)\bigr]} + \int \frac{\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}}{\sinh^2(\lambda x)}\,dx.$$

20. $(ax^n + bx\tanh^m y)\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int y^{-k}E\,dy, \qquad E = \exp\biggl[b(n-1)\int y^{-k}\tanh^m y\,dy\biggr].$$

21. $(ax^n + bx\tanh^m y)\dfrac{\partial w}{\partial x} + \tanh^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int \frac{E\,dy}{\tanh^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int \frac{\tanh^m y\,dy}{\tanh^k(\lambda y)}\biggr].$$

22. $(ax^n y^m + bx)\dfrac{\partial w}{\partial x} + \tanh^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int \frac{y^m E\,dy}{\tanh^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int \frac{dy}{\tanh^k(\lambda y)}\biggr].$$

23. $(ax^n\tanh^m y + bx)\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $k \neq 1$:

$$\Xi = x^{1-n}E + (n-1)a\int y^{-k}E\tanh^m y\,dy, \qquad E = \exp\biggl[\frac{b(n-1)}{1-k}y^{1-k}\biggr].$$

2°. Principal integral for $k = 1$:

$$\Xi = (xy^{-b})^{1-n} + (n-1)a\int y^{(n-1)b-1}\tanh^m y\,dy.$$

► **Coefficients of equations contain hyperbolic cotangent.**

24. $\dfrac{\partial w}{\partial x} + a\coth(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda y - a\ln\bigl|\sinh(\lambda x)\bigr|.$

25. $\dfrac{\partial w}{\partial x} + a\coth(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\lambda x - \ln\bigl[\cosh(\lambda y)\bigr].$

26. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + a\lambda - a(a+\lambda)\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}}{y - a\coth(\lambda x)} + \displaystyle\int \bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}\,dx.$

27. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + 3a\lambda - \lambda^2 - a(a+\lambda)\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}}{\cosh^2(\lambda x)\bigl[y - a\coth(\lambda x) + \lambda\tanh(\lambda x)\bigr]} + \int \frac{\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}}{\cosh^2(\lambda x)}\,dx.$$

► **Coefficients of equations contain different hyperbolic functions.**

28. $\dfrac{\partial w}{\partial x} + a\sinh(\lambda x)\cosh(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = 2\lambda\arctan\bigl(e^{\mu y}\bigr) - a\mu\cosh(\lambda x).$

29. $\dfrac{\partial w}{\partial x} + a\cosh(\lambda x)\sinh(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda\ln\bigl|\tanh\bigl(\tfrac{1}{2}\mu y\bigr)\bigr| - a\mu\sinh(\lambda x).$

30. $\dfrac{\partial w}{\partial x} + \bigl[y^2 - 2\lambda^2\tanh^2(\lambda x) - 2\lambda^2\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\sinh^2(\lambda x)\cosh^2(\lambda x)}{y - \lambda\tanh(\lambda x) - \lambda\coth(\lambda x)} + \int \sinh^2(\lambda x)\cosh^2(\lambda x)\,dx.$$

31. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + \lambda(a+b) - 2ab - a(a+\lambda)\tanh^2(\lambda x)$

$\qquad - b(b+\lambda)\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\sinh(\lambda x)\bigr]^{\frac{2b}{\lambda}}\bigl[\cosh(\lambda x)\bigr]^{\frac{2a}{\lambda}}}{y - a\tanh(\lambda x) - b\coth(\lambda x)} + \int \bigl[\sinh(\lambda x)\bigr]^{\frac{2b}{\lambda}}\bigl[\cosh(\lambda x)\bigr]^{\frac{2a}{\lambda}}\,dx.$$

32. $\sinh(\lambda y)\dfrac{\partial w}{\partial x} + a\cosh(\beta x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \beta\cosh(\lambda y) - a\lambda\sinh(\beta x).$

33. $\big[ax^n\cosh^m(\lambda y) + bx\big]\dfrac{\partial w}{\partial x} + \sinh^k(\beta y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int\frac{\cosh^m(\lambda y)E\,dy}{\sinh^k(\beta y)}, \qquad E = \exp\left[b(n-1)\int\frac{dy}{\sinh^k(\beta y)}\right].$$

1.1.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\dfrac{\partial w}{\partial x} + \big[a\ln^k(\lambda x) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int\ln^k(\lambda x)\,dx.$

2. $\dfrac{\partial w}{\partial x} + \big[a\ln^k(\lambda y) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int\frac{dy}{a\ln^k(\lambda y) + b}.$

3. $\dfrac{\partial w}{\partial x} + a\ln^k(\lambda x)\ln^n(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\displaystyle\int\ln^k(\lambda x)\,dx - \int\frac{dy}{\ln^n(\mu y)}.$

4. $\dfrac{\partial w}{\partial x} + a\ln^k(x + \lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x - \int\frac{dz}{1 + a\lambda\ln^k z}, \qquad z = x + \lambda y.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

5. $\dfrac{\partial w}{\partial x} + ax^n\ln^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{a}{n+1}x^{n+1} - \displaystyle\int\frac{dy}{\ln^k(\lambda y)}.$

6. $\dfrac{\partial w}{\partial x} + ay^n\ln^k(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{1-n}y^{1-n} - a\displaystyle\int\ln^k(\lambda x)\,dx.$

7. $\dfrac{\partial w}{\partial x} + \left[y^2 + a\ln(\beta x)y - ab\ln(\beta x) - b^2\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{(2b-a)x}E}{y-b} + \int e^{(2b-a)x}E\,dx, \qquad E = \exp\bigl[ax\ln(\beta x)\bigr].$$

8. $\dfrac{\partial w}{\partial x} + \left[y^2 + ax\ln^m(bx)y + a\ln^m(bx)\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[a\int x\ln^m(bx)\,dx\right].$$

9. $\dfrac{\partial w}{\partial x} + (ax^n y^2 - abx^{n+1}y\ln x + b\ln x + b)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx\ln x} + a\int x^n E\,dx, \qquad E = \exp\left[\frac{ab}{n+2}x^{n+2}\left(\ln x - \frac{1}{n+2}\right)\right].$$

10. $\dfrac{\partial w}{\partial x} - \left[(n+1)x^n y^2 - ax^{n+1}(\ln x)^m y + a(\ln x)^m\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{x^{-2(n+1)}E}{y - x^{-n-1}} - (n+1)\int x^{-n-2}E\,dx, \qquad E = \exp\left[a\int x^{n+1}(\ln x)^m\,dx\right].$$

11. $\dfrac{\partial w}{\partial x} + \left[a(\ln x)^n y^2 + bmx^{m-1} - ab^2x^{2m}(\ln x)^n\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^m} + a\int(\ln x)^n E\,dx, \qquad E = \exp\left[2ab\int x^m(\ln x)^n\,dx\right].$$

12. $\dfrac{\partial w}{\partial x} + \left[a(\ln x)^n y^2 - abx(\ln x)^{n+1}y + b\ln x + b\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx\ln x} + a\int(\ln x)^n E\,dx, \qquad E = \exp\left[ab\int x(\ln x)^{n+1}\,dx\right].$$

13. $\dfrac{\partial w}{\partial x} + \left[a(\ln x)^k(y - bx^n - c)^2 + bnx^{n-1}\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y - bx^n - c} + a\displaystyle\int(\ln x)^k\,dx.$

14. $\dfrac{\partial w}{\partial x}+\big[a(\ln x)^n y^2+b(\ln x)^m y+bc(\ln x)^m-ac^2(\ln x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+c}+a\int(\ln x)^n E\,dx,\qquad E=\exp\biggl\{\int\bigl[b(\ln x)^m-2ac(\ln x)^n\bigr]\,dx\biggr\}.$$

15. $x\dfrac{\partial w}{\partial x}+(ay+b\ln x)^2\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\ln x-\int\frac{dv}{av^2+b},\qquad v=ay+b\ln x.$$

16. $x\dfrac{\partial w}{\partial x}+\big[xy^2-A^2x\ln^2(\beta x)+A\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{-2Ax}E}{y-A\ln(\beta x)}+\int e^{-2Ax}E\,dx,\qquad E=\exp\bigl[2Ax\ln(\beta x)\bigr].$$

17. $x\dfrac{\partial w}{\partial x}+\big[xy^2-A^2x\ln^{2k}(\beta x)+kA\ln^{k-1}(\beta x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-A\ln^k(\beta x)}+\int E\,dx,\qquad E=\exp\biggl[2A\int\ln^k(\beta x)\,dx\biggr].$$

18. $x\dfrac{\partial w}{\partial x}+(ax^ny^2+b-ab^2x^n\ln^2 x)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-b\ln x}+\int ax^{n-1}E\,dx,\qquad E=\exp\biggl[\frac{2abx^n}{n^2}(n\ln x-1)\biggr].$$

19. $x\dfrac{\partial w}{\partial x}+\big[a\ln^m(\lambda x)y^2+ky+ab^2x^{2k}\ln^m(\lambda x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\biggl(\dfrac{y}{bx^k}\biggr)-ab\displaystyle\int x^{k-1}\ln^m(\lambda x)\,dx.$

20. $x\dfrac{\partial w}{\partial x}+\big[ax^n(y+b\ln x)^2-b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{y+b\ln x}+\dfrac{a}{n}x^n.$

21. $x\dfrac{\partial w}{\partial x}+\big[ax^{2n}(\ln x)y^2+(bx^n\ln x-n)y+c\ln x\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\int x^{n-1}\ln x\,dx,\qquad v=x^ny.$$

22. $x^k \frac{\partial w}{\partial x} + (ay^n \ln^m x + by \ln^s x)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{1-n}E + (n-1)a\int x^{-k}E\ln^m x\,dx, \qquad E = \exp\Bigl[b(n-1)\int x^{-k}\ln^s x\,dx\Bigr].$$

23. $(a\ln x + b)\frac{\partial w}{\partial x} + \bigl[y^2 + c(\ln x)^n y - \lambda^2 + \lambda c(\ln x)^n\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y+\lambda} + \int \frac{E\,dx}{a\ln x + b}, \qquad E = \exp\biggl[\int \frac{c(\ln x)^n - 2\lambda}{a\ln x + b}\biggr]dx.$$

24. $(a\ln x + b)\frac{\partial w}{\partial x} + \bigl[(\ln x)^n y^2 + cy - \lambda^2(\ln x)^n + c\lambda\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y+\lambda} + \int \frac{(\ln x)^n E\,dx}{a\ln x + b}, \qquad E = \exp\biggl[\int \frac{c - 2\lambda(\ln x)^n}{a\ln x + b}\,dx\biggr].$$

25. $x^2\ln(ax)\frac{\partial w}{\partial x} - \bigl[x^2y^2\ln(ax) + 1\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{x}{\ln(ax)\bigl[xy\ln(ax) - 1\bigr]} - \displaystyle\int \frac{dx}{\ln^2(ax)}.$

26. $\ln^k(\lambda x)\frac{\partial w}{\partial x} + (ay^n + by\ln^m x)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{1-n}E + (n-1)a\int \frac{E\,dx}{\ln^k(\lambda x)}, \qquad E = \exp\biggl[b(n-1)\int \frac{\ln^m x\,dx}{\ln^k(\lambda x)}\biggr].$$

27. $\ln^k(\lambda x)\frac{\partial w}{\partial x} + (ay^n\ln^m x + by)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{1-n}E + (n-1)a\int \frac{E\ln^m x\,dx}{\ln^k(\lambda x)}, \qquad E = \exp\biggl[b(n-1)\int \frac{dx}{\ln^k(\lambda x)}\biggr].$$

1.1.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $\frac{\partial w}{\partial x} + \bigl[a\sin^k(\lambda x) + b\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int \sin^k(\lambda x)\,dx.$

2. $\dfrac{\partial w}{\partial x}+\bigl[a\sin^k(\lambda y)+b\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=x-\displaystyle\int\frac{dy}{a\sin^k(\lambda y)+b}.$

3. $\dfrac{\partial w}{\partial x}+a\sin^k(\lambda x)\sin^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int\sin^k(\lambda x)\,dx-\int\frac{dy}{\sin^n(\mu y)}.$

4. $\dfrac{\partial w}{\partial x}+a\sin^k(x+\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x-\int\frac{dz}{1+a\lambda\sin^k z},\qquad z=x+\lambda y.$$

5. $\dfrac{\partial w}{\partial x}+\bigl[y^2-a^2+a\lambda\sin(\lambda x)+a^2\sin^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+a\cos(\lambda x)}+\int E\,dx,\qquad E=\exp\biggl[-\frac{2a}{\lambda}\sin(\lambda x)\biggr].$$

6. $\dfrac{\partial w}{\partial x}+\bigl[y^2+a\sin(\beta x)y+ab\sin(\beta x)-b^2\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+b}+\int E\,dx,\qquad E=\exp\biggl[-2bx-\frac{a}{\beta}\cos(\beta x)\biggr].$$

7. $\dfrac{\partial w}{\partial x}+\bigl[y^2+ax\sin^m(bx)y+a\sin^m(bx)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x(xy+1)}+\int x^{-2}E\,dx,\qquad E=\exp\biggl[a\int x\sin^m(bx)\,dx\biggr].$$

8. $\dfrac{\partial w}{\partial x}+\bigl[\lambda\sin(\lambda x)y^2+\lambda\sin^3(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+\cos(\lambda x)}+\lambda\int E\sin(\lambda x)\,dx,\qquad E=\exp\bigl[\tfrac{1}{2}\cos(2\lambda x)\bigr].$$

9. $2\dfrac{\partial w}{\partial x}+\bigl\{[\lambda+a-a\sin(\lambda x)]y^2+\lambda-a-a\sin(\lambda x)\bigr\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\tan\bigl(\frac{1}{2}\lambda x+\frac{1}{4}\pi\bigr)}+\frac{1}{2}\int\bigl[\lambda+a-a\sin(\lambda x)\bigr]E\,dx,$$
$$E=\frac{1}{1-\sin(\lambda x)}\exp\biggl[\frac{a}{\lambda}\sin(\lambda x)\biggr].$$

10. $\dfrac{\partial w}{\partial x} + \bigl\{[\lambda + a\sin^2(\lambda x)]y^2 + \lambda - a + a\sin^2(\lambda x)\bigr\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + \cot(\lambda x)} + \int \bigl[\lambda + a\sin^2(\lambda x)\bigr] E\,dx, \qquad E = \frac{1}{\sin^2(\lambda x)} \exp\biggl[\frac{a}{2\lambda}\cos(2\lambda x)\biggr].$$

11. $\dfrac{\partial w}{\partial x} - \bigl[(k+1)x^k y^2 - a x^{k+1}(\sin x)^m y + a(\sin x)^m\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y - 1)} - (k+1)\int \frac{E\,dx}{x^{k+2}}, \qquad E = \exp\biggl[a\int x^{k+1}(\sin x)^m\,dx\biggr].$$

12. $\dfrac{\partial w}{\partial x} + \bigl[a\sin^k(\lambda x + \mu)(y - bx^n - c)^2 + y - bx^n + bnx^{n-1} - c\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{e^x}{y - bx^n - c} + a\displaystyle\int e^x \sin^k(\lambda x + \mu)\,dx.$

13. $x\dfrac{\partial w}{\partial x} + \bigl[a\sin^m(\lambda x)y^2 + ky + ab^2x^{2k}\sin^m(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\biggl(\dfrac{y}{bx^k}\biggr) - ab\displaystyle\int x^{k-1}\sin^m(\lambda x)\,dx.$

14. $\bigl[a\sin(\lambda x) + b\bigr]\dfrac{\partial w}{\partial x} + \bigl[y^2 + c\sin(\mu x)y - k^2 + ck\sin(\mu x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + k} + \int \frac{E\,dx}{a\sin(\lambda x) + b}, \qquad E = \exp\biggl[\int \frac{c\sin(\mu x) - 2k}{a\sin(\lambda x) + b}\,dx\biggr].$$

► **Coefficients of equations contain cosine.**

15. $\dfrac{\partial w}{\partial x} + \bigl[a\cos^k(\lambda x) + b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int \cos^k(\lambda x)\,dx.$

16. $\dfrac{\partial w}{\partial x} + \bigl[a\cos^k(\lambda y) + b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\cos^k(\lambda y) + b}.$

17. $\dfrac{\partial w}{\partial x} + a\cos^k(\lambda x)\cos^n(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\displaystyle\int \cos^k(\lambda x)\,dx - \int \frac{dy}{\cos^n(\mu y)}.$

18. $\dfrac{\partial w}{\partial x}+a\cos^k(x+\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x-\int\frac{dz}{1+a\lambda\cos^k z},\qquad z=x+\lambda y.$$

19. $\dfrac{\partial w}{\partial x}+\bigl[y^2-a^2+a\lambda\cos(\lambda x)+a^2\cos^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\sin(\lambda x)}+\int E\,dx,\qquad E=\exp\biggl[-\frac{2a}{\lambda}\cos(\lambda x)\biggr].$$

20. $\dfrac{\partial w}{\partial x}+\bigl[\lambda\cos(\lambda x)y^2+\lambda\cos^3(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\sin(\lambda x)}+\lambda\int E\cos(\lambda x)\,dx,\qquad E=\exp\bigl[-\tfrac{1}{2}\cos(2\lambda x)\bigr].$$

21. $2\dfrac{\partial w}{\partial x}+\bigl\{[\lambda+a+a\cos(\lambda x)]y^2+\lambda-a+a\cos(\lambda x)\bigr\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\tan(\frac{1}{2}\lambda x)}+\frac{1}{2}\int\bigl[\lambda+a+a\cos(\lambda x)\bigr]E\,dx,$$
$$E=\frac{1}{1+\cos(\lambda x)}\exp\biggl[-\frac{a}{\lambda}\cos(\lambda x)\biggr].$$

22. $\dfrac{\partial w}{\partial x}+\bigl\{[\lambda+a\cos^2(\lambda x)]y^2+\lambda-a+a\cos^2(\lambda x)\bigr\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\tan(\lambda x)}+\int\bigl[\lambda+a\cos^2(\lambda x)\bigr]E\,dx,\qquad E=\frac{1}{\cos^2(\lambda x)}\exp\biggl[-\frac{a}{2\lambda}\cos(2\lambda x)\biggr].$$

23. $(ax^ny^m+bx)\dfrac{\partial w}{\partial x}+\cos^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+a(n-1)\int\frac{y^mE\,dy}{\cos^k(\lambda y)},\qquad E=\exp\biggl[b(n-1)\int\frac{dy}{\cos^k(\lambda y)}\biggr].$$

24. $(ax^n+bx\cos^m y)\dfrac{\partial w}{\partial x}+y^k\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+a(n-1)\int y^{-k}\,E\,dy,\qquad E=\exp\biggl[b(n-1)\int\frac{\cos^m y\,dy}{y^k}\biggr].$$

25. $\displaystyle (ax^n + bx\cos^m y)\frac{\partial w}{\partial x} + \cos^k(\lambda y)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + a(n-1)\int \frac{E\,dy}{\cos^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int \frac{\cos^m y\,dy}{\cos^k(\lambda y)}\biggr].$$

26. $\displaystyle (ax^n\cos^m y + bx)\frac{\partial w}{\partial x} + \cos^k(\lambda y)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + a(n-1)\int \frac{\cos^m y\,E\,dy}{\cos^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int \frac{dy}{\cos^k(\lambda y)}\biggr].$$

► **Coefficients of equations contain tangent.**

27. $\displaystyle \frac{\partial w}{\partial x} + \bigl[a\tan^k(\lambda x) + b\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = y - bx - a\int \tan^k(\lambda x)\,dx.$

28. $\displaystyle \frac{\partial w}{\partial x} + \bigl[a\tan^k(\lambda y) + b\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = x - \int \frac{dy}{a\tan^k(\lambda y) + b}.$

29. $\displaystyle \frac{\partial w}{\partial x} + a\tan^k(\lambda x)\tan^n(\mu y)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = a\int \tan^k(\lambda x)\,dx - \int \cot^n(\mu y)\,dy.$

30. $\displaystyle \frac{\partial w}{\partial x} + \bigl[y^2 + a\lambda + a(\lambda - a)\tan^2(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \frac{\bigl[\cos(\lambda x)\bigr]^{-2a/\lambda}}{y - a\tan(\lambda x)} + \int \bigl[\cos(\lambda x)\bigr]^{-2a/\lambda}\,dx.$

31. $\displaystyle \frac{\partial w}{\partial x} + \bigl[y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\tan^2(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\cos(\lambda x)\bigr]^{-2a/\lambda}}{\sin^2(\lambda x)\bigl[y - a\tan(\lambda x) + \lambda\cot(\lambda x)\bigr]} + \int \frac{\bigl[\cos(\lambda x)\bigr]^{-2a/\lambda}}{\sin^2(\lambda x)}\,dx.$$

32. $\displaystyle \frac{\partial w}{\partial x} + \bigl[y^2 + ax\tan^k(bx)y + a\tan^k(bx)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\biggl[a\int x\tan^k(bx)\,dx\biggr].$$

33. $\dfrac{\partial w}{\partial x}-\big[(k+1)x^k y^2-ax^{k+1}(\tan x)^m y+a(\tan x)^m\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x^{k+1}(x^{k+1}y-1)}-(k+1)\int\frac{E\,dx}{x^{k+2}},\qquad E=\exp\biggl[a\int x^{k+1}(\tan x)^m\,dx\biggr].$$

34. $\dfrac{\partial w}{\partial x}+\big[a\tan^n(\lambda x)y^2-ab^2\tan^{n+2}(\lambda x)+b\lambda\tan^2(\lambda x)+b\lambda\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-b\tan(\lambda x)}+a\int E\tan^n(\lambda x)\,dx,\qquad E=\exp\biggl[2ab\int\tan^{n+1}(\lambda x)\,dx\biggr].$$

35. $\dfrac{\partial w}{\partial x}+\big[a\tan^k(\lambda x+\mu)(y-bx^n-c)^2+y-bx^n+bnx^{n-1}-c\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{e^x}{y-bx^n-c}+a\displaystyle\int e^x\tan^k(\lambda x+\mu)\,dx.$

36. $x\dfrac{\partial w}{\partial x}+\big[a\tan^m(\lambda x)y^2+ky+ab^2x^{2k}\tan^m(\lambda x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\Bigl(\dfrac{1}{b}x^{-k}y\Bigr)-ab\displaystyle\int x^{k-1}\tan^m(\lambda x)\,dx.$

37. $\big[a\tan(\lambda x)+b\big]\dfrac{\partial w}{\partial x}+\big[y^2+c\tan(\mu x)y-k^2+ck\tan(\mu x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+k}+\int\frac{E\,dx}{a\tan(\lambda x)+b},\qquad E=\exp\biggl[\int\frac{c\tan(\mu x)-2k}{a\tan(\lambda x)+b}\,dx\biggr].$$

38. $(ax^ny^m+bx)\dfrac{\partial w}{\partial x}+\tan^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{y^mE\,dy}{\tan^k(\lambda y)},\qquad E=\exp\biggl[b(n-1)\int\frac{dy}{\tan^k(\lambda y)}\biggr].$$

39. $(ax^n+bx\tan^m y)\dfrac{\partial w}{\partial x}+y^k\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int y^{-k}E\,dy,\qquad E=\exp\biggl[b(n-1)\int y^{-k}\tan^m y\,dy\biggr].$$

40. $(ax^n+bx\tan^m y)\dfrac{\partial w}{\partial x}+\tan^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{E\,dy}{\tan^k(\lambda y)},\qquad E=\exp\biggl[b(n-1)\int\frac{\tan^m y\,dy}{\tan^k(\lambda y)}\biggr].$$

41. $(ax^n \tan^m y + bx)\dfrac{\partial w}{\partial x} + \tan^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int \frac{\tan^m yE\,dy}{\tan^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int \frac{dy}{\tan^k(\lambda y)}\biggr].$$

► **Coefficients of equations contain cotangent.**

42. $\dfrac{\partial w}{\partial x} + \bigl[a\cot^k(\lambda x) + b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int \cot^k(\lambda x)\,dx.$

43. $\dfrac{\partial w}{\partial x} + \bigl[a\cot^k(\lambda y) + b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\cot^k(\lambda y) + b}.$

44. $\dfrac{\partial w}{\partial x} + a\cot^k(x + \lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x - \int \frac{dz}{1 + a\lambda\cot^k z}, \qquad z = x + \lambda y.$$

45. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + a\lambda + a(\lambda - a)\cot^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{\bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}}{y + a\cot(\lambda x)} + \displaystyle\int \bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}\,dx.$

46. $\dfrac{\partial w}{\partial x} + \bigl[y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\cot^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}}{\cos^2(\lambda x)\bigl[y - \lambda\tan(\lambda x) + a\cot(\lambda x)\bigr]} + \int \frac{\bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}}{\cos^2(\lambda x)}\,dx.$$

47. $\dfrac{\partial w}{\partial x} + \bigl[y^2 - 2a\cot(ax)y + b^2 - a^2\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{\sin^{-2}(bx)}{y - a\cot(ax) + b\cot(bx)} - \dfrac{1}{b}\cot(bx).$

48. $\cot(\lambda x)\dfrac{\partial w}{\partial x} + a\cot(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\mu\ln\bigl|\cos(\lambda x)\bigr| - \lambda\ln\bigl|\cos(\mu y)\bigr|.$

49. $\cot(\mu y)\dfrac{\partial w}{\partial x}+a\cot(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\mu\ln\bigl|\sin(\lambda x)\bigr|-\lambda\ln\bigl|\sin(\mu y)\bigr|.$

50. $\cot(\mu y)\dfrac{\partial w}{\partial x}+a\cot^2(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda\ln\bigl|\sin(\mu y)\bigr|+a\mu\cot(\lambda x)+a\lambda\mu x.$

51. $\cot(y+a)\dfrac{\partial w}{\partial x}+c\cot(x+b)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=c\ln\bigl|\sin(x+b)\bigr|-\ln\bigl|\sin(y+a)\bigr|.$

52. $\cot(\lambda x)\cot(\mu y)\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda\ln\bigl|\sin(\mu y)\bigr|+a\mu\ln\bigl|\cos(\lambda x)\bigr|.$

53. $\cot(\lambda x)\cot(\mu y)\dfrac{\partial w}{\partial x}+a\cot(\nu x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\mu\displaystyle\int\frac{\cot(\nu x)}{\cot(\lambda x)}\,dx-\ln\bigl|\sin(\mu y)\bigr|.$

► **Coefficients of equations contain different trigonometric functions.**

54. $\dfrac{\partial w}{\partial x}+a\sin^k(\lambda x)\cos^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int\sin^k(\lambda x)\,dx-\int\frac{dy}{\cos^n(\mu y)}$. In the special case $a=1$, $k=1$, and $n=-1$ we have $\Xi=\mu\cos(\lambda x)+\lambda\sin(\mu y).$

55. $\dfrac{\partial w}{\partial x}+\bigl[y^2-y\tan x+a(1-a)\cot^2 x\bigr]\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $a\neq\frac{1}{2}$:

$$\Xi=\frac{(\sin x)^{-2a}\cos x}{y+a\cot x}+\frac{1}{1-2a}(\sin x)^{1-2a}.$$

2°. Principal integral for $a=\frac{1}{2}$:

$$\Xi=\frac{\cos x}{y\sin x+\frac{1}{2}\cos x}+\ln|\sin x|.$$

56. $\dfrac{\partial w}{\partial x}+(y^2-my\tan x+b^2\cos^{2m}x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\left(\dfrac{1}{b}y\cos^{-m}x\right)-b\displaystyle\int\cos^m x\,dx.$

57. $\displaystyle \frac{\partial w}{\partial x} + (y^2 + my\cot x + b^2\sin^m x)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \arctan\left(\frac{1}{b}y\sin^{-m}x\right) - b\int \sin^m x\,dx.$

58. $\displaystyle \frac{\partial w}{\partial x} + \bigl[y^2 - 2\lambda^2\tan^2(\lambda x) - 2\lambda^2\cot^2(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\sin^2(\lambda x)\cos^2(\lambda x)}{y - \lambda\cot(\lambda x) + \lambda\tan(\lambda x)} + \frac{1}{8}x - \frac{1}{8\lambda}\sin(\lambda x)\cos(\lambda x)\cos(2\lambda x).$$

59. $\displaystyle \frac{\partial w}{\partial x} + \bigl[y^2 + \lambda(a+b) + 2ab + a(\lambda - a)\tan^2(\lambda x) + b(\lambda - b)\cot^2(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\tan(\lambda x) + b\cot(\lambda x)} + \int E\,dx, \qquad E = \bigl[\cos(\lambda x)\bigr]^{-\frac{2a}{\lambda}}\bigl[\sin(\lambda x)\bigr]^{-\frac{2b}{\lambda}}.$$

60. $\displaystyle \frac{\partial w}{\partial x} + \bigl[\lambda\sin(\lambda x)y^2 + a\cos^n(\lambda x)y - a\cos^{n-1}(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)\bigl[y\cos(\lambda x) - 1\bigr]} + \lambda\int \frac{E\sin(\lambda x)}{\cos^2(\lambda x)}\,dx, \qquad E = \exp\left[a\int \cos^n(\lambda x)\,dx\right].$$

61. $\displaystyle \frac{\partial w}{\partial x} + \bigl[\lambda\sin(\lambda x)y^2 + a\sin(\lambda x)y - a\tan(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)\bigl[y\cos(\lambda x) - 1\bigr]} + \lambda\int \frac{E\sin(\lambda x)}{\cos^2(\lambda x)}\,dx, \qquad E = \exp\left[-\frac{a}{\lambda}\cos(\lambda x)\right].$$

62. $\displaystyle \frac{\partial w}{\partial x} + \bigl[\lambda\sin(\lambda x)y^2 + ax^n\cos(\lambda x)y - ax^n\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)\bigl[y\cos(\lambda x) - 1\bigr]} + \lambda\int \frac{E\sin(\lambda x)}{\cos^2(\lambda x)}\,dx, \qquad E = \exp\left[a\int x^n\cos(\lambda x)\,dx\right].$$

63. $\displaystyle \frac{\partial w}{\partial x} + \bigl[Ae^{\lambda x}\cos(ay) + Be^{\mu x}\sin(ay) + Ae^{\lambda x}\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \tan\frac{ay}{2}\exp\left(-\frac{aB}{\mu}e^{\mu x}\right) - aA\int \exp\left(\lambda x - \frac{aB}{\mu}e^{\mu x}\right)dx.$

64. $\displaystyle \sin^{n+1}(2x)\frac{\partial w}{\partial x} + (ay^2\sin^{2n}x + b\cos^{2n}x)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + n2^{n+1}v + b} - 2^{-n-1}\ln\tan x, \qquad v = y\tan^n x.$$

1.1.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $\dfrac{\partial w}{\partial x} + \big[a\arcsin^k(\lambda x) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int \arcsin^k(\lambda x)\,dx.$

2. $\dfrac{\partial w}{\partial x} + \big[a\arcsin^k(\lambda y) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\arcsin^k(\lambda y) + b}.$

3. $\dfrac{\partial w}{\partial x} + k\arcsin^n(ax + by + c)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{a + bk\arcsin^n v} - x, \qquad v = ax + by + c.$$

4. $\dfrac{\partial w}{\partial x} + a\arcsin^k(\lambda x)\arcsin^n(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\displaystyle\int \arcsin^k(\lambda x)\,dx - \int \frac{dy}{\arcsin^n(\mu y)}.$

5. $\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda(\arcsin x)^n y - a^2 + a\lambda(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{-2ax}E}{y + a} + \int e^{-2ax}E\,dx, \qquad E = \exp\left[\lambda\int(\arcsin x)^n\,dx\right].$$

6. $\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda x(\arcsin x)^n y + \lambda(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy + 1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[\lambda\int x(\arcsin x)^n\,dx\right].$$

7. $\dfrac{\partial w}{\partial x} - \big[(k+1)x^k y^2 - \lambda(\arcsin x)^n(x^{k+1}y - 1)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y - 1)} - (k+1)\int x^{-k-2}E\,dx, \qquad E = \exp\left[\lambda\int x^{k+1}(\arcsin x)^n\,dx\right].$$

8. $\dfrac{\partial w}{\partial x} + \bigl[\lambda(\arcsin x)^n y^2 + ay + ab - b^2\lambda(\arcsin x)^n\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{ax}E}{y+b} + \lambda\int e^{ax}(\arcsin x)^n E\,dx, \qquad E = \exp\biggl[-2b\lambda\int(\arcsin x)^n\,dx\biggr].$$

9. $\dfrac{\partial w}{\partial x} + \bigl[\lambda(\arcsin x)^n y^2 - b\lambda x^m(\arcsin x)^n y + bmx^{m-1}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx^m} + \lambda\int(\arcsin x)^n E\,dx, \qquad E = \exp\biggl[b\lambda\int x^m(\arcsin x)^n\,dx\biggr].$$

10. $\dfrac{\partial w}{\partial x} + \bigl[\lambda(\arcsin x)^n y^2 + bmx^{m-1} - \lambda b^2x^{2m}(\arcsin x)^n\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx^m} + \lambda\int(\arcsin x)^n E\,dx, \qquad E = \exp\biggl[2b\lambda\int x^m(\arcsin x)^n\,dx\biggr].$$

11. $\dfrac{\partial w}{\partial x} + \bigl[\lambda(\arcsin x)^n(y - ax^m - b)^2 + amx^{m-1}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y-ax^m-b} + \lambda\displaystyle\int(\arcsin x)^n\,dx.$

12. $x\dfrac{\partial w}{\partial x} + \bigl[\lambda(\arcsin x)^n y^2 + ky + \lambda b^2x^{2k}(\arcsin x)^n\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\biggl(\dfrac{y}{bx^k}\biggr) - \lambda b\displaystyle\int x^{k-1}(\arcsin x)^n\,dx.$

► **Coefficients of equations contain arccosine.**

13. $\dfrac{\partial w}{\partial x} + \bigl[a\arccos^k(\lambda x) + b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int\arccos^k(\lambda x)\,dx.$

14. $\dfrac{\partial w}{\partial x} + \bigl[a\arccos^k(\lambda y) + b\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int\frac{dy}{a\arccos^k(\lambda y) + b}.$

15. $\dfrac{\partial w}{\partial x} + k\arccos^n(ax+by+c)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{a + bk\arccos^n v} - x, \qquad v = ax + by + c.$$

16. $\dfrac{\partial w}{\partial x}+a\arccos^k(\lambda x)\arccos^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int\arccos^k(\lambda x)\,dx-\int\frac{dy}{\arccos^n(\mu y)}.$

17. $\dfrac{\partial w}{\partial x}+\big[y^2+\lambda(\arccos x)^n y-a^2+a\lambda(\arccos x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{-2ax}E}{y+a}+\int e^{-2ax}E\,dx,\qquad E=\exp\left[\lambda\int(\arccos x)^n\,dx\right].$$

18. $\dfrac{\partial w}{\partial x}+\big[y^2+\lambda x(\arccos x)^n y+\lambda(\arccos x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x(xy+1)}+\int x^{-2}E\,dx,\qquad E=\exp\left[\lambda\int x(\arccos x)^n\,dx\right].$$

19. $\dfrac{\partial w}{\partial x}-\big[(k+1)x^k y^2-\lambda(\arccos x)^n(x^{k+1}y-1)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x^{k+1}(x^{k+1}y-1)}-(k+1)\int x^{-k-2}E\,dx,\qquad E=\exp\left[\lambda\int x^{k+1}(\arccos x)^n\,dx\right].$$

20. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arccos x)^n y^2+ay+ab-b^2\lambda(\arccos x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{ax}E}{y+b}+\lambda\int e^{ax}(\arccos x)^n E\,dx,\qquad E=\exp\left[-2b\lambda\int(\arccos x)^n\,dx\right].$$

21. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arccos x)^n y^2-b\lambda x^m(\arccos x)^n y+bmx^{m-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\arccos x)^n E\,dx,\qquad E=\exp\left[b\lambda\int x^m(\arccos x)^n\,dx\right].$$

22. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arccos x)^n y^2+bmx^{m-1}-\lambda b^2x^{2m}(\arccos x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\arccos x)^n E\,dx,\qquad E=\exp\left[2b\lambda\int x^m(\arccos x)^n\,dx\right].$$

23. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arccos x)^n(y-ax^m-b)^2+amx^{m-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{y-ax^m-b}+\lambda\displaystyle\int(\arccos x)^n\,dx.$

24. $\displaystyle x\frac{\partial w}{\partial x}+\bigl[\lambda(\arccos x)^n y^2+ky+\lambda b^2x^{2k}(\arccos x)^n\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle \Xi=\arctan\left(\frac{y}{bx^k}\right)-\lambda b\int x^{k-1}(\arccos x)^n\,dx.$

► **Coefficients of equations contain arctangent.**

25. $\displaystyle \frac{\partial w}{\partial x}+\bigl[a\arctan^k(\lambda x)+b\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle \Xi=y-bx-a\int\arctan^k(\lambda x)\,dx.$

26. $\displaystyle \frac{\partial w}{\partial x}+\bigl[a\arctan^k(\lambda y)+b\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle \Xi=x-\int\frac{dy}{a\arctan^k(\lambda y)+b}.$

27. $\displaystyle \frac{\partial w}{\partial x}+k\arctan^n(ax+by+c)\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{a+bk\arctan^n v}-x,\qquad v=ax+by+c.$$

28. $\displaystyle \frac{\partial w}{\partial x}+a\arctan^k(\lambda x)\arctan^n(\mu y)\frac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle \Xi=a\int\arctan^k(\lambda x)\,dx-\int\frac{dy}{\arctan^n(\mu y)}.$

29. $\displaystyle \frac{\partial w}{\partial x}+\bigl[y^2+\lambda(\arctan x)^n y-a^2+a\lambda(\arctan x)^n\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{-2ax}E}{y+a}+\int e^{-2ax}E\,dx,\qquad E=\exp\left[\lambda\int(\arctan x)^n\,dx\right].$$

30. $\displaystyle \frac{\partial w}{\partial x}+\bigl[y^2+\lambda x(\arctan x)^n y+\lambda(\arctan x)^n\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x(xy+1)}+\int x^{-2}E\,dx,\qquad E=\exp\left[\lambda\int x(\arctan x)^n\,dx\right].$$

31. $\displaystyle \frac{\partial w}{\partial x}-\bigl[(k+1)x^k y^2-\lambda(\arctan x)^n(x^{k+1}y-1)\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x^{k+1}(x^{k+1}y-1)}-(k+1)\int x^{-k-2}E\,dx,\qquad E=\exp\left[\lambda\int x^{k+1}(\arctan x)^n\,dx\right].$$

32. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2+ay+ab-b^2\lambda(\arctan x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{ax}E}{y+b}+\lambda\int e^{ax}(\arctan x)^n E\,dx,\qquad E=\exp\biggl[-2b\lambda\int(\arctan x)^n\,dx\biggr].$$

33. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2-b\lambda x^m(\arctan x)^n y+bmx^{m-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\arctan x)^n E\,dx,\qquad E=\exp\biggl[b\lambda\int x^m(\arctan x)^n\,dx\biggr].$$

34. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2+bmx^{m-1}-\lambda b^2x^{2m}(\arctan x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\arctan x)^n E\,dx,\qquad E=\exp\biggl[2b\lambda\int x^m(\arctan x)^n\,dx\biggr].$$

35. $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n(y-ax^m-b)^2+amx^{m-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle\Xi=\frac{1}{y-ax^m-b}+\lambda\int(\arctan x)^n\,dx.$

36. $x\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2+ky+\lambda b^2x^{2k}(\arctan x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle\Xi=\arctan\Bigl(\frac{y}{bx^k}\Bigr)-\lambda b\int x^{k-1}(\arctan x)^n\,dx.$

► **Coefficients of equations contain arccotangent.**

37. $\dfrac{\partial w}{\partial x}+\big[a\,\mathrm{arccot}^k(\lambda x)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle\Xi=y-bx-a\int\mathrm{arccot}^k(\lambda x)\,dx.$

38. $\dfrac{\partial w}{\partial x}+\big[a\,\mathrm{arccot}^k(\lambda y)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle\Xi=x-\int\frac{dy}{a\,\mathrm{arccot}^k(\lambda y)+b}.$

39. $\dfrac{\partial w}{\partial x}+k\,\mathrm{arccot}^n(ax+by+c)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{a+bk\,\mathrm{arccot}^n v}-x,\qquad v=ax+by+c.$$

40. $\dfrac{\partial w}{\partial x}+a\,\mathrm{arccot}^k(\lambda x)\,\mathrm{arccot}^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int \mathrm{arccot}^k(\lambda x)\,dx-\int\frac{dy}{\mathrm{arccot}^n(\mu y)}.$

41. $\dfrac{\partial w}{\partial x}+\bigl[y^2+\lambda(\mathrm{arccot}\,x)^n y-a^2+a\lambda(\mathrm{arccot}\,x)^n\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{-2ax}E}{y+a}+\int e^{-2ax}E\,dx,\qquad E=\exp\biggl[\lambda\int(\mathrm{arccot}\,x)^n\,dx\biggr].$$

42. $\dfrac{\partial w}{\partial x}+\bigl[y^2+\lambda x(\mathrm{arccot}\,x)^n y+\lambda(\mathrm{arccot}\,x)^n\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x(xy+1)}+\int x^{-2}E\,dx,\qquad E=\exp\biggl[\lambda\int x(\mathrm{arccot}\,x)^n\,dx\biggr].$$

43. $\dfrac{\partial w}{\partial x}-\bigl[(k+1)x^k y^2-\lambda(\mathrm{arccot}\,x)^n(x^{k+1}y-1)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x^{k+1}(x^{k+1}y-1)}-(k+1)\int x^{-k-2}E\,dx,\qquad E=\exp\biggl[\lambda\int x^{k+1}(\mathrm{arccot}\,x)^n\,dx\biggr].$$

44. $\dfrac{\partial w}{\partial x}+\bigl[\lambda(\mathrm{arccot}\,x)^n y^2+ay+ab-b^2\lambda(\mathrm{arccot}\,x)^n\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{ax}E}{y+b}+\lambda\int e^{ax}(\mathrm{arccot}\,x)^n E\,dx,\qquad E=\exp\biggl[-2b\lambda\int(\mathrm{arccot}\,x)^n\,dx\biggr].$$

45. $\dfrac{\partial w}{\partial x}+\bigl[\lambda(\mathrm{arccot}\,x)^n y^2-b\lambda x^m(\mathrm{arccot}\,x)^n y+bmx^{m-1}\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\mathrm{arccot}\,x)^n E\,dx,\qquad E=\exp\biggl[b\lambda\int x^m(\mathrm{arccot}\,x)^n\,dx\biggr].$$

46. $\dfrac{\partial w}{\partial x}+\bigl[\lambda(\mathrm{arccot}\,x)^n y^2+bmx^{m-1}-\lambda b^2x^{2m}(\mathrm{arccot}\,x)^n\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\mathrm{arccot}\,x)^n E\,dx,\qquad E=\exp\biggl[2b\lambda\int x^m(\mathrm{arccot}\,x)^n\,dx\biggr].$$

47. $\dfrac{\partial w}{\partial x}+\bigl[\lambda(\mathrm{arccot}\,x)^n(y-ax^m-b)^2+amx^{m-1}\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{y-ax^m-b}+\lambda\displaystyle\int(\mathrm{arccot}\,x)^n\,dx.$

48. $x\dfrac{\partial w}{\partial x}+\big[\lambda(\operatorname{arccot} x)^n y^2+ky+\lambda b^2x^{2k}(\operatorname{arccot} x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\operatorname{arccot}\left(\dfrac{y}{bx^k}\right)-\lambda b\displaystyle\int x^{k-1}(\operatorname{arccot} x)^n\,dx.$

1.1.7 Equations Containing Arbitrary Functions of x

◆ *Notation:* $f=f(x)$, $g=g(x)$, *and* $h=h(x)$ *are arbitrary functions, and* a, b, k, n, *and* λ *are arbitrary parameters.*

► **Equations contain arbitrary and power-law functions.**

1. $\dfrac{\partial w}{\partial x}+\big[f(x)y+g(x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=e^{-F}y-\int e^{-F}g(x)\,dx,\qquad F=\int f(x)\,dx.$$

2. $\dfrac{\partial w}{\partial x}+\big[f(x)y+g(x)y^k\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=e^{-F}y^{1-k}-(1-k)\int e^{-F}g(x)\,dx,\qquad F=(1-k)\int f(x)\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

3. $\dfrac{\partial w}{\partial x}+(y^2+fy-a^2-af)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{2ax}E}{y-a}+\int e^{2ax}E\,dx,\qquad E=\exp\left(\int f\,dx\right).$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

4. $\dfrac{\partial w}{\partial x}+(y^2+xfy+f)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x(xy+1)}+\int x^{-2}E\,dx,\qquad E=\exp\left(\int xf\,dx\right).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

5. $\dfrac{\partial w}{\partial x} - \bigl[(k+1)x^k y^2 - x^{k+1} f y + f\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y-1)} - (k+1)\int x^{-k-2}E\,dx, \qquad E = \exp\biggl(\int x^{k+1} f\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

6. $\dfrac{\partial w}{\partial x} + (fy^2 + ay - ab - b^2 f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{ax}E}{y-b} + \int e^{ax} f E\,dx, \qquad E = \exp\biggl(2b\int f\,dx\biggr).$$

7. $\dfrac{\partial w}{\partial x} + (fy^2 - ax^n f y + anx^{n-1})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-ax^n} + \int fE\,dx, \qquad E = \exp\biggl(a\int x^n f\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

8. $\dfrac{\partial w}{\partial x} + (fy^2 + anx^{n-1} - a^2x^{2n} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-ax^n} + \int fE\,dx, \qquad E = \exp\biggl(2a\int x^n f\,dx\biggr).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

9. $\dfrac{\partial w}{\partial x} + (fy^2 + gy - a^2 f - ag)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-a} + \int fE\,dx, \qquad E = \exp\int(2af+g)\,dx.$$

10. $\dfrac{\partial w}{\partial x} + (fy^2 + gy + anx^{n-1} - ax^n g - a^2x^{2n} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-ax^n} + \int fE\,dx, \qquad E = \exp\biggl[\int(2ax^n f + g)\,dx\biggr].$$

11. $\dfrac{\partial w}{\partial x} + \big[fy^2 - ax^n gy + anx^{n-1} + a^2x^{2n}(g-f)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ax^n} + \int fE\,dx, \qquad E = \exp\left[a\int x^n(2f-g)\,dx\right].$$

12. $x\dfrac{\partial w}{\partial x} + (fy^2 + ny + ax^{2n}f)\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $a > 0$:

$$\Xi = \arctan\left(\frac{y}{\sqrt{a}\,x^n}\right) - \sqrt{a}\int x^{n-1}f\,dx.$$

2°. Principal integral for $a < 0$:

$$\Xi = \operatorname{arctanh}\left(\frac{y}{\sqrt{|a|}\,x^n}\right) + \sqrt{|a|}\int x^{n-1}f\,dx.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

13. $x\dfrac{\partial w}{\partial x} + \big[x^{2n}fy^2 + (ax^nf - n)y + bf\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{v^2 + av + b} - \int x^{n-1}f\,dx, \qquad v = x^n y.$$

▶ **Equations contain arbitrary and exponential functions.**

14. $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + ae^{\lambda x}fy + \lambda f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{-2\lambda x}E}{ay + \lambda e^{-\lambda x}} + \int e^{-\lambda x}E\,dx, \quad E = \exp\left(a\int e^{\lambda x}f\,dx\right).$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

15. $\dfrac{\partial w}{\partial x} + (fy^2 - ae^{\lambda x}fy + a\lambda e^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\left(a\int e^{\lambda x}f\,dx\right).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

16. $\dfrac{\partial w}{\partial x} + (fy^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl(2a\int e^{\lambda x} f\,dx\biggr).$$

17. $\dfrac{\partial w}{\partial x} + (fy^2 + \lambda y + ae^{2\lambda x} f)\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $a > 0$:

$$\Xi = \arctan\biggl(\frac{e^{-\lambda x} y}{\sqrt{a}}\biggr) - \sqrt{a}\int e^{\lambda x} f\,dx.$$

2°. Principal integral for $a < 0$:

$$\Xi = \operatorname{arctanh}\biggl(\frac{e^{-\lambda x} y}{\sqrt{|a|}}\biggr) + \sqrt{|a|}\int e^{\lambda x} f\,dx.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

18. $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - (ae^{\lambda x} + b)fy + a\lambda e^{\lambda x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x} - b} + \int fE\,dx, \qquad E = \exp\biggl[\int (ae^{\lambda x} + b) f\,dx\biggr].$$

19. $\dfrac{\partial w}{\partial x} + \bigl[e^{\lambda x} fy^2 + (af - \lambda)y + be^{-\lambda x} f\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v^2 + av + b} - \int f(x)\,dx, \qquad v = e^{\lambda x} y.$$

20. $\dfrac{\partial w}{\partial x} + (fy^2 + gy + a\lambda e^{\lambda x} - ae^{\lambda x} g - a^2 e^{2\lambda x} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl[\int (2ae^{\lambda x} f + g)\,dx\biggr].$$

21. $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - ae^{\lambda x} gy + a\lambda e^{\lambda x} + a^2 e^{2\lambda x}(g - f)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl[a\int e^{\lambda x}(2f - g)\,dx\biggr].$$

22. $\dfrac{\partial w}{\partial x}+\bigl(fy^2+2a\lambda xe^{\lambda x^2}-a^2fe^{2\lambda x^2}\bigr)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-ae^{\lambda x^2}}+\int fE\,dx,\qquad E=\exp\biggl(2a\int e^{\lambda x^2}f\,dx\biggr).$$

23. $\dfrac{\partial w}{\partial x}+\bigl(fy^2+2\lambda xy+afe^{2\lambda x^2}\bigr)\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $a>0$:

$$\Xi=\arctan\biggl(\frac{e^{-\lambda x^2}y}{\sqrt{a}}\biggr)-\sqrt{a}\int e^{\lambda x^2}f\,dx.$$

2°. Principal integral for $a<0$:

$$\Xi=\operatorname{arctanh}\biggl(\frac{e^{-\lambda x^2}y}{\sqrt{|a|}}\biggr)+\sqrt{|a|}\int e^{\lambda x^2}f\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

24. $\dfrac{\partial w}{\partial x}+\bigl[f(x)e^{\lambda y}+g(x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=e^{-\lambda y}E+\lambda\int f(x)E\,dx,\qquad E=\exp\biggl[\lambda\int g(x)\,dx\biggr].$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

► Equations contain arbitrary and hyperbolic functions.

25. $\dfrac{\partial w}{\partial x}+\bigl[fy^2-a^2f+a\lambda\sinh(\lambda x)-a^2f\sinh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\cosh(\lambda x)}+\int fE\,dx,\qquad E=\exp\biggl[2a\int f\cosh(\lambda x)\,dx\biggr].$$

26. $\dfrac{\partial w}{\partial x}+\bigl[fy^2-a(af+\lambda)\tanh^2(\lambda x)+a\lambda\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\tanh(\lambda x)}+\int fE\,dx,\qquad E=\exp\biggl[2a\int f\tanh(\lambda x)\,dx\biggr].$$

27. $\dfrac{\partial w}{\partial x}+\bigl[fy^2-a(af+\lambda)\coth^2(\lambda x)+a\lambda\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\coth(\lambda x)}+\int fE\,dx,\qquad E=\exp\biggl[2a\int f\coth(\lambda x)\,dx\biggr].$$

► **Equations contain arbitrary and logarithmic functions.**

28. $\dfrac{\partial w}{\partial x} - \big[ay^2 \ln x - axy(\ln x - 1)f + f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(\ln x - 1)[axy(\ln x - 1) - 1]} - \int \frac{E \ln x\, dx}{x^2(\ln x - 1)^2},$$
$$E = \exp\left[a \int xf(\ln x - 1)\, dx\right].$$

29. $\dfrac{\partial w}{\partial x} + \big[fy^2 - ax(\ln x)fy + a\ln x + a\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ax\ln x} + \int fE\, dx, \qquad E = \exp\left(a \int xf \ln x\, dx\right).$$

30. $x\dfrac{\partial w}{\partial x} + \big[fy^2 + a - a^2(\ln x)^2 f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\ln x} + \int x^{-1} fE\, dx, \qquad E = \exp\left(2a \int x^{-1} f \ln x\, dx\right).$$

31. $x\dfrac{\partial w}{\partial x} + \big[(y + a\ln x)^2 f - a\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y + a\ln x} + \displaystyle\int \frac{f(x)}{x}\, dx.$

► **Equations contain arbitrary and trigonometric functions.**

32. $\dfrac{\partial w}{\partial x} + \big[\lambda \sin(\lambda x) y^2 + f\cos(\lambda x)y - f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)[\cos(\lambda x)y - 1]} + \lambda \int \frac{\sin(\lambda x)}{\cos^2(\lambda x)} E\, dx, \qquad E = \exp\left[\int f\cos(\lambda x)\, dx\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

33. $\dfrac{\partial w}{\partial x} + \big[fy^2 - a^2 f + a\lambda\sin(\lambda x) + a^2 f\sin^2(\lambda x)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + a\cos(\lambda x)} + \int fE\, dx, \qquad E = \exp\left[-2a \int f\cos(\lambda x)\, dx\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

34. $\displaystyle \frac{\partial w}{\partial x} + \bigl[fy^2 - a^2 f + a\lambda\cos(\lambda x) + a^2 f\cos^2(\lambda x)\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\sin(\lambda x)} + \int fE\,dx, \qquad E = \exp\left[2a\int f\sin(\lambda x)\,dx\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

35. $\displaystyle \frac{\partial w}{\partial x} + \bigl[fy^2 - a(af - \lambda)\tan^2(\lambda x) + a\lambda\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\tan(\lambda x)} + \int fE\,dx, \qquad E = \exp\left[2a\int f\tan(\lambda x)\,dx\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

36. $\displaystyle \frac{\partial w}{\partial x} + \bigl[fy^2 - a(af - \lambda)\cot^2(\lambda x) + a\lambda\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + a\cot(\lambda x)} + \int fE\,dx, \qquad E = \exp\left[-2a\int f\cot(\lambda x)\,dx\right].$$

► **Equations contain arbitrary functions and their derivatives.**

37. $\displaystyle \frac{\partial w}{\partial x} + (fy^2 - fgy + g'_x)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - g} + \int fE\,dx, \qquad E = \exp\left(\int fg\,dx\right).$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

38. $\displaystyle \frac{\partial w}{\partial x} - (f'_x y^2 - fgy + g)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{f(fy - 1)} - \int \frac{f'_x E}{f^2}\,dx, \qquad E = \exp\left(\int fg\,dx\right).$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

39. $\displaystyle \frac{\partial w}{\partial x} + \bigl[g(y - f)^2 + f'_x\bigr]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \frac{1}{y - f} + \int g\,dx.$

40. $\dfrac{\partial w}{\partial x} + \left(\dfrac{f'_x}{g} y^2 - \dfrac{g'_x}{f}\right) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{f(fy+g)} + \int \frac{f'_x \, dx}{f^2 g}.$$

41. $f^2 \dfrac{\partial w}{\partial x} + \left[f'_x y^2 - g(y-f)\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{f^2 E}{y-f} + \int f'_x E \, dx, \qquad E = \exp\biggl(-\int \frac{g \, dx}{f^2}\biggr).$$

42. $\dfrac{\partial w}{\partial x} + \left(y^2 - \dfrac{f''_{xx}}{f}\right) \dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{f(fy+f'_x)} + \displaystyle\int \frac{dx}{f^2}.$

43. $g \dfrac{\partial w}{\partial x} + \left[afgy^3 + (bfg^3 + g'_x)y + cfg^4\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^3 + bv + c} - \int fg^2 \, dx, \qquad v = \frac{y}{g}.$$

44. $\dfrac{\partial w}{\partial x} + \left[fy^3 + 3fhy^2 + (g + 3fh^2)y + fh^3 + gh - h'_x\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{(y+h)^2} + 2\int fE \, dx, \qquad E = \exp\biggl(2\int g \, dx\biggr).$$

45. $\dfrac{\partial w}{\partial x} + \left[\dfrac{g'_x}{f^2(ag+b)^3} y^3 + \dfrac{f'_x}{f} y + fg'_x\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v^3 - av + 1} - \frac{1}{a} \ln|ag+b|, \qquad v = \frac{y}{f(ag+b)}.$$

46. $\dfrac{\partial w}{\partial x} + \left[(y-f)(y-g)\Bigl(y - \dfrac{af+bg}{a+b}\Bigr)h + \dfrac{y-g}{f-g} f'_x + \dfrac{y-f}{g-f} g'_x\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = E|y-f|^a |y-g|^b \left|y - \frac{af+bg}{a+b}\right|^{-a-b}, \qquad E = \exp\biggl[-\frac{ab}{a+b} \int (f-g)^2 h \, dx\biggr].$$

47. $\dfrac{\partial w}{\partial x}+(fy^2+g'_x y+afe^{2g})\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $a>0$:

$$\Xi=\arctan\left(\frac{e^{-g}y}{\sqrt{a}}\right)-\sqrt{a}\int fe^g\,dx.$$

2°. Principal integral for $a<0$:

$$\Xi=\operatorname{arctanh}\left(\frac{e^{-g}y}{\sqrt{|a|}}\right)+\sqrt{|a|}\int fe^g\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

48. $\dfrac{\partial w}{\partial x}+(f'_x y^2+ae^{\lambda x}fy+ae^{\lambda x})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{f(fy+1)}+\int\frac{f'_x E}{f^2}\,dx,\qquad E=\exp\left(a\int e^{\lambda x}f\,dx\right).$$

1.1.8 Equations Containing Arbitrary Functions of Different Arguments

► **Equations contain arbitrary functions of x and arbitrary functions of y.**

1. $f(x)\dfrac{\partial w}{\partial x}+g(y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\displaystyle\int\frac{dx}{f(x)}-\int\frac{dy}{g(y)}.$

⊙ *Literature*: E. Kamke (1965).

2. $[f(x)+g(y)]\dfrac{\partial w}{\partial x}+f'_x(x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=f(x)e^{-y}-\displaystyle\int e^{-y}g(y)\,dy.$

3. $[x^n f(y)+xg(y)]\dfrac{\partial w}{\partial x}+h(y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)\int\frac{f(y)E}{h(y)}\,dy,\qquad E=\exp\left[(n-1)\int\frac{g(y)}{h(y)}\,dy\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

4. $\big[f(y)+amx^n y^{m-1}\big]\dfrac{\partial w}{\partial x}-\big[g(x)+anx^{n-1}y^m\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\displaystyle\int f(y)\,dy+\int g(x)\,dx+ax^n y^m.$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

5. $\big[e^{\alpha x}f(y)+c\beta\big]\dfrac{\partial w}{\partial x}-\big[e^{\beta y}g(x)+c\alpha\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\displaystyle\int e^{-\beta y}f(y)\,dy+\int e^{-\alpha x}g(x)\,dx-ce^{-\alpha x-\beta y}.$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

► Equations contain one arbitrary function of complicated argument.

6. $\dfrac{\partial w}{\partial x}+f(ax+by+c)\dfrac{\partial w}{\partial y}=0,\qquad b\neq 0.$

Principal integral:

$$\Xi=\int\frac{dv}{a+bf(v)}-x,\qquad v=ax+by+c.$$

7. $\dfrac{\partial w}{\partial x}+f\Big(\dfrac{y}{x}\Big)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{f(v)-v}-\ln|x|,\qquad v=\frac{y}{x}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

8. $\dfrac{\partial w}{\partial x}+\big[f(y+ax^n+b)-anx^{n-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{f(v)}-x,\qquad v=y+ax^n+b.$$

9. $x\dfrac{\partial w}{\partial x}+yf(x^n y^m)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{v\big[mf(v)+n\big]}-\ln|x|,\qquad v=x^n y^m.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

10. $y^{m-1}\dfrac{\partial w}{\partial x}+x^{n-1}f(ax^n+by^m)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{an+bmf(v)}-\frac{1}{n}x^n,\qquad v=ax^n+by^m.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

11. $\dfrac{\partial w}{\partial x} + e^{-\lambda x} f(e^{\lambda x} y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{f(v) + \lambda v} - x, \qquad v = e^{\lambda x} y.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

12. $\dfrac{\partial w}{\partial x} + e^{\lambda y} f(e^{\lambda y} x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v[\lambda v f(v) + 1]} - \ln|x|, \qquad v = e^{\lambda y} x.$$

13. $\dfrac{\partial w}{\partial x} + y f(e^{\alpha x} y^m)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v[\alpha + m f(v)]} - x, \qquad v = e^{\alpha x} y^m.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

14. $x\dfrac{\partial w}{\partial x} + f(x^n e^{\alpha y})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v[n + \alpha f(v)]} - \ln|x|, \qquad v = x^n e^{\alpha y}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

15. $\dfrac{\partial w}{\partial x} + e^{\lambda x - \beta y} f(a e^{\lambda x} + b e^{\beta y})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{a\lambda + b\beta f(v)} - \frac{1}{\lambda} e^{\lambda x}, \qquad v = a e^{\lambda x} + b e^{\beta y}.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

16. $\dfrac{\partial w}{\partial x} + \bigl[f(y + a e^{\lambda x} + b) - a\lambda e^{\lambda x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{f(v)} - x, \qquad v = y + a e^{\lambda x} + b.$$

17. $\alpha xy\dfrac{\partial w}{\partial x}+\big[\alpha f(x^n e^{\alpha y})-ny\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=yE-\frac{1}{\alpha}\int v^{-1}E\,dv,\qquad v=x^n e^{\alpha y},\quad E=\exp\biggl[\frac{n}{\alpha^2}\int\frac{dv}{vf(v)}\biggr].$$

18. $mx(\ln y)\dfrac{\partial w}{\partial x}+\big[yf(x^n y^m)-ny\ln y\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=E\ln y-\frac{1}{m}\int v^{-1}E\,dv,\qquad v=x^n y^m,\quad E=\exp\biggl[\frac{n}{m}\int\frac{dv}{vf(v)}\biggr].$$

19. $\dfrac{\partial w}{\partial x}+\big[f(y+a\tan x)-a\tan^2 x\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{a+f(v)}-x,\qquad v=y+a\tan x.$$

20. $e^{\lambda x}\dfrac{\partial w}{\partial x}+f(\lambda x+\ln y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{e^v\,dv}{f(v)+\lambda e^v}-x,\qquad v=\lambda x+\ln y.$$

21. $\dfrac{\partial w}{\partial x}+e^{\lambda y}f(\lambda y+\ln x)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{\lambda e^v f(v)+1}-\ln x,\qquad v=\lambda y+\ln x.$$

► **Equations contain several arbitrary functions.**

22. $mx\dfrac{\partial w}{\partial x}-\big[ny-xy^k f(x)g(x^n y^m)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int v^{\frac{1-k-m}{m}}\frac{dv}{g(v)}-\int x^{\frac{n(1-k)}{m}}f(x)\,dx,\qquad v=x^n y^m.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

23. $y^n\dfrac{\partial w}{\partial x}-\big[ax^n+g(x)f(y^{n+1}+ax^{n+1})\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{f(v)}+(n+1)\int g(x)\,dx,\qquad v=y^{n+1}+ax^{n+1}.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

24. $\Big[f\Big(\frac{y}{x}\Big)+x^a h\Big(\frac{y}{x}\Big)\Big]\frac{\partial w}{\partial x}+\Big[g\Big(\frac{y}{x}\Big)+yx^{a-1}h\Big(\frac{y}{x}\Big)\Big]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = x^{-a}E + a\int \frac{h(v)E\,dv}{g(v)-vf(v)}, \qquad v=\frac{y}{x},$$

where $E=\exp\biggl[a\int\frac{f(v)\,dv}{g(v)-vf(v)}\biggr]$.

25. $\big[f(ax+by)+bxg(ax+by)\big]\frac{\partial w}{\partial x}+\big[h(ax+by)-axg(ax+by)\big]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = xE - \int \frac{f(v)E\,dv}{af(v)+bh(v)}, \qquad v=ax+by,$$

where $E=\exp\biggl[-b\int\frac{g(v)\,dv}{af(v)+bh(v)}\biggr]$.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

26. $\big[f(ax+by)+byg(ax+by)\big]\frac{\partial w}{\partial x}+\big[h(ax+by)-ayg(ax+by)\big]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = yE - \int \frac{h(v)E\,dv}{af(v)+bh(v)}, \qquad v=ax+by,$$

where $E=\exp\biggl[a\int\frac{g(v)\,dv}{af(v)+bh(v)}\biggr]$.

27. $x\big[f(x^ny^m)+mx^kg(x^ny^m)\big]\frac{\partial w}{\partial x}+y\big[h(x^ny^m)-nx^kg(x^ny^m)\big]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = x^{-k}E + km\int \frac{g(v)E\,dv}{v\big[nf(v)+mh(v)\big]}, \qquad v=x^ny^m,$$

where $E=\exp\biggl\{k\int\frac{f(v)\,dv}{v\big[nf(v)+mh(v)\big]}\biggr\}$.

28. $x\big[f(x^ny^m)+my^kg(x^ny^m)\big]\frac{\partial w}{\partial x}+y\big[h(x^ny^m)-ny^kg(x^ny^m)\big]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = y^{-k}E - kn\int \frac{g(v)E\,dv}{v\big[nf(v)+mh(v)\big]}, \qquad v=x^ny^m,$$

where $E=\exp\biggl\{k\int\frac{h(v)\,dv}{v\big[nf(v)+mh(v)\big]}\biggr\}$.

29. $x\big[sf(x^n y^m) - mg(x^k y^s)\big]\dfrac{\partial w}{\partial x} + y\big[ng(x^k y^s) - kf(x^n y^m)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{vg(v)} - \int \frac{dz}{zf(z)}, \qquad v = x^k y^s, \quad z = x^n y^m.$$

30. $f_y\dfrac{\partial w}{\partial x} - f_x\dfrac{\partial w}{\partial y} = 0.$

Here f_x and f_y are the partial derivatives of the function $f = f(x, y)$ with respect to x and y.

General solution: $w = \Phi(f(x, y))$, where $\Phi = \Phi(\xi)$ is an arbitrary function.

⊙ *Literature*: E. Kamke (1965).

31. $f(x,\, y)\dfrac{\partial w}{\partial x} - g(x,\, y)\dfrac{\partial w}{\partial y} = 0,$ **where** $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y}.$

Principal integral:

$$\Xi = \int_{y_0}^{y} f(x_0,\, t)\, dt + \int_{x_0}^{x} g(t,\, y)\, dt,$$

where x_0 and y_0 are arbitrary constants.

32. $x\dfrac{\partial w}{\partial x} + \big[xf(x)g(x^n e^y) - n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{vg(v)} - \int f(x)\, dx, \qquad v = x^n e^y.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

33. $m\dfrac{\partial w}{\partial x} + \big[my^k f(x)g(e^{\alpha x} y^m) - \alpha y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int v^{\frac{1-k-m}{m}} \frac{dv}{g(v)} - m\int f(x) \exp\left[\frac{\alpha(1-k)}{m}x\right] dx, \qquad v = e^{\alpha x} y^m.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

34. $\big[f(ax+by)+be^{\lambda y}g(ax+by)\big]\dfrac{\partial w}{\partial x}+\big[h(ax+by)-ae^{\lambda y}g(ax+by)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = e^{-\lambda y}E - \lambda a\int \frac{g(v)E\, dv}{af(v) + bh(v)}, \qquad v = ax + by,$$

where $E = \exp\left[\lambda \displaystyle\int \frac{h(v)\, dv}{af(v) + bh(v)}\right].$

35. $\boldsymbol{\big[f(ax+by)+be^{\alpha x}g(ax+by)\big]\frac{\partial w}{\partial x}+\big[h(ax+by)-ae^{\alpha x}g(ax+by)\big]\frac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi = e^{-\alpha x}E + \alpha b\int \frac{g(v)E\,dv}{af(v)+bh(v)}, \qquad v = ax+by,$$

where $E = \exp\left[\alpha\int \frac{f(v)\,dv}{af(v)+bh(v)}\right]$.

36. $\boldsymbol{x\big[f(x^n e^{\alpha y}) + \alpha y g(x^n e^{\alpha y})\big]\frac{\partial w}{\partial x} + \big[h(x^n e^{\alpha y}) - nyg(x^n e^{\alpha y})\big]\frac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = yE - \int \frac{h(v)E\,dv}{v\big[nf(v)+\alpha h(v)\big]}, \qquad v = x^n e^{\alpha y},$$

where $E = \exp\left\{n\int \frac{g(v)\,dv}{v\big[nf(v)+\alpha h(v)\big]}\right\}$.

37. $\boldsymbol{\big[f(e^{\alpha x}y^m) + mxg(e^{\alpha x}y^m)\big]\frac{\partial w}{\partial x} + y\big[h(e^{\alpha x}y^m) - \alpha xg(e^{\alpha x}y^m)\big]\frac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = xE - \int \frac{f(v)E\,dv}{v\big[\alpha f(v)+mh(v)\big]}, \qquad v = e^{\alpha x}y^m,$$

where $E = \exp\left\{-m\int \frac{g(v)\,dv}{v\big[\alpha f(v)+mh(v)\big]}\right\}$.

38. $\boldsymbol{x\frac{\partial w}{\partial x} + \big[xyf(x)g(x^n\ln y) - ny\ln y\big]\frac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int x^n f(x)\,dx, \qquad v = x^n \ln y.$$

39. $\boldsymbol{x\big[f(x^n y^m)+mg(x^n y^m)\ln y\big]\frac{\partial w}{\partial x}+y\big[h(x^n y^m)-ng(x^n y^m)\ln y\big]\frac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi = E\ln y - \int \frac{h(v)E\,dv}{v\big[nf(v)+mh(v)\big]}, \qquad v = x^n y^m,$$

where $E = \exp\left\{n\int \frac{g(v)\,dv}{v\big[nf(v)+mh(v)\big]}\right\}$.

40. $x\big[f(x^n y^m)+mg(x^n y^m)\ln x\big]\dfrac{\partial w}{\partial x}+y\big[h(x^n y^m)-ng(x^n y^m)\ln x\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = E\ln x - \int \frac{f(v)E\,dv}{v\big[nf(v)+mh(v)\big]}, \qquad v = x^n y^m,$$

where $E = \exp\left\{-m\displaystyle\int \frac{g(v)\,dv}{v\big[nf(v)+mh(v)\big]}\right\}.$

41. $\cos y\,\dfrac{\partial w}{\partial x} + \big[f(x)g(\sin x\sin y) - \cot x\sin y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int f(x)\sin x\,dx, \qquad v = \sin x\sin y.$$

42. $\sin 2x\,\dfrac{\partial w}{\partial x} + \big[\sin 2x\cos^2 y\,f(x)g(\tan x\tan y) - \sin 2y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int f(x)\tan x\,dx, \qquad v = \tan x\tan y.$$

43. $x\dfrac{\partial w}{\partial x} + \big[x\cos^2 y\,f(x)g(x^{2n}\tan y) - n\sin 2y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int x^{2n} f(x)\,dx, \qquad v = x^{2n}\tan y.$$

44. $\dfrac{\partial w}{\partial x} + \big[\cos^2 y\,f(x)g(e^{2x}\tan y) - \sin 2y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int e^{2x} f(x)\,dx, \qquad v = e^{2x}\tan y.$$

1.2 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = h(x,y)$

◆ *The solutions given below contain an arbitrary function* $\Phi = \Phi(z)$.

1.2.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x and y.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c.$

The *equation of a cylindrical surface*. Two forms of the general solution:

$$w = \frac{c}{a}x + \Phi(bx - ay), \qquad w = \frac{c}{b}y + \Phi(bx - ay).$$

⊙ *Literature*: E. Kamke (1965).

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \alpha x + \beta y + \gamma.$

General solution: $w = \dfrac{\alpha}{2a}x^2 + \dfrac{\gamma}{a}x + \dfrac{\beta}{2b}y^2 + \Phi(bx - ay).$

3. $ax\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \alpha x + \beta y + \gamma.$

General solution: $w = \dfrac{\alpha}{a}x + \dfrac{\gamma}{a}\ln|x| + \dfrac{\beta}{2b}y^2 + \Phi\bigl(ay - b\ln|x|\bigr).$

4. $ax\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = c.$

General solution: $w = \dfrac{c}{a}\ln|x| + \Phi(bx - ay).$

5. $(ax + b)\dfrac{\partial w}{\partial x} + (cy + d)\dfrac{\partial w}{\partial y} = \alpha x + \beta y + \gamma.$

General solution:

$$w = \frac{\alpha}{a}x + \frac{a\gamma - b\alpha}{a^2}\ln|ax + b| + \frac{\beta}{c}y - \frac{d\beta}{c^2}\ln|cy + d| + \Phi\bigl(|ax + b|^c|cy + d|^{-a}\bigr).$$

6. $ay\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \alpha x + \beta y + \gamma.$

General solution: $w = \dfrac{\beta}{a}x + \dfrac{\alpha x + \gamma}{b}y - \dfrac{a\alpha}{3b^2}y^3 + \Phi\bigl(2bx - ay^2\bigr).$

7. $ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = c.$

General solution: $w = \dfrac{c}{\sqrt{ab}}\ln\bigl|\sqrt{ab}\,x + ay\bigr| + \Phi\bigl(ay^2 - bx^2\bigr).$

8. $ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = cx + ky.$

General solution: $w = \dfrac{bkx + acy}{ab} + \Phi\bigl(ay^2 - bx^2\bigr).$

▶ **Coefficients of equations are quadratic in x and y.**

9. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cx^2 + dy^2 + kxy + n.$

General solution:

$$w = \frac{1}{6a^2 b}\big[b(2ac - bk)x^3 + 2a^2 dy^3 + 3abx(kxy + 2n)\big] + \Phi(bx - ay).$$

10. $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = cx^2 + dy^2 + kxy + n.$

General solution:

$$w = \begin{cases} \dfrac{1}{2ab}(bcx^2 + ady^2) + \dfrac{n}{a}\ln|x| + \dfrac{k}{a+b}xy + \Phi\big(|x|^{-b/a}y\big) & \text{if } a + b \neq 0, \\ \dfrac{1}{2a}(cx^2 - dy^2) + \dfrac{1}{a}(kxy + n)\ln|x| + \Phi(xy) & \text{if } a + b = 0. \end{cases}$$

11. $\displaystyle ay\frac{\partial w}{\partial x} + bx\frac{\partial w}{\partial y} = cxy + d.$

General solution: $\displaystyle w = \frac{c}{2a}x^2 + \frac{d}{\sqrt{ab}}\ln\big|\sqrt{ab}\,x + ay\big| + \Phi(ay^2 - bx^2).$

12. $\displaystyle ax^2\frac{\partial w}{\partial x} + by^2\frac{\partial w}{\partial y} = cx^2 + dy^2 + kxy + nx + my + s.$

General solution:

$$w = \frac{c}{a}x - \frac{s}{ax} - \frac{dy^2}{ax - by} + \frac{kxy}{ax - by}\ln\left|\frac{ax}{y}\right| + \frac{n}{a}\ln|x| + \frac{m}{b}\ln|y| + \Phi\left(\frac{ax - by}{xy}\right).$$

13. $\displaystyle x^2\frac{\partial w}{\partial x} + axy\frac{\partial w}{\partial y} = by^2.$

General solution:

$$w = \begin{cases} \dfrac{b}{2a - 1}\dfrac{y^2}{x} + \Phi(|x|^{-a}y) & \text{if } a \neq \tfrac{1}{2}, \\ b\dfrac{y^2}{x}\ln|x| + \Phi(|x|^{-1/2}y) & \text{if } a = \tfrac{1}{2}. \end{cases}$$

14. $\displaystyle ay^2\frac{\partial w}{\partial x} + bx^2\frac{\partial w}{\partial y} = cx^2 + d.$

This is a special case of equation 1.2.7.14 with $k = 2$, $f(x) = a$, $g(x) = bx^2$, and $h(x) = cx^2 + d$.

15. $\displaystyle ay^2\frac{\partial w}{\partial x} + bxy\frac{\partial w}{\partial y} = cx^2 + dy^2.$

General solution:

$$w = \frac{ac + bd}{ab}x - \frac{c}{b}\sqrt{\frac{ay^2 - bx^2}{b}}\,\arctan\left(x\sqrt{\frac{b}{ay^2 - bx^2}}\right) + \Phi(ay^2 - bx^2).$$

► **Coefficients of equations contain other power-law functions.**

16. $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a\sqrt{x^2+y^2}.}$

General solution: $w = a\sqrt{x^2+y^2} + \Phi\left(\dfrac{y}{x}\right).$

17. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cxy^2 + dx^2y + k.}$

General solution: $w = \dfrac{cxy^2}{a+2b} + \dfrac{dx^2y}{2a+b} + \dfrac{k}{a}\ln|x| + \Phi\big(|x|^{-b/a}y\big).$

18. $\boldsymbol{ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = cx^2y + d.}$

General solution: $w = \dfrac{c}{3a}x^3 + \dfrac{d}{\sqrt{ab}}\ln\big|\sqrt{ab}\,x + ay\big| + \Phi(ay^2 - bx^2).$

19. $\boldsymbol{(ax+b)\dfrac{\partial w}{\partial x} + (cy+d)\dfrac{\partial w}{\partial y} = kx^3 + ny^3.}$

General solution:

$$w = \frac{k}{a}\left(\frac{1}{3}x^3 - \frac{b}{2a}x^2 + \frac{b^2}{a^2}x - \frac{b^3}{a^3}\ln|ax+b|\right) + \frac{n}{c}\left(\frac{1}{3}y^3 - \frac{d}{2c}y^2 + \frac{d^2}{c^2}y - \frac{d^3}{c^3}\ln|cy+d|\right) + \Phi\big(|ax+b|^c|cy+d|^{-a}\big).$$

20. $\boldsymbol{x^2\dfrac{\partial w}{\partial x} + xy\dfrac{\partial w}{\partial y} = y^2(ax+by).}$

General solution: $w = \dfrac{(ax+by)y^2}{2x} + \Phi\left(\dfrac{y}{x}\right).$

21. $\boldsymbol{ax^3\dfrac{\partial w}{\partial x} + by^3\dfrac{\partial w}{\partial y} = cx + d.}$

General solution: $w = -\dfrac{2cx+d}{2ax^2} + \Phi\left(\dfrac{ax^2-by^2}{x^2y^2}\right).$

► **Coefficients of equations contain arbitrary powers of x and y.**

22. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n + dy^m.}$

General solution: $w = \Phi(bx - ay) + \dfrac{c}{a(n+1)}x^{n+1} + \dfrac{d}{b(m+1)}y^{m+1}.$

23. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n y.$

General solution:

$$w = \begin{cases} \dfrac{c[a(n+2)y - bx]x^{n+1}}{a^2(n+1)(n+2)} + \Phi(bx - ay) & \text{if } n \neq -1, -2; \\ \dfrac{bc}{a^2}x\bigl(1 - \ln|x|\bigr) + \dfrac{c}{a}y\ln|x| + \Phi(bx - ay) & \text{if } n = -1; \\ \dfrac{bc}{a^2}\bigl(1 + \ln|x|\bigr) - \dfrac{cy}{ax} + \Phi(bx - ay) & \text{if } n = -2. \end{cases}$$

24. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a(x^2 + y^2)^k.$

General solution: $w = \dfrac{a}{2k}(x^2 + y^2)^k + \Phi\left(\dfrac{y}{x}\right).$

25. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cx^n y^m.$

General solution:

$$w = \begin{cases} \dfrac{c}{an + bm}x^n y^m + \Phi\bigl(|y|^a|x|^{-b}\bigr) & \text{if } an + bm \neq 0, \\ \dfrac{c}{a}x^n y^m \ln|x| + \Phi\bigl(|y|^a|x|^{-b}\bigr) & \text{if } an + bm = 0. \end{cases}$$

26. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cx^n + dy^m.$

General solution: $w = \dfrac{c}{an}x^n + \dfrac{d}{bm}y^m + \Phi\bigl(y^a x^{-b}\bigr).$

27. $mx\dfrac{\partial w}{\partial x} + ny\dfrac{\partial w}{\partial y} = (ax^n + by^m)^k.$

General solution: $w = \dfrac{1}{mnk}(ax^n + by^m)^k + \Phi\bigl(y^m x^{-n}\bigr).$

28. $ax^n\dfrac{\partial w}{\partial x} + by^m\dfrac{\partial w}{\partial y} = cx^k + dy^s.$

This is a special case of equation 1.2.7.20. General solution:

$$w = \frac{c}{a(k-n+1)}x^{k-n+1} + \frac{d}{b(s-m+1)}y^{s-m+1} + \Phi(u),$$
$$u = \frac{1}{a(1-n)}x^{1-n} - \frac{1}{b(1-m)}y^{1-m}.$$

29. $ax^n\dfrac{\partial w}{\partial x} + bx^m y\dfrac{\partial w}{\partial y} = cx^k y^s + d.$

This is a special case of equation 1.2.7.34 with $f(x) = ax^n$, $g(x) = bx^m$, and $h(x, y) = cx^k y^s + d$.

30. $\boldsymbol{ax^n \dfrac{\partial w}{\partial x} + \big(bx^m y + cx^k\big)\dfrac{\partial w}{\partial y} = sx^p y^q + d.}$

This is a special case of equation 1.2.7.35 with $f(x) = ax^n$, $g_1(x) = bx^m$, $g_0(x) = cx^k$, and $h(x,y) = sx^p y^q + d$.

31. $\boldsymbol{ax^n \dfrac{\partial w}{\partial x} + \big(bx^m y^k + cx^l y\big)\dfrac{\partial w}{\partial y} = sx^p y^q + d.}$

This is a special case of equation 1.2.7.36 with $f(x) = ax^n$, $g_1(x) = cx^l$, $g_0(x) = bx^m$, and $h(x,y) = sx^p y^q + d$.

32. $\boldsymbol{ay^k \dfrac{\partial w}{\partial x} + bx^n \dfrac{\partial w}{\partial y} = cx^m + d.}$

This is a special case of equation 1.2.7.14 with $f(x) = a$, $g(x) = bx^n$, and $h(x) = cx^m + d$.

1.2.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = ce^{\lambda x} + de^{\mu y}.}$

General solution: $w = \dfrac{c}{a\lambda}e^{\lambda x} + \dfrac{d}{b\mu}e^{\mu y} + \Phi(bx - ay)$.

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = ce^{\alpha x + \beta y}.}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\alpha + b\beta}e^{\alpha x + \beta y} + \Phi(bx - ay) & \text{if } a\alpha + b\beta \neq 0, \\ \dfrac{c}{a}xe^{\alpha x + \beta y} + \Phi(bx - ay) & \text{if } a\alpha + b\beta = 0. \end{cases}$$

3. $\boldsymbol{ae^{\lambda x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = c.}$

General solution: $w = -\dfrac{c}{a\lambda}e^{-\lambda x} + \Phi(u)$, where $u = b\beta e^{-\lambda x} - a\lambda e^{-\beta y}$.

4. $\boldsymbol{ae^{\lambda y}\dfrac{\partial w}{\partial x} + be^{\beta x}\dfrac{\partial w}{\partial y} = c.}$

General solution: $w = \dfrac{c(\beta x - \lambda y)}{u} + \Phi(u)$, where $u = a\beta e^{\lambda y} - b\lambda e^{\beta x}$.

5. $\boldsymbol{ae^{\alpha x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = ce^{\gamma x - \beta y}.}$

Introduce the notation $u = \dfrac{1}{\beta b}e^{\beta y} - \dfrac{1}{\alpha a}e^{\alpha x}$.

General solution:

$$w = \begin{cases} \dfrac{c}{a(\gamma-\alpha)} e^{(\gamma-\alpha)x} \left[e^{-\beta y} + \dfrac{b\beta e^{-\alpha x}}{a(\gamma - 2\alpha)} \right] + \Phi(u) & \text{if } \gamma \neq \alpha,\ 2\alpha, \\ \dfrac{c}{a} \left[x e^{-\beta y} - \dfrac{b\beta}{a\alpha^2} (\alpha x + 1) e^{-\alpha x} \right] + \Phi(u) & \text{if } \gamma = \alpha, \\ \dfrac{c}{a\alpha} \left[e^{\alpha x - \beta y} + \dfrac{b\beta}{a\alpha} (\alpha x - 1) \right] + \Phi(u) & \text{if } \gamma = 2\alpha. \end{cases}$$

6. $\boldsymbol{a e^{\alpha x} \dfrac{\partial w}{\partial x} + b e^{\beta y} \dfrac{\partial w}{\partial y} = c e^{\gamma x - 2\beta y}.}$

Introduce the notation $u = \dfrac{1}{\beta b} e^{\beta y} - \dfrac{1}{\alpha a} e^{\alpha x}$.

1°. General solution for $\gamma \neq \alpha$, $\gamma \neq 2\alpha$, and $\gamma \neq 3\alpha$:

$$w = \frac{c}{a(\gamma-\alpha)} \left[e^{-2\beta y} + \frac{2b\beta}{a(\gamma-2\alpha)} e^{-\alpha x - \beta y} + \frac{2b^2\beta^2}{a^2(\gamma-2\alpha)(\gamma-3\alpha)} e^{-2\alpha x} \right] e^{(\gamma-\alpha)x} + \Phi(u).$$

2°. General solution for $\gamma = \alpha$:

$$w = \frac{c}{a} \left[x e^{-2\beta y} - \frac{2b\beta}{a\alpha^2} (\alpha x + 1) e^{-\alpha x - \beta y} + \frac{b^2\beta^2}{a^2\alpha^3} \left(\alpha x + \frac{3}{2} \right) e^{-2\alpha x} \right] + \Phi(u).$$

3°. General solution for $\gamma = 2\alpha$:

$$w = \frac{c}{a\alpha} \left[e^{\alpha x - \beta y} + \frac{2b\beta}{a\alpha} (\alpha x - 1) \right] e^{-\beta y} + \Phi(u).$$

4°. General solution for $\gamma = 3\alpha$:

$$w = \frac{c}{a\alpha} \left[\frac{1}{2} e^{2(\alpha x - \beta y)} + \frac{b\beta}{a\alpha} e^{\alpha x - \beta y} + \frac{b^2\beta^2}{a^2\alpha^2} \left(\alpha x - \frac{3}{2} \right) \right] + \Phi(u).$$

7. $\boldsymbol{a e^{\alpha x} \dfrac{\partial w}{\partial x} + b e^{\beta y} \dfrac{\partial w}{\partial y} = c e^{\gamma x} + s e^{\mu y}.}$

This is a special case of equation 1.2.7.20 with $f(x) = a e^{\alpha x}$, $g(y) = b e^{\beta y}$, $h_1(x) = c e^{\gamma x}$, and $h_2(y) = s e^{\mu y}$.

8. $\boldsymbol{a e^{\beta x} \dfrac{\partial w}{\partial x} + \left(b e^{\gamma x} + c e^{\lambda y} \right) \dfrac{\partial w}{\partial y} = s e^{\mu x} + k e^{\delta y} + p.}$

This is a special case of equation 1.2.7.37 with $f(x) = a e^{\beta x}$, $g_1(x) = b e^{\gamma x}$, $g_0(x) = c$, and $h(x, y) = s e^{\mu x} + k e^{\delta y} + p$.

9. $\boldsymbol{a e^{\beta x} \dfrac{\partial w}{\partial x} + \left(b e^{\gamma x} + c e^{\lambda y} \right) \dfrac{\partial w}{\partial y} = s e^{\mu x + \delta y} + k.}$

This is a special case of equation 1.2.7.37 with $f(x) = a e^{\beta x}$, $g_1(x) = b e^{\gamma x}$, $g_0(x) = c$, and $h(x, y) = s e^{\mu x + \delta y} + k$.

10. $\boldsymbol{ae^{\beta x}\frac{\partial w}{\partial x} + be^{\gamma x+\lambda y}\frac{\partial w}{\partial y} = ce^{\mu x+\delta y} + k.}$

This is a special case of equation 1.2.7.37 with $f(x) = ae^{\beta x}$, $g_1(x) \equiv 0$, $g_0(x) = be^{\gamma x}$, and $h(x,y) = ce^{\mu x+\delta y} + k$.

11. $\boldsymbol{ae^{\lambda y}\frac{\partial w}{\partial x} + be^{\beta x}\frac{\partial w}{\partial y} = ce^{\gamma x} + d.}$

This is a special case of equation 1.2.7.16 with $f(x) = a$, $g(x) = be^{\beta x}$, and $h(x) = ce^{\gamma x} + d$.

► **Coefficients of equations contain exponential and power-law functions.**

12. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cye^{\lambda x} + kxe^{\mu y}.}$

General solution: $w = \dfrac{c}{a\lambda}e^{\lambda x}\left(y - \dfrac{b}{a\lambda}\right) + \dfrac{k}{b\mu}e^{\mu y}\left(x - \dfrac{a}{b\mu}\right) + \Phi(bx - ay)$.

13. $\boldsymbol{\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = ax^k e^{\lambda y}.}$

This is a special case of equation 1.2.7.5 with $f(x) = ax^k$.

14. $\boldsymbol{\frac{\partial w}{\partial x} + (ay + be^{\lambda x})\frac{\partial w}{\partial y} = ce^{\beta x}.}$

This is a special case of equation 1.2.7.6 with $f(x) = be^{\lambda x}$ and $g(x) = ce^{\beta x}$.

15. $\boldsymbol{\frac{\partial w}{\partial x} + (ae^{\lambda x}y + be^{\beta x}y^k)\frac{\partial w}{\partial y} = ce^{\mu x}.}$

This is a special case of equation 1.2.7.12 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_2(x) = be^{\beta x}$, and $h(x) = ce^{\mu x}$.

16. $\boldsymbol{\frac{\partial w}{\partial x} + (ax^k + bx^n e^{\lambda y})\frac{\partial w}{\partial y} = ce^{\beta x}.}$

This is a special case of equation 1.2.7.13 with $f(x) = 1$, $g_1(x) = ax^k$, $g_2(x) = bx^n$, and $h(x) = ce^{\beta x}$.

17. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = axe^{\lambda x+\mu y}.}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}e^{\lambda x+\mu y} + \Phi\left(\dfrac{y}{x}\right)$.

18. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = aye^{\lambda x} + bxe^{\mu y}.}$

General solution: $w = \dfrac{ay}{\lambda x}e^{\lambda x} + \dfrac{bx}{\mu y}e^{\mu y} + \Phi\left(\dfrac{y}{x}\right)$.

19. $\boldsymbol{ax^k\frac{\partial w}{\partial x} + be^{\lambda y}\frac{\partial w}{\partial y} = cx^n + s.}$

This is a special case of equation 1.2.7.13 with $f(x) = ax^k$, $g_1(x) = 0$, $g_2(x) = b$, and $h(x) = cx^n + s$.

20. $ay^k \dfrac{\partial w}{\partial x} + be^{\lambda x} \dfrac{\partial w}{\partial y} = ce^{\mu x} + s.$

This is a special case of equation 1.2.7.14 with $f(x) = a$, $g(x) = be^{\lambda x}$, and $h(x) = ce^{\mu x} + s$.

21. $ae^{\lambda x} \dfrac{\partial w}{\partial x} + by^k \dfrac{\partial w}{\partial y} = cx^n + s.$

This is a special case of equation 1.2.7.12 with $f(x) = ae^{\lambda x}$, $g_1(x) = 0$, $g_2(x) = b$, and $h(x) = cx^n + s$.

22. $ae^{\lambda y} \dfrac{\partial w}{\partial x} + bx^k \dfrac{\partial w}{\partial y} = ce^{\mu x} + s.$

This is a special case of equation 1.2.7.16 with $f(x) = a$, $g(x) = bx^k$, and $h(x) = ce^{\mu x} + s$.

1.2.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = c \sinh(\lambda x) + k \sinh(\mu y).$

General solution: $w = \dfrac{c}{a\lambda} \cosh(\lambda x) + \dfrac{k}{b\mu} \cosh(\mu y) + \Phi(bx - ay).$

2. $a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = c \sinh(\lambda x + \mu y).$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu} \cosh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a} x \sinh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

3. $x \dfrac{\partial w}{\partial x} + y \dfrac{\partial w}{\partial y} = ax \sinh(\lambda x + \mu y).$

General solution: $w = \dfrac{ax}{\lambda x + \mu y} \cosh(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right).$

4. $a \dfrac{\partial w}{\partial x} + b \sinh^n(\lambda x) \dfrac{\partial w}{\partial y} = c \sinh^m(\mu x) + s \sinh^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b \sinh^n(\lambda x)$, and $h(x, y) = c \sinh^m(\mu x) + s \sinh^k(\beta y)$.

5. $a \dfrac{\partial w}{\partial x} + b \sinh^n(\lambda y) \dfrac{\partial w}{\partial y} = c \sinh^m(\mu x) + s \sinh^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b \sinh^n(\lambda y)$, $h_1(x) = c \sinh^m(\mu x)$, and $h_2(y) = s \sinh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cosine.**

6. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\cosh(\lambda x) + k\cosh(\mu y).}$

General solution: $w = \dfrac{c}{a\lambda}\sinh(\lambda x) + \dfrac{k}{b\mu}\sinh(\mu y) + \Phi(bx - ay).$

7. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\cosh(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\sinh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\cosh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

8. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cosh(\lambda x + \mu y).}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\sinh(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right).$

9. $\boldsymbol{a\frac{\partial w}{\partial x} + b\cosh^n(\lambda x)\frac{\partial w}{\partial y} = c\cosh^m(\mu x) + s\cosh^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^n(\lambda x)$, and $h(x,y) = c\cosh^m(\mu x) + s\cosh^k(\beta y)$.

10. $\boldsymbol{a\frac{\partial w}{\partial x} + b\cosh^n(\lambda y)\frac{\partial w}{\partial y} = c\cosh^m(\mu x) + s\cosh^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\cosh^n(\lambda y)$, $h_1(x) = c\cosh^m(\mu x)$, and $h_2(y) = s\cosh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic tangent.**

11. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\tanh(\lambda x) + k\tanh(\mu y).}$

General solution: $w = \dfrac{c}{a\lambda}\ln\bigl[\cosh(\lambda x)\bigr] + \dfrac{k}{b\mu}\ln\bigl[\cosh(\mu y)\bigr] + \Phi(bx - ay).$

12. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\tanh(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\ln\bigl[\cosh(\lambda x + \mu y)\bigr] + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\tanh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

13. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\tanh(\lambda x + \mu y).}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\ln\bigl[\cosh(\lambda x + \mu y)\bigr] + \Phi\left(\dfrac{y}{x}\right).$

14. $\displaystyle a\frac{\partial w}{\partial x} + b\tanh^n(\lambda x)\frac{\partial w}{\partial y} = c\tanh^m(\mu x) + s\tanh^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\tanh^n(\lambda x)$, and $h(x,y) = c\tanh^m(\mu x) + s\tanh^k(\beta y)$.

15. $\displaystyle a\frac{\partial w}{\partial x} + b\tanh^n(\lambda y)\frac{\partial w}{\partial y} = c\tanh^m(\mu x) + s\tanh^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\tanh^n(\lambda y)$, $h_1(x) = c\tanh^m(\mu x)$, and $h_2(y) = s\tanh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cotangent.**

16. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\coth(\lambda x) + k\coth(\mu y).$

General solution: $\displaystyle w = \frac{c}{a\lambda}\ln\bigl|\sinh(\lambda x)\bigr| + \frac{k}{b\mu}\ln\bigl|\sinh(\mu y)\bigr| + \Phi(bx - ay).$

17. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\coth(\lambda x + \mu y).$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\ln\bigl|\sinh(\lambda x + \mu y)\bigr| + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\coth(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

18. $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\coth(\lambda x + \mu y).$

General solution: $\displaystyle w = \frac{ax}{\lambda x + \mu y}\ln\bigl|\sinh(\lambda x + \mu y)\bigr| + \Phi\Bigl(\frac{y}{x}\Bigr).$

19. $\displaystyle a\frac{\partial w}{\partial x} + b\coth^n(\lambda x)\frac{\partial w}{\partial y} = c\coth^m(\mu x) + s\coth^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\coth^n(\lambda x)$, and $h(x,y) = c\coth^m(\mu x) + s\coth^k(\beta y)$.

20. $\displaystyle a\frac{\partial w}{\partial x} + b\coth^n(\lambda y)\frac{\partial w}{\partial y} = c\coth^m(\mu x) + s\coth^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\coth^n(\lambda y)$, $h_1(x) = c\coth^m(\mu x)$, and $h_2(y) = s\coth^k(\beta y)$.

► **Coefficients of equations contain different hyperbolic functions.**

21. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\sinh(\lambda x) + k\cosh(\mu y).$

General solution: $\displaystyle w = \frac{c}{a\lambda}\cosh(\lambda x) + \frac{k}{b\mu}\sinh(\mu y) + \Phi(bx - ay).$

22. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \tanh(\lambda x) + k\coth(\mu y).}$

General solution: $w = \dfrac{1}{a\lambda}\ln\bigl|\cosh(\lambda x)\bigr| + \dfrac{k}{b\mu}\ln\bigl|\sinh(\mu y)\bigr| + \Phi(bx - ay).$

23. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \sinh(\lambda x) + k\tanh(\mu y).}$

General solution: $w = \dfrac{1}{a\lambda}\cosh(\lambda x) + \dfrac{k}{b\mu}\ln\bigl|\cosh(\mu y)\bigr| + \Phi(bx - ay).$

24. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cosh(\mu y)\dfrac{\partial w}{\partial y} = \sinh(\lambda x).}$

General solution: $w = \dfrac{1}{a\lambda}\cosh(\lambda x) + \Phi(u)$, where $u = b\mu x - 2a\arctan\Bigl(\tanh\dfrac{\mu x}{2}\Bigr).$

25. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sinh(\mu y)\dfrac{\partial w}{\partial y} = \cosh(\lambda x).}$

General solution: $w = \dfrac{1}{a\lambda}\sinh(\lambda x) + \Phi(u)$, where $u = b\mu x - a\ln\Bigl|\tanh\dfrac{\mu x}{2}\Bigr|.$

1.2.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln(\lambda x + \beta y).}$

General solution:

$$w = \begin{cases} \dfrac{c(\lambda x + \beta y)}{a\lambda + b\beta}\bigl[\ln(\lambda x + \beta y) - 1\bigr] + \Phi(bx - ay) & \text{if } a\lambda \neq -b\beta, \\ \dfrac{c}{a}x\ln(\lambda x + \beta y) + \Phi(bx - ay) & \text{if } a\lambda = -b\beta. \end{cases}$$

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln(\lambda x) + k\ln(\beta y).}$

General solution: $w = \dfrac{c}{a}x\bigl[\ln(\lambda x) - 1\bigr] + \dfrac{k}{b}y\bigl[\ln(\beta y) - 1\bigr] + \Phi(bx - ay).$

3. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\ln(\lambda x)\ln(\beta y)\dfrac{\partial w}{\partial y} = c\ln(\gamma x).}$

General solution:

$$w = \frac{c}{a}x\bigl[\ln(\gamma x) - 1\bigr] + \Phi(u), \qquad \text{where} \quad u = bx\bigl[\ln(\lambda x) - 1\bigr] - a\int\frac{dy}{\ln(\beta y)}.$$

4. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\ln^n(\lambda x)\dfrac{\partial w}{\partial y} = c\ln^m(\mu x) + s\ln^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^n(\lambda x)$, and $h(x,y) = c\ln^m(\mu x) + s\ln^k(\beta y)$.

5. $\displaystyle a\frac{\partial w}{\partial x} + b\ln^n(\lambda y)\frac{\partial w}{\partial y} = c\ln^m(\mu x) + s\ln^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\ln^n(\lambda y)$, $h_1(x) = c\ln^m(\mu x)$, and $h_2(y) = s\ln^k(\beta y)$.

6. $\displaystyle a\ln^n(\lambda x)\frac{\partial w}{\partial x} + b\ln^k(\beta y)\frac{\partial w}{\partial y} = c\ln^m(\gamma x).$

General solution:

$$w = \frac{c}{a}\int \frac{\ln^m(\gamma x)}{\ln^n(\lambda x)}\,dx + \Phi(u), \qquad \text{where} \quad u = b\int \frac{dx}{\ln^n(\lambda x)} - a\int \frac{dy}{\ln^k(\beta y)}.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

7. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cx^n + s\ln^k(\lambda y).$

General solution: $\displaystyle w = \frac{c}{a(n+1)}x^{n+1} + \frac{s}{b}\int \ln^k(\lambda y)\,dy + \Phi(bx - ay).$

8. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = by^2 + cx^n y + s\ln^k(\lambda x).$

This is a special case of equation 1.2.7.3 with $f(x) = b$, $g(x) = cx^n$, and $h(x) = s\ln^k(\lambda x)$.

9. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = b\ln^k(\lambda x)\ln^n(\beta y).$

This is a special case of equation 1.2.7.18 with $f(x) = b\ln^k(\lambda x)$ and $g(y) = \ln^n(\beta y)$.

10. $\displaystyle \frac{\partial w}{\partial x} + (ay + bx^n)\frac{\partial w}{\partial y} = c\ln^k(\lambda x).$

This is a special case of equation 1.2.7.6 with $f(x) = bx^n$ and $g(x) = c\ln^k(\lambda x)$.

11. $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = x^k(n\ln x + m\ln y).$

This is a special case of equation 1.2.7.28 with $f(u) = \ln u$.

12. $\displaystyle ax^k\frac{\partial w}{\partial x} + by^n\frac{\partial w}{\partial y} = c\ln^m(\lambda x) + s\ln^l(\beta y).$

General solution:

$$w = \frac{c}{a}\int x^{-k}\ln^m(\lambda x)\,dx + \frac{s}{b}\int y^{-n}\ln^l(\beta y)\,dy + \Phi(u), \qquad u = \frac{b}{1-k}x^{1-k} - \frac{a}{1-n}y^{1-n}.$$

1.2.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\sin(\lambda x) + k\sin(\mu y).$

General solution: $\displaystyle w = -\frac{c}{a\lambda}\cos(\lambda x) - \frac{k}{b\mu}\cos(\mu y) + \Phi(bx - ay).$

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\sin(\lambda x + \mu y).$

General solution:

$$w = \begin{cases} -\dfrac{c}{a\lambda + b\mu}\cos(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\sin(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

3. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\sin(\lambda x + \mu y).$

General solution: $w = -\dfrac{ax}{\lambda x + \mu y}\cos(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right).$

4. $a\dfrac{\partial w}{\partial x} + b\sin^n(\lambda x)\dfrac{\partial w}{\partial y} = c\sin^m(\mu x) + s\sin^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\sin^n(\lambda x)$, and $h(x,y) = c\sin^m(\mu x) + s\sin^k(\beta y)$.

5. $a\dfrac{\partial w}{\partial x} + b\sin^n(\lambda y)\dfrac{\partial w}{\partial y} = c\sin^m(\mu x) + s\sin^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\sin^n(\lambda y)$, $h_1(x) = c\sin^m(\mu x)$, and $h_2(y) = s\sin^k(\beta y)$.

► Coefficients of equations contain cosine.

6. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cos(\lambda x) + k\cos(\mu y).$

General solution: $w = \dfrac{c}{a\lambda}\sin(\lambda x) + \dfrac{k}{b\mu}\sin(\mu y) + \Phi(bx - ay).$

7. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cos(\lambda x + \mu y).$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\sin(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\cos(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

8. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\cos(\lambda x + \mu y).$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\sin(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right).$

9. $a\dfrac{\partial w}{\partial x} + b\cos^n(\lambda x)\dfrac{\partial w}{\partial y} = c\cos^m(\mu x) + s\cos^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cos^n(\lambda x)$, and $h(x,y) = c\cos^m(\mu x) + s\cos^k(\beta y)$.

10. $a\dfrac{\partial w}{\partial x} + b\cos^n(\lambda y)\dfrac{\partial w}{\partial y} = c\cos^m(\mu x) + s\cos^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\cos^n(\lambda y)$, $h_1(x) = c\cos^m(\mu x)$, and $h_2(y) = s\cos^k(\beta y)$.

▶ **Coefficients of equations contain tangent.**

11. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\tan(\lambda x)+k\tan(\mu y).$

General solution: $w=-\dfrac{c}{a\lambda}\ln\bigl|\cos(\lambda x)\bigr|-\dfrac{k}{b\mu}\ln\bigl|\cos(\mu y)\bigr|+\Phi(bx-ay).$

12. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\tan(\lambda x+\mu y).$

General solution:

$$w=\begin{cases}-\dfrac{c}{a\lambda+b\mu}\ln\bigl|\cos(\lambda x+\mu y)\bigr|+\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \dfrac{c}{a}x\tan(\lambda x+\mu y)+\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

13. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\tan(\lambda x+\mu y).$

General solution: $w=-\dfrac{ax}{\lambda x+\mu y}\ln\bigl|\cos(\lambda x+\mu y)\bigr|+\Phi\left(\dfrac{y}{x}\right).$

14. $a\dfrac{\partial w}{\partial x}+b\tan^n(\lambda x)\dfrac{\partial w}{\partial y}=c\tan^m(\mu x)+s\tan^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\tan^n(\lambda x)$, and $h(x,y)=c\tan^m(\mu x)+s\tan^k(\beta y)$.

15. $a\dfrac{\partial w}{\partial x}+b\tan^n(\lambda y)\dfrac{\partial w}{\partial y}=c\tan^m(\mu x)+s\tan^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x)=a$, $g(y)=b\tan^n(\lambda y)$, $h_1(x)=c\tan^m(\mu x)$, and $h_2(y)=s\tan^k(\beta y)$.

▶ **Coefficients of equations contain cotangent.**

16. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\cot(\lambda x)+k\cot(\mu y).$

General solution: $w=\dfrac{c}{a\lambda}\ln\bigl|\sin(\lambda x)\bigr|+\dfrac{k}{b\mu}\ln\bigl|\sin(\mu y)\bigr|+\Phi(bx-ay).$

17. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\cot(\lambda x+\mu y).$

General solution:

$$w=\begin{cases}\dfrac{c}{a\lambda+b\mu}\ln\bigl|\sin(\lambda x+\mu y)\bigr|+\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \dfrac{c}{a}x\cot(\lambda x+\mu y)+\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

18. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\cot(\lambda x+\mu y).$

General solution: $w=\dfrac{ax}{\lambda x+\mu y}\ln\bigl|\sin(\lambda x+\mu y)\bigr|+\Phi\left(\dfrac{y}{x}\right).$

19. $a\dfrac{\partial w}{\partial x} + b\cot^n(\lambda x)\dfrac{\partial w}{\partial y} = c\cot^m(\mu x) + s\cot^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cot^n(\lambda x)$, and $h(x,y) = c\cot^m(\mu x) + s\cot^k(\beta y)$.

20. $a\dfrac{\partial w}{\partial x} + b\cot^n(\lambda y)\dfrac{\partial w}{\partial y} = c\cot^m(\mu x) + s\cot^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\cot^n(\lambda y)$, $h_1(x) = c\cot^m(\mu x)$, and $h_2(y) = s\cot^k(\beta y)$.

► **Coefficients of equations contain different trigonometric functions.**

21. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \sin(\lambda x) + c\cos(\mu y) + k.$

General solution: $w = \dfrac{k}{a}x - \dfrac{1}{a\lambda}\cos(\lambda x) + \dfrac{c}{b\mu}\sin(\mu y) + \Phi(bx - ay).$

22. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \tan(\lambda x) + c\sin(\mu y) + k.$

General solution: $w = \dfrac{k}{a}x - \dfrac{1}{a\lambda}\ln\bigl|\cos(\lambda x)\bigr| - \dfrac{c}{b\mu}\cos(\mu y) + \Phi(bx - ay).$

23. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \sin(\lambda x)\cos(\mu y) + c.$

General solution:

$$w = \begin{cases} \dfrac{c}{a}x - \dfrac{\cos(\lambda x - \mu y)}{2(a\lambda - b\mu)} - \dfrac{\cos(\lambda x + \mu y)}{2(a\lambda + b\mu)} + \Phi(bx - ay) & \text{if } a\lambda \pm b\mu \neq 0, \\ \dfrac{c}{a}x + \dfrac{x}{2a}\sin\Bigl[\dfrac{\mu}{a}(bx - ay)\Bigr] - \dfrac{\cos(\lambda x + \mu y)}{2(a\lambda + b\mu)} + \Phi(bx - ay) & \text{if } a\lambda - b\mu = 0, \\ \dfrac{c}{a}x - \dfrac{x}{2a}\sin\Bigl[\dfrac{\mu}{a}(bx - ay)\Bigr] - \dfrac{\cos(\lambda x - \mu y)}{2(a\lambda - b\mu)} + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

24. $a\dfrac{\partial w}{\partial x} + b\sin(\mu y)\dfrac{\partial w}{\partial y} = \cos(\lambda x) + c.$

General solution: $w = \dfrac{c}{a}x + \dfrac{1}{a\lambda}\sin(\lambda x) + \Phi(u)$, where $u = b\mu x - a\ln\bigl|\tan\bigl(\tfrac{1}{2}\mu y\bigr)\bigr|$.

25. $a\dfrac{\partial w}{\partial x} + b\tan(\mu y)\dfrac{\partial w}{\partial y} = \sin(\lambda x) + c.$

General solution: $w = \dfrac{c}{a}x - \dfrac{1}{a\lambda}\cos(\lambda x) + \Phi(u)$, where $u = b\mu x - a\ln\bigl|\sin(\mu y)\bigr|$.

26. $a\dfrac{\partial w}{\partial x} + b\tan(\mu y)\dfrac{\partial w}{\partial y} = \cot(\lambda x) + c.$

General solution: $w = \dfrac{c}{a}x - \dfrac{1}{a\lambda}\ln\bigl|\sin(\lambda x)\bigr| + \Phi(u)$, where $u = b\mu x - a\ln\bigl|\sin(\mu y)\bigr|$.

1.2.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arcsin\dfrac{x}{\lambda} + k\arcsin\dfrac{y}{\beta}.}$

General solution:

$$w = \frac{c}{a}\Big(x\arcsin\frac{x}{\lambda} + \sqrt{\lambda^2 - x^2}\Big) + \frac{k}{b}\Big(y\arcsin\frac{y}{\beta} + \sqrt{\beta^2 - y^2}\Big) + \Phi(bx - ay).$$

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arcsin(\lambda x + \beta y).}$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \frac{c}{a\lambda + b\beta}\Big[(\lambda x + \beta y)\arcsin(\lambda x + \beta y) + \sqrt{1 - (\lambda x + \beta y)^2}\Big] + \Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \frac{c}{a}x\arcsin(\lambda x + \beta y) + \Phi(bx - ay).$$

3. $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\arcsin(\lambda x + \beta y).}$

General solution: $w = ax\left[\arcsin(\lambda x + \beta y) + \dfrac{\sqrt{1 - (\lambda x + \beta y)^2}}{\lambda x + \beta y}\right] + \Phi\Big(\dfrac{y}{x}\Big).$

4. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y} = c\arcsin^m(\mu x) + s\arcsin^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arcsin^n(\lambda x)$, and $h(x, y) = c\arcsin^m(\mu x) + s\arcsin^k(\beta y)$.

5. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda y)\dfrac{\partial w}{\partial y} = c\arcsin^m(\mu x) + s\arcsin^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\arcsin^n(\lambda y)$, $h_1(x) = c\arcsin^m(\mu x)$, and $h_2(y) = s\arcsin^k(\beta y)$.

► **Coefficients of equations contain arccosine.**

6. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arccos\dfrac{x}{\lambda} + k\arccos\dfrac{y}{\beta}.}$

General solution:

$$w = \frac{c}{a}\Big(x\arccos\frac{x}{\lambda} - \sqrt{\lambda^2 - x^2}\Big) + \frac{k}{b}\Big(y\arccos\frac{y}{\beta} - \sqrt{\beta^2 - y^2}\Big) + \Phi(bx - ay).$$

7. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arccos(\lambda x + \beta y).}$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \frac{c}{a\lambda + b\beta}\Big[(\lambda x + \beta y)\arccos(\lambda x + \beta y) - \sqrt{1 - (\lambda x + \beta y)^2}\,\Big] + \Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \frac{c}{a}x\arccos(\lambda x + \beta y) + \Phi(bx - ay).$$

8. $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\arccos(\lambda x + \beta y).}$

General solution: $w = ax\left[\arccos(\lambda x + \beta y) - \dfrac{\sqrt{1 - (\lambda x + \beta y)^2}}{\lambda x + \beta y}\right] + \Phi\left(\dfrac{y}{x}\right).$

9. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\arccos^n(\lambda x)\dfrac{\partial w}{\partial y} = c\arccos^m(\mu x) + s\arccos^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arccos^n(\lambda x)$, and $h(x,y) = c\arccos^m(\mu x) + s\arccos^k(\beta y)$.

10. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\arccos^n(\lambda y)\dfrac{\partial w}{\partial y} = c\arccos^m(\mu x) + s\arccos^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\arccos^n(\lambda y)$, $h_1(x) = c\arccos^m(\mu x)$, and $h_2(y) = s\arccos^k(\beta y)$.

► Coefficients of equations contain arctangent.

11. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arctan\dfrac{x}{\lambda} + k\arctan\dfrac{y}{\beta}.}$

General solution:

$$w = \frac{c}{a}\left[x\arctan\frac{x}{\lambda} - \frac{\lambda}{2}\ln(\lambda^2 + x^2)\right] + \frac{k}{b}\left[y\arctan\frac{y}{\beta} - \frac{\beta}{2}\ln(\beta^2 + y^2)\right] + \Phi(bx - ay).$$

12. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arctan(\lambda x + \beta y).}$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \frac{c}{a\lambda + b\beta}\left\{(\lambda x + \beta y)\arctan(\lambda x + \beta y) - \frac{1}{2}\ln\big[1 + (\lambda x + \beta y)^2\big]\right\} + \Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \frac{c}{a}x\arctan(\lambda x + \beta y) + \Phi(bx - ay).$$

13. $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\arctan(\lambda x + \beta y).$

General solution:

$$w = ax\left\{\arctan(\lambda x + \beta y) - \frac{1}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\} + \Phi\left(\frac{y}{x}\right).$$

14. $\displaystyle a\frac{\partial w}{\partial x} + b\arctan^n(\lambda x)\frac{\partial w}{\partial y} = c\arctan^m(\mu x) + s\arctan^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arctan^n(\lambda x)$, and $h(x, y) = c\arctan^m(\mu x) + s\arctan^k(\beta y)$.

15. $\displaystyle a\frac{\partial w}{\partial x} + b\arctan^n(\lambda y)\frac{\partial w}{\partial y} = c\arctan^m(\mu x) + s\arctan^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\arctan^n(\lambda y)$, $h_1(x) = c\arctan^m(\mu x)$, and $h_2(y) = s\arctan^k(\beta y)$.

► Coefficients of equations contain arccotangent.

16. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\,\mathrm{arccot}\,\frac{x}{\lambda} + k\,\mathrm{arccot}\,\frac{y}{\beta}.$

General solution:

$$w = \frac{c}{a}\left[x\,\mathrm{arccot}\,\frac{x}{\lambda} + \frac{\lambda}{2}\ln(\lambda^2 + x^2)\right] + \frac{k}{b}\left[y\,\mathrm{arccot}\,\frac{y}{\beta} + \frac{\beta}{2}\ln(\beta^2 + y^2)\right] + \Phi(bx - ay).$$

17. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\,\mathrm{arccot}(\lambda x + \beta y).$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \frac{c}{a\lambda + b\beta}\left\{(\lambda x + \beta y)\,\mathrm{arccot}(\lambda x + \beta y) + \frac{1}{2}\ln\left[1 + (\lambda x + \beta y)^2\right]\right\} + \Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \frac{c}{a}x\,\mathrm{arccot}(\lambda x + \beta y) + \Phi(bx - ay).$$

18. $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\,\mathrm{arccot}(\lambda x + \beta y).$

General solution:

$$w = ax\left\{\mathrm{arccot}(\lambda x + \beta y) + \frac{1}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\} + \Phi\left(\frac{y}{x}\right).$$

19. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y} = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\,\mathrm{arccot}^n(\lambda x)$, and $h(x, y) = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)$.

20. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda y)\frac{\partial w}{\partial y} = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\,\mathrm{arccot}^n(\lambda y)$, $h_1(x) = c\,\mathrm{arccot}^m(\mu x)$, and $h_2(y) = s\,\mathrm{arccot}^k(\beta y)$.

1.2.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of x.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = f(x).$

General solution: $w = \dfrac{1}{a}\displaystyle\int f(x)\,dx + \Phi(bx - ay).$

⊙ *Literature*: E. Kamke (1965).

2. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)y.$

General solution: $w = \displaystyle\int_{x_0}^{x} (y - ax + at) f(t)\,dt + \Phi(y - ax)$, where x_0 may be taken as arbitrary.

3. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)y^2 + g(x)y + h(x).$

General solution:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x) + \Phi(y - ax),$$

where

$$\varphi(x) = \int f(x)\,dx, \quad \psi(x) = \int \bigl[g(x) - 2a\varphi(x)\bigr]\,dx, \quad \chi(x) = \int \bigl[h(x) - a\psi(x)\bigr]\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

4. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)y^k.$

General solution: $w = \displaystyle\int_{x_0}^{x} (y - ax + at)^k f(t)\,dt + \Phi(y - ax)$, where x_0 may be taken as arbitrary.

5. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)e^{\lambda y}.$

General solution: $w = e^{\lambda(y - ax)}\displaystyle\int f(x)e^{a\lambda x}\,dx + \Phi(y - ax).$

6. $\dfrac{\partial w}{\partial x} + [ay + f(x)]\dfrac{\partial w}{\partial y} = g(x).$

General solution: $w = \displaystyle\int g(x)\,dx + \Phi(u)$, where $u = e^{-ax}y - \displaystyle\int f(x)e^{-ax}\,dx.$

7. $\dfrac{\partial w}{\partial x} + \bigl[ay + f(x)\bigr]\dfrac{\partial w}{\partial y} = g(x)y^k.$

This is a special case of equation 1.2.7.19 with $h(y) = y^k$.

8. $f(x)\dfrac{\partial w}{\partial x} + y^k \dfrac{\partial w}{\partial y} = g(x).$

General solution:

$$w = \int \frac{g(x)}{f(x)}\,dx + \Phi(u), \qquad \text{where} \quad u = \begin{cases} \dfrac{1}{k-1} y^{1-k} + \displaystyle\int \frac{dx}{f(x)} & \text{if} \quad k \neq 1, \\ y \exp\biggl[-\displaystyle\int \frac{dx}{f(x)}\biggr] & \text{if} \quad k = 1. \end{cases}$$

9. $f(x)\dfrac{\partial w}{\partial x} + (y+a)\dfrac{\partial w}{\partial y} = by + c.$

General solution: $w = by + (c - ab)\ln|y+a| + \Phi(u)$, where $u = (y+a)\exp\biggl[-\displaystyle\int \frac{dx}{f(x)}\biggr]$.

10. $f(x)\dfrac{\partial w}{\partial x} + (y + ax)\dfrac{\partial w}{\partial y} = g(x).$

General solution:

$$w = \int \frac{g(x)}{f(x)}\,dx + \Phi\biggl(e^{-z} y - a\int \frac{xe^{-z}}{f(x)}\,dx\biggr), \qquad \text{where} \quad z = \int \frac{dx}{f(x)}.$$

11. $f(x)\dfrac{\partial w}{\partial x} + \bigl[g_1(x)y + g_0(x)\bigr]\dfrac{\partial w}{\partial y} = h_2(x)y^2 + h_1(x)y + h_0(x).$

General solution:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x) + \Phi(u), \qquad u = e^{-G} y - \int e^{-G}\frac{g_0}{f}\,dx,$$

where

$$\varphi(x) = e^{-2G}\int e^{2G}\frac{h_2}{f}\,dx, \qquad G = G(x) = \int \frac{g_1}{f}\,dx,$$
$$\psi(x) = e^{-G}\int e^{G}\frac{h_1 - 2g_0\varphi}{f}\,dx, \quad \chi(x) = \int \frac{h_0 - g_0\psi}{f}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

12. $f(x)\dfrac{\partial w}{\partial x} + \bigl[g_1(x)y + g_2(x)y^k\bigr]\dfrac{\partial w}{\partial y} = h(x).$

General solution: $w = \displaystyle\int \frac{h(x)}{f(x)}\,dx + \Phi(u)$, where

$$u = e^{-G} y^{1-k} - (1-k)\int e^{-G}\frac{g_2(x)}{f(x)}\,dx, \qquad G = (1-k)\int \frac{g_1(x)}{f(x)}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

13. $f(x)\dfrac{\partial w}{\partial x} + \bigl[g_1(x) + g_2(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y} = h(x).$

General solution: $w = \displaystyle\int \frac{h(x)}{f(x)}\,dx + \Phi(u)$, where

$$u = e^{-\lambda y}E(x) + \lambda\int \frac{g_2(x)}{f(x)}E(x)\,dx, \qquad E(x) = \exp\biggl[\lambda\int \frac{g_1(x)}{f(x)}\,dx\biggr].$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

14. $\boldsymbol{f(x)y^k\dfrac{\partial w}{\partial x} + g(x)\dfrac{\partial w}{\partial y} = h(x).}$

General solution: $w = \Phi(u) + \displaystyle\int_{x_0}^{x} \frac{h(t)}{f(t)}\bigl[u + E(t)\bigr]^{-\frac{k}{k+1}}\,dt$, where

$$u = y^{k+1} - E(x), \qquad E(x) = (k+1)\int \frac{g(x)}{f(x)}\,dx, \qquad x_0 \text{ may be taken as arbitrary.}$$

15. $\boldsymbol{f(x)y^k\dfrac{\partial w}{\partial x} + \bigl[g_1(x)y^{k+1} + g_0(x)\bigr]\dfrac{\partial w}{\partial y} = h_2(x)y^{3k+2} + h_1(x)y^{2k+1} + h_0(x)y^k.}$

The substitution $z = y^{k+1}$ leads to an equation of the form 1.2.7.11:

$$f(x)\frac{\partial w}{\partial x} + (k+1)\bigl[g_1(x)z + g_0(x)\bigr]\frac{\partial w}{\partial z} = h_2(x)z^2 + h_1(x)z + h_0(x).$$

16. $\boldsymbol{f(x)e^{\lambda y}\dfrac{\partial w}{\partial x} + g(x)\dfrac{\partial w}{\partial y} = h(x).}$

General solution:

$$w = \Phi(u) + \int_{x_0}^{x} \frac{h(t)\,dt}{f(t)\bigl[u + E(t)\bigr]}, \qquad u = e^{\lambda y} - E(x), \quad E(x) = \lambda\int \frac{g(x)}{f(x)}\,dx,$$

where x_0 may be taken as arbitrary.

► Equations contain arbitrary functions of x and arbitrary functions of y.

17. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = f(x) + g(y).}$

General solution: $w = \dfrac{1}{a}\displaystyle\int f(x)\,dx + \dfrac{1}{b}\int g(y)\,dy + \Phi(bx - ay)$.

18. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)g(y).}$

General solution: $w = \displaystyle\int_{x_0}^{x} f(t)g(y - ax + at)\,dt + \Phi(y - ax)$, where x_0 may be taken as arbitrary.

19. $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[ay + f(x)\bigr]\dfrac{\partial w}{\partial y} = g(x)h(y).}$

General solution:

$$w = \int g(x)\,h\left(e^{ax}u + e^{ax}\int f(x)e^{-ax}\,dx\right)dx + \Phi(u), \qquad u = e^{-ax}y - \int f(x)e^{-ax}\,dx.$$

In the integration, u is considered a parameter.

20. $\boldsymbol{f(x)\frac{\partial w}{\partial x} + g(y)\frac{\partial w}{\partial y} = h_1(x) + h_2(y).}$

General solution:

$$w = \int \frac{h_1(x)}{f(x)}dx + \int \frac{h_2(y)}{g(y)}dy + \Phi\biggl(\int \frac{dx}{f(x)} - \int \frac{dy}{g(y)}\biggr).$$

21. $\boldsymbol{f_1(x)\frac{\partial w}{\partial x} + \bigl[f_2(x)y + f_3(x)y^k\bigr]\frac{\partial w}{\partial y} = g(x)h(y).}$

The transformation $\xi = \int \frac{f_2(x)}{f_1(x)}\,dx,\ \eta = y^{1-k}$ leads to an equation of the form 1.2.7.19:

$$\frac{\partial w}{\partial \xi} + \bigl[(1-k)\eta + F(\xi)\bigr]\frac{\partial w}{\partial \eta} = G(\xi)H(\eta),$$

where $F(\xi) = (1-k)\frac{f_3(x)}{f_2(x)},\ G(\xi) = \frac{g(x)}{f_2(x)},$ and $H(\eta) = h(y)$.

22. $\boldsymbol{f_1(x)g_1(y)\frac{\partial w}{\partial x} + f_2(x)g_2(y)\frac{\partial w}{\partial y} = h_1(x)h_2(y).}$

The transformation $\xi = \int \frac{f_2(x)}{f_1(x)}\,dx,\ \eta = \int \frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.2.7.18:

$$\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} = F(\xi)G(\eta), \qquad \text{where} \quad F(\xi) = \frac{h_1(x)}{f_2(x)}, \quad G(\eta) = \frac{h_2(y)}{g_1(y)}.$$

23. $\boldsymbol{f_1(x)g_1(y)\frac{\partial w}{\partial x} + f_2(x)g_2(y)\frac{\partial w}{\partial y} = h_1(x) + h_2(y).}$

This is a special case of equation 1.2.7.38 with $h(x,y) = h_1(x) + h_2(y)$.

► Equations contain arbitrary functions of complicated arguments.

24. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = f(\alpha x + \beta y).}$

General solution:

$$w = \begin{cases} \dfrac{1}{a\alpha + b\beta}\displaystyle\int f(z)\,dz + \Phi(bx - ay) & \text{if } a\alpha + b\beta \neq 0, \\ \dfrac{1}{a}xf(\alpha x + \beta y) + \Phi(bx - ay) & \text{if } a\alpha + b\beta = 0, \end{cases}$$

where $z = \alpha x + \beta y$.

25. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = xf\Bigl(\frac{y}{x}\Bigr) + yg\Bigl(\frac{y}{x}\Bigr).}$

General solution: $w = xf\Bigl(\frac{y}{x}\Bigr) + yg\biggl(\frac{y}{x}\biggr) + \Phi\Bigl(\frac{y}{x}\Bigr)$.

26. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = f(x^2+y^2)$.

General solution: $w = \Phi\Big(\dfrac{y}{x}\Big) + \dfrac{1}{2}\displaystyle\int f(\xi)\frac{d\xi}{\xi}$, where $\xi = x^2+y^2$.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

27. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = xf\Big(\dfrac{y}{x}\Big) + g(x^2+y^2)$.

General solution: $w = \Phi\Big(\dfrac{y}{x}\Big) + xf\Big(\dfrac{y}{x}\Big) + \dfrac{1}{2}\displaystyle\int g(\xi)\frac{d\xi}{\xi}$, where $\xi = x^2+y^2$.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

28. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = x^k f(x^n y^m)$.

General solution:

$$w = \begin{cases} \dfrac{1}{a}\displaystyle\int x^{k-1} f\Big(x^{\frac{an+bm}{a}} u^{\frac{m}{a}}\Big)\,dx + \Phi(u) & \text{if } an \neq -bm, \\ \dfrac{1}{ak} x^k f(x^n y^m) + \Phi(u) & \text{if } an = -bm,\ k \neq 0, \\ \dfrac{1}{a} f(x^n y^m) \ln|x| + \Phi(u) & \text{if } an = -bm,\ k = 0, \end{cases}$$

where $u = y^a x^{-b}$. In the integration, u is considered a parameter.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

29. $mx\dfrac{\partial w}{\partial x} + ny\dfrac{\partial w}{\partial y} = f(ax^n + by^m)$.

General solution: $w = \Phi\big(y^m x^{-n}\big) + \dfrac{1}{nm}\displaystyle\int f(\xi)\frac{d\xi}{\xi}$, where $\xi = ax^n + by^m$.

30. $x^2\dfrac{\partial w}{\partial x} + xy\dfrac{\partial w}{\partial y} = y^k f(\alpha x + \beta y)$.

General solution: $w = \dfrac{y^k}{x(\alpha x+\beta y)^{k-1}}\displaystyle\int z^{k-2} f(z)\,dz + \Phi\Big(\frac{y}{x}\Big)$, where $z = \alpha x + \beta y$.

31. $\dfrac{f(x)}{f'(x)}\dfrac{\partial w}{\partial x} + \dfrac{g(y)}{g'(y)}\dfrac{\partial w}{\partial y} = h\big(f(x)+g(y)\big)$.

General solution:

$$w = \Phi(u) + \int h(\xi)\frac{d\xi}{\xi}, \qquad \text{where} \quad u = \frac{g(y)}{f(x)}, \quad \xi = f(x) + g(y).$$

► **Equations contain arbitrary functions of two variables.**

32. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x,y)$.

General solution: $w = \displaystyle\int_{x_0}^{x} f(t, y - ax + at)\,dt + \Phi(y - ax)$, where x_0 may be taken as arbitrary.

33. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=f(x,y).}$

General solution:

$$w=\frac{1}{a}\int\frac{1}{x}f\bigl(x,\,u^{1/a}x^{b/a}\bigr)\,dx+\Phi(u),\qquad\text{where}\quad u=y^a x^{-b}.$$

In the integration, u is considered a parameter.

34. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+g(x)y\dfrac{\partial w}{\partial y}=h(x,y).}$

General solution:

$$w=\Phi(u)+\int\frac{h(x,\,uG)}{f}\,dx,\qquad\text{where}\quad u=\frac{y}{G},\quad G=\exp\biggl(\int\frac{g}{f}\,dx\biggr).$$

In the integration, u is considered a parameter.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

35. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)\bigr]\dfrac{\partial w}{\partial y}=h(x,y).}$

General solution:

$$w=\Phi(u)+\int\frac{h(x,\,uG+Q)}{f}\,dx,\qquad u=\frac{y-Q}{G},$$

where $G=\exp\biggl(\displaystyle\int\frac{g_1}{f}\,dx\biggr)$ and $Q=G\displaystyle\int\frac{g_0\,dx}{fG}$. In the integration, u is considered a parameter.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

36. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)y^k\bigr]\dfrac{\partial w}{\partial y}=h(x,y).}$

For $k=1$, see equation 1.2.7.34. For $k\neq 1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 1.2.7.35:

$$f(x)\frac{\partial w}{\partial x}+(1-k)\bigl[g_1(x)\xi+g_0(x)\bigr]\frac{\partial w}{\partial \xi}=h\Bigl(x,\,\xi^{\frac{1}{1-k}}\Bigr).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

37. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)+g_0(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y}=h(x,y).}$

The substitution $z=e^{-\lambda y}$ leads to an equation of the form 1.2.7.35:

$$f(x)\frac{\partial w}{\partial x}-\lambda\bigl[g_1(x)z+g_0(x)\bigr]\frac{\partial w}{\partial z}=h\biggl(x,\,-\frac{1}{\lambda}\ln z\biggr).$$

38. $\boldsymbol{f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h(x,y).}$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.2.7.32:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi,\eta),\qquad\text{where}\quad F(\xi,\eta)=\frac{h(x,y)}{f_2(x)g_1(y)}.$$

1.3 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h(x,y)w$

◆ *The solutions given below contain an arbitrary function* $\Phi=\Phi(z)$.

1.3.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x and y.**

1. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw.$

Two forms of the representation of the general solution:

$$w=\exp\Bigl(\frac{c}{a}x\Bigr)\Phi(bx-ay),\qquad w=\exp\Bigl(\frac{c}{b}y\Bigr)\Phi(bx-ay).$$

2. $a\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=bw.$

General solution: $w=|y|^b\Phi\bigl(|y|^ae^{-x}\bigr)$.

⊙ *Literature*: E. Kamke (1965).

3. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=aw.$

Differential equation for homogeneous functions of order a with two independent variables.
General solution: $w=x^a\Phi(y/x)$.

⊙ *Literature*: E. Kamke (1965).

4. $x\Bigl(a\dfrac{\partial w}{\partial x}-b\dfrac{\partial w}{\partial y}\Bigr)=cyw.$

General solution: $w=\exp\Bigl\{\dfrac{c}{a^2}\bigl[(bx+ay)\ln x-bx\bigr]\Bigr\}\Phi(bx+ay)$.

5. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=axw.$

General solution: $w=e^{ax}\Phi\Bigl(\dfrac{y}{x}\Bigr)$.

6. $(x-a)\dfrac{\partial w}{\partial x}+(y-b)\dfrac{\partial w}{\partial y}=w.$

Differential equation of a conic surface with the vertex at the point $(a,b,0)$.

General solution: $w=(x-a)\Phi\Bigl(\dfrac{y-b}{x-a}\Bigr)$.

7. $(y+ax)\dfrac{\partial w}{\partial x}+(y-ax)\dfrac{\partial w}{\partial y}=bw.$

General solution:

$$w=\xi^{\frac{b}{a+1}}\Phi\biggl(\ln\sqrt{\xi}+\frac{a+1}{2}\int\frac{dv}{v^2+(a-1)v+a}\biggr),$$

where $\xi=y^2+(a-1)xy+ax^2$ and $v=y/x$.

► **Coefficients of equations are quadratic in x and y.**

8. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=(x^2-y^2)w.$

General solution: $\displaystyle w=\exp\left[\frac{1}{3ab}(bx^3-ay^3)\right]\Phi(bx-ay).$

9. $\displaystyle x^2\frac{\partial w}{\partial x}+axy\frac{\partial w}{\partial y}=by^2w.$

General solution:

$$w=\begin{cases}\exp\left(\dfrac{b}{2a-1}\dfrac{y^2}{x}\right)\Phi(x^{-a}y) & \text{if } a\neq\frac{1}{2},\\ \exp\left(b\dfrac{y^2}{x}\ln x\right)\Phi(x^{-1/2}y) & \text{if } a=\frac{1}{2}.\end{cases}$$

10. $\displaystyle ax^2\frac{\partial w}{\partial x}+by^2\frac{\partial w}{\partial y}=(x+cy)w.$

General solution: $\displaystyle w=x^{1/a}y^{c/b}\Phi\left(\frac{b}{x}-\frac{a}{y}\right).$

11. $\displaystyle x^2\frac{\partial w}{\partial x}+ay^2\frac{\partial w}{\partial y}=(bx^2+cxy+dy^2)w.$

General solution: $\displaystyle w=\exp\left(\frac{dy^2+abxy-bx^2}{ay-x}-\frac{cxy}{ay-x}\ln\left|\frac{x}{y}\right|\right)\Phi\left(\frac{x-ay}{xy}\right).$

12. $\displaystyle y^2\frac{\partial w}{\partial x}+ax^2\frac{\partial w}{\partial y}=(bx^2+cy^2)w.$

General solution: $\displaystyle w=\exp\left(cx+\frac{b}{a}y\right)\Phi\left(ax^3-y^3\right).$

13. $\displaystyle xy\frac{\partial w}{\partial x}+ay^2\frac{\partial w}{\partial y}=(bx+cy+d)w.$

General solution:

$$w=\begin{cases}x^c\exp\left[\dfrac{(1-a)d-abx}{a(a-1)y}\right]\Phi\left(x^{-a}y\right) & \text{if } a\neq 1,\\ \exp\left[\left(\dfrac{bx}{y}+c\right)\ln|x|-\dfrac{d}{y}\right]\Phi\left(\dfrac{y}{x}\right) & \text{if } a=1.\end{cases}$$

14. $\displaystyle x(ay+b)\frac{\partial w}{\partial x}+(ay^2-bx)\frac{\partial w}{\partial y}=ayw.$

General solution: $\displaystyle w=(x+y)\Phi\left(\frac{ax-b}{x+y}+a\ln\left|\frac{x+y}{x}\right|\right).$

15. $\displaystyle x(ky-x+a)\frac{\partial w}{\partial x}-y(kx-y+a)\frac{\partial w}{\partial y}=b(y-x)w.$

General solution: $\displaystyle w=(x+y-a)^b\Phi\left(\frac{(x+y-a)^k}{xy}\right).$

► **Coefficients of equations contain other power-law functions.**

16. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=(cx^3+dy^3)w.$

General solution: $w=\exp\left(\dfrac{bcx^4+ady^4}{4ab}\right)\Phi(bx-ay).$

17. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=a\sqrt{x^2+y^2}\,w.$

General solution: $w=\exp\left(a\sqrt{x^2+y^2}\right)\Phi\left(\dfrac{y}{x}\right).$

18. $x^2\dfrac{\partial w}{\partial x}+xy\dfrac{\partial w}{\partial y}=y^2(ax+by)w.$

General solution: $w=\exp\left[\dfrac{(ax+by)y^2}{2x}\right]\Phi\left(\dfrac{y}{x}\right).$

19. $x^2y\dfrac{\partial w}{\partial x}+axy^2\dfrac{\partial w}{\partial y}=(bxy+cx+dy+k)w.$

General solution:

$$w=\begin{cases} x^b\exp\left[-\dfrac{k}{(a+1)xy}-\dfrac{d}{x}-\dfrac{c}{ay}\right]\Phi\left(x^{-a}y\right) & \text{if } a\neq-1,\\ \exp\left[\left(\dfrac{k}{xy}+b\right)\ln|x|+\dfrac{c}{y}-\dfrac{d}{y}\right]\Phi(xy) & \text{if } a=-1.\end{cases}$$

20. $axy^2\dfrac{\partial w}{\partial x}+bx^2y\dfrac{\partial w}{\partial y}=(any^2+bmx^2)w.$

General solution: $w=x^ny^m\Phi(ay^2-bx^2).$

21. $x^3\dfrac{\partial w}{\partial x}+ay^3\dfrac{\partial w}{\partial y}=x^2(bx+cy)w.$

General solution:

$$w=\exp\left(c\sqrt{\frac{x^2y^2}{x^2-ay^2}}\,\ln\left|\sqrt{\frac{x^2}{y^2}-a}+\frac{x}{y}\right|+bx\right)\Phi\left(\frac{x^2-ay^2}{x^2y^2}\right).$$

► **Coefficients of equations contain arbitrary powers of x and y.**

22. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=(cx^n+dy^m)w.$

General solution: $w=\Phi(bx-ay)\exp\left[\dfrac{c}{a(n+1)}x^{n+1}+\dfrac{d}{b(m+1)}y^{m+1}\right].$

23. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n yw.}$

General solution:

$$w = \begin{cases} \exp\left\{\dfrac{c[a(n+2)y - bx]x^{n+1}}{a^2(n+1)(n+2)}\right\}\Phi(bx - ay) & \text{if } n \neq -1, -2; \\ \exp\left[\dfrac{bc}{a^2}x(1 - \ln x) + \dfrac{c}{a}y\ln x\right]\Phi(bx - ay) & \text{if } n = -1; \\ \exp\left[\dfrac{bc}{a^2}(1 + \ln x) - \dfrac{cy}{ax}\right]\Phi(bx - ay) & \text{if } n = -2. \end{cases}$$

24. $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a(x^2 + y^2)^k w.}$

General solution: $w = \exp\left[\dfrac{a}{2k}(x^2 + y^2)^k\right]\Phi\left(\dfrac{y}{x}\right)$.

25. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cx^n y^m w.}$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{an + bm}x^n y^m\right)\Phi\big(y^a x^{-b}\big) & \text{if } an + bm \neq 0, \\ \exp\left(\dfrac{c}{a}x^n y^m \ln x\right)\Phi\big(y^a x^{-b}\big) & \text{if } an + bm = 0. \end{cases}$$

26. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = (cx^n + ky^m)w.}$

General solution: $w = \exp\left(\dfrac{c}{an}x^n + \dfrac{k}{bm}y^m\right)\Phi\big(y^a x^{-b}\big)$.

27. $\boldsymbol{mx\dfrac{\partial w}{\partial x} + ny\dfrac{\partial w}{\partial y} = (ax^n + by^m)^k w.}$

General solution: $w = \exp\left[\dfrac{1}{mnk}(ax^n + by^m)^k\right]\Phi\big(y^m x^{-n}\big)$.

28. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x} + by^m\dfrac{\partial w}{\partial y} = (cx^k + dy^s)w.}$

This is a special case of equation 1.3.7.19. General solution:

$$w = \exp\left[\frac{cx^{k-n+1}}{a(k-n+1)} + \frac{dy^{s-m+1}}{b(s-m+1)}\right]\Phi(u), \qquad u = \frac{x^{1-n}}{a(1-n)} - \frac{y^{1-m}}{b(1-m)}.$$

29. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x} + bx^m y\dfrac{\partial w}{\partial y} = \big(cx^k y^s + d\big)w.}$

This is a special case of equation 1.3.7.32 with $f(x) = ax^n$, $g(x) = bx^m$, and $h(x, y) = cx^k y^s + d$.

30. $\boldsymbol{ax^n \dfrac{\partial w}{\partial x} + \big(bx^m y + cx^k\big)\dfrac{\partial w}{\partial y} = \big(sx^p y^q + d\big)w.}$

This is a special case of equation 1.3.7.33 with $f(x) = ax^n$, $g_1(x) = bx^m$, $g_0(x) = cx^k$, and $h(x,y) = sx^p y^q + d$.

31. $\boldsymbol{ax^n \dfrac{\partial w}{\partial x} + bx^m y^k \dfrac{\partial w}{\partial y} = \big(cx^p y^q + s\big)w.}$

This is a special case of equation 1.3.7.34 with $f(x) = ax^n$, $g_1(x) \equiv 0$, $g_0(x) = bx^m$, and $h(x,y) = cx^p y^q + s$.

32. $\boldsymbol{ay^k \dfrac{\partial w}{\partial x} + bx^n \dfrac{\partial w}{\partial y} = \big(cx^m + s\big)w.}$

This is a special case of equation 1.3.7.14 with $f(x) = a$, $g(x) = bx^n$, and $h(x) = cx^m + s$.

33. $\boldsymbol{x\big[x^n + (2n-1)y^n\big]\dfrac{\partial w}{\partial x} + y\big[y^n + (2n-1)x^n\big]\dfrac{\partial w}{\partial y} = kn(x^n + y^n)w.}$

General solution: $w = (x^n - y^n)^k \Phi\left(\dfrac{(x^n - y^n)^2}{xy}\right)$.

34. $\boldsymbol{x\big[(n-2)y^n - 2x^n\big]\dfrac{\partial w}{\partial x} + y\big[2y^n - (n-2)x^n\big]\dfrac{\partial w}{\partial y}}$
$\boldsymbol{= \big\{[a(n-2) + 2b]y^n - [2a + b(n-2)]x^n\big\}w.}$

General solution: $w = x^a y^b \Phi\left(\dfrac{x^n + y^n}{x^2 y^2}\right)$.

1.3.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = ce^{\alpha x + \beta y}w.}$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{a\alpha + b\beta}e^{\alpha x + \beta y}\right)\Phi(bx - ay) & \text{if } a\alpha + b\beta \neq 0, \\ \exp\left(\dfrac{c}{a}xe^{\alpha x + \beta y}\right)\Phi(bx - ay) & \text{if } a\alpha + b\beta = 0. \end{cases}$$

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big(ce^{\lambda x} + ke^{\mu y}\big)w.}$

General solution: $w = \exp\left(\dfrac{c}{a\lambda}e^{\lambda x} + \dfrac{k}{b\mu}e^{\mu y}\right)\Phi(bx - ay)$.

3. $\boldsymbol{ae^{\lambda x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = cw.}$

General solution: $w = \exp\left(-\dfrac{c}{a\lambda}e^{-\lambda x}\right)\Phi\big(b\beta e^{-\lambda x} - a\lambda e^{-\beta y}\big)$.

4. $ae^{\lambda y}\dfrac{\partial w}{\partial x} + be^{\beta x}\dfrac{\partial w}{\partial y} = cw.$

General solution: $w = \exp\left[\dfrac{c(\beta x - \lambda y)}{a\beta e^{\lambda y} - b\lambda e^{\beta x}}\right]\Phi\bigl(a\beta e^{\lambda y} - b\lambda e^{\beta x}\bigr).$

5. $ae^{\lambda x}\dfrac{\partial w}{\partial x} + be^{\beta x}\dfrac{\partial w}{\partial y} = ce^{\gamma y}w.$

This is a special case of equation 1.3.7.33 with $f(x) = ae^{\lambda x}$, $g_1(x) \equiv 0$, $g_0(x) = be^{\beta x}$, and $h(x, y) = ce^{\gamma y}$.

6. $ae^{\lambda x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = \bigl(ce^{\gamma x} + se^{\delta y}\bigr)w.$

This is a special case of equation 1.3.7.19 with $f(x) = ae^{\lambda x}$, $g(y) = be^{\beta y}$, $h_1(x) = ce^{\gamma x}$, and $h_2(y) = se^{\delta y}$.

7. $ae^{\beta x}\dfrac{\partial w}{\partial x} + \bigl(be^{\gamma x} + ce^{\lambda y}\bigr)\dfrac{\partial w}{\partial y} = \bigl(se^{\mu x} + ke^{\delta y} + p\bigr)w.$

This is a special case of equation 1.3.7.35 with $f(x) = ae^{\beta x}$, $g_1(x) = be^{\gamma x}$, $g_0(x) = c$, and $h(x, y) = se^{\mu x} + ke^{\delta y} + p$.

8. $ae^{\beta x}\dfrac{\partial w}{\partial x} + \bigl(be^{\gamma x} + ce^{\lambda y}\bigr)\dfrac{\partial w}{\partial y} = \bigl(se^{\mu x+\delta y} + k\bigr)w.$

This is a special case of equation 1.3.7.35 with $f(x) = ae^{\beta x}$, $g_1(x) = be^{\gamma x}$, $g_0(x) = c$, and $h(x, y) = se^{\mu x+\delta y} + k$.

9. $ae^{\beta x}\dfrac{\partial w}{\partial x} + be^{\gamma x+\lambda y}\dfrac{\partial w}{\partial y} = \bigl(ce^{\mu x+\delta y} + k\bigr)w.$

This is a special case of equation 1.3.7.35 with $f(x) = ae^{\beta x}$, $g_1(x) \equiv 0$, $g_0(x) = be^{\gamma x}$, and $h(x, y) = ce^{\mu x+\delta y} + k$.

10. $ae^{\lambda y}\dfrac{\partial w}{\partial x} + be^{\beta x}\dfrac{\partial w}{\partial y} = \bigl(ce^{\mu x} + k\bigr)w.$

This is a special case of equation 1.3.7.15 with $f(x) = a$, $g(x) = be^{\beta x}$, and $h(x) = ce^{\mu x} + k$.

► Coefficients of equations contain exponential and power-law functions.

11. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \bigl(cye^{\lambda x} + kxe^{\mu y}\bigr)w.$

General solution: $w = \exp\left[\dfrac{c}{a\lambda}e^{\lambda x}\left(y - \dfrac{b}{a\lambda}\right) + \dfrac{k}{b\mu}e^{\mu y}\left(x - \dfrac{a}{b\mu}\right)\right]\Phi(bx - ay).$

12. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = axe^{\lambda x+\mu y}w.$

General solution: $w = \exp\left(\dfrac{ax}{\lambda x + \mu y}e^{\lambda x+\mu y}\right)\Phi\left(\dfrac{y}{x}\right).$

13. $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=\bigl(aye^{\lambda x}+bxe^{\mu y}\bigr)w.$

General solution: $w=\exp\left(\dfrac{ay}{\lambda x}e^{\lambda x}+\dfrac{bx}{\mu y}e^{\mu y}\right)\Phi\left(\dfrac{y}{x}\right)$.

14. $\displaystyle ax^k\frac{\partial w}{\partial x}+be^{\lambda y}\frac{\partial w}{\partial y}=\bigl(cx^n+s\bigr)w.$

This is a special case of equation 1.3.7.13 with $f(x)=ax^k$, $g_1(x)=0$, $g_2(x)=b$, and $h(x)=cx^n+s$.

15. $\displaystyle ay^k\frac{\partial w}{\partial x}+be^{\lambda x}\frac{\partial w}{\partial y}=\bigl(ce^{\mu x}+s\bigr)w.$

This is a special case of equation 1.3.7.14 with $f(x)=a$, $g(x)=be^{\lambda x}$, and $h(x)=ce^{\mu x}+s$.

16. $\displaystyle ae^{\lambda x}\frac{\partial w}{\partial x}+by^k\frac{\partial w}{\partial y}=\bigl(cx^n+s\bigr)w.$

This is a special case of equation 1.3.7.12 with $f(x)=ae^{\lambda x}$, $g_1(x)=0$, $g_2(x)=b$, and $h(x)=cx^n+s$.

17. $\displaystyle ae^{\lambda y}\frac{\partial w}{\partial x}+bx^k\frac{\partial w}{\partial y}=\bigl(ce^{\mu x}+s\bigr)w.$

This is a special case of equation 1.3.7.15 with $f(x)=a$, $g(x)=bx^k$, and $h(x)=ce^{\mu x}+s$.

1.3.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\bigl[c\sinh(\lambda x)+k\sinh(\mu y)\bigr]w.$

General solution: $w=\exp\left[\dfrac{c}{a\lambda}\cosh(\lambda x)+\dfrac{k}{b\mu}\cosh(\mu y)\right]\Phi(bx-ay)$.

2. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\sinh(\lambda x+\mu y)w.$

General solution:

$$w=\begin{cases}\exp\left[\dfrac{c}{a\lambda+b\mu}\cosh(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \exp\left[\dfrac{c}{a}x\sinh(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

3. $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=ax\sinh(\lambda x+\mu y)w.$

General solution: $w=\exp\left[\dfrac{ax}{\lambda x+\mu y}\cosh(\lambda x+\mu y)\right]\Phi\left(\dfrac{y}{x}\right)$.

4. $a\dfrac{\partial w}{\partial x}+b\sinh^n(\lambda x)\dfrac{\partial w}{\partial y}=\bigl[c\sinh^m(\mu x)+s\sinh^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\sinh^n(\lambda x)$, and $h(x,y)=c\sinh^m(\mu x)+s\sinh^k(\beta y)$.

5. $a\dfrac{\partial w}{\partial x}+b\sinh^n(\lambda y)\dfrac{\partial w}{\partial y}=\bigl[c\sinh^m(\mu x)+s\sinh^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\sinh^n(\lambda y)$, $h_1(x)=c\sinh^m(\mu x)$, and $h_2(y)=s\sinh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cosine.**

6. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\bigl[c\cosh(\lambda x)+k\cosh(\mu y)\bigr]w.$

General solution: $w=\exp\left[\dfrac{c}{a\lambda}\sinh(\lambda x)+\dfrac{k}{b\mu}\sinh(\mu y)\right]\Phi(bx-ay).$

7. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\cosh(\lambda x+\mu y)w.$

General solution:

$$w=\begin{cases}\exp\left[\dfrac{c}{a\lambda+b\mu}\sinh(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \exp\left[\dfrac{c}{a}x\cosh(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

8. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\cosh(\lambda x+\mu y)w.$

General solution: $w=\exp\left[\dfrac{ax}{\lambda x+\mu y}\sinh(\lambda x+\mu y)\right]\Phi\left(\dfrac{y}{x}\right).$

9. $a\dfrac{\partial w}{\partial x}+b\cosh^n(\lambda x)\dfrac{\partial w}{\partial y}=\bigl[c\cosh^m(\mu x)+s\cosh^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\cosh^n(\lambda x)$, and $h(x,y)=c\cosh^m(\mu x)+s\cosh^k(\beta y)$.

10. $a\dfrac{\partial w}{\partial x}+b\cosh^n(\lambda y)\dfrac{\partial w}{\partial y}=\bigl[c\cosh^m(\mu x)+s\cosh^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\cosh^n(\lambda y)$, $h_1(x)=c\cosh^m(\mu x)$, and $h_2(y)=s\cosh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic tangent.**

11. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\bigl[c\tanh(\lambda x)+k\tanh(\mu y)\bigr]w.$

General solution: $w=\exp\left[\dfrac{c}{a\lambda}\ln\cosh(\lambda x)+\dfrac{k}{b\mu}\ln\cosh(\mu y)\right]\Phi(bx-ay).$

12. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\tanh(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{c}{a\lambda + b\mu}\ln\cosh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \exp\left[\dfrac{c}{a}x\tanh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

13. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\tanh(\lambda x + \mu y)w.$

General solution: $w = \exp\left[\dfrac{ax}{\lambda x + \mu y}\ln\cosh(\lambda x + \mu y)\right]\Phi\left(\dfrac{y}{x}\right).$

14. $a\dfrac{\partial w}{\partial x} + b\tanh^n(\lambda x)\dfrac{\partial w}{\partial y} = \left[c\tanh^m(\mu x) + s\tanh^k(\beta y)\right]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\tanh^n(\lambda x)$, and $h(x,y) = c\tanh^m(\mu x) + s\tanh^k(\beta y)$.

15. $a\dfrac{\partial w}{\partial x} + b\tanh^n(\lambda y)\dfrac{\partial w}{\partial y} = \left[c\tanh^m(\mu x) + s\tanh^k(\beta y)\right]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\tanh^n(\lambda y)$, $h_1(x) = c\tanh^m(\mu x)$, and $h_2(y) = s\tanh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cotangent.**

16. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \left[c\coth(\lambda x) + k\coth(\mu y)\right]w.$

General solution: $w = \exp\left(\dfrac{c}{a\lambda}\ln\left|\sinh(\lambda x)\right| + \dfrac{k}{b\mu}\ln\left|\sinh(\mu y)\right|\right)\Phi(bx - ay).$

17. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\coth(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{a\lambda + b\mu}\ln\left|\sinh(\lambda x + \mu y)\right|\right)\Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \exp\left[\dfrac{c}{a}x\coth(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

18. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\coth(\lambda x + \mu y)w.$

General solution: $w = \exp\left(\dfrac{ax}{\lambda x + \mu y}\ln\left|\sinh(\lambda x + \mu y)\right|\right)\Phi\left(\dfrac{y}{x}\right).$

19. $a\dfrac{\partial w}{\partial x} + b\coth^n(\lambda x)\dfrac{\partial w}{\partial y} = \left[c\coth^m(\mu x) + s\coth^k(\beta y)\right]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\coth^n(\lambda x)$, and $h(x,y) = c\coth^m(\mu x) + s\coth^k(\beta y)$.

20. $a\dfrac{\partial w}{\partial x} + b\coth^n(\lambda y)\dfrac{\partial w}{\partial y} = \big[c\coth^m(\mu x) + s\coth^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\coth^n(\lambda y)$, $h_1(x) = c\coth^m(\mu x)$, and $h_2(y) = s\coth^k(\beta y)$.

► **Coefficients of equations contain different hyperbolic functions.**

21. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big[c\sinh(\lambda x) + k\cosh(\mu y)\big]w.$

General solution: $w = \exp\left[\dfrac{c}{a\lambda}\cosh(\lambda x) + \dfrac{k}{b\mu}\sinh(\mu y)\right]\Phi(bx - ay).$

22. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big[\tanh(\lambda x) + k\coth(\mu y)\big]w.$

General solution: $w = \cosh^{1/a\lambda}(\lambda x)\sinh^{k/b\mu}(\mu y)\Phi(bx - ay).$

23. $\dfrac{\partial w}{\partial x} + a\sinh(\mu y)\dfrac{\partial w}{\partial y} = b\cosh(\lambda x)w.$

General solution: $w = \exp\left[\dfrac{b}{\lambda}\sinh(\lambda x)\right]\Phi\left(a\mu x - \ln\left|\tanh\dfrac{\mu y}{2}\right|\right).$

24. $\dfrac{\partial w}{\partial x} + a\sinh(\mu y)\dfrac{\partial w}{\partial y} = b\tanh(\lambda x)w.$

General solution: $w = \cosh^{b/\lambda}(\lambda x)\Phi\left(a\mu x - \ln\left|\tanh\dfrac{\mu y}{2}\right|\right).$

25. $a\sinh(\lambda x)\dfrac{\partial w}{\partial x} + b\cosh(\mu y)\dfrac{\partial w}{\partial y} = w.$

General solution: $w = \tanh^{1/a\lambda}\left(\dfrac{\lambda x}{2}\right)\Phi\left(2a\arctan\left(\tanh\dfrac{\mu y}{2}\right) + \dfrac{b\mu}{\lambda}\ln\left|\coth\dfrac{\lambda x}{2}\right|\right).$

26. $a\tanh(\lambda x)\dfrac{\partial w}{\partial x} + b\coth(\mu y)\dfrac{\partial w}{\partial y} = w.$

General solution: $w = \sinh^{1/a\lambda}(\lambda x)\Phi\big(\cosh^{a\lambda}(\mu y)\sinh^{-b\mu}(\lambda x)\big).$

1.3.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln(\lambda x + \beta y)w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{c(\lambda x + \beta y)}{a\lambda + b\beta}\big(\ln(\lambda x + \beta y) - 1\big)\right]\Phi(bx - ay) & \text{if } a\lambda \neq -b\beta, \\ \exp\left[\dfrac{c}{a}x\ln(\lambda x + \beta y)\right]\Phi(bx - ay) & \text{if } a\lambda = -b\beta. \end{cases}$$

2. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\big[c\ln(\lambda x)+k\ln(\beta y)\big]w.$

General solution: $w=\exp\left[\dfrac{c}{a}x\big(\ln(\lambda x)-1\big)+\dfrac{k}{b}y\big(\ln(\beta y)-1\big)\right]\Phi(bx-ay).$

3. $a\dfrac{\partial w}{\partial x}+b\ln^n(\lambda x)\dfrac{\partial w}{\partial y}=\big[c\ln^m(\mu x)+s\ln^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\ln^n(\lambda x)$, and $h(x,y)=c\ln^m(\mu x)+s\ln^k(\beta y)$.

4. $a\dfrac{\partial w}{\partial x}+b\ln^n(\lambda y)\dfrac{\partial w}{\partial y}=\big[c\ln^m(\mu x)+s\ln^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\ln^n(\lambda y)$, $h_1(x)=c\ln^m(\mu x)$, and $h_2(y)=s\ln^k(\beta y)$.

5. $\ln(\beta y)\dfrac{\partial w}{\partial x}+a\ln(\lambda x)\dfrac{\partial w}{\partial y}=bw\ln(\beta y).$

General solution: $w=e^{bx}\Phi(u)$, where $u=ax\big[1-\ln(\lambda x)\big]+y\big[\ln(\beta y)-1\big]$.

6. $a\ln^n(\lambda x)\dfrac{\partial w}{\partial x}+b\ln^k(\beta y)\dfrac{\partial w}{\partial y}=c\ln^m(\gamma x)w.$

General solution:

$$w=\Phi(u)\exp\left[\frac{c}{a}\int\frac{\ln^m(\gamma x)}{\ln^n(\lambda x)}\,dx\right],\qquad \text{where } u=b\int\frac{dx}{\ln^n(\lambda x)}-a\int\frac{dy}{\ln^k(\beta y)}.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

7. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\big[cx^n+s\ln^k(\lambda y)\big]w.$

General solution: $w=\Phi(bx-ay)\exp\left[\dfrac{c}{a(n+1)}x^{n+1}+\dfrac{s}{b}\displaystyle\int\ln^k(\lambda y)\,dy\right].$

8. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=\big[by^2+cx^ny+s\ln^k(\lambda x)\big]w.$

This is a special case of equation 1.3.7.3 with $f(x)=b$, $g(x)=cx^n$, and $h(x)=s\ln^k(\lambda x)$.

9. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=b\ln^k(\lambda x)\ln^n(\beta y)w.$

This is a special case of equation 1.3.7.17 with $f(x)=b\ln^k(\lambda x)$ and $g(y)=\ln^n(\beta y)$.

10. $\dfrac{\partial w}{\partial x}+(ay+bx^n)\dfrac{\partial w}{\partial y}=c\ln^k(\lambda x)w.$

This is a special case of equation 1.3.7.6 with $f(x)=bx^n$ and $g(x)=c\ln^k(\lambda x)$.

11. $ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=x^k(n\ln x+m\ln y)w.$

This is a special case of equation 1.3.7.26 with $f(u)=\ln u$.

12. $\displaystyle ax^k\frac{\partial w}{\partial x}+by^n\frac{\partial w}{\partial y}=\big[c\ln^m(\lambda x)+s\ln^l(\beta y)\big]w.$

General solution:

$$w=\Phi(u)\exp\left[\frac{c}{a}\int x^{-k}\ln^m(\lambda x)\,dx+\frac{s}{b}\int y^{-n}\ln^l(\beta y)\,dy\right],$$

$$u=\frac{b}{1-k}x^{1-k}-\frac{a}{1-n}y^{1-n}.$$

1.3.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\sin(\lambda x+\mu y)w.$

General solution:

$$w=\begin{cases}\exp\left[-\dfrac{c}{a\lambda+b\mu}\cos(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \exp\left[\dfrac{c}{a}x\sin(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

2. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\big[c\sin(\lambda x)+k\sin(\mu y)\big]w.$

General solution: $\displaystyle w=\exp\left[-\frac{c}{a\lambda}\cos(\lambda x)-\frac{k}{b\mu}\cos(\mu y)\right]\Phi(bx-ay).$

3. $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=ax\sin(\lambda x+\mu y)w.$

General solution: $\displaystyle w=\exp\left[-\frac{ax}{\lambda x+\mu y}\cos(\lambda x+\mu y)\right]\Phi\left(\frac{y}{x}\right).$

4. $\displaystyle a\frac{\partial w}{\partial x}+b\sin^n(\lambda x)\frac{\partial w}{\partial y}=\big[c\sin^m(\mu x)+s\sin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\sin^n(\lambda x)$, and $h(x,y)=c\sin^m(\mu x)+s\sin^k(\beta y)$.

5. $\displaystyle a\frac{\partial w}{\partial x}+b\sin^n(\lambda y)\frac{\partial w}{\partial y}=\big[c\sin^m(\mu x)+s\sin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\sin^n(\lambda y)$, $h_1(x)=c\sin^m(\mu x)$, and $h_2(y)=s\sin^k(\beta y)$.

► **Coefficients of equations contain cosine.**

6. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\cos(\lambda x+\mu y)w.$

General solution:

$$w=\begin{cases}\exp\left[\dfrac{c}{a\lambda+b\mu}\sin(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \exp\left[\dfrac{c}{a}x\cos(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

7. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big[c\cos(\lambda x) + k\cos(\mu y)\big]w.$

General solution: $w = \exp\left[\dfrac{c}{a\lambda}\sin(\lambda x) + \dfrac{k}{b\mu}\sin(\mu y)\right]\Phi(bx - ay).$

8. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\cos(\lambda x + \mu y)w.$

General solution: $w = \exp\left[\dfrac{ax}{\lambda x + \mu y}\sin(\lambda x + \mu y)\right]\Phi\left(\dfrac{y}{x}\right).$

9. $a\dfrac{\partial w}{\partial x} + b\cos^n(\lambda x)\dfrac{\partial w}{\partial y} = \big[c\cos^m(\mu x) + s\cos^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cos^n(\lambda x)$, and $h(x,y) = c\cos^m(\mu x) + s\cos^k(\beta y)$.

10. $a\dfrac{\partial w}{\partial x} + b\cos^n(\lambda y)\dfrac{\partial w}{\partial y} = \big[c\cos^m(\mu x) + s\cos^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\cos^n(\lambda y)$, $h_1(x) = c\cos^m(\mu x)$, and $h_2(y) = s\cos^k(\beta y)$.

► **Coefficients of equations contain tangent.**

11. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\tan(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left(-\dfrac{c}{a\lambda + b\mu}\ln\left|\cos(\lambda x + \mu y)\right|\right)\Phi(bx - ay) & \text{if } a\lambda \neq -b\mu, \\ \exp\left[\dfrac{c}{a}x\tan(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda = -b\mu. \end{cases}$$

12. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big[c\tan(\lambda x) + k\tan(\mu y)\big]w.$

General solution: $w = \exp\left(-\dfrac{c}{a\lambda}\ln\left|\cos(\lambda x)\right| - \dfrac{k}{b\mu}\ln\left|\cos(\mu y)\right|\right)\Phi(bx - ay).$

13. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\tan(\lambda x + \mu y)w.$

General solution: $w = \exp\left(-\dfrac{ax}{\lambda x + \mu y}\ln\left|\cos(\lambda x + \mu y)\right|\right)\Phi\left(\dfrac{y}{x}\right).$

14. $a\dfrac{\partial w}{\partial x} + b\tan^n(\lambda x)\dfrac{\partial w}{\partial y} = \big[c\tan^m(\mu x) + s\tan^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\tan^n(\lambda x)$, and $h(x,y) = c\tan^m(\mu x) + s\tan^k(\beta y)$.

15. $a\dfrac{\partial w}{\partial x} + b\tan^n(\lambda y)\dfrac{\partial w}{\partial y} = \big[c\tan^m(\mu x) + s\tan^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\tan^n(\lambda y)$, $h_1(x) = c\tan^m(\mu x)$, and $h_2(y) = s\tan^k(\beta y)$.

► **Coefficients of equations contain cotangent.**

16. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\cot(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{a\lambda + b\mu}\ln\left|\sin(\lambda x + \mu y)\right|\right)\Phi(bx - ay) & \text{if } a\lambda \neq -b\mu, \\ \exp\left[\dfrac{c}{a}x\cot(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda = -b\mu. \end{cases}$$

17. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c\cot(\lambda x) + k\cot(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left(\frac{c}{a\lambda}\ln\left|\sin(\lambda x)\right| + \frac{k}{b\mu}\ln\left|\sin(\mu y)\right|\right)\Phi(bx - ay).$

18. $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cot(\lambda x + \mu y)w.$

General solution: $\displaystyle w = \exp\left(\frac{ax}{\lambda x + \mu y}\ln\left|\sin(\lambda x + \mu y)\right|\right)\Phi\left(\frac{y}{x}\right).$

19. $\displaystyle a\frac{\partial w}{\partial x} + b\cot^n(\lambda x)\frac{\partial w}{\partial y} = \big[c\cot^m(\mu x) + s\cot^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cot^n(\lambda x)$, and $h(x, y) = c\cot^m(\mu x) + s\cot^k(\beta y)$.

20. $\displaystyle a\frac{\partial w}{\partial x} + b\cot^n(\lambda y)\frac{\partial w}{\partial y} = \big[c\cot^m(\mu x) + s\cot^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\cot^n(\lambda y)$, $h_1(x) = c\cot^m(\mu x)$, and $h_2(y) = s\cot^k(\beta y)$.

► **Coefficients of equations contain different trigonometric functions.**

21. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \big[b\sin(\lambda x) + k\cos(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left[\frac{k}{a\mu}\sin(\mu y) - \frac{b}{\lambda}\cos(\lambda x)\right]\Phi(ax - y).$

22. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \big[b\sin(\lambda x) + k\tan(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left[-\frac{b}{\lambda}\cos(\lambda x)\right]\cos^{k/a\mu}(\mu y)\Phi(ax - y).$

23. $\displaystyle \frac{\partial w}{\partial x} + a\sin(\mu y)\frac{\partial w}{\partial y} = bw\tan(\lambda x).$

General solution: $\displaystyle w = \cos^{-b/\lambda}(\lambda x)\Phi\left(a\mu x - \ln\left|\tan\frac{\mu y}{2}\right|\right).$

24. $\dfrac{\partial w}{\partial x}+a\tan(\mu y)\dfrac{\partial w}{\partial y}=bw\sin(\lambda x).$

General solution: $w=\exp\left[-\dfrac{b}{\lambda}\cos(\lambda x)\right]\Phi\Big(a\mu x-\ln\big|\sin(\mu y)\big|\Big).$

25. $\sin(\lambda x)\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=bw\cos(\mu y).$

General solution: $w=\exp\left[\dfrac{b}{a\mu}\sin(\mu y)\right]\Phi\left(\lambda y+b\ln\left|\cot\dfrac{\lambda x}{2}\right|\right).$

26. $\cot(\lambda x)\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=bw\tan(\mu y).$

General solution: $w=\cos^{-b/a\mu}(\mu y)\Phi\Big(\lambda y+b\ln\big|\cos(\lambda x)\big|\Big).$

1.3.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\left(c\arcsin\dfrac{x}{\lambda}+k\arcsin\dfrac{y}{\beta}\right)w.$

General solution:

$$w=\exp\left[\frac{c}{a}\left(x\arcsin\frac{x}{\lambda}+\sqrt{\lambda^2-x^2}\right)+\frac{k}{b}\left(y\arcsin\frac{y}{\beta}+\sqrt{\beta^2-y^2}\right)\right]\Phi(bx-ay).$$

2. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arcsin(\lambda x+\beta y)w.$

1°. General solution for $a\lambda+b\beta\neq 0$:

$$w=\exp\left[\frac{c(\lambda x+\beta y)}{a\lambda+b\beta}\arcsin(\lambda x+\beta y)+\frac{\sqrt{1-(\lambda x+\beta y)^2}}{a\lambda+b\beta}\right]\Phi(bx-ay).$$

2°. General solution for $a\lambda+b\beta=0$:

$$w=\exp\left[\frac{c}{a}x\arcsin(\lambda x+\beta y)\right]\Phi(bx-ay).$$

3. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\arcsin(\lambda x+\beta y)w.$

General solution: $w=\exp\left[ax\arcsin(\lambda x+\beta y)+ax\dfrac{\sqrt{1-(\lambda x+\beta y)^2}}{\lambda x+\beta y}\right]\Phi\left(\dfrac{y}{x}\right).$

4. $a\dfrac{\partial w}{\partial x}+b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y}=\big[c\arcsin^m(\mu x)+s\arcsin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arcsin^n(\lambda x)$, and $h(x,y)=c\arcsin^m(\mu x)+s\arcsin^k(\beta y)$.

5. $a\dfrac{\partial w}{\partial x}+b\arcsin^n(\lambda y)\dfrac{\partial w}{\partial y}=\big[c\arcsin^m(\mu x)+s\arcsin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\arcsin^n(\lambda y)$, $h_1(x)=c\arcsin^m(\mu x)$, and $h_2(y)=s\arcsin^k(\beta y)$.

► **Coefficients of equations contain arccosine.**

6. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\Big(c\arccos\frac{x}{\lambda}+k\arccos\frac{y}{\beta}\Big)w.$

General solution:

$$w=\exp\left[\frac{c}{a}\Big(x\arccos\frac{x}{\lambda}-\sqrt{\lambda^2-x^2}\Big)+\frac{k}{b}\Big(y\arccos\frac{y}{\beta}-\sqrt{\beta^2-y^2}\Big)\right]\Phi(bx-ay).$$

7. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\arccos(\lambda x+\beta y)w.$

1°. General solution for $a\lambda+b\beta\neq 0$:

$$w=\exp\left[\frac{c(\lambda x+\beta y)}{a\lambda+b\beta}\arccos(\lambda x+\beta y)-\frac{\sqrt{1-(\lambda x+\beta y)^2}}{a\lambda+b\beta}\right]\Phi(bx-ay).$$

2°. General solution for $a\lambda+b\beta=0$:

$$w=\exp\left[\frac{c}{a}x\arccos(\lambda x+\beta y)\right]\Phi(bx-ay).$$

8. $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=ax\arccos(\lambda x+\beta y)w.$

General solution: $\displaystyle w=\exp\left[ax\arccos(\lambda x+\beta y)-ax\frac{\sqrt{1-(\lambda x+\beta y)^2}}{\lambda x+\beta y}\right]\Phi\Big(\frac{y}{x}\Big).$

9. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda x)\frac{\partial w}{\partial y}=\big[c\arccos^m(\mu x)+s\arccos^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arccos^n(\lambda x)$, and $h(x,y)=c\arccos^m(\mu x)+s\arccos^k(\beta y)$.

10. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda y)\frac{\partial w}{\partial y}=\big[c\arccos^m(\mu x)+s\arccos^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\arccos^n(\lambda y)$, $h_1(x)=c\arccos^m(\mu x)$, and $h_2(y)=s\arccos^k(\beta y)$.

► **Coefficients of equations contain arctangent.**

11. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\Big(c\arctan\frac{x}{\lambda}+k\arctan\frac{y}{\beta}\Big)w.$

General solution:

$$w=\exp\left\{\frac{c}{a}\Big[x\arctan\frac{x}{\lambda}-\frac{\lambda}{2}\ln\big(\lambda^2+x^2\big)\Big]+\frac{k}{b}\Big[y\arctan\frac{y}{\beta}-\frac{\beta}{2}\ln\big(\beta^2+y^2\big)\Big]\right\}\Phi(bx-ay).$$

12. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arctan(\lambda x + \beta y)w.$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \exp\biggl\{\frac{c(\lambda x + \beta y)}{a\lambda + b\beta}\arctan(\lambda x + \beta y) - \frac{\ln[1 + (\lambda x + \beta y)^2]}{2(a\lambda + b\beta)}\biggr\}\Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \exp\Bigl[\frac{c}{a}x\arctan(\lambda x + \beta y)\Bigr]\Phi(bx - ay).$$

13. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\arctan(\lambda x + \beta y)w.$

General solution:

$$w = \exp\biggl\{ax\arctan(\lambda x + \beta y) - \frac{ax}{2(\lambda x + \beta y)}\ln\Bigl[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\Bigr]\biggr\}\Phi\Bigl(\frac{y}{x}\Bigr).$$

14. $a\dfrac{\partial w}{\partial x} + b\arctan^n(\lambda x)\dfrac{\partial w}{\partial y} = \bigl[c\arctan^m(\mu x) + s\arctan^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arctan^n(\lambda x)$, and $h(x,y) = c\arctan^m(\mu x) + s\arctan^k(\beta y)$.

15. $a\dfrac{\partial w}{\partial x} + b\arctan^n(\lambda y)\dfrac{\partial w}{\partial y} = \bigl[c\arctan^m(\mu x) + s\arctan^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\arctan^n(\lambda y)$, $h_1(x) = c\arctan^m(\mu x)$, and $h_2(y) = s\arctan^k(\beta y)$.

► **Coefficients of equations contain arccotangent.**

16. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \Bigl(c\,\mathrm{arccot}\,\dfrac{x}{\lambda} + k\,\mathrm{arccot}\,\dfrac{y}{\beta}\Bigr)w.$

General solution:

$$w = \exp\biggl\{\frac{c}{a}\Bigl[x\,\mathrm{arccot}\,\frac{x}{\lambda} + \frac{\lambda}{2}\ln\bigl(\lambda^2 + x^2\bigr)\Bigr] + \frac{k}{b}\Bigl[y\,\mathrm{arccot}\,\frac{y}{\beta} + \frac{\beta}{2}\ln\bigl(\beta^2 + y^2\bigr)\Bigr]\biggr\}\Phi(bx - ay).$$

17. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\,\mathrm{arccot}(\lambda x + \beta y)w.$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \exp\biggl\{\frac{c(\lambda x + \beta y)}{a\lambda + b\beta}\mathrm{arccot}(\lambda x + \beta y) + \frac{\ln[1 + (\lambda x + \beta y)^2]}{2(a\lambda + b\beta)}\biggr\}\Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \exp\Bigl[\frac{c}{a}x\,\mathrm{arccot}(\lambda x + \beta y)\Bigr]\Phi(bx - ay).$$

18. $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\,\mathrm{arccot}(\lambda x + \beta y)w.$

General solution:

$$w = \exp\left\{ax\,\mathrm{arccot}(\lambda x + \beta y) + \frac{ax}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\}\Phi\left(\frac{y}{x}\right).$$

19. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y} = \left[c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)\right]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\,\mathrm{arccot}^n(\lambda x)$, and $h(x,y) = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)$.

20. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda y)\frac{\partial w}{\partial y} = \left[c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)\right]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\,\mathrm{arccot}^n(\lambda y)$, $h_1(x) = c\,\mathrm{arccot}^m(\mu x)$, and $h_2(y) = s\,\mathrm{arccot}^k(\beta y)$.

1.3.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of x.**

1. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = f(x)w.$

General solution: $\displaystyle w = \exp\left[\frac{1}{a}\int f(x)\,dx\right]\Phi(bx - ay).$

2. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x)yw.$

General solution: $\displaystyle w = \exp\left[\int_{x_0}^{x}(y - ax + at)f(t)\,dt\right]\Phi(y - ax)$, where x_0 may be chosen arbitrarily.

3. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \left[f(x)y^2 + g(x)y + h(x)\right]w.$

General solution:

$$w = \exp\left[\varphi(x)y^2 + \psi(x)y + \chi(x)\right]\Phi(y - ax),$$

where

$$\varphi(x) = \int f(x)\,dx, \quad \psi(x) = \int\left[g(x) - 2a\varphi(x)\right]dx, \quad \chi(x) = \int\left[h(x) - a\psi(x)\right]dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

4. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x)y^k w.$

General solution: $\displaystyle w = \exp\left[\int_{x_0}^{x}(y - ax + at)^k f(t)\,dt\right]\Phi(y - ax)$, where x_0 can be chosen arbitrarily.

5. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)e^{\lambda y}w.$

General solution: $w = \exp\left[e^{\lambda(y-ax)}\int f(x)e^{a\lambda x}\,dx\right]\Phi(y-ax).$

6. $\dfrac{\partial w}{\partial x} + \bigl[ay + f(x)\bigr]\dfrac{\partial w}{\partial y} = g(x)w.$

General solution: $w = \exp\left[\int g(x)\,dx\right]\Phi(u)$, where $u = e^{-ax}y - \int f(x)e^{-ax}\,dx.$

7. $\dfrac{\partial w}{\partial x} + \bigl[ay + f(x)\bigr]\dfrac{\partial w}{\partial y} = g(x)y^k w.$

This is a special case of equation 1.3.7.18 with $h(y) = y^k$.

8. $f(x)\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = g(x)w.$

General solution:

$$w = \exp\left[\int \frac{g(x)}{f(x)}\,dx\right]\Phi(u), \qquad \text{where} \quad u = \begin{cases} \dfrac{1}{k-1}y^{1-k} + \displaystyle\int \frac{dx}{f(x)} & \text{if } k \neq 1, \\ y\exp\left[-\displaystyle\int \frac{dx}{f(x)}\right] & \text{if } k = 1. \end{cases}$$

9. $f(x)\dfrac{\partial w}{\partial x} + (y+a)\dfrac{\partial w}{\partial y} = (by+c)w.$

General solution: $w = (y+a)^{c-ab}e^{by}\Phi(u)$, where $u = (y+a)\exp\left[-\int \dfrac{dx}{f(x)}\right].$

10. $f(x)\dfrac{\partial w}{\partial x} + (y+ax)\dfrac{\partial w}{\partial y} = g(x)w.$

General solution:

$$w = \exp\left[\int \frac{g(x)}{f(x)}\,dx\right]\Phi\left(e^{-z}y - a\int \frac{xe^{-z}}{f(x)}\,dx\right), \qquad \text{where} \quad z = \int \frac{dx}{f(x)}.$$

11. $f(x)\dfrac{\partial w}{\partial x} + \bigl[g_1(x)y + g_0(x)\bigr]\dfrac{\partial w}{\partial y} = \bigl[h_2(x)y^2 + h_1(x)y + h_0(x)\bigr]w.$

General solution:

$$w = \exp\bigl[\varphi(x)y^2 + \psi(x)y + \chi(x)\bigr]\Phi(u), \qquad u = e^{-G}y - \int e^{-G}\frac{g_0}{f}\,dx,$$

where

$$\varphi(x) = e^{-2G}\int e^{2G}\frac{h_2}{f}\,dx, \qquad G = G(x) = \int \frac{g_1}{f}\,dx,$$
$$\psi(x) = e^{-G}\int e^{G}\frac{h_1 - 2g_0\varphi}{f}\,dx, \quad \chi(x) = \int \frac{h_0 - g_0\psi}{f}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

12. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_2(x)y^k\bigr]\dfrac{\partial w}{\partial y}=h(x)w.$

General solution: $w=\exp\biggl[\displaystyle\int\frac{h(x)}{f(x)}\,dx\biggr]\Phi(u)$, where

$$u=e^{-G}y^{1-k}-(1-k)\int e^{-G}\frac{g_2(x)}{f(x)}\,dx,\qquad G=(1-k)\int\frac{g_1(x)}{f(x)}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

13. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)+g_2(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y}=h(x)w.$

General solution: $w=\exp\biggl[\displaystyle\int\frac{h(x)}{f(x)}\,dx\biggr]\Phi(u)$, where

$$u=e^{-\lambda y}E(x)+\lambda\int\frac{g_2(x)}{f(x)}E(x)\,dx,\qquad E(x)=\exp\biggl[\lambda\int\frac{g_1(x)}{f(x)}\,dx\biggr].$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

14. $f(x)y^k\dfrac{\partial w}{\partial x}+g(x)\dfrac{\partial w}{\partial y}=h(x)w.$

General solution: $w=\Phi(u)\exp\biggl\{\displaystyle\int_{x_0}^{x}\frac{h(t)}{f(t)}\bigl[u+E(t)\bigr]^{-\frac{k}{k+1}}\,dt\biggr\}$, where

$u=y^{k+1}-E(x)$, $\quad E(x)=(k+1)\displaystyle\int\frac{g(x)}{f(x)}\,dx$, where x_0 may be chosen arbitrarily.

15. $f(x)e^{\lambda y}\dfrac{\partial w}{\partial x}+g(x)\dfrac{\partial w}{\partial y}=h(x)w.$

General solution:

$$w=\Phi(u)\exp\biggl\{\int_{x_0}^{x}\frac{h(t)\,dt}{f(t)\bigl[u+E(t)\bigr]}\biggr\},\qquad u=e^{\lambda y}-E(x),\qquad E(x)=\lambda\int\frac{g(x)}{f(x)}\,dx,$$

where x_0 may be chosen arbitrarily.

► Equations contain arbitrary functions of x and arbitrary functions of y.

16. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\bigl[f(x)+g(y)\bigr]w.$

General solution: $w=\exp\biggl[\dfrac{1}{a}\displaystyle\int f(x)\,dx+\frac{1}{b}\int g(y)\,dy\biggr]\Phi(bx-ay).$

17. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=f(x)g(y)w.$

General solution: $w=\exp\biggl[\displaystyle\int_{x_0}^{x}f(t)g(y-ax+at)\,dt\biggr]\Phi(y-ax)$, where x_0 may be taken as arbitrary.

18. $\dfrac{\partial w}{\partial x}+[ay+f(x)]\dfrac{\partial w}{\partial y}=g(x)h(y)w.$

The substitutions $w=\pm e^u$ lead to an equation of the form 1.2.7.19:

$$\frac{\partial u}{\partial x}+[ay+f(x)]\frac{\partial u}{\partial y}=g(x)h(y).$$

19. $f(x)\dfrac{\partial w}{\partial x}+g(y)\dfrac{\partial w}{\partial y}=\big[h_1(x)+h_2(y)\big]w.$

General solution:

$$w=\exp\biggl[\int\frac{h_1(x)}{f(x)}\,dx+\int\frac{h_2(y)}{g(y)}\,dy\biggr]\Phi\biggl(\int\frac{dx}{f(x)}\,dx-\int\frac{dy}{g(y)}\,dy\biggr).$$

20. $f_1(x)\dfrac{\partial w}{\partial x}+\big[f_2(x)y+f_3(x)y^k\big]\dfrac{\partial w}{\partial y}=g(x)h(y)w.$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=y^{1-k}$ leads to an equation of the form 1.3.7.18:

$$\frac{\partial w}{\partial \xi}+\big[(1-k)\eta+F(\xi)\big]\frac{\partial w}{\partial \eta}=G(\xi)H(\eta)w,$$

where $F(\xi)=(1-k)\dfrac{f_3(x)}{f_2(x)},\ G(\xi)=\dfrac{g(x)}{f_2(x)},$ and $H(\eta)=h(y).$

21. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h_1(x)h_2(y)w.$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.3.7.17:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi)G(\eta)w,\qquad \text{where}\quad F(\xi)=\frac{h_1(x)}{f_2(x)},\quad G(\eta)=\frac{h_2(y)}{g_1(y)}.$$

22. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=\big[h_1(x)+h_2(y)\big]w.$

This is a special case of equation 1.3.7.36 with $h(x,y)=h_1(x)+h_2(y)$.

▶ Equations contain arbitrary functions of complicated arguments.

23. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(\alpha x+\beta y)w.$

General solution:

$$w=\begin{cases}\exp\biggl[\dfrac{1}{a\alpha+b\beta}\displaystyle\int f(u)\,du\biggr]\Phi(bx-ay) & \text{if } a\alpha+b\beta\neq 0,\\[2ex] \exp\biggl[\dfrac{1}{a}xf(\alpha x+\beta y)\biggr]\Phi(bx-ay) & \text{if } a\alpha+b\beta=0,\end{cases}$$

where $u=\alpha x+\beta y$.

24. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=xf\Big(\dfrac{y}{x}\Big)w.$

General solution: $w=\exp\Big[xf\Big(\dfrac{y}{x}\Big)\Big]\Phi\Big(\dfrac{y}{x}\Big).$

25. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=f(x^2+y^2)w.$

General solution: $w=\Phi\Big(\dfrac{y}{x}\Big)\exp\Big[\dfrac{1}{2}\displaystyle\int f(\xi)\dfrac{d\xi}{\xi}\Big]$, where $\xi=x^2+y^2$.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

26. $ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=x^kf(x^ny^m)w.$

General solution:

$$w=\begin{cases}\exp\Big[\dfrac{1}{a}\displaystyle\int x^{k-1}f\Big(x^{\frac{an+bm}{a}}u^{\frac{m}{a}}\Big)\,dx\Big]\Phi(u) & \text{if } an\neq -bm;\\ \exp\Big[\dfrac{1}{ak}x^kf(x^ny^m)\Big]\Phi(u) & \text{if } an=-bm,\ k\neq 0;\\ \exp\Big[\dfrac{1}{a}f(x^ny^m)\ln x\Big]\Phi(u) & \text{if } an=-bm,\ k=0,\end{cases}$$

where $u=y^ax^{-b}$. In the integration, u is considered a parameter.

27. $mx\dfrac{\partial w}{\partial x}+ny\dfrac{\partial w}{\partial y}=f(ax^n+by^m)w.$

General solution: $w=\Phi\big(y^mx^{-n}\big)\exp\Big[\dfrac{1}{nm}\displaystyle\int f(\xi)\dfrac{d\xi}{\xi}\Big]$, where $\xi=ax^n+by^m$.

28. $x^2\dfrac{\partial w}{\partial x}+xy\dfrac{\partial w}{\partial y}=y^kf(\alpha x+\beta y)w.$

General solution:

$$w=\exp\Big[\frac{y^k}{x(\alpha x+\beta y)^{k-1}}\int z^{k-2}f(z)\,dz\Big]\Phi\Big(\frac{y}{x}\Big),\qquad \text{where}\quad z=\alpha x+\beta y.$$

29. $\dfrac{f(x)}{f'(x)}\dfrac{\partial w}{\partial x}+\dfrac{g(y)}{g'(y)}\dfrac{\partial w}{\partial y}=h\big(f(x)+g(y)\big)w.$

General solution:

$$w=\Phi(u)\exp\Big[\int h(\xi)\frac{d\xi}{\xi}\Big],\qquad \text{where}\quad u=\frac{g(y)}{f(x)},\quad \xi=f(x)+g(y).$$

► **Equations contain arbitrary functions of two variables.**

30. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=f(x,y)w.$

General solution: $w=\exp\Big[\displaystyle\int_{x_0}^{x}f(t,y-ax+at)\,dt\Big]\Phi(y-ax)$, where x_0 may be taken as arbitrary.

31. $ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=f(x,y)w.$

General solution:

$$w=\exp\left[\frac{1}{a}\int\frac{1}{x}f\bigl(x,\,u^{1/a}x^{b/a}\bigr)\,dx\right]\Phi(u),\qquad\text{where}\quad u=y^ax^{-b}.$$

In the integration, u is considered a parameter.

32. $f(x)\dfrac{\partial w}{\partial x}+g(x)y\dfrac{\partial w}{\partial y}=h(x,y)w.$

General solution:

$$w=\Phi(u)\exp\left[\int\frac{h(x,\,uG)}{f(x)}\,dx\right],\qquad\text{where}\quad u=\frac{y}{G},\quad G=\exp\biggl(\int\frac{g}{f}\,dx\biggr).$$

In the integration, u is considered a parameter.

33. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w.$

General solution:

$$w=\Phi(u)\exp\left[\int\frac{h(x,\,uG+Q)}{f(x)}\,dx\right],\quad u=\frac{y-Q}{G},$$

where $G=\exp\biggl(\displaystyle\int\frac{g_1}{f}\,dx\biggr)$ and $Q=G\displaystyle\int\frac{g_0\,dx}{fG}$. In the integration, u is considered a parameter.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

34. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)y^k\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w.$

For $k=1$, see equation 1.3.7.32. For $k\neq 1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 1.3.7.33:

$$f(x)\frac{\partial w}{\partial x}+(1-k)\bigl[g_1(x)\xi+g_0(x)\bigr]\frac{\partial w}{\partial \xi}=h\Bigl(x,\,\xi^{\frac{1}{1-k}}\Bigr)w.$$

35. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)+g_0(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w.$

The substitution $z=e^{-\lambda y}$ leads to an equation of the form 1.3.7.33:

$$f(x)\frac{\partial w}{\partial x}-\lambda\bigl[g_1(x)z+g_0(x)\bigr]\frac{\partial w}{\partial z}=h\Bigl(x,\,-\frac{1}{\lambda}\ln z\Bigr)w.$$

36. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h(x,y)w.$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.3.7.30:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi,\eta)w,\qquad\text{where}\quad F(\xi,\eta)=\frac{h(x,y)}{f_2(x)g_1(y)}.$$

1.4 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = h_1(x,y)w + h_0(x,y)$

◆ *The solutions given below contain an arbitrary function* $\Phi = \Phi(z)$.

1.4.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x and y.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + d.$

General solution: $w = -\dfrac{d}{c} + e^{cx/a}\Phi(bx - ay)$.

2. $(x-a)\dfrac{\partial w}{\partial x} + (y-b)\dfrac{\partial w}{\partial y} = w - c.$

Differential equation of a conic surface with the vertex at the point (a,b,c).

General solution: $w = c + (x-a)\Phi\Bigl(\dfrac{y-b}{x-a}\Bigr)$.

⊙ *Literature*: E. Kamke (1965).

3. $(ax+b)\dfrac{\partial w}{\partial x} + (cx+d)\dfrac{\partial w}{\partial y} = \alpha w + \beta.$

General solution:

$$w = \begin{cases} -\dfrac{\beta}{\alpha} + (ax+b)^{\alpha/a}\Phi\bigl(a(cx-ay) + (ad-bc)\ln|ax+b|\bigr) & \text{if } a \neq 0, \\ -\dfrac{\beta}{\alpha} + e^{\alpha x/b}\Phi\bigl(x(cx+2d) - 2by\bigr) & \text{if } a = 0. \end{cases}$$

4. $(ax+b)\dfrac{\partial w}{\partial x} + (cy+d)\dfrac{\partial w}{\partial y} = \alpha w + \beta.$

General solution:

$$w = \begin{cases} -\dfrac{\beta}{\alpha} + (ax+b)^{\alpha/a}\Phi\bigl((ax+b)^{-c/a}(cy+d)\bigr) & \text{if } a \neq 0, \\ -\dfrac{\beta}{\alpha} + e^{\alpha x/b}\Phi\bigl((cy+d)e^{-cx/b}\bigr) & \text{if } a = 0. \end{cases}$$

5. $(ax+b)\dfrac{\partial w}{\partial x} + (cy+d)\dfrac{\partial w}{\partial y} = \alpha w + \beta y + \gamma x.$

1°. General solution for $a \neq 0$, $a \neq \alpha$, and $c \neq \alpha$:

$$w = \frac{\gamma(\alpha x + b)}{\alpha(a-\alpha)} - \frac{\beta(\alpha y + d)}{\alpha(\alpha - c)} + (ax+b)^{\alpha/a}\Phi\bigl((ax+b)^{-c/a}(cy+d)\bigr).$$

2°. General solution for $a \neq 0$, $a = \alpha$, and $c \neq \alpha$:

$$w = \frac{\gamma\bigl[b + (ax+b)\ln|ax+b|\bigr]}{a^2} - \frac{\beta(ay+d)}{a(a-c)} + (ax+b)\Phi\bigl((ax+b)^{-c/a}(cy+d)\bigr).$$

3°. General solution for $a \neq 0$ and $a = c = \alpha$:

$$w = \frac{b\gamma + d\beta[\gamma(ax+b) + \beta(ay+d)]\ln|ax+b|}{a^2} + (ax+b)\Phi\biggl(\frac{ay+d}{ax+b}\biggr).$$

4°. General solution for $a = 0$ and $c \neq \alpha$:

$$w = -\frac{\gamma(\alpha x+b)}{\alpha^2} - \frac{\beta(\alpha y+d)}{\alpha(\alpha-c)} + e^{\alpha x/b}\Phi\bigl((cy+d)e^{-cx/b}\bigr).$$

5°. General solution for $a = 0$ and $c = \alpha$:

$$w = \frac{(d\beta - b\gamma)(cx+b)}{bc^2} + \frac{\beta}{b}xy + e^{cx/b}\Phi\bigl((cy+d)e^{-cx/b}\bigr).$$

6. $\boldsymbol{(ax+b)\dfrac{\partial w}{\partial x} + (cx+dy)\dfrac{\partial w}{\partial y} = \alpha w + \beta.}$

1°. General solution for $a \neq 0$ and $a \neq d$:

$$w = -\frac{\beta}{\alpha} + (ax+b)^{\alpha/a}\Phi\bigl(\bigl[c(dx+b) + d(d-a)y\bigr](ax+b)^{-d/a}\bigr).$$

2°. General solution for $a \neq 0$ and $a = d$:

$$w = -\frac{\beta}{\alpha} + (ax+b)^{\alpha/a}\Phi\biggl(\frac{bc - a^2y}{ax+b} + c\ln|ax+b|\biggr).$$

3°. General solution for $a = 0$:

$$w = -\frac{\beta}{\alpha} + e^{\alpha x/b}\Phi\bigl(\bigl[bc + d(cx+dy)\bigr]e^{-dx/b}\bigr).$$

7. $\boldsymbol{(a_1x+a_0)\dfrac{\partial w}{\partial x} + (b_2y+b_1x+b_0)\dfrac{\partial w}{\partial y} = (c_2y+c_1x+c_0)w + k_2y+k_1x+k_0.}$

This is a special case of equation 1.4.7.22 with $f(x) = a_1x + a_0$, $g_1(x) = b_2$, $g_0(x) = b_1x + b_0$, $h(x,y) = c_2y + c_1x + c_0$, and $F(x,y) = k_2y + k_1x + k_0$.

8. $\boldsymbol{ay\dfrac{\partial w}{\partial x} + (b_1x+b_0)\dfrac{\partial w}{\partial y} = (c_1x+c_0)w + s_1x + s_0.}$

This is a special case of equation 1.4.7.11 with $k = 1$, $f_1(x) = a$, $f_2(x) = b_1x + b_0$, $g(x) = c_1x + c_0$, and $h(x) = s_1x + s_0$.

► **Coefficients of equations are quadratic in $\boldsymbol{x}$ and $\boldsymbol{y}$.**

9. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \beta xy + \gamma.}$

General solution: $w = -\dfrac{\gamma}{c} - \dfrac{\beta}{c^3}\bigl[(cx+a)(cy+b) + ab\bigr] + e^{cx/a}\Phi(bx - ay).$

10. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + x(\beta x + \gamma y) + \delta.}$

General solution:

$$w = -\frac{\delta}{c} - \frac{1}{c^3}\big[\beta(cx+a)^2 + \gamma(cx+a)(cy+b) + a(a\beta + b\gamma)\big] + e^{cx/a}\Phi(bx - ay).$$

11. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = w + ax^2 + by^2 + c.}$

General solution: $w = ax^2 + by^2 - c + x\Phi(y/x)$.

12. $\boldsymbol{ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = cw + x(\beta x + \gamma y) + \delta.}$

1°. General solution for $c \neq 2a$ and $c \neq a + b$:

$$w = -\frac{\delta}{c} + \frac{\beta}{2a - c}x^2 + \frac{\gamma}{a + b - c}xy + x^{c/a}\Phi\big(y|x|^{-b/a}\big).$$

2°. General solution for $c = 2a$ and $a \neq b$:

$$w = -\frac{\delta}{c} + \frac{\beta}{a}x^2 \ln|x| - \frac{\gamma}{a - b}xy + x^2\Phi\big(y|x|^{-b/a}\big).$$

3°. General solution for $c = a + b$ and $a \neq b$:

$$w = -\frac{\delta}{c} + \frac{\beta}{a - b}x^2 + \frac{\gamma}{a}xy \ln|x| + x\Phi\big(y|x|^{-b/a}\big).$$

4°. General solution for $c = 2a$ and $a = b$:

$$w = -\frac{\delta}{c} + \frac{1}{a}x(\beta x + \gamma y)\ln|x| + \Phi\Big(\frac{y}{x}\Big).$$

13. $\boldsymbol{ay\frac{\partial w}{\partial x} + (b_2x^2 + b_1x + b_0)\frac{\partial w}{\partial y} = (c_2x^2 + c_1x + c_0)w + s_2x^2 + s_1x + s_0.}$

This is a special case of equation 1.4.7.11 with $k = 1$, $f_1(x) = a$, $f_2(x) = b_2x^2 + b_1x + b_0$, $g(x) = c_2x^2 + c_1x + c_0$, and $h(x) = s_2x^2 + s_1x + s_0$.

14. $\boldsymbol{ay^2\frac{\partial w}{\partial x} + (b_1x^2 + b_0)\frac{\partial w}{\partial y} = (c_1x^2 + c_0)w + s_1x^2 + s_0.}$

This is a special case of equation 1.4.7.11 with $k = 2$, $f_1(x) = a$, $f_2(x) = b_1x^2 + b_0$, $g(x) = c_1x^2 + c_0$, and $h(x) = s_1x^2 + s_0$.

15. $\boldsymbol{(a_1x^2 + a_0)\frac{\partial w}{\partial x} + (y + b_2x^2 + b_1x + b_0)\frac{\partial w}{\partial y}}$

$$\boldsymbol{= (c_2y + c_1x + c_0)w + k_{22}y^2 + k_{12}xy + k_{11}x^2 + k_0.}$$

This is a special case of equation 1.4.7.22 with $f(x) = a_1x^2 + a_0$, $g_1(x) = 1$, $g_0(x) = b_2x^2 + b_1x + b_0$, $h(x, y) = c_2y + c_1x + c_0$, and $F(x, y) = k_{22}y^2 + k_{12}xy + k_{11}x^2 + k_0$.

16. $\boldsymbol{(a_1x^2 + a_0)\frac{\partial w}{\partial x} + (b_2y^2 + b_1xy)\frac{\partial w}{\partial y}}$

$$\boldsymbol{= (c_2y^2 + c_1x^2)w + s_{22}y^2 + s_{12}xy + s_{11}x^2 + s_0.}$$

This is a special case of equation 1.4.7.23 with $k = 2$, $f(x) = a_1x^2 + a_0$, $g_1(x) = b_1x$, $g_0(x) = b_2$, $h(x, y) = c_2y^2 + c_1x^2$, and $F(x, y) = s_{22}y^2 + s_{12}xy + s_{11}x^2 + s_0$.

► **Coefficients of equations contain square roots.**

17. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{xy}+\gamma.}$

1°. General solution for $2\alpha\neq a+b$:

$$w=\frac{2\beta}{a+b-2\alpha}\sqrt{xy}-\frac{\gamma}{\alpha}+x^{\alpha/a}\Phi\bigl(y|x|^{-b/a}\bigr).$$

2°. General solution for $2\alpha=a+b$:

$$w=\frac{\beta}{a}\sqrt{xy}\,\ln|x|-\frac{2\gamma}{a+b}+\sqrt{xy}\,\Phi\bigl(y|x|^{-b/a}\bigr).$$

3°. General solution for $\alpha=a=-b$:

$$w=\frac{1}{a}\bigl(\beta\sqrt{xy}+\gamma\bigr)+x\Phi(xy).$$

18. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=\lambda\sqrt{xy}\,w+\beta xy+\gamma.}$

1°. General solution for $b\neq -a$:

$$w=-\frac{\beta}{\lambda}\sqrt{xy}-\frac{\beta(a+b)}{2\lambda^2}+\exp\left(\frac{2\lambda}{a+b}\sqrt{xy}\right)\Phi\bigl(x^{-b/a}y\bigr).$$

2°. General solution for $b=-a$:

$$w=-\frac{\beta}{\lambda}\sqrt{xy}+\exp\left(\frac{\lambda}{a}\sqrt{xy}\,\ln|x|\right)\Phi(xy).$$

19. $\boldsymbol{ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

This is a special case of equation 1.4.7.11 with $k=1$, $f_1(x)=a$, $f_2(x)=bx$, $g(x)=\alpha$, and $h(x)=\beta\sqrt{x}+\gamma$.

20. $\boldsymbol{ay\dfrac{\partial w}{\partial x}+b\sqrt{x}\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

This is a special case of equation 1.4.7.11 with $k=1$, $f_1(x)=a$, $f_2(x)=b\sqrt{x}$, $g(x)=\alpha$, and $h(x)=\beta\sqrt{x}+\gamma$.

21. $\boldsymbol{a\sqrt{x}\dfrac{\partial w}{\partial x}+b\sqrt{y}\dfrac{\partial w}{\partial y}=\alpha w+\beta x+\gamma y+\delta.}$

General solution:

$$w=-\frac{a\beta\sqrt{x}+b\gamma\sqrt{y}}{\alpha^2}-\frac{\beta x+\gamma y+\delta}{\alpha}-\frac{a^2\beta+b^2\gamma}{2\alpha^3}+\exp\left(\frac{2\alpha}{a}\sqrt{x}\right)\Phi\Bigl(b\sqrt{x}-a\sqrt{y}\Bigr).$$

22. $\boldsymbol{a\sqrt{x}\dfrac{\partial w}{\partial x}+b\sqrt{y}\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

General solution: $w=-\dfrac{\beta\sqrt{x}+\gamma}{\alpha}-\dfrac{a\beta}{2\alpha^2}+\exp\left(\dfrac{2\alpha}{a}\sqrt{x}\right)\Phi\Bigl(b\sqrt{x}-a\sqrt{y}\Bigr).$

23. $\boldsymbol{a\sqrt{y}\dfrac{\partial w}{\partial x}+b\sqrt{x}\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

This is a special case of equation 1.4.7.11 with $k=1/2$, $f_1(x)=a$, $f_2(x)=b\sqrt{x}$, $g(x)=\alpha$, and $h(x)=\beta\sqrt{x}+\gamma$.

► **Coefficients of equations contain arbitrary powers of x and y.**

24. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+kx^ny^m.$

Two forms of the representation of the general solution:

$$w=\exp\Bigl(\frac{c}{a}x\Bigr)\biggl[\Phi(bx-ay)+\frac{k}{a^{m+1}}\int x^n(bx-u)^m\exp\Bigl(-\frac{c}{a}x\Bigr)\,dx\biggr],$$
$$w=\exp\Bigl(\frac{c}{b}y\Bigr)\biggl[\Phi(bx-ay)+\frac{k}{b^{n+1}}\int y^m(ay+u)^n\exp\Bigl(-\frac{c}{b}y\Bigr)\,dy\biggr],$$

where $u=bx-ay$. In the integration, u is considered a parameter.

25. $a\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=bw+cx^ny^m.$

General solution:

$$w=y^b\biggl[\Phi(y^ae^{-x})+c\int y^{m-b-1}(a\ln y-\ln u)^n\,dy\biggr],\quad \text{where}\quad u=y^ae^{-x}.$$

In the integration, u is considered a parameter.

26. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=axw+bx^ny^m.$

General solution: $w=e^{ax}\biggl[\Phi\Bigl(\dfrac{y}{x}\Bigr)+bx^{-m}y^m\displaystyle\int x^{m+n-1}e^{-ax}\,dx\biggr].$

27. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=a\sqrt{x^2+y^2}\,w+bx^ny^m.$

General solution:

$$w=\exp\Bigl(a\sqrt{x^2+y^2}\Bigr)\biggl[\Phi\Bigl(\frac{y}{x}\Bigr)+bx^{-m}y^m\int x^{m+n-1}\exp\Bigl(-ax\sqrt{1+u^2}\Bigr)\,dx\biggr],\quad u=\frac{y}{x}.$$

In the integration, u is considered a parameter.

28. $ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=cx^ny^mw+px^ky^s.$

1°. General solution for $an+bm\neq 0$:

$$w=\exp\Bigl(\frac{c}{an+bm}x^ny^m\Bigr)\Bigl[\Phi\bigl(y^ax^{-b}\bigr)+\psi(x,y)\Bigr],$$
$$\psi(x,y)=px^{-\frac{bs}{a}}y^s\int x^{\frac{bs+ak-a}{a}}\exp\Bigl(-\frac{c}{an+bm}u^{\frac{m}{a}}x^{\frac{an+bm}{a}}\Bigr)\,dx,$$

where $u=y^ax^{-b}$. In the integration, u is considered a parameter.

2°. General solution for $an+bm=0$:

$$w=\exp\Bigl(\frac{c}{a}x^ny^m\ln x\Bigr)\Bigl[\Phi\bigl(y^ax^{-b}\bigr)+\psi(x,y)\Bigr],$$
$$\psi(x,y)=\begin{cases} pk^{-2}x^{\frac{ak-bs}{a}}y^s\exp\Bigl(-\dfrac{c}{a}x^{-\frac{bm}{a}}y^m\Bigr)(k\ln x-1) & \text{if } k\neq 0,\\ \frac{1}{2}px^{-\frac{bs}{a}}y^s\exp\Bigl(-\dfrac{c}{a}x^{-\frac{bm}{a}}y^m\Bigr)(\ln x)^2 & \text{if } k=0.\end{cases}$$

29. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = (cx^n + py^m)w + qx^k y^s.}$

General solution:

$$w = \exp\left(\frac{cx^n}{an} + \frac{py^m}{bm}\right)\left[\Phi\big(y^a x^{-b}\big) + qx^{-\frac{bs}{a}} y^s \int x^{\frac{ak-a+bs}{a}} \exp\left(-\frac{cx^n}{an} - \frac{p}{bm} u^{\frac{m}{a}} x^{\frac{bm}{a}}\right) dx\right],$$

where $u = y^a x^{-b}$. In the integration, u is considered a parameter.

30. $\boldsymbol{x^2\dfrac{\partial w}{\partial x} + axy\dfrac{\partial w}{\partial y} = by^2 w + cx^n y^m.}$

1°. General solution for $a \neq 1/2$:

$$w = \exp\left(\frac{b}{2a-1}\frac{y^2}{x}\right)\left[\Phi(x^{-a}y) + cx^{-am}y^m \int x^{am+n-2} \exp\left(-\frac{b}{2a-1}u^2 x^{2a-1}\right) dx\right],$$

where $u = x^{-a}y$. In the integration, u is considered a parameter.

2°. General solution for $a = 1/2$:

$$w = \exp\left(b\frac{y^2}{x}\ln x\right)\Phi(x^{-1/2}y) + \frac{2cx^n y^m}{(m+2n-2)x - by^2}.$$

31. $\boldsymbol{x^2\dfrac{\partial w}{\partial x} + xy\dfrac{\partial w}{\partial y} = y^2(ax+by)w + cx^n y^m.}$

General solution:

$$w = \exp\left[\frac{(ax+by)y^2}{2x}\right]\left\{\Phi\left(\frac{y}{x}\right) + cx^{-m}y^m \int x^{m+n-2} \exp\left[-\frac{(a+bu)u^2x^2}{2}\right] dx\right\},$$

where $u = y/x$. In the integration, u is considered a parameter.

32. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x} + bx^m y\dfrac{\partial w}{\partial y} = cx^p y^q w + sx^\gamma y^\delta + d.}$

This is a special case of equation 1.4.7.21 with $f(x) = ax^n$, $g(x) = bx^m$, $h(x,y) = cx^p y^q$, and $F(x,y) = sx^\gamma y^\delta + d$.

33. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x} + \big(bx^m y + cx^k\big)\dfrac{\partial w}{\partial y} = sx^p y^q w + d.}$

This is a special case of equation 1.4.7.22 with $f(x) = ax^n$, $g_1(x) = bx^m$, $g_0(x) = cx^k$, $h(x,y) = sx^p y^q$, and $F(x,y) = d$.

34. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x} + bx^m y^k\dfrac{\partial w}{\partial y} = cw + sx^p y^q + d.}$

This is a special case of equation 1.4.7.23 with $f(x) = ax^n$, $g_1(x) \equiv 0$, $g_0(x) = bx^m$, $h(x,y) = c$, and $F(x,y) = sx^p y^q + d$.

35. $\boldsymbol{ay^k\dfrac{\partial w}{\partial x} + bx^n\dfrac{\partial w}{\partial y} = cw + sx^m.}$

This is a special case of equation 1.4.7.11 with $f_1(x) = a$, $f_2(x) = bx^n$, $g(x) = c$, and $h(x) = sx^m$.

1.4.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\big(ce^{\lambda x}+se^{\mu y}\big)w+ke^{\nu x}.$

General solution:

$$w=\exp\Big(\frac{c}{a\lambda}e^{\lambda x}+\frac{s}{b\mu}e^{\mu y}\Big)\bigg[\Phi(bx-ay)+\frac{k}{a}\int\exp\Big(\nu x-\frac{c}{a\lambda}e^{\lambda x}-\frac{s}{b\mu}e^{\frac{\mu bx-\mu u}{a}}\Big)\,dx\bigg],$$

where $u=bx-ay$. In the integration, u is considered a parameter.

2. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=ce^{\alpha x+\beta y}w+ke^{\gamma x}.$

1°. General solution for $a\alpha+b\beta\neq 0$:

$$w=\exp\bigg(\frac{c}{a\alpha+b\beta}e^{\alpha x+\beta y}\bigg)\bigg\{\Phi(bx-ay)+\frac{k}{a}\int\exp\bigg[\gamma x-\frac{c}{a\alpha+b\beta}e^{\frac{(a\alpha+b\beta)x-\beta u}{a}}\bigg]\,dx\bigg\},$$

where $u=bx-ay$. In the integration, u is considered a parameter.

2°. General solution for $a\alpha+b\beta=0$:

$$w=\exp\bigg(\frac{c}{a}xe^{\alpha x+\beta y}\bigg)\Phi(bx-ay)+\frac{ke^{\gamma x}}{a\gamma-ce^{\alpha x+\beta y}}.$$

3. $ae^{\lambda x}\dfrac{\partial w}{\partial x}+be^{\beta x}\dfrac{\partial w}{\partial y}=ce^{\gamma y}w+se^{\mu x+\delta y}.$

This is a special case of equation 1.4.7.22 with $f(x)=ae^{\lambda x}$, $g_1(x)\equiv 0$, $g_0(x)=be^{\beta x}$, $h(x,y)=ce^{\gamma y}$, and $F(x,y)=se^{\mu x+\delta y}$.

4. $ae^{\beta x}\dfrac{\partial w}{\partial x}+\big(be^{\gamma x}+ce^{\lambda y}\big)\dfrac{\partial w}{\partial y}=sw+ke^{\mu x+\delta y}.$

This is a special case of equation 1.4.7.24 with $f(x)=ae^{\beta x}$, $g_1(x)=be^{\gamma x}$, $g_0(x)=c$, $h(x,y)=s$, and $F(x,y)=ke^{\mu x+\delta y}$.

5. $ae^{\beta x}\dfrac{\partial w}{\partial x}+\big(be^{\gamma x}+ce^{\lambda y}\big)\dfrac{\partial w}{\partial y}=se^{\mu x+\delta y}w+k.$

This is a special case of equation 1.4.7.24 with $f(x)=ae^{\beta x}$, $g_1(x)=be^{\gamma x}$, $g_0(x)=c$, $h(x,y)=se^{\mu x+\delta y}$, and $F(x,y)=k$.

6. $ae^{\beta x}\dfrac{\partial w}{\partial x}+be^{\gamma x+\lambda y}\dfrac{\partial w}{\partial y}=ce^{\sigma y}w+ke^{\mu x+\delta y}+d.$

This is a special case of equation 1.4.7.24 with $f(x)=ae^{\beta x}$, $g_1(x)\equiv 0$, $g_0(x)=be^{\gamma x}$, $h(x,y)=ce^{\sigma y}$, and $F(x,y)=ke^{\mu x+\delta y}+d$.

7. $ae^{\lambda y}\dfrac{\partial w}{\partial x}+bx^{\beta x}\dfrac{\partial w}{\partial y}=cw+se^{\gamma x}.$

This is a special case of equation 1.4.7.12 with $f_1(x)=a$, $f_2(x)=bx^{\beta x}$, $g(x)=c$, and $h(x)=se^{\gamma x}$.

8. $\boldsymbol{ae^{\lambda y}\frac{\partial w}{\partial x} + bx^{\beta x}\frac{\partial w}{\partial y} = ce^{\gamma x}w + s.}$

This is a special case of equation 1.4.7.12 with $f_1(x) = a$, $f_2(x) = bx^{\beta x}$, $g(x) = ce^{\gamma x}$, and $h(x) = s$.

► **Coefficients of equations contain exponential and power-law functions.**

9. $\boldsymbol{\frac{\partial w}{\partial x} + (ae^{\lambda x}y + bx^n)\frac{\partial w}{\partial y} = cw + ke^{\gamma x}.}$

This is a special case of equation 1.4.7.7 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_0(x) = bx^n$, $h_1(x) = c$, and $h_0(x) = ke^{\gamma x}$.

10. $\boldsymbol{\frac{\partial w}{\partial x} + (ae^{\lambda x}y + be^{\beta x})\frac{\partial w}{\partial y} = cw + ke^{\gamma x}.}$

This is a special case of equation 1.4.7.7 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_0(x) = be^{\beta x}$, $h_1(x) = c$, and $h_0(x) = ke^{\gamma x}$.

11. $\boldsymbol{\frac{\partial w}{\partial x} + (ae^{\lambda x}y + be^{\beta x})\frac{\partial w}{\partial y} = cw + kx^n.}$

This is a special case of equation 1.4.7.7 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_0(x) = be^{\beta x}$, $h_1(x) = c$, and $h_0(x) = kx^n$.

12. $\boldsymbol{\frac{\partial w}{\partial x} + (ae^{\lambda y} + bx^k)\frac{\partial w}{\partial y} = cw + ke^{\gamma x}.}$

This is a special case of equation 1.4.7.10 with $f(x)=1$, $g_1(x)=bx^k$, $g_0(x)=a$, $h_2(x)=c$, $h_1(x) = 0$, and $h_0(x) = ke^{\gamma x}$.

13. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = axe^{\lambda x+\mu y}w + be^{\nu x}.}$

General solution:

$$w = \exp\left(\frac{ax}{\lambda x + \mu y}e^{\lambda x+\mu y}\right)\left\{\Phi\left(\frac{y}{x}\right) + b\int \exp\left[\nu x - \frac{a}{\lambda + \mu u}e^{(\lambda+\mu u)x}\right]\frac{dx}{x}\right\},$$

where $u = y/x$. In the integration, u is considered a parameter.

14. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \big(aye^{\lambda x} + bxe^{\mu y}\big)w + ce^{\nu x}.}$

General solution:

$$w = \exp\left(\frac{ay}{\lambda x}e^{\lambda x} + \frac{bx}{\mu y}e^{\mu y}\right)\left[\Phi\left(\frac{y}{x}\right) + c\int \exp\left(\nu x - \frac{au}{\lambda}e^{\lambda x} - \frac{b}{\mu u}e^{\mu u x}\right)\frac{dx}{x}\right],$$

where $u = y/x$. In the integration, u is considered a parameter.

15. $\boldsymbol{ay^k\frac{\partial w}{\partial x} + be^{\lambda x}\frac{\partial w}{\partial y} = w + ce^{\beta x}.}$

This is a special case of equation 1.4.7.11 with $f_1(x) = a$, $f_2(x) = be^{\lambda x}$, $g(x) = 1$, and $h(x) = ce^{\beta x}$.

16. $$ae^{\lambda x}\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}=w+ce^{\lambda x}.$$

This is a special case of equation 1.4.7.7 with $f(x)=ae^{\lambda x}$, $g_1(x)=b$, $g_0(x)=0$, $h_1(x)=1$, and $h_0(x)=ce^{\lambda x}$.

17. $$ae^{\lambda y}\frac{\partial w}{\partial x}+bx^k\frac{\partial w}{\partial y}=w+ce^{\beta x}.$$

This is a special case of equation 1.4.7.12 with $f_1(x)=a$, $f_2(x)=bx^k$, $g(x)=1$, and $h(x)=ce^{\beta x}$.

18. $$ae^{\lambda y}\frac{\partial w}{\partial x}+be^{\beta x}\frac{\partial w}{\partial y}=w+cx^k.$$

This is a special case of equation 1.4.7.12 with $f_1(x)=a$, $f_2(x)=be^{\beta x}$, $g(x)=1$, and $h(x)=cx^k$.

1.4.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=cw+\sinh^k(\lambda x)\sinh^n(\beta y).$$

This is a special case of equation 1.4.7.13 with $f(x)=\sinh^k(\lambda x)$ and $g(y)=\sinh^n(\beta y)$.

2. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\sinh^k(\lambda x)w+s\sinh^n(\beta x).$$

This is a special case of equation 1.4.7.1 with $f(x)=\sinh^k(\lambda x)$ and $g(y)=\sinh^n(\beta x)$.

3. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\bigl[c_1\sinh^{n_1}(\lambda_1 x)+c_2\sinh^{n_2}(\lambda_2 y)\bigr]w+s_1\sinh^{k_1}(\beta_1 x)+s_2\sinh^{k_2}(\beta_2 y).$$

This is a special case of equation 1.4.7.16 with $f(x)=c_1\sinh^{n_1}(\lambda_1 x)$, $g(y)=c_2\sinh^{n_2}(\lambda_2 y)$, $p(x)=s_1\sinh^{k_1}(\beta_1 x)$, and $q(y)=s_2\sinh^{k_2}(\beta_2 y)$.

4. $$a\sinh^n(\lambda x)\frac{\partial w}{\partial x}+b\sinh^m(\mu x)\frac{\partial w}{\partial y}=c\sinh^k(\nu x)w+p\sinh^s(\beta y).$$

This is a special case of equation 1.4.7.22 with $f(x)=a\sinh^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\sinh^m(\mu x)$, $h(x,y)=c\sinh^k(\nu x)$, and $F(x,y)=p\sinh^s(\beta y)$.

5. $$a\sinh^n(\lambda x)\frac{\partial w}{\partial x}+b\sinh^m(\mu x)\frac{\partial w}{\partial y}=c\sinh^k(\nu y)w+p\sinh^s(\beta x).$$

This is a special case of equation 1.4.7.22 with $f(x)=a\sinh^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\sinh^m(\mu x)$, $h(x,y)=c\sinh^k(\nu y)$, and $F(x,y)=p\sinh^s(\beta x)$.

► **Coefficients of equations contain hyperbolic cosine.**

6. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=cw+\cosh^k(\lambda x)\cosh^n(\beta y).$$

This is a special case of equation 1.4.7.13 with $f(x)=\cosh^k(\lambda x)$ and $g(y)=\cosh^n(\beta y)$.

7. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cosh^k(\lambda x)w + s\cosh^n(\beta x).}$

This is a special case of equation 1.4.7.1 with $f(x) = \cosh^k(\lambda x)$ and $g(y) = \cosh^n(\beta x)$.

8. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \bigl[c_1\cosh^{n_1}(\lambda_1 x) + c_2\cosh^{n_2}(\lambda_2 y)\bigr]w + s_1\cosh^{k_1}(\beta_1 x) + s_2\cosh^{k_2}(\beta_2 y).}$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1\cosh^{n_1}(\lambda_1 x)$, $g(y) = c_2\cosh^{n_2}(\lambda_2 y)$, $p(x) = s_1\cosh^{k_1}(\beta_1 x)$, and $q(y) = s_2\cosh^{k_2}(\beta_2 y)$.

9. $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\cosh(\lambda x + \mu y)w + b\cosh(\nu x).}$

General solution:

$$w = \exp\left[\frac{ax\sinh(\lambda x + \mu y)}{\lambda x + \mu y}\right]\left[\Phi\left(\frac{y}{x}\right) + b\int\cosh(\nu x)\exp\left(-\frac{a\sinh[(\lambda + \mu u)x]}{\lambda + \mu u}\right)\frac{dx}{x}\right],$$

where $u = y/x$. In the integration, u is considered a parameter.

10. $\boldsymbol{a\cosh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cosh^m(\mu x)\dfrac{\partial w}{\partial y} = c\cosh^k(\nu x)w + p\cosh^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\cosh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^m(\mu x)$, $h(x,y) = c\cosh^k(\nu x)$, and $F(x,y) = p\cosh^s(\beta y)$.

11. $\boldsymbol{a\cosh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cosh^m(\mu x)\dfrac{\partial w}{\partial y} = c\cosh^k(\nu y)w + p\cosh^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\cosh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^m(\mu x)$, $h(x,y) = c\cosh^k(\nu y)$, and $F(x,y) = p\cosh^s(\beta x)$.

► Coefficients of equations contain hyperbolic tangent.

12. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \tanh^k(\lambda x)\tanh^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \tanh^k(\lambda x)$ and $g(y) = \tanh^n(\beta y)$.

13. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\tanh^k(\lambda x)w + s\tanh^n(\beta x).}$

This is a special case of equation 1.4.7.1 with $f(x) = \tanh^k(\lambda x)$ and $g(y) = \tanh^n(\beta x)$.

14. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \bigl[c_1\tanh^{n_1}(\lambda_1 x) + c_2\tanh^{n_2}(\lambda_2 y)\bigr]w + s_1\tanh^{k_1}(\beta_1 x) + s_2\tanh^{k_2}(\beta_2 y).}$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1\tanh^{n_1}(\lambda_1 x)$, $g(y) = c_2\tanh^{n_2}(\lambda_2 y)$, $p(x) = s_1\tanh^{k_1}(\beta_1 x)$, and $q(y) = s_2\tanh^{k_2}(\beta_2 y)$.

15. $\boldsymbol{a\tanh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\tanh^m(\mu x)\dfrac{\partial w}{\partial y} = c\tanh^k(\nu x)w + p\tanh^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\tanh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\tanh^m(\mu x)$, $h(x,y) = c\tanh^k(\nu x)$, and $F(x,y) = p\tanh^s(\beta y)$.

16. $\displaystyle a\tanh^n(\lambda x)\frac{\partial w}{\partial x}+b\tanh^m(\mu x)\frac{\partial w}{\partial y}=c\tanh^k(\nu y)w+p\tanh^s(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x)=a\tanh^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\tanh^m(\mu x)$, $h(x,y)=c\tanh^k(\nu y)$, and $F(x,y)=p\tanh^s(\beta x)$.

► **Coefficients of equations contain hyperbolic cotangent.**

17. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=cw+\coth^k(\lambda x)\coth^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x)=\coth^k(\lambda x)$ and $g(y)=\coth^n(\beta y)$.

18. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\coth^k(\lambda x)w+s\coth^n(\beta x).$

This is a special case of equation 1.4.7.1 with $f(x)=\coth^k(\lambda x)$ and $g(y)=\coth^n(\beta x)$.

19. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\big[c_1\coth^{n_1}(\lambda_1 x)+c_2\coth^{n_2}(\lambda_2 y)\big]w+s_1\coth^{k_1}(\beta_1 x)+s_2\coth^{k_2}(\beta_2 y).$

This is a special case of equation 1.4.7.16 in which $f(x)=c_1\coth^{n_1}(\lambda_1 x)$, $g(y)=c_2\coth^{n_2}(\lambda_2 y)$, $p(x)=s_1\coth^{k_1}(\beta_1 x)$, and $q(y)=s_2\coth^{k_2}(\beta_2 y)$.

20. $\displaystyle a\coth^n(\lambda x)\frac{\partial w}{\partial x}+b\coth^m(\mu x)\frac{\partial w}{\partial y}=c\coth^k(\nu x)w+p\coth^s(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x)=a\coth^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\coth^m(\mu x)$, $h(x,y)=c\coth^k(\nu x)$, and $F(x,y)=p\coth^s(\beta y)$.

21. $\displaystyle a\coth^n(\lambda x)\frac{\partial w}{\partial x}+b\coth^m(\mu x)\frac{\partial w}{\partial y}=c\coth^k(\nu y)w+p\coth^s(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x)=a\coth^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\coth^m(\mu x)$, $h(x,y)=c\coth^k(\nu y)$, and $F(x,y)=p\coth^s(\beta x)$.

► **Coefficients of equations contain different hyperbolic functions.**

22. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=w+c_1\sinh^k(\lambda x)+c_2\cosh^n(\beta y).$

This is a special case of equation 1.4.7.16 with $f(x)=0$, $g(y)=1$, $p(x)=c_1\sinh^k(\lambda x)$, and $q(y)=c_2\cosh^n(\beta y)$.

23. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=cw+\sinh^k(\lambda x)\cosh^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x)=\sinh^k(\lambda x)$ and $g(y)=\cosh^n(\beta y)$.

24. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=cw+k\tanh(\lambda x)+s\coth(\mu y).$

General solution:

$$w=e^{cx/a}\left\{\Phi(bx-ay)-\frac{1}{a}\int_0^x\left[s\coth\left(\frac{b\mu}{a}(x-t)-\mu y\right)-k\tanh(\lambda t)\right]e^{-ct/a}\,dt\right\}.$$

25. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sinh(\lambda x)\dfrac{\partial w}{\partial y} = cw + k\cosh(\mu y).}$

General solution:

$$w = e^{cx/a}\left\{\int_0^x \cosh\left[\mu y + \frac{b\mu}{a\lambda}\Big(\cosh(\lambda t) - \cosh(\lambda x)\Big)\right]e^{-ct/a}\,dt + \Phi\big(a\lambda y - b\cosh(\lambda x)\big)\right\}.$$

26. $\boldsymbol{a\sinh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cosh^m(\mu x)\dfrac{\partial w}{\partial y} = c\cosh^k(\nu x)w + p\sinh^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sinh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^m(\mu x)$, $h(x,y) = c\cosh^k(\nu x)$, and $F(x,y) = p\sinh^s(\beta y)$.

27. $\boldsymbol{a\tanh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\coth^m(\mu x)\dfrac{\partial w}{\partial y} = c\tanh^k(\nu y)w + p\coth^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\tanh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\coth^m(\mu x)$, $h(x,y) = c\tanh^k(\nu y)$, and $F(x,y) = p\coth^s(\beta x)$.

1.4.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \ln^k(\lambda x)\ln^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \ln^k(\lambda x)$ and $g(y) = \ln^n(\beta y)$.

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln^k(\lambda x)w + s\ln^n(\beta x).}$

This is a special case of equation 1.4.7.1 with $f(x) = \ln^k(\lambda x)$ and $g(y) = \ln^n(\beta x)$.

3. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big[c_1\ln^{n_1}(\lambda_1 x) + c_2\ln^{n_2}(\lambda_2 y)\big]w + s_1\ln^{k_1}(\beta_1 x) + s_2\ln^{k_2}(\beta_2 y).}$

This is a special case of equation 1.4.7.16 with $f(x) = c_1\ln^{n_1}(\lambda_1 x)$, $g(y) = c_2\ln^{n_2}(\lambda_2 y)$, $p(x) = s_1\ln^{k_1}(\beta_1 x)$, and $q(y) = s_2\ln^{k_2}(\beta_2 y)$.

4. $\boldsymbol{a\ln(\lambda x)\dfrac{\partial w}{\partial x} + b\ln(\mu y)\dfrac{\partial w}{\partial y} = cw + k.}$

General solution:

$$w = -\frac{k}{c} + \Phi(u)\exp\left[\frac{c}{a}\int\frac{dx}{\ln(\lambda x)}\right], \qquad u = b\int\frac{dx}{\ln(\lambda x)} - a\int\frac{dy}{\ln(\mu y)}.$$

5. $\boldsymbol{a\ln^n(\lambda x)\dfrac{\partial w}{\partial x} + b\ln^m(\mu x)\dfrac{\partial w}{\partial y} = c\ln^k(\nu x)w + p\ln^s(\beta y) + q.}$

This is a special case of equation 1.4.7.22 with $f(x) = a\ln^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^m(\mu x)$, $h(x,y) = c\ln^k(\nu x)$, and $F(x,y) = p\ln^s(\beta y) + q$.

6. $\boldsymbol{a\ln^n(\lambda x)\dfrac{\partial w}{\partial x} + b\ln^m(\mu x)\dfrac{\partial w}{\partial y} = c\ln^k(\nu y)w + p\ln^s(\beta x) + q.}$

This is a special case of equation 1.4.7.22 with $f(x) = a\ln^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^m(\mu x)$, $h(x,y) = c\ln^k(\nu y)$, and $F(x,y) = p\ln^s(\beta x) + q$.

► **Coefficients of equations contain logarithmic and power-law functions.**

7. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=w+c_1x^k+c_2\ln^n(\beta y).$

This is a special case of equation 1.4.7.16 with $f(x)=0$, $g(y)=1$, $p(x)=c_1x^k$, and $q(y)=c_2\ln^n(\beta y)$.

8. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+x^k\ln^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x)=x^k$ and $g(y)=\ln^n(\beta y)$.

9. $ax^k\dfrac{\partial w}{\partial x}+bx^n\dfrac{\partial w}{\partial y}=cw+s\ln^m(\beta x).$

This is a special case of equation 1.4.7.7 with $f(x)=ax^k$, $g_1(x)=0$, $g_0(x)=bx^n$, $h_1(x)=c$, and $h_0(x)=s\ln^m(\beta x)$.

10. $ax^n\dfrac{\partial w}{\partial x}+by^k\dfrac{\partial w}{\partial y}=cw+s\ln^m(\beta x).$

This is a special case of equation 1.4.7.23 with $f(x)=ax^n$, $g_1(x)=0$, $g_0(x)=b$, $h(x,y)=c$, and $F(x,y)=s\ln^m(\beta x)$.

11. $ax^k\dfrac{\partial w}{\partial x}+b\ln^n(\lambda x)\dfrac{\partial w}{\partial y}=cw+sx^m.$

This is a special case of equation 1.4.7.7 with $f(x)=ax^k$, $g_1(x)=0$, $g_0(x)=b\ln^n(\lambda x)$, $h_1(x)=c$, and $h_0(x)=sx^m$.

12. $ay^k\dfrac{\partial w}{\partial x}+bx^n\dfrac{\partial w}{\partial y}=cw+s\ln^m(\beta x).$

This is a special case of equation 1.4.7.11 with $f_1(x)=a$, $f_2(x)=bx^n$, $g(x)=c$, and $h(x)=s\ln^m(\beta x)$.

13. $ay^k\dfrac{\partial w}{\partial x}+b\ln^n(\lambda x)\dfrac{\partial w}{\partial y}=cw+sx^m.$

This is a special case of equation 1.4.7.11 with $f_1(x)=a$, $f_2(x)=b\ln^n(\lambda x)$, $g(x)=c$, and $h(x)=sx^m$.

1.4.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+k\sin(\lambda x+\mu y).$

General solution:

$$w=e^{cx/a}\Phi(bx-ay)-\frac{k}{c^2+(a\lambda+b\mu)^2}\bigl[(a\lambda+b\mu)\cos(\lambda x+\mu y)+c\sin(\lambda x+\mu y)\bigr].$$

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1 \sin^k(\lambda x) + c_2 \sin^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 \sin^k(\lambda x)$, and $q(y) = c_2 \sin^n(\beta y)$.

3. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \sin^k(\lambda x)\sin^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \sin^k(\lambda x)$ and $g(y) = \sin^n(\beta y)$.

4. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cw + k\sin(\lambda x + \mu y).}$

General solution:

$$w = x^{c/a}\left[\frac{k}{a}\int_0^x t^{-(a+c)/a}\sin\bigl(\lambda t + \mu t^{b/a}x^{-b/a}y\bigr)\,dt + \Phi\bigl(x^{-b/a}y\bigr)\right].$$

5. $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\sin(\lambda x + \mu y)w + b\sin(\nu x).}$

General solution:

$$w = \exp\left[-\frac{ax}{\lambda x + \mu y}\cos(\lambda x + \mu y)\right]\left\{\Phi\Bigl(\frac{y}{x}\Bigr) + b\int \sin(\nu x)\exp\left(\frac{a}{\lambda + \mu u}\cos\bigl[(\lambda + \mu u)x\bigr]\right)dx\right\},$$

where $u = y/x$. In the integration, u is considered a parameter.

6. $\boldsymbol{a\sin^n(\lambda x)\dfrac{\partial w}{\partial x} + b\sin^m(\mu x)\dfrac{\partial w}{\partial y} = c\sin^k(\nu x)w + p\sin^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sin^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\sin^m(\mu x)$, $h(x,y) = c\sin^k(\nu x)$, and $F(x,y) = p\sin^s(\beta y)$.

7. $\boldsymbol{a\sin^n(\lambda x)\dfrac{\partial w}{\partial x} + b\sin^m(\mu x)\dfrac{\partial w}{\partial y} = c\sin^k(\nu y)w + p\sin^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sin^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\sin^m(\mu x)$, $h(x,y) = c\sin^k(\nu y)$, and $F(x,y) = p\sin^s(\beta x)$.

► **Coefficients of equations contain cosine.**

8. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + k\cos(\lambda x + \mu y).}$

General solution:

$$w = e^{cx/a}\Phi(bx - ay) + \frac{k}{c^2 + (a\lambda + b\mu)^2}\bigl[(a\lambda + b\mu)\sin(\lambda x + \mu y) - c\cos(\lambda x + \mu y)\bigr].$$

9. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1 \cos^k(\lambda x) + c_2 \cos^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 \cos^k(\lambda x)$, and $q(y) = c_2 \cos^n(\beta y)$.

10. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \cos^k(\lambda x)\cos^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \cos^k(\lambda x)$ and $g(y) = \cos^n(\beta y)$.

11. $\boldsymbol{ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = cw + k\cos(\lambda x + \mu y).}$

General solution:

$$w = x^{c/a}\left[\frac{k}{a}\int_0^x t^{-(a+c)/a}\cos\bigl(\lambda t + \mu t^{b/a}x^{-b/a}y\bigr)\,dt + \Phi\bigl(x^{-b/a}y\bigr)\right].$$

12. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cos(\lambda x + \mu y)w + b\cos(\nu x).}$

General solution:

$$w = \exp\left[\frac{ax}{\lambda x + \mu y}\sin(\lambda x + \mu y)\right]\left\{\Phi\Bigl(\frac{y}{x}\Bigr) + b\int\cos(\nu x)\exp\biggl(-\frac{a}{\lambda + \mu u}\sin\bigl[(\lambda + \mu u)x\bigr]\biggr)\,dx\right\},$$

where $u = y/x$. In the integration, u is considered a parameter.

13. $\boldsymbol{a\cos^n(\lambda x)\frac{\partial w}{\partial x} + b\cos^m(\mu x)\frac{\partial w}{\partial y} = c\cos^k(\nu x)w + p\cos^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\cos^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cos^m(\mu x)$, $h(x,y) = c\cos^k(\nu x)$, and $F(x,y) = p\cos^s(\beta y)$.

► **Coefficients of equations contain tangent.**

14. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + k\tan(\lambda x + \mu y).}$

General solution:

$$w = e^{cx/a}\left\{\Phi(bx - ay) + \frac{k}{a}\int_0^x \tan\left[\Bigl(\lambda + \frac{b\mu}{a}\Bigr)t + \mu\Bigl(y - \frac{bx}{a}\Bigr)\right]e^{-ct/a}\,dt\right\}.$$

15. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = w + c_1\tan^k(\lambda x) + c_2\tan^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1\tan^k(\lambda x)$, and $q(y) = c_2\tan^n(\beta y)$.

16. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \tan^k(\lambda x)\tan^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \tan^k(\lambda x)$ and $g(y) = \tan^n(\beta y)$.

17. $\boldsymbol{a\frac{\partial w}{\partial x} + b\tan(\mu y)\frac{\partial w}{\partial y} = c\tan(\lambda x)w + k\tan(\nu x).}$

General solution:

$$w = \bigl|\cos(\lambda x)\bigr|^{-c/a\lambda}\left[\frac{k}{a}\int\bigl|\cos(\lambda x)\bigr|^{c/a\lambda}\tan(\nu x)\,dx + \Phi\bigl(b\mu x - a\ln|\sin(\mu y)|\bigr)\right].$$

18. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cw + k\tan(\lambda x + \mu y).}$

General solution: $w = x^{c/a}\left[\dfrac{k}{a}\displaystyle\int_0^x t^{-(a+c)/a}\tan\bigl(\lambda t + \mu t^{b/a}x^{-b/a}y\bigr)\,dt + \Phi\bigl(x^{-b/a}y\bigr)\right].$

19. $\boldsymbol{a\tan^n(\lambda x)\dfrac{\partial w}{\partial x} + b\tan^m(\mu x)\dfrac{\partial w}{\partial y} = c\tan^k(\nu x)w + p\tan^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\tan^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\tan^m(\mu x)$, $h(x,y) = c\tan^k(\nu x)$, and $F(x,y) = p\tan^s(\beta y)$.

20. $\boldsymbol{a\tan^n(\lambda x)\dfrac{\partial w}{\partial x} + b\tan^m(\mu x)\dfrac{\partial w}{\partial y} = c\tan^k(\nu y)w + p\tan^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\tan^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\tan^m(\mu x)$, $h(x,y) = c\tan^k(\nu y)$, and $F(x,y) = p\tan^s(\beta x)$.

► Coefficients of equations contain cotangent.

21. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + k\cot(\lambda x + \mu y).}$

General solution:

$$w = e^{cx/a}\left\{\Phi(bx - ay) + \frac{k}{a}\int_0^x \cot\left[\left(\lambda + \frac{b\mu}{a}\right)t + \mu\left(y - \frac{bx}{a}\right)\right]e^{-ct/a}\,dt\right\}.$$

22. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1\cot^k(\lambda x) + c_2\cot^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1\cot^k(\lambda x)$, and $q(y) = c_2\cot^n(\beta y)$.

23. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \cot^k(\lambda x)\cot^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \cot^k(\lambda x)$ and $g(y) = \cot^n(\beta y)$.

24. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cot(\mu y)\dfrac{\partial w}{\partial y} = c\cot(\lambda x)w + k\cot(\nu x).}$

General solution:

$$w = \bigl|\sin(\lambda x)\bigr|^{c/a\lambda}\left[\frac{k}{a}\int\bigl|\sin(\lambda x)\bigr|^{-c/a\lambda}\cot(\nu x)\,dx + \Phi\bigl(b\mu x + a\ln|\cos(\mu y)|\bigr)\right].$$

25. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cw + k\cot(\lambda x + \mu y).}$

General solution: $w = x^{c/a}\left[\dfrac{k}{a}\displaystyle\int_0^x t^{-(a+c)/a}\cot\bigl(\lambda t + \mu t^{b/a}x^{-b/a}y\bigr)\,dt + \Phi\bigl(x^{-b/a}y\bigr)\right].$

26. $\boldsymbol{a\cot^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cot^m(\mu x)\dfrac{\partial w}{\partial y} = c\cot^k(\nu x)w + p\cot^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\cot^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cot^m(\mu x)$, $h(x,y) = c\cot^k(\nu x)$, and $F(x,y) = p\cot^s(\beta y)$.

27. $\boldsymbol{a\cot^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cot^m(\mu x)\dfrac{\partial w}{\partial y} = c\cot^k(\nu y)w + p\cot^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\cot^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cot^m(\mu x)$, $h(x,y) = c\cot^k(\nu y)$, and $F(x,y) = p\cot^s(\beta x)$.

► **Coefficients of equations contain different trigonometric functions.**

28. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1\sin^k(\lambda x) + c_2\cos^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1\sin^k(\lambda x)$, and $q(y) = c_2\cos^n(\beta y)$.

29. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \sin^k(\lambda x)\cos^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \sin^k(\lambda x)$ and $g(y) = \cos^n(\beta y)$.

30. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sin(\mu y)\dfrac{\partial w}{\partial y} = c\sin(\lambda x)w + k\cos(\nu x) + s.}$

General solution:

$$w = \exp\left(-\frac{c}{a\lambda}\cos(\lambda x)\right)\left[\frac{1}{a}\int\bigl(s + k\cos(\nu x)\bigr)\exp\left(\frac{c}{a\lambda}\cos(\lambda x)\right)dx + \Phi\left(b\mu x - a\ln\left|\tan\frac{\mu}{2}y\right|\right)\right].$$

31. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sin(\mu y)\dfrac{\partial w}{\partial y} = c\sin(\lambda x)w + k\tan(\nu x) + s.}$

General solution:

$$w = \exp\left(-\frac{c}{a\lambda}\cos(\lambda x)\right)\left[\frac{1}{a}\int\bigl(s + k\tan(\nu x)\bigr)\exp\left(\frac{c}{a\lambda}\cos(\lambda x)\right)dx + \Phi\left(b\mu x - a\ln\left|\tan\frac{\mu}{2}y\right|\right)\right].$$

32. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\tan(\mu y)\dfrac{\partial w}{\partial y} = c\tan(\lambda x)w + k\cot(\nu x) + s.}$

General solution:

$$w = \bigl|\cos(\lambda x)\bigr|^{-c/a\lambda}\left[\frac{1}{a}\int\bigl(s + k\cot(\nu x)\bigr)\bigl|\cos(\lambda x)\bigr|^{c/a\lambda}dx + \Phi\bigl(b\mu x - a\ln\bigl|\sin(\mu y)\bigr|\bigr)\right].$$

33. $\boldsymbol{a\sin^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cos^m(\mu x)\dfrac{\partial w}{\partial y} = c\cos^k(\nu x)w + p\sin^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sin^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cos^m(\mu x)$, $h(x,y) = c\cos^k(\nu x)$, and $F(x,y) = p\sin^s(\beta y)$.

34. $\boldsymbol{a\tan^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cot^m(\mu x)\dfrac{\partial w}{\partial y} = c\tan^k(\nu y)w + p\cot^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\tan^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cot^m(\mu x)$, $h(x,y) = c\tan^k(\nu y)$, and $F(x,y) = p\cot^s(\beta x)$.

1.4.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = w + c_1 \arcsin^k(\lambda x) + c_2 \arcsin^n(\beta y).$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 \arcsin^k(\lambda x)$, and $q(y) = c_2 \arcsin^n(\beta y)$.

2. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \arcsin^k(\lambda x) \arcsin^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x) = \arcsin^k(\lambda x)$ and $g(y) = \arcsin^n(\beta y)$.

3. $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c_1 \arcsin(\lambda_1 x) + c_2 \arcsin(\lambda_2 y)\big]w + s_1 \arcsin^n(\beta_1 x) + s_2 \arcsin^k(\beta_2 y).$$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1 \arcsin(\lambda_1 x)$, $g(y) = c_2 \arcsin(\lambda_2 y)$, $p(x) = s_1 \arcsin^n(\beta_1 x)$, and $q(y) = s_2 \arcsin^k(\beta_2 y)$.

4. $\displaystyle a\frac{\partial w}{\partial x} + b \arcsin^m(\mu x)\frac{\partial w}{\partial y} = c \arcsin^k(\nu x) w + p \arcsin^n(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b \arcsin^m(\mu x)$, $h(x,y) = c \arcsin^k(\nu x)$, and $F(x,y) = p \arcsin^n(\beta y)$.

5. $\displaystyle a\frac{\partial w}{\partial x} + b \arcsin^m(\mu x)\frac{\partial w}{\partial y} = c \arcsin^k(\nu y) w + p \arcsin^n(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b \arcsin^m(\mu x)$, $h(x,y) = c \arcsin^k(\nu y)$, and $F(x,y) = p \arcsin^n(\beta x)$.

► **Coefficients of equations contain arccosine.**

6. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = w + c_1 \arccos^k(\lambda x) + c_2 \arccos^n(\beta y).$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 \arccos^k(\lambda x)$, and $q(y) = c_2 \arccos^n(\beta y)$.

7. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \arccos^k(\lambda x) \arccos^n(\beta y).$

This is a special case of equation 1.4.7.13 in which $f(x) = \arccos^k(\lambda x)$ and $g(y) = \arccos^n(\beta y)$.

8. $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c_1 \arccos(\lambda_1 x) + c_2 \arccos(\lambda_2 y)\big]w + s_1 \arccos^n(\beta_1 x) + s_2 \arccos^k(\beta_2 y).$$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1 \arccos(\lambda_1 x)$, $g(y) = c_2 \arccos(\lambda_2 y)$, $p(x) = s_1 \arccos^n(\beta_1 x)$, and $q(y) = s_2 \arccos^k(\beta_2 y)$.

9. $a\dfrac{\partial w}{\partial x}+b\arccos^m(\mu x)\dfrac{\partial w}{\partial y}=c\arccos^k(\nu x)w+p\arccos^n(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arccos^m(\mu x)$, $h(x,y)=c\arccos^k(\nu x)$, and $F(x,y)=p\arccos^n(\beta y)$.

10. $a\dfrac{\partial w}{\partial x}+b\arccos^m(\mu x)\dfrac{\partial w}{\partial y}=c\arccos^k(\nu y)w+p\arccos^n(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arccos^m(\mu x)$, $h(x,y)=c\arccos^k(\nu y)$, and $F(x,y)=p\arccos^n(\beta x)$.

► **Coefficients of equations contain arctangent.**

11. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=w+c_1\arctan^k(\lambda x)+c_2\arctan^n(\beta y).$

This is a special case of equation 1.4.7.16 with $f(x)=0$, $g(y)=1$, $p(x)=c_1\arctan^k(\lambda x)$, and $q(y)=c_2\arctan^n(\beta y)$.

12. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+\arctan^k(\lambda x)\arctan^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x)=\arctan^k(\lambda x)$ and $g(y)=\arctan^n(\beta y)$.

13. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\bigl[c_1\arctan(\lambda_1 x)+c_2\arctan(\lambda_2 y)\bigr]w$
$$+s_1\arctan^n(\beta_1 x)+s_2\arctan^k(\beta_2 y).$$

This is a special case of equation 1.4.7.16 in which $f(x)=c_1\arctan(\lambda_1 x)$, $g(y)=c_2\arctan(\lambda_2 y)$, $p(x)=s_1\arctan^n(\beta_1 x)$, and $q(y)=s_2\arctan^k(\beta_2 y)$.

14. $a\dfrac{\partial w}{\partial x}+b\arctan^m(\mu x)\dfrac{\partial w}{\partial y}=c\arctan^k(\nu x)w+s\arctan^n(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arctan^m(\mu x)$, $h(x,y)=c\arctan^k(\nu x)$, and $F(x,y)=s\arctan^n(\beta y)$.

15. $a\dfrac{\partial w}{\partial x}+b\arctan^m(\mu x)\dfrac{\partial w}{\partial y}=c\arctan^k(\nu y)w+s\arctan^n(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arctan^m(\mu x)$, $h(x,y)=c\arctan^k(\nu y)$, and $F(x,y)=s\arctan^n(\beta x)$.

► **Coefficients of equations contain arccotangent.**

16. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=w+c_1\operatorname{arccot}^k(\lambda x)+c_2\operatorname{arccot}^n(\beta y).$

This is a special case of equation 1.4.7.16 with $f(x)=0$, $g(y)=1$, $p(x)=c_1\operatorname{arccot}^k(\lambda x)$, and $q(y)=c_2\operatorname{arccot}^n(\beta y)$.

17. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+\operatorname{arccot}^k(\lambda x)\operatorname{arccot}^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x)=\operatorname{arccot}^k(\lambda x)$ and $g(y)=\operatorname{arccot}^n(\beta y)$.

18. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\bigl[c_1\operatorname{arccot}(\lambda_1 x)+c_2\operatorname{arccot}(\lambda_2 y)\bigr]w + s_1\operatorname{arccot}^n(\beta_1 x)+s_2\operatorname{arccot}^k(\beta_2 y).$

This is a special case of equation 1.4.7.16 in which $f(x)=c_1\operatorname{arccot}(\lambda_1 x)$, $g(y)=c_2\operatorname{arccot}(\lambda_2 y)$, $p(x)=s_1\operatorname{arccot}^n(\beta_1 x)$, and $q(y)=s_2\operatorname{arccot}^k(\beta_2 y)$.

19. $\displaystyle a\operatorname{arccot}^n(\lambda x)\frac{\partial w}{\partial x}+b\operatorname{arccot}^m(\mu x)\frac{\partial w}{\partial y}=c\operatorname{arccot}^k(\nu x)w+p\operatorname{arccot}^s(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x)=a\operatorname{arccot}^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\operatorname{arccot}^m(\mu x)$, $h(x,y)=c\operatorname{arccot}^k(\nu x)$, and $F(x,y)=p\operatorname{arccot}^s(\beta y)$.

20. $\displaystyle a\operatorname{arccot}^n(\lambda x)\frac{\partial w}{\partial x}+b\operatorname{arccot}^m(\mu x)\frac{\partial w}{\partial y}=c\operatorname{arccot}^k(\nu y)w+p\operatorname{arccot}^s(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x)=a\operatorname{arccot}^n(\lambda x)$, $g_1(x)\equiv 0$, $g_0(x)=b\operatorname{arccot}^m(\mu x)$, $h(x,y)=c\operatorname{arccot}^k(\nu y)$, and $F(x,y)=p\operatorname{arccot}^s(\beta x)$.

1.4.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of x.**

1. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=f(x)w+g(x).$

General solution:

$$w=\exp\biggl[\frac{1}{a}\int f(x)\,dx\biggr]\biggl\{\Phi(bx-ay)+\frac{1}{a}\int g(x)\exp\biggl[-\frac{1}{a}\int f(x)\,dx\biggr]\,dx\biggr\}.$$

2. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=(cy+k)w+f(x).$

General solution:

$$w=\exp\biggl\{\frac{x}{2a^2}\bigl[2a(cy+k)-bcx\bigr]\biggr\}\biggl\{\Phi(bx-ay) +\frac{1}{a}\int_{x_0}^{x}\exp\biggl\{-\frac{t}{2a^2}\bigl[2a(cy+k)+bc(t-2x)\bigr]\biggr\}g(t)\,dt\biggr\},$$

where x_0 may be taken as arbitrary.

3. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=f(x)yw+g(x).$

General solution:

$$w=F(x,u)\biggl[\Phi(u)+\frac{1}{a}\int_{x_0}^{x}\frac{g(t)\,dt}{F(t,u)}\biggr],\qquad u=bx-ay,$$

where $\displaystyle F(x,u)=\exp\biggl[\frac{1}{a^2}\int_{x_0}^{x}(b\tau-u)f(\tau)\,d\tau\biggr].$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

4. $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = f(x)w + g(x).}$

General solution:

$$w = \exp\left[\frac{1}{a}\int \frac{f(x)\,dx}{x}\right]\left\{\Phi\big(x^{-b/a}y\big) + \frac{1}{a}\int \frac{g(x)}{x}\exp\left[-\frac{1}{a}\int \frac{f(x)\,dx}{x}\right]dx\right\}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

5. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + (ay+b)\dfrac{\partial w}{\partial y} = cw + g(x).}$

General solution:

$$w = \exp\left[c\int \frac{dx}{f(x)}\right]\left\{\int \frac{g(x)}{f(x)}\exp\left[-c\int \frac{dx}{f(x)}\right]dx + \Phi\left(a\int \frac{dx}{f(x)} - \ln|ay+b|\right)\right\}.$$

6. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + g(x)\dfrac{\partial w}{\partial y} = h(x)w + p(x).}$

General solution:

$$w = \exp\left[\int \frac{h(x)}{f(x)}dx\right]\left\{\int \frac{p(x)}{f(x)}\exp\left[-\int \frac{h(x)}{f(x)}dx\right]dx + \Phi\left(\int \frac{g(x)}{f(x)}dx - y\right)\right\}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

7. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \big[g_1(x)y + g_0(x)\big]\dfrac{\partial w}{\partial y} = h_1(x)w + h_0(x).}$

This is a special case of equation 1.4.7.22 with $h(x,y) = h_1(x)$ and $F(x,y) = h_0(x)$. General solution:

$$w = H(x)\left[\Phi(u) + \int \frac{h_0(x)\,dx}{f(x)H(x)}\right], \qquad u = yG(x) - S(x),$$

where

$$H(x) = \exp\left[\int \frac{h_1(x)}{f(x)}\,dx\right], \quad G(x) = \exp\left[-\int \frac{g_1(x)}{f(x)}\,dx\right], \quad S(x) = \int G(x)\frac{g_0(x)}{f(x)}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

8. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \big[g_1(x)y + g_0(x)\big]\dfrac{\partial w}{\partial y} = h_2(x)w + h_1(x)y + h_0(x).}$

This is a special case of equation 1.4.7.22 with $h(x,y) = h_2(x)$ and $F(x,y) = h_1(x)y + h_0(x)$.

9. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \big[g_1(x)y + g_0(x)y^k\big]\dfrac{\partial w}{\partial y} = h_2(x)w + h_1(x)y^n + h_0(x).}$

This is a special case of equation 1.4.7.23 with $h(x,y) = h_2(x)$ and $F(x,y) = h_1(x)y^n + h_0(x)$.

10. $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \bigl[g_1(x) + g_0(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y} = h_2(x)w + h_1(x)e^{\beta y} + h_0(x).}$

This is a special case of equation 1.4.7.24 with $h(x,y) = h_2(x)$ and $F(x,y) = h_1(x)e^{\beta y} + h_0(x)$.

11. $\boldsymbol{f_1(x)y^k\dfrac{\partial w}{\partial x} + f_2(x)\dfrac{\partial w}{\partial y} = g(x)w + h(x).}$

General solution:

$$w = \Phi(u)G(x,u) + G(x,u)\int_{x_0}^{x} \frac{h(t)}{f_1(t)}\bigl[u + F(t)\bigr]^{-\frac{k}{k+1}}\frac{dt}{G(t,u)},$$

where

$$u = y^{k+1} - F(x), \quad F(x) = (k+1)\int \frac{f_2(x)}{f_1(x)}\,dx,$$

$$G(x,u) = \exp\biggl\{\int_{x_0}^{x} \frac{g(t)}{f_1(t)}\bigl[u + F(t)\bigr]^{-\frac{k}{k+1}}\,dt\biggr\},$$

and x_0 may be taken as arbitrary.

12. $\boldsymbol{f_1(x)e^{\lambda y}\dfrac{\partial w}{\partial x} + f_2(x)\dfrac{\partial w}{\partial y} = g(x)w + h(x).}$

General solution:

$$w = \Phi(u)G(x,u) + G(x,u)\int_{x_0}^{x} \frac{h(t)\,dt}{f_1(t)\bigl[u + F(t)\bigr]G(t,u)},$$

where

$$u = e^{\lambda y} - F(x), \quad F(x) = \lambda\int \frac{f_2(x)}{f_1(x)}\,dx, \quad G(x,u) = \exp\biggl\{\int_{x_0}^{x} \frac{g(t)\,dt}{f_1(t)\bigl[u + F(t)\bigr]}\biggr\},$$

and x_0 may be taken as arbitrary.

► Equations contain arbitrary functions of x and arbitrary functions of y.

13. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + f(x)g(y).}$

General solution:

$$w = e^{cx/a}\biggl[\Phi(bx - ay) + \frac{1}{a}\int_{x_0}^{x} f(t)g\biggl(\frac{b(t-x) + ay}{a}\biggr)e^{-ct/a}\,dt\biggr],$$

where x_0 may be taken as arbitrary.

14. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + xf(y) + yg(x).}$

General solution:

$$w = e^{cx/a}\biggl\{\Phi(bx - ay) + \frac{1}{a^2}\int_{x_0}^{x}\biggl[atf\biggl(\frac{b(t-x) + ay}{a}\biggr) + \bigl[b(t-x) + ay\bigr]g(t)\biggr]e^{-ct/a}\,dt\biggr\}.$$

15. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(x)w+g(x)h(y).}$

General solution:

$$w=F(x)\left[\Phi(u)+\frac{1}{a}\int_{x_0}^{x}\frac{g(t)}{F(t)}g\left(\frac{bt-u}{a}\right)dt\right],\qquad u=bx-ay,$$

where $F(x)=\exp\left[\dfrac{1}{a}\displaystyle\int f(x)\,dx\right]$.

16. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\left[f(x)+g(y)\right]w+p(x)+q(y).}$

General solution:

$$w=\exp\left[\frac{1}{a}\int f(x)\,dx+\frac{1}{b}\int g(y)\,dy\right]\left\{\Phi(bx-ay)\right.$$
$$\left.+\frac{1}{a}\int\left[p(x)+q\left(\frac{bx-u}{a}\right)\right]\exp\left\{-\frac{1}{a}\int\left[f(x)+g\left(\frac{bx-u}{a}\right)\right]dx\right\}dx\right\},$$

where $u=bx-ay$. In the integration, u is considered a parameter.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

17. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=cw+f(x)g(y).}$

General solution:

$$w=x^{c/a}\left[\Phi\left(x^{-b/a}y\right)+\frac{1}{a}\int_{x_0}^{x}t^{-(a+c)/a}f(t)g\left(t^{b/a}x^{-b/a}y\right)dt\right].$$

18. $\boldsymbol{f_1(x)\dfrac{\partial w}{\partial x}+f_2(y)\dfrac{\partial w}{\partial y}=aw+g_1(x)+g_2(y).}$

General solution:

$$w=E_1(x)\Phi(u)+E_1(x)\int\frac{g_1(x)\,dx}{f_1(x)E_1(x)}+E_2(y)\int\frac{g_2(y)\,dy}{f_2(y)E_2(y)},$$

where

$$E_1(x)=\exp\left[a\int\frac{dx}{f_1(x)}\right],\quad E_2(y)=\exp\left[a\int\frac{dy}{f_2(y)}\right],\quad u=\int\frac{dx}{f_1(x)}-\int\frac{dy}{f_2(y)}.$$

► Equations contain arbitrary functions of two variables.

19. $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=xf\left(\dfrac{y}{x}\right)w+g(x,y).}$

General solution:

$$w=\exp\left[xf\left(\frac{y}{x}\right)\right]\left\{\Phi\left(\frac{y}{x}\right)+\int_{x_0}^{x}g(t,ut)\exp\left[-tf(u)\right]\frac{dt}{t}\right\},\quad u=\frac{y}{x},$$

where x_0 may be taken as arbitrary.

20. $ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=f(x,y)w+g(x,y).$

General solution:

$$w=\exp\left[\frac{1}{a}\int\frac{1}{x}f\bigl(x,\,u^{1/a}x^{b/a}\bigr)\,dx\right]\left\{\Phi(u)\right.$$
$$\left.+\frac{1}{a}\int\frac{1}{x}g\bigl(x,\,u^{1/a}x^{b/a}\bigr)\exp\left[-\frac{1}{a}\int\frac{1}{x}f\bigl(x,\,u^{1/a}x^{b/a}\bigr)\,dx\right]dx\right\},$$

where $u=y^ax^{-b}$. In the integration, u is considered a parameter.

21. $f(x)\dfrac{\partial w}{\partial x}+g(x)y\dfrac{\partial w}{\partial y}=h(x,y)w+F(x,y).$

General solution:

$$w=H(x,u)\left[\Phi(u)+\int\frac{F(x,\,uG)}{f(x)H(x,u)}\,dx\right],\qquad u=\frac{y}{G},$$

where $G=G(x)=\exp\left[\int\dfrac{g(x)}{f(x)}dx\right]$ and $H(x,u)=\exp\left[\int\dfrac{h(x,\,uG)}{f(x)}dx\right]$. In the integration, u is considered a parameter.

22. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w+F(x,y).$

General solution:

$$w=H(x,u)\left[\Phi(u)+\int\frac{F(x,\,uG+Q)}{f(x)H(x,u)}\,dx\right],$$

where

$$u=\frac{y-Q}{G},\quad H(x,u)=\exp\left[\int\frac{h(x,\,uG+Q)}{f(x)}dx\right],$$
$$G=G(x)=\exp\left[\int\frac{g_1(x)}{f(x)}dx\right],\quad Q=Q(x)=G(x)\int\frac{g_0(x)\,dx}{f(x)G(x)}.$$

In the integration, u is considered a parameter.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

23. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)y^k\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w+F(x,y).$

For $k=1$, see equation 1.4.7.21. For $k\neq 1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 1.4.7.22:

$$f(x)\frac{\partial w}{\partial x}+(1-k)\bigl[g_1(x)\xi+g_0(x)\bigr]\frac{\partial w}{\partial \xi}=h\Bigl(x,\,\xi^{\frac{1}{1-k}}\Bigr)w+F\Bigl(x,\,\xi^{\frac{1}{1-k}}\Bigr).$$

24. $f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)+g_0(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w+F(x,y).$

The substitution $z=e^{-\lambda y}$ leads to an equation of the form 1.4.7.22:

$$f(x)\frac{\partial w}{\partial x}-\lambda\bigl[g_1(x)z+g_0(x)\bigr]\frac{\partial w}{\partial z}=h\Bigl(x,\,-\frac{1}{\lambda}\ln z\Bigr)w+F\Bigl(x,\,-\frac{1}{\lambda}\ln z\Bigr).$$

Chapter 2

First-Order Equations with Three or More Independent Variables

2.1 Equations of the Form $f(x,y,z)\frac{\partial w}{\partial x} + g(x,y,z)\frac{\partial w}{\partial y} + h(x,y,z)\frac{\partial w}{\partial z} = 0$

◆ *For brevity, only an integral basis*

$$u_1 = u_1(x,y), \qquad u_2 = u_2(x,y)$$

of an equation will often be presented in Section 2.1. The general solution of the equation is given by

$$w = \Phi(u_1, u_2),$$

where $\Phi = \Phi(u_1, u_2)$ *is an arbitrary function of two variables.*

2.1.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x, y, and z.**

1. $a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + c\frac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = bx - ay$, $u_2 = cx - az$.

⊙ *Literature*: E. Kamke (1965).

2. $\frac{\partial w}{\partial x} + ax\frac{\partial w}{\partial y} + by\frac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = ax^2 - 2y$, $u_2 = 3z + bx(ax^2 - 3y)$.

3. $a\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} + cz\frac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = |y|^a e^{-bx}$, $u_2 = |z|^a e^{-cx}$.

⊙ *Literature*: E. Kamke (1965).

4. $\dfrac{\partial w}{\partial x} + az\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = by^2 - az^2, \quad u_2 = \begin{cases} \left(by + \sqrt{ab}\,z\right)\exp\left(-\sqrt{ab}\,x\right) & \text{if } ab > 0, \\ by\cos\left(\sqrt{|ab|}\,x\right) + \sqrt{|ab|}\,z\sin\left(\sqrt{|ab|}\,x\right) & \text{if } ab < 0. \end{cases}$$

⊙ *Literature*: E. Kamke (1965).

5. $x\dfrac{\partial w}{\partial x} + ay\dfrac{\partial w}{\partial y} + bz\dfrac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = |x|^a/y,\ u_2 = |x|^b/z.$

⊙ *Literature*: E. Kamke (1965).

6. $x\dfrac{\partial w}{\partial x} + az\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = by^2 - az^2, \quad u_2 = \begin{cases} |x|^{\sqrt{ab}}\left(by - \sqrt{ab}\,z\right) & \text{if } ab > 0, \\ |x|^{\sqrt{-ab}}\exp\left(-\arctan\dfrac{\sqrt{-ab}\,z}{by}\right) & \text{if } ab < 0. \end{cases}$$

⊙ *Literature*: E. Kamke (1965).

7. $x\dfrac{\partial w}{\partial x} + (ax + by)\dfrac{\partial w}{\partial y} + (\alpha x + \beta y + \gamma z)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.1.21 with $s_1 = 1$, $s_2 = b$, and $s_3 = \gamma$.

8. $abx\dfrac{\partial w}{\partial x} + (ay + bz)\left(b\dfrac{\partial w}{\partial y} - a\dfrac{\partial w}{\partial z}\right) = 0.$

Integral basis: $u_1 = ay + bz,\ u_2 = x\exp\left(-\dfrac{ay}{ay + bz}\right).$

9. $abx\dfrac{\partial w}{\partial x} + b(ay + bz)\dfrac{\partial w}{\partial y} + a(ay - bz)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.1.21 with $s_1 = 1$ and $s_{2,3} = \pm\sqrt{2}$.

Integral basis:

$$u_1 = \left[ay + (\sqrt{2} - 1)bz\right]|x|^{-\sqrt{2}}, \quad u_2 = \left[ay - (\sqrt{2} + 1)bz\right]|x|^{\sqrt{2}}.$$

Particular solution: $w = a^2y^2 - 2abyz - b^2z^2.$

10. $b^2cy\dfrac{\partial w}{\partial x} + a^2cx\dfrac{\partial w}{\partial y} - ab(ax + by)\dfrac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = ax + by + cz,\ u_2 = a^2x^2 - b^2y^2.$

11. $\boldsymbol{cz\dfrac{\partial w}{\partial x}+(ax+by)\dfrac{\partial w}{\partial y}+(ax+by+cz)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.1.21.

1°. Integral basis for $a\neq b$:

$$u_1=z-x-y,$$
$$u_2=\frac{2acz+(b-c-\rho)(ax+by)}{2acz+(b-c+\rho)(ax+by)}\bigl[acz^2+(b-c)(ax+by)z-(ax+by)^2\bigr]^{\frac{\rho}{b+c}},$$

where $\rho^2=4ac+(b-c)^2\neq 0$.

2°. Integral basis for $a=b$:

$$u_1=z-x-y,\quad u_2=[a(x+y)+cz]\exp\left[\left(\frac{1}{a}+\frac{1}{c}\right)\frac{cy-ax}{z-x-y}\right].$$

⊙ *Literature*: E. Kamke (1965).

12. $\boldsymbol{b^2cz\dfrac{\partial w}{\partial x}-a^2cx\dfrac{\partial w}{\partial y}+ab^2y\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.1.21 with $s_1=-1$, $s_2=\frac{1}{2}\bigl(1+i\sqrt{3}\,\bigr)$, and $s_3=\frac{1}{2}\bigl(1-i\sqrt{3}\,\bigr)$.

13. $\boldsymbol{(x+a)\dfrac{\partial w}{\partial x}+(y+b)\dfrac{\partial w}{\partial y}+(z+c)\dfrac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{x+a}{z+c}$, $u_2=\dfrac{y+b}{z+c}$.

14. $\boldsymbol{2bc(ax-by)\dfrac{\partial w}{\partial x}-ac(ax-by-cz)\dfrac{\partial w}{\partial y}-ab(ax-by-3cz)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.1.21 with $s_1=0$, $s_2=2$, and $s_3=4$.

15. $\boldsymbol{bc(y-z)\dfrac{\partial w}{\partial x}+ac(z-x)\dfrac{\partial w}{\partial y}+ab(x-y)\dfrac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=ax+by+cz$, $u_2=ax^2+by^2+cz^2$.

16. $\boldsymbol{bc(by-2cz)\dfrac{\partial w}{\partial x}+ac(3cz-ax)\dfrac{\partial w}{\partial y}+ab(2ax-3by)\dfrac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=3ax+2by+cz$, $u_2=a^2x^2+b^2y^2+c^2z^2$.

17. $\boldsymbol{2bc(by-cz)\dfrac{\partial w}{\partial x}-ac(4ax-3by-cz)\dfrac{\partial w}{\partial y}+3ab(4ax-by-3cz)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.1.21. Integral basis:

$$u_1=3ax-3by-cz,\quad u_2=\frac{(8ax-5by-3cz)^2}{2ax-by-cz}.$$

18. $\displaystyle (ax+y-z)\frac{\partial w}{\partial x}-(x+ay-z)\frac{\partial w}{\partial y}+(a-1)(y-x)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.1.21 with $s_1=0$ and $s_{2,3}=\pm\sqrt{(a+3)(a-1)}$. One of the integrals: $u_1=x+y+z$.

⊙ *Literature*: E. Kamke (1965).

19. $\displaystyle 2bc(3ax-2by+cz)\frac{\partial w}{\partial x}-2ac(2ax-5by+3cz)\frac{\partial w}{\partial y}+ab(2ax-6by+11cz)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.1.21 with $s_1=3abc$, $s_2=6abc$, and $s_3=18abc$.

Integral basis:

$$u_1=\frac{(2ax+2by+cz)^2}{2ax-by-2cz},\qquad u_2=\frac{(2ax-by-2cz)^3}{ax-2by+2cz}.$$

20. $\displaystyle (Ax+cy+bz)\frac{\partial w}{\partial x}+(cx+By+az)\frac{\partial w}{\partial y}+(bx+ay+Cz)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.1.21, where s is the root of the cubic equation

$$(A-s)(B-s)(C-s)-[a^2(A-s)+b^2(B-s)+c^2(C-s)]+2abc=0.$$

⊙ *Literature*: E. Kamke (1965).

21. $\displaystyle (a_1x+b_1y+c_1z+d_1)\frac{\partial w}{\partial x}+(a_2x+b_2y+c_2z+d_2)\frac{\partial w}{\partial y}+(a_3x+b_3y+c_3z+d_3)\frac{\partial w}{\partial z}=0.$

An integral basis u_1, u_2 of this equation is determined by the solution of the cubic equation

$$\begin{vmatrix} a_1-s & b_1 & c_1\\ a_2 & b_2-s & c_2\\ a_3 & b_3 & c_3-s\end{vmatrix}=0, \tag{1}$$

the solution of the linear algebraic system

$$\begin{cases}\alpha a_1+\beta a_2+\gamma a_3=\alpha s,\\ \alpha b_1+\beta b_2+\gamma b_3=\beta s,\\ \alpha c_1+\beta c_2+\gamma c_3=\gamma s,\end{cases} \tag{2}$$

and the value of the coefficient

$$D=\alpha d_1+\beta d_2+\gamma d_3. \tag{3}$$

First, one finds the roots s of the cubic equation (1) and determines the solutions α, β, γ of the linear system (2). Then one calculates the coefficient D of (3). Three cases are possible.

1. If all $s=D=0$, then one of the integrals has the form

$$u_1=\alpha x+\beta y+\gamma z.$$

Another integral, u_2, can be obtained by using the transformation specified in Section 13.1.3 (see paragraph *The method of reducing the number of independent variables*).

2. If equation (1) has two different nonzero roots, s_1 and s_2, then two sets of numbers α_k, β_k, γ_k $(k=1,2)$, not all equal to zero within each set, can be found from system (2). Then one of the integrals is

$$u_1=\frac{(\alpha_1x+\beta_1y+\gamma_1z+D_1/s_1)^{s_2}}{(\alpha_2x+\beta_2y+\gamma_2z+D_2/s_2)^{s_1}}.$$

3. If equation (1) has three different roots, s_1, s_2, and s_3, other than zero, then the integrals

$$u_1=\frac{(\alpha_1x+\beta_1y+\gamma_1z+D_1/s_1)^{s_2}}{(\alpha_2x+\beta_2y+\gamma_2z+D_2/s_2)^{s_1}},\qquad u_2=\frac{(\alpha_1x+\beta_1y+\gamma_1z+D_1/s_1)^{s_3}}{(\alpha_3x+\beta_3y+\gamma_3z+D_3/s_3)^{s_1}}$$

form an integral basis. If there are multiple or zero roots, then we can take the u_k which is not constant to be one of the integrals and then use the transformation specified in Section 13.1.3 (for $u=u_k$).

⊙ *Literature*: E. Kamke (1965).

▶ Coefficients of equations are quadratic in x, y, and z.

22. $$\frac{\partial w}{\partial x}+\left(a_1xy+b_1x^2+c_1x\right)\frac{\partial w}{\partial y}+\left(a_2xy+b_2x^2+c_2x\right)\frac{\partial w}{\partial z}=0.$$

This is a special case of equation 2.1.7.4 with $f_1(x)=a_1x$, $f_2(x)=b_1x^2+c_1x$, $g_1(x)=a_2x$, and $g_2(x)=b_2x^2+c_2x$.

23. $$\frac{\partial w}{\partial x}+\left(a_1xy+b_1x^2+c_1x\right)\frac{\partial w}{\partial y}+\left(a_2xz+b_2x^2+c_2x\right)\frac{\partial w}{\partial z}=0.$$

This is a special case of equation 2.1.7.5 with $f_1(x)=a_1x$, $f_2(x)=b_1x^2+c_1x$, $g_1(x)=a_2x$, and $g_2(x)=b_2x^2+c_2x$.

24. $$\frac{\partial w}{\partial x}+\left(a_1xy+b_1x^2+c_1x\right)\frac{\partial w}{\partial y}+\left(a_2yz+b_2y^2+c_2y\right)\frac{\partial w}{\partial z}=0.$$

This is a special case of equation 2.1.7.12 with $f_1(x)=a_1x$, $f_2(x)=b_1x^2+c_1x$, $g_1(y)=a_2y$, and $g_2(y)=b_2y^2+c_2y$.

25. $$\frac{\partial w}{\partial x}+\left(a_1xy+b_1y^2\right)\frac{\partial w}{\partial y}+\left(a_2xz+b_2z^2\right)\frac{\partial w}{\partial z}=0.$$

This is a special case of equation 2.1.7.8 with $k=m=2$, $f_1(x)=a_1x$, $f_2(x)=b_1$, $g_1(x)=a_2x$, and $g_2(x)=b_2$.

26. $\boldsymbol{a\frac{\partial w}{\partial x} + xz\frac{\partial w}{\partial y} - xy\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = y^2 + z^2, \quad u_2 = y\sin\left(\frac{x^2}{2a}\right) + z\cos\left(\frac{x^2}{2a}\right).$$

The function $u_2 = x^2 + 2a\arctan(z/y)$ can also be taken to be the second integral.

⊙ *Literature*: E. Kamke (1965).

27. $\boldsymbol{cx\frac{\partial w}{\partial x} + cy\frac{\partial w}{\partial y} + (ax^2 + by^2)\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = ax^2 + by^2 - 2cz,\ u_2 = y/x.$

28. $\boldsymbol{cz\frac{\partial w}{\partial x} - a(2ax - b)y\frac{\partial w}{\partial y} + a(2ax - b)z\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = yz,\ u_2 = ax(ax - b) - cz.$

29. $\boldsymbol{acx^2\frac{\partial w}{\partial x} - acxy\frac{\partial w}{\partial y} - b^2y^2\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = xy,\ u_2 = 3acxyz - b^2y^3.$

30. $\boldsymbol{ax^2\frac{\partial w}{\partial x} + by^2\frac{\partial w}{\partial y} + cz^2\frac{\partial w}{\partial z} = 0.}$

Any two of the functions

$$u_1 = \frac{1}{by} - \frac{1}{ax}, \quad u_2 = \frac{1}{cz} - \frac{1}{by}, \quad u_3 = \frac{1}{ax} - \frac{1}{cz}$$

form an integral basis.

⊙ *Literature*: E. Kamke (1965).

31. $\boldsymbol{abx^2\frac{\partial w}{\partial x} + cz^2\frac{\partial w}{\partial y} + 2abxz\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = \dfrac{x^2}{z},\ u_2 = by - \dfrac{cz^2}{3ax}.$

32. $\boldsymbol{bcxy\frac{\partial w}{\partial x} + a^2cx^2\frac{\partial w}{\partial y} - by(2ax + cz)\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = a^2x^2 - by^2,\ u_2 = x(ax + cz).$

33. $\boldsymbol{bcxy\frac{\partial w}{\partial x} + c^2yz\frac{\partial w}{\partial y} + b^2y^2\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = b^2y^2 - c^2z^2,\ u_2 = \dfrac{by + cz}{x}.$

34. $\boldsymbol{xy\frac{\partial w}{\partial x} + y(y - a)\frac{\partial w}{\partial y} + z(y - a)\frac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = \dfrac{y}{z},\ u_2 = \dfrac{y - a}{x}.$

⊙ *Literature*: E. Kamke (1965).

35. $\boldsymbol{by^2\frac{\partial w}{\partial x}-axy\frac{\partial w}{\partial y}+cxz\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=ax^2+by^2,\ u_2=y^cz^a.$

36. $\boldsymbol{cxz\frac{\partial w}{\partial x}+2axy\frac{\partial w}{\partial y}-(2ax+cz)z\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=x(ax+cz),\ u_2=xyz.$

37. $\boldsymbol{cxz\frac{\partial w}{\partial x}+cyz\frac{\partial w}{\partial y}+abxy\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{y}{x},\ u_2=cz^2-abxy.$

38. $\boldsymbol{cxz\frac{\partial w}{\partial x}-cyz\frac{\partial w}{\partial y}+(by^2-ax)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=xy,\ u_2=2ax+by^2+cz^2.$

39. $\boldsymbol{cxz\frac{\partial w}{\partial x}-cyz\frac{\partial w}{\partial y}+(ax^2+by^2)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=xy,\ u_2=ax^2-by^2-cz^2.$

40. $\boldsymbol{xz\frac{\partial w}{\partial x}+yz\frac{\partial w}{\partial y}+(ax^2+ay^2+bz^2)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{y}{x},\ u_2=\dfrac{a(x^2+y^2)+(b-1)z^2}{(x^2+y^2)^b}.$

⊙ *Literature*: E. Kamke (1965).

41. $\boldsymbol{2cxz\frac{\partial w}{\partial x}+2cyz\frac{\partial w}{\partial y}+(cz^2-ax^2-by^2)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{ax^2+by^2+cz^2}{x},\ u_2=\dfrac{ax^2+by^2+cz^2}{y}.$

42. $\boldsymbol{bcyz\frac{\partial w}{\partial x}+acxz\frac{\partial w}{\partial y}+abxy\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=ax^2-by^2,\ u_2=cz^2-by^2.$

43. $\boldsymbol{bc(x^2-a^2)\frac{\partial w}{\partial x}+c(bxy+acz)\frac{\partial w}{\partial y}+b(cxz+aby)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{by+cz}{x-a},\ u_2=\dfrac{by-cz}{x+a}.$

⊙ *Literature*: E. Kamke (1965).

44. $\boldsymbol{bx(by+c)\frac{\partial w}{\partial x}+(b^2y^2-acx)\frac{\partial w}{\partial y}+b^2yz\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{z}{ax+by},\ u_2=\dfrac{ax-c}{ax+by}+\ln\left|\dfrac{ax+by}{x}\right|.$

45. $x(by-cz)\dfrac{\partial w}{\partial x}+y(cz-ax)\dfrac{\partial w}{\partial y}+z(ax-by)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=ax+by+cz,\ u_2=xyz.$

46. $a(y+\beta)(z+\gamma)\dfrac{\partial w}{\partial x}-b(x+\alpha)(z+\gamma)\dfrac{\partial w}{\partial y}-c(x+\alpha)(y+\beta)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b(x+\alpha)^2+a(y+\beta)^2,\ u_2=c(x+\alpha)^2+a(z+\gamma)^2.$

47. $bc(acxz+b^2y^2)\dfrac{\partial w}{\partial x}+ac(bcyz-2a^2x^2)\dfrac{\partial w}{\partial y}-ab(2abxy+c^2z^2)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=2acxz-b^2y^2,\ u_2=a^2x^2+bcyz.$

48. $a(y^2+z^2)\dfrac{\partial w}{\partial x}+x(bz-ay)\dfrac{\partial w}{\partial y}-x(by+az)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=x^2+y^2+z^2,\ u_2=2a\arctan(y/z)+b\ln\bigl(y^2+z^2\bigr).$

⊙ *Literature*: E. Kamke (1965).

49. $b(by+cz)^2\dfrac{\partial w}{\partial x}-ax(by+2cz)\dfrac{\partial w}{\partial y}+abxz\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=z(by+cz),\ u_2=ax^2+b^2y^2-c^2z^2.$

50. $(f_0x-f_1)\dfrac{\partial w}{\partial x}+(f_0y-f_2)\dfrac{\partial w}{\partial y}+(f_0z-f_3)\dfrac{\partial w}{\partial z}=0,$

$$f_n=a_n+b_nx+c_ny+d_nz.$$

Hesse's equation ($n=0,\ 1,\ 2,\ 3$).

By introducing the homogeneous coordinates $x=\xi/\tau$, $y=\eta/\tau$, and $z=\zeta/\tau$, we arrive at an equation with linear coefficients for $w=w(\tau,\xi,\eta,\zeta)$ having four independent variables:

$$g_0\frac{\partial v}{\partial\tau}+g_1\frac{\partial v}{\partial\xi}+g_2\frac{\partial v}{\partial\eta}+g_3\frac{\partial v}{\partial\zeta}=0,$$

where $g_n=a_n\tau+b_n\xi+c_n\eta+d_n\zeta$. For the solution of this equation, see 2.4.9.12 with $n=4$.

⊙ *Literature*: E. Kamke (1965).

► **Coefficients of equations contain other powers of x, y, and z.**

51. $2b^2xz\dfrac{\partial w}{\partial x}+by(b^2z^2+1)\dfrac{\partial w}{\partial y}+axy(bz+1)^2\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=bz-axy,\quad u_2=\frac{bz-axy}{(bz-axy+1)^2}\ln\left|\frac{axy}{bz+1}\right|-\frac{1}{(bz+1)(bz-axy+1)}-\frac{1}{2}\ln|x|.$$

52. $bcxy^2\dfrac{\partial w}{\partial x}+2bcy^3\dfrac{\partial w}{\partial y}+2(cyz-ax^2)^2\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=\dfrac{x^2}{y},\ u_2=y\exp\biggl(\dfrac{by}{cyz-ax^2}\biggr).$

53. $\boldsymbol{bc^2y^2z\frac{\partial w}{\partial x}+ac^2xz^2\frac{\partial w}{\partial y}-abxy^2\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=ax^2+c^2z^2,\ u_2=by^3+c^2z^3.$

54. $\boldsymbol{x(by^2-cz^2)\frac{\partial w}{\partial x}+y(cz^2-ax^2)\frac{\partial w}{\partial y}+z(ax^2-by^2)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=ax^2+by^2+cz^2,\ u_2=xyz.$

55. $\boldsymbol{by(3ax^2+by^2+cz^2)\frac{\partial w}{\partial x}-2ax(ax^2+cz^2)\frac{\partial w}{\partial y}+2abxyz\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{ax^2+by^2+cz^2}{z},\ u_2=\dfrac{2ax^2+by^2}{z^2}.$

56. $\boldsymbol{b[a(a^2x^2+b^2y^2-1)x+by]\frac{\partial w}{\partial x}}$

$$\boldsymbol{+a[b(a^2x^2+b^2y^2-1)y-ax]\frac{\partial w}{\partial y}+2abz\frac{\partial w}{\partial z}=0.}$$

The transformation $ax=\xi,\ by=\eta$ leads to an equation of the same form with $a=b=1$. Integral basis:

$$u_1=\frac{a^2x^2+b^2y^2-1}{a^2x^2+b^2y^2}\exp\left(2\arctan\frac{by}{ax}\right),\quad u_2=z\exp\left(2\arctan\frac{by}{ax}\right).$$

57. $\boldsymbol{x(b^3y^3-2a^3x^3)\frac{\partial w}{\partial x}+y(2b^3y^3-a^3x^3)\frac{\partial w}{\partial y}+9z(a^3x^3-b^3y^3)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=x^3y^3z,\ u_2=\dfrac{ax}{b^2y^2}+\dfrac{by}{a^2x^2}.$

58. $\boldsymbol{ax^2(abxy-c^2z^2)\frac{\partial w}{\partial x}+axy(abxy-c^2z^2)\frac{\partial w}{\partial y}+byz(bcyz+2a^2x^2)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{y}{x},\ u_2=\dfrac{a^2bx^2y+b^2cy^2z+ac^2xz^2}{yz}.$

59. $\boldsymbol{x(cz^4-by^4)\frac{\partial w}{\partial x}+y(ax^4-2cz^4)\frac{\partial w}{\partial y}+z(2by^4-ax^4)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=ax^4+by^4+cz^4,\ u_2=x^2yz.$

60. $\boldsymbol{x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+a\sqrt{x^2+y^2}\,\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=y/x,\ u_2=a\sqrt{x^2+y^2}-z.$

⊙ *Literature*: E. Kamke (1965).

61. $\boldsymbol{x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+\big(z-a\sqrt{x^2+y^2+z^2}\,\big)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=y/x,\ u_2=x^{a-1}\big(z+\sqrt{x^2+y^2+z^2}\,\big).$

⊙ *Literature*: E. Kamke (1965).

62. $\displaystyle z\sqrt{y^2+z^2}\,\frac{\partial w}{\partial x}+az\sqrt{x^2+z^2}\,\frac{\partial w}{\partial y}-\big(x\sqrt{y^2+z^2}+ay\sqrt{x^2+z^2}\,\big)\frac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=r^2,\quad u_2=a\arcsin(x/r)-\arcsin(y/r),\quad \text{where}\quad r^2=x^2+y^2+z^2.$$

⊙ *Literature*: E. Kamke (1965).

63. $\displaystyle (y-z)\sqrt{f(x)}\,\frac{\partial w}{\partial x}+(z-x)\sqrt{f(y)}\,\frac{\partial w}{\partial y}+(x-y)\sqrt{f(z)}\,\frac{\partial w}{\partial z}=0,$
where $f(t)=a_6t^6+a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0.$

Integral basis:

$$u_1=\left[\frac{(y-z)\sqrt{f(x)}+(z-x)\sqrt{f(y)}+(x-y)\sqrt{f(z)}}{(y-z)(z-x)(x-y)}\right]^2-a_6(x+y+z)^2-a_5(x+y+z),$$
$$u_2=\left[\frac{y^2z^2(y-z)\sqrt{f(x)}+z^2x^2(z-x)\sqrt{f(y)}+x^2y^2(x-y)\sqrt{f(z)}}{xyz(y-z)(z-x)(x-y)}\right]^2$$
$$-a_0\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-a_1\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

⊙ *Literature*: E. Kamke (1965).

► **Coefficients of equations contain arbitrary powers of x, y, and z.**

64. $\displaystyle \frac{\partial w}{\partial x}+ax^ny^m\frac{\partial w}{\partial y}+bx^\nu y^\mu z^\lambda\frac{\partial w}{\partial z}=0.$

1°. Integral basis for $m\neq 1$ and $n\neq -1$:

$$u_1=\frac{1}{k}y^k-\frac{a}{n+1}x^{n+1},\qquad k=1-m;$$
$$u_2=\begin{cases}\dfrac{z^{1-\lambda}}{1-\lambda}-b\displaystyle\int x^\nu\left(ku_1+\frac{ak}{n+1}x^{n+1}\right)^{\mu/k}dx & \text{if } \lambda\neq 1,\\ \ln|z|-b\displaystyle\int x^\nu\left(ku_1+\frac{ak}{n+1}x^{n+1}\right)^{\mu/k}dx & \text{if } \lambda=1.\end{cases}$$

In the integration, u_1 is considered a parameter.

2°. Integral basis for $m=1$ and $n\neq -1$:

$$u_1=\ln|y|-\frac{a}{s}x^s,\qquad s=n+1;$$
$$u_2=\begin{cases}\dfrac{z^{1-\lambda}}{1-\lambda}-by^\mu\exp\left(-\dfrac{a\mu}{s}x^s\right)\displaystyle\int x^\nu\exp\left(\frac{a\mu}{s}x^s\right)dx & \text{if } \lambda\neq 1,\\ \ln|z|-by^\mu\exp\left(-\dfrac{a\mu}{s}x^s\right)\displaystyle\int x^\nu\exp\left(\frac{a\mu}{s}x^s\right)dx & \text{if } \lambda=1.\end{cases}$$

3°. Integral basis for $m\neq 1$ and $n=-1$:

$$u_1=\frac{1}{k}y^k-a\ln|x|,\qquad k=1-m;$$

$$u_2=\begin{cases}\dfrac{z^{1-\lambda}}{1-\lambda}-b\displaystyle\int x^\nu\bigl(ku_1+ak\ln|x|\bigr)^{\mu/k}\,dx & \text{if } \lambda\neq 1,\\ \ln|z|-b\displaystyle\int x^\nu\bigl(ku_1+ak\ln|x|\bigr)^{\mu/k}\,dx & \text{if } \lambda=1.\end{cases}$$

In the integration, u_1 is considered a parameter.

4°. Integral basis for $m=1$ and $n=-1$:

$$u_1=x^{-a}y,\qquad u_2=\begin{cases}\dfrac{z^{1-\lambda}}{1-\lambda}-\dfrac{bx^{\nu+1}y^\mu}{a\mu+\nu+1} & \text{if } \lambda\neq 1,\ a\mu+\nu\neq -1;\\ \dfrac{z^{1-\lambda}}{1-\lambda}-by^\mu x^{-a\mu}\ln|x| & \text{if } \lambda\neq 1,\ a\mu+\nu=-1;\\ \ln|z|-\dfrac{bx^{\nu+1}y^\mu}{a\mu+\nu+1} & \text{if } \lambda=1,\ a\mu+\nu\neq -1;\\ \ln|z|-by^\mu x^{-a\mu}\ln|x| & \text{if } \lambda=1,\ a\mu+\nu=-1.\end{cases}$$

65. $\boldsymbol{\dfrac{\partial w}{\partial x}+\bigl(a_1x^{n_1}y+b_1x^{m_1}\bigr)\dfrac{\partial w}{\partial y}+\bigl(a_2x^{n_2}y+b_2x^{m_2}\bigr)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.7.4 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1x^{m_1}$, $g_1(x)=a_2x^{n_2}$, and $g_2(x)=b_2x^{m_2}$.

66. $\boldsymbol{\dfrac{\partial w}{\partial x}+\bigl(a_1x^{n_1}y+b_1x^{m_1}\bigr)\dfrac{\partial w}{\partial y}+\bigl(a_2x^{n_2}z+b_2x^{m_2}\bigr)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.7.5 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1x^{m_1}$, $g_1(x)=a_2x^{n_2}$, and $g_2(x)=b_2x^{m_2}$.

67. $\boldsymbol{\dfrac{\partial w}{\partial x}+\bigl(a_1x^{n_1}y+b_1x^{m_1}\bigr)\dfrac{\partial w}{\partial y}+\bigl(a_2y^{n_2}z+b_2y^{m_2}\bigr)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.7.12 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1x^{m_1}$, $g_1(y)=a_2y^{n_2}$, and $g_2(y)=b_2y^{m_2}$.

68. $\boldsymbol{\dfrac{\partial w}{\partial x}+\bigl(a_1x^{n_1}y+b_1x^{m_1}y^{k_1}\bigr)\dfrac{\partial w}{\partial y}+\bigl(a_2x^{n_2}z+b_2x^{m_2}z^{k_2}\bigr)\dfrac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.7.8.

69. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x}+by^m\dfrac{\partial w}{\partial y}+cz^l\dfrac{\partial w}{\partial z}=0.}$

1°. Integral basis for $n\neq 1$ and $m\neq 1$:

$$u_1=\frac{b}{1-n}x^{1-n}-\frac{a}{1-m}y^{1-m},\qquad u_2=\begin{cases}\dfrac{c}{1-n}x^{1-n}-\dfrac{a}{1-l}z^{1-l} & \text{if } l\neq 1,\\ \dfrac{c}{1-n}x^{1-n}-a\ln|z| & \text{if } l=1.\end{cases}$$

2°. Integral basis for $m=l=1$:

$$u_1=x^{-c/a}z,\qquad u_2=\begin{cases}\dfrac{c}{1-n}x^{1-n}-a\ln|y| & \text{if } n\neq 1,\\ x^{-b/a}y & \text{if } n=1.\end{cases}$$

70. $\boldsymbol{ay^m \dfrac{\partial w}{\partial x} + bx^n \dfrac{\partial w}{\partial y} + cz^l \dfrac{\partial w}{\partial z} = 0.}$

1°. Integral basis for $n \neq -1$ and $m \neq -1$:

$$u_1 = \frac{b}{n+1}x^{n+1} - \frac{a}{m+1}y^{m+1},$$

$$u_2 = \begin{cases} c\displaystyle\int \left[\frac{b(m+1)}{a(n+1)}x^{n+1} - \frac{m+1}{a}u_1\right]^{-\frac{m}{m+1}} dx - \frac{a}{1-l}z^{1-l} & \text{if } l \neq 1; \\ c\displaystyle\int \left[\frac{b(m+1)}{a(n+1)}x^{n+1} - \frac{m+1}{a}u_1\right]^{-\frac{m}{m+1}} dx - a\ln|z| & \text{if } l = 1. \end{cases}$$

In the integration, u_1 is considered a parameter.

2°. Integral basis for $n \neq -1$ and $m = -1$:

$$u_1 = \frac{b}{n+1}x^{n+1} - a\ln|y|,$$

$$u_2 = \begin{cases} cy\exp\left[-\dfrac{b}{a(n+1)}x^{n+1}\right]\displaystyle\int \exp\left[\frac{b}{a(n+1)}x^{n+1}\right] dx - \frac{a}{1-l}z^{1-l} & \text{if } l \neq 1; \\ cy\exp\left[-\dfrac{b}{a(n+1)}x^{n+1}\right]\displaystyle\int \exp\left[\frac{b}{a(n+1)}x^{n+1}\right] dx - a\ln|z| & \text{if } l = 1. \end{cases}$$

3°. Integral basis for $n = m = -1$:

$$u_1 = x^{-b/a}y, \qquad u_2 = \begin{cases} (a+b)cxy - \dfrac{a^2}{1-l}z^{1-l} & \text{if } l \neq 1,\ a+b \neq 0; \\ cxy\ln|x| - \dfrac{a}{1-l}z^{1-l} & \text{if } l \neq 1,\ a+b = 0; \\ (a+b)cxy - a^2\ln|z| & \text{if } l = 1,\ a+b \neq 0; \\ cxy\ln|x| - a\ln|z| & \text{if } l = 1,\ a+b = 0. \end{cases}$$

71. $\boldsymbol{x(y^n - z^n)\dfrac{\partial w}{\partial x} + y(z^n - x^n)\dfrac{\partial w}{\partial y} + z(x^n - y^n)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = xyz,\ u_2 = x^n + y^n + z^n$.

⊙ *Literature*: E. Kamke (1965).

2.1.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + be^{\alpha x}\dfrac{\partial w}{\partial y} + ce^{\beta y}\dfrac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = \dfrac{b}{\alpha}e^{\alpha x} - ay,\ u_2 = \alpha z - c\displaystyle\int \frac{e^{\beta y}\,dy}{ay + u_1}$. In the integration, u_1 is considered a parameter.

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + be^{\alpha x}\dfrac{\partial w}{\partial y} + ce^{\gamma z}\dfrac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = \dfrac{b}{\alpha}e^{\alpha x} - ay,\ u_2 = cx + \dfrac{a}{\gamma}e^{-\gamma z}$.

3. $a\dfrac{\partial w}{\partial x}+be^{\beta y}\dfrac{\partial w}{\partial y}+ce^{\gamma z}\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx+\dfrac{a}{\beta}e^{-\beta y}$, $u_2=-\dfrac{c}{\beta}e^{-\beta y}+\dfrac{b}{\gamma}e^{-\gamma z}$.

4. $\dfrac{\partial w}{\partial x}+\bigl(A_1e^{\alpha_1 x}+B_1e^{\nu_1 x+\lambda y}\bigr)\dfrac{\partial w}{\partial y}+\bigl(A_2e^{\alpha_2 x}+B_2e^{\nu_2 x+\beta z}\bigr)\dfrac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.10 with $f_1(x)=A_1e^{\alpha_1 x}$, $f_2(x)=B_1e^{\nu_1 x}$, $g_1(x)=A_2e^{\alpha_2 x}$, and $g_2(x)=B_2e^{\nu_2 x}$.

5. $ae^{\alpha x}\dfrac{\partial w}{\partial x}+be^{\beta y}\dfrac{\partial w}{\partial y}+ce^{\gamma z}\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=-\dfrac{1}{a\alpha}e^{-\alpha x}+\dfrac{1}{b\beta}e^{-\beta y}$, $u_2=-\dfrac{1}{b\beta}e^{-\beta y}+\dfrac{1}{c\gamma}e^{-\gamma z}$.

6. $ae^{\beta y}\dfrac{\partial w}{\partial x}+be^{\alpha x}\dfrac{\partial w}{\partial y}+ce^{\gamma z}\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=-\dfrac{1}{a\alpha}e^{\alpha x}+\dfrac{1}{b\beta}e^{\beta y}$, $u_2=\dfrac{\beta y-\alpha x}{b\beta e^{\alpha x}-a\alpha e^{\beta y}}+\dfrac{1}{c\gamma}e^{-\gamma z}$.

7. $(a_1+a_2e^{\alpha x})\dfrac{\partial w}{\partial x}+(b_1+b_2e^{\beta y})\dfrac{\partial w}{\partial y}+(c_1+c_2e^{\gamma z})\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{1}{a_1\alpha}\bigl[\alpha x-\ln(a_1+a_2e^{\alpha x})\bigr]-\frac{1}{b_1\beta}\bigl[\beta y-\ln(b_1+b_2e^{\beta y})\bigr],$$
$$u_2=\frac{1}{a_1\alpha}\bigl[\alpha x-\ln(a_1+a_2e^{\alpha x})\bigr]-\frac{1}{c_1\gamma}\bigl[\gamma z-\ln(c_1+c_2e^{\gamma z})\bigr].$$

8. $e^{\beta y}(a_1+a_2e^{\alpha x})\dfrac{\partial w}{\partial x}+e^{\alpha x}(b_1+b_2e^{\beta y})\dfrac{\partial w}{\partial y}+ce^{\beta y+\gamma z}\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{1}{a_2\alpha}\ln(a_1+a_2e^{\alpha x})-\frac{1}{b_1\beta}\ln(b_1+b_2e^{\beta y}),\quad u_2=\frac{1}{a_1\alpha}\bigl[\alpha x-\ln(a_1+a_2e^{\alpha x})\bigr]+\frac{1}{c\gamma}e^{-\gamma z}.$$

► **Coefficients of equations contain exponential and power-law functions.**

9. $aye^{\alpha x}\dfrac{\partial w}{\partial x}+be^{\beta y}\dfrac{\partial w}{\partial y}+ce^{\gamma z}\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=-\dfrac{1}{a\alpha}e^{-\alpha x}+\dfrac{\beta y+1}{b\beta^2}e^{-\beta y}$, $u_2=-\dfrac{1}{b\beta}e^{-\beta y}+\dfrac{1}{c\gamma}e^{-\gamma z}$.

10. $axe^{\alpha x}\dfrac{\partial w}{\partial x}+bye^{\beta y}\dfrac{\partial w}{\partial y}+cze^{\gamma z}\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=-\frac{1}{a\alpha}\left(x^{-1}e^{-\alpha x}+\int x^{-2}e^{-\alpha x}\,dx\right)+\frac{1}{b\beta}\left(y^{-1}e^{-\beta y}+\int y^{-2}e^{-\beta y}\,dy\right),$$
$$u_2=-\frac{1}{a\alpha}\left(x^{-1}e^{-\alpha x}+\int x^{-2}e^{-\alpha x}\,dx\right)+\frac{1}{c\gamma}\left(z^{-1}e^{-\gamma z}+\int z^{-2}e^{-\gamma z}\,dz\right).$$

11. $\dfrac{\partial w}{\partial x}+\left[y^2+ae^{\alpha x}(\alpha-ae^{\alpha x})\right]\dfrac{\partial w}{\partial y}+\left[z^2+bz+ce^{\beta x}(\beta-b-ce^{\beta x})\right]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{E_1}{y-ae^{\alpha x}}+\int E_1\,dx,\qquad E_1=\exp\left(\frac{2a}{\alpha}e^{\alpha x}\right),$$
$$u_2=\frac{E_2}{z-ce^{\beta x}}+\int E_2\,dx,\qquad E_2=\exp\left(\frac{2c}{\beta}e^{\beta x}+bx\right).$$

12. $\dfrac{\partial w}{\partial x}+\left[y^2+by+ae^{\alpha x}(\alpha-b-ae^{\alpha x})\right]\dfrac{\partial w}{\partial y}+\left[z^2+ce^{\beta x}(z-k)-k^2\right]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{E_1}{y-ae^{\alpha x}}+\int E_1\,dx,\qquad E_1=\exp\left(\frac{2a}{\alpha}e^{\alpha x}+bx\right),$$
$$u_2=\frac{E_2}{z-k}+\int E_2\,dx,\qquad E_2=\exp\left(\frac{c}{\beta}e^{\beta x}+2kx\right).$$

13. $\dfrac{\partial w}{\partial x}+\left(ay^2e^{\alpha x}+be^{-\alpha x}\right)\dfrac{\partial w}{\partial y}+\left[de^{\beta x}z^2+ce^{\gamma x}(\gamma-cde^{(\beta+\gamma)x})\right]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\int\frac{dv}{av^2+\alpha v+b}-x,\qquad v=e^{\alpha x}y;$$
$$u_2=\frac{E}{z-ce^{\gamma x}}+d\int e^{\beta x}E\,dx,\qquad E=\exp\left[\frac{2cd}{\beta+\gamma}e^{(\beta+\gamma)x}\right].$$

14. $\dfrac{\partial w}{\partial x}+\left[be^{\alpha x}y^2+ae^{\beta x}(\beta-abe^{(\alpha+\beta)x})\right]\dfrac{\partial w}{\partial y}+\left(cz^2e^{\gamma x}+dz+ke^{-\gamma x}\right)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\int\frac{dv}{cv^2+(d+\gamma)v+k}-x,\qquad v=e^{\gamma x}z,$$
$$u_2=\frac{E}{y-ae^{\beta x}}+b\int e^{\alpha x}E\,dx,\qquad E=\exp\left[\frac{2ab}{\alpha+\beta}e^{(\alpha+\beta)x}\right].$$

15. $\dfrac{\partial w}{\partial x}+\left(ae^{\alpha x}y^2+by+ce^{-\alpha x}\right)\dfrac{\partial w}{\partial y}+\left[e^{\beta x}z^2+de^{\gamma x}(z+\beta e^{-\beta x})\right]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\int\frac{dv}{av^2+(b+\alpha)v+c}-x,\qquad v=e^{\alpha x}y,$$
$$u_2=\frac{E}{z+\beta e^{-\beta x}}+\int Ee^{\beta x}\,dx,\qquad E=\exp\left(\frac{d}{\gamma}e^{\gamma x}-2\beta x\right).$$

16. $\frac{\partial w}{\partial x} + \left[e^{\alpha x}y^2 + aye^{\beta x} + a\alpha e^{(\beta-\alpha)x}\right]\frac{\partial w}{\partial y} + \left[\gamma e^{\gamma x}z^2 + be^{\delta x}(z + e^{-\gamma x})\right]\frac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \frac{E_1}{y + \alpha e^{-\alpha x}} + \int e^{\alpha x}E_1\,dx, \qquad E_1 = \exp\left(\frac{a}{\beta}e^{\beta x} - 2\alpha x\right),$$
$$u_2 = \frac{E_2}{z + e^{-\gamma x}} + \gamma\int e^{\gamma x}E_2\,dx, \qquad E_2 = \exp\left(\frac{b}{\delta}e^{\delta x} - 2\gamma x\right).$$

17. $\frac{\partial w}{\partial x} + \left[\alpha e^{\alpha x}y^2 + ae^{\beta x}(y + e^{-\alpha x})\right]\frac{\partial w}{\partial y} + \left[e^{\gamma x}(z - be^{\delta x})^2 + b\delta e^{\delta x}\right]\frac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \frac{E}{y + e^{-\alpha x}} + \alpha\int e^{\alpha x}E\,dx, \quad u_2 = \frac{1}{z - be^{-\delta x}} + \frac{1}{\gamma}e^{\gamma x},$$

where $E = \exp\left(\frac{a}{\beta}e^{\beta x} - 2\alpha x\right)$.

18. $x\frac{\partial w}{\partial x} + (a_1e^{\alpha x}y^2 + \beta y + a_1b_1^2x^{2\beta}e^{\alpha x})\frac{\partial w}{\partial y} + \left[a_2x^{2n}z^2e^{\lambda x} + (b_2x^ne^{\lambda x} - n)z + ce^{\lambda x}\right]\frac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \arctan\frac{y}{b_1x^\beta} - a_1b_1\int x^{\beta-1}e^{\alpha x}\,dx,$$
$$u_2 = \int\frac{dv}{a_2v^2 + b_2v + c} - \int x^{n-1}e^{\lambda x}\,dx, \qquad v = x^nz.$$

19. $\frac{\partial w}{\partial x} + \left(a_1e^{\lambda_1 x}y + b_1e^{\beta_1 x}y^k\right)\frac{\partial w}{\partial y} + \left(a_2e^{\lambda_2 x}z + b_2e^{\beta_2 x}z^m\right)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.8 with $f_1(x) = a_1e^{\lambda_1 x}$, $f_2(x) = b_1e^{\beta_1 x}$, $g_1(x) = a_2e^{\lambda_2 x}$, and $g_2(x) = b_2e^{\beta_2 x}$.

20. $\frac{\partial w}{\partial x} + \left(a_1e^{\beta_1 x}y + b_1e^{\gamma_1 x}y^k\right)\frac{\partial w}{\partial y} + \left(a_2e^{\beta_2 x} + b_2e^{\gamma_2 x + \lambda z}\right)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.9 with $f_1(x) = a_1e^{\beta_1 x}$, $f_2(x) = b_1e^{\gamma_1 x}$, $g_1(x) = a_2e^{\beta_2 x}$, and $g_2(x) = b_2e^{\gamma_2 x}$.

21. $\frac{\partial w}{\partial x} + \left(a_1x^n + b_1x^me^{\lambda y}\right)\frac{\partial w}{\partial y} + \left(a_2x^k + b_2x^se^{\beta z}\right)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.10 with $f_1(x) = a_1x^n$, $f_2(x) = b_1x^m$, $g_1(x) = a_2x^k$, and $g_2(x) = b_2x^s$.

22. $(ax^n e^{\lambda y} + bxy^m)\dfrac{\partial w}{\partial x} + e^{\mu y}\dfrac{\partial w}{\partial y} + (cy^l z^k + dy^p z)\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = x^{1-n}E_1 + a(n-1)\int e^{(\lambda-\mu)y}E_1\,dy, \qquad E_1 = \exp\biggl[b(n-1)\int y^m e^{-\mu y}\,dy\biggr],$$
$$u_2 = z^{1-k}E_2 + c(k-1)\int y^l e^{-\mu y}E_2\,dy, \qquad E_2 = \exp\biggl[d(k-1)\int y^p e^{-\mu y}\,dy\biggr].$$

23. $\dfrac{\partial w}{\partial x} + \bigl(y^2 + 2a\alpha e^{\alpha x^2} - a^2 e^{2\alpha x^2}\bigr)\dfrac{\partial w}{\partial y} + \bigl(ce^{-2\beta x^2}z^2 + 2\beta xz + b^2 c\bigr)\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \frac{E}{y - ae^{\alpha x^2}} + \int E\,dx, \qquad E = \exp\biggl(2a\int e^{\alpha x^2}\,dx\biggr),$$
$$u_2 = \arctan\biggl(\frac{1}{b}e^{-\beta x^2}z\biggr) - bc\int e^{-\beta x^2}\,dx.$$

24. $\dfrac{\partial w}{\partial x} + \bigl(ae^{-2\alpha x^2}y^2 + 2\alpha xy + ab^2\bigr)\dfrac{\partial w}{\partial y} + \bigl(cx^\beta z^2 + 2\gamma xz + cd^2 x^\beta e^{2\gamma x^2}\bigr)\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \arctan\biggl(\frac{1}{b}e^{-\alpha x^2}y\biggr) - ab\int e^{-\alpha x^2}\,dx, \quad u_2 = \arctan\biggl(\frac{1}{d}e^{-\gamma x^2}z\biggr) - cd\int x^\beta e^{\gamma x^2}\,dx.$$

2.1.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\sinh(\lambda x)\dfrac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = bx - ay$, $u_2 = a\lambda z - c\cosh(\lambda x)$.

2. $a\dfrac{\partial w}{\partial x} + b\sinh(\beta y)\dfrac{\partial w}{\partial y} + c\sinh(\lambda x)\dfrac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = b\beta x - a\ln\Bigl|\tanh\dfrac{\beta y}{2}\Bigr|$, $u_2 = a\lambda z - c\cosh(\lambda x)$.

3. $a\dfrac{\partial w}{\partial x} + b\sinh(\beta y)\dfrac{\partial w}{\partial y} + c\sinh(\gamma z)\dfrac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = b\beta x - a\ln\Bigl|\tanh\dfrac{\beta y}{2}\Bigr|$, $u_2 = c\gamma x - a\ln\Bigl|\tanh\dfrac{\gamma z}{2}\Bigr|$.

4. $a\sinh(\lambda x)\dfrac{\partial w}{\partial x} + b\sinh(\beta y)\dfrac{\partial w}{\partial y} + c\sinh(\gamma z)\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \frac{1}{a\lambda}\ln\Bigl|\tanh\frac{\lambda x}{2}\Bigr| - \frac{1}{b\beta}\ln\Bigl|\tanh\frac{\beta y}{2}\Bigr|, \qquad u_2 = \frac{1}{a\lambda}\ln\Bigl|\tanh\frac{\lambda x}{2}\Bigr| - \frac{1}{c\gamma}\ln\Bigl|\tanh\frac{\gamma z}{2}\Bigr|.$$

5. $a\sinh(\beta y)\dfrac{\partial w}{\partial x}+b\sinh(\lambda x)\dfrac{\partial w}{\partial y}+c\sinh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=b\beta\cosh(\lambda x)-a\lambda\cosh(\beta y),$$
$$u_2=c|\lambda|\gamma\int\frac{dx}{\sqrt{[b\beta\cosh(\lambda x)-u_1]^2-a^2\lambda^2}}+2\,\mathrm{sign}(a)\,\mathrm{arctanh}(e^{\gamma z}).$$

In the integration, u_1 is considered a parameter.

6. $a\sinh(\beta y)\dfrac{\partial w}{\partial x}+b\sinh(\lambda x)\dfrac{\partial w}{\partial y}+c\sinh(\lambda x)\sinh(\beta y)\sinh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=\dfrac{1}{a\lambda}\cosh(\lambda x)-\dfrac{1}{b\beta}\cosh(\beta y),\ u_2=\dfrac{1}{a\lambda}\cosh(\lambda x)-\dfrac{1}{c\gamma}\ln\left|\tanh\dfrac{\gamma z}{2}\right|.$

► **Coefficients of equations contain hyperbolic cosine.**

7. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\cosh(\beta x)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx-ay,\ u_2=a\beta z-c\sinh(\beta x).$

8. $a\dfrac{\partial w}{\partial x}+b\cosh(\beta x)\dfrac{\partial w}{\partial y}+c\cosh(\lambda x)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=a\beta y-b\sinh(\beta x),\ u_2=a\lambda z-c\sinh(\lambda x).$

9. $a\dfrac{\partial w}{\partial x}+b\cosh(\beta y)\dfrac{\partial w}{\partial y}+c\cosh(\lambda x)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-2a\arctan\left|\tanh\dfrac{\beta y}{2}\right|,\ u_2=a\lambda z-c\sinh(\lambda x).$

10. $a\dfrac{\partial w}{\partial x}+b\cosh(\beta y)\dfrac{\partial w}{\partial y}+c\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-2a\arctan\left|\tanh\dfrac{\beta y}{2}\right|,\ u_2=c\gamma x-2a\arctan\left|\tanh\dfrac{\gamma z}{2}\right|.$

11. $a\cosh(\lambda x)\dfrac{\partial w}{\partial x}+b\cosh(\beta y)\dfrac{\partial w}{\partial y}+c\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{2}{a\lambda}\arctan\bigl(e^{\lambda x}\bigr)-\frac{2}{b\beta}\arctan\bigl(e^{\beta y}\bigr),\qquad u_2=\frac{2}{a\lambda}\arctan\bigl(e^{\lambda x}\bigr)-\frac{2}{c\gamma}\arctan\bigl(e^{\gamma z}\bigr).$$

12. $a\cosh(\beta y)\dfrac{\partial w}{\partial x}+b\cosh(\lambda x)\dfrac{\partial w}{\partial y}+c\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=b\beta\sinh(\lambda x)-a\lambda\sinh(\beta y),$$
$$u_2=c|\lambda|\gamma\int\frac{dx}{\sqrt{[b\beta\sinh(\lambda x)-u_1]^2+a^2\lambda^2}}-2\,\mathrm{sign}(a)\arctan(e^{\gamma z}).$$

In the integration, u_1 is considered a parameter.

► **Coefficients of equations contain hyperbolic tangent.**

13. $a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\tanh(\gamma z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx-ay,\ u_2=c\gamma x-a\ln\bigl|\sinh(\gamma z)\bigr|.$

14. $a\frac{\partial w}{\partial x}+b\tanh(\beta x)\frac{\partial w}{\partial y}+c\tanh(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=a\beta y-b\ln\bigl|\sinh(\beta x)\bigr|,\ u_2=a\lambda z-c\ln\bigl|\sinh(\lambda x)\bigr|.$

15. $a\frac{\partial w}{\partial x}+b\tanh(\beta y)\frac{\partial w}{\partial y}+c\tanh(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl|\sinh(\beta y)\bigr|,\ u_2=a\lambda z-c\ln\bigl|\cosh(\lambda x)\bigr|.$

16. $a\frac{\partial w}{\partial x}+b\tanh(\beta y)\frac{\partial w}{\partial y}+c\tanh(\gamma z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl|\sinh(\beta y)\bigr|,\ u_2=c\gamma x-a\ln\bigl|\sinh(\gamma z)\bigr|.$

17. $a\tanh(\lambda x)\frac{\partial w}{\partial x}+b\tanh(\beta y)\frac{\partial w}{\partial y}+c\tanh(\gamma z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=\dfrac{\sinh^{a\lambda}(\beta y)}{\sinh^{b\beta}(\lambda x)},\ u_2=\dfrac{\sinh^{a\lambda}(\gamma z)}{\sinh^{c\gamma}(\lambda x)}.$

18. $a\tanh(\beta y)\frac{\partial w}{\partial x}+b\tanh(\lambda x)\frac{\partial w}{\partial y}+c\tanh(\gamma z)\frac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{\cosh^{a\lambda}(\beta y)}{\cosh^{b\beta}(\lambda x)},\qquad u_2=c\gamma\int\coth\Bigl\{\operatorname{arccosh}^{1/a\lambda}\bigl[u_1\cosh^{b\beta}(\lambda x)\bigr]\Bigr\}\,dx-a\ln\bigl|\sinh(\gamma z)\bigr|.$$

In the integration, u_1 is considered a parameter.

► **Coefficients of equations contain hyperbolic cotangent.**

19. $a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\coth(\lambda z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx-ay,\ u_2=c\lambda x-a\ln\bigl[\cosh(\lambda z)\bigr].$

20. $a\frac{\partial w}{\partial x}+b\coth(\beta x)\frac{\partial w}{\partial y}+c\coth(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=a\beta y-b\ln\bigl[\cosh(\beta x)\bigr],\ u_2=a\lambda z-c\ln\bigl[\cosh(\lambda x)\bigr].$

21. $a\frac{\partial w}{\partial x}+b\coth(\beta y)\frac{\partial w}{\partial y}+c\coth(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl[\cosh(\beta y)\bigr],\ u_2=a\lambda z-c\ln\bigl|\sinh(\lambda x)\bigr|.$

22. $a\dfrac{\partial w}{\partial x}+b\coth(\beta y)\dfrac{\partial w}{\partial y}+c\coth(\lambda z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl[\cosh(\beta y)\bigr]$, $u_2=c\lambda x-a\ln\bigl[\cosh(\lambda z)\bigr]$.

23. $a\coth(\lambda x)\dfrac{\partial w}{\partial x}+b\coth(\beta y)\dfrac{\partial w}{\partial y}+c\coth(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=\dfrac{\cosh^{a\lambda}(\beta y)}{\cosh^{b\beta}(\lambda x)}$, $u_2=\dfrac{\cosh^{a\lambda}(\gamma z)}{\cosh^{c\gamma}(\lambda x)}$.

24. $a\coth(\beta y)\dfrac{\partial w}{\partial x}+b\coth(\lambda x)\dfrac{\partial w}{\partial y}+c\coth(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{\sinh^{a\lambda}(\beta y)}{\sinh^{b\beta}(\lambda x)},\qquad u_2=c\gamma\int\tanh\Bigl\{\operatorname{arcsinh}^{1/a\lambda}\bigl[u_1\sinh^{b\beta}(\lambda x)\bigr]\Bigr\}\,dx-a\ln\bigl[\cosh(\gamma z)\bigr].$$

In the integration, u_1 is considered a parameter.

► **Coefficients of equations contain different hyperbolic functions.**

25. $a\sinh(\lambda x)\dfrac{\partial w}{\partial x}+b\sinh(\beta y)\dfrac{\partial w}{\partial y}+c\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right|-\frac{1}{b\beta}\ln\left|\tanh\frac{\beta y}{2}\right|,\qquad u_2=\frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right|-\frac{2}{c\gamma}\arctan\bigl(e^{\gamma z}\bigr).$$

26. $a\sinh(\lambda x)\dfrac{\partial w}{\partial x}+b\cosh(\beta y)\dfrac{\partial w}{\partial y}+c\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right|-\frac{2}{b\beta}\arctan\bigl(e^{\beta y}\bigr),\qquad u_2=\frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right|-\frac{2}{c\gamma}\arctan\bigl(e^{\gamma z}\bigr).$$

27. $a\sinh(\beta y)\dfrac{\partial w}{\partial x}+b\sinh(\lambda x)\dfrac{\partial w}{\partial y}+c\sinh(\lambda x)\sinh(\beta y)\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=\dfrac{1}{a\lambda}\cosh(\lambda x)-\dfrac{1}{b\beta}\cosh(\beta y)$, $u_2=\dfrac{1}{a\lambda}\cosh(\lambda x)-\dfrac{2}{c\gamma}\arctan\bigl(e^{\gamma z}\bigr)$.

28. $a\cosh(\beta y)\dfrac{\partial w}{\partial x}+b\tanh(\lambda x)\dfrac{\partial w}{\partial y}+c\cosh(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=b\beta\ln\bigl|\cosh(\lambda x)\bigr|-a\lambda\sinh(\beta y),$$
$$u_2=c|\lambda|\gamma\int\frac{dx}{\sqrt{[b\beta\ln|\cosh(\lambda x)|-u_1]^2+a^2\lambda^2}}-2\operatorname{sign}(a)\arctan(e^{\gamma z}).$$

In the integration, u_1 is considered a parameter.

29. $\boldsymbol{a\coth(\beta y)\dfrac{\partial w}{\partial x} + b\tanh(\lambda x)\dfrac{\partial w}{\partial y} + c\tanh(\gamma z)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = \frac{\sinh^{a\lambda}(\beta y)}{\cosh^{b\beta}(\lambda x)}, \quad u_2 = c\gamma \int \coth\Bigl\{\operatorname{arcsinh}^{1/a\lambda}\bigl[u_1 \cosh^{b\beta}(\lambda x)\bigr]\Bigr\}\,dx - a\ln\bigl|\sinh(\gamma z)\bigr|.$$

In the integration, u_1 is considered a parameter.

30. $\boldsymbol{a\coth(\beta y)\dfrac{\partial w}{\partial x} + b\tanh(\lambda x)\dfrac{\partial w}{\partial y} + c\coth(\gamma z)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = \frac{\sinh^{a\lambda}(\beta y)}{\cosh^{b\beta}(\lambda x)}, \quad u_2 = c\gamma \int \coth\Bigl\{\operatorname{arcsinh}^{1/a\lambda}\bigl[u_1 \cosh^{b\beta}(\lambda x)\bigr]\Bigr\}\,dx - a\ln\bigl|\cosh(\gamma z)\bigr|.$$

In the integration, u_1 is considered a parameter.

2.1.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\ln(\beta y)\ln(\lambda z)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = bx - ay,\ u_2 = cy\bigl[1 - \ln(\beta y)\bigr] + b\displaystyle\int \frac{dz}{\ln(\lambda z)}.$

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\ln(\beta x)\dfrac{\partial w}{\partial y} + c\ln(\lambda x)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis: $u_1 = bx\bigl[1 - \ln(\beta x)\bigr] + ay,\ u_2 = cx\bigl[1 - \ln(\lambda x)\bigr] + az.$

3. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\ln(\beta x)\ln(\lambda y)\dfrac{\partial w}{\partial y} + c\ln(\mu x)\ln(\gamma z)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = bx\bigl[1 - \ln(\beta x)\bigr] + a\int \frac{dy}{\ln(\lambda y)}, \quad u_2 = cx\bigl[1 - \ln(\mu x)\bigr] + a\int \frac{dz}{\ln(\gamma z)}.$$

4. $\boldsymbol{a\ln(\beta x)\dfrac{\partial w}{\partial x} + b\ln(\lambda y)\dfrac{\partial w}{\partial y} + c\ln(\gamma z)\dfrac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = b\int \frac{dx}{\ln(\beta x)} - a\int \frac{dy}{\ln(\lambda y)}, \quad u_2 = c\int \frac{dx}{\ln(\beta x)} - a\int \frac{dz}{\ln(\gamma z)}.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

5. $\dfrac{\partial w}{\partial x}+ax^n\dfrac{\partial w}{\partial y}+b\ln^k(\lambda x)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=y-\dfrac{a}{n+1}x^{n+1},\ u_2=z-b\displaystyle\int\ln^k(\lambda x)\,dx.$

6. $\dfrac{\partial w}{\partial x}+\big[ay+c\ln^k(\lambda x)\big]\dfrac{\partial w}{\partial y}+\big[bz+s\ln^n(\beta x)\big]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=ye^{-ax}-c\int\ln^k(\lambda x)e^{-ax}\,dx,\quad u_2=ze^{-bx}-s\int\ln^n(\beta x)e^{-bx}\,dx.$$

7. $ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}+\big[c\ln^n(\lambda x)+s\ln^k(\beta y)\big]\dfrac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.21 with $f(x,y)=c\ln^n(\lambda x)+s\ln^k(\beta y)$.

8. $ax\ln(\lambda x)\dfrac{\partial w}{\partial x}+by\ln(\beta y)\dfrac{\partial w}{\partial y}+cz\ln(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\ln\big|\ln(\lambda x)\big|-a\ln\big|\ln(\beta y)\big|,\ u_2=c\ln\big|\ln(\lambda x)\big|-a\ln\big|\ln(\gamma z)\big|.$

9. $ax\ln(\lambda x)\dfrac{\partial w}{\partial x}+by\ln(\beta y)\dfrac{\partial w}{\partial y}+cz\ln(\lambda x)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\ln\big|\ln(\lambda x)\big|-a\ln\big|\ln(\beta y)\big|,\ u_2=|x|^c|z|^{-a}.$

10. $ax(\ln x)^n\dfrac{\partial w}{\partial x}+by(\ln y)^m\dfrac{\partial w}{\partial y}+cz(\ln z)^k\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{b}{1-n}(\ln x)^{1-n}-\frac{a}{1-m}(\ln y)^{1-m},\quad u_2=\frac{c}{1-n}(\ln x)^{1-n}-\frac{a}{1-k}(\ln z)^{1-k}.$$

2.1.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\sin(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx-ay,\ u_2=c\gamma x-a\ln\Big|\tan\dfrac{\gamma z}{2}\Big|.$

2. $a\dfrac{\partial w}{\partial x}+b\sin(\beta y)\dfrac{\partial w}{\partial y}+c\sin(\lambda x)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\Big|\tan\dfrac{\beta y}{2}\Big|,\ u_2=a\lambda z+c\cos(\lambda x).$

3. $a\dfrac{\partial w}{\partial x}+b\sin(\beta y)\dfrac{\partial w}{\partial y}+c\sin(\gamma z)\dfrac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\Big|\tan\dfrac{\beta y}{2}\Big|,\ u_2=c\gamma x-a\ln\Big|\tan\dfrac{\gamma z}{2}\Big|.$

4. $\displaystyle a\frac{\partial w}{\partial x}+b\sin(\lambda x)\sin(\beta y)\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=cx-az,\ u_2=\cos(\lambda x)+\dfrac{a\lambda}{b\beta}\ln\Bigl|\tan\dfrac{\beta y}{2}\Bigr|.$

5. $\displaystyle a\frac{\partial w}{\partial x}+b\sin^n(\lambda x)\sin^m(\beta y)\frac{\partial w}{\partial y}+c\sin^k(\mu x)\sin^l(\gamma z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b\sin^n(\lambda x)$, $g(y)=\sin^m(\beta y)$, $f_3(x)=c\sin^k(\mu x)$, and $h(z)=\sin^l(\gamma z)$.

► Coefficients of equations contain cosine.

6. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\cos(\lambda z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx-ay,\ u_2=c\lambda x-a\ln\bigl|\tan\bigl(\tfrac{1}{2}\lambda z+\tfrac{\pi}{4}\bigr)\bigr|.$

7. $\displaystyle a\frac{\partial w}{\partial x}+b\cos(\beta x)\frac{\partial w}{\partial y}+c\cos(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=a\beta y-b\sin(\beta x),\ u_2=a\lambda z-c\sin(\lambda x).$

8. $\displaystyle a\frac{\partial w}{\partial x}+b\cos(\beta y)\frac{\partial w}{\partial y}+c\cos(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl|\tan\bigl(\tfrac{1}{2}\beta y+\tfrac{\pi}{4}\bigr)\bigr|,\ u_2=a\lambda z-c\sin(\lambda x).$

9. $\displaystyle a\frac{\partial w}{\partial x}+b\cos(\beta y)\frac{\partial w}{\partial y}+c\cos(\lambda z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl|\tan\bigl(\tfrac{1}{2}\beta y+\tfrac{\pi}{4}\bigr)\bigr|,\ u_2=c\lambda x-a\ln\bigl|\tan\bigl(\tfrac{1}{2}\lambda z+\tfrac{\pi}{4}\bigr)\bigr|.$

10. $\displaystyle a\frac{\partial w}{\partial x}+b\cos^n(\lambda x)\cos^m(\beta y)\frac{\partial w}{\partial y}+c\cos^k(\mu x)\cos^l(\gamma z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b\cos^n(\lambda x)$, $g(y)=\cos^m(\beta y)$, $f_3(x)=c\cos^k(\mu x)$, and $h(z)=\cos^l(\gamma z)$.

► Coefficients of equations contain tangent.

11. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\tan(\lambda z)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=bx-ay,\ u_2=c\lambda x-a\ln\bigl|\sin(\lambda z)\bigr|.$

12. $\displaystyle a\frac{\partial w}{\partial x}+b\tan(\beta x)\frac{\partial w}{\partial y}+c\tan(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=a\beta y+b\ln\bigl|\cos(\beta x)\bigr|,\ u_2=a\lambda z+c\ln\bigl|\cos(\lambda x)\bigr|.$

13. $\displaystyle a\frac{\partial w}{\partial x}+b\tan(\beta y)\frac{\partial w}{\partial y}+c\tan(\lambda x)\frac{\partial w}{\partial z}=0.$

Integral basis: $u_1=b\beta x-a\ln\bigl|\sin(\beta y)\bigr|,\ u_2=a\lambda z+c\ln\bigl|\cos(\lambda x)\bigr|.$

14. $\boldsymbol{a\frac{\partial w}{\partial x}+b\tan(\beta y)\frac{\partial w}{\partial y}+c\tan(\lambda z)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=b\beta x-a\ln\bigl|\sin(\beta y)\bigr|,\ u_2=c\lambda x-a\ln\bigl|\sin(\lambda z)\bigr|.$

15. $\boldsymbol{\mu\nu\tan(\lambda x)\frac{\partial w}{\partial x}+\lambda\nu\tan(\mu y)\frac{\partial w}{\partial y}+\lambda\mu\tan(\nu z)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{\sin(\lambda x)}{\sin(\mu y)},\ u_2=\dfrac{\sin(\mu y)}{\sin(\nu z)}.$

► **Coefficients of equations contain cotangent.**

16. $\boldsymbol{a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\cot(\lambda z)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=bx-ay,\ u_2=c\lambda x+a\ln\bigl|\cos(\lambda z)\bigr|.$

17. $\boldsymbol{a\frac{\partial w}{\partial x}+b\cot(\beta x)\frac{\partial w}{\partial y}+c\cot(\lambda x)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=a\beta y-b\ln\bigl|\sin(\beta x)\bigr|,\ u_2=a\lambda z-c\ln\bigl|\sin(\lambda x)\bigr|.$

18. $\boldsymbol{a\frac{\partial w}{\partial x}+b\cot(\beta y)\frac{\partial w}{\partial y}+c\cot(\lambda x)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=b\beta x+a\ln\bigl|\cos(\beta y)\bigr|,\ u_2=a\lambda z-c\ln\bigl|\sin(\lambda x)\bigr|.$

19. $\boldsymbol{a\frac{\partial w}{\partial x}+b\cot(\beta y)\frac{\partial w}{\partial y}+c\cot(\lambda z)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=b\beta x+a\ln\bigl|\cos(\beta y)\bigr|,\ u_2=c\lambda x+a\ln\bigl|\cos(\lambda z)\bigr|.$

20. $\boldsymbol{\mu\nu\cot(\lambda x)\frac{\partial w}{\partial x}+\lambda\nu\cot(\mu y)\frac{\partial w}{\partial y}+\lambda\mu\cot(\nu z)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=\dfrac{\cos(\lambda x)}{\cos(\mu y)},\ u_2=\dfrac{\cos(\mu y)}{\cos(\nu z)}.$

► **Coefficients of equations contain different trigonometric functions.**

21. $\boldsymbol{a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+\bigl[c\sin^n(\lambda x)+s\cos^k(\beta y)\bigr]\frac{\partial w}{\partial z}=0.}$

This is a special case of equation 2.1.7.18 with $f(x,y)=c\sin^n(\lambda x)+s\cos^k(\beta y)$.

22. $\boldsymbol{a\frac{\partial w}{\partial x}+b\sin(\beta y)\frac{\partial w}{\partial y}+c\cos(\lambda x)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=b\beta x-a\ln\Bigl|\tan\dfrac{\beta y}{2}\Bigr|,\ u_2=a\lambda z-c\sin(\lambda x).$

23. $\boldsymbol{\frac{\partial w}{\partial x}+a\sin^n(\lambda x)\frac{\partial w}{\partial y}+b\cos^k(\beta x)\frac{\partial w}{\partial z}=0.}$

Integral basis: $u_1=y-a\displaystyle\int\sin^n(\lambda x)\,dx,\ u_2=z-b\int\cos^k(\beta x)\,dx.$

24. $\dfrac{\partial w}{\partial x} + a\cos^n(\lambda x)\dfrac{\partial w}{\partial y} + b\sin^k(\beta y)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.12 with $f_1(x) = 0$, $f_2(x) = a\cos^n(\lambda x)$, $g_1(y) = 0$, and $g_2(y) = b\sin^k(\beta y)$.

25. $a\dfrac{\partial w}{\partial x} + b\tan(\beta y)\dfrac{\partial w}{\partial y} + c\cot(\lambda x)\dfrac{\partial w}{\partial z} = 0.$

Integral basis: $u_1 = b\beta x - a\ln\bigl|\sin(\beta y)\bigr|,\ u_2 = a\lambda z - c\ln\bigl|\sin(\lambda x)\bigr|.$

26. $\dfrac{\partial w}{\partial x} + a\cot^n(\lambda x)\dfrac{\partial w}{\partial y} + b\tan^k(\beta y)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.12 with $f_1(x) = 0$, $f_2(x) = a\cot^n(\lambda x)$, $g_1(y) = 0$, and $g_2(y) = b\tan^k(\beta y)$.

2.1.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\arcsin^n(\lambda x)\arcsin^k(\beta z)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\arcsin^n(\lambda x)$, $g(y) = 1$, and $h(z) = \arcsin^k(\beta z)$.

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\arcsin^n(\lambda x)\arcsin^m(\beta y)\arcsin^k(\gamma z)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.19 with $f(x,y) = c\arcsin^n(\lambda x)\arcsin^m(\beta y)$ and $g(z) = \arcsin^k(\gamma z)$.

3. $a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y} + c\arcsin^k(\beta x)\dfrac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = ay - b\int \arcsin^n(\lambda x)\,dx, \quad u_2 = az - c\int \arcsin^k(\beta x)\,dx.$$

4. $a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y} + c\arcsin^k(\beta z)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b\arcsin^n(\lambda x)$, $f_3(x)=1$, $g(y) = 1$, and $h(z) = c\arcsin^k(\beta z)$.

5. $a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda y)\dfrac{\partial w}{\partial y} + c\arcsin^k(\beta z)\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.15 with $f(x) = a$, $g(y) = b\arcsin^n(\lambda y)$, and $h(z) = c\arcsin^k(\beta z)$.

► **Coefficients of equations contain arccosine.**

6. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arccos^n(\lambda x)\arccos^k(\beta z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\arccos^n(\lambda x)$, $g(y)=1$, and $h(z)=\arccos^k(\beta z)$.

7. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arccos^n(\lambda x)\arccos^m(\beta y)\arccos^k(\gamma z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.19 with $f(x,y)=c\arccos^n(\lambda x)\arccos^m(\beta y)$, and $g(z)=\arccos^k(\gamma z)$.

8. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda x)\frac{\partial w}{\partial y}+c\arccos^k(\beta x)\frac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=ay-b\int\arccos^n(\lambda x)\,dx,\quad u_2=az-c\int\arccos^k(\beta x)\,dx.$$

9. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda x)\frac{\partial w}{\partial y}+c\arccos^k(\beta z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b\arccos^n(\lambda x)$, $f_3(x)=1$, $g(y)=1$, and $h(z)=c\arccos^k(\beta z)$.

10. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda y)\frac{\partial w}{\partial y}+c\arccos^k(\beta z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.15 with $f(x)=a$, $g(y)=b\arccos^n(\lambda y)$, and $h(z)=c\arccos^k(\beta z)$.

► **Coefficients of equations contain arctangent.**

11. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arctan^n(\lambda x)\arctan^k(\beta z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\arctan^n(\lambda x)$, $g(y)=1$, and $h(z)=\arctan^k(\beta z)$.

12. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arctan^n(\lambda x)\arctan^m(\beta y)\arctan^k(\gamma z)\frac{\partial w}{\partial z}=0.$

This is a special case of equation 2.1.7.19 with $f(x,y)=c\arctan^n(\lambda x)\arctan^m(\beta y)$ and $g(z)=\arctan^k(\gamma z)$.

13. $\displaystyle a\frac{\partial w}{\partial x}+b\arctan^n(\lambda x)\frac{\partial w}{\partial y}+c\arctan^k(\beta x)\frac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=ay-b\int\arctan^n(\lambda x)\,dx,\quad u_2=az-c\int\arctan^k(\beta x)\,dx.$$

14. $\displaystyle a\frac{\partial w}{\partial x} + b\arctan^n(\lambda x)\frac{\partial w}{\partial y} + c\arctan^k(\beta z)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b\arctan^n(\lambda x)$, $f_3(x)=1$, $g(y)=1$, and $h(z)=c\arctan^k(\beta z)$.

15. $\displaystyle a\frac{\partial w}{\partial x} + b\arctan^n(\lambda y)\frac{\partial w}{\partial y} + c\arctan^k(\beta z)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.15 with $f(x)=a$, $g(y)=b\arctan^n(\lambda y)$, and $h(z)=c\arctan^k(\beta z)$.

► **Coefficients of equations contain arccotangent.**

16. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^k(\beta z)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\,\mathrm{arccot}^n(\lambda x)$, $g(y)=1$, and $h(z)=\mathrm{arccot}^k(\beta z)$.

17. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^m(\beta y)\,\mathrm{arccot}^k(\gamma z)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.19 with $f(x,y)=c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^m(\beta y)$ and $g(z)=\mathrm{arccot}^k(\gamma z)$.

18. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y} + c\,\mathrm{arccot}^k(\beta x)\frac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = ay - b\int \mathrm{arccot}^n(\lambda x)\,dx, \quad u_2 = az - c\int \mathrm{arccot}^k(\beta x)\,dx.$$

19. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y} + c\,\mathrm{arccot}^k(\beta z)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.16 with $f_1(x)=a$, $f_2(x)=b\,\mathrm{arccot}^n(\lambda x)$, $f_3(x)=1$, $g(y)=1$, and $h(z)=c\,\mathrm{arccot}^k(\beta z)$.

20. $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda y)\frac{\partial w}{\partial y} + c\,\mathrm{arccot}^k(\beta z)\frac{\partial w}{\partial z} = 0.$

This is a special case of equation 2.1.7.15 with $f(x)=a$, $g(y)=b\,\mathrm{arccot}^n(\lambda y)$, and $h(z)=c\,\mathrm{arccot}^k(\beta z)$.

2.1.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of x.**

1. $\displaystyle \frac{\partial w}{\partial x} + f(x)\frac{\partial w}{\partial y} + g(x)\frac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = y - \int f(x)\,dx, \quad u_2 = z - \int g(x)\,dx.$$

2. $\dfrac{\partial w}{\partial x}+f(x)(y+a)\dfrac{\partial w}{\partial y}+g(x)(z+b)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\ln|y+a|-\int f(x)\,dx,\quad u_2=\ln|z+b|-\int g(x)\,dx.$$

3. $\dfrac{\partial w}{\partial x}+\big[ay+f(x)\big]\dfrac{\partial w}{\partial y}+\big[bz+g(x)\big]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=ye^{-ax}-\int f(x)e^{-ax}\,dx,\quad u_2=ze^{-bx}-\int g(x)e^{-bx}\,dx.$$

4. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)y+g_2(x)\big]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=yF(x)-\int f_2(x)F(x)\,dx,\qquad F(x)=\exp\biggl[-\int f_1(x)\,dx\biggr],$$
$$u_2=z-\varphi(x)y+\int\big[f_2(x)\varphi(x)-g_2(x)\big]\,dx,\qquad \varphi(x)=F(x)\int\frac{g_1(x)}{F(x)}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

5. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)z+g_2(x)\big]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=yF(x)-\int f_2(x)F(x)\,dx,\qquad F(x)=\exp\biggl[-\int f_1(x)\,dx\biggr],$$
$$u_2=zG(x)-\int g_2(x)G(x)\,dx,\qquad G(x)=\exp\biggl[-\int g_1(x)\,dx\biggr].$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

6. $\dfrac{\partial w}{\partial x}+\big[f_2(x)y+f_1(x)z+f_0(x)\big]\dfrac{\partial w}{\partial y}+\big[g_2(x)y+g_1(x)z+g_0(x)\big]\dfrac{\partial w}{\partial z}=0.$

One of the integrals has the form

$$u_1=\varphi(x)y+\psi(x)z+\chi(x),$$

where the functions $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by solving the following system of first-order ordinary differential equations:

$$\begin{aligned}&\varphi'_x+f_2\varphi+g_2\psi=0,\\&\psi'_x+f_1\varphi+g_1\psi=0,\\&\chi'_x+f_0\varphi+g_0\psi=0.\end{aligned}$$

In some cases, this system can be integrated in quadrature. For example, this can be done for $g_2 \equiv 0$ (or $f_1 \equiv 0$); in this case, one should begin the integration with the first (resp., second) equation. In the general case, the integration of the system can be reduced to the solution of a second-order linear ordinary differential equation which follows from the first two equations.

The general solution of the equation in question can be obtained using the technique described in Section 13.1.3 (see paragraph *The method of reducing the number of independent variables*).

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

7. $\displaystyle \frac{\partial w}{\partial x} + \big[y^2 - a^2 + a\lambda\sinh(\lambda x) - a^2\sinh^2(\lambda x)\big]\frac{\partial w}{\partial y} + f(x)\sinh(\gamma z)\frac{\partial w}{\partial z} = 0.$

Integral basis:

$$u_1 = \int f(x)\,dx - \frac{1}{\gamma}\ln\left|\tanh\frac{\gamma z}{2}\right|,$$

$$u_2 = \frac{E}{y - a\cosh(\lambda x)} + \int E\,dx, \qquad E = \exp\left[\frac{2a}{\lambda}\sinh(\lambda x)\right].$$

8. $\displaystyle \frac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)y^k\big]\frac{\partial w}{\partial y} + \big[g_1(x)z + g_2(x)z^m\big]\frac{\partial w}{\partial z} = 0.$

1°. For $k \neq 1$ and $m \neq 1$, the transformation $\xi = y^{1-k}$, $\eta = z^{1-m}$ leads to an equation of the form 2.1.7.5:

$$\frac{\partial w}{\partial x} + (1-k)\big[f_1(x)\xi + f_2(x)\big]\frac{\partial w}{\partial \xi} + (1-m)\big[g_1(x)\eta + g_2(x)\big]\frac{\partial w}{\partial \eta} = 0.$$

2°. For $k \neq 1$ and $m = 1$, the substitution $\xi = y^{1-k}$ leads to an equation of the form 2.1.7.5:

$$\frac{\partial w}{\partial x} + (1-k)\big[f_1(x)\xi + f_2(x)\big]\frac{\partial w}{\partial \xi} + z\big[g_1(x) + g_2(x)\big]\frac{\partial w}{\partial z} = 0.$$

3°. For $k = m = 1$, see equation 2.1.7.5.

9. $\displaystyle \frac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)y^k\big]\frac{\partial w}{\partial y} + \big[g_1(x) + g_2(x)e^{\lambda z}\big]\frac{\partial w}{\partial z} = 0.$

The transformation $\xi = y^{1-k}$, $\eta = e^{-\lambda z}$ leads to an equation of the form 2.1.7.5:

$$\frac{\partial w}{\partial x} + (1-k)\big[f_1(x)\xi + f_2(x)\big]\frac{\partial w}{\partial \xi} - \lambda\big[g_1(x)\eta + g_2(x)\big]\frac{\partial w}{\partial \eta} = 0.$$

10. $\displaystyle \frac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\frac{\partial w}{\partial y} + \big[g_1(x) + g_2(x)e^{\beta z}\big]\frac{\partial w}{\partial z} = 0.$

The transformation $\xi = e^{-\lambda y}$, $\eta = e^{-\beta z}$ leads to an equation of the form 2.1.7.5:

$$\frac{\partial w}{\partial x} - \lambda\big[f_1(x)\xi + f_2(x)\big]\frac{\partial w}{\partial \xi} - \beta\big[g_1(x)\eta + g_2(x)\big]\frac{\partial w}{\partial \eta} = 0.$$

► **Coefficients of equations contain arbitrary functions of different variables.**

11. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+[z+f(x)g(y)]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\frac{y}{x},\quad u_2=\frac{z}{x}-\int x^{-2}f(x)g(u_1x)\,dx.$$

In the integration, u_1 is considered a parameter.

12. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(y)z+g_2(y)\big]\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=yF(x)-\int f_2(x)F(x)\,dx,\qquad F(x)=\exp\biggl[-\int f_1(x)\,dx\biggr],$$
$$u_2=zG(x,u_1)-\int_{x_0}^{x}\bar g_2(t,u_1)G(t,u_1)\,dt,\qquad G(x,u_1)=\exp\biggl[-\int_{x_0}^{x}\bar g_1(t,u_1)\,dt\biggr].$$

Here $\bar g_1(x,u_1)\equiv g_1(y)$, $\bar g_2(x,u_1)\equiv g_2(y)$ (y is expressed via x and u_1 from the first integral), and x_0 is an arbitrary number.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

13. $\dfrac{\partial w}{\partial x}+\big[y^2-a^2+a\lambda\sinh(\lambda x)-a^2\sinh^2(\lambda x)\big]\dfrac{\partial w}{\partial y}+f(x)g(z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\int f(x)\,dx-\int\frac{dz}{g(z)},$$
$$u_2=\frac{E}{y-a\cosh(\lambda x)}+\int E\,dx,\qquad E=\exp\biggl[\frac{2a}{\lambda}\sinh(\lambda x)\biggr].$$

14. $f(x)\dfrac{\partial w}{\partial x}+z^k\dfrac{\partial w}{\partial y}+g(y)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\int g(y)\,dy-\frac{1}{k+1}z^{k+1},\quad u_2=\int\frac{dx}{f(x)}-\int\biggl[(k+1)\int g(y)\,dy-(k+1)u_1\biggr]^{-\frac{k}{k+1}}dy.$$

In the integration, u_1 is considered a parameter.

15. $f(x)\dfrac{\partial w}{\partial x}+g(y)\dfrac{\partial w}{\partial y}+h(z)\dfrac{\partial w}{\partial z}=0.$

Integral basis:

$$u_1=\int\frac{dx}{f(x)}-\int\frac{dy}{g(y)},\quad u_2=\int\frac{dx}{f(x)}-\int\frac{dz}{h(z)}.$$

16. $\boldsymbol{f_1(x)\frac{\partial w}{\partial x} + f_2(x)g(y)\frac{\partial w}{\partial y} + f_3(x)h(z)\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = \int \frac{f_2(x)}{f_1(x)}dx - \int \frac{dy}{g(y)}, \quad u_2 = \int \frac{f_3(x)}{f_1(x)}dx - \int \frac{dz}{h(z)}.$$

17. $\boldsymbol{a \sinh(\beta y)\frac{\partial w}{\partial x} + b \sinh(\gamma x)\frac{\partial w}{\partial y} + f_1(x)f_2(z)\sinh(\beta y)\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = \frac{1}{a\gamma}\cosh(\gamma x) - \frac{1}{b\beta}\cosh(\beta y), \quad u_2 = \frac{1}{a}\int f_1(x)\,dx - \int \frac{dz}{f_2(z)}.$$

► **Coefficients of equations contain arbitrary functions of two variables.**

18. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + f(x,y)\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = bx - ay, \quad u_2 = bz - \int_{y_0}^{y} f\left(x + \frac{a(t-y)}{b},\, t\right) dt,$$

where y_0 may be taken as arbitrary.

19. $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + f(x,y)g(z)\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = bx - ay, \quad u_2 = b\int \frac{dz}{g(z)} - \int_{y_0}^{y} f\left(x + \frac{a(t-y)}{b},\, t\right) dt.$$

20. $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + [z + f(x,y)]\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = \frac{y}{x}, \quad u_2 = \frac{z}{x} - \int_{x_0}^{x} f\left(t,\, \frac{y}{x}t\right) t^{-2}\,dt,$$

where x_0 may be taken as arbitrary.

⊙ *Literature*: E. Kamke (1965).

21. $\boldsymbol{ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} + f(x,y)\frac{\partial w}{\partial z} = 0.}$

Integral basis:

$$u_1 = x^b y^{-a}, \quad u_2 = bz - \int_{y_0}^{y} t^{-1} f\bigl(xy^{-a/b}t^{a/b},\, t\bigr)\,dt.$$

22. $\boldsymbol{ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+f(x,y)g(z)\frac{\partial w}{\partial z}=0.}$

Integral basis:

$$u_1=x^by^{-a},\quad u_2=b\int\frac{dz}{g(z)}-\int_{y_0}^{y}t^{-1}f\bigl(xy^{-a/b}t^{a/b},\,t\bigr)\,dt.$$

23. $\boldsymbol{\frac{\partial w}{\partial x}+\bigl[f_1(x)y+f_2(x)\bigr]\frac{\partial w}{\partial y}+\bigl[g(x,y)z+h(x,y)\bigr]\frac{\partial w}{\partial z}=0.}$

Integral basis:

$$u_1=yF(x)-\int f_2(x)F(x)\,dx,\qquad F(x)=\exp\biggl[-\int f_1(x)\,dx\biggr],$$

$$u_2=zG(x,u_1)-\int_{x_0}^{x}\bar h(t,u_1)G(t,u_1)\,dt,\qquad G(x,u_1)=\exp\biggl[-\int_{x_0}^{x}\bar g(t,u_1)\,dt\biggr].$$

Here $\bar g(x,u_1)\equiv g(x,y)$ and $\bar h(x,u_1)\equiv h(x,y)$ (y is expressed via x and u_1 from the first integral), and x_0 is an arbitrary number.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

24. $\boldsymbol{\frac{\partial w}{\partial x}+\bigl[f_1(x)y+f_2(x)y^k\bigr]\frac{\partial w}{\partial y}+\bigl[g(x,y)z+h(x,y)z^m\bigr]\frac{\partial w}{\partial z}=0.}$

1°. For $k\neq 1$ and $m\neq 1$, the transformation $\xi=y^{1-k}$, $\eta=z^{1-m}$ leads to an equation of the form 2.1.7.23:

$$\frac{\partial w}{\partial x}+(1-k)\bigl[f_1(x)\xi+f_2(x)\bigr]\frac{\partial w}{\partial \xi}+(1-m)\bigl[\bar g(x,\xi)\eta+\bar h(x,\xi)\bigr]\frac{\partial w}{\partial \eta}=0,$$

where $\bar g(x,\xi)\equiv g\bigl(x,\xi^{\frac{1}{1-k}}\bigr)$ and $\bar h(x,\xi)\equiv h\bigl(x,\xi^{\frac{1}{1-k}}\bigr)$.

2°. For $k\neq 1$ and $m=1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 2.1.7.23:

$$\frac{\partial w}{\partial x}+(1-k)\bigl[f_1(x)\xi+f_2(x)\bigr]\frac{\partial w}{\partial \xi}+z\bigl[\bar g(x,\xi)+\bar h(x,\xi)\bigr]\frac{\partial w}{\partial z}=0.$$

3°. For $k=m=1$, see equation 2.1.7.23.

25. $\boldsymbol{\frac{\partial w}{\partial x}+\bigl[f_1(x)y+f_2(x)y^k\bigr]\frac{\partial w}{\partial y}+\bigl[g(x,y)+h(x,y)e^{\lambda z}\bigr]\frac{\partial w}{\partial z}=0.}$

The transformation $\xi=y^{1-k}$, $\eta=e^{-\lambda z}$ leads to an equation of the form 2.1.7.23:

$$\frac{\partial w}{\partial x}+(1-k)\bigl[f_1(x)\xi+f_2(x)\bigr]\frac{\partial w}{\partial \xi}-\lambda\bigl[\bar g(x,\xi)\eta+\bar h(x,\xi)\bigr]\frac{\partial w}{\partial \eta}=0,$$

where $\bar g(x,\xi)\equiv g\bigl(x,\xi^{\frac{1}{1-k}}\bigr)$ and $\bar h(x,\xi)\equiv h\bigl(x,\xi^{\frac{1}{1-k}}\bigr)$.

26. $\dfrac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} + \big[g(x,y)z + h(x,y)z^k\big]\dfrac{\partial w}{\partial z} = 0.$

The transformation $\xi = e^{-\lambda y}$, $\eta = z^{1-k}$ leads to an equation of the form 2.1.7.23:

$$\frac{\partial w}{\partial x} - \lambda\big[f_1(x)\xi + f_2(x)\big]\frac{\partial w}{\partial \xi} + (1-k)\big[\bar g(x,\xi)\eta + \bar h(x,\xi)\big]\frac{\partial w}{\partial \eta} = 0,$$

where $\bar g(x,\xi) \equiv g\big(x, -\frac{1}{\lambda}\ln\xi\big)$ and $\bar h(x,\xi) \equiv h\big(x, -\frac{1}{\lambda}\ln\xi\big)$.

27. $\dfrac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} + \big[g(x,y) + h(x,y)e^{\beta z}\big]\dfrac{\partial w}{\partial z} = 0.$

The transformation $\xi = e^{-\lambda y}$, $\eta = e^{-\beta z}$ leads to an equation of the form 2.1.7.23:

$$\frac{\partial w}{\partial x} - \lambda\big[f_1(x)\xi + f_2(x)\big]\frac{\partial w}{\partial \xi} - \beta\big[\bar g(x,\xi)\eta + \bar h(x,\xi)\big]\frac{\partial w}{\partial \eta} = 0,$$

where $\bar g(x,\xi) \equiv g\big(x, -\frac{1}{\lambda}\ln\xi\big)$ and $\bar h(x,\xi) \equiv h\big(x, -\frac{1}{\lambda}\ln\xi\big)$.

28. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x} + f_2(x)g_2(y)\dfrac{\partial w}{\partial y} + \big[h_1(x,y)z + h_2(x,y)z^m\big]\dfrac{\partial w}{\partial z} = 0.$

The transformation $\xi = \displaystyle\int \frac{f_2(x)}{f_1(x)}\,dx$, $\eta = \displaystyle\int \frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.1.7.24 with $f_1 \equiv 0$, $f_2 \equiv 1$, and $k = 0$:

$$\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} + \big[\bar h_1(\xi,\eta)z + \bar h_2(\xi,\eta)z^m\big]\frac{\partial w}{\partial z} = 0,$$

where $\bar h_1(\xi,\eta) \equiv \dfrac{h_1(x,y)}{f_2(x)g_1(y)}$ and $\bar h_2(\xi,\eta) \equiv \dfrac{h_2(x,y)}{f_2(x)g_1(y)}$.

29. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x} + f_2(x)g_2(y)\dfrac{\partial w}{\partial y} + \big[h_1(x,y) + h_2(x,y)e^{\lambda z}\big]\dfrac{\partial w}{\partial z} = 0.$

The transformation $\xi = \displaystyle\int \frac{f_2(x)}{f_1(x)}\,dx$, $\eta = \displaystyle\int \frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.1.7.25 with $f_1 \equiv 0$, $f_2 \equiv 1$, and $k = 0$:

$$\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} + \big[\bar h_1(\xi,\eta) + \bar h_2(\xi,\eta)e^{\lambda z}\big]\frac{\partial w}{\partial z} = 0,$$

where $\bar h_1(\xi,\eta) \equiv \dfrac{h_1(x,y)}{f_2(x)g_1(y)}$ and $\bar h_2(\xi,\eta) \equiv \dfrac{h_2(x,y)}{f_2(x)g_1(y)}$.

2.2 Equations of the Form $f_1\dfrac{\partial w}{\partial x} + f_2\dfrac{\partial w}{\partial y} + f_3\dfrac{\partial w}{\partial z} = f_4$, $f_n = f_n(x,y,z)$

◆ *The solutions given below contain arbitrary functions of two variables* $\Phi = \Phi(u_1, u_2)$*, where* $u_1 = u_1(x,y,z)$ *and* $u_2 = u_2(x,y,z)$ *are some functions.*

2.2.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x, y, and z.**

1. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=\alpha x+\beta y+\gamma z+\delta.$

General solution: $\displaystyle w=\frac{\alpha}{2a}x^2+\frac{\beta}{2b}y^2+\frac{\gamma}{2c}z^2+\frac{\delta}{a}x+\Phi(bx-ay,\,cy-bz).$

2. $\displaystyle \frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=cx+s.$

General solution: $w=\frac{1}{2}cx^2+sx+\Phi(u_1,u_2)$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}\bigl(by+\sqrt{ab}\,z\bigr)\exp\bigl(-\sqrt{ab}\,x\bigr) & \text{if } ab>0,\\ by\cos\bigl(\sqrt{|ab|}\,x\bigr)+\sqrt{|ab|}\,z\sin\bigl(\sqrt{|ab|}\,x\bigr) & \text{if } ab<0.\end{cases}$$

3. $\displaystyle \frac{\partial w}{\partial x}+(a_1x+a_0)\frac{\partial w}{\partial y}+(b_1x+b_0)\frac{\partial w}{\partial z}=\alpha x+\beta y+\gamma z+\delta.$

This is a special case of equation 2.2.7.1 with $f(x)=a_1x+a_0$, $g(x)=b_1x+b_0$, $h_2(x)=\beta$, $h_1(x)=\gamma$, and $h_0(x)=\alpha x+\delta$.

4. $\displaystyle \frac{\partial w}{\partial x}+(a_2y+a_1x+a_0)\frac{\partial w}{\partial y}+(b_2y+b_1x+b_0)\frac{\partial w}{\partial z}=c_2y+c_1z+c_0x+s.$

This is a special case of equation 2.2.7.4 with $f_1(x)=a_2$, $f_2(x)=a_1x+a_0$, $g_1(x)=b_2$, $g_2(x)=b_1x+b_0$, $h_2(x)=c_2$, $h_1(x)=c_1$, and $h_0(x)=c_0x+s$.

5. $\displaystyle \frac{\partial w}{\partial x}+(ay+k_1x+k_0)\frac{\partial w}{\partial y}+(bz+s_1x+s_0)\frac{\partial w}{\partial z}=c_1x+c_0.$

This is a special case of equation 2.2.7.3 with $f(x)=k_1x+k_0$, $g(x)=s_1x+s_0$, and $h(x)=c_1x+c_0$.

6. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=\alpha x+\beta y+\gamma z+\delta.$

General solution: $\displaystyle w=\frac{\alpha}{a}x+\frac{\beta}{b}y+\frac{\gamma}{c}z+\frac{\delta}{a}\ln|x|+\Phi\left(\frac{|y|^a}{|x|^b},\,\frac{|z|^a}{|x|^c}\right).$

7. $\displaystyle x\frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=c.$

General solution: $w=c\ln|x|+\Phi(u_1,u_2)$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}|x|^{\sqrt{ab}}\bigl(by-\sqrt{ab}\,z\bigr) & \text{if } ab>0,\\ |x|^{\sqrt{-ab}}\exp\left(-\arctan\dfrac{\sqrt{-ab}\,z}{by}\right) & \text{if } ab<0.\end{cases}$$

8. $\displaystyle abx\frac{\partial w}{\partial x} + b(ay+bz)\frac{\partial w}{\partial y} + a(ay-bz)\frac{\partial w}{\partial z} = c.$

General solution: $w = \dfrac{c}{ab}\ln|x| + \Phi(u_1, u_2)$, where

$$u_1 = \bigl[ay + (\sqrt{2}-1)bz\bigr]|x|^{-\sqrt{2}}, \quad u_2 = \bigl[ay - (\sqrt{2}+1)bz\bigr]|x|^{\sqrt{2}}.$$

Particular solution: $w = \dfrac{c}{ab}\ln|x| + \Phi\bigl(a^2y^2 - 2abyz - b^2z^2\bigr)$.

9. $\displaystyle (a_1x+a_0)\frac{\partial w}{\partial x} + (b_1y+b_0)\frac{\partial w}{\partial y} + (c_1z+c_0)\frac{\partial w}{\partial z} = \alpha x + \beta y + \gamma z + \delta.$

1°. General solution for $a_1b_1c_1 \neq 0$:

$$w = \frac{\alpha}{a_1}x + \frac{\beta}{b_1}y + \frac{\gamma}{c_1}z + \frac{1}{a_1}\left(\delta - \frac{\alpha a_0}{a_1} - \frac{\beta b_0}{b_1} - \frac{\gamma c_0}{c_1}\right)\ln\bigl|a_1x+a_0\bigr| + \Phi\left(\frac{|b_1y+b_0|^{a_1}}{|a_1x+a_0|^{b_1}}, \frac{|b_1y+b_0|^{c_1}}{|c_1z+c_0|^{b_1}}\right).$$

2°. General solution for $a_1b_1 \neq 0$ and $c_1 = 0$:

$$w = \frac{\alpha}{a_1}x + \frac{\beta}{b_1}y + \frac{\gamma}{2c_0}z^2 + \frac{1}{c_0}\left(\delta - \frac{\alpha a_0}{a_1} - \frac{\beta b_0}{b_1}\right)z + \Phi\left(\frac{|b_1y+b_0|^{a_1}}{|a_1x+a_0|^{b_1}}, |b_1y+b_0|^{c_0}e^{-b_1z}\right).$$

3°. General solution for $a_1 \neq 0$ and $b_1 = c_1 = 0$:

$$w = \frac{\alpha}{a_1}x + \frac{\beta}{2b_0}y^2 + \frac{\gamma}{2c_0}z^2 + \frac{1}{c_0}\left(\delta - \frac{\alpha a_0}{a_1}\right)z + \Phi\bigl(|a_1x+a_0|^{b_0}e^{-a_1y},\ c_0y - b_0z\bigr).$$

4°. For $a_1 = b_1 = c_1 = 0$, see equation 2.2.1.1.

► **Coefficients of equations are quadratic in x, y, and z.**

10. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + c\frac{\partial w}{\partial z} = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta.$

General solution: $w = \dfrac{\alpha}{3a}x^3 + \dfrac{\beta}{3b}y^3 + \dfrac{\gamma}{3c}z^3 + \dfrac{\delta}{a}x + \Phi(bx-ay,\ cy-bz)$.

11. $\displaystyle \frac{\partial w}{\partial x} + (a_1x^2+a_0)\frac{\partial w}{\partial y} + (b_1x^2+b_0)\frac{\partial w}{\partial z} = \alpha x + \beta y + \gamma z + \delta.$

This is a special case of equation 2.2.7.1 with $f(x) = a_1x^2 + a_0$, $g(x) = b_1x^2 + b_0$, $h_2(x) = \beta$, $h_1(x) = \gamma$, and $h_0(x) = \alpha x + \delta$.

12. $\displaystyle \frac{\partial w}{\partial x} + (ay + k_1x^2 + k_0)\frac{\partial w}{\partial y} + (bz + s_1x^2 + s_0)\frac{\partial w}{\partial z} = c_1x^2 + c_0.$

This is a special case of equation 2.2.7.3 with $f(x) = k_1x^2 + k_0$, $g(x) = s_1x^2 + s_0$, and $h(x) = c_1x^2 + c_0$.

13. $\displaystyle \frac{\partial w}{\partial x}+(a_2xy+a_1x^2+a_0)\frac{\partial w}{\partial y}+(b_2xy+b_1x^2+b_0)\frac{\partial w}{\partial z}=c_2y+c_1z+c_0x+s.$

This is a special case of equation 2.2.7.4 with $f_1(x)=a_2x$, $f_2(x)=a_1x^2+a_0$, $g_1(x)=b_2x$, $g_2(x)=b_1x^2+b_0$, $h_2(x)=c_2$, $h_1(x)=c_1$, and $h_0(x)=c_0x+s$.

14. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=x(\alpha x+\beta y+\gamma z).$

1°. General solution for $b\neq -a$ and $c\neq -a$:

$$w=\frac{\alpha}{2a}x^2+\frac{\beta}{a+b}xy+\frac{\gamma}{a+c}xz+\Phi\bigl(x|y|^{-a/b},\,x|z|^{-a/c}\bigr).$$

2°. General solution for $b=-a$ and $c\neq -a$:

$$w=\frac{1}{2a}x\bigl(\alpha x+2\beta y\ln|x|\bigr)+\frac{\gamma}{a+c}xz+\Phi\bigl(xy,\,x|z|^{-a/c}\bigr).$$

3°. General solution for $b=c=-a$:

$$w=\frac{1}{2a}x\bigl[\alpha x+2(\beta y+\gamma z)\ln|x|\bigr]+\Phi(xy,xz).$$

15. $\displaystyle ax^2\frac{\partial w}{\partial x}+bxy\frac{\partial w}{\partial y}+cxz\frac{\partial w}{\partial z}=\alpha x+\beta y+\gamma z.$

1°. General solution for $b\neq a$ and $c\neq a$:

$$w=\frac{\alpha}{a}\ln|x|+\frac{1}{x}\left(\frac{\beta y}{b-a}+\frac{\gamma z}{c-a}\right)+\Phi\bigl(x|y|^{-a/b},\,x|z|^{-a/c}\bigr).$$

2°. General solution for $b=a$ and $c\neq a$:

$$w=\frac{\alpha}{a}\ln|x|+\frac{1}{x}\left(\frac{\beta y}{a}\ln|x|+\frac{\gamma z}{c-a}\right)+\Phi\left(\frac{x}{y},\,x|z|^{-a/c}\right).$$

3°. General solution for $a=b=c$:

$$w=\frac{\ln|x|}{ax}(\alpha x+\beta y+\gamma z)+\Phi\left(\frac{x}{y},\,\frac{x}{z}\right).$$

16. $\displaystyle ax^2\frac{\partial w}{\partial x}+bxy\frac{\partial w}{\partial y}+cz^2\frac{\partial w}{\partial z}=ky^2.$

1°. General solution for $a\neq 2b$:

$$w=\frac{ky^2}{(2b-a)x}+\Phi\left(xy^{-a/b},\,\frac{c}{x}-\frac{a}{z}\right).$$

2°. General solution for $a=2b$:

$$w=\frac{ky^2\ln|x|}{ax}+\Phi\left(\frac{x}{y^2},\,\frac{c}{x}-\frac{a}{z}\right).$$

17. $ax^2\dfrac{\partial w}{\partial x}+by^2\dfrac{\partial w}{\partial y}+cz^2\dfrac{\partial w}{\partial z}=kxy.$

General solution: $w=\dfrac{kxy}{ax-by}\ln\left|\dfrac{ax}{y}\right|+\Phi\left(\dfrac{b}{x}-\dfrac{a}{y},\ \dfrac{c}{x}-\dfrac{a}{z}\right).$

18. $ax^2\dfrac{\partial w}{\partial x}+by^2\dfrac{\partial w}{\partial y}+cz^2\dfrac{\partial w}{\partial z}=\alpha x^2+\beta y^2+\gamma z^2.$

General solution: $w=\dfrac{\alpha}{a}x+\dfrac{\beta}{b}y+\dfrac{\gamma}{c}z+\Phi\left(\dfrac{b}{x}-\dfrac{a}{y},\ \dfrac{c}{x}-\dfrac{a}{z}\right).$

► **Coefficients of equations contain other powers in x, y, and z.**

19. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=xyz.$

General solution: $w=\frac{1}{2}x^2yz-\frac{1}{6}x^3(az+by)+\frac{1}{12}abx^4+\Phi(y-ax,\ z-bx).$

20. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\dfrac{\partial w}{\partial z}=kx^3+sy^2.$

General solution: $w=\dfrac{k}{4a}x^4+\dfrac{s}{3a}y^3+\Phi(bx-ay,cx-az).$

21. $a\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}+cz\dfrac{\partial w}{\partial z}=kx+s\sqrt{x}.$

General solution: $w=\dfrac{k}{2a}x^2+\dfrac{2s}{3a}x^{3/2}+\Phi\bigl(|y|^ae^{-bx},|z|^ae^{-cx}\bigr).$

22. $\dfrac{\partial w}{\partial x}+az\dfrac{\partial w}{\partial y}+by\dfrac{\partial w}{\partial z}=c\sqrt{x}+s.$

General solution: $w=\frac{2}{3}cx^{3/2}+sx+\Phi(u_1,u_2)$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}\bigl(by+\sqrt{ab}\,z\bigr)\exp\bigl(-\sqrt{ab}\,x\bigr) & \text{if } ab>0,\\ by\cos\bigl(\sqrt{|ab|}\,x\bigr)+\sqrt{|ab|}\,z\sin\bigl(\sqrt{|ab|}\,x\bigr) & \text{if } ab<0.\end{cases}$$

23. $ax^2\dfrac{\partial w}{\partial x}+by^2\dfrac{\partial w}{\partial y}+cz^2\dfrac{\partial w}{\partial z}=kxyz.$

General solution:

$$w=w_0(x,y,z)+\Phi\left(\frac{1}{ax}-\frac{1}{by},\ \frac{1}{ax}-\frac{1}{cz}\right),$$

where $w_0=w_0(x,y,z)$ is a particular solution,

$$w_0=kxyz\left[\frac{ax\ln(ax)}{(ax-by)(ax-cz)}+\frac{by\ln(by)}{(by-ax)(by-cz)}+\frac{cz\ln(cz)}{(cz-ax)(cz-by)}\right].$$

► **Coefficients of equations contain arbitrary powers of x, y, and z.**

24. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=\alpha x^n+\beta y^m+\gamma z^k.$

General solution:

$$w=\frac{\alpha}{a(n+1)}x^{n+1}+\frac{\beta}{b(m+1)}y^{m+1}+\frac{\gamma}{c(k+1)}z^{k+1}+\Phi(bx-ay,cx-az).$$

25. $\displaystyle a\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=\alpha x^n+\beta y^m+\gamma z^k.$

General solution: $\displaystyle w=\frac{\alpha}{a(n+1)}x^{n+1}+\frac{\beta}{bm}y^m+\frac{\gamma}{ck}z^k+\Phi\bigl(|y|^a e^{-bx},|z|^a e^{-cx}\bigr).$

26. $\displaystyle \frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=cx^n.$

General solution: $\displaystyle w=\frac{c}{n+1}x^{n+1}+\Phi(u_1,u_2),$ where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}\bigl(by+\sqrt{ab}\,z\bigr)\exp\bigl(-\sqrt{ab}\,x\bigr) & \text{if } ab>0,\\ by\cos\bigl(\sqrt{|ab|}\,x\bigr)+\sqrt{|ab|}\,z\sin\bigl(\sqrt{|ab|}\,x\bigr) & \text{if } ab<0.\end{cases}$$

27. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=\alpha x^n+\beta y^m+\gamma z^k.$

General solution: $\displaystyle w=\frac{\alpha}{an}x^n+\frac{\beta}{bm}y^m+\frac{\gamma}{ck}z^k+\Phi\left(\frac{|y|^a}{|x|^b},\frac{|z|^a}{|x|^c}\right).$

28. $\displaystyle x\frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=cx^n.$

General solution: $\displaystyle w=\frac{c}{n}x^n+\Phi(u_1,u_2),$ where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}|x|^{\sqrt{ab}}\bigl(by-\sqrt{ab}\,z\bigr) & \text{if } ab>0,\\ |x|^{\sqrt{-ab}}\exp\left(-\arctan\dfrac{\sqrt{-ab}\,z}{by}\right) & \text{if } ab<0.\end{cases}$$

29. $\displaystyle abx\frac{\partial w}{\partial x}+b(ay+bz)\frac{\partial w}{\partial y}+a(ay-bz)\frac{\partial w}{\partial z}=cx^n.$

General solution: $\displaystyle w=\frac{c}{abn}x^n+\Phi(u_1,u_2),$ where

$$u_1=\bigl[ay+(\sqrt{2}-1)bz\bigr]|x|^{-\sqrt{2}},\quad u_2=\bigl[ay-(\sqrt{2}+1)bz\bigr]|x|^{\sqrt{2}}.$$

Particular solution: $\displaystyle w=\frac{c}{abn}x^n+\Phi\bigl(a^2y^2-2abyz-b^2z^2\bigr).$

30. $\dfrac{\partial w}{\partial x}+ax^ny^m\dfrac{\partial w}{\partial y}+bx^\nu y^\mu z^\lambda\dfrac{\partial w}{\partial z}=cx^k.$

General solution:

$$w=\Phi(u_1,u_2)+\begin{cases}\dfrac{c}{k+1}x^{k+1} & \text{if } k\neq -1,\\ c\ln|x| & \text{if } k=-1,\end{cases}$$

where u_1, u_2 is the integral basis of the homogeneous equation 2.1.1.64.

31. $\dfrac{\partial w}{\partial x}+\big(a_1x^{n_1}y+b_1x^{m_1}\big)\dfrac{\partial w}{\partial y}+\big(a_2x^{n_2}y+b_2x^{m_2}\big)\dfrac{\partial w}{\partial z}=c_2x^{k_2}y+c_1x^{k_1}z.$

This is a special case of equation 2.2.7.4 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1x^{m_1}$, $g_1(x)=a_2x^{n_2}$, $g_2(x)=b_2x^{m_2}$, $h_2(x)=c_2x^{k_2}$, $h_1(x)=c_1x^{k_1}$, and $h_0(x)=0$.

32. $\dfrac{\partial w}{\partial x}+\big(a_1x^{n_1}y+b_1x^{m_1}\big)\dfrac{\partial w}{\partial y}+\big(a_2x^{n_2}z+b_2x^{m_2}\big)\dfrac{\partial w}{\partial z}=c_2x^{k_2}y+c_1x^{k_1}z.$

This is a special case of equation 2.2.7.5 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1x^{m_1}$, $g_1(x)=a_2x^{n_2}$, $g_2(x)=b_2x^{m_2}$, $h_2(x)=c_2x^{k_2}$, $h_1(x)=c_1x^{k_1}$, and $h_0(x)=0$.

33. $\dfrac{\partial w}{\partial x}+\big(a_1x^{n_1}y+b_1y^k\big)\dfrac{\partial w}{\partial y}+\big(a_2x^{n_2}z+b_2z^m\big)\dfrac{\partial w}{\partial z}=cx^s.$

This is a special case of equation 2.2.7.7 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1$, $g_1(x)=a_2x^{n_2}$, $g_2(x)=b_2$, and $h(x)=cx^s$.

34. $\dfrac{\partial w}{\partial x}+\big(a_1x^{n_1}y+b_1y^k\big)\dfrac{\partial w}{\partial y}+\big(a_2y^{n_2}z+b_2z^m\big)\dfrac{\partial w}{\partial z}=c_1x^{s_1}+c_2y^{s_2}.$

This is a special case of equation 2.2.7.23 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=b_1$, $g_1(x,y)=a_2y^{n_2}$, $g_2(x,y)=b_2$, and $h(x,y,z)=c_1x^{s_1}+c_2y^{s_2}$.

35. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+a\sqrt{x^2+y^2}\,\dfrac{\partial w}{\partial z}=bx^n.$

General solution: $w=\Phi\big(u_1,\,u_2\big)+w_0(x)$, where

$$u_1=\frac{y}{x},\quad u_2=a\sqrt{x^2+y^2}-z,\quad w_0(x)=\begin{cases}(b/n)x^n & \text{if } n\neq 0,\\ b\ln|x| & \text{if } n=0.\end{cases}$$

36. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+\big(z-a\sqrt{x^2+y^2+z^2}\,\big)\dfrac{\partial w}{\partial z}=bx^n.$

General solution: $w=\Phi\big(u_1,\,u_2\big)+w_0(x)$, where

$$u_1=\frac{y}{x},\quad u_2=|x|^{a-1}\big(z+\sqrt{x^2+y^2+z^2}\,\big),\quad w_0(x)=\begin{cases}(b/n)x^n & \text{if } n\neq 0,\\ b\ln|x| & \text{if } n=0.\end{cases}$$

2.2.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\dfrac{\partial w}{\partial y} + be^{\beta x}\dfrac{\partial w}{\partial z} = ce^{\gamma x}.$

This is a special case of equation 2.2.7.1 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\beta x}$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = ce^{\gamma x}$.

2. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\dfrac{\partial w}{\partial y} + be^{\beta y}\dfrac{\partial w}{\partial z} = ce^{\gamma y} + se^{\mu z}.$

This is a special case of equation 2.2.7.22 with $f_1(x) = 0$, $f_2(x) = ae^{\lambda x}$, $g_1(x, y) = 0$, $g_2(x, y) = be^{\beta y}$, and $h(x, y, z) = ce^{\gamma y} + se^{\mu z}$.

3. $\dfrac{\partial w}{\partial x} + ae^{\lambda y}\dfrac{\partial w}{\partial y} + be^{\beta z}\dfrac{\partial w}{\partial z} = ce^{\gamma x} + se^{\mu z}.$

This is a special case of equation 2.2.7.26 with $f_1(x) = 0$, $f_2(x) = a$, $g_1(x, y) = be^{\beta y}$, $g_2(x, y) = 0$, and $h(x, y, z) = ce^{\gamma x} + se^{\mu z}$.

4. $\dfrac{\partial w}{\partial x} + \big(A_1 e^{\alpha_1 x} + B_1 e^{\nu_1 x + \lambda y}\big)\dfrac{\partial w}{\partial y} + \big(A_2 e^{\alpha_2 x} + B_2 e^{\nu_2 x + \beta z}\big)\dfrac{\partial w}{\partial z} = ke^{\gamma z}.$

This is a special case of equation 2.2.7.9 with $f_1(x) = A_1 e^{\alpha_1 x}$, $f_2(x) = B_1 e^{\nu_1 x}$, $g_1(x) = A_2 e^{\alpha_2 x}$, $g_2(x) = B_2 e^{\nu_2 x}$, and $h(x) = ke^{\gamma z}$.

5. $ae^{\alpha x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} + ce^{\gamma z}\dfrac{\partial w}{\partial z} = ke^{\lambda x}.$

General solution:

$$w = \begin{cases} \Phi(u_1, u_2) + \dfrac{ke^{(\lambda-\alpha)x}}{a(\lambda-\alpha)} & \text{if } \lambda \neq \alpha, \\ \Phi(u_1, u_2) + \dfrac{k}{a}x & \text{if } \lambda = \alpha, \end{cases}$$

where $u_1 = -\dfrac{1}{a\alpha}e^{-\alpha x} + \dfrac{1}{b\beta}e^{-\beta y}$ and $u_2 = -\dfrac{1}{b\beta}e^{-\beta y} + \dfrac{1}{c\gamma}e^{-\gamma z}$.

6. $ae^{\beta y}\dfrac{\partial w}{\partial x} + be^{\alpha x}\dfrac{\partial w}{\partial y} + ce^{\gamma z}\dfrac{\partial w}{\partial z} = ke^{\lambda x}.$

General solution:

$$w = \begin{cases} \Phi(u_1, u_2) + \dfrac{k\alpha}{b\beta}\displaystyle\int \dfrac{e^{\lambda x}\,dx}{e^{\alpha x} + a\alpha u_1} & \text{if } \lambda \neq \alpha,\ \lambda \neq 0; \\ \Phi(u_1, u_2) + \dfrac{k}{b}y & \text{if } \lambda = \alpha \neq 0; \\ \Phi(u_1, u_2) - \dfrac{k}{c\gamma}e^{-\gamma z} & \text{if } \lambda = 0, \end{cases}$$

where $u_1 = -\dfrac{1}{a\alpha}e^{\alpha x} + \dfrac{1}{b\beta}e^{\beta y}$ and $u_2 = \dfrac{\beta y - \alpha x}{b\beta e^{\alpha x} - a\alpha e^{\beta y}} + \dfrac{1}{c\gamma}e^{-\gamma z}$. In the integration, u_1 is considered a parameter.

7. $\displaystyle (a_1+a_2e^{\alpha x})\frac{\partial w}{\partial x}+(b_1+b_2e^{\beta y})\frac{\partial w}{\partial y}+(c_1+c_2e^{\gamma z})\frac{\partial w}{\partial z}=k_1+k_2e^{\alpha x}.$

General solution: $w=\Phi(u_1,u_2)+\dfrac{k_1}{a_1}x+\dfrac{1}{\alpha}\left(\dfrac{k_2}{a_2}-\dfrac{k_1}{a_1}\right)\ln(a_1+a_2e^{\alpha x})$, where

$$u_1=\frac{1}{a_1\alpha}\big[\alpha x-\ln(a_1+a_2e^{\alpha x})\big]-\frac{1}{b_1\beta}\big[\beta y-\ln(b_1+b_2e^{\beta y})\big],$$
$$u_2=\frac{1}{a_1\alpha}\big[\alpha x-\ln(a_1+a_2e^{\alpha x})\big]-\frac{1}{c_1\gamma}\big[\gamma z-\ln(c_1+c_2e^{\gamma z})\big].$$

8. $\displaystyle e^{\beta y}(a_1+a_2e^{\alpha x})\frac{\partial w}{\partial x}+e^{\alpha x}(b_1+b_2e^{\beta y})\frac{\partial w}{\partial y}+ce^{\beta y+\gamma z}\frac{\partial w}{\partial z}=k_3e^{\beta y}(k_1+k_2e^{\alpha x}).$

General solution: $w=\Phi(u_1,u_2)+\dfrac{k_1k_3}{a_1}x+\dfrac{k_3}{\alpha}\left(\dfrac{k_2}{a_2}-\dfrac{k_1}{a_1}\right)\ln(a_1+a_2e^{\alpha x})$, where

$$u_1=\frac{1}{a_2\alpha}\ln(a_1+a_2e^{\alpha x})-\frac{1}{b_1\beta}\ln(b_1+b_2e^{\beta y}),\quad u_2=\frac{1}{a_1\alpha}\big[\alpha x-\ln(a_1+a_2e^{\alpha x})\big]+\frac{1}{c\gamma}e^{-\gamma z}.$$

► **Coefficients of equations contain exponential and power-law functions.**

9. $\displaystyle \frac{\partial w}{\partial x}+ax^n\frac{\partial w}{\partial y}+bx^m\frac{\partial w}{\partial z}=ce^{\lambda x}y+ke^{\beta x}z+se^{\gamma x}.$

This is a special case of equation 2.2.7.1 with $f(x)=ax^n$, $g(x)=bx^m$, $h_2(x)=ce^{\lambda x}$, $h_1(x)=ke^{\beta x}$, and $h_0(x)=se^{\gamma x}$.

10. $\displaystyle \frac{\partial w}{\partial x}+ae^{\lambda x}\frac{\partial w}{\partial y}+bx^m\frac{\partial w}{\partial z}=cx^ny+ke^{\beta x}z+se^{\gamma x}.$

This is a special case of equation 2.2.7.1 with $f(x)=ae^{\lambda x}$, $g(x)=bx^m$, $h_2(x)=cx^n$, $h_1(x)=ke^{\beta x}$, and $h_0(x)=se^{\gamma x}$.

11. $\displaystyle \frac{\partial w}{\partial x}+ae^{\lambda x}\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=ke^{\beta x}z+se^{\gamma x}.$

This is a special case of equation 2.2.7.4 with $f_1(x)=0$, $f_2(x)=ae^{\lambda x}$, $g_1(x)=b$, $g_2(x)=h_2(x)=0$, $h_1(x)=ke^{\beta x}$, and $h_0(x)=se^{\gamma x}$.

12. $\displaystyle \frac{\partial w}{\partial x}+ay^n\frac{\partial w}{\partial y}+bz^m\frac{\partial w}{\partial z}=ce^{\lambda x}+ke^{\beta y}+se^{\gamma z}.$

This is a special case of equation 2.2.7.10 with $f(x)=1$, $g(y)=ay^n$, $h(z)=bz^m$, $\varphi(x)=ce^{\lambda x}$, $\psi(y)=ke^{\beta y}$, and $\chi(z)=se^{\gamma z}$.

13. $\displaystyle \frac{\partial w}{\partial x}+ae^{\beta y}\frac{\partial w}{\partial y}+bz^m\frac{\partial w}{\partial z}=ce^{\lambda x}+ky^n+se^{\gamma z}.$

This is a special case of equation 2.2.7.10 with $f(x)=1$, $g(y)=ae^{\beta y}$, $h(z)=bz^m$, $\varphi(x)=ce^{\lambda x}$, $\psi(y)=ky^n$, and $\chi(z)=se^{\gamma z}$.

14. $\dfrac{\partial w}{\partial x} + \big(ae^{\alpha x}y^2 + be^{-\alpha x}\big)\dfrac{\partial w}{\partial y} + \big[de^{\beta x}z^2 + ce^{\gamma x}(\gamma - cde^{(\beta+\gamma)x})\big]\dfrac{\partial w}{\partial z} = ke^{\lambda x}.$

General solution: $w = \dfrac{k}{\lambda}e^{\lambda x} + \Phi(u_1, u_2)$, where u_1, u_2 is an integral basis of the homogeneous equation 2.1.2.15.

15. $\dfrac{\partial w}{\partial x} + \big(a_1e^{\lambda_1 x}y + b_1e^{\beta_1 x}y^k\big)\dfrac{\partial w}{\partial y} + \big(a_2e^{\lambda_2 x}z + b_2e^{\beta_2 x}z^m\big)\dfrac{\partial w}{\partial z} = cx^s.$

This is a special case of equation 2.2.7.7 with $f_1(x) = a_1e^{\lambda_1 x}$, $f_2(x) = b_1e^{\beta_1 x}$, $g_1(x) = a_2e^{\lambda_2 x}$, $g_2(x) = b_2e^{\beta_2 x}$, and $h(x) = cx^s$.

16. $\dfrac{\partial w}{\partial x} + \big(a_1e^{\beta_1 x}y + b_1e^{\gamma_1 x}y^k\big)\dfrac{\partial w}{\partial y} + \big(a_2e^{\beta_2 x} + b_2e^{\gamma_2 x + \lambda z}\big)\dfrac{\partial w}{\partial z} = cx^s.$

This is a special case of equation 2.2.7.8 with $f_1(x) = a_1e^{\beta_1 x}$, $f_2(x) = b_1e^{\gamma_1 x}$, $g_1(x) = a_2e^{\beta_2 x}$, $g_2(x) = b_2e^{\gamma_2 x}$, and $h(x) = cx^s$.

17. $\dfrac{\partial w}{\partial x} + \big(a_1x^n + b_1x^m e^{\lambda y}\big)\dfrac{\partial w}{\partial y} + \big(a_2x^k + b_2x^l e^{\beta z}\big)\dfrac{\partial w}{\partial z} = cx^s.$

This is a special case of equation 2.2.7.9 with $f_1(x) = a_1x^n$, $f_2(x) = b_1x^m$, $g_1(x) = a_2x^k$, $g_2(x) = b_2x^l$, and $h(x) = cx^s$.

2.2.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\sinh^k(\lambda x) + s.$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \sinh^k(\lambda x)\,dx + sx.$

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\sinh(\lambda x)\dfrac{\partial w}{\partial z} = k\sinh(\beta y) + s\sinh(\gamma z).$

General solution:

$$w = \Phi(u_1, u_2) + \frac{k}{b\beta}\cosh(\beta y) - \frac{s}{a}\int_0^x \sinh\left[\frac{c\gamma}{a\lambda}\big(\cosh(\lambda x) - \cosh(\lambda t)\big) - \gamma z\right] dt,$$

where $u_1 = bx - ay$ and $u_2 = a\lambda z - c\cosh(\lambda x)$.

3. $\dfrac{\partial w}{\partial x} + a\sinh^n(\beta x)\dfrac{\partial w}{\partial y} + b\sinh^k(\lambda x)\dfrac{\partial w}{\partial z} = c\sinh^m(\gamma x) + s.$

This is a special case of equation 2.2.7.1 with $f(x) = a\sinh^n(\beta x)$, $g(x) = b\sinh^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\sinh^m(\gamma x) + s$.

4. $a\dfrac{\partial w}{\partial x} + b\sinh(\beta y)\dfrac{\partial w}{\partial y} + c\sinh(\lambda x)\dfrac{\partial w}{\partial z} = k\sinh(\gamma z).$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a}\displaystyle\int_0^x \sinh\left\{\gamma z + \frac{c\gamma}{a\lambda}\big[\cosh(\lambda x) - \cosh(\lambda t)\big]\right\} dt,$
where $u_1 = b\beta x - a\ln\left|\tanh\dfrac{\beta y}{2}\right|$ and $u_2 = a\lambda z - c\cosh(\lambda x)$.

5. $$a_1 \sinh^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x} + b_1 \sinh^{m_1}(\beta_1 y)\frac{\partial w}{\partial y} + c_1 \sinh^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z} = a_2 \sinh^{n_2}(\lambda_2 x) + b_2 \sinh^{m_2}(\beta_2 y) + c_2 \sinh^{k_2}(\gamma_2 z).$$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1 \sinh^{n_1}(\lambda_1 x)$, $g(y) = b_1 \sinh^{m_1}(\beta_1 y)$, $h(z) = c_1 \sinh^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2 \sinh^{n_2}(\lambda_2 x)$, $\psi(y) = b_2 \sinh^{m_2}(\beta_2 y)$, and $\chi(z) = c_2 \sinh^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain hyperbolic cosine.**

6. $$\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} + b\frac{\partial w}{\partial z} = c\cosh^k(\lambda x) + s.$$

General solution: $w = \Phi(y - ax, z - bx) + c\int \cosh^k(\lambda x)\,dx + sx.$

7. $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + c\cosh(\lambda x)\frac{\partial w}{\partial z} = k\cosh(\beta y) + s\cosh(\gamma z).$$

General solution:

$$w = \Phi(u_1, u_2) + \frac{k}{b\beta}\sinh(\beta y) + \frac{s}{a}\int_0^x \cosh\left[\frac{c\gamma}{a\lambda}\big(\cosh(\lambda t) - \cosh(\lambda x)\big) + \gamma z\right] dt,$$

where $u_1 = bx - ay$ and $u_2 = a\lambda z - c\sinh(\lambda x)$.

8. $$\frac{\partial w}{\partial x} + a\cosh^n(\beta x)\frac{\partial w}{\partial y} + b\cosh^k(\lambda x)\frac{\partial w}{\partial z} = c\cosh^m(\gamma x) + s.$$

This is a special case of equation 2.2.7.1 with $f(x) = a\cosh^n(\beta x)$, $g(x) = b\cosh^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\cosh^m(\gamma x) + s$.

9. $$a\frac{\partial w}{\partial x} + b\cosh(\beta y)\frac{\partial w}{\partial y} + c\cosh(\lambda x)\frac{\partial w}{\partial z} = k\cosh(\gamma z).$$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a}\displaystyle\int_0^x \cosh\left\{\gamma z + \frac{c\gamma}{a\lambda}\big[\sinh(\lambda t) - \sinh(\lambda x)\big]\right\} dt,$

where $u_1 = b\beta x - 2a\arctan\left|\tanh\dfrac{\beta y}{2}\right|$ and $u_2 = a\lambda z - c\sinh(\lambda x)$.

10. $$a\frac{\partial w}{\partial x} + b\cosh(\beta y)\frac{\partial w}{\partial y} + c\cosh(\gamma z)\frac{\partial w}{\partial z} = p\cosh(\lambda x) + q.$$

General solution: $w = \Phi(u_1, u_2) + \dfrac{q}{a}x + \dfrac{p}{a\lambda}\sinh(\lambda x)$, where $u_1 = b\beta x - 2a\arctan\left|\tanh\dfrac{\beta y}{2}\right|$ and $u_2 = c\gamma x - 2a\arctan\left|\tanh\dfrac{\gamma z}{2}\right|$.

11. $$a_1 \cosh^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x} + b_1 \cosh^{m_1}(\beta_1 y)\frac{\partial w}{\partial y} + c_1 \cosh^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z} = a_2 \cosh^{n_2}(\lambda_2 x) + b_2 \cosh^{m_2}(\beta_2 y) + c_2 \cosh^{k_2}(\gamma_2 z).$$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1 \cosh^{n_1}(\lambda_1 x)$, $g(y) = b_1 \cosh^{m_1}(\beta_1 y)$, $h(z) = c_1 \cosh^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2 \cosh^{n_2}(\lambda_2 x)$, $\psi(y) = b_2 \cosh^{m_2}(\beta_2 y)$, and $\chi(z) = c_2 \cosh^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain hyperbolic tangent.**

12. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\tanh^k(\lambda x)+s.$

General solution: $w=\Phi(y-ax,z-bx)+c\displaystyle\int\tanh^k(\lambda x)\,dx+sx.$

13. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\tanh(\gamma z)\dfrac{\partial w}{\partial z}=k\tanh(\lambda x)+s\tanh(\beta y).$

General solution: $w=\Phi(u_1,u_2)+\dfrac{k}{a\lambda}\ln\bigl[\cosh(\lambda x)\bigr]+\dfrac{s}{b\beta}\ln\bigl[\cosh(\beta y)\bigr]$, where $u_1=bx-ay$ and $u_2=c\gamma x-a\ln\bigl|\sinh(\gamma z)\bigr|$.

14. $\dfrac{\partial w}{\partial x}+a\tanh^n(\beta x)\dfrac{\partial w}{\partial y}+b\tanh^k(\lambda x)\dfrac{\partial w}{\partial z}=c\tanh^m(\gamma x)+s.$

This is a special case of equation 2.2.7.1 with $f(x)=a\tanh^n(\beta x)$, $g(x)=b\tanh^k(\lambda x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=c\tanh^m(\gamma x)+s$.

15. $a\dfrac{\partial w}{\partial x}+b\tanh(\beta y)\dfrac{\partial w}{\partial y}+c\tanh(\lambda x)\dfrac{\partial w}{\partial z}=k\tanh(\gamma z).$

General solution:

$$w=\Phi(u_1,u_2)+\frac{k}{a}\int_0^x\tanh\biggl\{\gamma z+\frac{c\gamma}{a\lambda}\bigl[\ln|\cosh(\lambda x)|-\ln|\cosh(\lambda t)|\bigr]\biggr\}\,dt,$$

where $u_1=b\beta x-a\ln\bigl|\sinh(\beta y)\bigr|$ and $u_2=a\lambda z-c\ln\bigl[\cosh(\lambda x)\bigr]$.

16. $a\dfrac{\partial w}{\partial x}+b\tanh(\beta y)\dfrac{\partial w}{\partial y}+c\tanh(\gamma z)\dfrac{\partial w}{\partial z}=k\tanh(\lambda x).$

General solution: $w=\Phi(u_1,u_2)+\dfrac{k}{a\lambda}\ln\bigl[\cosh(\lambda x)\bigr]$, where $u_1=b\beta x-a\ln\bigl|\sinh(\beta y)\bigr|$ and $u_2=c\gamma x-a\ln\bigl|\sinh(\gamma z)\bigr|$.

17. $a\tanh(\lambda x)\dfrac{\partial w}{\partial x}+b\tanh(\beta y)\dfrac{\partial w}{\partial y}+c\tanh(\gamma z)\dfrac{\partial w}{\partial z}=k.$

General solution:

$$w=\Phi(u_1,u_2)+\frac{k}{a\lambda}\ln\bigl|\sinh(\lambda x)\bigr|,\qquad u_1=\frac{|\sinh(\beta y)|^{a\lambda}}{|\sinh(\lambda x)|^{b\beta}},\quad u_2=\frac{|\sinh(\gamma z)|^{a\lambda}}{|\sinh(\lambda x)|^{c\gamma}}.$$

18. $a_1\tanh^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_1\tanh^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y}+c_1\tanh^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$=a_2\tanh^{n_2}(\lambda_2 x)+b_2\tanh^{m_2}(\beta_2 y)+c_2\tanh^{k_2}(\gamma_2 z).$

This is a special case of equation 2.2.7.10 in which $f(x)=a_1\tanh^{n_1}(\lambda_1 x)$, $g(y)=b_1\tanh^{m_1}(\beta_1 y)$, $h(z)=c_1\tanh^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\tanh^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\tanh^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\tanh^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain hyperbolic cotangent.**

19. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\coth^k(\lambda x) + s.$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \coth^k(\lambda x)\,dx + sx.$

20. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\coth(\gamma z)\dfrac{\partial w}{\partial z} = k\coth(\lambda x) + s\coth(\beta y).$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a\lambda}\ln\bigl|\sinh(\lambda x)\bigr| + \dfrac{s}{b\beta}\ln\bigl|\sinh(\beta y)\bigr|$, where $u_1 = bx - ay$ and $u_2 = c\gamma x - a\ln\bigl[\cosh(\gamma z)\bigr]$.

21. $\dfrac{\partial w}{\partial x} + a\coth^n(\beta x)\dfrac{\partial w}{\partial y} + b\coth^k(\lambda x)\dfrac{\partial w}{\partial z} = c\coth^m(\gamma x) + s.$

This is a special case of equation 2.2.7.1 with $f(x) = a\coth^n(\beta x)$, $g(x) = b\coth^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\coth^m(\gamma x) + s$.

22. $a\dfrac{\partial w}{\partial x} + b\coth(\beta y)\dfrac{\partial w}{\partial y} + c\coth(\lambda x)\dfrac{\partial w}{\partial z} = k\coth(\gamma z).$

General solution:

$$w = \Phi(u_1, u_2) + \frac{k}{a}\int_0^x \coth\left\{\gamma z + \frac{c\gamma}{a\lambda}\bigl[\ln|\sinh(\lambda t)| - \ln|\sinh(\lambda x)|\bigr]\right\}dt,$$

where $u_1 = b\beta x - a\ln\bigl[\cosh(\beta y)\bigr]$ and $u_2 = a\lambda z - c\ln\bigl|\sinh(\lambda x)\bigr|$.

23. $a\dfrac{\partial w}{\partial x} + b\coth(\beta y)\dfrac{\partial w}{\partial y} + c\coth(\gamma z)\dfrac{\partial w}{\partial z} = k\coth(\lambda x).$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a\lambda}\ln\bigl|\sinh(\lambda x)\bigr|$, where $u_1 = b\beta x - a\ln\bigl[\cosh(\beta y)\bigr]$ and $u_2 = c\gamma x - a\ln\bigl[\cosh(\gamma z)\bigr]$.

24. $a_1\coth^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\coth^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\coth^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= a_2\coth^{n_2}(\lambda_2 x) + b_2\coth^{m_2}(\beta_2 y) + c_2\coth^{k_2}(\gamma_2 z).$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1\coth^{n_1}(\lambda_1 x)$, $g(y) = b_1\coth^{m_1}(\beta_1 y)$, $h(z) = c_1\coth^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2\coth^{n_2}(\lambda_2 x)$, $\psi(y) = b_2\coth^{m_2}(\beta_2 y)$, and $\chi(z) = c_2\coth^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain different hyperbolic functions.**

25. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\sinh^n(\lambda y)\dfrac{\partial w}{\partial z} = s\cosh^m(\beta x) + k\sinh^l(\gamma y).$

This is a special case of equation 2.2.7.18 with $f(x,y) = c\sinh^n(\lambda y)$ and $g(x,y) = s\cosh^m(\beta x) + k\sinh^l(\gamma y)$.

26. $\displaystyle \frac{\partial w}{\partial x} + a\sinh^n(\lambda x)\frac{\partial w}{\partial y} + b\cosh^m(\beta x)\frac{\partial w}{\partial z} = s\cosh^k(\gamma x).$

This is a special case of equation 2.2.7.1 with $f(x) = a\sinh^n(\lambda x)$, $g(x) = b\cosh^m(\beta x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = s\cosh^k(\gamma x)$.

27. $\displaystyle \frac{\partial w}{\partial x} + a\cosh^n(\lambda x)\frac{\partial w}{\partial y} + b\sinh^m(\beta y)\frac{\partial w}{\partial z} = s\sinh^k(\gamma z).$

This is a special case of equation 2.2.7.22 with $f_1(x) = 0$, $f_2(x) = a\cosh^n(\lambda x)$, $g_1(x,y) = 0$, $g_2(x,y) = b\sinh^m(\beta y)$, and $h(x,y,z) = s\sinh^k(\gamma z)$.

28. $\displaystyle \frac{\partial w}{\partial x} + a\tanh^n(\lambda x)\frac{\partial w}{\partial y} + b\coth^m(\beta x)\frac{\partial w}{\partial z} = s\coth^k(\gamma x).$

This is a special case of equation 2.2.7.1 with $f(x) = a\tanh^n(\lambda x)$, $g(x) = b\coth^m(\beta x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = s\coth^k(\gamma x)$.

29. $\displaystyle a\sinh(\lambda x)\frac{\partial w}{\partial x} + b\sinh(\beta y)\frac{\partial w}{\partial y} + c\sinh(\gamma z)\frac{\partial w}{\partial z} = k\cosh(\lambda x).$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a\lambda}\ln|\sinh(\lambda x)|$, where

$$u_1 = \frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right| - \frac{1}{b\beta}\ln\left|\tanh\frac{\beta y}{2}\right|, \quad u_2 = \frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right| - \frac{1}{c\gamma}\ln\left|\tanh\frac{\gamma z}{2}\right|.$$

2.2.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} + b\frac{\partial w}{\partial z} = c\ln^k(\lambda x) + s.$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \ln^k(\lambda x)\,dx + sx.$

2. $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + c\ln(\beta y)\ln(\gamma z)\frac{\partial w}{\partial z} = k\ln(\alpha x).$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a}x\big[\ln(\lambda x) - 1\big]$, where

$$u_1 = bx - ay, \quad u_2 = cy\big[1 - \ln(\beta y)\big] + b\int \frac{dz}{\ln(\gamma z)}.$$

3. $\displaystyle \frac{\partial w}{\partial x} + a\ln^n(\beta x)\frac{\partial w}{\partial y} + b\ln^k(\lambda x)\frac{\partial w}{\partial z} = c\ln^m(\gamma x) + s.$

This is a special case of equation 2.2.7.1 with $f(x) = a\ln^n(\beta x)$, $g(x) = b\ln^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\ln^m(\gamma x) + s$.

4. $\displaystyle \frac{\partial w}{\partial x} + a\ln^n(\lambda x)\frac{\partial w}{\partial y} + b\ln^m(\beta y)\frac{\partial w}{\partial z} = c\ln^k(\gamma y) + s\ln^l(\mu z).$

This is a special case of equation 2.2.7.22 with $f_1(x) = 0$, $f_2(x) = a\ln^n(\lambda x)$, $g_1(x,y) = 0$, $g_2(x,y) = b\ln^m(\beta y)$, and $h(x,y,z) = c\ln^k(\gamma y) + s\ln^l(\mu z)$.

5. $$a_1 \ln^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x} + b_1 \ln^{m_1}(\beta_1 y)\frac{\partial w}{\partial y} + c_1 \ln^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z} = a_2 \ln^{n_2}(\lambda_2 x) + b_2 \ln^{m_2}(\beta_2 y) + c_2 \ln^{k_2}(\gamma_2 z).$$

This is a special case of equation 2.2.7.10 with $f(x) = a_1 \ln^{n_1}(\lambda_1 x)$, $g(y) = b_1 \ln^{m_1}(\beta_1 y)$, $h(z) = c_1 \ln^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2 \ln^{n_2}(\lambda_2 x)$, $\psi(y) = b_2 \ln^{m_2}(\beta_2 y)$, and $\chi(z) = c_2 \ln^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain logarithmic and power-law functions.**

6. $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + cx^n \ln^k(\lambda y)\frac{\partial w}{\partial z} = sy^m \ln^l(\beta x).$$

This is a special case of equation 2.2.7.18 with $f(x,y) = cx^n \ln^k(\lambda y)$, and $g(x,y) = sy^m \ln^l(\beta x)$.

7. $$\frac{\partial w}{\partial x} + ax^n\frac{\partial w}{\partial y} + bx^m\frac{\partial w}{\partial z} = cy \ln^k(\lambda x) + sz \ln^l(\beta x).$$

This is a special case of equation 2.2.7.1 with $f(x) = ax^n$, $g(x) = bx^m$, $h_2(x) = c \ln^k(\lambda x)$, $h_1(x) = s \ln^l(\beta x)$, and $h_0(x) = 0$.

8. $$\frac{\partial w}{\partial x} + a \ln^n(\lambda x)\frac{\partial w}{\partial y} + by^m\frac{\partial w}{\partial z} = c \ln^k(\beta x) + s \ln^l(\gamma z).$$

This is a special case of equation 2.2.7.22 with $f_1(x) = 0$, $f_2(x) = a \ln^n(\lambda x)$, $g_1(x,y) = 0$, $g_2(x,y) = by^m$, and $h(x,y,z) = c \ln^k(\beta x) + s \ln^l(\gamma z)$.

9. $$a \ln^n(\lambda x)\frac{\partial w}{\partial x} + z\frac{\partial w}{\partial y} + b \ln^k(\beta y)\frac{\partial w}{\partial z} = cx^m + s \ln(\gamma y).$$

This is a special case of equation 2.2.7.11 with $f(x) = a \ln^n(\lambda x)$, $g(y) = b \ln^k(\beta y)$, $h_2(x) = cx^m$, and $h_1(y) = s \ln(\gamma y)$.

10. $$ax(\ln x)^n\frac{\partial w}{\partial x} + by(\ln y)^m\frac{\partial w}{\partial y} + cz(\ln z)^l\frac{\partial w}{\partial z} = k(\ln x)^s.$$

General solution:
$$w = \Phi(u_1,\, u_2) + \begin{cases} \dfrac{k}{a(s-n+1)}(\ln x)^{s-n+1} & \text{if } s+1 \neq n, \\ \dfrac{k}{a} \ln|\ln x| & \text{if } s+1 = n, \end{cases}$$
where
$$u_1 = \frac{(\ln x)^{1-n}}{a(n-1)} - \frac{(\ln y)^{1-m}}{b(m-1)}, \quad u_2 = \frac{(\ln x)^{1-n}}{a(n-1)} - \frac{(\ln z)^{1-l}}{c(l-1)}.$$

2.2.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $$\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} + b\frac{\partial w}{\partial z} = c \sin^k(\lambda x) + s.$$

General solution: $w = \Phi(y - ax, z - bx) + c\int \sin^k(\lambda x)\,dx + sx.$

2. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\sin(\gamma z)\dfrac{\partial w}{\partial z} = k\sin(\alpha x) + s\sin(\beta y).}$

General solution: $w = \Phi(u_1, u_2) - \dfrac{k}{a\alpha}\cos(\alpha x) - \dfrac{s}{b\beta}\cos(\beta y)$, where

$$u_1 = bx - ay, \quad u_2 = c\gamma x - a\ln\left|\tan\frac{\gamma z}{2}\right|.$$

3. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\sin^n(\lambda x)\dfrac{\partial w}{\partial y} + b\sin^m(\beta x)\dfrac{\partial w}{\partial z} = c\sin^k(\gamma x).}$

This is a special case of equation 2.2.7.1 with $f(x) = a\sin^n(\lambda x)$, $g(x) = b\sin^m(\beta x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\sin^k(\gamma x)$.

4. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\sin^n(\lambda x)\dfrac{\partial w}{\partial y} + b\sin^m(\beta y)\dfrac{\partial w}{\partial z} = c\sin^k(\gamma y) + s\sin^l(\mu z).}$

This is a special case of equation 2.2.7.22 with $f_1(x)=0$, $f_2(x)=a\sin^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y) = b\sin^m(\beta y)$, and $h(x,y,z) = c\sin^k(\gamma y) + s\sin^l(\mu z)$.

5. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sin(\beta y)\dfrac{\partial w}{\partial y} + c\sin(\lambda x)\dfrac{\partial w}{\partial z} = k\sin(\gamma z).}$

General solution:

$$w = \Phi(u_1, u_2) + \frac{k}{a}\int_0^x \sin\left\{\gamma z + \frac{c\gamma}{a\lambda}\big[\cos(\lambda x) - \cos(\lambda t)\big]\right\} dt,$$

where $u_1 = b\beta x - a\ln\left|\tan\dfrac{\beta y}{2}\right|$, $u_2 = a\lambda z + c\cos(\lambda x)$.

6. $\boldsymbol{a_1\sin^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\sin^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\sin^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}}$

$$\boldsymbol{= a_2\sin^{n_2}(\lambda_2 x) + b_2\sin^{m_2}(\beta_2 y) + c_2\sin^{k_2}(\gamma_2 z).}$$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1\sin^{n_1}(\lambda_1 x)$, $g(y) = b_1\sin^{m_1}(\beta_1 y)$, $h(z) = c_1\sin^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2\sin^{n_2}(\lambda_2 x)$, $\psi(y) = b_2\sin^{m_2}(\beta_2 y)$, and $\chi(z) = c_2\sin^{k_2}(\gamma_2 z)$.

► Coefficients of equations contain cosine.

7. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\cos^k(\lambda x) + s.}$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \cos^k(\lambda x)\,dx + sx.$

8. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\cos(\beta z)\dfrac{\partial w}{\partial z} = k\cos(\lambda x) + s\cos(\gamma y).}$

General solution: $w = \Phi(u_1, u_2) + \dfrac{k}{a\lambda}\sin(\lambda x) + \dfrac{s}{b\gamma}\sin(\gamma y)$, where

$$u_1 = bx - ay, \quad u_2 = c\beta x - a\ln\left|\sec(\beta z) + \tan(\beta z)\right|.$$

9. $\dfrac{\partial w}{\partial x} + a\cos^n(\beta x)\dfrac{\partial w}{\partial y} + b\cos^k(\lambda x)\dfrac{\partial w}{\partial z} = c\cos^m(\gamma x) + s.$

This is a special case of equation 2.2.7.1 with $f(x) = a\cos^n(\beta x)$, $g(x) = b\cos^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\cos^m(\gamma x) + s$.

10. $\dfrac{\partial w}{\partial x} + a\cos^n(\lambda x)\dfrac{\partial w}{\partial y} + b\cos^m(\beta y)\dfrac{\partial w}{\partial z} = c\cos^k(\gamma y) + s\cos^l(\mu z).$

This is a special case of equation 2.2.7.22 with $f_1(x)=0$, $f_2(x)=a\cos^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y) = b\cos^m(\beta y)$, and $h(x,y,z) = c\cos^k(\gamma y) + s\cos^l(\mu z)$.

11. $a\dfrac{\partial w}{\partial x} + b\cos(\beta y)\dfrac{\partial w}{\partial y} + c\cos(\lambda x)\dfrac{\partial w}{\partial z} = k\cos(\gamma z).$

General solution:

$$w = \Phi(u_1, u_2) + \frac{k}{a}\int_0^x \cos\left\{\gamma z + \frac{c\gamma}{a\lambda}\bigl[\sin(\lambda t) - \sin(\lambda x)\bigr]\right\} dt,$$

where $u_1 = b\beta x - a\ln\bigl|\sec(\beta y) + \tan(\beta y)\bigr|$, $u_2 = a\lambda z - c\sin(\lambda x)$.

12. $a_1\cos^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\cos^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\cos^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= a_2\cos^{n_2}(\lambda_2 x) + b_2\cos^{m_2}(\beta_2 y) + c_2\cos^{k_2}(\gamma_2 z).$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1\cos^{n_1}(\lambda_1 x)$, $g(y) = b_1\cos^{m_1}(\beta_1 y)$, $h(z) = c_1\cos^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2\cos^{n_2}(\lambda_2 x)$, $\psi(y) = b_2\cos^{m_2}(\beta_2 y)$, and $\chi(z) = c_2\cos^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain tangent.**

13. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\tan^k(\lambda x) + s.$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \tan^k(\lambda x)\,dx + sx.$

14. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\tan(\beta z)\dfrac{\partial w}{\partial z} = k\tan(\lambda x) + s\tan(\gamma y).$

General solution:

$$w = \Phi(u_1, u_2) - \frac{k}{a\lambda}\ln\bigl|\cos(\lambda x)\bigr| - \frac{s}{b\gamma}\ln\bigl|\cos(\gamma y)\bigr|,$$

where $u_1 = bx - ay$ and $u_2 = c\beta x - a\ln\bigl|\sin(\beta z)\bigr|$.

15. $\dfrac{\partial w}{\partial x} + a\tan^n(\beta x)\dfrac{\partial w}{\partial y} + b\tan^k(\lambda x)\dfrac{\partial w}{\partial z} = c\tan^m(\gamma x) + s.$

This is a special case of equation 2.2.7.1 with $f(x) = a\tan^n(\beta x)$, $g(x) = b\tan^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\tan^m(\gamma x) + s$.

16. $\dfrac{\partial w}{\partial x}+a\tan^n(\lambda x)\dfrac{\partial w}{\partial y}+b\tan^m(\beta y)\dfrac{\partial w}{\partial z}=c\tan^k(\gamma y)+s\tan^l(\mu z).$

This is a special case of equation 2.2.7.22 with $f_1(x)=0$, $f_2(x)=a\tan^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y)=b\tan^m(\beta y)$, and $h(x,y,z)=c\tan^k(\gamma y)+s\tan^l(\mu z)$.

17. $a_1\tan^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_1\tan^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y}+c_1\tan^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$=a_2\tan^{n_2}(\lambda_2 x)+b_2\tan^{m_2}(\beta_2 y)+c_2\tan^{k_2}(\gamma_2 z).$

This is a special case of equation 2.2.7.10 in which $f(x)=a_1\tan^{n_1}(\lambda_1 x)$, $g(y)=b_1\tan^{m_1}(\beta_1 y)$, $h(z)=c_1\tan^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\tan^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\tan^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\tan^{k_2}(\gamma_2 z)$.

► Coefficients of equations contain cotangent.

18. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\cot^k(\lambda x)+s.$

General solution: $w=\Phi(y-ax,z-bx)+c\displaystyle\int\cot^k(\lambda x)\,dx+sx.$

19. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\cot(\gamma z)\dfrac{\partial w}{\partial z}=k\cot(\lambda x)+s\cot(\beta y).$

General solution:

$$w=\Phi(u_1,u_2)+\frac{k}{a\lambda}\ln\bigl|\sin(\lambda x)\bigr|+\frac{s}{b\beta}\ln\bigl|\sin(\beta y)\bigr|,$$

where $u_1=bx-ay$, $u_2=c\gamma x+a\ln\bigl|\cos(\gamma z)\bigr|$.

20. $\dfrac{\partial w}{\partial x}+a\cot^n(\beta x)\dfrac{\partial w}{\partial y}+b\cot^k(\lambda x)\dfrac{\partial w}{\partial z}=c\cot^m(\gamma x)+s.$

This is a special case of equation 2.2.7.1 with $f(x)=a\cot^n(\beta x)$, $g(x)=b\cot^k(\lambda x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=c\cot^m(\gamma x)+s$.

21. $\dfrac{\partial w}{\partial x}+a\cot^n(\lambda x)\dfrac{\partial w}{\partial y}+b\cot^m(\beta y)\dfrac{\partial w}{\partial z}=c\cot^k(\gamma y)+s\cot^l(\mu z).$

This is a special case of equation 2.2.7.22 with $f_1(x)=0$, $f_2(x)=a\cot^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y)=b\cot^m(\beta y)$, and $h(x,y,z)=c\cot^k(\gamma y)+s\cot^l(\mu z)$.

22. $a\dfrac{\partial w}{\partial x}+b\cot(\beta y)\dfrac{\partial w}{\partial y}+c\cot(\lambda x)\dfrac{\partial w}{\partial z}=k\cot(\gamma z).$

General solution:

$$w=\Phi(u_1,u_2)+\frac{k}{a}\int_0^x\cot\biggl\{\gamma z+\frac{c\gamma}{a\lambda}\bigl[\ln|\sin(\lambda t)|-\ln|\sin(\lambda x)|\bigr]\biggr\}\,dt,$$

where $u_1=b\beta x+a\ln\bigl|\cos(\beta y)\bigr|$ and $u_2=a\lambda z-c\ln\bigl|\sin(\lambda x)\bigr|$.

23. $$a_1 \cot^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x} + b_1 \cot^{m_1}(\beta_1 y)\frac{\partial w}{\partial y} + c_1 \cot^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z} = a_2 \cot^{n_2}(\lambda_2 x) + b_2 \cot^{m_2}(\beta_2 y) + c_2 \cot^{k_2}(\gamma_2 z).$$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1 \cot^{n_1}(\lambda_1 x)$, $g(y) = b_1 \cot^{m_1}(\beta_1 y)$, $h(z) = c_1 \cot^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2 \cot^{n_2}(\lambda_2 x)$, $\psi(y) = b_2 \cot^{m_2}(\beta_2 y)$, and $\chi(z) = c_2 \cot^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain different trigonometric functions.**

24. $$\frac{\partial w}{\partial x} + a\sin^n(\lambda x)\frac{\partial w}{\partial y} + b\cos^m(\beta x)\frac{\partial w}{\partial z} = c\sin^k(\gamma x) + s.$$

This is a special case of equation 2.2.7.1 with $f(x) = a\sin^n(\lambda x)$, $g(x) = b\cos^m(\beta x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\sin^k(\gamma x) + s$.

25. $$\frac{\partial w}{\partial x} + a\cos^n(\lambda x)\frac{\partial w}{\partial y} + b\sin^m(\beta y)\frac{\partial w}{\partial z} = c\cos^k(\gamma y) + s\sin^l(\mu z).$$

This is a special case of equation 2.2.7.22 with $f_1(x)=0$, $f_2(x)=a\cos^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y) = b\sin^m(\beta y)$, and $h(x,y,z) = c\cos^k(\gamma y) + s\sin^l(\mu z)$.

26. $$\frac{\partial w}{\partial x} + a\cos^n(\lambda x)\frac{\partial w}{\partial y} + b\tan^m(\beta y)\frac{\partial w}{\partial z} = c\cos^k(\gamma y) + s\tan^l(\mu z).$$

This is a special case of equation 2.2.7.22 with $f_1(x)=0$, $f_2(x)=a\cos^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y) = b\tan^m(\beta y)$, and $h(x,y,z) = c\cos^k(\gamma y) + s\tan^l(\mu z)$.

27. $$a_1 \sin^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x} + b_1 \cos^{m_1}(\beta_1 y)\frac{\partial w}{\partial y} + c_1 \cos^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z} = a_2 \cos^{n_2}(\lambda_2 x) + b_2 \sin^{m_2}(\beta_2 y) + c_2 \cos^{k_2}(\gamma_2 z).$$

This is a special case of equation 2.2.7.10 with $f(x)=a_1\sin^{n_1}(\lambda_1 x)$, $g(y)=b_1\cos^{m_1}(\beta_1 y)$, $h(z)=c_1\cos^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\cos^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\sin^{m_2}(\beta_2 y)$, $\chi(z)=c_2\cos^{k_2}(\gamma_2 z)$.

28. $$a_1 \tan^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x} + b_1 \cot^{m_1}(\beta_1 y)\frac{\partial w}{\partial y} + c_1 \cot^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z} = a_2 \cot^{n_2}(\lambda_2 x) + b_2 \tan^{m_2}(\beta_2 y) + c_2 \cot^{k_2}(\gamma_2 z).$$

This is a special case of equation 2.2.7.10 in which $f(x) = a_1 \tan^{n_1}(\lambda_1 x)$, $g(y) = b_1 \cot^{m_1}(\beta_1 y)$, $h(z) = c_1 \cot^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2 \cot^{n_2}(\lambda_2 x)$, $\psi(y) = b_2 \tan^{m_2}(\beta_2 y)$, and $\chi(z) = c_2 \cot^{k_2}(\gamma_2 z)$.

2.2.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $$\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} + b\frac{\partial w}{\partial z} = c\arcsin^k(\lambda x) + s.$$

General solution: $w = \Phi(y - ax, z - bx) + c\int \arcsin^k(\lambda x)\,dx + sx.$

2. $a_1 \dfrac{\partial w}{\partial x} + a_2 \dfrac{\partial w}{\partial y} + a_3 \dfrac{\partial w}{\partial z} = b_1 \arcsin(\lambda_1 x) + b_2 \arcsin(\lambda_2 y) + b_3 \arcsin(\lambda_3 z)$.

This is a special case of equation 2.2.7.10 with $f(x) = a_1$, $g(y) = a_2$, $h(z) = a_3$, $\varphi(x) = b_1 \arcsin(\lambda_1 x)$, $\psi(y) = b_2 \arcsin(\lambda_2 y)$, and $\chi(z) = b_3 \arcsin(\lambda_3 z)$.

3. $a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} + c \arcsin^n(\lambda x) \arcsin^k(\beta z) \dfrac{\partial w}{\partial z} = s \arcsin^m(\gamma x)$.

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b$, $f_3(x) = c \arcsin^n(\lambda x)$, $f_4(x) = s \arcsin^m(\gamma x)$, $g(y) = 1$, and $h(z) = \arcsin^k(\beta z)$.

4. $a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} + c \arcsin^n(\lambda x) \arcsin^m(\beta y) \arcsin^k(\gamma z) \dfrac{\partial w}{\partial z} = s$.

This is a special case of equation 2.2.7.19 with $f(x, y) = c \arcsin^n(\lambda x) \arcsin^m(\beta y)$, $g(z) = \arcsin^k(\gamma z)$, and $h(x, y) = s$.

5. $a \dfrac{\partial w}{\partial x} + b \arcsin^n(\lambda x) \dfrac{\partial w}{\partial y} + c \arcsin^k(\beta z) \dfrac{\partial w}{\partial z} = s \arcsin^m(\gamma x)$.

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b \arcsin^n(\lambda x)$, $f_3(x) = 1$, $f_4(x) = s \arcsin^m(\gamma x)$, $g(y) = 1$, and $h(z) = c \arcsin^k(\beta z)$.

6. $a \dfrac{\partial w}{\partial x} + b \arcsin^n(\lambda y) \dfrac{\partial w}{\partial y} + c \arcsin^k(\beta z) \dfrac{\partial w}{\partial z} = s$.

This is a special case of equation 2.2.7.10 with $f(x) = a$, $g(y) = b \arcsin^n(\lambda y)$, $h(z) = c \arcsin^k(\beta z)$, $\varphi(x) = s$, and $\psi(y) = \chi(z) = 0$.

► **Coefficients of equations contain arccosine.**

7. $\dfrac{\partial w}{\partial x} + a \dfrac{\partial w}{\partial y} + b \dfrac{\partial w}{\partial z} = c \arccos^k(\lambda x) + s$.

General solution: $w = \Phi(y - ax, z - bx) + c \displaystyle\int \arccos^k(\lambda x)\,dx + sx$.

8. $a_1 \dfrac{\partial w}{\partial x} + a_2 \dfrac{\partial w}{\partial y} + a_3 \dfrac{\partial w}{\partial z} = b_1 \arccos(\lambda_1 x) + b_2 \arccos(\lambda_2 y) + b_3 \arccos(\lambda_3 z)$.

This is a special case of equation 2.2.7.10 with $f(x) = a_1$, $g(y) = a_2$, $h(z) = a_3$, $\varphi(x) = b_1 \arccos(\lambda_1 x)$, $\psi(y) = b_2 \arccos(\lambda_2 y)$, and $\chi(z) = b_3 \arccos(\lambda_3 z)$.

9. $a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} + c \arccos^n(\lambda x) \arccos^k(\beta z) \dfrac{\partial w}{\partial z} = s \arccos^m(\gamma x)$.

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b$, $f_3(x) = c \arccos^n(\lambda x)$, $f_4(x) = s \arccos^m(\gamma x)$, $g(y) = 1$, and $h(z) = \arccos^k(\beta z)$.

10. $a \dfrac{\partial w}{\partial x} + b \arccos^n(\lambda x) \dfrac{\partial w}{\partial y} + c \arccos^k(\beta z) \dfrac{\partial w}{\partial z} = s \arccos^m(\gamma x)$.

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b \arccos^n(\lambda x)$, $f_3(x) = 1$, $f_4(x) = s \arccos^m(\gamma x)$, $g(y) = 1$, and $h(z) = c \arccos^k(\beta z)$.

11. $a \dfrac{\partial w}{\partial x} + b \arccos^n(\lambda y) \dfrac{\partial w}{\partial y} + c \arccos^k(\beta z) \dfrac{\partial w}{\partial z} = s$.

This is a special case of equation 2.2.7.10 with $f(x) = a$, $g(y) = b \arccos^n(\lambda y)$, $h(z) = c \arccos^k(\beta z)$, $\varphi(x) = s$, and $\psi(y) = \chi(z) = 0$.

► **Coefficients of equations contain arctangent.**

12. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\arctan^k(\lambda x) + s.$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \arctan^k(\lambda x)\,dx + sx.$

13. $a_1\dfrac{\partial w}{\partial x} + a_2\dfrac{\partial w}{\partial y} + a_3\dfrac{\partial w}{\partial z} = b_1\arctan(\lambda_1 x) + b_2\arctan(\lambda_2 y) + b_3\arctan(\lambda_3 z).$

This is a special case of equation 2.2.7.10 with $f(x) = a_1$, $g(y) = a_2$, $h(z) = a_3$, $\varphi(x) = b_1\arctan(\lambda_1 x)$, $\psi(y) = b_2\arctan(\lambda_2 y)$, and $\chi(z) = b_3\arctan(\lambda_3 z)$.

14. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\arctan^n(\lambda x)\arctan^k(\beta z)\dfrac{\partial w}{\partial z} = s\arctan^m(\gamma x).$

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b$, $f_3(x) = c\arctan^n(\lambda x)$, $f_4(x) = s\arctan^m(\gamma x)$, $g(y) = 1$, and $h(z) = \arctan^k(\beta z)$.

15. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\arctan^n(\lambda x)\arctan^m(\beta y)\arctan^k(\gamma z)\dfrac{\partial w}{\partial z} = s.$

This is a special case of equation 2.2.7.19 with $f(x, y) = c\arctan^n(\lambda x)\arctan^m(\beta y)$, $g(z) = \arctan^k(\gamma z)$, and $h(x, y) = s$.

16. $a\dfrac{\partial w}{\partial x} + b\arctan^n(\lambda x)\dfrac{\partial w}{\partial y} + c\arctan^k(\beta z)\dfrac{\partial w}{\partial z} = s\arctan^m(\gamma x).$

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b\arctan^n(\lambda x)$, $f_3(x) = 1$, $f_4(x) = s\arctan^m(\gamma x)$, $g(y) = 1$, and $h(z) = c\arctan^k(\beta z)$.

► **Coefficients of equations contain arccotangent.**

17. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\,\mathrm{arccot}^k(\lambda x) + s.$

General solution: $w = \Phi(y - ax, z - bx) + c\displaystyle\int \mathrm{arccot}^k(\lambda x)\,dx + sx.$

18. $a_1\dfrac{\partial w}{\partial x} + a_2\dfrac{\partial w}{\partial y} + a_3\dfrac{\partial w}{\partial z} = b_1\,\mathrm{arccot}(\lambda_1 x) + b_2\,\mathrm{arccot}(\lambda_2 y) + b_3\,\mathrm{arccot}(\lambda_3 z).$

This is a special case of equation 2.2.7.10 with $f(x) = a_1$, $g(y) = a_2$, $h(z) = a_3$, $\varphi(x) = b_1\,\mathrm{arccot}(\lambda_1 x)$, $\psi(y) = b_2\,\mathrm{arccot}(\lambda_2 y)$, and $\chi(z) = b_3\,\mathrm{arccot}(\lambda_3 z)$.

19. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^k(\beta z)\dfrac{\partial w}{\partial z} = s\,\mathrm{arccot}^m(\gamma x).$

This is a special case of equation 2.2.7.12 with $f_1(x) = a$, $f_2(x) = b$, $f_3(x) = c\,\mathrm{arccot}^n(\lambda x)$, $f_4(x) = s\,\mathrm{arccot}^m(\gamma x)$, $g(y) = 1$, and $h(z) = \mathrm{arccot}^k(\beta z)$.

20. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^m(\beta y)\,\mathrm{arccot}^k(\gamma z)\dfrac{\partial w}{\partial z} = s.$

This is a special case of equation 2.2.7.19 with $f(x, y) = c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^m(\beta y)$, $g(z) = \mathrm{arccot}^k(\gamma z)$, and $h(x, y) = s$.

21. $a\dfrac{\partial w}{\partial x}+b\operatorname{arccot}^n(\lambda x)\dfrac{\partial w}{\partial y}+c\operatorname{arccot}^k(\beta z)\dfrac{\partial w}{\partial z}=s\operatorname{arccot}^m(\gamma x).$

This is a special case of equation 2.2.7.12 with $f_1(x)=a$, $f_2(x)=b\operatorname{arccot}^n(\lambda x)$, $f_3(x)=1$, $f_4(x)=s\operatorname{arccot}^m(\gamma x)$, $g(y)=1$, and $h(z)=c\operatorname{arccot}^k(\beta z)$.

2.2.7 Equations Containing Arbitrary Functions

► Coefficients of equations contain arbitrary functions of x.

◆ *Throughout Section 2.2.7, sometimes only a particular solution $\widetilde{w}$ of the nonhomogeneous equation and a basis u_1, u_2 of the corresponding homogeneous equation are presented. The general solution can be obtained as $w=\widetilde{w}+\Phi(u_1,u_2)$, where $\Phi(u_1,u_2)$ is an arbitrary function of two variables.*

1. $\dfrac{\partial w}{\partial x}+f(x)\dfrac{\partial w}{\partial y}+g(x)\dfrac{\partial w}{\partial z}=h_2(x)y+h_1(x)z+h_0(x).$

General solution:

$$w=H_2(x)y+H_1(x)z+H_0(x)-\int f(x)H_2(x)\,dx-\int g(x)H_1(x)\,dx+\Phi(u_1,u_2),$$

where

$$H_k(x)=\int h_k(x)\,dx\quad(k=0,\ 1,\ 2),\qquad u_1=y-\int f(x)\,dx,\qquad u_2=z-\int g(x)\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

2. $\dfrac{\partial w}{\partial x}+f(x)(y+a)\dfrac{\partial w}{\partial y}+g(x)(z+b)\dfrac{\partial w}{\partial z}=h(x).$

General solution:

$$w=\int h(x)\,dx+\Phi(u_1,u_2),\quad u_1=\ln|y+a|-\int f(x)\,dx,\quad u_2=\ln|z+b|-\int g(x)\,dx.$$

3. $\dfrac{\partial w}{\partial x}+[ay+f(x)]\dfrac{\partial w}{\partial y}+[bz+g(x)]\dfrac{\partial w}{\partial z}=h(x).$

General solution:

$$w=\int h(x)\,dx+\Phi(u_1,u_2),\quad u_1=ye^{-ax}-\int f(x)e^{-ax}\,dx,\quad u_2=ze^{-bx}-\int g(x)e^{-bx}\,dx.$$

4. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)y+g_2(x)\big]\dfrac{\partial w}{\partial z}=h_2(x)y+h_1(x)z+h_0(x).$

Particular solution:

$$\widetilde{w}=\varphi(x)y+\psi(x)z+\int\big[h_0(x)-f_2(x)\varphi(x)-g_2(x)\psi(x)\big]\,dx,$$

$$\varphi(x)=F(x)\int\frac{h_2(x)-g_1(x)\psi(x)}{F(x)}\,dx,\quad \psi(x)=\int h_1(x)\,dx,\quad F(x)=\exp\left[-\int f_1(x)\,dx\right].$$

For an integral basis u_1, u_2 of the homogeneous equation, see 2.1.7.4.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

5. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)z+g_2(x)\big]\dfrac{\partial w}{\partial z}=h_2(x)y+h_1(x)z+h_0(x).$

Particular solution:

$$\widetilde{w}=\varphi(x)y+\psi(x)z+\int\big[h_0(x)-f_2(x)\varphi(x)-g_2(x)\psi(x)\big]\,dx,$$
$$\varphi(x)=F(x)\int\frac{h_2(x)}{F(x)}dx,\quad F(x)=\exp\Big[-\int f_1(x)\,dx\Big],$$
$$\psi(x)=G(x)\int\frac{h_1(x)}{G(x)}dx,\quad G(x)=\exp\Big[-\int g_1(x)\,dx\Big].$$

For an integral basis u_1, u_2 of the homogeneous equation, see 2.1.7.5.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

6. $\dfrac{\partial w}{\partial x}+\big[y^2-a^2+a\lambda\sinh(\lambda x)-a^2\sinh^2(\lambda x)\big]\dfrac{\partial w}{\partial y}+f(x)\sinh(\gamma z)\dfrac{\partial w}{\partial z}=g(x).$

General solution: $w=\displaystyle\int g(x)\,dx+\Phi(u_1,u_2)$, where

$$u_1=\int f(x)\,dx-\frac{1}{\gamma}\ln\left|\tanh\frac{\gamma z}{2}\right|,\quad u_2=\frac{E}{y-a\cosh(\lambda x)}+\int E\,dx,\quad E=\exp\Big[\frac{2a}{\lambda}\sinh(\lambda x)\Big].$$

7. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)z+g_2(x)z^m\big]\dfrac{\partial w}{\partial z}=h(x).$

1°. For $k\neq 1$ and $m\neq 1$, the transformation

$$\xi=y^{1-k},\quad \eta=z^{1-m},\quad W=w-\int h(x)\,dx$$

leads to an equation of the form 2.1.7.5:

$$\frac{\partial W}{\partial x}+(1-k)\big[f_1(x)\xi+f_2(x)\big]\frac{\partial W}{\partial \xi}+(1-m)\big[g_1(x)\eta+g_2(x)\big]\frac{\partial W}{\partial \eta}=0.$$

2°. For $k\neq 1$ and $m=1$, the transformation $\xi=y^{1-k}$, $W=w-\displaystyle\int h(x)\,dx$ also leads to an equation of the form 2.1.7.5.

3°. For $k=m=1$ see equation 2.2.7.5 with $h_1(x)=h_2(x)=0$.

8. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)+g_2(x)e^{\lambda z}\big]\dfrac{\partial w}{\partial z}=h(x).$

The transformation

$$\xi=y^{1-k},\quad \eta=e^{-\lambda z},\quad W=w-\int h(x)\,dx$$

leads to an equation of the form 2.1.7.5:

$$\frac{\partial W}{\partial x}+(1-k)\big[f_1(x)\xi+f_2(x)\big]\frac{\partial W}{\partial \xi}-\lambda\big[g_1(x)\eta+g_2(x)\big]\frac{\partial W}{\partial \eta}=0.$$

9. $\dfrac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} + \big[g_1(x) + g_2(x)e^{\beta z}\big]\dfrac{\partial w}{\partial z} = h(x).$

The transformation

$$\xi = e^{-\lambda y}, \quad \eta = e^{-\beta z}, \quad W = w - \int h(x)\,dx$$

leads to an equation of the form 2.1.7.5:

$$\frac{\partial W}{\partial x} - \lambda\big[f_1(x)\xi + f_2(x)\big]\frac{\partial W}{\partial \xi} - \beta\big[g_1(x)\eta + g_2(x)\big]\frac{\partial W}{\partial \eta} = 0.$$

► **Coefficients of equations contain arbitrary functions of different variables.**

10. $f(x)\dfrac{\partial w}{\partial x} + g(y)\dfrac{\partial w}{\partial y} + h(z)\dfrac{\partial w}{\partial z} = \varphi(x) + \psi(y) + \chi(z).$

General solution:

$$w = \int \frac{\varphi(x)}{f(x)}dx + \int \frac{\psi(y)}{g(y)}dy + \int \frac{\chi(z)}{h(z)}dz + \Phi(u_1, u_2),$$

where

$$u_1 = \int \frac{dx}{f(x)} - \int \frac{dy}{g(y)}, \quad u_2 = \int \frac{dx}{f(x)} - \int \frac{dz}{h(z)}.$$

11. $f(x)\dfrac{\partial w}{\partial x} + z\dfrac{\partial w}{\partial y} + g(y)\dfrac{\partial w}{\partial z} = h_2(x) + h_1(y).$

General solution:

$$w = \int \frac{h_2(x)}{f(x)}dx + \int_{y_0}^{y} \frac{h_1(t)\,dt}{\sqrt{2G(t) - 2G(y) + z^2}} + \Phi(u_1, u_2),$$

where

$$G(y) = \int g(y)\,dy, \quad u_1 = G(y) - \frac{z^2}{2}, \quad u_2 = \int \frac{dx}{f(x)} - \int_{y_0}^{y} \frac{dt}{\sqrt{2G(t) - 2G(y) + z^2}}.$$

12. $f_1(x)\dfrac{\partial w}{\partial x} + f_2(x)g(y)\dfrac{\partial w}{\partial y} + f_3(x)h(z)\dfrac{\partial w}{\partial z} = f_4(x).$

General solution:

$$w = \int \frac{f_4(x)}{f_1(x)}dx + \Phi(u_1, u_2), \quad u_1 = \int \frac{f_2(x)}{f_1(x)}dx - \int \frac{dy}{g(y)}, \quad u_2 = \int \frac{f_3(x)}{f_1(x)}dx - \int \frac{dz}{h(z)}.$$

13. $\dfrac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)\big]\dfrac{\partial w}{\partial y} + \big[g_1(x)z + g_2(y)\big]\dfrac{\partial w}{\partial z} = h_1(x) + h_2(y).$

This is a special case of equation 2.2.7.22 with $g_1(x,y) = g_1(x)$, $g_2(x,y) = g_2(y)$, and $h(x,y,z) = h_1(x) + h_2(y)$.

14. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(y)z+g_2(x)z^m\big]\dfrac{\partial w}{\partial z}=h_1(x)+h_2(y).$

This is a special case of equation 2.2.7.23 with $g_1(x,y)=g_1(y)$, $g_2(x,y)=g_2(x)$, and $h(x,y,z)=h_1(x)+h_2(y)$.

15. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)+g_2(y)e^{\lambda z}\big]\dfrac{\partial w}{\partial z}=h_1(x)+h_2(y).$

This is a special case of equation 2.2.7.24 with $g_1(x,y)=g_1(x)$, $g_2(x,y)=g_2(y)$, and $h(x,y,z)=h_1(x)+h_2(y)$.

16. $\dfrac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y}+\big[g_1(y)z+g_2(x)z^k\big]\dfrac{\partial w}{\partial z}=h_1(x)+h_2(y).$

This is a special case of equation 2.2.7.25 with $g_1(x,y)=g_1(y)$, $g_2(x,y)=g_2(x)$, and $h(x,y,z)=h_1(x)+h_2(y)$.

17. $\dfrac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)+g_2(y)e^{\beta z}\big]\dfrac{\partial w}{\partial z}=h_1(x)+h_2(y).$

This is a special case of equation 2.2.7.26 with $g_1(x,y)=g_1(x)$, $g_2(x,y)=g_2(y)$, and $h(x,y,z)=h_1(x)+h_2(y)$.

► **Coefficients of equations contain arbitrary functions of two variables.**

18. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+f(x,y)\dfrac{\partial w}{\partial z}=g(x,y).$

General solution: $w=\Phi(u_1,u_2)+\dfrac{1}{b}\displaystyle\int_{y_0}^{y}g\left(x+\frac{a(t-y)}{b},\,t\right)dt,$ where

$$u_1=bx-ay,\quad u_2=bz-\int_{y_0}^{y}f\left(x+\frac{a(t-y)}{b},\,t\right)dt,$$

and y_0 may be taken as arbitrary.

19. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+f(x,y)g(z)\dfrac{\partial w}{\partial z}=h(x,y).$

General solution: $w=\Phi(u_1,u_2)+\dfrac{1}{b}\displaystyle\int_{y_0}^{y}h\left(x+\frac{a(t-y)}{b},\,t\right)dt,$ where

$$u_1=bx-ay,\quad u_2=b\int\frac{dz}{g(z)}-\int_{y_0}^{y}f\left(x+\frac{a(t-y)}{b},\,t\right)dt.$$

20. $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+[z+f(x,y)]\dfrac{\partial w}{\partial z}=g(x,y).$

General solution: $w=\Phi(u_1,u_2)+\displaystyle\int_{y_0}^{y}g\left(\frac{xt}{y},t\right)\frac{dt}{t},$ where

$$u_1=\frac{y}{x},\quad u_2=\frac{z}{y}-\int_{y_0}^{y}f\left(\frac{xt}{y},t\right)\frac{dt}{t^2}.$$

21. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} + f(x,y)g(z)\dfrac{\partial w}{\partial z} = h(x,y).$

General solution: $w = \Phi(u_1,u_2) + \dfrac{1}{b}\displaystyle\int_{y_0}^{y} t^{-1} h\bigl(xy^{-a/b}t^{a/b},\, t\bigr)\,dt,$ where

$$u_1 = x^b y^{-a}, \quad u_2 = b\int \frac{dz}{g(z)} - \int_{y_0}^{y} t^{-1} f\bigl(xy^{-a/b}t^{a/b},\, t\bigr)\,dt.$$

22. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x,y)z + g_2(x,y)\bigr]\dfrac{\partial w}{\partial z} = h(x,y,z).$

General solution:

$$w = \Phi(u_1,u_2) + \int_{x_0}^{x} \bar{h}(t,u_1,u_2)\,dt,$$

where

$$u_1 = yF(x) - \int f_2(x)F(x)\,dx, \quad F(x) = \exp\biggl[-\int f_1(x)\,dx\biggr], \tag{1}$$

$$u_2 = zG(x,u_1) - \int_{x_0}^{x} \bar{g}_2(t,u_1)G(t,u_1)\,dt, \quad G(x,u_1) = \exp\biggl[-\int_{x_0}^{x} \bar{g}_1(t,u_1)\,dt\biggr]. \tag{2}$$

Here $\bar{g}_1(x,u_1) \equiv g_1(x,y)$, $\bar{g}_2(x,u_1) \equiv g_2(x,y)$, $\bar{h}(x,u_1,u_2) \equiv h(x,y,z)$ [in these functions, y must be expressed via x and u_1 from relation (1), and z must be expressed via x, u_1, and u_2 from relation (2)], and x_0 is an arbitrary number.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

23. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)y^k\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x,y)z + g_2(x,y)z^m\bigr]\dfrac{\partial w}{\partial z} = h(x,y,z).$

1°. For $k \neq 1$ and $m \neq 1$, the transformation $\xi = y^{1-k}$, $\eta = z^{1-m}$ leads to an equation of the form 2.2.7.22:

$$\frac{\partial w}{\partial x} + (1-k)\bigl[f_1(x)\xi + f_2(x)\bigr]\frac{\partial w}{\partial \xi} + (1-m)\bigl[\bar{g}_1(x,\xi)\eta + \bar{g}_2(x,\xi)\bigr]\frac{\partial w}{\partial \eta} = \bar{h}(x,\xi,\eta),$$

where

$$\bar{g}_1(x,\xi) \equiv g_1\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar{g}_2(x,\xi) \equiv g_2\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad h(x,\xi,\eta) \equiv h\bigl(x,\xi^{\frac{1}{1-k}},\eta^{\frac{1}{1-m}}\bigr).$$

2°. For $k \neq 1$ and $m = 1$, the substitution $\xi = y^{1-k}$ leads to an equation of the form 2.2.7.22.

3°. For $k = m = 1$, see equation 2.2.7.22.

24. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)y^k\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x,y) + g_2(x,y)e^{\lambda z}\bigr]\dfrac{\partial w}{\partial z} = h(x,y,z).$

The transformation $\xi = y^{1-k}$, $\eta = e^{-\lambda z}$ leads to an equation of the form 2.2.7.22:

$$\frac{\partial w}{\partial x} + (1-k)\bigl[f_1(x)\xi + f_2(x)\bigr]\frac{\partial w}{\partial \xi} - \lambda\bigl[\bar{g}_1(x,\xi)\eta + \bar{g}_2(x,\xi)\bigr]\frac{\partial w}{\partial \eta} = \bar{h}(x,\xi,\eta),$$

where

$$\bar{g}_1(x,\xi) \equiv g_1\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar{g}_2(x,\xi) \equiv g_2\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar{h}(x,\xi,\eta) \equiv h\bigl(x,\xi^{\frac{1}{1-k}},-\frac{1}{\lambda}\ln\eta\bigr).$$

25. $\dfrac{\partial w}{\partial x}+\left[f_1(x)+f_2(x)e^{\lambda y}\right]\dfrac{\partial w}{\partial y}+\left[g_1(x,y)z+g_2(x,y)z^k\right]\dfrac{\partial w}{\partial z}=h(x,y,z).$

The transformation $\xi=e^{-\lambda y}$, $\eta=z^{1-k}$ leads to an equation of the form 2.2.7.22:

$$\frac{\partial w}{\partial x}-\lambda\left[f_1(x)\xi+f_2(x)\right]\frac{\partial w}{\partial \xi}+(1-k)\left[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\right]\frac{\partial w}{\partial \eta}=\bar h(x,\xi,\eta),$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\bigl(x,-\frac{1}{\lambda}\ln\xi\bigr)$ and $\bar h(x,\xi,\eta)\equiv h\bigl(x,-\frac{1}{\lambda}\ln\xi,\eta^{\frac{1}{1-k}}\bigr)$.

26. $\dfrac{\partial w}{\partial x}+\left[f_1(x)+f_2(x)e^{\lambda y}\right]\dfrac{\partial w}{\partial y}+\left[g_1(x,y)+g_2(x,y)e^{\beta z}\right]\dfrac{\partial w}{\partial z}=h(x,y,z).$

The transformation $\xi=e^{-\lambda y}$, $\eta=e^{-\beta z}$ leads to an equation of the form 2.2.7.22:

$$\frac{\partial w}{\partial x}-\lambda\left[f_1(x)\xi+f_2(x)\right]\frac{\partial w}{\partial \xi}-\beta\left[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\right]\frac{\partial w}{\partial \eta}=\bar h(x,\xi,\eta),$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\bigl(x,-\frac{1}{\lambda}\ln\xi\bigr)$ and $\bar h(x,\xi,\eta)\equiv h\bigl(x,-\frac{1}{\lambda}\ln\xi,-\frac{1}{\beta}\ln\eta\bigr)$.

27. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}+\left[h_1(x,y)z+h_2(x,y)z^m\right]\dfrac{\partial w}{\partial z}=h_3(x,y,z).$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx$, $\eta=\displaystyle\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.2.7.23 with $f_1\equiv 0$, $f_2\equiv 1$, and $k=0$:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}+\left[\bar h_1(\xi,\eta)z+\bar h_2(\xi,\eta)z^m\right]\frac{\partial w}{\partial z}=\bar h_3(\xi,\eta,z),$$

where $\bar h_1(\xi,\eta)\equiv\dfrac{h_1(x,y)}{f_2(x)g_1(y)}$, $\bar h_2(\xi,\eta)\equiv\dfrac{h_2(x,y)}{f_2(x)g_1(y)}$, and $\bar h_3(\xi,\eta,z)\equiv\dfrac{h_3(x,y,z)}{f_2(x)g_1(y)}$.

28. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}+\left[h_1(x,y)+h_2(x,y)e^{\lambda z}\right]\dfrac{\partial w}{\partial z}=h_3(x,y,z).$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx$, $\eta=\displaystyle\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.2.7.24 with $f_1\equiv 0$, $f_2\equiv 1$, and $k=0$:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}+\left[\bar h_1(\xi,\eta)+\bar h_2(\xi,\eta)e^{\lambda z}\right]\frac{\partial w}{\partial z}=\bar h_3(x,y,z),$$

where $\bar h_1(\xi,\eta)\equiv\dfrac{h_1(x,y)}{f_2(x)g_1(y)}$, $\bar h_2(\xi,\eta)\equiv\dfrac{h_2(x,y)}{f_2(x)g_1(y)}$, and $\bar h_3(\xi,\eta,z)\equiv\dfrac{h_3(x,y,z)}{f_2(x)g_1(y)}$.

2.3 Equations of the Form $f_1\dfrac{\partial w}{\partial x}+f_2\dfrac{\partial w}{\partial y}+f_3\dfrac{\partial w}{\partial z}=f_4w,\quad f_n=f_n(x,y,z)$

◆ *The solutions given below contain arbitrary functions of two variables* $\Phi=\Phi(u_1,u_2)$, *where* $u_1=u_1(x,y,z)$ *and* $u_2=u_2(x,y,z)$ *are some functions.*

2.3.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x, y, and z.**

1. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=(\alpha x+\beta y+\gamma z+\delta)w.$

General solution: $\displaystyle w=\exp\left(\frac{\alpha}{2a}x^2+\frac{\beta}{2b}y^2+\frac{\gamma}{2c}z^2+\frac{\delta}{a}x\right)\Phi(bx-ay,\ cy-bz).$

2. $\displaystyle \frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=(cx+s)w.$

General solution: $w=\exp\bigl(\tfrac{1}{2}cx^2+sx\bigr)\Phi(u_1,u_2)$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}\bigl(by+\sqrt{ab}\,z\bigr)\exp\bigl(-\sqrt{ab}\,x\bigr) & \text{if } ab>0,\\ by\cos\bigl(\sqrt{|ab|}\,x\bigr)+\sqrt{|ab|}\,z\sin\bigl(\sqrt{|ab|}\,x\bigr) & \text{if } ab<0.\end{cases}$$

3. $\displaystyle \frac{\partial w}{\partial x}+(a_1x+a_0)\frac{\partial w}{\partial y}+(b_1x+b_0)\frac{\partial w}{\partial z}=(\alpha x+\beta y+\gamma z+\delta)w.$

This is a special case of equation 2.3.7.1 with $f(x)=a_1x+a_0$, $g(x)=b_1x+b_0$, $h_2(x)=\beta$, $h_1(x)=\gamma$, and $h_0(x)=\alpha x+\delta$.

4. $\displaystyle \frac{\partial w}{\partial x}+(a_2y+a_1x+a_0)\frac{\partial w}{\partial y}+(b_2y+b_1x+b_0)\frac{\partial w}{\partial z}=(c_2y+c_1z+c_0x+s)w.$

This is a special case of equation 2.3.7.4 with $f_1(x)=a_2$, $f_2(x)=a_1x+a_0$, $g_1(x)=b_2$, $g_2(x)=b_1x+b_0$, $h_2(x)=c_2$, $h_1(x)=c_1$, and $h_0(x)=c_0x+s$.

5. $\displaystyle \frac{\partial w}{\partial x}+(ay+k_1x+k_0)\frac{\partial w}{\partial y}+(bz+s_1x+s_0)\frac{\partial w}{\partial z}=(c_1x+c_0)w.$

This is a special case of equation 2.3.7.3 with $f(x)=k_1x+k_0$, $g(x)=s_1x+s_0$, and $h(x)=c_1x+c_0$.

6. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=(\alpha x+\beta y+\gamma z+\delta)w.$

General solution: $\displaystyle w=|x|^{\delta/a}\exp\left(\frac{\alpha}{a}x+\frac{\beta}{b}y+\frac{\gamma}{c}z\right)\Phi\left(\frac{|y|^a}{|x|^b},\ \frac{|z|^a}{|x|^c}\right).$

7. $\displaystyle x\frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=cw.$

General solution: $w=|x|^c\Phi(u_1,u_2)$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}|x|^{\sqrt{ab}}\bigl(by-\sqrt{ab}\,z\bigr) & \text{if } ab>0,\\ |x|^{\sqrt{-ab}}\exp\left(-\arctan\dfrac{\sqrt{-ab}\,z}{by}\right) & \text{if } ab<0.\end{cases}$$

8. $\displaystyle abx\frac{\partial w}{\partial x}+b(ay+bz)\frac{\partial w}{\partial y}+a(ay-bz)\frac{\partial w}{\partial z}=cw.$

General solution: $w=|x|^{c/(ab)}\Phi(u_1,u_2)$, where

$$u_1=\bigl[ay+(\sqrt{2}-1)bz\bigr]|x|^{-\sqrt{2}},\quad u_2=\bigl[ay-(\sqrt{2}+1)bz\bigr]|x|^{\sqrt{2}}.$$

Particular solution: $w=|x|^{c/(ab)}\Phi\bigl(a^2y^2-2abyz-b^2z^2\bigr)$.

9. $\displaystyle (a_1x+a_0)\frac{\partial w}{\partial x}+(b_1y+b_0)\frac{\partial w}{\partial y}+(c_1z+c_0)\frac{\partial w}{\partial z}=(\alpha x+\beta y+\gamma z+\delta)w.$

1°. General solution for $a_1b_1c_1\neq 0$:

$$w=\exp\left[\frac{\alpha}{a_1}x+\frac{\beta}{b_1}y+\frac{\gamma}{c_1}z+\frac{1}{a_1}\left(\delta-\frac{\alpha a_0}{a_1}-\frac{\beta b_0}{b_1}-\frac{\gamma c_0}{c_1}\right)\ln\bigl|a_1x+a_0\bigr|\right]\times\Phi\left(\frac{|b_1y+b_0|^{a_1}}{|a_1x+a_0|^{b_1}},\ \frac{|b_1y+b_0|^{c_1}}{|c_1z+c_0|^{b_1}}\right).$$

2°. General solution for $a_1b_1\neq 0$ and $c_1=0$:

$$w=\exp\left[\frac{\alpha}{a_1}x+\frac{\beta}{b_1}y+\frac{\gamma}{2c_0}z^2+\frac{1}{c_0}\left(\delta-\frac{\alpha a_0}{a_1}-\frac{\beta b_0}{b_1}\right)z\right]\Phi\left(\frac{|b_1y+b_0|^{a_1}}{|a_1x+a_0|^{b_1}},\ |b_1y+b_0|^{c_0}e^{-b_1z}\right).$$

3°. General solution for $a_1\neq 0$ and $b_1=c_1=0$:

$$w=\exp\left[\frac{\alpha}{a_1}x+\frac{\beta}{2b_0}y^2+\frac{\gamma}{2c_0}z^2+\frac{1}{c_0}\left(\delta-\frac{\alpha a_0}{a_1}\right)z\right]\Phi\bigl(|a_1x+a_0|^{b_0}e^{-a_1y},\ c_0y-b_0z\bigr).$$

4°. For $a_1=b_1=c_1=0$, see equation 2.3.1.1.

► **Coefficients of equations are quadratic in x, y, and z.**

10. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=(\lambda x^2+\beta y^2+\gamma z^2+\delta)w.$

General solution: $\displaystyle w=\exp\left(\frac{\lambda}{3a}x^3+\frac{\beta}{3b}y^3+\frac{\gamma}{3c}z^3+\frac{\delta}{a}x\right)\Phi(bx-ay,\ cy-bz).$

11. $\displaystyle \frac{\partial w}{\partial x}+(a_1x^2+a_0)\frac{\partial w}{\partial y}+(b_1x^2+b_0)\frac{\partial w}{\partial z}=(\lambda x+\beta y+\gamma z+\delta)w.$

This is a special case of equation 2.3.7.1 with $f(x)=a_1x^2+a_0$, $g(x)=b_1x^2+b_0$, $h_2(x)=\beta$, $h_1(x)=\gamma$, and $h_0(x)=\lambda x+\delta$.

12. $\displaystyle \frac{\partial w}{\partial x}+(ay+k_1x^2+k_0)\frac{\partial w}{\partial y}+(bz+s_1x^2+s_0)\frac{\partial w}{\partial z}=(c_1x^2+c_0)w.$

This is a special case of equation 2.3.7.3 with $f(x)=k_1x^2+k_0$, $g(x)=s_1x^2+s_0$, and $h(x)=c_1x^2+c_0$.

13. $\dfrac{\partial w}{\partial x} + (a_2 xy + a_1 x^2 + a_0)\dfrac{\partial w}{\partial y} + (b_2 xy + b_1 x^2 + b_0)\dfrac{\partial w}{\partial z} = (c_2 y + c_1 z + c_0 x + s)w.$

This is a special case of equation 2.3.7.4 with $f_1(x) = a_2 x$, $f_2(x) = a_1 x^2 + a_0$, $g_1(x) = b_2 x$, $g_2(x) = b_1 x^2 + b_0$, $h_2(x) = c_2$, $h_1(x) = c_1$, and $h_0(x) = c_0 x + s$.

14. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} + cz\dfrac{\partial w}{\partial z} = x(\lambda x + \beta y + \gamma z)w.$

1°. General solution for $b \neq -a$ and $c \neq -a$:

$$w = \exp\left(\frac{\lambda}{2a}x^2 + \frac{\beta}{a+b}xy + \frac{\gamma}{a+c}xz\right)\Phi\bigl(x|y|^{-a/b},\ x|z|^{-a/c}\bigr).$$

2°. General solution for $b = -a$ and $c \neq -a$:

$$w = \exp\left[\frac{1}{2a}x\bigl(\lambda x + 2\beta y \ln|x|\bigr) + \frac{\gamma}{a+c}xz\right]\Phi\bigl(xy,\ x|z|^{-a/c}\bigr).$$

3°. General solution for $b = c = -a$:

$$w = \exp\left\{\frac{1}{2a}x\bigl[\lambda x + 2(\beta y + \gamma z)\ln|x|\bigr]\right\}\Phi(xy, xz).$$

15. $ax^2\dfrac{\partial w}{\partial x} + bxy\dfrac{\partial w}{\partial y} + cxz\dfrac{\partial w}{\partial z} = (\lambda x + \beta y + \gamma z)w.$

1°. General solution for $b \neq a$ and $c \neq a$:

$$w = |x|^{\lambda/a}\exp\left[\frac{1}{x}\left(\frac{\beta y}{b-a} + \frac{\gamma z}{c-a}\right)\right]\Phi\bigl(x|y|^{-a/b},\ x|z|^{-a/c}\bigr).$$

2°. General solution for $b = a$ and $c \neq a$:

$$w = |x|^{\lambda/a}\exp\left[\frac{1}{x}\left(\frac{\beta y}{a}\ln|x| + \frac{\gamma z}{c-a}\right)\right]\Phi\left(\frac{x}{y},\ x|z|^{-a/c}\right).$$

3°. General solution for $a = b = c$:

$$w = \exp\left[\frac{\ln|x|}{ax}(\lambda x + \beta y + \gamma z)\right]\Phi\left(\frac{x}{y},\ \frac{x}{z}\right).$$

16. $ax^2\dfrac{\partial w}{\partial x} + bxy\dfrac{\partial w}{\partial y} + cz^2\dfrac{\partial w}{\partial z} = ky^2 w.$

1°. General solution for $a \neq 2b$:

$$w = \exp\left[\frac{ky^2}{(2b-a)x}\right]\Phi\left(x|y|^{-a/b},\ \frac{c}{x} - \frac{a}{z}\right).$$

2°. General solution for $a = 2b$:

$$w = \exp\left(\frac{ky^2\ln|x|}{ax}\right)\Phi\left(\frac{x}{y^2},\ \frac{c}{x} - \frac{a}{z}\right).$$

17. $\displaystyle ax^2\frac{\partial w}{\partial x}+by^2\frac{\partial w}{\partial y}+cz^2\frac{\partial w}{\partial z}=kxyw.$

General solution: $\displaystyle w=\exp\left(\frac{kxy}{ax-by}\ln\left|\frac{ax}{y}\right|\right)\Phi\left(\frac{b}{x}-\frac{a}{y},\,\frac{c}{x}-\frac{a}{z}\right).$

18. $\displaystyle ax^2\frac{\partial w}{\partial x}+by^2\frac{\partial w}{\partial y}+cz^2\frac{\partial w}{\partial z}=(\lambda x^2+\beta y^2+\gamma z^2)w.$

General solution: $\displaystyle w=\exp\left(\frac{\lambda}{a}x+\frac{\beta}{b}y+\frac{\gamma}{c}z\right)\Phi\left(\frac{b}{x}-\frac{a}{y},\,\frac{c}{x}-\frac{a}{z}\right).$

► **Coefficients of equations contain other powers of x, y, and z.**

19. $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=xyzw.$

General solution: $w=\exp\left[\frac{1}{2}x^2yz-\frac{1}{6}x^3(az+by)+\frac{1}{12}abx^4\right]\Phi(y-ax,\,z-bx).$

20. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=\left(kx^3+sy^2\right)w.$

General solution: $\displaystyle w=\exp\left(\frac{k}{4a}x^4+\frac{s}{3a}y^3\right)\Phi(bx-ay,cx-az).$

21. $\displaystyle a\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=\left(kx+s\sqrt{x}\,\right)w.$

General solution: $\displaystyle w=\exp\left(\frac{k}{2a}x^2+\frac{2s}{3a}x^{3/2}\right)\Phi\left(|y|^a e^{-bx},|z|^a e^{-cx}\right).$

22. $\displaystyle \frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=(c\sqrt{x}+s)w.$

General solution: $w=\exp\left(\frac{2}{3}cx^{3/2}+sx\right)\Phi(u_1,u_2)$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}\left(by+\sqrt{ab}\,z\right)\exp\left(-\sqrt{ab}\,x\right) & \text{if } ab>0,\\ by\cos\left(\sqrt{|ab|}\,x\right)+\sqrt{|ab|}\,z\sin\left(\sqrt{|ab|}\,x\right) & \text{if } ab<0.\end{cases}$$

23. $\displaystyle ax^2\frac{\partial w}{\partial x}+by^2\frac{\partial w}{\partial y}+cz^2\frac{\partial w}{\partial z}=kxyzw.$

General solution:

$$w=w_0(x,y,z)\Phi\left(\frac{1}{ax}-\frac{1}{by},\,\frac{1}{ax}-\frac{1}{cz}\right),$$

where

$$w_0(x,y,z)=\exp\left\{kxyz\left[\frac{ax\ln(ax)}{(ax-by)(ax-cz)}+\frac{by\ln(by)}{(by-ax)(by-cz)}+\frac{cz\ln(cz)}{(cz-ax)(cz-by)}\right]\right\}.$$

► **Coefficients of equations contain arbitrary powers of x, y, and z.**

24. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\dfrac{\partial w}{\partial z} = \big(\lambda x^n + \beta y^m + \gamma z^k\big)w.$

General solution:

$$w = \exp\left[\frac{\lambda}{a(n+1)}x^{n+1} + \frac{\beta}{b(m+1)}y^{m+1} + \frac{\gamma}{c(k+1)}z^{k+1}\right]\Phi(bx-ay,\, cx-az).$$

25. $a\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} + cz\dfrac{\partial w}{\partial z} = \big(\lambda x^n + \beta y^m + \gamma z^k\big)w.$

General solution: $w = \exp\left[\dfrac{\lambda}{a(n+1)}x^{n+1} + \dfrac{\beta}{bm}y^m + \dfrac{\gamma}{ck}z^k\right]\Phi\big(|y|^a e^{-bx},\, |z|^a e^{-cx}\big).$

26. $\dfrac{\partial w}{\partial x} + az\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial z} = cx^n w.$

General solution: $w = \exp\left(\dfrac{c}{n+1}x^{n+1}\right)\Phi(u_1, u_2)$, where

$$u_1 = by^2 - az^2, \quad u_2 = \begin{cases} \big(by + \sqrt{ab}\,z\big)\exp\big(-\sqrt{ab}\,x\big) & \text{if } ab > 0,\\ by\cos\big(\sqrt{|ab|}\,x\big) + \sqrt{|ab|}\,z\sin\big(\sqrt{|ab|}\,x\big) & \text{if } ab < 0. \end{cases}$$

27. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} + cz\dfrac{\partial w}{\partial z} = \big(\lambda x^n + \beta y^m + \gamma z^k\big)w.$

General solution: $w = \exp\left(\dfrac{\lambda}{an}x^n + \dfrac{\beta}{bm}y^m + \dfrac{\gamma}{ck}z^k\right)\Phi\left(\dfrac{|y|^a}{|x|^b}, \dfrac{|z|^a}{|x|^c}\right).$

28. $x\dfrac{\partial w}{\partial x} + az\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial z} = cx^n w.$

General solution: $w = \exp\left(\dfrac{c}{n}x^n\right)\Phi(u_1, u_2)$, where

$$u_1 = by^2 - az^2, \quad u_2 = \begin{cases} |x|^{\sqrt{ab}}\big(by - \sqrt{ab}\,z\big) & \text{if } ab > 0,\\ |x|^{\sqrt{-ab}}\exp\left(-\arctan\dfrac{\sqrt{-ab}\,z}{by}\right) & \text{if } ab < 0. \end{cases}$$

29. $abx\dfrac{\partial w}{\partial x} + b(ay + bz)\dfrac{\partial w}{\partial y} + a(ay - bz)\dfrac{\partial w}{\partial z} = cx^n w.$

General solution: $w = \exp\left(\dfrac{c}{abn}x^n\right)\Phi(u_1, u_2)$, where

$$u_1 = \big[ay + (\sqrt{2} - 1)bz\big]|x|^{-\sqrt{2}}, \quad u_2 = \big[ay - (\sqrt{2} + 1)bz\big]|x|^{\sqrt{2}}.$$

Particular solution: $w = \exp\left(\dfrac{c}{abn}x^n\right)\Phi\big(a^2y^2 - 2abyz - b^2z^2\big).$

30. $\dfrac{\partial w}{\partial x} + ax^n y^m \dfrac{\partial w}{\partial y} + bx^\nu y^\mu z^\lambda \dfrac{\partial w}{\partial z} = cx^k w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{k+1} x^{k+1}\right)\Phi(u_1, u_2) & \text{if } k \neq -1, \\ |x|^c \Phi(u_1, u_2) & \text{if } k = -1, \end{cases}$$

where u_1, u_2 are the integral basis of equation 2.1.1.64.

31. $\dfrac{\partial w}{\partial x} + \left(a_1 x^{n_1} y + b_1 x^{m_1}\right)\dfrac{\partial w}{\partial y} + \left(a_2 x^{n_2} y + b_2 x^{m_2}\right)\dfrac{\partial w}{\partial z} = (c_2 x^{k_2} y + c_1 x^{k_1} z)w.$

This is a special case of equation 2.3.7.4 with $f_1(x) = a_1 x^{n_1}$, $f_2(x) = b_1 x^{m_1}$, $g_1(x) = a_2 x^{n_2}$, $g_2(x) = b_2 x^{m_2}$, $h_2(x) = c_2 x^{k_2}$, $h_1(x) = c_1 x^{k_1}$, and $h_0(x) = 0$.

32. $\dfrac{\partial w}{\partial x} + \left(a_1 x^{n_1} y + b_1 x^{m_1}\right)\dfrac{\partial w}{\partial y} + \left(a_2 x^{n_2} z + b_2 x^{m_2}\right)\dfrac{\partial w}{\partial z} = (c_2 x^{k_2} y + c_1 x^{k_1} z)w.$

This is a special case of equation 2.3.7.5 with $f_1(x) = a_1 x^{n_1}$, $f_2(x) = b_1 x^{m_1}$, $g_1(x) = a_2 x^{n_2}$, $g_2(x) = b_2 x^{m_2}$, $h_2(x) = c_2 x^{k_2}$, $h_1(x) = c_1 x^{k_1}$, and $h_0(x) = 0$.

33. $\dfrac{\partial w}{\partial x} + \left(a_1 x^{n_1} y + b_1 y^k\right)\dfrac{\partial w}{\partial y} + \left(a_2 x^{n_2} z + b_2 z^m\right)\dfrac{\partial w}{\partial z} = cx^s w.$

This is a special case of equation 2.3.7.7 with $f_1(x) = a_1 x^{n_1}$, $f_2(x) = b_1$, $g_1(x) = a_2 x^{n_2}$, $g_2(x) = b_2$, and $h(x) = cx^s$.

34. $\dfrac{\partial w}{\partial x} + \left(a_1 x^{n_1} y + b_1 y^k\right)\dfrac{\partial w}{\partial y} + \left(a_2 y^{n_2} z + b_2 z^m\right)\dfrac{\partial w}{\partial z} = (c_1 x^{s_1} + c_2 y^{s_2} + c_3 z^{s_3})w.$

This is a special case of equation 2.3.7.23 with $f_1(x) = a_1 x^{n_1}$, $f_2(x) = b_1$, $g_1(x, y) = a_2 y^{n_2}$, $g_2(x, y) = b_2$, and $h(x, y, z) = c_1 x^{s_1} + c_2 y^{s_2} + c_3 z^{s_3}$.

35. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + a\sqrt{x^2 + y^2}\,\dfrac{\partial w}{\partial z} = bx^n w.$

General solution: $w = w_0(x)\Phi\bigl(u_1,\, u_2\bigr)$, where

$$u_1 = \frac{y}{x}, \quad u_2 = a\sqrt{x^2 + y^2} - z, \quad w_0(x) = \begin{cases} \exp\bigl(bx^n/n\bigr) & \text{if } n \neq 0, \\ |x|^b & \text{if } n = 0. \end{cases}$$

36. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + \left(z - a\sqrt{x^2 + y^2 + z^2}\,\right)\dfrac{\partial w}{\partial z} = bx^n w.$

General solution: $w = w_0(x)\Phi\bigl(u_1,\, u_2\bigr)$, where

$$u_1 = \frac{y}{x}, \quad u_2 = |x|^{a-1}\bigl(z + \sqrt{x^2 + y^2 + z^2}\,\bigr), \quad w_0(x) = \begin{cases} \exp\bigl(bx^n/n\bigr) & \text{if } n \neq 0, \\ |x|^b & \text{if } n = 0. \end{cases}$$

2.3.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\displaystyle \frac{\partial w}{\partial x} + ae^{\lambda x}\frac{\partial w}{\partial y} + be^{\beta x}\frac{\partial w}{\partial z} = ce^{\gamma x}w.$

This is a special case of equation 2.3.7.1 with $f(x)=ae^{\lambda x}$, $g(x)=be^{\beta x}$, $h_2(x)=h_1(x)=0$, and $h_0(x)=ce^{\gamma x}$.

2. $\displaystyle \frac{\partial w}{\partial x} + ae^{\lambda x}\frac{\partial w}{\partial y} + be^{\beta y}\frac{\partial w}{\partial z} = \big(ce^{\gamma y} + se^{\mu z}\big)w.$

This is a special case of equation 2.3.7.22 with $f_1(x)=0$, $f_2(x)=ae^{\lambda x}$, $g_1(x,y)=0$, $g_2(x,y)=be^{\beta y}$, and $h(x,y,z)=ce^{\gamma y}+se^{\mu z}$.

3. $\displaystyle \frac{\partial w}{\partial x} + ae^{\lambda y}\frac{\partial w}{\partial y} + be^{\beta y}\frac{\partial w}{\partial z} = \big(ce^{\gamma x} + se^{\mu z}\big)w.$

This is a special case of equation 2.3.7.26 with $f_1(x)=0$, $f_2(x)=a$, $g_1(x,y)=be^{\beta y}$, $g_2(x,y)=0$, and $h(x,y,z)=ce^{\gamma x}+se^{\mu z}$.

4. $\displaystyle \frac{\partial w}{\partial x} + \big(A_1e^{\alpha_1 x} + B_1e^{\nu_1 x+\lambda y}\big)\frac{\partial w}{\partial y} + \big(A_2e^{\alpha_2 x} + B_2e^{\nu_2 x+\beta z}\big)\frac{\partial w}{\partial z} = ke^{\gamma z}w.$

This is a special case of equation 2.3.7.9 with $f_1(x)=A_1e^{\alpha_1 x}$, $f_2(x)=B_1e^{\nu_1 x}$, $g_1(x)=A_2e^{\alpha_2 x}$, $g_2(x)=B_2e^{\nu_2 x}$, and $h(x)=ke^{\gamma z}$.

5. $\displaystyle ae^{\alpha x}\frac{\partial w}{\partial x} + be^{\beta y}\frac{\partial w}{\partial y} + ce^{\gamma z}\frac{\partial w}{\partial z} = ke^{\lambda x}w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{k}{a(\lambda-\alpha)}e^{(\lambda-\alpha)x}\right]\Phi(u_1,u_2) & \text{if } \lambda \neq \alpha, \\ \exp\left(\dfrac{k}{a}x\right)\Phi(u_1,u_2) & \text{if } \lambda = \alpha, \end{cases}$$

where $u_1 = -\dfrac{1}{a\alpha}e^{-\alpha x} + \dfrac{1}{b\beta}e^{-\beta y}$ and $u_2 = -\dfrac{1}{b\beta}e^{-\beta y} + \dfrac{1}{c\gamma}e^{-\gamma z}$.

6. $\displaystyle ae^{\beta y}\frac{\partial w}{\partial x} + be^{\alpha x}\frac{\partial w}{\partial y} + ce^{\gamma z}\frac{\partial w}{\partial z} = ke^{\lambda x}w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{k\alpha}{b\beta}\displaystyle\int \dfrac{e^{\lambda x}\,dx}{e^{\alpha x}+a\alpha u_1}\right)\Phi(u_1,u_2) & \text{if } \lambda \neq \alpha,\ \lambda \neq 0; \\ \exp\left(\dfrac{k}{b}y\right)\Phi(u_1,u_2) & \text{if } \lambda = \alpha \neq 0; \\ \exp\left(-\dfrac{k}{c\gamma}e^{-\gamma z}\right)\Phi(u_1,u_2) & \text{if } \lambda = 0, \end{cases}$$

where $u_1 = -\dfrac{1}{a\alpha}e^{\alpha x} + \dfrac{1}{b\beta}e^{\beta y}$ and $u_2 = \dfrac{\beta y - \alpha x}{b\beta e^{\alpha x} - a\alpha e^{\beta y}} + \dfrac{1}{c\gamma}e^{-\gamma z}$. In the integration, u_1 is considered a parameter.

7. $\displaystyle (a_1 + a_2 e^{\alpha x})\frac{\partial w}{\partial x} + (b_1 + b_2 e^{\beta y})\frac{\partial w}{\partial y} + (c_1 + c_2 e^{\gamma z})\frac{\partial w}{\partial z} = \big(k_1 + k_2 e^{\alpha x}\big)w.$

General solution: $w = \Phi(u_1, u_2)\exp\left[\frac{k_1}{a_1}x + \frac{1}{\alpha}\left(\frac{k_2}{a_2} - \frac{k_1}{a_1}\right)\ln(a_1 + a_2 e^{\alpha x})\right]$, where

$$u_1 = \frac{1}{a_1\alpha}\big[\alpha x - \ln(a_1 + a_2 e^{\alpha x})\big] - \frac{1}{b_1\beta}\big[\beta y - \ln(b_1 + b_2 e^{\beta y})\big],$$
$$u_2 = \frac{1}{a_1\alpha}\big[\alpha x - \ln(a_1 + a_2 e^{\alpha x})\big] - \frac{1}{c_1\gamma}\big[\gamma z - \ln(c_1 + c_2 e^{\gamma z})\big].$$

8. $\displaystyle e^{\beta y}(a_1 + a_2 e^{\alpha x})\frac{\partial w}{\partial x} + e^{\alpha x}(b_1 + b_2 e^{\beta y})\frac{\partial w}{\partial y} + c e^{\beta y + \gamma z}\frac{\partial w}{\partial z} = k_3 e^{\beta y}(k_1 + k_2 e^{\alpha x})w.$

General solution: $w = \Phi(u_1, u_2)\exp\left[\frac{k_1 k_3}{a_1}x + \frac{k_3}{\alpha}\left(\frac{k_2}{a_2} - \frac{k_1}{a_1}\right)\ln(a_1 + a_2 e^{\alpha x})\right]$, where

$$u_1 = \frac{1}{a_2\alpha}\ln(a_1 + a_2 e^{\alpha x}) - \frac{1}{b_1\beta}\ln(b_1 + b_2 e^{\beta y}), \quad u_2 = \frac{1}{a_1\alpha}\big[\alpha x - \ln(a_1 + a_2 e^{\alpha x})\big] + \frac{1}{c\gamma}e^{-\gamma z}.$$

► **Coefficients of equations contain exponential and power-law functions.**

9. $\displaystyle \frac{\partial w}{\partial x} + a x^n \frac{\partial w}{\partial y} + b x^m \frac{\partial w}{\partial z} = \big(c e^{\lambda x} y + k e^{\beta x} z + s e^{\gamma x}\big)w.$

This is a special case of equation 2.3.7.1 with $f(x) = ax^n$, $g(x) = bx^m$, $h_2(x) = ce^{\lambda x}$, $h_1(x) = ke^{\beta x}$, and $h_0(x) = se^{\gamma x}$.

10. $\displaystyle \frac{\partial w}{\partial x} + a e^{\lambda x} \frac{\partial w}{\partial y} + b x^m \frac{\partial w}{\partial z} = \big(c x^n y + k e^{\beta x} z + s e^{\gamma x}\big)w.$

This is a special case of equation 2.3.7.1 with $f(x) = ae^{\lambda x}$, $g(x) = bx^m$, $h_2(x) = cx^n$, $h_1(x) = ke^{\beta x}$, and $h_0(x) = se^{\gamma x}$.

11. $\displaystyle \frac{\partial w}{\partial x} + a e^{\lambda x} \frac{\partial w}{\partial y} + b y \frac{\partial w}{\partial z} = \big(k e^{\beta x} z + s e^{\gamma x}\big)w.$

This is a special case of equation 2.3.7.4 with $f_1(x) = 0$, $f_2(x) = ae^{\lambda x}$, $g_1(x) = b$, $g_2(x) = h_2(x) = 0$, $h_1(x) = ke^{\beta x}$, and $h_0(x) = se^{\gamma x}$.

12. $\displaystyle \frac{\partial w}{\partial x} + a y^n \frac{\partial w}{\partial y} + b z^m \frac{\partial w}{\partial z} = \big(c e^{\lambda x} + k e^{\beta y} + s e^{\gamma z}\big)w.$

This is a special case of equation 2.3.7.10 with $f(x) = 1$, $g(y) = ay^n$, $h(z) = bz^m$, $\varphi(x) = ce^{\lambda x}$, $\psi(y) = ke^{\beta y}$, and $\chi(z) = se^{\gamma z}$.

13. $\displaystyle \frac{\partial w}{\partial x} + a e^{\beta y} \frac{\partial w}{\partial y} + b z^m \frac{\partial w}{\partial z} = \big(c e^{\lambda x} + k y^n + s e^{\gamma z}\big)w.$

This is a special case of equation 2.3.7.10 with $f(x) = 1$, $g(y) = ae^{\beta y}$, $h(z) = bz^m$, $\varphi(x) = ce^{\lambda x}$, $\psi(y) = ky^n$, and $\chi(z) = se^{\gamma z}$.

14. $\dfrac{\partial w}{\partial x} + \big(a_1 e^{\lambda_1 x} y + b_1 e^{\beta_1 x} y^k\big)\dfrac{\partial w}{\partial y} + \big(a_2 e^{\lambda_2 x} z + b_2 e^{\beta_2 x} z^m\big)\dfrac{\partial w}{\partial z} = c x^s w.$

This is a special case of equation 2.3.7.7 with $f_1(x) = a_1 e^{\lambda_1 x}$, $f_2(x) = b_1 e^{\beta_1 x}$, $g_1(x) = a_2 e^{\lambda_2 x}$, $g_2(x) = b_2 e^{\beta_2 x}$, and $h(x) = c x^s$.

15. $\dfrac{\partial w}{\partial x} + \big(a_1 e^{\beta_1 x} y + b_1 e^{\gamma_1 x} y^k\big)\dfrac{\partial w}{\partial y} + \big(a_2 e^{\beta_2 x} + b_2 e^{\gamma_2 x + \lambda z}\big)\dfrac{\partial w}{\partial z} = c x^s w.$

This is a special case of equation 2.3.7.8 with $f_1(x) = a_1 e^{\beta_1 x}$, $f_2(x) = b_1 e^{\gamma_1 x}$, $g_1(x) = a_2 e^{\beta_2 x}$, $g_2(x) = b_2 e^{\gamma_2 x}$, and $h(x) = c x^s$.

16. $\dfrac{\partial w}{\partial x} + \big(a_1 x^n + b_1 x^m e^{\lambda y}\big)\dfrac{\partial w}{\partial y} + \big(a_2 x^k + b_2 x^l e^{\beta z}\big)\dfrac{\partial w}{\partial z} = c x^s w.$

This is a special case of equation 2.3.7.9 with $f_1(x) = a_1 x^n$, $f_2(x) = b_1 x^m$, $g_1(x) = a_2 x^k$, $g_2(x) = b_2 x^l$, and $h(x) = c x^s$.

2.3.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c \sinh^n(\beta x) w.$

General solution: $w = \Phi(y - ax, z - bx) \exp\left[c \int \sinh^n(\beta x)\, dx\right].$

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\sinh(\lambda x)\dfrac{\partial w}{\partial z} = \big[k \sinh(\beta y) + s \sinh(\gamma z)\big] w.$

General solution:

$$w = \Phi(u_1, u_2) \exp\left\{ \frac{k}{b\beta} \cosh(\beta y) - \frac{s}{a} \int_0^x \sinh\left[\frac{c\gamma}{a\lambda}\big(\cosh(\lambda x) - \cosh(\lambda t)\big) - \gamma z\right] dt \right\},$$

where $u_1 = bx - ay$ and $u_2 = a\lambda z - c\cosh(\lambda x)$.

3. $\dfrac{\partial w}{\partial x} + a\sinh^n(\beta x)\dfrac{\partial w}{\partial y} + b\sinh^k(\lambda x)\dfrac{\partial w}{\partial z} = c\sinh^m(\gamma x) w.$

This is a special case of equation 2.3.7.1 with $f(x) = a\sinh^n(\beta x)$, $g(x) = b\sinh^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\sinh^m(\gamma x)$.

4. $a\dfrac{\partial w}{\partial x} + b\sinh(\beta y)\dfrac{\partial w}{\partial y} + c\sinh(\lambda x)\dfrac{\partial w}{\partial z} = k\sinh(\gamma z) w.$

General solution:

$$w = \Phi(u_1, u_2) \exp\left\{ \frac{k}{a} \int_0^x \sinh\left(\gamma z + \frac{c\gamma}{a\lambda}\big[\cosh(\lambda x) - \cosh(\lambda t)\big]\right) dt \right\},$$

where $u_1 = b\beta x - a\ln\left|\tanh\dfrac{\beta y}{2}\right|$ and $u_2 = a\lambda z - c\cosh(\lambda x)$.

5. $$a_1\sinh^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x}+b_1\sinh^{m_1}(\beta_1 y)\frac{\partial w}{\partial y}+c_1\sinh^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z}=\bigl[a_2\sinh^{n_2}(\lambda_2 x)+b_2\sinh^{m_2}(\beta_2 y)+c_2\sinh^{k_2}(\gamma_2 z)\bigr]w.$$

This is a special case of equation 2.3.7.10 in which $f(x)=a_1\sinh^{n_1}(\lambda_1 x)$, $g(y)=b_1\sinh^{m_1}(\beta_1 y)$, $h(z)=c_1\sinh^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\sinh^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\sinh^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\sinh^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain hyperbolic cosine.**

6. $$\frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=c\cosh^n(\beta x)w.$$

General solution: $w=\Phi(y-ax,z-bx)\exp\Bigl[c\int\cosh^n(\beta x)\,dx\Bigr]$.

7. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\cosh(\lambda x)\frac{\partial w}{\partial z}=\bigl[k\cosh(\beta y)+s\cosh(\gamma z)\bigr]w.$$

General solution:

$$w=\Phi(u_1,u_2)\exp\biggl\{\frac{k}{b\beta}\sinh(\beta y)+\frac{s}{a}\int_0^x\cosh\biggl[\frac{c\gamma}{a\lambda}\bigl(\cosh(\lambda t)-\cosh(\lambda x)\bigr)+\gamma z\biggr]\,dt\biggr\},$$

where $u_1=bx-ay$ and $u_2=a\lambda z-c\sinh(\lambda x)$.

8. $$\frac{\partial w}{\partial x}+a\cosh^n(\beta x)\frac{\partial w}{\partial y}+b\cosh^k(\lambda x)\frac{\partial w}{\partial z}=c\cosh^m(\gamma x)w.$$

This is a special case of equation 2.3.7.1 with $f(x)=a\cosh^n(\beta x)$, $g(x)=b\cosh^k(\lambda x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=c\cosh^m(\gamma x)$.

9. $$a\frac{\partial w}{\partial x}+b\cosh(\beta y)\frac{\partial w}{\partial y}+c\cosh(\lambda x)\frac{\partial w}{\partial z}=k\cosh(\gamma z)w.$$

General solution:

$$w=\Phi(u_1,u_2)\exp\biggl\{\frac{k}{a}\int_0^x\cosh\biggl(\gamma z+\frac{c\gamma}{a\lambda}\bigl[\sinh(\lambda t)-\sinh(\lambda x)\bigr]\biggr)\,dt\biggr\},$$

where $u_1=b\beta x-2a\arctan\Bigl|\tanh\dfrac{\beta y}{2}\Bigr|$ and $u_2=a\lambda z-c\sinh(\lambda x)$.

10. $$a_1\cosh^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x}+b_1\cosh^{m_1}(\beta_1 y)\frac{\partial w}{\partial y}+c_1\cosh^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z}=\bigl[a_2\cosh^{n_2}(\lambda_2 x)+b_2\cosh^{m_2}(\beta_2 y)+c_2\cosh^{k_2}(\gamma_2 z)\bigr]w.$$

This is a special case of equation 2.3.7.10 in which $f(x)=a_1\cosh^{n_1}(\lambda_1 x)$, $g(y)=b_1\cosh^{m_1}(\beta_1 y)$, $h(z)=c_1\cosh^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\cosh^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\cosh^{m_2}(\beta_2 y)$, $\chi(z)=c_2\cosh^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain hyperbolic tangent.**

11. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\tanh^n(\beta x)w.$

General solution: $w = \Phi(y - ax, z - bx)\exp\left[c\int \tanh^n(\beta x)\,dx\right].$

12. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\tanh(\beta z)\dfrac{\partial w}{\partial z} = \bigl[k\tanh(\lambda x) + s\tanh(\gamma y)\bigr]w.$

General solution: $w = \cosh^{k/a\lambda}(\lambda x)\cosh^{s/b\gamma}(\gamma y)\Phi(u_1, u_2)$, where

$$u_1 = bx - ay, \quad u_2 = c\beta x - a\ln\bigl|\sinh(\beta z)\bigr|.$$

13. $\dfrac{\partial w}{\partial x} + a\tanh^n(\beta x)\dfrac{\partial w}{\partial y} + b\tanh^k(\lambda x)\dfrac{\partial w}{\partial z} = c\tanh^m(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x) = a\tanh^n(\beta x)$, $g(x) = b\tanh^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\tanh^m(\gamma x)$.

14. $a\dfrac{\partial w}{\partial x} + b\tanh(\beta y)\dfrac{\partial w}{\partial y} + c\tanh(\lambda x)\dfrac{\partial w}{\partial z} = k\tanh(\gamma z)w.$

General solution:

$$w = \Phi(u_1, u_2)\exp\left\{\frac{k}{a}\int_0^x \tanh\left(\gamma z + \frac{c\gamma}{a\lambda}\bigl[\ln|\cosh(\lambda x)| - \ln|\cosh(\lambda t)|\bigr]\right)dt\right\},$$

where $u_1 = b\beta x - a\ln\bigl|\sinh(\beta y)\bigr|$ and $u_2 = a\lambda z - c\ln\bigl|\cosh(\lambda x)\bigr|$.

15. $a\dfrac{\partial w}{\partial x} + b\tanh(\beta y)\dfrac{\partial w}{\partial y} + c\tanh(\gamma z)\dfrac{\partial w}{\partial z} = k\tanh(\lambda x)w.$

General solution: $w = \cosh^{k/a\lambda}(\lambda x)\Phi(u_1, u_2)$, where

$$u_1 = b\beta x - a\ln\bigl|\sinh(\beta y)\bigr|, \quad u_2 = c\gamma x - a\ln\bigl|\sinh(\gamma z)\bigr|.$$

16. $a_1\tanh^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\tanh^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\tanh^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= \bigl[a_2\tanh^{n_2}(\lambda_2 x) + b_2\tanh^{m_2}(\beta_2 y) + c_2\tanh^{k_2}(\gamma_2 z)\bigr]w.$

This is a special case of equation 2.3.7.10 in which $f(x) = a_1\tanh^{n_1}(\lambda_1 x)$, $g(y) = b_1\tanh^{m_1}(\beta_1 y)$, $h(z) = c_1\tanh^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2\tanh^{n_2}(\lambda_2 x)$, $\psi(y) = b_2\tanh^{m_2}(\beta_2 y)$, and $\chi(z) = c_2\tanh^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain hyperbolic cotangent.**

17. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\coth^n(\beta x)w.$

General solution: $w = \Phi(y - ax, z - bx)\exp\left[c\int \coth^n(\beta x)\,dx\right].$

18. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\coth(\beta z)\frac{\partial w}{\partial z}=[k\coth(\lambda x)+s\coth(\gamma y)]w.$

General solution: $w=|\sinh(\lambda x)|^{k/a\lambda}|\sinh(\gamma y)|^{s/b\gamma}\Phi(u_1,u_2)$, where

$$u_1=bx-ay,\quad u_2=c\beta x-a\ln\bigl[\cosh(\beta z)\bigr].$$

19. $\displaystyle \frac{\partial w}{\partial x}+a\coth^n(\beta x)\frac{\partial w}{\partial y}+b\coth^k(\lambda x)\frac{\partial w}{\partial z}=c\coth^m(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x)=a\coth^n(\beta x)$, $g(x)=b\coth^k(\lambda x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=c\coth^m(\gamma x)$.

20. $\displaystyle a\frac{\partial w}{\partial x}+b\coth(\beta y)\frac{\partial w}{\partial y}+c\coth(\lambda x)\frac{\partial w}{\partial z}=k\coth(\gamma z)w.$

General solution:

$$w=\Phi(u_1,u_2)\exp\biggl\{\frac{k}{a}\int_0^x\coth\biggl(\gamma z+\frac{c\gamma}{a\lambda}\bigl[\ln|\sinh(\lambda t)|-\ln|\sinh(\lambda x)|\bigr]\biggr)\,dt\biggr\},$$

where $u_1=b\beta x-a\ln\bigl[\cosh(\beta y)\bigr]$ and $u_2=a\lambda z-c\ln\bigl|\sinh(\lambda x)\bigr|$.

21. $\displaystyle a\frac{\partial w}{\partial x}+b\coth(\beta y)\frac{\partial w}{\partial y}+c\coth(\gamma z)\frac{\partial w}{\partial z}=k\coth(\lambda x)w.$

General solution: $w=\bigl|\sinh(\lambda x)\bigr|^{k/a\lambda}\Phi(u_1,u_2)$, where

$$u_1=b\beta x-a\ln\bigl[\cosh(\beta y)\bigr],\quad u_2=c\gamma x-a\ln\bigl[\cosh(\gamma z)\bigr].$$

22. $\displaystyle a_1\coth^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x}+b_1\coth^{m_1}(\beta_1 y)\frac{\partial w}{\partial y}+c_1\coth^{k_1}(\gamma_1 z)\frac{\partial w}{\partial z}$
$\displaystyle \qquad =\bigl[a_2\coth^{n_2}(\lambda_2 x)+b_2\coth^{m_2}(\beta_2 y)+c_2\coth^{k_2}(\gamma_2 z)\bigr]w.$

This is a special case of equation 2.3.7.10 in which $f(x)=a_1\coth^{n_1}(\lambda_1 x)$, $g(y)=b_1\coth^{m_1}(\beta_1 y)$, $h(z)=c_1\coth^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\coth^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\coth^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\coth^{k_2}(\gamma_2 z)$.

► Coefficients of equations contain different hyperbolic functions.

23. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\sinh^n(\lambda y)\frac{\partial w}{\partial z}=\bigl[s\cosh^m(\beta x)+k\sinh^l(\gamma y)\bigr]w.$

This is a special case of equation 2.3.7.18 with $f(x,y)=c\sinh^n(\lambda y)$ and $g(x,y)=s\cosh^m(\beta x)+k\sinh^l(\gamma y)$.

24. $\displaystyle \frac{\partial w}{\partial x}+a\sinh^n(\lambda x)\frac{\partial w}{\partial y}+b\cosh^m(\beta x)\frac{\partial w}{\partial z}=s\cosh^k(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x)=a\sinh^n(\lambda x)$, $g(x)=b\cosh^m(\beta x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=s\cosh^k(\gamma x)$.

25. $\displaystyle \frac{\partial w}{\partial x}+a\cosh^n(\lambda x)\frac{\partial w}{\partial y}+b\sinh^m(\beta y)\frac{\partial w}{\partial z}=s\sinh^k(\gamma z)w.$

This is a special case of equation 2.3.7.22 with $f_1(x)=0$, $f_2(x)=a\cosh^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y)=b\sinh^m(\beta y)$, and $h(x,y,z)=s\sinh^k(\gamma z)$.

26. $\dfrac{\partial w}{\partial x}+a\tanh^n(\lambda x)\dfrac{\partial w}{\partial y}+b\coth^m(\beta x)\dfrac{\partial w}{\partial z}=s\coth^k(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x)=a\tanh^n(\lambda x)$, $g(x)=b\coth^m(\beta x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=s\coth^k(\gamma x)$.

27. $a\sinh(\lambda x)\dfrac{\partial w}{\partial x}+b\sinh(\beta y)\dfrac{\partial w}{\partial y}+c\sinh(\gamma z)\dfrac{\partial w}{\partial z}=k\cosh(\lambda x)w.$

General solution: $w=\bigl|\sinh(\lambda x)\bigr|^{k/a\lambda}\Phi(u_1,u_2)$, where

$$u_1=\frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right|-\frac{1}{b\beta}\ln\left|\tanh\frac{\beta y}{2}\right|,\quad u_2=\frac{1}{a\lambda}\ln\left|\tanh\frac{\lambda x}{2}\right|-\frac{1}{c\gamma}\ln\left|\tanh\frac{\gamma z}{2}\right|.$$

2.3.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\ln^n(\beta x)w.$

General solution: $w=\Phi(y-ax,z-bx)\exp\left[c\displaystyle\int\ln^n(\beta x)\,dx\right].$

2. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\ln^n(\beta x)\dfrac{\partial w}{\partial z}=s\ln^m(\lambda y)w.$

This is a special case of equation 2.3.7.18 with $f(x,y)=c\ln^n(\beta x)$ and $g(x,y)=s\ln^m(\lambda y)$.

3. $\dfrac{\partial w}{\partial x}+a\ln^n(\beta x)\dfrac{\partial w}{\partial y}+b\ln^k(\lambda x)\dfrac{\partial w}{\partial z}=c\ln^m(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x)=a\ln^n(\beta x)$, $g(x)=b\ln^k(\lambda x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=c\ln^m(\gamma x)$.

4. $\dfrac{\partial w}{\partial x}+a\ln^n(\beta x)\dfrac{\partial w}{\partial y}+b\ln^k(\lambda y)\dfrac{\partial w}{\partial z}=c\ln^m(\gamma x)w.$

This is a special case of equation 2.3.7.22 with $f_1(x)=0$, $f_2(x)=a\ln^n(\beta x)$, $g_1(x,y)=0$, $g_2(x,y)=b\ln^k(\lambda y)$, and $h(x,y)=c\ln^m(\gamma x)$.

5. $a_1\ln^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_1\ln^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y}+c_1\ln^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$\qquad=\bigl[a_2\ln^{n_2}(\lambda_2 x)+b_2\ln^{m_2}(\beta_2 y)+c_2\ln^{k_2}(\gamma_2 z)\bigr]w.$

This is a special case of equation 2.3.7.10 with $f(x)=a_1\ln^{n_1}(\lambda_1 x)$, $g(y)=b_1\ln^{m_1}(\beta_1 y)$, $h(z)=c_1\ln^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\ln^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\ln^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\ln^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain logarithmic and power-law functions.**

6. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+cx^n\ln^k(\lambda y)\dfrac{\partial w}{\partial z}=sy^m\ln^l(\beta x)w.$

This is a special case of equation 2.3.7.18 with $f(x,y)=cx^n\ln^k(\lambda y)$ and $g(x,y)=sy^m\ln^l(\beta x)$.

7. $\dfrac{\partial w}{\partial x} + ax^n \dfrac{\partial w}{\partial y} + bx^m \dfrac{\partial w}{\partial z} = \bigl[cy \ln^k(\lambda x) + sz \ln^l(\beta x)\bigr] w.$

This is a special case of equation 2.3.7.1 with $f(x) = ax^n$, $g(x) = bx^m$, $h_2(x) = c \ln^k(\lambda x)$, $h_1(x) = s \ln^l(\beta x)$, and $h_0(x) = 0$.

8. $\dfrac{\partial w}{\partial x} + a \ln^n(\lambda x) \dfrac{\partial w}{\partial y} + by^m \dfrac{\partial w}{\partial z} = \bigl[c \ln^k(\beta x) + s \ln^l(\gamma z)\bigr] w.$

This is a special case of equation 2.3.7.22 with $f_1(x) = 0$, $f_2(x) = a \ln^n(\lambda x)$, $g_1(x, y) = 0$, $g_2(x, y) = by^m$, and $h(x, y, z) = c \ln^k(\beta x) + s \ln^l(\gamma z)$.

9. $a \ln^n(\lambda x) \dfrac{\partial w}{\partial x} + z \dfrac{\partial w}{\partial y} + b \ln^k(\beta y) \dfrac{\partial w}{\partial z} = \bigl[cx^m + s \ln(\gamma y)\bigr] w.$

This is a special case of equation 2.3.7.11 with $f(x) = a \ln^n(\lambda x)$, $g(y) = b \ln^k(\beta y)$, $h_2(x) = cx^m$, and $h_1(y) = s \ln(\gamma y)$.

10. $ax(\ln x)^n \dfrac{\partial w}{\partial x} + by(\ln y)^m \dfrac{\partial w}{\partial y} + cz(\ln z)^l \dfrac{\partial w}{\partial z} = k(\ln x)^s w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{k}{a(s-n+1)} (\ln x)^{s-n+1}\right] \Phi(u_1,\, u_2) & \text{if } \ s+1 \neq n, \\ (\ln x)^{k/a} \Phi(u_1,\, u_2) & \text{if } \ s+1 = n, \end{cases}$$

where

$$u_1 = \frac{(\ln x)^{1-n}}{a(n-1)} - \frac{(\ln y)^{1-m}}{b(m-1)}, \quad u_2 = \frac{(\ln x)^{1-n}}{a(n-1)} - \frac{(\ln z)^{1-l}}{c(l-1)}.$$

2.3.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $\dfrac{\partial w}{\partial x} + a \dfrac{\partial w}{\partial y} + b \dfrac{\partial w}{\partial z} = c \sin^n(\lambda x) w.$

General solution: $w = \Phi(y - ax, z - bx) \exp\left[c \int \sin^n(\lambda x)\, dx\right].$

2. $a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} + c \sin(\lambda z) \dfrac{\partial w}{\partial z} = \bigl[k \sin(\gamma x) + s \sin(\beta y)\bigr] w.$

General solution: $w = \Phi(u_1, u_2) \exp\left[-\dfrac{k}{a\gamma} \cos(\gamma x) - \dfrac{s}{b\beta} \cos(\beta y)\right]$, where

$$u_1 = bx - ay, \quad u_2 = c\lambda x - a \ln\left|\tan \frac{\lambda z}{2}\right|.$$

3. $\dfrac{\partial w}{\partial x} + a \sin^n(\lambda x) \dfrac{\partial w}{\partial y} + b \sin^m(\beta x) \dfrac{\partial w}{\partial z} = c \sin^k(\gamma x) w.$

This is a special case of equation 2.3.7.1 with $f(x) = a \sin^n(\lambda x)$, $g(x) = b \sin^m(\beta x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c \sin^k(\gamma x)$.

4. $\dfrac{\partial w}{\partial x} + a\sin^n(\lambda x)\dfrac{\partial w}{\partial y} + b\sin^m(\beta y)\dfrac{\partial w}{\partial z} = \big[c\sin^k(\gamma y) + s\sin^l(\mu z)\big]w.$

This is a special case of equation 2.3.7.22 with $f_1(x)=0$, $f_2(x)=a\sin^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y) = b\sin^m(\beta y)$, and $h(x,y,z) = c\sin^k(\gamma y) + s\sin^l(\mu z)$.

5. $a\dfrac{\partial w}{\partial x} + b\sin(\beta y)\dfrac{\partial w}{\partial y} + c\sin(\lambda x)\dfrac{\partial w}{\partial z} = k\sin(\gamma z).$

General solution:

$$w = \Phi(u_1, u_2)\exp\left\{\frac{k}{a}\int_0^x \sin\left(\gamma z + \frac{c\gamma}{a\lambda}\big[\cos(\lambda x) - \cos(\lambda t)\big]\right)dt\right\},$$

where $u_1 = b\beta x - a\ln\left|\tan\dfrac{\beta y}{2}\right|$ and $u_2 = a\lambda z + c\cos(\lambda x)$.

6. $a_1\sin^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\sin^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\sin^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= \big[a_2\sin^{n_2}(\lambda_2 x) + b_2\sin^{m_2}(\beta_2 y) + c_2\sin^{k_2}(\gamma_2 z)\big]w.$

This is a special case of equation 2.3.7.10 with $f(x)=a_1\sin^{n_1}(\lambda_1 x)$, $g(y)=b_1\sin^{m_1}(\beta_1 y)$, $h(z)=c_1\sin^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\sin^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\sin^{m_2}(\beta_2 y)$, $\chi(z)=c_2\sin^{k_2}(\gamma_2 z)$.

▶ Coefficients of equations contain cosine.

7. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\cos^n(\beta x)w.$

General solution: $w = \Phi(y - ax, z - bx)\exp\left[c\displaystyle\int \cos^n(\beta x)\,dx\right].$

8. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\cos(\beta z)\dfrac{\partial w}{\partial z} = \big[k\cos(\lambda x) + s\cos(\gamma y)\big]w.$

General solution: $w = \Phi(u_1, u_2)\exp\left[\dfrac{k}{a\lambda}\sin(\lambda x) + \dfrac{s}{b\gamma}\sin(\gamma y)\right]$, where

$$u_1 = bx - ay, \quad u_2 = c\beta x - a\ln\big|\sec(\beta z) + \tan(\beta z)\big|.$$

9. $\dfrac{\partial w}{\partial x} + a\cos^n(\beta x)\dfrac{\partial w}{\partial y} + b\cos^k(\lambda x)\dfrac{\partial w}{\partial z} = c\cos^m(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x) = a\cos^n(\beta x)$, $g(x) = b\cos^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\cos^m(\gamma x)$.

10. $\dfrac{\partial w}{\partial x} + a\cos^n(\beta x)\dfrac{\partial w}{\partial y} + b\cos^m(\lambda y)\dfrac{\partial w}{\partial z} = \big[c\cos^k(\gamma y) + s\cos^l(\mu z)\big]w.$

This is a special case of equation 2.3.7.22 with $f_1(x)=0$, $f_2(x)=a\cos^n(\beta x)$, $g_1(x,y)=0$, $g_2(x,y) = b\cos^m(\lambda y)$, and $h(x,y,z) = c\cos^k(\gamma y) + s\cos^l(\mu z)$.

11. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\cos(\beta y)\dfrac{\partial w}{\partial y}+c\cos(\lambda x)\dfrac{\partial w}{\partial z}=k\cos(\gamma z)w.}$

General solution:

$$w=\Phi(u_1,u_2)\exp\biggl\{\frac{k}{a}\int_0^x\cos\biggl(\gamma z+\frac{c\gamma}{a\lambda}\bigl[\sin(\lambda t)-\sin(\lambda x)\bigr]\biggr)\,dt\biggr\},$$

where $u_1=b\beta x-a\ln\bigl|\sec(\beta y)+\tan(\beta y)\bigr|$ and $u_2=a\lambda z-c\sin(\lambda x)$.

12. $\boldsymbol{a_1\cos^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_1\cos^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y}+c_1\cos^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}}$

$$\boldsymbol{=\bigl[a_2\cos^{n_2}(\lambda_2 x)+b_2\cos^{m_2}(\beta_2 y)+c_2\cos^{k_2}(\gamma_2 z)\bigr]w.}$$

This is a special case of equation 2.3.7.10 in which $f(x)=a_1\cos^{n_1}(\lambda_1 x)$, $g(y)=b_1\cos^{m_1}(\beta_1 y)$, $h(z)=c_1\cos^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\cos^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\cos^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\cos^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain tangent.**

13. $\boldsymbol{\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\tan^n(\beta x)w.}$

General solution: $w=\Phi(y-ax,z-bx)\exp\biggl[c\displaystyle\int\tan^n(\beta x)\,dx\biggr]$.

14. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\tan(\beta z)\dfrac{\partial w}{\partial z}=\bigl[k\tan(\lambda x)+s\tan(\gamma y)\bigr]w.}$

General solution:

$$w=\bigl|\cos(\lambda x)\bigr|^{-k/a\lambda}\bigl|\cos(\gamma y)\bigr|^{-s/b\gamma}\Phi(u_1,u_2),$$

where $u_1=bx-ay$ and $u_2=c\beta x-a\ln\bigl|\sin(\beta z)\bigr|$.

15. $\boldsymbol{\dfrac{\partial w}{\partial x}+a\tan^n(\beta x)\dfrac{\partial w}{\partial y}+b\tan^k(\lambda x)\dfrac{\partial w}{\partial z}=c\tan^m(\gamma x)w.}$

This is a special case of equation 2.3.7.1 with $f(x)=a\tan^n(\beta x)$, $g(x)=b\tan^k(\lambda x)$, $h_2(x)=h_1(x)=0$, and $h_0(x)=c\tan^m(\gamma x)$.

16. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\tan(\beta y)\dfrac{\partial w}{\partial y}+c\tan(\lambda x)\dfrac{\partial w}{\partial z}=k\tan(\gamma z)w.}$

General solution:

$$w=\Phi(u_1,u_2)\exp\biggl\{\frac{k}{a}\int_0^x\tan\biggl(\gamma z+\frac{c\gamma}{a\lambda}\bigl[\ln|\cos(\lambda x)|-\ln|\cos(\lambda t)|\bigr]\biggr)\,dt\biggr\},$$

where $u_1=b\beta x-a\ln\bigl|\sin(\beta y)\bigr|$ and $u_2=a\lambda z+c\ln\bigl|\cos(\lambda x)\bigr|$.

17. $\boldsymbol{a_1\tan^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_1\tan^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y}+c_1\tan^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}}$

$$\boldsymbol{=\bigl[a_2\tan^{n_2}(\lambda_2 x)+b_2\tan^{m_2}(\beta_2 y)+c_2\tan^{k_2}(\gamma_2 z)\bigr]w.}$$

This is a special case of equation 2.3.7.10 in which $f(x)=a_1\tan^{n_1}(\lambda_1 x)$, $g(y)=b_1\tan^{m_1}(\beta_1 y)$, $h(z)=c_1\tan^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\tan^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\tan^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\tan^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain cotangent.**

18. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\cot^n(\beta x)w.$

General solution: $w = \Phi(y - ax, z - bx)\exp\left[c\int \cot^n(\beta x)\,dx\right].$

19. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\cot(\beta z)\dfrac{\partial w}{\partial z} = \bigl[k\cot(\lambda x) + s\cot(\gamma y)\bigr]w.$

General solution: $w = \bigl|\sin(\lambda x)\bigr|^{k/a\lambda}\bigl|\sin(\gamma y)\bigr|^{s/b\gamma}\Phi(u_1, u_2)$, where

$$u_1 = bx - ay, \quad u_2 = c\beta x + a\ln\bigl|\cos(\beta z)\bigr|.$$

20. $\dfrac{\partial w}{\partial x} + a\cot^n(\beta x)\dfrac{\partial w}{\partial y} + b\cot^k(\lambda x)\dfrac{\partial w}{\partial z} = c\cot^m(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x) = a\cot^n(\beta x)$, $g(x) = b\cot^k(\lambda x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\cot^m(\gamma x)$.

21. $a\dfrac{\partial w}{\partial x} + b\cot(\beta y)\dfrac{\partial w}{\partial y} + c\cot(\lambda x)\dfrac{\partial w}{\partial z} = k\cot(\gamma z)w.$

General solution:

$$w = \Phi(u_1, u_2)\exp\left\{\frac{k}{a}\int_0^x \cot\left(\gamma z + \frac{c\gamma}{a\lambda}\bigl[\ln|\sin(\lambda t)| - \ln|\sin(\lambda x)|\bigr]\right)dt\right\},$$

where $u_1 = b\beta x + a\ln\bigl|\cos(\beta y)\bigr|$ and $u_2 = a\lambda z - c\ln\bigl|\sin(\lambda x)\bigr|$.

22. $a_1\cot^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\cot^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\cot^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= \bigl[a_2\cot^{n_2}(\lambda_2 x) + b_2\cot^{m_2}(\beta_2 y) + c_2\cot^{k_2}(\gamma_2 z)\bigr]w.$

This is a special case of equation 2.3.7.10 in which $f(x) = a_1\cot^{n_1}(\lambda_1 x)$, $g(y) = b_1\cot^{m_1}(\beta_1 y)$, $h(z) = c_1\cot^{k_1}(\gamma_1 z)$, $\varphi(x) = a_2\cot^{n_2}(\lambda_2 x)$, $\psi(y) = b_2\cot^{m_2}(\beta_2 y)$, and $\chi(z) = c_2\cot^{k_2}(\gamma_2 z)$.

► **Coefficients of equations contain different trigonometric functions.**

23. $\dfrac{\partial w}{\partial x} + a\sin^n(\lambda x)\dfrac{\partial w}{\partial y} + b\cos^m(\beta x)\dfrac{\partial w}{\partial z} = c\sin^k(\gamma x)w.$

This is a special case of equation 2.3.7.1 with $f(x) = a\sin^n(\lambda x)$, $g(x) = b\cos^m(\beta x)$, $h_2(x) = h_1(x) = 0$, and $h_0(x) = c\sin^k(\gamma x)$.

24. $\dfrac{\partial w}{\partial x} + a\cos^n(\lambda x)\dfrac{\partial w}{\partial y} + b\sin^m(\beta y)\dfrac{\partial w}{\partial z} = \bigl[c\cos^k(\gamma y) + s\sin^l(\mu z)\bigr]w.$

This is a special case of equation 2.3.7.22 with $f_1(x) = 0$, $f_2(x) = a\cos^n(\lambda x)$, $g_1(x,y) = 0$, $g_2(x,y) = b\sin^m(\beta y)$, and $h(x,y,z) = c\cos^k(\gamma y) + s\sin^l(\mu z)$.

25. $\dfrac{\partial w}{\partial x} + a\cos^n(\lambda x)\dfrac{\partial w}{\partial y} + b\tan^m(\beta y)\dfrac{\partial w}{\partial z} = \bigl[c\cos^k(\gamma y) + s\tan^l(\mu z)\bigr]w.$

This is a special case of equation 2.3.7.22 with $f_1(x)=0$, $f_2(x)=a\cos^n(\lambda x)$, $g_1(x,y)=0$, $g_2(x,y)=b\tan^m(\beta y)$, and $h(x,y,z)=c\cos^k(\gamma y)+s\tan^l(\mu z)$.

26. $a_1\sin^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\cos^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\cos^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= \bigl[a_2\cos^{n_2}(\lambda_2 x) + b_2\sin^{m_2}(\beta_2 y) + c_2\cos^{k_2}(\gamma_2 z)\bigr]w.$

This is a special case of equation 2.3.7.10 with $f(x)=a_1\sin^{n_1}(\lambda_1 x)$, $g(y)=b_1\cos^{m_1}(\beta_1 y)$, $h(z)=c_1\cos^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\cos^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\sin^{m_2}(\beta_2 y)$, $\chi(z)=c_2\cos^{k_2}(\gamma_2 z)$.

27. $a_1\tan^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_1\cot^{m_1}(\beta_1 y)\dfrac{\partial w}{\partial y} + c_1\cot^{k_1}(\gamma_1 z)\dfrac{\partial w}{\partial z}$
$= \bigl[a_2\cot^{n_2}(\lambda_2 x) + b_2\tan^{m_2}(\beta_2 y) + c_2\cot^{k_2}(\gamma_2 z)\bigr]w.$

This is a special case of equation 2.3.7.10 in which $f(x)=a_1\tan^{n_1}(\lambda_1 x)$, $g(y)=b_1\cot^{m_1}(\beta_1 y)$, $h(z)=c_1\cot^{k_1}(\gamma_1 z)$, $\varphi(x)=a_2\cot^{n_2}(\lambda_2 x)$, $\psi(y)=b_2\tan^{m_2}(\beta_2 y)$, and $\chi(z)=c_2\cot^{k_2}(\gamma_2 z)$.

2.3.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\arcsin^n(\beta x)w.$

General solution: $w=\Phi(y-ax,\,z-bx)\exp\left[c\displaystyle\int \arcsin^n(\beta x)\,dx\right].$

2. $a_1\dfrac{\partial w}{\partial x} + a_2\dfrac{\partial w}{\partial y} + a_3\dfrac{\partial w}{\partial z}$
$= \bigl[b_1\arcsin(\lambda_1 x) + b_2\arcsin(\lambda_2 y) + b_3\arcsin(\lambda_3 z)\bigr]w.$

This is a special case of equation 2.3.7.10 with $f(x)=a_1$, $g(y)=a_2$, $h(z)=a_3$, $\varphi(x)=b_1\arcsin(\lambda_1 x)$, $\psi(y)=b_2\arcsin(\lambda_2 y)$, and $\chi(z)=b_3\arcsin(\lambda_3 z)$.

3. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\arcsin^n(\lambda x)\arcsin^k(\beta z)\dfrac{\partial w}{\partial z} = s\arcsin^m(\gamma x)w.$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\arcsin^n(\lambda x)$, $f_4(x)=s\arcsin^m(\gamma x)$, $g(y)=1$, and $h(z)=\arcsin^k(\beta z)$.

4. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\arcsin^n(\lambda x)\arcsin^m(\beta y)\arcsin^k(\gamma z)\dfrac{\partial w}{\partial z} = sw.$

This is a special case of equation 2.3.7.19 with $f(x,y)=c\arcsin^n(\lambda x)\arcsin^m(\beta y)$, $g(z)=\arcsin^k(\gamma z)$, and $h(x,y)=s$.

5. $a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y} + c\arcsin^k(\beta z)\dfrac{\partial w}{\partial z} = s\arcsin^m(\gamma x)w.$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b\arcsin^n(\lambda x)$, $f_3(x)=1$, $f_4(x)=s\arcsin^m(\gamma x)$, $g(y)=1$, and $h(z)=c\arcsin^k(\beta z)$.

6. $\displaystyle a\frac{\partial w}{\partial x}+b\arcsin^n(\lambda y)\frac{\partial w}{\partial y}+c\arcsin^k(\beta z)\frac{\partial w}{\partial z}=sw.$

This is a special case of equation 2.3.7.10 with $f(x)=a$, $g(y)=b\arcsin^n(\lambda y)$, $h(z)=c\arcsin^k(\beta z)$, $\varphi(x)=s$, and $\psi(y)=\chi(z)=0$.

► **Coefficients of equations contain arccosine.**

7. $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=c\arccos^n(\beta x)w.$

General solution: $\displaystyle w=\Phi(y-ax,z-bx)\exp\biggl[c\int\arccos^n(\beta x)\,dx\biggr].$

8. $$a_1\frac{\partial w}{\partial x}+a_2\frac{\partial w}{\partial y}+a_3\frac{\partial w}{\partial z}=\bigl[b_1\arccos(\lambda_1 x)+b_2\arccos(\lambda_2 y)+b_3\arccos(\lambda_3 z)\bigr]w.$$

This is a special case of equation 2.3.7.10 with $f(x)=a_1$, $g(y)=a_2$, $h(z)=a_3$, $\varphi(x)=b_1\arccos(\lambda_1 x)$, $\psi(y)=b_2\arccos(\lambda_2 y)$, and $\chi(z)=b_3\arccos(\lambda_3 z)$.

9. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arccos^n(\lambda x)\arccos^k(\beta z)\frac{\partial w}{\partial z}=s\arccos^m(\gamma x)w.$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\arccos^n(\lambda x)$, $f_4(x)=s\arccos^m(\gamma x)$, $g(y)=1$, and $h(z)=\arccos^k(\beta z)$.

10. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arccos^n(\lambda x)\arccos^m(\beta y)\arccos^k(\gamma z)\frac{\partial w}{\partial z}=sw.$

This is a special case of equation 2.3.7.19 with $f(x,y)=c\arccos^n(\lambda x)\arccos^m(\beta y)$, $g(z)=\arccos^k(\gamma z)$, and $h(x,y)=s$.

11. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda x)\frac{\partial w}{\partial y}+c\arccos^k(\beta z)\frac{\partial w}{\partial z}=s\arccos^m(\gamma x)w.$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b\arccos^n(\lambda x)$, $f_3(x)=1$, $f_4(x)=s\arccos^m(\gamma x)$, $g(y)=1$, and $h(z)=c\arccos^k(\beta z)$.

12. $\displaystyle a\frac{\partial w}{\partial x}+b\arccos^n(\lambda y)\frac{\partial w}{\partial y}+c\arccos^k(\beta z)\frac{\partial w}{\partial z}=sw.$

This is a special case of equation 2.3.7.10 with $f(x)=a$, $g(y)=b\arccos^n(\lambda y)$, $h(z)=c\arccos^k(\beta z)$, $\varphi(x)=s$, and $\psi(y)=\chi(z)=0$.

► **Coefficients of equations contain arctangent.**

13. $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=c\arctan^n(\beta x)w.$

General solution: $\displaystyle w=\Phi(y-ax,z-bx)\exp\biggl[c\int\arctan^n(\beta x)\,dx\biggr].$

14. $$a_1\frac{\partial w}{\partial x}+a_2\frac{\partial w}{\partial y}+a_3\frac{\partial w}{\partial z}=\big[b_1\arctan(\lambda_1 x)+b_2\arctan(\lambda_2 y)+b_3\arctan(\lambda_3 z)\big]w.$$

This is a special case of equation 2.3.7.10 with $f(x)=a_1$, $g(y)=a_2$, $h(z)=a_3$, $\varphi(x)=b_1\arctan(\lambda_1 x)$, $\psi(y)=b_2\arctan(\lambda_2 y)$, and $\chi(z)=b_3\arctan(\lambda_3 z)$.

15. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arctan^n(\lambda x)\arctan^k(\beta z)\frac{\partial w}{\partial z}=s\arctan^m(\gamma x)w.$$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\arctan^n(\lambda x)$, $f_4(x)=s\arctan^m(\gamma x)$, $g(y)=1$, and $h(z)=\arctan^k(\beta z)$.

16. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\arctan^n(\lambda x)\arctan^m(\beta y)\arctan^k(\gamma z)\frac{\partial w}{\partial z}=sw.$$

This is a special case of equation 2.3.7.19 with $f(x,y)=c\arctan^n(\lambda x)\arctan^m(\beta y)$, $g(z)=\arctan^k(\gamma z)$, and $h(x,y)=s$.

17. $$a\frac{\partial w}{\partial x}+b\arctan^n(\lambda x)\frac{\partial w}{\partial y}+c\arctan^k(\beta z)\frac{\partial w}{\partial z}=s\arctan^m(\gamma x)w.$$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b\arctan^n(\lambda x)$, $f_3(x)=1$, $f_4(x)=s\arctan^m(\gamma x)$, $g(y)=1$, and $h(z)=c\arctan^k(\beta z)$.

► **Coefficients of equations contain arccotangent.**

18. $$\frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=c\,\mathrm{arccot}^n(\beta x)w.$$

General solution: $w=\Phi(y-ax,z-bx)\exp\Big[c\int\mathrm{arccot}^n(\beta x)\,dx\Big]$.

19. $$a_1\frac{\partial w}{\partial x}+a_2\frac{\partial w}{\partial y}+a_3\frac{\partial w}{\partial z}=\big[b_1\,\mathrm{arccot}(\lambda_1 x)+b_2\,\mathrm{arccot}(\lambda_2 y)+b_3\,\mathrm{arccot}(\lambda_3 z)\big]w.$$

This is a special case of equation 2.3.7.10 with $f(x)=a_1$, $g(y)=a_2$, $h(z)=a_3$, $\varphi(x)=b_1\,\mathrm{arccot}(\lambda_1 x)$, $\psi(y)=b_2\,\mathrm{arccot}(\lambda_2 y)$, and $\chi(z)=b_3\,\mathrm{arccot}(\lambda_3 z)$.

20. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^k(\beta z)\frac{\partial w}{\partial z}=s\,\mathrm{arccot}^m(\gamma x)w.$$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b$, $f_3(x)=c\,\mathrm{arccot}^n(\lambda x)$, $f_4(x)=s\,\mathrm{arccot}^m(\gamma x)$, $g(y)=1$, and $h(z)=\mathrm{arccot}^k(\beta z)$.

21. $$a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^m(\beta y)\,\mathrm{arccot}^k(\gamma z)\frac{\partial w}{\partial z}=sw.$$

This is a special case of equation 2.3.7.19 with $f(x,y)=c\,\mathrm{arccot}^n(\lambda x)\,\mathrm{arccot}^m(\beta y)$, $g(z)=\mathrm{arccot}^k(\gamma z)$, and $h(x,y)=s$.

22. $$a\frac{\partial w}{\partial x}+b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y}+c\,\mathrm{arccot}^k(\beta z)\frac{\partial w}{\partial z}=s\,\mathrm{arccot}^m(\gamma x)w.$$

This is a special case of equation 2.3.7.12 with $f_1(x)=a$, $f_2(x)=b\,\mathrm{arccot}^n(\lambda x)$, $f_3(x)=1$, $f_4(x)=s\,\mathrm{arccot}^m(\gamma x)$, $g(y)=1$, and $h(z)=c\,\mathrm{arccot}^k(\beta z)$.

2.3.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of x.**

◆ *Throughout Section 2.3.7, sometimes only a particular solution* $\widetilde{w}$ *of the equation and a basis* u_1, u_2 *of the corresponding "truncated" equation with a zero right-hand side are presented. The general solution can be obtained as* $w = \widetilde{w}\,\Phi(u_1,u_2)$, *where* $\Phi(u_1,u_2)$ *is an arbitrary function of two variables.*

1. $\displaystyle \frac{\partial w}{\partial x} + f(x)\frac{\partial w}{\partial y} + g(x)\frac{\partial w}{\partial z} = \big[h_2(x)y + h_1(x)z + h_0(x)\big]w.$

General solution:

$$w = \exp\left[H_2(x)y + H_1(x)z + H_0(x) - \int f(x)H_2(x)\,dx - \int g(x)H_1(x)\,dx\right]\Phi(u_1,u_2),$$

where

$$H_k(x) = \int h_k(x)\,dx \quad (k=0,1,2), \quad u_1 = y - \int f(x)\,dx, \quad u_2 = z - \int g(x)\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

2. $\displaystyle \frac{\partial w}{\partial x} + f(x)(y+a)\frac{\partial w}{\partial y} + g(x)(z+b)\frac{\partial w}{\partial z} = h(x)w.$

General solution:

$$w = \exp\left[\int h(x)\,dx\right]\Phi(u_1,u_2),$$
$$u_1 = \ln|y+a| - \int f(x)\,dx, \quad u_2 = \ln|z+b| - \int g(x)\,dx.$$

3. $\displaystyle \frac{\partial w}{\partial x} + [ay + f(x)]\frac{\partial w}{\partial y} + [bz + g(x)]\frac{\partial w}{\partial z} = h(x)w.$

General solution:

$$w = \exp\left[\int h(x)\,dx\right]\Phi(u_1,u_2),$$
$$u_1 = ye^{-ax} - \int f(x)e^{-ax}\,dx, \quad u_2 = ze^{-bx} - \int g(x)e^{-bx}\,dx.$$

4. $\displaystyle \frac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)\big]\frac{\partial w}{\partial y} + \big[g_1(x)y + g_2(x)\big]\frac{\partial w}{\partial z} = \big[h_2(x)y + h_1(x)z + h_0(x)\big]w.$

Particular solution:

$$\widetilde{w} = \exp\left\{\varphi(x)y + \psi(x)z + \int \big[h_0(x) - f_2(x)\varphi(x) - g_2(x)\psi(x)\big]\,dx\right\},$$
$$\varphi(x) = F(x)\int \frac{h_2(x) - g_1(x)\psi(x)}{F(x)}\,dx, \quad \psi(x) = \int h_1(x)\,dx, \quad F(x) = \exp\left[-\int f_1(x)\,dx\right].$$

For an integral basis u_1, u_2 of the corresponding "truncated" equation, see 2.1.7.4.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

5. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)z+g_2(x)\big]\dfrac{\partial w}{\partial z}=\big[h_2(x)y+h_1(x)z+h_0(x)\big]w.$

Particular solution:

$$\bar{w}=\exp\left\{\varphi(x)y+\psi(x)z+\int\big[h_0(x)-f_2(x)\varphi(x)-g_2(x)\psi(x)\big]\,dx\right\},$$

$$\varphi(x)=F(x)\int\frac{h_2(x)}{F(x)}dx,\quad F(x)=\exp\left[-\int f_1(x)\,dx\right],$$

$$\psi(x)=G(x)\int\frac{h_1(x)}{G(x)}dx,\quad G(x)=\exp\left[-\int g_1(x)\,dx\right].$$

For an integral basis u_1, u_2 of the corresponding "truncated" equation, see 2.1.7.5.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

6. $\dfrac{\partial w}{\partial x}+\big[y^2-a^2+a\lambda\sinh(\lambda x)-a^2\sinh^2(\lambda x)\big]\dfrac{\partial w}{\partial y}+f(x)\sinh(\gamma z)\dfrac{\partial w}{\partial z}=g(x)w.$

General solution: $w=\exp\left[\int g(x)\,dx\right]\Phi(u_1,u_2)$, where

$$u_1=\int f(x)\,dx-\frac{1}{\gamma}\ln\left|\tanh\frac{\gamma z}{2}\right|,\quad u_2=\frac{E}{y-a\cosh(\lambda x)}+\int E\,dx,\quad E=\exp\left[\frac{2a}{\lambda}\sinh(\lambda x)\right].$$

7. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)z+g_2(x)z^m\big]\dfrac{\partial w}{\partial z}=h(x)w.$

1°. For $k\neq 1$ and $m\neq 1$, the transformation

$$\xi=y^{1-k},\quad \eta=z^{1-m},\quad W=w\exp\left[-\int h(x)\,dx\right]$$

leads to an equation of the form 2.1.7.5:

$$\frac{\partial W}{\partial x}+(1-k)\big[f_1(x)\xi+f_2(x)\big]\frac{\partial W}{\partial \xi}+(1-m)\big[g_1(x)\eta+g_2(x)\big]\frac{\partial W}{\partial \eta}=0.$$

2°. For $k\neq 1$ and $m=1$, the transformation $\xi=y^{1-k}$, $W=w\exp\left[-\int h(x)\,dx\right]$ also leads to an equation of the form 2.1.7.5.

3°. For $k=m=1$, see equation 2.3.7.5 with $h_1(x)=h_2(x)=0$.

8. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)+g_2(x)e^{\lambda z}\big]\dfrac{\partial w}{\partial z}=h(x)w.$

The transformation

$$\xi=y^{1-k},\quad \eta=e^{-\lambda z},\quad W=w\exp\left[-\int h(x)\,dx\right]$$

leads to an equation of the form 2.1.7.5:

$$\frac{\partial W}{\partial x}+(1-k)\big[f_1(x)\xi+f_2(x)\big]\frac{\partial W}{\partial \xi}-\lambda\big[g_1(x)\eta+g_2(x)\big]\frac{\partial W}{\partial \eta}=0.$$

9. $\dfrac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} + \big[g_1(x) + g_2(x)e^{\beta z}\big]\dfrac{\partial w}{\partial z} = h(x)w.$

The transformation

$$\xi = e^{-\lambda y}, \quad \eta = e^{-\beta z}, \quad W = w \exp\Big[-\int h(x)\,dx\Big]$$

leads to an equation of the form 2.1.7.5:

$$\frac{\partial W}{\partial x} - \lambda\big[f_1(x)\xi + f_2(x)\big]\frac{\partial W}{\partial \xi} - \beta\big[g_1(x)\eta + g_2(x)\big]\frac{\partial W}{\partial \eta} = 0.$$

► **Coefficients of equations contain arbitrary functions of different variables.**

10. $f(x)\dfrac{\partial w}{\partial x} + g(y)\dfrac{\partial w}{\partial y} + h(z)\dfrac{\partial w}{\partial z} = \big[\varphi(x) + \psi(y) + \chi(z)\big]w.$

General solution:

$$w = \exp\Big[\int \frac{\varphi(x)}{f(x)}dx + \int \frac{\psi(y)}{g(y)}dy + \int \frac{\chi(z)}{h(z)}dz\Big]\Phi(u_1, u_2),$$

where

$$u_1 = \int \frac{dx}{f(x)} - \int \frac{dy}{g(y)}, \quad u_2 = \int \frac{dx}{f(x)} - \int \frac{dz}{h(z)}.$$

11. $f(x)\dfrac{\partial w}{\partial x} + z\dfrac{\partial w}{\partial y} + g(y)\dfrac{\partial w}{\partial z} = \big[h_2(x) + h_1(y)\big]w.$

General solution:

$$w = \exp\Big[\int \frac{h_2(x)}{f(x)}dx + \int_{y_0}^{y} \frac{h_1(t)\,dt}{\sqrt{2G(t) - 2G(y) + z^2}}\Big]\Phi(u_1, u_2),$$

where

$$G(y) = \int g(y)\,dy, \quad u_1 = G(y) - \frac{z^2}{2}, \quad u_2 = \int \frac{dx}{f(x)} - \int_{y_0}^{y} \frac{dt}{\sqrt{2G(t) - 2G(y) + z^2}}.$$

12. $f_1(x)\dfrac{\partial w}{\partial x} + f_2(x)g(y)\dfrac{\partial w}{\partial y} + f_3(x)h(z)\dfrac{\partial w}{\partial z} = f_4(x)w.$

General solution:

$$w = \exp\Big[\int \frac{f_4(x)}{f_1(x)}dx\Big]\Phi(u_1, u_2),$$

$$u_1 = \int \frac{f_2(x)}{f_1(x)}dx - \int \frac{dy}{g(y)}, \quad u_2 = \int \frac{f_3(x)}{f_1(x)}dx - \int \frac{dz}{h(z)}.$$

13. $\dfrac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)\big]\dfrac{\partial w}{\partial y} + \big[g_1(x)z + g_2(y)\big]\dfrac{\partial w}{\partial z} = \big[h_1(x) + h_2(y)\big]w.$

This is a special case of equation 2.3.7.22 with $g_1(x,y) = g_1(x)$, $g_2(x,y) = g_2(y)$, and $h(x,y,z) = h_1(x) + h_2(y)$.

14. $\dfrac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)y^k\big]\dfrac{\partial w}{\partial y} + \big[g_1(y)z + g_2(x)z^m\big]\dfrac{\partial w}{\partial z} = \big[h_1(x) + h_2(y)\big]w.$

This is a special case of equation 2.3.7.23 with $g_1(x,y) = g_1(y)$, $g_2(x,y) = g_2(x)$, and $h(x,y,z) = h_1(x) + h_2(y)$.

15. $\dfrac{\partial w}{\partial x} + \big[f_1(x)y + f_2(x)y^k\big]\dfrac{\partial w}{\partial y} + \big[g_1(x) + g_2(y)e^{\lambda z}\big]\dfrac{\partial w}{\partial z} = \big[h_1(x) + h_2(y)\big]w.$

This is a special case of equation 2.3.7.24 with $g_1(x,y) = g_1(x)$, $g_2(x,y) = g_2(y)$, and $h(x,y,z) = h_1(x) + h_2(y)$.

16. $\dfrac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} + \big[g_1(y)z + g_2(x)z^k\big]\dfrac{\partial w}{\partial z} = \big[h_1(x) + h_2(y)\big]w.$

This is a special case of equation 2.3.7.25 with $g_1(x,y) = g_1(y)$, $g_2(x,y) = g_2(x)$, and $h(x,y,z) = h_1(x) + h_2(y)$.

17. $\dfrac{\partial w}{\partial x} + \big[f_1(x) + f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} + \big[g_1(x) + g_2(y)e^{\beta z}\big]\dfrac{\partial w}{\partial z} = \big[h_1(x) + h_2(y)\big]w.$

This is a special case of equation 2.3.7.26 with $g_1(x,y) = g_1(x)$, $g_2(x,y) = g_2(y)$, and $h(x,y,z) = h_1(x) + h_2(y)$.

► **Coefficients of equations contain arbitrary functions of two variables.**

18. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + f(x,y)\dfrac{\partial w}{\partial z} = g(x,y)w.$

General solution: $w = \Phi(u_1, u_2)\exp\left[\dfrac{1}{b}\displaystyle\int_{y_0}^{y} g\left(x + \dfrac{a(t-y)}{b},\, t\right)dt\right]$, where

$$u_1 = bx - ay, \quad u_2 = bz - \int_{y_0}^{y} f\left(x + \frac{a(t-y)}{b},\, t\right)dt,$$

and y_0 may be taken as arbitrary.

19. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + f(x,y)g(z)\dfrac{\partial w}{\partial z} = h(x,y)w.$

General solution: $w = \Phi(u_1, u_2)\exp\left[\dfrac{1}{b}\displaystyle\int_{y_0}^{y} h\left(x + \dfrac{a(t-y)}{b},\, t\right)dt\right]$, where

$$u_1 = bx - ay, \quad u_2 = b\int\frac{dz}{g(z)} - \int_{y_0}^{y} f\left(x + \frac{a(t-y)}{b},\, t\right)dt.$$

20. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + [z + f(x,y)]\dfrac{\partial w}{\partial z} = g(x,y)w.$

General solution: $w = \Phi(u_1, u_2)\exp\left[\displaystyle\int_{y_0}^{y} g\left(\dfrac{xt}{y}, t\right)\dfrac{dt}{t}\right]$, where

$$u_1 = \frac{y}{x}, \quad u_2 = \frac{z}{y} - \int_{y_0}^{y} f\left(\frac{xt}{y}, t\right)\frac{dt}{t^2}.$$

21. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} + f(x,y)g(z)\dfrac{\partial w}{\partial z} = h(x,y)w.$

General solution: $w = \Phi(u_1,u_2)\exp\Bigl[\dfrac{1}{b}\displaystyle\int_{y_0}^{y} t^{-1}h\bigl(xy^{-a/b}t^{a/b},\,t\bigr)\,dt\Bigr]$, where

$$u_1 = x^b y^{-a}, \quad u_2 = b\int\frac{dz}{g(z)} - \int_{y_0}^{y} t^{-1}f\bigl(xy^{-a/b}t^{a/b},\,t\bigr)\,dt.$$

22. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x,y)z + g_2(x,y)\bigr]\dfrac{\partial w}{\partial z} = h(x,y,z)w.$

General solution:

$$w = \Phi(u_1,u_2)\exp\Bigl[\int_{x_0}^{x}\bar h(t,u_1,u_2)\,dt\Bigr],$$

where

$$u_1 = yF(x) - \int f_2(x)F(x)\,dx, \quad F(x) = \exp\Bigl[-\int f_1(x)\,dx\Bigr], \tag{1}$$

$$u_2 = zG(x,u_1) - \int_{x_0}^{x}\bar g_2(t,u_1)G(t,u_1)\,dt, \quad G(x,u_1) = \exp\Bigl[-\int_{x_0}^{x}\bar g_1(t,u_1)\,dt\Bigr]. \tag{2}$$

Here $\bar g_1(x,u_1) \equiv g_1(x,y)$, $\bar g_2(x,u_1) \equiv g_2(x,y)$, $\bar h(x,u_1,u_2) \equiv h(x,y,z)$ [in these functions, y must be expressed via x and u_1 from relation (1), and z must be expressed via x, u_1, and u_2 from relation (2)], and x_0 is an arbitrary number.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

23. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)y^k\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x,y)z + g_2(x,y)z^m\bigr]\dfrac{\partial w}{\partial z} = h(x,y,z)w.$

1°. For $k \neq 1$ and $m \neq 1$, the transformation $\xi = y^{1-k}$, $\eta = z^{1-m}$ leads to an equation of the form 2.3.7.22:

$$\frac{\partial w}{\partial x} + (1-k)\bigl[f_1(x)\xi + f_2(x)\bigr]\frac{\partial w}{\partial \xi} + (1-m)\bigl[\bar g_1(x,\xi)\eta + \bar g_2(x,\xi)\bigr]\frac{\partial w}{\partial \eta} = \bar h(x,\xi,\eta)w,$$

where

$$\bar g_1(x,\xi) \equiv g_1\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar g_2(x,\xi) \equiv g_2\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar h(x,\xi,\eta) \equiv h\bigl(x,\xi^{\frac{1}{1-k}},\eta^{\frac{1}{1-m}}\bigr).$$

2°. For $k \neq 1$ and $m = 1$, the substitution $\xi = y^{1-k}$ leads to an equation of the form 2.3.7.22.

3°. For $k = m = 1$, see equation 2.3.7.22.

24. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)y^k\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x,y) + g_2(x,y)e^{\lambda z}\bigr]\dfrac{\partial w}{\partial z} = h(x,y,z)w.$

The transformation $\xi = y^{1-k}$, $\eta = e^{-\lambda z}$ leads to an equation of the form 2.3.7.22:

$$\frac{\partial w}{\partial x} + (1-k)\bigl[f_1(x)\xi + f_2(x)\bigr]\frac{\partial w}{\partial \xi} - \lambda\bigl[\bar g_1(x,\xi)\eta + \bar g_2(x,\xi)\bigr]\frac{\partial w}{\partial \eta} = \bar h(x,\xi,\eta)w,$$

where

$$\bar g_1(x,\xi) \equiv g_1\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar g_2(x,\xi) \equiv g_2\bigl(x,\xi^{\frac{1}{1-k}}\bigr), \quad \bar h(x,\xi,\eta) \equiv h\bigl(x,\xi^{\frac{1}{1-k}},-\frac{1}{\lambda}\ln\eta\bigr).$$

25. $\dfrac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y}+\big[g_1(x,y)z+g_2(x,y)z^k\big]\dfrac{\partial w}{\partial z}=h(x,y,z)w.$

The transformation $\xi=e^{-\lambda y}$, $\eta=z^{1-k}$ leads to an equation of the form 2.3.7.22:

$$\frac{\partial w}{\partial x}-\lambda\big[f_1(x)\xi+f_2(x)\big]\frac{\partial w}{\partial \xi}+(1-k)\big[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\big]\frac{\partial w}{\partial \eta}=\bar h(x,\xi,\eta)w,$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\big(x,-\frac{1}{\lambda}\ln\xi\big)$ and $\bar h(x,\xi,\eta)\equiv h\big(x,-\frac{1}{\lambda}\ln\xi,\eta^{\frac{1}{1-k}}\big)$.

26. $\dfrac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y}+\big[g_1(x,y)+g_2(x,y)e^{\beta z}\big]\dfrac{\partial w}{\partial z}=h(x,y,z)w.$

The transformation $\xi=e^{-\lambda y}$, $\eta=e^{-\beta z}$ leads to an equation of the form 2.3.7.22:

$$\frac{\partial w}{\partial x}-\lambda\big[f_1(x)\xi+f_2(x)\big]\frac{\partial w}{\partial \xi}-\beta\big[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\big]\frac{\partial w}{\partial \eta}=\bar h(x,\xi,\eta)w,$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\big(x,-\frac{1}{\lambda}\ln\xi\big)$ and $\bar h(x,\xi,\eta)\equiv h\big(x,-\frac{1}{\lambda}\ln\xi,-\frac{1}{\beta}\ln\eta\big)$.

27. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}+\big[h_1(x,y)z+h_2(x,y)z^m\big]\dfrac{\partial w}{\partial z}=s(x,y,z)w.$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx$, $\eta=\displaystyle\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.3.7.23 with $f_1\equiv 0$, $f_2\equiv 1$, $k=0$:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}+\big[\bar h_1(\xi,\eta)z+\bar h_2(\xi,\eta)z^m\big]\frac{\partial w}{\partial z}=\bar h_3(\xi,\eta,z)w,$$

where $\bar h_1(\xi,\eta)\equiv\dfrac{h_1(x,y)}{f_2(x)g_1(y)}$, $\bar h_2(\xi,\eta)\equiv\dfrac{h_2(x,y)}{f_2(x)g_1(y)}$, and $\bar h_3(\xi,\eta,z)\equiv\dfrac{s(x,y,z)}{f_2(x)g_1(y)}$.

28. $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}+\big[h_1(x,y)+h_2(x,y)e^{\lambda z}\big]\dfrac{\partial w}{\partial z}=s(x,y,z)w.$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx$, $\eta=\displaystyle\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.3.7.24 with $f_1\equiv 0$, $f_2\equiv 1$, and $k=0$:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}+\big[\bar h_1(\xi,\eta)+\bar h_2(\xi,\eta)e^{\lambda z}\big]\frac{\partial w}{\partial z}=\bar h_3(x,y,z)w,$$

where $\bar h_1(\xi,\eta)\equiv\dfrac{h_1(x,y)}{f_2(x)g_1(y)}$, $\bar h_2(\xi,\eta)\equiv\dfrac{h_2(x,y)}{f_2(x)g_1(y)}$, and $\bar h_3(\xi,\eta,z)\equiv\dfrac{s(x,y,z)}{f_2(x)g_1(y)}$.

2.4 Equations of the Form $f_1\dfrac{\partial w}{\partial x}+f_2\dfrac{\partial w}{\partial y}+f_3\dfrac{\partial w}{\partial z}=f_4w+f_5$, $f_n=f_n(x,y,z)$

◆ *The solutions given below contain arbitrary functions of two variables* $\Phi=\Phi(u_1,u_2)$, *where* $u_1=u_1(x,y,z)$ *and* $u_2=u_2(x,y,z)$ *are some functions.*

2.4.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in x, y, and z.**

1. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\frac{\partial w}{\partial z}=(\alpha x+\beta)w+px+q.$

General solution:

$$w=\frac{1}{a}\exp\left[\frac{x}{2a}(\alpha x+2\beta)\right]\left\{\Phi(bx-ay,cx-az)+\int(px+q)\exp\left[-\frac{x}{2a}(\alpha x+2\beta)\right]dx\right\}.$$

2. $\displaystyle \frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=(cx+k)w+px+q.$

General solution: $w=\exp\bigl(\frac{1}{2}cx^2+kx\bigr)\left[\Phi(u_1,u_2)+\int(px+q)\exp\bigl(-\frac{1}{2}cx^2-kx\bigr)\,dx\right]$, where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}(by+\sqrt{ab}\,z)\exp\bigl(-\sqrt{ab}\,x\bigr) & \text{if } ab>0,\\ by\cos(\sqrt{-ab}\,x)+\sqrt{-ab}\,z\sin(\sqrt{-ab}\,x) & \text{if } ab<0.\end{cases}$$

3. $\displaystyle \frac{\partial w}{\partial x}+(a_1x+a_0)\frac{\partial w}{\partial y}+(b_1x+b_0)\frac{\partial w}{\partial z}=(c_1x+c_0)w+s_1x+s_0.$

This is a special case of equation 2.4.7.1 with $f(x)=a_1x+a_0$, $g(x)=b_1x+b_0$, $h(x)=c_1x+c_0$, and $p(x)=s_1x+s_0$.

4. $\displaystyle \frac{\partial w}{\partial x}+(b_1x+b_0)\frac{\partial w}{\partial y}+(c_1y+c_0)\frac{\partial w}{\partial z}=aw+s_1x+s_0.$

This is a special case of equation 2.4.7.8 with $f(x)=b_1x+b_0$, $g(y)=c_1y+c_0$, and $h(x)=s_1x+s_0$.

5. $\displaystyle \frac{\partial w}{\partial x}+(ay+k_1x+k_0)\frac{\partial w}{\partial y}+(bz+n_1x+n_0)\frac{\partial w}{\partial z}=(c_1x+c_0)w+s_1x+s_0.$

This is a special case of equation 2.4.7.3 with $f(x)=k_1x+k_0$, $g(x)=n_1x+n_0$, $h(x)=c_1x+c_0$, and $p(x)=s_1x+s_0$.

6. $\displaystyle \frac{\partial w}{\partial x}+(a_2y+a_1x+a_0)\frac{\partial w}{\partial y}+(b_3z+b_2y+b_1x+b_0)\frac{\partial w}{\partial z}$
$\displaystyle \qquad=(c_3z+c_2y+c_1x+c_0)w+s_3z+s_2y+s_1x+s_0.$

This is a special case of equation 2.4.7.20 with $f_1(x)=a_2$, $f_2(x)=a_1x+a_0$, $g_1(x,y)=b_3$, $g_2(x,y)=b_2y+b_1x+b_0$, $h_1(x,y,z)=c_3z+c_2y+c_1x+c_0$, and $h_2(x,y,z)=s_3z+s_2y+s_1x+s_0$.

7. $\displaystyle ax\frac{\partial w}{\partial x}+bx\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=(\alpha x+\beta)w+px+q.$

General solution: $\displaystyle w=\frac{1}{a}x^{\beta/a}e^{\alpha x/a}\left[\Phi\bigl(bx-ay,\ x^{-c}z^a\bigr)+\int(px+q)x^{-(a+\beta)/a}e^{-\alpha x/a}\,dx\right].$

8. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=(\alpha x+\beta)w+px+q.$

General solution: $\displaystyle w=\frac{1}{a}x^{\beta/a}e^{\alpha x/a}\left[\Phi\left(x^by^{-a},\,x^cz^{-a}\right)+\int(px+q)x^{-(a+\beta)/a}e^{-\alpha x/a}\,dx\right].$

9. $\displaystyle x\frac{\partial w}{\partial x}+az\frac{\partial w}{\partial y}+by\frac{\partial w}{\partial z}=(cx+k)w+px+q.$

General solution: $\displaystyle w=x^ke^{cx}\left[\Phi(u_1,u_2)+\int(px+q)x^{-k-1}e^{-cx}\,dx\right],$ where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}x^{\sqrt{ab}}(by-\sqrt{ab}\,z) & \text{if } ab>0,\\ x^{\sqrt{-ab}}\exp\left(-\arctan\dfrac{\sqrt{-ab}\,z}{by}\right) & \text{if } ab<0.\end{cases}$$

► **Coefficients of equations are quadratic in x, y, and z.**

10. $\displaystyle \frac{\partial w}{\partial x}+(a_1x^2+a_0)\frac{\partial w}{\partial y}+(b_1x^2+b_0)\frac{\partial w}{\partial z}=(c_1x+c_0)w+s_1x^2+s_0.$

This is a special case of equation 2.4.7.1 with $f(x)=a_1x^2+a_0$, $g(x)=b_1x^2+b_0$, $h(x)=c_1x+c_0$, and $p(x)=s_1x^2+s_0$.

11. $\displaystyle \frac{\partial w}{\partial x}+(b_1x^2+b_0)\frac{\partial w}{\partial y}+(c_1y^2+c_0)\frac{\partial w}{\partial z}=aw+s_1x^2+s_0.$

This is a special case of equation 2.4.7.8 with $f(x)=b_1x^2+b_0$, $g(y)=c_1y^2+c_0$, and $h(x)=s_1x^2+s_0$.

12. $\displaystyle \frac{\partial w}{\partial x}+(ay+k_1x^2+k_0)\frac{\partial w}{\partial y}+(bz+n_1x^2+n_0)\frac{\partial w}{\partial z}=(c_1x+c_0)w+s_1x+s_0.$

This is a special case of equation 2.4.7.3 with $f(x)=k_1x^2+k_0$, $g(x)=n_1x^2+n_0$, $h(x)=c_1x+c_0$, and $p(x)=s_1x+s_0$.

13. $\displaystyle \frac{\partial w}{\partial x}+(a_2xy+a_1x+a_0)\frac{\partial w}{\partial y}+(b_3yz+b_2y^2+b_1x^2+b_0)\frac{\partial w}{\partial z}$
$\displaystyle =(c_3z+c_2y+c_1x+c_0)w+s_1xy+s_2xz.$

This is a special case of equation 2.4.7.20 with $f_1(x)=a_2x$, $f_2(x)=a_1x+a_0$, $g_1(x,y)=b_3y$, $g_2(x,y)=b_2y^2+b_1x^2+b_0$, $h_1(x,y,z)=c_3z+c_2y+c_1x+c_0$, and $h_2(x,y,z)=s_1xy+s_2xz$.

14. $\displaystyle ax\frac{\partial w}{\partial x}+bx\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=kxw+sx^2.$

General solution: $\displaystyle w=e^{kx/a}\Phi\left(bx-ay,\,x^cz^{-a}\right)-\frac{s}{k^2}(kx+a).$

15. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=kxw+sx^2.$

General solution: $\displaystyle w=e^{kx/a}\Phi\left(x^by^{-a},\,x^cz^{-a}\right)-\frac{s}{k^2}(kx+a).$

16. $\boldsymbol{ax^2 \frac{\partial w}{\partial x} + by^2 \frac{\partial w}{\partial y} + cz^2 \frac{\partial w}{\partial z} = (kx+s)w + px + q.}$

General solution:

$$w = x^{k/a} e^{-sx/a} \left[\Phi\left(\frac{b}{x} - \frac{a}{y}, \; \frac{c}{x} - \frac{a}{z} \right) + \frac{1}{a} \int (px+q) x^{-(2a+k)/a} e^{sx/a} \, dx \right].$$

► **Coefficients of equations contain other powers of x, y, and z.**

17. $\boldsymbol{\frac{\partial w}{\partial x} + (a_1\sqrt{x} + a_0) \frac{\partial w}{\partial y} + (b_1\sqrt{x} + b_0) \frac{\partial w}{\partial z} = cw + s_1\sqrt{x} + s_0.}$

This is a special case of equation 2.4.7.1 with $f(x) = a_1\sqrt{x} + a_0$, $g(x) = b_1\sqrt{x} + b_0$, $h(x) = c$, and $p(x) = s_1\sqrt{x} + s_0$.

18. $\boldsymbol{\frac{\partial w}{\partial x} + (b_1 x^2 + b_0) \frac{\partial w}{\partial y} + (c_1 y^3 + c_0) \frac{\partial w}{\partial z} = aw + s_1 x^3 + s_0.}$

This is a special case of equation 2.4.7.8 with $f(x) = b_1 x^2 + b_0$, $g(y) = c_1 y^3 + c_0$, and $h(x) = s_1 x^3 + s_0$.

19. $\boldsymbol{\frac{\partial w}{\partial x} + (ay + kx^3) \frac{\partial w}{\partial y} + (bz + nx^3) \frac{\partial w}{\partial z} = cw + sx^2.}$

This is a special case of equation 2.4.7.3 with $f(x) = kx^3$, $g(x) = nx^3$, $h(x) = c$, and $p(x) = sx^2$.

20. $\boldsymbol{\frac{\partial w}{\partial x} + (a_1 xy + a_2 x^3) \frac{\partial w}{\partial y} + (b_1 yz + b_2 y^3) \frac{\partial w}{\partial z} = (c_1 z + c_2 y)w + s_1 x^2 y + s_2 x z^2.}$

This is a special case of equation 2.4.7.20 with $f_1(x) = a_1 x$, $f_2(x) = a_2 x^3$, $g_1(x,y) = b_1 y$, $g_2(x,y) = b_2 y^3$, $h_1(x,y,z) = c_1 z + c_2 y$, and $h_2(x,y,z) = s_1 x^2 y + s_2 x z^2$.

21. $\boldsymbol{ax^3 \frac{\partial w}{\partial x} + by^3 \frac{\partial w}{\partial y} + cz^3 \frac{\partial w}{\partial z} = xw + kx + s.}$

General solution: $w = \exp\left(-\frac{1}{ax} \right) \Phi\left(\frac{b}{x^2} - \frac{a}{y^2}, \; \frac{c}{x^2} - \frac{a}{z^2} \right) - \frac{s}{x} + as - k.$

► **Coefficients of equations contain arbitrary powers of x, y, and z.**

22. $\boldsymbol{a \frac{\partial w}{\partial x} + b \frac{\partial w}{\partial y} + c \frac{\partial w}{\partial z} = kx^n w + sx^m.}$

1°. General solution for $n \neq -1$:

$$w = \exp\left[\frac{k}{a(n+1)} x^{n+1} \right] \left\{ \Phi(bx - ay, \; cx - az) + \frac{s}{a} \int x^m \exp\left[-\frac{k}{a(n+1)} x^{n+1} \right] dx \right\}.$$

2°. General solution for $n = -1$:

$$w = \begin{cases} x^{k/a} \Phi(bx - ay, \; cx - az) + \dfrac{sx^{m+1}}{a(m+1) - k} & \text{if } a(m+1) \neq k, \\ x^{k/a} \Phi(bx - ay, \; cx - az) + \dfrac{s}{a} x^{k/a} \ln x & \text{if } a(m+1) = k. \end{cases}$$

23. $a\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}+cz\dfrac{\partial w}{\partial z}=kx^n w+sx^m.$

1°. General solution for $n\neq -1$:

$$w=E(x)\Phi\bigl(y^a e^{-bx},\,z^a e^{-cx}\bigr)+\frac{s}{a}E(x)\int\frac{x^m}{E(x)}\,dx,\qquad E(x)=\exp\left[\frac{k}{a(n+1)}x^{n+1}\right].$$

2°. General solution for $n=-1$:

$$w=\begin{cases} x^{k/a}\Phi\bigl(y^a e^{-bx},\,z^a e^{-cx}\bigr)+\dfrac{sx^{m+1}}{a(m+1)-k} & \text{if } a(m+1)\neq k,\\ x^{k/a}\Phi\bigl(y^a e^{-bx},\,z^a e^{-cx}\bigr)+\dfrac{s}{a}x^{k/a}\ln x & \text{if } a(m+1)=k.\end{cases}$$

24. $\dfrac{\partial w}{\partial x}+az\dfrac{\partial w}{\partial y}+by\dfrac{\partial w}{\partial z}=cx^n w+sx^m.$

1°. General solution for $n\neq -1$:

$$w=\exp\left(\frac{c}{n+1}x^{n+1}\right)\left[\Phi(u_1,u_2)+s\int x^m\exp\left(-\frac{c}{n+1}x^{n+1}\right)dx\right],$$

where

$$u_1=by^2-az^2,\quad u_2=\begin{cases}\bigl(by+\sqrt{ab}\,z\bigr)\exp\bigl(-\sqrt{ab}\,x\bigr) & \text{if } ab>0,\\ by\cos\bigl(\sqrt{|ab|}\,x\bigr)+\sqrt{|ab|}\,z\sin\bigl(\sqrt{|ab|}\,x\bigr) & \text{if } ab<0.\end{cases}$$

2°. General solution for $n=-1$:

$$w=\begin{cases} x^c\Phi(u_1,\,u_2)+\dfrac{s}{m+1-c}x^{m+1} & \text{if } m+1\neq c,\\ x^c\Phi(u_1,u_2)+sx^c\ln x & \text{if } m+1=c,\end{cases}$$

where u_1 and u_2 are defined in Item 1°.

25. $\dfrac{\partial w}{\partial x}+ax^n\dfrac{\partial w}{\partial y}+bx^m\dfrac{\partial w}{\partial z}=cx^k w+sx^l.$

This is a special case of equation 2.4.7.1 with $f(x)=ax^n$, $g(x)=bx^m$, $h(x)=cx^k$, and $p(x)=sx^l$.

26. $\dfrac{\partial w}{\partial x}+bx^n\dfrac{\partial w}{\partial y}+cy^m\dfrac{\partial w}{\partial z}=aw+sx^k.$

This is a special case of equation 2.4.7.8 with $f(x)=bx^n$, $g(y)=cy^m$, and $h(x)=sx^k$.

27. $\dfrac{\partial w}{\partial x}+(ay+\beta x^n)\dfrac{\partial w}{\partial y}+(bz+\gamma x^m)\dfrac{\partial w}{\partial z}=cx^k w+sx^l.$

This is a special case of equation 2.4.7.3 with $f(x)=\beta x^n$, $g(x)=\gamma x^m$, $h(x)=cx^k$, and $p(x)=sx^l$.

28. $\dfrac{\partial w}{\partial x} + (a_1 x^{n_1} y + a_2 x^{n_2}) \dfrac{\partial w}{\partial y} + (b_1 y^{m_1} z + b_2 y^{m_2}) \dfrac{\partial w}{\partial z} = cw + s_1 x y^{k_1} + s_2 x^{k_2} z.$

This is a special case of equation 2.4.7.20 with $f_1(x) = a_1 x^{n_1}$, $f_2(x) = a_2 x^{n_2}$, $g_1(x, y) = b_1 y^{m_1}$, $g_2(x, y) = b_2 y^{m_2}$, $h_1(x, y, z) = c$, and $h_2(x, y, z) = s_1 x y^{k_1} + s_2 x^{k_2} z$.

29. $\dfrac{\partial w}{\partial x} + \bigl(a_1 x^{\lambda_1} y + a_2 x^{\lambda_2} y^k\bigr) \dfrac{\partial w}{\partial y} + \bigl(b_1 x^{\beta_1} z + b_2 x^{\beta_2} z^m\bigr) \dfrac{\partial w}{\partial z} = c_1 x^{\gamma_1} w + c_2 y^{\gamma_2}.$

This is a special case of equation 2.4.7.21 with $f_1(x) = a_1 x^{\lambda_1}$, $f_2(x) = a_2 x^{\lambda_2}$, $g_1(x, y) = b_1 x^{\beta_1}$, $g_2(x, y) = b_2 x^{\beta_2}$, $h_1(x, y, z) = c_1 x^{\gamma_1}$, and $h_2(x, y, z) = c_2 y^{\gamma_2}$.

30. $\dfrac{\partial w}{\partial x} + \bigl(a_1 x^{\lambda_1} y + a_2 x^{\lambda_2} y^k\bigr) \dfrac{\partial w}{\partial y} + \bigl(b_1 y^{\beta_1} z + b_2 y^{\beta_2} z^m\bigr) \dfrac{\partial w}{\partial z} = c_1 x^{\gamma_1} w + c_2 z^{\gamma_2}.$

This is a special case of equation 2.4.7.21 with $f_1(x) = a_1 x^{\lambda_1}$, $f_2(x) = a_2 x^{\lambda_2}$, $g_1(x, y) = b_1 y^{\beta_1}$, $g_2(x, y) = b_2 y^{\beta_2}$, $h_1(x, y, z) = c_1 x^{\gamma_1}$, and $h_2(x, y, z) = c_2 z^{\gamma_2}$.

31. $x \dfrac{\partial w}{\partial x} + a y \dfrac{\partial w}{\partial y} + b z \dfrac{\partial w}{\partial z} = c x^n w + k x^m.$

General solution: $w = \exp\left(\dfrac{c}{n} x^n\right) \left[\Phi\left(\dfrac{x^a}{y},\ \dfrac{x^b}{z}\right) + k \displaystyle\int x^{m-1} \exp\left(-\dfrac{c}{n} x^n\right) dx\right].$

32. $x \dfrac{\partial w}{\partial x} + a z \dfrac{\partial w}{\partial y} + b y \dfrac{\partial w}{\partial z} = c x^n w + k x^m.$

1°. General solution for $n \neq 0$:

$$w = \exp\left(\frac{c}{n} x^n\right) \left[\Phi(u_1,\ u_2) + k \int x^{m-1} \exp\left(-\frac{c}{n} x^n\right) dx\right],$$

where

$$u_1 = b y^2 - a z^2, \quad u_2 = \begin{cases} |x|^{\sqrt{ab}} \bigl(b y - \sqrt{ab}\, z\bigr) & \text{if } ab > 0, \\ |x|^{\sqrt{-ab}} \exp\left(-\arctan \dfrac{\sqrt{-ab}\, z}{b y}\right) & \text{if } ab < 0. \end{cases}$$

2°. General solution for $n = 0$:

$$w = \begin{cases} x^c \Phi(u_1,\ u_2) + \dfrac{k}{m - c} x^m & \text{if } m \neq c, \\ x^c \Phi(u_1, u_2) + k x^c \ln x & \text{if } m = c, \end{cases}$$

where u_1 and u_2 are defined in Item 1°.

33. $b c x \dfrac{\partial w}{\partial x} + c(b y + c z) \dfrac{\partial w}{\partial y} + b(b y - c z) \dfrac{\partial w}{\partial z} = k x^n w + s x^m.$

1°. General solution for $n \neq 0$:

$$w = \exp\left(\frac{k}{bcn} x^n\right) \left[\Phi(u_1, u_2) + \frac{s}{bc} \int x^{m-1} \exp\left(-\frac{k}{bcn} x^n\right) dx\right],$$

where

$$u_1 = \big[by + (\sqrt{2} - 1)cz\big]|x|^{-\sqrt{2}}, \quad u_2 = \big[by - (\sqrt{2} + 1)cz\big]|x|^{\sqrt{2}}.$$

2°. General solution for $n = 0$:

$$w = \begin{cases} x^{\frac{k}{bc}}\Phi(u_1,\, u_2) + \dfrac{s}{bcm - k}x^m & \text{if } bcm \neq k, \\ x^m\Phi(u_1, u_2) + \dfrac{s}{bc}x^m \ln x & \text{if } bcm = k, \end{cases}$$

where u_1 and u_2 are defined in Item 1°.

34. $\displaystyle b_1 x^{n_1}\frac{\partial w}{\partial x} + b_2 y^{n_2}\frac{\partial w}{\partial y} + b_3 z^{n_3}\frac{\partial w}{\partial z} = aw + c_1 x^{k_1} + c_2 y^{k_2} + c_3 x^{k_3}.$

This is a special case of equation 2.4.7.16 with $f_1(x) = b_1 x^{n_1}$, $f_2(y) = b_2 y^{n_2}$, $f_3(z) = b_3 z^{n_3}$, $g_1(x) = c_1 x^{k_1}$, $g_2(y) = c_2 y^{k_2}$, and $g_3(z) = c_3 z^{k_3}$.

35. $\displaystyle a_1 x^{n_1}\frac{\partial w}{\partial x} + a_2 y^{n_2}\frac{\partial w}{\partial y} + a_3 z^{n_3}\frac{\partial w}{\partial z} = bx^k w + cx^m.$

This is a special case of equation 2.4.7.10 with $f_1(x) = a_1 x^{n_1}$, $f_2(x) = f_3(x) = 1$, $f_4(x) = bx^k$, $f_5(x) = cx^m$, $g(y) = a_2 y^{n_2}$, and $h(z) = a_3 z^{n_3}$.

2.4.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

1. $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} + b\frac{\partial w}{\partial z} = ce^{\beta x}w + ke^{\lambda x}.$

1°. General solution for $\beta \neq 0$:

$$w = \exp\left(\frac{c}{\beta}e^{\beta x}\right)\left[\Phi(y - ax, z - bx) + k\int \exp\left(\lambda x - \frac{c}{\beta}e^{\beta x}\right)dx\right].$$

2°. General solution for $\beta = 0$:

$$w = e^{cx}\Phi(y - ax, z - bx) + \frac{k}{\lambda - c}e^{\lambda x}.$$

2. $\displaystyle \frac{\partial w}{\partial x} + ae^{\beta x}\frac{\partial w}{\partial y} + be^{\lambda x}\frac{\partial w}{\partial z} = ce^{\gamma x}w + se^{\mu x}.$

This is a special case of equation 2.4.7.1 with $f(x) = ae^{\beta x}$, $g(x) = be^{\lambda x}$, $h(x) = ce^{\gamma x}$, and $p(x) = se^{\mu x}$.

3. $\displaystyle \frac{\partial w}{\partial x} + be^{\beta x}\frac{\partial w}{\partial y} + ce^{\lambda y}\frac{\partial w}{\partial z} = aw + se^{\gamma x}.$

This is a special case of equation 2.4.7.8 with $f(x) = be^{\beta x}$, $g(y) = ce^{\lambda y}$, and $h(x) = se^{\gamma x}$.

4. $\dfrac{\partial w}{\partial x} + ae^{\beta x}\dfrac{\partial w}{\partial y} + be^{\lambda z}\dfrac{\partial w}{\partial z} = cw + ke^{\gamma x}.$

General solution:

$$w = \begin{cases} e^{cx}\Phi(u_1,u_2) + \dfrac{k}{\gamma - c}e^{\gamma x} & \text{if } c \neq \gamma, \\ e^{\gamma x}\big[\Phi(u_1,u_2) + kx\big] & \text{if } c = \gamma, \end{cases}$$

where $u_1 = ae^{\beta x} - \beta y$ and $u_2 = b\lambda x + e^{-\lambda z}$.

5. $\dfrac{\partial w}{\partial x} + \big(a_1 e^{\sigma x} + a_2 e^{\lambda y}\big)\dfrac{\partial w}{\partial y} + \big(b_1 e^{\mu y} + b_2 e^{\beta z}\big)\dfrac{\partial w}{\partial z} = c_1 w + c_2 e^{\nu x}.$

This is a special case of equation 2.4.7.24 with $f_1(x) = a_1 e^{\sigma x}$, $f_2(x) = a_2$, $g_1(x,y) = b_1 e^{\mu y}$, $g_2(x,y) = b_2$, $h_1(x,y,z) = c_1$, and $h_2(x,y,z) = c_2 e^{\nu x}$.

6. $b_1 e^{\lambda_1 x}\dfrac{\partial w}{\partial x} + b_2 e^{\lambda_2 y}\dfrac{\partial w}{\partial y} + b_3 e^{\lambda_3 z}\dfrac{\partial w}{\partial z} = aw + c_1 e^{\beta_1 x} + c_2 e^{\beta_2 y} + c_3 e^{\beta_3 z}.$

This is a special case of equation 2.4.7.16 with $f_1(x) = b_1 e^{\lambda_1 x}$, $f_2(y) = b_2 e^{\lambda_2 y}$, $f_3(z) = b_3 e^{\lambda_3 z}$, $g_1(x) = c_1 e^{\beta_1 x}$, $g_2(y) = c_2 e^{\beta_2 y}$, and $g_3(z) = c_3 e^{\beta_3 z}$.

7. $a_1 e^{\sigma_1 x + \beta_1 y}\dfrac{\partial w}{\partial x} + a_2 e^{\sigma_2 x + \beta_2 y}\dfrac{\partial w}{\partial y}$

$$+ \big(b_1 e^{\nu_1 x + \mu_1 y} + b_2 e^{\nu_2 x + \mu_2 y + \lambda z}\big)\frac{\partial w}{\partial z} = c_1 w + c_2.$$

This is a special case of equation 2.4.7.27 with $f_1(x) = a_1 e^{\sigma_1 x}$, $g_1(y) = e^{\beta_1 y}$, $f_2(x) = a_2 e^{\sigma_2 x}$, $g_2(y) = e^{\beta_2 y}$, $h_1(x,y) = b_1 e^{\nu_1 x + \mu_1 y}$, $h_2(x,y) = b_2 e^{\nu_2 x + \mu_2 y}$, $\varphi_1(x,y,z) = c_1$, and $\varphi_2(x,y,z) = c_2$.

► Coefficients of equations contain exponential and power-law functions.

8. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = ce^{\beta x}w + kx^n.$

General solution: $w = \exp\left(\dfrac{c}{\beta}e^{\beta x}\right)\left[\Phi(y - ax, z - bx) + k\displaystyle\int x^n \exp\left(-\dfrac{c}{\beta}e^{\beta x}\right)dx\right].$

9. $\dfrac{\partial w}{\partial x} + ax^n\dfrac{\partial w}{\partial y} + be^{\lambda x}\dfrac{\partial w}{\partial z} = ce^{\gamma x}w + sx^k.$

This is a special case of equation 2.4.7.1 with $f(x) = ax^n$, $g(x) = be^{\lambda x}$, $h(x) = ce^{\gamma x}$, and $p(x) = sx^k$.

10. $\dfrac{\partial w}{\partial x} + be^{\beta x}\dfrac{\partial w}{\partial y} + cy^n\dfrac{\partial w}{\partial z} = aw + se^{\gamma x}.$

This is a special case of equation 2.4.7.8 with $f(x) = be^{\beta x}$, $g(y) = cy^n$, and $h(x) = se^{\gamma x}$.

11. $\dfrac{\partial w}{\partial x} + \big(a_1 y + a_2 xy^k\big)\dfrac{\partial w}{\partial y} + \big(b_1 x + b_2 e^{\beta y + \lambda z}\big)\dfrac{\partial w}{\partial z} = c_1 w + c_2 e^{\gamma x}.$

This is a special case of equation 2.4.7.22 with $f_1(x) = a_1$, $f_2(x) = a_2 x$, $g_1(x,y) = b_1 x$, $g_2(x,y) = b_2 e^{\beta y}$, $h_1(x,y,z) = c_1$, and $h_2(x,y,z) = c_2 e^{\gamma x}$.

12. $\dfrac{\partial w}{\partial x}+\big(a_1x+a_2e^{\lambda y}\big)\dfrac{\partial w}{\partial y}+\big(b_1z+b_2e^{\beta y}z^k\big)\dfrac{\partial w}{\partial z}=c_1w+c_2.$

This is a special case of equation 2.4.7.23 with $f_1(x)=a_1x$, $f_2(x)=a_2$, $g_1(x,y)=b_1$, $g_2(x,y)=b_2e^{\beta y}$, $h_1(x,y,z)=c_1$, and $h_2(x,y,z)=c_2$.

13. $\dfrac{\partial w}{\partial x}+\big(a_1e^{\mu x}+a_2e^{\lambda y}\big)\dfrac{\partial w}{\partial y}+\big(b_1e^{\nu y}+b_2e^{\beta z}\big)\dfrac{\partial w}{\partial z}=c_1w+c_2.$

This is a special case of equation 2.4.7.24 with $f_1(x)=a_1e^{\mu x}$, $f_2(x)=a_2$, $g_1(x,y)=b_1e^{\nu y}$, $g_2(x,y)=b_2$, $h_1(x,y,z)=c_1$, and $h_2(x,y,z)=c_2$.

14. $\dfrac{\partial w}{\partial x}+\big(a_1e^{\lambda_1 x}y+a_2e^{\lambda_2 x}\big)\dfrac{\partial w}{\partial y}+\big(b_1e^{\beta_1 x}z+b_2e^{\beta_2 x}\big)\dfrac{\partial w}{\partial z}=c_1e^{\gamma_1 x}w+c_2e^{\gamma_2 x}.$

This is a special case of equation 2.4.7.20 with $f_1(x)=a_1e^{\lambda_1 x}$, $f_2(x)=a_2e^{\lambda_2 x}$, $g_1(x,y)=b_1e^{\beta_1 x}$, $g_2(x,y)=b_2e^{\beta_2 x}$, $h_1(x,y,z)=c_1e^{\gamma_1 x}$, and $h_2(x,y,z)=c_2e^{\gamma_2 x}$.

15. $\dfrac{\partial w}{\partial x}+\big(a_1e^{\lambda_1 x}y+a_2e^{\lambda_2 x}y^k\big)\dfrac{\partial w}{\partial y}+\big(b_1e^{\beta_1 x}z+b_2e^{\beta_2 x}z^m\big)\dfrac{\partial w}{\partial z}$
$=c_1e^{\gamma_1 x}w+c_2e^{\gamma_2 y}.$

This is a special case of equation 2.4.7.21 with $f_1(x)=a_1e^{\lambda_1 x}$, $f_2(x)=a_2e^{\lambda_2 x}$, $g_1(x,y)=b_1e^{\beta_1 x}$, $g_2(x,y)=b_2e^{\beta_2 x}$, $h_1(x,y,z)=c_1e^{\gamma_1 x}$, and $h_2(x,y,z)=c_2e^{\gamma_2 y}$.

16. $\dfrac{\partial w}{\partial x}+\big(a_1e^{\lambda_1 x}y+a_2e^{\lambda_2 x}y^k\big)\dfrac{\partial w}{\partial y}+\big(b_1e^{\beta_1 y}z+b_2e^{\beta_2 y}z^m\big)\dfrac{\partial w}{\partial z}$
$=c_1e^{\gamma_1 x}w+c_2e^{\gamma_2 z}.$

This is a special case of equation 2.4.7.21 with $f_1(x)=a_1e^{\lambda_1 x}$, $f_2(x)=a_2e^{\lambda_2 x}$, $g_1(x,y)=b_1e^{\beta_1 y}$, $g_2(x,y)=b_2e^{\beta_2 y}$, $h_1(x,y,z)=c_1e^{\gamma_1 x}$, and $h_2(x,y,z)=c_2e^{\gamma_2 z}$.

17. $a_1e^{\beta y}\dfrac{\partial w}{\partial x}+a_2e^{\sigma x}\dfrac{\partial w}{\partial y}+\big(b_1x^ne^{\mu y}+b_2y^me^{\nu x+\lambda z}\big)\dfrac{\partial w}{\partial z}=c_1w+c_2.$

This is a special case of equation 2.4.7.27 with $f_1(x)=1$, $g_1(y)=a_1e^{\beta y}$, $f_2(x)=a_2e^{\sigma x}$, $g_2(y)=1$, $h_1(x,y)=b_1x^ne^{\mu y}$, $h_2(x,y)=b_2y^me^{\nu x}$, $\varphi_1(x,y,z)=c_1$, and $\varphi_2(x,y,z)=c_2$.

2.4.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

1. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\sinh^n(\beta x)w+k\sinh^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\sinh^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\biggl[c\int\sinh^n(\beta x)\,dx\biggr].$$

2. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\sinh(\beta z)\dfrac{\partial w}{\partial z}=\big[p\sinh(\lambda x)+q\big]w+k\sinh(\gamma x).$

General solution:

$$w=\frac{k}{a}\exp\biggl[\frac{p\cosh(\lambda x)+q\lambda x}{a\lambda}\biggr]\biggl\{\Phi(u_1,u_2)+\int\sinh(\gamma x)\exp\biggl[-\frac{p\cosh(\lambda x)+q\lambda x}{a\lambda}\biggr]dx\biggr\},$$

where $u_1=bx-ay$ and $u_2=c\beta x-a\ln\Bigl|\tanh\dfrac{\beta z}{2}\Bigr|$.

3. $\dfrac{\partial w}{\partial x}+a\sinh^n(\beta x)\dfrac{\partial w}{\partial y}+b\sinh^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+s\sinh^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\sinh^n(\beta x)$, $g(x)=b\sinh^k(\lambda x)$, $h(x)=c$, and $p(x)=s\sinh^m(\mu x)$.

4. $\dfrac{\partial w}{\partial x}+b\sinh^n(\beta x)\dfrac{\partial w}{\partial y}+c\sinh^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\sinh^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\sinh^n(\beta x)$, $g(y)=c\sinh^k(\lambda y)$, and $h(x)=s\sinh^m(\mu x)$.

5. $b_1\sinh^{n_1}(\lambda_1x)\dfrac{\partial w}{\partial x}+b_2\sinh^{n_2}(\lambda_2y)\dfrac{\partial w}{\partial y}+b_3\sinh^{n_3}(\lambda_3z)\dfrac{\partial w}{\partial z}$
$$=aw+c_1\sinh^{k_1}(\beta_1x)+c_2\sinh^{k_2}(\beta_2y)+c_3\sinh^{k_3}(\beta_3z).$$

This is a special case of equation 2.4.7.16 in which $f_1(x)=b_1\sinh^{n_1}(\lambda_1x)$, $f_2(y)=b_2\sinh^{n_2}(\lambda_2y)$, $f_3(z)=b_3\sinh^{n_3}(\lambda_3z)$, $g_1(x)=c_1\sinh^{k_1}(\beta_1x)$, $g_2(y)=c_2\sinh^{k_2}(\beta_2y)$, and $g_3(z)=c_3\sinh^{k_3}(\beta_3z)$.

► **Coefficients of equations contain hyperbolic cosine.**

6. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\cosh^n(\beta x)w+k\cosh^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\cosh^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\left[c\int\cosh^n(\beta x)\,dx\right].$$

7. $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}+c\cosh(\beta z)\dfrac{\partial w}{\partial z}=\bigl[p\cosh(\lambda x)+q\bigr]w+k\cosh(\gamma x).$

General solution:

$$w=\frac{k}{a}\exp\left[\frac{p\sinh(\lambda x)+q\lambda x}{a\lambda}\right]\left\{\Phi(u_1,u_2)+\int\cosh(\gamma x)\exp\left[-\frac{p\sinh(\lambda x)+q\lambda x}{a\lambda}\right]dx\right\},$$

where $u_1=bx-ay$ and $u_2=c\beta x-2a\arctan\left|\tanh\dfrac{\beta z}{2}\right|$.

8. $\dfrac{\partial w}{\partial x}+a\cosh^n(\beta x)\dfrac{\partial w}{\partial y}+b\cosh^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+s\cosh^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\cosh^n(\beta x)$, $g(x)=b\cosh^k(\lambda x)$, $h(x)=c$, and $p(x)=s\cosh^m(\mu x)$.

9. $\dfrac{\partial w}{\partial x}+b\cosh^n(\beta x)\dfrac{\partial w}{\partial y}+c\cosh^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\cosh^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\cosh^n(\beta x)$, $g(y)=c\cosh^k(\lambda y)$, and $h(x)=s\cosh^m(\mu x)$.

10. $\displaystyle b_1\cosh^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x}+b_2\cosh^{n_2}(\lambda_2 y)\frac{\partial w}{\partial y}+b_3\cosh^{n_3}(\lambda_3 z)\frac{\partial w}{\partial z}$
$\displaystyle =aw+c_1\cosh^{k_1}(\beta_1 x)+c_2\cosh^{k_2}(\beta_2 y)+c_3\cosh^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 in which $f_1(x)=b_1\cosh^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\cosh^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\cosh^{n_3}(\lambda_3 z)$, $g_1(x)=c_1\cosh^{k_1}(\beta_1 x)$, $g_2(y)=c_2\cosh^{k_2}(\beta_2 y)$, and $g_3(z)=c_3\cosh^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain hyperbolic tangent.**

11. $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=c\tanh^n(\beta x)w+k\tanh^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\tanh^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x)=\exp\biggl[c\int\tanh^n(\beta x)\,dx\biggr].$$

12. $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}+c\tanh(\beta z)\frac{\partial w}{\partial z}=\bigl[p\tanh(\lambda x)+q\bigr]w+k\tanh(\gamma x).$

General solution:

$$w=\frac{k}{a}e^{qx/a}\cosh^{p/a\lambda}(\lambda x)\biggl[\Phi(u_1,u_2)+\int e^{-qx/a}\cosh^{-p/a\lambda}(\lambda x)\tanh(\gamma x)\,dx\biggr],$$

where $u_1=bx-ay$ and $u_2=c\beta x-a\ln\bigl|\sinh(\beta z)\bigr|$.

13. $\displaystyle \frac{\partial w}{\partial x}+a\tanh^n(\beta x)\frac{\partial w}{\partial y}+b\tanh^k(\lambda x)\frac{\partial w}{\partial z}=cw+s\tanh^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\tanh^n(\beta x)$, $g(x)=b\tanh^k(\lambda x)$, $h(x)=c$, and $p(x)=s\tanh^m(\mu x)$.

14. $\displaystyle \frac{\partial w}{\partial x}+b\tanh^n(\beta x)\frac{\partial w}{\partial y}+c\tanh^k(\lambda y)\frac{\partial w}{\partial z}=aw+s\tanh^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\tanh^n(\beta x)$, $g(y)=c\tanh^k(\lambda y)$, and $h(x)=s\tanh^m(\mu x)$.

15. $\displaystyle b_1\tanh^{n_1}(\lambda_1 x)\frac{\partial w}{\partial x}+b_2\tanh^{n_2}(\lambda_2 y)\frac{\partial w}{\partial y}+b_3\tanh^{n_3}(\lambda_3 z)\frac{\partial w}{\partial z}$
$\displaystyle =aw+c_1\tanh^{k_1}(\beta_1 x)+c_2\tanh^{k_2}(\beta_2 y)+c_3\tanh^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 in which $f_1(x)=b_1\tanh^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\tanh^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\tanh^{n_3}(\lambda_3 z)$, $g_1(x)=c_1\tanh^{k_1}(\beta_1 x)$, $g_2(y)=c_2\tanh^{k_2}(\beta_2 y)$, and $g_3(z)=c_3\tanh^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain hyperbolic cotangent.**

16. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\coth^n(\beta x)w + k\coth^m(\lambda x).}$

General solution:

$$w = E(x)\Phi(y-ax, z-bx) + kE(x)\int \coth^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x) = \exp\left[c\int \coth^n(\beta x)\,dx\right].$$

17. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} + c\coth(\beta z)\dfrac{\partial w}{\partial z} = \big[p\coth(\lambda x) + q\big]w + k\coth(\gamma x).}$

General solution:

$$w = \frac{k}{a}e^{qx/a}\sinh^{p/a\lambda}(\lambda x)\left[\Phi(u_1,u_2) + \int e^{-qx/a}\sinh^{-p/a\lambda}(\lambda x)\coth(\gamma x)\,dx\right],$$

where $u_1 = bx - ay$ and $u_2 = c\beta x - a\ln\big[\cosh(\beta z)\big]$.

18. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\coth^n(\beta x)\dfrac{\partial w}{\partial y} + b\coth^k(\lambda x)\dfrac{\partial w}{\partial z} = cw + s\coth^m(\mu x).}$

This is a special case of equation 2.4.7.1 with $f(x) = a\coth^n(\beta x)$, $g(x) = b\coth^k(\lambda x)$, $h(x) = c$, and $p(x) = s\coth^m(\mu x)$.

19. $\boldsymbol{\dfrac{\partial w}{\partial x} + b\coth^n(\beta x)\dfrac{\partial w}{\partial y} + c\coth^k(\lambda y)\dfrac{\partial w}{\partial z} = aw + s\coth^m(\mu x).}$

This is a special case of equation 2.4.7.8 with $f(x) = b\coth^n(\beta x)$, $g(y) = c\coth^k(\lambda y)$, and $h(x) = s\coth^m(\mu x)$.

20. $\boldsymbol{b_1\coth^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_2\coth^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y} + b_3\coth^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}}$
$\boldsymbol{= aw + c_1\coth^{k_1}(\beta_1 x) + c_2\coth^{k_2}(\beta_2 y) + c_3\coth^{k_3}(\beta_3 z).}$

This is a special case of equation 2.4.7.16 in which $f_1(x) = b_1\coth^{n_1}(\lambda_1 x)$, $f_2(y) = b_2\coth^{n_2}(\lambda_2 y)$, $f_3(z) = b_3\coth^{n_3}(\lambda_3 z)$, $g_1(x) = c_1\coth^{k_1}(\beta_1 x)$, $g_2(y) = c_2\coth^{k_2}(\beta_2 y)$, and $g_3(z) = c_3\coth^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain different hyperbolic functions.**

21. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\sinh^n(\beta x)w + k\cosh^m(\lambda x).}$

General solution:

$$w = E(x)\Phi(y-ax, z-bx) + kE(x)\int \cosh^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x) = \exp\left[c\int \sinh^n(\beta x)\,dx\right].$$

22. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\tanh^n(\beta x)w+k\coth^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\coth^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x)=\exp\biggl[c\int\tanh^n(\beta x)\,dx\biggr].$$

23. $\dfrac{\partial w}{\partial x}+b\cosh^n(\beta x)\dfrac{\partial w}{\partial y}+c\sinh^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\cosh^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\cosh^n(\beta x)$, $g(y)=c\sinh^k(\lambda y)$, and $h(x)=s\cosh^m(\mu x)$.

24. $\dfrac{\partial w}{\partial x}+a\tanh^n(\beta x)\dfrac{\partial w}{\partial y}+b\coth^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+s\tanh^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\tanh^n(\beta x)$, $g(x)=b\coth^k(\lambda x)$, $h(x)=c$, and $p(x)=s\tanh^m(\mu x)$.

25. $b_1\sinh^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_2\cosh^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y}+b_3\sinh^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}$
$=aw+c_1\cosh^{k_1}(\beta_1 x)+c_2\sinh^{k_2}(\beta_2 y)+c_3\sinh^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 in which $f_1(x)=b_1\sinh^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\cosh^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\sinh^{n_3}(\lambda_3 z)$, $g_1(x)=c_1\cosh^{k_1}(\beta_1 x)$, $g_2(y)=c_2\sinh^{k_2}(\beta_2 y)$, and $g_3(z)=c_3\sinh^{k_3}(\beta_3 z)$.

2.4.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

1. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\ln^n(\beta x)w+k\ln^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\ln^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x)=\exp\biggl[c\int\ln^n(\beta x)\,dx\biggr].$$

2. $\dfrac{\partial w}{\partial x}+a\ln^n(\beta x)\dfrac{\partial w}{\partial y}+b\ln^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+s\ln^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\ln^n(\beta x)$, $g(x)=b\ln^k(\lambda x)$, $h(x)=c$, and $p(x)=s\ln^m(\mu x)$.

3. $\dfrac{\partial w}{\partial x}+b\ln^n(\beta x)\dfrac{\partial w}{\partial y}+c\ln^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\ln^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\ln^n(\beta x)$, $g(y)=c\ln^k(\lambda y)$, and $h(x)=s\ln^m(\mu x)$.

4. $a\ln(\alpha x)\dfrac{\partial w}{\partial x}+b\ln(\beta y)\dfrac{\partial w}{\partial y}+c\ln(\gamma z)\dfrac{\partial w}{\partial z}=pw+q\ln(\lambda x).$

General solution:

$$w=\frac{q}{a}\exp\left[\frac{p}{a}\int\frac{dx}{\ln(\alpha x)}\right]\left\{\Phi(u_1,u_2)+\int\frac{\ln(\lambda x)}{\ln(\alpha x)}\exp\left[-\frac{p}{a}\int\frac{dx}{\ln(\alpha x)}\right]dx\right\},$$

where

$$u_1=b\int\frac{dx}{\ln(\alpha x)}-a\int\frac{dy}{\ln(\beta y)},\quad u_2=c\int\frac{dx}{\ln(\alpha x)}-a\int\frac{dz}{\ln(\gamma z)}.$$

5. $b_1\ln^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_2\ln^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y}+b_3\ln^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}$
$=aw+c_1\ln^{k_1}(\beta_1 x)+c_2\ln^{k_2}(\beta_2 y)+c_3\ln^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 with $f_1(x)=b_1\ln^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\ln^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\ln^{n_3}(\lambda_3 z)$, $g_1(x)=c_1\ln^{k_1}(\beta_1 x)$, $g_2(y)=c_2\ln^{k_2}(\beta_2 y)$, and $g_3(z)=c_3\ln^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain logarithmic and power-law functions.**

6. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\ln^n(\beta x)w+kx^m.$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\frac{x^m}{E(x)}\,dx,\qquad E(x)=\exp\left[c\int\ln^n(\beta x)\,dx\right].$$

7. $\dfrac{\partial w}{\partial x}+ax^n\dfrac{\partial w}{\partial y}+b\ln^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+sx^m.$

This is a special case of equation 2.4.7.1 with $f(x)=ax^n$, $g(x)=b\ln^k(\lambda x)$, $h(x)=c$, and $p(x)=sx^m$.

8. $\dfrac{\partial w}{\partial x}+b\ln^n(\beta x)\dfrac{\partial w}{\partial y}+cy^k\dfrac{\partial w}{\partial z}=aw+s\ln^m(\lambda x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\ln^n(\beta x)$, $g(y)=cy^k$, and $h(x)=s\ln^m(\lambda x)$.

9. $b_1\ln^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_2\ln^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y}+b_3\ln^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}$
$=aw+c_1x^{k_1}+c_2y^{k_2}+c_3z^{k_3}.$

This is a special case of equation 2.4.7.16 with $f_1(x)=b_1\ln^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\ln^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\ln^{n_3}(\lambda_3 z)$, $g_1(x)=c_1x^{k_1}$, $g_2(y)=c_2y^{k_2}$, and $g_3(z)=c_3z^{k_3}$.

10. $ax(\ln x)^n\dfrac{\partial w}{\partial x}+by(\ln y)^m\dfrac{\partial w}{\partial y}+cz(\ln z)^l\dfrac{\partial w}{\partial z}=k(\ln x)^s w+p\ln(\nu x).$

Introduce the notation

$$u_1=\frac{(\ln x)^{1-n}}{a(n-1)}-\frac{(\ln y)^{1-m}}{b(m-1)},\quad u_2=\frac{(\ln x)^{1-n}}{a(n-1)}-\frac{(\ln z)^{1-l}}{c(l-1)}.$$

1°. General solution for $s+1\neq n$:

$$w=\exp\left[\frac{k(\ln x)^{s-n+1}}{a(s-n+1)}\right]\left\{\Phi(u_1,u_2)+\frac{p}{a}\int\frac{\ln(\nu x)}{x(\ln x)^n}\exp\left[-\frac{k(\ln x)^{s-n+1}}{a(s-n+1)}\right]dx\right\}.$$

2°. General solution for $s+1=n$:

$$w=(\ln x)^{\frac{k}{a}}\left\{\Phi(u_1,u_2)+\frac{p}{a}\int\frac{\ln(\nu x)}{x}(\ln x)^{\frac{k-an}{a}}\,dx\right\}.$$

2.4.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $\displaystyle\frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}+b\frac{\partial w}{\partial z}=c\sin^n(\beta x)w+k\sin^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\sin^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\left[c\int\sin^n(\beta x)\,dx\right].$$

2. $\displaystyle\frac{\partial w}{\partial x}+a\sin^n(\beta x)\frac{\partial w}{\partial y}+b\sin^k(\lambda x)\frac{\partial w}{\partial z}=cw+s\sin^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\sin^n(\beta x)$, $g(x)=b\sin^k(\lambda x)$, $h(x)=c$, and $p(x)=s\sin^m(\mu x)$.

3. $\displaystyle\frac{\partial w}{\partial x}+b\sin^n(\beta x)\frac{\partial w}{\partial y}+c\sin^k(\lambda y)\frac{\partial w}{\partial z}=aw+s\sin^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\sin^n(\beta x)$, $g(y)=c\sin^k(\lambda y)$, and $h(x)=s\sin^m(\mu x)$.

4. $\displaystyle a\frac{\partial w}{\partial x}+b\sin(\beta y)\frac{\partial w}{\partial y}+c\sin(\gamma x)\frac{\partial w}{\partial z}=p\sin(\mu x)w+q\sin(\lambda x).$

General solution:

$$w=\frac{q}{a}\exp\left[-\frac{p}{a\mu}\cos(\mu x)\right]\left\{\Phi(u_1,u_2)+\int\sin(\lambda x)\exp\left[\frac{p}{a\mu}\cos(\mu x)\right]dx\right\},$$

where $u_1=b\beta x-a\ln\left|\tan\dfrac{\beta y}{2}\right|$ and $u_2=a\gamma z+c\cos(\gamma x)$.

5. $\displaystyle a\frac{\partial w}{\partial x}+b\sin(\beta y)\frac{\partial w}{\partial y}+c\sin(\gamma z)\frac{\partial w}{\partial z}=p\sin(\mu x)w+q\sin(\lambda x).$

General solution:

$$w=\frac{q}{a}\exp\left[-\frac{p}{a\mu}\cos(\mu x)\right]\left\{\Phi(u_1,u_2)+\int\sin(\lambda x)\exp\left[\frac{p}{a\mu}\cos(\mu x)\right]dx\right\},$$

where $u_1=b\beta x-a\ln\left|\tan\dfrac{\beta y}{2}\right|$ and $u_2=c\gamma x-a\ln\left|\tan\dfrac{\gamma z}{2}\right|$.

6. $$b_1 \sin^{n_1}(\lambda_1 x) \frac{\partial w}{\partial x} + b_2 \sin^{n_2}(\lambda_2 y) \frac{\partial w}{\partial y} + b_3 \sin^{n_3}(\lambda_3 z) \frac{\partial w}{\partial z} = aw + c_1 \sin^{k_1}(\beta_1 x) + c_2 \sin^{k_2}(\beta_2 y) + c_3 \sin^{k_3}(\beta_3 z).$$

This is a special case of equation 2.4.7.16 in which $f_1(x) = b_1 \sin^{n_1}(\lambda_1 x)$, $f_2(y) = b_2 \sin^{n_2}(\lambda_2 y)$, $f_3(z) = b_3 \sin^{n_3}(\lambda_3 z)$, $g_1(x) = c_1 \sin^{k_1}(\beta_1 x)$, $g_2(y) = c_2 \sin^{k_2}(\beta_2 y)$, and $g_3(z) = c_3 \sin^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain cosine.**

7. $$\frac{\partial w}{\partial x} + a \frac{\partial w}{\partial y} + b \frac{\partial w}{\partial z} = c \cos^n(\beta x) w + k \cos^m(\lambda x).$$

General solution:

$$w = E(x)\Phi(y - ax, z - bx) + kE(x) \int \cos^m(\lambda x) \frac{dx}{E(x)}, \quad E(x) = \exp\left[c \int \cos^n(\beta x)\, dx\right].$$

8. $$\frac{\partial w}{\partial x} + a \cos^n(\beta x) \frac{\partial w}{\partial y} + b \cos^k(\lambda x) \frac{\partial w}{\partial z} = cw + s \cos^m(\mu x).$$

This is a special case of equation 2.4.7.1 with $f(x) = a \cos^n(\beta x)$, $g(x) = b \cos^k(\lambda x)$, $h(x) = c$, and $p(x) = s \cos^m(\mu x)$.

9. $$\frac{\partial w}{\partial x} + b \cos^n(\beta x) \frac{\partial w}{\partial y} + c \cos^k(\lambda y) \frac{\partial w}{\partial z} = aw + s \cos^m(\mu x).$$

This is a special case of equation 2.4.7.8 with $f(x) = b \cos^n(\beta x)$, $g(y) = c \cos^k(\lambda y)$, and $h(x) = s \cos^m(\mu x)$.

10. $$a \frac{\partial w}{\partial x} + b \cos(\beta y) \frac{\partial w}{\partial y} + c \cos(\gamma z) \frac{\partial w}{\partial z} = p \cos(\mu x) w + q \cos(\lambda x).$$

General solution:

$$w = \frac{q}{a} \exp\left[\frac{p}{a\mu} \sin(\mu x)\right] \left\{\Phi(u_1, u_2) + \int \cos(\lambda x) \exp\left[-\frac{p}{a\mu} \sin(\mu x)\right] dx\right\},$$

where $u_1 = b\beta x - a \ln\left|\sec(\beta y) + \tan(\beta y)\right|$ and $u_2 = c\gamma x - a \ln\left|\sec(\gamma z) + \tan(\gamma z)\right|$.

11. $$a \frac{\partial w}{\partial x} + b \cos(\beta y) \frac{\partial w}{\partial y} + c \cos(\gamma x) \frac{\partial w}{\partial z} = p \cos(\mu x) w + q \cos(\lambda x).$$

General solution:

$$w = \frac{q}{a} \exp\left[\frac{p}{a\mu} \sin(\mu x)\right] \left\{\Phi(u_1, u_2) + \int \cos(\lambda x) \exp\left[-\frac{p}{a\mu} \sin(\mu x)\right] dx\right\},$$

where $u_1 = b\beta x - a \ln\left|\sec(\beta y) + \tan(\beta y)\right|$ and $u_2 = a\gamma z - c \sin(\gamma x)$.

12. $$b_1 \cos^{n_1}(\lambda_1 x) \frac{\partial w}{\partial x} + b_2 \cos^{n_2}(\lambda_2 y) \frac{\partial w}{\partial y} + b_3 \cos^{n_3}(\lambda_3 z) \frac{\partial w}{\partial z} = aw + c_1 \cos^{k_1}(\beta_1 x) + c_2 \cos^{k_2}(\beta_2 y) + c_3 \cos^{k_3}(\beta_3 z).$$

This is a special case of equation 2.4.7.16 in which $f_1(x) = b_1 \cos^{n_1}(\lambda_1 x)$, $f_2(y) = b_2 \cos^{n_2}(\lambda_2 y)$, $f_3(z) = b_3 \cos^{n_3}(\lambda_3 z)$, $g_1(x) = c_1 \cos^{k_1}(\beta_1 x)$, $g_2(y) = c_2 \cos^{k_2}(\beta_2 y)$, and $g_3(z) = c_3 \cos^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain tangent.**

13. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\tan^n(\beta x)w+k\tan^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\tan^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\left[c\int\tan^n(\beta x)\,dx\right].$$

14. $\dfrac{\partial w}{\partial x}+a\tan^n(\beta x)\dfrac{\partial w}{\partial y}+b\tan^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+s\tan^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\tan^n(\beta x)$, $g(x)=b\tan^k(\lambda x)$, $h(x)=c$, and $p(x)=s\tan^m(\mu x)$.

15. $\dfrac{\partial w}{\partial x}+b\tan^n(\beta x)\dfrac{\partial w}{\partial y}+c\tan^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\tan^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\tan^n(\beta x)$, $g(y)=c\tan^k(\lambda y)$, and $h(x)=s\tan^m(\mu x)$.

16. $a\dfrac{\partial w}{\partial x}+b\tan(\beta y)\dfrac{\partial w}{\partial y}+c\tan(\gamma z)\dfrac{\partial w}{\partial z}=p\tan(\mu x)w+q\tan(\lambda x).$

General solution: $w=\left|\cos(\mu x)\right|^{-p/a\mu}\left[\Phi(u_1,u_2)+\dfrac{q}{a}\displaystyle\int\left|\cos(\mu x)\right|^{p/a\mu}\tan(\lambda x)\,dx\right],$
where $u_1=b\beta x-a\ln\left|\sin(\beta y)\right|$ and $u_2=c\gamma x-a\ln\left|\sin(\gamma z)\right|$.

17. $a\dfrac{\partial w}{\partial x}+b\tan(\beta y)\dfrac{\partial w}{\partial y}+c\tan(\gamma x)\dfrac{\partial w}{\partial z}=p\tan(\mu x)w+q\tan(\lambda x).$

General solution: $w=\left|\cos(\mu x)\right|^{-p/a\mu}\left[\Phi(u_1,u_2)+\dfrac{q}{a}\displaystyle\int\left|\cos(\mu x)\right|^{p/a\mu}\tan(\lambda x)\,dx\right],$
where $u_1=b\beta x-a\ln\left|\sin(\beta y)\right|$ and $u_2=a\gamma z+c\ln\left|\cos(\gamma x)\right|$.

18. $b_1\tan^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_2\tan^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y}+b_3\tan^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}$
$=aw+c_1\tan^{k_1}(\beta_1 x)+c_2\tan^{k_2}(\beta_2 y)+c_3\tan^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 in which $f_1(x)=b_1\tan^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\tan^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\tan^{n_3}(\lambda_3 z)$, $g_1(x)=c_1\tan^{k_1}(\beta_1 x)$, $g_2(y)=c_2\tan^{k_2}(\beta_2 y)$, and $g_3(z)=c_3\tan^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain cotangent.**

19. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\cot^n(\beta x)w+k\cot^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\cot^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\left[c\int\cot^n(\beta x)\,dx\right].$$

20. $\dfrac{\partial w}{\partial x}+a\cot^n(\beta x)\dfrac{\partial w}{\partial y}+b\cot^k(\lambda x)\dfrac{\partial w}{\partial z}=cw+s\cot^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\cot^n(\beta x)$, $g(x)=b\cot^k(\lambda x)$, $h(x)=c$, and $p(x)=s\cot^m(\mu x)$.

21. $\dfrac{\partial w}{\partial x}+b\cot^n(\beta x)\dfrac{\partial w}{\partial y}+c\cot^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\cot^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\cot^n(\beta x)$, $g(y)=c\cot^k(\lambda y)$, and $h(x)=s\cot^m(\mu x)$.

22. $a\dfrac{\partial w}{\partial x}+b\cot(\beta y)\dfrac{\partial w}{\partial y}+c\cot(\gamma z)\dfrac{\partial w}{\partial z}=p\cot(\mu x)w+q\cot(\lambda x).$

General solution: $w=\bigl|\sin(\mu x)\bigr|^{p/a\mu}\Bigl[\Phi(u_1,u_2)+\dfrac{q}{a}\displaystyle\int\bigl|\sin(\mu x)\bigr|^{-p/a\mu}\cot(\lambda x)\,dx\Bigr]$, where $u_1=b\beta x+a\ln\bigl|\cos(\beta y)\bigr|$ and $u_2=c\gamma x+a\ln\bigl|\cos(\gamma z)\bigr|$.

23. $a\dfrac{\partial w}{\partial x}+b\cot(\beta y)\dfrac{\partial w}{\partial y}+c\cot(\gamma x)\dfrac{\partial w}{\partial z}=p\cot(\mu x)w+q\cot(\lambda x).$

General solution: $w=\bigl|\sin(\mu x)\bigr|^{p/a\mu}\Bigl[\Phi(u_1,u_2)+\dfrac{q}{a}\displaystyle\int\bigl|\sin(\mu x)\bigr|^{-p/a\mu}\cot(\lambda x)\,dx\Bigr]$, where $u_1=b\beta x+a\ln\bigl|\cos(\beta y)\bigr|$ and $u_2=a\gamma z-c\ln\bigl|\sin(\gamma x)\bigr|$.

24. $b_1\cot^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x}+b_2\cot^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y}+b_3\cot^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}$
$=aw+c_1\cot^{k_1}(\beta_1 x)+c_2\cot^{k_2}(\beta_2 y)+c_3\cot^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 in which $f_1(x)=b_1\cot^{n_1}(\lambda_1 x)$, $f_2(y)=b_2\cot^{n_2}(\lambda_2 y)$, $f_3(z)=b_3\cot^{n_3}(\lambda_3 z)$, $g_1(x)=c_1\cot^{k_1}(\beta_1 x)$, $g_2(y)=c_2\cot^{k_2}(\beta_2 y)$, and $g_3(z)=c_3\cot^{k_3}(\beta_3 z)$.

► **Coefficients of equations contain different trigonometric functions.**

25. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\sin^n(\beta x)w+k\cos^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\cos^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\Bigl[c\int\sin^n(\beta x)\,dx\Bigr].$$

26. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\tan^n(\beta x)w+k\cot^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\cot^m(\lambda x)\frac{dx}{E(x)},\quad E(x)=\exp\Bigl[c\int\tan^n(\beta x)\,dx\Bigr].$$

27. $\dfrac{\partial w}{\partial x}+b\cos^n(\beta x)\dfrac{\partial w}{\partial y}+c\sin^k(\lambda y)\dfrac{\partial w}{\partial z}=aw+s\cos^m(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\cos^n(\beta x)$, $g(y)=c\sin^k(\lambda y)$, and $h(x)=s\cos^m(\mu x)$.

28. $\dfrac{\partial w}{\partial x} + a\tan^n(\beta x)\dfrac{\partial w}{\partial y} + b\cot^k(\lambda x)\dfrac{\partial w}{\partial z} = cw + s\tan^m(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x) = a\tan^n(\beta x)$, $g(x) = b\cot^k(\lambda x)$, $h(x) = c$, and $p(x) = s\tan^m(\mu x)$.

29. $b_1\sin^{n_1}(\lambda_1 x)\dfrac{\partial w}{\partial x} + b_2\cos^{n_2}(\lambda_2 y)\dfrac{\partial w}{\partial y} + b_3\sin^{n_3}(\lambda_3 z)\dfrac{\partial w}{\partial z}$
$= aw + c_1\cos^{k_1}(\beta_1 x) + c_2\sin^{k_2}(\beta_2 y) + c_3\sin^{k_3}(\beta_3 z).$

This is a special case of equation 2.4.7.16 in which $f_1(x) = b_1\sin^{n_1}(\lambda_1 x)$, $f_2(y) = b_2\cos^{n_2}(\lambda_2 y)$, $f_3(z) = b_3\sin^{n_3}(\lambda_3 z)$, $g_1(x) = c_1\cos^{k_1}(\beta_1 x)$, $g_2(y) = c_2\sin^{k_2}(\beta_2 y)$, and $g_3(z) = c_3\sin^{k_3}(\beta_3 z)$.

2.4.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

1. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c\arcsin^n(\beta x)w + k\arcsin^m(\lambda x).$

General solution:

$$w = E(x)\Phi(y - ax, z - bx) + kE(x)\int \arcsin^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x) = \exp\left[c\int \arcsin^n(\beta x)\,dx\right].$$

2. $\dfrac{\partial w}{\partial x} + a\arcsin^k(\lambda x)\dfrac{\partial w}{\partial y} + b\arcsin^m(\beta x)\dfrac{\partial w}{\partial z} = cw + s\arcsin^n(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x) = a\arcsin^k(\lambda x)$, $g(x) = b\arcsin^m(\beta x)$, $h(x) = c$, and $p(x) = s\arcsin^n(\mu x)$.

3. $\dfrac{\partial w}{\partial x} + b\arcsin^k(\lambda x)\dfrac{\partial w}{\partial y} + c\arcsin^m(\beta y)\dfrac{\partial w}{\partial z} = aw + s\arcsin^n(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x) = b\arcsin^k(\lambda x)$, $g(y) = c\arcsin^m(\beta y)$, and $h(x) = s\arcsin^n(\mu x)$.

4. $\dfrac{\partial w}{\partial x} + a\arcsin^k(\lambda x)\dfrac{\partial w}{\partial y} + b\arcsin^m(\beta z)\dfrac{\partial w}{\partial z} = cw + s\arcsin^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x) = 1$, $f_2(x) = a\arcsin^k(\lambda x)$, $f_3(x) = 1$, $f_4(x) = c$, $f_5(x) = s\arcsin^n(\mu x)$, $g(y) = 1$, and $h(z) = b\arcsin^m(\beta z)$.

5. $\dfrac{\partial w}{\partial x} + a\arcsin^k(\lambda y)\dfrac{\partial w}{\partial y} + b\arcsin^m(\beta x)\dfrac{\partial w}{\partial z} = cw + s\arcsin^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x) = f_2(x) = 1$, $f_3(x) = b\arcsin^m(\beta x)$, $f_4(x) = c$, $f_5(x) = s\arcsin^n(\mu x)$, $g(y) = a\arcsin^k(\lambda y)$, and $h(z) = 1$.

► **Coefficients of equations contain arccosine.**

6. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\arccos^n(\beta x)w+k\arccos^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\arccos^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x)=\exp\biggl[c\int\arccos^n(\beta x)\,dx\biggr].$$

7. $\dfrac{\partial w}{\partial x}+a\arccos^k(\lambda x)\dfrac{\partial w}{\partial y}+b\arccos^m(\beta x)\dfrac{\partial w}{\partial z}=cw+s\arccos^n(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\arccos^k(\lambda x)$, $g(x)=b\arccos^m(\beta x)$, $h(x)=c$, and $p(x)=s\arccos^n(\mu x)$.

8. $\dfrac{\partial w}{\partial x}+b\arccos^k(\lambda x)\dfrac{\partial w}{\partial y}+c\arccos^m(\beta y)\dfrac{\partial w}{\partial z}=aw+s\arccos^n(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x)=b\arccos^k(\lambda x)$, $g(y)=c\arccos^m(\beta y)$, and $h(x)=s\arccos^n(\mu x)$.

9. $\dfrac{\partial w}{\partial x}+a\arccos^k(\lambda x)\dfrac{\partial w}{\partial y}+b\arccos^m(\beta z)\dfrac{\partial w}{\partial z}=cw+s\arccos^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x)=1$, $f_2(x)=a\arccos^k(\lambda x)$, $f_3(x)=1$, $f_4(x)=c$, $f_5(x)=s\arccos^n(\mu x)$, $g(y)=1$, and $h(z)=b\arccos^m(\beta z)$.

10. $\dfrac{\partial w}{\partial x}+a\arccos^k(\lambda y)\dfrac{\partial w}{\partial y}+b\arccos^m(\beta x)\dfrac{\partial w}{\partial z}=cw+s\arccos^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x)=f_2(x)=1$, $f_3(x)=b\arccos^m(\beta x)$, $f_4(x)=c$, $f_5(x)=s\arccos^n(\mu x)$, $g(y)=a\arccos^k(\lambda y)$, and $h(z)=1$.

► **Coefficients of equations contain arctangent.**

11. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=c\arctan^n(\beta x)w+k\arctan^m(\lambda x).$

General solution:

$$w=E(x)\Phi(y-ax,z-bx)+kE(x)\int\arctan^m(\lambda x)\frac{dx}{E(x)},$$
$$E(x)=\exp\biggl[c\int\arctan^n(\beta x)\,dx\biggr].$$

12. $\dfrac{\partial w}{\partial x}+a\arctan^k(\lambda x)\dfrac{\partial w}{\partial y}+b\arctan^m(\beta x)\dfrac{\partial w}{\partial z}=cw+s\arctan^n(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x)=a\arctan^k(\lambda x)$, $g(x)=b\arctan^m(\beta x)$, $h(x)=c$, and $p(x)=s\arctan^n(\mu x)$.

13. $\dfrac{\partial w}{\partial x} + b \arctan^k(\lambda x)\dfrac{\partial w}{\partial y} + c \arctan^m(\beta y)\dfrac{\partial w}{\partial z} = aw + s \arctan^n(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x) = b \arctan^k(\lambda x)$, $g(y) = c \arctan^m(\beta y)$, and $h(x) = s \arctan^n(\mu x)$.

14. $\dfrac{\partial w}{\partial x} + a \arctan^k(\lambda x)\dfrac{\partial w}{\partial y} + b \arctan^m(\beta z)\dfrac{\partial w}{\partial z} = cw + s \arctan^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x) = 1$, $f_2(x) = a \arctan^k(\lambda x)$, $f_3(x) = 1$, $f_4(x) = c$, $f_5(x) = s \arctan^n(\mu x)$, $g(y) = 1$, and $h(z) = b \arctan^m(\beta z)$.

15. $\dfrac{\partial w}{\partial x} + a \arctan^k(\lambda y)\dfrac{\partial w}{\partial y} + b \arctan^m(\beta x)\dfrac{\partial w}{\partial z} = cw + s \arctan^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x) = f_2(x) = 1$, $f_3(x) = b \arctan^m(\beta x)$, $f_4(x) = c$, $f_5(x) = s \arctan^n(\mu x)$, $g(y) = a \arctan^k(\lambda y)$, and $h(z) = 1$.

► Coefficients of equations contain arccotangent.

16. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} = c \operatorname{arccot}^n(\beta x) w + k \operatorname{arccot}^m(\lambda x).$

General solution:

$$w = E(x)\Phi(y - ax, z - bx) + kE(x) \int \operatorname{arccot}^m(\lambda x) \frac{dx}{E(x)},$$
$$E(x) = \exp\left[c \int \operatorname{arccot}^n(\beta x)\, dx\right].$$

17. $\dfrac{\partial w}{\partial x} + a \operatorname{arccot}^k(\lambda x)\dfrac{\partial w}{\partial y} + b \operatorname{arccot}^m(\beta x)\dfrac{\partial w}{\partial z} = cw + s \operatorname{arccot}^n(\mu x).$

This is a special case of equation 2.4.7.1 with $f(x) = a \operatorname{arccot}^k(\lambda x)$, $g(x) = b \operatorname{arccot}^m(\beta x)$, $h(x) = c$, and $p(x) = s \operatorname{arccot}^n(\mu x)$.

18. $\dfrac{\partial w}{\partial x} + b \operatorname{arccot}^k(\lambda x)\dfrac{\partial w}{\partial y} + c \operatorname{arccot}^m(\beta y)\dfrac{\partial w}{\partial z} = aw + s \operatorname{arccot}^n(\mu x).$

This is a special case of equation 2.4.7.8 with $f(x) = b \operatorname{arccot}^k(\lambda x)$, $g(y) = c \operatorname{arccot}^m(\beta y)$, and $h(x) = s \operatorname{arccot}^n(\mu x)$.

19. $\dfrac{\partial w}{\partial x} + a \operatorname{arccot}^k(\lambda x)\dfrac{\partial w}{\partial y} + b \operatorname{arccot}^m(\beta z)\dfrac{\partial w}{\partial z} = cw + s \operatorname{arccot}^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x) = 1$, $f_2(x) = a \operatorname{arccot}^k(\lambda x)$, $f_3(x) = 1$, $f_4(x) = c$, $f_5(x) = s \operatorname{arccot}^n(\mu x)$, $g(y) = 1$, and $h(z) = b \operatorname{arccot}^m(\beta z)$.

20. $\dfrac{\partial w}{\partial x} + a \operatorname{arccot}^k(\lambda y)\dfrac{\partial w}{\partial y} + b \operatorname{arccot}^m(\beta x)\dfrac{\partial w}{\partial z} = cw + s \operatorname{arccot}^n(\mu x).$

This is a special case of equation 2.4.7.10 with $f_1(x) = f_2(x) = 1$, $f_3(x) = b \operatorname{arccot}^m(\beta x)$, $f_4(x) = c$, $f_5(x) = s \operatorname{arccot}^n(\mu x)$, $g(y) = a \operatorname{arccot}^k(\lambda y)$, and $h(z) = 1$.

2.4.7 Equations Containing Arbitrary Functions

▶ **Coefficients of equations contain arbitrary functions of x.**

1. $\dfrac{\partial w}{\partial x} + f(x)\dfrac{\partial w}{\partial y} + g(x)\dfrac{\partial w}{\partial z} = h(x)w + p(x).$

General solution: $w = \exp\Bigl(\int h(x)\,dx\Bigr)\Bigl[\Phi(u_1,u_2) + \int p(x)\exp\Bigl(-\int h(x)\,dx\Bigr)\,dx\Bigr],$

where $u_1 = y - \int f(x)\,dx$ and $u_2 = z - \int g(x)\,dx.$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

2. $\dfrac{\partial w}{\partial x} + (y+a)f(x)\dfrac{\partial w}{\partial y} + (z+b)g(x)\dfrac{\partial w}{\partial z} = h(x)w + p(x).$

General solution: $w = \exp\Bigl(\int h(x)\,dx\Bigr)\Bigl[\Phi(u_1,u_2) + \int p(x)\exp\Bigl(-\int h(x)\,dx\Bigr)\,dx\Bigr],$

where $u_1 = \ln|y+a| - \int f(x)\,dx$ and $u_2 = \ln|z+b| - \int g(x)\,dx.$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

3. $\dfrac{\partial w}{\partial x} + [ay + f(x)]\dfrac{\partial w}{\partial y} + [bz + g(x)]\dfrac{\partial w}{\partial z} = h(x)w + p(x).$

General solution: $w = \exp\Bigl(\int h(x)\,dx\Bigr)\Bigl[\Phi(u_1,u_2) + \int p(x)\exp\Bigl(-\int h(x)\,dx\Bigr)\,dx\Bigr],$

where $u_1 = ye^{-ax} - \int f(x)e^{-ax}\,dx$ and $u_2 = ze^{-bx} - \int g(x)e^{-bx}\,dx.$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

4. $\dfrac{\partial w}{\partial x} + f(x)y^k\dfrac{\partial w}{\partial y} + g(x)z^m\dfrac{\partial w}{\partial z} = h(x)w + p(x).$

General solution: $w = \exp\Bigl(\int h(x)\,dx\Bigr)\Bigl[\Phi(u_1,u_2) + \int p(x)\exp\Bigl(-\int h(x)\,dx\Bigr)\,dx\Bigr],$

where $u_1 = \dfrac{1}{1-k}y^{1-k} - \int f(x)\,dx$ and $u_2 = \dfrac{1}{1-m}z^{1-m} - \int g(x)\,dx.$

5. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x)y + f_2(x)y^k\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x)z + g_2(x)z^m\bigr]\dfrac{\partial w}{\partial z} = h_1(x)w + h_2(x).$

This is a special case of equation 2.4.7.21 in which $g_1(x,y) = g_1(x)$, $g_2(x,y) = g_2(x)$, $h_1(x,y,z) = h_1(x)$, and $h_2(x,y,z) = h_2(x)$.

6. $\dfrac{\partial w}{\partial x} + \bigl[f_1(x) + f_2(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y} + \bigl[g_1(x) + g_2(x)e^{\beta z}\bigr]\dfrac{\partial w}{\partial z} = h_1(x)w + h_2(x).$

This is a special case of equation 2.4.7.24 in which $g_1(x,y) = g_1(x)$, $g_2(x,y) = g_2(x)$, $h_1(x,y,z) = h_1(x)$, and $h_2(x,y,z) = h_2(x)$.

7. $\dfrac{\partial w}{\partial x}+\big[y^2-a^2+a\lambda\sinh(\lambda x)-a^2\sinh^2(\lambda x)\big]\dfrac{\partial w}{\partial y}+f(x)\sinh(\gamma z)\dfrac{\partial w}{\partial z}=g(x)w+h(x).$

General solution: $w=\exp\Big[\int g(x)\,dx\Big]\Big\{\Phi(u_1,u_2)+\int h(x)\exp\Big[-\int g(x)\,dx\Big]\,dx\Big\},$
where

$$u_1=\int f(x)\,dx-\frac{1}{\gamma}\ln\Big|\tanh\frac{\gamma z}{2}\Big|,$$
$$u_2=\frac{E}{y-a\cosh(\lambda x)}+\int E\,dx,\quad E=\exp\Big[\frac{2a}{\lambda}\sinh(\lambda x)\Big].$$

► **Coefficients of equations contain arbitrary functions of different variables.**

8. $\dfrac{\partial w}{\partial x}+f(x)\dfrac{\partial w}{\partial y}+g(y)\dfrac{\partial w}{\partial z}=aw+h(x).$

General solution: $w=e^{ax}\Big[\Phi(u,v)+\int e^{-ax}h(x)\,dx\Big]$, where

$$u=y-F(x),\quad v=z-\int_{x_0}^{x}g\big(u+F(t)\big)\,dt,\quad F(x)=\int f(x)\,dx,$$

and x_0 may be taken as arbitrary.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

9. $\dfrac{\partial w}{\partial x}+f(x)y^k\dfrac{\partial w}{\partial y}+g(y)z^m\dfrac{\partial w}{\partial z}=aw+h(x).$

The transformation

$$Y(y)=\begin{cases}\frac{1}{1-k}y^{1-k} & \text{if } k\neq 1,\\ \ln|y| & \text{if } k=1,\end{cases}\qquad Z(z)=\begin{cases}\frac{1}{1-m}z^{1-m} & \text{if } m\neq 1,\\ \ln|z| & \text{if } m=1,\end{cases}$$

leads to an equation of the form 2.4.7.8:

$$\frac{\partial w}{\partial x}+f(x)\frac{\partial w}{\partial Y}+\bar g(Y)\frac{\partial w}{\partial Z}=aw+h(x),\qquad \bar g(Y)\equiv g(y).$$

10. $f_1(x)\dfrac{\partial w}{\partial x}+f_2(x)g(y)\dfrac{\partial w}{\partial y}+f_3(x)h(z)\dfrac{\partial w}{\partial z}=f_4(x)w+f_5(x).$

General solution:

$$w=F(x)\Big[\Phi(u_1,u_2)+\int\frac{f_5(x)}{f_1(x)}\frac{dx}{F(x)}\Big],\quad F(x)=\exp\Big[\int\frac{f_4(x)}{f_1(x)}\,dx\Big],$$

where

$$u_1=\int\frac{f_2(x)}{f_1(x)}\,dx-\int\frac{dy}{g(y)},\quad u_2=\int\frac{f_3(x)}{f_1(x)}\,dx-\int\frac{dz}{h(z)}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

11. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)z+g_2(y)\big]\dfrac{\partial w}{\partial z}=h_1(x)w+h_2(y).$

This is a special case of equation 2.4.7.20 in which $g_1(x,y)=g_1(x)$, $g_2(x,y)=g_2(y)$, $h_1(x,y,z)=h_1(x)$, and $h_2(x,y,z)=h_2(y)$.

12. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(y)z+g_2(x)z^m\big]\dfrac{\partial w}{\partial z}=h_1(y)w+h_2(z).$

This is a special case of equation 2.4.7.21 in which $g_1(x,y)=g_1(y)$, $g_2(x,y)=g_2(x)$, $h_1(x,y,z)=h_1(y)$, and $h_2(x,y,z)=h_2(z)$.

13. $\dfrac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)+g_2(y)e^{\lambda z}\big]\dfrac{\partial w}{\partial z}=h_1(x)w+h_2(y).$

This is a special case of equation 2.4.7.22 in which $g_1(x,y)=g_1(x)$, $g_2(x,y)=g_2(y)$, $h_1(x,y,z)=h_1(x)$, $h_2(x,y,z)=h_2(y)$.

14. $\dfrac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y}+\big[g_1(y)z+g_2(x)z^k\big]\dfrac{\partial w}{\partial z}=h_1(z)w+h_2(y).$

This is a special case of equation 2.4.7.23 in which $g_1(x,y)=g_1(y)$, $g_2(x,y)=g_2(x)$, $h_1(x,y,z)=h_1(z)$, and $h_2(x,y,z)=h_2(y)$.

15. $\dfrac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y}+\big[g_1(x)+g_2(y)e^{\beta z}\big]\dfrac{\partial w}{\partial z}=h_1(x)w+h_2(y).$

This is a special case of equation 2.4.7.24 in which $g_1(x,y)=g_1(x)$, $g_2(x,y)=g_2(y)$, $h_1(x,y,z)=h_1(x)$, and $h_2(x,y,z)=h_2(y)$.

16. $f_1(x)\dfrac{\partial w}{\partial x}+f_2(y)\dfrac{\partial w}{\partial y}+f_3(z)\dfrac{\partial w}{\partial z}=aw+g_1(x)+g_2(y)+g_3(z).$

General solution:

$$w=E_1(x)\Phi(u_1,u_2)+E_1(x)\int\frac{g_1(x)\,dx}{f_1(x)E_1(x)}+E_2(y)\int\frac{g_2(y)\,dy}{f_2(y)E_2(y)}+E_3(z)\int\frac{g_3(z)\,dz}{f_3(z)E_3(z)},$$

where

$$E_1(x)=\exp\left[a\int\frac{dx}{f_1(x)}\right],\quad E_2(y)=\exp\left[a\int\frac{dy}{f_2(y)}\right],\quad E_3(z)=\exp\left[a\int\frac{dz}{f_3(z)}\right],$$
$$u_1=\int\frac{dx}{f_1(x)}-\int\frac{dy}{f_2(y)},\quad u_2=\int\frac{dx}{f_1(x)}-\int\frac{dz}{f_3(z)}.$$

► **Coefficients of equations contain arbitrary functions of two variables.**

17. $\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial z}=F(x,y,z)w+G(x,y,z).$

This is a special case of equation 2.4.7.20 with $f_1(x)\equiv 0$, $f_2(x)=a$, $g_1(x,y)\equiv 0$, and $g_2(x,y)=b$.

18. $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=aw+f(x,y,z).$

General solution:

$$w=x^a\left[\Phi\left(\frac{y}{x},\frac{z}{x}\right)+\int_{x_0}^{x}f(t,u_1t,u_2t)\frac{dt}{t^{a+1}}\right],\qquad u_1=\frac{y}{x},\quad u_2=\frac{z}{x},$$

where x_0 may be taken as arbitrary.

19. $\displaystyle ax\frac{\partial w}{\partial x}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=F(x,y,z)w+G(x,y,z).$

The substitution $x=e^{a\xi}$ leads to an equation of the form 2.4.7.20:

$$\frac{\partial w}{\partial \xi}+by\frac{\partial w}{\partial y}+cz\frac{\partial w}{\partial z}=F(e^{a\xi},y,z)w+G(e^{a\xi},y,z).$$

20. $\displaystyle \frac{\partial w}{\partial x}+\bigl[f_1(x)y+f_2(x)\bigr]\frac{\partial w}{\partial y}+\bigl[g_1(x,y)z+g_2(x,y)\bigr]\frac{\partial w}{\partial z}=h_1(x,y,z)w+h_2(x,y,z).$

General solution:

$$w=H(x,u,v)\left[\Phi(u,v)+\int_{x_0}^{x}\frac{\bar h_2(t,u,v)}{H(t,u,v)}\,dt\right],\qquad H(x,u,v)=\exp\left[\int_{x_0}^{x}\bar h_1(t,u,v)\,dt\right],$$

where

$$u=yF(x)-\int f_2(x)F(x)\,dx,\qquad F(x)=\exp\left[-\int f_1(x)\,dx\right],\tag{1}$$

$$v=zG(x,u)-\int_{x_0}^{x}\bar g_2(t,u)G(t,u)\,dt,\qquad G(x,u)=\exp\left[-\int_{x_0}^{x}\bar g_1(t,u)\,dt\right].\tag{2}$$

Here $\bar g_n(x,u)\equiv g_n(x,y)$, $\bar h_n(x,u,v)\equiv h_n(x,y,z)$ [in these functions, y must be expressed via x and u from relation (1), and z must be expressed via x, u, and v from relation (2)], and x_0 is an arbitrary number.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

21. $\displaystyle \frac{\partial w}{\partial x}+\bigl[f_1(x)y+f_2(x)y^k\bigr]\frac{\partial w}{\partial y}+\bigl[g_1(x,y)z+g_2(x,y)z^m\bigr]\frac{\partial w}{\partial z}=h_1(x,y,z)w+h_2(x,y,z).$

1°. For $k\neq 1$ and $m\neq 1$, the transformation $\xi=y^{1-k}$, $\eta=z^{1-m}$ leads to an equation of the form 2.4.7.20:

$$\frac{\partial w}{\partial x}+(1-k)\bigl[f_1(x)\xi+f_2(x)\bigr]\frac{\partial w}{\partial \xi}+(1-m)\bigl[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\bigr]\frac{\partial w}{\partial \eta}=\bar h_1(x,\xi,\eta)w+\bar h_2(x,\xi,\eta),$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\bigl(x,\xi^{\frac{1}{1-k}}\bigr)$ and $\bar h_{1,2}(x,\xi,\eta)\equiv h_{1,2}\bigl(x,\xi^{\frac{1}{1-k}},\eta^{\frac{1}{1-m}}\bigr)$.

2°. For $k\neq 1$ and $m=1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 2.4.7.20.

3°. For $k=m=1$, see equation 2.4.7.20.

22. $$\frac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\frac{\partial w}{\partial y}+\big[g_1(x,y)+g_2(x,y)e^{\lambda z}\big]\frac{\partial w}{\partial z}=h_1(x,y,z)w+h_2(x,y,z).$$

The transformation $\xi=y^{1-k}$, $\eta=e^{-\lambda z}$ leads to an equation of the form 2.4.7.20:

$$\frac{\partial w}{\partial x}+(1-k)\big[f_1(x)\xi+f_2(x)\big]\frac{\partial w}{\partial \xi}-\lambda\big[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\big]\frac{\partial w}{\partial \eta}=\bar h_1(x,\xi,\eta)w+\bar h_2(x,\xi,\eta),$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\big(x,\xi^{\frac{1}{1-k}}\big)$ and $\bar h_{1,2}(x,\xi,\eta)\equiv h_{1,2}\big(x,\xi^{\frac{1}{1-k}},-\frac{1}{\lambda}\ln\eta\big)$.

23. $$\frac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\frac{\partial w}{\partial y}+\big[g_1(x,y)z+g_2(x,y)z^k\big]\frac{\partial w}{\partial z}=h_1(x,y,z)w+h_2(x,y,z).$$

The transformation $\xi=e^{-\lambda y}$, $\eta=z^{1-k}$ leads to an equation of the form 2.4.7.20:

$$\frac{\partial w}{\partial x}-\lambda\big[f_1(x)\xi+f_2(x)\big]\frac{\partial w}{\partial \xi}+(1-k)\big[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\big]\frac{\partial w}{\partial \eta}=\bar h_1(x,\xi,\eta)w+\bar h_2(x,\xi,\eta),$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\big(x,-\frac{1}{\lambda}\ln\xi\big)$ and $\bar h_{1,2}(x,\xi,\eta)\equiv h_{1,2}\big(x,-\frac{1}{\lambda}\ln\xi,\eta^{\frac{1}{1-k}}\big)$.

24. $$\frac{\partial w}{\partial x}+\big[f_1(x)+f_2(x)e^{\lambda y}\big]\frac{\partial w}{\partial y}+\big[g_1(x,y)+g_2(x,y)e^{\beta z}\big]\frac{\partial w}{\partial z}=h_1(x,y,z)w+h_2(x,y,z).$$

The transformation $\xi=e^{-\lambda y}$, $\eta=e^{-\beta z}$ leads to an equation of the form 2.4.7.20:

$$\frac{\partial w}{\partial x}-\lambda\big[f_1(x)\xi+f_2(x)\big]\frac{\partial w}{\partial \xi}-\beta\big[\bar g_1(x,\xi)\eta+\bar g_2(x,\xi)\big]\frac{\partial w}{\partial \eta}=\bar h_1(x,\xi,\eta)w+\bar h_2(x,\xi,\eta),$$

where $\bar g_{1,2}(x,\xi)\equiv g_{1,2}\big(x,-\frac{1}{\lambda}\ln\xi\big)$ and $\bar h_{1,2}(x,\xi,\eta)\equiv h_{1,2}\big(x,-\frac{1}{\lambda}\ln\xi,-\frac{1}{\beta}\ln\eta\big)$.

25. $$f_0(x)\frac{\partial w}{\partial x}+\big[f_1(x)y+f_2(x)y^k\big]\frac{\partial w}{\partial y}+\big[g_1(x,y)z+g_2(x,y)z^m\big]\frac{\partial w}{\partial z}=h_1(x,y,z)w+h_2(x,y,z).$$

On dividing the equation by $f_0(x)$, we obtain an equation of the form 2.4.7.21.

26. $$f_1(x)g_1(y)\frac{\partial w}{\partial x}+f_2(x)g_2(y)\frac{\partial w}{\partial y}+\big[h_1(x,y)z+h_2(x,y)z^m\big]\frac{\partial w}{\partial z}=\varphi_1(x,y,z)w+\varphi_2(x,y,z).$$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx$, $\eta=\displaystyle\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.4.7.21 with $f_1\equiv 0$, $f_2\equiv 1$, and $k=0$:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}+\big[\bar h_1(\xi,\eta)z+\bar h_2(\xi,\eta)z^m\big]\frac{\partial w}{\partial z}=\bar\varphi_1(\xi,\eta,z)w+\bar\varphi_2(\xi,\eta,z),$$

where $\bar h_n(\xi,\eta)\equiv\dfrac{h_n(x,y)}{f_2(x)g_1(y)}$ and $\bar\varphi_n(\xi,\eta,z)\equiv\dfrac{\varphi_n(x,y,z)}{f_2(x)g_1(y)}$; $n=1,\,2$.

27. $\displaystyle f_1(x)g_1(y)\frac{\partial w}{\partial x}+f_2(x)g_2(y)\frac{\partial w}{\partial y}+\big[h_1(x,y)+h_2(x,y)e^{\lambda z}\big]\frac{\partial w}{\partial z}=\varphi_1(x,y,z)w+\varphi_2(x,y,z).$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 2.4.7.22 with $f_1\equiv 0$, $f_2\equiv 1$, and $k=0$:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}+\big[\bar h_1(\xi,\eta)+\bar h_2(\xi,\eta)e^{\lambda z}\big]\frac{\partial w}{\partial z}=\bar\varphi_1(\xi,\eta,z)w+\bar\varphi_2(\xi,\eta,z),$$

where $\bar h_n(\xi,\eta)\equiv\dfrac{h_n(x,y)}{f_2(x)g_1(y)}$ and $\bar\varphi_n(\xi,\eta,z)\equiv\dfrac{\varphi_n(x,y,z)}{f_2(x)g_1(y)}$; $n=1,\,2.$

28. $\displaystyle \frac{\partial w}{\partial x}+f_1(x,y,z)\frac{\partial w}{\partial y}+f_2(x,y,z)\frac{\partial w}{\partial z}=g(x)w+h(x).$

General solution:

$$w=G(x)\Phi(u_1,u_2)+G(x)\int\frac{h(x)}{G(x)}\,dx,\qquad G(x)=\exp\bigg[\int g(x)\,dx\bigg],$$

where u_1, u_2 is an integral basis of the corresponding "truncated" homogeneous equation with $g(x)=h(x)=0$: $\dfrac{\partial u}{\partial x}+f_1(x,y,z)\dfrac{\partial u}{\partial y}+f_2(x,y,z)\dfrac{\partial u}{\partial z}=0.$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

2.4.8 Underdetermined Equations Containing Operator div

1. $\operatorname{div}\mathbf{u}=0.$

Let $\mathbf{u}=(u,v,w)$ be a vector field, where u, v, and w are its components depending on rectangular Cartesian coordinates x,y,z. Then the homogeneous equation in question has the form

$$\operatorname{div}\mathbf{u}\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad (1)$$

This is an underdetermined equation, which contains three unknown functions $u=u(x,y,z)$, $v=v(x,y,z)$, and $w=w(x,y,z)$. A similar equation often occurs in various systems of coupled equations and is called the *continuity equation*. Here we study this equation separately.

Equation (1) is invariant with respect to the transformation

$$u=\bar u+\xi_1(y,z),\quad v=\bar v+\xi_2(x,z),\quad w=\bar w+\xi_3(x,y),$$

where $\xi_1(y,z)$, $\xi_2(x,z)$, and $\xi_3(x,y)$ are arbitrary functions, which will be called the *calibration functions*.

Consider various representations of the general solution of Eq. (1).

1°. Any two out of three functions u, v, and w can be assumed to be arbitrary, and the remaining function is determined by a simple single integration of Eq. (1).

This method permits one to obtain the general solution in the form

$$u=u(x,y,z)\ \text{is an arbitrary function,}$$
$$v=v(x,y,z)\ \text{is an arbitrary function,}$$
$$w=-\int(u_x+v_y)\,dz+\xi_3(x,y).$$

2°. Suppose that $\psi^{(1)}=\psi^{(1)}(x,y,z)$ and $\psi^{(1)}=\psi^{(2)}(x,y,z)$ are two arbitrary twice continuously differentiable functions.

2.1. The general solution of Eq. (1) can be represented, for example, in the simple form

$$\mathbf{u}=(u,v,w),\quad u=\psi^{(1)}_y,\quad v=-\psi^{(1)}_x+\psi^{(2)}_z,\quad w=-\psi^{(2)}_y. \tag{2}$$

Remark 2.1. In hydrodynamics, $\psi^{(1)}$ and $\psi^{(2)}$ can be treated as two *stream functions*, which allow one to reduce the original three-dimensional equations of motion of an incompressible fluid with velocity components u, v, and w to equations for $\psi^{(1)}$ and $\psi^{(2)}$ (Polyanin & Zaitsev, 2012, p. 1248). In the special case of $\psi^{(2)}\equiv 0$ in (2), we have a usual representation of the fluid velocity components for two-dimensional flows in the plane (x,y) with $w=0$ in terms of one stream function.

Remark 2.2. Formulas (2) are invariant under the following transformation of the stream functions:

$$\psi^{(1)}=\bar\psi^{(1)}+\theta_z(x,z),\qquad \psi^{(2)}=\bar\psi^{(2)}+\theta_x(x,z),$$

where $\theta(x,z)$ is an arbitrary (calibration) function.

2.2. The general solution of Eq. (1) can also be represented in the form

$$\begin{aligned}u&=a_1\psi^{(1)}_y-a_3\psi^{(1)}_z+b_1\psi^{(2)}_y-b_3\psi^{(2)}_z,\\ v&=a_2\psi^{(1)}_z-a_1\psi^{(1)}_x+b_2\psi^{(2)}_z-b_1\psi^{(2)}_x,\\ w&=a_3\psi^{(1)}_x-a_2\psi^{(1)}_y+b_3\psi^{(2)}_x-b_2\psi^{(2)}_y,\end{aligned} \tag{3}$$

where a_i and b_i are arbitrary constants such that $a_1^2+a_2^2+a_3^2\neq 0$ and $b_1^2+b_2^2+b_3^2\neq 0$. Formulas (3) contain excessively many arbitrary constants. (Some of them can be set to 1, 0, or -1.) This gives us some freedom in representing the results. For example, by setting $a_2=a_3=b_1=b_3=0$ and $a_1=b_2=1$ in (3), we obtain the simple representation (2).

3°. The general solution of Eq. (1) can be represented in the vector form

$$\mathbf{u}=\operatorname{curl}\boldsymbol{\Psi}=(\Psi_{3y}-\Psi_{2z},\Psi_{1z}-\Psi_{3x},\Psi_{2x}-\Psi_{1y}), \tag{4}$$

where $\boldsymbol{\Psi}=(\Psi_1,\Psi_2,\Psi_3)$ is an arbitrary vector function.

Remark 2.3. The representation of the solution in the form (2) corresponds to the special choice of the components of the vector $\boldsymbol{\Psi}$ in the form

$$\Psi_1=\psi^{(2)},\quad \Psi_2=0,\quad \Psi_3=\psi^{(1)},$$

and the representation of the solution in the form (3) corresponds to the special case in which

$$\Psi_1=a_2\psi^{(1)}+b_2\psi^{(2)},\quad \Psi_2=a_3\psi^{(1)}+b_3\psi^{(2)},\quad \Psi_3=a_1\psi^{(1)}+b_1\psi^{(2)}.$$

Remark 2.4. The representation of the solution in the form (4) corresponds to the choice of the stream functions $\psi^{(1)}$ and $\psi^{(2)}$ in (2) in the form

$$\psi^{(1)} = \Psi_3 - \frac{\partial}{\partial z}\int_{y_0}^{y} \Psi_2(x,\bar{y},z)\,d\bar{y}, \quad \psi^{(2)} = \Psi_1 - \frac{\partial}{\partial x}\int_{y_0}^{y} \Psi_2(x,\bar{y},z)\,d\bar{y}, \tag{5}$$

where y_0 is an arbitrary constant. The representation (5) is unique under the conditions $\psi^{(1)}|_{y=y_0} = \Psi_3|_{y=y_0}$ and $\psi^{(2)}|_{y=y_0} = \Psi_1|_{y=y_0}$.

2. $\operatorname{div}\mathbf{u} = f(\mathbf{x})$.

The nonhomogeneous equation in the Cartesian coordinates has the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = f(\mathbf{x}), \tag{1}$$

where $\mathbf{u} = (u, v, w)$ and $\mathbf{x} = (x, y, z)$.

Particular solutions of equation (1) can be found by various methods.

1°. The simplest particular solutions are obtained if one sets any two out of the three functions u, v, and w to be zero. For example, this method gives a particular solution of the form

$$u_p = 0, \quad v_p = 0, \quad w_p = \int f(\mathbf{x})\,dz, \tag{2}$$

where $\int f(\mathbf{x})\,dz \equiv \int_{z_0}^{z} f(x,y,\bar{z})\,d\bar{z}$. From now on, the subscript p indicates that particular solutions are considered.

2°. One often seeks a particular solution in the form of the gradient of a scalar function,

$$\mathbf{u}_p = \nabla\varphi; \qquad \text{i.e.,} \qquad u_p = \varphi_x, \quad v_p = \varphi_y, \quad w_p = \varphi_z.$$

As a result, for φ one obtains the Poisson equation

$$\Delta\varphi = f(\mathbf{x}), \tag{3}$$

where Δ is the Laplace operator. For solutions of Eq. (3), see Section 10.2.

The general solution of the nonhomogeneous equation (1) can be represented as the sum of the general solution of the homogeneous equation (see Eq. 2.4.8.1) and a particular solution of the nonhomogeneous equation. For example, the general solution of the nonhomogeneous equation (1) can be represented in the form

$$u = \psi_y^{(1)}, \quad v = -\psi_x^{(1)} + \psi_z^{(2)}, \quad w = -\psi_y^{(2)} + \int f(\mathbf{x})\,dz.$$

3. $\operatorname{div}\mathbf{u} + \mathbf{f}(\mathbf{x})\cdot\mathbf{u} + g(\mathbf{x}) = 0$.

In the Cartesian coordinates x, y, z, the equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + f_1(\mathbf{x})u + f_2(\mathbf{x})v + f_3(\mathbf{x})w + g(\mathbf{x}) = 0,$$

where $\mathbf{u} = (u, v, w)$, $\mathbf{f} = (f_1, f_2, f_3)$, and $\mathbf{x} = (x, y, z)$.

This underdetermined equation is easy to integrate. (One arbitrarily specifies any two out of three components of the vector **u**, and for the remaining component we have a linear first-order ODE in which two spatial variables occur as parameters.) For example, if we arbitrarily specify the second and third components of the vector **u**, then the general solution can be represented in the form

$$u=E\Big(A-\int FE^{-1}\,dx\Big),\qquad v=v(\mathbf{x}),\qquad w=w(\mathbf{x}),$$
$$F=v_y+w_z+f_2v+f_3w+g,\qquad E=\exp\Big(-\int f_1\,dx\Big),$$

where $v(\mathbf{x})$, $w(\mathbf{x})$, and $A=A(y,z)$ are arbitrary functions.

2.4.9 Equations with Four or More Independent Variables

► **Equations containing power-law functions.**

1. $\displaystyle \frac{\partial w}{\partial x_1}+a\frac{\partial w}{\partial x_2}+b\frac{\partial w}{\partial x_3}+c\frac{\partial w}{\partial x_4}=0.$

Integral basis: $u_1=ax_1-x_2,\ u_2=bx_1-x_3,\ u_3=cx_1-x_4.$

2. $\displaystyle \frac{\partial w}{\partial x_1}+ax_1\frac{\partial w}{\partial x_2}+bx_1\frac{\partial w}{\partial x_3}+cx_1\frac{\partial w}{\partial x_4}=0.$

Integral basis: $u_1=x_2-\frac{1}{2}ax_1^2,\ u_2=x_3-\frac{1}{2}bx_1^2,\ u_3=x_4-\frac{1}{2}cx_1^2.$

3. $\displaystyle \frac{\partial w}{\partial x_1}+ax_2\frac{\partial w}{\partial x_2}+bx_3\frac{\partial w}{\partial x_3}+cx_4\frac{\partial w}{\partial x_4}=0.$

Integral basis: $u_1=ax_1-\ln|x_2|,\ u_2=bx_1-\ln|x_3|,\ u_3=cx_1-\ln|x_4|.$

4. $\displaystyle \beta\gamma\delta\frac{\partial w}{\partial x_1}+\alpha\gamma\delta(\gamma x_3-\delta x_4)\frac{\partial w}{\partial x_2}+\alpha\beta\delta(\alpha x_1+\beta x_2+\gamma x_3)\frac{\partial w}{\partial x_3}$
$\displaystyle \qquad+\alpha\beta\gamma(\alpha x_1+\beta x_2+\delta x_4)\frac{\partial w}{\partial x_4}=0.$

Integral basis:

$$u_1=\beta x_2-\gamma x_3-\delta x_4,\quad u_2=(\gamma x_3-\delta x_4)e^{-\alpha x_1},$$
$$u_3=(\alpha\gamma x_1x_3-\alpha\delta x_1x_4-\alpha x_1-\beta x_2-\delta x_4-1)e^{-\alpha x_1}.$$

5. $\displaystyle \alpha\beta\gamma x_1\frac{\partial w}{\partial x_1}+\beta\gamma(\beta x_3+\gamma x_4)\frac{\partial w}{\partial x_2}+\alpha\gamma(\alpha x_2+\gamma x_4)\frac{\partial w}{\partial x_3}$
$\displaystyle \qquad+\alpha\beta(\alpha x_2+\beta x_3)\frac{\partial w}{\partial x_4}=0.$

Integral basis:

$$u_1=x_1(\alpha x_2-\beta x_3),\quad u_2=x_1(\alpha x_2-\gamma x_4),\quad u_3=(\alpha x_2+\beta x_3+\gamma x_4)x_1^{-2}.$$

6. $\beta\gamma\delta(\beta x_2+\gamma x_3+\delta x_4)\dfrac{\partial w}{\partial x_1}+\alpha\gamma\delta(\alpha x_1+\gamma x_3+\delta x_4)\dfrac{\partial w}{\partial x_2}$
$+\alpha\beta\delta(\alpha x_1+\beta x_2+\delta x_4)\dfrac{\partial w}{\partial x_3}+\alpha\beta\gamma(\alpha x_1+\beta x_2+\gamma x_3)\dfrac{\partial w}{\partial x_4}=0.$

Integral basis:

$$u_1=\frac{\delta x_4-\beta x_2}{\delta x_4-\alpha x_1},\quad u_2=\frac{\delta x_4-\gamma x_3}{\delta x_4-\alpha x_1},\quad u_3=(\delta x_4-\alpha x_1)^3(\alpha x_1+\beta x_2+\gamma x_3+\delta x_4).$$

7. $x_1x_3\dfrac{\partial w}{\partial x_1}+x_2x_3\dfrac{\partial w}{\partial x_2}+x_3^2\dfrac{\partial w}{\partial x_3}+(x_1x_2+ax_3x_4)\dfrac{\partial w}{\partial x_4}=0.$

Integral basis:

$$u_1=\frac{x_2}{x_1},\quad u_2=\frac{x_3}{x_1},\quad u_3=\begin{cases}x_1^{1-a}\dfrac{x_2}{x_3}+(a-1)x_4x_1^{-a} & \text{if } a\neq 1,\\ \dfrac{x_4}{x_1}-\dfrac{x_2\ln x_1}{x_3} & \text{if } a=1.\end{cases}$$

8. $(\gamma\delta x_3x_4-\alpha\beta x_1x_2^2)\dfrac{\partial w}{\partial x_1}+\alpha\gamma x_2x_3\dfrac{\partial w}{\partial x_2}+\alpha\gamma x_3^2\dfrac{\partial w}{\partial x_3}+\alpha\gamma x_3x_4\dfrac{\partial w}{\partial x_4}=0.$

Integral basis: $u_1=\dfrac{x_3}{x_2},\ u_2=\dfrac{x_4}{x_2},\ u_3=\left(\alpha x_1-\dfrac{\gamma\delta x_3x_4}{\beta x_2^2}\right)\exp\left(\dfrac{\beta x_2^2}{\gamma x_3}\right).$

9. $\beta\gamma\delta x_2x_3x_4\dfrac{\partial w}{\partial x_1}+\alpha\gamma\delta x_1x_3x_4\dfrac{\partial w}{\partial x_2}+\alpha\beta\delta x_1x_2x_4\dfrac{\partial w}{\partial x_3}+\alpha\beta\gamma x_1x_2x_3\dfrac{\partial w}{\partial x_4}=0.$

Integral basis: $u_1=\alpha x_1^2-\beta x_2^2,\ u_2=\beta x_2^2-\gamma x_3^2,\ u_3=\gamma x_3^2-\delta x_4^2.$

10. $\displaystyle\sum_{k=1}^{n}x_k\frac{\partial w}{\partial x_k}=aw.$

Equation for homogeneous functions of order a. General solution:

$$w=x_n^a\Phi\left(\frac{x_1}{x_n},\ \frac{x_2}{x_n},\ \dots,\ \frac{x_{n-1}}{x_n}\right).$$

⊙ *Literature*: E. Kamke (1965).

11. $\displaystyle\sum_{k=1}^{n}(X-x_k)\frac{\partial w}{\partial x_k}=0,\qquad X=\sum_{k=1}^{n}x_k.$

Integral basis: $u_\nu=\dfrac{X-nx_\nu}{X-nx_{\nu+1}};\ \nu=1,\,2,\,\dots,\,n-1.$

12. $\displaystyle\sum_{k=1}^{n}\left(a_{k0}+\sum_{l=1}^{n}a_{kl}x_l\right)\frac{\partial w}{\partial x_k}=0.$

Let $s_1,\dots,s_n$ be the roots of the characteristic determinant

$$\begin{vmatrix}a_{11}-s & a_{12} & a_{13} & \dots & a_{1n}\\ a_{21} & a_{22}-s & a_{23} & \dots & a_{2n}\\ a_{31} & a_{32} & a_{33}-s & \dots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-s\end{vmatrix}.$$

For each s_i there exist n numbers b_{ij} not equal to zero simultaneously such that

$$\sum_{j=1}^{n} a_{jm} b_{ij} = b_{im} s_i; \quad m = 1,\, 2,\, \dots,\, n,$$

and

$$d_i = \sum_{m=1}^{n} a_{m0} b_{im}.$$

1°. If for some i we have $s_i = d_i = 0$, then one of the integrals is given by

$$u_i = \sum_{j=1}^{n} b_{ij} x_j.$$

2°. If an s_i and an s_j are distinct nonzero roots, then one of the integrals is

$$u_j = \frac{\left(\dfrac{d_i}{s_i} + \sum\limits_{m=1}^{n} b_{im} x_m\right)^{s_j}}{\left(\dfrac{d_j}{s_j} + \sum\limits_{m=1}^{n} b_{jm} x_m\right)^{s_i}}.$$

If all the roots s_i are different, this formula provides an integral basis.

3°. If there are multiple roots among s_i, then one can reduce the number of independent variables by using the substitution specified in Section 13.1.3 (see paragraph *The method of reducing the number of independent variables*).

13. $\displaystyle \sum_{k=1}^{n} (A_0 x_k - A_k) \frac{\partial w}{\partial x_k} = 0, \quad \text{where} \quad A_k = a_{k0} + \sum_{l=1}^{n} a_{kl} x_l.$

Hesse's equation. Introduce the homogeneous coordinates $x_1 = \xi_1/\xi_0, \dots, x_n = \xi_n/\xi_0$ to reduce Hesse's equation to a constant coefficient equation for $w = w(\xi_0, \xi_1, \dots, \xi_n)$ with $n + 1$ independent variables:

$$\sum_{k=0}^{n} B_k \frac{\partial w}{\partial \xi_k} = 0, \quad \text{where} \quad B_k = \sum_{l=0}^{n} a_{kl} \xi_l.$$

For the solution of this equation, see 2.4.9.12.

⊙ *Literature*: E. Kamke (1965).

14. $\displaystyle \frac{\partial w}{\partial x_1} + \sum_{k=2}^{n} a_k x_1^{m_k} \frac{\partial w}{\partial x_k} = bw + c.$

This is a special case of equation 2.4.9.32 with $f_k(x_1) = a_k x_1^{m_k}$, $g(x_1) = b$, and $h(x_1) = c$.

15. $\displaystyle \frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl(a_k x_k+b_k x_1^{m_k}\bigr)\frac{\partial w}{\partial x_k}=c_1 w+c_2.$

This is a special case of equation 2.4.9.33 with $f_k(x_1)=b_k x_1^{m_k}$, $g(x_1)=c_1$, and $h(x_1)=c_2$.

16. $\displaystyle \sum_{k=1}^{n} a_k x_k^{m_k}\frac{\partial w}{\partial x_k}=bw+\sum_{k=1}^{n} c_k x_k^{s_k}.$

This is a special case of equation 2.4.9.34 with $f_k(x_k)=a_k x_k^{m_k}$ and $g_k(x_k)=c_k x_k^{s_k}$.

17. $\displaystyle \frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl(b_k x_{k-1}^{m_k} x_k+c_k x_k^{a_k}\bigr)\frac{\partial w}{\partial x_k}=s_1 w+s_2.$

This is a special case of equation 2.4.9.36 with $f_k(\dots)=b_k x_{k-1}^{m_k}$, $g_k(\dots)=c_k$, $h_1(\dots)=s_1$, and $h_2(\dots)=s_2$.

► **Other equations containing arbitrary parameters.**

18. $\displaystyle \frac{\partial w}{\partial x_1}+\sum_{k=2}^{n} a_k e^{\lambda_k x_1}\frac{\partial w}{\partial x_k}=bw+c.$

This is a special case of equation 2.4.9.32 with $f_k(x_1)=a_k e^{\lambda_k x_1}$, $g(x_1)=b$, and $h(x_1)=c$.

19. $\displaystyle \sum_{k=1}^{n} a_k e^{\lambda_k x_k}\frac{\partial w}{\partial x_k}=bw+c.$

General solution:

$$w=\begin{cases}-\dfrac{c}{b}+\exp\Bigl(-\dfrac{b}{a_n\lambda_n}e^{-\lambda_n x_n}\Bigr)\Phi(u_1,u_2,\dots,u_{n-1}) & \text{if } b\neq 0,\\[2ex] -\dfrac{c}{a_n\lambda_n}e^{-\lambda_n x_n}+\Phi(u_1,u_2,\dots,u_{n-1}) & \text{if } b=0,\end{cases}$$

where $u_m=a_m\lambda_m e^{-\lambda_n x_n}-a_n\lambda_n e^{-\lambda_m x_m}$; $m=1,\,\dots,\,n-1$.

20. $\displaystyle \frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl(a_k e^{\beta_k x_{k-1}}+b_k e^{\lambda_k x_k}\bigr)\frac{\partial w}{\partial x_k}=c_1 w+c_2.$

This is a special case of equation 2.4.9.37.

21. $\displaystyle \frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl(a_k x_k+b_k e^{\lambda_k x_1}\bigr)\frac{\partial w}{\partial x_k}=c_1 w+c_2.$

This is a special case of equation 2.4.9.33 in which $f_k(x_1)=b_k e^{\lambda_k x_1}$, $g(x_1)=c_1$, and $h(x_1)=c_2$.

22. $\displaystyle \frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl(a_k x_{k-1}^{\beta_k}+b_k e^{\lambda_k x_k}\bigr)\frac{\partial w}{\partial x_k}=c_1 w+c_2.$

This is a special case of equation 2.4.9.37.

23. $\displaystyle\sum_{k=1}^{n} a_k \sinh\bigl(\lambda_k x_k\bigr)\frac{\partial w}{\partial x_k}=bw+c.$

General solution:

$$w=\begin{cases}-\dfrac{c}{b}+\bigl[\tanh\bigl(\tfrac12\lambda_n x_n\bigr)\bigr]^{\frac{b}{a_n\lambda_n}}\Phi(u_1,u_2,\dots,u_{n-1}) & \text{if } b\neq 0,\\[1ex] -\dfrac{c}{a_n\lambda_n}\ln\bigl|\tanh\bigl(\tfrac12\lambda_n x_n\bigr)\bigr|+\Phi(u_1,u_2,\dots,u_{n-1}) & \text{if } b=0,\end{cases}$$

where $u_m=\coth^{a_m\lambda_m}\bigl(\tfrac12\lambda_n x_n\bigr)\tanh^{a_n\lambda_n}\bigl(\tfrac12\lambda_m x_m\bigr);\ m=1,\ \dots,\ n-1.$

24. $\displaystyle\frac{\partial w}{\partial x_1}+\sum_{k=2}^{n} a_k \cosh(\lambda_k x_1)\frac{\partial w}{\partial x_k}=bw+c.$

This is a special case of equation 2.4.9.32 with $f_k(x_1)=a_k\cosh(\lambda_k x_1)$, $g(x_1)=b$, and $h(x_1)=c$.

25. $\displaystyle\sum_{k=1}^{n} a_k \cosh\bigl(\lambda_k x_k\bigr)\frac{\partial w}{\partial x_k}=bw+c.$

General solution:

$$w=\begin{cases}-\dfrac{c}{b}+\exp\left[\dfrac{2b}{a_n\lambda_n}\arctan\left(\tanh\dfrac{\lambda_n x_n}{2}\right)\right]\Phi(u_1,u_2,\dots,u_{n-1}) & \text{if } b\neq 0,\\[1ex] \dfrac{2c}{a_n\lambda_n}\arctan\left(\tanh\dfrac{\lambda_n x_n}{2}\right)+\Phi(u_1,u_2,\dots,u_{n-1}) & \text{if } b=0.\end{cases}$$

The functions u_m, $m=1,\ \dots,\ n-1$, are expressed as $u_m=a_m\lambda_m\arctan\bigl[\tanh\bigl(\tfrac12\lambda_n x_n\bigr)\bigr]-a_n\lambda_n\arctan\bigl[\tanh\bigl(\tfrac12\lambda_m x_m\bigr)\bigr]$.

26. $\displaystyle\sum_{k=1}^{n} a_k \tanh\bigl(\lambda_k x_k\bigr)\frac{\partial w}{\partial x_k}=bw+c.$

This is a special case of equation 2.4.9.34 with $f_k(x_k)=a_k\tanh\bigl(\lambda_k x_k\bigr)$.

27. $\displaystyle\sum_{k=1}^{n} a_k \ln\bigl(\lambda_k x_k\bigr)\frac{\partial w}{\partial x_k}=bw+c.$

This is a special case of equation 2.4.9.34 with $f_k(x_k)=a_k\ln\bigl(\lambda_k x_k\bigr)$.

28. $\displaystyle\frac{\partial w}{\partial x_1}+\sum_{k=2}^{n} a_k \sin(\lambda_k x_1)\frac{\partial w}{\partial x_k}=bw+c.$

This is a special case of equation 2.4.9.32 with $f_k(x_1)=a_k\sin(\lambda_k x_1)$, $g(x_1)=b$, and $h(x_1)=c$.

29. $\displaystyle\sum_{k=1}^{n} a_k \sin(\lambda_k x_k)\frac{\partial w}{\partial x_k} = bw + c.$

General solution:

$$w = \begin{cases} -\dfrac{c}{b} + \tan^{\frac{b}{a_n\lambda_n}}\left(\dfrac{\lambda_n x_n}{2}\right)\Phi(u_1, u_2, \ldots, u_{n-1}) & \text{if } b \neq 0, \\ \dfrac{c}{a_n\lambda_n}\ln\left|\tan\dfrac{\lambda_n x_n}{2}\right| + \Phi(u_1, u_2, \ldots, u_{n-1}) & \text{if } b = 0, \end{cases}$$

where $u_m = \cot^{a_m\lambda_m}\bigl(\tfrac{1}{2}\lambda_n x_n\bigr)\tan^{a_n\lambda_n}\bigl(\tfrac{1}{2}\lambda_m x_m\bigr);\ m = 1,\ \ldots,\ n-1.$

30. $\displaystyle\sum_{k=1}^{n} a_k \cos(\lambda_k x_k)\frac{\partial w}{\partial x_k} = bw + c.$

General solution:

$$w = \begin{cases} -\dfrac{c}{b} + \bigl[\sec(\lambda_n x_n) + \tan(\lambda_n x_n)\bigr]^{\frac{c}{a_n\lambda_n}}\Phi(u_1, u_2, \ldots, u_{n-1}) & \text{if } b \neq 0, \\ \dfrac{c}{a_n\lambda_n}\ln\bigl|\sec(\lambda_n x_n) + \tan(\lambda_n x_n)\bigr| + \Phi(u_1, u_2, \ldots, u_{n-1}) & \text{if } b = 0, \end{cases}$$

where

$$u_m = a_m\lambda_m \ln\bigl|\sec(\lambda_n x_n) + \tan(\lambda_n x_n)\bigr| + a_n\lambda_n \ln\bigl|\sec(\lambda_m x_m) + \tan(\lambda_m x_m)\bigr|;$$
$$m = 1, \ldots, n-1.$$

31. $\displaystyle\sum_{k=1}^{n} a_k \tan(\lambda_k x_k)\frac{\partial w}{\partial x_k} = bw + c.$

This is a special case of equation 2.4.9.34 with $f_k(x_k) = a_k\tan\bigl(\lambda_k x_k\bigr)$, $g_1(x_1) = c$, and $g_2(x_2) = g_n(x_n) = 0$.

► **Equations containing arbitrary functions.**

32. $\displaystyle\frac{\partial w}{\partial x_1} + \sum_{k=2}^{n} f_k(x_1)\frac{\partial w}{\partial x_k} = g(x_1)w + h(x_1).$

General solution:

$$w = G(x_1)\left[\Phi(u_1, u_2, \ldots, u_{n-1}) + \int\frac{h(x_1)}{G(x_1)}\,dx\right], \qquad G(x_1) = \exp\left[\int g(x_1)\,dx_1\right],$$

where $u_m = x_{m+1} - \displaystyle\int f_{m+1}(x_1)\,dx_1;\ m = 1,\ 2,\ \ldots,\ n-1.$

33. $\displaystyle \frac{\partial w}{\partial x_1} + \sum_{k=2}^{n} \big[a_k x_k + f_k(x_1)\big] \frac{\partial w}{\partial x_k} = g(x_1)w + h(x_1).$

General solution:

$$w = G(x_1)\left[\Phi(u_1, u_2, \dots, u_{n-1}) + \int \frac{h(x_1)}{G(x_1)}\,dx\right], \qquad G(x_1) = \exp\left[\int g(x_1)\,dx_1\right],$$

where

$$u_m = x_{m+1}\exp(-a_{m+1}x_1) - \int f_{m+1}(x_1)\exp(-a_{m+1}x_1)\,dx_1; \quad m = 1,\,\dots,\,n-1.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

34. $\displaystyle \sum_{k=1}^{n} f_k(x_k) \frac{\partial w}{\partial x_k} = bw + \sum_{k=1}^{n} g_k(x_k).$

Introduce the notation

$$u_m = \int \frac{dx_m}{f_m(x_m)} - \int \frac{dx_n}{f_n(x_n)}; \qquad m = 1,\,\dots,\,n-1.$$

1°. General solution for $b = 0$:

$$w = \Phi(u_1, u_2, \dots, u_{n-1}) + \sum_{k=1}^{n} \int \frac{g_k(x_k)}{f_k(x_k)}\,dx_k.$$

2°. General solution for $b \neq 0$:

$$w = F_1(x_1)\Phi(u_1, u_2, \dots, u_{n-1}) + \sum_{k=1}^{n} F_k(x_k) \int \frac{g_k(x_k)\,dx_k}{f_k(x_k)F_k(x_k)},$$

$$F_k(x_k) = \exp\left[b \int \frac{dx_k}{f_k(x_k)}\right].$$

35. $\displaystyle \frac{\partial w}{\partial x_1} + \sum_{k=2}^{n} \big[x_k f_k(x_1, x_2, \dots, x_{k-1}) + g_k(x_1, x_2, \dots, x_{k-1})\big] \frac{\partial w}{\partial x_k}$
$\displaystyle = h_1(x_1, x_2, \dots, x_k)w + h_2(x_1, x_2, \dots, x_k).$

Change the variables $x_1, x_2, x_3, \dots, x_n$ for $x_1, u_2, x_3, \dots, x_n$, where

$$u_2 = x_2 F_2(x_1) - \int g_2(x_1)F_2(x_1)\,dx_1, \qquad F_2(x_1) = \exp\left[-\int f_2(x_1)\,dx_1\right], \quad (1)$$

to obtain the equation

$$\frac{\partial w}{\partial x_1} + \sum_{k=3}^{n} \big[x_k \bar f_k(x_1, u_2, x_3, \dots, x_{k-1}) + \bar g_k(x_1, u_2, x_3, \dots, x_{k-1})\big] \frac{\partial w}{\partial x_k}$$
$$= \bar h_1(x_1, u_2, x_3, \dots, x_k)w + \bar h_2(x_1, u_2, x_3, \dots, x_k), \quad (2)$$

whose coefficients are defined as $f_k(x_1, x_2, x_3, \dots, x_{k-1}) \equiv \bar f_k(x_1, u_2, x_3, \dots, x_{k-1})$, etc.

Equation (2) is similar to the original one, but contains fewer independent variables, $x_1, x_3, \dots, x_n$ (there is no derivative of w with respect to u_2 in the transformed equation, and hence, u_2 can be treated as a parameter). By applying transformations of the form (1) successively, one can reduce the original partial differential equation to a first-order linear ordinary differential equation for x_1, the coefficients of which depend on the parameters $u_2, \dots, u_n$.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

36. $$\frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl[x_k f_k(x_1,x_2,\dots,x_{k-1})+x_k^{a_k}g_k(x_1,x_2,\dots,x_{k-1})\bigr]\frac{\partial w}{\partial x_k}=h_1(x_1,x_2,\dots,x_k)w+h_2(x_1,x_2,\dots,x_k).$$

1°. If $a_2, \dots, a_n$ are not all equal to unity simultaneously, then the transformation $z_k = x_k^{1-a_k}$ $(k = 2, \dots, n)$ leads to an equation of the form 2.4.9.35:

$$\frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}(1-a_k)\bigl[z_k\widetilde{f}_k(x_1,z_2,\dots,z_{k-1})+\widetilde{g}_k(x_1,z_2,\dots,z_{k-1})\bigr]\frac{\partial w}{\partial z_k}=\widetilde{h}_1(x_1,z_2,\dots,z_k)w+\widetilde{h}_2(x_1,z_2,\dots,z_k),$$

where $\widetilde{f}_k(x_1,z_2,\dots,z_{k-1})\equiv f_k(x_1,x_2,\dots,x_n)$, etc.

2°. If an $a_m=1$ and the other $a_k\neq 1$ $(k\neq m)$, then the transformation $z_m=x_m$, $z_k=x_k^{1-a_k}$ $(k\neq m)$ leads to an equation of the form 2.4.9.35.

37. $$\frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}\bigl[f_k(x_1,x_2,\dots,x_{k-1})+\exp(\lambda_k x_k)g_k(x_1,x_2,\dots,x_{k-1})\bigr]\frac{\partial w}{\partial x_k}=h_1(x_1,x_2,\dots,x_k)w+h_2(x_1,x_2,\dots,x_k).$$

The transformation $z_k = \exp(-\lambda_k x_k)$ $(k = 2, \dots, n)$ leads to an equation of the form 2.4.9.35:

$$\frac{\partial w}{\partial x_1}-\sum_{k=2}^{n}\lambda_k\bigl[z_k\widetilde{f}_k(x_1,z_2,\dots,z_{k-1})+\widetilde{g}_k(x_1,z_2,\dots,z_{k-1})\bigr]\frac{\partial w}{\partial z_k}=\widetilde{h}_1(x_1,z_2,\dots,z_k)w+\widetilde{h}_2(x_1,z_2,\dots,z_k),$$

where $\widetilde{f}_k(x_1,z_2,\dots,z_{k-1})\equiv f_k(x_1,x_2,\dots,x_n)$, etc.

38. $$\sum_{k=1}^{n}(f_k-f_0x_k)\frac{\partial w}{\partial x_k}+\sum_{k=1}^{m}\varphi_k(y_1,\dots,y_m)\frac{\partial w}{\partial y_k}=0,\qquad f_k=a_{k0}+\sum_{l=1}^{n}a_{kl}x_l.$$

By applying the Hesse technique (see equation 2.4.9.13), one can make the first $n+1$ coefficients linear. On introducing the homogeneous coordinates $x_1=\xi_1/\xi_0, \dots, x_n=\xi_n/\xi_0$, we arrive at the equation

$$\sum_{k=0}^{n}g_k\frac{\partial w}{\partial \xi_k}+\sum_{k=1}^{m}\varphi_k(y_1,\dots,y_m)\frac{\partial w}{\partial y_k}=0,\quad\text{where}\quad g_k=\sum_{l=0}^{n}a_{kl}\xi_l.$$

In particular, if $m = 1$ and $\varphi_1 = \varphi(y_1)$, then $t = \displaystyle\int \frac{dy_1}{\varphi(y_1)}$ can be chosen to be the independent variable in the characteristic equations, with the latter forming a linear system $\xi_k'(t) = g_k$ $(k = 0, \ldots, n)$.

⊙ *Literature*: E. Kamke (1965).

Chapter 3

Second-Order Parabolic Equations with One Space Variable

3.1 Constant Coefficient Equations

3.1.1 Heat Equation $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2}$

This equation is often encountered in the theory of heat and mass transfer. It describes one-dimensional unsteady thermal processes in quiescent media or solids with constant thermal diffusivity. A similar equation is used in studying corresponding one-dimensional unsteady mass-exchange processes with constant diffusivity.

► **Particular solutions (A, B, and μ are arbitrary constants).**

$$
\begin{aligned}
w(x) &= Ax + B,\\
w(x,t) &= A(x^2 + 2at) + B,\\
w(x,t) &= A(x^3 + 6atx) + B,\\
w(x,t) &= A(x^4 + 12atx^2 + 12a^2t^2) + B,\\
w(x,t) &= A(x^5 + 20atx^3 + 60a^2t^2x) + B,\\
w(x,t) &= A(x^6 + 30atx^4 + 180a^2t^2x^2 + 120a^3t^3) + B,\\
w(x,t) &= A(x^7 + 42atx^5 + 420a^2t^2x^3 + 840a^3t^3x) + B,\\
w(x,t) &= x^{2n} + \sum_{k=1}^{n} \frac{(2n)(2n-1)\dots(2n-2k+1)}{k!}(at)^k x^{2n-2k},\\
w(x,t) &= x^{2n+1} + \sum_{k=1}^{n} \frac{(2n+1)(2n)\dots(2n-2k+2)}{k!}(at)^k x^{2n-2k+1},\\
w(x,t) &= A\exp(a\mu^2 t \pm \mu x) + B,\\
w(x,t) &= A\exp(-a\mu^2 t)\cos(\mu x) + B,
\end{aligned}
$$

$$w(x,t) = A\exp(-a\mu^2 t)\sin(\mu x) + B,$$
$$w(x,t) = A\frac{1}{\sqrt{t}}\exp\biggl(-\frac{x^2}{4at}\biggr) + B,$$
$$w(x,t) = A\frac{x}{t^{3/2}}\exp\biggl(-\frac{x^2}{4at}\biggr) + B,$$
$$w(x,t) = A\exp(-\mu x)\cos(\mu x - 2a\mu^2 t) + B,$$
$$w(x,t) = A\exp(-\mu x)\sin(\mu x - 2a\mu^2 t) + B,$$
$$w(x,t) = A\,\mathrm{erf}\biggl(\frac{x}{2\sqrt{at}}\biggr) + B,$$
$$w(x,t) = A\,\mathrm{erfc}\biggl(\frac{x}{2\sqrt{at}}\biggr) + B,$$
$$w(x,t) = A\biggl[\sqrt{\frac{t}{\pi}}\exp\biggl(-\frac{x^2}{4at}\biggr) - \frac{x}{2\sqrt{a}}\,\mathrm{erfc}\biggl(\frac{x}{2\sqrt{at}}\biggr)\biggr] + B,$$

where n is a positive integer, $\mathrm{erf}\, z \equiv \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \exp(-\xi^2)\,d\xi$ the error function (probability integral), and $\mathrm{erfc}\, z = 1 - \mathrm{erf}\, z$ the complementary error function (complementary probability integral).

Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\sqrt{\pi a t}}\exp\biggl(-\frac{x^2}{4at}\biggr).$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

► Formulas allowing the construction of particular solutions.

Suppose $w = w(x,t)$ is a solution of the heat equation. Then the functions

$$w_1 = Aw(\pm\lambda x + C_1,\ \lambda^2 t + C_2),$$
$$w_2 = A\exp(\lambda x + a\lambda^2 t)w(x + 2a\lambda t + C_1,\ t + C_2),$$
$$w_3 = \frac{A}{\sqrt{|\delta + \beta t|}}\exp\biggl[-\frac{\beta x^2}{4a(\delta + \beta t)}\biggr]w\biggl(\pm\frac{x}{\delta + \beta t},\ \frac{\gamma + \lambda t}{\delta + \beta t}\biggr), \qquad \lambda\delta - \beta\gamma = 1,$$

where A, C_1, C_2, β, δ, and λ are arbitrary constants, are also solutions of this equation. The last formula with $\beta = 1$, $\gamma = -1$, $\delta = \lambda = 0$ was obtained with the Appell transformation.

⊙ *Literature*: W. Miller, Jr. (1977), P. J. Olver (1986).

► Infinite series solutions.

A solution involving an arbitrary function of the space variable:

$$w(x,t) = f(x) + \sum_{n=1}^{\infty} \frac{(at)^n}{n!} f_x^{(2n)}(x), \qquad f_x^{(m)}(x) = \frac{d^m}{dx^m} f(x),$$

where $f(x)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = f(x)$. The sum is finite if $f(x)$ is a polynomial.

Solutions involving arbitrary functions of time:

$$w(x,t) = g(t) + \sum_{n=1}^{\infty} \frac{1}{a^n (2n)!} x^{2n} g_t^{(n)}(t),$$

$$w(x,t) = xh(t) + x \sum_{n=1}^{\infty} \frac{1}{a^n (2n+1)!} x^{2n} h_t^{(n)}(t),$$

where $g(t)$ and $h(t)$ are infinitely differentiable functions. The sums are finite if $g(t)$ and $h(t)$ are polynomials. The first solution satisfies the boundary condition of the first kind $w(0,t) = g(t)$ and the second solution the boundary condition of the second kind $\partial_x w(0,t) = h(t)$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Transformations allowing separation of variables.

Table 3.1 presents transformations that reduce the heat equation to separable equations (the identity transformation with $\xi = x$ and $g = 1$ is omitted).

TABLE 3.1

Transformations of the form $\xi = f(x,t)$, $w = g(\xi,t)\, u(\xi,t)$ for which the equation $\partial_t w - \partial_{xx} w = 0$ admits multiplicatively separable particular solutions with $u(\xi,t) = \varphi(t)\,\psi(\xi)$

No	Function $\xi = f(x,t)$	Factor $g = g(\xi,t)$	Function $\varphi = \varphi(t)$; λ is arbitrary	Equation for $\psi = \psi(\xi)$
1	$\xi = \dfrac{x}{\sqrt{t}}$	$g = 1$	$\varphi = t^{\lambda}$	$\psi''_{\xi\xi} + \frac{1}{2}\xi\psi'_{\xi} - \lambda\psi = 0$
2	$\xi = x - \frac{1}{2}t^2$	$g = \exp\bigl(-\frac{1}{2}\xi t\bigr)$	$\varphi = \exp\bigl(-\frac{1}{12}t^3 + \lambda t\bigr)$	$\psi''_{\xi\xi} + \bigl(\frac{1}{2}\xi - \lambda\bigr)\psi = 0$
3	$\xi = \dfrac{x}{\sqrt{1+t^2}}$	$g = \exp\bigl(-\frac{1}{4}\xi^2 t\bigr)$	$\varphi = \dfrac{\exp(\lambda \arctan t)}{(1+t^2)^{1/4}}$	$\psi''_{\xi\xi} + \bigl(\frac{1}{4}\xi^2 - \lambda\bigr)\psi = 0$

Remark 3.1. In general, the solution of the equation for ψ in the first row of Table 3.1 is expressed in terms of degenerate hypergeometric functions. In the special case $\lambda = \frac{1}{2}n$ ($n = 0, 1, 2, \dots$), the equation admits solutions of the form $\psi(\xi) = (i/2)^n H_n(i\xi/2)$, where $H_n(z)$ is the nth Hermite polynomial, $i^2 = -1$. The solution of the equation for ψ in the second row of Table 3.1 is expressed in terms of Bessel functions, and that in the third row, in terms of parabolic cylinder functions.

⊙ *Literature*: E. Kalnins and W. Miller, Jr. (1974), W. Miller, Jr. (1977).

► Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t)=\frac{1}{2\sqrt{\pi at}}\int_{-\infty}^{\infty}\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr]f(\xi)\,d\xi.$$

Example 3.1. The initial temperatures in the domains $|x|<x_0$ and $|x|>x_0$ are constant and equal to w_1 and w_2, respectively, i.e.,

$$f(x)=\begin{cases} w_1 & \text{for } |x|<x_0,\\ w_2 & \text{for } |x|>x_0.\end{cases}$$

Solution:

$$w=\frac{1}{2}(w_1-w_2)\biggl[\operatorname{erf}\biggl(\frac{x_0-x}{2\sqrt{at}}\biggr)+\operatorname{erf}\biggl(\frac{x_0+x}{2\sqrt{at}}\biggr)\biggr]+w_2.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0\le x<\infty$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w&=f(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g(t) &&\text{at}\quad x=0 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(x,t)&=\frac{1}{2\sqrt{\pi at}}\int_0^{\infty}\biggl\{\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr]-\exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr]\biggr\}f(\xi)\,d\xi\\ &\quad+\frac{x}{2\sqrt{\pi a}}\int_0^t\exp\biggl[-\frac{x^2}{4a(t-\tau)}\biggr]\frac{g(\tau)\,d\tau}{(t-\tau)^{3/2}}.\end{aligned}$$

Example 3.2. The initial temperature is linearly dependent on the space coordinate, $f(x)=w_0+bx$. The temperature at the boundary is zero, $g(t)=0$.

Solution:

$$w=w_0\operatorname{erf}\biggl(\frac{x}{2\sqrt{at}}\biggr)+bx.$$

The case of uniform initial temperature with $f(x)=w_0$ corresponds to the value $b=0$.

Example 3.3. The initial temperature is zero, $f(x)=0$. The temperature at the boundary increases linearly with time, $g(t)=At$.

Solution:

$$w=At\biggl[\biggl(1+\frac{x^2}{2at}\biggr)\operatorname{erfc}\biggl(\frac{x}{2\sqrt{at}}\biggr)-\frac{x}{\sqrt{\pi at}}\exp\biggl(-\frac{x^2}{4at}\biggr)\biggr].$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

▶ **Domain: $0 \le x < \infty$. Second boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \frac{1}{2\sqrt{\pi a t}} \int_0^\infty \left\{ \exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right] \right\} f(\xi)\, d\xi$$
$$- \sqrt{\frac{a}{\pi}} \int_0^t \exp\left[-\frac{x^2}{4a(t-\tau)}\right] \frac{g(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

Example 3.4. The initial temperature is zero, $f(x) = 0$. A constant thermal flux is maintained at the boundary all the time, $g(t) = -Q$.

Solution:

$$w = 2Q\sqrt{\frac{at}{\pi}} \exp\left(-\frac{x^2}{4at}\right) - Qx \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right).$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

▶ **Domain: $0 \le x < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^\infty f(\xi) G(x,\xi,t)\, d\xi - a \int_0^t g(\tau) G(x,0,t-\tau)\, d\tau,$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}} \left\{ \exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right] \right.$$
$$\left. - 2k \int_0^\infty \exp\left[-\frac{(x+\xi+\eta)^2}{4at} - k\eta\right] d\eta \right\}.$$

The improper integral may be calculated by the formula

$$\int_0^\infty \exp\left[-\frac{(x+\xi+\eta)^2}{4at} - k\eta\right] d\eta = \sqrt{\pi a t}\, \exp\left[ak^2 t + k(x+\xi)\right] \operatorname{erfc}\left(\frac{x+\xi}{2\sqrt{at}} + k\sqrt{at}\right).$$

Example 3.5. The initial temperature is uniform, $f(x) = w_0$. The temperature of the contacting medium is zero, $g(t) = 0$.

Solution:

$$w = w_0 \left[\operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right) + \exp(kx + ak^2 t) \operatorname{erfc}\left(\frac{x}{2\sqrt{at}} + k\sqrt{at}\right) \right].$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l$. First boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},$$
$$w = g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\biggl(\frac{n\pi x}{l}\biggr) \exp\biggl(-\frac{an^2\pi^2 t}{l^2}\biggr) M_n(t),$$

where

$$M_n(t) = \int_0^l f(\xi)\sin\biggl(\frac{n\pi\xi}{l}\biggr)\,d\xi + \frac{an\pi}{l}\int_0^t \exp\biggl(\frac{an^2\pi^2\tau}{l^2}\biggr)\bigl[g_1(\tau) - (-1)^n g_2(\tau)\bigr]\,d\tau.$$

Remark 3.2. Using the relations [see Prudnikov, Brychkov, and Marichev (1986)]

$$\sum_{n=1}^{\infty} \frac{\sin n\xi}{n} = \frac{\pi - \xi}{2} \quad (0 < \xi < 2\pi); \qquad \sum_{n=1}^{\infty} (-1)^{n-1}\frac{\sin n\xi}{n} = \frac{\xi}{2} \quad (-\pi < \xi < \pi),$$

one can transform the solution to

$$w(x,t) = g_1(t) + \frac{x}{l}\bigl[g_2(t) - g_1(t)\bigr] + \frac{2}{l}\sum_{n=1}^{\infty} \sin(\lambda_n x)\exp(-a\lambda_n^2 t)R_n(t), \qquad \lambda_n = \frac{n\pi}{l},$$

where

$$\begin{aligned} R_n(t) &= \int_0^l f(\xi)\sin(\lambda_n\xi)\,d\xi - \frac{1}{\lambda_n}\exp(a\lambda_n^2 t)\bigl[g_1(t) - (-1)^n g_2(t)\bigr] \\ &\quad + a\lambda_n \int_0^t \exp(a\lambda_n^2\tau)\bigl[g_1(\tau) - (-1)^n g_2(\tau)\bigr]\,d\tau \\ &= \int_0^l f(\xi)\sin(\lambda_n\xi)\,d\xi - \frac{1}{\lambda_n}\bigl[g_1(0) - (-1)^n g_2(0)\bigr] \\ &\quad - \frac{1}{\lambda_n}\int_0^t \exp(a\lambda_n^2\tau)\bigl[g_1'(\tau) - (-1)^n g_2'(\tau)\bigr]\,d\tau. \end{aligned}$$

Note that another representation of the solution is given in Section 3.1.2 (see the first boundary value problem for $0 \le x \le l$).

Example 3.6. The initial temperature is uniform, $f(x) = w_0$. Both ends are maintained at zero temperature, $g_1(t) = g_2(t) = 0$.

Solution:

$$w = \frac{4w_0}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\biggl[\frac{(2n+1)\pi x}{l}\biggr] \exp\biggl[-\frac{a(2n+1)^2\pi^2 t}{l^2}\biggr].$$

Example 3.7. The initial temperature is zero, $f(x) = 0$. The ends are maintained at uniform temperatures, $g_1(t) = w_1$ and $g_2(t) = w_2$.

Solution:

$$w = w_1 + (w_2 - w_1)\frac{x}{l} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n w_2 - w_1}{n} \sin\biggl(\frac{n\pi x}{l}\biggr) \exp\biggl(-\frac{an^2\pi^2 t}{l^2}\biggr).$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ \partial_x w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \int_0^l f(\xi) G(x,\xi,t)\,d\xi - a\int_0^t g_1(\tau) G(x,0,t-\tau)\,d\tau + a\int_0^t g_2(\tau) G(x,l,t-\tau)\,d\tau,$$

where

$$G(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\Bigl(\frac{n\pi x}{l}\Bigr)\cos\Bigl(\frac{n\pi\xi}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr).$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l$. Third boundary value problem ($k_1 > 0$ and $k_2 > 0$).**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \int_0^l f(\xi) G(x,\xi,t)\,d\xi - a\int_0^t g_1(\tau) G(x,0,t-\tau)\,d\tau + a\int_0^t g_2(\tau) G(x,l,t-\tau)\,d\tau,$$

where

$$G(x,\xi,t) = \sum_{n=1}^{\infty} \frac{1}{\|y_n\|^2} y_n(x) y_n(\xi)\exp(-a\mu_n^2 t),$$

$$y_n(x) = \cos(\mu_n x) + \frac{k_1}{\mu_n}\sin(\mu_n x), \qquad \|y_n\|^2 = \frac{k_2}{2\mu_n^2}\frac{\mu_n^2 + k_1^2}{\mu_n^2 + k_2^2} + \frac{k_1}{2\mu_n^2} + \frac{l}{2}\Bigl(1 + \frac{k_1^2}{\mu_n^2}\Bigr).$$

Here, the μ_n are positive roots of the transcendental equation $\dfrac{\tan(\mu l)}{\mu} = \dfrac{k_1 + k_2}{\mu^2 - k_1 k_2}$.

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l$. Mixed boundary value problems.**

1°. The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \int_0^l f(\xi)G(x,\xi,t)\,d\xi + a\int_0^t g_1(\tau)\Lambda(x,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau,$$

where

$$\begin{aligned} G(x,\xi,t) &= \frac{2}{l}\sum_{n=0}^{\infty}\sin\left[\frac{\pi(2n+1)x}{2l}\right]\sin\left[\frac{\pi(2n+1)\xi}{2l}\right]\exp\left[-\frac{a\pi^2(2n+1)^2t}{4l^2}\right],\\ \Lambda(x,t) &= \frac{\partial}{\partial\xi}G(x,\xi,t)\Big|_{\xi=0}. \end{aligned}$$

2°. The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \int_0^l f(\xi)G(x,\xi,t)\,d\xi - a\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau - a\int_0^t g_2(\tau)H(x,t-\tau)\,d\tau,$$

where

$$\begin{aligned} G(x,\xi,t) &= \frac{2}{l}\sum_{n=0}^{\infty}\cos\left[\frac{\pi(2n+1)x}{2l}\right]\cos\left[\frac{\pi(2n+1)\xi}{2l}\right]\exp\left[-\frac{a\pi^2(2n+1)^2t}{4l^2}\right],\\ H(x,t) &= \frac{\partial}{\partial\xi}G(x,\xi,t)\Big|_{\xi=l}. \end{aligned}$$

Note that Section 3.1.2 (see the mixed boundary value problems for $0 \le x \le l$) also gives other forms of representation of solutions to mixed boundary value problems.

Example 3.8. The initial temperature is zero, $f(x) = 0$. The left end is heat insulated, and the right end is maintained at a constant temperature, $g_1(t) = 0$ and $g_2(t) = A$.

Solution:

$$w = A + \frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\cos\left[\frac{\pi(2n+1)x}{2l}\right]\exp\left[-\frac{a\pi^2(2n+1)^2t}{4l^2}\right].$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), A. V. Bitsadze and D. F. Kalinichenko (1985).

► Problems without initial conditions.

In applications, problems are encountered in which the process is studied at a time instant fairly remote from the initial instant and, in this case, the initial conditions do not practically affect the distribution of the desired quantity at the observation instant. In such problems, no initial condition is stated, and the boundary conditions are assumed to be prescribed for all preceding time instants, $-\infty < t$. However, in addition, the boundedness condition in the entire domain is imposed on the solution.

As an example, consider the first boundary value problem for the half-space $0 \le x < \infty$ with the boundary conditions

$$w = g(t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty.$$

Solution:

$$w(x,t) = \frac{x}{2\sqrt{\pi a}} \int_{-\infty}^{t} \frac{g(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4a(t-\tau)}\right] d\tau.$$

Example 3.9. The temperature at the boundary is a harmonic function of time, i.e.,

$$g(t) = w_0 \cos(\omega t + \beta).$$

Solution:

$$w = w_0 \exp\left(-\sqrt{\frac{\omega}{2a}}\, x\right) \cos\left(\omega t - \sqrt{\frac{\omega}{2a}}\, x + \beta\right).$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin, et al. (1964), A. N. Tikhonov and A. A. Samarskii (1990).

► Conjugate heat and mass transfer problems.

In such problems, one deals with two (or more) domains, V_1 and V_2, with interface S. The domains are filled by different media. Each of the media is characterized by its own thermal conductivity, λ_1 and λ_2, and thermal diffusivity, a_1 and a_2. The processes in each of the media are described by appropriate (different) equations of heat and mass transfer. The thermal equilibrium conditions express the equality of the temperatures and of the thermal fluxes at the interface. Below we consider a typical example of a conjugate problem (a more detailed analysis of such problems is beyond the scope of this handbook).

Consider two semiinfinite solids (two semiinfinite quiescent media) with temperature distributions, $w_1 = w_1(x,t)$ and $w_2 = w_2(x,t)$, governed by the equations

$$\frac{\partial w_1}{\partial t} = a_1 \frac{\partial^2 w_1}{\partial x^2} \qquad \text{(in the range } 0 < x < \infty),$$
$$\frac{\partial w_2}{\partial t} = a_2 \frac{\partial^2 w_2}{\partial x^2} \qquad \text{(in the range } -\infty < x < 0).$$

Either solid has its own temperature profile at the initial instant $t = 0$, and conjugate boundary conditions are imposed at the interface $x = 0$; specifically,

$$\begin{aligned} w_1 &= f_1(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w_2 &= f_2(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w_1 &= w_2 && \text{at} \quad x = 0 \quad \text{(boundary condition)},\\ \lambda_1 \partial_x w_1 &= \lambda_2 \partial_x w_2 && \text{at} \quad x = 0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w_1(x,t) = \frac{1}{2\sqrt{\pi a_1 t}} \int_0^\infty f_1(\xi)\left\{\exp\left[-\frac{(x-\xi)^2}{4a_1 t}\right] + \exp\left[-\frac{(x+\xi)^2}{4a_1 t}\right]\right\} d\xi$$
$$- \sqrt{\frac{a_1}{\pi\lambda_1^2}} \int_0^t \exp\left[-\frac{x^2}{4a_1(t-\tau)}\right] \frac{g(\tau)\,d\tau}{\sqrt{t-\tau}},$$
$$w_2(x,t) = \frac{1}{2\sqrt{\pi a_2 t}} \int_0^\infty f_2(-\xi)\left\{\exp\left[-\frac{(x-\xi)^2}{4a_2 t}\right] + \exp\left[-\frac{(x+\xi)^2}{4a_2 t}\right]\right\} d\xi$$
$$+ \sqrt{\frac{a_2}{\pi\lambda_2^2}} \int_0^t \exp\left[-\frac{x^2}{4a_2(t-\tau)}\right] \frac{g(\tau)\,d\tau}{\sqrt{t-\tau}}.$$

The function $g(t)$ is given by

$$g(t) = \frac{\lambda_1\lambda_2}{\pi(\lambda_1\sqrt{a_2} + \lambda_2\sqrt{a_1}\,)} \frac{d}{dt} \int_0^t \frac{F(\tau)\,d\tau}{\sqrt{\tau(t-\tau)}},$$

where

$$F(t) = \frac{1}{\sqrt{a_1}} \int_0^\infty f_1(\xi) \exp\left(-\frac{\xi^2}{4a_1 t}\right) d\xi - \frac{1}{\sqrt{a_2}} \int_0^\infty f_2(-\xi) \exp\left(-\frac{\xi^2}{4a_2 t}\right) d\xi.$$

Example 3.10. The initial temperatures are uniform, $f_1(x) = A$ and $f_2(x) = B$.
Solution:

$$\frac{w_1(x,t) - B}{A - B} = \frac{K}{1+K}\left[1 + \frac{1}{K}\,\mathrm{erf}\left(\frac{x}{2\sqrt{a_1 t}}\right)\right],$$
$$\frac{w_2(x,t) - B}{A - B} = \frac{K}{1+K}\,\mathrm{erfc}\left(\frac{|x|}{2\sqrt{a_1 t}}\right),$$

where the quantity $K = \dfrac{\lambda_1}{\lambda_2}\sqrt{\dfrac{a_2}{a_1}}$ characterizes the thermal activity of the first medium with respect to the second medium.

⊙ *Literature*: A. V. Lykov (1967), H. S. Carslaw and J. C. Jaeger (1984).

3.1.2 Equation of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

This sort of equation describes one-dimensional unsteady thermal processes in quiescent media or solids with constant thermal diffusivity in the presence of a volume thermal source dependent on the space coordinate and time.

► **Domain: $-\infty < x < \infty$. Cauchy problem.**

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x,\xi,t)\,d\xi + \int_0^t \int_{-\infty}^{\infty} \Phi(\xi,\tau) G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi at}}\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr].$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $0 \le x < \infty$. First boundary value problem.

The following conditions are prescribed:

$$w=f(x)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$w=g(t)\quad\text{at}\quad x=0\quad\text{(boundary condition)}.$$

Solution:

$$w(x,t)=\int_0^\infty f(\xi)G(x,\xi,t)\,d\xi+\int_0^t g(\tau)H(x,t-\tau)\,d\tau + \int_0^t\!\!\int_0^\infty \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi at}}\biggl\{\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr]-\exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr]\biggr\},$$
$$H(x,t)=\frac{x}{2\sqrt{\pi a}\,t^{3/2}}\exp\biggl(-\frac{x^2}{4at}\biggr).$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $0 \le x < \infty$. Second boundary value problem.

The following conditions are prescribed:

$$w=f(x)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$\partial_x w=g(t)\quad\text{at}\quad x=0\quad\text{(boundary condition)}.$$

Solution:

$$w(x,t)=\int_0^\infty G(x,\xi,t)f(\xi)\,d\xi-a\int_0^t g(\tau)G(x,0,t-\tau)\,d\tau + \int_0^t\!\!\int_0^\infty \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi at}}\biggl\{\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr]+\exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr]\biggr\}.$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x < \infty$. Third boundary value problem.

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^\infty f(\xi) G(x,\xi,t)\, d\xi - a \int_0^t g(\tau) G(x,0,t-\tau)\, d\tau + \int_0^t \int_0^\infty \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau,$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}} \biggl\{ \exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] + \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr] - 2k \int_0^\infty \exp\biggl[-\frac{(x+\xi+\eta)^2}{4at} - k\eta\biggr] d\eta \biggr\}.$$

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le l$. First boundary value problem.

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},$$
$$w = g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^l f(\xi) G(x,\xi,t)\, d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau + a \int_0^t g_1(\tau) H_1(x,t-\tau)\, d\tau - a \int_0^t g_2(\tau) H_2(x,t-\tau)\, d\tau.$$

Two forms of representation of the Green's function:

$$G(x,\xi,t) = \frac{2}{l} \sum_{n=1}^\infty \sin\Bigl(\frac{n\pi x}{l}\Bigr) \sin\Bigl(\frac{n\pi \xi}{l}\Bigr) \exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr)$$
$$= \frac{1}{2\sqrt{\pi a t}} \sum_{n=-\infty}^{\infty} \biggl\{ \exp\biggl[-\frac{(x-\xi+2nl)^2}{4at}\biggr] - \exp\biggl[-\frac{(x+\xi+2nl)^2}{4at}\biggr] \biggr\}.$$

The first series converges rapidly at large t and the second series at small t. The functions H_1 and H_2 are expressed in terms of the Green's function as

$$H_1(x,t) = \frac{\partial}{\partial \xi} G(x,\xi,t)\Big|_{\xi=0}, \qquad H_2(x,t) = \frac{\partial}{\partial \xi} G(x,\xi,t)\Big|_{\xi=l}.$$

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = &\int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\ &- a\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau. \end{aligned}$$

Two forms of representation of the Green's function:

$$\begin{aligned} G(x,\xi,t) &= \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)\\ &= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right] + \exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}. \end{aligned}$$

The first series converges rapidly at large t and the second series at small t.

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le x \le l$. Third boundary value problem ($k_1 > 0$, $k_2 > 0$).**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

The solution is given by the formula presented in Section 3.1.1 (see the third boundary value problem for $0 \le x \le l$) with the additional term

$$\int_0^t\int_0^l \Phi(\xi,\tau)\,G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

which takes into account the nonhomogeneity of the equation.

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l$. Mixed boundary value problems.**

1°. The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ w &= g_1(t) \quad &\text{at}\quad x=0 \quad &\text{(boundary condition)},\\ \partial_x w &= g_2(t) \quad &\text{at}\quad x=l \quad &\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\ &\quad + a\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=0} d\tau + a\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau. \end{aligned}$$

Two forms of representation of the Green's function:

$$\begin{aligned} G(x,\xi,t) &= \frac{2}{l}\sum_{n=0}^{\infty}\sin\left[\frac{\pi(2n+1)x}{2l}\right]\sin\left[\frac{\pi(2n+1)\xi}{2l}\right]\exp\left[-\frac{a\pi^2(2n+1)^2t}{4l^2}\right]\\ &= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}(-1)^n\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right]-\exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}. \end{aligned}$$

The first series converges rapidly at large t and the second series at small t.

2°. The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ \partial_x w &= g_1(t) \quad &\text{at}\quad x=0 \quad &\text{(boundary condition)},\\ w &= g_2(t) \quad &\text{at}\quad x=l \quad &\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\ &\quad - a\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau - a\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau. \end{aligned}$$

Two forms of representation of the Green's function:

$$\begin{aligned} G(x,\xi,t) &= \frac{2}{l}\sum_{n=0}^{\infty}\cos\left[\frac{\pi(2n+1)x}{2l}\right]\cos\left[\frac{\pi(2n+1)\xi}{2l}\right]\exp\left[-\frac{a\pi^2(2n+1)^2t}{4l^2}\right]\\ &= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}(-1)^n\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right]+\exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}. \end{aligned}$$

The first series converges rapidly at large t and the second series at small t.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), A. V. Bitsadze and D. F. Kalinichenko (1985).

3.1.3 Equation of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw + \Phi(x,t)$

Homogeneous equations of this form describe one-dimensional unsteady mass transfer in a quiescent medium with a first-order volume chemical reaction; the cases $b<0$ and $b>0$ correspond to absorption and release of substance, respectively. A similar equation is used to analyze appropriate one-dimensional thermal processes in which volume heat release ($b>0$) proportional to temperature occurs in the medium. Furthermore, this equation governs heat transfer in a one-dimensional rod whose lateral surface exchanges heat with the ambient medium having constant temperature; $b>0$ if the temperature of the medium is greater than that of the rod, and $b<0$ otherwise.

► **Homogeneous equation ($\Phi \equiv 0$).**

1°. Particular solutions:

$$
\begin{aligned}
w(x) &= Ae^{\lambda x} + Be^{-\lambda x}, \qquad \lambda = \sqrt{-b/a},\\
w(x,t) &= (Ax+B)e^{bt},\\
w(x,t) &= \bigl[A(x^2+2at)+B\bigr]e^{bt},\\
w(x,t) &= \bigl[A(x^3+6atx)+B\bigr]e^{bt},\\
w(x,t) &= \bigl[A(x^4+12atx^2+12a^2t^2)+B\bigr]e^{bt},\\
w(x,t) &= \bigl[A(x^5+20atx^3+60a^2t^2x)+B\bigr]e^{bt},\\
w(x,t) &= \bigl[A(x^6+30atx^4+180a^2t^2x^2+120a^3t^3)+B\bigr]e^{bt},\\
w(x,t) &= A\exp\bigl[(a\mu^2+b)t \pm \mu x\bigr] + Be^{bt},\\
w(x,t) &= A\frac{1}{\sqrt{t}}\exp\Bigl(-\frac{x^2}{4at}+bt\Bigr) + Be^{bt},\\
w(x,t) &= A\frac{x}{t^{3/2}}\exp\Bigl(-\frac{x^2}{4at}+bt\Bigr) + Be^{bt},\\
w(x,t) &= A\exp\bigl[(b-a\mu^2)t\bigr]\cos(\mu x) + Be^{bt},\\
w(x,t) &= A\exp\bigl[(b-a\mu^2)t\bigr]\sin(\mu x) + Be^{bt},\\
w(x,t) &= A\exp(-\mu x+bt)\cos(\mu x-2a\mu^2t) + Be^{bt},\\
w(x,t) &= A\exp(-\mu x+bt)\sin(\mu x-2a\mu^2t) + Be^{bt},\\
w(x,t) &= A\exp(-\mu x)\cos(\beta x-2a\beta\mu t), \quad \beta=\sqrt{\mu^2+b/a},\\
w(x,t) &= A\exp(-\mu x)\sin(\beta x-2a\beta\mu t), \quad \beta=\sqrt{\mu^2+b/a},\\
w(x,t) &= Ae^{bt}\operatorname{erf}\Bigl(\frac{x}{2\sqrt{at}}\Bigr) + Be^{bt},\\
w(x,t) &= Ae^{bt}\operatorname{erfc}\Bigl(\frac{x}{2\sqrt{at}}\Bigr) + Be^{bt},
\end{aligned}
$$

where A, B, and μ are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}(x,t)=\frac{1}{2\sqrt{\pi at}}\exp\biggl(-\frac{x^2}{4at}+bt\biggr).$$

► **Reduction to the heat equation. Remarks on the Green's functions.**

The substitution $w(x,t)=e^{bt}u(x,t)$ leads to the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+e^{-bt}\Phi(x,t),$$

which is discussed in Section 3.1.2 in detail. The initial condition for the new variable u remains the same, and the nonhomogeneous part in the boundary conditions is multiplied by e^{-bt}. Taking this into account, one can easily solve the original equation subject to the initial and boundary conditions considered in Section 3.1.2.

In all the boundary value problems that are dealt with in the current subsection, the Green's function can be represented in the form

$$G_b(x,\xi,t)=e^{bt}G_0(x,\xi,t),$$

where $G_0(x,\xi,t)$ is the Green's function for the heat equation that corresponds to $b=0$.

► **Domain: $-\infty<x<\infty$. Cauchy problem.**

An initial condition is prescribed:

$$w=f(x)\qquad\text{at}\quad t=0.$$

Solution:

$$w(x,t)=\int_{-\infty}^{\infty}f(\xi)G(x,\xi,t)\,d\xi+\int_0^t\int_{-\infty}^{\infty}\Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi at}}\exp\biggl[-\frac{(x-\xi)^2}{4at}+bt\biggr].$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0\le x<\infty$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}&w=f(x)\quad\text{at}\quad t=0\quad\text{(initial condition)},\\&w=g(t)\quad\text{at}\quad x=0\quad\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned}w(x,t)=&\int_0^{\infty}f(\xi)G(x,\xi,t)\,d\xi+\int_0^t g(\tau)H(x,t-\tau)\,d\tau\\&+\int_0^t\int_0^{\infty}\Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,\end{aligned}$$

where

$$G(x,\xi,t) = \frac{e^{bt}}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\},$$
$$H(x,t) = \frac{x e^{bt}}{2\sqrt{\pi a}\, t^{3/2}} \exp\left(-\frac{x^2}{4at}\right).$$

⊙ *Literature*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

► **Domain: $0 \le x < \infty$. Second boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^\infty G(x,\xi,t) f(\xi)\, d\xi - a\int_0^t g(\tau) G(x,0,t-\tau)\, d\tau + \int_0^t \int_0^\infty \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau,$$

where

$$G(x,\xi,t) = \frac{e^{bt}}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}.$$

► **Domain: $0 \le x < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t) = \frac{e^{bt}}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right] - 2k\int_0^\infty \exp\left[-\frac{(x+\xi+\eta)^2}{4at} - k\eta\right] d\eta\right\}.$$

► **Domain: $0 \le x \le l$. First boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},$$
$$w = g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+\,a\int_0^t g_1(\tau)H_1(x,t-\tau)\,d\tau - a\int_0^t g_2(\tau)H_2(x,t-\tau)\,d\tau,$$

where

$$G(x,\xi,t) = \frac{2}{l}e^{bt}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right),$$
$$H_1(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)\bigg|_{\xi=0},\quad H_2(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)\bigg|_{\xi=l}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at}\quad t=0 && \text{(initial condition)},\\ \partial_x w &= g_1(t) && \text{at}\quad x=0 && \text{(boundary condition)},\\ \partial_x w &= g_2(t) && \text{at}\quad x=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$-\,a\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau,$$

where

$$G(x,\xi,t) = e^{bt}\left[\frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at}\quad t=0 && \text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(t) && \text{at}\quad x=0 && \text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(t) && \text{at}\quad x=l && \text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t) = e^{bt}\sum_{n=1}^{\infty}\frac{1}{\|y_n\|^2}y_n(x)y_n(\xi)\exp(-a\mu_n^2 t),$$

$$y_n(x) = \cos(\mu_n x) + \frac{k_1}{\mu_n}\sin(\mu_n x), \qquad \|y_n\|^2 = \frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2} + \frac{k_1}{2\mu_n^2} + \frac{l}{2}\left(1+\frac{k_1^2}{\mu_n^2}\right).$$

Here, the μ_n are positive roots of the transcendental equation $\dfrac{\tan(\mu l)}{\mu} = \dfrac{k_1+k_2}{\mu^2 - k_1 k_2}$.

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Mixed boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\ &\quad + a\int_0^t g_1(\tau)\Lambda(x,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau, \end{aligned}$$

where

$$G(x,\xi,t) = \frac{2}{l}e^{bt}\sum_{n=0}^{\infty}\sin\left[\frac{\pi(2n+1)x}{2l}\right]\sin\left[\frac{\pi(2n+1)\xi}{2l}\right]\exp\left[-\frac{a\pi^2(2n+1)^2 t}{4l^2}\right],$$

$$\Lambda(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)\Big|_{\xi=0}.$$

► **Domain: $0 \le x < \infty$. A problem with $\Phi = 0$ and without an initial condition.**

The following conditions are prescribed:

$$w = A\cos(\omega t+\gamma) \quad \text{at} \quad x=0, \qquad w \to 0 \quad \text{as} \quad x \to \infty.$$

Solution:

$$w = Ae^{-\lambda x}\cos(\omega t - \beta x + \gamma),$$

where

$$\lambda = \left(\frac{\sqrt{\omega^2+b^2}-b}{2a}\right)^{1/2}, \qquad \beta = \left(\frac{\sqrt{\omega^2+b^2}+b}{2a}\right)^{1/2}.$$

3.1.4 Equation of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + \Phi(x,t)$

This equation is encountered in one-dimensional nonstationary problems of convective mass transfer in a continuous medium that moves with a constant velocity; the case $\Phi \equiv 0$ means that there is no absorption or release of substance.

► **Homogeneous equation ($\Phi \equiv 0$).**

1°. Particular solutions:

$$\begin{aligned}
w(x) &= Ae^{-\lambda x} + B, \qquad \lambda = b/a,\\
w(x,t) &= Ax + Abt + B,\\
w(x,t) &= A(x+bt)^2 + 2Aat + B,\\
w(x,t) &= A(x+bt)^3 + 6Aatx + B,\\
w(x,t) &= A\exp\bigl[(a\mu^2 + b\mu)t + \mu x\bigr] + B,\\
w(x,t) &= A\frac{1}{\sqrt{t}}\exp\left[-\frac{(x+bt)^2}{4at}\right] + B,\\
w(x,t) &= A\exp(-a\mu^2 t)\cos(\mu x + b\mu t) + B,\\
w(x,t) &= A\exp(-a\mu^2 t)\sin(\mu x + b\mu t) + B,\\
w(x,t) &= A\exp(-\mu x)\cos\bigl[\beta x + \beta(b - 2a\mu)t\bigr] + B, \qquad \beta = \sqrt{\mu^2 - (b/a)\mu},\\
w(x,t) &= A\exp(-\mu x)\sin\bigl[\beta x + \beta(b - 2a\mu)t\bigr] + B, \qquad \beta = \sqrt{\mu^2 - (b/a)\mu},\\
w(x,t) &= A\operatorname{erf}\left(\frac{x+bt}{2\sqrt{at}}\right) + B,\\
w(x,t) &= A\operatorname{erfc}\left(\frac{x+bt}{2\sqrt{at}}\right) + B,
\end{aligned}$$

where A, B, and μ are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(x+bt)^2}{4at}\right].$$

► **Reduction to the heat equation. Remarks on the Green's function.**

1°. The substitution

$$w(x,t) = \exp(-\beta t - \mu x)u(x,t), \qquad \beta = \frac{b^2}{4a}, \quad \mu = \frac{b}{2a} \tag{1}$$

leads to the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \exp(\beta t + \mu x)\Phi(x,t), \tag{2}$$

which is considered in Section 3.1.2 in detail.

2°. On passing from t, x to the new variables t, $z = x + bt$, we obtain the nonhomogeneous heat equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + \Phi(z - bt, t),$$

which is treated in Section 3.1.2.

3°. For all first boundary value problems, the Green's function can be represented as

$$G_b(x, \xi, t) = \exp\left[\frac{b}{2a}(\xi - x) - \frac{b^2}{4a}t\right] G_0(x, \xi, t),$$

where $G_0(x, \xi, t)$ is the Green's function for the heat equation that corresponds to $b = 0$.

► **Domain: $-\infty < x < \infty$. Cauchy problem.**

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x, t) = \int_{-\infty}^{\infty} f(\xi)G(x, \xi, t)\, d\xi + \int_0^t \int_{-\infty}^{\infty} \Phi(\xi, \tau)G(x, \xi, t - \tau)\, d\xi\, d\tau,$$

where

$$G(x, \xi, t) = \frac{1}{2\sqrt{\pi a t}} \exp\left[\frac{b(\xi - x)}{2a} - \frac{b^2 t}{4a} - \frac{(x - \xi)^2}{4at}\right].$$

► **Domain: $0 \le x < \infty$. First boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x, t) = \int_0^{\infty} f(\xi)G(x, \xi, t)\, d\xi + a\int_0^t g(\tau)\Lambda(x, t - \tau)\, d\tau$$
$$+ \int_0^t \int_0^{\infty} \Phi(\xi, \tau)G(x, \xi, t - \tau)\, d\xi\, d\tau,$$

where

$$G(x, \xi, t) = \frac{1}{2\sqrt{\pi a t}} \exp\left[\frac{b(\xi - x)}{2a} - \frac{b^2 t}{4a}\right] \left\{\exp\left[-\frac{(x - \xi)^2}{4at}\right] - \exp\left[-\frac{(x + \xi)^2}{4at}\right]\right\},$$
$$\Lambda(x, t) = \frac{\partial}{\partial \xi} G(x, \xi, t)\Big|_{\xi=0}.$$

► **Domain: $0 \le x < \infty$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_x w &= g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}. \end{aligned}$$

Substitution (1) reduces the considered equation to the nonhomogeneous heat equation (2) with the following initial and boundary conditions:

$$\begin{aligned} u &= \exp(\mu x) f(x) \quad \text{at} \quad t = 0,\\ \partial_x u - \mu u &= \exp(\beta t) g(t) \quad \text{at} \quad x = 0. \end{aligned} \tag{3}$$

See Section 3.1.2 for the solution of the third boundary value problem (2)–(3) for $0 \le x < \infty$.

► **Domain: $0 \le x < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_x w - kw &= g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}. \end{aligned}$$

Substitution (1) reduces the considered equation to the nonhomogeneous heat equation (2) with the following initial and boundary conditions:

$$\begin{aligned} u &= \exp(\mu x) f(x) \quad \text{at} \quad t = 0,\\ \partial_x u - (k+\mu) u &= \exp(\beta t) g(t) \quad \text{at} \quad x = 0. \end{aligned} \tag{4}$$

See Section 3.1.2 for the solution of the third boundary value problem (2), (4) for $0 \le x < \infty$.

► **Domain: $0 \le x \le l$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w &= g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},\\ w &= g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \int_0^l f(\xi) G(x,\xi,t)\, d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau\\ &\quad + a \int_0^t g_1(\tau) H_1(x,t-\tau)\, d\tau - a \int_0^t g_2(\tau) H_2(x,t-\tau)\, d\tau, \end{aligned}$$

where

$$G(x,\xi,t) = \frac{2}{l}\exp\left[\frac{b}{2a}(\xi - x) - \frac{b^2}{4a}t\right]\sum_{n=1}^{\infty}\sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\exp\left(-\frac{a\pi^2 n^2}{l^2}t\right),$$

$$H_1(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)\Big|_{\xi=0}, \quad H_2(x,t) = \frac{\partial}{\partial\xi}G(x,\xi,t)\Big|_{\xi=l}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_x w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)}, \\ \partial_x w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau \\ &\quad - a\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(x,\xi,t) &= \frac{b}{a(e^{bl/a}-1)}\exp\left(\frac{b\xi}{a}\right) \\ &\quad + \frac{2}{l}\exp\left[\frac{b(\xi - x)}{2a} - \frac{b^2 t}{4a}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)}{1+\mu_n^2}\exp\left(-\frac{a\pi^2 n^2}{l^2}t\right), \\ y_n(x) &= \cos\left(\frac{\pi n x}{l}\right) + \mu_n \sin\left(\frac{\pi n x}{l}\right), \qquad \mu_n = \frac{bl}{2a\pi n}. \end{aligned}$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_x w - k_1 w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)}, \\ \partial_x w + k_2 w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t) = \exp\left[\frac{b(\xi - x)}{2a} - \frac{b^2 t}{4a}\right] \sum_{n=1}^{\infty} \frac{1}{B_n} y_n(x) y_n(\xi) \exp(-a\mu_n^2 t),$$

$$y_n(x) = \cos(\mu_n x) + \frac{2ak_1 + b}{2a\mu_n} \sin(\mu_n x),$$

$$B_n = \frac{2ak_2 - b}{4a\mu_n^2} \frac{4a^2\mu_n^2 + (2ak_1 + b)^2}{4a^2\mu_n^2 + (2ak_2 - b)^2} + \frac{2ak_1 + b}{4a\mu_n^2} + \frac{l}{2} + \frac{l(2ak_1 + b)^2}{8a^2\mu_n^2},$$

and the μ_n are positive roots of the transcendental equation

$$\frac{\tan(\mu l)}{\mu} = \frac{4a^2(k_1 + k_2)}{4a^2\mu^2 - (2ak_1 + b)(2ak_2 - b)}.$$

3.1.5 Equation of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + \Phi(x,t)$

For $\Phi \equiv 0$, this equation describes one-dimensional unsteady convective mass transfer with a first-order volume chemical reaction in a continuous medium that moves with a constant velocity. A similar equation is used for the analysis of the corresponding one-dimensional thermal processes in a moving medium with volume heat release proportional to temperature.

► **Homogeneous equation ($\Phi \equiv 0$).**

1°. Particular solutions:

$$w(x,t) = e^{ct}(Ax + Abt + B),$$
$$w(x,t) = e^{ct}\left[A(x + bt)^2 + 2Aat + B\right],$$
$$w(x,t) = e^{ct}\left[A(x + bt)^3 + 6Aatx + B\right],$$
$$w(x,t) = Ae^{-\lambda x + ct} + Be^{ct}, \qquad \lambda = b/a,$$
$$w(x,t) = A\exp\left[(a\mu^2 + b\mu + c)t + \mu x\right] + Be^{ct},$$
$$w(x,t) = A\frac{1}{\sqrt{t}}\exp\left[-\frac{(x + bt)^2}{4at} + ct\right] + Be^{ct},$$
$$w(x,t) = A\exp(ct - a\mu^2 t)\cos(\mu x + b\mu t) + Be^{ct},$$
$$w(x,t) = A\exp(ct - a\mu^2 t)\sin(\mu x + b\mu t) + Be^{ct},$$
$$w(x,t) = A\exp(-\mu x)\cos\left[\beta x + \beta(b - 2a\mu)t\right], \qquad \beta = \sqrt{\mu^2 - (b/a)\mu + c/a},$$
$$w(x,t) = A\exp(-\mu x)\sin\left[\beta x + \beta(b - 2a\mu)t\right], \qquad \beta = \sqrt{\mu^2 - (b/a)\mu + c/a},$$
$$w(x,t) = Ae^{ct}\operatorname{erf}\left(\frac{x + bt}{2\sqrt{at}}\right) + Be^{ct},$$
$$w(x,t) = Ae^{ct}\operatorname{erfc}\left(\frac{x + bt}{2\sqrt{at}}\right) + Be^{ct},$$

where A, B, and μ are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\sqrt{\pi a t}} \exp\biggl[-\frac{(x+bt)^2}{4at} + ct\biggr].$$

▶ **Reduction to the heat equation. Remarks on the Green's functions.**

1°. The substitution

$$w(x,t) = \exp(-\beta t - \mu x)u(x,t), \qquad \beta = -c + \frac{b^2}{4a}, \quad \mu = \frac{b}{2a} \tag{1}$$

leads to the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \exp(\beta t + \mu x)\Phi(x,t), \tag{2}$$

which is considered in Section 3.1.2 in detail.

2°. The transformation

$$w(x,t) = e^{ct} v(z,t), \quad z = x + bt,$$

leads to the nonhomogeneous heat equation

$$\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial z^2} + e^{-ct}\Phi(z - bt, t),$$

which is treated in Section 3.1.2.

3°. For all first boundary value problems, the Green's function can be represented as

$$G_{b,c}(x,\xi,t) = \exp\biggl[\frac{b}{2a}(\xi - x) + \Bigl(c - \frac{b^2}{4a}\Bigr)t\biggr] G_{0,0}(x,\xi,t),$$

where $G_{0,0}(x,\xi,t)$ is the Green's function for the heat equation corresponding to $b = c = 0$.

▶ **Domain: $-\infty < x < \infty$. Cauchy problem.**

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} f(\xi)G(x,\xi,t)\,d\xi + \int_0^t \int_{-\infty}^{\infty} \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}} \exp\biggl[\frac{b}{2a}(\xi - x) + \Bigl(c - \frac{b^2}{4a}\Bigr)t - \frac{(x-\xi)^2}{4at}\biggr].$$

► **Domain: $0 \le x < \infty$. First boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^\infty f(\xi)G(x,\xi,t)\,d\xi + a\int_0^t g(\tau)\Lambda(x,t-\tau)\,d\tau + \int_0^t\int_0^\infty \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi a t}}\exp\left[\frac{b(\xi-x)}{2a}+\left(c-\frac{b^2}{4a}\right)t\right]\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]-\exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\},$$
$$\Lambda(x,t)=\frac{\partial}{\partial\xi}G(x,\xi,t)\Big|_{\xi=0}.$$

► **Domain: $0 \le x < \infty$. Second boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Substitution (1) reduces the considered equation to the nonhomogeneous heat equation (2) with the following initial and boundary conditions:

$$u = \exp(\mu x)f(x) \quad \text{at} \quad t = 0,$$
$$\partial_x u - \mu u = \exp(\beta t)g(t) \quad \text{at} \quad x = 0. \tag{3}$$

See Section 3.1.2 for the solution of the third boundary value problem (2)–(3) for $0 \le x < \infty$.

► **Domain: $0 \le x < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Substitution (1) reduces the considered equation to the nonhomogeneous heat equation (2) with the following initial and boundary conditions:

$$u = \exp(\mu x)f(x) \quad \text{at} \quad t = 0,$$
$$\partial_x u - (k+\mu)u = \exp(\beta t)g(t) \quad \text{at} \quad x = 0. \tag{4}$$

See Section 3.1.2 for the solution of the third boundary value problem (2), (4) for $0 \le x < \infty$.

► **Domain: $0 \le x \le l$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = {} & \int_0^l f(\xi) G(x,\xi,t)\,d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\,d\xi\,d\tau \\ & + a\int_0^t g_1(\tau) H_1(x,t-\tau)\,d\tau - a\int_0^t g_2(\tau) H_2(x,t-\tau)\,d\tau, \end{aligned}$$

where

$$G(x,\xi,t) = \frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a} + \left(c - \frac{b^2}{4a}\right)t\right] \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi n \xi}{l}\right) \exp\left(-\frac{a\pi^2 n^2}{l^2}t\right),$$

$$H_1(x,t) = \frac{\partial}{\partial \xi} G(x,\xi,t)\Big|_{\xi=0}, \quad H_2(x,t) = \frac{\partial}{\partial \xi} G(x,\xi,t)\Big|_{\xi=l}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ \partial_x w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = {} & \int_0^l f(\xi) G(x,\xi,t)\,d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\,d\xi\,d\tau \\ & - a\int_0^t g_1(\tau) G(x,0,t-\tau)\,d\tau + a\int_0^t g_2(\tau) G(x,l,t-\tau)\,d\tau, \end{aligned}$$

where

$$G(x,\xi,t) = A\exp\left(\frac{b\xi}{a} + ct\right) + \frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a} + \left(c - \frac{b^2}{4a}\right)t\right] \sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{1+\mu_n^2} \exp\left(-\frac{a\pi^2 n^2}{l^2}t\right),$$

$$A = \frac{b}{a(e^{bl/a}-1)}, \quad y_n(x) = \cos\left(\frac{\pi n x}{l}\right) + \mu_n \sin\left(\frac{\pi n x}{l}\right), \quad \mu_n = \frac{bl}{2a\pi n}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Third boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},\\
\partial_x w + k_2 w &= g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\begin{aligned}
G(x,\xi,t) &= \exp\left[\frac{b(\xi - x)}{2a} + \left(c - \frac{b^2}{4a}\right)t\right] \sum_{n=1}^{\infty} \frac{1}{B_n} y_n(x) y_n(\xi) \exp(-a\mu_n^2 t),\\
y_n(x) &= \cos(\mu_n x) + \frac{2ak_1 + b}{2a\mu_n} \sin(\mu_n x),\\
B_n &= \frac{2ak_2 - b}{4a\mu_n^2} \frac{4a^2\mu_n^2 + (2ak_1 + b)^2}{4a^2\mu_n^2 + (2ak_2 - b)^2} + \frac{2ak_1 + b}{4a\mu_n^2} + \frac{l}{2} + \frac{l(2ak_1 + b)^2}{8a^2\mu_n^2},
\end{aligned}
$$

and the μ_n are positive roots of the transcendental equation

$$
\frac{\tan(\mu l)}{\mu} = \frac{4a^2(k_1 + k_2)}{4a^2\mu^2 - (2ak_1 + b)(2ak_2 - b)}.
$$

3.2 Heat Equation with Axial or Central Symmetry and Related Equations

3.2.1 Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$

This is a sourceless heat equation that describes one-dimensional unsteady thermal processes having axial symmetry. It is often represented in the equivalent form

$$
\frac{\partial w}{\partial t} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right).
$$

A similar equation is used for the analysis of the corresponding one-dimensional unsteady diffusion processes.

► **Particular solutions (A, B, and μ are arbitrary constants).**

$$
\begin{aligned}
w(r) &= A + B \ln r,\\
w(r,t) &= A + B(r^2 + 4at),\\
w(r,t) &= A + B(r^4 + 16atr^2 + 32a^2t^2),
\end{aligned}
$$

$$w(r,t) = A + B\biggl(r^{2n} + \sum_{k=1}^{n} \frac{4^k[n(n-1)\dots(n-k+1)]^2}{k!}(at)^k r^{2n-2k}\biggr),$$
$$w(r,t) = A + B\bigl(4at\ln r + r^2\ln r - r^2\bigr),$$
$$w(r,t) = A + \frac{B}{t}\exp\Bigl(-\frac{r^2}{4at}\Bigr),$$
$$w(r,t) = A + B\int_1^{\zeta} e^{-z}\frac{dz}{z}, \quad \zeta = \frac{r^2}{4at},$$
$$w(r,t) = A + B\exp(-a\mu^2 t)J_0(\mu r),$$
$$w(r,t) = A + B\exp(-a\mu^2 t)Y_0(\mu r),$$
$$w(r,t) = A + \frac{B}{t}\exp\Bigl(-\frac{r^2+\mu^2}{4t}\Bigr)I_0\Bigl(\frac{\mu r}{2t}\Bigr),$$
$$w(r,t) = A + \frac{B}{t}\exp\Bigl(-\frac{r^2+\mu^2}{4t}\Bigr)K_0\Bigl(\frac{\mu r}{2t}\Bigr),$$

where n is an arbitrary positive integer, $J_0(z)$ and $Y_0(z)$ are Bessel functions, and $I_0(z)$ and $K_0(z)$ are modified Bessel functions.

Suppose $w = w(r,t)$ is a solution of the original equation. Then the functions

$$w_1 = Aw(\pm\lambda r,\ \lambda^2 t + C),$$
$$w_2 = \frac{A}{\delta+\beta t}\exp\biggl[-\frac{\beta r^2}{4a(\delta+\beta t)}\biggr]w\biggl(\pm\frac{r}{\delta+\beta t},\ \frac{\gamma+\lambda t}{\delta+\beta t}\biggr), \qquad \lambda\delta - \beta\gamma = 1,$$

where A, C, β, δ, and λ are arbitrary constants, are also solutions of this equation. The second formula usually may be encountered with $\beta = 1$, $\gamma = -1$, and $\delta = \lambda = 0$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

► Particular solutions in the form of an infinite series.

A solution containing an arbitrary function of the space variable:

$$w(r,t) = f(r) + \sum_{n=1}^{\infty} \frac{(at)^n}{n!}\boldsymbol{L}^n[f(r)], \qquad \boldsymbol{L} \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr},$$

where $f(r)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(r,0) = f(r)$. The sum is finite if $f(r)$ is a polynomial that contains only even powers.

A solution containing an arbitrary function of time:

$$w(r,t) = g(t) + \sum_{n=1}^{\infty} \frac{1}{(4a)^n(n!)^2} r^{2n} g_t^{(n)}(t),$$

where $g(t)$ is any infinitely differentiable function. This solution is bounded at $r = 0$ and possesses the properties

$$w(0,t) = g(t), \qquad \partial_r w(0,t) = 0.$$

▶ **Domain: $0 \le r \le R$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g(t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ |w| &\neq \infty &&\text{at}\quad r=0 &&\text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_0^R f(\xi)G(r,\xi,t)\,d\xi - a\int_0^t g(\tau)\Lambda(r,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t) = \sum_{n=1}^{\infty} \frac{2\xi}{R^2 J_1^2(\mu_n)} J_0\Big(\mu_n \frac{r}{R}\Big) J_0\Big(\mu_n \frac{\xi}{R}\Big) \exp\Big(-\frac{a\mu_n^2 t}{R^2}\Big), \quad \Lambda(r,t) = \frac{\partial}{\partial \xi} G(r,\xi,t)\Big|_{\xi=R},$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu)=0$. Below are the numerical values of the first ten roots:

$$\begin{aligned} &\mu_1 = 2.4048, && \mu_2 = 5.5201, && \mu_3 = 8.6537, && \mu_4 = 11.7915, && \mu_5 = 14.9309,\\ &\mu_6 = 18.0711, && \mu_7 = 21.2116, && \mu_8 = 24.3525, && \mu_9 = 27.4935, && \mu_{10} = 30.6346. \end{aligned}$$

The zeroes of the Bessel function $J_0(\mu)$ may be approximated by the formula

$$\mu_n = 2.4 + 3.13(n-1) \qquad (n = 1,\ 2,\ 3,\ \ldots),$$

which is accurate within 0.3%. As $n \to \infty$, we have $\mu_{n+1} - \mu_n \to \pi$.

Example 3.11. The initial temperature of the cylinder is uniform, $f(r) = w_0$, and its lateral surface is maintained all the time at a constant temperature, $g(t) = w_R$.

Solution:

$$\frac{w(r,t) - w_R}{w_0 - w_R} = \sum_{n=1}^{\infty} \frac{2}{\mu_n J_1(\mu_n)} \exp\Big(-\mu_n^2 \frac{at}{R^2}\Big) J_0\Big(\mu_n \frac{r}{R}\Big).$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ **Domain: $0 \le r \le R$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g(t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ |w| &\neq \infty &&\text{at}\quad r=0 &&\text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_0^R f(\xi)G(r,\xi,t)\,d\xi + a\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t) = \frac{2}{R^2}\xi + \frac{2}{R^2}\sum_{n=1}^{\infty} \frac{\xi}{J_0^2(\mu_n)} J_0\Bigl(\frac{\mu_n r}{R}\Bigr) J_0\Bigl(\frac{\mu_n \xi}{R}\Bigr) \exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu) = 0$. Below are the numerical values of the first ten roots:

$$\begin{aligned} &\mu_1 = 3.8317, &&\mu_2 = 7.0156, &&\mu_3 = 10.1735, &&\mu_4 = 13.3237, &&\mu_5 = 16.4706,\\ &\mu_6 = 19.6159, &&\mu_7 = 22.7601, &&\mu_8 = 25.9037, &&\mu_9 = 29.0468, &&\mu_{10} = 32.1897. \end{aligned}$$

As $n \to \infty$, we have $\mu_{n+1} - \mu_n \to \pi$.

Example 3.12. The initial temperature of the cylinder is uniform, $f(r) = w_0$. The lateral surface is maintained at constant thermal flux, $g(t) = g_R$.

Solution:

$$w(r,t) = w_0 + g_R R\biggl[2\frac{at}{R^2} - \frac{1}{4} + \frac{r^2}{2R^2} - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} \exp\Bigl(-\mu_n^2 \frac{at}{R^2}\Bigr) J_0\Bigl(\mu_n \frac{r}{R}\Bigr)\biggr].$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le r \le R$. Third boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ \partial_r w + kw &= g(t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\ |w| &\neq \infty &&\text{at}\quad r = 0 &&\text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_0^R f(\xi) G(r,\xi,t)\, d\xi + a\int_0^t g(\tau) G(r,R,t-\tau)\, d\tau.$$

Here,

$$G(r,\xi,t) = \frac{2}{R^2}\sum_{n=1}^{\infty} \frac{\mu_n^2 \xi}{(k^2R^2 + \mu_n^2) J_0^2(\mu_n)} J_0\Bigl(\frac{\mu_n r}{R}\Bigr) J_0\Bigl(\frac{\mu_n \xi}{R}\Bigr) \exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$

where the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu) - kR J_0(\mu) = 0.$$

The numerical values of the first six roots μ_n can be found in Carslaw and Jaeger (1984).

Example 3.13. The initial temperature of the cylinder is uniform, $f(r) = w_0$. The temperature of the environment is also uniform and is equal to w_R, which corresponds to $g(t) = kw_R$.

Solution:

$$\frac{w(r,t) - w_0}{w_R - w_0} = 1 - \sum_{n=1}^{\infty} A_n \exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr) J_0\Bigl(\frac{\mu_n r}{R}\Bigr), \quad A_n = \frac{2kR}{(k^2R^2 + \mu_n^2) J_0(\mu_n)}.$$

⊙ *Literature*: A. V. Lykov (1967), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $R_1 \le r \le R_2$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) \quad \text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ w &= g_1(t) \quad \text{at} \quad r = R_1 \quad &&\text{(boundary condition)},\\ w &= g_2(t) \quad \text{at} \quad r = R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi + a\int_0^t g_1(\tau)\Lambda_1(r,t-\tau)\,d\tau - a\int_0^t g_2(\tau)\Lambda_2(r,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t) = \frac{\pi^2}{2R_1^2}\sum_{n=1}^{\infty}\frac{\mu_n^2 J_0^2(s\mu_n)\xi}{J_0^2(\mu_n) - J_0^2(s\mu_n)}\Psi_n(r)\Psi_n(\xi)\exp\left(-\frac{a\mu_n^2 t}{R_1^2}\right),$$

$$\Psi_n(r) = Y_0(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right) - J_0(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right), \qquad s = \frac{R_2}{R_1},$$

$$\Lambda_1(r,t) = \frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R_1}, \quad \Lambda_2(r,t) = \frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R_2},$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions; the μ_n are positive roots of the transcendental equation

$$J_0(\mu)Y_0(s\mu) - J_0(s\mu)Y_0(\mu) = 0.$$

The numerical values of the first five roots $\mu_n = \mu_n(s)$ range in the interval $1.4 \le s \le 4.0$ and can be found in Carslaw and Jaeger (1984). See also Abramowitz and Stegun (1964).

Example 3.14. The initial temperature of the hollow cylinder is zero, and its interior and exterior surfaces are held all the time at constant temperatures, $g_1(t) = w_1$ and $g_2(t) = w_2$.

Solution:

$$\begin{aligned} w(r,t) &= \frac{1}{\ln s}\left(w_1 \ln\frac{R_2}{r} + w_2\ln\frac{r}{R_1}\right)\\ &\quad - \pi\sum_{n=1}^{\infty}\frac{J_0(\mu_n)[w_2 J_0(\mu_n) - w_1 J_0(s\mu_n)]}{J_0^2(\mu_n) - J_0^2(s\mu_n)}\exp\left(-\frac{a\mu_n^2 t}{R_1^2}\right)\Psi_n(r). \end{aligned}$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $R_1 \le r \le R_2$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) \quad \text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ \partial_r w &= g_1(t) \quad \text{at} \quad r = R_1 \quad &&\text{(boundary condition)},\\ \partial_r w &= g_2(t) \quad \text{at} \quad r = R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t)=\int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi-a\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau+a\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t)=\frac{2\xi}{R_2^2-R_1^2}+\frac{\pi^2}{2R_1^2}\sum_{n=1}^{\infty}\frac{\mu_n^2J_1^2(s\mu_n)\xi}{J_1^2(\mu_n)-J_1^2(s\mu_n)}\Psi_n(r)\Psi_n(\xi)\exp\left(-\frac{a\mu_n^2 t}{R_1^2}\right),$$

$$\Psi_n(r)=Y_1(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right)-J_1(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right),\quad s=\frac{R_2}{R_1},$$

where $J_k(z)$ and $Y_k(z)$ are Bessel functions ($k=0,\ 1$), and the μ_n are positive roots of the transcendental equation

$$J_1(\mu)Y_1(s\mu)-J_1(s\mu)Y_1(\mu)=0.$$

The numerical values of the first five roots $\mu_n=\mu_n(s)$ can be found in Abramowitz and Stegun (1964).

⊙ *Literature*: A. V. Lykov (1967), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1\le r\le R_2$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w&=f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w-k_1w&=g_1(t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ \partial_r w+k_2w&=g_2(t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t)=\int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi-a\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau+a\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2}{B_n}[k_2J_0(\lambda_nR_2)-\lambda_nJ_1(\lambda_nR_2)]^2\xi H_n(r)H_n(\xi)\exp(-a\lambda_n^2t),$$

where

$$B_n=(\lambda_n^2+k_2^2)[k_1J_0(\lambda_nR_1)+\lambda_nJ_1(\lambda_nR_1)]^2-(\lambda_n^2+k_1^2)[k_2J_0(\lambda_nR_2)-\lambda_nJ_1(\lambda_nR_2)]^2,$$
$$H_n(r)=[k_1Y_0(\lambda_nR_1)+\lambda_nY_1(\lambda_nR_1)]J_0(\lambda_nr)-[k_1J_0(\lambda_nR_1)+\lambda_nJ_1(\lambda_nR_1)]Y_0(\lambda_nr),$$

and the λ_n are positive roots of the transcendental equation

$$[k_1J_0(\lambda R_1)+\lambda J_1(\lambda R_1)][k_2Y_0(\lambda R_2)-\lambda Y_1(\lambda R_2)]\\ -[k_2J_0(\lambda R_2)-\lambda J_1(\lambda R_2)][k_1Y_0(\lambda R_1)+\lambda Y_1(\lambda R_1)]=0.$$

⊙ *Literature*: A. V. Lykov (1967), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r < \infty$. Cauchy type problem.**

This problem is encountered in the theory of diffusion wake behind a drop or a solid particle.

Given the initial condition

$$w = f(r) \quad \text{at} \quad t = 0,$$

the equation has the following bounded solution:

$$w(r,t) = \frac{1}{2a}\int_0^\infty \frac{\xi}{t}\exp\biggl(-\frac{r^2+\xi^2}{4at}\biggr) I_0\biggl(\frac{r\xi}{2at}\biggr) f(\xi)\,d\xi,$$

where $I_0(\xi)$ is the modified Bessel function.

⊙ *Literature*: W. G. L. Sutton (1943), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985).

► **Domain: $R \le r < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_r w - kw &= g(t) && \text{at} \quad r = R && \text{(boundary condition)},\\ |w| &\ne \infty && \text{at} \quad r \to \infty && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_R^\infty f(\xi)G(r,\xi,t)\,d\xi - a\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau,$$

where

$$\begin{aligned} G(r,\xi,t) &= \xi\int_0^\infty \exp\bigl(-au^2t\bigr)F(r,u)F(\xi,u)u\,du,\\ F(r,u) &= \frac{J_0(ur)[uY_1(uR)+kY_0(uR)] - Y_0(ur)[uJ_1(uR)+kJ_0(uR)]}{\sqrt{[uJ_1(uR)+kJ_0(uR)]^2+[uY_1(uR)+kY_0(uR)]^2}}. \end{aligned}$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

3.2.2 Equation of the Form $\frac{\partial w}{\partial t} = a\Bigl(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\Bigr) + \Phi(r,t)$

This equation is encountered in plane problems of heat conduction with heat release (the function Φ is proportional to the amount of heat released per unit time in the volume under consideration). The equation describes one-dimensional unsteady thermal processes having axial symmetry.

► **Domain: $0 \le r \le R$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g(t) && \text{at} \quad r = R && \text{(boundary condition)},\\ |w| &\neq \infty && \text{at} \quad r = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_0^R f(\xi) G(r,\xi,t)\,d\xi - a\int_0^t g(\tau)\Lambda(r,t-\tau)\,d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(r,\xi,t) = \sum_{n=1}^{\infty} \frac{2\xi}{R^2 J_1^2(\mu_n)} J_0\left(\mu_n \frac{r}{R}\right) J_0\left(\mu_n \frac{\xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$
$$\Lambda(r,t) = \frac{\partial}{\partial \xi} G(r,\xi,t)\Big|_{\xi=R}.$$

Here, the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten roots μ_n are given in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_r w &= g(t) && \text{at} \quad r = R && \text{(boundary condition)},\\ |w| &\neq \infty && \text{at} \quad r = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) &= \int_0^R f(\xi) G(r,\xi,t)\,d\xi + a\int_0^t g(\tau) G(r,R,t-\tau)\,d\tau \\ &\quad + \int_0^t \int_0^R \Phi(\xi,\tau) G(r,\xi,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t) = \frac{2}{R^2}\xi + \frac{2}{R^2}\sum_{n=1}^{\infty} \frac{\xi}{J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu) = 0$. The numerical values of the first ten roots μ_n can be found in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

⊙ *Literature*: A. V. Lykov (1967), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r \le R$. Third boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r) &&\text{at} \quad t = 0 &&\text{(initial condition)},\\
\partial_r w + kw &= g(t) &&\text{at} \quad r = R &&\text{(boundary condition)},\\
|w| &\neq \infty &&\text{at} \quad r = 0 &&\text{(boundedness condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,t) = {} & \int_0^R f(\xi)G(r,\xi,t)\,d\xi + a\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau\\
& + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\xi,t) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\mu_n^2\xi}{(k^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\biggl(\frac{\mu_n r}{R}\biggr)J_0\biggl(\frac{\mu_n \xi}{R}\biggr)\exp\biggl(-\frac{a\mu_n^2 t}{R^2}\biggr),
$$

where the μ_n are positive roots of the transcendental equation

$$
\mu J_1(\mu) - kRJ_0(\mu) = 0.
$$

The numerical values of the first six roots μ_n can be found in Carslaw and Jaeger (1984).

⊙ *Literature*: A. V. Lykov (1967), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r) &&\text{at} \quad t = 0 &&\text{(initial condition)},\\
w &= g_1(t) &&\text{at} \quad r = R_1 &&\text{(boundary condition)},\\
w &= g_2(t) &&\text{at} \quad r = R_2 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,t) = {} & \int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi + \int_0^t\int_{R_1}^{R_2} \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau\\
& + a\int_0^t g_1(\tau)\Lambda_1(r,t-\tau)\,d\tau - a\int_0^t g_2(\tau)\Lambda_2(r,t-\tau)\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\xi,t) = \sum_{n=1}^{\infty} A_n\xi\Psi_n(r)\Psi_n(\xi)\exp\biggl(-\frac{a\mu_n^2 t}{R_1^2}\biggr), \quad \Lambda_1(r,t) = \frac{\partial G}{\partial\xi}\bigg|_{\xi=R_1}, \quad \Lambda_2(r,t) = \frac{\partial G}{\partial\xi}\bigg|_{\xi=R_2},
$$

$$
A_n = \frac{\pi^2\mu_n^2 J_0^2(s\mu_n)}{2R_1^2\bigl[J_0^2(\mu_n) - J_0^2(s\mu_n)\bigr]}, \quad \Psi_n(r) = Y_0(\mu_n)J_0\biggl(\frac{\mu_n r}{R_1}\biggr) - J_0(\mu_n)Y_0\biggl(\frac{\mu_n r}{R_1}\biggr), \quad s = \frac{R_2}{R_1},
$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions, and the μ_n are positive roots of the transcendental equation

$$J_0(\mu)Y_0(s\mu) - J_0(s\mu)Y_0(\mu) = 0.$$

The numerical values of the first five roots $\mu_n = \mu_n(s)$ can be found in Carslaw and Jaeger (1984).

► Domain: $R_1 \le r \le R_2$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w &= g_1(t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ \partial_r w &= g_2(t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) = &\int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi + \int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau \\ &- a\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t) = \frac{2\xi}{R_2^2 - R_1^2} + \frac{\pi^2}{2R_1^2}\sum_{n=1}^{\infty} \frac{\mu_n^2 J_1^2(s\mu_n)\xi}{J_1^2(\mu_n) - J_1^2(s\mu_n)} \Psi_n(r)\Psi_n(\xi)\exp\left(-\frac{a\mu_n^2 t}{R_1^2}\right),$$

$$\Psi_n(r) = Y_1(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right) - J_1(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right), \qquad s = \frac{R_2}{R_1},$$

where $J_k(z)$ and $Y_k(z)$ are Bessel functions of order $k = 0,\ 1$, and the μ_n are positive roots of the transcendental equation

$$J_1(\mu)Y_1(s\mu) - J_1(s\mu)Y_1(\mu) = 0.$$

The numerical values of the first five roots $\mu_n = \mu_n(s)$ can be found in Abramowitz and Stegun (1964).

► Domain: $R_1 \le r \le R_2$. Third boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w - k_1 w &= g_1(t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ \partial_r w + k_2 w &= g_2(t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution is given by the formula from Section 3.2.1 (see the third boundary value problem for $R_1 \le r \le R_2$) with the additional term

$$\int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,$$

which takes into account the nonhomogeneity of the equation.

► **Domain: $0 \le r < \infty$. Cauchy type problem.**

The bounded solution of this equation subject to the initial condition

$$w = f(r) \quad \text{at} \quad t = 0$$

is given by the relations

$$w(r,t) = \int_0^\infty G(r,\xi,t) f(\xi)\, d\xi + \int_0^t \int_0^\infty G(r,\xi,t-\tau)\Phi(\xi,\tau)\, d\xi\, d\tau,$$
$$G(r,\xi,t) = \frac{\xi}{2at} \exp\left(-\frac{r^2+\xi^2}{4at}\right) I_0\left(\frac{r\xi}{2at}\right),$$

where $I_0(z)$ is the modified Bessel function.

⊙ *Literature*: W. G. L. Sutton (1943), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R \le r < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_r w - kw &= g(t) && \text{at} \quad r = R && \text{(boundary condition)},\\ |w| &\neq \infty && \text{at} \quad r \to \infty && \text{(boundedness condition)}. \end{aligned}$$

The solution is given by the formula from Section 3.2.1 (see the third boundary value problem for $R \le r < \infty$) with the additional term

$$\int_0^t \int_R^\infty \Phi(\xi,\tau) G(r,\xi,t-\tau)\, d\xi\, d\tau,$$

which takes into account the nonhomogeneity of the equation.

3.2.3 Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right)$

This is a sourceless heat equation that describes unsteady heat processes with central symmetry. It is often represented in the equivalent form

$$\frac{\partial w}{\partial t} = \frac{a}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial w}{\partial r}\right).$$

A similar equation is used for the analysis of the corresponding one-dimensional unsteady diffusion processes.

► **Particular solutions (A, B, and μ are arbitrary constants).**

$$
\begin{aligned}
w(r) &= A + B\frac{1}{r},\\
w(r,t) &= A + B(r^2 + 6at),\\
w(r,t) &= A + B(r^4 + 20atr^2 + 60a^2t^2),\\
w(r,t) &= A + B\left[r^{2n} + \sum_{k=1}^{n} \frac{(2n+1)(2n)\dots(2n-2k+2)}{k!}(at)^k r^{2n-2k}\right],\\
w(r,t) &= A + 2aB\frac{t}{r} + Br,\\
w(r,t) &= Ar^{-1}\exp(a\mu^2 t \pm \mu r) + B,\\
w(r,t) &= A + \frac{B}{t^{3/2}}\exp\left(-\frac{r^2}{4at}\right),\\
w(r,t) &= A + \frac{B}{r\sqrt{t}}\exp\left(-\frac{r^2}{4at}\right),\\
w(r,t) &= Ar^{-1}\exp(-a\mu^2 t)\cos(\mu r) + B,\\
w(r,t) &= Ar^{-1}\exp(-a\mu^2 t)\sin(\mu r) + B,\\
w(r,t) &= Ar^{-1}\exp(-\mu r)\cos(\mu r - 2a\mu^2 t) + B,\\
w(r,t) &= Ar^{-1}\exp(-\mu r)\sin(\mu r - 2a\mu^2 t) + B,\\
w(r,t) &= \frac{A}{r}\operatorname{erf}\left(\frac{r}{2\sqrt{at}}\right) + B,\\
w(r,t) &= \frac{A}{r}\operatorname{erfc}\left(\frac{r}{2\sqrt{at}}\right) + B,
\end{aligned}
$$

where n is an arbitrary positive integer.

► **Reduction to a constant coefficient equation. Some formulas.**

1°. The substitution $u(r,t) = rw(r,t)$ brings the original equation with variable coefficients to the constant coefficient equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial r^2},$$

which is discussed in Section 3.1.1 in detail.

2°. Suppose $w = w(r,t)$ is a solution of the original equation. Then the functions

$$
\begin{aligned}
w_1 &= Aw(\pm\lambda r,\ \lambda^2 t + C),\\
w_2 &= \frac{A}{|\delta + \beta t|^{3/2}}\exp\left[-\frac{\beta r^2}{4a(\delta + \beta t)}\right] w\left(\pm\frac{r}{\delta + \beta t}, \frac{\gamma + \lambda t}{\delta + \beta t}\right), \qquad \lambda\delta - \beta\gamma = 1,
\end{aligned}
$$

where A, C, β, δ, and λ are arbitrary constants, are also solutions of this equation. The second formula may usually be encountered with $\beta = 1$, $\gamma = -1$, and $\delta = \lambda = 0$.

► **Infinite series particular solutions.**

A solution containing an arbitrary function of the space variable:

$$w(r,t) = f(r) + \sum_{n=1}^{\infty} \frac{(at)^n}{n!} \boldsymbol{L}^n\bigl[f(r)\bigr], \qquad \boldsymbol{L} \equiv \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr},$$

where $f(r)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(r,0) = f(r)$. The sum is finite if $f(r)$ is a polynomial that contains only even powers.

A solution containing an arbitrary function of time:

$$w(r,t) = g(t) + \sum_{n=1}^{\infty} \frac{1}{a^n(2n+1)!} r^{2n} g_t^{(n)}(t),$$

where $g(t)$ is any infinitely differentiable function. This solution is bounded at $r = 0$ and possesses the properties

$$w(0,t) = g(t), \qquad \partial_r w(0,t) = 0.$$

► **Domain: $0 \le r \le R$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w &= g(t) && \text{at} \quad r = R \quad \text{(boundary condition)},\\ |w| &\ne \infty && \text{at} \quad r = 0 \quad \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \frac{2}{R} \sum_{n=1}^{\infty} \frac{1}{r} \sin\left(\frac{\pi n r}{R}\right) \exp\left(-\frac{a\pi^2 n^2 t}{R^2}\right) M_n(t),$$

where

$$M_n(t) = \int_0^R \xi f(\xi) \sin\left(\frac{\pi n \xi}{R}\right) d\xi - (-1)^n a\pi n \int_0^t g(\tau) \exp\left(\frac{a\pi^2 n^2 \tau}{R^2}\right) d\tau.$$

Remark 3.3. Using the relation [see Prudnikov, Brychkov, and Marichev (1986)]

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nz}{n} = \frac{z}{2} \qquad (-\pi < z < \pi),$$

we rewrite the solution in the form

$$w(r,t) = g(t) + \frac{2}{Rr} \sum_{n=1}^{\infty} \sin(\lambda_n r) \exp(-a\lambda_n^2 t) H_n(t), \qquad \lambda_n = \frac{\pi n}{R},$$

where

$$\begin{aligned} H_n(t) &= \int_0^R \xi f(\xi) \sin(\lambda_n \xi)\, d\xi + (-1)^n \frac{R}{\lambda_n} g(t) \exp(a\lambda_n^2 t) - (-1)^n a\pi n \int_0^t g(\tau) \exp(a\lambda_n^2 \tau)\, d\tau \\ &= \int_0^R \xi f(\xi) \sin(\lambda_n \xi)\, d\xi + (-1)^n \frac{R}{\lambda_n} g(0) + (-1)^n \frac{R}{\lambda_n} \int_0^t g'_\tau(\tau) \exp(a\lambda_n^2 \tau)\, d\tau. \end{aligned}$$

Example 3.15. The initial temperature is uniform, $f(r) = w_0$, and the surface of the sphere is maintained at constant temperature, $g(t) = w_R$.

Solution:

$$\frac{w(r,t) - w_R}{w_0 - w_R} = \frac{2R}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{\pi n r}{R}\right) \exp\left(-\frac{a\pi^2 n^2 t}{R^2}\right).$$

The average temperature $\overline{w}$ depends on time t as follows:

$$\frac{\overline{w} - w_R}{w_0 - w_R} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{a\pi^2 n^2 t}{R^2}\right), \qquad \overline{w} = \frac{1}{V} \int_v w\, dv,$$

where V is the volume of the sphere of radius R.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $0 \le r \le R$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w &= g(t) && \text{at} \quad r = R && \text{(boundary condition)}, \\ |w| &\neq \infty && \text{at} \quad r = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_0^R f(\xi) G(r, \xi, t)\, d\xi + a \int_0^t g(\tau) G(r, R, t - \tau)\, d\tau,$$

where

$$G(r, \xi, t) = \frac{3\xi^2}{R^3} + \frac{2\xi}{Rr} \sum_{n=1}^{\infty} \frac{\mu_n^2 + 1}{\mu_n^2} \sin\left(\frac{\mu_n r}{R}\right) \sin\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right).$$

Here, the μ_n are positive roots of the transcendental equation $\tan\mu - \mu = 0$. The first five roots are

$$\mu_1 = 4.4934, \quad \mu_2 = 7.7253, \quad \mu_3 = 10.9041, \quad \mu_4 = 14.0662, \quad \mu_5 = 17.2208.$$

Example 3.16. The initial temperature of the sphere is uniform, $f(r) = w_0$. The thermal flux at the sphere surface is a maintained constant, $g(t) = g_R$.

Solution:

$$w(r,t) = w_0 + g_R R \left[\frac{3at}{R^2} + \frac{5r^2 - 3R^2}{10R^2} - \sum_{n=1}^{\infty} \frac{2R}{\mu_n^3 \cos(\mu_n)} \frac{1}{r} \sin\left(\frac{\mu_n r}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right) \right],$$

⊙ *Literature*: A. V. Lykov (1967), A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $0 \le r \le R$. Third boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\
\partial_r w + kw &= g(t) \quad &&\text{at} \quad r = R \quad &&\text{(boundary condition)},\\
|w| &\neq \infty \quad &&\text{at} \quad r = 0 \quad &&\text{(boundedness condition)}.
\end{aligned}
$$

Solution:

$$
w(r,t) = \int_0^R f(\xi)G(r,\xi,t)\,d\xi + a\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau.
$$

Here,

$$
G(r,\xi,t) = \frac{2\xi}{Rr}\sum_{n=1}^{\infty}\frac{\mu_n^2 + (kR-1)^2}{\mu_n^2 + kR(kR-1)}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),
$$

where the μ_n are positive roots of the transcendental equation

$$
\mu\cot\mu + kR - 1 = 0.
$$

The numerical values of the first six roots μ_n can be found in Carslaw and Jaeger (1984).

Example 3.17. The initial temperature of the sphere is uniform, $f(r) = w_0$. The temperature of the ambient medium is zero, $g(t) = 0$.

Solution:

$$
w(r,t) = \frac{2kR^2 w_0}{r}\sum_{n=1}^{\infty}\frac{\sin\mu_n[\mu_n^2 + (kR-1)^2]}{\mu_n^2[\mu_n^2 + kR(kR-1)]}\sin\left(\frac{\mu_n r}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),
$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\
w &= g_1(t) \quad &&\text{at} \quad r = R_1 \quad &&\text{(boundary condition)},\\
w &= g_2(t) \quad &&\text{at} \quad r = R_2 \quad &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
w(r,t) = \int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi + a\int_0^t g_1(\tau)\Lambda_1(r,t-\tau)\,d\tau - a\int_0^t g_2(\tau)\Lambda_2(r,t-\tau)\,d\tau,
$$

where

$$
G(r,\xi,t) = \frac{2\xi}{(R_2-R_1)r}\sum_{n=1}^{\infty}\sin\left[\frac{\pi n(r-R_1)}{R_2-R_1}\right]\sin\left[\frac{\pi n(\xi-R_1)}{R_2-R_1}\right]\exp\left[-\frac{\pi^2 n^2 at}{(R_2-R_1)^2}\right],
$$

$$
\Lambda_1(r,t) = \frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R_1}, \quad \Lambda_2(r,t) = \frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R_2}.
$$

Example 3.18. The initial temperature is zero. The temperatures of the interior and exterior surfaces of the spherical layer are maintained constants, $g_1(t) = w_1$ and $g_2(t) = w_2$.

Solution:

$$w(r,t) = \frac{R_1 w_1}{r} + \frac{(r-R_1)(R_2 w_2 - R_1 w_1)}{r(R_2 - R_1)} + \frac{2}{r}\sum_{n=1}^{\infty} \frac{(-1)^n R_2 w_2 - R_1 w_1}{\pi n} \sin\left[\frac{\pi n(r-R_1)}{R_2 - R_1}\right] \exp\left[-\frac{\pi^2 n^2 a t}{(R_2 - R_1)^2}\right].$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $R_1 \le r \le R_2$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ \partial_r w &= g_1(t) &&\text{at}\quad r = R_1 &&\text{(boundary condition)},\\ \partial_r w &= g_2(t) &&\text{at}\quad r = R_2 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_{R_1}^{R_2} f(\xi) G(r,\xi,t)\,d\xi - a\int_0^t g_1(\tau) G(r,R_1,t-\tau)\,d\tau + a\int_0^t g_2(\tau) G(r,R_2,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t) = \frac{3\xi^2}{R_2^3 - R_1^3} + \frac{2\xi}{(R_2 - R_1)r}\sum_{n=1}^{\infty} \frac{(1 + R_2^2\lambda_n^2)\Psi_n(r)\Psi_n(\xi)\exp(-a\lambda_n^2 t)}{\lambda_n^2\left[R_1^2 + R_2^2 + R_1 R_2(1 + R_1 R_2 \lambda_n^2)\right]},$$

$$\Psi_n(r) = \sin[\lambda_n(r-R_1)] + R_1\lambda_n\cos[\lambda_n(r-R_1)],$$

where the λ_n are positive roots of the transcendental equation

$$(\lambda^2 R_1 R_2 + 1)\tan[\lambda(R_2 - R_1)] - \lambda(R_2 - R_1) = 0.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $R_1 \le r \le R_2$. Third boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ \partial_r w - k_1 w &= g_1(t) &&\text{at}\quad r = R_1 &&\text{(boundary condition)},\\ \partial_r w + k_2 w &= g_2(t) &&\text{at}\quad r = R_2 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_{R_1}^{R_2} f(\xi) G(r,\xi,t)\,d\xi - a\int_0^t g_1(\tau) G(r,R_1,t-\tau)\,d\tau + a\int_0^t g_2(\tau) G(r,R_2,t-\tau)\,d\tau.$$

Here,

$$G(r,\xi,t)=\frac{2\xi}{r}\sum_{n=1}^{\infty}\frac{(b_2^2+R_2^2\lambda_n^2)\Psi_n(r)\Psi_n(\xi)\exp(-a\lambda_n^2 t)}{(R_2-R_1)(b_1^2+R_1^2\lambda_n^2)(b_2^2+R_2^2\lambda_n^2)+(b_1R_2+b_2R_1)(b_1b_2+R_1R_2\lambda_n^2)},$$

$$\Psi_n(r)=b_1\sin[\lambda_n(r-R_1)]+R_1\lambda_n\cos[\lambda_n(r-R_1)],\quad b_1=k_1R_1+1,\quad b_2=k_2R_2-1,$$

where the λ_n are positive roots of the transcendental equation

$$(b_1b_2-R_1R_2\lambda^2)\sin[\lambda(R_2-R_1)]+\lambda(R_1b_2+R_2b_1)\cos[\lambda(R_2-R_1)]=0.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $0\le r<\infty$. Cauchy type problem.

The bounded solution of this equation subject to the initial condition

$$w=f(r)\quad\text{at}\quad t=0$$

has the form

$$w(r,t)=\frac{1}{2r\sqrt{\pi at}}\int_0^{\infty}\xi\left\{\exp\left[-\frac{(r-\xi)^2}{4at}\right]-\exp\left[-\frac{(r+\xi)^2}{4at}\right]\right\}f(\xi)\,d\xi.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

▶ Domain: $R\le r<\infty$. First boundary value problem.

The following conditions are prescribed:

$$w=f(r)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$w=g(t)\quad\text{at}\quad r=R\quad\text{(boundary condition)}.$$

Solution:

$$w(r,t)=\frac{1}{2r\sqrt{\pi at}}\int_R^{\infty}\xi f(\xi)\left\{\exp\left[-\frac{(r-\xi)^2}{4at}\right]-\exp\left[-\frac{(r+\xi-2R)^2}{4at}\right]\right\}d\xi$$
$$+\frac{2R}{r\sqrt{\pi}}\int_z^{\infty}g\left(t-\frac{(r-R)^2}{4a\tau^2}\right)\exp(-\tau^2)\,d\tau,\qquad z=\frac{r-R}{2\sqrt{at}}.$$

Example 3.19. The temperature of the ambient medium is uniform at the initial instant $t=0$ and the boundary of the domain is held at constant temperature, that is, $f(r)=w_0$ and $g(t)=w_R$.
Solution:

$$\frac{w-w_0}{w_R-w_0}=\frac{R}{r}\operatorname{erfc}\left(\frac{r-R}{2\sqrt{at}}\right),$$

where $\operatorname{erfc} z=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_z^{\infty}\exp(-\xi^2)\,d\xi$ is the error function.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

3.2.4 Equation of the Form $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial w}{\partial r}\right) + \Phi(r,t)$

This equation is encountered in heat conduction problems with heat release; the function Φ is proportional to the amount of heat released per unit time in the volume under consideration. The equation describes one-dimensional unsteady thermal processes having central symmetry.

The substitution $u(r,t) = rw(r,t)$ brings the original nonhomogeneous equation with variable coefficients to the nonhomogeneous constant coefficient equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial r^2} + r\Phi(r,t),$$

which is considered in Section 3.1.2.

► Domain: $0 \le r \le R$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} &w = f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ &w = g(t) && \text{at} \quad r = R && \text{(boundary condition)},\\ &w \neq \infty && \text{at} \quad r = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \int_0^R f(\xi)G(r,\xi,t)\,d\xi - a\int_0^t g(\tau)\Lambda(r,t-\tau)\,d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(r,\xi,t) = \frac{2\xi}{Rr}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi r}{R}\right)\sin\left(\frac{n\pi\xi}{R}\right)\exp\left(-\frac{an^2\pi^2 t}{R^2}\right), \qquad \Lambda(r,t) = \frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R}.$$

► Domain: $0 \le r \le R$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_r w &= g(t) && \text{at} \quad r = R && \text{(boundary condition)},\\ w &\neq \infty && \text{at} \quad r = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) = {} & \int_0^R f(\xi)G(r,\xi,t)\,d\xi + a\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau \\ & + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau, \end{aligned}$$

where

$$G(r,\xi,t)=\frac{3\xi^2}{R^3}+\frac{2\xi}{Rr}\sum_{n=1}^{\infty}\frac{\mu_n^2+1}{\mu_n^2}\sin\Bigl(\frac{\mu_n r}{R}\Bigr)\sin\Bigl(\frac{\mu_n \xi}{R}\Bigr)\exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr).$$

Here, the μ_n are positive roots of the transcendental equation $\tan\mu-\mu=0$. The values of the first five roots μ_n can be found in Section 3.2.3 (see the second boundary value problem for $0\le r\le R$).

⊙ *Literature*: A. V. Lykov (1967), A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $0\le r\le R$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w&=f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w+kw&=g(t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ w&\ne\infty &&\text{at}\quad r=0 &&\text{(boundedness condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,t)=&\int_0^R f(\xi)G(r,\xi,t)\,d\xi+a\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau\\ &+\int_0^t\!\!\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.\end{aligned}$$

Here,

$$G(r,\xi,t)=\frac{2\xi}{Rr}\sum_{n=1}^{\infty}\frac{\mu_n^2+(kR-1)^2}{\mu_n^2+kR(kR-1)}\sin\Bigl(\frac{\mu_n r}{R}\Bigr)\sin\Bigl(\frac{\mu_n \xi}{R}\Bigr)\exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$

where the μ_n are positive roots of the transcendental equation $\mu\cot\mu+kR-1=0$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1\le r\le R_2$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w&=f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g_1(t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ w&=g_2(t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,t)=&\int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi+\int_0^t\!\!\int_{R_1}^{R_2}\Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau\\ &+a\int_0^t g_1(\tau)\Lambda_1(r,t-\tau)\,d\tau-a\int_0^t g_2(\tau)\Lambda_2(r,t-\tau)\,d\tau,\end{aligned}$$

where

$$G(r,\xi,t)=\frac{2\xi}{(R_2-R_1)r}\sum_{n=1}^{\infty}\sin\left[\frac{\pi n(r-R_1)}{R_2-R_1}\right]\sin\left[\frac{\pi n(\xi-R_1)}{R_2-R_1}\right]\exp\left[-\frac{\pi^2 n^2 at}{(R_2-R_1)^2}\right],$$

$$\Lambda_1(r,t)=\frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R_1},\quad \Lambda_2(r,t)=\frac{\partial}{\partial\xi}G(r,\xi,t)\Big|_{\xi=R_2}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g_1(t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ \partial_r w &= g_2(t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) = \int_{R_1}^{R_2} f(\xi)G(r,\xi,t)\,d\xi &+ \int_0^t\int_{R_1}^{R_2}\Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau\\ &- a\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau + a\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau, \end{aligned}$$

where the function $G(r,\xi,t)$ is the same as in Section 3.2.3 (see the second boundary value problem for $R_1 \le r \le R_2$).

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w - k_1 w &= g_1(t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ \partial_r w + k_2 w &= g_2(t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}. \end{aligned}$$

The solution is given by the formula of Section 3.2.3 (see the third boundary value problem for $R_1 \le r \le R_2$) with the additional term

$$\int_0^t\int_{R_1}^{R_2}\Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,$$

which takes into account the nonhomogeneity of the equation.

► **Domain: $0 \le r < \infty$. Cauchy type problem.**

The bounded solution of this equation subject to the initial condition

$$w = f(r) \quad \text{at} \quad t = 0$$

is given by

$$w(r,t) = \int_0^\infty f(\xi)G(r,\xi,t)\,d\xi + \int_0^t \int_0^\infty \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,$$
$$G(r,\xi,t) = \frac{\xi}{2r\sqrt{\pi a t}}\left\{\exp\left[-\frac{(r-\xi)^2}{4at}\right] - \exp\left[-\frac{(r+\xi)^2}{4at}\right]\right\}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $R \le r < \infty$. First boundary value problem.**

The following conditions are prescribed:

$$w = f(r) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

Solution:

$$w(r,t)=\int_R^\infty f(\xi)G(r,\xi,t)\,d\xi+\int_0^t\int_R^\infty \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau+a\int_0^t g(\tau)\Lambda(r,t-\tau)\,d\tau,$$

where

$$G(r,\xi,t) = \frac{\xi}{2r\sqrt{\pi a t}}\left\{\exp\left[-\frac{(r-\xi)^2}{4at}\right] - \exp\left[-\frac{(r+\xi-2R)^2}{4at}\right]\right\},$$
$$\Lambda(r,t) = \frac{\partial}{\partial\xi}G(r,\xi,t)\bigg|_{\xi=R}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

3.2.5 Equation of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{1-2\beta}{x}\dfrac{\partial w}{\partial x}$

This dimensionless equation is encountered in problems of the diffusion boundary layer. For $\beta = 0$, $\beta = \frac{1}{2}$, or $\beta = -\frac{1}{2}$, see the equations of Sections 3.2.1, 3.1.1, or 3.2.3, respectively.

► **Particular solutions (A, B, and μ are arbitrary constants).**

$$w(x) = A + Bx^{2\beta},$$
$$w(x,t) = A + 4(1-\beta)Bt + Bx^2,$$
$$w(x,t) = A + 16(2-\beta)(1-\beta)Bt^2 + 8(2-\beta)Btx^2 + Bx^4,$$
$$w(x,t) = x^{2n} + \sum_{p=1}^{n} \frac{4^p}{p!} s_{n,p} s_{n-\beta,p} t^p x^{2(n-p)}, \quad s_{q,p} = q(q-1)\dots(q-p+1),$$
$$w(x,t) = A + 4(1+\beta)Btx^{2\beta} + Bx^{2\beta+2},$$
$$w(x,t) = A + Bt^{\beta-1} \exp\left(-\frac{x^2}{4t}\right),$$
$$w(x,t) = A + B\frac{x^{2\beta}}{t^{\beta+1}} \exp\left(-\frac{x^2}{4t}\right),$$
$$w(x,t) = A + B\gamma\left(\beta, \frac{x^2}{4t}\right),$$
$$w(x,t) = A + B\exp(-\mu^2 t)x^\beta J_\beta(\mu x),$$
$$w(x,t) = A + B\exp(-\mu^2 t)x^\beta Y_\beta(\mu x),$$
$$w(x,t) = A + B\frac{x^\beta}{t} \exp\left(-\frac{x^2+\mu^2}{4t}\right) I_\beta\left(\frac{\mu x}{2t}\right),$$
$$w(x,t) = A + B\frac{x^\beta}{t} \exp\left(-\frac{x^2+\mu^2}{4t}\right) I_{-\beta}\left(\frac{\mu x}{2t}\right),$$
$$w(x,t) = A + B\frac{x^\beta}{t} \exp\left(-\frac{x^2+\mu^2}{4t}\right) K_\beta\left(\frac{\mu x}{2t}\right),$$

where n is an arbitrary positive integer, $\gamma(\beta,z)$ is the incomplete gamma function, $J_\beta(z)$ and $Y_\beta(z)$ are Bessel functions, and $I_\beta(z)$ and $K_\beta(z)$ are modified Bessel functions.

⊙ *Literature*: W. G. L. Sutton (1943), A. D. Polyanin (2001).

► **Infinite series solutions.**

A solution containing an arbitrary function of the space variable:

$$w(x,t) = f(x) + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \boldsymbol{L}^n[f(x)], \qquad \boldsymbol{L} \equiv \frac{d^2}{dx^2} + \frac{1-2\beta}{x}\frac{d}{dx},$$

where $f(x)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = f(x)$. The sum is finite if $f(x)$ is a polynomial that contains only even powers.

A solution containing an arbitrary function of time:

$$w(x,t) = g(t) + \sum_{n=1}^{\infty} \frac{1}{4^n n!\,(1-\beta)(2-\beta)\dots(n-\beta)} x^{2n} g_t^{(n)}(t),$$

where $g(t)$ is any infinitely differentiable function. This solution is bounded at $x=0$ and possesses the properties

$$w(0,t)=g(t), \qquad \partial_x w(0,t)=0.$$

► **Formulas and transformations for constructing particular solutions.**

Suppose $w=w(x,t)$ is a solution of the original equation. Then the functions

$$w_1 = Aw(\pm\lambda x,\ \lambda^2 t + C),$$
$$w_2 = A|a+bt|^{\beta-1}\exp\left[-\frac{bx^2}{4(a+bt)}\right] w\left(\pm\frac{x}{a+bt},\ \frac{c+kt}{a+bt}\right), \qquad ak-bc=1,$$

where A, C, a, b, and c are arbitrary constants, are also solutions of this equation. The second formula usually may be encountered with $a=k=0$, $b=1$, and $c=-1$.

The substitution $w=x^{2\beta}u(x,t)$ brings the equation with parameter β to an equation of the same type with parameter $-\beta$:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{1+2\beta}{x}\frac{\partial u}{\partial x}.$$

► **Domain: $0\le x<\infty$. First boundary value problem.**

The following conditions are prescribed:

$$w=f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$w=g(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)}.$$

Solution for $0<\beta<1$:

$$w(x,t)=\frac{x^\beta}{2t}\int_0^\infty f(\xi)\xi^{1-\beta}\exp\left(-\frac{x^2+\xi^2}{4t}\right) I_\beta\left(\frac{\xi x}{2t}\right) d\xi$$
$$+\frac{x^{2\beta}}{2^{2\beta+1}\Gamma(\beta+1)}\int_0^t g(\tau)\exp\left[-\frac{x^2}{4(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{1+\beta}}.$$

Example 3.20. For $f(x)=w_0$ and $g(t)=w_1$, where $f(x)=\text{const}$ and $g(t)=\text{const}$, we have

$$w=\frac{(w_0-w_1)}{\Gamma(\beta)}\gamma\left(\beta,\frac{x^2}{4t}\right)+w_1, \qquad \gamma(\beta,z)=\int_0^z \xi^{\beta-1}e^{-\xi}\,d\xi.$$

Here, $\gamma(\beta,z)$ is the incomplete gamma function and $\Gamma(\beta)=\gamma(\beta,\infty)$ is the gamma function.

⊙ *Literature*: W. G. L. Sutton (1943).

► **Domain: $0\le x<\infty$. Second boundary value problem.**

The following conditions are prescribed:

$$w=f(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\left(x^{1-2\beta}\partial_x w\right)=g(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)}.$$

Solution for $0 < \beta < 1$:

$$w(x,t) = \frac{x^\beta}{2t}\int_0^\infty f(\xi)\xi^{1-\beta}\exp\biggl(-\frac{x^2+\xi^2}{4t}\biggr)I_{-\beta}\biggl(\frac{\xi x}{2t}\biggr)\,d\xi \\ - \frac{2^{2\beta-1}}{\Gamma(1-\beta)}\int_0^t g(\tau)\exp\biggl[-\frac{x^2}{4(t-\tau)}\biggr]\frac{d\tau}{(t-\tau)^{1-\beta}}.$$

► **Domain: $0 \le x < \infty$. Third boundary value problem.**

The following conditions are prescribed:

$$w = 0 \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\bigl[x^{1-2\beta}\partial_x w + k(w_0 - w)\bigr] = 0 \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution for $0 < \beta < 1$:

$$w(x,t) = w_0\frac{2^{2\beta-1}k}{\Gamma(1-\beta)}\int_0^t \varphi(\tau)\exp\biggl[-\frac{x^2}{4(t-\tau)}\biggr]\frac{d\tau}{(t-\tau)^{1-\beta}},$$

where the function $\varphi(t)$ is given as the series

$$\varphi(t) = \sum_{n=0}^\infty \frac{(-\mu t^\beta)^n}{\Gamma(n\beta+1)}, \qquad \mu = \frac{2^{2\beta-1}k\Gamma(\beta)}{\Gamma(1-\beta)},$$

which is convergent for any t.

⊙ *Literature*: W. G. L. Sutton (1943).

3.2.6 Equation of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{1-2\beta}{x}\dfrac{\partial w}{\partial x} + \Phi(x,t)$

This equation is encountered in problems of a diffusion boundary layer with sources/sinks of substance. For $\beta = 0$, $\beta = \frac{1}{2}$, or $\beta = -\frac{1}{2}$, see the equations of Sections 3.2.2, 3.1.2, or 3.2.4, respectively.

► **Domain: $0 \le x < \infty$. First boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution for $0 < \beta < 1$:

$$w(x,t) = \frac{x^\beta}{2t}\int_0^\infty f(\xi)\xi^{1-\beta}\exp\biggl(-\frac{x^2+\xi^2}{4t}\biggr)I_\beta\biggl(\frac{\xi x}{2t}\biggr)\,d\xi \\ + \frac{x^{2\beta}}{2^{2\beta+1}\Gamma(\beta+1)}\int_0^t g(\tau)\exp\biggl[-\frac{x^2}{4(t-\tau)}\biggr]\frac{d\tau}{(t-\tau)^{1+\beta}} \\ + \frac{1}{2}\int_0^t\int_0^\infty \Phi(\xi,\tau)\frac{x^\beta\xi^{1-\beta}}{t-\tau}\exp\biggl[-\frac{x^2+\xi^2}{4(t-\tau)}\biggr]I_\beta\biggl(\frac{\xi x}{2(t-\tau)}\biggr)\,d\xi\,d\tau.$$

► **Domain: $0 \le x < \infty$. Second boundary value problem.**

The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\bigl(x^{1-2\beta}\partial_x w\bigr) = g(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution for $0 < \beta < 1$:

$$\begin{aligned} w(x,t) &= \frac{x^\beta}{2t}\int_0^\infty f(\xi)\xi^{1-\beta}\exp\biggl(-\frac{x^2+\xi^2}{4t}\biggr)I_{-\beta}\biggl(\frac{\xi x}{2t}\biggr)\,d\xi \\ &\quad - \frac{2^{2\beta-1}}{\Gamma(1-\beta)}\int_0^t g(\tau)\exp\biggl[-\frac{x^2}{4(t-\tau)}\biggr]\frac{d\tau}{(t-\tau)^{1-\beta}} \\ &\quad + \frac{1}{2}\int_0^t\int_0^\infty \Phi(\xi,\tau)\frac{x^\beta\xi^{1-\beta}}{t-\tau}\exp\biggl[-\frac{x^2+\xi^2}{4(t-\tau)}\biggr]I_{-\beta}\biggl(\frac{\xi x}{2(t-\tau)}\biggr)\,d\xi\,d\tau. \end{aligned}$$

⊙ *Reference for Section* 3.2.6: W. G. L. Sutton (1943).

3.3 Equations Containing Power Functions and Arbitrary Parameters

3.3.1 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)w$

► **The function f depends on the space coordinate x alone.**

Such equations are encountered in problems of heat and mass transfer with heat release (or volume chemical reaction). The one-dimensional Schrödinger equation can be reduced to this form by the change of variable $t \to -iht$ [the function $-f(x)$ describes the potential against the space coordinate; see Section 3.9.2].

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)w.$

This equation is a special case of equation 3.8.9 with $s(x) = 1$, $p(x) = a$, $q(x) = -bx - c$, and $\Phi(x,t) = 0$. Also, it is a special case of equation 3.8.1.6 with $f(t) = b$ and $g(t) = c$.

1°. Particular solutions (A and μ are arbitrary constants):

$$\begin{aligned} w(x,t) &= A\exp\bigl(btx + \tfrac{1}{3}ab^2t^3 + ct\bigr), \\ w(x,t) &= A(x + abt^2)\exp\bigl(btx + \tfrac{1}{3}ab^2t^3 + ct\bigr), \\ w(x,t) &= A\exp\bigl[x(bt+\mu) + \tfrac{1}{3}ab^2t^3 + ab\mu t^2 + (a\mu^2 + c)t\bigr], \\ w(x,t) &= A\exp(-\mu t)\sqrt{\xi}\,J_{1/3}\biggl(\frac{2}{3b\sqrt{a}}\xi^{3/2}\biggr), \quad \xi = bx + c + \mu, \\ w(x,t) &= A\exp(-\mu t)\sqrt{\xi}\,Y_{1/3}\biggl(\frac{2}{3b\sqrt{a}}\xi^{3/2}\biggr), \quad \xi = bx + c + \mu, \end{aligned}$$

where $J_{1/3}(z)$ and $Y_{1/3}(z)$ are Bessel functions of the first and second kind of order $1/3$.

2°. The transformation

$$w(x,t) = u(z,t)\exp\bigl(btx + \tfrac{1}{3}ab^2t^3 + ct\bigr), \quad z = x + abt^2$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \frac{1}{2\sqrt{\pi at}}\exp\bigl(btx + \tfrac{1}{3}ab^2t^3 + ct\bigr)\int_{-\infty}^{\infty}\exp\biggl[-\frac{(x + abt^2 - \xi)^2}{4at}\biggr]f(\xi)\,d\xi.$$

⊙ *Literature*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998), see also W. Miller, Jr. (1977).

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} - (bx^2 + c)w, \qquad b > 0.$

This equation is a special case of equation 3.8.9 with $s(x) = 1$, $p(x) = a$, $q(x) = bx^2 + c$, and $\Phi(x,t) = 0$. In addition, it is a special case of equation 3.8.1.7 with $f(t) = -c$.

1°. Particular solutions (A and μ are arbitrary constants):

$$w(x,t) = A\exp\biggl[\bigl(\sqrt{ab} - c\bigr)t + \frac{\sqrt{b}}{2\sqrt{a}}x^2\biggr],$$

$$w(x,t) = A\exp\biggl[-\bigl(\sqrt{ab} + c\bigr)t - \frac{\sqrt{b}}{2\sqrt{a}}x^2\biggr],$$

$$w(x,t) = A\exp\biggl(-\mu t - \frac{\sqrt{b}}{2\sqrt{a}}x^2\biggr)\Phi\biggl(\frac{c-\mu}{4\sqrt{ab}} + \frac{1}{4},\ \frac{1}{2};\ \sqrt{\frac{b}{a}}x^2\biggr),$$

$$w(x,t) = A\exp\biggl(-\mu t - \frac{\sqrt{b}}{2\sqrt{a}}x^2\biggr)x\Phi\biggl(\frac{c-\mu}{4\sqrt{ab}} + \frac{3}{4},\ \frac{3}{2};\ \sqrt{\frac{b}{a}}x^2\biggr),$$

where $\Phi(\alpha,\ \beta;\ z) = 1 + \sum\limits_{m=1}^{\infty}\dfrac{\alpha(\alpha+1)\dots(\alpha+m-1)}{\beta(\beta+1)\dots(\beta+m-1)}\dfrac{z^m}{m!}$ is the degenerate hypergeometric function.

2°. In quantum mechanics the following particular solution is encountered:

$$w(x,t) = e^{-[c+\sqrt{ab}\,(2n+1)]t}\psi_n(\xi), \quad \psi_n(\xi) = \frac{1}{\pi^{1/4}\sqrt{2^n n!\,x_0}}e^{-\frac{1}{2}\xi^2}H_n(\xi),$$

$$\xi = \frac{x}{x_0}, \quad x_0 = \Bigl(\frac{a}{b}\Bigr)^{1/4},$$

where $H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}\bigl(e^{-\xi^2}\bigr)$ are the Hermite polynomials, $n = 0,\ 1,\ 2,\ \dots$ These solutions satisfy the normalization condition

$$\int_{-\infty}^{\infty}|\psi_n(\xi)|^2 d\xi = 1.$$

3°. The transformation (A is any number)

$$w(x,t) = u(z,\tau)\exp\left[\frac{\sqrt{b}}{2\sqrt{a}}x^2 + (\sqrt{ab} - c)t\right],$$
$$z = x\exp\bigl(2\sqrt{ab}\,t\bigr), \quad \tau = \frac{\sqrt{a}}{4\sqrt{b}}\exp\bigl(4\sqrt{ab}\,t\bigr) + A,$$

leads to the constant coefficient equation $\partial_\tau u = \partial_{zz} u$, which is considered in Section 3.1.1.

⊙ *Literature*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998), see also W. Miller, Jr. (1977).

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx^2 - c)w, \qquad b > 0.$

This is a special case of equation 3.8.9 with $s(x) = 1$, $p(x) = a$, $q(x) = c - bx^2$, and $\Phi(x,t) = 0$. The transformation

$$w(x,t) = \frac{1}{\sqrt{\left|\cos\bigl(2\sqrt{ab}\,t\bigr)\right|}}\exp\left[\frac{\sqrt{b}}{2\sqrt{a}}x^2\tan\bigl(2\sqrt{ab}\,t\bigr) - ct\right]u(z,\tau),$$
$$z = \frac{x}{\cos\bigl(2\sqrt{ab}\,t\bigr)}, \quad \tau = \frac{\sqrt{a}}{2\sqrt{b}}\tan\bigl(2\sqrt{ab}\,t\bigr),$$

leads to the constant coefficient equation $\partial_\tau u = \partial_{zz} u$, which is considered in Section 3.1.1.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx^2 + cx + k)w.$

The substitution $z = x + c/(2b)$ leads to an equation of the form 3.3.1.2 (for $b < 0$) or 3.3.1.3 (for $b > 0$).

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \bigl(b + cx^{-2}\bigr)w.$

1°. Particular solutions

$$w(x,t) = e^{(b-a\mu^2)t}\sqrt{x}\,\bigl[AJ_\nu(\mu x) + BY_\nu(\mu x)\bigr], \qquad \nu^2 = \frac{1}{4} - \frac{c}{a},$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions; A, B, and μ are arbitrary constants.

2°. Domain: $0 \le x < \infty$. First boundary value problem.
The following conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = 0 \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \frac{e^{bt}}{2at}\int_0^\infty \sqrt{x\xi}\,\exp\left(-\frac{x^2+\xi^2}{4at}\right)I_\nu\left(\frac{\xi x}{2at}\right)f(\xi)\,d\xi, \qquad \nu^2 = \frac{1}{4} - \frac{c}{a},$$

where $-\frac{3}{4}a < c < \frac{1}{4}a$.

3°. The transformation

$$w(x,t) = e^{bt}x^k u(x,\tau), \quad \tau = at,$$

where k is a root of the quadratic equation $ak^2 - ak + c = 0$, leads to an equation of the form 3.2.5:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{2k}{x}\frac{\partial u}{\partial x}.$$

See also W. Miller, Jr. (1977).

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \bigl(-bx^2 + c + kx^{-2}\bigr)w, \qquad b > 0.$

This is a special case of equation 3.8.1.2 with $f(x) = -bx^2 - c + kx^{-2}$. The transformation ($A$ is any number)

$$w(x,t) = u(z,\tau)\exp\left[\frac{\sqrt{b}}{2\sqrt{a}}x^2 + (\sqrt{ab}+c)t\right],$$
$$z = x\exp\bigl(2\sqrt{ab}\,t\bigr), \quad \tau = \frac{1}{4\sqrt{ab}}\exp\bigl(4\sqrt{ab}\,t\bigr) + A,$$

leads to an equation of the form 3.3.1.5:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2} + \frac{k}{a}z^{-2}u.$$

See also W. Miller, Jr. (1977).

7. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx^2 - c + kx^{-2})w, \qquad b > 0.$

This is a special case of equation 3.8.1.2 with $f(x) = bx^2 - c + kx^{-2}$. The transformation

$$w(x,t) = \frac{1}{\sqrt{\left|\cos\bigl(2\sqrt{ab}\,t\bigr)\right|}}\exp\left[\frac{\sqrt{b}}{2\sqrt{a}}x^2\tan\bigl(2\sqrt{ab}\,t\bigr) - ct\right]u(z,\tau),$$
$$z = \frac{x}{\cos\bigl(2\sqrt{ab}\,t\bigr)}, \quad \tau = \frac{\sqrt{a}}{2\sqrt{b}}\tan\bigl(2\sqrt{ab}\,t\bigr),$$

leads to an equation of the form 3.3.1.5:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2} + \frac{k}{a}z^{-2}u.$$

► **The function f depends on time t alone.**

8. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \bigl(bt + c\bigr)w.$

This is a special case of equation 3.8.1.1 with $f(t) = bt + c$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$\begin{aligned}
w(x,t) &= (Ax+B)\exp\bigl(\tfrac{1}{2}bt^2+ct\bigr),\\
w(x,t) &= A(x^2+2at)\exp\bigl(\tfrac{1}{2}bt^2+ct\bigr),\\
w(x,t) &= A\exp\bigl[\mu x+\tfrac{1}{2}bt^2+(c+a\mu^2)t\bigr],\\
w(x,t) &= A\exp\bigl[\tfrac{1}{2}bt^2+(c-a\mu^2)t\bigr]\cos(\mu x),\\
w(x,t) &= A\exp\bigl[\tfrac{1}{2}bt^2+(c-a\mu^2)t\bigr]\sin(\mu x).
\end{aligned}$$

2°. The substitution $w(x,t)=u(x,t)\exp\bigl(\frac{1}{2}bt^2+ct\bigr)$ leads to a constant coefficient equation, $\partial_t u = a\partial_{xx}u$, which is considered in Section 3.1.1.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^k w.$

This is a special case of equation 3.8.1.1 with $f(t)=bt^k$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$\begin{aligned}
w(x,t) &= (Ax+B)\exp\Bigl(\frac{b}{k+1}t^{k+1}\Bigr),\\
w(x,t) &= A(x^2+2at)\exp\Bigl(\frac{b}{k+1}t^{k+1}\Bigr),\\
w(x,t) &= A\exp\Bigl(\mu x+a\mu^2 t+\frac{b}{k+1}t^{k+1}\Bigr),\\
w(x,t) &= A\exp\Bigl(\frac{b}{k+1}t^{k+1}-a\mu^2 t\Bigr)\cos(\mu x),\\
w(x,t) &= A\exp\Bigl(\frac{b}{k+1}t^{k+1}-a\mu^2 t\Bigr)\sin(\mu x).
\end{aligned}$$

2°. The substitution $w(x,t)=u(x,t)\exp\bigl(\frac{b}{k+1}t^{k+1}\bigr)$ leads to a constant coefficient equation, $\partial_t u = a\,\partial_{xx}u$, which is considered in Section 3.1.1.

► **The function f depends on both x and t.**

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx+ct+d)w.$

This is a special case of equation 3.8.1.6 with $f(t)=b$ and $g(t)=ct+d$.

1°. Particular solutions (A and μ are arbitrary constants):

$$\begin{aligned}
w(x,t) &= A\exp\bigl(btx+\tfrac{1}{3}ab^2t^3+\tfrac{1}{2}ct^2+dt\bigr),\\
w(x,t) &= A(x+abt^2)\exp\bigl(btx+\tfrac{1}{3}ab^2t^3+\tfrac{1}{2}ct^2+dt\bigr),\\
w(x,t) &= A\exp(\tfrac{1}{2}ct^2-\mu t)\sqrt{\xi}\,J_{1/3}\biggl(\frac{2}{3b\sqrt{a}}\xi^{3/2}\biggr), \quad \xi=bx+d+\mu,\\
w(x,t) &= A\exp(\tfrac{1}{2}ct^2-\mu t)\sqrt{\xi}\,Y_{1/3}\biggl(\frac{2}{3b\sqrt{a}}\xi^{3/2}\biggr), \quad \xi=bx+d+\mu,
\end{aligned}$$

where $J_{1/3}(\xi)$ and $Y_{1/3}(\xi)$ are Bessel functions of the first and second kind.

2°. The transformation

$$w(x,t) = u(z,t)\exp\bigl(btx + \tfrac{1}{3}ab^2t^3 + \tfrac{1}{2}ct^2 + dt\bigr), \quad z = x + abt^2$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(bt + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = bt + c$.

1°. Particular solutions (A and μ are arbitrary constants):

$$\begin{aligned} w(x,t) &= A\exp\bigl[x\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + a\bigl(\tfrac{1}{20}b^2t^5 + \tfrac{1}{4}bct^4 + \tfrac{1}{3}c^2t^3\bigr)\bigr], \\ w(x,t) &= A\bigl[x + a\bigl(\tfrac{1}{3}bt^3 + ct^2\bigr)\bigr]\exp\bigl[x\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + a\phi(t)\bigr], \\ w(x,t) &= A\exp\bigl[x\bigl(\tfrac{1}{2}bt^2 + ct + \mu\bigr) + a\mu\bigl(\tfrac{1}{3}bt^3 + ct^2 + \mu t\bigr) + a\phi(t)\bigr], \end{aligned}$$

where $\phi(t) = \frac{1}{20}b^2t^5 + \frac{1}{4}bct^4 + \frac{1}{3}c^2t^3$.

2°. The transformation

$$w(x,t) = u(z,t)\exp\bigl[x\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + a\bigl(\tfrac{1}{20}b^2t^5 + \tfrac{1}{4}bct^4 + \tfrac{1}{3}c^2t^3\bigr)\bigr], \quad z = x + a\bigl(\tfrac{1}{3}bt^3 + ct^2\bigr)$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bxt + cx + dt + e)w.$

This is a special case of equation 3.8.1.6 with $f(t) = bt + c$ and $g(t) = dt + e$.

1°. Particular solution:

$$w(x,t) = \exp\bigl[x\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + a\bigl(\tfrac{1}{20}b^2t^5 + \tfrac{1}{4}bct^4 + \tfrac{1}{3}c^2t^3\bigr) + \tfrac{1}{2}dt^2 + et\bigr].$$

2°. The transformation

$$\begin{aligned} w(x,t) &= u(z,t)\exp\bigl[x\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + a\bigl(\tfrac{1}{20}b^2t^5 + \tfrac{1}{4}bct^4 + \tfrac{1}{3}c^2t^3\bigr) + \tfrac{1}{2}dt^2 + et\bigr], \\ z &= x + a(bt^2 + 2ct) \end{aligned}$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (-bx^2 + ct + d)w.$

This is a special case of equation 3.8.1.7 with $f(t) = ct + d$.

1°. Particular solutions (A is an arbitrary constant):

$$\begin{aligned} w(x,t) &= A\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}\,x^2 + \frac{1}{2}ct^2 + \bigl(\sqrt{ab} + d\bigr)t\right], \\ w(x,t) &= Ax\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}\,x^2 + \frac{1}{2}ct^2 + \bigl(3\sqrt{ab} + d\bigr)t\right]. \end{aligned}$$

2°. The transformation (A is any number)

$$w(x,t) = u(z,\tau)\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}\,x^2 + \frac{1}{2}ct^2 + \bigl(\sqrt{ab}+d\bigr)t\right],$$
$$z = x\exp\bigl(2\sqrt{ab}\,t\bigr), \qquad \tau = \frac{1}{4}\sqrt{\frac{b}{a}}\,\exp\bigl(4\sqrt{ab}\,t\bigr) + A$$

leads to the constant coefficient equation $\partial_\tau u = \partial_{zz} u$, which is considered in Section 3.1.1.

14. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(-bx + ct + d)w.$

This is a special case of equation 3.8.1.8 with $f(t) = ct + d$.

15. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bxt^k w.$

This is a special case of equation 3.8.1.3 with $f(t) = bt^k$.

1°. Particular solutions (A and μ are arbitrary constants):

$$w(x,t) = A\exp\left[\frac{b}{k+1}xt^{k+1} + \frac{ab^2}{(k+1)^2(2k+3)}t^{2k+3}\right],$$
$$w(x,t) = A\left[x + \frac{2ab}{(k+1)(k+2)}t^{k+2}\right]\exp\left[\frac{b}{k+1}xt^{k+1} + \frac{ab^2}{(k+1)^2(2k+3)}t^{2k+3}\right],$$
$$w(x,t) = A\exp\left[\frac{b}{k+1}xt^{k+1} + \mu x + \frac{ab^2}{(k+1)^2(2k+3)}t^{2k+3} + \frac{2ab\mu}{(k+1)(k+2)}t^{k+2} + a\mu^2 t\right].$$

2°. The transformation

$$w(x,t) = u(z,t)\exp\left[\frac{b}{k+1}xt^{k+1} + \frac{ab^2}{(k+1)^2(2k+3)}t^{2k+3}\right], \quad z = x + \frac{2ab}{(k+1)(k+2)}t^{k+2}$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz} u$, which is considered in Section 3.1.1.

16. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx^2t^n + cxt^m + dt^k)w.$

This is a special case of equation 3.8.7.5 with $n(t) = a$, $f(t) = g(t) = 0$, $h(t) = bt^n$, $s(t) = ct^m$, and $p(t) = dt^k$.

3.3.2 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x}$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bt + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = bt + c$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = 2Ax + A(bt^2 + 2ct) + B,$$
$$w(x,t) = A\bigl(x + \tfrac{1}{2}bt^2 + ct\bigr)^2 + 2aAt + B,$$
$$w(x,t) = A\exp\bigl[\mu x + \tfrac{1}{2}\mu bt^2 + (a\mu^2 + \mu c)t\bigr] + B.$$

2°. The substitution $z = x + \frac{1}{2}bt^2 + ct$ leads to a constant coefficient equation, $\partial_t w = a\partial_{zz} w$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x}.$

This equation is a special case of equation 3.8.2.2 with $f(x) = bx$ and a special case of equation 3.8.2.3 with $f(t) = b$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x) = A\int \exp\Bigl(-\frac{b}{2a}x^2\Bigr)\,dx + B,$$
$$w(x,t) = Axe^{bt} + B,$$
$$w(x,t) = Abx^2e^{2bt} + Aae^{2bt} + B,$$
$$w(x,t) = A\exp\bigl(2b\mu xe^{bt} + 2ab\mu^2e^{2bt}\bigr) + B.$$

2°. On passing from t, x to the new variables (A and B are any numbers)

$$\tau = \frac{A^2}{2b}e^{2bt} + B, \quad z = Axe^{bt},$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = a\partial_{zz}w$, which is considered in Section 3.1.1.

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \Bigl[\frac{2\pi a}{b}\bigl(e^{2bt} - 1\bigr)\Bigr]^{-1/2}\int_{-\infty}^{\infty}\exp\biggl[-\frac{b\bigl(xe^{bt} - \xi\bigr)^2}{2a\bigl(e^{2bt} - 1\bigr)}\biggr]f(\xi)\,d\xi.$$

⊙ *Literature*: W. Miller, Jr. (1977), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bt^2 + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = bt^2 + c$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = A\bigl(x + \tfrac{1}{3}bt^3 + ct\bigr) + B,$$
$$w(x,t) = A\bigl(x + \tfrac{1}{3}bt^3 + ct\bigr)^2 + 2aAt + B,$$
$$w(x,t) = A\exp\bigl[\mu x + \tfrac{1}{3}\mu bt^3 + \mu(a\mu + c)t\bigr] + B.$$

2°. On passing from t, x to the new variables t, $z = x + \frac{1}{3}bt^3 + ct$, we obtain a constant coefficient equation, $\partial_t w = a\partial_{zz}w$, which is considered in Section 3.1.1.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + x(bt + c)\frac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = bt + c$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = Ax\exp\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + B,$$
$$w(x,t) = Ax^2\exp\bigl(bt^2 + 2ct\bigr) + 2Aa\int \exp\bigl(bt^2 + 2ct\bigr)\,dt + B,$$
$$w(x,t) = A\exp\Bigl[\mu x\exp\bigl(\tfrac{1}{2}bt^2 + ct\bigr) + a\mu^2\int \exp\bigl(bt^2 + 2ct\bigr)\,dt\Bigr] + B.$$

2°. On passing from t, x to the new variables (A is any number)

$$\tau = \int \exp\bigl(bt^2 + 2ct\bigr)\,dt + A, \quad z = x\exp\bigl(\tfrac{1}{2}bt^2 + ct\bigr),$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = a\partial_{zz}w,$ which is considered in Section 3.1.1.

5. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + (ax + bt + c)\frac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.7 with $f(t) = a$ and $g(t) = bt + c$. See also equation 3.8.2.4.

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{x}{t}\frac{\partial w}{\partial x}.$

Ilkovič's equation. It describes heat transfer to the surface of a growing drop that flows out of a thin capillary into a fluid solution (the mass rate of flow of the fluid moving in the capillary is assumed constant). This equation is a special case of equation 3.8.2.3 with $f(t) = b/t$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = Axt^b + B,$$
$$w(x,t) = A(2b+1)x^2t^{2b} + 2Aat^{2b+1} + B,$$
$$w(x,t) = A\exp\left(\mu xt^b + \frac{a\mu^2}{2b+1}t^{2b+1}\right) + B.$$

2°. On passing from t, x to the new variables

$$\tau = \frac{1}{2b+1}t^{2b+1}, \quad z = xt^b,$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = a\partial_{zz}w,$ which is considered in Section 3.1.1.

3°. The solution of the original equation in the important special case where the drop surface has a time-invariant temperature w_s and the heat exchange occurs with an infinite medium having an initial temperature w_0, namely,

$$\begin{aligned} w &= w_0 \quad \text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ w &= w_s \quad \text{at} \quad x = 0 \quad &&\text{(boundary condition)},\\ w &\to w_0 \quad \text{at} \quad x \to \infty \quad &&\text{(boundary condition)}, \end{aligned}$$

is expressed in terms of the error function as follows:

$$\frac{w - w_s}{w_0 - w_s} = \operatorname{erf}\left(\frac{\sqrt{2b+1}}{2\sqrt{a}}\frac{x}{\sqrt{t}}\right), \quad \operatorname{erf}\xi = \frac{2}{\sqrt{\pi}}\int_0^\xi \exp\bigl(-\zeta^2\bigr)\,d\zeta.$$

⊙ *Literature*: Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bt^k x + ct^m)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.7 with $f(t) = bt^k$ and $g(t) = ct^m$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left(cx + \dfrac{b}{x}\right)\dfrac{\partial w}{\partial x}.$

On passing from t, x to the new variables z, τ by the formulas

$$z = xe^{ct}, \quad \tau = \frac{a}{2c}e^{2ct} + \text{const},$$

we obtain a simpler equation of the form 3.2.5:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2} + \frac{\mu}{z}\frac{\partial w}{\partial z}, \qquad \mu = \frac{b}{a}.$$

For $\mu = 1$ and $\mu = 2$, see also equations from Sections 3.2.1 and 3.2.3.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left(ct^n x + \dfrac{b}{x}\right)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = ct^n$.

3.3.3 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w + h(x,t)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (cx + d)w.$

This is a special case of equation 3.8.7.4 with $n(t) = a$, $f(t) = 0$, $g(t) = b$, $h(t) = c$, and $s(t) = d$.

1°. Particular solutions (A and μ are arbitrary constants):

$$w(x,t) = A\exp\left[ctx - \frac{b}{2a}x + \frac{1}{3}ac^2t^3 + \left(d - \frac{b^2}{4a}\right)t\right],$$
$$w(x,t) = A(x + act^2)\exp\left[ctx - \frac{b}{2a}x + \frac{1}{3}ac^2t^3 + \left(d - \frac{b^2}{4a}\right)t\right],$$
$$w(x,t) = A\exp\left[x\left(ct + \mu - \frac{b}{2a}\right) + \frac{1}{3}ac^2t^3 + ac\mu t^2 + \left(a\mu^2 + d - \frac{b^2}{4a}\right)t\right],$$
$$w(x,t) = A\exp\left(-\mu t - \frac{b}{2a}x\right)\sqrt{\xi}\,J_{1/3}\left(\frac{2}{3c\sqrt{a}}\xi^{3/2}\right), \quad \xi = cx + \mu + d - \frac{b^2}{4a},$$
$$w(x,t) = A\exp\left(-\mu t - \frac{b}{2a}x\right)\sqrt{\xi}\,Y_{1/3}\left(\frac{2}{3c\sqrt{a}}\xi^{3/2}\right), \quad \xi = cx + \mu + d - \frac{b^2}{4a},$$

where $J_{1/3}(\xi)$ and $Y_{1/3}(\xi)$ are the Bessel functions of the first and second kind of order $1/3$, respectively.

2°. The transformation

$$w(x,t) = u(z,t)\exp\left[ctx - \frac{b}{2a}x + \frac{1}{3}ac^2t^3 + \left(d - \frac{b^2}{4a}\right)t\right], \quad z = x + act^2$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bx\frac{\partial w}{\partial x} + (cx + d)w.$

This is a special case of equation 3.8.7.4 with $n(t) = a$, $f(t) = b$, $g(t) = 0$, $h(t) = c$, and $s(t) = d$.

1°. Particular solutions (A and μ are arbitrary constants):

$$w(x,t) = A\exp\left[-\frac{c}{b}x + \left(d + \frac{ac^2}{b^2}\right)t\right],$$
$$w(x,t) = A\left(x - \frac{2ac}{b^2}\right)\exp\left[-\frac{c}{b}x + \left(b + d + \frac{ac^2}{b^2}\right)t\right],$$
$$w(x,t) = A\exp\left[\frac{a\mu^2}{2b}e^{2bt} + \mu e^{bt}\left(x - \frac{2ac}{b^2}\right) - \frac{c}{b}x + \left(d + \frac{ac^2}{b^2}\right)t\right].$$

See 3.3.4.7 for more complicated solutions.

2°. The transformation

$$w(x,t) = u(z,\tau)\exp\left[-\frac{c}{b}x + \left(d + \frac{ac^2}{b^2}\right)t\right], \quad \tau = \frac{a}{2b}e^{2bt}, \quad z = e^{bt}\left(x - \frac{2ac}{b^2}\right),$$

leads to a constant coefficient equation, $\partial_\tau u = \partial_{zz}u$, which is considered in Section 3.1.1.

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx + c)\frac{\partial w}{\partial x} + (dx + e)w.$

For $b = 0$, see equation 3.3.3.1. For $b \neq 0$, the substitution $z = x + c/b$ leads to an equation of the form 3.3.3.2:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + bz\frac{\partial w}{\partial z} + (dz + k)w, \qquad k = e - \frac{cd}{b}.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + 2(ax+b)\dfrac{\partial w}{\partial x} + (a^2x^2 + 2abx + c)w.$

This equation is a special case of equation 3.8.6.5 and a special case of equation 3.8.7.5. The substitution $w(x,t) = u(x,t)\exp\bigl(-\frac{1}{2}ax^2 - bx\bigr)$ leads to a constant coefficient equation of the form 3.1.3 with $\Phi \equiv 0$, namely, $\partial_t u = \partial_{xx} u + (c - a - b^2)u$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + (ax+b)\dfrac{\partial w}{\partial x} + (cx^2 + dx + e)w.$

This equation is a special case of equation 3.8.6.5 and a special case of equation 3.8.7.5.

1°. The substitution

$$w(x,t) = u(x,t)\exp\bigl(\tfrac{1}{2}Ax^2\bigr),$$

where A is a root of the quadratic equation $A^2 + aA + c = 0$, yields an equation of the form 3.8.7.3,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \bigl[(2A+a)x + b\bigr]\frac{\partial u}{\partial x} + \bigl[(Ab+d)x + A + e\bigr]u,$$

which is reduced to a constant coefficient equation.

2°. The substitution

$$w(x,t) = u(x,t)\exp\bigl(\tfrac{1}{2}Ax^2 + Bx + Ct\bigr)$$

leads to an equation of the analogous form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \bigl[(2A+a)x + 2B + b\bigr]\frac{\partial u}{\partial x} + \bigl[(A^2 + Aa + c)x^2 + (2AB + Ab + Ba + d)x + B^2 + Bb + A - C + e\bigr]u.$$

By appropriately choosing the coefficients A, B, and C, one can simplify the original equation in various ways.

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + (ax + bt + c)\dfrac{\partial w}{\partial x} + (sx^2 + ptx + qt^2 + kx + lt + m)w.$

This is a special case of equation 3.8.7.5.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bt^k x + ct^m)\dfrac{\partial w}{\partial x} + st^n w.$

This is a special case of equation 3.8.3.6 with $f(t) = bt^k$, $g(t) = ct^m$, and $h(t) = st^n$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{b}{x}\dfrac{\partial w}{\partial x} + \Bigl(c + \dfrac{k}{x^2}\Bigr)w.$

1°. The transformation

$$w(x,t) = e^{ct}x^{\lambda}u(x,\tau), \quad \tau = at,$$

where λ is a root of the quadratic equation $a\lambda^2 + (b-a)\lambda + k = 0$, leads to an equation of the form 3.2.5:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \Bigl(2\lambda + \frac{b}{a}\Bigr)\frac{1}{x}\frac{\partial u}{\partial x}.$$

2°. If $w(x,t)$ is a solution of the original equation, then the functions

$$w_1 = Ae^{c(1-a^2)\tau} w(\pm ax, a^2\tau), \quad \tau = t + B,$$
$$w_2 = A\tau^{\lambda-1} \exp\biggl(-\frac{x^2}{4a\tau} + c\tau + \frac{c}{a^2\tau}\biggr) w\biggl(\pm\frac{x}{a\tau}, -\frac{1}{a^2\tau}\biggr), \quad \lambda = \frac{1}{2} - \frac{b}{2a},$$

where A and B are arbitrary constants, are also solutions of this equation.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{b}{x}\dfrac{\partial w}{\partial x} + \Bigl(c + \dfrac{k}{x^2}\Bigr)w + \Phi(x,t).$

The transformation

$$w(x,t) = e^{ct}x^\lambda u(x,\tau), \quad \tau = at,$$

where λ is a root of the quadratic equation $a\lambda^2 + (b-a)\lambda + k = 0$, leads to an equation of the form 3.2.6:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \biggl(2\lambda + \frac{b}{a}\biggr)\frac{1}{x}\frac{\partial u}{\partial x} + \Psi(x,\tau), \qquad \Psi(x,\tau) = \frac{1}{a}e^{-ct}x^{-\lambda}\Phi(x,t).$$

3.3.4 Equations of the Form $\dfrac{\partial w}{\partial t} = (ax+b)\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w$

1. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 3.8.6.1 with $f(x) = ax$. See also equation 3.3.6.6 with $n = 0$.

1°. Particular solutions (A, B, C, and μ are arbitrary constants):

$$w(x) = Ax + B,$$
$$w(x,t) = 2Aatx + Ax^2 + B,$$
$$w(x,t) = Aa^2t^2x + Aatx^2 + \tfrac{1}{6}Ax^3 + B,$$
$$w(x,t) = 2Aa^3t^3x + 3Aa^2t^2x^2 + Aatx^3 + \tfrac{1}{12}Ax^4 + B,$$
$$w(x,t) = x^n + \sum_{k=1}^{n-1} \frac{[n(n-1)\dots(n-k)]^2}{n(n-k)k!}(at)^k x^{n-k},$$
$$w(x,t) = A\exp\biggl(-\frac{x}{at+B}\biggr) + C,$$
$$w(x,t) = \frac{Ax}{(at+B)^2}\exp\biggl(-\frac{x}{at+B}\biggr) + C,$$
$$w(x,t) = Aat + A(x\ln x - x) + B,$$
$$w(x,t) = Aa^2t^2 + 2Aat(x\ln x - x) + A(x^2\ln x - \tfrac{5}{2}x^2) + B,$$
$$w(x,t) = e^{\mu t}\sqrt{x}\,\biggl[AJ_1\biggl(\frac{2}{\sqrt{a}}\sqrt{-\mu x}\biggr) + BY_1\biggl(\frac{2}{\sqrt{a}}\sqrt{-\mu x}\biggr)\biggr] \quad \text{for } \mu < 0,$$
$$w(x,t) = e^{\mu t}\sqrt{x}\,\biggl[AI_1\biggl(\frac{2}{\sqrt{a}}\sqrt{\mu x}\biggr) + BK_1\biggl(\frac{2}{\sqrt{a}}\sqrt{\mu x}\biggr)\biggr] \quad \text{for } \mu > 0,$$

where $J_1(z)$ and $Y_1(z)$ are Bessel functions, and $I_1(z)$ and $K_1(z)$ are modified Bessel functions.

2°. A solution containing an arbitrary function of the space variable:

$$w(x,t) = f(x) + \sum_{n=1}^{\infty} \frac{(at)^n}{n!} \boldsymbol{L}^n[f(x)], \qquad \boldsymbol{L} \equiv x\frac{d^2}{dx^2},$$

where $f(x)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = f(x)$. The sum is finite if $f(x)$ is a polynomial.

3°. A solution containing an arbitrary function of time:

$$w(x,t) = A + xg(t) + \sum_{n=2}^{\infty} \frac{1}{n[(n-1)!]^2 a^{n-1}} x^n g_t^{(n-1)}(t),$$

where $g(t)$ is any infinitely differentiable function and A is an arbitrary number. This solution possesses the properties

$$w(0,t) = A, \qquad \partial_x w(0,t) = g(t).$$

4°. Suppose $w = w(x,t)$ is a solution of the original equation. Then the functions

$$w_1 = Aw(\lambda x,\ \lambda t + C),$$
$$w_2 = A\exp\left[-\frac{\beta x}{a(\delta+\beta t)}\right] w\left(\frac{x}{(\delta+\beta t)^2},\ \frac{\gamma+\lambda t}{\delta+\beta t}\right), \qquad \lambda\delta - \beta\gamma = 1,$$

where A, C, β, δ, and λ are arbitrary constants, are also solutions of the equation.

2. $\displaystyle \frac{\partial w}{\partial t} = ax\frac{\partial^2 w}{\partial x^2} + (-bx + c)w.$

The transformation

$$w(x,t) = u(z,\tau)\exp\left(\sqrt{\frac{b}{a}}\,x + ct\right), \quad z = x\exp\bigl(2\sqrt{ab}\,t\bigr), \quad \tau = \frac{1}{2}\sqrt{\frac{a}{b}}\exp\bigl(2\sqrt{ab}\,t\bigr)$$

leads to a simpler equation of the form 3.3.4.1:

$$\frac{\partial u}{\partial \tau} = z\frac{\partial^2 u}{\partial z^2}.$$

3. $\displaystyle \frac{\partial w}{\partial t} = ax\frac{\partial^2 w}{\partial x^2} + bxt^n w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = 0$, $h(t) = bt^n$, and $s(t) = 0$.

4. $\displaystyle \frac{\partial w}{\partial t} = ax\frac{\partial^2 w}{\partial x^2} + (bt^n x + ct^m)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = 0$, $h(t) = bt^n$, and $s(t) = ct^m$.

5. $\displaystyle \frac{\partial w}{\partial t} = a\Big[(x+b)\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x}\Big].$

This equation describes heat transfer in a quiescent medium (solid body) in the case of thermal diffusivity as a linear function of the space coordinate.

1°. The original equation can be rewritten in a form more suitable for applications,

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left[(x+b)\frac{\partial w}{\partial x}\right].$$

2°. The substitution $x = \frac{1}{4}z^2 - b$ leads to the equation

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial z^2} + \frac{1}{z}\frac{\partial w}{\partial z}\right),$$

which is considered in Section 3.2.1.

6. $\displaystyle \frac{\partial w}{\partial t} = ax\frac{\partial^2 w}{\partial x^2} + bx\frac{\partial w}{\partial x} + (cx+d)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = b$, $h(t) = c$, and $s(t) = d$.

7. $\displaystyle \frac{\partial w}{\partial t} = (a_2x+b_2)\frac{\partial^2 w}{\partial x^2} + (a_1x+b_1)\frac{\partial w}{\partial x} + (a_0x+b_0)w.$

This is a special case of equation 3.8.6.5 with $f(x) = a_2x + b_2$, $g(x) = a_1x + b_1$, $h(x) = a_0x + b_0$, and $\Phi \equiv 0$.

Particular solutions of the original equation are presented in Table 3.2, where the function

$$\mathcal{J}(a,b;x) = C_1\Phi(a,b;x) + C_2\Psi(a,b;x), \qquad C_1,\ C_2 \text{ are any numbers},$$

is an arbitrary solution of the degenerate hypergeometric equation

$$xy''_{xx} + (b-x)y'_x - ay = 0,$$

and the function

$$Z_\nu(x) = C_1J_\nu(x) + C_2Y_\nu(x), \qquad C_1,\ C_2 \text{ are any numbers},$$

is an arbitrary solution of the Bessel equation

$$x^2y''_{xx} + xy'_x + (x^2 - \nu^2)y = 0.$$

For the degenerate hypergeometric functions $\Phi(a,b;x)$ and $\Psi(a,b;x)$, see Section 30.9 and books by Bateman and Erdélyi (1953, Vol. 1) and Abramowitz and Stegun (1964).

For the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$, see Section 30.6 and Bateman and Erdélyi (1953, Vol. 2), Abramowitz and Stegun (1964), Nikiforov and Uvarov (1988), and Temme (1996).

Remark 3.4. For $a_2 = 0$ the original equation is a special case of equation 3.8.7.4 and can be reduced to the constant coefficient heat equation that is considered in Section 3.1.1. In this case, a number of solutions are not displayed in Table 3.2.

TABLE 3.2
Particular solutions of equation 3.3.4.7 for different values
of the determining parameters (μ is an arbitrary number)

Particular solution: $w(x,t)=\exp(hx-\mu t)F(\xi)$, where $\xi=(x+\gamma)/p$					
Constraints	h	p	γ	F	Parameters
$a_2\neq 0$, $D\neq 0$	$\dfrac{D-a_1}{2a_2}$	$-\dfrac{a_2}{A(h)}$	$\dfrac{b_2}{a_2}$	$\mathcal{J}(a,b;\xi)$	$a=B(h)/A(h)$, $b=(a_2b_1-a_1b_2)a_2^{-2}$
$a_2=0$, $a_1\neq 0$	$-\dfrac{a_0}{a_1}$	1	$\dfrac{2b_2h+b_1}{a_1}$	$\mathcal{J}\bigl(a,\frac{1}{2};k\xi^2\bigr)$	$a=B(h)/(2a_1)$, $k=-a_1/(2b_2)$
$a_2\neq 0$, $a_1^2=4a_0a_2$	$-\dfrac{a_1}{2a_2}$	a_2	$\dfrac{b_2}{a_2}$	$\xi^\alpha Z_{2\alpha}\bigl(\beta\sqrt{\xi}\bigr)$	$\alpha=\dfrac{1}{2}-\dfrac{2b_2h+b_1}{2a_2}$, $\beta=2\sqrt{B(h)}$
$a_2=a_1=0$, $a_0\neq 0$	$-\dfrac{b_1}{2b_2}$	1	$\dfrac{4(b_0+\mu)b_2-b_1^2}{4a_0b_2}$	$\xi^{1/2}Z_{1/3}\bigl(k\xi^{3/2}\bigr)$	$k=\dfrac{2}{3}\Bigl(\dfrac{a_0}{b_2}\Bigr)^{1/2}$
Notation: $D^2=a_1^2-4a_0a_2$, $A(h)=2a_2h+a_1$, $B(h)=b_2h^2+b_1h+b_0+\mu$					

3.3.5 Equations of the Form $\dfrac{\partial w}{\partial t}=(ax^2+bx+c)\dfrac{\partial^2 w}{\partial x^2}+f(x,t)\dfrac{\partial w}{\partial x}+g(x,t)w$

1. $\dfrac{\partial w}{\partial t}=ax^2\dfrac{\partial^2 w}{\partial x^2}+bx\dfrac{\partial w}{\partial x}+cw.$

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t)=\bigl(A\ln|x|+B\bigr)|x|^n\exp\bigl[(c-an^2)t\bigr],$$
$$w(x,t)=A\bigl(2at+\ln^2|x|\bigr)|x|^n\exp\bigl[(c-an^2)t\bigr],$$
$$w(x,t)=A|x|^\mu\exp\bigl[(c+a\mu^2-2an\mu)t\bigr],$$

where $n=\frac{1}{2}(a-b)/a$.

2°. The transformation

$$w(x,t)=|x|^n\exp\bigl[(c-an^2)t\bigr]u(z,t),\quad z=\ln|x|,\qquad n=\frac{a-b}{2a},$$

leads to a constant coefficient equation, $\partial_t u=a\partial_{zz}u$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t}=ax^2\dfrac{\partial^2 w}{\partial x^2}+(bt^n+c)w.$

This is a special case of equation 3.8.4.2 with $f(t)=bt^n+c$.

The transformation

$$w(x,t)=u(z,t)\exp\left(\frac{b}{n+1}t^{n+1}+ct\right),\quad z=\ln|x|$$

leads to a constant coefficient equation of the form 3.1.4:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial z^2} - a\frac{\partial u}{\partial z}.$$

3. $\boldsymbol{\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x} + (cx^k + s)w.}$

This is a special case of equation 3.8.6.5 with $f(x) = ax^2$, $g(x) = bx$, $h(x) = cx^k + s$, and $\Phi \equiv 0$. For $c = 0$, see equation 3.3.5.1.

Particular solutions for $c \neq 0$:

$$w(x,t) = Ae^{-\mu t}x^{\frac{a-b}{2a}}J_\nu\left(\frac{2}{k}\sqrt{\frac{c}{a}}\,x^{\frac{k}{2}}\right), \quad \nu = \frac{1}{ak}\sqrt{(a-b)^2 - 4a(s+\mu)},$$
$$w(x,t) = Ae^{-\mu t}x^{\frac{a-b}{2a}}Y_\nu\left(\frac{2}{k}\sqrt{\frac{c}{a}}\,x^{\frac{k}{2}}\right), \quad \nu = \frac{1}{ak}\sqrt{(a-b)^2 - 4a(s+\mu)},$$

where A and μ are arbitrary constants, and $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

4. $\boldsymbol{\dfrac{\partial w}{\partial t} = a_2x^2\dfrac{\partial^2 w}{\partial x^2} + (a_1x^2 + b_1x)\dfrac{\partial w}{\partial x} + (a_0x^2 + b_0x + c_0)w.}$

This is a special case of equation 3.8.6.5 with $f(x) = a_2x^2$, $g(x) = a_1x^2 + b_1x$, $h(x) = a_0x^2 + b_0x + c_0$, and $\Phi \equiv 0$.

1°. Particular solutions for $a_1^2 \neq 4a_0a_2$:

$$\begin{aligned} w(x,t) &= A\exp(-\nu t + \mu x)x^k\Phi\Big(\alpha,\, 2k + \frac{b_1}{a_2};\, -\gamma x\Big), \\ w(x,t) &= A\exp(-\nu t + \mu x)x^k\Psi\Big(\alpha,\, 2k + \frac{b_1}{a_2};\, -\gamma x\Big), \end{aligned} \tag{1}$$

where A and ν are arbitrary constants,

$$\mu = \frac{\sqrt{a_1^2 - 4a_0a_2} - a_1}{2a_2}, \quad \alpha = \frac{(b_1 + 2a_2k)\mu + b_0 + a_1k}{2a_2\mu + a_1}, \quad \gamma = 2\mu + \frac{a_1}{a_2},$$

$k = k(\nu)$ is a root of the quadratic equation $a_2k^2 + (b_1 - a_2)k + c_0 + \nu = 0$, and $\Phi(\alpha, \beta; z)$ and $\Psi(\alpha, \beta; z)$ are the degenerate hypergeometric functions. [For the degenerate hypergeometric functions, see Section 30.9 and the books by Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 1)].

2°. Particular solutions for $a_1^2 = 4a_0a_2$:

$$\begin{aligned} w(x,t) &= A\exp\Big(-\nu t - \frac{a_1}{2a_2}x\Big)x^k\xi^m J_{2m}\big(2\sqrt{p\xi}\,\big), \quad \xi = \frac{x}{a_2}, \\ w(x,t) &= A\exp\Big(-\nu t - \frac{a_1}{2a_2}x\Big)x^k\xi^m Y_{2m}\big(2\sqrt{p\xi}\,\big), \quad \xi = \frac{x}{a_2}, \end{aligned} \tag{2}$$

where A and ν are arbitrary constants,

$$m = \frac{1}{2} - k - \frac{b_1}{2a_2}, \quad p = -\frac{a_1}{2a_2}(b_1 + 2a_2k) + b_0 + a_1k = 0,$$

$k = k(\nu)$ is a root of the quadratic equation $a_2k^2 + (b_1 - a_2)k + c_0 + \nu = 0$, and $J_m(z)$ and $Y_m(z)$ are Bessel functions. [For the Bessel functions, see Section 30.6 and the books by Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 2)].

Remark 3.5. In solutions (1) and (2), the parameter k can be regarded as arbitrary, and then $\nu = -a_2k^2 - (b_1 - a_2)k - c_0$.

5. $\displaystyle \frac{\partial w}{\partial t} = a_2x^2\frac{\partial^2 w}{\partial x^2} + (a_1x^{k+1} + b_1x)\frac{\partial w}{\partial x} + (a_0x^{2k} + b_0x^k + c_0)w.$

This is a special case of equation 3.8.6.5 with $f(x) = a_2x^2$, $g(x) = a_1x^{k+1} + b_1x$, $h(x) = a_0x^{2k} + b_0x^k + c_0$, and $\Phi \equiv 0$.

The substitution $\xi = x^k$ leads to an equation of the form 3.3.5.4:

$$\frac{\partial w}{\partial t} = a_2k^2\xi^2\frac{\partial^2 w}{\partial \xi^2} + k\bigl(a_1\xi^2 + \beta\xi\bigr)\frac{\partial w}{\partial \xi} + (a_0\xi^2 + b_0\xi + c_0)w,$$

where $\beta = b_1 + a_2(k-1)$.

6. $\displaystyle \frac{\partial w}{\partial t} = (ax^2 + b)\frac{\partial^2 w}{\partial x^2} + ax\frac{\partial w}{\partial x} + cw.$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2+b}}$ leads to the constant coefficient equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + cw,$$

which is considered in Section 3.1.3.

3.3.6 Equations of the Form $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x,t)\dfrac{\partial w}{\partial x} + h(x,t)w$

1. $\displaystyle \frac{\partial w}{\partial t} = ax^3\frac{\partial^2 w}{\partial x^2} + bxt^k\frac{\partial w}{\partial x} + ct^m w.$

This is a special case of equation 3.8.8.7 with $n = 3$, $f(t) = a$, $g(t) = bt^k$, and $h(t) = ct^m$.

2. $\displaystyle \frac{\partial w}{\partial t} = ax^4\frac{\partial^2 w}{\partial x^2} + bw.$

This is a special case of equation 3.3.7.6 with $n = m = 0$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = e^{bt}(Ax + B),$$
$$w(x,t) = e^{bt}\Bigl(2Aatx + \frac{A}{x} + B\Bigr),$$
$$w(x,t) = Ax\exp\Bigl[(b + a\mu^2)t + \frac{\mu}{x}\Bigr].$$

2°. The transformation $w(x,t) = xe^{bt}u(\xi,t)$, $\xi = 1/x$ leads to a constant coefficient equation, $\partial_t u = a\partial_{\xi\xi}u$, which is considered in Section 3.1.1.

3. $\dfrac{\partial w}{\partial t} = (x^2 + a^2)^2 \dfrac{\partial^2 w}{\partial x^2} + \Phi(x,t).$

Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} &w = f(x) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ &w = g(t) &&\text{at}\quad x = 0 &&\text{(boundary condition)},\\ &w = h(t) &&\text{at}\quad x = l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \int_0^t \int_0^l G(x,\xi,t-\tau)\Phi(\xi,\tau)\,d\xi\,d\tau + \int_0^l G(x,\xi,t)f(\xi)\,d\xi \\ &\quad + a^4 \int_0^t g(\tau)\Lambda_1(x,t-\tau)\,d\tau - (a^2+l^2)^2 \int_0^t h(\tau)\Lambda_2(x,t-\tau)\,d\tau. \end{aligned}$$

Here, the Green's function G is given by

$$\begin{gathered} G(x,\xi,t) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)\exp(-\mu_n^2 t)}{\|y_n\|^2(\xi^2+a^2)^2}, \quad \mu_n^2 = \left[\frac{\pi n a}{\arctan(l/a)}\right]^2 - a^2, \\ y_n(x) = \sqrt{x^2+a^2}\,\sin\left[\pi n \frac{\arctan(x/a)}{\arctan(l/a)}\right], \quad \|y_n\|^2 = \frac{\arctan(l/a)}{2a}, \end{gathered}$$

and the functions Λ_1 and Λ_2 are expressed via the Green's function as follows:

$$\Lambda_1(x,t) = \frac{\partial}{\partial \xi} G(x,\xi,t)\Big|_{\xi=0}, \quad \Lambda_2(x,t) = \frac{\partial}{\partial \xi} G(x,\xi,t)\Big|_{\xi=l}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

4. $\dfrac{\partial w}{\partial t} = (x-a_1)^2(x-a_2)^2 \dfrac{\partial^2 w}{\partial x^2} - bw, \qquad a_1 \neq a_2.$

The transformation

$$w(x,t) = (x-a_2)e^{-bt}u(\xi,\tau), \quad \xi = \ln\left|\frac{x-a_1}{x-a_2}\right|, \quad \tau = (a_1-a_2)^2 t$$

leads to a constant coefficient equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial u}{\partial \xi},$$

which is considered in Section 3.1.4.

5. $\dfrac{\partial w}{\partial t} = (a_2 x^2 + a_1 x + a_0)^2 \dfrac{\partial^2 w}{\partial x^2} + bw.$

The transformation

$$w(x,t) = \exp\left[(a_2 a_0 - \tfrac{1}{4}a_1^2 + b)t\right]\sqrt{|a_2x^2+a_1x+a_0|}\,u(\xi,t), \quad \xi = \int \frac{dx}{a_2x^2+a_1x+a_0}$$

leads to a constant coefficient equation, $\partial_t u = \partial_{\xi\xi} u$, which is considered in Section 3.1.1.

6. $\dfrac{\partial w}{\partial t} = ax^{1-n}\dfrac{\partial^2 w}{\partial x^2}.$

This equation is encountered in diffusion boundary layer problems (see equation 3.9.1.3) and is a special case of 3.8.6.1 with $f(x) = ax^{1-n}$. In addition, it is a special case of equation 3.3.5.1 with $n = -1$ and is an equation of the form 3.3.6.2 for $n = -3$ (in both cases the equation is reduced to a constant coefficient equation). For $n = 0$, see equation 3.3.4.1.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$
\begin{aligned}
w(x) &= Ax + B,\\
w(x,t) &= Aan(n+1)t + Ax^{n+1} + B,\\
w(x,t) &= Aa(n+1)(n+2)tx + Ax^{n+2} + B,\\
w(x,t) &= A\left[an(n+1)t^2 + 2tx^{n+1} + \frac{x^{2n+2}}{a(n+1)(2n+1)}\right] + B,\\
w(x,t) &= A\left[a(n+1)(n+2)t^2x + 2tx^{n+2} + \frac{x^{2n+3}}{a(n+1)(2n+3)}\right] + B,\\
w(x,t) &= A + Bt^{-\frac{n}{n+1}}\exp\left[-\frac{x^{n+1}}{a(n+1)^2t}\right],\\
w(x,t) &= A + Bxt^{-\frac{n+2}{n+1}}\exp\left[-\frac{x^{n+1}}{a(n+1)^2t}\right],\\
w(x,t) &= e^{\mu t}\sqrt{x}\left[AJ_{\frac{1}{2q}}\left(\frac{\sqrt{-\mu}}{\sqrt{a}\,q}x^q\right) + BY_{\frac{1}{2q}}\left(\frac{\sqrt{-\mu}}{\sqrt{a}\,q}x^q\right)\right] \qquad \text{for } \mu < 0,\\
w(x,t) &= e^{\mu t}\sqrt{x}\left[AI_{\frac{1}{2q}}\left(\frac{\sqrt{\mu}}{\sqrt{a}\,q}x^q\right) + BK_{\frac{1}{2q}}\left(\frac{\sqrt{\mu}}{\sqrt{a}\,q}x^q\right)\right] \qquad \text{for } \mu > 0,
\end{aligned}
$$

where $q = \frac{1}{2}(n+1)$, $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, and $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions.

Suppose $2/(n+1) = 2m+1$, where m is an integer. We have the following particular solutions (A, B, and μ are arbitrary constants):

$$
\begin{aligned}
w(x,t) &= e^{\mu t}x(x^{1-2q}D)^{m+1}\left[A\exp\left(\frac{\sqrt{\mu}}{\sqrt{a}\,q}x^q\right) + B\exp\left(-\frac{\sqrt{\mu}}{\sqrt{a}\,q}x^q\right)\right] \qquad \text{for } m \ge 0,\\
w(x,t) &= e^{\mu t}x(x^{1-2q}D)^{-m}\left[A\exp\left(\frac{\sqrt{\mu}}{\sqrt{a}\,q}x^q\right) + B\exp\left(-\frac{\sqrt{\mu}}{\sqrt{a}\,q}x^q\right)\right] \qquad \text{for } m < 0,
\end{aligned}
$$

where $D = \dfrac{d}{dx}$, $q = \dfrac{n+1}{2} = \dfrac{1}{2m+1}$.

2°. Suppose $w = w(x,t)$ is a solution of the original equation. Then the functions

$$
\begin{aligned}
w_1 &= Aw(\lambda x, \lambda^{n+1}t + C),\\
w_2 &= \frac{A}{|\delta+\beta t|^{nk}}\exp\left[-\frac{\beta k^2 x^{n+1}}{a(\delta+\beta t)}\right]w\left(\frac{x}{(\delta+\beta t)^{2k}}, \frac{\gamma+\lambda t}{\delta+\beta t}\right), \quad k = \frac{1}{n+1}, \quad \lambda\delta - \beta\gamma = 1,
\end{aligned}
$$

where A, C, β, δ, and λ are arbitrary constants, are also solutions of the equation.

3°. Domain: $0 \le x < \infty$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} &w = w_0 \quad \text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ &w = w_1 \quad \text{at} \quad x = 0 \quad &&\text{(boundary condition)},\\ &w \to w_0 \quad \text{at} \quad x \to \infty \quad &&\text{(boundary condition)}, \end{aligned}$$

where $w_0 = \text{const}$ and $w_1 = \text{const}$.

Solution:

$$\frac{w - w_1}{w_0 - w_1} = \frac{1}{\Gamma(k)}\gamma\left(k,\ \frac{k^2 x^{n+1}}{at}\right), \qquad k = \frac{1}{n+1},$$

where $\Gamma(k) = \gamma(k, \infty)$ is the gamma function and $\gamma(k, \zeta) = \int_0^\zeta \zeta^{k-1} e^{-\zeta}\, d\zeta$ is the incomplete gamma function.

4°. The transformation

$$\tau = \tfrac{1}{4}a(n+1)^2 t, \quad \xi = x^{\frac{n+1}{2}}$$

leads to the equation

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial \xi^2} + \frac{1-2k}{\xi}\frac{\partial w}{\partial \xi}, \qquad k = \frac{1}{n+1},$$

which is considered in Section 3.2.5.

5°. Two discrete transformations are worth mentioning. They preserve the form of the original equation, but the parameter n is changed.

5.1. The point transformation

$$z = \frac{1}{x}, \quad u = \frac{w}{x} \qquad \text{(transformation } \mathcal{F}\text{)}$$

leads to a similar equation,

$$\frac{\partial u}{\partial t} = a z^{n+3} \frac{\partial^2 u}{\partial z^2}. \tag{1}$$

The transformation $\mathcal{F}$ changes the equation parameter in accordance with the rule $n \overset{\mathcal{F}}{\Longrightarrow} -n-2$. The second application of the transformation $\mathcal{F}$ leads to the original equation.

5.2. Using the Bäcklund transformation (see 3.8.6.1, Item 5.2)

$$\xi = x^n, \quad w = \frac{\partial v}{\partial \xi} \qquad \text{(transformation } \mathcal{H}\text{)}$$

and integrating the resulting equation with respect to ξ, we obtain

$$\frac{\partial v}{\partial t} = a n^2 \xi^{\frac{n-1}{n}} \frac{\partial^2 v}{\partial \xi^2}. \tag{2}$$

The transformation $\mathcal{H}$ changes the equation parameter in accordance with the rule $n \overset{\mathcal{H}}{\Longrightarrow} \dfrac{1}{n}$. The second application of the transformation $\mathcal{H}$ leads to the original equation.

The composition of transformations $\mathcal{G} = \mathcal{H} \circ \mathcal{F}$ changes the equation parameter in accordance with the rule $n \overset{\mathcal{G}}{\Longrightarrow} -\dfrac{1}{n+2}$.

The original equation reduces to a constant coefficient equation for $n=-3$ (see 3.3.6.2). Substituting $n = -3$ into (2) yields the equation

$$\frac{\partial v}{\partial t} = A\xi^{4/3}\frac{\partial^2 v}{\partial \xi^2},$$

which also can be reduced to a constant coefficient equation.

Likewise, using the transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$, one may find some other equations of the given type that are reduced to a constant coefficient heat equation.

7. $\boldsymbol{\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x}.}$

This is a special case of equation 3.8.4.5 with $f(t) = b$. On passing from t, x to the new variables

$$z = xe^{bt}, \quad \tau = \frac{a}{b(2-n)}e^{b(2-n)t} + \text{const},$$

we obtain an equation of the form 3.3.6.6:

$$\frac{\partial w}{\partial \tau} = z^n\frac{\partial^2 w}{\partial z^2}.$$

8. $\boldsymbol{\dfrac{\partial w}{\partial t} = a\Big(x^n\dfrac{\partial^2 w}{\partial x^2} + nx^{n-1}\dfrac{\partial w}{\partial x}\Big).}$

This equation describes heat transfer in a quiescent medium (solid body) in the case where thermal diffusivity is a power-law function of the coordinate. The equation can be rewritten in the form

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right),$$

which is more customary for applications.

1°. For $n = 2$, see equation 3.3.5.1. For $n \neq 2$, by passing from t, x to the new variables $\tau = \frac{1}{4}a(2-n)^2 t$, $z = x^{\frac{2-n}{2}}$, we obtain an equation of the form 3.2.5:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2} + \frac{n}{2-n}\frac{1}{z}\frac{\partial w}{\partial z}.$$

2°. The transformation

$$w(x,t) = x^{1-n}u(\xi,t), \quad \xi = x^{3-2n}$$

leads to a similar equation

$$\frac{\partial u}{\partial t} = b\frac{\partial}{\partial \xi}\left(\xi^{\frac{4-3n}{3-2n}}\frac{\partial u}{\partial \xi}\right), \qquad b = a(3-2n)^2.$$

9. $\dfrac{\partial w}{\partial t} = a\Big(x^{2m}\dfrac{\partial^2 w}{\partial x^2} + mx^{2m-1}\dfrac{\partial w}{\partial x}\Big).$

This is a special case of equation 3.8.4.7 with $f(t) = g(t) = 0$.

The substitution

$$\xi = \begin{cases} \dfrac{1}{1-m}x^{1-m} & \text{if } m \neq 1, \\ \ln|x| & \text{if } m = 1, \end{cases}$$

leads to a constant coefficient equation, $\partial_t w = a\partial_{\xi\xi} w$, which is considered in Section 3.1.1.

10. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(bt^m + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = bt^m + c$.

3.3.7 Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,t)\dfrac{\partial^2 w}{\partial x^2} + g(x,t)\dfrac{\partial w}{\partial x} + h(x,t)w$

1. $\dfrac{\partial w}{\partial t} = axt\dfrac{\partial^2 w}{\partial x^2} + (bx + ct^n)w.$

This is a special case of equation 3.8.8.1 with $f(t) = at$, $g(t) = 0$, $h(t) = b$, and $s(t) = ct^n$.

2. $\dfrac{\partial w}{\partial t} = axt^k\dfrac{\partial^2 w}{\partial x^2} + bxt^m\dfrac{\partial w}{\partial x} + ct^n w.$

This is a special case of equation 3.8.8.1 with $f(t) = at^k$, $g(t) = bt^m$, $h(t) = 0$, and $s(t) = ct^n$.

3. $\dfrac{\partial w}{\partial t} = x\Big(at^k\dfrac{\partial^2 w}{\partial x^2} + bt^m\dfrac{\partial w}{\partial x} + ct^n w\Big).$

This is a special case of equation 3.8.8.1 with $f(t) = at^k$, $g(t) = bt^m$, $h(t) = ct^n$, and $s(t) = 0$.

4. $\dfrac{\partial w}{\partial t} = ax^2t^k\dfrac{\partial^2 w}{\partial x^2} + bt^m x\dfrac{\partial w}{\partial x} + ct^n w.$

This is a special case of equation 3.8.8.2 with $f(t) = at^k$, $g(t) = bt^m$, and $h(t) = ct^n$.

5. $\dfrac{\partial w}{\partial t} = ax^3t^m\dfrac{\partial^2 w}{\partial x^2} + bxt^k\dfrac{\partial w}{\partial x} + ct^l w.$

This is a special case of equation 3.8.8.7 with $n = 3$, $f(t) = at^m$, $g(t) = bt^k$, and $h(t) = ct^l$.

6. $\dfrac{\partial w}{\partial t} = ax^4t^n\dfrac{\partial^2 w}{\partial x^2} + bt^m w.$

This is a special case of equation 3.8.8.4 with $f(t) = at^n$ and $g(t) = bt^m$.

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = (Ax+B)\exp\biggl(\frac{b}{m+1}t^{m+1}\biggr),$$
$$w(x,t) = A\biggl(\frac{2a}{n+1}t^{n+1}x+\frac{1}{x}\biggr)\exp\biggl(\frac{b}{m+1}t^{m+1}\biggr),$$
$$w(x,t) = Ax\exp\biggl(\frac{b}{m+1}t^{m+1}+\frac{a\lambda^2}{n+1}t^{n+1}+\frac{\lambda}{x}\biggr).$$

2°. The transformation

$$w(x,t) = x\exp\biggl(\frac{b}{m+1}t^{m+1}\biggr)u(\xi,\tau), \quad \xi=\frac{1}{x}, \quad \tau=\frac{a}{n+1}t^{n+1}$$

leads to a constant coefficient equation, $\partial_\tau u = \partial_{\xi\xi}u$, which is considered in Section 3.1.1.

7. $\dfrac{\partial w}{\partial t} = at^n\dfrac{\partial^2 w}{\partial x^2} + (bt^m x + ct^i)\dfrac{\partial w}{\partial x} + (dt^l x + et^p)w.$

This is a special case of equation 3.8.7.4 (hence, it can be reduced to a constant coefficient equation, which is considered in Section 3.1.1).

8. $\dfrac{\partial w}{\partial t} = at^n\dfrac{\partial^2 w}{\partial x^2} + (bt^m x + ct^i)\dfrac{\partial w}{\partial x} + (dt^l x^2 + et^p x + st^q)w.$

This is a special case of equation 3.8.7.5.

9. $\dfrac{\partial w}{\partial t} = ax^n t^m\dfrac{\partial^2 w}{\partial x^2} + bxt^k\dfrac{\partial w}{\partial x} + ct^p w.$

This is a special case of equation 3.8.8.7 with $f(t)=at^m$, $g(t)=bt^k$, and $h(t)=ct^p$.

3.3.8 Liquid-Film Mass Transfer Equation $(1-y^2)\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial y^2}$

This equation describes steady-state heat and mass transfer in a fluid film with a parabolic velocity profile. The variables have the following physical meanings: w is a dimensionless temperature (concentration); x and y are dimensionless coordinates measured, respectively, along and across the film ($y=0$ corresponds to the free surface of the film and $y=1$ to the solid surface the film flows down), and $\mathrm{Pe}=1/a$ is the Peclet number. Mixed boundary conditions are usually encountered in practical applications.

► **Particular solutions (A, B, k, and λ are arbitrary constants).**

$$w(x,y) = kx - \frac{k}{12a}y^4 + \frac{k}{2a}y^2 + Ay + B,$$
$$w(x,y) = A\exp\bigl(-a\lambda^2 x\bigr)\exp\bigl(-\tfrac{1}{2}\lambda y^2\bigr)\,\Phi\bigl(\tfrac{1}{4}-\tfrac{1}{4}\lambda,\ \tfrac{1}{2};\ \lambda y^2\bigr),$$
$$w(x,y) = A\exp\bigl(-a\lambda^2 x\bigr)y\exp\bigl(-\tfrac{1}{2}\lambda y^2\bigr)\,\Phi\bigl(\tfrac{3}{4}-\tfrac{1}{4}\lambda,\ \tfrac{3}{2};\ \lambda y^2\bigr),$$

where $\Phi(\alpha,\ \beta;\ z) = 1+\sum\limits_{m=1}^{\infty}\dfrac{\alpha(\alpha+1)\dots(\alpha+m-1)}{\beta(\beta+1)\dots(\beta+m-1)}\dfrac{z^m}{m!}$ is the degenerate hypergeometric function.

► **Mass exchange between fluid film and gas.**

The mass exchange between a fluid film and the gas above the free surface, provided that the admixture concentration at the film surface is constant and there is no mass transfer through the solid surface, meets the boundary conditions

$$\begin{aligned} w &= 0 \quad \text{at} \quad x = 0 \quad (0 < y < 1),\\ w &= 1 \quad \text{at} \quad y = 0 \quad (x > 0),\\ \partial_y w &= 0 \quad \text{at} \quad y = 1 \quad (x > 0). \end{aligned}$$

The solution of the original equation subject to these boundary conditions is given by

$$\begin{aligned} w(x,y) &= 1 - \sum_{m=1}^{\infty} A_m \exp\bigl(-a\lambda_m^2 x\bigr) F_m(y),\\ F_m(y) &= y \exp\bigl(-\tfrac{1}{2}\lambda_m y^2\bigr)\Phi\bigl(\tfrac{3}{4} - \tfrac{1}{4}\lambda_m,\ \tfrac{3}{2};\ \lambda_m y^2\bigr), \end{aligned} \tag{1}$$

where the function F_m and the coefficients A_m and λ_m are independent of the parameter a.

The eigenvalues λ_m are solutions of the transcendental equation

$$\lambda_m \Phi\bigl(\tfrac{3}{4} - \tfrac{1}{4}\lambda_m,\ \tfrac{3}{2};\ \lambda_m\bigr) - \Phi\bigl(\tfrac{3}{4} - \tfrac{1}{4}\lambda_m,\ \tfrac{1}{2};\ \lambda_m\bigr) = 0.$$

The series coefficients A_m are calculated from

$$A_m = \frac{\displaystyle\int_0^1 (1-y^2) F_m(y)\,dy}{\displaystyle\int_0^1 (1-y^2)\bigl[F_m(y)\bigr]^2\,dy}, \qquad \text{where} \quad m = 1,\ 2,\ \dots$$

Table 3.3 shows the first ten eigenvalues λ_m and coefficients A_m.

TABLE 3.3
Eigenvalues λ_m and coefficients A_m in solution (1)

m	λ_m	A_m	m	λ_m	A_m
1	2.2631	1.3382	6	22.3181	−0.1873
2	6.2977	−0.5455	7	26.3197	0.1631
3	10.3077	0.3589	8	30.3209	−0.1449
4	14.3128	−0.2721	9	34.3219	0.1306
5	18.3159	0.2211	10	38.3227	−0.1191

The solution asymptotics as $ax \to 0$ is given by

$$w = \operatorname{erfc}\left(\frac{y}{2\sqrt{ax}}\right),$$

where $\operatorname{erfc} z = \displaystyle\int_z^{\infty} \exp\bigl(-\xi^2\bigr)\,d\xi$ is the complementary error function.

► **Dissolution of a plate by a laminar fluid film.**

The dissolution of a plate by a laminar fluid film, provided that the concentration at the solid surface is constant and there is no mass flux from the film into the gas, satisfies the boundary conditions

$$
\begin{aligned}
w &= 0 \quad \text{at} \quad x = 0 \quad (0 < y < 1),\\
\partial_y w &= 0 \quad \text{at} \quad y = 0 \quad (x > 0),\\
w &= 1 \quad \text{at} \quad y = 1 \quad (x > 0).
\end{aligned}
$$

The solution of the original equation subject to these boundary conditions is given by

$$
\begin{aligned}
w(x,y) &= 1 - \sum_{m=0}^{\infty} A_m \exp\bigl(-a\lambda_m^2 x\bigr) G_m(y),\\
G_m(y) &= \exp\bigl(-\tfrac{1}{2}\lambda_m y^2\bigr)\,\Phi\bigl(\tfrac{1}{4} - \tfrac{1}{4}\lambda_m,\ \tfrac{1}{2};\ \lambda_m y^2\bigr),
\end{aligned}
\tag{2}
$$

where the functions G_m and the constants A_m and λ_m are independent of the parameter a.

The eigenvalues λ_m are solutions of the transcendental equation

$$\Phi\bigl(\tfrac{1}{4} - \tfrac{1}{4}\lambda_m,\ \tfrac{1}{2};\ \lambda_m\bigr) = 0.$$

The following approximate relation is convenient to calculate λ_m:

$$\lambda_m = 4m + 1.68 \qquad (m = 0,\ 1,\ 2,\ \ldots). \tag{3}$$

The maximum error of this formula is less than 0.2%.

The coefficients A_m are approximated by

$$A_0 = 1.2, \quad A_m = (-1)^m 2.27\,\lambda_m^{-7/6} \qquad \text{for} \quad m = 1,\ 2,\ 3,\ \ldots,$$

where the eigenvalues λ_m are defined by (3). The maximum error of the expressions for A_m is less than 0.1%.

The solution asymptotics as $ax \to 0$ is given by

$$w = \frac{1}{\Gamma\bigl(\frac{1}{3}\bigr)}\Gamma\bigl(\tfrac{1}{3},\ \tfrac{2}{9}\zeta\bigr), \quad \zeta = \frac{(1-y)^3}{ax},$$

where $\Gamma(\alpha, z) = \displaystyle\int_z^{\infty} e^{-\xi}\xi^{\alpha-1}\,d\xi$ is the incomplete gamma function, $\Gamma(\alpha) = \Gamma(\alpha, 0)$ is the gamma function, and $\Gamma\bigl(\frac{1}{3}\bigr) \approx 2.679$.

⊙ *References for Section* 3.3.8: Z. Rotem and J. E. Neilson (1966), E. J. Davis (1973), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

3.3.9 Equations of the Form $f(x,y)\frac{\partial w}{\partial x}+g(x,y)\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}+h(x,y)$

1. $ax^n\frac{\partial w}{\partial x}+bx^k y\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}.$

This is a special case of equation 3.9.1.1 with $f(x)=ax^n$ and $g(x)=bx^k$.

The transformation

$$t=\frac{1}{a}\int\exp\left[-\frac{2b}{a(k-n+1)}x^{k-n+1}\right]\frac{dx}{x^n},\qquad z=y\exp\left[-\frac{b}{a(k-n+1)}x^{k-n+1}\right]$$

leads to a constant coefficient equation, $\partial_t w=\partial_{zz}w$, which is considered in Section 3.1.1.

2. $ax^n\frac{\partial w}{\partial x}+bx^k y\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}-cx^m w.$

This is a special case of equation 3.9.1.2 with $f(x)=ax^n$, $g(x)=bx^k$, and $h(x)=cx^m$.

3. $ax^m y^{n-1}\frac{\partial w}{\partial x}+bx^k y^n\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}.$

This is a special case of equation 3.9.1.3 with $f(x)=ax^m$ and $g(x)=bx^k$.

The transformation

$$t=\frac{1}{a}\int\exp\left[-\frac{b(n+1)}{a(k-m+1)}x^{k-m+1}\right]\frac{dx}{x^m},\qquad z=y\exp\left[-\frac{b}{a(k-m+1)}x^{k-m+1}\right]$$

leads to a simpler equation of the form 3.3.6.6:

$$\frac{\partial w}{\partial t}=z^{1-n}\frac{\partial^2 w}{\partial z^2}.$$

4. $a\left(\frac{y}{\sqrt{x}}\right)^n\frac{\partial w}{\partial x}+\frac{b}{\sqrt{x}}\left(\frac{y}{\sqrt{x}}\right)^k\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}.$

This is a special case of equation 3.9.1.4 with $f(z)=az^n$ and $g(z)=bz^k$.

◆ *See Section 3.9.1 for other equations of this form.*

3.4 Equations Containing Exponential Functions and Arbitrary Parameters

3.4.1 Equations of the Form $\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+f(x,t)w$

1. $\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+\left(be^{\beta t}+c\right)w.$

This is a special case of equation 3.8.1.1 with $f(t)=be^{\beta t}+c$.

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = (Ax+B)\exp\left(\frac{b}{\beta}e^{\beta t} + ct\right),$$
$$w(x,t) = A(x^2+2at)\exp\left(\frac{b}{\beta}e^{\beta t} + ct\right),$$
$$w(x,t) = A\exp\left(\lambda x + a\lambda^2 t + \frac{b}{\beta}e^{\beta t} + ct\right).$$

2°. The substitution $w(x,t) = u(x,t)\exp\left(\frac{b}{\beta}e^{\beta t} + ct\right)$ leads to a constant coefficient equation, $\partial_t u = a\partial_{xx}u$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta t} + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = be^{\beta t}$.

1°. Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp\left(cxt + \frac{b}{\beta}e^{\beta t} + \frac{1}{3}ac^2t^3\right),$$
$$w(x,t) = A\bigl(x+act^2\bigr)\exp\left(cxt + \frac{b}{\beta}e^{\beta t} + \frac{1}{3}ac^2t^3\right),$$
$$w(x,t) = A\exp\left[x(ct+\lambda) + \frac{b}{\beta}e^{\beta t} + \frac{1}{3}ac^2t^3 + ac\lambda t^2 + a\lambda^2 t\right].$$

2°. The transformation

$$w(x,t) = u(z,t)\exp\left(cxt + \frac{b}{\beta}e^{\beta t} + \frac{1}{3}ac^2t^3\right), \quad z = x + act^2$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta x} - c)w.$

This is a special case of equation 3.8.1.2 with $f(x) = be^{\beta x} - c$.

Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp(-\lambda t)J_\nu\left(\frac{2\sqrt{b}}{\beta\sqrt{a}}e^{\beta x/2}\right), \quad \nu = \frac{2}{\beta}\sqrt{\frac{c-\lambda}{a}},$$
$$w(x,t) = A\exp(-\lambda t)Y_\nu\left(\frac{2\sqrt{b}}{\beta\sqrt{a}}e^{\beta x/2}\right), \quad \nu = \frac{2}{\beta}\sqrt{\frac{c-\lambda}{a}},$$

where $J_\nu(\xi)$ and $Y_\nu(\xi)$ are the Bessel functions of the first and second kind, respectively.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bxe^{\beta t} + c)w.$

This is a special case of equation 3.8.1.6 with $f(t) = be^{\beta t}$ and $g(t) = c$.

1°. Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp\left(\frac{b}{\beta}xe^{\beta t} + \frac{ab^2}{2\beta^3}e^{2\beta t} + ct\right),$$
$$w(x,t) = A\left(x + \frac{2ab}{\beta^2}e^{\beta t}\right)\exp\left(\frac{b}{\beta}xe^{\beta t} + \frac{ab^2}{2\beta^3}e^{2\beta t} + ct\right),$$
$$w(x,t) = A\exp\left[x\left(\frac{b}{\beta}e^{\beta t} + \lambda\right) + \frac{ab^2}{2\beta^3}e^{2\beta t} + \frac{2ab\lambda}{\beta^2}e^{\beta t} + (a\lambda^2 + c)t\right].$$

2°. The transformation

$$w(x,t) = u(z,t)\exp\left(\frac{b}{\beta}xe^{\beta t} + \frac{ab^2}{2\beta^3}e^{2\beta t} + ct\right), \quad z = x + \frac{2ab}{\beta^2}e^{\beta t}$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(be^{\beta t} + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = be^{\beta t} + c$.

1°. Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp\left[x\left(\frac{b}{\beta}e^{\beta t} + ct\right) + a\phi(t)\right],$$
$$w(x,t) = A\left[x + a\left(\frac{2b}{\beta^2}e^{\beta t} + ct^2\right)\right]\exp\left[x\left(\frac{b}{\beta}e^{\beta t} + ct\right) + a\phi(t)\right],$$
$$w(x,t) = A\exp\left[x\left(\frac{b}{\beta}e^{\beta t} + ct + \lambda\right) + a\lambda\left(\frac{2b}{\beta^2}e^{\beta t} + ct^2 + \lambda t\right) + a\phi(t)\right],$$

where $\phi(t) = \frac{1}{2}b^2\beta^{-3}e^{2\beta t} + 2bc\beta^{-3}(\beta t - 1)e^{\beta t} + \frac{1}{3}c^2t^3$.

2°. The transformation

$$w(x,t) = u(z,t)\exp\left[x\left(\frac{b}{\beta}e^{\beta t} + ct\right) + a\phi(t)\right], \quad z = x + a\left(\frac{2b}{\beta^2}e^{\beta t} + ct^2\right)$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x^2(be^{\beta t} + c)w.$

This is a special case of equation 3.8.7.5 with $n(t) = a$, $f(t) = g(t) = s(t) = p(t) = 0$, and $h(t) = be^{\beta t} + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta t} + ce^{\lambda t})w.$

This is a special case of equation 3.8.1.1 with $f(t) = be^{\beta t} + ce^{\lambda t}$.

1°. Particular solutions (A, B, and ν are arbitrary constants):

$$w(x,t) = (Ax + B)\exp\biggl(\frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\biggr),$$
$$w(x,t) = A(x^2 + 2at)\exp\biggl(\frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\biggr),$$
$$w(x,t) = A\exp\biggl(\nu x + a\nu^2 t + \frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\biggr).$$

2°. The substitution $w(x,t) = u(x,t)\exp\biggl(\dfrac{b}{\beta}e^{\beta t} + \dfrac{c}{\lambda}e^{\lambda t}\biggr)$ leads to a constant coefficient equation, $\partial_t u = a\partial_{xx}u$, which is considered in Section 3.1.1.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta x} + ce^{\lambda t} + d)w.$

The substitution $w(x,t) = u(x,t)\exp\Bigl(\dfrac{c}{\lambda}e^{\lambda t}\Bigr)$ leads to an equation of the form 3.4.1.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + (be^{\beta x} + d)u.$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta x + \lambda t} + c)w.$

For $\beta = 0$, see equation 3.4.1.1; for $\lambda = 0$, see equation 3.4.1.3.

For $\beta \neq 0$, the transformation

$$w(x,t) = u(z,t)e^{\mu x}, \quad z = x + \frac{\lambda}{\beta}t, \qquad \text{where} \quad \mu = \frac{\lambda}{2a\beta},$$

leads to an equation of the form 3.4.1.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial z^2} + (be^{\beta z} + c + a\mu^2)u.$$

3.4.2 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x}$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta t} + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = be^{\beta t} + c$.

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = Ax + A\left(\frac{b}{\beta}e^{\beta t} + ct\right) + B,$$
$$w(x,t) = A\left(x + \frac{b}{\beta}e^{\beta t} + ct\right)^2 + 2aAt + B,$$
$$w(x,t) = A\exp\left[\lambda x + \lambda\frac{b}{\beta}e^{\beta t} + (a\lambda^2 + c\lambda)t\right] + B.$$

2°. On passing from t, x to the new variables t, $z = x + \dfrac{b}{\beta}e^{\beta t} + ct$, we obtain a constant coefficient equation, $\partial_t w = a\partial_{zz}w$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta x} + c)\dfrac{\partial w}{\partial x}$.

This is a special case of equation 3.8.2.2 with $f(x) = be^{\beta x} + c$.

1°. Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp(-\lambda t + k\beta x)\,\Phi\left(k,\ 2k + 1 + \frac{c}{a\beta};\ -\frac{b}{a\beta}e^{\beta x}\right),$$
$$w(x,t) = A\exp(-\lambda t + k\beta x)\,\Psi\left(k,\ 2k + 1 + \frac{c}{a\beta};\ -\frac{b}{a\beta}e^{\beta x}\right), \qquad (1)$$

where $k = k(\lambda)$ is a root of the quadratic equation $a\beta^2k^2 + c\beta k + \lambda = 0$; $\Phi(\alpha,\nu;z)$ and $\Psi(\alpha,\nu;z)$ are degenerate hypergeometric functions. [Regarding the degenerate hypergeometric functions, see Section 30.9 and the books by Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 1).]

Remark 3.6. In solutions (1), the parameter k can be considered arbitrary, and then we have $\lambda = -a\beta^2k^2 - c\beta k$.

2°. Other particular solutions (A and B are arbitrary constants):

$$w(x) = A + B\int F(x)\,dx, \quad F(x) = \exp\left(-\frac{b}{a\beta}e^{\beta x} - \frac{c}{a}x\right),$$
$$w(x,t) = Aat + A\int F(x)\left(\int\frac{dx}{F(x)}\right)dx,$$
$$w(x,t) = AatG(x) + A\int F(x)\left(\int\frac{G(x)\,dx}{F(x)}\right)dx, \quad G(x) = \int F(x)\,dx.$$

3°. The substitution $z = e^{\beta x}$ leads to an equation of the form 3.3.5.4:

$$\frac{\partial w}{\partial t} = a\beta^2z^2\frac{\partial^2 w}{\partial z^2} + \beta z(bz + c + a\beta)\frac{\partial w}{\partial z}.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(be^{\beta t} + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = be^{\beta t} + c$.

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = AxF(t) + B, \quad F(t) = \exp\left(\frac{b}{\beta}e^{\beta t} + ct\right),$$
$$w(x,t) = Ax^2F^2(t) + 2Aa\int F^2(t)\,dt + B,$$
$$w(x,t) = A\exp\left[\lambda xF(t) + a\lambda^2\int F^2(t)\,dt\right] + B.$$

2°. On passing from t, x to the new variables (A is any number)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t), \qquad \text{where} \quad F(t) = \exp\left(\frac{b}{\beta}e^{\beta t} + ct\right),$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = a\partial_{zz}w$, which is considered in Section 3.1.1.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta t} + ce^{\lambda t})\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = be^{\beta t} + ce^{\lambda t}$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = Ax + A\left(\frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\right) + B,$$
$$w(x,t) = A\left(x + \frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\right)^2 + 2aAt + B,$$
$$w(x,t) = A\exp\left(\mu x + \mu\frac{b}{\beta}e^{\beta t} + \mu\frac{c}{\lambda}e^{\lambda t} + a\mu^2 t\right) + B.$$

2°. On passing from t, x to the new variables t, $z = x + \dfrac{b}{\beta}e^{\beta t} + \dfrac{c}{\lambda}e^{\lambda t}$, we obtain a constant coefficient equation, $\partial_t w = a\partial_{zz}w$, which is considered in Section 3.1.1.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta t + \lambda x} + c)\dfrac{\partial w}{\partial x}.$

For $\beta = 0$, see equation 3.4.2.2; for $\lambda = 0$, see equation 3.4.2.1.

For $\lambda \neq 0$, the substitution $z = x + (\beta/\lambda)t$ leads to an equation of the form 3.4.2.2:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + \left(be^{\lambda z} + c - \frac{\beta}{\lambda}\right)\frac{\partial w}{\partial z}.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (be^{\beta t} + cx)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = be^{\beta t}$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bxe^{\beta t} + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.1.6 with $f(t) = be^{\beta t}$ and $g(t) = c$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bxe^{\beta t} + ce^{\lambda t})\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.1.6 with $f(t) = be^{\beta t}$ and $g(t) = ce^{\lambda t}$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(be^{\beta t} + ce^{\lambda t})\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = be^{\beta t} + ce^{\lambda t}$.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = AxF(t) + B, \quad F(t) = \exp\left(\frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\right),$$
$$w(x,t) = Ax^2F^2(t) + 2Aa\int F^2(t)\,dt + B,$$
$$w(x,t) = A\exp\left[\mu x F(t) + a\mu^2\int F^2(t)\,dt\right] + B.$$

2°. On passing from t, x to the new variables (A is any number)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t), \qquad \text{where} \quad F(t) = \exp\left(\frac{b}{\beta}e^{\beta t} + \frac{c}{\lambda}e^{\lambda t}\right),$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = a\partial_{zz}w$, which is considered in Section 3.1.1.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left(cxe^{\beta t} + \dfrac{b}{x}\right)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = ce^{\beta t}$.

3.4.3 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (ce^{\beta t} + s)w.$

The substitution

$$w(x,t) = u(x,t)\exp\left[-\frac{b}{2a}x + \frac{c}{\beta}e^{\beta t} + \left(s - \frac{b^2}{4a}\right)t\right]$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{xx}u$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (ce^{\beta x} + s)w.$

The substitution $w(x,t) = u(x,t)\exp\left(-\frac{b}{2a}x\right)$ leads to an equation of the form 3.4.1.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \left(ce^{\beta x} + s - \frac{b^2}{4a}\right)u.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (ce^{\beta t} + se^{\mu t})w.$

The substitution

$$w(x,t) = u(x,t)\exp\left(-\frac{b}{2a}x + \frac{c}{\beta}e^{\beta t} + \frac{s}{\mu}e^{\mu t} - \frac{b^2}{4a}t\right)$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{xx}u$, which is considered in Section 3.1.1.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (ce^{\beta x} + se^{\mu t})w.$

The substitution $w(x,t) = u(x,t)\exp\left(\frac{s}{\mu}e^{\mu t}\right)$ leads to an equation of the form 3.4.3.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + ce^{\beta x}u.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta x\dfrac{\partial w}{\partial x} + ce^{2\beta t}w.$

On passing from t, x to the new variables (A and B are any numbers)

$$\tau = \frac{A^2}{2\beta}e^{2\beta t} + B, \quad z = Axe^{\beta t},$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation of the form 3.1.3:

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial z^2} + cA^{-2}w.$$

6. $\dfrac{\partial w}{\partial t} = a_2\dfrac{\partial^2 w}{\partial x^2} + \left(a_1 e^{\beta x} + b_1\right)\dfrac{\partial w}{\partial x} + \left(a_0 e^{2\beta x} + b_0 e^{\beta x} + c_0\right)w.$

The substitution $z = e^{\beta x}$ leads to an equation of the form 3.3.5.4:

$$\frac{\partial w}{\partial t} = a_2\beta^2 z^2\frac{\partial^2 w}{\partial z^2} + \beta z\left(a_1 z + b_1 + a_2\beta\right)\frac{\partial w}{\partial z} + \left(a_0 z^2 + b_0 z + c_0\right)w.$$

3.4.4 Equations of the Form $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w$

1. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + \left(be^{\beta t} + cx\right)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = 0$, $h(t) = c$, and $s(t) = be^{\beta t}$.

2. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + \left(bxe^{\beta t} + c\right)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = 0$, $h(t) = be^{\beta t}$, and $s(t) = c$.

3. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x} + \left(ce^{\beta t} + d\right)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = b$, $h(t) = 0$, and $s(t) = ce^{\beta t} + d$.

4. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + \big(be^{\beta t} + c\big)w.$

This is a special case of equation 3.8.8.2 with $f(t) = a$, $g(t) = 0$, and $h(t) = be^{\beta t} + c$.

5. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + \big(be^{\beta t} + c\big)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = be^{\beta t} + c$.

6. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x\big(be^{\beta t} + c\big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = be^{\beta t} + c$.

7. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + bxe^{\beta t}\dfrac{\partial w}{\partial x} + ce^{\mu t}w.$

This is a special case of equation 3.8.8.7 with $f(t) = a$, $g(t) = be^{\beta t}$, and $h(t) = ce^{\mu t}$.

3.4.5 Equations of the Form $\dfrac{\partial w}{\partial t} = ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w$

1. $\dfrac{\partial w}{\partial t} = ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 3.8.6.1 with $f(x) = ae^{\beta x}$.

Particular solutions (A, B, and μ are arbitrary constants):

$$
\begin{aligned}
w(x) &= Ax + B,\\
w(x,t) &= A\big(a\beta^2 t + e^{-\beta x}\big) + B,\\
w(x,t) &= A\big(a\beta^3 tx + \beta xe^{-\beta x} + 2e^{-\beta x}\big) + B,\\
w(x,t) &= A\big(2a^2\beta^4 t^2 + 4a\beta^2 te^{-\beta x} + e^{-2\beta x}\big) + B,\\
w(x,t) &= A\exp(-\mu t)\, J_0\left(\frac{2}{\beta}\sqrt{\frac{\mu}{a}}\exp\big(-\tfrac{1}{2}\beta x\big)\right),\\
w(x,t) &= A\exp(-\mu t)\, Y_0\left(\frac{2}{\beta}\sqrt{\frac{\mu}{a}}\exp\big(-\tfrac{1}{2}\beta x\big)\right),
\end{aligned}
$$

where $J_0(\xi)$ and $Y_0(\xi)$ are Bessel functions.

2. $\dfrac{\partial w}{\partial t} = ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2} + bw.$

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$
\begin{aligned}
w(x,t) &= e^{bt}(Ax + B),\\
w(x,t) &= Ae^{bt}\big(a\beta^2 t + e^{-\beta x}\big) + Be^{bt},\\
w(x,t) &= Ae^{bt}\big(a\beta^3 tx + \beta xe^{-\beta x} + 2e^{-\beta x}\big) + Be^{bt},\\
w(x,t) &= Ae^{bt}\big(2a^2\beta^4 t^2 + 4a\beta^2 te^{-\beta x} + e^{-2\beta x}\big),\\
w(x,t) &= A\exp(-\mu t)\, J_0\left(\frac{2}{\beta}\sqrt{\frac{\mu+b}{a}}\exp\big(-\tfrac{1}{2}\beta x\big)\right),\\
w(x,t) &= A\exp(-\mu t)\, Y_0\left(\frac{2}{\beta}\sqrt{\frac{\mu+b}{a}}\exp\big(-\tfrac{1}{2}\beta x\big)\right),
\end{aligned}
$$

where $J_0(\xi)$ and $Y_0(\xi)$ are Bessel functions.

2°. The substitution $w(x,t) = e^{bt}u(x,t)$ leads to an equation of the form 3.4.5.1: $\partial_t u = ae^{\beta x}\partial_{xx}u$.

3. $\dfrac{\partial w}{\partial t} = ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2} + (be^{\mu t} + c)w.$

1°. Particular solutions (A, B, μ are arbitrary constants):

$$
\begin{aligned}
w(x) &= (Ax+B)\exp\Big(\frac{b}{\mu}e^{\mu t} + ct\Big),\\
w(x,t) &= A\big(a\beta^2 t + e^{-\beta x}\big)\exp\Big(\frac{b}{\mu}e^{\mu t} + ct\Big),\\
w(x,t) &= A\big(a\beta^3 tx + \beta x e^{-\beta x} + 2e^{-\beta x}\big)\exp\Big(\frac{b}{\mu}e^{\mu t} + ct\Big),\\
w(x,t) &= A\big(2a^2\beta^4 t^2 + 4a\beta^2 te^{-\beta x} + e^{-2\beta x}\big)\exp\Big(\frac{b}{\mu}e^{\mu t} + ct\Big),\\
w(x,t) &= A\exp\Big[\frac{b}{\mu}e^{\mu t} + (c-\mu)t\Big]J_0\bigg(\frac{2}{\beta}\sqrt{\frac{\mu}{a}}\exp\big(-\tfrac{1}{2}\beta x\big)\bigg),\\
w(x,t) &= A\exp\Big[\frac{b}{\mu}e^{\mu t} + (c-\mu)t\Big]Y_0\bigg(\frac{2}{\beta}\sqrt{\frac{\mu}{a}}\exp\big(-\tfrac{1}{2}\beta x\big)\bigg).
\end{aligned}
$$

2°. The substitution $w(x,t) = \exp\big(\frac{b}{\mu}e^{\mu t} + ct\big)u(x,t)$ leads to an equation of the form 3.4.5.1: $\partial_t u = ae^{\beta x}\partial_{xx}u$.

4. $\dfrac{\partial w}{\partial t} = a\Big(e^{\beta x}\dfrac{\partial^2 w}{\partial x^2} + \beta e^{\beta x}\dfrac{\partial w}{\partial x}\Big).$

This equation describes heat transfer in a quiescent medium (solid body) in the case where thermal diffusivity is an exponential function of the coordinate. The equation can be rewritten in the divergence form

$$
\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\Big(e^{\beta x}\frac{\partial w}{\partial x}\Big),
$$

which is more customary for applications.

1°. Particular solutions (A, B, C, and μ are arbitrary constants):

$$
\begin{aligned}
w(x,t) &= A\exp\Big(-\frac{e^{-\beta x}}{a\beta^2 t + C}\Big) + B,\\
w(x,t) &= Aa\beta^2 t - A(\beta x+1)e^{-\beta x} + B,\\
w(x,t) &= 2Aa\beta^2 te^{-\beta x} + Ae^{-2\beta x} + B,\\
w(x,t) &= Aa^2\beta^4 t^2 - 2Aa\beta^2 t(\beta x+1)e^{-\beta x} - A(\beta x + \tfrac{5}{2})e^{-2\beta x} + B,\\
w(x,t) &= Aa^2\beta^4 t^2 e^{-\beta x} + Aa\beta^2 te^{-2\beta x} + \tfrac{1}{6}Ae^{-3\beta x} + B,\\
w(x,t) &= 2Aa^3\beta^6 t^3 e^{-\beta x} + 3Aa^2\beta^4 t^2 e^{-2\beta x} + Aa\beta^2 te^{-3\beta x} + \tfrac{1}{12}Ae^{-4\beta x} + B,
\end{aligned}
$$

$$w(x,t)=e^{-n\beta x}+\sum_{k=1}^{n-1}\frac{\bigl[n(n-1)\dots(n-k)\bigr]^2}{n(n-k)k!}(a\beta^2t)^k e^{(k-n)\beta x},$$

$$w(x,t)=e^{\mu t-\frac12\beta x}\biggl[AJ_1\biggl(\frac{2\sqrt{-\mu}}{\beta\sqrt{a}}e^{-\frac12\beta x}\biggr)+BY_1\biggl(\frac{2\sqrt{-\mu}}{\beta\sqrt{a}}e^{-\frac12\beta x}\biggr)\biggr]\quad\text{for }\ \mu<0,$$

$$w(x,t)=e^{\mu t-\frac12\beta x}\biggl[AI_1\biggl(\frac{2\sqrt{\mu}}{\beta\sqrt{a}}e^{-\frac12\beta x}\biggr)+BK_1\biggl(\frac{2\sqrt{\mu}}{\beta\sqrt{a}}e^{-\frac12\beta x}\biggr)\biggr]\quad\text{for }\ \mu>0,$$

where $J_1(z)$ and $Y_1(z)$ are Bessel functions, and $I_1(z)$ and $K_1(z)$ are modified Bessel functions.

2°. A solution containing an arbitrary function of the space variable:

$$w(x,t)=f(x)+\sum_{n=1}^{\infty}\frac{(at)^n}{n!}\boldsymbol{L}^n\bigl[f(x)\bigr],\qquad \boldsymbol{L}\equiv\frac{d}{dx}\biggl(e^{\beta x}\frac{d}{dx}\biggr),$$

where $f(x)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(x,0)=f(x)$.

3°. A solution containing an arbitrary function of time:

$$w(x,t)=A+e^{-\beta x}g(t)+\sum_{n=2}^{\infty}\frac{1}{n[(n-1)!]^2(a\beta^2)^{n-1}}e^{-\beta nx}g_t^{(n-1)}(t),$$

where $g(t)$ is any infinitely differentiable function. If $g(t)$ is a polynomial, then the series has finitely many terms.

4°. The transformation (C_1, C_2, and C_3 are any numbers)

$$w(x,t)=u(\xi,\tau)\exp\biggl[-\frac{e^{-\beta x}}{a\beta^2(t+C_1)}\biggr],\quad \xi=x-\frac{1}{\beta}\ln\frac{C_2}{(t+C_1)^2},\quad \tau=C_3-\frac{C_2}{t+C_1},$$

leads to the same equation, up to the notation,

$$\frac{\partial u}{\partial\tau}=a\frac{\partial}{\partial\xi}\biggl(e^{\beta\xi}\frac{\partial u}{\partial\xi}\biggr).$$

5°. The substitution $z=e^{-\beta x}$ leads to an equation of the form 3.3.4.1:

$$\frac{\partial w}{\partial t}=a\beta^2z\frac{\partial^2 w}{\partial z^2}.$$

6°. A series solution of the original equation (under constant values of w at the boundary and at the initial instant) can be found in Lykov (1967).

5. $\displaystyle\boldsymbol{\frac{\partial w}{\partial t}=ae^{2\beta x}\frac{\partial^2 w}{\partial x^2}+\sqrt{a}\,e^{\beta x}\bigl(\sqrt{a}\,\beta e^{\beta x}+b\bigr)\frac{\partial w}{\partial x}}$.

This is a special case of equation 3.8.5.2 with $f(t)=b$ and $g(t)=0$. The substitution $\xi=\dfrac{1}{\sqrt{a}\,\beta}(1-e^{-\beta x})$ leads to a constant coefficient equation of the form 3.1.4 with $\Phi\equiv 0$:

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial\xi^2}+b\frac{\partial w}{\partial\xi}.$$

6. $\dfrac{\partial w}{\partial t} = ae^{2\beta x}\dfrac{\partial^2 w}{\partial x^2} + \sqrt{a}\, e^{\beta x}\big(\sqrt{a}\,\beta e^{\beta x} + be^{\mu t}\big)\dfrac{\partial w}{\partial x} + ce^{\nu t}w.$

This is a special case of equation 3.8.5.2 with $f(t) = be^{\mu t}$ and $g(t) = ce^{\nu t}$.

3.4.6 Other Equations

1. $\dfrac{\partial w}{\partial t} = ae^{\beta t}\dfrac{\partial^2 w}{\partial x^2} + be^{\mu t}\dfrac{\partial w}{\partial x} + ce^{\nu t}w.$

This is a special case of equation 3.8.7.3 with $f(t) = ae^{\beta t}$, $g(t) = be^{\mu t}$, and $h(t) = ce^{\nu t}$.

2. $\dfrac{\partial w}{\partial t} = ae^{\beta t}\dfrac{\partial^2 w}{\partial x^2} + bxe^{\mu t}\dfrac{\partial w}{\partial x} + cxe^{\nu t}w.$

This is a special case of equation 3.8.7.4 with $n(t)=ae^{\beta t}$, $f(t)=be^{\mu t}$, $g(t)=0$, $h(t)=ce^{\nu t}$, and $s(t) = 0$.

3. $\dfrac{\partial w}{\partial t} = ae^{\beta x+\mu t}\dfrac{\partial^2 w}{\partial x^2} + be^{\nu t}w.$

The transformation

$$w(x,t) = \exp\left(\frac{b}{\nu}e^{\nu t}\right)u(x,\tau), \quad \tau = \frac{a}{\mu}e^{\mu t}$$

leads to an equation of the form 3.4.5.1: $\partial_\tau u = e^{\beta x}\partial_{xx}u$.

3.5 Equations Containing Hyperbolic Functions and Arbitrary Parameters

3.5.1 Equations Containing a Hyperbolic Cosine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cosh^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\cosh^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cosh^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\cosh^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\cosh^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\cosh^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c\cosh^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\cosh^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cosh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\cosh^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\cosh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\cosh^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c\cosh^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\cosh^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Big(cx\cosh^k \omega t + \dfrac{b}{x}\Big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\cosh^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\cosh^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\cosh^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\cosh^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\cosh^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b\cosh^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\cosh^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(b\cosh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\cosh^k \omega t + c$.

3.5.2 Equations Containing a Hyperbolic Sine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\sinh^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\sinh^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\sinh^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\sinh^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\sinh^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\sinh^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c\sinh^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\sinh^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\sinh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\sinh^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\sinh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\sinh^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c\sinh^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\sinh^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Big(cx\sinh^k \omega t + \dfrac{b}{x}\Big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\sinh^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\sinh^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\sinh^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\sinh^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\sinh^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b\sinh^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\sinh^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(b\sinh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\sinh^k \omega t + c$.

3.5.3 Equations Containing a Hyperbolic Tangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\tanh^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\tanh^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\tanh^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\tanh^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\tanh^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\tanh^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c\tanh^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\tanh^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\tanh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\tanh^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\tanh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\tanh^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c\tanh^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\tanh^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Big(cx\tanh^k \omega t + \dfrac{b}{x}\Big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\tanh^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\tanh^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\tanh^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\tanh^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\tanh^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b\tanh^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\tanh^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(b\tanh^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\tanh^k \omega t + c$.

3.5.4 Equations Containing a Hyperbolic Cotangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\coth^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\coth^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b \coth^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\coth^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b \coth^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\coth^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c \coth^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\coth^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b \coth^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\coth^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b \coth^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\coth^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c \coth^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\coth^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Big(cx \coth^k \omega t + \dfrac{b}{x}\Big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\coth^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx \coth^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\coth^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b \coth^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\coth^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b \coth^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\coth^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(b \coth^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\coth^k \omega t + c$.

3.6 Equations Containing Logarithmic Functions and Arbitrary Parameters

3.6.1 Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$

1. $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (b\ln t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\ln t + c$.

The substitution $w(x,t) = u(x,t)\exp(bt\ln t - bt + ct)$ leads to the constant coefficient equation $\partial_t u = a\partial^2_{xx} u$, which is considered in Section 3.1.1.

2. $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx + c\ln t)w.$

This is a special case of equation 3.8.1.6 with $f(t) = b$ and $g(t) = c\ln t$.

3. $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + x(b\ln^k t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\ln^k t + c$.

4. $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (-bx^2 + c\ln^k t)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\ln^k t$.

5. $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (b\ln t + c)\frac{\partial w}{\partial x}.$

The change of variable $z = x + bt\ln t - bt + ct$ leads to the constant coefficient equation $\partial_t w = a\partial^2_{zz} w$ that is considered in Section 3.1.1.

6. $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + x(b\ln t + c)\frac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\ln t + c$.

On passing from t, x to the new variables (A and B are any numbers)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t), \qquad \text{where} \quad F(t) = B\exp(bt\ln t - bt + ct),$$

we arrive at the constant coefficient equation $\partial_\tau w = a\partial_{zz} w$ for $w(\tau, z)$; this equation is considered in Section 3.1.1.

3.6.2 Equations of the Form $\frac{\partial w}{\partial t} = ax^k\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$

1. $\frac{\partial w}{\partial t} = ax\frac{\partial^2 w}{\partial x^2} + (bx + c\ln t)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = 0$, $h(t) = b$, and $s(t) = c\ln t$.

2. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\ln t + c)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = 0$, $h(t) = b\ln t$, and $s(t) = c$.

3. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x} + (c\ln t + d)w.$

This is a special case of equation 3.8.8.1 with $f(t) = a$, $g(t) = b$, $h(t) = 0$, and $s(t) = c\ln t + d$.

4. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\ln t + c)w.$

This is a special case of equation 3.8.8.2 with $f(t) = a$, $g(t) = 0$, and $h(t) = b\ln t + c$.

5. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + b\ln x\, w.$

This is a special case of equation 3.8.8.3 with $n(t) = a$, $f(t) = g(t) = h(t) = p(t) = 0$, and $s(t) = b$.

6. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + bt^k\ln x\, w.$

This is a special case of equation 3.8.8.3 with $n(t) = a$, $f(t) = g(t) = h(t) = p(t) = 0$, and $s(t) = bt^k$.

7. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + b\ln^2 x\, w.$

This is a special case of equation 3.8.8.3 with $n(t) = a$, $f(t) = g(t) = s(t) = p(t) = 0$, and $h(t) = b$. See also equation 3.8.6.5.

8. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + bt^k\ln^2 x\, w.$

This is a special case of equation 3.8.8.3 with $n(t) = a$, $f(t) = g(t) = s(t) = p(t) = 0$, and $h(t) = bt^k$.

9. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + \big(b\ln^2 x + c\ln x\ln t + d\ln^2 t\big)w.$

This is a special case of equation 3.8.8.3 with $n(t) = a$, $f(t) = g(t) = 0$, $h(t) = b$, $s(t) = c\ln t$, and $p(t) = d\ln^2 t$.

10. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b\ln t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\ln t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + (b\ln t + c)w.$

This is a special case of equation 3.8.8.7 with $f(t) = a$, $g(t) = 0$, and $h(t) = b\ln t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(b\ln t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\ln t + c$.

3.7 Equations Containing Trigonometric Functions and Arbitrary Parameters

3.7.1 Equations Containing a Cosine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cos^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\cos^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cos^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\cos^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\cos^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\cos^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c\cos^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\cos^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cos^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\cos^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\cos^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\cos^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c\cos^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\cos^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left(cx\cos^k \omega t + \dfrac{b}{x}\right)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\cos^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\cos^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\cos^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\cos^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\cos^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b\cos^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\cos^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n \dfrac{\partial^2 w}{\partial x^2} + x(b\cos^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\cos^k \omega t + c$.

3.7.2 Equations Containing a Sine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\sin^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\sin^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\sin^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\sin^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\sin^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\sin^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c\sin^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\sin^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\sin^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\sin^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\sin^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\sin^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c\sin^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\sin^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Big(cx\sin^k \omega t + \dfrac{b}{x}\Big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\sin^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\sin^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\sin^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\sin^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\sin^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4 \dfrac{\partial^2 w}{\partial x^2} + (b \sin^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b \sin^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n \dfrac{\partial^2 w}{\partial x^2} + x(b \sin^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b \sin^k \omega t + c$.

3.7.3 Equations Containing a Tangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b \tan^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b \tan^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b \tan^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b \tan^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b \tan^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b \tan^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c \tan^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c \tan^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b \tan^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b \tan^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b \tan^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b \tan^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c \tan^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c \tan^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left(cx \tan^k \omega t + \dfrac{b}{x}\right)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c \tan^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx \tan^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b \tan^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2 \dfrac{\partial^2 w}{\partial x^2} + (b\tan^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\tan^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4 \dfrac{\partial^2 w}{\partial x^2} + (b\tan^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\tan^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n \dfrac{\partial^2 w}{\partial x^2} + x(b\tan^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\tan^k \omega t + c$.

3.7.4 Equations Containing a Cotangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cot^k \omega t + c)w.$

This is a special case of equation 3.8.1.1 with $f(t) = b\cot^k \omega t + c$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cot^k \omega t + cx)w.$

This is a special case of equation 3.8.1.6 with $f(t) = c$ and $g(t) = b\cot^k \omega t$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\cot^k \omega t + c)w.$

This is a special case of equation 3.8.1.3 with $f(t) = b\cot^k \omega t + c$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (c\cot^k \omega t - bx^2)w.$

This is a special case of equation 3.8.1.7 with $f(t) = c\cot^k \omega t$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\cot^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.1 with $f(t) = b\cot^k \omega t + c$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x(b\cot^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.3 with $f(t) = b\cot^k \omega t + c$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (c\cot^k \omega t + s)w.$

This is a special case of equation 3.8.3.1 with $f(t) = b$ and $g(t) = c\cot^k \omega t + s$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left(cx\cot^k \omega t + \dfrac{b}{x}\right)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.2.6 with $f(t) = c\cot^k \omega t$.

9. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + (bx\cot^k \omega t + c)w.$

This is a special case of equation 3.8.4.1 with $f(t) = b\cot^k \omega t$ and $g(t) = c$.

10. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + (b\cot^k \omega t + c)w.$

This is a special case of equation 3.8.4.2 with $f(t) = b\cot^k \omega t + c$.

11. $\dfrac{\partial w}{\partial t} = ax^4\dfrac{\partial^2 w}{\partial x^2} + (b\cot^k \omega t + c)w.$

This is a special case of equation 3.8.8.4 with $f(t) = a$ and $g(t) = b\cot^k \omega t + c$.

12. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + x(b\cot^k \omega t + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.4.5 with $f(t) = b\cot^k \omega t + c$.

3.8 Equations Containing Arbitrary Functions

3.8.1 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)w$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w.$

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = (Ax+B)\exp\Bigl[\int f(t)\,dt\Bigr],$$
$$w(x,t) = A(x^2+2at)\exp\Bigl[\int f(t)\,dt\Bigr],$$
$$w(x,t) = A\exp\Bigl[\lambda x + a\lambda^2 t + \int f(t)\,dt\Bigr],$$
$$w(x,t) = A\cos(\lambda x)\exp\Bigl[-a\lambda^2 t + \int f(t)\,dt\Bigr],$$
$$w(x,t) = A\sin(\lambda x)\exp\Bigl[-a\lambda^2 t + \int f(t)\,dt\Bigr].$$

2°. The substitution $w(x,t) = u(x,t)\exp\Bigl[\int f(t)\,dt\Bigr]$ leads to a constant coefficient equation, $\partial_t u = a\partial_{xx} u$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)w.$

This is a special case of equation 3.8.9 with $s(x) \equiv 1$, $p(x) = a = \text{const}$, $q(x) = -f(x)$, and $\Phi \equiv 0$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + xf(t)w.$

1°. Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp\Bigl[xF(t) + a\int F^2(t)\,dt\Bigr], \quad F(t) = \int f(t)\,dt,$$

$$w(x,t) = A\Bigl[x + 2a\int F(t)\,dt\Bigr]\exp\Bigl[xF(t) + a\int F^2(t)\,dt\Bigr],$$

$$w(x,t) = A\exp\Bigl[xF(t) + \lambda x + a\lambda^2 t + a\int F^2(t)\,dt + 2a\lambda\int F(t)\,dt\Bigr].$$

2°. The transformation

$$w(x,t) = u(z,t)\exp\Bigl[xF(t) + a\int F^2(t)\,dt\Bigr], \quad z = x + 2a\int F(t)\,dt,$$

where $F(t) = \displaystyle\int f(t)\,dt$, leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x^2 f(t)w.$

This is a special case of equation 3.8.7.5.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \bigl[f(x) + g(t)\bigr]w.$

1°. There are particular solutions in the product form (λ is an arbitrary constant)

$$w(x,t) = \exp\Bigl[\lambda t + \int g(t)\,dt\Bigr]\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + \bigl[f(x) - \lambda\bigr]\varphi = 0.$$

2°. The substitution $w(x,t) = u(x,t)\exp\Bigl[\displaystyle\int g(t)\,dt\Bigr]$ leads to an equation of the form 3.8.1.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x)u.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \bigl[xf(t) + g(t)\bigr]w.$

1°. Particular solutions (A and λ are arbitrary constants):

$$w(x,t) = A\exp\Bigl[xF(t) + a\int F^2(t)\,dt + \int g(t)\,dt\Bigr], \quad F(t) = \int f(t)\,dt,$$

$$w(x,t) = A\Bigl[x + 2a\int F(t)\,dt\Bigr]\exp\Bigl[xF(t) + a\int F^2(t)\,dt + \int g(t)\,dt\Bigr],$$

$$w(x,t) = \exp\Bigl[xF(t) + \lambda x + a\lambda^2 t + 2a\lambda\int F(t)\,dt + a\int F^2(t)\,dt + \int g(t)\,dt\Bigr].$$

2°. The transformation

$$w(x,t) = u(z,t)\exp\Big[xF(t) + a\int F^2(t)\,dt + \int g(t)\,dt\Big], \quad z = x + 2a\int F(t)\,dt,$$

where $F(t) = \displaystyle\int f(t)\,dt$, leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + [-bx^2 + f(t)]w.$

1°. Particular solutions (A is an arbitrary constant):

$$w(x,t) = A\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}x^2 + \sqrt{ab}\,t + \int f(t)\,dt\right],$$
$$w(x,t) = Ax\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}x^2 + 3\sqrt{ab}\,t + \int f(t)\,dt\right].$$

2°. The transformation (C is any number)

$$w(x,t) = u(z,\tau)\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}x^2 + \sqrt{ab}\,t + \int f(t)\,dt\right],$$
$$z = x\exp\big(2\sqrt{ab}\,t\big), \quad \tau = \frac{1}{4}\sqrt{\frac{a}{b}}\exp\big(4\sqrt{ab}\,t\big) + C$$

leads to a constant coefficient equation, $\partial_\tau u = \partial_{zz}u$, which is considered in Section 3.1.1.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + x[-bx + f(t)]w.$

1°. Particular solution (A and B are arbitrary constants):

$$w(x,t) = \exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}\,x^2 + xF(t) + \sqrt{ab}\,t + a\int_A^t F^2(\tau)\,d\tau\right],$$
$$F(t) = \exp\big(2\sqrt{ab}\,t\big)\int_B^t f(\tau)\exp\big(-2\sqrt{ab}\,\tau\big)\,d\tau.$$

2°. The transformation

$$w(x,t) = \exp\left(\frac{1}{2}\sqrt{\frac{b}{a}}\,x^2\right)u(z,\tau), \quad z = x\exp\big(2\sqrt{ab}\,t\big), \quad \tau = \frac{1}{4\sqrt{ab}}\exp\big(4\sqrt{ab}\,t\big)$$

leads to an equation of the form 3.8.1.6:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial z^2} + \left[z\Phi(\tau) + \frac{1}{4\tau}\right]u,$$

where $\Phi(\tau) = \dfrac{1}{(n\tau)^{3/2}}f\left(\dfrac{\ln\tau + \ln n}{n}\right)$, $n = 4\sqrt{ab}$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[x^2 f(t) + x g(t) + h(t)\big] w.$

This is a special case of equation 3.8.7.5.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x - bt) w.$

On passing from t, x to the new variables t, $\xi = x - bt$, we obtain an equation of the form 3.8.6.5:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial \xi^2} + b\frac{\partial w}{\partial \xi} + f(\xi)w.$$

3.8.2 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x}$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x}.$

This equation describes heat transfer in a moving medium where the velocity of motion is an arbitrary function of time.

1°. Particular solutions (A, B, and μ are arbitrary constants):

$$w(x,t) = Ax + A\int f(t)\,dt + B,$$
$$w(x,t) = A\left[x + \int f(t)\,dt\right]^2 + 2aAt + B,$$
$$w(x,t) = A\exp\left[\lambda x + a\lambda^2 t + \lambda\int f(t)\,dt\right] + B.$$

2°. On passing from t, x to the new variables t, $z = x + \displaystyle\int f(t)\,dt$, we obtain a constant coefficient equation, $\partial_t w = a\partial_{zz} w$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.8.6.4. This equation describes heat transfer in a moving medium where the velocity of motion is an arbitrary function of the coordinate.

1°. The equation has particular solutions of the form

$$w(x,t) = e^{-\lambda t}u(x),$$

where the function $u = u(x)$ is determined by solving the following ordinary differential equation with parameter λ:

$$au''_{xx} + f(x)u'_x + \lambda u = 0.$$

Other particular solutions (A and B are arbitrary constants):

$$w(x) = A + B\int F(x)\,dx, \quad F(x) = \exp\biggl[-\frac{1}{a}\int f(x)\,dx\biggr],$$
$$w(x,t) = Aat + A\int\biggl(\int\frac{dx}{F(x)}\biggr)F(x)\,dx + B,$$
$$w(x,t) = Aat\Phi(x) + A\int\biggl(\int\frac{\Phi(x)\,dx}{F(x)}\biggr)F(x)\,dx, \quad \Phi(x) = \int F(x)\,dx.$$

More sophisticated solutions are specified below.

2°. The original equation admits particular solutions of the form

$$w_n(x,t) = \sum_{i=0}^{n} t^i \varphi_{n,i}(x) \tag{1}$$

for any $f(x)$. Substituting the expression of (1) into the original equation and matching the coefficients of like powers of t, we arrive at the following system of ordinary differential equations for $\varphi_{n,i} = \varphi_{n,i}(x)$:

$$a\varphi''_{n,n} + f(x)\varphi'_{n,n} = 0,$$
$$a\varphi''_{n,i} + f(x)\varphi'_{n,i} = (i+1)\varphi_{n,i+1}; \quad i = 0,\,1,\,\dots,\,n-1,$$

where the prime denotes the derivative with respect to x. Integrating these equations successively in order of decreasing number i, we obtain the solution (A and B are any numbers):

$$\begin{aligned}\varphi_{n,n}(x) &= A + B\int F(x)\,dx, \quad F(x) = \exp\biggl[-\frac{1}{a}\int f(x)\,dx\biggr],\\ \varphi_{n,i}(x) &= n(n-1)\dots(i+1)\boldsymbol{L}_f^{n-i}\bigl[\varphi_{n,n}(x)\bigr]; \quad i = 0,\,1,\,\dots,\,n-1.\end{aligned} \tag{2}$$

Here, the integral operator $\boldsymbol{L}_f$ is introduced as follows:

$$\boldsymbol{L}_f[y(x)] \equiv \frac{1}{a}\int F(x)\biggl(\int\frac{y(x)\,dx}{F(x)}\biggr)dx. \tag{3}$$

The powers of the operator are defined by $\boldsymbol{L}_f^i[y(x)] = \boldsymbol{L}_f\bigl[\boldsymbol{L}_f^{i-1}[y(x)]\bigr]$.

Formulas (1)–(3) give an exact analytical solution of the original equation for arbitrary $f(x)$.

A linear combination of particular solutions

$$w(x,t) = \sum_{n=0}^{N} C_n w_n(x,t) \qquad (C_n \text{ are arbitrary constants})$$

is also a particular solution of the original equation.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x}.$

Generalized Ilkovič equation. The equation describes mass transfer to the surface of a growing drop that flows out of a thin capillary into a fluid solution (the mass rate of flow of the fluid moving through the capillary is an arbitrary function of time).

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = AxF(t) + B, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

$$w(x,t) = Ax^2F^2(t) + 2Aa\int F^2(t)\,dt + B,$$

$$w(x,t) = A\exp\left[\lambda xF(t) + a\lambda^2\int F^2(t)\,dt\right] + B.$$

2°. On passing from t, x to the new variables (A is any number)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t), \qquad \text{where} \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = a\partial_{zz}w$, which is considered in Section 3.1.1.

3°. Consider the special case where the heat exchange occurs with a semiinfinite medium; the medium has a uniform temperature w_0 at the initial instant $t=0$ and the boundary $x=0$ is maintained at a constant temperature w_1 all the time. In this case, the original equation subject to the initial and boundary conditions

$$\begin{aligned} &w = w_0 \quad \text{at} \quad t = 0 \quad &&(\text{initial condition}),\\ &w = w_1 \quad \text{at} \quad x = 0 \quad &&(\text{boundary condition}),\\ &w \to w_0 \quad \text{at} \quad x \to \infty \quad &&(\text{boundary condition}) \end{aligned}$$

has the solution

$$\frac{w - w_1}{w_0 - w_1} = \operatorname{erf}\left(\frac{z}{2\sqrt{a\tau}}\right), \quad \operatorname{erf}\xi = \frac{2}{\sqrt{\pi}}\int_0^\xi \exp\bigl(-\zeta^2\bigr)\,d\zeta,$$

$$\tau = \int_0^t F^2(\zeta)\,d\zeta, \quad z = xF(t), \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

where $\operatorname{erf}\xi$ is the error function.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x - bt)\dfrac{\partial w}{\partial x}.$

1°. Particular solutions (A and B are arbitrary constants):

$$w(x,t) = A + B\int F(z)\,dz, \quad F(z) = \exp\left[-\frac{1}{a}\int f(z)\,dz - \frac{b}{a}z\right],$$

$$w(x,t) = Aat + A\int\left(\int\frac{dz}{F(z)}\right)F(z)\,dz,$$

$$w(x,t) = Aat\Phi(z) + A\int\left(\int\frac{\Phi(z)\,dz}{F(z)}\right)F(z)\,dx, \quad \Phi(z) = \int F(z)\,dz,$$

where $z = x - bt$.

2°. On passing from t, x to the new variables t, $z = x - bt$, we obtain a separable equation of the form 3.8.2.2:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + \bigl[f(z) + b\bigr]\frac{\partial w}{\partial z}.$$

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \frac{1}{\sqrt{t}}f\Bigl(\frac{x}{\sqrt{t}}\Bigr)\frac{\partial w}{\partial x}.$

1°. On passing from t, x to the new variables $\tau = \ln t$, $\xi = x/\sqrt{t}$, we obtain a separable equation of the form 3.8.2.2:

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial \xi^2} + \bigl[f(\xi) + \tfrac{1}{2}\xi\bigr]\frac{\partial w}{\partial \xi}.$$

2°. Consider the special case where the heat exchange occurs with a semiinfinite medium; the medium has a uniform temperature w_0 at the initial instant $t = 0$ and the boundary $x = 0$ is maintained at a constant temperature w_1 all the time. In this case, the original equation subject to the initial and boundary conditions

$$\begin{aligned} &w = w_0 \quad \text{at} \quad t = 0 \qquad &&\text{(initial condition)},\\ &w = w_1 \quad \text{at} \quad x = 0 \qquad &&\text{(boundary condition)},\\ &w \to w_0 \quad \text{at} \quad x \to \infty \qquad &&\text{(boundary condition)} \end{aligned}$$

has the solution

$$\frac{w - w_0}{w_1 - w_0} = \frac{\int_\xi^\infty \exp\bigl[-\Phi(\xi)\bigr]\,d\xi}{\int_0^\infty \exp\bigl[-\Phi(\xi)\bigr]\,d\xi}, \qquad \Phi(\xi) = \frac{1}{4a}\xi^2 + \frac{1}{a}\int_0^\xi f(\xi)\,d\xi,$$

where $\xi = x/\sqrt{t}$.

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \Bigl[xf(t) + \frac{b}{x}\Bigr]\frac{\partial w}{\partial x}.$

1°. Particular solutions (A and B are arbitrary constants):

$$w(x,t) = Ax^n \exp\Bigl[n\int f(t)\,dt\Bigr] + B, \quad n = 1 - \frac{b}{a},$$

$$w(x,t) = Ax^2F(t) + 2A(a+b)\int F(t)\,dt + B, \quad F(t) = \exp\Bigl[2\int f(t)\,dt\Bigr].$$

2°. On passing from t, x to the new variables (A is any number)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t), \qquad \text{where} \quad F(t) = \exp\Bigl[\int f(t)\,dt\Bigr],$$

for the function $w(\tau, z)$ we obtain a simpler equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial z^2} + \frac{b}{z}\frac{\partial w}{\partial z},$$

which is considered in Sections 3.2.1, 3.2.3, and 3.2.5.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation (A, B, and C are any numbers)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t) + \int g(t)F(t)\,dt + C, \qquad F(t) = B\exp\biggl[\int f(t)\,dt\biggr],$$

leads to a constant coefficient equation, $\partial_\tau w = a\partial_{zz} w$, which is considered in Section 3.1.1.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{f(t)}{x}\dfrac{\partial w}{\partial x}.$

Particular solutions:

$$w = \frac{A}{\sqrt{t}}\exp\biggl[-\frac{x^2}{4at} - \frac{1}{2a}\int \frac{f(t)}{t}\,dt\biggr] + B,$$
$$w = x^2 + 2at + 2\int f(t)\,dt + A,$$
$$w = x^4 + p(t)x^2 + q(t),$$

where

$$p(t) = 12at + 4\int f(t)\,dt + A, \quad q(t) = 2\int \big[a + f(t)\big]p(t)\,dt + B,$$

with A and B being arbitrary constants. The second and third solutions are special cases of a solution having the form

$$w = x^{2n} + A_{2n-2}(t)x^{2n-2} + \cdots + A_2(t)x^2 + A_0(t),$$

which contains n arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Bigl[xf(t) + \dfrac{g(t)}{x}\Bigr]\dfrac{\partial w}{\partial x}.$

On passing from t, x to the new variables (A is any number)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t), \qquad \text{where} \quad F(t) = \exp\biggl[\int f(t)\,dt\biggr],$$

for the function $w(\tau, z)$ we obtain a simpler equation of the form 3.8.2.8:

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial z^2} + \frac{\varphi(\tau)}{z}\frac{\partial w}{\partial z}.$$

The function $\varphi = \varphi(\tau)$ is defined parametrically as

$$\varphi = \frac{g(t)}{F(t)}, \quad \tau = \int F^2(t)\,dt + A.$$

3.8.3 Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(t)w.$

This is a special case of equation 3.8.7.3.

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = \bigl[Ax + AF(t) + B\bigr]\exp\biggl[\int g(t)\,dt\biggr], \quad F(t) = \int f(t)\,dt,$$

$$w(x,t) = A\bigl\{\bigl[x + F(t)\bigr]^2 + 2at\bigr\}\exp\biggl[\int g(t)\,dt\biggr],$$

$$w(x,t) = A\exp\biggl[a\lambda^2 t + \int g(t)\,dt \pm \lambda F(t) \pm \lambda x\biggr],$$

$$w(x,t) = A\exp\biggl[-a\lambda^2 t + \int g(t)\,dt\biggr]\cos\bigl[\lambda x + \lambda F(t)\bigr],$$

$$w(x,t) = A\exp\biggl[-a\lambda^2 t + \int g(t)\,dt\biggr]\sin\bigl[\lambda x + \lambda F(t)\bigr].$$

2°. The transformation

$$w(x,t) = u(z,t)\exp\biggl[\int g(t)\,dt\biggr], \quad z = x + \int f(t)\,dt$$

leads to a constant coefficient equation, $\partial_t u = a\partial_{zz}u$, which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + g(t)w.$

1°. Particular solutions (A and B are arbitrary constants):

$$w(x,t) = \biggl[A + B\int F(x)\,dx\biggr]G(t),$$

$$w(x,t) = A\biggl[at + \int F(x)\biggl(\int \frac{dx}{F(x)}\biggr)dx\biggr]G(t),$$

$$w(x,t) = A\biggl[at\Psi(x) + \int F(x)\biggl(\int \frac{\Psi(x)\,dx}{F(x)}\biggr)dx\biggr]G(t).$$

The following notation is used here:

$$G(t) = \exp\biggl[\int g(t)\,dt\biggr], \quad F(x) = \exp\biggl[-\frac{1}{a}\int f(x)\,dx\biggr], \quad \Psi(x) = \int F(x)\,dx.$$

2°. The substitution $w(x,t) = u(x,t)\exp\biggl[\int g(t)\,dt\biggr]$ leads to an equation of the form 3.8.2.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x}.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + g(x)w.$

This is a special case of equation 3.8.6.5.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation (A, B, and C are any numbers)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t) + C, \quad w(t,x) = u(\tau,z)\exp\left[\int g(t)\,dt\right],$$

where $F(t) = B\exp\left[\int f(t)\,dt\right]$, leads to a constant coefficient equation, $\partial_\tau u = a\partial_{zz}u$, which is considered in Section 3.1.1.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left[xf(t) + \dfrac{b}{x}\right]\dfrac{\partial w}{\partial x} + g(t)w.$

The substitution $w(x,t) = u(x,t)\exp\left[\int g(t)\,dt\right]$ leads to an equation of the form 3.8.2.6:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \left[xf(t) + \frac{b}{x}\right]\frac{\partial u}{\partial x}.$$

For the special case $b = 0$, see equation 3.8.2.3.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + [xf(t) + g(t)]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation (A, B, and C are any numbers)

$$\tau = \int F^2(t)\,dt + A, \quad z = xF(t) + \int g(t)F(t)\,dt + C, \quad w(t,x) = u(\tau,z)\exp\left[\int h(t)\,dt\right],$$

where $F(t) = B\exp\left[\int f(t)\,dt\right]$, leads to a constant coefficient equation, $\partial_\tau u = a\partial_{zz}u$, which is considered in Section 3.1.1.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + [xf(t) + g(t)]\dfrac{\partial w}{\partial x} + [xh(t) + s(t)]w.$

This is a special case of equation 3.8.7.4 with $n(t) = a$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + [xf(t) + g(t)]\dfrac{\partial w}{\partial x} + [x^2h(t) + xs(t) + p(t)]w.$

This is a special case of equation 3.8.7.5 with $n(t) = a$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{f(t)}{x}\dfrac{\partial w}{\partial x} + g(t)w.$

1°. Particular solutions (A, B, and C are arbitrary constants):

$$w = \frac{A}{\sqrt{t}}\exp\left[-\frac{x^2}{4at} - \frac{1}{2a}\int\frac{f(t)}{t}\,dt + \int g(t)\,dt\right],$$

$$w = A\exp\left[\int g(t)\,dt\right]\left[x^2 + 2at + 2\int f(t)\,dt + B\right],$$

$$w = A\exp\left[\int g(t)\,dt\right]\left[x^4 + p(t)x^2 + q(t)\right],$$

where

$$p(t) = 12at + 4\int f(t)\,dt + B, \quad q(t) = 2\int\left[a + f(t)\right]p(t)\,dt + C.$$

2°. The substitution $w = \exp\left[\int g(t)\,dt\right]u(x,t)$ leads to an equation of the form 3.8.2.8 for $u = u(x,t)$.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left[xf(t) + \dfrac{g(t)}{x}\right]\dfrac{\partial w}{\partial x} + h(t)w.$

The substitution $w = \exp\left[\int h(t)\,dt\right]u(x,t)$ leads to an equation of the form 3.8.2.9 for $u = u(x,t)$.

3.8.4 Equations of the Form $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w$

1. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial^2 w}{\partial x^2} + [xf(t) + g(t)]w.$

This is a special case of equation 3.8.8.1.

2. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + f(t)w.$

This is a special case of equation 3.8.8.2. The transformation

$$w(x,t) = u(z,t)\exp\left[\int f(t)\,dt\right], \quad z = \ln|x|$$

leads to a constant coefficient equation of the form 3.1.4:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial z^2} - a\frac{\partial u}{\partial z}.$$

3. $\dfrac{\partial w}{\partial t} = ax^2\dfrac{\partial^2 w}{\partial x^2} + \ln x\, f(t)w.$

This is a special case of equation 3.8.8.3.

4. $\dfrac{\partial w}{\partial t} = ax^2 \dfrac{\partial^2 w}{\partial x^2} + \ln^2 x\, f(t)w.$

This is a special case of equation 3.8.8.3.

5. $\dfrac{\partial w}{\partial t} = ax^n \dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x}.$

1°. Particular solutions (A and B are arbitrary constants):

$$w(x,t) = Ax \exp\left[\int f(t)\,dt\right] + B,$$
$$w(x,t) = Ax^{2-n}F(t) + Aa(n-1)(n-2)\int F(t)\,dt,$$

where

$$F(t) = \exp\left[(2-n)\int f(t)\,dt\right].$$

2°. On passing from t, x to the new variables

$$z = xF(t), \quad \tau = a\int F^{2-n}(t)\,dt,$$

where

$$F(t) = \exp\left[\int f(t)\,dt\right],$$

we obtain an equation of the form 3.3.6.6:

$$\frac{\partial w}{\partial \tau} = z^n \frac{\partial^2 w}{\partial z^2}.$$

6. $\dfrac{\partial w}{\partial t} = ax^n \dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x} + bw.$

The substitution $w(x,t) = e^{bt}u(x,t)$ leads to an equation of the form 3.8.4.5:

$$\frac{\partial u}{\partial t} = ax^n \frac{\partial^2 u}{\partial x^2} + xf(t)\frac{\partial u}{\partial x}.$$

7. $\dfrac{\partial w}{\partial t} = ax^{2n} \dfrac{\partial^2 w}{\partial x^2} + \sqrt{a}\,x^n\big[\sqrt{a}\,nx^{n-1} + f(t)\big]\dfrac{\partial w}{\partial x} + g(t)w.$

The substitution

$$\xi = \frac{1}{\sqrt{a}}\begin{cases} \dfrac{x^{1-n}}{1-n} & \text{if } n \neq 1, \\ \ln|x| & \text{if } n = 1 \end{cases}$$

leads to a special case of equation 3.8.7.3, namely,

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + f(t)\frac{\partial w}{\partial \xi} + g(t)w.$$

3.8.5 Equations of the Form $\frac{\partial w}{\partial t} = ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$

1. $\frac{\partial w}{\partial t} = ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + f(t)w.$

The substitution $w(x,t)=u(x,t)\exp\Big[\int f(t)\,dt\Big]$ leads to an equation of the form 3.4.5.1:

$$\frac{\partial u}{\partial t} = ae^{\beta x}\frac{\partial^2 u}{\partial x^2}.$$

2. $\frac{\partial w}{\partial t} = ae^{2\beta x}\frac{\partial^2 w}{\partial x^2} + \sqrt{a}\,e^{\beta x}\big[\sqrt{a}\,\beta e^{\beta x} + f(t)\big]\frac{\partial w}{\partial x} + g(t)w.$

The substitution $\xi = \frac{1}{\beta\sqrt{a}}\big(1 - e^{-\beta x}\big)$ leads to a special case of equation 3.8.7.3:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + f(t)\frac{\partial w}{\partial \xi} + g(t)w.$$

3. $\frac{\partial w}{\partial t} = ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + f(x)\frac{\partial w}{\partial x}.$

This is a special case of equation 3.8.6.4.

1°. The equation has particular solutions of the form

$$w(x,t) = e^{-\lambda t}u(x), \tag{1}$$

where the function $u(x)$ is determined by solving the following linear ordinary differential equation with parameter λ:

$$ae^{\beta x}u''_{xx} + f(x)u'_x + \lambda u = 0. \tag{2}$$

2°. Other particular solutions (A and B are arbitrary constants):

$$w(x) = A + B\int F(x)\,dx, \quad F(x) = \exp\Big[-\frac{1}{a}\int e^{-\beta x}f(x)\,dx\Big],$$

$$w(x,t) = Aat + A\int F(x)\bigg(\int \frac{dx}{e^{\beta x}F(x)}\bigg)dx,$$

$$w(x,t) = Aat\Phi(x) + A\int F(x)\bigg(\int \frac{\Phi(x)\,dx}{e^{\beta x}F(x)}\bigg)dx, \quad \Phi(x) = \int F(x)\,dx.$$

4. $\frac{\partial w}{\partial t} = ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + f(x)\frac{\partial w}{\partial x} + g(t)w.$

This is a special case of equation 3.8.6.6.

The substitution $w(x,t) = u(x,t)\exp\Big[\int g(t)\,dt\Big]$ leads to an equation of the form 3.8.5.3:

$$\frac{\partial u}{\partial t} = ae^{\beta x}\frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x}.$$

3.8.6 Equations of the Form $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x,t)\dfrac{\partial w}{\partial x} + h(x,t)w$

1. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2}.$

This is an equation of the form 3.8.9 with $s(x) = 1/f(x)$, $p(x) \equiv 1$, $q(x) \equiv 0$, and $\Phi(x,t) \equiv 0$.

1°. The equation has particular solutions of the form

$$w(x,t) = e^{-\lambda t}u(x), \tag{1}$$

where the function $u(x)$ is determined by solving the following linear ordinary differential equation with parameter λ:

$$f(x)u''_{xx} + \lambda u = 0. \tag{2}$$

The procedure for constructing solutions to specific boundary value problems for the original equations with the help of particular solutions of the form (1) is described in detail in Sections 15.1.1 and 15.1.2.

The main problem here is to investigate the auxiliary equation (2), which is far from always admitting a closed-form solution; therefore, recourse to numerical solution methods is often necessary. Many specific solvable equations of the form (2) can be found in the handbooks by Murphy (1960), Kamke (1977), and Polyanin and Zaitsev (2003).

2°. Particular solutions (A, B, and x_0 are arbitrary constants):

$$\begin{aligned}
w(x) &= Ax + B,\\
w(x,t) &= At + AF(x), \quad F(x) = \int_{x_0}^{x} \frac{x-\xi}{f(\xi)}\,d\xi,\\
w(x,t) &= Atx + AG(x), \quad G(x) = \int_{x_0}^{x} \frac{x-\xi}{f(\xi)}\xi\,d\xi,\\
w(x,t) &= At^2 + 2AtF(x) + 2A\int_{x_0}^{x} \frac{x-\xi}{f(\xi)}F(\xi)\,d\xi,\\
w(x,t) &= At^2x + 2AtG(x) + 2A\int_{x_0}^{x} \frac{x-\xi}{f(\xi)}G(\xi)\,d\xi.
\end{aligned}$$

More sophisticated solutions are specified below in Item 3°.

3°. For any function $f(x)$, the original equation admits exact analytical solutions of the form

$$w_n(x,t) = t^n + \sum_{i=0}^{n-1} t^i \varphi_{n,i}(x). \tag{3}$$

Substituting expression (3) into the original equation and matching the coefficients of like powers of t, we arrive at the following system of ordinary differential equations for $\varphi_{n,i} = \varphi_{n,i}(x)$:

$$\begin{aligned}
&f(x)\varphi''_{n,i} = (i+1)\varphi_{n,i+1},\\
&i = 0,\ 1,\ \ldots,\ n-1; \quad \varphi_{n,n} \equiv 1,
\end{aligned}$$

where the prime stands for the differentiation with respect to x. Integrating these equations successively in order of decreasing number i, we obtain

$$\varphi_{n,i}(x) = n(n-1)\dots(i+1)\boldsymbol{L}_f^{n-i}[1]. \tag{4}$$

Here, the integral operator $\boldsymbol{L}_f$ is introduced as follows:

$$\boldsymbol{L}_f[y(x)] \equiv \int\left(\int \frac{y(x)}{f(x)}dx\right)dx = \int_{x_0}^{x} \frac{x-\xi}{f(\xi)}y(\xi)\,d\xi + Ax + B, \tag{5}$$

where x_0, A, and B are arbitrary constants. The powers of the operator are defined by the usual relation $\boldsymbol{L}_f^i[y(x)] = \boldsymbol{L}_f\big[\boldsymbol{L}_f^{i-1}[y(x)]\big]$; generally speaking, the constants A and B are not the same in repeated actions of $\boldsymbol{L}_f$ in this formula.

Formulas (3) and (4) determine an exact analytical solution of the original equation for arbitrary $f(x)$.

A linear combination of particular solutions (3),

$$w(x,t) = \sum_{n=0}^{N} C_n w_n(x,t) \qquad (C_n \text{ are arbitrary constants})$$

is also a particular solution of the original equation.

The original equation also admits other exact analytical solutions, specifically,

$$w_n(x,t) = t^n x + \sum_{i=0}^{n-1} t^i \phi_{n,i}(x), \quad \phi_{n,i}(x) = n(n-1)\dots(i+1)\boldsymbol{L}_f^{n-i}[x],$$

where n is a positive integer and the operator $\boldsymbol{L}_f$ is given by relation (5). A linear combination of these solutions with a linear combination of solutions (3) is also a solution of the original equation.

For the structure of other particular solutions, see equation 3.8.6.5 (Remark 3.7).

4°. The equation admits the following infinite-series solution that contains an arbitrary function of the coordinate:

$$w(x,t) = \Theta(x) + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \boldsymbol{L}^n[\Theta(x)], \qquad \boldsymbol{L} \equiv f(x)\frac{d^2}{dx^2},$$

where $\Theta(x)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = \Theta(x)$.

5°. Below are two discrete transformations that preserve the form of the original equation; the function f is subject to changes.

5.1. The transformation

$$z = \frac{1}{x}, \quad u = \frac{w}{x} \qquad \text{(point transformation)}$$

leads to a similar equation

$$\frac{\partial u}{\partial t} = z^4 f\left(\frac{1}{z}\right)\frac{\partial^2 u}{\partial z^2}.$$

5.2. First, we perform the change of variable

$$\xi = \int \frac{dx}{f(x)}$$

to obtain the equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial \xi}\left[F(\xi)\frac{\partial w}{\partial \xi}\right],$$

where the function $F = F(\xi)$ is defined parametrically as

$$F = \frac{1}{f(x)}, \quad \xi = \int \frac{dx}{f(x)}.$$

Introducing the new dependent variable $v = v(\xi, t)$ by the formula

$$w = \frac{\partial v}{\partial \xi} \qquad \text{(Bäcklund transformation)}$$

and integrating the resulting equation with respect to ξ, we arrive at the desired equation

$$\frac{\partial v}{\partial t} = F(\xi)\frac{\partial^2 v}{\partial \xi^2}.$$

(Here, the function v is defined up to an arbitrary additive term that depends on t.)

For power-law and exponential functions the above transformation acts as follows:

$$f(x) = bx^n \quad \Longrightarrow \quad F(\xi) = A\xi^{\frac{n}{n-1}},$$
$$f(x) = be^{-\beta x} \quad \Longrightarrow \quad F(\xi) = \beta\xi,$$

where $A = \dfrac{1}{b}\big[b(1-n)\big]^{\frac{n}{n-1}}$.

2. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + \Phi(x,t).$

This is an equation of the form 3.8.9 with $s(x) = 1/f(x)$, $p(x) \equiv 1$, and $q(x) \equiv 0$. For $\Phi(x,t) \equiv 0$, see equation 3.8.6.1.

1°. For

$$\Phi(x,t) = g_n(x)t^n \qquad (n = 0,\, 1,\, 2,\, \dots)$$

and arbitrary functions $f(x)$ and $g_n(x)$, the original equation has a particular solution of the form

$$\bar{w}_n(x,t) = \sum_{i=0}^{n} t^i \psi_{n,i}(x). \tag{1}$$

The functions $\psi_{n,i} = \psi_{n,i}(x)$ are calculated by the formulas

$$\psi_{n,i}(x) = \begin{cases} -\boldsymbol{L}_f\big[g_n(x)\big] & \text{if } i = n, \\ -n(n-1)\dots(i+1)\boldsymbol{L}_f^{n-i+1}\big[g_n(x)\big] & \text{if } i = 0,\, 1,\, \dots,\, n-1 \end{cases} \tag{2}$$

with the aid of the integral operator $\boldsymbol{L}_f$ that is defined by relation (5) in equation 3.8.6.1.

2°. If the nonhomogeneous part of the equation can be represented in the form

$$\Phi(x,t)=\sum_{n=1}^{N} g_n(x)t^n,$$

then there is a particular solution that is the sum of particular solutions of the form (1):

$$\bar{w}(x,t)=\sum_{n=1}^{N} \bar{w}_n(x,t).$$

For example, if

$$\Phi(x,t)=g(x)t+h(x),$$

where $g(x)$ and $h(x)$ are arbitrary functions, the original equation has a solution of the form

$$\bar{w}(x,t)=-t\psi(x)-\int_{x_0}^{x}\frac{\psi(\xi)+h(\xi)}{f(\xi)}(x-\xi)\,d\xi,$$
$$\psi(x)=\int_{x_0}^{x}\frac{g(\xi)}{f(\xi)}(x-\xi)\,d\xi,\quad x_0 \text{ is any.}$$

For the structure of particular solutions for other $\Phi(x,t)$, see equation 3.8.6.5, Item 3°.

By summing different solutions of the homogeneous equation (see equation 3.8.6.1) and any particular solution of the nonhomogeneous equation, one can obtain a wide class of particular solutions of the nonhomogeneous equation.

3. $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\Big[f(x)\dfrac{\partial w}{\partial x}\Big].$

This is a special case of equation 3.8.6.4 with $g(x)=f'_x(x)$. The equation describes heat transfer in a quiescent medium (solid body) in the case where the thermal diffusivity $f(x)$ is a coordinate dependent function.

1°. Particular solutions (A and B are arbitrary constants):

$$w(x)=A+B\int\frac{dx}{f(x)},$$
$$w(x,t)=At+A\int\frac{x\,dx}{f(x)}+B,$$
$$w(x,t)=At\varphi(x)+A\int\Big(\int\varphi(x)\,dx\Big)\frac{dx}{f(x)}+B,\quad \varphi(x)=\int\frac{dx}{f(x)},$$
$$w(x,t)=At^2+2At\psi(x)+2A\int\Big(\int\psi(x)\,dx\Big)\frac{dx}{f(x)}+B,\quad \psi(x)=\int\frac{x\,dx}{f(x)},$$
$$w(x,t)=At^2\varphi(x)+2AtI(x)+2A\int\Big(\int I(x)\,dx\Big)\frac{dx}{f(x)}+B,$$
$$I(x)=\int\Big(\int\varphi(x)\,dx\Big)\frac{dx}{f(x)}.$$

2°. A solution in the form of an infinite series:

$$w(x,t) = \Theta(x) + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \boldsymbol{L}^n \bigl[\Theta(x)\bigr], \qquad \boldsymbol{L} \equiv \frac{d}{dx}\left[f(x)\frac{d}{dx}\right].$$

It contains an arbitrary function of the space variable, $\Theta = \Theta(x)$. This solution satisfies the initial condition $w(x,0) = \Theta(x)$.

3°. The transformation

$$w(x,t) = \varphi(x)u(\xi,t), \quad \xi = -\int \varphi^2(x)\,dx, \qquad \varphi(x) = \int \frac{dx}{f(x)},$$

leads to the analogous equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \xi}\left[F(\xi)\frac{\partial u}{\partial \xi}\right],$$

where the function $F(\xi)$ is defined parametrically as

$$F(\xi) = f(x)\psi^4(x), \quad \xi = -\int \varphi^2(x)\,dx, \qquad \varphi(x) = \int \frac{dx}{f(x)}.$$

4°. The substitution $z = \displaystyle\int \frac{dx}{f(x)}$ leads to an equation of the form 3.8.6.1:

$$\frac{\partial w}{\partial t} = g(z)\frac{\partial^2 w}{\partial z^2},$$

where the function $g(z)$ is defined parametrically as

$$g(z) = \frac{1}{f(x)}, \quad z = \int \frac{dx}{f(x)}.$$

4. $\boldsymbol{\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x}}$**.**

1°. This equation can be rewritten in the form

$$s(x)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right], \tag{1}$$

where

$$s(x) = \frac{1}{f(x)}\exp\left[\int \frac{g(x)}{f(x)}\,dx\right], \quad p(x) = \exp\left[\int \frac{g(x)}{f(x)}\,dx\right].$$

For solutions of equation (1), see Section 3.8.9 with $q(x) \equiv 0$.

2°. There are particular solutions of the form

$$w(x,t) = e^{-\lambda t}u(x), \tag{2}$$

where the function $u(x)$ is identified by solving the following linear ordinary differential equation with parameter λ:

$$f(x)u''_{xx} + g(x)u'_x + \lambda u = 0. \tag{3}$$

A procedure for constructing solutions to specific boundary value problems for the original equation with the aid of particular solutions (2) is described in detail in Sections 15.1.1 and 15.1.2. A good deal of specific solvable equations of the form (3) can be found in Murphy (1960), Kamke (1977), and Polyanin and Zaitsev (2003).

3°. Other particular solutions (A and B are arbitrary constants):

$$w(x) = A + B\int F(x)\,dx, \quad F(x) = \exp\biggl[-\int \frac{g(x)}{f(x)}dx\biggr],$$
$$w(x,t) = At + A\int F(x)\biggl(\int \frac{dx}{f(x)F(x)}\biggr)dx,$$
$$w(x,t) = At\Phi(x) + A\int F(x)\biggl(\int \frac{\Phi(x)\,dx}{f(x)F(x)}\biggr)dx, \quad \Phi(x) = \int F(x)\,dx.$$

More sophisticated solutions are presented below in Item 4°.

4°. For any $f(x)$ and $g(x)$, the original equation admits particular solutions of the form

$$w_n(x,t) = \sum_{i=0}^{n} t^i \varphi_{n,i}(x). \tag{4}$$

Substituting expression (4) into the original equation and matching the coefficients of like powers of t, we arrive at the following system of ordinary differential equations for $\varphi_{n,i} = \varphi_{n,i}(x)$:

$$f(x)\varphi''_{n,n} + g(x)\varphi'_{n,n} = 0,$$
$$f(x)\varphi''_{n,i} + g(x)\varphi'_{n,i} = (i+1)\varphi_{n,i+1}, \quad i = 0,\,1,\,\ldots,\,n-1,$$

where the prime stands for the differentiation with respect to x. Integrating these equations successively in order of decreasing number i, we obtain (A and B are any numbers)

$$\begin{aligned} &\varphi_{n,n}(x) = A + B\int F(x)\,dx, \quad F(x) = \exp\biggl[-\int \frac{g(x)}{f(x)}dx\biggr],\\ &\varphi_{n,i}(x) = n(n-1)\ldots(i+1)\boldsymbol{L}_f^{n-i}\bigl[\varphi_{n,n}(x)\bigr]; \quad i = 0,\,1,\,\ldots,\,n-1. \end{aligned} \tag{5}$$

Here, the integral operator $\boldsymbol{L}_f$ is introduced as follows:

$$\boldsymbol{L}_f[y(x)] \equiv \int F(x)\biggl(\int \frac{y(x)\,dx}{f(x)F(x)}\biggr)dx. \tag{6}$$

The powers of the operator are defined as $\boldsymbol{L}_f^i[y(x)] = \boldsymbol{L}_f\bigl[\boldsymbol{L}_f^{i-1}[y(x)]\bigr]$.

Formulas (4)–(6) determine an exact analytical solution of the original equation for arbitrary $f(x)$.

A linear combination of particular solutions (4),

$$w(x,t) = \sum_{n=0}^{N} C_n w_n(x,t) \qquad (C_n \text{ are arbitrary numbers}),$$

is also a particular solution of the homogeneous equation.

For the structure of other particular solutions, see equation 3.8.6.5 (Remark 3.7).

5°. The substitution $\xi = \int \varphi(x)\,dx$, $\varphi(x) = \exp\left[-\int \frac{g(x)}{f(x)}dx\right]$ leads to an equation of the form 3.8.6.1:

$$\frac{\partial w}{\partial t} = F(\xi)\frac{\partial^2 w}{\partial \xi^2},$$

where the function $F = F(\xi)$ is determined by eliminating x from the relations

$$F = f(x)\varphi^2(x), \quad \xi = \int \varphi(x)\,dx.$$

6°. An infinite series solution containing an arbitrary function of the coordinate:

$$w(x,t) = \Theta(x) + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \boldsymbol{L}^n[\Theta(x)], \qquad \boldsymbol{L} \equiv f(x)\frac{d^2}{dx^2} + g(x)\frac{d}{dx},$$

where $\Theta(x)$ is any infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = \Theta(x)$.

5. $\boldsymbol{\frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x)\frac{\partial w}{\partial x} + h(x)w + \Phi(x,t).}$

1°. This equation can be rewritten in the form

$$s(x)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right] - q(x)w + s(x)\Phi(x,t), \tag{1}$$

where

$$s(x) = \frac{1}{f(x)}\exp\left[\int \frac{g(x)}{f(x)}dx\right], \quad p(x) = \exp\left[\int \frac{g(x)}{f(x)}dx\right],$$
$$q(x) = -\frac{h(x)}{f(x)}\exp\left[\int \frac{g(x)}{f(x)}dx\right].$$

For solutions of equation (1), see Section 3.8.9.

2°. Consider the homogeneous equation, i.e., the case $\Phi(x,t) \equiv 0$.

2.1. There are particular solutions of the form

$$w(x,t) = e^{-\lambda t}u(x),$$

where the function $u(x)$ is determined by solving the following linear ordinary differential equation with parameter λ:

$$f(x)u''_{xx} + g(x)u'_x + \big[h(x) + \lambda\big]u = 0.$$

2.2. Suppose we know a nontrivial particular solution $w_0 = w_0(x)$ of the ordinary differential equation

$$f(x)w''_0 + g(x)w'_0 + h(x)w_0 = 0 \tag{2}$$

that corresponds to the stationary case ($\partial_t w \equiv 0$). Then the functions

$$w(x) = Aw_0 + Bw_0 \int \frac{F}{w_0^2}dx, \quad F = \exp\biggl(-\int \frac{g}{f}dx\biggr),$$
$$w(x,t) = Atw_0 + Aw_0 \int \frac{F}{w_0^2}\biggl(\int \frac{w_0^2}{fF}\,dx\biggr)\,dx,$$
$$w(x,t) = Atw_0\Psi + Aw_0 \int \frac{F}{w_0^2}\biggl(\int \frac{w_0^2\Psi}{fF}dx\biggr)\,dx, \quad \Psi = \int \frac{F}{w_0^2}dx,$$

where A and B are arbitrary constants, are also particular solutions of the original equation.

By performing the change of variable $w(x,t) = w_0(x)u(x,t)$, we arrive at the simpler equation

$$\frac{\partial u}{\partial t} = f(x)\frac{\partial^2 u}{\partial x^2} + \biggl[2f(x)\frac{w_0'(x)}{w_0(x)} + g(x)\biggr]\frac{\partial u}{\partial x}.$$

It determines a wide class of more complicated analytical solutions of the original equation.

It follows, with reference to the results of Item 4° from 3.8.6.4, that any nontrivial particular solution of the auxiliary linear ordinary differential equation (2) generates infinitely many particular solutions of the original partial differential equation.

For the structure of other particular solutions, see the remark at the end of Item 3°.

2.3. Let a particular nonstationary solution $w_0 = w_0(x,t)$ ($\partial_t w_0 \not\equiv 0$) of the homogeneous equation be known. Then the functions

$$w_n(x,t) = \frac{\partial^n w_0}{\partial t^n}(x,t),$$

obtained by differentiating the solution w_0 with respect to t, are also particular solutions of the equation in question.

In addition, a new particular solution can be sought in the form

$$\bar{w}(x,t) = \int_{t_0}^{t} w_0(x,\tau)\,d\tau + \phi(x), \tag{3}$$

where the unknown function $\phi(x)$ is determined on substituting expression (3) into the original equation. On constructing solution (3), one can use the above approach to construct another solution, and so on.

2.4. Case $h(x) = h =$ const. Particular solutions (A and B are arbitrary constants):

$$w(x,t) = e^{ht}\biggl[A + B\int F(x)\,dx\biggr], \quad F(x) = \exp\biggl[-\int \frac{g(x)}{f(x)}dx\biggr],$$
$$w(x,t) = Ae^{ht}\biggl[t + \int F(x)\biggl(\int \frac{dx}{f(x)F(x)}\biggr)dx\biggr],$$
$$w(x,t) = Ae^{ht}\biggl[t\Psi(x) + \int F(x)\biggl(\int \frac{\Psi(x)\,dx}{f(x)F(x)}\biggr)dx\biggr], \quad \Psi(x) = \int F(x)\,dx.$$

The substitution $w(x,t) = e^{ht}v(x,t)$ leads to an equation of the form 3.8.6.4:

$$\frac{\partial v}{\partial t} = f(x)\frac{\partial^2 v}{\partial x^2} + g(x)\frac{\partial v}{\partial x}.$$

3°. The structure of particular solutions $\bar{w}(x,t)$ of the nonhomogeneous equation 3.8.6.5 for some functions $\Phi(x,t)$ is presented in Table 3.4.

Remark 3.7. The homogeneous equation (with $\Phi \equiv 0$) admits all the particular solutions specified in Table 3.4. In this case, n should be assumed an integer and β and λ arbitrary numbers.

TABLE 3.4
Structure of particular solutions of linear nonhomogeneous equations of special form

No	Functions $\Phi(x,t)$	Form of particular solutions $\bar{w}(x,t)$	Remarks
1	$\varphi(x)t^n$	$\sum\limits_{m=0}^{n} \psi_m(x)t^m$	n is an integer; the equations for $\psi_m(x)$ are solved consecutively, starting with $m=n$
2	$\varphi(x)e^{\beta t}$	$\psi(x)e^{\beta t}$	$\psi(x)$ is governed by a single equation
3	$\varphi(x)t^n e^{\beta t}$	$e^{\beta t}\sum\limits_{m=0}^{n} \psi_m(x)t^m$	n is an integer; the equations for $\psi_m(x)$ are solved consecutively, starting with $m=n$
4	$\varphi(x)\sinh(\beta t)$	$\psi(x)e^{\beta t} + \chi(x)e^{-\beta t}$	the equations for $\psi(x)$ and $\chi(x)$ are independent
5	$\varphi(x)\cosh(\beta t)$	$\psi(x)e^{\beta t} + \chi(x)e^{-\beta t}$	the equations for $\psi(x)$ and $\chi(x)$ are independent
6	$\varphi(x)\sin(\beta t)$	$\psi(x)\sin(\beta t) + \chi(x)\cos(\beta t)$	$\psi(x)$ and $\chi(x)$ are determined by a system of equations
7	$\varphi(x)\cos(\beta t)$	$\psi(x)\sin(\beta t) + \chi(x)\cos(\beta t)$	$\psi(x)$ and $\chi(x)$ are determined by a system of equations
8	$\varphi(x)e^{\lambda t}\sin(\beta t)$	$\psi(x)e^{\lambda t}\sin(\beta t) + \chi(x)e^{\lambda t}\cos(\beta t)$	$\psi(x)$ and $\chi(x)$ are determined by a system of equations
9	$\varphi(x)e^{\lambda t}\cos(\beta t)$	$\psi(x)e^{\lambda t}\sin(\beta t) + \chi(x)e^{\lambda t}\cos(\beta t)$	$\psi(x)$ and $\chi(x)$ are determined by a system of equations

6. $\displaystyle \frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x)\frac{\partial w}{\partial x} + h(t)w.$

1°. Particular solutions (A and B are arbitrary constants):

$$w(x,t) = \left[A + B\int F(x)\,dx\right]H(t),$$

$$w(x,t) = A\left[t + \int F(x)\left(\int \frac{dx}{f(x)F(x)}\right)dx\right]H(t),$$

$$w(x,t) = A\left[t\Psi(x) + \int F(x)\left(\int \frac{\Psi(x)\,dx}{f(x)F(x)}\right)dx\right]H(t),$$

where

$$H(t) = \exp\left[\int h(t)\,dt\right], \quad F(x) = \exp\left[-\int \frac{g(x)}{f(x)}dx\right], \quad \Psi(x) = \int F(x)\,dx.$$

2°. The substitution $w(x,t) = u(x,t)\exp\left[\int h(t)\,dt\right]$ leads to an equation of the form 3.8.6.4:

$$\frac{\partial u}{\partial t} = f(x)\frac{\partial^2 u}{\partial x^2} + g(x)\frac{\partial u}{\partial x}.$$

7. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + [h_1(x) + h_2(t)]w.$

The substitution $w(x,t) = u(x,t)\exp\left[\int h_2(t)\,dt\right]$ leads to an equation of the form 3.8.6.5:

$$\frac{\partial u}{\partial t} = f(x)\frac{\partial^2 u}{\partial x^2} + g(x)\frac{\partial u}{\partial x} + h_1(x)u.$$

8. $\dfrac{\partial w}{\partial t} = f^2(x)\dfrac{\partial^2 w}{\partial x^2} + f(x)[f'_x(x) + g(t)]\dfrac{\partial w}{\partial x} + h(t)w.$

The change of variable $\xi = \displaystyle\int \frac{dx}{f(x)}$ leads to a special case of equation 3.8.7.3:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + g(t)\frac{\partial w}{\partial \xi} + h(t)w.$$

9. $\dfrac{\partial w}{\partial t} = f^2\dfrac{\partial^2 w}{\partial x^2} + f\big(f'_x + 2g + \varphi\big)\dfrac{\partial w}{\partial x} + (fg'_x + g^2 + g\varphi + \psi)w,$

where $f = f(x),\ g = g(x),\ \varphi = \varphi(t),\ \psi = \psi(t).$

The transformation

$$w(x,t) = u(\xi,t)\exp\left(-\int \frac{g}{f}dx\right), \quad \xi = \int \frac{dx}{f(x)}$$

leads to a special case of equation 3.8.7.3:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + \varphi(t)\frac{\partial u}{\partial \xi} + \psi(t)u.$$

3.8.7 Equations of the Form $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + g(x,t)\dfrac{\partial w}{\partial x} + h(x,t)w$

1. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + g(t)\dfrac{\partial w}{\partial x}.$

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = Ax + A\int g(t)\,dt + B,$$

$$w(x,t) = A\left[x + \int g(t)\,dt\right]^2 + 2A\int f(t)\,dt + B,$$

$$w(x,t) = A\exp\left[\lambda x + \lambda^2\int f(t)\,dt + \lambda\int g(t)\,dt\right].$$

2°. On passing from t, x to the new variables (A and B are any numbers)

$$\tau = \int f(t)\,dt + A, \quad z = x + \int g(t)\,dt + B,$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = \partial_{zz} w,$ which is considered in Section 3.1.1.

2. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + xg(t)\dfrac{\partial w}{\partial x}.$

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = AxG(t) + B, \quad G(t) = \exp\biggl[\int g(t)\,dt\biggr],$$
$$w(x,t) = Ax^2G^2(t) + 2A\int f(t)G^2(t)\,dt + B,$$
$$w(x,t) = A\exp\biggl[\lambda xG(t) + \lambda^2\int f(t)G^2(t)\,dt\biggr].$$

2°. On passing from t, x to the new variables (A is any number)

$$\tau = \int f(t)G^2(t)\,dt + A, \quad z = xG(t), \qquad \text{where} \quad G(t) = \exp\biggl[\int g(t)\,dt\biggr],$$

for the function $w(\tau, z)$ we obtain a constant coefficient equation, $\partial_\tau w = \partial_{zz} w,$ which is considered in Section 3.1.1.

3. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + g(t)\dfrac{\partial w}{\partial x} + h(t)w.$

This is a special case of equation 3.8.7.4.

The transformation

$$w(x,t) = u(z,\tau)\exp\biggl[\int h(t)\,dt\biggr], \quad z = x + \int g(t)\,dt, \quad \tau = \int f(t)\,dt$$

leads to a constant coefficient equation, $\partial_\tau u = \partial_{zz} u,$ which is considered in Section 3.1.1.

4. $\dfrac{\partial w}{\partial t} = n(t)\dfrac{\partial^2 w}{\partial x^2} + \bigl[xf(t) + g(t)\bigr]\dfrac{\partial w}{\partial x} + \bigl[xh(t) + s(t)\bigr]w.$

Let us perform the transformation

$$w(x,t) = \exp\bigl[x\alpha(t) + \beta(t)\bigr]u(z,\tau), \quad \tau = \varphi(t), \quad z = x\psi(t) + \chi(t), \qquad (1)$$

where the unknown functions $\alpha(t)$, $\beta(t)$, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are chosen so that the resulting equation is as simple as possible. For the new dependent variable $u(z,\tau)$ we have

$$\varphi'_t\frac{\partial u}{\partial \tau} = n\psi^2\frac{\partial^2 u}{\partial z^2} + \bigl[x(f\psi - \psi'_t) + 2n\psi\alpha + g\psi - \chi'_t\bigr]\frac{\partial u}{\partial z}$$
$$+ \bigl[x(f\alpha + h - \alpha'_t) + n\alpha^2 + g\alpha + s - \beta'_t\bigr]u.$$

Let the unknown functions satisfy the system of ordinary differential equations

$$\varphi'_t = n\psi^2, \tag{2}$$
$$\psi'_t = f\psi, \tag{3}$$
$$\chi'_t = 2n\alpha\psi + g\psi, \tag{4}$$
$$\alpha'_t = f\alpha + h, \tag{5}$$
$$\beta'_t = n\alpha^2 + g\alpha + s. \tag{6}$$

Then the original equation can be reduced with the transformation (1)–(6) to the constant coefficient equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2},$$

which is discussed in Section 3.1.1 in detail.

System (2)–(6) can be solved successively. To this end, we start with equation (3), for example, in order $(3) \to (2) \to (5) \to (6) \to (4)$. As a result, we obtain

$$\psi = C_1 \exp\biggl(\int f\,dt\biggr), \quad C_1 \neq 0,$$
$$\varphi = \int n\psi^2\,dt + C_2,$$
$$\alpha = \psi \int \frac{h}{\psi} dt + C_3\psi,$$
$$\beta = \int \bigl(n\alpha^2 + g\alpha + s\bigr)\,dt + C_4,$$
$$\chi = \int (2n\alpha + g)\psi\,dt + C_5,$$

where C_1, C_2, C_3, C_4, and C_5 are arbitrary constants.

Remark 3.8. Likewise, one can simplify the nonhomogeneous equation with an additional term $\Phi(x,t)$ on the right-hand side.

5. $\displaystyle \frac{\partial w}{\partial t} = n(t)\frac{\partial^2 w}{\partial x^2} + \bigl[xf(t) + g(t)\bigr]\frac{\partial w}{\partial x} + \bigl[x^2 h(t) + xs(t) + p(t)\bigr]w.$

The substitution $w(x,t) = \exp\bigl[\varphi(t)x^2\bigr]u(x,t)$ leads to an equation

$$\frac{\partial u}{\partial t} = n\frac{\partial^2 u}{\partial x^2} + \bigl[x(4n\varphi + f) + g\bigr]\frac{\partial u}{\partial x} + \bigl[x^2(h + 2f\varphi + 4n\varphi^2 - \varphi'_t) + x(s + 2g\varphi) + p + 2n\varphi\bigr]u. \tag{1}$$

We choose the function $\varphi = \varphi(t)$ so that it is a (particular) solution of the Riccati ordinary differential equation

$$\varphi'_t = 4n\varphi^2 + 2f\varphi + h. \tag{2}$$

Then the transformed equation (1) becomes an equation of the form 3.8.7.4:

$$\frac{\partial u}{\partial t} = n\frac{\partial^2 u}{\partial x^2} + \bigl[x(4n\varphi + f) + g\bigr]\frac{\partial u}{\partial x} + \bigl[x(s + 2g\varphi) + p + 2n\varphi\bigr]u.$$

A number of specific solvable Riccati equations (2) can be found in Murphy (1960), Kamke (1977), and Polyanin and Zaitsev (2003).

In the special case where

$$n = af, \quad h = bf, \qquad \text{with} \quad a, b = \text{const}, \quad f = f(t),$$

the roots of the quadratic equation $4a\varphi^2 + 2\varphi + b = 0$ ($\varphi = \text{const}$) are particular solutions of equation (2).

3.8.8 Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,t)\dfrac{\partial^2 w}{\partial x^2} + g(x,t)\dfrac{\partial w}{\partial x} + h(x,t)w$

1. $\dfrac{\partial w}{\partial t} = xf(t)\dfrac{\partial^2 w}{\partial x^2} + xg(t)\dfrac{\partial w}{\partial x} + \big[xh(t) + s(t)\big]w.$

Let us perform the transformation

$$\tau = \varphi(t), \quad z = x\psi(t), \quad w(x,t) = u(z,\tau)\exp\left[x\alpha(t) + \int s(t)\,dt\right], \tag{1}$$

where the unknown functions $\varphi(t)$, $\psi(t)$, and $\alpha(t)$ are chosen so that the resulting equation is as simple as possible. For the new dependent variable $u(z,\tau)$ we have

$$\varphi'_t\frac{\partial u}{\partial \tau} = zf\psi\frac{\partial^2 u}{\partial z^2} + \frac{z}{\psi}\big(2f\psi\alpha + g\psi - \psi'_t\big)\frac{\partial u}{\partial z} + \frac{z}{\psi}\big(f\alpha^2 + g\alpha + h - \alpha'_t\big)u.$$

Let the unknown functions satisfy the system of ordinary differential equations

$$\varphi'_t = f\psi, \tag{2}$$
$$\psi'_t = 2f\alpha\psi + g\psi, \tag{3}$$
$$\alpha'_t = f\alpha^2 + g\alpha + h. \tag{4}$$

Then the original equation can be reduced with the transformation (1)–(4) to a constant coefficient equation of the form 3.3.4.1:

$$\frac{\partial u}{\partial \tau} = z\frac{\partial^2 u}{\partial z^2}.$$

Let us solve system (2)–(4) successively, starting with equation (4) in the order $(4) \to (3) \to (2)$.

The Riccati equation (4) can be solved separately. A lot of specific solvable Riccati equations can be found in Murphy (1960), Kamke (1977), and Polyanin and Zaitsev (2003).

Suppose a solution $\alpha = \alpha(t)$ of equation (4) is known. Then the solutions of equations (2) and (3) can be found in the form

$$\psi(t) = C_1\exp\left[\int(2f\alpha + g)\,dt\right], \quad \varphi(t) = \int f\psi\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

Remark 3.9. The transformation (1)–(4) can also be used to simplify the nonhomogeneous equation with an additional term $\Phi(x,t)$ on the right-hand side.

2. $\dfrac{\partial w}{\partial t} = x^2 f(t)\dfrac{\partial^2 w}{\partial x^2} + xg(t)\dfrac{\partial w}{\partial x} + h(t)w.$

The substitution $x = \pm e^{\xi}$ leads to an equation of the form 3.8.7.3:

$$\frac{\partial w}{\partial t} = f(t)\frac{\partial^2 w}{\partial \xi^2} + \bigl[g(t) - f(t)\bigr]\frac{\partial w}{\partial \xi} + h(t)w.$$

3. $\dfrac{\partial w}{\partial t} = x^2 n(t)\dfrac{\partial^2 w}{\partial x^2} + x\bigl[f(t)\ln x + g(t)\bigr]\dfrac{\partial w}{\partial x} + \bigl[h(t)\ln^2 x + s(t)\ln x + p(t)\bigr]w.$

The substitution $z = \ln x$ leads to an equation of the form 3.8.7.5:

$$\frac{\partial w}{\partial t} = n(t)\frac{\partial^2 w}{\partial z^2} + \bigl[zf(t) + g(t) - n(t)\bigr]\frac{\partial w}{\partial z} + \bigl[z^2 h(t) + zs(t) + p(t)\bigr]w.$$

4. $\dfrac{\partial w}{\partial t} = x^4 f(t)\dfrac{\partial^2 w}{\partial x^2} + g(t)w.$

1°. Particular solutions (A, B, and λ are arbitrary constants):

$$w(x,t) = (Ax + B)\exp\left[\int g(t)\,dt\right],$$
$$w(x,t) = \left[2Ax\int f(t)\,dt + Bx + \frac{A}{x}\right]\exp\left[\int g(t)\,dt\right],$$
$$w(x,t) = Ax\exp\left[\lambda^2\int f(t)\,dt + \int g(t)\,dt + \frac{\lambda}{x}\right].$$

2°. The transformation

$$w(x,t) = x\exp\left[\int g(t)\,dt\right]u(\xi,\tau), \quad \xi = \frac{1}{x}, \quad \tau = \int f(t)\,dt$$

leads to a constant coefficient equation, $\partial_\tau u = \partial_{\xi\xi} u$, which is considered in Section 3.1.1.

5. $\dfrac{\partial w}{\partial t} = (x - a_1)^2 (x - a_2)^2 f(t)\dfrac{\partial^2 w}{\partial x^2} + g(t)w, \qquad a_1 \neq a_2.$

The transformation

$$w(x,t) = (x - a_2)\exp\left[\int g(t)\,dt\right]u(\xi,\tau), \quad \xi = \ln\left|\frac{x - a_1}{x - a_2}\right|, \quad \tau = (a_1 - a_2)^2\int f(t)\,dt$$

leads to a constant coefficient equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial u}{\partial \xi},$$

which is considered in Section 3.1.4.

6. $\dfrac{\partial w}{\partial t} = (ax^2 + bx + c)^2 f(t)\dfrac{\partial^2 w}{\partial x^2} + g(t)w.$

The transformation

$$w(x,t)=|ax^2+bx+c|^{1/2}\exp\biggl[\int g(t)\,dt\biggr]u(\xi,\tau),\quad \xi=\int\frac{dx}{ax^2+bx+c},\quad \tau=\int f(t)\,dt$$

leads to a constant coefficient equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} + \bigl(ac - \tfrac{1}{4}b^2\bigr)u,$$

which is considered in Section 3.1.3.

7. $\dfrac{\partial w}{\partial t} = x^n f(t)\dfrac{\partial^2 w}{\partial x^2} + xg(t)\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)\exp\biggl[\int h(t)\,dt\biggr],\quad z = xG(t),\quad \tau = \int f(t)G^{2-n}(t)\,dt,$$

where $G(t) = \exp\biggl[\int g(t)\,dt\biggr]$, leads to an equation of the form 3.3.6.6:

$$\frac{\partial u}{\partial \tau} = z^n\frac{\partial^2 u}{\partial z^2}.$$

8. $\dfrac{\partial w}{\partial t} = e^{\beta x} f(t)\dfrac{\partial^2 w}{\partial x^2} + g(t)w.$

The transformation

$$w(x,t) = \exp\biggl[\int g(t)\,dt\biggr]u(x,\tau),\quad \tau = \int f(t)\,dt$$

leads to an equation of the form 3.4.5.1:

$$\frac{\partial u}{\partial \tau} = e^{\beta x}\frac{\partial^2 u}{\partial x^2}.$$

9. $\dfrac{\partial w}{\partial t} = f(x)g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)w.$

1°. Particular solutions (A, B, and x_0 are arbitrary constants):

$$\begin{aligned}
w(x,t) &= (Ax + B)H(t),\\
w(x,t) &= A\bigl[M(t) + F(x)\bigr]H(t),\\
w(x,t) &= A\bigl[M(t)x + \Psi(x)\bigr]H(t),\\
w(x,t) &= A\biggl[M^2(t) + 2M(t)F(x) + 2\int_{x_0}^{x}\frac{x-\xi}{f(\xi)}F(\xi)\,d\xi\biggr]H(t),\\
w(x,t) &= A\biggl[M^2(t)x + 2M(t)\Psi(x) + 2\int_{x_0}^{x}\frac{x-\xi}{f(\xi)}\Psi(\xi)\,d\xi\biggr]H(t).
\end{aligned}$$

Here we use the shorthand notation

$$H(t)=\exp\left[\int h(t)\,dt\right],\quad M(t)=\int g(t)\,dt,\quad F(x)=\int_{x_0}^{x}\frac{x-\xi}{f(\xi)}d\xi,\quad \Psi(x)=\int_{x_0}^{x}\frac{x-\xi}{f(\xi)}\xi\,d\xi.$$

2°. The transformation

$$w(x,t)=\exp\left[\int h(t)\,dt\right]u(x,\tau),\qquad \tau=\int g(t)\,dt$$

leads to a simpler equation

$$\frac{\partial u}{\partial \tau}=f(x)\frac{\partial^2 u}{\partial x^2}.$$

A wide class of exact analytical solutions to this equation is specified in 3.8.6.1.

3.8.9 Equations of the Form $s(x)\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[p(x)\dfrac{\partial w}{\partial x}\right]-q(x)w+\Phi(x,t)$

Equations of this form are often encountered in heat and mass transfer theory and chemical engineering sciences. Throughout this subsection, we assume that the functions s, p, p'_x, and q are continuous and $s>0$, $p>0$, and $x_1\le x\le x_2$.

► General formulas for solving linear nonhomogeneous boundary value problems.

The solution of the equation in question under the initial condition

$$w=f(x)\quad\text{at}\quad t=0 \tag{1}$$

and the arbitrary linear nonhomogeneous boundary conditions

$$\begin{aligned} a_1\partial_x w+b_1 w&=g_1(t)\quad\text{at}\quad x=x_1,\\ a_2\partial_x w+b_2 w&=g_2(t)\quad\text{at}\quad x=x_2, \end{aligned} \tag{2}$$

can be represented as the sum

$$\begin{aligned} w(x,t)&=\int_0^t\int_{x_1}^{x_2}\Phi(\xi,\tau)\mathcal{G}(x,\xi,t-\tau)\,d\xi\,d\tau+\int_{x_1}^{x_2}s(\xi)f(\xi)\mathcal{G}(x,\xi,t)\,d\xi\\ &\quad+p(x_1)\int_0^t g_1(\tau)\Lambda_1(x,t-\tau)\,d\tau+p(x_2)\int_0^t g_2(\tau)\Lambda_2(x,t-\tau)\,d\tau. \end{aligned} \tag{3}$$

Here, the modified Green's function is given by

$$\mathcal{G}(x,\xi,t)=\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)}{\|y_n\|^2}\exp(-\lambda_n t),\qquad \|y_n\|^2=\int_{x_1}^{x_2}s(x)y_n^2(x)\,dx, \tag{4}$$

where the λ_n and $y_n(x)$ are the eigenvalues and corresponding eigenfunctions of the following Sturm–Liouville problem for a second-order linear ordinary differential equation:

$$\begin{aligned} &[p(x)y'_x]'_x+[\lambda s(x)-q(x)]y=0,\\ &a_1y'_x+b_1y=0\quad\text{at}\quad x=x_1,\\ &a_2y'_x+b_2y=0\quad\text{at}\quad x=x_2. \end{aligned} \tag{5}$$

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ that occur in the integrands of the last two terms in solution (3) are expressed via the Green's function (4). Appropriate formulas will be given below when considering specific boundary value problems.

► **General properties of the Sturm–Liouville problem (5).**

1°. There are infinitely many eigenvalues. All eigenvalues are real and different and can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$ (therefore, there can exist only finitely many negative eigenvalues). Each eigenvalue is of multiplicity 1.

2°. The eigenfunctions are determined up to a constant multiplier. Each eigenfunction $y_n(x)$ has exactly $n-1$ zeros in the open interval (x_1, x_2).

3°. Eigenfunctions $y_n(x)$ and $y_m(x)$, $n \neq m$, are orthogonal with weight $s(x)$ on the interval $x_1 \le x \le x_2$:

$$\int_{x_1}^{x_2} s(x) y_n(x) y_m(x)\, dx = 0 \quad \text{for} \quad n \neq m.$$

4°. An arbitrary function $F(x)$ that has a continuous derivative and satisfies the boundary conditions of the Sturm–Liouville problem can be expanded into an absolutely and uniformly convergent series in eigenfunctions:

$$F(x) = \sum_{n=1}^{\infty} F_n y_n(x), \quad F_n = \frac{1}{\|y_n\|^2} \int_{x_1}^{x_2} s(x) F(x) y_n(x)\, dx,$$

where the norm $\|y_n\|^2$ is defined in (4).

5°. If the conditions

$$q(x) \ge 0, \quad a_1 b_1 \le 0, \quad a_2 b_2 \ge 0 \tag{6}$$

are satisfied, there are no negative eigenvalues. If $q \equiv 0$ and $b_1 = b_2 = 0$, then $\lambda_1 = 0$ is the least eigenvalue, to which there corresponds the eigenfunction $\varphi_1 =$ const. Otherwise, all eigenvalues are positive, provided that conditions (6) are satisfied.

6°. The following asymptotic relation holds for large eigenvalues as $n \to \infty$:

$$\lambda_n = \frac{\pi^2 n^2}{\Delta^2} + O(1), \quad \Delta = \int_{x_1}^{x_2} \sqrt{\frac{s(x)}{p(x)}}\, dx. \tag{7}$$

Special, boundary value condition-dependent properties of the Sturm–Liouville problem are presented below.

Remark 3.10. Equation (5) can be reduced to one with $p(x) \equiv 1$ and $s(x) \equiv 1$ by the change of variables

$$\zeta = \int \sqrt{\frac{s(x)}{p(x)}}\, dx, \quad u(\zeta) = \bigl[p(x)s(x)\bigr]^{1/4} y(x).$$

The boundary conditions transform into boundary conditions of the same type.

► **First boundary value problem: the case of $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$.**

The solution of the first boundary value problem with the initial condition (1) and the boundary conditions

$$w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by formulas (3)–(4) with

$$\Lambda_1(x,t) = \frac{\partial}{\partial \xi} \mathcal{G}(x,\xi,t)\Big|_{\xi=x_1}, \quad \Lambda_2(x,t) = -\frac{\partial}{\partial \xi} \mathcal{G}(x,\xi,t)\Big|_{\xi=x_2}.$$

Some special properties of the Sturm–Liouville problem are worth mentioning.

1°. For $n \to \infty$, the asymptotic relation (7) can be used to estimate eigenvalues λ_n. The corresponding eigenfunctions $y_n(x)$ satisfy the asymptotic relation

$$\frac{y_n(x)}{\|y_n\|} = \left[\frac{4}{\Delta^2 p(x)s(x)}\right]^{1/4} \sin\left[\frac{\pi n}{\Delta} \int_{x_1}^{x} \sqrt{\frac{s(x)}{p(x)}}\,dx\right] + O\left(\frac{1}{n}\right), \quad \Delta = \int_{x_1}^{x_2} \sqrt{\frac{s(x)}{p(x)}}\,dx.$$

2°. For $q \ge 0$, the following upper estimate (Rayleigh principle) holds for the least eigenvalue:

$$\lambda_1 \le \frac{\displaystyle\int_{x_1}^{x_2} \left[p(x)(z'_x)^2 + q(x)z^2\right] dx}{\displaystyle\int_{x_1}^{x_2} s(x)z^2\,dx}, \tag{8}$$

where $z = z(x)$ is any twice differentiable function that satisfies the conditions $z(x_1) = z(x_2) = 0$. The equality in (8) is attained for $z = y_1(x)$, where $y_1(x)$ is the eigenfunction of the Sturm–Liouville problem corresponding to the eigenvalue λ_1. To obtain particular estimates, one may set $z = (x - x_1)(x_2 - x)$ or $z = \sin[\pi(x - x_1)/(x_2 - x_1)]$ in (8).

3°. Suppose

$$0 < p_{\min} \le p(x) \le p_{\max}, \quad 0 < q_{\min} \le q(x) \le q_{\max}, \quad 0 < s_{\min} \le s(x) \le s_{\max}.$$

Then the following double-ended estimate holds for the eigenvalues:

$$\frac{p_{\min}}{s_{\max}} \frac{\pi^2 n^2}{(x_2 - x_1)^2} + \frac{q_{\min}}{s_{\max}} \le \lambda_n \le \frac{p_{\max}}{s_{\min}} \frac{\pi^2 n^2}{(x_2 - x_1)^2} + \frac{q_{\max}}{s_{\min}}.$$

4°. In engineering calculations, the approximate formula

$$\lambda_n = \frac{\pi^2 n^2}{\Delta^2} + \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \frac{q(x)}{s(x)}\,dx, \qquad \text{where} \quad \Delta = \int_{x_1}^{x_2} \sqrt{\frac{s(x)}{p(x)}}\,dx, \tag{9}$$

may be used to determine eigenvalues. This formula is exact if $p(x)s(x) = \text{const}$ and $q(x)/s(x) = \text{const}$ (in particular, for constant $p = p_0$, $q = q_0$, and $s = s_0$) and provides correct asymptotics (7) for any $p(x)$, $q(x)$, and $s(x)$. Furthermore, for $p(x) = \text{const}$ and $s(x) = \text{const}$, relation (9) gives two correct first terms as $n \to \infty$; the same holds true if $p(x)s(x) = \text{const}$.

5°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. Then the following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi n}{x_2 - x_1} + \frac{1}{\pi n} Q(x_1, x_2) + O\left(\frac{1}{n^2}\right),$$
$$y_n(x) = \sin\frac{\pi n(x - x_1)}{x_2 - x_1} - \frac{1}{\pi n}\big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\big]\cos\frac{\pi n(x - x_1)}{x_2 - x_1} + O\left(\frac{1}{n^2}\right),$$

where

$$Q(u, v) = \frac{1}{2}\int_u^v q(x)\,dx. \tag{10}$$

► **Second boundary value problem: the case of $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$.**

The solution of the second boundary value problem with the initial condition (1) and the boundary conditions

$$\partial_x w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$\partial_x w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by formulas (3)–(4) with

$$\Lambda_1(x, t) = -\mathcal{G}(x, x_1, t), \quad \Lambda_2(x, t) = \mathcal{G}(x, x_2, t).$$

Some special properties of the Sturm–Liouville problem are worth mentioning.

1°. For $q > 0$, the upper estimate (8) holds for the least eigenvalue, with $z = z(x)$ being any twice differentiable function that satisfies the conditions $z'_x(x_1) = z'_x(x_2) = 0$. The equality in (8) is attained for $z = y_1(x)$, where $y_1(x)$ is the eigenfunction of the Sturm–Liouville problem corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. Then the following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(n-1)}{x_2 - x_1} + \frac{1}{\pi(n-1)} Q(x_1, x_2) + O\left(\frac{1}{n^2}\right),$$
$$y_n(x) = \cos\frac{\pi(n-1)(x - x_1)}{x_2 - x_1} + \frac{1}{\pi(n-1)}\Big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\Big]\sin\frac{\pi(n-1)(x - x_1)}{x_2 - x_1} + O\left(\frac{1}{n^2}\right),$$

where the function $Q(u, v)$ is defined by (10).

► **Third boundary value problem: the case of $a_1 = a_2 = 1$, $b_1 \neq 0$, and $b_2 \neq 0$.**

The solution of the third boundary value problem with the initial condition (1) and boundary conditions (2), with $a_1 = a_2 = 1$, is given by relations (3)–(4) in which

$$\Lambda_1(x,t) = -\mathcal{G}(x, x_1, t), \quad \Lambda_2(x,t) = \mathcal{G}(x, x_2, t).$$

Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. Then the following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(n-1)}{x_2 - x_1} + \frac{1}{\pi(n-1)}\Big[Q(x_1, x_2) - b_1 + b_2\Big] + O\Big(\frac{1}{n^2}\Big),$$
$$y_n(x) = \cos\frac{\pi(n-1)(x - x_1)}{x_2 - x_1} + \frac{1}{\pi(n-1)}\Big\{(x_1 - x)\big[Q(x, x_2) + b_2\big] + (x_2 - x)\big[Q(x_1, x) - b_1\big]\Big\}\sin\frac{\pi(n-1)(x - x_1)}{x_2 - x_1} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is defined by (10).

► **Mixed boundary value problem: the case of $a_1 = b_2 = 0$ and $a_2 = b_1 = 1$.**

The solution of the mixed boundary value problem with the initial condition (1) and the boundary conditions

$$w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$\partial_x w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by relations (3)–(4) with

$$\Lambda_1(x,t) = \frac{\partial}{\partial\xi}\mathcal{G}(x, \xi, t)\Big|_{\xi = x_1}, \quad \Lambda_2(x,t) = \mathcal{G}(x, x_2, t).$$

Below are some special properties of the Sturm–Liouville problem.

1°. For $q \geq 0$, the upper estimate (8) holds for the least eigenvalue, with $z = z(x)$ being any twice differentiable function that satisfies the conditions $z(x_1) = 0$ and $z'_x(x_2) = 0$. The equality in (8) is attained for $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. Then the following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(2n-1)}{2(x_2 - x_1)} + \frac{1}{\pi(2n-1)}Q(x_1, x_2) + O\Big(\frac{1}{n^2}\Big),$$
$$y_n(x) = \sin\frac{\pi(2n-1)(x - x_1)}{2(x_2 - x_1)} - \frac{2}{\pi(2n-1)}\big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\big]\cos\frac{\pi(2n-1)(x - x_1)}{2(x_2 - x_1)} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is defined by (10).

► **Mixed boundary value problem: the case of $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$.**

The solution of the mixed boundary value problem with the initial condition (1) and the boundary conditions

$$\partial_x w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by formulas (3)–(4) in which

$$\Lambda_1(x,t) = -\mathcal{G}(x,x_1,t), \quad \Lambda_2(x,t) = -\frac{\partial}{\partial\xi}\mathcal{G}(x,\xi,t)\Big|_{\xi=x_2}.$$

Below are some special properties of the Sturm–Liouville problem.

1°. For $q \ge 0$, the upper estimate (8) holds for the least eigenvalue, with $z = z(x)$ being any twice differentiable function that satisfies the conditions $z'_x(x_1) = 0$ and $z(x_2) = 0$. The equality in (8) is attained for $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. Then the following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(2n-1)}{2(x_2-x_1)} + \frac{2}{\pi(2n-1)}Q(x_1,x_2) + O\left(\frac{1}{n^2}\right),$$
$$y_n(x) = \cos\frac{\pi(2n-1)(x-x_1)}{2(x_2-x_1)} + \frac{2}{\pi(2n-1)}\big[(x_1-x)Q(x,x_2)$$
$$+ (x_2-x)Q(x_1,x)\big]\sin\frac{\pi(2n-1)(x-x_1)}{2(x_2-x_1)} + O\left(\frac{1}{n^2}\right),$$

where the function $Q(u,v)$ is defined by relation (10).

⊙ References for Section 3.8.9: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), E. Kamke (1977), V. A. Marchenko (1986), V. S. Vladimirov (1988), B. M. Levitan and I. S. Sargsyan (1988), L. D. Akulenko and S. V. Nesterov (1997), A. D. Polyanin (2001).

3.9 Equations of Special Form

3.9.1 Equations of the Diffusion (Thermal) Boundary Layer

1. $f(x)\dfrac{\partial w}{\partial x} + g(x)y\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial y^2}.$

This equation is encountered in diffusion boundary layer problems (mass exchange of drops and bubbles with flow).

The transformation (A and B are any numbers)

$$t = \int \frac{h^2(x)}{f(x)}\,dx + A, \quad z = yh(x), \qquad \text{where} \quad h(x) = B\exp\left[-\int\frac{g(x)}{f(x)}\,dx\right],$$

leads to a constant coefficient equation, $\partial_t w = \partial_{zz} w$, which is considered in Section 3.1.1.

⊙ *Literature*: V. G. Levich (1962), A. D. Polyanin and V. V. Dilman (1994), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

2. $\displaystyle f(x)\frac{\partial w}{\partial x}+g(x)y\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}-h(x)w.$

This equation is encountered in diffusion boundary layer problems with a first-order volume chemical reaction (usually $h\equiv\text{const}$).

The transformation (A and B are any numbers)

$$w(x,y)=u(t,z)\exp\left[-\int\frac{h(x)}{f(x)}dx\right],\quad t=\int\frac{\varphi^2(x)}{f(x)}dx+A,\quad z=y\varphi(x),$$

where $\varphi(x)=B\exp\left[-\int\frac{g(x)}{f(x)}dx\right]$, leads to a constant coefficient equation, $\partial_t u=\partial_{zz}u$, which is considered in Section 3.1.1.

⊙ *Literature*: Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985).

3. $\displaystyle f(x)y^{n-1}\frac{\partial w}{\partial x}+g(x)y^n\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}.$

This equation is encountered in diffusion boundary layer problems (mass exchange of solid particles, drops, and bubbles with flow).

The transformation (A and B are any numbers)

$$t=\int\frac{h^{n+1}(x)}{f(x)}dx+A,\quad z=yh(x),\qquad \text{where}\quad h(x)=B\exp\left[-\int\frac{g(x)}{f(x)}dx\right],$$

leads to a simpler equation of the form 3.3.6.6:

$$\frac{\partial w}{\partial t}=z^{1-n}\frac{\partial^2 w}{\partial z^2}.$$

⊙ *Literature*: V. G. Levich (1962), A. D. Polyanin and V. V. Dilman (1994), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

4. $\displaystyle f\Big(\frac{y}{\sqrt{x}}\Big)\frac{\partial w}{\partial x}+\frac{1}{\sqrt{x}}g\Big(\frac{y}{\sqrt{x}}\Big)\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}.$

This is a generalization of the problem of thermal boundary layer on a flat plate.

1°. By passing from x, y to the new variables $t=\ln x$, $\xi=y/\sqrt{x}$, we arrive at the separable equation

$$f(\xi)\frac{\partial w}{\partial t}+\big[g(\xi)-\tfrac{1}{2}\xi f(\xi)\big]\frac{\partial w}{\partial \xi}=\frac{\partial^2 w}{\partial \xi^2}.$$

There are particular solutions of the form

$$w(t,\xi)=Ae^{\beta t}\phi(\xi),$$

where the function $\phi(\xi)$ satisfies the ordinary differential equation

$$\phi''_{\xi\xi}=\big[g(\xi)-\tfrac{1}{2}\xi f(\xi)\big]\phi'_\xi+\beta f(\xi)\phi.$$

2°. The solution of the original equation with the boundary conditions

$$x = 0, \;\; w = w_0; \quad y = 0, \;\; w = w_1; \quad y \to \infty, \;\; w \to w_0$$

(w_0 and w_1 are some constants) is given by

$$\frac{w - w_0}{w_1 - w_0} = \frac{\int_\xi^\infty \exp\bigl[-\Psi(\xi)\bigr]\,d\xi}{\int_0^\infty \exp\bigl[-\Psi(\xi)\bigr]\,d\xi}, \qquad \Psi(\xi) = \int_0^\xi \bigl[\tfrac{1}{2}\xi f(\xi) - g(\xi)\bigr]\,d\xi,$$

where $\xi = y/\sqrt{x}$. It is assumed that the inequality $\xi f(\xi) > 2g(\xi)$ holds for $\xi > 0$.

3°. The equation of a thermal boundary layer on a flat plate corresponds to

$$f(\xi) = \mathsf{Pr}\, F'_\xi(\xi), \qquad g(\xi) = \tfrac{1}{2}\mathsf{Pr}\bigl[\xi F'_\xi(\xi) - F(\xi)\bigr],$$

where $F(\xi)$ is the Blasius solution in the problem of translational flow past a flat plate and Pr is the Prandtl number (x is the coordinate measured along the plate and y is the transverse coordinate to the plate surface). In this case the formulas in Item 2° transform into Polhausen's solution. See Schlichting (1981) for details.

⊙ *Literature*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

5. $\displaystyle f(x)\frac{\partial w}{\partial x} + \Bigl[g(x)y - \frac{b}{y}\Bigr]\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2}.$

For $b = 1$, equations of this sort govern the concentration distribution in the internal region of the diffusion wake behind a moving particle or drop.

The transformation (A and B are any numbers)

$$t = \int \frac{h^2(x)}{f(x)}\,dx + A, \quad z = yh(x), \qquad \text{where} \quad h(x) = B\exp\biggl[-\int \frac{g(x)}{f(x)}\,dx\biggr],$$

leads to an equation of the form 3.2.5:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + \frac{b}{z}\frac{\partial w}{\partial z}.$$

⊙ *Literature*: Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

6. $\displaystyle f(x)\frac{\partial w}{\partial x} + \Bigl[g(x)y - \frac{b}{y}\Bigr]\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} + h(x)w.$

The substitution $w(x,y) = u(x,y)\exp\Bigl[\int \frac{h(x)}{f(x)}dx\Bigr]$ leads to an equation of the form 3.9.1.5:

$$f(x)\frac{\partial u}{\partial x} + \Bigl[g(x)y - \frac{b}{y}\Bigr]\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}.$$

⊙ *Literature*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

7. $\boldsymbol{f(x)y^{n-1}\frac{\partial w}{\partial x} + \Big[g(x)y^n - \frac{b}{y}\Big]\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2}}$**.**

The transformation (A and B are any numbers)

$$t = \int \frac{h^{n+1}(x)}{f(x)}\,dx + A, \quad \xi = \frac{2}{n+1}\big[yh(x)\big]^{\frac{n+1}{2}},$$

where $h(x) = B\exp\left[-\int \frac{g(x)}{f(x)}\,dx\right]$, leads to the equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{1-2\beta}{\xi}\frac{\partial w}{\partial \xi}, \qquad \beta = \frac{1-b}{n+1},$$

which is considered in Section 3.2.5 (see also equations in Sections 3.2.1 and 3.2.3).

3.9.2 One-Dimensional Schrödinger Equation $i\hbar\frac{\partial w}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 w}{\partial x^2} + U(x)w$

► **Eigenvalue problem. Cauchy problem.**

Schrödinger's equation is the basic equation of quantum mechanics; w is the wave function, $i^2 = -1$, $\hbar$ is Planck's constant, m is the mass of the particle, and $U(x)$ is the potential energy of the particle in the force field.

1°. In discrete spectrum problems, the particular solutions are sought in the form

$$w(x,t) = \exp\left(-\frac{iE_n}{\hbar}t\right)\psi_n(x),$$

where the eigenfunctions ψ_n and the respective energies E_n have to be determined by solving the eigenvalue problem

$$\begin{gathered}\frac{d^2\psi_n}{dx^2} + \frac{2m}{\hbar^2}\big[E_n - U(x)\big]\psi_n = 0,\\ \psi_n \to 0 \text{ at } x \to \pm\infty, \qquad \int_{-\infty}^{\infty} |\psi_n|^2\,dx = 1.\end{gathered} \tag{1}$$

The last relation is the normalizing condition for ψ_n.

2°. In the cases where the eigenfunctions $\psi_n(x)$ form an orthonormal basis in $L_2(\mathbb{R})$, the solution of the Cauchy problem for Schrödinger's equation with the initial condition

$$w = f(x) \quad \text{at} \quad t = 0 \tag{2}$$

is given by

$$w(x,t) = \int_{-\infty}^{\infty} G(x,\xi,t)f(\xi)\,d\xi, \qquad G(x,\xi,t) = \sum_{n=0}^{\infty}\psi_n(x)\psi_n(\xi)\exp\left(-\frac{iE_n}{\hbar}t\right).$$

Various potentials $U(x)$ are considered below and particular solutions of the boundary value problem (1) or the Cauchy problem for Schrödinger's equation are presented. In some cases, nonnormalized eigenfunctions $\Psi_n(x)$ are given instead of normalized eigenfunctions $\psi_n(x)$; the former differ from the latter by a constant multiplier.

► Free particle: $U(x) = 0$.

The solution of the Cauchy problem with the initial condition (2) is given by

$$w(x,t) = \frac{1}{2\sqrt{i\pi\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4i\tau}\right] f(\xi)\,d\xi,$$

$$\tau = \frac{\hbar t}{2m}, \quad \sqrt{ia} = \begin{cases} e^{\pi i/4}\sqrt{|a|} & \text{if } a > 0, \\ e^{-\pi i/4}\sqrt{|a|} & \text{if } a < 0. \end{cases}$$

⊙ *Literature*: W. Miller, Jr. (1977).

► Linear potential (motion in a uniform external field): $U(x) = ax$.

Solution of the Cauchy problem with the initial condition (2):

$$w(x,t) = \frac{1}{2\sqrt{i\pi\tau}} \exp\bigl(-ib\tau x - \tfrac{1}{3} i b^2 \tau^3\bigr) \int_{-\infty}^{\infty} \exp\left[-\frac{(x + b\tau^2 - \xi)^2}{4i\tau}\right] f(\xi)\,d\xi,$$

$$\tau = \frac{\hbar t}{2m}, \quad b = \frac{2am}{\hbar^2}.$$

See also W. Miller, Jr. (1977).

► Linear harmonic oscillator: $U(x) = \frac{1}{2} m\omega^2 x^2$.

Eigenvalues:

$$E_n = \hbar\omega\bigl(n + \tfrac{1}{2}\bigr), \qquad n = 0,\ 1,\ \ldots$$

Normalized eigenfunctions:

$$\psi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!\, x_0}} \exp\bigl(-\tfrac{1}{2}\xi^2\bigr) H_n(\xi), \qquad \xi = \frac{x}{x_0}, \quad x_0 = \sqrt{\frac{\hbar}{m\omega}},$$

where $H_n(\xi)$ are the Hermite polynomials.

The functions $\psi_n(x)$ form an orthonormal basis in $L_2(\mathbb{R})$.

⊙ *Literature*: S. G. Krein (1972), W. Miller, Jr. (1977), A. N. Tikhonov and A. A. Samarskii (1990).

► Isotropic free particle: $U(x) = a/x^2$.

Here, the variable $x \geq 0$ plays the role of the radial coordinate, and $a > 0$. The equation with $U(x) = a/x^2$ results from Schrödinger's equation for a free particle with n space coordinates if one passes to spherical (cylindrical) coordinates and separates the angular variables.

The solution of Schrödinger's equation satisfying the initial condition (2) has the form

$$w(x,t) = \frac{\exp\bigl[-\frac{1}{2} i\pi(\mu+1)\operatorname{sign} t\bigr]}{2|\tau|} \int_0^{\infty} \sqrt{xy}\, \exp\left(i\frac{x^2+y^2}{4\tau}\right) J_\mu\left(\frac{xy}{2|\tau|}\right) f(y)\,dy,$$

$$\tau = \frac{\hbar t}{2m}, \quad \mu = \sqrt{\frac{2am}{\hbar^2} + \frac{1}{4}} \geq 1,$$

where $J_\mu(\xi)$ is the Bessel function.

⊙ *Literature*: W. Miller, Jr. (1977).

► **Isotropic harmonic oscillator:** $U(x) = \frac{1}{2}m\omega^2 x^2 + ax^{-2}$**.**

Here, the variable $x \geq 0$ plays the role of the radial coordinate, and $a > 0$. The equation with this $U(x)$ results from Schrödinger's equation for a harmonic oscillator with n space coordinates if one passes to spherical (cylindrical) coordinates and separates the angular variables.

Eigenvalues:

$$E_n = -\hbar\omega(2n + \mu + 1), \quad \mu = \sqrt{\frac{2am}{\hbar^2} + \frac{1}{4}} \geq 1, \qquad n = 0,\, 1,\, \ldots$$

Normalized eigenfunctions:

$$\psi_n(x) = \sqrt{\frac{2n!}{\Gamma(n+1+\mu)x_0}}\, \xi^{\frac{2\mu+1}{2}} \exp\bigl(-\tfrac{1}{2}\xi^2\bigr) L_n^\mu(\xi^2), \qquad \xi = \frac{x}{x_0}, \quad x_0 = \sqrt{\frac{\hbar}{m\omega}},$$

where $L_n^\mu(z)$ is the nth generalized Laguerre polynomial with parameter μ. The norm $|\psi_n(x)|^2$ refers to the semiaxis $x \geq 0$.

The functions $\psi_n(x)$ form an orthonormal basis in $L_2(\mathbb{R}_+)$.

⊙ *Literature*: W. Miller, Jr. (1977).

► **Morse potential:** $U(x) = U_0(e^{-2x/a} - 2e^{-x/a})$**.**

Eigenvalues:

$$E_n = -U_0\left[1 - \frac{1}{\beta}(n + \tfrac{1}{2})\right]^2, \quad \beta = \frac{a\sqrt{2mU_0}}{\hbar}, \qquad 0 \leq n < \beta - 2.$$

Eigenfunctions:

$$\psi_n(x) = \xi^s e^{-\xi/2}\Phi(-n,\, 2s+1,\, \xi), \qquad \xi = 2\beta e^{-x/a}, \quad s = \frac{a\sqrt{-2mE_n}}{\hbar},$$

where $\Phi(a, b, \xi)$ is the degenerate hypergeometric function.

In this case the number of eigenvalues (energy levels) E_n and eigenfunctions ψ_n is finite: $n = 0,\, 1,\, \ldots,\, n_{\max}$.

⊙ *Literature*: S. G. Krein (1964), L. D. Landau and E. M. Lifshitz (1974).

► **Potential with a hyperbolic function:** $U(x) = -U_0 \cosh^{-2}(x/a)$**.**

Eigenvalues:

$$E_n = -\frac{\hbar^2}{2ma^2}(s-n)^2, \quad s = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{8mU_0a^2}{\hbar^2}}\right), \qquad 0 \leq n < s.$$

Eigenfunctions:

$$\psi_n(x)=\begin{cases}\left(\cosh\dfrac{x}{a}\right)^{-2s}F\left(\dfrac{\beta-s}{2},\ -\dfrac{\beta+s}{2},\ \dfrac{1}{2},\ -\sinh^2\dfrac{x}{a}\right) & \text{for even } n,\\[2ex] \sinh\dfrac{x}{a}\left(\cosh\dfrac{x}{a}\right)^{-2s}F\left(\dfrac{1+\beta-s}{2},\ \dfrac{1-\beta-s}{2},\ \dfrac{3}{2},\ -\sinh^2\dfrac{x}{a}\right) & \text{for odd } n,\end{cases}$$

where $F(a,b,c,\xi)$ is the hypergeometric function and $\beta=\dfrac{a}{\hbar}\sqrt{-2mE_n}$.

The number of eigenvalues (energy levels) E_n and eigenfunctions ψ_n is finite in this case: $n=0,\ 1,\ \ldots,\ n_{\max}$.

⊙ *Literature*: S. G. Krein (1964).

► Potential with a trigonometric function: $U(x)=U_0\cot^2(\pi x/a)$.

Eigenvalues:

$$E_n=\frac{\pi^2\hbar^2}{2ma^2}(n^2+2ns-s),\qquad s=\frac{1}{2}\left(-1+\sqrt{1+\frac{8mU_0a^2}{\pi^2\hbar^2}}\right),\qquad n=0,\ 1,\ \ldots$$

Eigenfunctions:

$$\psi_n(x)=\begin{cases}\cos\dfrac{\pi x}{a}\left(\sin\dfrac{\pi x}{a}\right)^{-2s}F\left(\dfrac{1-n-s}{2},\ \dfrac{n+1}{2},\ \dfrac{3}{2},\ \cos^2\dfrac{\pi x}{a}\right) & \text{for even } n,\\[2ex] \left(\sin\dfrac{\pi x}{a}\right)^{-2s}F\left(-\dfrac{n+s}{2},\ \dfrac{n}{2},\ \dfrac{1}{2},\ \cos^2\dfrac{\pi x}{a}\right) & \text{for odd } n,\end{cases}$$

where $F(a,b,c,\xi)$ is the hypergeometric function.

In particular, if $a=\pi\hbar/\sqrt{2m}$, $U_0=2$, and $n=k-1$, we have

$$E_k=k^2-2,\qquad \psi_k(x)=k\cos\frac{k\pi x}{a}-\sin\frac{k\pi x}{a}\cot\frac{\pi x}{a},\qquad k=1,\ 2,\ \ldots$$

⊙ *Literature*: S. G. Krein (1964), L. D. Landau and E. M. Lifshitz (1974).

Chapter 4

Second-Order Parabolic Equations with Two Space Variables

4.1 Heat Equation $\frac{\partial w}{\partial t} = a\Delta_2 w$

4.1.1 Boundary Value Problems in Cartesian Coordinates

In rectangular Cartesian coordinates, the two-dimensional sourceless heat equation has the form

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right).$$

It governs two-dimensional unsteady heat transfer processes in quiescent media or solid bodies with constant thermal diffusivity a. A similar equation is used to study analogous two-dimensional unsteady mass transfer phenomena with constant diffusivity; in this case the equation is called a diffusion equation.

► **Particular solutions:**

$$\begin{aligned}
w(x,y) &= Axy + C_1x + C_2y + C_3,\\
w(x,y,t) &= Ax^2 + By^2 + 2a(A+B)t,\\
w(x,y,t) &= A(x^2+2at)(y^2+2at) + B,\\
w(x,y,t) &= A\exp\bigl[k_1x + k_2y + (k_1^2+k_2^2)at\bigr] + B,\\
w(x,y,t) &= A\cos(k_1x+C_1)\cos(k_2y+C_2)\exp\bigl[-(k_1^2+k_2^2)at\bigr],\\
w(x,y,t) &= A\cos(k_1x+C_1)\sinh(k_2y+C_2)\exp\bigl[-(k_1^2-k_2^2)at\bigr],\\
w(x,y,t) &= A\cos(k_1x+C_1)\cosh(k_2y+C_2)\exp\bigl[-(k_1^2-k_2^2)at\bigr],\\
w(x,y,t) &= A\exp(-\mu x-\lambda y)\cos(\mu x - 2a\mu^2t + C_1)\cos(\lambda y - 2a\lambda^2t + C_2),\\
w(x,y,t) &= \frac{A}{t-t_0}\exp\left[-\frac{(x-x_0)^2+(y-y_0)^2}{4a(t-t_0)}\right],
\end{aligned}$$

$$w(x,y,t) = A\,\mathrm{erf}\left(\frac{x-x_0}{2\sqrt{at}}\right)\mathrm{erf}\left(\frac{y-y_0}{2\sqrt{at}}\right) + B,$$

where A, B, C_1, C_2, C_3, k_1, k_2, x_0, y_0, and t_0 are arbitrary constants.

Fundamental solution:

$$\mathscr{E}(x,y,t) = \frac{1}{4\pi at}\exp\left(-\frac{x^2+y^2}{4at}\right).$$

► **Formulas to construct particular solutions. Remarks on the Green's functions.**

1°. Apart from usual separable solutions $w(x,y,t) = f_1(x)f_2(y)f_3(t)$, the equation in question has more sophisticated solutions in the product form

$$w(x,y,t) = u(x,t)v(y,t),$$

where $u = u(x,t)$ and $v = v(y,t)$ are solutions of the one-dimensional heat equations

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial y^2},$$

considered in Section 3.1.1.

2°. Suppose $w = w(x,y,t)$ is a solution of the heat equation. Then the functions

$$w_1 = Aw(\pm\lambda x + C_1,\ \pm\lambda y + C_2,\ \lambda^2 t + C_3),$$
$$w_2 = Aw(x\cos\beta - y\sin\beta + C_1,\ x\sin\beta + y\cos\beta + C_2,\ t + C_3),$$
$$w_3 = A\exp\bigl[\lambda_1 x + \lambda_2 y + a(\lambda_1^2+\lambda_2^2)t\bigr]\,w(x + 2a\lambda_1 t + C_1,\ y + 2a\lambda_2 t + C_2,\ t + C_3),$$
$$w_4 = \frac{A}{\delta+\beta t}\exp\left[-\frac{\beta(x^2+y^2)}{4a(\delta+\beta t)}\right]w\left(\frac{x}{\delta+\beta t}, \frac{y}{\delta+\beta t}, \frac{\gamma+\lambda t}{\delta+\beta t}\right), \qquad \lambda\delta - \beta\gamma = 1,$$

where A, C_1, C_2, C_3, β, δ, λ, λ_1 and λ_2 are arbitrary constants, are also solutions of this equation. The signs at λ's in the formula for w_1 are taken arbitrarily, independently of each other.

⊙ *Literature*: W. Miller, Jr. (1977).

3°. In all two-dimensional boundary value problems discussed in Section 4.1.1, the Green's function can be represented in the product form

$$G(x,y,\xi,\eta,t) = G_1(x,\xi,t)\,G_2(y,\eta,t),$$

where $G_1(x,\xi,t)$ and $G_2(y,\eta,t)$ are the Green's functions of the corresponding one-dimensional boundary value problems (these functions are specified in Sections 3.1.1 and 3.1.2).

Example 4.1. The Green's function of the first boundary value problem for a semiinfinite strip ($0 \le x \le l$, $0 \le y < \infty$), considered in Section 4.1.1, is the product of two one-dimensional Green's functions. The first Green's function is that of the first boundary value problem on a closed interval ($0 \le x \le l$) presented in Section 3.1.2. The second Green's function is that of the first boundary value problem on a semiinfinite interval ($0 \le y < \infty$) presented in Section 3.1.2, where x and ξ must be renamed y and η, respectively.

► Transformations that allow separation of variables.

Table 4.1 lists possible transformations that allow reduction of the two-dimensional heat equation to a separable equation. All transformations of the independent variables have the form $(x, y, t) \longmapsto (\xi, \eta, t)$. The transformations that can be obtained by interchange of independent variables, $x \rightleftharpoons y$, are omitted.

The anharmonic oscillator functions are solutions of the second-order ordinary differential equation $F''_{zz} + (az^4 + bz^2 + c)F = 0$. The Ince polynomials are the 2π-periodic solutions of the Whittaker–Hill equation $F''_{zz} + k\sin 2z\, F'_z + (a - bk\cos 2z)F = 0$; see Arscott (1964, 1967).

⊙ *Literature*: W. Miller, Jr. (1977).

► Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x, y) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x, y, t) = \frac{1}{4\pi at} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \exp\left[-\frac{(x - \xi)^2 + (y - \eta)^2}{4at}\right] d\xi\, d\eta.$$

Example 4.2. The initial temperature is piecewise-constant and equal to w_1 in the domain $|x| < x_0$, $|y| < y_0$ and w_2 in the domain $|x| > x_0$, $|y| > y_0$, specifically,

$$f(x, y) = \begin{cases} w_1 & \text{for } |x| < x_0,\ |y| < y_0, \\ w_2 & \text{for } |x| > x_0,\ |y| > y_0. \end{cases}$$

Solution:

$$w = \frac{1}{4}(w_1 - w_2)\left[\operatorname{erf}\left(\frac{x_0 - x}{2\sqrt{at}}\right) + \operatorname{erf}\left(\frac{x_0 + x}{2\sqrt{at}}\right)\right]\left[\operatorname{erf}\left(\frac{y_0 - y}{2\sqrt{at}}\right) + \operatorname{erf}\left(\frac{y_0 + y}{2\sqrt{at}}\right)\right] + w_2.$$

If the initial temperature distribution $f(x, y)$ is an infinitely differentiable function in both arguments, then the solution can be represented in the series form

$$w(x, y, t) = f(x, y) + \sum_{n=1}^{\infty} \frac{(at)^n}{n!} \boldsymbol{L}^n \bigl[f(x, y)\bigr], \quad \boldsymbol{L} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Such a representation is useful for small t.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x < \infty$, $-\infty < y < \infty$. First boundary value problem.

A half-plane is considered. The following conditions are prescribed:

$$w = f(x, y) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(y, t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

TABLE 4.1
Transformations $(x, y, t) \longmapsto (\xi, \eta, t)$ that allow solutions with $\mathcal{R}$-separated variables, $w = \exp[\mathcal{R}(\xi, \eta, t)]\, f(\xi)\, g(\eta)\, h(t)$, for the two-dimensional heat equation $\partial_t w = \partial_{xx} w + \partial_{yy} w$. Everywhere, the function $h(t)$ is exponential

No	Transformations	Factor $\exp \mathcal{R}$	Function $f(\xi)$	Function $g(\eta)$
1	$x = \xi$, $y = \eta$	$\mathcal{R} = 0$	Exponential function	Exponential function
2	$x = \xi$, $y = \eta\sqrt{\lvert t\rvert}$	$\mathcal{R} = 0$	Exponential function	Hermite function
3	$x = \xi\sqrt{\lvert t\rvert}$, $y = \eta\sqrt{\lvert t\rvert}$	$\mathcal{R} = 0$	Hermite function	Hermite function
4	$x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$	$\mathcal{R} = 0$	Parabolic cylinder function	Parabolic cylinder function
5	$x = \xi\cos\eta$, $y = \xi\sin\eta$	$\mathcal{R} = 0$	Bessel function	Exponential function
6	$x = \cosh\xi\cos\eta$, $y = \sinh\xi\sin\eta$	$\mathcal{R} = 0$	Modified Mathieu function	Mathieu function
7	$x = \sqrt{\lvert t\rvert}\,\xi\cos\eta$, $y = \sqrt{\lvert t\rvert}\,\xi\sin\eta$	$\mathcal{R} = 0$	Laguerre function	Exponential function
8	$x = \sqrt{\lvert t\rvert}\,\cosh\xi\cos\eta$, $y = \sqrt{\lvert t\rvert}\,\sinh\xi\sin\eta$	$\mathcal{R} = 0$	Ince polynomial	Ince polynomial
9	$x = \xi$, $y = \eta + at^2$	$\mathcal{R} = -a\eta t$	Exponential function	Airy function
10	$x = \xi$, $y = \eta t + b/t$	$\mathcal{R} = -\frac{1}{4}\eta^2 t + \frac{1}{2}b\eta/t$	Exponential function	Airy function
11	$x = \xi$, $y = \eta\sqrt{1 + t^2}$	$\mathcal{R} = -\frac{1}{4}\eta^2 t$	Exponential function	Parabolic cylinder function
12	$x = \xi$, $y = \eta\sqrt{\lvert 1 - t^2\rvert}$	$\mathcal{R} = -\frac{1}{4}\varepsilon\eta^2 t$, $\varepsilon = \operatorname{sign}(1 - t^2)$	Exponential function	Hermite function
13	$x = \xi t$, $y = \eta t$	$\mathcal{R} = -\frac{1}{4}(\xi^2 + \eta^2)t$	Exponential function	Exponential function
14	$x = \xi + at^2$, $y = \eta + bt^2$	$\mathcal{R} = -(a\xi + b\eta)t$	Airy function	Airy function
15	$x = \xi t + a/t$, $y = \eta t + b/t$	$\mathcal{R} = -\frac{1}{4}(\xi^2 + \eta^2)t + \frac{1}{2}(a\xi + b\eta)/t$	Airy function	Airy function

TABLE 4.1
(*continued*)

No	Transformations	Factor $\exp\mathcal{R}$	Function $f(\xi)$	Function $g(\eta)$
16	$x = \frac{1}{2}(\xi^2 - \eta^2)t$, $y = \xi\eta t$	$\mathcal{R} = -\frac{1}{16}(\xi^2 + \eta^2)^2 t$	Parabolic cylinder function	Parabolic cylinder function
17	$x = \xi\sqrt{1+t^2}$, $y = \eta\sqrt{1+t^2}$	$\mathcal{R} = -\frac{1}{4}(\xi^2 + \eta^2)t$	Parabolic cylinder function	Parabolic cylinder function
18	$x = \xi\sqrt{\lvert 1-t^2\rvert}$, $y = \eta\sqrt{\lvert 1-t^2\rvert}$	$\mathcal{R} = -\frac{1}{4}\varepsilon(\xi^2 + \eta^2)t$, $\varepsilon = \operatorname{sign}(1-t^2)$	Hermite function	Hermite function
19	$x = \frac{1}{2}(\xi^2 - \eta^2) + at^2$, $y = \xi\eta$	$\mathcal{R} = -\frac{1}{2}a(\xi^2 - \eta^2)t$	Anharmonic oscillator function	Anharmonic oscillator function
20	$x = \frac{1}{2}(\xi^2 - \eta^2)t + a/t$, $y = \xi\eta t$	$\mathcal{R} = -\frac{1}{16}(\xi^2 + \eta^2)^2 t + \frac{1}{4}a(\xi^2 - \eta^2)/t$	Anharmonic oscillator function	Anharmonic oscillator function
21	$x = \xi t\cos\eta$, $y = \xi t\sin\eta$	$\mathcal{R} = -\frac{1}{4}\xi^2 t$	Bessel function	Exponential function
22	$x = t\cosh\xi\cos\eta$, $y = t\sinh\xi\sin\eta$	$\mathcal{R} = -\frac{1}{4}(\sinh^2\xi + \cos^2\eta)t$	Modified Mathieu function	Mathieu function
23	$x = \sqrt{1+t^2}\,\xi\cos\eta$, $y = \sqrt{1+t^2}\,\xi\sin\eta$	$\mathcal{R} = -\frac{1}{4}\xi^2 t$	Whittaker function	Exponential function
24	$x = \sqrt{\lvert 1-t^2\rvert}\,\xi\cos\eta$, $y = \sqrt{\lvert 1-t^2\rvert}\,\xi\sin\eta$	$\mathcal{R} = -\frac{1}{4}\varepsilon\xi^2 t$, $\varepsilon = \operatorname{sign}(1-t^2)$	Laguerre function	Exponential function
25	$x = \sqrt{1+t^2}\,\cosh\xi\cos\eta$, $y = \sqrt{1+t^2}\,\sinh\xi\sin\eta$	$\mathcal{R} = -\frac{1}{4}(\sinh^2\xi + \cos^2\eta)t$	Ince polynomial	Ince polynomial
26	$x = \sqrt{\lvert 1-t^2\rvert}\,\cosh\xi\cos\eta$, $y = \sqrt{\lvert 1-t^2\rvert}\,\sinh\xi\sin\eta$	$\mathcal{R} = -\frac{1}{4}\varepsilon(\sinh^2\xi + \cos^2\eta)t$, $\varepsilon = \operatorname{sign}(1-t^2)$	Ince polynomial	Ince polynomial

Solution:

$$w(x,y,t) = \int_0^\infty \int_{-\infty}^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi + a\int_0^t \int_{-\infty}^\infty g(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau,$$

where

$$G(x,y,\xi,\eta,t)=\frac{1}{4\pi at}\left\{\exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right]-\exp\left[-\frac{(x+\xi)^2+(y-\eta)^2}{4at}\right]\right\}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0\le x<\infty$, $-\infty<y<\infty$. Second boundary value problem.

A half-plane is considered. The following conditions are prescribed:

$$w=f(x,y)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$\partial_x w=g(y,t)\quad\text{at}\quad x=0\quad\text{(boundary condition)}.$$

Solution:

$$w(x,y,t)=\int_0^\infty\int_{-\infty}^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi-a\int_0^t\int_{-\infty}^\infty g(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau,$$

where

$$G(x,y,\xi,\eta,t)=\frac{1}{4\pi at}\left\{\exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(x+\xi)^2+(y-\eta)^2}{4at}\right]\right\}.$$

► Domain: $0\le x<\infty$, $-\infty<y<\infty$. Third boundary value problem.

A half-plane is considered. The following conditions are prescribed:

$$w=f(x,y)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$\partial_x w-kw=g(y,t)\quad\text{at}\quad x=0\quad\text{(boundary condition)}.$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t)=\frac{1}{4\pi at}\exp\left[-\frac{(y-\eta)^2}{4at}\right]\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]+\exp\left[-\frac{(x+\xi)^2}{4at}\right]-2k\int_0^\infty\exp\left[-\frac{(x+\xi+s)^2}{4at}-ks\right]ds\right\}.$$

► Domain: $0\le x<\infty$, $0\le y<\infty$. First boundary value problem.

A quadrant of the plane is considered. The following conditions are prescribed:

$$w=f(x,y)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$w=g_1(y,t)\quad\text{at}\quad x=0\quad\text{(boundary condition)},$$
$$w=g_2(x,t)\quad\text{at}\quad y=0\quad\text{(boundary condition)}.$$

Solution:

$$w(x,y,t) = \int_0^\infty \int_0^\infty f(\xi,\eta)\, G(x,y,\xi,\eta,t)\, d\xi\, d\eta + a\int_0^t \int_0^\infty g_1(\eta,\tau) \left[\frac{\partial}{\partial\xi} G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\, d\tau + a\int_0^t \int_0^\infty g_2(\xi,\tau) \left[\frac{\partial}{\partial\eta} G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\, d\tau,$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\} \times \left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right] - \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.$$

Example 4.3. The initial temperature is uniform, $f(x,y) = w_0$. The boundary is maintained at zero temperature, $g_1(y,t) = g_2(x,t) = 0$.

Solution:

$$w = w_0 \operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right) \operatorname{erf}\left(\frac{y}{2\sqrt{at}}\right).$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x < \infty$, $0 \le y < \infty$. Second boundary value problem.

A quadrant of the plane is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_x w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\ \partial_y w &= g_2(x,t) && \text{at} \quad y=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,y,t) = \int_0^\infty \int_0^\infty f(\xi,\eta)\, G(x,y,\xi,\eta,t)\, d\xi\, d\eta - a\int_0^t \int_0^\infty g_1(\eta,\tau)\, G(x,y,0,\eta,t-\tau)\, d\eta\, d\tau - a\int_0^t \int_0^\infty g_2(\xi,\tau)\, G(x,y,\xi,0,t-\tau)\, d\xi\, d\tau,$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\} \times \left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right] + \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.$$

► Domain: $0 \le x < \infty$, $0 \le y < \infty$. Third boundary value problem.

A quadrant of the plane is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_y w - k_2 w &= g_2(x,t) && \text{at} \quad y=0 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\begin{aligned}
G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}&\biggl\{\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] + \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr]\\
&- 2k_1\sqrt{\pi at}\,\exp\bigl[ak_1^2 t + k_1(x+\xi)\bigr]\operatorname{erfc}\biggl(\frac{x+\xi}{2\sqrt{at}} + k_1\sqrt{at}\biggr)\biggr\}\\
&\times\biggl\{\exp\biggl[-\frac{(y-\eta)^2}{4at}\biggr] + \exp\biggl[-\frac{(y+\eta)^2}{4at}\biggr]\\
&- 2k_2\sqrt{\pi at}\,\exp\bigl[ak_2^2 t + k_2(y+\eta)\bigr]\operatorname{erfc}\biggl(\frac{y+\eta}{2\sqrt{at}} + k_2\sqrt{at}\biggr)\biggr\}.
\end{aligned}
$$

Example 4.4. The initial temperature is constant, $f(x,y) = w_0$. The temperature of the environment is zero, $g_1(y,t) = g_2(x,t) = 0$.

Solution:

$$
\begin{aligned}
w = w_0&\biggl[\operatorname{erf}\biggl(\frac{x}{2\sqrt{at}}\biggr) + \exp(k_1x + ak_1^2t)\operatorname{erfc}\biggl(\frac{x}{2\sqrt{at}} + k_1\sqrt{at}\biggr)\biggr]\\
\times&\biggl[\operatorname{erf}\biggl(\frac{y}{2\sqrt{at}}\biggr) + \exp(k_2y + ak_2^2t)\operatorname{erfc}\biggl(\frac{y}{2\sqrt{at}} + k_2\sqrt{at}\biggr)\biggr].
\end{aligned}
$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x < \infty$, $0 \le y < \infty$. Mixed boundary value problems.

1°. A quadrant of the plane is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_y w &= g_2(x,t) && \text{at} \quad y=0 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \int_0^\infty\!\!\int_0^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta\\
&+ a\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\biggr]_{\xi=0} d\eta\,d\tau\\
&- a\int_0^t\!\!\int_0^\infty g_2(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau,
\end{aligned}
$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}$$
$$\times \left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right] + \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.$$

2°. A quadrant of the plane is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w - kw &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w &= g_2(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,t) &= \int_0^\infty\int_0^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta\\ &\quad - a\int_0^t\int_0^\infty g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau\\ &\quad + a\int_0^t\int_0^\infty g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau,\end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}\left\{\exp\left[\frac{(y-\eta)^2}{4at}\right] - \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}$$
$$\times \left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right.$$
$$\left. - 2k\sqrt{\pi at}\,\exp\bigl[ak^2t + k(x+\xi)\bigr]\operatorname{erfc}\left(\frac{x+\xi}{2\sqrt{at}} + k\sqrt{at}\right)\right\}.$$

Example 4.5. The initial temperature is uniform, $f(x,y) = w_0$. Heat exchange with the environment of zero temperature occurs at one side and the other side is maintained at zero temperature: $g_1(y,t) = g_2(x,t) = 0$.

Solution:

$$w = w_0\left[\operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right) + \exp(kx + ak^2t)\operatorname{erfc}\left(\frac{x}{2\sqrt{at}} + k\sqrt{at}\right)\right]\operatorname{erf}\left(\frac{y}{2\sqrt{at}}\right).$$

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. First boundary value problem.**

A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w &= g_2(y,t) &&\text{at}\quad x=l &&\text{(boundary condition)},\\ w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \int_0^\infty\!\int_0^l f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta\\
&\quad + a\int_0^t\!\int_0^\infty g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau\\
&\quad - a\int_0^t\!\int_0^\infty g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l} d\eta\,d\tau\\
&\quad + a\int_0^t\!\int_0^l g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,\xi,\eta,t) &= G_1(x,\xi,t)\,G_2(y,\eta,t),\\
G_1(x,\xi,t) &= \frac{2}{l}\sum_{n=1}^\infty \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right),\\
G_2(y,\eta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]-\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.
\end{aligned}
$$

Example 4.6. The initial temperature is uniform, $f(x,y)=w_0$. The boundary is maintained at zero temperature, $g_1(y,t)=g_2(y,t)=g_3(x,t)=0$.

Solution:

$$
w = \frac{4w_0}{\pi}\,\mathrm{erf}\left(\frac{y}{2\sqrt{at}}\right)\sum_{n=0}^\infty \frac{1}{2n+1}\sin\left[\frac{(2n+1)\pi x}{l}\right]\exp\left[-\frac{\pi^2(2n+1)^2 at}{l^2}\right].
$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x \le l$, $0 \le y < \infty$. Second boundary value problem.

A semiinfinite strip is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,t) &&\text{at}\quad x=l &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \int_0^\infty\!\int_0^l f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta\\
&\quad - a\int_0^t\!\int_0^\infty g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau\\
&\quad + a\int_0^t\!\int_0^\infty g_2(\eta,\tau)\,G(x,y,l,\eta,t-\tau)\,d\eta\,d\tau\\
&\quad - a\int_0^t\!\int_0^l g_3(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau,
\end{aligned}
$$

where

$$G(x,y,\xi,\eta,t) = G_1(x,\xi,t)\,G_2(y,\eta,t),$$
$$G_1(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right),$$
$$G_2(y,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right] + \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.$$

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. Third boundary value problem.**

A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(y,t) &&\text{at}\quad x=l &&\text{(boundary condition)},\\ \partial_y w - k_3 w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\begin{aligned} G(x,y,\xi,\eta,t) &= G_1(x,\xi,t)\,G_2(y,\eta,t),\\ G_1(x,\xi,t) &= \sum_{n=1}^{\infty}\frac{1}{\|y_n\|^2}y_n(x)y_n(\xi)\exp(-a\mu_n^2 t),\\ G_2(y,\eta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right] + \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right.\\ &\quad\left. - 2k_3\int_0^{\infty}\exp\left[-\frac{(y+\eta+\zeta)^2}{4at} - k_3\zeta\right]d\zeta\right\}. \end{aligned}$$

Here

$$y_n(x) = \cos(\mu_n x) + \frac{k_1}{\mu_n}\sin(\mu_n x), \qquad \|y_n\|^2 = \frac{k_2}{2\mu_n^2}\,\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2} + \frac{k_1}{2\mu_n^2} + \frac{l}{2}\left(1 + \frac{k_1^2}{\mu_n^2}\right),$$

and the μ_n are positive roots of the transcendental equation $\dfrac{\tan(\mu l)}{\mu} = \dfrac{k_1+k_2}{\mu^2 - k_1 k_2}$.

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. Mixed boundary value problems.**

1°. A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w &= g_2(y,t) &&\text{at}\quad x=l &&\text{(boundary condition)},\\ \partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) &= \int_0^\infty\!\!\int_0^l f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta \\
&\quad + a\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau \\
&\quad - a\int_0^t\!\!\int_0^\infty g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l} d\eta\,d\tau \\
&\quad - a\int_0^t\!\!\int_0^l g_3(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,\xi,\eta,t) &= G_1(x,\xi,t)\,G_2(y,\eta,t), \\
G_1(x,\xi,t) &= \frac{2}{l}\sum_{n=1}^\infty \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right), \\
G_2(y,\eta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.
\end{aligned}$$

2°. A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,t) &&\text{at}\quad x=l &&\text{(boundary condition)},\\
w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) &= \int_0^\infty\!\!\int_0^l f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta \\
&\quad - a\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau \\
&\quad + a\int_0^t\!\!\int_0^\infty g_2(\eta,\tau)\,G(x,y,l,\eta,t-\tau)\,d\eta\,d\tau \\
&\quad + a\int_0^t\!\!\int_0^l g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,\xi,\eta,t) &= G_1(x,\xi,t)\,G_2(y,\eta,t), \\
G_1(x,\xi,t) &= \frac{1}{l}+\frac{2}{l}\sum_{n=1}^\infty \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right), \\
G_2(y,\eta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]-\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}.
\end{aligned}$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. First boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \int_0^{l_1}\!\!\int_0^{l_2} f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad + a\int_0^t\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau\\
&\quad - a\int_0^t\!\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1} d\eta\,d\tau\\
&\quad + a\int_0^t\!\!\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\
&\quad - a\int_0^t\!\!\int_0^{l_1} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=l_2} d\xi\,d\tau,
\end{aligned}
$$

where

$$
G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{n\pi x}{l_1}\sin\frac{n\pi\xi}{l_1}\sin\frac{m\pi y}{l_2}\sin\frac{m\pi\eta}{l_2}\exp\left[-\pi^2\left(\frac{n^2}{l_1^2}+\frac{m^2}{l_2^2}\right)at\right].
$$

Example 4.7. The initial temperature is uniform, $f(x,y)=w_0$. The boundary is maintained at zero temperature, $g_1(y,t)=g_2(y,t)=g_3(x,t)=g_4(x,t)=0$.

Solution:

$$
\begin{aligned}
w = \frac{16w_0}{\pi^2}&\left\{\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left[\frac{(2n+1)\pi x}{l_1}\right]\exp\left[-\frac{\pi^2(2n+1)^2 at}{l_1^2}\right]\right\}\\
&\times\left\{\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\left[\frac{(2m+1)\pi y}{l_2}\right]\exp\left[-\frac{\pi^2(2m+1)^2 at}{l_2^2}\right]\right\}.
\end{aligned}
$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Second boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) = {} & \int_0^{l_1}\!\int_0^{l_2} f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
& - a\int_0^t\!\int_0^{l_2} g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau \\
& + a\int_0^t\!\int_0^{l_2} g_2(\eta,\tau)\,G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau \\
& - a\int_0^t\!\int_0^{l_1} g_3(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau \\
& + a\int_0^t\!\int_0^{l_1} g_4(\xi,\tau)\,G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,\xi,\eta,t) = {} & \frac{1}{l_1 l_2}\left[1 + 2\sum_{n=1}^{\infty}\exp\left(-\frac{\pi^2 n^2 at}{l_1^2}\right)\cos\frac{n\pi x}{l_1}\cos\frac{n\pi\xi}{l_1}\right] \\
& \times \left[1 + 2\sum_{m=1}^{\infty}\exp\left(-\frac{\pi^2 m^2 at}{l_2^2}\right)\cos\frac{m\pi y}{l_2}\cos\frac{m\pi\eta}{l_2}\right].
\end{aligned}
$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x \le l_1,\ 0 \le y \le l_2$. Third boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)}, \\
\partial_x w - k_1 w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)}, \\
\partial_x w + k_2 w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)}, \\
\partial_y w - k_3 w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)}, \\
\partial_y w + k_4 w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\begin{aligned}
& G(x,y,\xi,\eta,t) = \left\{\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2}\exp(-a\mu_n^2 t)\right\}\left\{\sum_{m=1}^{\infty}\frac{\psi_m(y)\psi_m(\eta)}{\|\psi_m\|^2}\exp(-a\lambda_m^2 t)\right\}, \\
& \varphi_n(x) = \cos(\mu_n x) + \frac{k_1}{\mu_n}\sin(\mu_n x), \qquad \|\varphi_n\|^2 = \frac{k_2}{2\mu_n^2}\,\frac{\mu_n^2 + k_1^2}{\mu_n^2 + k_2^2} + \frac{k_1}{2\mu_n^2} + \frac{l_1}{2}\left(1 + \frac{k_1^2}{\mu_n^2}\right), \\
& \psi_m(y) = \cos(\lambda_m y) + \frac{k_3}{\lambda_m}\sin(\lambda_m y), \quad \|\psi_m\|^2 = \frac{k_4}{2\lambda_m^2}\,\frac{\lambda_m^2 + k_3^2}{\lambda_m^2 + k_4^2} + \frac{k_3}{2\lambda_m^2} + \frac{l_2}{2}\left(1 + \frac{k_3^2}{\lambda_m^2}\right).
\end{aligned}
$$

Here, the μ_n and λ_m are positive roots of the transcendental equations

$$
\frac{\tan(\mu l_1)}{\mu} = \frac{k_1 + k_2}{\mu^2 - k_1 k_2}, \qquad \frac{\tan(\lambda l_2)}{\lambda} = \frac{k_3 + k_4}{\lambda^2 - k_3 k_4}.
$$

► **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2$. Mixed boundary value problems.**

1°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) = {} & \int_0^{l_1}\!\int_0^{l_2} f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&+ a\int_0^t\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau\\
&- a\int_0^t\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1} d\eta\,d\tau\\
&- a\int_0^t\!\int_0^{l_1} g_3(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau\\
&+ a\int_0^t\!\int_0^{l_1} g_4(\xi,\tau)\,G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,\xi,\eta,t) = {} & \frac{4}{l_1 l_2}\left[\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l_1}\sin\frac{n\pi\xi}{l_1}\exp\left(-\frac{\pi^2 n^2 at}{l_1^2}\right)\right]\\
&\times\left[\frac{1}{2}+\sum_{m=1}^{\infty}\cos\frac{m\pi y}{l_2}\cos\frac{m\pi\eta}{l_2}\exp\left(-\frac{\pi^2 m^2 at}{l_2^2}\right)\right].
\end{aligned}$$

2°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \int_0^{l_1}\int_0^{l_2} f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&\quad + a\int_0^t\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau \\
&\quad + a\int_0^t\int_0^{l_2} g_2(\eta,\tau)\,G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau \\
&\quad + a\int_0^t\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\
&\quad + a\int_0^t\int_0^{l_1} g_4(\xi,\tau)\,G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2}&\left\{\sum_{n=0}^{\infty}\sin\left[\frac{\pi(2n+1)x}{2l_1}\right]\sin\left[\frac{\pi(2n+1)\xi}{2l_1}\right]\exp\left[-\frac{a\pi^2(2n+1)^2t}{4l_1^2}\right]\right\} \\
\times&\left\{\sum_{m=0}^{\infty}\sin\left[\frac{\pi(2m+1)y}{2l_2}\right]\sin\left[\frac{\pi(2m+1)\eta}{2l_2}\right]\exp\left[-\frac{a\pi^2(2m+1)^2t}{4l_2^2}\right]\right\}.
\end{aligned}
$$

4.1.2 Problems in Polar Coordinates

The sourceless heat equation with two space variables in the polar coordinate system r, φ has the form

$$
\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right), \qquad r = \sqrt{x^2+y^2}.
$$

One-dimensional problems with axial symmetry that have solutions of the form $w = w(r,t)$ are considered in Section 3.2.1.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le 2\pi$. Cauchy problem.**

An initial condition is prescribed:

$$
w = f(r,\varphi) \quad \text{at} \quad t = 0.
$$

Solution:

$$
w(r,\varphi,t) = \frac{1}{4\pi a t}\int_0^{2\pi}\int_0^{\infty} \xi \exp\left[-\frac{r^2+\xi^2-2r\xi\cos(\varphi-\eta)}{4at}\right] f(\xi,\eta)\,d\xi\,d\eta.
$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$w = f(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$w = g(\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

Solution:

$$w(r,\varphi,t) = \int_0^{2\pi}\int_0^R f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta - aR\int_0^t\int_0^{2\pi} g(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau.$$

Here,

$$G(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$A_0 = 1, \quad A_n = 2 \quad (n=1,\,2,\,\ldots),$$

where $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument), and μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$w = f(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_r w = g(\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

Solution:

$$w(r,\varphi,t) = \int_0^{2\pi}\int_0^R f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta + aR\int_0^t\int_0^{2\pi} g(\eta,\tau)\,G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau.$$

Here,

$$G(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2 - n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$A_0 = 1, \quad A_n = 2 \quad (n=1,\,2,\,\ldots),$$

where $J_n(\xi)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$w = f(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_r w + kw = g(\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2+k^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at),$$
$$A_0=1, \quad A_n=2 \quad (n=1,\,2,\,\dots).$$

Here, $J_n(\xi)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$w = f(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$w = g_1(\varphi,t) \quad \text{at} \quad r=R_1 \quad \text{(boundary condition)},$$
$$w = g_2(\varphi,t) \quad \text{at} \quad r=R_2 \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &\quad + aR_1\int_0^t\int_0^{2\pi} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R_1} d\eta\,d\tau \\ &\quad - aR_2\int_0^t\int_0^{2\pi} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R_2} d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t) = \frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_nB_{nm}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at),$$
$$A_n = \begin{cases} 1/2 & \text{if } n=0, \\ 1 & \text{if } n\neq 0, \end{cases} \qquad B_{nm} = \frac{\mu_{nm}^2 J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)},$$
$$Z_n(\mu_{nm}r) = J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where $J_n(r)$ and $Y_n(r)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)}, \\ \partial_r w &= g_1(\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)}, \\ \partial_r w &= g_2(\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta)\, G(r,\varphi,\xi,\eta,t)\xi\, d\xi\, d\eta \\ &\quad - aR_1 \int_0^t \int_0^{2\pi} g_1(\eta,\tau)\, G(r,\varphi,R_1,\eta,t-\tau)\, d\eta\, d\tau \\ &\quad + aR_2 \int_0^t \int_0^{2\pi} g_2(\eta,\tau)\, G(r,\varphi,R_2,\eta,t-\tau)\, d\eta\, d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,\xi,\eta,t) &= \frac{1}{\pi(R_2^2-R_1^2)} \\ &\quad + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r) Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at)}{(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)}, \\ Z_n(\mu_{nm}r) &= J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r), \end{aligned}$$

where $A_0 = 1$ and $A_n = 2$ for $n = 1,\ 2,\ \dots$; $J_n(r)$ and $Y_n(r)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2) - Y_n'(\mu R_1)J_n'(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)}, \\ \partial_r w - k_1 w &= g_1(\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)}, \\ \partial_r w + k_2 w &= g_2(\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta)\, G(r,\varphi,\xi,\eta,t)\xi\, d\xi\, d\eta \\ &\quad - aR_1 \int_0^t \int_0^{2\pi} g_1(\eta,\tau)\, G(r,\varphi,R_1,\eta,t-\tau)\, d\eta\, d\tau \\ &\quad + aR_2 \int_0^t \int_0^{2\pi} g_2(\eta,\tau)\, G(r,\varphi,R_2,\eta,t-\tau)\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{B_{nm}}\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$B_{nm}=(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1),$$

$$Z_n(\mu_{nm}r)=\bigl[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r)-\bigl[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r),$$

where $A_0=1$ and $A_n=2$ for $n=1,\,2,\,\ldots$; $J_n(r)$ and $Y_n(r)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation

$$\bigl[\mu J_n'(\mu R_1)-k_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2)+k_2Y_n(\mu R_2)\bigr]=\bigl[\mu Y_n'(\mu R_1)-k_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2)+k_2J_n(\mu R_2)\bigr].$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0\le r<\infty$, $0\le\varphi\le\varphi_0$. First boundary value problem.**

A wedge domain is considered. The following conditions are prescribed:

$$\begin{aligned}
&w=f(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
&w=g_1(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
&w=g_2(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,t)&=\int_0^{\varphi_0}\!\!\int_0^{\infty}f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad+a\int_0^t\!\!\int_0^{\infty}g_1(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau\\
&\quad-a\int_0^t\!\!\int_0^{\infty}g_2(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{a\varphi_0t}\exp\biggl(-\frac{r^2+\xi^2}{4at}\biggr)\sum_{n=1}^{\infty}I_{n\pi/\varphi_0}\Bigl(\frac{r\xi}{2at}\Bigr)\sin\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\sin\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr),$$

where $I_\nu(r)$ are modified Bessel functions.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$. Second boundary value problem.**

A wedge domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ r^{-1}\partial_\varphi w &= g_1(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_2(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \int_0^{\varphi_0}\!\!\int_0^\infty f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad - a\int_0^t\!\!\int_0^\infty g_1(\xi,\tau)\,G(r,\varphi,\xi,0,t-\tau)\,d\xi\,d\tau\\ &\quad + a\int_0^t\!\!\int_0^\infty g_2(\xi,\tau)\,G(r,\varphi,\xi,\varphi_0,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,\xi,\eta,t) &= \frac{1}{a\varphi_0 t}\exp\biggl(-\frac{r^2+\xi^2}{4at}\biggr)\biggl[\frac{1}{2}I_0\Bigl(\frac{r\xi}{2at}\Bigr)\\ &\quad + \sum_{n=1}^\infty I_{n\pi/\varphi_0}\Bigl(\frac{r\xi}{2at}\Bigr)\cos\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\cos\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr)\biggr], \end{aligned}$$

where $I_\nu(r)$ are modified Bessel functions.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. First boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ w &= g_2(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\ w &= g_3(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \int_0^{\varphi_0}\!\!\int_0^R f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad - aR\int_0^t\!\!\int_0^{\varphi_0} g_1(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\biggr]_{\xi=R}d\eta\,d\tau\\ &\quad + a\int_0^t\!\!\int_0^R g_2(\xi,\tau)\frac{1}{\xi}\biggl[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\biggr]_{\eta=0}d\xi\,d\tau\\ &\quad - a\int_0^t\!\!\int_0^R g_3(\xi,\tau)\frac{1}{\xi}\biggl[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\biggr]_{\eta=\varphi_0}d\xi\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{R^2\varphi_0[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\sin\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr)\exp(-\mu_{nm}^2at),$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

Example 4.8. The initial temperature is uniform, $f(r,\varphi)=w_0$. The boundary is maintained at zero temperature, $g_1(\varphi,t)=g_2(r,t)=g_3(r,t)=0$.

Solution:

$$w=\frac{8w_0}{\pi R^2}\sum_{n=0}^{\infty}\frac{\sin(s_n\varphi)}{2n+1}\sum_{m=1}^{\infty}\exp(-\mu_{nm}^2at)\frac{J_{s_n}(\mu_{nm}r)}{[J'_{s_n}(\mu_{nm}R)]^2}\int_0^R J_{s_n}(\mu_{nm}\xi)\xi\,d\xi,\quad s_n=\frac{(2n+1)\pi}{\varphi_0},$$

where the μ_{nm} are positive roots of the transcendental equation $J_{s_n}(\mu R)=0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0\le r\le R$, $0\le\varphi\le\varphi_0$. Second boundary value problem.

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w&=g_1(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w&=g_2(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w&=g_3(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,t)&=\int_0^{\varphi_0}\int_0^R f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad+aR\int_0^t\int_0^{\varphi_0} g_1(\eta,\tau)\,G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau\\
&\quad-a\int_0^t\int_0^R g_2(\xi,\tau)\,G(r,\varphi,\xi,0,t-\tau)\,d\xi\,d\tau\\
&\quad+a\int_0^t\int_0^R g_3(\xi,\tau)\,G(r,\varphi,\xi,\varphi_0,t-\tau)\,d\xi\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(r,\varphi,\xi,\eta,t)&=\frac{2}{R^2\varphi_0}+4\varphi_0\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_{nm}^2J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{(R^2\varphi_0^2\mu_{nm}^2-n^2\pi^2)[J_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\\
&\qquad\times\cos\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\cos\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr)\exp(-\mu_{nm}^2at),
\end{aligned}$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation $J'_{n\pi/\varphi_0}(\mu R)=0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Mixed boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w - kw &= g(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_\varphi w &= 0 && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ \partial_\varphi w &= 0 && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) = &\int_0^{\varphi_0}\int_0^R f(\xi,\eta)\,G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+ aR\int_0^t\int_0^{\varphi_0} g(\eta,\tau)\,G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}J_{s_n}(\mu_{nm}r)J_{s_n}(\mu_{nm}\xi)\cos(s_n\varphi)\cos(s_n\eta)\exp(-\mu_{nm}^2 at),$$

$$s_n=\frac{n\pi}{\varphi_0},\qquad A_{nm}=\frac{4\mu_{nm}^2}{\varphi_0(\mu_{nm}^2R^2+k^2R^2-s_n^2)\bigl[J_{s_n}(\mu_{nm}R)\bigr]^2},$$

where $J_{s_n}(r)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation

$$\mu J'_{s_n}(\mu R) + kJ_{s_n}(\mu R) = 0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le \varphi_0$. Different boundary value problems.**

Some problems for this domain were studied in Budak, Samarskii, and Tikhonov (1980).

4.1.3 Axisymmetric Problems

In the case of angular symmetry, the two-dimensional sourceless heat equation in the cylindrical coordinate system has the form

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right), \qquad r=\sqrt{x^2+y^2}.$$

This equation governs two-dimensional unsteady thermal processes in quiescent media or solid bodies (bounded by coordinate surfaces of the cylindrical system) in the case where the initial and boundary conditions are independent of the angular coordinate. A similar equation is used to study analogous two-dimensional unsteady mass transfer phenomena.

► Particular solutions. Remarks on the Green's functions.

1°. Apart from usual separable solutions $w(r,z,t)=f_1(r)f_2(z)f_3(t)$, the equation in question has more sophisticated solutions in the product form

$$w(r,z,t) = u(r,t)v(z,t),$$

where $u = u(r,t)$ and $v = v(z,t)$ are solutions of the simpler one-dimensional equations

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad \text{(see Section 3.2.1 for particular solutions of this equation)},$$

$$\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial z^2} \qquad \text{(see Section 3.1.1 for particular solutions of this equation)}.$$

2°. For all two-dimensional boundary value problems considered in Section 4.1.3, the Green's function can be represented in the product form

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$

where $G_1(r,\xi,t)$ and $G_2(z,\eta,t)$ are the Green's functions of appropriate one-dimensional boundary value problems.

► Domain: $0 \le r \le R$, $0 \le z < \infty$. First boundary value problem.

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi\int_0^\infty\!\!\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ &\quad - 2\pi aR\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau\\ &\quad + 2\pi a\int_0^t\!\!\int_0^R \xi g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,z,\xi,\eta,t) &= G_1(r,\xi,t)\,G_2(z,\eta,t),\\ G_1(r,\xi,t) &= \frac{1}{\pi R^2}\sum_{n=1}^\infty \frac{1}{J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),\\ G_2(z,\eta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] - \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\}, \end{aligned}$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

Example 4.9. The initial temperature is the same at every point of the cylinder, $f(r,z) = w_0$. The lateral surface and the end face are maintained at zero temperature, $g_1(r,t) = g_2(z,t) = 0$.

Solution:

$$w(r,z,t) = \frac{2w_0}{R}\operatorname{erf}\left(\frac{z}{2\sqrt{at}}\right)\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n R)}\exp(-\lambda_n^2 at), \qquad \lambda_n = \frac{\mu_n}{R}.$$

Example 4.10. The initial temperature of the cylinder is everywhere zero, $f(r,z) = 0$. The lateral surface $r = R$ is maintained at a constant temperature w_0, and the end face $z = 0$ at zero temperature.

Solution:

$$\begin{aligned} w(r,z,t) = w_0 - \frac{w_0}{R}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n R)}&\left[2\exp(-\lambda_n^2 at)\operatorname{erf}\left(\frac{z}{2\sqrt{at}}\right)\right.\\ &\left.+\exp(\lambda_n z)\operatorname{erfc}\left(\frac{z}{2\sqrt{at}}+\lambda_n\sqrt{at}\right)+\exp(-\lambda_n z)\operatorname{erfc}\left(\frac{z}{2\sqrt{at}}-\lambda_n\sqrt{at}\right)\right], \end{aligned}$$

where the λ_n are positive zeros of the Bessel function, $J_0(\lambda R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le r \le R$, $0 \le z < \infty$. Second boundary value problem.

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ \partial_r w &= g_1(z,t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\ \partial_z w &= g_2(r,t) &&\text{at}\quad z = 0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) = 2\pi&\int_0^{\infty}\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ &+2\pi aR\int_0^t\int_0^{\infty} g_1(\eta,\tau)\,G(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\ &-2\pi a\int_0^t\int_0^R \xi g_2(\xi,\tau)\,G(r,z,\xi,0,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$\begin{gathered} G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),\\ G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),\\ G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right]+\exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\}, \end{gathered}$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

► Domain: $0 \le r \le R,\ 0 \le z < \infty$. Third boundary value problem.

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) \quad &&\text{at} \quad t=0 \quad &&\text{(initial condition)},\\ \partial_r w + k_1 w &= g_1(z,t) \quad &&\text{at} \quad r=R \quad &&\text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(r,t) \quad &&\text{at} \quad z=0 \quad &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\begin{aligned} G(r,z,\xi,\eta,t) &= G_1(r,\xi,t)\,G_2(z,\eta,t),\\ G_1(r,\xi,t) &= \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{\mu_n^2}{(k_1^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\biggl(\frac{\mu_n r}{R}\biggr)J_0\biggl(\frac{\mu_n\xi}{R}\biggr)\exp\biggl(-\frac{a\mu_n^2 t}{R^2}\biggr),\\ G_2(z,\eta,t) &= \frac{1}{2\sqrt{\pi a t}}\biggl\{\exp\biggl[-\frac{(z-\eta)^2}{4at}\biggr]+\exp\biggl[-\frac{(z+\eta)^2}{4at}\biggr]\\ &\qquad - 2k_2\int_0^\infty \exp\biggl[-\frac{(z+\eta+s)^2}{4at}-k_2 s\biggr]ds\biggr\}. \end{aligned}$$

Here, $J_0(\mu)$ is the zeroth Bessel function, and the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0.$$

Example 4.11. The initial temperature is the same at every point of the cylinder, $f(r,z)=w_0$. At the lateral surface and the end face, heat exchange of the cylinder with the zero temperature environment occurs, $g_1(z,t)=g_2(r,t)=0$.

Solution:

$$w(r,z,t)=\frac{2w_0k_1}{R}\biggl[\operatorname{erf}\biggl(\frac{z}{2\sqrt{at}}\biggr)+\exp(k_2z+k_2^2at)\operatorname{erfc}\biggl(\frac{z}{2\sqrt{at}}+k_2\sqrt{at}\biggr)\biggr]\sum_{n=1}^{\infty}\frac{J_0(\nu_n r)\exp(-\nu_n^2 at)}{(k_1^2+\nu_n^2)J_0(\nu_n R)},$$

where the ν_n are positive roots of the transcendental equation $\nu J_1(\nu R) - k_1 J_0(\nu R) = 0$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le r \le R,\ 0 \le z < \infty$. Mixed boundary value problems.

1°. A semiinfinite circular cylinder is considered. The following boundary conditions are prescribed:

$$\begin{aligned} w &= f(r,z) \quad &&\text{at} \quad t=0 \quad &&\text{(initial condition)},\\ w &= g_1(z,t) \quad &&\text{at} \quad r=R \quad &&\text{(boundary condition)},\\ \partial_z w &= g_2(r,t) \quad &&\text{at} \quad z=0 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi\int_0^\infty\!\!\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ &\quad - 2\pi a R\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,z,\xi,\eta,t-\tau)\biggr]_{\xi=R}d\eta\,d\tau\\ &\quad - 2\pi a\int_0^t\!\!\int_0^R \xi g_2(\xi,\tau)\,G(r,z,\xi,0,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$

$$G_1(r,\xi,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] + \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

2°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi\int_0^\infty\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ &\quad + 2\pi a R\int_0^t\int_0^\infty g_1(\eta,\tau)\,G(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\ &\quad + 2\pi a\int_0^t\int_0^R \xi g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau. \end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$

$$G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] - \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

3°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w + kw &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,z,t)$ is determined by the formula in Item 2° (see above) where

$$G(r,z,\xi,\eta,t)=G_1(r,\xi,t)\,G_2(z,\eta,t),$$

$$G_1(r,\xi,t)=\frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{\mu_n^2}{(k^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z,\eta,t)=\frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right]-\exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu)-kRJ_0(\mu)=0.$$

Example 4.12. The initial temperature is the same at every point of the cylinder, $f(r,z)=w_0$. Heat exchange of the cylinder with the zero temperature environment occurs at the lateral surface, $g_1(z,t)=0$. The end face is maintained at zero temperature, $g_2(r,t)=0$.

Solution:

$$w(r,z,t)=\frac{2w_0k}{R}\operatorname{erf}\left(\frac{z}{2\sqrt{at}}\right)\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{(k^2+\lambda_n^2)J_0(\lambda_n R)}\exp(-\lambda_n^2at),\qquad \lambda_n=\frac{\mu_n}{R}.$$

Example 4.13. The initial temperature of the cylinder is everywhere zero, $f(r,z)=0$. Heat exchange of the cylinder with the zero temperature environment occurs at the lateral surface, $g_1(z,t)=0$. The end face is maintained at a constant temperature, $g_2(r,t)=w_0$.

Solution:

$$w(r,z,t)=\frac{kw_0}{R}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{(\lambda_n^2+k^2)J_0(\lambda_n R)}\biggl[2\exp(-\lambda_n z)$$
$$+\exp(\lambda_n z)\operatorname{erfc}\left(\lambda_n\sqrt{at}+\frac{z}{2\sqrt{at}}\right)-\exp(-\lambda_n z)\operatorname{erfc}\left(\lambda_n\sqrt{at}-\frac{z}{2\sqrt{at}}\right)\biggr],\quad \lambda_n=\frac{\mu_n}{R}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0\le r\le R$, $0\le z\le l$. First boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w&=g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w&=g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t)&=2\pi\int_0^l\int_0^R\xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&-2\pi aR\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,z,\xi,\eta,t-\tau)\right]_{\xi=R}d\eta\,d\tau\\
&+2\pi a\int_0^t\int_0^R\xi g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau\\
&-2\pi a\int_0^t\int_0^R\xi g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\right]_{\eta=l}d\xi\,d\tau.
\end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\Bigl(\frac{\mu_n r}{R}\Bigr)J_0\Bigl(\frac{\mu_n \xi}{R}\Bigr)\exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$
$$G_2(z,\eta,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\Bigl(\frac{n\pi z}{l}\Bigr)\sin\Bigl(\frac{n\pi\eta}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr),$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

Example 4.14. The initial temperature is the same at every point of the cylinder, $f(r,z) = w_0$. The lateral surface and the end faces are maintained at zero temperature, $g_1(z,t) = g_2(r,t) = g_3(r,t) = 0$.

Solution:

$$w = \frac{8w_0}{\pi}\Biggl\{\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\biggl[\frac{(2n+1)\pi z}{l}\biggr]\exp\biggl[-\frac{a(2n+1)^2\pi^2 t}{l^2}\biggr]\Biggr\}$$
$$\times\Biggl\{\sum_{n=1}^{\infty}\frac{1}{\mu_n J_1(\mu_n)}J_0\Bigl(\frac{\mu_n r}{R}\Bigr)\exp\Bigl(-\frac{\mu_n^2 at}{R^2}\Bigr)\Biggr\},$$

where μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le r \le R$, $0 \le z \le l$. Second boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_r w &= g_1(z,t) && \text{at} \quad r = R && \text{(boundary condition)},\\
\partial_z w &= g_2(r,t) && \text{at} \quad z = 0 && \text{(boundary condition)},\\
\partial_z w &= g_3(r,t) && \text{at} \quad z = l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) &= 2\pi\int_0^l\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&\quad + 2\pi aR\int_0^t\int_0^l g_1(\eta,\tau)\,G(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\
&\quad - 2\pi a\int_0^t\int_0^R \xi g_2(\xi,\tau)\,G(r,z,\xi,0,t-\tau)\,d\xi\,d\tau\\
&\quad + 2\pi a\int_0^t\int_0^R \xi g_3(\xi,\tau)\,G(r,z,\xi,l,t-\tau)\,d\xi\,d\tau.
\end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\biggl(\frac{\mu_n r}{R}\biggr)J_0\biggl(\frac{\mu_n\xi}{R}\biggr)\exp\biggl(-\frac{a\mu_n^2 t}{R^2}\biggr),$$
$$G_2(z,\eta,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\biggl(\frac{n\pi z}{l}\biggr)\cos\biggl(\frac{n\pi\eta}{l}\biggr)\exp\biggl(-\frac{an^2\pi^2 t}{l^2}\biggr),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

▶ **Domain: $0 \le r \le R$, $0 \le z \le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w + k_1 w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w + k_3 w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{\mu_n^2}{(k_1^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\biggl(\frac{\mu_n r}{R}\biggr)J_0\biggl(\frac{\mu_n\xi}{R}\biggr)\exp\biggl(-\frac{a\mu_n^2 t}{R^2}\biggr),$$
$$G_2(z,\eta,t) = \sum_{m=1}^{\infty}\frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2}\exp(-a\lambda_m^2 t),\qquad \varphi_m(z) = \cos(\lambda_m z) + \frac{k_2}{\lambda_m}\sin(\lambda_m z),$$
$$\|\varphi_m\|^2 = \frac{k_3}{2\lambda_m^2}\,\frac{\lambda_m^2+k_2^2}{\lambda_m^2+k_3^2} + \frac{k_2}{2\lambda_m^2} + \frac{l}{2}\biggl(1+\frac{k_2^2}{\lambda_m^2}\biggr),$$

and the μ_n and λ_m are positive roots of the transcendental equations

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0,\qquad \frac{\tan(\lambda l)}{\lambda} = \frac{k_2+k_3}{\lambda^2 - k_2 k_3}.$$

▶ **Domain: $0 \le r \le R$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ \partial_z w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= 2\pi \int_0^l \int_0^R \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta \\
&\quad - 2\pi a R \int_0^t \int_0^l g_1(\eta,\tau) \left[\frac{\partial}{\partial \xi} G(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\, d\tau \\
&\quad - 2\pi a \int_0^t \int_0^R \xi g_2(\xi,\tau)\, G(r,z,\xi,0,t-\tau)\, d\xi\, d\tau \\
&\quad + 2\pi a \int_0^t \int_0^R \xi g_3(\xi,\tau)\, G(r,z,\xi,l,t-\tau)\, d\xi\, d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,z,\xi,\eta,t) &= G_1(r,\xi,t)\, G_2(z,\eta,t), \\
G_1(r,\xi,t) &= \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{1}{J_1^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right), \\
G_2(z,\eta,t) &= \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi z}{l}\right) \cos\left(\frac{n\pi \eta}{l}\right) \exp\left(-\frac{an^2\pi^2 t}{l^2}\right),
\end{aligned}
$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\
\partial_r w &= g_1(z,t) && \text{at} \quad r = R && \text{(boundary condition)}, \\
w &= g_2(r,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\
w &= g_3(r,t) && \text{at} \quad z = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= 2\pi \int_0^l \int_0^R \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta \\
&\quad + 2\pi a R \int_0^t \int_0^l g_1(\eta,\tau)\, G(r,z,R,\eta,t-\tau)\, d\eta\, d\tau \\
&\quad + 2\pi a \int_0^t \int_0^R \xi g_2(\xi,\tau) \left[\frac{\partial}{\partial \eta} G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\, d\tau \\
&\quad - 2\pi a \int_0^t \int_0^R \xi g_3(\xi,\tau) \left[\frac{\partial}{\partial \eta} G(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\, d\tau.
\end{aligned}
$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\biggl(\frac{\mu_n r}{R}\biggr)J_0\biggl(\frac{\mu_n \xi}{R}\biggr)\exp\biggl(-\frac{a\mu_n^2 t}{R^2}\biggr),$$
$$G_2(z,\eta,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\biggl(\frac{n\pi z}{l}\biggr)\sin\biggl(\frac{n\pi\eta}{l}\biggr)\exp\biggl(-\frac{an^2\pi^2 t}{l^2}\biggr),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n)=0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

▶ **Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. First boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) &= 2\pi\int_0^l\int_{R_1}^{R_2}\xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&+ 2\pi aR_1\int_0^t\int_0^l g_1(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,z,\xi,\eta,t-\tau)\biggr]_{\xi=R_1}d\eta\,d\tau\\
&- 2\pi aR_2\int_0^t\int_0^l g_2(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,z,\xi,\eta,t-\tau)\biggr]_{\xi=R_2}d\eta\,d\tau\\
&+ 2\pi a\int_0^t\int_{R_1}^{R_2}\xi g_3(\xi,\tau)\biggl[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\biggr]_{\eta=0}d\xi\,d\tau\\
&- 2\pi a\int_0^t\int_{R_1}^{R_2}\xi g_4(\xi,\tau)\biggl[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\biggr]_{\eta=l}d\xi\,d\tau.
\end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(z,\eta,t)\,G_2(r,\xi,t),$$
$$G_1(z,\eta,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\biggl(\frac{n\pi z}{l}\biggr)\sin\biggl(\frac{n\pi\eta}{l}\biggr)\exp\biggl(-\frac{an^2\pi^2 t}{l^2}\biggr),$$
$$G_2(r,\xi,t) = \frac{\pi}{4R_1^2}\sum_{n=1}^{\infty}\frac{\mu_n^2 J_0^2(s\mu_n)}{J_0^2(\mu_n)-J_0^2(s\mu_n)}\Psi_n(r)\Psi_n(\xi)\exp\biggl(-\frac{a\mu_n^2 t}{R_1^2}\biggr),$$
$$\Psi_n(r) = Y_0(\mu_n)J_0\biggl(\frac{\mu_n r}{R_1}\biggr) - J_0(\mu_n)Y_0\biggl(\frac{\mu_n r}{R_1}\biggr), \qquad s=\frac{R_2}{R_1},$$

where $J_0(\mu)$ and $Y_0(\mu)$ are Bessel functions, the μ_n are positive roots of the transcendental equation

$$J_0(\mu)Y_0(s\mu) - J_0(s\mu)Y_0(\mu) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) = 2\pi &\int_0^l \int_{R_1}^{R_2} \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta\\
&- 2\pi a R_1 \int_0^t \int_0^l g_1(\eta,\tau)\, G(r,z,R_1,\eta,t-\tau)\, d\eta\, d\tau\\
&+ 2\pi a R_2 \int_0^t \int_0^l g_2(\eta,\tau)\, G(r,z,R_2,\eta,t-\tau)\, d\eta\, d\tau\\
&- 2\pi a \int_0^t \int_{R_1}^{R_2} \xi g_3(\xi,\tau)\, G(r,z,\xi,0,t-\tau)\, d\xi\, d\tau\\
&+ 2\pi a \int_0^t \int_{R_1}^{R_2} \xi g_4(\xi,\tau)\, G(r,z,\xi,l,t-\tau)\, d\xi\, d\tau.
\end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(z,\eta,t)\, G_2(r,\xi,t),$$

$$G_1(z,\eta,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi z}{l}\right) \cos\left(\frac{n\pi \eta}{l}\right) \exp\left(-\frac{an^2\pi^2 t}{l^2}\right),$$

$$G_2(r,\xi,t) = \frac{1}{\pi(R_2^2 - R_1^2)} + \frac{\pi}{4R_1^2}\sum_{n=1}^{\infty} \frac{\mu_n^2 J_1^2(s\mu_n)}{J_1^2(\mu_n) - J_1^2(s\mu_n)} \Psi_n(r)\Psi_n(\xi) \exp\left(-\frac{a\mu_n^2 t}{R_1^2}\right),$$

$$\Psi_n(r) = Y_1(\mu_n) J_0\left(\frac{\mu_n r}{R_1}\right) - J_1(\mu_n) Y_0\left(\frac{\mu_n r}{R_1}\right), \qquad s = \frac{R_2}{R_1},$$

where $J_k(\mu)$ and $Y_k(\mu)$ are Bessel functions of order $k = 0,\ 1$ and the μ_n are positive roots of the transcendental equation

$$J_1(\mu)Y_1(s\mu) - J_1(s\mu)Y_1(\mu) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Third boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w - k_1 w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w + k_2 w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\ \partial_z w - k_3 w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w + k_4 w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

For the solution of this problem, see the last paragraph in Section 4.2.3 with $\Phi \equiv 0$.

4.2 Heat Equation with a Source $\dfrac{\partial w}{\partial t} = a\Delta_2 w + \Phi(x,y,t)$

4.2.1 Problems in Cartesian Coordinates

In the rectangular Cartesian coordinate system, the heat equation has the form

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \Phi(x,y,t).$$

It governs two-dimensional unsteady thermal processes in quiescent media or solids with constant thermal diffusivity in the cases where there are volume thermal sources or sinks.

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.**

An initial condition is prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned} w(x,y,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta \\ &\quad + \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi a t}\exp\left[-\frac{(x-\xi)^2 + (y-\eta)^2}{4at}\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► Domain: $0 \le x < \infty$, $-\infty < y < \infty$. First boundary value problem.

A half-plane is considered. The following conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(y,t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(x,y,t) &= \int_0^\infty \int_{-\infty}^\infty f(\xi,\eta)\, G(x,y,\xi,\eta,t)\, d\eta\, d\xi \\ &+ a \int_0^t \int_{-\infty}^\infty g(\eta,\tau) \Big[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta,t-\tau)\Big]_{\xi=0} d\eta\, d\tau \\ &+ \int_0^t \int_0^\infty \int_{-\infty}^\infty \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t-\tau)\, d\eta\, d\xi\, d\tau, \end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi a t}\bigg\{\exp\bigg[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\bigg] - \exp\bigg[-\frac{(x+\xi)^2+(y-\eta)^2}{4at}\bigg]\bigg\}.$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x < \infty$, $-\infty < y < \infty$. Second boundary value problem.

A half-plane is considered. The following conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w = g(y,t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(x,y,t) &= \int_0^\infty \int_{-\infty}^\infty f(\xi,\eta)\, G(x,y,\xi,\eta,t)\, d\eta\, d\xi \\ &- a \int_0^t \int_{-\infty}^\infty g(\eta,\tau)\, G(x,y,0,\eta,t-\tau)\, d\eta\, d\tau \\ &+ \int_0^t \int_0^\infty \int_{-\infty}^\infty \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t-\tau)\, d\eta\, d\xi\, d\tau, \end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi a t}\bigg\{\exp\bigg[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\bigg] + \exp\bigg[-\frac{(x+\xi)^2+(y-\eta)^2}{4at}\bigg]\bigg\}.$$

► **Domain: $0 \le x < \infty$, $-\infty < y < \infty$. Third boundary value problem.**

A half-plane is considered. The following conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(y,t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi at} \exp\biggl[-\frac{(y-\eta)^2}{4at}\biggr] \biggl\{\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] + \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr] - 2k\int_0^\infty \exp\biggl[-\frac{(x+\xi+s)^2}{4at} - ks\biggr]\,ds\biggr\}.$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. First boundary value problem.**

A quadrant of the plane is considered. The following conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(y,t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},$$
$$w = g_2(x,t) \quad \text{at} \quad y = 0 \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(x,y,t) &= \int_0^\infty\!\!\int_0^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta \\ &+ a\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\Bigl[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\Bigr]_{\xi=0}\,d\eta\,d\tau \\ &+ a\int_0^t\!\!\int_0^\infty g_2(\xi,\tau)\Bigl[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\Bigr]_{\eta=0}\,d\xi\,d\tau \\ &+ \int_0^t\!\!\int_0^\infty\!\!\int_0^\infty \Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}\biggl\{\exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] - \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr]\biggr\} \times \biggl\{\exp\biggl[-\frac{(y-\eta)^2}{4at}\biggr] - \exp\biggl[-\frac{(y+\eta)^2}{4at}\biggr]\biggr\}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. Second boundary value problem.**

A quadrant of the plane is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_y w &= g_2(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,t) = &\int_0^\infty\!\!\int_0^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta\\ &- a\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau\\ &- a\int_0^t\!\!\int_0^\infty g_2(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau\\ &+\int_0^t\!\!\int_0^\infty\!\!\int_0^\infty \Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}&\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]+\exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}\\ &\times\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}. \end{aligned}$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. Third boundary value problem.**

A quadrant of the plane is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_y w - k_2 w &= g_2(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\begin{aligned} G(x,y,\xi,\eta,t) = \frac{1}{4\pi at}&\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]+\exp\left[-\frac{(x+\xi)^2}{4at}\right]\right.\\ &\left.-2k_1\sqrt{\pi at}\,\exp\bigl[ak_1^2t+k_1(x+\xi)\bigr]\operatorname{erfc}\left(\frac{x+\xi}{2\sqrt{at}}+k_1\sqrt{at}\right)\right\}\\ &\times\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right.\\ &\left.-2k_2\sqrt{\pi at}\,\exp\bigl[ak_2^2t+k_2(y+\eta)\bigr]\operatorname{erfc}\left(\frac{y+\eta}{2\sqrt{at}}+k_2\sqrt{at}\right)\right\}. \end{aligned}$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. Mixed boundary value problem.**

A quadrant of the plane is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g_1(y,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_y w &= g_2(x,t) \quad \text{at} \quad y=0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,t) = &\int_0^\infty\!\!\int_0^\infty f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta\\ &+ a\int_0^t\!\!\int_0^\infty g_1(\eta,\tau)\Big[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\Big]_{\xi=0}\,d\eta\,d\tau\\ &- a\int_0^t\!\!\int_0^\infty g_2(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau\\ &+ \int_0^t\!\!\int_0^\infty\!\!\int_0^\infty \Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(x,y,\xi,\eta,t) = &\frac{1}{4\pi at}\bigg\{\exp\bigg[-\frac{(x-\xi)^2}{4at}\bigg] - \exp\bigg[-\frac{(x+\xi)^2}{4at}\bigg]\bigg\}\\ &\times\bigg\{\exp\bigg[-\frac{(y-\eta)^2}{4at}\bigg] + \exp\bigg[-\frac{(y+\eta)^2}{4at}\bigg]\bigg\}. \end{aligned}$$

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. First boundary value problem.**

A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g_1(y,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ w &= g_2(y,t) \quad \text{at} \quad x=l \quad \text{(boundary condition)},\\ w &= g_3(x,t) \quad \text{at} \quad y=0 \quad \text{(boundary condition)}. \end{aligned}$$

The solution is given by the formula of Section 4.1.1 (see the first boundary value problem for $0 \le x \le l$, $0 \le y < \infty$) with the additional term

$$\int_0^t\!\!\int_0^l\!\!\int_0^\infty \Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,$$

which takes into account the equation's nonhomogeneity.

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. Second boundary value problem.**

A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w &= g_1(y,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w &= g_2(y,t) \quad \text{at} \quad x=l \quad \text{(boundary condition)},\\ \partial_y w &= g_3(x,t) \quad \text{at} \quad y=0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,y,t)=\int_0^\infty\int_0^l f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\xi\,d\eta+\int_0^t\int_0^\infty\int_0^l \Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau$$
$$-a\int_0^t\int_0^\infty g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau+a\int_0^t\int_0^\infty g_2(\eta,\tau)\,G(x,y,l,\eta,t-\tau)\,d\eta\,d\tau$$
$$-a\int_0^t\int_0^l g_3(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,y,\xi,\eta,t) = G_1(x,\xi,t)\,G_2(y,\eta,t),$$
$$G_1(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^\infty \cos\Bigl(\frac{n\pi x}{l}\Bigr)\cos\Bigl(\frac{n\pi\xi}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr),$$
$$G_2(y,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\Bigl\{\exp\Bigl[-\frac{(y-\eta)^2}{4at}\Bigr]+\exp\Bigl[-\frac{(y+\eta)^2}{4at}\Bigr]\Bigr\}.$$

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. Third boundary value problem.**

A semiinfinite strip is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(y,t) &&\text{at}\quad x=l &&\text{(boundary condition)},\\ \partial_y w - k_3 w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,z,\xi,\eta,t)=G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t)=\frac{1}{\pi R^2}\sum_{n=1}^\infty \frac{\mu_n^2}{(k_1^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\Bigl(\frac{\mu_n r}{R}\Bigr)J_0\Bigl(\frac{\mu_n\xi}{R}\Bigr)\exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$
$$G_2(z,\eta,t)=\sum_{m=1}^\infty \frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2}\exp(-a\lambda_m^2 t),\qquad \varphi_m(z)=\cos(\lambda_m z)+\frac{k_2}{\lambda_m}\sin(\lambda_m z),$$
$$\|\varphi_m\|^2=\frac{k_3}{2\lambda_m^2}\frac{\lambda_m^2+k_2^2}{\lambda_m^2+k_3^2}+\frac{k_2}{2\lambda_m^2}+\frac{l}{2}\Bigl(1+\frac{k_2^2}{\lambda_m^2}\Bigr),$$

and the μ_n and λ_m are positive roots of the transcendental equations

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0,\qquad \frac{\tan(\lambda l)}{\lambda} = \frac{k_2+k_3}{\lambda^2-k_2k_3}.$$

► **Domain: $0 \le x \le l$, $0 \le y < \infty$. Mixed boundary value problem.**

A semiinfinite strip is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,t) && \text{at} \quad x=l && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)}.
\end{aligned}
$$

The solution is given by the formula of Section 4.1.1 (see Item 1° of mixed boundary value problems for $0 \le x \le l$, $0 \le y < \infty$) with the additional term

$$
\int_0^t \int_0^l \int_0^\infty \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t-\tau)\, d\eta\, d\xi\, d\tau,
$$

which takes into account the equation's nonhomogeneity.

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. First boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution is given by the formula of Section 4.1.1 (see the first boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term

$$
\int_0^t \int_0^{l_1} \int_0^{l_2} \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t-\tau)\, d\eta\, d\xi\, d\tau,
$$

which takes into account the equation's nonhomogeneity.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Second boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$w(x,y,t)=\int_0^{l_1}\!\int_0^{l_2} f(\xi,\eta)\,G(x,y,\xi,\eta,t)\,d\eta\,d\xi+\int_0^t\!\int_0^{l_1}\!\int_0^{l_2}\Phi(\xi,\eta,\tau)\,G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau$$
$$-a\int_0^t\!\int_0^{l_2} g_1(\eta,\tau)\,G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau+a\int_0^t\!\int_0^{l_2} g_2(\eta,\tau)\,G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau$$
$$-a\int_0^t\!\int_0^{l_1} g_3(\xi,\tau)\,G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau+a\int_0^t\!\int_0^{l_1} g_4(\xi,\tau)\,G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,y,\xi,\eta,t)=\frac{1}{l_1 l_2}\left[1+2\sum_{n=1}^{\infty}\exp\left(-\frac{\pi^2 n^2 at}{l_1^2}\right)\cos\frac{n\pi x}{l_1}\cos\frac{n\pi\xi}{l_1}\right]$$
$$\times\left[1+2\sum_{m=1}^{\infty}\exp\left(-\frac{\pi^2 m^2 at}{l_2^2}\right)\cos\frac{m\pi y}{l_2}\cos\frac{m\pi\eta}{l_2}\right].$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Third boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\ \partial_y w - k_3 w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\ \partial_y w + k_4 w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t)=\left\{\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2}\exp(-a\mu_n^2 t)\right\}\left\{\sum_{m=1}^{\infty}\frac{\psi_m(y)\psi_m(\eta)}{\|\psi_m\|^2}\exp(-a\lambda_m^2 t)\right\},$$
$$\varphi_n(x)=\cos(\mu_n x)+\frac{k_1}{\mu_n}\sin(\mu_n x),\qquad \|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\,\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{l_1}{2}\left(1+\frac{k_1^2}{\mu_n^2}\right),$$
$$\psi_m(y)=\cos(\lambda_m y)+\frac{k_3}{\lambda_m}\sin(\lambda_m y),\quad \|\psi_m\|^2=\frac{k_4}{2\lambda_m^2}\,\frac{\lambda_m^2+k_3^2}{\lambda_m^2+k_4^2}+\frac{k_3}{2\lambda_m^2}+\frac{l_2}{2}\left(1+\frac{k_3^2}{\lambda_m^2}\right).$$

Here, the μ_n and λ_m are positive roots of the transcendental equations

$$\frac{\tan(\mu l_1)}{\mu}=\frac{k_1+k_2}{\mu^2-k_1k_2},\qquad \frac{\tan(\lambda l_2)}{\lambda}=\frac{k_3+k_4}{\lambda^2-k_3k_4}.$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Mixed boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\ \partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\ \partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}. \end{aligned}$$

The solution is given by the formula of Section 4.1.1 (see Item 1° of mixed boundary value problems for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term

$$\int_0^t \int_0^{l_1} \int_0^{l_2} \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t-\tau)\, d\eta\, d\xi\, d\tau,$$

which takes into account the equation's nonhomogeneity.

4.2.2 Problems in Polar Coordinates

The heat equation with a volume source in the polar coordinate system r, φ is written as

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right) + \Phi(r,\varphi,t).$$

Solutions of the form $w = w(r,t)$ that are independent of the angular coordinate φ and govern plane thermal processes with central symmetry are presented in Section 3.2.2.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le 2\pi$. Cauchy problem.**

An initial condition is prescribed:

$$w = f(r,\varphi) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \int_0^{2\pi}\int_0^{\infty} f(\xi,\eta)\, G(r,\varphi,\xi,\eta,t)\, \xi\, d\xi\, d\eta \\ &\quad + \int_0^t \int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\tau)\, G(r,\varphi,\xi,\eta,t-\tau)\, \xi\, d\xi\, d\eta\, d\tau, \end{aligned}$$

where

$$G(r,\varphi,\xi,\eta,t) = \frac{1}{4\pi a t} \exp\left[-\frac{r^2 + \xi^2 - 2r\xi\cos(\varphi-\eta)}{4at}\right].$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$w = f(r, \varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(\varphi, t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(r, \varphi, t) &= \int_0^{2\pi}\int_0^R f(\xi, \eta)\, G(r, \varphi, \xi, \eta, t)\xi\, d\xi\, d\eta \\ &\quad - aR\int_0^t\int_0^{2\pi} g(\eta, \tau)\left[\frac{\partial}{\partial \xi}G(r, \varphi, \xi, \eta, t-\tau)\right]_{\xi=R} d\eta\, d\tau \\ &\quad + \int_0^t\int_0^{2\pi}\int_0^R \Phi(\xi, \eta, \tau)\, G(r, \varphi, \xi, \eta, t-\tau)\xi\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r, \varphi, \xi, \eta, t) = \frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2} J_n(\mu_{nm}r) J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$A_0 = 1, \quad A_n = 2 \quad (n = 1,\, 2,\, \ldots),$$

where $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument), and μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$w = f(r, \varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_r w = g(\varphi, t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(r, \varphi, t) &= \int_0^{2\pi}\int_0^R f(\xi, \eta)\, G(r, \varphi, \xi, \eta, t)\xi\, d\xi\, d\eta \\ &\quad + aR\int_0^t\int_0^{2\pi} g(\eta, \tau)\, G(r, \varphi, R, \eta, t-\tau)\, d\eta\, d\tau \\ &\quad + \int_0^t\int_0^{2\pi}\int_0^R \Phi(\xi, \eta, \tau)\, G(r, \varphi, \xi, \eta, t-\tau)\xi\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r, \varphi, \xi, \eta, t) = \frac{1}{\pi R^2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 - n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$A_0 = 1, \quad A_n = 2 \quad (n = 1,\, 2,\, \ldots),$$

where $J_n(\xi)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$w = f(r,\varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_r w + kw = g(\varphi,t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2+k^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$A_0 = 1, \quad A_n = 2 \quad (n = 1,\ 2,\ \ldots).$$

Here, $J_n(\xi)$ are Bessel functions, and μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Different boundary value problems.**

1°. The solution of the first boundary value problem for an annular domain is given by the formula in Section 4.1.2 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$) with the additional term

$$\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\tau)\,G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau, \tag{1}$$

which allows for the equation's nonhomogeneity.

2°. The solution of the second boundary value problem for an annular domain is given by the formula in Section 4.1.2 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$) with the additional term (1); the Green's function is also taken from Section 4.1.2.

3°. The solution of the third boundary value problem for an annular domain is given by the formula in Section 4.1.2 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$) with the additional term (1); the Green's function is also taken from Section 4.1.2.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$. Different boundary value problems.**

1°. The solution of the first boundary value problem for a wedge domain is given by the formula in Section 4.1.2 (see the first boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$) with the additional term

$$\int_0^t\int_0^{\varphi_0}\int_0^{\infty}\Phi(\xi,\eta,\tau)\,G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau, \tag{2}$$

which allows for the equation's nonhomogeneity.

2°. The solution of the second boundary value problem for a wedge domain is given by the formula in Section 4.1.2 (see the second boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$) with the additional term (2); the Green's function is also taken from Section 4.1.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Different boundary value problems.**

1°. The solution of the first boundary value problem for a sector of a circle is given by the formula of Section 4.1.2 (see the first boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$) with the additional term

$$\int_0^t \int_0^{\varphi_0} \int_0^R \Phi(\xi,\eta,\tau)\, G(r,\varphi,\xi,\eta,t-\tau)\xi \, d\xi \, d\eta \, d\tau, \tag{3}$$

which allows for the equation's nonhomogeneity.

2°. The solution of the mixed boundary value problem for a sector of a circle is given by the formula of Section 4.1.2 (see the mixed boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$) with the additional term (3); the Green's function is also taken from Section 4.1.2.

4.2.3 Axisymmetric Problems

In the case of axial symmetry, the heat equation in the cylindrical coordinate system is written as

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) + \Phi(r,z,t),$$

provided there are heat sources or sinks.

One-dimensional axisymmetric problems that have solutions of the form $w = w(r,t)$ can be found in Section 3.2.2.

► **Domain: $0 \le r \le R$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
w &= g_1(z,t) && \text{at} \quad r = R && \text{(boundary condition)},\\
w &= g_2(r,t) && \text{at} \quad z = 0 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) &= 2\pi \int_0^\infty \int_0^R \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta\\
&\quad - 2\pi a R \int_0^t \int_0^\infty g_1(\eta,\tau) \left[\frac{\partial}{\partial \xi} G(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\, d\tau\\
&\quad + 2\pi a \int_0^t \int_0^R \xi g_2(\xi,\tau) \left[\frac{\partial}{\partial \eta} G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\, d\tau\\
&\quad + 2\pi \int_0^t \int_0^\infty \int_0^R \xi \Phi(\xi,\eta,\tau)\, G(r,z,\xi,\eta,t-\tau)\, d\xi\, d\eta\, d\tau.
\end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$
$$G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] - \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le r \le R$, $0 \le z < \infty$. Second boundary value problem.

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ \partial_z w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi\int_0^\infty\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ &+ 2\pi a R\int_0^t\int_0^\infty g_1(\eta,\tau)\,G(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\ &- 2\pi a\int_0^t\int_0^R \xi g_2(\xi,\tau)\,G(r,z,\xi,0,t-\tau)\,d\xi\,d\tau\\ &+ 2\pi\int_0^t\int_0^\infty\int_0^R \xi\Phi(\xi,\eta,\tau)\,G(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.\end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$
$$G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$
$$G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] + \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$, $0 \le z < \infty$. Third boundary value problem.**

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ \partial_r w + k_1 w &= g_1(z, t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(r, t) &&\text{at}\quad z = 0 &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r, z, t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r, z, \xi, \eta, t) = G_1(r, \xi, t)\, G_2(z, \eta, t),$$

$$G_1(r, \xi, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2}{(k_1^2 R^2 + \mu_n^2) J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$\begin{aligned} G_2(z, \eta, t) = \frac{1}{2\sqrt{\pi a t}} \biggl\{ &\exp\left[-\frac{(z-\eta)^2}{4at}\right] + \exp\left[-\frac{(z+\eta)^2}{4at}\right] \\ &- 2k_2 \int_0^\infty \exp\left[-\frac{(z+\eta+s)^2}{4at} - k_2 s\right] ds \biggr\}. \end{aligned}$$

Here, $J_0(\mu)$ is the zeroth Bessel function, and the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r \le R$, $0 \le z < \infty$. Mixed boundary value problems.**

1°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ w &= g_1(z, t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\ \partial_z w &= g_2(r, t) &&\text{at}\quad z = 0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r, z, t) = {} & 2\pi \int_0^\infty \int_0^R \xi f(\xi, \eta)\, G(r, z, \xi, \eta, t)\, d\xi\, d\eta \\ & - 2\pi a R \int_0^t \int_0^\infty g_1(\eta, \tau) \left[\frac{\partial}{\partial \xi} G(r, z, \xi, \eta, t-\tau)\right]_{\xi=R} d\eta\, d\tau \\ & - 2\pi a \int_0^t \int_0^R \xi g_2(\xi, \tau)\, G(r, z, \xi, 0, t-\tau)\, d\xi\, d\tau \\ & + 2\pi \int_0^t \int_0^\infty \int_0^R \xi \Phi(\xi, \eta, \tau)\, G(r, z, \xi, \eta, t-\tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$

$$G_1(r,\xi,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] + \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

2°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi\int_0^\infty\int_0^R \xi f(\xi,\eta)\,G(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ &+ 2\pi a R\int_0^t\int_0^\infty g_1(\eta,\tau)\,G(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\ &+ 2\pi a\int_0^t\int_0^R \xi g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\ &+ 2\pi\int_0^t\int_0^\infty\int_0^R \xi\Phi(\xi,\eta,\tau)\,G(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\,G_2(z,\eta,t),$$

$$G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z,\eta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\eta)^2}{4at}\right] - \exp\left[-\frac{(z+\eta)^2}{4at}\right]\right\},$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

3°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + kw &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r, z, t)$ is determined by the formula in Item 2° (see above) where

$$G(r, z, \xi, \eta, t) = G_1(r, \xi, t)\, G_2(z, \eta, t),$$

$$G_1(r, \xi, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2}{(k^2R^2 + \mu_n^2) J_0^2(\mu_n)} J_0\Bigl(\frac{\mu_n r}{R}\Bigr) J_0\Bigl(\frac{\mu_n \xi}{R}\Bigr) \exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$

$$G_2(z, \eta, t) = \frac{1}{2\sqrt{\pi a t}} \biggl\{ \exp\biggl[-\frac{(z-\eta)^2}{4at}\biggr] - \exp\biggl[-\frac{(z+\eta)^2}{4at}\biggr] \biggr\},$$

where the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu) - kRJ_0(\mu) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r, z) &&\text{at} \quad t = 0 &&\text{(initial condition)},\\ w &= g_1(z, t) &&\text{at} \quad r = R &&\text{(boundary condition)},\\ w &= g_2(r, t) &&\text{at} \quad z = 0 &&\text{(boundary condition)},\\ w &= g_3(r, t) &&\text{at} \quad z = l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r, z, t) &= 2\pi \int_0^l \int_0^R \xi f(\xi, \eta)\, G(r, z, \xi, \eta, t)\, d\xi\, d\eta \\ &\quad - 2\pi a R \int_0^t \int_0^l g_1(\eta, \tau) \biggl[\frac{\partial}{\partial \xi} G(r, z, \xi, \eta, t-\tau)\biggr]_{\xi=R} d\eta\, d\tau \\ &\quad + 2\pi a \int_0^t \int_0^R \xi g_2(\xi, \tau) \biggl[\frac{\partial}{\partial \eta} G(r, z, \xi, \eta, t-\tau)\biggr]_{\eta=0} d\xi\, d\tau \\ &\quad - 2\pi a \int_0^t \int_0^R \xi g_3(\xi, \tau) \biggl[\frac{\partial}{\partial \eta} G(r, z, \xi, \eta, t-\tau)\biggr]_{\eta=l} d\xi\, d\tau \\ &\quad + 2\pi \int_0^t \int_0^l \int_0^R \xi \Phi(\xi, \eta, \tau)\, G(r, z, \xi, \eta, t-\tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r, z, \xi, \eta, t) = G_1(r, \xi, t)\, G_2(z, \eta, t),$$

$$G_1(r, \xi, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{1}{J_1^2(\mu_n)} J_0\Bigl(\frac{\mu_n r}{R}\Bigr) J_0\Bigl(\frac{\mu_n \xi}{R}\Bigr) \exp\Bigl(-\frac{a\mu_n^2 t}{R^2}\Bigr),$$

$$G_2(z, \eta, t) = \frac{2}{l} \sum_{n=1}^{\infty} \sin\Bigl(\frac{n\pi z}{l}\Bigr) \sin\Bigl(\frac{n\pi \eta}{l}\Bigr) \exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr),$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi \int_0^l \int_0^R \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta \\ &\quad + 2\pi a R \int_0^t \int_0^l g_1(\eta,\tau)\, G(r,z,R,\eta,t-\tau)\, d\eta\, d\tau \\ &\quad - 2\pi a \int_0^t \int_0^R \xi g_2(\xi,\tau)\, G(r,z,\xi,0,t-\tau)\, d\xi\, d\tau \\ &\quad + 2\pi a \int_0^t \int_0^R \xi g_3(\xi,\tau)\, G(r,z,\xi,l,t-\tau)\, d\xi\, d\tau \\ &\quad + 2\pi \int_0^t \int_0^l \int_0^R \xi \Phi(\xi,\eta,\tau)\, G(r,z,\xi,\eta,t-\tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,z,\xi,\eta,t) &= G_1(r,\xi,t)\, G_2(z,\eta,t),\\ G_1(r,\xi,t) &= \frac{1}{\pi R^2} + \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{1}{J_0^2(\mu_n)} J_0\Big(\frac{\mu_n r}{R}\Big) J_0\Big(\frac{\mu_n \xi}{R}\Big) \exp\Big(-\frac{a\mu_n^2 t}{R^2}\Big),\\ G_2(z,\eta,t) &= \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \cos\Big(\frac{n\pi z}{l}\Big) \cos\Big(\frac{n\pi\eta}{l}\Big) \exp\Big(-\frac{an^2\pi^2 t}{l^2}\Big), \end{aligned}$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + k_1 w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w + k_3 w &= g_3(r,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r, z, t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r, z, \xi, \eta, t) = G_1(r, \xi, t)\, G_2(z, \eta, t),$$

$$G_1(r, \xi, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2}{(k_1^2 R^2 + \mu_n^2) J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z, \eta, t) = \sum_{m=1}^{\infty} \frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2} \exp(-a\lambda_m^2 t), \qquad \varphi_m(z) = \cos(\lambda_m z) + \frac{k_2}{\lambda_m}\sin(\lambda_m z),$$

$$\|\varphi_m\|^2 = \frac{k_3}{2\lambda_m^2} \frac{\lambda_m^2 + k_2^2}{\lambda_m^2 + k_3^2} + \frac{k_2}{2\lambda_m^2} + \frac{l}{2}\left(1 + \frac{k_2^2}{\lambda_m^2}\right),$$

and the μ_n and λ_m are positive roots of the transcendental equations

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0, \qquad \frac{\tan(\lambda l)}{\lambda} = \frac{k_2 + k_3}{\lambda^2 - k_2 k_3}.$$

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g_1(z, t) && \text{at} \quad r = R && \text{(boundary condition)},\\ \partial_z w &= g_2(r, t) && \text{at} \quad z = 0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r, t) && \text{at} \quad z = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r, z, t) = {} & 2\pi \int_0^l \int_0^R \xi f(\xi, \eta)\, G(r, z, \xi, \eta, t)\, d\xi\, d\eta \\ & - 2\pi a R \int_0^t \int_0^l g_1(\eta, \tau) \left[\frac{\partial}{\partial \xi} G(r, z, \xi, \eta, t - \tau)\right]_{\xi = R} d\eta\, d\tau \\ & - 2\pi a \int_0^t \int_0^R \xi g_2(\xi, \tau)\, G(r, z, \xi, 0, t - \tau)\, d\xi\, d\tau \\ & + 2\pi a \int_0^t \int_0^R \xi g_3(\xi, \tau)\, G(r, z, \xi, l, t - \tau)\, d\xi\, d\tau \\ & + 2\pi \int_0^t \int_0^l \int_0^R \xi \Phi(\xi, \eta, \tau)\, G(r, z, \xi, \eta, t - \tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r, z, \xi, \eta, t) = G_1(r, \xi, t)\, G_2(z, \eta, t),$$

$$G_1(r, \xi, t) = \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{1}{J_1^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z, \eta, t) = \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi z}{l}\right) \cos\left(\frac{n\pi \eta}{l}\right) \exp\left(-\frac{a n^2 \pi^2 t}{l^2}\right),$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu_n) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ \partial_r w &= g_1(z,t) &&\text{at} \quad r = R &&\text{(boundary condition)},\\ w &= g_2(r,t) &&\text{at} \quad z = 0 &&\text{(boundary condition)},\\ w &= g_3(r,t) &&\text{at} \quad z = l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi \int_0^l \int_0^R \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta \\ &+ 2\pi a R \int_0^t \int_0^l g_1(\eta,\tau)\, G(r,z,R,\eta,t-\tau)\, d\eta\, d\tau \\ &+ 2\pi a \int_0^t \int_0^R \xi g_2(\xi,\tau) \left[\frac{\partial}{\partial\eta} G(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\, d\tau \\ &- 2\pi a \int_0^t \int_0^R \xi g_3(\xi,\tau) \left[\frac{\partial}{\partial\eta} G(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\, d\tau \\ &+ 2\pi \int_0^t \int_0^l \int_0^R \xi \Phi(\xi,\eta,\tau)\, G(r,z,\xi,\eta,t-\tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\, G_2(z,\eta,t),$$

$$G_1(r,\xi,t) = \frac{1}{\pi R^2} + \frac{1}{\pi R^2} \sum_{n=1}^{\infty} \frac{1}{J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

$$G_2(z,\eta,t) = \frac{2}{l} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right) \exp\left(-\frac{an^2\pi^2 t}{l^2}\right),$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu_n) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. First boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,z) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ w &= g_1(z,t) &&\text{at} \quad r = R_1 &&\text{(boundary condition)},\\ w &= g_2(z,t) &&\text{at} \quad r = R_2 &&\text{(boundary condition)},\\ w &= g_3(r,t) &&\text{at} \quad z = 0 &&\text{(boundary condition)},\\ w &= g_4(r,t) &&\text{at} \quad z = l &&\text{(boundary condition)}. \end{aligned}$$

The solution to the first boundary value problem for a hollow circular cylinder of interior radius R_1, exterior radius R_2, and length l is given by the formula of Section 4.1.3 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le z \le l$) with the term

$$2\pi \int_0^t \int_0^l \int_{R_1}^{R_2} \xi\Phi(\xi,\eta,\tau)\, G(r,z,\xi,\eta,t-\tau)\, d\xi\, d\eta\, d\tau \tag{1}$$

added; this term takes into account the equation's nonhomogeneity.

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Second boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
\partial_z w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

The solution to the second boundary value problem for a finite hollow circular cylinder is given by the formula of Section 4.1.3 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le z \le l$) with the additional term (1). The Green's function is the same as in Section 4.1.3.

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Third boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w - k_1 w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w + k_2 w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
\partial_z w - k_3 w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w + k_4 w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) = 2\pi &\int_0^l \int_{R_1}^{R_2} \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta\\
&- 2\pi a R_1 \int_0^t \int_0^l g_1(\eta,\tau)\, G(r,z,R_1,\eta,t-\tau)\, d\eta\, d\tau\\
&+ 2\pi a R_2 \int_0^t \int_0^l g_2(\eta,\tau)\, G(r,z,R_2,\eta,t-\tau)\, d\eta\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&- 2\pi a \int_0^t \int_{R_1}^{R_2} \xi g_3(\xi,\tau)\, G(r,z,\xi,0,t-\tau)\, d\xi\, d\tau \\
&+ 2\pi a \int_0^t \int_{R_1}^{R_2} \xi g_4(\xi,\tau)\, G(r,z,\xi,l,t-\tau)\, d\xi\, d\tau \\
&+ 2\pi \int_0^t \int_0^l \int_{R_1}^{R_2} \xi \Phi(\xi,\eta,\tau)\, G(r,\xi,z,\eta,t-\tau)\, d\xi\, d\eta\, d\tau.
\end{aligned}
$$

Here, the Green's function is given by

$$
\begin{aligned}
&G(r,z,\xi,\eta,t) = G_1(r,\xi,t)\, G_2(z,\eta,t), \\
&G_1(r,\xi,t) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{B_n} \big[k_2 J_0(\lambda_n R_2) - \lambda_n J_1(\lambda_n R_2)\big]^2 H_n(r) H_n(\xi) \exp(-\lambda_n^2 a t), \\
&G_2(z,\eta,t) = \sum_{m=1}^{\infty} \frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2} \exp\big(-\mu_m^2 a t\big),
\end{aligned}
$$

with

$$
\begin{aligned}
&B_n = (\lambda_n^2 + k_2^2)\big[k_1 J_0(\lambda_n R_1) + \lambda_n J_1(\lambda_n R_1)\big]^2 - (\lambda_n^2 + k_1^2)\big[k_2 J_0(\lambda_n R_2) - \lambda_n J_1(\lambda_n R_2)\big]^2, \\
&H_n(r) = \big[k_1 Y_0(\lambda_n R_1) + \lambda_n Y_1(\lambda_n R_1)\big] J_0(\lambda_n r) - \big[k_1 J_0(\lambda_n R_1) + \lambda_n J_1(\lambda_n R_1)\big] Y_0(\lambda_n r), \\
&\varphi_m(z) = \mu_m \cos(\mu_m z) + k_3 \sin(\mu_m z), \qquad \|\varphi_m\|^2 = \frac{k_4}{2} \frac{\mu_m^2 + k_3^2}{\mu_m^2 + k_4^2} + \frac{k_3}{2} + \frac{l}{2}\big(\mu_m^2 + k_3^2\big),
\end{aligned}
$$

where $J_0(\lambda)$, $J_1(\lambda)$, $Y_0(\lambda)$, and $Y_1(\lambda)$ are Bessel functions, the λ_n are positive roots of the transcendental equation

$$
\begin{aligned}
&\big[k_1 J_0(\lambda R_1) + \lambda J_1(\lambda R_1)\big]\big[k_2 Y_0(\lambda R_2) - \lambda Y_1(\lambda R_2)\big] \\
&\qquad - \big[k_2 J_0(\lambda R_2) - \lambda J_1(\lambda R_2)\big]\big[k_1 Y_0(\lambda R_1) + \lambda Y_1(\lambda R_1)\big] = 0,
\end{aligned}
$$

and the μ_m are positive roots of the transcendental equation

$$
\frac{\tan \mu l}{\mu} = \frac{k_3 + k_4}{\mu^2 - k_3 k_4}.
$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Mixed boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\
\partial_r w - k_1 w &= g_1(z,t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\
\partial_r w + k_2 w &= g_2(z,t) && \text{at} \quad r = R_2 && \text{(boundary condition)}, \\
w &= g_3(r,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\
w &= g_4(r,t) && \text{at} \quad z = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$\begin{aligned} w(r,z,t) &= 2\pi \int_0^l \int_{R_1}^{R_2} \xi f(\xi,\eta)\, G(r,z,\xi,\eta,t)\, d\xi\, d\eta \\ &\quad - 2\pi a R_1 \int_0^t \int_0^l g_1(\eta,\tau)\, G(r,z,R_1,\eta,t-\tau)\, d\eta\, d\tau \\ &\quad + 2\pi a R_2 \int_0^t \int_0^l g_2(\eta,\tau)\, G(r,z,R_2,\eta,t-\tau)\, d\eta\, d\tau \\ &\quad + 2\pi a \int_0^t \int_{R_1}^{R_2} \xi g_3(\xi,\tau) \Big[\frac{\partial}{\partial\eta} G(r,z,\xi,\eta,t-\tau)\Big]_{\eta=0} d\xi\, d\tau \\ &\quad - 2\pi a \int_0^t \int_{R_1}^{R_2} \xi g_4(\xi,\tau) \Big[\frac{\partial}{\partial\eta} G(r,z,\xi,\eta,t-\tau)\Big]_{\eta=l} d\xi\, d\tau \\ &\quad + 2\pi \int_0^t \int_0^l \int_{R_1}^{R_2} \xi \Phi(\xi,\eta,\tau)\, G(r,\xi,z,\eta,t-\tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r,\xi,z,\eta,t) = G_1(r,\xi,t)\, \frac{2}{l} \sum_{m=1}^{\infty} \sin\Big(\frac{\pi m z}{l}\Big) \sin\Big(\frac{\pi m \eta}{l}\Big) \exp\Big(-\frac{\pi^2 m^2 a t}{l^2}\Big),$$

where the expression of $G_1(r,\xi,t)$ is specified in the previous paragraph (for the third boundary value problem).

◆ *Section 5.2.2 presents solutions of other boundary value problems; a more general, three-dimensional equation is discussed there.*

4.3 Other Equations

4.3.1 Equations Containing Arbitrary Parameters

1. $\displaystyle \frac{\partial w}{\partial t} = a\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + (bx + cy + k)w.$

The transformation

$$w(x,y,t) = u(\xi,\eta,t) \exp\big[(bx+cy+k)t + \tfrac{1}{3}a(b^2+c^2)t^3\big], \quad \xi = x + abt^2, \quad \eta = y + act^2$$

leads to the two-dimensional heat equation $\partial_t u = a\big(\partial_{\xi\xi} u + \partial_{\eta\eta} u\big)$.

See also Niederer (1973) and Boyer (1974).

2. $\displaystyle \frac{\partial w}{\partial t} = a\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) - \big(bx^2 + by^2 + k\big)w, \qquad b > 0.$

The transformation (C is an arbitrary constant)

$$w(x,y,t) = u(\xi,\eta,\tau) \exp\Big[\frac{1}{2}\sqrt{\frac{b}{a}}\,(x^2+y^2) + \big(2\sqrt{ab} - k\big)\, t\Big],$$

$$\xi = x \exp\big(2\sqrt{ab}\, t\big), \quad \eta = y \exp\big(2\sqrt{ab}\, t\big), \quad \tau = \frac{1}{4\sqrt{ab}} \exp\big(4\sqrt{ab}\, t\big) + C$$

leads to the two-dimensional heat equation $\partial_\tau u = a\bigl(\partial_{\xi\xi} u + \partial_{\eta\eta} u\bigr)$.

See also Niederer (1973) and Boyer (1974).

3. $\displaystyle \frac{\partial w}{\partial t} = a\Bigl(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Bigr) + \bigl(bx^2 + by^2 - k\bigr)w, \qquad b > 0.$

The transformation

$$w(x,y,t) = \frac{1}{\cos\bigl(2\sqrt{ab}\,t\bigr)} \exp\biggl[\frac{\sqrt{b}}{2\sqrt{a}} \tan\bigl(2\sqrt{ab}\,t\bigr)\bigl(x^2+y^2\bigr) - kt\biggr] u(\xi,\eta,\tau),$$
$$\xi = \frac{x}{\cos\bigl(2\sqrt{ab}\,t\bigr)}, \quad \eta = \frac{y}{\cos\bigl(2\sqrt{ab}\,t\bigr)}, \quad \tau = \frac{\sqrt{a}}{2\sqrt{b}} \tan\bigl(2\sqrt{ab}\,t\bigr)$$

leads to the two-dimensional heat equation $\partial_\tau u = \partial_{\xi\xi} u + \partial_{\eta\eta} u$.

See also Niederer (1973) and Boyer (1974).

4. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \bigl(ax^{-2} + by^{-2}\bigr)w.$

This is a special case of equation 4.3.2.7. Boyer (1976) showed that this equation admits the separation of variables into 25 systems of coordinates for $ab = 0$ and 15 systems of coordinates for $ab \neq 0$.

5. $\displaystyle \frac{\partial w}{\partial t} = a\Bigl(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Bigr) + (bt^n x + ct^m y + st^k)w.$

This is a special case of equation 4.3.2.2 with $f(t) = bt^n$, $g(t) = ct^m$, and $h(t) = st^k$.

6. $\displaystyle \frac{\partial w}{\partial t} = a\Bigl(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Bigr) + \bigl[-b(x^2+y^2) + c_1 t^{n_1} x + c_2 t^{n_2} y + st^k\bigr]w.$

This is a special case of equation 4.3.2.3 with $f(t) = c_1 t^{n_1}$, $g(t) = c_2 t^{n_2}$, and $h(t) = st^k$.

7. $\displaystyle \frac{\partial w}{\partial t} = a\Bigl(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Bigr) + b_1\frac{\partial w}{\partial x} + b_2\frac{\partial w}{\partial y} + cw.$

This equation describes an unsteady temperature (concentration) field in a medium moving with a constant velocity, provided there is volume release (absorption) of heat proportional to temperature.

The substitution

$$w(x,y,t) = \exp\bigl(A_1 x + A_2 y + Bt\bigr)U(x,y,t),$$
$$A_1 = -\frac{b_1}{2a}, \quad A_2 = -\frac{b_2}{2a}, \quad B = c - \frac{b_1^2 + b_2^2}{4a},$$

leads to the two-dimensional heat equation $\partial_t U = a\Delta_2 U$ that is considered in Section 4.1.1.

8. $\displaystyle \frac{\partial w}{\partial t} = a\Bigl(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Bigr) + b_1\frac{\partial w}{\partial x} + b_2\frac{\partial w}{\partial y} + (c_1 x + c_2 y + k)w.$

The transformation

$$w(x,y,t) = \exp\bigl[(c_1 x + c_2 y)t + \tfrac{1}{3}a(c_1^2 + c_2^2)t^3 + \tfrac{1}{2}(b_1 c_1 + b_2 c_2)t^2 + kt\bigr]U(\xi,\eta,t),$$
$$\xi = x + ac_1 t^2 + b_1 t, \quad \eta = y + ac_2 t^2 + b_2 t$$

leads to the two-dimensional heat equation $\partial_t U = a(\partial_{\xi\xi} U + \partial_{\eta\eta} U)$ that is considered in Section 4.1.1.

9. $\dfrac{\partial w}{\partial t}=a\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\Big)+b_1t^{n_1}\dfrac{\partial w}{\partial x}+b_2t^{n_2}\dfrac{\partial w}{\partial y}+(c_1t^{m_1}x+c_2t^{m_2}y+st^k)w.$

This is a special case of equation 4.3.2.5. The equation can be reduced to the two-dimensional heat equation treated in Section 4.1.1.

10. $i\hbar\dfrac{\partial w}{\partial t}+\dfrac{\hbar^2}{2m}\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\Big)=0.$

Two-dimensional Schrödinger equation, $i^2=-1$.

Fundamental solution:

$$\mathscr{E}(x,y,t)=-\frac{im}{2\pi\hbar^2 t}\exp\left[\frac{im}{2\hbar t}(x^2+y^2)-i\frac{\pi}{2}\right].$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

4.3.2 Equations Containing Arbitrary Functions

1. $\dfrac{\partial w}{\partial t}=a\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\Big)+f(t)w.$

This equation describes two-dimensional thermal phenomena in quiescent media or solids with constant thermal diffusivities in the case of unsteady volume heat release proportional to temperature.

The substitution $w(x,y,t)=\exp\big[\int f(t)\,dt\big]U(x,y,t)$ leads to the two-dimensional heat equation $\partial_t U=a(\partial_{xx}U+\partial_{yy}U)$ treated in Section 4.1.1.

2. $\dfrac{\partial w}{\partial t}=a\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\Big)+[xf(t)+yg(t)+h(t)]w.$

The transformation

$$w(x,y,t)=u(\xi,\eta,t)\exp\left[xF(t)+yG(t)+H(t)+a\int F^2(t)\,dt+a\int G^2(t)\,dt\right],$$
$$\xi=x+2a\int F(t)\,dt,\quad \eta=y+2a\int G(t)\,dt,$$

where

$$F(t)=\int f(t)\,dt,\quad G(t)=\int g(t)\,dt,\quad H(t)=\int h(t)\,dt,$$

leads to the two-dimensional heat equation $\partial_t u=a\big(\partial_{\xi\xi}u+\partial_{\eta\eta}u\big)$.

3. $\dfrac{\partial w}{\partial t}=a\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\Big)+[-b(x^2+y^2)+f(t)x+g(t)y+h(t)]w.$

1°. Case $b>0$. The transformation

$$w(x,y,t)=u(\xi,\eta,\tau)\exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}\,(x^2+y^2)\right],$$
$$\xi=x\exp\big(2\sqrt{ab}\,t\big),\quad \eta=y\exp\big(2\sqrt{ab}\,t\big),\quad \tau=\frac{1}{4\sqrt{ab}}\exp\big(4\sqrt{ab}\,t\big)$$

leads to an equation of the form 4.3.2.2:

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + \big[F(\tau)\xi + G(\tau)\eta + H(\tau)\big]u,$$

$$F(\tau) = \frac{1}{(c\tau)^{3/2}} f\left(\frac{\ln(c\tau)}{c}\right), \quad G(\tau) = \frac{1}{(c\tau)^{3/2}} g\left(\frac{\ln(c\tau)}{c}\right),$$

$$H(\tau) = \frac{1}{c\tau} h\left(\frac{\ln(c\tau)}{c}\right) + \frac{1}{2\tau}, \quad c = 4\sqrt{ab}.$$

2°. Case $b < 0$. The transformation

$$w(x,y,t) = v(\bar\xi, \bar\eta, \bar\tau) \exp\left[\frac{\sqrt{-b}}{2\sqrt{a}} \tan\big(2\sqrt{-ab}\,t\big)\,(x^2+y^2)\right],$$

$$\bar\xi = \frac{x}{\cos\big(2\sqrt{-ab}\,t\big)}, \quad \bar\eta = \frac{y}{\cos\big(2\sqrt{-ab}\,t\big)}, \quad \bar\tau = \frac{1}{2\sqrt{-ab}} \tan\big(2\sqrt{-ab}\,t\big)$$

also leads to an equation of the form 2.3.2.2 (the transformed equation is not written out here).

4. $\dfrac{\partial w}{\partial t} = a_1(t)\dfrac{\partial^2 w}{\partial x^2} + a_2(t)\dfrac{\partial^2 w}{\partial y^2} + \Phi(x,y,t).$

This is a special case of equation 4.3.2.8. Let $0 < a_1(t) < \infty$ and $0 < a_2(t) < \infty$.

For the first, second, third, and mixed boundary value problems treated in rectangular, finite, or infinite domains ($x_1 \le x \le x_2$, $y_1 \le y \le y_2$), the Green's function admits incomplete separation of variables and can be represented in the product form (see Section 17.5.2)

$$G(x,y,\xi,\eta,t,\tau) = G_1(x,\xi,T_1)\,G_2(y,\eta,T_2),$$

$$T_1 = \int_\tau^t a_1(\eta)\,d\eta, \quad T_2 = \int_\tau^t a_2(\eta)\,d\eta,$$

where $G_1 = G_1(x,\xi,t)$ is the auxiliary Green's function that corresponds to the one-dimensional heat equation with $a_1(t) = 1$, $a_2(t) = 0$, and $\Phi(x,y,t) = 0$ and with homogeneous boundary conditions at $x = x_1$ and $x = x_2$ (the functions G_1 for various boundary value problems can be found in Sections 3.1.1 and 3.1.2). Similarly, $G_2 = G_2(y,\eta,t)$ is the auxiliary Green's function corresponding to the one-dimensional heat equation with $a_1(t) = 0$, $a_2(t) = 1$, and $\Phi(x,y,t) = 0$ and with homogeneous boundary conditions at $y = y_1$ and $y = y_2$. Note that the Green's functions G_1 and G_2 are introduced for $\tau = 0$.

See Section 17.1.1 for the solution of various one-dimensional boundary value problems with the help of the Green's function.

Example 4.15. Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem. An initial condition is prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,y,t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t,\tau)\, d\xi\, d\eta\, d\tau + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi,\eta)\, G(x,y,\xi,\eta,t,0)\, d\xi\, d\eta,$$

where

$$G(x,y,\xi,\eta,t,\tau) = \frac{1}{4\pi\sqrt{T_1 T_2}} \exp\left[-\frac{(x-\xi)^2}{4T_1} - \frac{(y-\eta)^2}{4T_2}\right], \quad T_1 = \int_\tau^t a_1(\eta)\, d\eta, \quad T_2 = \int_\tau^t a_2(\eta)\, d\eta.$$

Example 4.16. Domain: $0 \le x < \infty,\ 0 \le y < \infty$. Second boundary value problem. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g_1(y,t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ \partial_y w &= g_2(x,t) && \text{at} \quad y = 0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,t) = &\int_0^t \int_0^{\infty} \int_0^{\infty} \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t,\tau)\, d\xi\, d\eta\, d\tau \\ &+ \int_0^{\infty} \int_0^{\infty} f(\xi,\eta)\, G(x,y,\xi,\eta,t,0)\, d\xi\, d\eta \\ &- \int_0^t \int_0^{\infty} a_1(\tau) g_1(\eta,\tau)\, G(x,y,0,\eta,t,\tau)\, d\eta\, d\tau \\ &- \int_0^t \int_0^{\infty} a_2(\tau) g_2(\xi,\tau)\, G(x,y,\xi,0,t,\tau)\, d\xi\, d\tau, \end{aligned}$$

where

$$G(x,y,\xi,\eta,t,\tau) = G_1(x,\xi,T_1)\, G_2(y,\eta,T_2),$$

$$G_1(x,\xi,T_1) = \frac{1}{2\sqrt{\pi T_1}} \left\{ \exp\left[-\frac{(x-\xi)^2}{4T_1}\right] + \exp\left[-\frac{(x+\xi)^2}{4T_1}\right] \right\}, \quad T_1 = \int_\tau^t a_1(\eta)\, d\eta,$$

$$G_2(y,\eta,T_2) = \frac{1}{2\sqrt{\pi T_2}} \left\{ \exp\left[-\frac{(y-\eta)^2}{4T_2}\right] + \exp\left[-\frac{(y+\eta)^2}{4T_2}\right] \right\}, \quad T_2 = \int_\tau^t a_2(\eta)\, d\eta.$$

Example 4.17. Domain: $0 \le x \le l_1,\ 0 \le y \le l_2$. First boundary value problem. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g_1(y,t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= g_2(y,t) && \text{at} \quad x = l_1 && \text{(boundary condition)},\\ w &= h_1(x,t) && \text{at} \quad y = 0 && \text{(boundary condition)},\\ w &= h_2(x,t) && \text{at} \quad y = l_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \int_0^t \int_0^{l_1} \int_0^{l_2} \Phi(\xi,\eta,\tau)\, G(x,y,\xi,\eta,t,\tau)\, d\eta\, d\xi\, d\tau \\
&+ \int_0^{l_1} \int_0^{l_2} f(\xi,\eta)\, G(x,y,\xi,\eta,t,0)\, d\eta\, d\xi \\
&+ \int_0^t \int_0^{l_2} a_1(\tau) g_1(\eta,\tau) \left[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta,t,\tau)\right]_{\xi=0} d\eta\, d\tau \\
&- \int_0^t \int_0^{l_2} a_1(\tau) g_2(\eta,\tau) \left[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta,t,\tau)\right]_{\xi=l_1} d\eta\, d\tau \\
&+ \int_0^t \int_0^{l_1} a_2(\tau) h_1(\xi,\tau) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta,t,\tau)\right]_{\eta=0} d\xi\, d\tau \\
&- \int_0^t \int_0^{l_1} a_2(\tau) h_2(\xi,\tau) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta,t,\tau)\right]_{\eta=l_2} d\xi\, d\tau,
\end{aligned}
$$

where

$$
G(x,y,\xi,\eta,t,\tau) = G_1(x,\xi,T_1)\, G_2(y,\eta,T_2),
$$

$$
G_1(x,\xi,T_1) = \frac{2}{l_1} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l_1}\right) \sin\left(\frac{n\pi \xi}{l_1}\right) \exp\left(-\frac{n^2\pi^2 T_1}{l_1{}^2}\right), \qquad T_1 = \int_\tau^t a_1(\eta)\, d\eta,
$$

$$
G_2(y,\eta,T_2) = \frac{2}{l_2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{l_2}\right) \sin\left(\frac{n\pi \eta}{l_2}\right) \exp\left(-\frac{n^2\pi^2 T_2}{l_2{}^2}\right), \qquad T_2 = \int_\tau^t a_2(\eta)\, d\eta.
$$

5. $$\frac{\partial w}{\partial t} = a_1(t)\frac{\partial^2 w}{\partial x^2} + a_2(t)\frac{\partial^2 w}{\partial y^2} + [b_1(t)x + c_1(t)]\frac{\partial w}{\partial x} + [b_2(t)y + c_2(t)]\frac{\partial w}{\partial y} + [s_1(t)x + s_2(t)y + p(t)]w.$$

The transformation

$$
w(x,y,t) = \exp\bigl[f_1(t)x + f_2(t)y + g(t)\bigr]\, u(\xi,\eta,t), \quad \xi = h_1(t)x + r_1(t), \quad \eta = h_2(t)y + r_2(t),
$$

where

$$
\begin{aligned}
h_k(t) &= A_k \exp\left[\int b_k(t)\, dt\right], \\
f_k(t) &= h_k(t) \int \frac{s_k(t)}{h_k(t)}\, dt + B_k h_k(t), \\
r_k(t) &= \int \bigl[2a_k(t) f_k(t) + c_k(t)\bigr] h_k(t)\, dt + C_k, \\
g(t) &= \int \bigl[a_1(t) f_1^2(t) + a_2(t) f_2^2(t) + c_1(t) f_1(t) + c_2(t) f_2(t) + p(t)\bigr]\, dt + D,
\end{aligned}
$$

($k = 1,\, 2$; A_k, B_k, C_k, and D are arbitrary constants), leads to an equation of the form 4.3.2.4:

$$
\frac{\partial u}{\partial t} = a_1(t) h_1^2(t) \frac{\partial^2 u}{\partial \xi^2} + a_2(t) h_2^2(t) \frac{\partial^2 u}{\partial \eta^2}.
$$

6. $\dfrac{\partial w}{\partial t} = a_1(t)\dfrac{\partial^2 w}{\partial x^2} + a_2(t)\dfrac{\partial^2 w}{\partial y^2} + [b_1(t)x + c_1(t)]\dfrac{\partial w}{\partial x} + [b_2(t)y + c_2(t)]\dfrac{\partial w}{\partial y}$
$+ [s_1(t)x^2 + s_2(t)y^2 + p_1(t)x + p_2(t)y + q(t)]w.$

The substitution

$$w(x, y, t) = \exp\bigl[f_1(t)x^2 + f_2(t)y^2\bigr]u(x, y, t),$$

where the functions $f_1 = f_1(t)$ and $f_2 = f_2(t)$ are solutions of the Riccati equations

$$f_1' = 4a_1(t)f_1^2 + 2b_1(t)f_1 + s_1(t),$$
$$f_2' = 4a_2(t)f_2^2 + 2b_2(t)f_2 + s_2(t),$$

leads to an equation of the form 4.3.2.5 for $u = u(x, y, t)$.

7. $\dfrac{\partial w}{\partial t} = a_1(x)\dfrac{\partial^2 w}{\partial x^2} + a_2(y)\dfrac{\partial^2 w}{\partial y^2} + b_1(x)\dfrac{\partial w}{\partial x} + b_2(y)\dfrac{\partial w}{\partial y}$
$+ [c_1(x) + c_2(y)]w + \Phi(x, y, t).$

Domain: $x_1 \le x \le x_2$, $y_1 \le y \le y_2$. Different boundary value problems:

$$\begin{aligned}
w &= f(x, y) && \text{at} \quad t = 0 && \text{(initial condition)},\\
s_1\partial_x w - k_1 w &= g_1(y, t) && \text{at} \quad x = x_1 && \text{(boundary condition)},\\
s_2\partial_x w + k_2 w &= g_2(y, t) && \text{at} \quad x = x_2 && \text{(boundary condition)},\\
s_3\partial_y w - k_3 w &= g_3(x, t) && \text{at} \quad y = y_1 && \text{(boundary condition)},\\
s_4\partial_y w + k_4 w &= g_4(x, t) && \text{at} \quad y = y_2 && \text{(boundary condition)}.
\end{aligned}$$

By choosing appropriate parameters s_n and k_n ($n = 1, 2, 3, 4$), one obtains the first, second, third, or mixed boundary value problem. If the domain is infinite, say, $x_2 = \infty$, then the corresponding boundary condition should be omitted; this is also true for $x_1 = -\infty$, $y_1 = -\infty$, or $y_2 = \infty$.

The Green's function admits incomplete separation of variables; specifically, it can be represented in the product form (see Section 17.5.2)

$$G(x, y, \xi, \eta, t) = G_1(x, \xi, t)\, G_2(y, \eta, t).$$

Here $G_1 = G_1(x, \xi, t)$ and $G_2 = G_2(y, \eta, t)$ are auxiliary Green's functions determined by solving the following simpler one-dimensional problems with homogeneous boundary conditions:

$$\begin{aligned}
&\frac{\partial G_1}{\partial t} = a_1(x)\frac{\partial^2 G_1}{\partial x^2} + b_1(x)\frac{\partial G_1}{\partial x} + c_1(x)\, G_1, && \frac{\partial G_2}{\partial t} = a_2(y)\frac{\partial^2 G_2}{\partial y^2} + b_2(y)\frac{\partial G_2}{\partial y} + c_2(y)\, G_2,\\
&G_1 = \delta(x-\xi) \quad \text{at} \quad t = 0, && G_2 = \delta(y-\eta) \quad \text{at} \quad t = 0,\\
&s_1\partial_x G_1 - k_1 G_1 = 0 \quad \text{at} \quad x = x_1, && s_3\partial_y G_2 - k_3 G_2 = 0 \quad \text{at} \quad y = y_1,\\
&s_2\partial_x G_1 + k_2 G_1 = 0 \quad \text{at} \quad x = x_2, && s_4\partial_y G_2 + k_4 G_2 = 0 \quad \text{at} \quad y = y_2,
\end{aligned}$$

where ξ and η are free parameters and $\delta(x)$ is the Dirac delta function.

The equation for G_1 coincides with equation 3.8.6.5, which can be reduced to the equation in Section 3.8.9 (where the expression for the Green's function can also be found). In the general case, the equation for G_2 differs from that for G_1 only in notation.

8. $\displaystyle \frac{\partial w}{\partial t} = a_1(x,t)\frac{\partial^2 w}{\partial x^2} + a_2(y,t)\frac{\partial^2 w}{\partial y^2} + b_1(x,t)\frac{\partial w}{\partial x} + b_2(y,t)\frac{\partial w}{\partial y} + [c_1(x,t) + c_2(y,t)]w + \Phi(x,y,t).$

Suppose that this equation is supplemented with the same initial and boundary conditions as equation 4.3.2.7. Then the Green's function for this problem admits incomplete separation of variables and can be represented in the product form (see Section 17.5.2)

$$G(x,y,\xi,\eta,t,\tau) = G_1(x,\xi,t,\tau)\,G_2(y,\eta,t,\tau).$$

Here $G_1 = G_1(x,\xi,t,\tau)$ and $G_2 = G_2(y,\eta,t,\tau)$, $t \ge \tau$, are auxiliary Green's functions determined by solving the following simpler boundary value problems with homogeneous boundary conditions:

$$\begin{aligned} &\frac{\partial G_1}{\partial t} = a_1(x,t)\frac{\partial^2 G_1}{\partial x^2} + b_1(x,t)\frac{\partial G_1}{\partial x} + c_1(x,t)\,G_1, \\ &\qquad G_1 = \delta(x-\xi) \qquad\quad \text{at} \quad t=\tau, \\ &\qquad s_1\partial_x G_1 - k_1 G_1 = 0 \quad \text{at} \quad x = x_1, \\ &\qquad s_2\partial_x G_1 + k_2 G_1 = 0 \quad \text{at} \quad x = x_2, \end{aligned}$$

$$\begin{aligned} &\frac{\partial G_2}{\partial t} = a_2(y,t)\frac{\partial^2 G_2}{\partial y^2} + b_2(y,t)\frac{\partial G_2}{\partial y} + c_2(y,t)\,G_2, \\ &\qquad G_2 = \delta(y-\eta) \qquad\quad \text{at} \quad t=\tau, \\ &\qquad s_3\partial_y G_2 - k_3 G_2 = 0 \quad \text{at} \quad y = y_1, \\ &\qquad s_4\partial_y G_2 + k_4 G_2 = 0 \quad \text{at} \quad y = y_2, \end{aligned}$$

where ξ, η, and τ are free parameters, and $\delta(x)$ is the Dirac delta function.

See Section 17.1.1 for the solution of various one-dimensional boundary value problems with the help of the Green's function.

Chapter 5

Second-Order Parabolic Equations with Three or More Space Variables

5.1 Heat Equation $\frac{\partial w}{\partial t} = a\Delta_3 w$

5.1.1 Problems in Cartesian Coordinates

The three-dimensional sourceless heat equation in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

It governs three-dimensional thermal phenomena in quiescent media or solids with constant thermal diffusivity. A similar equation is used to study the corresponding three-dimensional unsteady mass-exchange processes with constant diffusivity.

► **Particular solutions:**

$$w(x,y,z,t) = Ax^2 + By^2 + Cz^2 + 2a(A+B+C)t,$$

$$w(x,y,z,t) = A(x^2+2at)(y^2+2at)(z^2+2at) + B,$$

$$w(x,y,z,t) = A\exp\bigl[k_1x + k_2y + k_3z + (k_1^2+k_2^2+k_3^2)at\bigr] + B,$$

$$w(x,y,z,t) = A\cos(k_1x+C_1)\cos(k_2y+C_2)\cos(k_3z+C_3)\exp\bigl[-(k_1^2+k_2^2+k_3^2)at\bigr],$$

$$w(x,y,z,t) = A\cos(k_1x+C_1)\cos(k_2y+C_2)\sinh(k_3z+C_3)\exp\bigl[-(k_1^2+k_2^2-k_3^2)at\bigr],$$

$$w(x,y,z,t) = A\cos(k_1x+C_1)\cos(k_2y+C_2)\cosh(k_3z+C_3)\exp\bigl[-(k_1^2+k_2^2-k_3^2)at\bigr],$$

$$w(x,y,z,t) = Ae^{-k_1x-k_2y-k_3z}\cos(k_1x-2ak_1^2t)\cos(k_2y-2ak_2^2t)\cos(k_3z-2ak_3^2t),$$

$$w(x,y,z,t) = \frac{A}{(t-t_0)^{3/2}}\exp\left[-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a(t-t_0)}\right],$$

$$w(x,y,z,t) = A\operatorname{erf}\left(\frac{x-x_0}{2\sqrt{at}}\right)\operatorname{erf}\left(\frac{y-y_0}{2\sqrt{at}}\right)\operatorname{erf}\left(\frac{z-z_0}{2\sqrt{at}}\right) + B,$$

where A, B, C, C_1, C_2, C_3, k_1, k_2, k_3, x_0, y_0, z_0, and t_0 are arbitrary constants.

Fundamental solution:

$$\mathscr{E}(x,y,z,t) = \frac{1}{8(\pi a t)^{3/2}} \exp\biggl(-\frac{x^2+y^2+z^2}{4at}\biggr).$$

► **Formulas to construct particular solutions. Remarks on the Green's functions.**

1°. Apart from usual solutions with separated variables,

$$w(x,y,z,t) = f_1(x) f_2(y) f_3(z) f_4(t),$$

the equation in question admits more sophisticated solutions in the product form

$$w(x,y,z,t) = u_1(x,t) u_2(y,t) u_3(z,t),$$

where the functions $u_1 = u_1(x,t)$, $u_2 = u_2(y,t)$, and $u_3 = u_3(y,t)$ are solutions of the one-dimensional heat equations

$$\frac{\partial u_1}{\partial t} = a\frac{\partial^2 u_1}{\partial x^2}, \qquad \frac{\partial u_2}{\partial t} = a\frac{\partial^2 u_2}{\partial y^2}, \qquad \frac{\partial u_3}{\partial t} = a\frac{\partial^2 u_3}{\partial z^2},$$

treated in Section 3.1.1.

2°. Suppose $w = w(x,y,z,t)$ is a solution of the three-dimensional heat equation. Then the functions

$$w_1 = Aw(\pm\lambda x + C_1,\ \pm\lambda y + C_2,\ \pm\lambda z + C_3,\ \lambda^2 t + C_4),$$

$$w_2 = A\exp\bigl[\lambda_1 x + \lambda_2 y + \lambda_3 z + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)at\bigr] w(x + 2a\lambda_1 t,\ y + 2a\lambda_2 t,\ z + 2a\lambda_3 t,\ t),$$

$$w_3 = \frac{A}{|\delta+\beta t|^{3/2}} \exp\biggl[-\frac{\beta(x^2+y^2+z^2)}{4a(\delta+\beta t)}\biggr] w\biggl(\frac{x}{\delta+\beta t}, \frac{y}{\delta+\beta t}, \frac{z}{\delta+\beta t}, \frac{\gamma+\lambda t}{\delta+\beta t}\biggr), \quad \lambda\delta-\beta\gamma=1,$$

where A, C_1, C_2, C_3, C_4, λ, λ_1, λ_2, λ_3, β, and δ are arbitrary constants, are also solutions of this equation. The signs at λ in the formula for w_1 can be taken independently of one another.

3°. For the three-dimensional boundary value problems considered in Section 5.1.1, the Green's function can be represented in the product form

$$G(x,y,z,\xi,\eta,\zeta,t) = G_1(x,\xi,t)\, G_2(y,\eta,t)\, G_3(z,\zeta,t),$$

where $G_1(x,\xi,t)$, $G_2(y,\eta,t)$, $G_3(z,\zeta,t)$ are the Green's functions of the corresponding one-dimensional boundary value problems; these functions can be found in Sections 3.1.1 and 3.1.2.

Example 5.1. The Green's function of the mixed boundary value problem for a semiinfinite layer ($-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z < l$) with the initial and boundary conditions

$$w = f(x,y,z) \quad \text{at} \quad t = 0,$$

$$w = g_1(x,z,t) \quad \text{at} \quad y = 0, \qquad \partial_z w = g_2(x,y,t) \quad \text{at} \quad z = 0, \qquad \partial_z w = g_3(x,y,t) \quad \text{at} \quad z = l,$$

is the product of three one-dimensional Green's functions from Section 3.1.2 (in which one needs to carry out obvious renaming of variables): (i) for the Cauchy problem with $-\infty < x < \infty$, (ii) for the first boundary value problem with $0 \le y < \infty$, and (iii) for the second boundary value problem with $0 \le z < l$.

▶ Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,y,z,t) = \frac{1}{8(\pi at)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta) \exp\left[-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4at}\right] d\xi\, d\eta\, d\zeta.$$

Example 5.2. The initial temperature is constant and is equal to w_1 in the domain $|x| < x_0$, $|y| < y_0$, $|z| < z_0$ and is equal to w_2 in the domain $|x| > x_0$, $|y| > y_0$, $|z| > z_0$; specifically,

$$f(x,y,z) = \begin{cases} w_1 & \text{for } |x| < x_0,\ |y| < y_0,\ |z| < z_0, \\ w_2 & \text{for } |x| > x_0,\ |y| > y_0,\ |z| > z_0. \end{cases}$$

Solution:

$$w = \frac{1}{8}(w_1 - w_2)\left[\operatorname{erf}\left(\frac{x_0 - x}{2\sqrt{at}}\right) + \operatorname{erf}\left(\frac{x_0 + x}{2\sqrt{at}}\right)\right]$$
$$\times \left[\operatorname{erf}\left(\frac{y_0 - y}{2\sqrt{at}}\right) + \operatorname{erf}\left(\frac{y_0 + y}{2\sqrt{at}}\right)\right]\left[\operatorname{erf}\left(\frac{z_0 - z}{2\sqrt{at}}\right) + \operatorname{erf}\left(\frac{z_0 + z}{2\sqrt{at}}\right)\right] + w_2.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ Domain: $0 \le x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. First boundary value problem.

A half-space is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(y,z,t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta$$
$$+ a\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau,$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{8(\pi at)^{3/2}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}\exp\left[-\frac{(y-\eta)^2+(z-\zeta)^2}{4at}\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Second boundary value problem.**

A half-space is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_x w = g(y,z,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta - a\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\eta,\zeta,\tau)\,G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau,$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{8(\pi a t)^{3/2}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}\exp\left[-\frac{(y-\eta)^2+(z-\zeta)^2}{4at}\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Third boundary value problem.**

A half-space is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(y,z,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)}.$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{8(\pi a t)^{3/2}}\exp\left[-\frac{(y-\eta)^2+(z-\zeta)^2}{4at}\right]\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right] - 2k\sqrt{\pi a t}\,\exp\big[k^2at + k(x+\xi)\big]\,\operatorname{erfc}\left(\frac{x+\xi}{2\sqrt{at}} + k\sqrt{at}\right)\right\}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$. First boundary value problem.**

An infinite layer is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$w = g_1(x,y,t) \quad \text{at} \quad z=0 \quad \text{(boundary condition)},$$
$$w = g_2(x,y,t) \quad \text{at} \quad z=l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,y,z,t) = \int_0^l \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta$$
$$+ a\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\xi,\eta,\tau) \left[\frac{\partial}{\partial\zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\, d\eta\, d\tau$$
$$- a\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(\xi,\eta,\tau) \left[\frac{\partial}{\partial\zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\, d\eta\, d\tau,$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{2\pi alt} \exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right] \sum_{n=1}^{\infty} \sin\frac{n\pi z}{l} \sin\frac{n\pi\zeta}{l} \exp\left(-\frac{n^2\pi^2 at}{l^2}\right),$$

or

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{8(\pi at)^{3/2}} \exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right]$$
$$\times \sum_{n=-\infty}^{\infty} \left\{\exp\left[-\frac{(2nl+z-\zeta)^2}{4at}\right] - \exp\left[-\frac{(2nl+z+\zeta)^2}{4at}\right]\right\}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$. Second boundary value problem.**

An infinite layer is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_z w = g_1(x,y,t) \quad \text{at} \quad z=0 \quad \text{(boundary condition)},$$
$$\partial_z w = g_2(x,y,t) \quad \text{at} \quad z=l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,y,z,t) = \int_0^l \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta$$
$$- a\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,0,t-\tau)\, d\xi\, d\eta\, d\tau$$
$$+ a\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,l,t-\tau)\, d\xi\, d\eta\, d\tau,$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{4\pi alt} \exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right]$$
$$\times \left[1 + 2\sum_{n=1}^{\infty} \cos\frac{n\pi z}{l} \cos\frac{n\pi\zeta}{l} \exp\left(-\frac{n^2\pi^2 at}{l^2}\right)\right],$$

or

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{(2\sqrt{\pi a t}\,)^3}\exp\biggl[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\biggr]\\ \times\sum_{n=-\infty}^{\infty}\biggl\{\exp\biggl[-\frac{(z-\zeta+2nl)^2}{4at}\biggr]+\exp\biggl[-\frac{(z+\zeta+2nl)^2}{4at}\biggr]\biggr\}.$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $-\infty<x<\infty$, $-\infty<y<\infty$, $0\le z\le l$. Third boundary value problem.**

An infinite layer is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_z w-k_1w&=g_1(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w+k_2w&=g_2(x,y,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{4\pi at}\exp\biggl[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\biggr]\sum_{n=1}^{\infty}\frac{\varphi_n(z)\varphi_n(\zeta)}{\|\varphi_n\|^2}\exp(-a\mu_n^2t),$$

$$\varphi_n(z)=\cos(\mu_n z)+\frac{k_1}{\mu_n}\sin(\mu_n z),\qquad \|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{l}{2}\biggl(1+\frac{k_1^2}{\mu_n^2}\biggr).$$

Here, the μ_n are positive roots of the transcendental equation $\dfrac{\tan(\mu l)}{\mu}=\dfrac{k_1+k_2}{\mu^2-k_1k_2}$.

► **Domain: $-\infty<x<\infty$, $-\infty<y<\infty$, $0\le z\le l$. Mixed boundary value problem.**

An infinite layer is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g_1(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w&=g_2(x,y,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t)&=\int_0^l\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &\quad+a\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_1(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0}d\xi\,d\eta\,d\tau\\ &\quad+a\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_2(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau,\end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{2\pi alt}\exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right] \\ \times\sum_{n=0}^{\infty}\sin\left[\frac{(2n+1)\pi z}{2l}\right]\sin\left[\frac{(2n+1)\pi\zeta}{2l}\right]\exp\left[-\frac{(2n+1)^2\pi^2 at}{4l^2}\right],$$

or

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{(2\sqrt{\pi at}\,)^3}\exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4at}\right] \\ \times\sum_{n=-\infty}^{\infty}(-1)^n\left\{\exp\left[-\frac{(z-\zeta+2nl)^2}{4at}\right]-\exp\left[-\frac{(z+\zeta+2nl)^2}{4at}\right]\right\}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$. First boundary value problem.

A semiinfinite layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\ w &= g_2(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ w &= g_3(x,y,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_0^l\int_0^\infty\int_{-\infty}^\infty f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\ &+ a\int_0^t\int_0^l\int_{-\infty}^\infty g_1(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau \\ &+ a\int_0^t\int_0^\infty\int_{-\infty}^\infty g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau \\ &- a\int_0^t\int_0^\infty\int_{-\infty}^\infty g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{2\pi alt}\exp\left[-\frac{(x-\xi)^2}{4at}\right]\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]-\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\} \\ \times\sum_{n=1}^{\infty}\sin\frac{n\pi z}{l}\sin\frac{n\pi\zeta}{l}\exp\left(-\frac{n^2\pi^2 at}{l^2}\right).$$

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$. Second boundary value problem.**

A semiinfinite layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_y w &= g_1(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\ \partial_z w &= g_2(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w &= g_3(x,y,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) = &\int_0^l\int_0^\infty\int_{-\infty}^\infty f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &-a\int_0^t\int_0^l\int_{-\infty}^\infty g_1(\xi,\zeta,\tau)\,G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\ &-a\int_0^t\int_0^\infty\int_{-\infty}^\infty g_2(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\ &+a\int_0^t\int_0^\infty\int_{-\infty}^\infty g_3(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau,\end{aligned}$$

where

$$\begin{aligned} G(x,y,z,\xi,\eta,\zeta,t) = &\frac{1}{4\pi alt}\exp\left[-\frac{(x-\xi)^2}{4at}\right]\left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}\\ &\times\left[1+2\sum_{n=1}^\infty\cos\frac{n\pi z}{l}\cos\frac{n\pi\zeta}{l}\exp\left(-\frac{n^2\pi^2at}{l^2}\right)\right].\end{aligned}$$

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$. Third boundary value problem.**

A semiinfinite layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_y w - k_1 w &= g_1(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w + k_3 w &= g_3(x,y,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{4\pi at}\exp\left[-\frac{(x-\xi)^2}{4at}\right]H(y,\eta,t)\sum_{n=1}^\infty\frac{\varphi_n(z)\varphi_n(\zeta)}{\|\varphi_n\|^2}\exp(-a\mu_n^2t),$$

$$H(y,\eta,t)=\exp\left[-\frac{(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(y+\eta)^2}{4at}\right]-2k_1\int_0^\infty\exp\left[-\frac{(y+\eta+s)^2}{4at}-k_1s\right]ds.$$

Here,

$$\varphi_n(z)=\cos(\mu_nz)+\frac{k_2}{\mu_n}\sin(\mu_nz),\qquad \|\varphi_n\|^2=\frac{k_3}{2\mu_n^2}\frac{\mu_n^2+k_2^2}{\mu_n^2+k_3^2}+\frac{k_2}{2\mu_n^2}+\frac{l}{2}\left(1+\frac{k_2^2}{\mu_n^2}\right),$$

with the μ_n being positive roots of the transcendental equation $\dfrac{\tan(\mu l)}{\mu} = \dfrac{k_2 + k_3}{\mu^2 - k_2 k_3}$.

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A semiinfinite layer is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
w &= g_1(x,z,t) &&\text{at} \quad y=0 &&\text{(boundary condition)},\\
\partial_z w &= g_2(x,y,t) &&\text{at} \quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(x,y,t) &&\text{at} \quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \int_0^l \int_0^\infty \int_{-\infty}^\infty f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta\\
&+ a\int_0^t \int_0^l \int_{-\infty}^\infty g_1(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau\\
&- a\int_0^t \int_0^\infty \int_{-\infty}^\infty g_2(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,0,t-\tau)\, d\xi\, d\eta\, d\tau\\
&+ a\int_0^t \int_0^\infty \int_{-\infty}^\infty g_3(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,l,t-\tau)\, d\xi\, d\eta\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{4\pi a l t} \exp\left[-\frac{(x-\xi)^2}{4at}\right] \left\{\exp\left[-\frac{(y-\eta)^2}{4at}\right] - \exp\left[-\frac{(y+\eta)^2}{4at}\right]\right\}\\
\times \left[1 + 2\sum_{n=1}^\infty \cos\frac{n\pi z}{l} \cos\frac{n\pi\zeta}{l} \exp\left(-\frac{n^2\pi^2 a t}{l^2}\right)\right].
\end{aligned}
$$

2°. A semiinfinite layer is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
\partial_y w &= g_1(x,z,t) &&\text{at} \quad y=0 &&\text{(boundary condition)},\\
w &= g_2(x,y,t) &&\text{at} \quad z=0 &&\text{(boundary condition)},\\
w &= g_3(x,y,t) &&\text{at} \quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \int_0^l \int_0^\infty \int_{-\infty}^\infty f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta\\
&- a\int_0^t \int_0^l \int_{-\infty}^\infty g_1(\xi,\zeta,\tau)\, G(x,y,z,\xi,0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau\\
&+ a\int_0^t \int_0^\infty \int_{-\infty}^\infty g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\, d\eta\, d\tau\\
&- a\int_0^t \int_0^\infty \int_{-\infty}^\infty g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\, d\eta\, d\tau,
\end{aligned}
$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{2\pi alt} \exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] \biggl\{\exp\biggl[-\frac{(y-\eta)^2}{4at}\biggr] + \exp\biggl[-\frac{(y+\eta)^2}{4at}\biggr]\biggr\}$$

$$\times \sum_{n=1}^{\infty} \sin\frac{n\pi z}{l} \sin\frac{n\pi\zeta}{l} \exp\biggl(-\frac{n^2\pi^2 at}{l^2}\biggr).$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. First boundary value problem.**

An octant is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g_1(y,z,t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= g_2(x,z,t) && \text{at} \quad y = 0 && \text{(boundary condition)},\\ w &= g_3(x,y,t) && \text{at} \quad z = 0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_0^\infty\!\int_0^\infty\!\int_0^\infty G(x,y,z,\xi,\eta,\zeta,t) f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta\\ &+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_1(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=0} d\eta\,d\zeta\,d\tau\\ &+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_2(\xi,\zeta,\tau)\biggl[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\eta=0} d\xi\,d\zeta\,d\tau\\ &+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_3(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0} d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{\bigl(2\sqrt{\pi at}\,\bigr)^3} H(x,\xi,t)H(y,\eta,t)H(z,\zeta,t),$$

$$H(x,\xi,t) = \exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] - \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr].$$

Example 5.3. The initial temperature is uniform, $f(x,y,z) = w_0$. The faces are maintained at zero temperature, $g_1 = g_2 = g_3 = 0$.

Solution:

$$w = w_0 \operatorname{erf}\biggl(\frac{x}{2\sqrt{at}}\biggr) \operatorname{erf}\biggl(\frac{y}{2\sqrt{at}}\biggr) \operatorname{erf}\biggl(\frac{z}{2\sqrt{at}}\biggr).$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. Second boundary value problem.**

An octant is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g_1(y,z,t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ \partial_y w &= g_2(x,z,t) && \text{at} \quad y = 0 && \text{(boundary condition)},\\ \partial_z w &= g_3(x,y,t) && \text{at} \quad z = 0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,y,z,t) = \int_0^\infty\int_0^\infty\int_0^\infty G(x,y,z,\xi,\eta,\zeta,t) f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta$$
$$- a\int_0^t\int_0^\infty\int_0^\infty g_1(\eta,\zeta,\tau)\,G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau$$
$$- a\int_0^t\int_0^\infty\int_0^\infty g_2(\xi,\zeta,\tau)\,G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau$$
$$- a\int_0^t\int_0^\infty\int_0^\infty g_3(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau,$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{\big(2\sqrt{\pi a t}\,\big)^3} H(x,\xi,t)H(y,\eta,t)H(z,\zeta,t),$$
$$H(x,\xi,t) = \exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] + \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr].$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. Third boundary value problem.**

An octant is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_y w - k_2 w &= g_2(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\ \partial_z w - k_3 w &= g_3(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{\big(2\sqrt{\pi a t}\,\big)^3} H(x,\xi,t;k_1)H(y,\eta,t;k_2)H(z,\zeta,t;k_3),$$
$$H(x,\xi,t;k) = \exp\biggl[-\frac{(x-\xi)^2}{4at}\biggr] + \exp\biggl[-\frac{(x+\xi)^2}{4at}\biggr]$$
$$- 2k\sqrt{\pi a t}\,\exp\bigl[ak^2t + k(x+\xi)\bigr]\operatorname{erfc}\biggl(\frac{x+\xi}{2\sqrt{at}} + k\sqrt{at}\biggr).$$

Example 5.4. The initial temperature is uniform, $f(x,y,z) = w_0$. The temperature of the contacting media is zero, $g_1 = g_2 = g_3 = 0$.

Solution:

$$w = w_0\biggl[\operatorname{erf}\biggl(\frac{x}{2\sqrt{at}}\biggr) + \exp(k_1x + k_1^2at)\operatorname{erfc}\biggl(\frac{x}{2\sqrt{at}} + k_1\sqrt{at}\biggr)\biggr]$$
$$\times\biggl[\operatorname{erf}\biggl(\frac{y}{2\sqrt{at}}\biggr) + \exp(k_2y + k_2^2at)\operatorname{erfc}\biggl(\frac{y}{2\sqrt{at}} + k_2\sqrt{at}\biggr)\biggr]$$
$$\times\biggl[\operatorname{erf}\biggl(\frac{z}{2\sqrt{at}}\biggr) + \exp(k_3z + k_3^2at)\operatorname{erfc}\biggl(\frac{z}{2\sqrt{at}} + k_3\sqrt{at}\biggr)\biggr].$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ **Domain: $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. Mixed boundary value problems.**

1°. An octant is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_y w &= g_2(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\ \partial_z w &= g_3(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_0^\infty\!\int_0^\infty\!\int_0^\infty G(x,y,z,\xi,\eta,\zeta,t) f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta \\ &+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau \\ &- a\int_0^t\!\int_0^\infty\!\int_0^\infty g_2(\xi,\zeta,\tau)\,G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\ &- a\int_0^t\!\int_0^\infty\!\int_0^\infty g_3(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{\left(2\sqrt{\pi a t}\right)^3}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]-\exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}H(y,\eta,t)H(z,\zeta,t),$$

$$H(y,\eta,t)=\exp\left[-\frac{(y-\eta)^2}{4at}\right]+\exp\left[-\frac{(y+\eta)^2}{4at}\right].$$

2°. An octant is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w &= g_2(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\ \partial_z w &= g_3(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_0^\infty\!\int_0^\infty\!\int_0^\infty G(x,y,z,\xi,\eta,\zeta,t) f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta \\ &+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau \\ &+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_2(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau \\ &- a\int_0^t\!\int_0^\infty\!\int_0^\infty g_3(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{\left(2\sqrt{\pi a t}\,\right)^3} H(x,\xi,t) H(y,\eta,t) \left\{ \exp\left[-\frac{(z-\zeta)^2}{4at}\right] + \exp\left[-\frac{(z+\zeta)^2}{4at}\right] \right\},$$

$$H(x,\xi,t) = \exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right].$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$. First boundary value problem.**

An infinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t) &= \int_{-\infty}^{\infty}\int_0^{l_2}\int_0^{l_1} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta\\
&+ a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau\\
&- a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\, d\zeta\, d\tau\\
&+ a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_1} g_3(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau\\
&- a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_1} g_4(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=l_2} d\xi\, d\zeta\, d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= \frac{1}{2\sqrt{\pi a t}} \exp\left[-\frac{(z-\zeta)^2}{4at}\right] H_1(x,\xi,t) H_2(y,\eta,t),\\
H_1(x,\xi,t) &= \frac{2}{l_1}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l_1}\right)\sin\left(\frac{\pi n \xi}{l_1}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right),\\
H_2(y,\eta,t) &= \frac{2}{l_2}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n y}{l_2}\right)\sin\left(\frac{\pi n \eta}{l_2}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_2^2}\right).
\end{aligned}$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$. Second boundary value problem.**

An infinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t) = &\int_{-\infty}^{\infty}\int_0^{l_2}\int_0^{l_1} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta\\
&- a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_2} g_1(\eta,\zeta,\tau)\, G(x,y,z,0,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau\\
&+ a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_2} g_2(\eta,\zeta,\tau)\, G(x,y,z,l_1,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau\\
&- a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_1} g_3(\xi,\zeta,\tau)\, G(x,y,z,\xi,0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau\\
&+ a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_1} g_4(\xi,\zeta,\tau)\, G(x,y,z,\xi,l_2,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= \frac{1}{2\sqrt{\pi a t}}\exp\biggl[-\frac{(z-\zeta)^2}{4at}\biggr] H_1(x,\xi,t) H_2(y,\eta,t),\\
H_1(x,\xi,t) &= \frac{1}{l_1}\biggl[1+2\sum_{n=1}^{\infty}\cos\Bigl(\frac{\pi n x}{l_1}\Bigr)\cos\Bigl(\frac{\pi n \xi}{l_1}\Bigr)\exp\Bigl(-\frac{\pi^2 n^2 a t}{l_1^2}\Bigr)\biggr],\\
H_2(y,\eta,t) &= \frac{1}{l_2}\biggl[1+2\sum_{n=1}^{\infty}\cos\Bigl(\frac{\pi n y}{l_2}\Bigr)\cos\Bigl(\frac{\pi n \eta}{l_2}\Bigr)\exp\Bigl(-\frac{\pi^2 n^2 a t}{l_2^2}\Bigr)\biggr].
\end{aligned}$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$. Third boundary value problem.**

An infinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w + k_2 w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w - k_3 w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w + k_4 w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}$$

The solution $w(x, y, z, t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right]H_1(x,\xi,t)H_2(y,\eta,t),$$

$$H_1(x,\xi,t)=\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2}\exp(-a\mu_n^2 t),\quad H_2(y,\eta,t)=\sum_{m=1}^{\infty}\frac{\psi_m(y)\psi_m(\eta)}{\|\psi_m\|^2}\exp(-a\lambda_m^2 t).$$

Here,

$$\varphi_n(x)=\cos(\mu_n x)+\frac{k_1}{\mu_n}\sin(\mu_n x),\qquad \|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{l_1}{2}\left(1+\frac{k_1^2}{\mu_n^2}\right),$$

$$\psi_m(y)=\cos(\lambda_m y)+\frac{k_3}{\lambda_m}\sin(\lambda_m y),\quad \|\psi_m\|^2=\frac{k_4}{2\lambda_m^2}\frac{\lambda_m^2+k_3^2}{\lambda_m^2+k_4^2}+\frac{k_3}{2\lambda_m^2}+\frac{l_2}{2}\left(1+\frac{k_3^2}{\lambda_m^2}\right);$$

the μ_n and λ_m are positive roots of the transcendental equations

$$\frac{\tan(\mu l_1)}{\mu}=\frac{k_1+k_2}{\mu^2-k_1k_2},\qquad \frac{\tan(\lambda l_2)}{\lambda}=\frac{k_3+k_4}{\lambda^2-k_3k_4}.$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$. Mixed boundary value problem.**

An infinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\ w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\ \partial_y w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\ \partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) = {} & \int_{-\infty}^{\infty}\int_0^{l_2}\int_0^{l_1} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\ & + a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau \\ & - a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\,d\zeta\,d\tau \\ & - a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_1} g_3(\xi,\zeta,\tau)\,G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\ & + a\int_0^t\int_{-\infty}^{\infty}\int_0^{l_1} g_4(\xi,\zeta,\tau)\,G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{2}{l_1l_2\sqrt{\pi at}}\exp\biggl[-\frac{(z-\zeta)^2}{4at}\biggr]\biggl[\sum_{n=1}^{\infty}\sin\Bigl(\frac{\pi nx}{l_1}\Bigr)\sin\Bigl(\frac{\pi n\xi}{l_1}\Bigr)\exp\Bigl(-\frac{\pi^2n^2at}{l_1^2}\Bigr)\biggr]$$
$$\times\biggl[\frac{1}{2}+\sum_{m=1}^{\infty}\cos\Bigl(\frac{\pi mx}{l_2}\Bigr)\cos\Bigl(\frac{\pi m\xi}{l_2}\Bigr)\exp\Bigl(-\frac{\pi^2m^2at}{l_2^2}\Bigr)\biggr].$$

► **Domain: $0\le x\le l_1$, $0\le y\le l_2$, $0\le z<\infty$. First boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$\begin{aligned}
&w=f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
&w=g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
&w=g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
&w=g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
&w=g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
&w=g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t)&=\int_0^{\infty}\!\!\int_0^{l_2}\!\!\int_0^{l_1} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+a\int_0^t\!\!\int_0^{\infty}\!\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=0}d\eta\,d\zeta\,d\tau\\
&-a\int_0^t\!\!\int_0^{\infty}\!\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=l_1}d\eta\,d\zeta\,d\tau\\
&+a\int_0^t\!\!\int_0^{\infty}\!\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\biggl[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\eta=0}d\xi\,d\zeta\,d\tau\\
&-a\int_0^t\!\!\int_0^{\infty}\!\!\int_0^{l_1} g_4(\xi,\zeta,\tau)\biggl[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\eta=l_2}d\xi\,d\zeta\,d\tau\\
&+a\int_0^t\!\!\int_0^{l_2}\!\!\int_0^{l_1} g_5(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0}d\xi\,d\eta\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t)&=G_1(x,\xi,t;l_1)\,G_1(y,\eta,t;l_2)\,G_2(z,\zeta,t),\\
G_1(x,\xi,t;l)&=\frac{2}{l}\sum_{n=1}^{\infty}\sin\Bigl(\frac{\pi nx}{l}\Bigr)\sin\Bigl(\frac{\pi n\xi}{l}\Bigr)\exp\Bigl(-\frac{\pi^2n^2at}{l^2}\Bigr),\\
G_2(z,\zeta,t)&=\frac{1}{2\sqrt{\pi at}}\biggl\{\exp\biggl[-\frac{(z-\zeta)^2}{4at}\biggr]-\exp\biggl[-\frac{(z+\zeta)^2}{4at}\biggr]\biggr\}.
\end{aligned}$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$. Second boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t) &= \int_0^\infty\!\int_0^{l_2}\!\int_0^{l_1} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&\quad - a\int_0^t\!\int_0^\infty\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\,G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&\quad + a\int_0^t\!\int_0^\infty\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\,G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&\quad - a\int_0^t\!\int_0^\infty\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\,G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&\quad + a\int_0^t\!\int_0^\infty\!\int_0^{l_1} g_4(\xi,\zeta,\tau)\,G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&\quad - a\int_0^t\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= G_1(x,\xi,t)\,G_2(y,\eta,t)\,G_3(z,\zeta,t),\\
G_1(x,\xi,t) &= \frac{1}{l_1}\left[1+2\sum_{n=1}^\infty \cos\left(\frac{\pi n x}{l_1}\right)\cos\left(\frac{\pi n \xi}{l_1}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right)\right],\\
G_2(y,\eta,t) &= \frac{1}{l_2}\left[1+2\sum_{n=1}^\infty \cos\left(\frac{\pi n y}{l_2}\right)\cos\left(\frac{\pi n \eta}{l_2}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_2^2}\right)\right],\\
G_3(z,\zeta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]+\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}.
\end{aligned}$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$. Third boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) \quad &&\text{at} \quad t=0 \quad &&\text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(y,z,t) \quad &&\text{at} \quad x=0 \quad &&\text{(boundary condition)},\\
\partial_x w + k_2 w &= g_2(y,z,t) \quad &&\text{at} \quad x=l_1 \quad &&\text{(boundary condition)},\\
\partial_y w - k_3 w &= g_3(x,z,t) \quad &&\text{at} \quad y=0 \quad &&\text{(boundary condition)},\\
\partial_y w + k_4 w &= g_4(x,z,t) \quad &&\text{at} \quad y=l_2 \quad &&\text{(boundary condition)},\\
\partial_z w - k_5 w &= g_5(x,y,t) \quad &&\text{at} \quad z=0 \quad &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= H_1(x,\xi,t)H_2(y,\eta,t)H_3(z,\zeta,t),\\
H_1(x,\xi,t) &= \sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2}\exp(-a\mu_n^2 t), \quad \varphi_n(x)=\cos(\mu_n x)+\frac{k_1}{\mu_n}\sin(\mu_n x),\\
H_2(y,\eta,t) &= \sum_{m=1}^{\infty}\frac{\psi_m(y)\psi_m(\eta)}{\|\psi_m\|^2}\exp(-a\lambda_m^2 t), \quad \psi_m(y)=\cos(\lambda_m y)+\frac{k_3}{\lambda_m}\sin(\lambda_m y),\\
H_3(z,\zeta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]+\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}\\
&\quad - k_5\exp\bigl[k_5^2 at + k_5(z+\zeta)\bigr]\operatorname{erfc}\left(\frac{z+\zeta}{2\sqrt{at}}+k_5\sqrt{at}\right).
\end{aligned}
$$

Here,

$$
\|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{l_1}{2}\left(1+\frac{k_1^2}{\mu_n^2}\right), \quad \|\psi_m\|^2=\frac{k_4}{2\lambda_m^2}\frac{\lambda_m^2+k_3^2}{\lambda_m^2+k_4^2}+\frac{k_3}{2\lambda_m^2}+\frac{l_2}{2}\left(1+\frac{k_3^2}{\lambda_m^2}\right),
$$

and the μ_n and λ_m are positive roots of the transcendental equations

$$
\frac{\tan(\mu l_1)}{\mu}=\frac{k_1+k_2}{\mu^2-k_1k_2}, \qquad \frac{\tan(\lambda l_2)}{\lambda}=\frac{k_3+k_4}{\lambda^2-k_3k_4}.
$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$. Mixed boundary value problems.**

1°. A semiinfinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) \quad &&\text{at} \quad t=0 \quad &&\text{(initial condition)},\\
w &= g_1(y,z,t) \quad &&\text{at} \quad x=0 \quad &&\text{(boundary condition)},\\
w &= g_2(y,z,t) \quad &&\text{at} \quad x=l_1 \quad &&\text{(boundary condition)},\\
w &= g_3(x,z,t) \quad &&\text{at} \quad y=0 \quad &&\text{(boundary condition)},\\
w &= g_4(x,z,t) \quad &&\text{at} \quad y=l_2 \quad &&\text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) \quad &&\text{at} \quad z=0 \quad &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \int_0^\infty\!\!\int_0^{l_2}\!\!\int_0^{l_1} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ a\int_0^t\!\!\int_0^\infty\!\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau \\
&- a\int_0^t\!\!\int_0^\infty\!\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\,d\zeta\,d\tau \\
&+ a\int_0^t\!\!\int_0^\infty\!\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau \\
&- a\int_0^t\!\!\int_0^\infty\!\!\int_0^{l_1} g_4(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=l_2} d\xi\,d\zeta\,d\tau \\
&- a\int_0^t\!\!\int_0^{l_2}\!\!\int_0^{l_1} g_5(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= H_1(x,\xi,t;l_1)H_2(y,\eta,t;l_2)H_3(z,\zeta,t), \\
H_1(x,\xi,t;l_1) &= \frac{2}{l_1}\sum_{n=1}^\infty \sin\left(\frac{\pi n x}{l_1}\right)\sin\left(\frac{\pi n \xi}{l_1}\right)\exp\left(-\frac{\pi^2 n^2 at}{l_1^2}\right), \\
H_2(y,\eta,t;l_2) &= \frac{2}{l_2}\sum_{n=1}^\infty \sin\left(\frac{\pi n y}{l_2}\right)\sin\left(\frac{\pi n \eta}{l_2}\right)\exp\left(-\frac{\pi^2 n^2 at}{l_2^2}\right), \\
H_3(z,\zeta,t) &= \frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]+\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}.
\end{aligned}
$$

2°. A semiinfinite cylindrical domain of a rectangular cross-section is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) = {} & \int_0^\infty \int_0^{l_2} \int_0^{l_1} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\
& - a \int_0^t \int_0^\infty \int_0^{l_2} g_1(\eta,\zeta,\tau)\, G(x,y,z,0,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau \\
& + a \int_0^t \int_0^\infty \int_0^{l_2} g_2(\eta,\zeta,\tau)\, G(x,y,z,l_1,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau \\
& - a \int_0^t \int_0^\infty \int_0^{l_1} g_3(\xi,\zeta,\tau)\, G(x,y,z,\xi,0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
& + a \int_0^t \int_0^\infty \int_0^{l_1} g_4(\xi,\zeta,\tau)\, G(x,y,z,\xi,l_2,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
& + a \int_0^t \int_0^{l_2} \int_0^{l_1} g_5(\xi,\eta,\tau) \left[\frac{\partial}{\partial\zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\, d\eta\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= H_1(x,\xi,t;l_1) H_2(y,\eta,t;l_2) H_3(z,\zeta,t), \\
H_1(x,\xi,t;l_1) &= \frac{1}{l_1}\left[1 + 2\sum_{n=1}^{\infty} \cos\left(\frac{\pi n x}{l_1}\right) \cos\left(\frac{\pi n \xi}{l_1}\right) \exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right)\right], \\
H_2(y,\eta,t;l_2) &= \frac{1}{l_2}\left[1 + 2\sum_{n=1}^{\infty} \cos\left(\frac{\pi n y}{l_2}\right) \cos\left(\frac{\pi n \eta}{l_2}\right) \exp\left(-\frac{\pi^2 n^2 a t}{l_2^2}\right)\right], \\
H_3(z,\zeta,t) &= \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] - \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}.
\end{aligned}
$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. First boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\
w &= g_1(y,z,t) && \text{at} \quad x = 0 && \text{(boundary condition)}, \\
w &= g_2(y,z,t) && \text{at} \quad x = l_1 && \text{(boundary condition)}, \\
w &= g_3(x,z,t) && \text{at} \quad y = 0 && \text{(boundary condition)}, \\
w &= g_4(x,z,t) && \text{at} \quad y = l_2 && \text{(boundary condition)}, \\
w &= g_5(x,y,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\
w &= g_6(x,y,t) && \text{at} \quad z = l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) = {} & \int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\
& + a\int_0^t\int_0^{l_3}\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau \\
& - a\int_0^t\int_0^{l_3}\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\, d\zeta\, d\tau \\
& + a\int_0^t\int_0^{l_3}\int_0^{l_1} g_3(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau \\
& - a\int_0^t\int_0^{l_3}\int_0^{l_1} g_4(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=l_2} d\xi\, d\zeta\, d\tau \\
& + a\int_0^t\int_0^{l_2}\int_0^{l_1} g_5(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\, d\eta\, d\tau \\
& - a\int_0^t\int_0^{l_2}\int_0^{l_1} g_6(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l_3} d\xi\, d\eta\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= G_1(x,\xi,t)\, G_2(y,\eta,t)\, G_3(z,\zeta,t), \\
G_1(x,\xi,t) &= \frac{2}{l_1}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l_1}\right)\sin\left(\frac{\pi n \xi}{l_1}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right), \\
G_2(y,\eta,t) &= \frac{2}{l_2}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n y}{l_2}\right)\sin\left(\frac{\pi n \eta}{l_2}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_2^2}\right), \\
G_3(z,\zeta,t) &= \frac{2}{l_3}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n z}{l_3}\right)\sin\left(\frac{\pi n \zeta}{l_3}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_3^2}\right).
\end{aligned}
$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Second boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\
\partial_x w &= g_1(y,z,t) && \text{at} \quad x = 0 && \text{(boundary condition)}, \\
\partial_x w &= g_2(y,z,t) && \text{at} \quad x = l_1 && \text{(boundary condition)}, \\
\partial_y w &= g_3(x,z,t) && \text{at} \quad y = 0 && \text{(boundary condition)}, \\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y = l_2 && \text{(boundary condition)}, \\
\partial_z w &= g_5(x,y,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\
\partial_z w &= g_6(x,y,t) && \text{at} \quad z = l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&\quad - a\int_0^t\int_0^{l_3}\int_0^{l_2} g_1(\eta,\zeta,\tau)\,G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&\quad + a\int_0^t\int_0^{l_3}\int_0^{l_2} g_2(\eta,\zeta,\tau)\,G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&\quad - a\int_0^t\int_0^{l_3}\int_0^{l_1} g_3(\xi,\zeta,\tau)\,G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
&\quad + a\int_0^t\int_0^{l_3}\int_0^{l_1} g_4(\xi,\zeta,\tau)\,G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
&\quad - a\int_0^t\int_0^{l_2}\int_0^{l_1} g_5(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau \\
&\quad + a\int_0^t\int_0^{l_2}\int_0^{l_1} g_6(\xi,\eta,\tau)\,G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= G_1(x,\xi,t)\,G_2(y,\eta,t)\,G_3(z,\zeta,t), \\
G_1(x,\xi,t) &= \frac{1}{l_1}\left[1+2\sum_{n=1}^{\infty}\cos\left(\frac{\pi n x}{l_1}\right)\cos\left(\frac{\pi n\xi}{l_1}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right)\right], \\
G_2(y,\eta,t) &= \frac{1}{l_2}\left[1+2\sum_{n=1}^{\infty}\cos\left(\frac{\pi n y}{l_2}\right)\cos\left(\frac{\pi n\eta}{l_2}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_2^2}\right)\right], \\
G_3(z,\zeta,t) &= \frac{1}{l_3}\left[1+2\sum_{n=1}^{\infty}\cos\left(\frac{\pi n z}{l_3}\right)\cos\left(\frac{\pi n\zeta}{l_3}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_3^2}\right)\right].
\end{aligned}
$$

► **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3$. Third boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\
\partial_x w - k_1 w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)}, \\
\partial_x w + k_2 w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)}, \\
\partial_y w - k_3 w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)}, \\
\partial_y w + k_4 w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}, \\
\partial_z w - k_5 w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)}, \\
\partial_z w + k_6 w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta,t) = H_1(x,\xi,t)H_2(y,\eta,t)H_3(z,\zeta,t),$$
$$H_1(x,\xi,t) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2} \exp(-a\mu_n^2 t),$$
$$H_2(y,\eta,t) = \sum_{m=1}^{\infty} \frac{\psi_m(y)\psi_m(\eta)}{\|\psi_m\|^2} \exp(-a\lambda_m^2 t),$$
$$H_3(z,\zeta,t) = \sum_{s=1}^{\infty} \frac{\rho_s(z)\rho_s(\zeta)}{\|\rho_s\|^2} \exp(-a\nu_s^2 t).$$

Here,

$$\varphi_n(x)=\cos(\mu_n x)+\frac{k_1}{\mu_n}\sin(\mu_n x), \quad \|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{l_1}{2}\left(1+\frac{k_1^2}{\mu_n^2}\right),$$
$$\psi_m(y)=\cos(\lambda_m y)+\frac{k_3}{\lambda_m}\sin(\lambda_m y), \quad \|\psi_m\|^2=\frac{k_4}{2\lambda_m^2}\frac{\lambda_m^2+k_3^2}{\lambda_m^2+k_4^2}+\frac{k_3}{2\lambda_m^2}+\frac{l_2}{2}\left(1+\frac{k_3^2}{\lambda_m^2}\right),$$
$$\rho_s(x)=\cos(\nu_s x)+\frac{k_5}{\nu_s}\sin(\nu_s x), \quad \|\rho_s\|^2=\frac{k_6}{2\nu_s^2}\frac{\nu_s^2+k_5^2}{\nu_s^2+k_6^2}+\frac{k_5}{2\nu_s^2}+\frac{l_3}{2}\left(1+\frac{k_5^2}{\nu_s^2}\right).$$

The μ_n, λ_m, and ν_s are positive roots of the transcendental equations

$$\frac{\tan(\mu l_1)}{\mu} = \frac{k_1+k_2}{\mu^2-k_1k_2}, \quad \frac{\tan(\lambda l_2)}{\lambda} = \frac{k_3+k_4}{\lambda^2-k_3k_4}, \quad \frac{\tan(\nu l_3)}{\nu} = \frac{k_5+k_6}{\nu^2-k_5k_6}.$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Mixed boundary value problems.**

1°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\ w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\ w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\ w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\ \partial_z w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) = {} & \int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\
& + a\int_0^t\!\int_0^{l_3}\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau \\
& - a\int_0^t\!\int_0^{l_3}\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\, d\zeta\, d\tau \\
& + a\int_0^t\!\int_0^{l_3}\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau \\
& - a\int_0^t\!\int_0^{l_3}\!\int_0^{l_1} g_4(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=l_2} d\xi\, d\zeta\, d\tau \\
& - a\int_0^t\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,0,t-\tau)\, d\xi\, d\eta\, d\tau \\
& + a\int_0^t\!\int_0^{l_2}\!\int_0^{l_1} g_6(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,l_3,t-\tau)\, d\xi\, d\eta\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= G_1(x,\xi,t)\, G_2(y,\eta,t)\, G_3(z,\zeta,t), \\
G_1(x,\xi,t) &= \frac{2}{l_1}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l_1}\right) \sin\left(\frac{\pi n \xi}{l_1}\right) \exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right), \\
G_2(y,\eta,t) &= \frac{2}{l_2}\sum_{k=1}^{\infty} \sin\left(\frac{\pi k y}{l_2}\right) \sin\left(\frac{\pi k \eta}{l_2}\right) \exp\left(-\frac{\pi^2 k^2 a t}{l_2^2}\right), \\
G_3(z,\zeta,t) &= \frac{1}{l_3} + \frac{2}{l_3}\sum_{m=1}^{\infty} \cos\left(\frac{\pi m z}{l_3}\right) \cos\left(\frac{\pi m \zeta}{l_3}\right) \exp\left(-\frac{\pi^2 m^2 a t}{l_3^2}\right).
\end{aligned}
$$

2°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\
w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)}, \\
w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)}, \\
\partial_y w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)}, \\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}, \\
\partial_z w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)}, \\
\partial_z w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) = {} & \int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\
& + a\int_0^t\int_0^{l_3}\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau \\
& - a\int_0^t\int_0^{l_3}\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\, d\zeta\, d\tau \\
& - a\int_0^t\int_0^{l_3}\int_0^{l_2} g_3(\xi,\zeta,\tau)\, G(x,y,z,\xi,0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
& + a\int_0^t\int_0^{l_3}\int_0^{l_1} g_4(\xi,\zeta,\tau)\, G(x,y,z,\xi,l_2,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
& - a\int_0^t\int_0^{l_2}\int_0^{l_1} g_5(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,0,t-\tau)\, d\xi\, d\eta\, d\tau \\
& + a\int_0^t\int_0^{l_2}\int_0^{l_1} g_6(\xi,\eta,\tau)\, G(x,y,z,\xi,\eta,l_3,t-\tau)\, d\xi\, d\eta\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) &= G_1(x,\xi,t)\, G_2(y,\eta,t)\, G_3(z,\zeta,t), \\
G_1(x,\xi,t) &= \frac{2}{l_1}\sum_{n=1}^{\infty}\sin\left(\frac{\pi n x}{l_1}\right)\sin\left(\frac{\pi n \xi}{l_1}\right)\exp\left(-\frac{\pi^2 n^2 a t}{l_1^2}\right), \\
G_2(y,\eta,t) &= \frac{1}{l_2} + \frac{2}{l_2}\sum_{k=1}^{\infty}\cos\left(\frac{\pi k y}{l_2}\right)\cos\left(\frac{\pi k \eta}{l_2}\right)\exp\left(-\frac{\pi^2 k^2 a t}{l_2^2}\right), \\
G_3(z,\zeta,t) &= \frac{1}{l_3} + \frac{2}{l_3}\sum_{m=1}^{\infty}\cos\left(\frac{\pi m z}{l_3}\right)\cos\left(\frac{\pi m \zeta}{l_3}\right)\exp\left(-\frac{\pi^2 m^2 a t}{l_3^2}\right).
\end{aligned}
$$

5.1.2 Problems in Cylindrical Coordinates

The three-dimensional sourceless heat equation in the cylindrical coordinate system has the form

$$\frac{\partial w}{\partial t} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}\right], \qquad r = \sqrt{x^2+y^2}.$$

It is used to describe nonsymmetric unsteady processes in moving media or solids with cylindrical or plane boundaries. A similar equation is used to study the corresponding three-dimensional unsteady mass-exchange processes with constant diffusivity.

One-dimensional problems with axial symmetry that have solutions of the form $w = w(r,t)$ are discussed in Section 3.2.1. Two-dimensional problems whose solutions have the form $w = w(r,\varphi,t)$ or $w = w(r,z,t)$ are considered in Sections 4.1.2 and 4.1.3.

► **Remarks on the Green's functions.**

For the three-dimensional problems dealt with in Section 5.1.2, the Green's function can be represented in the product form

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

where $G_1(r,\varphi,\xi,\eta,t)$ is the Green's function of the two-dimensional boundary value problem (such functions are presented in Section 4.1.2), and $G_2(z,\zeta,t)$ is the Green's function of the corresponding one-dimensional boundary value problem (such functions can be found in Sections 3.1.1 and 3.1.2).

Example 5.5. The Green's function of the first boundary value problem for a semiinfinite circular cylinder ($0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with the initial and boundary conditions

$$w = f(r,\varphi,z) \quad \text{at} \quad t = 0,$$
$$w = g_1(\varphi,z,t) \quad \text{at} \quad r = R, \qquad w = g_2(r,\varphi,t) \quad \text{at} \quad z = 0,$$

is the product of the two-dimensional Green's function of the first boundary value problem of Section 4.1.2 (for $0 \le r \le R$, $0 \le \varphi \le 2\pi$) and the one-dimensional Green's function of the first boundary value problem of Section 3.1.2 (for $0 \le z < \infty$), in which one should perform obvious renaming of variables.

General formulas that enable one to obtain solutions of basic boundary value problems with the help of the Green's function can be found in Sections 17.4.1 and 17.5.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. First boundary value problem.**

An infinite circular cylinder is considered. The following conditions are prescribed:

$$w = f(r,\varphi,z) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(\varphi,z,t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

Solution:

$$w(r,\varphi,z,t) = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta$$
$$- aR\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$
$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right], \qquad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Second boundary value problem.**

An infinite circular cylinder is considered. The following conditions are prescribed:

$$w = f(r,\varphi,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_r w = g(\varphi,z,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

Solution:

$$w(r,\varphi,z,t) = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta$$
$$+ aR\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\eta,\zeta,\tau)\,G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$
$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi R^2}+\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at),$$
$$G_2(z,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right], \qquad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R)=0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Third boundary value problem.**

An infinite circular cylinder is considered. The following conditions are prescribed:

$$w = f(r,\varphi,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_r w + kw = g(\varphi,z,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$
$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2+k^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at),$$
$$G_2(z,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right], \qquad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots\end{cases}$$

Here, the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^\infty\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &\quad - aR\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\ &\quad + a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2}\sum_{n=0}^\infty\sum_{m=1}^\infty \frac{A_n}{[J_n'(\mu_{nm}R)]^2} J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] - \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\},\quad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Second boundary value problem.**

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^\infty\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &\quad + aR\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 - n^2)[J_n(\mu_{nm} R)]^2} \cos[n(\varphi-\eta)] \exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] + \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\},\quad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Third boundary value problem.

A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + k_1 w &= g(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2)[J_n(\mu_{nm} R)]^2} \cos[n(\varphi-\eta)] \exp(-\mu_{nm}^2 at),$$

$$\begin{aligned} G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\biggl\{&\exp\left[-\frac{(z-\zeta)^2}{4at}\right] + \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\\ &- 2k_2\int_0^{\infty}\exp\left[-\frac{(z+\zeta+s)^2}{4at} - k_2 s\right] ds\biggr\}. \end{aligned}$$

Here, $A_0 = 1$ and $A_n = 2$ for $n = 1,\,2,\,\dots$; the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0.$$

► Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Mixed boundary value problems.

1°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \int_0^\infty\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta - aR\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau - a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2}\sum_{n=0}^\infty\sum_{m=1}^\infty \frac{A_n}{[J_n'(\mu_{nm}R)]^2} J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] + \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\},\quad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

2°. A semiinfinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \int_0^\infty\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta + aR\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau + a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2} + \frac{1}{\pi}\sum_{n=0}^\infty\sum_{m=1}^\infty \frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] - \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\},\quad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \int_0^l\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&\quad - aR\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\
&\quad + a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau\\
&\quad - a\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right), \qquad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
\partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) = &\int_0^l \int_0^{2\pi} \int_0^R \xi f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\ &+ aR \int_0^t \int_0^l \int_0^{2\pi} g_1(\eta,\zeta,\tau)\, G(r,\varphi,z,R,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau \\ &- a \int_0^t \int_0^{2\pi} \int_0^R \xi g_2(\xi,\eta,\tau)\, G(r,\varphi,z,\xi,\eta,0,t-\tau)\, d\xi\, d\eta\, d\tau \\ &+ a \int_0^t \int_0^{2\pi} \int_0^R \xi g_3(\xi,\eta,\tau)\, G(r,\varphi,z,\xi,\eta,l,t-\tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\, G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2} + \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 - n^2)[J_n(\mu_{nm}R)]^2} \cos[n(\varphi-\eta)] \exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi\xi}{l}\right) \exp\left(-\frac{an^2\pi^2 t}{l^2}\right), \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n=1,\,2,\,\dots, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) &&\text{at}\quad t = 0 &&\text{(initial condition)}, \\ \partial_r w + k_1 w &= g(\varphi,z,t) &&\text{at}\quad r = R &&\text{(boundary condition)}, \\ \partial_z w - k_2 w &= g_2(r,\varphi,t) &&\text{at}\quad z = 0 &&\text{(boundary condition)}, \\ \partial_z w + k_3 w &= g_3(r,\varphi,t) &&\text{at}\quad z = l &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t) \sum_{s=1}^{\infty} \frac{h_s(z) h_s(\zeta)}{\|h_s\|^2} \exp(-a\lambda_s^2 t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2)[J_n(\mu_{nm}R)]^2} \cos[n(\varphi-\eta)] \exp(-\mu_{nm}^2 at),$$

$$h_s(z) = \cos(\lambda_s z) + \frac{k_2}{\lambda_s} \sin(\lambda_s z), \qquad \|h_s\|^2 = \frac{k_3}{2\lambda_s^2} \frac{\lambda_s^2 + k_2^2}{\lambda_s^2 + k_3^2} + \frac{k_2}{2\lambda_s^2} + \frac{l}{2}\left(1 + \frac{k_2^2}{\lambda_s^2}\right).$$

Here, $A_0 = 1$ and $A_n = 2$ for $n = 1,\,2,\,\dots$; the $J_n(\xi)$ are Bessel functions; and the μ_{nm} and λ_s are positive roots of the transcendental equations

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0, \qquad \frac{\tan(\lambda l)}{\lambda} = \frac{k_2 + k_3}{\lambda^2 - k_2 k_3}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^l\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &\quad - aR\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\ &\quad + a\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t)=\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi z}{l}\right)\cos\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right),\quad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned}w(r,\varphi,z,t) &= \int_0^l\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\&+aR\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\&+a\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}d\xi\,d\eta\,d\tau\\&-a\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}d\xi\,d\eta\,d\tau.\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi R^2}+\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at),$$

$$G_2(z,\zeta,t)=\frac{2}{l}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2t}{l^2}\right),\quad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R)=0$.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. First boundary value problem.**

An infinite hollow circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned}&w=f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\&w=g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\&w=g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned}w(r,\varphi,z,t) &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\&+aR_1\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}d\eta\,d\zeta\,d\tau\\&-aR_2\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}d\eta\,d\zeta\,d\tau.\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left[-\frac{(z-\zeta)^2}{4at}\right] G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n B_{nm} Z_n(\mu_{nm} r) Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 a t),$$

$$A_n = \begin{cases} 1/2 & \text{for } n=0, \\ 1 & \text{for } n\neq 0, \end{cases} \qquad B_{nm} = \frac{\mu_{nm}^2 J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1) - J_n^2(\mu_{nm}R_2)},$$

$$Z_n(\mu_{nm} r) = J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Second boundary value problem.**

An infinite hollow circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)}, \\ \partial_r w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\, d\xi\, d\eta\, d\zeta \\ &\quad - aR_1\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g_1(\eta,\zeta,\tau)\, G(r,\varphi,z,R_1,\xi,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau \\ &\quad + aR_2\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g_2(\eta,\zeta,\tau)\, G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left[-\frac{(z-\zeta)^2}{4at}\right] G_1(r,\varphi,\xi,\eta,t),$$

$$\begin{aligned} G_1(r,\varphi,\xi,\eta,t) &= \frac{1}{\pi(R_2^2 - R_1^2)} \\ &\quad + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n \mu_{nm}^2 Z_n(\mu_{nm}r) Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 a t)}{(\mu_{nm}^2 R_2^2 - n^2) Z_n^2(\mu_{nm}R_2) - (\mu_{nm}^2 R_1^2 - n^2) Z_n^2(\mu_{nm}R_1)}, \end{aligned}$$

$$Z_n(\mu_{nm} r) = J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where $A_0 = 1$ and $A_n = 2$ for $n = 1,\ 2,\ \ldots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions (the prime denotes a derivative with respect to the argument); and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2) - Y_n'(\mu R_1)J_n'(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Third boundary value problem.**

An infinite hollow circular cylinder is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w - k_1 w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w + k_2 w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right]G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at)}{(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)},$$

$$
\begin{aligned}
Z_n(\mu_{nm}r)={}&\bigl[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r)\\
&-\bigl[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r).
\end{aligned}
$$

Here, $A_0=1$ and $A_n=2$ for $n=1,\,2,\,\dots$; $J_n(r)$ and $Y_n(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation

$$
\begin{aligned}
&\bigl[\mu J_n'(\mu R_1)-k_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2)+k_2Y_n(\mu R_2)\bigr]\\
&\qquad=\bigl[\mu Y_n'(\mu R_1)-k_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2)+k_2J_n(\mu R_2)\bigr].
\end{aligned}
$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite hollow circular cylinder is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) ={}& \int_0^\infty\!\int_0^{2\pi}\!\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+aR_1\int_0^t\!\int_0^\infty\!\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}d\eta\,d\zeta\,d\tau\\
&-aR_2\int_0^t\!\int_0^\infty\!\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}d\eta\,d\zeta\,d\tau\\
&+a\int_0^t\!\int_0^{2\pi}\!\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] - \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n B_{nm} Z_n(\mu_{nm}r) Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$A_n = \begin{cases} 1/2 & \text{for } n=0, \\ 1 & \text{for } n\neq 0, \end{cases} \qquad B_{nm} = \frac{\mu_{nm}^2 J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1) - J_n^2(\mu_{nm}R_2)},$$

$$Z_n(\mu_{nm}r) = J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Second boundary value problem.**

A semiinfinite hollow circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)}, \\ \partial_r w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}, \\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^\infty\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\ &\quad - aR_1\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\ &\quad + aR_2\int_0^t\int_0^\infty\int_0^{2\pi} g_2(\eta,\zeta,\tau)\,G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\ &\quad - a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] + \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$\begin{aligned} G_1(r,\varphi,\xi,\eta,t) &= \frac{1}{\pi(R_2^2-R_1^2)} \\ &\quad + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at)}{(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)}, \end{aligned}$$

$$Z_n(\mu_{nm}r) = J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where $A_0 = 1$ and $A_n = 2$ for $n = 1,\ 2,\ \dots$, the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2) - Y_n'(\mu R_1)J_n'(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z < \infty$. Third boundary value problem.**

A semiinfinite hollow circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w - k_1 w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w + k_2 w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\ \partial_z w - k_3 w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula from the previous paragraph (for the second boundary value problem) in which

$$\begin{gathered} G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(z,\zeta,t)\,G_2(r,\varphi,\xi,\eta,t),\\ G_1(z,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]+\exp\left[-\frac{(z+\zeta)^2}{4at}\right]-2k_3\int_0^\infty \exp\left[-\frac{(z+\zeta+s)^2}{4at}-k_3 s\right]ds\right\},\\ G_2(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^\infty\sum_{m=1}^\infty \frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at)}{(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)},\\ \begin{aligned} Z_n(\mu_{nm}r)=&\big[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\big]Y_n(\mu_{nm}r)\\ &-\big[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\big]J_n(\mu_{nm}r), \end{aligned} \end{gathered}$$

where $A_0 = 1$ and $A_n = 2$ for $n = 1,\ 2,\ \dots$, the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$\begin{aligned} &\big[\mu J_n'(\mu R_1) - k_1 J_n(\mu R_1)\big]\big[\mu Y_n'(\mu R_2) + k_2 Y_n(\mu R_2)\big]\\ &\qquad = \big[\mu Y_n'(\mu R_1) - k_1 Y_n(\mu R_1)\big]\big[\mu J_n'(\mu R_2) + k_2 J_n(\mu R_2)\big]. \end{aligned}$$

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z < \infty$. Mixed boundary value problems.**

1°. A semiinfinite hollow circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \int_0^\infty\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta$$
$$+ aR_1\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=R_1} d\eta\,d\zeta\,d\tau$$
$$- aR_2\int_0^t\int_0^\infty\int_0^{2\pi} g_2(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=R_2} d\eta\,d\zeta\,d\tau$$
$$- a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi at}}\biggl\{\exp\biggl[-\frac{(z-\zeta)^2}{4at}\biggr] + \exp\biggl[-\frac{(z+\zeta)^2}{4at}\biggr]\biggr\}G_1(r,\varphi,\xi,\eta,t),$$
$$G_1(r,\varphi,\xi,\eta,t) = \frac{\pi}{2}\sum_{n=0}^\infty\sum_{m=1}^\infty A_n B_{nm} Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$A_n = \begin{cases} 1/2 & \text{for } n=0,\\ 1 & \text{for } n\neq 0,\end{cases}\qquad B_{nm} = \frac{\mu_{nm}^2 J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1) - J_n^2(\mu_{nm}R_2)},$$
$$Z_n(\mu_{nm}r) = J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

2°. A semiinfinite hollow circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\ w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \int_0^\infty\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta$$
$$- aR_1\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau$$
$$+ aR_2\int_0^t\int_0^\infty\int_0^{2\pi} g_2(\eta,\zeta,\tau)\,G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau$$
$$+ a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]-\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi(R_2^2-R_1^2)}$$

$$+\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at)}{(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)},$$

$$Z_n(\mu_{nm}r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where $A_0=1$ and $A_n=2$ for $n=1,\ 2,\ \ldots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions (the prime denotes a derivative with respect to the argument); and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.$$

► **Domain: $R_1\le r\le R_2$, $0\le\varphi\le 2\pi$, $0\le z\le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
&w=f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
&w=g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
&w=g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
&w=g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
&w=g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t)&=\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+aR_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}d\eta\,d\zeta\,d\tau\\
&-aR_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}d\eta\,d\zeta\,d\tau\\
&+a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau\\
&-a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\left[\frac{2}{l}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)\right],$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n B_{nm} Z_n(\mu_{nm}r) Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$A_n = \begin{cases} 1/2 & \text{for } n=0, \\ 1 & \text{for } n\neq 0, \end{cases} \qquad B_{nm} = \frac{\mu_{nm}^2 J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1) - J_n^2(\mu_{nm}R_2)},$$

$$Z_n(\mu_{nm}r) = J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at } t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at } r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(\varphi,z,t) && \text{at } r=R_2 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at } z=0 && \text{(boundary condition)},\\ \partial_z w &= g_4(r,\varphi,t) && \text{at } z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\ &\quad - aR_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\ &\quad + aR_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\,G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\ &\quad + a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\left[\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi z}{l}\right)\cos\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2t}{l^2}\right)\right],$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi(R_2^2-R_1^2)}+\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at)}{(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)},$$

$$Z_n(\mu_{nm}r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where $A_0=1$ and $A_n=2$ for $n=1,\ 2,\ \ldots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions (the prime denotes a derivative with respect to the argument); and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.$$

► **Domain: $R_1\le r\le R_2,\ 0\le\varphi\le 2\pi,\ 0\le z\le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w-k_1w&=g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w+k_2w&=g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w-k_3w&=g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w+k_4w&=g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t).$$

Here, the first factor has the form

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at)}{(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)},$$

$$Z_n(\mu_{nm}r)=\bigl[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r)-\bigl[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r),$$

where $A_0=1$ and $A_n=2$ for $n=1,\ 2,\ \ldots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation

$$\bigl[\mu J_n'(\mu R_1)-k_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2)+k_2Y_n(\mu R_2)\bigr]=\bigl[\mu Y_n'(\mu R_1)-k_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2)+k_2J_n(\mu R_2)\bigr].$$

The second factor is given by

$$G_2(z,\zeta,t)=\sum_{s=1}^{\infty}\frac{h_s(z)h_s(\zeta)}{\|h_s\|^2}\exp(-a\lambda_s^2t),$$

$$h_s(z)=\cos(\lambda_sz)+\frac{k_3}{\lambda_s}\sin(\lambda_sz),\qquad \|h_s\|^2=\frac{k_4}{2\lambda_s^2}\frac{\lambda_s^2+k_3^2}{\lambda_s^2+k_4^2}+\frac{k_3}{2\lambda_s^2}+\frac{l}{2}\left(1+\frac{k_3^2}{\lambda_s^2}\right),$$

where the λ_s are positive roots of the transcendental equation $\dfrac{\tan(\lambda l)}{\lambda} = \dfrac{k_3 + k_4}{\lambda^2 - k_3 k_4}$.

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+ aR_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1} d\eta\,d\zeta\,d\tau\\
&- aR_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2} d\eta\,d\zeta\,d\tau\\
&- a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+ a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\left[\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}\cos\Bigl(\frac{n\pi z}{l}\Bigr)\cos\Bigl(\frac{n\pi\zeta}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2t}{l^2}\Bigr)\right],$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_nB_{nm}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at),$$

$$A_n=\begin{cases}1/2 & \text{for } n=0,\\ 1 & \text{for } n\neq 0,\end{cases}\qquad B_{nm}=\frac{\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)},$$

$$Z_n(\mu_{nm}r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\ w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\ &\quad - aR_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\,G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\ &\quad + aR_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\,G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\ &\quad + a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau\\ &\quad - a\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=G_1(r,\varphi,\xi,\eta,t)\left[\frac{2}{l}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2t}{l^2}\right)\right],$$

$$\begin{aligned} G_1(r,\varphi,\xi,\eta,t) &= \frac{1}{\pi(R_2^2-R_1^2)}\\ &\quad +\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2at)}{(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)}, \end{aligned}$$

$$Z_n(\mu_{nm}r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where $A_0=1$ and $A_n=2$ for $n=1,\ 2,\ \ldots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions (the prime denotes a derivative with respect to the argument); and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.$$

► **Domain: $0\le r<\infty$, $0\le\varphi\le\varphi_0$, $-\infty<z<\infty$. First boundary value problem.**

A dihedral angle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ w &= g_2(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \int_{-\infty}^{\infty}\int_0^{\varphi_0}\int_0^{\infty} f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\, d\xi\, d\eta\, d\zeta$$
$$+ a\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_1(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau$$
$$- a\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\, d\zeta\, d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right] G_1(r,\varphi,\xi,\eta,t),$$
$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right),$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $-\infty < z < \infty$. Second boundary value problem.**

A dihedral angle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ r^{-1}\partial_\varphi w &= g_1(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_2(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \int_{-\infty}^{\infty}\int_0^{\varphi_0}\int_0^{\infty} f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\, d\xi\, d\eta\, d\zeta$$
$$- a\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_1(\xi,\zeta,\tau)\, G(r,\varphi,z,\xi,0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau$$
$$+ a\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_2(\xi,\zeta,\tau)\, G(r,\varphi,z,\xi,\varphi_0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right] G_1(r,\varphi,\xi,\eta,t),$$
$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\left[\frac{1}{2}I_0\left(\frac{r\xi}{2at}\right) + \sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\eta}{\varphi_0}\right)\right],$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$. First boundary value problem.**

The upper half of a dihedral angle is considered. The following conditions are prescribed:

$$
\begin{aligned}
&w = f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
&w = g_1(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
&w = g_2(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\
&w = g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \int_0^\infty\!\int_0^{\varphi_0}\!\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_1(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau\\
&- a\int_0^t\!\int_0^\infty\!\int_0^\infty g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\,d\zeta\,d\tau\\
&+ a\int_0^t\!\int_0^{\varphi_0}\!\int_0^\infty g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right] - \exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),
$$

$$
G_1(r,\varphi,\xi,\eta,t) = \frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right),
$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$. Second boundary value problem.**

The upper half of a dihedral angle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
r^{-1}\partial_\varphi w &= g_1(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w &= g_2(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \int_0^\infty\!\int_0^{\varphi_0}\!\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&- a\int_0^t\!\int_0^\infty\!\int_0^\infty g_1(\xi,\zeta,\tau)\,G(r,\varphi,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&+ a\int_0^t\!\int_0^\infty\!\int_0^\infty g_2(\xi,\zeta,\tau)\,G(r,\varphi,z,\xi,\varphi_0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&- a\int_0^t\!\int_0^{\varphi_0}\!\int_0^\infty g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]+\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\left[\frac{1}{2}I_0\left(\frac{r\xi}{2at}\right)+\sum_{n=1}^{\infty}I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\eta}{\varphi_0}\right)\right],$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$. Mixed boundary value problems.**

1°. The upper half of a dihedral angle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ w &= g_2(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^\infty\int_0^{\varphi_0}\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\ &+ a\int_0^t\int_0^\infty\int_0^\infty g_1(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0}d\xi\,d\zeta\,d\tau\\ &- a\int_0^t\int_0^\infty\int_0^\infty g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\zeta\,d\tau\\ &- a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]+\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t)=\frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\sum_{n=1}^{\infty}I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right),$$

where the $I_\nu(r)$ are modified Bessel functions.

2°. The upper half of a dihedral angle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ r^{-1}\partial_\varphi w &= g_1(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_2(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)},\\ w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned}w(r,\varphi,z,t) &= \int_0^\infty\int_0^{\varphi_0}\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&- a\int_0^t\int_0^\infty\int_0^\infty g_1(\xi,\zeta,\tau)\,G(r,\varphi,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&+ a\int_0^t\int_0^\infty\int_0^\infty g_2(\xi,\zeta,\tau)\,G(r,\varphi,z,\xi,\varphi_0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&+ a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau.\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{1}{2\sqrt{\pi at}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]-\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$\begin{aligned}G_1(r,\varphi,\xi,\eta,t)&=\frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\left[\frac{1}{2}I_0\left(\frac{r\xi}{2at}\right)\right.\\
&\left.+\sum_{n=1}^\infty I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\eta}{\varphi_0}\right)\right],\end{aligned}$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. First boundary value problem.**

A wedge domain of finite thickness is considered. The following conditions are prescribed:

$$\begin{aligned}
&w = f(r,\varphi,z) && \text{at}\quad t=0 && \text{(initial condition)},\\
&w = g_1(r,z,t) && \text{at}\quad \varphi=0 && \text{(boundary condition)},\\
&w = g_2(r,z,t) && \text{at}\quad \varphi=\varphi_0 && \text{(boundary condition)},\\
&w = g_3(r,\varphi,t) && \text{at}\quad z=0 && \text{(boundary condition)},\\
&w = g_4(r,\varphi,t) && \text{at}\quad z=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}w(r,\varphi,z,t) &= \int_0^l\int_0^{\varphi_0}\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+ a\int_0^t\int_0^l\int_0^\infty g_1(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0}d\xi\,d\zeta\,d\tau\\
&- a\int_0^t\int_0^l\int_0^\infty g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\zeta\,d\tau\\
&+ a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau\\
&- a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau.\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\biggl[\frac{2}{l}\sum_{n=1}^{\infty}\sin\Bigl(\frac{n\pi z}{l}\Bigr)\sin\Bigl(\frac{n\pi\zeta}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr)\biggr],$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{a\varphi_0 t}\exp\Bigl(-\frac{r^2+\xi^2}{4at}\Bigr)\sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\Bigl(\frac{r\xi}{2at}\Bigr)\sin\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\sin\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr),$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. Second boundary value problem.**

A wedge domain of finite thickness is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ r^{-1}\partial_\varphi w &= g_1(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_2(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^l\int_0^{\varphi_0}\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\ &\quad - a\int_0^t\int_0^l\int_0^\infty g_1(\xi,\zeta,\tau)\,G(r,\varphi,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\ &\quad + a\int_0^t\int_0^l\int_0^\infty g_2(\xi,\zeta,\tau)\,G(r,\varphi,z,\xi,\varphi_0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\ &\quad + a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_4(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,z,\xi,\eta,\zeta,t) &= G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),\\ G_1(r,\varphi,\xi,\eta,t) &= \frac{1}{a\varphi_0 t}\exp\Bigl(-\frac{r^2+\xi^2}{4at}\Bigr)\biggl[\frac{1}{2}I_0\Bigl(\frac{r\xi}{2at}\Bigr)\\ &\quad + \sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\Bigl(\frac{r\xi}{2at}\Bigr)\cos\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\cos\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr)\biggr],\\ G_2(z,\zeta,t) &= \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\Bigl(\frac{n\pi z}{l}\Bigr)\cos\Bigl(\frac{n\pi\zeta}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr), \end{aligned}$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A wedge domain of finite thickness is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ w &= g_2(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^l\int_0^{\varphi_0}\int_0^\infty f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\ &\quad + a\int_0^t\int_0^l\int_0^\infty g_1(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^l\int_0^\infty g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_3(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\ &\quad + a\int_0^t\int_0^{\varphi_0}\int_0^\infty g_4(\xi,\eta,\tau)\,G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{a\varphi_0 t}\exp\left(-\frac{r^2+\xi^2}{4at}\right)\sum_{n=1}^\infty I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right),$$

$$G_2(z,\zeta,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^\infty \cos\left(\frac{n\pi z}{l}\right)\cos\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right),$$

where the $I_\nu(r)$ are modified Bessel functions.

2°. A wedge domain of finite thickness is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ r^{-1}\partial_\varphi w &= g_1(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_2(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)},\\ w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) = {} & \int_0^l \int_0^{\varphi_0} \int_0^{\infty} f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\, d\xi\, d\eta\, d\zeta \\
& - a\int_0^t \int_0^l \int_0^{\infty} g_1(\xi,\zeta,\tau)\, G(r,\varphi,z,\xi,0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
& + a\int_0^t \int_0^l \int_0^{\infty} g_2(\xi,\zeta,\tau)\, G(r,\varphi,z,\xi,\varphi_0,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
& + a\int_0^t \int_0^{\varphi_0} \int_0^{\infty} g_3(\xi,\eta,\tau) \left[\frac{\partial}{\partial\zeta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} \xi\, d\xi\, d\eta\, d\tau \\
& - a\int_0^t \int_0^{\varphi_0} \int_0^{\infty} g_4(\xi,\eta,\tau) \left[\frac{\partial}{\partial\zeta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} \xi\, d\xi\, d\eta\, d\tau.
\end{aligned}
$$

Here,

$$
G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\left[\frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{l}\right) \sin\left(\frac{n\pi\zeta}{l}\right) \exp\left(-\frac{an^2\pi^2 t}{l^2}\right)\right],
$$

$$
\begin{aligned}
G_1(r,\varphi,\xi,\eta,t) = {} & \frac{1}{a\varphi_0 t} \exp\left(-\frac{r^2+\xi^2}{4at}\right) \left[\frac{1}{2} I_0\left(\frac{r\xi}{2at}\right)\right. \\
& \left. + \sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\left(\frac{r\xi}{2at}\right) \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\eta}{\varphi_0}\right)\right],
\end{aligned}
$$

where the $I_\nu(r)$ are modified Bessel functions.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $-\infty < z < \infty$. First boundary value problem.**

An infinite cylindrical sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,\varphi,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\
w &= g_1(\varphi,z,t) && \text{at} \quad r = R && \text{(boundary condition)}, \\
w &= g_2(r,z,t) && \text{at} \quad \varphi = 0 && \text{(boundary condition)}, \\
w &= g_3(r,z,t) && \text{at} \quad \varphi = \varphi_0 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) = {} & \int_{-\infty}^{\infty} \int_0^{\varphi_0} \int_0^{R} f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\, d\xi\, d\eta\, d\zeta \\
& - aR\int_0^t \int_{-\infty}^{\infty} \int_0^{\varphi_0} g_1(\eta,\zeta,\tau) \left[\frac{\partial}{\partial\xi} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\, d\zeta\, d\tau \\
& + a\int_0^t \int_{-\infty}^{\infty} \int_0^{R} g_2(\xi,\zeta,\tau) \frac{1}{\xi} \left[\frac{\partial}{\partial\eta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau \\
& - a\int_0^t \int_{-\infty}^{\infty} \int_0^{R} g_3(\xi,\zeta,\tau) \frac{1}{\xi} \left[\frac{\partial}{\partial\eta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\, d\zeta\, d\tau.
\end{aligned}
$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right]G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{4}{R^2\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\times\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\exp(-\mu_{nm}^2 at),$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite cylindrical sector is considered. The following conditions are prescribed:

$$\begin{aligned} &w = f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ &w = g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ &w = g_2(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\ &w = g_3(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\ &w = g_4(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \int_0^\infty\int_0^{\varphi_0}\int_0^R f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\ &\quad - aR\int_0^t\int_0^\infty\int_0^{\varphi_0} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}d\eta\,d\zeta\,d\tau\\ &\quad + a\int_0^t\int_0^\infty\int_0^R g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0}d\xi\,d\zeta\,d\tau\\ &\quad - a\int_0^t\int_0^\infty\int_0^R g_3(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\zeta\,d\tau\\ &\quad + a\int_0^t\int_0^{\varphi_0}\int_0^R g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(z-\zeta)^2}{4at}\right]-\exp\left[-\frac{(z+\zeta)^2}{4at}\right]\right\}G_1(r,\varphi,\xi,\eta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{4}{R^2\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\times\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\exp(-\mu_{nm}^2 at),$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

► Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$. Mixed boundary value problem.

A semiinfinite cylindrical sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
w &= g_1(\varphi, z, t) && \text{at} \quad r = R && \text{(boundary condition)},\\
w &= g_2(r, z, t) && \text{at} \quad \varphi = 0 && \text{(boundary condition)},\\
w &= g_3(r, z, t) && \text{at} \quad \varphi = \varphi_0 && \text{(boundary condition)},\\
\partial_z w &= g_4(r, \varphi, t) && \text{at} \quad z = 0 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r, \varphi, z, t) &= \int_0^\infty \int_0^{\varphi_0} \int_0^R f(\xi, \eta, \zeta)\, G(r, \varphi, z, \xi, \eta, \zeta, t)\xi \, d\xi \, d\eta \, d\zeta\\
&\quad - aR \int_0^t \int_0^\infty \int_0^{\varphi_0} g_1(\eta, \zeta, \tau) \left[\frac{\partial}{\partial \xi} G(r, \varphi, z, \xi, \eta, \zeta, t-\tau) \right]_{\xi=R} d\eta \, d\zeta \, d\tau\\
&\quad + a \int_0^t \int_0^\infty \int_0^R g_2(\xi, \zeta, \tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, z, \xi, \eta, \zeta, t-\tau) \right]_{\eta=0} d\xi \, d\zeta \, d\tau\\
&\quad - a \int_0^t \int_0^\infty \int_0^R g_3(\xi, \zeta, \tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, z, \xi, \eta, \zeta, t-\tau) \right]_{\eta=\varphi_0} d\xi \, d\zeta \, d\tau\\
&\quad - a \int_0^t \int_0^{\varphi_0} \int_0^R g_4(\xi, \eta, \tau)\, G(r, \varphi, z, \xi, \eta, 0, t-\tau)\xi \, d\xi \, d\eta \, d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r, \varphi, z, \xi, \eta, \zeta, t) &= \frac{1}{2\sqrt{\pi a t}} \left\{ \exp\left[-\frac{(z-\zeta)^2}{4at} \right] + \exp\left[-\frac{(z+\zeta)^2}{4at} \right] \right\} G_1(r, \varphi, \xi, \eta, t),\\
G_1(r, \varphi, \xi, \eta, t) &= \frac{4}{R^2 \varphi_0} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{J_{n\pi/\varphi_0}(\mu_{nm} r) J_{n\pi/\varphi_0}(\mu_{nm} \xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm} R)]^2}\\
&\qquad \times \sin\left(\frac{n\pi\varphi}{\varphi_0} \right) \sin\left(\frac{n\pi\eta}{\varphi_0} \right) \exp(-\mu_{nm}^2 a t),
\end{aligned}
$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

► Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. First boundary value problem.

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
w &= g_1(\varphi, z, t) && \text{at} \quad r = R && \text{(boundary condition)},\\
w &= g_2(r, z, t) && \text{at} \quad \varphi = 0 && \text{(boundary condition)},\\
w &= g_3(r, z, t) && \text{at} \quad \varphi = \varphi_0 && \text{(boundary condition)},\\
w &= g_4(r, \varphi, t) && \text{at} \quad z = 0 && \text{(boundary condition)},\\
w &= g_5(r, \varphi, t) && \text{at} \quad z = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \int_0^l\int_0^{\varphi_0}\int_0^R f(\xi,\eta,\zeta)\,G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&\quad - aR\int_0^t\int_0^l\int_0^{\varphi_0} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau \\
&\quad + a\int_0^t\int_0^l\int_0^R g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau \\
&\quad - a\int_0^t\int_0^l\int_0^R g_3(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\,d\zeta\,d\tau \\
&\quad + a\int_0^t\int_0^{\varphi_0}\int_0^R g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} \xi\,d\xi\,d\eta\,d\tau \\
&\quad - a\int_0^t\int_0^{\varphi_0}\int_0^R g_5(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} \xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t), \\
G_1(r,\varphi,\xi,\eta,t) &= \frac{4}{R^2\varphi_0}\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2} \\
&\quad \times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\exp(-\mu_{nm}^2 at), \\
G_2(z,\zeta,t) &= \frac{2}{l}\sum_{n=1}^\infty \sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi\zeta}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right),
\end{aligned}
$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

► Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. Mixed boundary value problem.

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
w &= g_3(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_5(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \int_0^l \int_0^{\varphi_0} \int_0^R f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\xi \, d\xi \, d\eta \, d\zeta \\
&- aR \int_0^t \int_0^l \int_0^{\varphi_0} g_1(\eta,\zeta,\tau) \left[\frac{\partial}{\partial \xi} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau) \right]_{\xi=R} d\eta \, d\zeta \, d\tau \\
&+ a \int_0^t \int_0^l \int_0^R g_2(\xi,\zeta,\tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau) \right]_{\eta=0} d\xi \, d\zeta \, d\tau \\
&- a \int_0^t \int_0^l \int_0^R g_3(\xi,\zeta,\tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau) \right]_{\eta=\varphi_0} d\xi \, d\zeta \, d\tau \\
&- a \int_0^t \int_0^{\varphi_0} \int_0^R g_4(\xi,\eta,\tau)\, G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi \, d\xi \, d\eta \, d\tau \\
&+ a \int_0^t \int_0^{\varphi_0} \int_0^R g_5(\xi,\eta,\tau)\, G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi \, d\xi \, d\eta \, d\tau.
\end{aligned}
$$

Here,

$$
G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t) \left[\frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi z}{l}\right) \cos\left(\frac{n\pi \zeta}{l}\right) \exp\left(-\frac{an^2\pi^2 t}{l^2}\right) \right],
$$

$$
\begin{aligned}
G_1(r,\varphi,\xi,\eta,t) = \frac{4}{R^2\varphi_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{J_{n\pi/\varphi_0}(\mu_{nm} r) J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm} R)]^2} \\
\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\eta}{\varphi_0}\right) \exp(-\mu_{nm}^2 a t),
\end{aligned}
$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

5.1.3 Problems in Spherical Coordinates

The heat equation in the spherical coordinate system has the form

$$
\frac{\partial w}{\partial t} = a \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial w}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial w}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 w}{\partial \varphi^2} \right],
$$
$$
r = \sqrt{x^2 + y^2 + z^2}.
$$

This representation is convenient to describe three-dimensional heat and mass exchange phenomena in domains bounded by coordinate surfaces of the spherical coordinate system.

One-dimensional problems with central symmetry that have solutions of the form $w = w(r,t)$ are discussed in Section 3.2.3.

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(r,\theta,\varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)}, \\
w &= g(\theta,\varphi,t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$w(r,\theta,\varphi,t)=\int_0^{2\pi}\int_0^{\pi}\int_0^{R} f(\xi,\eta,\zeta)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta$$
$$-aR^2\int_0^t\int_0^{2\pi}\int_0^{\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}\sin\eta\,d\eta\,d\zeta\,d\tau,$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{2\pi R^2\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\exp(-\lambda_{nm}^2at),$$
$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{(2n+1)(n-k)!}{(n+k)!\left[J'_{n+1/2}(\lambda_{nm}R)\right]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as follows:

$$P_n^k(\mu)=(1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_n(\mu),\qquad P_n(\mu)=\frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n;$$

and the λ_{nm} are positive roots of the transcendental equation $J_{n+1/2}(\lambda R)=0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. Second boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$w=f(r,\theta,\varphi)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$\partial_r w=g(\theta,\varphi,t)\quad\text{at}\quad r=R\quad\text{(boundary condition)}.$$

Solution:

$$w(r,\theta,\varphi,t)=\int_0^{2\pi}\int_0^{\pi}\int_0^{R} f(\xi,\eta,\zeta)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta$$
$$+aR^2\int_0^t\int_0^{2\pi}\int_0^{\pi} g(\eta,\zeta,\tau)\,G(r,\theta,\varphi,R,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau,$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{3}{4\pi R^3}+\frac{1}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\exp(-\lambda_{nm}^2at),$$
$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{\lambda_{nm}^2(2n+1)(n-k)!}{(n+k)!\left[R^2\lambda_{nm}^2-n(n+1)\right]\left[J_{n+1/2}(\lambda_{nm}R)\right]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the previous paragraph), and the λ_{nm} are positive roots of the transcendental equation

$$2\lambda R J'_{n+1/2}(\lambda R) - J_{n+1/2}(\lambda R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$w = f(r,\theta,\varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_r w + kw = g(\theta,\varphi,t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n} A_s B_{nms} J_{n+1/2}(\lambda_{nm} r) J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta) P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\exp(-\lambda_{nm}^2 at),$$
$$A_s=\begin{cases}1 & \text{if } s=0,\\ 2 & \text{if } s\neq 0,\end{cases}\qquad B_{nms}=\frac{\lambda_{nm}^2(2n+1)(n-s)!}{(n+s)!\big[R^2\lambda_{nm}^2+(kR+n)(kR-n-1)\big]\big[J_{n+1/2}(\lambda_{nm}R)\big]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda R J'_{n+1/2}(\lambda R) + \big(kR - \tfrac{1}{2}\big) J_{n+1/2}(\lambda R) = 0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$w = f(r,\theta,\varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(\theta,\varphi,t) \quad \text{at} \quad r = R_1 \quad \text{(boundary condition)},$$
$$w = g_2(\theta,\varphi,t) \quad \text{at} \quad r = R_2 \quad \text{(boundary condition)}.$$

Solution:

$$w(r,\theta,\varphi,t)=\int_0^{2\pi}\!\int_0^{\pi}\!\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\, G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\, d\xi\, d\eta\, d\zeta$$
$$+aR_1^2\int_0^t\!\int_0^{2\pi}\!\int_0^{\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}\sin\eta\, d\eta\, d\zeta\, d\tau$$
$$-aR_2^2\int_0^t\!\int_0^{2\pi}\!\int_0^{\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}\sin\eta\, d\eta\, d\zeta\, d\tau,$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{\pi}{8\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}Z_{n+1/2}(\lambda_{nm}r)Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\exp(-\lambda_{nm}^2at).$$

Here,

$$Z_{n+1/2}(\lambda_{nm}r)=J_{n+1/2}(\lambda_{nm}R_1)Y_{n+1/2}(\lambda_{nm}r)-Y_{n+1/2}(\lambda_{nm}R_1)J_{n+1/2}(\lambda_{nm}r),$$
$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{\lambda_{nm}^2(2n+1)(n-k)!\,J_{n+1/2}^2(\lambda_{nm}R_2)}{(n+k)!\big[J_{n+1/2}^2(\lambda_{nm}R_1)-J_{n+1/2}^2(\lambda_{nm}R_2)\big]},$$

where the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as follows:

$$P_n^k(\mu)=(1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_n(\mu),\qquad P_n(\mu)=\frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n;$$

and the λ_{nm} are positive roots of the transcendental equation

$$Z_{n+1/2}(\lambda R_2)=0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1\le r\le R_2$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. Second boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w&=g_1(\theta,\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ \partial_r w&=g_2(\theta,\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t)&=\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f(\xi,\eta,\zeta)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &\quad -aR_1^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_1(\eta,\zeta,\tau)\,G(r,\theta,\varphi,R_1,\eta,\zeta,t-\tau)\,\sin\eta\,d\eta\,d\zeta\,d\tau\\ &\quad +aR_2^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_2(\eta,\zeta,\tau)\,G(r,\theta,\varphi,R_2,\eta,\zeta,t-\tau)\,\sin\eta\,d\eta\,d\zeta\,d\tau,\end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{3}{4\pi(R_2^3-R_1^3)}+\frac{1}{4\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}\frac{A_k}{B_{nmk}}Z_{n+1/2}(\lambda_{nm}r)$$
$$\times Z_{n+1/2}(\lambda_{nm}\xi)P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\exp(-\lambda_{nm}^2at).$$

Here,

$$A_k = \begin{cases} 1 & \text{for } k = 0, \\ 2 & \text{for } k \neq 0, \end{cases} \qquad B_{nmk} = \frac{(n+k)!}{(2n+1)(n-k)!} \int_{R_1}^{R_2} r Z^2_{n+1/2}(\lambda_{nm} r)\, dr,$$

$$Z_{n+1/2}(\lambda r) = \left[\lambda J'_{n+1/2}(\lambda R_1) - \frac{1}{2R_1} J_{n+1/2}(\lambda R_1)\right] Y_{n+1/2}(\lambda r) - \left[\lambda Y'_{n+1/2}(\lambda R_1) - \frac{1}{2R_1} Y_{n+1/2}(\lambda R_1)\right] J_{n+1/2}(\lambda r),$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the previous paragraph), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2) - \frac{1}{2R_2} Z_{n+1/2}(\lambda R_2) = 0.$$

The integrals that determine the coefficients B_{nmk} can be expressed in terms of the Bessel functions and their derivatives; see Budak, Samarskii, and Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\theta,\varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w - k_1 w &= g_1(\theta,\varphi,t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ \partial_r w + k_2 w &= g_2(\theta,\varphi,t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = \frac{1}{4\pi\sqrt{r\xi}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=0}^{n} \frac{A_s}{B_{nms}} Z_{n+1/2}(\lambda_{nm} r) Z_{n+1/2}(\lambda_{nm}\xi) \times P_n^s(\cos\theta) P_n^s(\cos\eta) \cos[s(\varphi-\zeta)] \exp(-\lambda_{nm}^2 at).$$

Here,

$$A_s = \begin{cases} 1 & \text{for } s = 0, \\ 2 & \text{for } s \neq 0, \end{cases} \qquad B_{nms} = \frac{(n+s)!}{(2n+1)(n-s)!} \int_{R_1}^{R_2} r Z^2_{n+1/2}(\lambda_{nm} r)\, dr,$$

$$Z_{n+1/2}(\lambda r) = \left[\lambda J'_{n+1/2}(\lambda R_1) - \left(k_1 + \frac{1}{2R_1}\right) J_{n+1/2}(\lambda R_1)\right] Y_{n+1/2}(\lambda r) - \left[\lambda Y'_{n+1/2}(\lambda R_1) - \left(k_1 + \frac{1}{2R_1}\right) Y_{n+1/2}(\lambda R_1)\right] J_{n+1/2}(\lambda r),$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2) + \left(k_2 - \frac{1}{2R_2}\right) Z_{n+1/2}(\lambda R_2) = 0.$$

The integrals that determine the coefficients B_{nms} can be expressed in terms of the Bessel functions and their derivatives.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ **Domain: $0 \le r < \infty$, $0 \le \theta \le \theta_0$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A cone is considered. The following conditions are prescribed:

$$w = f(r,\theta,\varphi) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(r,\varphi,t) \quad \text{at} \quad \theta = \theta_0 \quad \text{(boundary condition)}.$$

Solution:

$$w(r,\theta,\varphi,t) = \int_0^{2\pi}\int_0^{\theta_0}\int_0^{\infty} f(\xi,\eta,\zeta)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2 \sin\eta\,d\xi\,d\eta\,d\zeta$$
$$-a\int_0^t\int_0^{2\pi}\int_0^{\infty} g(\xi,\zeta,\tau)\left[\sin\eta\frac{\partial}{\partial\eta}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\theta_0} d\xi\,d\zeta\,d\tau,$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = -\frac{1}{4\pi at\sqrt{r\xi}}\sum_{m=0}^{\infty}\sum_{\nu}\frac{A_m(2\nu+1)}{B_{m\nu}}\exp\left(-\frac{r^2+\xi^2}{4at}\right)I_{\nu+1/2}\left(\frac{r\xi}{2at}\right)$$
$$\times P_\nu^{-m}(\cos\theta)P_\nu^{-m}(\cos\eta)\cos[m(\varphi-\zeta)],$$

$$A_m = \begin{cases} 1 & \text{for } m=0, \\ 2 & \text{for } m\neq 0, \end{cases} \qquad B_{m\nu} = \left[(1-\mu)^2\frac{d}{d\mu}P_\nu^{-m}(\mu)\frac{d}{d\nu}P_\nu^{-m}(\mu)\right]_{\mu=\cos\theta_0}.$$

Here, $P_\nu^{-m}(\mu)$ is the modified Legendre function expressed as

$$P_\nu^{-m}(\mu) = \frac{1}{\Gamma(1+m)}\left(\frac{1-\mu}{1+\mu}\right)^{m/2} F\bigl(-\nu,\,\nu+1,\,1+m;\,\tfrac{1}{2}-\tfrac{1}{2}\mu\bigr),$$

where $F(a,b,c;\mu)$ is the Gaussian hypergeometric function and $\Gamma(z)$ is the gamma function. The summation with respect to ν is performed over all roots of the equation $P_\nu^{-m}(\cos\theta_0) = 0$ that are greater than $-1/2$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

5.2 Heat Equation with Source $\dfrac{\partial w}{\partial t} = a\Delta_3 w + \Phi(x,y,z,t)$

5.2.1 Problems in Cartesian Coordinates

In the Cartesian coordinate system, the three-dimensional heat equation with a volume source has the form

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \Phi(x,y,z,t).$$

It describes three-dimensional unsteady thermal phenomena in quiescent media or solids with constant thermal diffusivity. A similar equation is used to study the corresponding three-dimensional mass transfer processes with constant diffusivity.

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.**

An initial condition is prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta,\tau)\, G(x,y,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{8(\pi a t)^{3/2}} \exp\left[-\frac{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}{4at}\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. First boundary value problem.**

A half-space is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(x,y,z) \quad \text{at} \quad t = 0 \quad \text{(initial condition)}, \\ w &= g(y,z,t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\ &\quad + a\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \Phi(\xi,\eta,\zeta,\tau)\, G(x,y,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{1}{8(\pi a t)^{3/2}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\} \exp\left[-\frac{(y-\eta)^2 + (z-\zeta)^2}{4at}\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Second boundary value problem.

A half-space is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_x w = g(y,z,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(x,y,z,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(\xi,\eta,\zeta)\,G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\ &\quad - a\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\eta,\zeta,\tau)\,G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \Phi(\xi,\eta,\zeta,\tau)\,G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau, \end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{8(\pi a t)^{3/2}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]+\exp\left[-\frac{(x+\xi)^2}{4at}\right]\right\}\exp\left[-\frac{(y-\eta)^2+(z-\zeta)^2}{4at}\right].$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► Domain: $0 \le x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Third boundary value problem.

A half-space is considered. The following conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_x w - kw = g(y,z,t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)}.$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\begin{aligned} G(x,y,z,\xi,\eta,\zeta,t)=\frac{1}{8(\pi a t)^{3/2}}\exp\left[-\frac{(y-\eta)^2+(z-\zeta)^2}{4at}\right]\biggl\{\exp\left[-\frac{(x-\xi)^2}{4at}\right]+\exp\left[-\frac{(x+\xi)^2}{4at}\right] \\ -2k\sqrt{\pi a t}\,\exp\bigl[k^2 at+k(x+\xi)\bigr]\operatorname{erfc}\left(\frac{x+\xi}{2\sqrt{at}}+k\sqrt{at}\right)\biggr\}. \end{aligned}$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$. Different boundary value problems.

1°. The solution $w(x,y,z,t)$ of the first boundary value problem for an infinite layer is given by the formula in Section 5.1.1 (see the first boundary value problem for $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$) with the additional term

$$\int_0^t\int_0^l\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta,\tau)\,G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau, \qquad (1)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x,y,z,t)$ of the second boundary value problem for an infinite layer is given by the formula in Section 5.1.1 (see the second boundary value problem for $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$) with the additional term (1); the Green's function is also taken from Section 5.1.1.

3°. The solution $w(x,y,z,t)$ of the third boundary value problem for an infinite layer is given by the formula in Section 5.1.1 (see the third boundary value problem for $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$) with the additional term (1); the Green's function is also taken from Section 5.1.1.

4°. The solution $w(x,y,z,t)$ of a mixed boundary value problem for an infinite layer is given by the formula in Section 5.1.1 (see the mixed boundary value problem for $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le l$) with the additional term (1); the Green's function is also taken from Section 5.1.1.

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(x,y,z,t)$ of the first boundary value problem for a semiinfinite layer is given by the formula in Section 5.1.1 (see the first boundary value problem for $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^\infty \int_{-\infty}^\infty \Phi(\xi,\eta,\zeta,\tau)\, G(x,y,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{2}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x,y,z,t)$ of the second boundary value problem for a semiinfinite layer is given by the formula in Section 5.1.1 (see the second boundary value problem for $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$) with the additional term (2); the Green's function is also taken from Section 5.1.1.

3°. The solution $w(x,y,z,t)$ of the third boundary value problem for a semiinfinite layer is given by the formula in Section 5.1.1 (see the third boundary value problem for $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$) with the additional term (2); the Green's function is also taken from Section 5.1.1.

4°. The solutions $w(x,y,z,t)$ of mixed boundary value problems for a semiinfinite layer are given by the formulas in Section 5.1.1 (see the mixed boundary value problems for $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z \le l$) with additional terms of the form (2); the Green's function is also taken from Section 5.1.1.

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. Different boundary value problems.**

1°. The solution $w(x,y,z,t)$ of the first boundary value problem for the first octant is given by the formula in Section 5.1.1 (see the first boundary value problem for $0 \le x < \infty$,

$0 \le y < \infty$, $0 \le z < \infty$) with the additional term

$$\int_0^t \int_0^\infty \int_0^\infty \int_0^\infty \Phi(\xi,\eta,\zeta,\tau)\, G(x,y,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{3}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x,y,z,t)$ of the second boundary value problem for the first octant is given by the formula in Section 5.1.1 (see the second boundary value problem for $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$) with the additional term (3); the Green's function is also taken from Section 5.1.1.

3°. The solution $w(x,y,z,t)$ of the third boundary value problem for the first octant is given by the formula in Section 5.1.1 (see the third boundary value problem for $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$) with the additional term (3); the Green's function is also taken from Section 5.1.1.

4°. The solutions $w(x,y,z,t)$ of mixed boundary value problems for the first octant are given by the formulas in Section 5.1.1 (see the mixed boundary value problems for $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$) with additional terms of the form (3); the Green's function is also taken from Section 5.1.1.

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$. Different boundary value problems.**

1°. The solution $w(x,y,z,t)$ of the first boundary value problem in an infinite rectangular domain is given by the formula in Section 5.1.1 (see the first boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$) with the additional term

$$\int_0^t \int_{-\infty}^\infty \int_0^{l_2} \int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau)\, G(x,y,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{4}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x,y,z,t)$ of the second boundary value problem in an infinite rectangular domain is given by the formula in Section 5.1.1 (see the second boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$) with the additional term (4); the Green's function is also taken from Section 5.1.1.

3°. The solution $w(x,y,z,t)$ of the third boundary value problem in an infinite rectangular domain is given by the formula in Section 5.1.1 (see the third boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$) with the additional term (4); the Green's function is also taken from Section 5.1.1.

4°. The solution $w(x,y,z,t)$ of a mixed boundary value problem in an infinite rectangular domain is given by the formula in Section 5.1.1 (see the mixed boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $-\infty < z < \infty$) with the additional term (4); the Green's function is also taken from Section 5.1.1.

► **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z < \infty$. Different boundary value problems.**

1°. The solution $w(x, y, z, t)$ of the first boundary value problem in a semiinfinite rectangular domain is given by the formula of Section 5.1.1 (see the first boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$) with the additional term

$$\int_0^t \int_0^\infty \int_0^{l_2} \int_0^{l_1} \Phi(\xi, \eta, \zeta, \tau)\, G(x, y, z, \xi, \eta, \zeta, t - \tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{5}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x, y, z, t)$ of the second boundary value problem in a semiinfinite rectangular domain is given by the formula in Section 5.1.1 (see the second boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$) with the additional term (5); the Green's function is also taken from Section 5.1.1.

3°. The solution $w(x, y, z, t)$ of the third boundary value problem in a semiinfinite rectangular domain is given by the formula in Section 5.1.1 (see the third boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$) with the additional term (5); the Green's function is also taken from Section 5.1.1.

4°. The solutions $w(x, y, z, t)$ of mixed boundary value problems in a semiinfinite rectangular domain are given by the formulas in Section 5.1.1 (see the mixed boundary value problems for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z < \infty$) with additional terms of the form (5); the Green's function is also taken from Section 5.1.1.

► **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3$. Different boundary value problems.**

1°. The solution $w(x, y, z, t)$ of the first boundary value problem for a rectangular parallelepiped is given by the formula in Section 5.1.1 (see the first boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$) with the additional term

$$\int_0^t \int_0^{l_3} \int_0^{l_2} \int_0^{l_1} \Phi(\xi, \eta, \zeta, \tau)\, G(x, y, z, \xi, \eta, \zeta, t - \tau)\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{6}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x, y, z, t)$ of the second boundary value problem for a rectangular parallelepiped is given by the formula in Section 5.1.1 (see the second boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$) with the additional term (6); the Green's function is also taken from Section 5.1.1.

3°. The solution $w(x, y, z, t)$ of the third boundary value problem for a rectangular parallelepiped is given by the formula in Section 5.1.1 (see the third boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$) with the additional term (6); the Green's function is also taken from Section 5.1.1.

4°. The solutions $w(x,y,z,t)$ of mixed boundary value problems for a rectangular parallelepiped are given by the formulas in Section 5.1.1 (see the mixed boundary value problems for $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$) with additional terms of the form (6); the Green's function is also taken from Section 5.1.1.

⊙ *Literature*: A. G. Butkovskiy (1979), H. S. Carslaw and J. C. Jaeger (1984).

5.2.2 Problems in Cylindrical Coordinates

In the cylindrical coordinate system, the heat equation with a volume source is written as

$$\frac{\partial w}{\partial t} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}\right] + \Phi(r,\varphi,z,t).$$

This representation is used to describe nonsymmetric unsteady thermal (diffusion) processes in quiescent media or solids bounded by cylindrical surfaces and planes.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. First boundary value problem.**

An infinite circular cylinder is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f(r,\varphi,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g(\varphi,z,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) = {} & \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \xi f(\xi,\eta,\zeta)\, G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ & - aR\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial \xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\ & + \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = G_1(r,\varphi,\xi,\eta,t)\,G_2(z,\zeta,t),$$

$$G_1(r,\varphi,\xi,\eta,t) = \frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$

$$G_2(z,\zeta,t) = \frac{1}{2\sqrt{\pi a t}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right], \qquad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n=1,\,2,\,\dots, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Second boundary value problem.**

An infinite circular cylinder is considered. The following conditions are prescribed:

$$w = f(r, \varphi, z) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_r w = g(\varphi, z, t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(r, \varphi, z, t) &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \xi f(\xi, \eta, \zeta)\, G(r, \varphi, z, \xi, \eta, \zeta, t)\, d\xi\, d\eta\, d\zeta \\ &+ aR\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\eta, \zeta, \tau)\, G(r, \varphi, z, R, \eta, \zeta, t-\tau)\, d\eta\, d\zeta\, d\tau \\ &+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \Phi(\xi, \eta, \zeta, \tau)\, G(r, \varphi, z, \xi, \eta, \zeta, t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau. \end{aligned}$$

Here,

$$G(r, \varphi, z, \xi, \eta, \zeta, t) = G_1(r, \varphi, \xi, \eta, t)\, G_2(z, \zeta, t),$$
$$G_1(r, \varphi, \xi, \eta, t) = \frac{1}{\pi R^2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 - n^2)[J_n(\mu_{nm} R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$G_2(z, \zeta, t) = \frac{1}{2\sqrt{\pi at}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right], \qquad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n=1,\ 2,\ \ldots, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Third boundary value problem.**

An infinite circular cylinder is considered. The following conditions are prescribed:

$$w = f(r, \varphi, z) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_r w + kw = g(\varphi, z, t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

The solution $w(r, \varphi, z, t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r, \varphi, z, \xi, \eta, \zeta, t) = G_1(r, \varphi, \xi, \eta, t)\, G_2(z, \zeta, t),$$
$$G_1(r, \varphi, \xi, \eta, t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)}{(\mu_{nm}^2 R^2 + k^2 R^2 - n^2)[J_n(\mu_{nm} R)]^2}\cos[n(\varphi-\eta)]\exp(-\mu_{nm}^2 at),$$
$$G_2(z, \zeta, t) = \frac{1}{2\sqrt{\pi at}}\exp\left[-\frac{(z-\zeta)^2}{4at}\right], \qquad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n=1,\ 2,\ \ldots \end{cases}$$

Here, the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Different boundary value problems.**

1°. The solution of the first boundary value problem for a semiinfinite circular cylinder is given by the formula in Section 5.1.2 (see the first boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with the additional term

$$\int_0^t \int_0^\infty \int_0^{2\pi} \int_0^R \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{1}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution of the second boundary value problem for a semiinfinite circular cylinder is given by the formula in Section 5.1.2 (see the second boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with the additional term (1); the Green's function is also taken from Section 5.1.2.

3°. The solution of the third boundary value problem for a semiinfinite circular cylinder is the sum of the solution presented in Section 5.1.2 (see the third boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) and expression (1); the Green's function is also taken from Section 5.1.2.

4°. The solutions of mixed boundary value problems for a semiinfinite circular cylinder are given by the formulas in Section 5.1.2 (see the mixed boundary value problems for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with additional terms of the form (1); the Green's function is also taken from Section 5.1.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for a circular cylinder of finite length is given by the formula in Section 5.1.2 (see the first boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^{2\pi} \int_0^R \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{2}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\varphi,z,t)$ of the second boundary value problem for a circular cylinder of finite length is given by the formula in Section 5.1.2 (see the second boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term (2); the Green's function is also taken from Section 5.1.2.

3°. The solution $w(r,\varphi,z,t)$ of the third boundary value problem for a circular cylinder of finite length is the sum of the solution presented in Section 5.1.2 (see the third boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) and expression (2); the Green's function is also taken from Section 5.1.2.

4°. The solutions $w(r,\varphi,z,t)$ of mixed boundary value problems for a circular cylinder of finite length are given by the formulas in Section 5.1.2 (see the mixed boundary value problems for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with additional terms of the form (2); the Green's function is also taken from Section 5.1.2.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Different boundary value problems.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for an infinite hollow cylinder is given by the formula in Section 5.1.2 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$) with the additional term

$$\int_0^t \int_{-\infty}^{\infty} \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \qquad (3)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\varphi,z,t)$ of the second boundary value problem for an infinite hollow cylinder is given by the formula in Section 5.1.2 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$) with the additional term (3); the Green's function is also taken from Section 5.1.2.

3°. The solution $w(r,\varphi,z,t)$ of the third boundary value problem for an infinite hollow cylinder is the sum of the solution presented in Section 5.1.2 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$) and expression (3); the Green's function is also taken from Section 5.1.2.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Different boundary value problems.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for a semiinfinite hollow cylinder is given by the formula in Section 5.1.2 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with the additional term

$$\int_0^t \int_0^{\infty} \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \qquad (4)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\varphi,z,t)$ of the second boundary value problem for a semiinfinite hollow cylinder is given by the formula in Section 5.1.2 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with the additional term (4); the Green's function is also taken from Section 5.1.2.

3°. The solution $w(r,\varphi,z,t)$ of the third boundary value problem for a semiinfinite hollow cylinder is the sum of the solution presented in Section 5.1.2 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) and expression (4); the Green's function is also taken from Section 5.1.2.

4°. The solutions $w(r,\varphi,z,t)$ of mixed boundary value problems for a semiinfinite hollow cylinder are given by the formulas in Section 5.1.2 (see the mixed boundary value problems for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$) with additional terms of the form (4); the Green's function is also taken from Section 5.1.2.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for a hollow cylinder of finite length is given by the formula in Section 5.1.2 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{5}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\varphi,z,t)$ of the second boundary value problem for a hollow cylinder of finite length is given by the formula in Section 5.1.2 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term (5); the Green's function is also taken from Section 5.1.2.

3°. The solution $w(r,\varphi,z,t)$ of the third boundary value problem for a hollow cylinder of finite length is the sum of the solution specified in Section 5.1.2 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) and expression (5); the Green's function is also taken from Section 5.1.2.

4°. The solutions $w(r,\varphi,z,t)$ of mixed boundary value problems for a hollow cylinder of finite length are given by the formulas in Section 5.1.2 (see the mixed boundary value problems for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with additional terms of the form (5); the Green's function is also taken from Section 5.1.2.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $-\infty < z < \infty$. Different boundary value problems.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for an infinite wedge domain is given by the formula in Section 5.1.2 (see the first boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $-\infty < z < \infty$) with the additional term

$$\int_0^t \int_{-\infty}^{\infty} \int_0^{\varphi_0} \int_0^{\infty} \Phi(\xi,\eta,\zeta,\tau)\, G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{6}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r, \varphi, z, t)$ of the second boundary value problem for an infinite wedge domain is given by the formula in Section 5.1.2 (see the second boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $-\infty < z < \infty$) with the additional term (6); the Green's function is also taken from Section 5.1.2.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$. Different boundary value problems.**

1°. The solution $w(r, \varphi, z, t)$ of the first boundary value problem for a semiinfinite wedge domain is given by the formula in Section 5.1.2 (see the first boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$) with the additional term

$$\int_0^t \int_0^\infty \int_0^{\varphi_0} \int_0^\infty \Phi(\xi, \eta, \zeta, \tau)\, G(r, \varphi, z, \xi, \eta, \zeta, t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{7}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r, \varphi, z, t)$ of the second boundary value problem for a semiinfinite wedge domain is given by the formula in Section 5.1.2 (see the second boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$) with the additional term (7); the Green's function is also taken from Section 5.1.2.

3°. The solutions $w(r, \varphi, z, t)$ of mixed boundary value problems for a semiinfinite wedge domain are given by the formulas in Section 5.1.2 (see the mixed boundary value problems for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z < \infty$) with additional terms of the form (7); the Green's functions are taken from Section 5.1.2.

► **Domain: $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(r, \varphi, z, t)$ of the first boundary value problem for a wedge domain of finite height is given by the formula in Section 5.1.2 (see the first boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^{\varphi_0} \int_0^\infty \Phi(\xi, \eta, \zeta, \tau)\, G(r, \varphi, z, \xi, \eta, \zeta, t-\tau)\xi\, d\xi\, d\eta\, d\zeta\, d\tau, \tag{8}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r, \varphi, z, t)$ of the second boundary value problem for a wedge domain of finite height is given by the formula in Section 5.1.2 (see the second boundary value problem for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$) with the additional term (8); the Green's function is also taken from Section 5.1.2.

3°. The solutions $w(r, \varphi, z, t)$ of mixed boundary value problems for a wedge domain of finite height are given by the formulas in Section 5.1.2 (see the mixed boundary value problems for $0 \le r < \infty$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$) with additional terms of the form (8); the Green's functions are taken from Section 5.1.2.

► **Different boundary value problems for a cylindrical sector.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for an unbounded cylindrical sector is given by the formula in Section 5.1.2 (see the first boundary value problem for $0\le r\le R$, $0\le\varphi\le\varphi_0$, $-\infty<z<\infty$) with the additional term

$$\int_0^t\int_{-\infty}^{\infty}\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\zeta,\tau)\,G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau,$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for a semibounded cylindrical sector is given by the formula in Section 5.1.2 (see the first boundary value problem for $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z<\infty$) with the additional term

$$\int_0^t\int_0^{\infty}\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\zeta,\tau)\,G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau, \tag{9}$$

which allows for the equation's nonhomogeneity; the Green's function is also taken from Section 5.1.2.

3°. The solution $w(r,\varphi,z,t)$ of the mixed boundary value problem for a semibounded cylindrical sector is given by the formula in Section 5.1.2 (see the mixed boundary value problem for $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z<\infty$) with the additional term (9); the Green's function is also taken from Section 5.1.2.

4°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for a cylindrical sector of finite height is given by the formula in Section 5.1.2 (see the first boundary value problem for $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z\le l$) with the additional term

$$\int_0^t\int_0^{l}\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\zeta,\tau)\,G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau, \tag{10}$$

which allows for the equation's nonhomogeneity; the Green's function is also taken from Section 5.1.2.

5°. The solution $w(r,\varphi,z,t)$ of a mixed boundary value problem for a cylindrical sector of finite height is given by the formula in Section 5.1.2 (see the mixed boundary value problem for $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z\le l$) with the additional term (10); the Green's function is also taken from Section 5.1.2.

5.2.3 Problems in Spherical Coordinates

In the spherical coordinate system, the heat equation with a volume source has the form

$$\frac{\partial w}{\partial t}=a\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}\right]+\Phi(r,\theta,\varphi,t).$$

One-dimensional problems with central symmetry that have solutions of the form $w=w(r,t)$ are discussed in Section 3.2.4.

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$w = f(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$w = g(\theta,\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \int_0^{2\pi}\!\int_0^{\pi}\!\int_0^R f(\xi,\eta,\zeta)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta \\ &\quad - aR^2\int_0^t\!\int_0^{2\pi}\!\int_0^{\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}\sin\eta\,d\eta\,d\zeta\,d\tau \\ &\quad + \int_0^t\!\int_0^{2\pi}\!\int_0^{\pi}\!\int_0^R \Phi(\xi,\eta,\zeta,\tau)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(r,\theta,\varphi,\xi,\eta,\zeta,t) &= \frac{1}{2\pi R^2\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n} A_k B_{nmk} J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi) \\ &\quad \times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\exp(-\lambda_{nm}^2 at), \end{aligned}$$

$$A_k = \begin{cases} 1 & \text{for } k=0, \\ 2 & \text{for } k\ne 0, \end{cases} \qquad B_{nmk} = \frac{(2n+1)(n-k)!}{(n+k)!\left[J'_{n+1/2}(\lambda_{nm}R)\right]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as follows:

$$P_n^k(\mu) = (1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_n(\mu), \qquad P_n(\mu) = \frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n;$$

and the λ_{nm} are positive roots of the transcendental equation $J_{n+1/2}(\lambda R) = 0$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$w = f(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},$$
$$\partial_r w = g(\theta,\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \int_0^{2\pi}\!\int_0^{\pi}\!\int_0^R f(\xi,\eta,\zeta)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta \\ &\quad + aR^2\int_0^t\!\int_0^{2\pi}\!\int_0^{\pi} g(\eta,\zeta,\tau)\,G(r,\theta,\varphi,R,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau \\ &\quad + \int_0^t\!\int_0^{2\pi}\!\int_0^{\pi}\!\int_0^R \Phi(\xi,\eta,\zeta,\tau)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau, \end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{3}{4\pi R^3}+\frac{1}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\exp(-\lambda_{nm}^2at),$$
$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{\lambda_{nm}^2(2n+1)(n-k)!}{(n+k)!\big[R^2\lambda_{nm}^2-n(n+1)\big]\big[J_{n+1/2}(\lambda_{nm}R)\big]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the previous paragraph), and the λ_{nm} are positive roots of the transcendental equation

$$2\lambda RJ'_{n+1/2}(\lambda R)-J_{n+1/2}(\lambda R)=0.$$

► **Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. Third boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$w=f(r,\theta,\varphi)\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$\partial_r w+kw=g(\theta,\varphi,t)\quad\text{at}\quad r=R\quad\text{(boundary condition)}.$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n}A_sB_{nms}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\exp(-\lambda_{nm}^2at),$$
$$A_s=\begin{cases}1 & \text{if } s=0,\\ 2 & \text{if } s\neq 0,\end{cases}\quad B_{nms}=\frac{\lambda_{nm}^2(2n+1)(n-s)!}{(n+s)!\big[R^2\lambda_{nm}^2+(kR+n)(kR-n-1)\big]\big[J_{n+1/2}(\lambda_{nm}R)\big]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda RJ'_{n+1/2}(\lambda R)+\big(kR-\tfrac{1}{2}\big)J_{n+1/2}(\lambda R)=0.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1\le r\le R_2$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. Different boundary value problems.**

1°. The solution $w(r,\theta,\varphi,t)$ of the first boundary value problem for a spherical layer is given by the formula in Section 5.1.3 (see the first boundary value problem for $R_1\le r\le R_2$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$) with the additional term

$$\int_0^t\int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)\,G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau,\qquad(1)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions. The Green's function is also taken from Section 5.1.3.

2°. The solution $w(r,\theta,\varphi,t)$ of the second boundary value problem for a spherical layer is given by the formula in Section 5.1.3 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) with the additional term (1); the Green's function is also taken from Section 5.1.3.

3°. The solution $w(r,\theta,\varphi,t)$ of the third boundary value problem for a spherical layer is the sum of the solution specified in Section 5.1.3 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) and expression (1); the Green's function is also taken from Section 5.1.3.

► **Domain: $0 \le r < \infty$, $0 \le \theta \le \theta_0$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

The solution $w(r,\theta,\varphi,t)$ of the first boundary value problem for an infinite cone is given by the formula in Section 5.1.3 (see the first boundary value problem for $0 \le r < \infty$, $0 \le \theta \le \theta_0$, $0 \le \varphi \le 2\pi$) with the additional term

$$\int_0^t \int_0^{2\pi} \int_0^{\theta_0} \int_0^\infty \Phi(\xi,\eta,\zeta,\tau)\, G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta\, d\tau,$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions. The Green's function is also taken from Section 5.1.3.

5.3 Other Equations with Three Space Variables

5.3.1 Equations Containing Arbitrary Parameters

1. $\displaystyle \frac{\partial w}{\partial t} = a\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) + (b_1 x + b_2 y + b_3 z + c)w.$

The transformation

$$w(x,y,z,t) = \exp\big[(b_1x + b_2y + b_3z)t + \tfrac{1}{3}a(b_1^2 + b_2^2 + b_3^2)t^3 + ct\big]u(\xi,\eta,\zeta,t),$$
$$\xi = x + ab_1t^2, \quad \eta = y + ab_2t^2, \quad \zeta = z + ab_3t^2$$

leads to the three-dimensional heat equation $\partial_t u = a(\partial_{\xi\xi}u + \partial_{\eta\eta}u + \partial_{\zeta\zeta}u)$ that is dealt with in Section 5.1.1.

2. $\displaystyle \frac{\partial w}{\partial t} = a\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) - \big[b(x^2+y^2+z^2) + c\big]w, \qquad b > 0.$

The transformation (A is any number)

$$w(x,y,z,t) = \exp\left[\frac{\sqrt{ab}}{2a}(x^2+y^2+z^2) + \big(3\sqrt{ab} - c\big)\,t\right]u(\xi,\eta,\zeta,\tau),$$
$$\xi = x\exp\big(2\sqrt{ab}\,t\big), \quad \eta = y\exp\big(2\sqrt{ab}\,t\big),$$
$$\zeta = z\exp\big(2\sqrt{ab}\,t\big), \quad \tau = \frac{1}{4\sqrt{ab}}\exp\big(4\sqrt{ab}\,t\big) + A$$

leads to the three-dimensional heat equation $\partial_\tau u = a(\partial_{\xi\xi}u + \partial_{\eta\eta}u + \partial_{\zeta\zeta}u)$ that is dealt with in Section 5.1.1.

3. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) + \left[-b(x^2+y^2+z^2) + c_1 x + c_2 y + c_3 z + s\right] w.$

This is a special case of equation 5.3.2.3 with $f_k(t) = c_k$, $g(t) = s$.

4. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) + b_1\dfrac{\partial w}{\partial x} + b_2\dfrac{\partial w}{\partial y} + b_3\dfrac{\partial w}{\partial z} + cw.$

This equation governs the nonstationary temperature (concentration) field in a medium moving with a constant velocity, provided there is volume release (absorption) of heat proportional to temperature (concentration).

The substitution

$$w(x,y,z,t) = \exp\bigl(A_1 x + A_2 y + A_3 z + Bt\bigr) U(x,y,z,t),$$

where

$$A_1 = -\frac{b_1}{2a}, \quad A_2 = -\frac{b_2}{2a}, \quad A_3 = -\frac{b_3}{2a}, \quad B = c - \frac{1}{4a}\bigl(b_1^2 + b_2^2 + b_3^2\bigr),$$

leads to the three-dimensional heat equation $\partial_t U = a\Delta_3 U$ that is considered in Section 5.1.1.

5. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) - by\dfrac{\partial w}{\partial x}.$

This equation is encountered in problems of convective heat and mass transfer in a simple shear flow.

Fundamental solution:

$$\begin{aligned}\mathscr{E}(x,y,z,\xi,\eta,\zeta,t) = {}& \frac{1}{(4\pi a t)^{3/2}\bigl(1 + \frac{1}{12}b^2t^2\bigr)^{1/2}} \\ & \times \exp\left\{-\frac{\bigl[x - \xi - \frac{1}{2}bt(y+\eta)\bigr]^2}{4at\bigl(1 + \frac{1}{12}b^2t^2\bigr)} - \frac{(y-\eta)^2 + (z-\zeta)^2}{4at}\right\}.\end{aligned}$$

⊙ *Literature*: E. A. Novikov (1958).

6. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) + b_1 x\dfrac{\partial w}{\partial x} + b_2 y\dfrac{\partial w}{\partial x} + b_3 z\dfrac{\partial w}{\partial z} + \Phi(x,y,z,t).$

This equation is encountered in problems of convective heat and mass transfer in a linear shear flow.

Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.
An initial condition is prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned}w(x,y,z,t) = {}& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta,t)\, dx\, dy\, dz \\ & + \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta,\tau)\, G(x,y,z,\xi,\eta,\zeta,t-\tau)\, dx\, dy\, dz\, d\tau,\end{aligned}$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = H(x,\xi,t;b_1)H(y,\eta,t;b_2)H(z,\zeta,t;b_3),$$
$$H(x,\xi,t;b) = \left[\frac{2\pi a}{b}(e^{2bt}-1)\right]^{-1/2} \exp\left[-\frac{b\big(xe^{bt}-\xi\big)^2}{2a(e^{2bt}-1)}\right].$$

7. $\dfrac{\partial w}{\partial t} = a\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\Big) + (b_1x+c_1)\dfrac{\partial w}{\partial x} + (b_2y+c_2)\dfrac{\partial w}{\partial y} + (b_3z+c_3)\dfrac{\partial w}{\partial z} + (s_1x+s_2y+s_3z+p)w.$

This is a special case of equation 5.3.2.5.

8. $i\hbar\dfrac{\partial w}{\partial t} + \dfrac{\hbar^2}{2m}\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\Big) = 0.$

Three-dimensional Schrödinger equation, $i^2 = -1$.

Fundamental solution:

$$\mathscr{E}(x,y,z,t) = -\frac{i}{\hbar}\left(\frac{m}{2\pi\hbar t}\right)^{3/2} \exp\left[i\frac{m}{2\hbar t}(x^2+y^2+z^2) - i\frac{3\pi}{4}\right].$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\Big(ax^n\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(by^m\dfrac{\partial w}{\partial y}\Big) + \dfrac{\partial}{\partial z}\Big(cz^k\dfrac{\partial w}{\partial z}\Big).$

This equation describes unsteady heat and mass transfer processes in inhomogeneous (anisotropic) media. It admits separable solutions, as well as solutions with incomplete separation of variables (see Section 17.5.2). In addition, for $n \neq 2$, $m \neq 2$, $k \neq 2$ there are particular solutions of the form

$$w = w(\xi,t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(\xi,t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi}, \qquad A = 2\left(\frac{1}{2-n} + \frac{1}{2-m} + \frac{1}{2-k}\right) - 1.$$

For solutions of this equation, see Sections 3.2.1, 3.2.3, and 3.2.5.

5.3.2 Equations Containing Arbitrary Functions

1. $\dfrac{\partial w}{\partial t} = a\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\Big) + f(t)w.$

This equation describes three-dimensional unsteady thermal phenomena in quiescent media or solids with constant thermal diffusivity, provided there is unsteady volume heat release proportional to temperature.

The substitution $w(x,y,z,t) = \exp\big[\int f(t)\,dt\big]U(x,y,z,t)$ leads to the usual heat equation $\partial_t U = a\Delta_3 U$ that is dealt with in Section 5.1.1.

2. $\dfrac{\partial w}{\partial t} = a\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\Big) + \big[xf_1(t) + yf_2(t) + zf_3(t) + g(t)\big]w.$

The transformation

$$w(x,y,z,t) = u(\xi,\eta,\zeta,t)\exp\bigg\{xF_1(t) + yF_2(t) + zF_3(t) + a\int\big[F_1^2(t) + F_2^2(t) + F_3^2(t)\big]\,dt + \int g(t)\,dt\bigg\},$$
$$\xi = x + 2a\int F_1(t)\,dt, \quad \eta = y + 2a\int F_2(t)\,dt,$$
$$\zeta = z + 2a\int F_3(t)\,dt, \quad F_k(t) = \int f_k(t)\,dt,$$

leads to the three-dimensional heat equation $\partial_t u = a\big(\partial_{\xi\xi}u + \partial_{\eta\eta}u + \partial_{\zeta\zeta}u\big)$ that is dealt with in Section 5.1.1.

3. $\dfrac{\partial w}{\partial t} = a\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\Big) + \big[-b(x^2+y^2+z^2) + xf_1(t) + yf_2(t) + zf_3(t) + g(t)\big]w.$

1°. Case $b > 0$. The transformation

$$w(x,y,z,t)=u(\xi,\eta,\zeta,\tau)\exp\bigg[\frac{\sqrt{ab}}{2a}\,(x^2+y^2+z^2)\bigg],$$
$$\xi=x\exp\big(2\sqrt{ab}\,t\big), \quad \eta=y\exp\big(2\sqrt{ab}\,t\big),$$
$$\zeta=z\exp\big(2\sqrt{ab}\,t\big), \quad \tau=\frac{1}{4\sqrt{ab}}\exp\big(4\sqrt{ab}\,t\big)+A,$$

where A is an arbitrary constant, leads to an equation of the form 5.3.2.2:

$$\frac{\partial u}{\partial t} = a\bigg(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2}\bigg) + \big[\xi F_1(\tau) + \eta F_2(\tau) + \zeta F_3(\tau) + G(\tau)\big]u,$$
$$F_k(\tau) = \frac{1}{(c\tau)^{3/2}} f_k\bigg(\frac{\ln(c\tau)}{c}\bigg), \quad G(\tau) = \frac{1}{c\tau} g\bigg(\frac{\ln(c\tau)}{c}\bigg) + \frac{3}{4\tau}, \quad c = 4\sqrt{ab}, \quad k = 1,\ 2,\ 3.$$

2°. Case $b < 0$. The transformation

$$w(x,y,z,t) = v(\xi_1,\eta_1,\zeta_1,\tau_1)\exp\bigg[\frac{\sqrt{-ab}}{2a}(x^2+y^2+z^2)\tan\big(2\sqrt{-ab}\,t\big)\bigg],$$
$$\xi_1 = \frac{x}{\cos\big(2\sqrt{-ab}\,t\big)}, \quad \eta_1 = \frac{y}{\cos\big(2\sqrt{-ab}\,t\big)},$$
$$\zeta_1 = \frac{z}{\cos\big(2\sqrt{-ab}\,t\big)}, \quad \tau_1 = \frac{1}{2\sqrt{-ab}}\tan\big(2\sqrt{-ab}\,t\big)$$

also leads to an equation of the form 5.3.2.2 for $v = v(\xi_1,\eta_1,\zeta_1,\tau_1)$ (the equation for v is not written out here).

4. $\displaystyle \frac{\partial w}{\partial t} = a_1(t)\frac{\partial^2 w}{\partial x^2} + a_2(t)\frac{\partial^2 w}{\partial y^2} + a_3(t)\frac{\partial^2 w}{\partial z^2} + \Phi(x, y, z, t).$

Here, $0 < a_k(t) < \infty$; $k = 1,\ 2,\ 3$.

Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem. An initial condition is prescribed:

$$w = f(x, y, z) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned} w(x, y, z, t) &= \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi, \eta, \zeta, \tau)\, G(x, y, z, \xi, \eta, \zeta, t, \tau)\, d\xi\, d\eta\, d\zeta\, d\tau \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta, \zeta)\, G(x, y, z, \xi, \eta, \zeta, t, 0)\, d\xi\, d\eta\, d\zeta, \end{aligned}$$

where

$$G(x, y, z, \xi, \eta, \zeta, t, \tau) = \frac{1}{8\pi^{3/2}\sqrt{T_1 T_2 T_3}} \exp\left[-\frac{(x-\xi)^2}{4T_1} - \frac{(y-\eta)^2}{4T_2} - \frac{(z-\zeta)^2}{4T_3}\right],$$

$$T_1 = \int_\tau^t a_1(\sigma)\, d\sigma, \quad T_2 = \int_\tau^t a_2(\sigma)\, d\sigma, \quad T_3 = \int_\tau^t a_3(\sigma)\, d\sigma.$$

See also the more general equation 5.4.3.3, where other boundary value problems are considered.

5. $\displaystyle \frac{\partial w}{\partial t} = a_1(t)\frac{\partial^2 w}{\partial x^2} + a_2(t)\frac{\partial^2 w}{\partial y^2} + a_3(t)\frac{\partial^2 w}{\partial z^2}$
$\displaystyle \qquad + \big[b_1(t)x + c_1(t)\big]\frac{\partial w}{\partial x} + \big[b_2(t)y + c_2(t)\big]\frac{\partial w}{\partial y} + \big[b_3(t)z + c_3(t)\big]\frac{\partial w}{\partial z}$
$\displaystyle \qquad + \big[s_1(t)x + s_2(t)y + s_3(t)z + p(t)\big]w.$

The transformation

$$\begin{aligned} &w(x, y, z, t) = \exp\big[f_1(t)x + f_2(t)y + f_3(t)z + g(t)\big]u(\xi, \eta, \zeta, t), \\ &\xi = h_1(t)x + r_1(t), \quad \eta = h_2(t)y + r_2(t), \quad \zeta = h_3(t)z + r_3(t), \end{aligned}$$

where

$$\begin{aligned} h_k(t) &= A_k \exp\left[\int b_k(t)\, dt\right], \\ f_k(t) &= h_k(t)\int \frac{s_k(t)}{h_k(t)}\, dt + B_k h_k(t), \\ r_k(t) &= \int \big[2a_k(t)f_k(t) + c_k(t)\big]h_k(t)\, dt + C_k, \\ g(t) &= \int \sum_{k=1}^{3}\big[a_k(t)f_k^2(t) + c_k(t)f_k(t)\big]\, dt + \int p(t)\, dt + D, \end{aligned}$$

($k = 1,\ 2,\ 3$; A_k, B_k, C_k, D are arbitrary constants) leads to an equation of the form 5.3.2.4:

$$\frac{\partial u}{\partial t} = a_1(t)h_1^2(t)\frac{\partial^2 u}{\partial \xi^2} + a_2(t)h_2^2(t)\frac{\partial^2 u}{\partial \eta^2} + a_3(t)h_3^2(t)\frac{\partial^2 u}{\partial \zeta^2}.$$

6. $\dfrac{\partial w}{\partial t}=a_1(t)\dfrac{\partial^2 w}{\partial x^2}+a_2(t)\dfrac{\partial^2 w}{\partial y^2}+a_3(t)\dfrac{\partial^2 w}{\partial z^2}$
$+\big[b_1(t)x+c_1(t)\big]\dfrac{\partial w}{\partial x}+\big[b_2(t)y+c_2(t)\big]\dfrac{\partial w}{\partial y}+\big[b_3(t)z+c_3(t)\big]\dfrac{\partial w}{\partial z}$
$+\big[s_1(t)x^2+s_2(t)y^2+s_3(t)z^2+p_1(t)x+p_2(t)y+p_3(t)z+q(t)\big]w.$

The substitution

$$w(x,y,z,t)=\exp\big[f_1(t)x^2+f_2(t)y^2+f_3(t)z^2\big]u(x,y,z,t),$$

where the functions $f_k=f_k(t)$ are solutions of the respective Riccati equations

$$f_k'=4a_k(t)f_k^2+2b_k(t)f_k+s_k(t)\qquad (k=1,\,2,\,3),$$

leads to an equation of the form 5.3.2.5 for $u=u(x,y,z,t)$.

7. $\dfrac{\partial w}{\partial t}+\sum\limits_{n=1}^{2}\big[f_n(t)+g_n(t)x_3\big]\dfrac{\partial w}{\partial x_n}-a\dfrac{\partial w}{\partial x_3}=\sum\limits_{n,m=1}^{2}K_{nm}(t)\dfrac{\partial^2 w}{\partial x_n\partial x_m}+K_{33}(t)\dfrac{\partial^2 w}{\partial x_3^2}.$

Equation of turbulent diffusion. It describes the diffusion of an admixture in a horizontal stream whose velocity components are linear functions of the height.

Fundamental solution:

$$\mathscr{E}(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3)=\frac{1}{(4\pi)^{3/2}\sqrt{\det|T|}}\exp\biggl[-\frac{1}{4}\sum_{i,j=1}^{3}T_{ij}^{-1}(t)y_iy_j\biggr],$$
$$T_{ij}(t)=\int_0^t S_{ij}(\tau)\,d\tau.$$

Here, the following notation is used ($n,\ m=1,\ 2$):

$$y_n=x_n-\xi_n-F_n(t)-x_3G_n(t)-a\int_0^t(t-\tau)g_n(\tau)\,d\tau,\quad y_3=x_3-\xi_3+at,$$
$$S_{nm}(t)=K_{nm}(t)+K_{33}(t)\,G_n(t)\,G_m(t),\quad S_{n3}(t)=S_{3n}(t)=-K_{33}(t)\,G_n(t),$$
$$S_{33}(t)=K_{33}(t),\quad F_n(t)=\int_0^t f_n(t)\,dt,\quad G_n(t)=\int_0^t g_n(t)\,dt,$$

where $\det|T|$ is the determinant of the matrix **T** with entries $T_{ij}(t)$ and $T_{ij}^{-1}(t)$ are the entries of the inverse of **T**. The inequalities $T_{11}(t)>0$, $T_{11}(t)T_{22}(t)-T_{12}^2(t)>0$, and $\det|T|>0$ are assumed to hold.

⊙ *Literature*: E. A. Novikov (1958).

5.3.3 Equations of the Form $\rho(x,y,z)\dfrac{\partial w}{\partial t}=\operatorname{div}[a(x,y,z)\nabla w]-q(x,y,z)w+\Phi(x,y,z,t)$

Equations of this form are often encountered in the theory of heat and mass transfer. For brevity, the following notation is used:

$$\operatorname{div}\big[a(\mathbf{r})\nabla w\big]=\frac{\partial}{\partial x}\biggl[a(\mathbf{r})\frac{\partial w}{\partial x}\biggr]+\frac{\partial}{\partial y}\biggl[a(\mathbf{r})\frac{\partial w}{\partial y}\biggr]+\frac{\partial}{\partial z}\biggl[a(\mathbf{r})\frac{\partial w}{\partial z}\biggr],\qquad \mathbf{r}=\{x,y,z\}.$$

The problems presented in this subsection are assumed to refer to a simply connected bounded domain V with smooth boundary S. It is also assumed that $\rho(\mathbf{r}) > 0$, $a(\mathbf{r}) > 0$, and $q(\mathbf{r}) \geq 0$.

► **First boundary value problem.**

The following conditions are prescribed:

$$w = f(\mathbf{r}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(\mathbf{r}, t) \quad \text{for} \quad \mathbf{r} \in S \quad \text{(boundary condition)}.$$

Solution:

$$\begin{aligned} w(\mathbf{r}, t) = {} & \int_0^t \int_V \Phi(\boldsymbol{\xi}, \tau) \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t-\tau)\, dV_\xi\, d\tau + \int_V f(\boldsymbol{\xi}) \rho(\boldsymbol{\xi}) \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t)\, dV_\xi \\ & - \int_0^t \int_S g(\boldsymbol{\xi}, \tau) a(\boldsymbol{\xi}) \left[\frac{\partial}{\partial N_\xi} \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t-\tau) \right] dS_\xi\, d\tau. \end{aligned} \tag{1}$$

Here, the modified Green's function is given by

$$\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t) = \sum_{n=1}^{\infty} \frac{u_n(\mathbf{r}) u_n(\boldsymbol{\xi})}{\|u_n\|^2} \exp(-\lambda_n t), \qquad \|u_n\|^2 = \int_V \rho(\mathbf{r}) u_n^2(\mathbf{r})\, dV, \qquad \boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}, \tag{2}$$

where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the following elliptic second-order equation with a homogeneous boundary condition of the first kind:

$$\operatorname{div}\big[a(\mathbf{r}) \nabla u\big] - q(\mathbf{r}) u + \lambda \rho(\mathbf{r}) u = 0, \tag{3}$$
$$u = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{4}$$

The integration in solution (1) is carried out with respect to ξ_1, ξ_2, ξ_3; $\frac{\partial}{\partial N_\xi}$ denotes the derivative along the outward normal to the surface S with respect to ξ_1, ξ_2, and ξ_3.

General properties of the Sturm–Liouville problem (3)–(4):

1°. There are countably many eigenvalues. All eigenvalues are real and can be ordered so that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$; consequently, there can exist only finitely many negative eigenvalues.

2°. For $\rho(\mathbf{r}) > 0$, $a(\mathbf{r}) > 0$, and $q(\mathbf{r}) \geq 0$, all eigenvalues are positive: $\lambda_n > 0$.

3°. The eigenfunctions are defined up to a constant multiplier. Any two eigenfunctions $u_n(\mathbf{r})$ and $u_m(\mathbf{r})$ corresponding to different eigenvalues λ_n and λ_m are orthogonal with weight $\rho(\mathbf{r})$ in the domain V:

$$\int_V \rho(\mathbf{r}) u_n(\mathbf{r}) u_m(\mathbf{r})\, dV = 0 \quad \text{for} \quad n \neq m.$$

4°. An arbitrary function $F(\mathbf{r})$ that is twice continuously differentiable and satisfies the boundary condition of the Sturm–Liouville problem ($F = 0$ for $\mathbf{r} \in S$) can be expanded into an absolutely and uniformly convergent series in the eigenvalues:

$$F(\mathbf{r}) = \sum_{n=1}^{\infty} F_n u_n(\mathbf{r}), \quad F_n = \frac{1}{\|u_n\|^2} \int_V F(\mathbf{r})\rho(\mathbf{r})u_n(\mathbf{r})\,dV,$$

where the formula for $\|u_n\|^2$ is given in (2).

Remark 5.1. In a three-dimensional problem, to each eigenvalue λ_n there generally correspond finitely many linearly independent eigenfunctions $u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(m)}$. These function can always be replaced by their linear combinations

$$\bar{u}_n^{(k)} = A_{k,1}u_n^{(1)} + \dots + A_{k,k-1}u_n^{(k-1)} + u_n^{(k)}, \qquad k = 1,\, 2,\, \dots,\, m,$$

such that $\bar{u}_n^{(1)}, \bar{u}_n^{(2)}, \dots, \bar{u}_n^{(m)}$ are now pairwise orthogonal. Thus, without loss of generality, we assume that all eigenfunctions are orthogonal.

► **Second boundary value problem.**

The following conditions are prescribed:

$$w = f(\mathbf{r}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\frac{\partial w}{\partial N} = g(\mathbf{r}, t) \quad \text{for} \quad \mathbf{r} \in S \quad \text{(boundary condition)}.$$

Solution:

$$w(\mathbf{r}, t) = \int_0^t \int_V \Phi(\boldsymbol{\xi}, \tau)\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t - \tau)\,dV_\xi\,d\tau + \int_V f(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t)\,dV_\xi$$
$$+ \int_0^t \int_S g(\boldsymbol{\xi}, \tau)a(\boldsymbol{\xi})\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t - \tau)\,dS_\xi\,d\tau. \tag{5}$$

Here, the modified Green's function is given by (2), where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the elliptic second-order equation (3) with a homogeneous boundary condition of the second kind,

$$\frac{\partial u}{\partial N} = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{6}$$

For $q(\mathbf{r}) > 0$ the general properties of the eigenvalue problem (3), (6) are the same as for the first boundary value problem (with all $\lambda_n > 0$).

► **Third boundary value problem.**

The following conditions are prescribed:

$$w = f(\mathbf{r}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\frac{\partial w}{\partial N} + k(\mathbf{r})w = g(\mathbf{r}, t) \quad \text{for} \quad \mathbf{r} \in S \quad \text{(boundary condition)}.$$

The solution of the third boundary value problem is given by formulas (5) and (2), where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation (3) with a homogeneous boundary condition of the third kind,

$$\frac{\partial u}{\partial N} + k(\mathbf{r})u = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{7}$$

For $q(\mathbf{r}) \ge 0$ and $k(\mathbf{r}) > 0$ the general properties of the eigenvalue problem (3), (7) are the same as for the first boundary value problem.

Let $k(\mathbf{r}) = k = \text{const}$. Denote the Green's functions of the second and third boundary value problems by $G_2(\mathbf{r}, \boldsymbol{\xi}, t)$ and $G_3(\mathbf{r}, \boldsymbol{\xi}, t, k)$, respectively. If $q(\mathbf{r}) > 0$, then the following limiting relation holds: $G_2(\mathbf{r}, \boldsymbol{\xi}, t) = \lim\limits_{k\to 0} G_3(\mathbf{r}, \boldsymbol{\xi}, t, k)$.

⊙ *Literature for Section* 5.3.3: V. S. Vladimirov (1988), A. D. Polyanin (2000a, 2000c).

5.4 Equations with n Space Variables

5.4.1 Equations of the Form $\dfrac{\partial w}{\partial t} = a\Delta_n w + \Phi(x_1, \dots, x_n, t)$

This is an n-dimensional nonhomogeneous heat equation. In the Cartesian system of coordinates, it is represented as

$$\frac{\partial w}{\partial t} = a\sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} + \Phi(\mathbf{x}, t), \qquad \mathbf{x} = \{x_1, \dots, x_n\}.$$

The solutions of various problems for this equation can be constructed on the basis of incomplete separation of variables (see Sections 16.2.2 and 17.5.2) taking into account the results of Sections 3.1.1 and 3.1.2. Some examples of solving such problems can be found below.

► **Homogeneous equation ($\Phi \equiv 0$).**

1°. Particular solutions:

$$w(\mathbf{x}, t) = A\exp\left(\sum_{m=1}^{n} k_m x_m + at\sum_{m=1}^{n} k_m^2\right),$$

$$w(\mathbf{x}, t) = A\exp\left(-at\sum_{m=1}^{n} k_m^2\right)\prod_{m=1}^{n} \cos(k_m x_m + C_m),$$

$$w(\mathbf{x}, t) = A\exp\left(-\sum_{m=1}^{n} k_m x_m\right)\prod_{m=1}^{n} \cos(k_m x_m - 2ak_m^2 t + C_m),$$

$$w(\mathbf{x}, t) = \frac{A}{(t-t_0)^{n/2}}\exp\left[-\frac{1}{4a(t-t_0)}\sum_{m=1}^{n} (x_m - C_m)^2\right],$$

$$w(\mathbf{x}, t) = A\prod_{m=1}^{n} \operatorname{erf}\left(\frac{x_m - C_m}{2\sqrt{at}}\right),$$

where $\mathbf{x} = \{x_1, \ldots, x_n\}$; A, k_m, C_m, and t_0 are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}(\mathbf{x}, t) = \frac{1}{\left(2\sqrt{\pi a t}\,\right)^n} \exp\left(-\frac{|\mathbf{x}|^2}{4at}\right), \qquad |\mathbf{x}|^2 = \sum_{k=1}^n x_k^2.$$

3°. Suppose $w = w(x_1, \ldots, x_n, t)$ is a solution of the homogeneous equation. Then the functions

$$w_1 = Aw(\pm\lambda x_1 + C_1,\ \ldots,\ \pm\lambda x_n + C_n,\ \lambda^2 t + C_{n+1}),$$
$$w_2 = A\exp\left(\sum_{k=1}^n \lambda_k x_k + at\sum_{k=1}^n \lambda_k^2\right) w(x_1 + 2a\lambda_1 t + C_1,\ \ldots,\ x_n + 2a\lambda_n t + C_n,\ t + C_{n+1}),$$
$$w_3 = \frac{A}{|\delta+\beta t|^{n/2}} \exp\left[-\frac{\beta}{4a(\delta+\beta t)} \sum_{k=1}^n x_k^2\right] w\left(\frac{x_1}{\delta+\beta t}, \ldots, \frac{x_n}{\delta+\beta t}, \frac{\gamma+\lambda t}{\delta+\beta t}\right), \quad \lambda\delta - \beta\gamma = 1,$$

where A, $C_1, \ldots, C_{n+1}$, λ, $\lambda_1, \ldots, \lambda_n$ β, and δ are arbitrary constants, are also solutions of the equation. The signs at λ in the formula for w_1 can be taken independently of one another.

▶ Domain: $\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1, \ldots, n\}$. Cauchy problem.

An initial condition is prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0.$$

Solution:

$$w(\mathbf{x}, t) = \frac{1}{\left(2\sqrt{\pi a t}\,\right)^n} \int_{\mathbb{R}^n} f(\mathbf{y}) \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4at}\right) dV_y$$
$$+ \int_0^t \int_{\mathbb{R}^n} \frac{\Phi(\mathbf{y}, \tau)}{\left(2\sqrt{\pi a(t-\tau)}\,\right)^n} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4a(t-\tau)}\right) dV_y\, d\tau,$$

where the following notation is used:

$$\mathbf{y} = \{y_1, \ldots, y_n\}, \ \ |\mathbf{x} - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}, \ \ dV_y = dy_1\, dy_2 \ldots dy_n.$$

⊙ *Literature*: V. S. Vladimirov (1988).

▶ Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\ \ldots,\ n\}$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} &w = f(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ &w = g_k(\mathbf{x}, t) && \text{at} \quad x_k = 0 && \text{(boundary conditions)},\\ &w = h_k(\mathbf{x}, t) && \text{at} \quad x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

Solution:

$$w(\mathbf{x},t) = \int_0^t \int_V \Phi(\mathbf{y},\tau)\, G(\mathbf{x},\mathbf{y},t-\tau)\, dV_y\, d\tau + \int_V f(\mathbf{y})\, G(\mathbf{x},\mathbf{y},t)\, dV_y$$
$$+ a\sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau) \right]_{y_k=0} dS_y^{(k)}\, d\tau$$
$$- a\sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[h_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau) \right]_{y_k=l_k} dS_y^{(k)}\, d\tau,$$

where the following notation is used:

$$dS_y^{(k)} = dy_1 \ldots dy_{k-1}\, dy_{k+1} \ldots dy_n,$$
$$S^{(k)} = \{0 \le y_m \le l_m \ \text{ for } \ m = 1,\ \ldots,\ k-1,\ k+1,\ \ldots,\ n\}.$$

The Green's function can be represented in the product form

$$G(\mathbf{x},\mathbf{y},t) = \prod_{k=1}^n G_k(x_k,y_k,t), \tag{1}$$

where the $G_k(x_k,y_k,t)$ are the Green's functions of the respective one-dimensional boundary value problems (see the first boundary value problem for $0 \le x \le l$ in Section 3.1.2):

$$G_k(x_k,y_k,t) = \frac{2}{l_k} \sum_{m=1}^\infty \sin\left(\frac{m\pi x}{l_k}\right) \sin\left(\frac{m\pi\xi}{l_k}\right) \exp\left(-\frac{am^2\pi^2 t}{l_k^2}\right).$$

► **Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\ \ldots,\ n\}$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_{x_k} w &= g_k(\mathbf{x},t) && \text{at} \quad x_k = 0 && \text{(boundary conditions)},\\ \partial_{x_k} w &= h_k(\mathbf{x},t) && \text{at} \quad x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

Solution:

$$w(\mathbf{x},t) = \int_0^t \int_V \Phi(\mathbf{y},\tau)\, G(\mathbf{x},\mathbf{y},t-\tau)\, dV_y\, d\tau + \int_V f(\mathbf{y})\, G(\mathbf{x},\mathbf{y},t)\, dV_y$$
$$- a\sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y},\tau)\, G(\mathbf{x},\mathbf{y},t-\tau) \right]_{y_k=0} dS_y^{(k)}\, d\tau$$
$$+ a\sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[h_k(\mathbf{y},\tau)\, G(\mathbf{x},\mathbf{y},t-\tau) \right]_{y_k=l_k} dS_y^{(k)}\, d\tau. \tag{2}$$

The Green's function can be represented as the product (1) of the corresponding one-dimensional Green's functions of the form (see the second boundary value problem for $0 \le x \le l$ in Section 3.1.2):

$$G_k(x_k,y_k,t) = \frac{1}{l_k} + \frac{2}{l_k} \sum_{m=1}^\infty \cos\left(\frac{m\pi x}{l_k}\right) \cos\left(\frac{m\pi\xi}{l_k}\right) \exp\left(-\frac{am^2\pi^2 t}{l_k^2}\right).$$

5.4.2 Other Equations Containing Arbitrary Parameters

1. $\dfrac{\partial w}{\partial t} = a\sum_{k=1}^{n}\dfrac{\partial^2 w}{\partial x_k^2} + \Big(c + \sum_{k=1}^{n} b_k x_k\Big)w.$

This is a special case of equation 5.4.3.1. The transformation

$$w(x_1,\dots,x_n,t) = \exp\biggl(t\sum_{k=1}^{n} b_k x_k + \tfrac{1}{3}at^3\sum_{k=1}^{n} b_k^2 + ct\biggr)u(\xi_1,\dots,\xi_n,t), \qquad \xi_k = x_k + ab_k t^2$$

leads to the n-dimensional heat equation $\partial_t u = a\sum_{k=1}^{n}\partial_{\xi_k\xi_k}u$ that is dealt with in Section 5.4.1.

2. $\dfrac{\partial w}{\partial t} = a\sum_{k=1}^{n}\dfrac{\partial^2 w}{\partial x_k^2} - \Big(c + b\sum_{k=1}^{n} x_k^2\Big)w, \qquad b > 0.$

The transformation (A is any number)

$$w(x_1,\dots,x_n,t) = u(\xi_1,\dots,\xi_n,\tau)\exp\biggl[\frac{1}{2}\sqrt{\frac{b}{a}}\sum_{k=1}^{n} x_k^2 + \bigl(n\sqrt{ab} - c\bigr)\,t\biggr],$$

$$\xi_1 = x_1\exp\bigl(2\sqrt{ab}\,t\bigr), \quad \dots, \quad \xi_n = x_n\exp\bigl(2\sqrt{ab}\,t\bigr), \quad \tau = \frac{1}{4\sqrt{ab}}\exp\bigl(4\sqrt{ab}\,t\bigr) + A$$

leads to the n-dimensional heat equation $\partial_\tau u = a\sum_{k=1}^{n}\partial_{\xi_k\xi_k}u$ that is dealt with in Section 5.4.1.

3. $\dfrac{\partial w}{\partial t} = a\sum_{k=1}^{n}\dfrac{\partial^2 w}{\partial x_k^2} + \Big(-b\sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} c_k x_k + s\Big)w.$

This is a special case of equation 5.4.3.2 with $f_k(t) = c_k$ and $g(t) = s$.

4. $\dfrac{\partial w}{\partial t} = a\sum_{k=1}^{n}\dfrac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} b_k\dfrac{\partial w}{\partial x_k} + cw.$

The substitution

$$w(x_1,\dots,x_n,t) = \exp\biggl(At - \frac{1}{2a}\sum_{k=1}^{n} b_k x_k\biggr)U(x_1,\dots,x_n,t), \qquad A = c - \frac{1}{4a}\sum_{k=1}^{n} b_k^2,$$

leads to the n-dimensional heat equation $\partial_t U = a\Delta_n U$ that is dealt with in Section 5.4.1.

5. $\dfrac{\partial w}{\partial t} = a\sum_{k=1}^{n}\dfrac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n}(b_k x_k + c_k)\dfrac{\partial w}{\partial x_k} + \Big(\sum_{k=1}^{n} s_k x_k + p\Big)w.$

This is a special case of equation 5.4.3.4.

6. $\displaystyle i\hbar\frac{\partial w}{\partial t}+\frac{\hbar^2}{2m}\sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2}=0.$

This is the *n-dimensional Schrödinger equation,* $i^2=-1$.

Fundamental solution:

$$\mathscr{E}(\mathbf{x},t)=-\frac{i}{\hbar}\left(\frac{m}{2\pi\hbar t}\right)^{n/2}\exp\left(i\frac{m}{2\hbar t}|\mathbf{x}|^2-i\frac{\pi n}{4}\right),\qquad |\mathbf{x}|^2=x_1^2+\cdots+x_n^2.$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

5.4.3 Equations Containing Arbitrary Functions

1. $\displaystyle \frac{\partial w}{\partial t}=a\sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2}+\Big[\sum_{k=1}^{n}x_k f_k(t)+g(t)\Big]w.$

The transformation

$$w(x_1,\dots,x_n,t)=\exp\left[\sum_{k=1}^{n}x_kF_k(t)+a\sum_{k=1}^{n}\int F_k^2(t)\,dt+G(t)\right]u(\xi_1,\dots,\xi_n,t),$$

$$\xi_k=x_k+2a\int F_k(t)\,dt,\qquad F_k(t)=\int f_k(t)\,dt,\quad G(t)=\int g(t)\,dt,$$

leads to the n-dimensional heat equation $\partial_t u=a\sum\limits_{k=1}^{n}\partial_{\xi_k\xi_k}u$ that is discussed in Section 5.4.1.

2. $\displaystyle \frac{\partial w}{\partial t}=a\sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2}+\Big[-b\sum_{k=1}^{n}x_k^2+\sum_{k=1}^{n}x_k f_k(t)+g(t)\Big]w.$

1°. Case $b>0$. The transformation

$$w(x_1,\dots,x_n,t)=u(\xi_1,\dots,\xi_n,\tau)\exp\left(\frac{1}{2}\sqrt{\frac{b}{a}}\sum_{k=1}^{n}x_k^2\right),$$

$$\xi_1=x_1\exp\big(2\sqrt{ab}\,t\big),\quad \dots,\quad \xi_n=x_n\exp\big(2\sqrt{ab}\,t\big),\quad \tau=\frac{1}{4\sqrt{ab}}\exp\big(4\sqrt{ab}\,t\big)+C,$$

where C is an arbitrary constant, leads to an equation of the form 5.4.3.1:

$$\frac{\partial u}{\partial t}=a\sum_{k=1}^{n}\frac{\partial^2 u}{\partial \xi_k^2}+\left[\sum_{k=1}^{n}\xi_kF_k(\tau)+G(\tau)\right]u,$$

$$F_k(\tau)=\frac{1}{(s\tau)^{3/2}}f_k\left(\frac{\ln(s\tau)}{s}\right),\quad G(\tau)=\frac{1}{s\tau}g\left(\frac{\ln(s\tau)}{s}\right)+\frac{n}{4\tau},\quad s=4\sqrt{ab}.$$

2°. Case $b<0$. The transformation

$$w(x_1,\dots,x_n,t)=v(z_1,\dots,z_n,\tau)\exp\left[\frac{\sqrt{-b}}{2\sqrt{a}}\tan\big(2\sqrt{-ab}\,t\big)\sum_{k=1}^{n}x_k^2\right],$$

$$z_1=\frac{x_1}{\cos\big(2\sqrt{-ab}\,t\big)},\quad \dots,\quad z_n=\frac{x_n}{\cos\big(2\sqrt{-ab}\,t\big)},\quad \tau=\frac{1}{2\sqrt{-ab}}\tan\big(2\sqrt{-ab}\,t\big)$$

also leads to an equation of the form 5.4.3.1 (this equation is not specified here).

3. $\displaystyle \frac{\partial w}{\partial t} = \sum_{k=1}^{n} a_k(t) \frac{\partial^2 w}{\partial x_k^2} + \Phi(x_1, \dots, x_n, t).$

The solutions of various problems for this equation can be constructed on the basis of incomplete separation of variables (see Sections 16.2.2 and 17.5.2) taking into account the results of Sections 3.1.1 and 3.1.2. Some examples of solving such problems are given below. It is assumed that $0 < a_k(t) < \infty$, $k = 1, \dots, n$.

1°. Domain: $\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1, \dots, n\}$. Cauchy problem.
An initial condition is prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0.$$

Solution:

$$w(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \tau)\, G(\mathbf{x}, \mathbf{y}, t, \tau)\, dV_y\, d\tau + \int_{\mathbb{R}^n} f(\mathbf{y})\, G(\mathbf{x}, \mathbf{y}, t, 0)\, dV_y,$$

where

$$G(\mathbf{x}, \mathbf{y}, t, \tau) = \frac{1}{2^n \pi^{n/2} \sqrt{T_1 T_2 \dots T_n}} \exp\biggl[-\sum_{k=1}^{n} \frac{(x_k - y_k)^2}{4T_k}\biggr], \quad T_k = \int_\tau^t a_k(\eta)\, d\eta,$$
$$\mathbf{x} = \{x_1, \dots, x_n\}, \quad \mathbf{y} = \{y_1, \dots, y_n\}, \quad dV_y = dy_1\, dy_2\, \dots\, dy_n.$$

2°. Domain: $V = \{0 \le x_k \le l_k;\ k = 1, \dots, n\}$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g_k(\mathbf{x}, t) && \text{at} \quad x_k = 0 && \text{(boundary conditions)},\\ w &= h_k(\mathbf{x}, t) && \text{at} \quad x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(\mathbf{x}, t) &= \int_0^t \int_V \Phi(\mathbf{y}, \tau)\, G(\mathbf{x}, \mathbf{y}, t, \tau)\, dV_y\, d\tau + \int_V f(\mathbf{y})\, G(\mathbf{x}, \mathbf{y}, t)\, dV_y\\ &\quad + \sum_{k=1}^{n} \int_0^t \int_{S^{(k)}} a_k(\tau) \biggl[g_k(\mathbf{y}, \tau) \frac{\partial}{\partial y_k} G(\mathbf{x}, \mathbf{y}, t, \tau)\biggr]_{y_k=0} dS_y^{(k)}\, d\tau\\ &\quad - \sum_{k=1}^{n} \int_0^t \int_{S^{(k)}} a_k(\tau) \biggl[h_k(\mathbf{y}, \tau) \frac{\partial}{\partial y_k} G(\mathbf{x}, \mathbf{y}, t, \tau)\biggr]_{y_k=l_k} dS_y^{(k)}\, d\tau, \end{aligned}$$

where the following notation is used:

$$\begin{aligned} dS_y^{(k)} &= dy_1 \dots dy_{k-1}\, dy_{k+1} \dots dy_n,\\ S^{(k)} &= \{0 \le y_m \le l_m \text{ for } m = 1, \dots, k-1, k+1, \dots, n\}. \end{aligned}$$

The Green's function can be represented in the product form

$$G(\mathbf{x}, \mathbf{y}, t, \tau) = \prod_{k=1}^{n} G_k(x_k, y_k, t, \tau), \tag{1}$$

where the $G_k(x_k, y_k, t, \tau)$ are the Green's functions of the respective boundary value problems,

$$G_k(x_k, y_k, t, \tau)=\frac{2}{l_k}\sum_{m=1}^{\infty}\sin\Bigl(\frac{m\pi x_k}{l_k}\Bigr)\sin\Bigl(\frac{m\pi y_k}{l_k}\Bigr)\exp\Bigl(-\frac{m^2\pi^2 T_k}{l_k^2}\Bigr),\quad T_k=\int_\tau^t a_k(\sigma)\,d\sigma. \tag{2}$$

3°. Domain: $V=\{0\le x_k\le l_k;\ k=1,\ \dots,\ n\}$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_{x_k} w &= g_k(\mathbf{x},t) &&\text{at}\quad x_k=0 &&\text{(boundary conditions)},\\ \partial_{x_k} w &= h_k(\mathbf{x},t) &&\text{at}\quad x_k=l_k &&\text{(boundary conditions)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(\mathbf{x},t) &= \int_0^t\int_V \Phi(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t,\tau)\,dV_y\,d\tau + \int_V f(\mathbf{y})\,G(\mathbf{x},\mathbf{y},t)\,dV_y\\ &\quad-\sum_{k=1}^n\int_0^t\int_{S^{(k)}} a_k(\tau)\bigl[g_k(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t,\tau)\bigr]_{y_k=0}\,dS_y^{(k)}\,d\tau\\ &\quad+\sum_{k=1}^n\int_0^t\int_{S^{(k)}} a_k(\tau)\bigl[h_k(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t,\tau)\bigr]_{y_k=l_k}\,dS_y^{(k)}\,d\tau.\end{aligned}$$

The Green's function can be represented as the product (1) of the corresponding one-dimensional Green's functions

$$\begin{aligned} G_k(x_k, y_k, t, \tau) &= \frac{1}{l_k}+\frac{2}{l_k}\sum_{m=1}^{\infty}\cos\Bigl(\frac{m\pi x_k}{l_k}\Bigr)\cos\Bigl(\frac{m\pi y_k}{l_k}\Bigr)\exp\Bigl(-\frac{m^2\pi^2 T_k}{l_k^2}\Bigr),\\ T_k &= \int_\tau^t a_k(\sigma)\,d\sigma.\end{aligned}$$

⊙ *Literature*: A. D. Polyanin (2000a, 2000b).

4. $\displaystyle \frac{\partial w}{\partial t}=\sum_{k=1}^n a_k(t)\frac{\partial^2 w}{\partial x_k^2}+\sum_{k=1}^n\bigl[b_k(t)x_k+c_k(t)\bigr]\frac{\partial w}{\partial x_k}+\Bigl[\sum_{k=1}^n s_k(t)x_k+p(t)\Bigr]w.$

Let us perform the transformation

$$w(x_1,\dots,x_n,t)=\exp\Biggl[\sum_{k=1}^n f_k(t)x_k+g(t)\Biggr]u(z_1,\dots,z_n,t),\quad z_k=h_k(t)x_k+r_k(t),$$

where the functions $f_k(t)$, $g(t)$, $h_k(t)$, and $r_k(t)$ are given by the following expressions

(A_k, B_k, C_k, and D are arbitrary constants):

$$
\begin{aligned}
h_k(t) &= A_k \exp\left[\int b_k(t)\,dt\right],\\
f_k(t) &= h_k(t)\int \frac{s_k(t)}{h_k(t)}\,dt + B_k h_k(t),\\
r_k(t) &= \int \bigl[2a_k(t)f_k(t) + c_k(t)\bigr] h_k(t)\,dt + C_k,\\
g(t) &= \int \left[p(t) + \sum_{k=1}^{n} a_k(t) f_k^2(t) + \sum_{k=1}^{n} c_k(t) f_k(t)\right] dt + D.
\end{aligned}
$$

As a result, we arrive at an equation of the form 5.4.3.3 for the new dependent variable $u = u(z_1, \ldots, z_n, t)$:

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{n} a_k(t) h_k^2(t) \frac{\partial^2 u}{\partial z_k^2}.$$

5. $\displaystyle \frac{\partial w}{\partial t} = \sum_{k=1}^{n} a_k(t)\frac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} \bigl[b_k(t)x_k + c_k(t)\bigr]\frac{\partial w}{\partial x_k} + \Bigl[\sum_{k=1}^{n} s_k(t)x_k^2 + \sum_{k=1}^{n} p_k(t)x_k + q(t)\Bigr]w.$

The substitution

$$w(x_1, \ldots, x_n, t) = \exp\left[\sum_{k=1}^{n} f_k(t) x_k^2\right] u(x_1, \ldots, x_n, t),$$

where the functions $f_k = f_k(t)$ are solutions of the Riccati equation

$$f_k' = 4a_k(t) f_k^2 + 2b_k(t) f_k + s_k(t) \qquad (k = 1, \ldots, n),$$

leads to an equation of the form 5.4.3.4 for $u = u(x_1, \ldots, x_n, t)$.

6. $\displaystyle \frac{\partial w}{\partial t} - \sum_{k=1}^{n} \left[a_k(x_k,t)\frac{\partial^2 w}{\partial x_k^2} + b_k(x_k,t)\frac{\partial w}{\partial x_k} + c_k(x_k,t)w\right] = \Phi(x_1, \ldots, x_n, t).$

Here, $0 < a_k(x_k, t) < \infty$ for all k. We introduce the notation $\mathbf{x} = \{x_1, \ldots, x_n\}$, $\mathbf{y} = \{y_1, \ldots, y_n\}$ and consider the domain $V = \{\alpha_k \le x_k \le \beta_k,\ k = 1, \ldots, n\}$, which is an n-dimensional parallelepiped.

1°. First boundary value problem. The following conditions are prescribed:

$$
\begin{aligned}
&w = f(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\
&w = g_k(\mathbf{x}, t) && \text{at} \quad x_k = \alpha_k && \text{(boundary conditions)},\\
&w = h_k(\mathbf{x}, t) && \text{at} \quad x_k = \beta_k && \text{(boundary conditions)}.
\end{aligned}
$$

Solution:

$$w(\mathbf{x},t) = \int_0^t \int_V \Phi(\mathbf{y},\tau)\, G(\mathbf{x},\mathbf{y},t,\tau)\, dV_y\, d\tau + \int_V f(\mathbf{y})\, G(\mathbf{x},\mathbf{y},t,0)\, dV_y$$
$$+ \sum_{k=1}^{n} \int_0^t \int_{S^{(k)}} a_k(\alpha_k,\tau) \left[g_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t,\tau) \right]_{y_k=\alpha_k} dS_y^{(k)}\, d\tau$$
$$- \sum_{k=1}^{n} \int_0^t \int_{S^{(k)}} a_k(\beta_k,\tau) \left[h_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t,\tau) \right]_{y_k=\beta_k} dS_y^{(k)}\, d\tau,$$

where

$$dV_y = dy_1\, dy_2\, \ldots\, dy_n, \qquad dS_y^{(k)} = dy_1 \ldots dy_{k-1}\, dy_{k+1} \ldots dy_n,$$
$$S^{(k)} = \{\alpha_m \le y_m \le \beta_m \ \text{ for } \ m = 1,\, \ldots,\, k-1,\, k+1,\, \ldots,\, n\}.$$

The Green's function can be represented in the product form

$$G(\mathbf{x},\mathbf{y},t,\tau) = \prod_{k=1}^{n} G_k(x_k,y_k,t,\tau). \tag{1}$$

Here, the $G_k = G_k(x_k,y_k,t,\tau)$ are auxiliary Green's functions that, for $t > \tau \ge 0$, satisfy the one-dimensional linear homogeneous equations

$$\frac{\partial G_k}{\partial t} - a_k(x_k,t) \frac{\partial^2 G_k}{\partial x_k^2} - b_k(x_k,t) \frac{\partial G_k}{\partial x_k} - c_k(x_k,t)\, G_k = 0 \qquad (k = 1,\, \ldots,\, n) \tag{2}$$

with nonhomogeneous initial conditions of a special form,

$$G_k = \delta(x_k - y_k) \quad \text{at} \quad t = \tau, \tag{3}$$

and homogeneous boundary conditions of the first kind,

$$G_k = 0 \quad \text{at} \quad x_k = \alpha_k,$$
$$G_k = 0 \quad \text{at} \quad x_k = \beta_k.$$

In determining the function G_k, the quantities y_k and τ play the role of parameters; $\delta(x)$ is the Dirac delta function.

2°. The second and third boundary value problems. The following conditions are prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_{x_k} w - s_k w = g_k(\mathbf{x},t) \quad \text{at} \quad x_k = \alpha_k \quad \text{(boundary conditions)},$$
$$\partial_{x_k} w + p_k w = h_k(\mathbf{x},t) \quad \text{at} \quad x_k = \beta_k \quad \text{(boundary conditions)}.$$

The second boundary value problem corresponds to $s_k = p_k = 0$.

Solution:

$$w(\mathbf{x},t)=\int_0^t\int_V \Phi(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t,\tau)\,dV_y\,d\tau+\int_V f(\mathbf{y})\,G(\mathbf{x},\mathbf{y},t,0)\,dV_y$$
$$-\sum_{k=1}^n\int_0^t\int_{S^{(k)}} a_k(\alpha_k,\tau)\bigl[g_k(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t,\tau)\bigr]_{y_k=\alpha_k}dS_y^{(k)}\,d\tau$$
$$+\sum_{k=1}^n\int_0^t\int_{S^{(k)}} a_k(\beta_k,\tau)\bigl[h_k(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t,\tau)\bigr]_{y_k=\beta_k}dS_y^{(k)}\,d\tau.$$

The Green's function can be represented as the product (1) of the corresponding one-dimensional Green's functions satisfying the linear equations (2) with the initial conditions (3) and the homogeneous boundary conditions

$$\partial_{x_k}G_k-s_kG_k=0\quad\text{at}\quad x_k=\alpha_k,$$
$$\partial_{x_k}G_k+p_kG_k=0\quad\text{at}\quad x_k=\beta_k.$$

⊙ *Literature*: A. D. Polyanin (2000a, 2000b).

7. $\displaystyle\frac{\partial w}{\partial t}=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Bigl[a_{ij}(x_1,\dots,x_n)\frac{\partial w}{\partial x_j}\Bigr]-q(x_1,\dots,x_n)w+\Phi(x_1,\dots,x_n,t).$

The problems considered below are assumed to refer to a bounded domain V with smooth surface S. We introduce the brief notation $\mathbf{x}=\{x_1,\dots,x_n\}$ and assume that the condition

$$\sum_{i,j=1}^n a_{ij}(\mathbf{x})\lambda_i\lambda_j\ge c\sum_{i=1}^n\lambda_i^2,\quad c>0,$$

is satisfied; this condition imposes the requirement that the differential operator on the right-hand side of the equation is elliptic.

1°. First boundary value problem. The following conditions are prescribed:

$$w=f(\mathbf{x})\quad\text{at}\quad t=0\quad\text{(initial condition)},$$
$$w=g(\mathbf{x},t)\quad\text{for}\quad \mathbf{x}\in S\quad\text{(boundary condition)}.$$

Solution:

$$w(\mathbf{x},t)=\int_0^t\int_V \Phi(\mathbf{y},\tau)\,G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau+\int_V f(\mathbf{y})\,G(\mathbf{x},\mathbf{y},t)\,dV_y$$
$$-\int_0^t\int_S g(\mathbf{y},\tau)\Bigl[\frac{\partial}{\partial M_y}G(\mathbf{x},\mathbf{y},t-\tau)\Bigr]\,dS_y\,d\tau.\qquad(1)$$

The Green's function is given by

$$G(\mathbf{x},\mathbf{y},t)=\sum_{n=1}^\infty\frac{u_n(\mathbf{x})u_n(\mathbf{y})}{\|u_n\|^2}\exp(-\lambda_n t),$$
$$\|u_n\|^2=\int_V u_n^2(\mathbf{x})\,dV,\qquad \mathbf{y}=\{y_1,\dots,y_n\},\qquad(2)$$

where the λ_n and $u_n(\mathbf{x})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the elliptic second-order equation with a homogeneous boundary condition of the first kind

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j}\right] - q(\mathbf{x})u + \lambda u = 0, \tag{3}$$

$$u = 0 \quad \text{for} \quad \mathbf{x} \in S. \tag{4}$$

The integration in solution (1) is carried out with respect to $y_1, \ldots, y_n$; $\frac{\partial}{\partial M_y}$ is the differential operator defined as

$$\frac{\partial G}{\partial M_y} \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{y})N_j\frac{\partial G}{\partial y_i}, \tag{5}$$

where $\mathbf{N} = \{N_1, \ldots, N_n\}$ is the unit outward normal to the surface S. In the special case where $a_{ii}(\mathbf{x}) = 1$ and $a_{ij}(\mathbf{x}) = 0$ for $i \neq j$, the operator of (5) coincides with the usual operator of differentiation along the direction of the outward normal to the surface S.

General properties of the Sturm–Liouville problem (3)–(4):

1. There are countably many eigenvalues. All eigenvalues are real and can be ordered so that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$; consequently, there can exist only finitely many negative eigenvalues.

2. For $q(\mathbf{x}) \geq 0$, all eigenvalues are positive: $\lambda_n > 0$.

3. The eigenfunctions are defined up to a constant multiplier. Any two eigenfunctions $u_n(\mathbf{x})$ and $u_m(\mathbf{x})$ corresponding to different eigenvalues λ_n and λ_m are orthogonal in the domain V:

$$\int_V u_n(\mathbf{x})u_m(\mathbf{x})\,dV = 0 \quad \text{for} \quad n \neq m.$$

Remark 5.2. To each eigenvalue λ_n there generally correspond finitely many linearly independent eigenfunctions $u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(m)}$. These functions can always be replaced by their linear combinations

$$\bar{u}_n^{(k)} = A_{k,1}u_n^{(1)} + \cdots + A_{k,k-1}u_n^{(k-1)} + u_n^{(k)}, \qquad k = 1,\, 2,\, \ldots,\, m,$$

such that $\bar{u}_n^{(1)}, \bar{u}_n^{(2)}, \ldots, \bar{u}_n^{(m)}$ are now pairwise orthogonal. Thus, without loss of generality, we assume that all eigenfunctions are orthogonal.

2°. Second boundary value problem. The following conditions are prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$

$$\frac{\partial w}{\partial M_x} = g(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in S \quad \text{(boundary condition)}.$$

Here, the left-hand side of the boundary condition is determined with the help of (5), where G, y, $\mathbf{y}$, and y_k must be replaced by w, x, $\mathbf{x}$, and x_k, respectively.

Solution:

$$w(\mathbf{x}, t) = \int_0^t \int_V \Phi(\mathbf{y}, \tau)\, G(\mathbf{x}, \mathbf{y}, t-\tau)\, dV_y\, d\tau + \int_V f(\mathbf{y})\, G(\mathbf{x}, \mathbf{y}, t)\, dV_y$$
$$+ \int_0^t \int_S g(\mathbf{y}, \tau)\, G(\mathbf{x}, \mathbf{y}, t-\tau)\, dS_y\, d\tau. \tag{6}$$

Here, the Green's function is defined by (2), where the λ_n and $u_n(\mathbf{x})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the elliptic second-order equation (3) with a homogeneous boundary condition of the second kind:

$$\frac{\partial u}{\partial M_x} = 0 \quad \text{for} \quad \mathbf{x} \in S. \tag{7}$$

For $q(\mathbf{x}) > 0$, the general properties of the eigenvalue problem (3), (7) are the same as for the first boundary value problem (see Item 1°). For $q(\mathbf{x}) \equiv 0$ the zero eigenvalue $\lambda_0 = 0$ arises which corresponds to the eigenfunction $u_0 = \text{const}$.

It should be noted that the Green's function of the second boundary value problem can be expressed in terms of the Green's function of the third boundary value problem (see Item 3°).

3°. Third boundary value problem. The following conditions are prescribed:

$$\begin{aligned} w &= f(\mathbf{x}) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)}, \\ \frac{\partial w}{\partial M_x} + k(\mathbf{x})w &= g(\mathbf{x}, t) \quad &&\text{for} \quad \mathbf{x} \in S \quad &&\text{(boundary condition)}. \end{aligned}$$

The solution of the third boundary value problem is given by relations (6) and (2), where the λ_n and $u_n(\mathbf{x})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation (3) with a homogeneous boundary condition of the third kind:

$$\frac{\partial u}{\partial M_x} + k(\mathbf{x})u = 0 \quad \text{for} \quad \mathbf{x} \in S. \tag{8}$$

For $q(\mathbf{x}) \ge 0$ and $k(\mathbf{x}) > 0$, the general properties of the eigenvalue problem (3), (8) are the same as for the first boundary value problem (see Item 1°).

Let $k(\mathbf{x}) = k = \text{const}$. Denote the Green's functions of the second and third boundary value problems by $G_2(\mathbf{x}, \mathbf{y}, t)$ and $G_3(\mathbf{x}, \mathbf{y}, t, k)$, respectively. Then the following relations hold:

$$G_2(\mathbf{x}, \mathbf{y}, t) = \begin{cases} \lim\limits_{k \to 0} G_3(\mathbf{x}, \mathbf{y}, t, k) & \text{if } \; q(\mathbf{x}) > 0; \\ \dfrac{1}{V_0} + \lim\limits_{k \to 0} G_3(\mathbf{x}, \mathbf{y}, t, k) & \text{if } \; q(\mathbf{x}) \equiv 0; \end{cases}$$

where $V_0 = \int_V dV$ is the volume of the domain in question.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), A. D. Polyanin (2000a, 2000b).

Chapter 6

Second-Order Hyperbolic Equations with One Space Variable

6.1 Constant Coefficient Equations

6.1.1 Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2}$

This equation is also known as the *equation of vibration of a string*. It is often encountered in elasticity, aerodynamics, acoustics, and electrodynamics.

► **General solution. Some formulas.**

1°. General solution:

$$w(x,t) = \varphi(x+at) + \psi(x-at),$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary functions.

Physical interpretation: The solution represents two traveling waves that propagate, respectively, to the left and right along the x-axis at a constant speed a.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2a}\vartheta\big(at - |x|\big), \qquad \vartheta(z) = \begin{cases} 0 & \text{for } z \le 0, \\ 1 & \text{for } z > 0. \end{cases}$$

3°. Infinite series solutions containing arbitrary functions of the space variable:

$$w(x,t) = f(x) + \sum_{n=1}^{\infty} \frac{(at)^{2n}}{(2n)!} f_x^{(2n)}(x), \qquad f_x^{(m)}(x) = \frac{d^m}{dx^m} f(x),$$

$$w(x,t) = tg(x) + t \sum_{n=1}^{\infty} \frac{(at)^{2n}}{(2n+1)!} g_x^{(2n)}(x),$$

where $f(x)$ and $g(x)$ are any infinitely differentiable functions. The first solution satisfies the initial conditions $w(x,0) = f(x)$ and $\partial_t w(x,0) = 0$, and the second $w(x,0) = 0$ and $\partial_t w(x,0) = g(x)$. The sums are finite if $f(x)$ and $g(x)$ are polynomials.

4°. Infinite series solutions containing arbitrary functions of time:

$$w(x,t) = f(t) + \sum_{n=1}^{\infty} \frac{1}{a^{2n}(2n)!} x^{2n} f_t^{(2n)}(t), \qquad f_t^{(m)}(t) = \frac{d^m}{dt^m} f(t),$$

$$w(x,t) = xg(t) + x \sum_{n=1}^{\infty} \frac{1}{a^{2n}(2n+1)!} x^{2n} g_t^{(2n)}(t),$$

where $f(t)$ and $g(t)$ are any infinitely differentiable functions. The sums are finite if $f(t)$ and $g(t)$ are polynomials. The first solution satisfies the boundary condition of the first kind $w(0,t) = f(t)$, and the second solution satisfies the boundary condition of the second kind $\partial_x w(0,t) = g(t)$.

5°. If $w(x,t)$ is a solution of the wave equation, then the functions

$$w_1 = Aw(\pm\lambda x + C_1, \pm\lambda t + C_2),$$

$$w_2 = Aw\left(\frac{x - vt}{\sqrt{1-(v/a)^2}}, \frac{t - va^{-2}x}{\sqrt{1-(v/a)^2}}\right),$$

$$w_3 = Aw\left(\frac{x}{x^2 - a^2t^2}, \frac{t}{x^2 - a^2t^2}\right),$$

are also solutions of the equation everywhere these functions are defined (A, C_1, C_2, v, and λ are arbitrary constants). The signs at λ's in the formula for w_1 are taken arbitrarily, independently of each other. The function w_2 results from the invariance of the wave equation under the *Lorentz transformations*.

⊙ *Literature*: G. N. Polozhii (1964), A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $-\infty < x < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$

$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution (*D'Alembert's formula*):

$$w(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi.$$

► **Domain: $0 \le x < \infty$. First boundary value problem.**

1°. Problem with a homogeneous boundary condition:

$$w = f(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$

$$\partial_t w = g(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$

$$w = 0 \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \begin{cases} \dfrac{1}{2}[f(x+at)+f(x-at)] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} g(\xi)\,d\xi & \text{if } t < \dfrac{x}{a}, \\ \dfrac{1}{2}[f(x+at)-f(at-x)] + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} g(\xi)\,d\xi & \text{if } t > \dfrac{x}{a}. \end{cases}$$

2°. Problem with a nonhomogeneous boundary condition:

$$\begin{aligned} w &= f(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= h(t) && \text{at} \quad x=0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \begin{cases} \dfrac{1}{2}[f(x+at)+f(x-at)] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} g(\xi)\,d\xi & \text{if } t < \dfrac{x}{a}, \\ \dfrac{1}{2}[f(x+at)-f(at-x)] + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} g(\xi)\,d\xi + h\Big(t-\dfrac{x}{a}\Big) & \text{if } t > \dfrac{x}{a}. \end{cases}$$

In the domain $t < x/a$ the boundary conditions have no effect on the solution and the expression of $w(x,t)$ coincides with D'Alembert's solution for an infinite line (see the Cauchy problem above).

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► Domain: $0 \le x < \infty$. Second boundary value problem.

1°. Problem with a homogeneous boundary condition:

$$\begin{aligned} w &= f(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w &= 0 && \text{at} \quad x=0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \begin{cases} \dfrac{1}{2}[f(x+at)+f(x-at)] + \dfrac{1}{2a}[G(x+at)-G(x-at)] & \text{if } t < \dfrac{x}{a}, \\ \dfrac{1}{2}[f(x+at)+f(at-x)] + \dfrac{1}{2a}[G(x+at)+G(at-x)] & \text{if } t > \dfrac{x}{a}, \end{cases}$$

where $G(z) = \displaystyle\int_0^z g(\xi)\,d\xi$.

2°. Problem with a nonhomogeneous boundary condition:

$$\begin{aligned} w &= f(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w &= h(t) && \text{at} \quad x=0 \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t)=\begin{cases}\dfrac{1}{2}[f(x+at)+f(x-at)]+\dfrac{1}{2a}[G(x+at)-G(x-at)] & \text{if } t<\dfrac{x}{a},\\ \dfrac{1}{2}[f(x+at)+f(at-x)]+\dfrac{1}{2a}[G(x+at)+G(at-x)]-aH\Big(t-\dfrac{x}{a}\Big) & \text{if } t>\dfrac{x}{a},\end{cases}$$

where $G(z)=\int_0^z g(\xi)\,d\xi$ and $H(z)=\int_0^z h(\xi)\,d\xi$. In the domain $t<x/a$ the boundary conditions have no effect on the solution, and the expression of $w(x,t)$ coincides with D'Alembert's solution for an infinite line (see the Cauchy problem above).

⊙ *Literature*: B. M. Budak, A. N. Tikhonov, and A. A. Samarskii (1980).

► Domain: $0\le x<\infty$. A problem without a initial condition for $\Phi=0$.

The following conditions are prescribed:

$$w=A\cos(\omega t+\gamma)\quad\text{at}\quad x=0,\qquad w\to 0\quad\text{as}\quad x\to 0.$$

Solution:

$$w=Ae^{-\lambda x}\cos(\omega t-\beta x+\gamma),$$

where

$$\lambda=\left(\frac{\sqrt{\omega^2+b^2}-b}{2a}\right)^{1/2},\quad \beta=\left(\frac{\sqrt{\omega^2+b^2}+b}{2a}\right)^{1/2}.$$

► Domain: $0\le x\le l$. First boundary value problem.

1°. Vibration of a string with rigidly fixed ends. The following conditions are prescribed:

$$\begin{aligned} w&=f(x) && \text{at}\quad t=0 && \text{(initial condition)},\\ \partial_t w&=g(x) && \text{at}\quad t=0 && \text{(initial condition)},\\ w&=0 && \text{at}\quad x=0 && \text{(boundary condition)},\\ w&=0 && \text{at}\quad x=l && \text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(x,t)=\sum_{n=1}^{\infty}\bigl[A_n\cos(\lambda_n at)+B_n\sin(\lambda_n at)\bigr]\sin(\lambda_n x),\qquad \lambda_n=\frac{\pi n}{l},$$

$$A_n=\frac{2}{l}\int_0^l f(x)\sin(\lambda_n x)\,dx,\qquad B_n=\frac{2}{a\pi n}\int_0^l g(x)\sin(\lambda_n x)\,dx.$$

Example 6.1. The initial shape of the string is a triangle with base $0\le x\le l$ and height h at $x=c$, i.e.,

$$f(x)=\begin{cases}\dfrac{hx}{c} & \text{if } 0\le x\le c,\\ \dfrac{h(l-x)}{l-c} & \text{if } c\le x\le l.\end{cases}$$

The initial velocities of the string points are zero, $g(x) = 0$.

Solution:

$$w(x,t) = \frac{2hl^2}{\pi^2 c(l-c)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right).$$

Example 6.2. Initially, the string has the shape of a parabola symmetric about the center of the string with elevation h, so that

$$f(x) = \frac{4h}{l^2} x(l-x).$$

The initial velocities of the string points are zero, $g(x) = 0$.

Solution:

$$w(x,t) = \frac{32h}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left[\frac{(2n+1)\pi x}{l}\right] \cos\left[\frac{(2n+1)\pi at}{l}\right].$$

2°. For the solution of the first boundary value problem with a nonhomogeneous boundary condition, see Section 6.1.2 with $\Phi(x,t) \equiv 0$.

⊙ *Literature*: B. M. Budak, A. N. Tikhonov, and A. A. Samarskii (1980), A. V. Bitsadze and D. F. Kalinichenko (1985).

► Domain: $0 \le x \le l$. Second boundary value problem.

1°. Longitudinal vibration of an elastic rod with free ends. The following conditions are prescribed:

$$\begin{aligned}
w &= f(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_t w &= g(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_x w &= 0 && \text{at} \quad x = 0 \quad \text{(boundary condition)},\\
\partial_x w &= 0 && \text{at} \quad x = l \quad \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \bigl[A_n \cos(\lambda_n at) + B_n \sin(\lambda_n at)\bigr] \cos(\lambda_n x),$$

$$\lambda_n = \frac{\pi n}{l}, \qquad A_0 = \frac{1}{l} \int_0^l f(x)\,dx, \qquad B_0 = \frac{1}{l} \int_0^l g(x)\,dx,$$

$$A_n = \frac{2}{l} \int_0^l f(x) \cos(\lambda_n x)\,dx, \qquad B_n = \frac{2}{a\pi n} \int_0^l g(x) \cos(\lambda_n x)\,dx.$$

2°. For the solution of the second boundary value problem with a nonhomogeneous boundary condition, see Section 6.1.2 with $\Phi(x,t) \equiv 0$.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $0 \le x \le l$. Third boundary value problem.**

1°. Longitudinal vibration of an elastic rod with clamped ends in the case of equal stiffness coefficients. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_t w &= g(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_x w - kw &= 0 && \text{at} \quad x = 0 && \text{(boundary condition)},\\
\partial_x w + kw &= 0 && \text{at} \quad x = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
w(x,t) = \sum_{n=1}^{\infty} \bigl[A_n \cos(\lambda_n a t) + B_n \sin(\lambda_n a t)\bigr] \sin(\lambda_n x + \varphi_n),
$$

where

$$
A_n = \frac{1}{\|X_n\|^2} \int_0^l \sin(\lambda_n x + \varphi_n) f(x)\,dx, \quad B_n = \frac{1}{a\lambda_n \|X_n\|^2} \int_0^l \sin(\lambda_n x + \varphi_n) g(x)\,dx,
$$

$$
\varphi_n = \arctan\frac{\lambda_n}{k}, \quad \|X_n\|^2 = \int_0^l \sin^2(\lambda_n x + \varphi_n)\,dx = \frac{l}{2} + \frac{k}{k^2 + \lambda_n^2};
$$

the λ_n are positive roots of the transcendental equation $\cot(\lambda l) = \dfrac{1}{2}\left(\dfrac{\lambda}{k} - \dfrac{k}{\lambda}\right)$.

2°. Longitudinal vibration of an elastic rod with clamped ends in the case of different stiffness coefficients. The following conditions are prescribed:

$$
\begin{aligned}
w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_t w &= g(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_x w - k_1 w &= 0 && \text{at} \quad x = 0 && \text{(boundary condition)},\\
\partial_x w + k_2 w &= 0 && \text{at} \quad x = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
w(x,t) = \sum_{n=1}^{\infty} \bigl[A_n \cos(\lambda_n a t) + B_n \sin(\lambda_n a t)\bigr] \sin(\lambda_n x + \varphi_n),
$$

where

$$
A_n = \frac{1}{\|X_n\|^2} \int_0^l \sin(\lambda_n x + \varphi_n) f(x)\,dx, \quad B_n = \frac{1}{a\lambda_n \|X_n\|^2} \int_0^l \sin(\lambda_n x + \varphi_n) g(x)\,dx,
$$

$$
\varphi_n = \arctan\frac{\lambda_n}{k_1}, \quad \|X_n\|^2 = \int_0^l \sin^2(\lambda_n x + \varphi_n)\,dx = \frac{l}{2} + \frac{(\lambda_n^2 + k_1 k_2)(k_1 + k_2)}{2(\lambda_n^2 + k_1^2)(\lambda_n^2 + k_2^2)};
$$

the λ_n are positive roots of the transcendental equation $\cot(\lambda l) = \dfrac{\lambda^2 - k_1 k_2}{\lambda(k_1 + k_2)}$.

3°. For the solution of the third boundary value problem with nonhomogeneous boundary conditions, see Section 6.1.2 (the third boundary value problem for $0 \le x \le l$ with $\Phi(x,t) \equiv 0$).

⊙ *Literature*: B. M. Budak, A. N. Tikhonov, and A. A. Samarskii (1980).

► **Domain: $0 \le x \le l$. Mixed boundary value problem.**

1°. Longitudinal vibration of an elastic rod with one end rigidly fixed and the other free. The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w &= 0 && \text{at} \quad x = 0 \quad \text{(boundary condition)},\\ \partial_x w &= 0 && \text{at} \quad x = l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \sum_{n=0}^{\infty} \bigl[A_n \cos(\lambda_n at) + B_n \sin(\lambda_n at)\bigr] \sin(\lambda_n x), \qquad \lambda_n = \frac{\pi(2n+1)}{2l},$$

$$A_n = \frac{2}{l} \int_0^l f(x) \sin(\lambda_n x)\, dx, \qquad B_n = \frac{2}{al\lambda_n} \int_0^l g(x) \sin(\lambda_n x)\, dx.$$

2°. For the solution of the mixed boundary value problem with nonhomogeneous boundary conditions, see Section 6.1.2 (the mixed boundary value problem for $0 \le x \le l$ with $\Phi(x,t) \equiv 0$).

⊙ *Literature*: M. M. Smirnov (1975), A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Goursat problem.**

The boundary conditions are prescribed to the equation characteristics:

$$\begin{aligned} w &= f(x) && \text{for} \quad x - at = 0 \quad (0 \le x \le b),\\ w &= g(x) && \text{for} \quad x + at = 0 \quad (0 \le x \le c), \end{aligned}$$

where $f(0) = g(0)$.

Solution:

$$w(x,t) = f\left(\frac{x+at}{2}\right) + g\left(\frac{x-at}{2}\right) - f(0).$$

The solution propagation domain is bounded by four lines:

$$x - at = 0, \quad x + at = 0, \quad x - at = 2c, \quad x + at = 2b.$$

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

6.1.2 Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

► **Domain: $-\infty < x < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0,\\ \partial_t w &= g(x) && \text{at} \quad t = 0. \end{aligned}$$

Solution:

$$w(x,t)=\frac{1}{2}[f(x-at)+f(x+at)]+\frac{1}{2a}\int_{x-at}^{x+at}g(\xi)\,d\xi+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau.$$

► **Domain: $0\le x<\infty$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w&=f(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=g(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=h(t) &&\text{at}\quad x=0 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(x,t)=w_1(x,t)+\frac{1}{2a}w_2(x,t),$$

where

$$w_1(x,t)=\begin{cases}\dfrac{1}{2}[f(x+at)+f(x-at)]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}g(\xi)\,d\xi & \text{if } t<\dfrac{x}{a},\\[2ex] \dfrac{1}{2}[f(x+at)-f(at-x)]+\dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at}g(\xi)\,d\xi+h\Big(t-\dfrac{x}{a}\Big) & \text{if } t>\dfrac{x}{a},\end{cases}$$

$$w_2(x,t)=\begin{cases}\displaystyle\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau & \text{if } t<\dfrac{x}{a},\\[2ex] \displaystyle\int_0^{t-x/a}\int_{a(t-\tau)-x}^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau+\int_{t-x/a}^{t}\int_{x-a(t-\tau)}^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau & \text{if } t>\dfrac{x}{a}.\end{cases}$$

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $0\le x<\infty$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w&=f(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=g(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w&=h(t) &&\text{at}\quad x=0 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(x,t)=w_1(x,t)+\frac{1}{2a}w_2(x,t),$$

where

$$w_1(x,t)=\begin{cases}\dfrac{1}{2}[f(x+at)+f(x-at)]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}g(\xi)\,d\xi & \text{if } t<\dfrac{x}{a},\\ \dfrac{1}{2}[f(x+at)+f(at-x)]+\dfrac{1}{2a}\displaystyle\int_{0}^{x+at}g(\xi)\,d\xi & \\ \qquad+\dfrac{1}{2a}\displaystyle\int_0^{at-x}g(\xi)\,d\xi-a\int_0^{t-x/a}h(\xi)\,d\xi & \text{if } t>\dfrac{x}{a},\end{cases}$$

$$w_2(x,t)=\begin{cases}\displaystyle\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau & \text{if } t<\dfrac{x}{a},\\ \displaystyle\int_0^{t-x/a}\int_0^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau+\int_0^{t-x/a}\int_0^{a(t-\tau)-x}\Phi(\xi,\tau)\,d\xi\,d\tau & \\ \qquad+\displaystyle\int_{t-x/a}^{t}\int_{x-a(t-\tau)}^{x+a(t-\tau)}\Phi(\xi,\tau)\,d\xi\,d\tau & \text{if } t>\dfrac{x}{a}.\end{cases}$$

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► Domain: $0\le x\le l$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w&=f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w&=g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(x,t)=\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi+\int_0^l f_1(\xi)G(x,\xi,t)\,d\xi+\int_0^t\int_0^l\Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+a^2\int_0^t g_1(\tau)\Big[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\Big]_{\xi=0}d\tau-a^2\int_0^t g_2(\tau)\Big[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\Big]_{\xi=l}d\tau,$$

where

$$G(x,\xi,t)=\frac{2}{a\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\Big(\frac{n\pi x}{l}\Big)\sin\Big(\frac{n\pi\xi}{l}\Big)\sin\Big(\frac{n\pi at}{l}\Big).$$

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) &&\text{at}\quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) &&\text{at}\quad t=0 \quad \text{(initial condition)},\\
\partial_x w &= g_1(t) &&\text{at}\quad x=0 \quad \text{(boundary condition)},\\
\partial_x w &= g_2(t) &&\text{at}\quad x=l \quad \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t) = \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi + \int_0^t\!\!\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\
-a^2\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau,
\end{aligned}$$

where

$$G(x,\xi,t) = \frac{t}{l} + \frac{2}{a\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\Bigl(\frac{n\pi x}{l}\Bigr)\cos\Bigl(\frac{n\pi\xi}{l}\Bigr)\sin\Bigl(\frac{n\pi a t}{l}\Bigr).$$

► **Domain: $0 \le x \le l$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) &&\text{at}\quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) &&\text{at}\quad t=0 \quad \text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(t) &&\text{at}\quad x=0 \quad \text{(boundary condition)},\\
\partial_x w + k_2 w &= g_2(t) &&\text{at}\quad x=l \quad \text{(boundary condition)}.
\end{aligned}$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t) = \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n\|u_n\|^2}\sin(\lambda_n x + \varphi_n)\sin(\lambda_n\xi + \varphi_n)\sin(\lambda_n a t),$$

$$\varphi_n = \arctan\frac{\lambda_n}{k_1}, \qquad \|u_n\|^2 = \frac{l}{2} + \frac{(\lambda_n^2 + k_1k_2)(k_1+k_2)}{2(\lambda_n^2 + k_1^2)(\lambda_n^2 + k_2^2)};$$

the λ_n are positive roots of the transcendental equation $\cot(\lambda l) = \dfrac{\lambda^2 - k_1k_2}{\lambda(k_1+k_2)}$.

► **Domain: $0 \le x \le l$. Mixed boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) &&\text{at}\quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) &&\text{at}\quad t=0 \quad \text{(initial condition)},\\
w &= g_1(t) &&\text{at}\quad x=0 \quad \text{(boundary condition)},\\
\partial_x w &= g_2(t) &&\text{at}\quad x=l \quad \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(x,t)=\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi+\int_0^l f_1(\xi)G(x,\xi,t)\,d\xi+\int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+a^2\int_0^t g_1(\tau)\Big[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\Big]_{\xi=0}d\tau+a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau,$$

where

$$G(x,\xi,t)=\frac{2}{al}\sum_{n=0}^{\infty}\frac{1}{\lambda_n}\sin(\lambda_n x)\sin(\lambda_n\xi)\sin(\lambda_n at),\qquad \lambda_n=\frac{\pi(2n+1)}{2l}.$$

6.1.3 Equation of the Form $\frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2}-bw+\Phi(x,t)$

This equation with $\Phi(x,t)\equiv 0$ and $b>0$ is encountered in quantum field theory and a number of applications and is referred to as the *Klein–Gordon equation*.

► **Solutions of the homogeneous equation ($\Phi\equiv 0$).**

1°. Particular solutions:

$$w(x,t)=\exp(\pm\mu t)(Ax+B),\quad b=-\mu^2,$$
$$w(x,t)=\exp(\pm\lambda x)(At+B),\quad b=a^2\lambda^2,$$
$$w(x,t)=\cos(\lambda x)[A\cos(\mu t)+B\sin(\mu t)],\quad b=-a^2\lambda^2+\mu^2,$$
$$w(x,t)=\sin(\lambda x)[A\cos(\mu t)+B\sin(\mu t)],\quad b=-a^2\lambda^2+\mu^2,$$
$$w(x,t)=\exp(\pm\mu t)[A\cos(\lambda x)+B\sin(\lambda x)],\quad b=-a^2\lambda^2-\mu^2,$$
$$w(x,t)=\exp(\pm\lambda x)[A\cos(\mu t)+B\sin(\mu t)],\quad b=a^2\lambda^2+\mu^2,$$
$$w(x,t)=\exp(\pm\lambda x)[A\exp(\mu t)+B\exp(-\mu t)],\quad b=a^2\lambda^2-\mu^2,$$
$$w(x,t)=AJ_0(\xi)+BY_0(\xi),\quad \xi=\frac{\sqrt{b}}{a}\sqrt{a^2(t+C_1)^2-(x+C_2)^2},\quad b>0,$$
$$w(x,t)=AI_0(\xi)+BK_0(\xi),\quad \xi=\frac{\sqrt{-b}}{a}\sqrt{a^2(t+C_1)^2-(x+C_2)^2},\quad b<0,$$

where A, B, C_1, and C_2 are arbitrary constants, $J_0(\xi)$ and $Y_0(\xi)$ are Bessel functions, and $I_0(\xi)$ and $K_0(\xi)$ are modified Bessel functions.

2°. Fundamental solutions:

$$\mathscr{E}(x,t)=\frac{\vartheta(at-|x|)}{2a}J_0\Big(\frac{c}{a}\sqrt{a^2t^2-x^2}\Big)\quad\text{for}\quad b=c^2>0,$$
$$\mathscr{E}(x,t)=\frac{\vartheta(at-|x|)}{2a}I_0\Big(\frac{c}{a}\sqrt{a^2t^2-x^2}\Big)\quad\text{for}\quad b=-c^2<0,$$

where $\vartheta(z)$ is the Heaviside unit step function ($\vartheta=0$ for $z<0$ and $\vartheta=1$ for $z\ge 0$), $J_0(z)$ is the Bessel function, and $I_0(z)$ is the modified Bessel function.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

► **Some formulas and transformations of the homogeneous equation ($\Phi \equiv 0$).**

1°. Suppose $w = w(x,t)$ is a solution of the Klein–Gordon equation. Then the functions

$$w_1 = Aw(x + C_1, \pm t + C_2),$$
$$w_2 = Aw(-x + C_1, \pm t + C_2),$$
$$w_3 = Aw\left(\frac{x - vt}{\sqrt{1-(v/a)^2}}, \frac{t - va^{-2}x}{\sqrt{1-(v/a)^2}}\right),$$

where A, C_1, C_2, and v are arbitrary constants, are also solutions of this equation.

2°. Table 6.1 lists transformations of the independent variables that allow separation of variables in the Klein–Gordon equation.

TABLE 6.1
Orthogonal coordinates $u = u(x,t)$, $v = v(x,t)$ admitting separable solutions $w = F(u)G(v)$ of the Klein–Gordon equation ($a = 1$; A_1, A_2, B_1, B_2, and λ are arbitrary constants)

No	Relation between x, t and u, v	Function $F = F(u)$ (differential equation)	Function $G = G(v)$ (differential equation)
1	$x = u,\ t = v$	$F = A_1 e^{u\sqrt{\lambda+b}} + A_2 e^{-u\sqrt{\lambda+b}}$	$G = B_1 e^{v\sqrt{\lambda}} + B_2 e^{-v\sqrt{\lambda}}$
2	$x = u\sinh v$, $t = u\cosh v$	$F = \sqrt{u}\,\bigl[A_1 J_\sigma\bigl(u\sqrt{b}\bigr) + A_2 Y_\sigma\bigl(u\sqrt{b}\bigr)\bigr]$, $\sigma = \frac{1}{2}\sqrt{1+\lambda^2}$	$G = B_1 e^{\lambda v} + B_2 e^{-\lambda v}$
3	$x = uv$, $t = \frac{1}{2}(u^2 + v^2)$	$F = A_1 D_\lambda(\beta u) + A_2 D_\lambda(-\beta u)$, $\beta = (-4b)^{1/4}$	$G = B_1 D_\lambda(\beta v) + B_2 D_\lambda(-\beta v)$, $\beta = (-4b)^{1/4}$
4	$x = \frac{1}{2}(u^2 + v^2)$, $t = uv$	$F = A_1 D_\lambda(\beta u) + A_2 D_\lambda(-\beta u)$, $\beta = (4b)^{1/4}$	$G = B_1 D_\lambda(\beta v) + B_2 D_\lambda(-\beta v)$, $\beta = (4b)^{1/4}$
5	$x = -\frac{1}{2}(u-v)^2 + u + v$, $t = \frac{1}{2}(u-v)^2 + u + v$	$F = \sqrt{U}\,\bigl[A_1 J_{\frac{1}{3}}(\xi) + A_2 Y_{\frac{1}{3}}(\xi)\bigr]$, $U = u + \lambda,\ \xi = \frac{2}{3}\sqrt{b}\,U^{3/2}$	$G = \sqrt{V}\,\bigl[B_1 J_{\frac{1}{3}}(\eta) + B_2 Y_{\frac{1}{3}}(\eta)\bigr]$, $V = v + \lambda,\ \eta = \frac{2}{3}\sqrt{b}\,V^{3/2}$
6	$t + x = \cosh\bigl[\frac{1}{2}(u-v)\bigr]$, $t - x = \sinh\bigl[\frac{1}{2}(u+v)\bigr]$	$F'' + (\lambda + b\sinh u)F = 0$	$G'' + (\lambda + b\sinh v)G = 0$
7	$x = \sinh(u-v) - \frac{1}{2}e^{u+v}$, $t = \sinh(u-v) + \frac{1}{2}e^{u+v}$	$F = A_1 J_\lambda(\beta e^u) + A_2 Y_\lambda(\beta e^u)$, $\beta = \sqrt{b}$	$G = B_1 I_\lambda(\beta e^v) + B_1 K_\lambda(\beta e^v)$, $\beta = \sqrt{b}$
8	$x = \cosh(u-v) - \frac{1}{2}e^{u+v}$, $t = \cosh(u-v) + \frac{1}{2}e^{u+v}$	$F = A_1 J_\lambda(\beta e^u) + A_2 Y_\lambda(\beta e^u)$, $\beta = \sqrt{b}$	$G = B_1 J_\lambda(\beta e^v) + B_1 Y_\lambda(\beta e^v)$, $\beta = \sqrt{b}$
9	$x = \cosh u \sinh v$, $t = \sinh u \cosh v$	$F'' + (\lambda + \frac{1}{2}b\cosh 2u)F = 0$, modified Mathieu equation	$G'' + (\lambda - \frac{1}{2}b\cosh 2v)G = 0$, modified Mathieu equation
10	$x = \sinh u \sinh v$, $t = \cosh u \cosh v$	$F'' + (\lambda + \frac{1}{2}b\cosh 2u)F = 0$, modified Mathieu equation	$G'' + (\lambda + \frac{1}{2}b\cosh 2v)G = 0$, modified Mathieu equation
11	$x = \sin u \sin v$, $t = \cos u \cos v$	$F'' + (\lambda - \frac{1}{2}b\cos 2u)F = 0$, Mathieu equation	$G'' + (\lambda - \frac{1}{2}b\cos 2v)G = 0$, Mathieu equation

Notation: $J_\sigma(z)$ and $Y_\sigma(z)$ are Bessel functions, $I_\sigma(z)$ and $K_\sigma(z)$ are modified Bessel functions, and $D_\lambda(z)$ is the parabolic cylinder function.

⊙ *Literature*: E. Kalnins (1975), W. Miller, Jr. (1977).

► **Domain: $-\infty < x < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution for $b = -c^2 < 0$:

$$w(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{ct}{2a}\int_{x-at}^{x+at} \frac{I_1\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}} f(\xi)\,d\xi$$
$$+ \frac{1}{2a}\int_{x-at}^{x+at} I_0\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr) g(\xi)\,d\xi$$
$$+ \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} I_0\bigl(c\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\,\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau,$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions of the first kind.

Solution for $b = c^2 > 0$:

$$w(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] - \frac{ct}{2a}\int_{x-at}^{x+at} \frac{J_1\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}} f(\xi)\,d\xi$$
$$+ \frac{1}{2a}\int_{x-at}^{x+at} J_0\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr) g(\xi)\,d\xi$$
$$+ \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} J_0\bigl(c\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\,\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau,$$

where $J_0(z)$ and $J_1(z)$ are Bessel functions of the first kind.

⊙ *Literature*: B. M. Budak, A. N. Tikhonov, and A. A. Samarskii (1980).

► **Domain: $0 \le x \le l$. First boundary value problem.**

The following conditions are prescribed:

$$w = f_0(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},$$
$$w = g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+ a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=0} d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau,$$

where

$$G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}\sin(\lambda_n x)\sin(\lambda_n\xi)\frac{\sin\bigl(t\sqrt{a^2\lambda_n^2+b}\,\bigr)}{\sqrt{a^2\lambda_n^2+b}},\quad \lambda_n=\frac{\pi n}{l}.$$

Remark 6.1. Let $b<0$ and $a^2\lambda_n^2+b<0$ for $n=1,\dots,m$ and $a^2\lambda_n^2+b>0$ for $n=m+1,\,m+2,\dots$ In this case the Green's function is modified and acquires the form

$$G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{m}\sin(\lambda_n x)\sin(\lambda_n\xi)\frac{\sinh\bigl(t\sqrt{|a^2\lambda_n^2+b|}\,\bigr)}{\sqrt{|a^2\lambda_n^2+b|}}$$
$$+\frac{2}{l}\sum_{n=m+1}^{\infty}\sin(\lambda_n x)\sin(\lambda_n\xi)\frac{\sin\bigl(t\sqrt{a^2\lambda_n^2+b}\,\bigr)}{\sqrt{a^2\lambda_n^2+b}},\quad \lambda_n=\frac{\pi n}{l}.$$

Analogously, the Green's functions for the second, third, and mixed boundary value problems are modified in similar cases.

⊙ *Literature*: A. G. Butkovskiy (1979).

► Domain: $0\le x\le l$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w&=g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w&=g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(x,t)=\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi+\int_0^l f_1(\xi)G(x,\xi,t)\,d\xi+\int_0^t\!\!\int_0^l\Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$-a^2\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau+a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau,$$

where

$$G(x,\xi,t)=\frac{1}{l\sqrt{b}}\sin\bigl(t\sqrt{b}\,\bigr)+\frac{2}{l}\sum_{n=1}^{\infty}\cos(\lambda_n x)\cos(\lambda_n\xi)\frac{\sin\bigl(t\sqrt{a^2\lambda_n^2+b}\,\bigr)}{\sqrt{a^2\lambda_n^2+b}},\quad \lambda_n=\frac{\pi n}{l}.$$

► Domain: $0\le x\le l$. Third boundary value problem.

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w-k_1w&=g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w+k_2w&=g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t)=\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)\sin\bigl(t\sqrt{a^2\lambda_n^2+b}\,\bigr)}{\|y_n\|^2\sqrt{a^2\lambda_n^2+b}},$$

$$y_n(x)=\cos(\lambda_n x)+\frac{k_1}{\lambda_n}\sin(\lambda_n x),\quad \|y_n\|^2=\frac{k_2}{2\lambda_n^2}\frac{\lambda_n^2+k_1^2}{\lambda_n^2+k_2^2}+\frac{k_1}{2\lambda_n^2}+\frac{l}{2}\biggl(1+\frac{k_1^2}{\lambda_n^2}\biggr).$$

Here, the λ_n are positive roots of the transcendental equation $\dfrac{\tan(\lambda l)}{\lambda}=\dfrac{k_1+k_2}{\lambda^2-k_1k_2}$.

► **Domain: $0\le x\le l$. Mixed boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w&=g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t)=&\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi+\int_0^l f_1(\xi)G(x,\xi,t)\,d\xi+\int_0^t\!\!\int_0^l\Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\
&+a^2\int_0^t g_1(\tau)\biggl[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\biggr]_{\xi=0}d\tau+a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau,
\end{aligned}$$

where

$$G(x,\xi,t)=\frac{2}{l}\sum_{n=0}^{\infty}\sin(\lambda_n x)\sin(\lambda_n\xi)\frac{\sin\bigl(t\sqrt{a^2\lambda_n^2+b}\,\bigr)}{\sqrt{a^2\lambda_n^2+b}},\quad \lambda_n=\frac{\pi(2n+1)}{2l}.$$

6.1.4 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2}=a^2\dfrac{\partial^2 w}{\partial x^2}-b\dfrac{\partial w}{\partial x}+\Phi(x,t)$

► **Reduction to the nonhomogeneous Klein–Gordon equation.**

The substitution $w(x,t)=\exp\bigl(\tfrac{1}{2}bx/a^2\bigr)u(x,t)$ leads the nonhomogeneous Klein–Gordon equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}-\frac{b^2}{4a^2}u+\exp\biggl(-\frac{bx}{2a^2}\biggr)\Phi(x,t),$$

which is discussed in Section 6.1.3.

► Domain: $-\infty < x < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned}
w(x,t) = {} & \frac{1}{2} f(x+at) \exp\left(-\frac{bt}{2a}\right) + \frac{1}{2} f(x-at) \exp\left(\frac{bt}{2a}\right) \\
& - \frac{\sigma t}{2a} \exp\left(\frac{bx}{2a^2}\right) \int_{x-at}^{x+at} \exp\left(-\frac{b\xi}{2a^2}\right) \frac{J_1\left(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\,\right)}{\sqrt{t^2-(x-\xi)^2/a^2}} f(\xi)\, d\xi \\
& + \frac{1}{2a} \exp\left(\frac{bx}{2a^2}\right) \int_{x-at}^{x+at} \exp\left(-\frac{b\xi}{2a^2}\right) J_0\left(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\,\right) g(\xi)\, d\xi \\
& + \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \exp\left[\frac{b(x-\xi)}{2a^2}\right] J_0\left(\sigma\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\,\right) \Phi(\xi,\tau)\, d\xi\, d\tau,
\end{aligned}$$

where $J_0(z)$ and $J_1(z)$ are Bessel functions of the first kind, and $\sigma = \frac{1}{2}|b|/a$.

► Domain: $0 \le x \le l$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\
w &= g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},\\
w &= g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t) = {} & \frac{\partial}{\partial t} \int_0^l f_0(\xi) G(x,\xi,t)\, d\xi + \int_0^l f_1(\xi) G(x,\xi,t)\, d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau \\
& + a^2 \int_0^t g_1(\tau) \left[\frac{\partial}{\partial \xi} G(x,\xi,t-\tau)\right]_{\xi=0} d\tau - a^2 \int_0^t g_2(\tau) \left[\frac{\partial}{\partial \xi} G(x,\xi,t-\tau)\right]_{\xi=l} d\tau,
\end{aligned}$$

where

$$G(x,\xi,t) = \frac{2}{l} \exp\left[\frac{b}{2a^2}(x-\xi)\right] \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi n \xi}{l}\right) \frac{\sin(\lambda_n t)}{\lambda_n},$$
$$\lambda_n = \sqrt{\frac{a^2\pi^2 n^2}{l^2} + \frac{b^2}{4a^2}}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

▶ **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = {} & \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau\\ & -a^2\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(x,\xi,t) &= \frac{bt}{a^2\bigl[1-\exp(-bl/a^2)\bigr]}\exp\left(-\frac{b\xi}{a^2}\right) + \frac{2}{l}\exp\left[\frac{b(x-\xi)}{2a^2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)\sin(\lambda_n t)}{\lambda_n(1+\mu_n^2)},\\ y_n(x) &= \cos\left(\frac{\pi n x}{l}\right) - \frac{bl}{2a^2\pi n}\sin\left(\frac{\pi n x}{l}\right), \quad \lambda_n = \sqrt{\frac{a^2\pi^2n^2}{l^2}+\frac{b^2}{4a^2}}, \quad \mu_n = \frac{bl}{2a^2\pi n}. \end{aligned}$$

⊙ *Literature*: A. G. Butkovskiy (1979).

▶ **Domain: $0 \le x \le l$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(x) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(t) \quad \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(t) \quad \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t) = \exp\left[\frac{b(x-\xi)}{2a^2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)\sin(a\lambda_n t)}{a\lambda_n B_n}.$$

Here,

$$\begin{aligned} y_n(x) &= \cos(\mu_n x) + \frac{2a^2k_1-b}{2a^2\mu_n}\sin(\mu_n x), \qquad \lambda_n = \sqrt{\mu_n^2+\frac{b^2}{4a^4}},\\ B_n &= \frac{2a^2k_2+b}{4a^2\mu_n^2}\,\frac{4a^4\mu_n^2+(2a^2k_1-b)^2}{4a^4\mu_n^2+(2a^2k_2+b)^2} + \frac{2a^2k_1-b}{4a^2\mu_n^2} + \frac{l}{2} + \frac{l(2a^2k_1-b)^2}{8a^4\mu_n^2}, \end{aligned}$$

where the μ_n are positive roots of the transcendental equation

$$\frac{\tan(\mu l)}{\mu} = \frac{4a^4(k_1+k_2)}{4a^4\mu^2 - (2a^2k_1 - b)(2a^2k_2 + b)}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

6.1.5 Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + \Phi(x,t)$

► **Reduction to the nonhomogeneous Klein–Gordon equation.**

The substitution $w(x,t) = \exp\bigl(-\frac{1}{2}a^{-2}bx\bigr)u(x,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \bigl(c - \tfrac{1}{4}a^{-2}b^2\bigr)u + \exp\bigl(\tfrac{1}{2}a^{-2}bx\bigr)\Phi(x,t),$$

which is discussed in Section 6.1.3.

► **Domain: $-\infty < x < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution for $c - \frac{1}{4}a^{-2}b^2 = \sigma^2 > 0$:

$$\begin{aligned}
w(x,t) &= \frac{1}{2}f(x+at)\exp\left(\frac{bt}{2a}\right) + \frac{1}{2}f(x-at)\exp\left(-\frac{bt}{2a}\right)\\
&\quad + \frac{\sigma t}{2a}\exp\left(-\frac{bx}{2a^2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)\frac{I_1\bigl(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi\\
&\quad + \frac{1}{2a}\exp\left(-\frac{bx}{2a^2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)I_0\bigl(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)g(\xi)\,d\xi\\
&\quad + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\left[\frac{b(\xi-x)}{2a^2}\right]I_0\bigl(\sigma\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\,\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau,
\end{aligned}$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions of the first kind.

Solution for $c - \frac{1}{4}a^{-2}b^2 = -\sigma^2 < 0$:

$$\begin{aligned}
w(x,t) &= \frac{1}{2}f(x+at)\exp\left(\frac{bt}{2a}\right) + \frac{1}{2}f(x-at)\exp\left(-\frac{bt}{2a}\right)\\
&\quad - \frac{\sigma t}{2a}\exp\left(-\frac{bx}{2a^2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)\frac{J_1\bigl(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi\\
&\quad + \frac{1}{2a}\exp\left(-\frac{bx}{2a^2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)J_0\bigl(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)g(\xi)\,d\xi\\
&\quad + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\left[\frac{b(\xi-x)}{2a^2}\right]J_0\bigl(\sigma\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\,\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau,
\end{aligned}$$

where $J_0(z)$ and $J_1(z)$ are Bessel functions of the first kind.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► Domain: $0 \le x \le l$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(t) && \text{at} \quad x=0 && \text{(boundary condition)},\\ w &= g_2(t) && \text{at} \quad x=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = &\int_0^t\!\!\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau + \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi\\ &+a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=0} d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau. \end{aligned}$$

Let $a^2\pi^2 + \frac{1}{4}a^{-2}b^2l^2 - cl^2 > 0$. Then

$$G(x,\xi,t) = \frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a^2}\right]\sum_{n=1}^{\infty}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}},$$

$$\lambda_n = \frac{a^2\pi^2n^2}{l^2} + \frac{b^2}{4a^2} - c.$$

Let

$$\begin{aligned} &a^2\pi^2n^2 + \tfrac{1}{4}a^{-2}b^2l^2 - cl^2 \le 0 \quad \text{at} \quad n=1,\,\dots,\,m;\\ &a^2\pi^2n^2 + \tfrac{1}{4}a^{-2}b^2l^2 - cl^2 > 0 \quad \text{at} \quad n=m+1,\,m+2,\,\dots \end{aligned}$$

Then

$$\begin{aligned} G(x,\xi,t) = &\frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a^2}\right]\sum_{n=1}^{m}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sinh\bigl(t\sqrt{\beta_n}\,\bigr)}{\sqrt{\beta_n}}\\ &+\frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a^2}\right]\sum_{n=m+1}^{\infty}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}},\\ &\beta_n = c - \frac{a^2\pi^2n^2}{l^2} - \frac{b^2}{4a^2}, \quad \lambda_n = \frac{a^2\pi^2n^2}{l^2} + \frac{b^2}{4a^2} - c. \end{aligned}$$

For $\beta_n = 0$ the ratio $\sinh\bigl(t\sqrt{\beta_n}\,\bigr)/\sqrt{\beta_n}$ must be replaced by t.

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le l$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w &= g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_x w &= g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = {}&\int_0^t\!\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau + \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi\\ &-a^2\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau. \end{aligned}$$

For $c<0$,

$$\begin{aligned} G(x,\xi,t) = {}&\frac{b}{a^2\bigl(e^{bl/a^2}-1\bigr)}\exp\left(\frac{b\xi}{a^2}\right)\frac{\sin\bigl(t\sqrt{|c|}\,\bigr)}{\sqrt{|c|}}\\ &+\frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a^2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)}{1+\mu_n^2}\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}, \qquad \lambda_n = \frac{a^2\pi^2n^2}{l^2}+\frac{b^2}{4a^2}-c,\\ y_n(x) = {}&\cos\left(\frac{\pi n x}{l}\right)+\mu_n\sin\left(\frac{\pi n x}{l}\right), \qquad \mu_n = \frac{bl}{2a^2\pi n}. \end{aligned}$$

For $c>0$,

$$\begin{aligned} G(x,\xi,t) = {}&\frac{b}{a^2\bigl(e^{bl/a^2}-1\bigr)}\exp\left(\frac{b\xi}{a^2}\right)\frac{\sinh\bigl(t\sqrt{c}\,\bigr)}{\sqrt{c}}\\ &+\frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a^2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)}{1+\mu_n^2}\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}, \end{aligned}$$

where the λ_n, $y_n(x)$, and μ_n were specified previously. If the inequality $\lambda_n<0$ holds for several first values $n=1,\dots,m$, then the $\sqrt{\lambda_n}$ in the corresponding terms of the series should be replaced by $\sqrt{|\lambda_n|}$, and the sines by the hyperbolic sines.

► **Domain: $0 \le x \le l$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w - k_1 w &= g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_x w + k_2 w &= g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,\xi,t)=\exp\left[\frac{b(\xi-x)}{2a^2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{B_n\sqrt{\lambda_n}}.$$

Here,

$$y_n(x)=\cos(\mu_n x)+\frac{2a^2k_1+b}{2a^2\mu_n}\sin(\mu_n x),\qquad \lambda_n=a^2\mu_n^2+\frac{b^2}{4a^2}-c,$$
$$B_n=\frac{2a^2k_2-b}{4a^2\mu_n^2}\,\frac{4a^4\mu_n^2+(2a^2k_1+b)^2}{4a^4\mu_n^2+(2a^2k_2-b)^2}+\frac{2a^2k_1+b}{4a^2\mu_n^2}+\frac{l}{2}+\frac{l(2a^2k_1+b)^2}{8a^4\mu_n^2},$$

where the μ_n are positive roots of the transcendental equation

$$\frac{\tan(\mu l)}{\mu}=\frac{4a^4(k_1+k_2)}{4a^4\mu^2-(2a^2k_1+b)(2a^2k_2-b)}.$$

6.2 Wave Equations with Axial or Central Symmetry

6.2.1 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2}=a^2\left(\dfrac{\partial^2 w}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial w}{\partial r}\right)$

This is the one-dimensional wave equation with axial symmetry, where $r=\sqrt{x^2+y^2}$ is the radial coordinate. In the problems considered for $0\le r\le R$ the solutions bounded at $r=0$ are sought (this is not specifically stated below).

► **Domain: $0\le r\le R$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,t)=\frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi+\int_0^R f_1(\xi)G(r,\xi,t)\,d\xi\\
-a^2\int_0^t g(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R}d\tau,
\end{aligned}$$

where

$$G(r,\xi,t)=\frac{2\xi}{aR}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1^2(\lambda_n)}J_0\!\left(\frac{\lambda_n r}{R}\right)J_0\!\left(\frac{\lambda_n\xi}{R}\right)\sin\!\left(\frac{\lambda_n at}{R}\right).$$

Here, the λ_n are positive zeros of the Bessel function, $J_0(\lambda)=0$. The numerical values of the first ten λ_n are specified in Section 3.2.1 (see the first boundary value problem for $0\le r\le R$).

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g(t) && \text{at} \quad r=R && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R f_1(\xi)G(r,\xi,t)\,d\xi + a^2\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau,$$

where

$$G(r,\xi,t) = \frac{2t\xi}{R^2} + \frac{2\xi}{aR}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_0^2(\lambda_n)} J_0\left(\frac{\lambda_n r}{R}\right) J_0\left(\frac{\lambda_n \xi}{R}\right)\sin\left(\frac{\lambda_n a t}{R}\right).$$

Here, the λ_n are positive zeros of the first-order Bessel function, $J_1(\lambda)=0$. The numerical values of the first ten roots λ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

⊙ *Literature*: M. M. Smirnov (1975), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + kw &= g(t) && \text{at} \quad r=R && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t) = \frac{2\xi}{aR}\sum_{n=1}^{\infty}\frac{\lambda_n}{(k^2R^2+\lambda_n^2)J_0^2(\lambda_n)} J_0\left(\frac{\lambda_n r}{R}\right) J_0\left(\frac{\lambda_n \xi}{R}\right)\sin\left(\frac{\lambda_n a t}{R}\right).$$

Here, the λ_n are positive roots of the transcendental equation

$$\lambda J_1(\lambda) - kRJ_0(\lambda) = 0.$$

The numerical values of the first six roots λ_n can be found in Carslaw and Jaeger (1984); see also Abramowitz and Stegun (1964).

► **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ w &= g_2(t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) = {} & \frac{\partial}{\partial t}\int_{R_1}^{R_2} f_0(\xi)G(r,\xi,t)\,d\xi + \int_{R_1}^{R_2} f_1(\xi)G(r,\xi,t)\,d\xi \\ & + a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_1} d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_2} d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t) = \sum_{n=1}^{\infty} A_n\xi\Psi_n(r)\Psi_n(\xi)\sin\left(\frac{\lambda_n at}{R_1}\right), \quad A_n = \frac{\pi^2\lambda_n J_0^2(s\lambda_n)}{2aR_1\left[J_0^2(\lambda_n) - J_0^2(s\lambda_n)\right]},$$

$$\Psi_n(r) = Y_0(\lambda_n)J_0\left(\frac{\lambda_n r}{R_1}\right) - J_0(\lambda_n)Y_0\left(\frac{\lambda_n r}{R_1}\right), \quad s = \frac{R_2}{R_1},$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions, and the λ_n are positive roots of the transcendental equation

$$J_0(\lambda)Y_0(s\lambda) - J_0(s\lambda)Y_0(\lambda) = 0.$$

The numerical values of the first five roots $\lambda_n = \lambda_n(s)$ can be found in Abramowitz and Stegun (1964) and Carslaw and Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) = {} & \frac{\partial}{\partial t}\int_{R_1}^{R_2} f_0(\xi)G(r,\xi,t)\,d\xi + \int_{R_1}^{R_2} f_1(\xi)G(r,\xi,t)\,d\xi \\ & - a^2\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t)=\frac{2t\xi}{R_2^2-R_1^2}+\sum_{n=1}^{\infty}A_n\xi\Psi_n(r)\Psi_n(\xi)\sin\left(\frac{\lambda_n at}{R_1}\right),\quad A_n=\frac{\pi^2\lambda_n J_1^2(s\lambda_n)}{2aR_1\left[J_1^2(\lambda_n)-J_1^2(s\lambda_n)\right]},$$

$$\Psi_n(r)=Y_1(\lambda_n)J_0\left(\frac{\lambda_n r}{R_1}\right)-J_1(\lambda_n)Y_0\left(\frac{\lambda_n r}{R_1}\right),\quad s=\frac{R_2}{R_1},$$

where $J_k(z)$ and $Y_k(z)$ are Bessel functions ($k=0,\,1$); the λ_n are positive roots of the transcendental equation

$$J_1(\lambda)Y_1(s\lambda)-J_1(s\lambda)Y_1(\lambda)=0.$$

The numerical values of the first five roots $\lambda_n=\lambda_n(s)$ can be found in Abramowitz and Stegun (1964).

► **Domain: $R_1\le r\le R_2$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(r) && \text{at} && t=0 && \text{(initial condition)},\\
\partial_t w&=f_1(r) && \text{at} && t=0 && \text{(initial condition)},\\
\partial_r w-k_1w&=g_1(t) && \text{at} && r=R_1 && \text{(boundary condition)},\\
\partial_r w+k_2w&=g_2(t) && \text{at} && r=R_2 && \text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t)=\frac{\pi^2}{2a}\sum_{n=1}^{\infty}\frac{\mu_n}{B_n}\left[k_2J_0(\mu_nR_2)-\mu_nJ_1(\mu_nR_2)\right]^2\xi H_n(r)H_n(\xi)\sin(\mu_n at).$$

Here,

$$B_n=(\mu_n^2+k_2^2)\left[k_1J_0(\mu_nR_1)+\mu_nJ_1(\mu_nR_1)\right]^2-(\mu_n^2+k_1^2)\left[k_2J_0(\mu_nR_2)-\mu_nJ_1(\mu_nR_2)\right]^2,$$
$$H_n(r)=\left[k_1Y_0(\mu_nR_1)+\mu_nY_1(\mu_nR_1)\right]J_0(\mu_nr)-\left[k_1J_0(\mu_nR_1)+\mu_nJ_1(\mu_nR_1)\right]Y_0(\mu_nr);$$

$J_k(z)$ and $Y_k(z)$ are Bessel functions ($k=0,\,1$); and the μ_n are positive roots of the transcendental equation

$$\begin{aligned}
&\left[k_1J_0(\mu R_1)+\mu J_1(\mu R_1)\right]\left[k_2Y_0(\mu R_2)-\mu Y_1(\mu R_2)\right]\\
&\qquad-\left[k_2J_0(\mu R_2)-\mu J_1(\mu R_2)\right]\left[k_1Y_0(\mu R_1)+\mu Y_1(\mu R_1)\right]=0.
\end{aligned}$$

6.2.2 Equation of the Form $\frac{\partial^2 w}{\partial t^2}=a^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\Phi(r,t)$

► **Domain: $0\le r\le R$. Different boundary value problems.**

1°. The solution to the first boundary value problem for a circle of radius R is given by the formula from Section 6.2.1 (see the first boundary value problem for $0\le r\le R$) with the additional term

$$\int_0^t \int_0^R \Phi(\xi,\tau) G(r,\xi,t-\tau)\, d\xi\, d\tau, \qquad (1)$$

which allows for the equation's nonhomogeneity.

2°. The solution to the second boundary value problem for a circle of radius R is given by the formula from Section 6.2.1 (see the second boundary value problem for $0 \le r \le R$) with the additional term (1); the Green's function is also taken from Section 6.2.1.

3°. The solution to the third boundary value problem for a circle of radius R is the sum of the solution presented in Section 6.2.1 (see the third boundary value problem for $0 \le r \le R$) and expression (1); the Green's function is also taken from Section 6.2.1.

► **Domain: $R_1 \le r \le R_2$. Different boundary value problems.**

1°. The solution to the first boundary value problem for an annular domain is given by the formula from Section 6.2.1 (see the first boundary value problem for $R_1 \le r \le R_2$) with the additional term

$$\int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau) G(r,\xi,t-\tau)\, d\xi\, d\tau, \qquad (2)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution to the second boundary value problem for an annular domain is given by the formula from Section 6.2.1 (see the second boundary value problem for $R_1 \le r \le R_2$) with the additional term (2); the Green's function is also taken from Section 6.2.1.

3°. The solution to the third boundary value problem for an annular domain is the sum of the solution presented in Section 6.2.1 (see the third boundary value problem for $R_1 \le r \le R_2$) and expression (2); the Green's function is also taken from Section 6.2.1.

6.2.3 Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right)$

This is the equation of one-dimensional vibration of a gas with central symmetry, where $r = \sqrt{x^2+y^2+z^2}$ is the radial coordinate. In the problems considered for $0 \le r \le R$ the solutions bounded at $r = 0$ are sought; this is not specifically stated below.

► **General solution:**

$$w(t,r) = \frac{\varphi(r+at) + \psi(r-at)}{r},$$

where $\varphi(r_1)$ and $\psi(r_2)$ are arbitrary functions.

► Reduction to a constant coefficient equation.

The substitution $u(r,t) = rw(r,t)$ leads to the constant coefficient equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial r^2},$$

which is discussed in Section 6.1.1.

► Domain: $0 \le r < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$w = f(r) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(r) \quad \text{at} \quad t = 0.$$

Solution:

$$w(r,t) = \frac{1}{2r}\big[(r-at)f\big(|r-at|\big) + (r+at)f\big(|r+at|\big)\big] + \frac{1}{2ar}\int_{r-at}^{r+at} \xi g\big(|\xi|\big)\,d\xi.$$

Solution at the center $r = 0$:

$$w(0,t) = atf'(at) + f(at) + tg(at).$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le r \le R$. First boundary value problem.

The following conditions are prescribed:

$$w = f_0(r) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_t w = f_1(r) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g(t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}.$$

Solution:

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R f_1(\xi)G(r,\xi,t)\,d\xi$$
$$- a^2\int_0^t g(\tau)\left[\frac{\partial}{\partial \xi}G(r,\xi,t-\tau)\right]_{\xi=R} d\tau,$$

where

$$G(r,\xi,t) = \frac{2\xi}{\pi a r}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi r}{R}\right)\sin\left(\frac{n\pi \xi}{R}\right)\sin\left(\frac{an\pi t}{R}\right).$$

► **Domain: $0 \le r \le R$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w &= g(t) && \text{at} \quad r = R && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R f_1(\xi)G(r,\xi,t)\,d\xi + a^2\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau,$$

where

$$G(r,\xi,t) = \frac{3t\xi^2}{R^3} + \frac{2\xi}{ar}\sum_{n=1}^{\infty}\frac{\mu_n^2+1}{\mu_n^3}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n \xi}{R}\right)\sin\left(\frac{\mu_n a t}{R}\right).$$

Here, the μ_n are positive roots of the transcendental equation $\tan\mu - \mu = 0$. The numerical values of the first five roots μ_n are specified in Section 3.2.3 (see the second boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w + kw &= g(t) && \text{at} \quad r = R && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t) = \frac{2\xi}{ar}\sum_{n=1}^{\infty}\frac{\mu_n^2+(kR-1)^2}{\mu_n\big[\mu_n^2+kR(kR-1)\big]}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n \xi}{R}\right)\sin\left(\frac{\mu_n a t}{R}\right).$$

Here, the μ_n are positive roots of the transcendental equation

$$\mu\cot\mu + kR - 1 = 0.$$

The numerical values of the first six roots μ_n can be found in Carslaw and Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ w &= g_1(t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ w &= g_2(t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t)=\frac{\partial}{\partial t}\int_{R_1}^{R_2} f_0(\xi)G(r,\xi,t)\,d\xi+\int_{R_1}^{R_2} f_1(\xi)G(r,\xi,t)\,d\xi$$
$$+a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_1}d\tau-a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_2}d\tau,$$

where

$$G(r,\xi,t)=\frac{2\xi}{\pi ar}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{\pi n(r-R_1)}{R_2-R_1}\right]\sin\left[\frac{\pi n(\xi-R_1)}{R_2-R_1}\right]\sin\left(\frac{\pi nat}{R_2-R_1}\right).$$

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t)=\frac{\partial}{\partial t}\int_{R_1}^{R_2} f_0(\xi)G(r,\xi,t)\,d\xi+\int_{R_1}^{R_2} f_1(\xi)G(r,\xi,t)\,d\xi$$
$$-a^2\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau+a^2\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau,$$

where

$$G(r,\xi,t)=\frac{3t\xi^2}{R_2^3-R_1^3}+\frac{2\xi}{a(R_2-R_1)r}\sum_{n=1}^{\infty}\frac{(1+R_2^2\lambda_n^2)\Psi_n(r)\Psi_n(\xi)\sin(\lambda_n at)}{\lambda_n^3\left[R_1^2+R_2^2+R_1R_2(1+R_1R_2\lambda_n^2)\right]},$$
$$\Psi_n(r)=\sin[\lambda_n(r-R_1)]+R_1\lambda_n\cos[\lambda_n(r-R_1)].$$

Here, the λ_n are positive roots of the transcendental equation

$$(\lambda^2R_1R_2+1)\tan[\lambda(R_2-R_1)]-\lambda(R_2-R_1)=0.$$

► **Domain: $R_1 \le r \le R_2$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w-k_1w &= g_1(t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w+k_2w &= g_2(t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t)=\frac{2\xi}{a(R_2-R_1)r}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{(b_2^2+R_2^2\lambda_n^2)\Psi_n(r)\Psi_n(\xi)\sin(\lambda_n at)}{(b_1^2+R_1^2\lambda_n^2)(b_2^2+R_2^2\lambda_n^2)+(b_1R_2+b_2R_1)(b_1b_2+R_1R_2\lambda_n^2)},$$
$$\Psi_n(r)=b_1\sin[\lambda_n(r-R_1)]+R_1\lambda_n\cos[\lambda_n(r-R_1)],\quad b_1=k_1R_1+1,\quad b_2=k_2R_2-1.$$

Here, the λ_n are positive roots of the transcendental equation

$$(b_1b_2-R_1R_2\lambda^2)\sin[\lambda(R_2-R_1)]+\lambda(R_1b_2+R_2b_1)\cos[\lambda(R_2-R_1)]=0.$$

6.2.4 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2}=a^2\left(\dfrac{\partial^2 w}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial w}{\partial r}\right)+\Phi(r,t)$

► **Reduction to a nonhomogeneous constant coefficient equation.**

The substitution $u(r,t)=rw(r,t)$ leads to the nonhomogeneous constant coefficient equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial r^2}+r\Phi(r,t),$$

which is discussed in Section 6.1.2.

► **Domain: $0\le r<\infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w=f(r)\quad\text{at}\quad t=0,$$
$$\partial_t w=g(r)\quad\text{at}\quad t=0.$$

Solution:

$$w(r,t)=\frac{1}{2r}\bigl[(r-at)f\bigl(|r-at|\bigr)+(r+at)f\bigl(|r+at|\bigr)\bigr]+\frac{1}{2ar}\int_{r-at}^{r+at}\xi g\bigl(|\xi|\bigr)\,d\xi$$
$$+\frac{1}{2ar}\int_0^t d\tau\int_{r-a(t-\tau)}^{r+a(t-\tau)}\xi\Phi\bigl(|\xi|,\tau\bigr)\,d\xi.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0\le r\le R$. Different boundary value problems.**

1°. The solution to the first boundary value problem for a sphere of radius R is given by the formula from Section 6.2.3 (see the first boundary value problem for $0\le r\le R$) with the additional term

$$\int_0^t\int_0^R\Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau, \tag{1}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution to the second boundary value problem for a sphere of radius R is given by the formula from Section 6.2.3 (see the second boundary value problem for $0 \le r \le R$) with the additional term (1); the Green's function is also taken from Section 6.2.3.

3°. The solution to the third boundary value problem for a sphere of radius R is the sum of the solution presented in Section 6.2.3 (see the third boundary value problem for $0 \le r \le R$) and expression (1); the Green's function is also taken from Section 6.2.3.

► **Domain: $R_1 \le r \le R_2$. Different boundary value problems.**

1°. The solution to the first boundary value problem for a spherical layer is given by the formula from Section 6.2.3 (see the first boundary value problem for $R_1 \le r \le R_2$) with the additional term

$$\int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau) G(r,\xi,t-\tau)\, d\xi\, d\tau, \tag{2}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution to the second boundary value problem for a spherical layer is given by the formula from Section 6.2.3 (see the second boundary value problem for $R_1 \le r \le R_2$) with the additional term (2); the Green's function is also taken from Section 6.2.3.

3°. The solution to the third boundary value problem for a spherical layer is the sum of the solution presented in Section 6.2.3 (see the third boundary value problem for $R_1 \le r \le R_2$) and expression (2); the Green's function is also taken from Section 6.2.3.

6.2.5 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2} = a^2\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\right) - bw + \Phi(r,t)$

For $b > 0$ and $\Phi \equiv 0$, this is the Klein–Gordon equation describing one-dimensional wave phenomena with axial symmetry. In the problems considered for $0 \le r \le R$ the solutions bounded at $r = 0$ are sought; this is not specifically stated below.

► **Domain: $0 \le r \le R$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w &= g(t) \quad \text{at} \quad r = R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) &= \frac{\partial}{\partial t}\int_0^R f_0(\xi) G(r,\xi,t)\, d\xi + \int_0^R f_1(\xi) G(r,\xi,t)\, d\xi \\ &\quad - a^2 \int_0^t g(\tau)\left[\frac{\partial}{\partial \xi} G(r,\xi,t-\tau)\right]_{\xi=R} d\tau + \int_0^t \int_0^R \Phi(\xi,\tau) G(r,\xi,t-\tau)\, d\xi\, d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t)=\frac{2\xi}{R^2}\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}},\qquad \lambda_n=\frac{a^2\mu_n^2}{R^2}+b,$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu)=0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0\le r\le R$).

► Domain: $0 \le r \le R$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w&=g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,t)&=\frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi+\int_0^R f_1(\xi)G(r,\xi,t)\,d\xi\\ &\quad+a^2\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau+\int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.\end{aligned}$$

Here,

$$\begin{aligned} G(r,\xi,t)&=\frac{2\xi\sin\left(t\sqrt{b}\,\right)}{R^2\sqrt{b}}+\frac{2\xi}{R^2}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}},\\ \lambda_n&=\frac{a^2\mu_n^2}{R^2}+b,\end{aligned}$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu)=0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

► Domain: $0 \le r \le R$. Third boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w+kw&=g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\mu_n^2\xi}{(k^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n \xi}{R}\right)\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}},\qquad \lambda_n=\frac{a^2\mu_n^2}{R^2}+b.$$

Here, the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu) - kRJ_0(\mu) = 0.$$

The numerical values of the first six roots μ_n can be found in Abramowitz and Stegun (1964) and Carslaw and Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)},\\ \partial_t w &= f_1(r) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)},\\ w &= g_1(t) \quad \text{at} \quad r=R_1 \quad &&\text{(boundary condition)},\\ w &= g_2(t) \quad \text{at} \quad r=R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) =& \int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau) G(r,\xi,t-\tau)\,d\xi\,d\tau \\ &+ \frac{\partial}{\partial t}\int_{R_1}^{R_2} f_0(\xi)G(r,\xi,t)\,d\xi + \int_{R_1}^{R_2} f_1(\xi)G(r,\xi,t)\,d\xi \\ &+ a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_1} d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_2} d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t) = \frac{\pi^2}{2R_1^2}\sum_{n=1}^{\infty} \frac{\mu_n^2 J_0^2(s\mu_n)\xi}{J_0^2(\mu_n) - J_0^2(s\mu_n)} \Psi_n(r)\Psi_n(\xi) \frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}, \qquad \lambda_n = \frac{a^2\mu_n^2}{R_1^2} + b,$$

$$\Psi_n(r) = Y_0(\mu_n) J_0\left(\frac{\mu_n r}{R_1}\right) - J_0(\mu_n) Y_0\left(\frac{\mu_n r}{R_1}\right), \qquad s = \frac{R_2}{R_1},$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions and the μ_n are positive roots of the transcendental equation

$$J_0(\mu)Y_0(s\mu) - J_0(s\mu)Y_0(\mu) = 0.$$

The numerical values of the first five roots $\mu_n = \mu_n(s)$ can be found in Abramowitz and Stegun (1964) and Carslaw and Jaeger (1984).

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)},\\ \partial_t w &= f_1(r) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)},\\ \partial_r w &= g_1(t) \quad \text{at} \quad r=R_1 \quad &&\text{(boundary condition)},\\ \partial_r w &= g_2(t) \quad \text{at} \quad r=R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) = {} & \int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau) G(r,\xi,t-\tau)\, d\xi\, d\tau \\ & + \frac{\partial}{\partial t} \int_{R_1}^{R_2} f_0(\xi) G(r,\xi,t)\, d\xi + \int_{R_1}^{R_2} f_1(\xi) G(r,\xi,t)\, d\xi \\ & - a^2 \int_0^t g_1(\tau) G(r,R_1,t-\tau)\, d\tau + a^2 \int_0^t g_2(\tau) G(r,R_2,t-\tau)\, d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t) = \frac{2\xi \sin\bigl(t\sqrt{b}\,\bigr)}{(R_2^2 - R_1^2)\sqrt{b}} + \frac{\pi^2}{2R_1^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 J_1^2(s\mu_n)\xi}{J_1^2(\mu_n) - J_1^2(s\mu_n)} \Psi_n(r)\Psi_n(\xi) \frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}},$$

$$\Psi_n(r) = Y_1(\mu_n) J_0\biggl(\frac{\mu_n r}{R_1}\biggr) - J_1(\mu_n) Y_0\biggl(\frac{\mu_n r}{R_1}\biggr), \quad \lambda_n = \frac{a^2\mu_n^2}{R_1^2} + b, \quad s = \frac{R_2}{R_1},$$

where $J_k(z)$ and $Y_k(z)$ are Bessel functions ($k = 0,\ 1$); the μ_n are positive roots of the transcendental equation

$$J_1(\mu) Y_1(s\mu) - J_1(s\mu) Y_1(\mu) = 0.$$

The numerical values of the first five roots $\mu_n = \mu_n(s)$ can be found in Abramowitz and Stegun (1964).

► **Domain: $R_1 \le r \le R_2$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w - k_1 w &= g_1(t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ \partial_r w + k_2 w &= g_2(t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\beta_n^2 \bigl[k_2 J_0(\beta_n R_2) - \beta_n J_1(\beta_n R_2)\bigr]^2}{B_n \sqrt{a^2\beta_n^2 + b}}\, \xi H_n(r) H_n(\xi) \sin\Bigl(t\sqrt{a^2\beta_n^2 + b}\Bigr).$$

Here,

$$\begin{aligned} B_n &= (\beta_n^2 + k_2^2)\bigl[k_1 J_0(\beta_n R_1) + \beta_n J_1(\beta_n R_1)\bigr]^2 - (\beta_n^2 + k_1^2)\bigl[k_2 J_0(\beta_n R_2) - \beta_n J_1(\beta_n R_2)\bigr]^2, \\ H_n(r) &= \bigl[k_1 Y_0(\beta_n R_1) + \beta_n Y_1(\beta_n R_1)\bigr] J_0(\beta_n r) - \bigl[k_1 J_0(\beta_n R_1) + \beta_n J_1(\beta_n R_1)\bigr] Y_0(\beta_n r), \end{aligned}$$

where the β_n are positive roots of the transcendental equation

$$\begin{aligned} & \bigl[k_1 J_0(\beta R_1) + \beta J_1(\beta R_1)\bigr]\bigl[k_2 Y_0(\beta R_2) - \beta Y_1(\beta R_2)\bigr] \\ & \qquad - \bigl[k_2 J_0(\beta R_2) - \beta J_1(\beta R_2)\bigr]\bigl[k_1 Y_0(\beta R_1) + \beta Y_1(\beta R_1)\bigr] = 0. \end{aligned}$$

6.2.6 Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right) - bw + \Phi(r,t)$

For $b > 0$ and $\Phi \equiv 0$, this is the Klein–Gordon equation describing one-dimensional wave phenomena with central symmetry. In the problems considered for $0 \le r \le R$ the solutions bounded at $r = 0$ are sought; this is not specifically stated below.

► **Domain: $0 \le r \le R$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) &= \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R f_1(\xi)G(r,\xi,t)\,d\xi\\ &\quad - a^2\int_0^t g(\tau)\left[\frac{\partial}{\partial \xi}G(r,\xi,t-\tau)\right]_{\xi=R} d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau, \end{aligned}$$

where

$$G(r,\xi,t) = \frac{2\xi}{Rr}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi r}{R}\right)\sin\left(\frac{n\pi\xi}{R}\right)\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}}, \qquad \lambda_n = \frac{a^2\pi^2 n^2}{R^2} + b.$$

► **Domain: $0 \le r \le R$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t) &= \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R f_1(\xi)G(r,\xi,t)\,d\xi\\ &\quad + a^2\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(r,\xi,t) &= \frac{3\xi^2\sin\left(t\sqrt{b}\,\right)}{R^3\sqrt{b}} + \frac{2\xi}{Rr}\sum_{n=1}^{\infty}\frac{\mu_n^2+1}{\mu_n^2\sqrt{\lambda_n}}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n\xi}{R}\right)\sin\left(t\sqrt{\lambda_n}\,\right),\\ \lambda_n &= \frac{a^2\mu_n^2}{R^2} + b. \end{aligned}$$

Here, the μ_n are positive roots of the transcendental equation $\tan\mu - \mu = 0$; for the numerical values of the first five roots μ_n, see Section 3.2.3 (the second boundary value problem with $0 \le r \le R$).

▶ Domain: $0 \le r \le R$. Third boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ \partial_t w &= f_1(r) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ \partial_r w + kw &= g(t) \quad &\text{at}\quad r=R \quad &\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\xi,t)=\frac{2\xi}{Rr}\sum_{n=1}^{\infty}\frac{\mu_n^2+(kR-1)^2}{\mu_n^2+kR(kR-1)}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n \xi}{R}\right)\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}},\qquad \lambda_n=\frac{a^2\mu_n^2}{R^2}+b.$$

Here, the μ_n are positive roots of the transcendental equation $\mu\cot\mu + kR - 1 = 0$. The numerical values of the first five roots μ_n can be found in Carslaw and Jaeger (1984).

▶ Domain: $R_1 \le r \le R_2$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ \partial_t w &= f_1(r) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ w &= g_1(t) \quad &\text{at}\quad r=R_1 \quad &\text{(boundary condition)},\\ w &= g_2(t) \quad &\text{at}\quad r=R_2 \quad &\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,t)=&\int_0^t\int_{R_1}^{R_2}\Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau\\ &+\frac{\partial}{\partial t}\int_{R_1}^{R_2}f_0(\xi)G(r,\xi,t)\,d\xi+\int_{R_1}^{R_2}f_1(\xi)G(r,\xi,t)\,d\xi\\ &+a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_1}d\tau-a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R_2}d\tau, \end{aligned}$$

where

$$\begin{aligned} G(r,\xi,t)&=\frac{2\xi}{(R_2-R_1)r}\sum_{n=1}^{\infty}\sin\left[\frac{\pi n(r-R_1)}{R_2-R_1}\right]\sin\left[\frac{\pi n(\xi-R_1)}{R_2-R_1}\right]\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}},\\ \lambda_n&=\frac{a^2\pi^2n^2}{(R_2-R_1)^2}+b. \end{aligned}$$

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,t) = {} & \int_0^t \int_{R_1}^{R_2} \Phi(\xi,\tau) G(r,\xi,t-\tau)\,d\xi\,d\tau \\
& + \frac{\partial}{\partial t}\int_{R_1}^{R_2} f_0(\xi)G(r,\xi,t)\,d\xi + \int_{R_1}^{R_2} f_1(\xi)G(r,\xi,t)\,d\xi \\
& - a^2\int_0^t g_1(\tau)G(r,R_1,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(r,R_2,t-\tau)\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\xi,t)=\frac{3\xi^2\sin\bigl(t\sqrt{b}\,\bigr)}{(R_2^3-R_1^3)\sqrt{b}}+\frac{2\xi}{(R_2-R_1)r}\sum_{n=1}^{\infty}\frac{(1+R_2^2\lambda_n^2)\Psi_n(r)\Psi_n(\xi)\sin\bigl(t\sqrt{a^2\lambda_n^2+b}\,\bigr)}{\lambda_n^2\bigl[R_1^2+R_2^2+R_1R_2(1+R_1R_2\lambda_n^2)\bigr]\sqrt{a^2\lambda_n^2+b}},
$$

$$
\Psi_n(r)=\sin[\lambda_n(r-R_1)]+R_1\lambda_n\cos[\lambda_n(r-R_1)],
$$

where the λ_n are positive roots of the transcendental equation

$$
(\lambda^2R_1R_2+1)\tan[\lambda(R_2-R_1)]-\lambda(R_2-R_1)=0.
$$

► **Domain: $R_1 \le r \le R_2$. Third boundary value problem.**

The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w - k_1 w &= g_1(t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w + k_2 w &= g_2(t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
G(r,\xi,t)=\frac{2\xi}{r}\sum_{n=1}^{\infty}\frac{(b_2^2+R_2^2\lambda_n^2)\Psi_n(r)\Psi_n(\xi)\sin\bigl(\beta_n t\bigr)}{\beta_n\bigl[(R_2-R_1)(b_1^2+R_1^2\lambda_n^2)(b_2^2+R_2^2\lambda_n^2)+(b_1R_2+b_2R_1)(b_1b_2+R_1R_2\lambda_n^2)\bigr]},
$$

$$
\Psi_n(r)=b_1\sin[\lambda_n(r-R_1)]+R_1\lambda_n\cos[\lambda_n(r-R_1)],
$$

$$
b_1=k_1R_1+1,\quad b_2=k_2R_2-1,\quad \beta_n=\sqrt{a^2\lambda_n^2+b}.
$$

Here, the λ_n are positive roots of the transcendental equation

$$
(b_1b_2-R_1R_2\lambda^2)\sin[\lambda(R_2-R_1)]+\lambda(R_1b_2+R_2b_1)\cos[\lambda(R_2-R_1)]=0.
$$

6.3 Equations Containing Power Functions and Arbitrary Parameters

6.3.1 Equations of the Form $\frac{\partial^2 w}{\partial t^2} = (ax+b)\frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial x} + kw + \Phi(x,t)$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) + \Phi(x,t).$

For $\Phi(x,t) \equiv 0$, this equation governs small-amplitude free vibration of a hanging heavy homogeneous thread (a^2 is the acceleration due to gravity, w the deflection of the thread from the vertical axis, and x the vertical coordinate).

1°. The substitution $x = \frac{1}{4}r^2$ leads to the equation

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \Phi\left(\tfrac{1}{4}r^2, t\right),$$

which is discussed in Sections 6.2.1–6.2.2.

2°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)},\\ w &\neq \infty && \text{at} \quad x = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi \\ &\quad - a^2 l\int_0^t g(\tau)\left[\frac{\partial}{\partial \xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau, \end{aligned}$$

where

$$G(x,\xi,t) = \frac{2}{a\sqrt{l}}\sum_{n=1}^{\infty}\frac{1}{\mu_n J_1^2(\mu_n)} J_0\left(\mu_n\sqrt{\frac{x}{l}}\right) J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)\sin\left(\frac{\mu_n a t}{2\sqrt{l}}\right).$$

Here, the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

⊙ *Literature*: M. M. Smirnov (1975).

3°. Domain: $0 \le x \le l$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_x w &= g(t) && \text{at} \quad x = l \quad \text{(boundary condition)},\\
w &\neq \infty && \text{at} \quad x = 0 \quad \text{(boundedness condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t) &= \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi\\
&\quad + a^2 l\int_0^t g(\tau)G(x,l,t-\tau)\,d\tau + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,\xi,t) = \frac{t}{l} + \frac{2}{a\sqrt{l}}\sum_{n=1}^{\infty}\frac{1}{\mu_n J_0^2(\mu_n)} J_0\left(\mu_n\sqrt{\frac{x}{l}}\right) J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)\sin\left(\frac{\mu_n at}{2\sqrt{l}}\right).$$

Here, the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

4°. Domain: $0 \le x \le l$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_x w + kw &= g(t) && \text{at} \quad x = l \quad \text{(boundary condition)},\\
w &\neq \infty && \text{at} \quad x = 0 \quad \text{(boundedness condition)}.
\end{aligned}$$

The solution $w(x,t)$ is given by the formula in Item 3° with

$$G(x,\xi,t) = \frac{2}{a\sqrt{l}}\sum_{n=1}^{\infty}\frac{\mu_n}{(4k^2 l + \mu_n^2)J_0^2(\mu_n)} J_0\left(\mu_n\sqrt{\frac{x}{l}}\right) J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)\sin\left(\frac{\mu_n at}{2\sqrt{l}}\right).$$

Here, the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu) - 2k\sqrt{l}\,J_0(\mu) = 0.$$

The numerical values of the first six roots μ_n can be found in Carslaw and Jaeger (1984).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) - bw + \Phi(x,t).$

For $b < 0$ and $\Phi(x,t) \equiv 0$, this equation describes small-amplitude vibration of a heavy homogeneous thread that rotates at a constant angular velocity $\omega = \sqrt{|b|}$ about the vertical axis (a^2 is the acceleration due to gravity).

1°. The substitution $x = \frac{1}{4}r^2$ leads to the equation

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - bw + \Phi\left(\tfrac{1}{4}r^2, t\right),$$

which is discussed in Section 6.2.5.

2°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)},\\ w &\neq \infty && \text{at} \quad x = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi\\ &\quad - a^2 l\int_0^t g(\tau)\left[\frac{\partial}{\partial \xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau + \int_0^t\!\!\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$G(x,\xi,t) = \frac{1}{l}\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\mu_n\sqrt{\frac{x}{l}}\right)J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)\frac{\sin\left(t\sqrt{\lambda_n}\,\right)}{\sqrt{\lambda_n}}, \qquad \lambda_n = \frac{a^2\mu_n^2}{4l} + b,$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

⊙ *Literature*: M. M. Smirnov (1975).

3°. Domain: $0 \le x \le l$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)},\\ w &\neq \infty && \text{at} \quad x = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) &= \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi\\ &\quad + a^2 l\int_0^t g(\tau)G(x,l,t-\tau)\,d\tau + \int_0^t\!\!\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$G(r,\xi,t)=\frac{\sin\bigl(t\sqrt{b}\bigr)}{l\sqrt{b}}+\frac{1}{l}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\biggl(\mu_n\sqrt{\frac{x}{l}}\biggr)J_0\biggl(\mu_n\sqrt{\frac{\xi}{l}}\biggr)\frac{\sin\bigl(t\sqrt{\lambda_n}\bigr)}{\sqrt{\lambda_n}},$$

$$\lambda_n=\frac{a^2\mu_n^2}{4l}+b,$$

where the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu)=0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

4°. Domain: $0\le x\le l$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w&=f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w+kw&=g(t) &&\text{at}\quad x=l &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is given by the formula in Item 3° with

$$G(r,\xi,t)=\sum_{n=1}^{\infty}\frac{\mu_n^2}{l(4k^2l+\mu_n^2)J_0^2(\mu_n)}J_0\biggl(\mu_n\sqrt{\frac{x}{l}}\biggr)J_0\biggl(\mu_n\sqrt{\frac{\xi}{l}}\biggr)\frac{\sin\bigl(t\sqrt{\lambda_n}\bigr)}{\sqrt{\lambda_n}},\quad \lambda_n=\frac{a^2\mu_n^2}{4l}+b.$$

Here, the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu)-2k\sqrt{l}\,J_0(\mu)=0.$$

The numerical values of the first six roots μ_n can be found in Abramowitz and Stegun (1964) and Carslaw and Jaeger (1984).

3. $\dfrac{\partial^2 w}{\partial t^2}=a^2\dfrac{\partial}{\partial x}\Bigl[(l-x)\dfrac{\partial w}{\partial x}\Bigr].$

This equation governs small-amplitude free vibration of a heavy homogeneous thread of length l (a^2 is the acceleration due to gravity, w the deflection of the thread from the vertical axis, and x the vertical coordinate).

The change of variable $z=l-x$ leads a special case of equation 6.3.1.1 with $b=0$ and $\Phi\equiv 0$.

4. $\dfrac{\partial^2 w}{\partial t^2}=a^2\Bigl(\dfrac{2}{2n+1}x\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial w}{\partial x}\Bigr),\qquad n=1,\ 2,\ \dots$

General solution:

$$w(x,t)=\frac{\partial^{n-1}}{\partial x^{n-1}}\left[\frac{\Phi\bigl(\sqrt{2(2n+1)x}+at\bigr)+\Psi\bigl(\sqrt{2(2n+1)x}-at\bigr)}{\sqrt{x}}\right],$$

where Φ and Ψ are arbitrary functions.

⊙ *Literature*: M. M. Smirnov (1975).

5. $\dfrac{\partial^2 w}{\partial t^2} = (ax+b)\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial w}{\partial x} + cw + \Phi(x,t).$

The substitution $z = ax + b$ leads to an equation of the form 6.3.1.2:

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial}{\partial z}\left(z\frac{\partial w}{\partial z}\right) + cw + \Phi\left(\frac{z-b}{a}, t\right).$$

6. $\dfrac{\partial^2 w}{\partial t^2} = (ax+b)\dfrac{\partial^2 w}{\partial x^2} + \tfrac{1}{2}a\dfrac{\partial w}{\partial x} + cw + \Phi(x,t).$

The substitution $z = 2\sqrt{ax+b}$ leads to the equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial z^2} + cw + \Phi\left(\frac{z^2-4b}{4a}, t\right),$$

which is considered in Section 6.1.3.

7. $\dfrac{\partial^2 w}{\partial t^2} = (a_2x+b_2)\dfrac{\partial^2 w}{\partial x^2} + (a_1x+b_1)\dfrac{\partial w}{\partial x} + (a_0x+b_0)w.$

This is a special case of equation 6.5.3.4 with $f(x) = a_2x + b_2$, $g(x) = a_1x + b_1$, $h(x) = a_0x + b_0$, and $\Phi \equiv 0$.

Particular solutions:

$$w(x,t) = \exp(kx)F\left(\frac{x+q}{p}\right)\left[A\sin\left(t\sqrt{\mu}\right) + B\cos\left(t\sqrt{\mu}\right)\right] \quad \text{for} \quad \mu > 0,$$
$$w(x,t) = \exp(kx)F\left(\frac{x+q}{p}\right)\left[A\sinh\left(t\sqrt{-\mu}\right) + B\cosh\left(t\sqrt{-\mu}\right)\right] \quad \text{for} \quad \mu < 0,$$

where A, B, and μ are arbitrary constants; the coefficients k, p, q and the function $F = F(\xi)$ are listed in Table 6.2, in which

$$\mathcal{J}(\alpha,\beta;x) = C_1\Phi(\alpha,\beta;x) + C_2\Psi(\alpha,\beta;x), \qquad C_1,\ C_2 \text{ are arbitrary numbers,}$$

is an arbitrary solution of the degenerate hypergeometric equation $xy''_{xx} + (\beta - x)y'_x - \alpha y = 0$, and

$$Z_\nu(x) = C_1J_\nu(x) + C_2Y_\nu(x), \qquad C_1,\ C_2 \text{ are arbitrary numbers,}$$

is an arbitrary solution of the Bessel equation $x^2y''_{xx} + xy'_x + (x^2 - \nu^2)y = 0$.

For details about the degenerate hypergeometric functions $\Phi(a,b;x)$ and $\Psi(a,b;x)$, see Section 30.9 as well as the books by Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 1). For the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$, see Section 30.6 as well as the books by Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 2).

TABLE 6.2

The coefficients k, p, q and the function $F = F(\xi)$ determining the form of particular solutions to equation 6.3.1.7. Notation: $E(k) = b_2k^2 + b_1k + b_0 + \mu$

Conditions	k	p	q	$F = F(\xi)$	Parameters
$a_2 \neq 0,\ D \neq 0$ $D \equiv a_1^2 - 4a_0a_2$	$\dfrac{\sqrt{D} - a_1}{2a_2}$	$-\dfrac{a_2}{2a_2k + a_1}$	$\dfrac{b_2}{a_2}$	$\mathcal{J}(\alpha, \beta; \xi)$	$\alpha = E(k)/(2a_2k + a_1)$, $\beta = (a_2b_1 - a_1b_2)a_2^{-2}$
$a_2 = 0$, $a_1 \neq 0$	$-\dfrac{a_0}{a_1}$	1	$\dfrac{2b_2k + b_1}{a_1}$	$\mathcal{J}\bigl(\alpha, \frac{1}{2}; \sigma\xi^2\bigr)$	$\alpha = E(k)/(2a_1)$, $\sigma = -a_1/(2b_2)$
$a_2 \neq 0$, $a_1^2 = 4a_0a_2$	$-\dfrac{a_1}{2a_2}$	a_2	$\dfrac{b_2}{a_2}$	$\xi^\alpha Z_{2\alpha}\bigl(\sigma\sqrt{\xi}\bigr)$	$\alpha = \dfrac{1}{2} - \dfrac{2b_2k + b_1}{2a_2}$, $\sigma = 2\sqrt{E(k)}$
$a_2 = a_1 = 0$, $a_0 \neq 0$	$-\dfrac{b_1}{2b_2}$	1	$\dfrac{4(b_0 + \mu)b_2 - b_1^2}{4a_0b_2}$	$\xi^{1/2}Z_{1/3}\bigl(\sigma\xi^{3/2}\bigr)$	$\sigma = \dfrac{2}{3}\Bigl(\dfrac{a_0}{b_2}\Bigr)^{1/2}$

6.3.2 Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = (ax^2 + b)\dfrac{\partial^2 w}{\partial x^2} + cx\dfrac{\partial w}{\partial x} + kw + \Phi(x, t)$

1. $\dfrac{\partial^2 w}{\partial t^2} = x^2\dfrac{\partial^2 w}{\partial x^2} + \Phi(x, t).$

This is a special case of equation 6.3.2.2 with $a = 1$ and $b = c = 0$.

1°. Domain: $1 \leq x \leq a$. First boundary value problem.
The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x) &&\text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(x) &&\text{at} \quad t = 0 \quad \text{(initial condition)},\\
w &= g_1(t) &&\text{at} \quad x = 1 \quad \text{(boundary condition)},\\
w &= g_2(t) &&\text{at} \quad x = a \quad \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x, t) &= \int_0^t \int_1^a \Phi(\xi, \tau)G(x, \xi, t - \tau)\, d\xi\, d\tau\\
&\quad + \frac{\partial}{\partial t}\int_1^a f_0(\xi)G(x, \xi, t)\, d\xi + \int_1^a f_1(\xi)G(x, \xi, t)\, d\xi\\
&\quad + \int_0^t g_1(\tau)\left[\frac{\partial}{\partial \xi}G(x, \xi, t - \tau)\right]_{\xi=1} d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial \xi}G(x, \xi, t - \tau)\right]_{\xi=a} d\tau,
\end{aligned}
$$

where

$$
G(x, \xi, t) = \frac{2\sqrt{x}}{\xi^{3/2}\ln a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\sin(\mu_n \ln x)\sin(\mu_n \ln \xi)\sin(\lambda_n t), \qquad \mu_n = \frac{\pi n}{\ln a},\ \lambda_n = \sqrt{\mu_n^2 + \tfrac{1}{4}}.
$$

$2°$. Domain: $1 \le x \le a$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w &= g_1(t) &&\text{at}\quad x=1 &&\text{(boundary condition)},\\ \partial_x w &= g_2(t) &&\text{at}\quad x=a &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = &\int_0^t \int_1^a \Phi(\xi,\tau) G(x,\xi,t-\tau)\,d\xi\,d\tau \\ &+ \frac{\partial}{\partial t}\int_1^a f_0(\xi)G(x,\xi,t)\,d\xi + \int_1^a f_1(\xi)G(x,\xi,t)\,d\xi \\ &- \int_0^t g_1(\tau)G(x,1,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(x,a,t-\tau)\,d\tau, \end{aligned}$$

where

$$G(x,\xi,t) = \frac{at}{(a-1)\xi^2} + \frac{8\sqrt{x}}{\xi^{3/2}\ln a}\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n(1+\mu_n^2)}\varphi_n(x)\varphi_n(\xi)\sin(\lambda_n t),$$

$$\varphi_n(x) = \cos(\mu_n \ln x) - \frac{1}{2\mu_n}\sin(\mu_n\ln x),\quad \mu_n = \frac{\pi n}{\ln a},\quad \lambda_n = \sqrt{\mu_n^2 + \tfrac{1}{4}}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

2. $\dfrac{\partial^2 w}{\partial t^2} = ax^2\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x} + cw + \Phi(x,t).$

The substitution $x = ke^z$ $(k \neq 0)$ leads to the constant coefficient equation $\partial_{tt}w = a\partial_{zz}w + (b-a)\partial_z w + cw + \Phi(ke^z,t)$, which is discussed in Section 6.1.5.

3. $\dfrac{\partial^2 w}{\partial t^2} = (ax^2+b)\dfrac{\partial^2 w}{\partial x^2} + ax\dfrac{\partial w}{\partial x} + cw.$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2+b}}$ leads to the constant coefficient equation $\partial_{tt}w = \partial_{zz}w + cw$, which is discussed in Section 6.1.3.

4. $\dfrac{\partial^2 w}{\partial t^2} = a^2\dfrac{\partial}{\partial x}\Big[(l^2-x^2)\dfrac{\partial w}{\partial x}\Big] + \Phi(x,t).$

Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ w &\neq \infty &&\text{at}\quad x=l &&\text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(x,t)=\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi+\int_0^l f_1(\xi)G(x,\xi,t)\,d\xi$$
$$+a^2l^2\int_0^t g(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=0}d\tau+\int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau.$$

Here,

$$G(x,\xi,t)=\frac{1}{al}\sum_{n=1}^{\infty}\frac{4n-1}{\lambda_n}P_{2n-1}\left(\frac{x}{l}\right)P_{2n-1}\left(\frac{\xi}{l}\right)\sin(\lambda_n at),\qquad \lambda_n=\sqrt{2n(2n-1)},$$

where $P_k(x)=\dfrac{1}{2^k k!}\dfrac{d^k}{dx^k}\left[(x^2-1)^k\right]$ are the Legendre polynomials.

⊙ *Literature*: M. M. Smirnov (1975).

6.3.3 Other Equations

1. $\displaystyle \frac{\partial^2 w}{\partial t^2}=a^2\left[\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial w}{\partial r}-\frac{n(n+1)}{r^2}w\right],\qquad n=1,\ 2,\ 3,\ \dots$

General solution:

$$w(r,t)=r^n\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n\left[\frac{\Phi(r+at)+\Psi(r-at)}{r}\right],$$

where $\Phi(r_1)$ and $\Psi(r_2)$ are arbitrary functions.

⊙ *Literature*: M. M. Smirnov (1975).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial x^2}+\frac{\alpha}{x}\frac{\partial w}{\partial x}.$

The hyperbolic Euler–Poisson–Darboux equation.

1°. For $\alpha=1$ and $\alpha=2$, see Sections 6.2.1–6.2.4. For $\alpha\neq 1$, the substitution $z=x^{1-\alpha}$ leads to an equation of the form 6.5.3.1:

$$\frac{\partial^2 w}{\partial t^2}=(1-\alpha)^2 z^{\frac{2\alpha}{\alpha-1}}\frac{\partial^2 w}{\partial z^2}.$$

2°. Suppose $w_\alpha=w_\alpha(x,t)$ is a solution of the equation in question for a fixed value of the parameter α. Then the functions $\widetilde{w}_\alpha$ defined by the relations

$$\widetilde{w}_\alpha=\frac{\partial w_\alpha}{\partial t},$$
$$\widetilde{w}_\alpha=x\frac{\partial w_\alpha}{\partial x}+t\frac{\partial w_\alpha}{\partial t},$$
$$\widetilde{w}_\alpha=2xt\frac{\partial w_\alpha}{\partial x}+(x^2+t^2)\frac{\partial w_\alpha}{\partial t}+\alpha t w_\alpha$$

are also solutions of this equation.

3°. Suppose $w_\alpha = w_\alpha(x,t)$ is a solution of the equation in question for a fixed value of the parameter α. Using this w_α, one can construct solutions of the equation with other values of the parameter by the formulas

$$\begin{aligned}
w_{2-\alpha} &= x^{\alpha-1} w_\alpha,\\
w_{\alpha-2} &= x\frac{\partial w_\alpha}{\partial x} + (\alpha-1)w_\alpha,\\
w_{\alpha-2} &= xt\frac{\partial w_\alpha}{\partial x} + x^2\frac{\partial w_\alpha}{\partial t} + (\alpha-1)tw_\alpha,\\
w_{\alpha-2} &= x(x^2+t^2)\frac{\partial w_\alpha}{\partial x} + 2x^2t\frac{\partial w_\alpha}{\partial t} + \left[x^2+(\alpha-1)t^2\right] w_\alpha,\\
w_{\alpha+2} &= \frac{1}{x}\frac{\partial w_\alpha}{\partial x},\\
w_{\alpha+2} &= \frac{t}{x}\frac{\partial w_\alpha}{\partial x} + \frac{\partial w_\alpha}{\partial t},\\
w_{\alpha+2} &= \frac{x^2+t^2}{x}\frac{\partial w_\alpha}{\partial x} + 2t\frac{\partial w_\alpha}{\partial t} + \alpha w_\alpha.
\end{aligned}$$

⊙ The results of Items 2° and 3° were obtained by A. V. Aksenov (2001).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{2a}{x}\frac{\partial w}{\partial x} + b^2 w, \qquad 0 < 2a < 1.$

General solution:

$$\begin{aligned}
w(x,t) &= \int_0^1 \frac{\Phi\bigl(t + x(2\xi-1)\bigr)}{[\xi(1-\xi)]^{1-a}} \bar{J}_{a-1}\bigl(2bx\sqrt{\xi(1-\xi)}\,\bigr)\,d\xi\\
&\quad + x^{1-2a}\int_0^1 \frac{\Psi\bigl(t + x(2\xi-1)\bigr)}{[\xi(1-\xi)]^{a}} \bar{J}_{-a}\bigl(2bx\sqrt{\xi(1-\xi)}\,\bigr)\,d\xi,
\end{aligned}$$

where $\Phi(\xi_1)$ and $\Psi(\xi_2)$ are arbitrary functions; $\bar{J}_{-\nu}(z) = \Gamma(1-\nu)2^{-\nu}z^\nu J_{-\nu}(z)$; $J_{-\nu}(z)$ is the Bessel function.

⊙ *Literature*: M. M. Smirnov (1975).

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^4\frac{\partial^2 w}{\partial x^2} + \Phi(x,t).$

The transformation $z = 1/x$, $u = w/x$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial z^2} + z\Phi\left(\frac{1}{z}, t\right),$$

which is discussed in Section 6.1.2.

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = (ax+b)^4\frac{\partial^2 w}{\partial x^2}.$

The transformation

$$u = \frac{w}{ax+b}, \quad z = at + \frac{1}{ax+b}, \quad y = -at + \frac{1}{ax+b}$$

leads to the equation $\partial_{zy}u = 0$. Thus, the general solution of the original equation has the form

$$w = (ax+b)[f(z)+g(y)],$$

where $f = f(z)$ and $g = g(y)$ are arbitrary functions.

⊙ *Literature*: N. H. Ibragimov (1994).

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = (a^2 - x^2)^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x,t).$

Domain: $-l \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x=l && \text{(boundary condition)},\\ w &= 0 && \text{at} \quad x=-l && \text{(boundary condition)}. \end{aligned}$$

Solution for $0 < l < a$:

$$w(x,t) = \frac{\partial}{\partial t}\int_{-l}^{l} f(\xi)G(x,\xi,t)\,d\xi + \int_{-l}^{l} g(\xi)G(x,\xi,t)\,d\xi + \int_0^t \int_{-l}^{l} \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t) = \frac{2a}{k(\xi^2-a^2)^2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\varphi_n(x)\varphi_n(\xi)\sin(\lambda_n t),$$

$$\varphi_n(x) = \sqrt{a^2-x^2}\,\sin\left(\frac{\pi n}{2} + \frac{\pi n}{2k}\ln\frac{a+x}{a-x}\right), \quad \lambda_n = \frac{a}{k}\sqrt{\pi^2 n^2 + k^2}, \quad k = \ln\frac{a+l}{a-l}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = (x-a_1)^2(x-a_2)^2\frac{\partial^2 w}{\partial x^2}, \qquad a_1 \neq a_2.$

The transformation

$$w(x,t) = (x-a_2)u(\xi,\tau), \quad \xi = \ln\left|\frac{x-a_1}{x-a_2}\right|, \quad \tau = |a_1-a_2|t$$

leads to the constant coefficient equation $\partial_{\tau\tau}u = \partial_{\xi\xi}u - \partial_\xi u$, which is discussed in Section 6.1.4.

8. $\displaystyle \frac{\partial^2 w}{\partial t^2} = (ax^2+bx+c)^2\frac{\partial^2 w}{\partial x^2}.$

The transformation

$$w(x,t) = u(z,t)\sqrt{|ax^2+bx+c|}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

leads to the constant coefficient equation $\partial_{tt}u = \partial_{zz}u + \left(ac - \tfrac{1}{4}b^2\right)u$, which is discussed in Section 6.1.3.

9. $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial}{\partial x}\left(x^m \dfrac{\partial w}{\partial x}\right) + \Phi(x,t).$

1°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

1.1. Case $0 < m < 1$:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,t) = {}& \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l f_1(\xi)G(x,\xi,t)\,d\xi \\ & - a^2 l^m \int_0^t g(\tau)\left[\frac{\partial}{\partial \xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau. \end{aligned} \tag{1}$$

Here,

$$G(x,\xi,t) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)\sin(\lambda_n a t)}{a\|y_n\|^2\lambda_n}, \qquad \lambda_n = \frac{\mu_n}{2}(2-m)l^{\frac{m-2}{2}}, \tag{2}$$

where

$$y_n(x) = x^{\frac{1-m}{2}} J_p\biggl(\mu_n\Bigl(\frac{x}{l}\Bigr)^{\frac{2-m}{2}}\biggr), \qquad \|y_n\|^2 = \int_0^l y_n^2(x)\,dx, \qquad p = \left|\frac{1-m}{2-m}\right|;$$

the μ_n are positive zeros of the Bessel function, $J_p(\mu) = 0$.

1.2. Case $1 \le m < 2$:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &\ne \infty && \text{at} \quad x = 0 && \text{(boundedness condition)},\\ w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

The solution is given by the formulas presented in Item 1.1.

⊙ *Literature*: M. M. Smirnov (1975).

2°. Domain: $0 \le x \le l$. Mixed boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ (x^m \partial_x w) &= 0 && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

The solution for $0 < m < 1$ is given by relations (1) and (2) with

$$y_n(x) = x^{\frac{1-m}{2}} J_{-p}\left(\mu_n \left(\frac{x}{l}\right)^{\frac{2-m}{2}}\right), \quad \|y_n\|^2 = \int_0^l y_n^2(x)\,dx, \quad p = \frac{1-m}{2-m};$$

the μ_n are positive zeros of the Bessel function, $J_{-p}(\mu) = 0$.

3°. For $\Phi \equiv 0$, the change of variable $z = x^{1-m}$ leads to an equation of the form 6.3.3.10:

$$\frac{\partial^2 w}{\partial t^2} = a^2(1-m)^2 z^{\frac{m}{m-1}} \frac{\partial^2 w}{\partial z^2}.$$

10. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a^2 x^m \frac{\partial^2 w}{\partial x^2}.$

1°. Particular solutions (A_1, A_2, B_1, B_2, and μ are arbitrary constants):

$$w(x,t) = \sqrt{x}\left[A_1 J_{\frac{1}{2q}}(\mu x^q) + A_2 Y_{\frac{1}{2q}}(\mu x^q)\right]\left[B_1 \sin(aq\mu t) + B_2 \cos(aq\mu t)\right],$$

$$w(x,t) = \sqrt{x}\left[A_1 I_{\frac{1}{2q}}(\mu x^q) + A_2 K_{\frac{1}{2q}}(\mu x^q)\right]\left[B_1 \sinh(aq\mu t) + B_2 \cosh(aq\mu t)\right],$$

where $q = \frac{1}{2}(2-m)$; $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions; $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions.

2°. Below are discrete transformations that preserve the form of the original equation; what changes is the parameter n.

2.1. The point transformation

$$z = \frac{1}{x}, \quad u = \frac{w}{x} \qquad \text{(transformation } \mathcal{P}\text{)}$$

leads to a similar equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 z^{4-m} \frac{\partial^2 u}{\partial z^2}.$$

The transformation $\mathcal{P}$ changes the equation parameter in accordance with the rule $m \overset{\mathcal{P}}{\Longrightarrow} 4-m$. The double application of the transformation $\mathcal{P}$ yields the original equation.

2.2. Suppose $w = w(x,t)$ is a solution of the original equation. Then the function $v = v(\xi,\tau)$, which is related to the solution $w = w(x,t)$ by the Bäcklund transformation

$$v(\xi,\tau) = \frac{\partial}{\partial x} w(x,t), \quad x = \xi^{\frac{1}{1-m}}, \quad \tau = |1-m|t \qquad \text{(transformation } \mathcal{B}\text{)},$$

is a solution of a similar equation

$$\frac{\partial^2 v}{\partial \tau^2} = a^2 \xi^{\frac{m}{m-1}} \frac{\partial^2 v}{\partial \xi^2}.$$

The transformation $\mathcal{B}$ changes the equation parameters in accordance with the rule $m \overset{\mathcal{B}}{\Longrightarrow} \dfrac{m}{m-1}$. The double application of the transformation $\mathcal{B}$ yields the original equation.

2.3. The composition of transformations $\mathcal{F} = \mathcal{B} \circ \mathcal{P}$ changes the equation parameter as follows:

$$m \overset{\mathcal{F}}{\Longrightarrow} \frac{4-m}{3-m} \overset{\mathcal{F}}{\Longrightarrow} \frac{8-3m}{5-2m} \overset{\mathcal{F}}{\Longrightarrow} \frac{12-5m}{7-3m} \overset{\mathcal{F}}{\Longrightarrow} \frac{16-7m}{9-4m} \overset{\mathcal{F}}{\Longrightarrow} \cdots$$

The n-fold application of the transformation $\mathcal{F}$ yields the equation with parameter

$$m \overset{\mathcal{F}^n}{\Longrightarrow} \frac{4n-(2n-1)m}{2n+1-nm}. \tag{1}$$

2.4. The composition of transformations $\mathcal{G} = \mathcal{P} \circ \mathcal{B}$ changes the equation parameter as follows:

$$m \overset{\mathcal{G}}{\Longrightarrow} \frac{4-3m}{1-m} \overset{\mathcal{G}}{\Longrightarrow} \frac{8-5m}{3-2m} \overset{\mathcal{G}}{\Longrightarrow} \frac{12-7m}{5-3m} \overset{\mathcal{G}}{\Longrightarrow} \frac{16-9m}{7-4m} \overset{\mathcal{G}}{\Longrightarrow} \cdots$$

The n-fold application of the transformation $\mathcal{G}$ yields the equation with parameter

$$m \overset{\mathcal{G}^n}{\Longrightarrow} \frac{4n-(2n+1)m}{2n-1-nm}. \tag{2}$$

2.5. Setting $m = 0$ in (1) and (2), we arrive at two families of equations

$$\frac{\partial^2 w}{\partial t^2} = a^2 x^{\frac{4n}{2n+1}} \frac{\partial^2 w}{\partial x^2} \quad \text{at} \quad n = 1,\ 2,\ \ldots;$$
$$\frac{\partial^2 w}{\partial t^2} = a^2 x^{\frac{4n}{2n-1}} \frac{\partial^2 w}{\partial x^2} \quad \text{at} \quad n = 1,\ 2,\ \ldots;$$

whose solutions can be obtained with the aid of the wave equation; for this constant coefficient wave equation, see Section 6.1.1.

3°. Below are some useful transformations that lead to other equations.

3.1. The substitution $\xi = x^{1-m}$ leads to an equation of the form 6.3.3.9:

$$\frac{\partial^2 w}{\partial t^2} = a^2(1-m)^2 \frac{\partial}{\partial \xi}\left(\xi^{\frac{m}{m-1}} \frac{\partial w}{\partial \xi}\right).$$

3.2. The transformation $\tau = \frac{1}{2}a|2-m|t$, $\xi = x^{\frac{2-m}{2}}$ leads to an equation of the form 6.3.3.2:

$$\frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial \xi^2} + \frac{m}{m-2}\frac{1}{\xi}\frac{\partial w}{\partial \xi}.$$

11. $\displaystyle \frac{\partial^2 w}{\partial t^2} = t^m \frac{\partial^2 w}{\partial x^2}.$

1°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution for $m > 0$:

$$w(x,t) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \int_0^1 f\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}} (2\xi - 1)\right) [\xi(1-\xi)]^{\beta-1} d\xi$$
$$+ \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)} t \int_0^1 g\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}} (2\xi - 1)\right) [\xi(1-\xi)]^{-\beta} d\xi,$$

where

$$\beta = \frac{m}{2(m+2)}, \qquad \Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds.$$

⊙ *Literature*: M. M. Smirnov (1975).

2°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= 0 && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution for $m > -1$:

$$w(x,t) = \sqrt{t} \sum_{n=1}^{\infty} \Big[A_n J_{-p}\Big(2p\lambda_n t^{\frac{1}{2p}}\Big) + B_n J_p\Big(2p\lambda_n t^{\frac{1}{2p}}\Big) \Big] \sin(\lambda_n x),$$
$$A_n = \Gamma(1-p)(\lambda_n p)^p \frac{2}{l} \int_0^l f(x) \sin(\lambda_n x)\, dx, \quad p = \frac{1}{m+2},$$
$$B_n = \Gamma(1+p)(\lambda_n p)^{-p} \frac{2}{l} \int_0^l g(x) \sin(\lambda_n x)\, dx, \quad \lambda_n = \frac{\pi n}{l},$$

where $\Gamma(p)$ is the gamma function.

⊙ *Literature*: M. M. Smirnov (1975).

12. $\displaystyle \frac{\partial^2 w}{\partial t^2} = t^m \frac{\partial^2 w}{\partial x^2} + b t^{\frac{m-2}{2}} \frac{\partial w}{\partial x}, \qquad m \ge 2.$

Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0,\\ \partial_t w &= g(x) && \text{at} \quad t = 0. \end{aligned}$$

1°. Solution for $|b| < \frac{1}{2}m$:

$$w(x,t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 f\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}} (2\xi - 1)\right) \xi^{\beta-1} (1-\xi)^{\alpha-1} d\xi$$
$$+ \frac{\Gamma(2-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} t \int_0^1 g\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}} (2\xi - 1)\right) \xi^{-\alpha} (1-\xi)^{-\beta} d\xi,$$

where

$$\alpha = \frac{m-2b}{2(m+2)}, \quad \beta = \frac{m+2b}{2(m+2)}, \quad \Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds.$$

2°. Solution for $b = \frac{1}{2}m$:

$$w(x,t) = f\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}}\right) + \frac{2t}{m+2} \int_0^1 g\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}} (2\xi - 1)\right)(1-\xi)^{-\frac{m}{m+2}} d\xi.$$

3°. Solution for $b = -\frac{1}{2}m$:

$$w(x,t) = f\left(x - \frac{2}{m+2} t^{\frac{m+2}{2}}\right) + \frac{2t}{m+2} \int_0^1 g\left(x + \frac{2}{m+2} t^{\frac{m+2}{2}} (2\xi - 1)\right)(1-\xi)^{-\frac{m}{m+2}} d\xi.$$

⊙ *Literature*: M. M. Smirnov (1975).

13. $\displaystyle (b+x)^2 \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left[(b+x)^2 \frac{\partial w}{\partial x}\right].$

General solution:

$$w(x,t) = \frac{f(x+at) + g(x-at)}{b+x},$$

where $f(y)$ and $g(z)$ are arbitrary functions.

6.4 Equations Containing the First Time Derivative

6.4.1 Equations of the Form $\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + \Phi(x,t)$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x,t).$

For $\Phi(x,t) \equiv 0$, this equation governs free transverse vibration of a string, and also longitudinal vibration of a rod in a resisting medium with a velocity-proportional resistance coefficient.

1°. The substitution $w(x,t) = \exp\left(-\frac{1}{2}kt\right)u(x,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \tfrac{1}{4}k^2 u + \exp\left(\tfrac{1}{2}kt\right)\Phi(x,t),$$

which is considered in Section 6.1.3.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2a}\vartheta\left(at - |x|\right) \exp\left(-\tfrac{1}{2}kt\right) I_0\left(\tfrac{1}{2}k\sqrt{t^2 - x^2/a^2}\,\right),$$

where $\vartheta(z)$ is the Heaviside unit step function and $I_0(z)$ is the modified Bessel function.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned}
w(x,t) &= \tfrac{1}{2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\bigl[f(x+at)+f(x-at)\bigr] \\
&+ \frac{kt}{4a}\exp\bigl(-\tfrac{1}{2}kt\bigr)\int_{x-at}^{x+at}\frac{I_1\bigl(\frac{1}{2}k\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi \\
&+ \frac{1}{2a}\exp\bigl(-\tfrac{1}{2}kt\bigr)\int_{x-at}^{x+at} I_0\bigl(\tfrac{1}{2}k\sqrt{t^2-(x-\xi)^2/a^2}\,\bigr)\bigl[g(\xi)+\tfrac{1}{2}kf(\xi)\bigr]\,d\xi \\
&+ \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\bigl[-\tfrac{1}{2}k(t-\tau)\bigr]I_0\bigl(\tfrac{1}{2}k\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\,\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau,
\end{aligned}$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions of the first kind.

4°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\
w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\
w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t) &= \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau \\
&+ \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l\bigl[f_1(\xi)+kf_0(\xi)\bigr]G(x,\xi,t)\,d\xi \\
&+ a^2\int_0^t g_1(\tau)\biggl[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\biggr]_{\xi=0}d\tau - a^2\int_0^t g_2(\tau)\biggl[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\biggr]_{\xi=l}d\tau,
\end{aligned}$$

where

$$G(x,\xi,t) = \frac{2}{l}\exp\left(-\frac{kt}{2}\right)\sum_{n=1}^{\infty}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sin(\lambda_n t)}{\lambda_n}, \qquad \lambda_n = \sqrt{\frac{a^2\pi^2n^2}{l^2} - \frac{k^2}{4}}.$$

Example 6.3. Consider the homogeneous equation ($\Phi \equiv 0$). The initial shape of the string is a triangle with base $0 \le x \le l$ and height h at $x = c$, that is,

$$f(x) = \begin{cases} \dfrac{hx}{c} & \text{if } 0 \le x \le c, \\[2mm] \dfrac{h(l-x)}{l-c} & \text{if } c \le x \le l. \end{cases}$$

The initial velocities of the string points are zero, $g(x) = 0$.

Solution:

$$w(x,t) = \frac{2hl^2}{\pi^2 c(l-c)} \exp\left(-\tfrac{1}{2}kt\right) \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \Theta_n(t),$$

where

$$\Theta_n(t) = \begin{cases} \cos(\lambda_n t) + \dfrac{k}{2\lambda_n}\sin(\lambda_n t) & \text{if } k < \dfrac{2\pi na}{l}, \\ 1 + \dfrac{kt}{2} & \text{if } k = \dfrac{2\pi na}{l}, \\ \cosh(\lambda_n t) + \dfrac{k}{2\lambda_n}\sinh(\lambda_n t) & \text{if } k > \dfrac{2\pi na}{l}, \end{cases} \qquad \lambda_n = \sqrt{\left|\frac{a^2 n^2 \pi^2}{l^2} - \frac{k^2}{4}\right|}.$$

⊙ *Literature*: M. M. Smirnov (1975), B. M. Budak, A. N. Tikhonov, and A. A. Samarskii (1980).

5°. For the second and third boundary value problems on the interval $0 \le x \le l$, see equation 6.4.1.2 (Items 5° and 6° with $b = 0$).

6°. Domain: $0 \le x < \infty$. A problem without initial conditions for $\Phi = 0$. The following boundary conditions are prescribed:

$$w = A\cos(\omega t + \gamma) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty.$$

Solution:

$$w = Ae^{-\lambda x}\cos(\omega t - \beta x + \gamma),$$

where

$$\lambda = \left(\frac{\omega\sqrt{k^2+\omega^2} - \omega^2}{2a^2}\right)^{1/2}, \qquad \beta = \left(\frac{\omega\sqrt{k^2+\omega^2} + \omega^2}{2a^2}\right)^{1/2}.$$

2. $\boldsymbol{\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a^2\dfrac{\partial^2 w}{\partial x^2} + bw + \Phi(x,t).}$

Telegraph equation (with $k > 0$, $b < 0$, and $\Phi(x,t) \equiv 0$).

1°. The substitution $w(x,t) = \exp\left(-\tfrac{1}{2}kt\right)u(x,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \left(b + \tfrac{1}{4}k^2\right)u + \exp\left(\tfrac{1}{2}kt\right)\Phi(x,t),$$

which is considered in Section 6.1.3.

2°. Fundamental solutions:

$$\mathscr{E}(x,t) = \frac{1}{2a}\vartheta\left(at - |x|\right)\exp\left(-\tfrac{1}{2}kt\right)I_0\left(c\sqrt{t^2 - x^2/a^2}\right) \quad \text{for} \quad b + \tfrac{1}{4}k^2 = c^2 > 0,$$
$$\mathscr{E}(x,t) = \frac{1}{2a}\vartheta\left(at - |x|\right)\exp\left(-\tfrac{1}{2}kt\right)J_0\left(c\sqrt{t^2 - x^2/a^2}\right) \quad \text{for} \quad b + \tfrac{1}{4}k^2 = -c^2 < 0,$$

where $\vartheta(z)$ is the Heaviside unit step function, $J_0(z)$ and $J_1(z)$ are Bessel functions, and $I_0(z)$ and $I_1(z)$ are modified Bessel functions.

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution for $b + \frac{1}{4}k^2 = c^2 > 0$:

$$\begin{aligned}
w(x,t) &= \tfrac{1}{2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\bigl[f(x+at)+f(x-at)\bigr] \\
&+ \frac{ct}{2a}\exp\bigl(-\tfrac{1}{2}kt\bigr)\int_{x-at}^{x+at}\frac{I_1\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi \\
&+ \frac{1}{2a}\exp\bigl(-\tfrac{1}{2}kt\bigr)\int_{x-at}^{x+at} I_0\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\bigr)\bigl[g(\xi)+\tfrac{1}{2}kf(\xi)\bigr]\,d\xi \\
&+ \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\bigl[-\tfrac{1}{2}k(t-\tau)\bigr]I_0\bigl(c\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau.
\end{aligned}$$

Solution for $b + \frac{1}{4}k^2 = -c^2 < 0$:

$$\begin{aligned}
w(x,t) &= \tfrac{1}{2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\bigl[f(x+at)+f(x-at)\bigr] \\
&- \frac{ct}{2a}\exp\bigl(-\tfrac{1}{2}kt\bigr)\int_{x-at}^{x+at}\frac{J_1\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\bigr)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi \\
&+ \frac{1}{2a}\exp\bigl(-\tfrac{1}{2}kt\bigr)\int_{x-at}^{x+at} J_0\bigl(c\sqrt{t^2-(x-\xi)^2/a^2}\bigr)\bigl[g(\xi)+\tfrac{1}{2}kf(\xi)\bigr]\,d\xi \\
&+ \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\bigl[-\tfrac{1}{2}k(t-\tau)\bigr]J_0\bigl(c\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\bigr)\Phi(\xi,\tau)\,d\xi\,d\tau.
\end{aligned}$$

4°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) && \text{at} \quad t = 0 \quad \text{(initial condition)}, \\
\partial_t w &= f_1(x) && \text{at} \quad t = 0 \quad \text{(initial condition)}, \\
w &= g_1(t) && \text{at} \quad x = 0 \quad \text{(boundary condition)}, \\
w &= g_2(t) && \text{at} \quad x = l \quad \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t) &= \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau \\
&+ \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l \bigl[f_1(\xi)+kf_0(\xi)\bigr]G(x,\xi,t)\,d\xi \\
&+ a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=0}d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=l}d\tau.
\end{aligned}$$

Let $a^2\pi^2 - bl^2 - \frac{1}{4}k^2l^2 > 0$. Then

$$G(x,\xi,t)=\frac{2}{l}\exp\left(-\frac{kt}{2}\right)\sum_{n=1}^{\infty}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}},\quad \lambda_n=\frac{a^2\pi^2n^2}{l^2}-b-\frac{k^2}{4}.$$

Let $a^2\pi^2n^2 - bl^2 - \frac{1}{4}k^2l^2 \le 0$ for $n = 1,\dots,m$ and $a^2\pi^2n^2 - bl^2 - \frac{1}{4}k^2l^2 > 0$ for $n = m+1, m+2,\dots$ Then

$$G(x,\xi,t) = \frac{2}{l}\exp\left(-\frac{kt}{2}\right)\sum_{n=1}^{m}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sinh\left(t\sqrt{\beta_n}\right)}{\sqrt{\beta_n}}$$
$$+\frac{2}{l}\exp\left(-\frac{kt}{2}\right)\sum_{n=m+1}^{\infty}\sin\left(\frac{\pi nx}{l}\right)\sin\left(\frac{\pi n\xi}{l}\right)\frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}},$$
$$\beta_n = b+\frac{k^2}{4}-\frac{a^2\pi^2n^2}{l^2},\qquad \lambda_n=\frac{a^2\pi^2n^2}{l^2}-b-\frac{k^2}{4}.$$

5°. Domain: $0 \le x \le l$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(t) && \text{at} \quad x=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(x,t) = \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l \bigl[f_1(\xi)+kf_0(\xi)\bigr]G(x,\xi,t)\,d\xi$$
$$-a^2\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau.$$

For $p = b + \frac{1}{4}k^2 < 0$,

$$G(x,\xi,t) = \exp\left(-\tfrac{1}{2}kt\right)\left[\frac{\sin\left(t\sqrt{|p|}\right)}{l\sqrt{|p|}} + \frac{2}{l}\sum_{n=1}^{\infty}\cos(\mu_n x)\cos(\mu_n\xi)\frac{\sin\left(t\sqrt{a^2\mu_n^2-p}\right)}{\sqrt{a^2\mu_n^2-p}}\right],$$
$$\mu_n = \pi n/l.$$

For $p = b + \frac{1}{4}k^2 > 0$,

$$G(x,\xi,t) = \exp\left(-\tfrac{1}{2}kt\right)\left[\frac{\sinh\left(t\sqrt{p}\right)}{l\sqrt{p}} + \frac{2}{l}\sum_{n=1}^{\infty}\cos(\mu_n x)\cos(\mu_n\xi)\frac{\sin\left(t\sqrt{a^2\mu_n^2-p}\right)}{\sqrt{a^2\mu_n^2-p}}\right],$$
$$\mu_n = \pi n/l.$$

If the inequality $a^2\mu_n^2 - p < 0$ holds for several first values $n = 1,\dots,m$, then the expressions $\sqrt{a^2\mu_n^2-p}$ should be replaced by $\sqrt{|a^2\mu_n^2-p|}$ and the sines by the hyperbolic sines in the corresponding terms of the series.

6°. Domain: $0 \le x \le l$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_x w - s_1 w &= g_1(t) && \text{at} \quad x=0 \quad \text{(boundary condition)},\\ \partial_x w + s_2 w &= g_2(t) && \text{at} \quad x=l \quad \text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is determined by the formula in Item 5° with

$$G(x,\xi,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)\sin\bigl(t\sqrt{a^2\mu_n^2-p}\,\bigr)}{B_n\sqrt{a^2\mu_n^2-p}}, \qquad p = b + \tfrac{1}{4}k^2,$$

$$y_n(x) = \cos(\mu_n x) + \frac{s_1}{\mu_n}\sin(\mu_n x), \quad B_n = \frac{s_2}{2\mu_n^2}\frac{\mu_n^2+s_1^2}{\mu_n^2+s_2^2} + \frac{s_1}{2\mu_n^2} + \frac{l}{2}\left(1+\frac{s_1^2}{\mu_n^2}\right).$$

Here, the μ_n are positive roots of the transcendental equation $\dfrac{\tan(\mu l)}{\mu} = \dfrac{s_1+s_2}{\mu^2 - s_1 s_2}$.

If the inequality $a^2\mu_n^2 - p < 0$ holds for several first values $n = 1, \dots, m$, then the expressions $\sqrt{a^2\mu_n^2-p}$ should be replaced by $\sqrt{|a^2\mu_n^2-p|}$ and the sines by the hyperbolic sines in the corresponding terms of the series.

7°. Domain: $0 \le x < \infty$. A problem without initial conditions for $\Phi = 0$.
The following boundary conditions are prescribed:

$$w = A\cos(\omega t + \gamma) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty.$$

Solution:

$$w = Ae^{-\lambda x}\cos(\omega t - \beta x + \gamma),$$

where

$$\lambda = \left[\frac{\sqrt{k^2\omega^2 + (\omega^2+b)^2} - \omega^2 - b}{2a^2}\right]^{1/2}, \qquad \beta = \left[\frac{\sqrt{k^2\omega^2 + (\omega^2+b)^2} + \omega^2 + b}{2a^2}\right]^{1/2}.$$

3. $\boldsymbol{\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a^2\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + cw + \Phi(x,t).}$

1°. The substitution $w(x,t) = \exp\bigl(-\tfrac{1}{2}a^{-2}bx - \tfrac{1}{2}kt\bigr)u(x,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \bigl(c + \tfrac{1}{4}k^2 - \tfrac{1}{4}a^{-2}b^2\bigr)u + \exp\bigl(\tfrac{1}{2}a^{-2}bx + \tfrac{1}{2}kt\bigr)\Phi(x,t),$$

which is discussed in Section 6.1.3.

2°. Fundamental solutions:

$$\mathscr{E}(x,t) = \frac{1}{2a}\vartheta\bigl(at-|x|\bigr)\exp\left(-\frac{bx}{2a^2}-\frac{kt}{2}\right)I_0\left(\sigma\sqrt{t^2-\frac{x^2}{a^2}}\right) \quad \text{if} \quad c+\frac{k^2}{4}-\frac{b^2}{4a^2}=\sigma^2>0,$$

$$\mathscr{E}(x,t) = \frac{1}{2a}\vartheta\bigl(at-|x|\bigr)\exp\left(-\frac{bx}{2a^2}-\frac{kt}{2}\right)J_0\left(\sigma\sqrt{t^2-\frac{x^2}{a^2}}\right) \quad \text{if} \quad c+\frac{k^2}{4}-\frac{b^2}{4a^2}=-\sigma^2<0,$$

where $\vartheta(z)$ is the Heaviside unit step function, $J_0(z)$ and $J_1(z)$ are Bessel functions, and $I_0(z)$ and $I_1(z)$ are modified Bessel functions.

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution for $c + \frac{1}{4}k^2 - \frac{1}{4}a^{-2}b^2 = \sigma^2 > 0$:

$$w(x,t) = \frac{1}{2}\exp\left(-\frac{kt}{2}\right)\left[f(x+at)\exp\left(\frac{bt}{2a}\right) + f(x-at)\exp\left(-\frac{bt}{2a}\right)\right]$$
$$+\frac{\sigma t}{2a}\exp\left(-\frac{bx}{2a^2} - \frac{kt}{2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)\frac{I_1\left(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\right)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi$$
$$+\frac{1}{2a}\exp\left(-\frac{bx}{2a^2} - \frac{kt}{2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)I_0\left(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\right)\left[g(\xi) + \tfrac{1}{2}kf(\xi)\right]d\xi$$
$$+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\left[\frac{b(\xi-x)}{2a^2} - \frac{k(t-\tau)}{2}\right]I_0\left(\sigma\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\right)\Phi(\xi,\tau)\,d\xi\,d\tau.$$

Solution for $c + \frac{1}{4}k^2 - \frac{1}{4}a^{-2}b^2 = -\sigma^2 < 0$:

$$w(x,t) = \frac{1}{2}\exp\left(-\frac{kt}{2}\right)\left[f(x+at)\exp\left(\frac{bt}{2a}\right) + f(x-at)\exp\left(-\frac{bt}{2a}\right)\right]$$
$$-\frac{\sigma t}{2a}\exp\left(-\frac{bx}{2a^2} - \frac{kt}{2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)\frac{J_1\left(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\right)}{\sqrt{t^2-(x-\xi)^2/a^2}}f(\xi)\,d\xi$$
$$+\frac{1}{2a}\exp\left(-\frac{bx}{2a^2} - \frac{kt}{2}\right)\int_{x-at}^{x+at}\exp\left(\frac{b\xi}{2a^2}\right)J_0\left(\sigma\sqrt{t^2-(x-\xi)^2/a^2}\right)\left[g(\xi) + \tfrac{1}{2}kf(\xi)\right]d\xi$$
$$+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\exp\left[\frac{b(\xi-x)}{2a^2} - \frac{k(t-\tau)}{2}\right]J_0\left(\sigma\sqrt{(t-\tau)^2-(x-\xi)^2/a^2}\right)\Phi(\xi,\tau)\,d\xi\,d\tau.$$

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

4°. Domain: $0 \le x \le l$. First boundary value problem.
The following conditions are prescribed:

$$w = f_0(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$w = g_1(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)},$$
$$w = g_2(t) \quad \text{at} \quad x = l \quad \text{(boundary condition)}.$$

Solution:

$$w(x,t) = \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+\frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l\left[f_1(\xi) + kf_0(\xi)\right]G(x,\xi,t)\,d\xi$$
$$+a^2\int_0^t g_1(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=0}d\tau - a^2\int_0^t g_2(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=l}d\tau.$$

Let $a^2\pi^2 + \frac{1}{4}a^{-2}b^2l^2 - cl^2 - \frac{1}{4}k^2l^2 > 0$. Then

$$G(x,\xi,t) = \frac{2}{l}\exp\left[\frac{b(\xi - x)}{2a^2} - \frac{kt}{2}\right]\sum_{n=1}^{\infty}\sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}},$$

$$\lambda_n = \frac{a^2\pi^2n^2}{l^2} + \frac{b^2}{4a^2} - c - \frac{k^2}{4}.$$

Let

$$a^2\pi^2n^2 + \tfrac{1}{4}a^{-2}b^2l^2 - cl^2 - \tfrac{1}{4}k^2l^2 \le 0 \quad \text{for} \quad n = 1, \dots, m;$$
$$a^2\pi^2n^2 + \tfrac{1}{4}a^{-2}b^2l^2 - cl^2 - \tfrac{1}{4}k^2l^2 > 0 \quad \text{for} \quad n = m+1, m+2, \dots$$

Then

$$G(x,\xi,t) = \frac{2}{l}\exp\left[\frac{b(\xi - x)}{2a^2} - \frac{kt}{2}\right]\sum_{n=1}^{m}\sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\frac{\sinh\left(t\sqrt{\beta_n}\right)}{\sqrt{\beta_n}}$$
$$+ \frac{2}{l}\exp\left[\frac{b(\xi - x)}{2a^2} - \frac{kt}{2}\right]\sum_{n=m+1}^{\infty}\sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}},$$

where $\beta_n = c + \dfrac{k^2}{4} - \dfrac{a^2\pi^2n^2}{l^2} - \dfrac{b^2}{4a^2}$ and $\lambda_n = \dfrac{a^2\pi^2n^2}{l^2} + \dfrac{b^2}{4a^2} - c - \dfrac{k^2}{4}$.

⊙ *Literature*: A. G. Butkovskiy (1979).

5°. Domain: $0 \le x \le l$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g_1(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ \partial_x w &= g_2(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau$$
$$+ \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l \left[f_1(\xi) + kf_0(\xi)\right]G(x,\xi,t)\,d\xi$$
$$- a^2\int_0^t g_1(\tau)G(x,0,t-\tau)\,d\tau + a^2\int_0^t g_2(\tau)G(x,l,t-\tau)\,d\tau.$$

For $p = c + \frac{1}{4}k^2 < 0$,

$$G(x,\xi,t) = A\exp\left(\frac{b\xi}{a^2} - \frac{kt}{2}\right)\frac{\sin\left(t\sqrt{|p|}\right)}{\sqrt{|p|}}$$
$$+ \frac{2}{l}\exp\left[\frac{b(\xi - x)}{2a^2} - \frac{kt}{2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)}{1+\mu_n^2}\frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}},$$

where

$$A=\frac{b}{a^2\big(e^{bl/a^2}-1\big)},\qquad \lambda_n=\frac{a^2\pi^2n^2}{l^2}+\frac{b^2}{4a^2}-c-\frac{k^2}{4},$$
$$y_n(x)=\cos\left(\frac{\pi nx}{l}\right)+\mu_n\sin\left(\frac{\pi nx}{l}\right),\qquad \mu_n=\frac{bl}{2a^2\pi n}.$$

For $p=c+\frac{1}{4}k^2>0$,

$$G(x,\xi,t)=A\exp\left(\frac{b\xi}{a^2}-\frac{kt}{2}\right)\frac{\sinh\big(t\sqrt{p}\,\big)}{\sqrt{p}}$$
$$+\frac{2}{l}\exp\left[\frac{b(\xi-x)}{2a^2}-\frac{kt}{2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)}{1+\mu_n^2}\frac{\sin\big(t\sqrt{\lambda_n}\,\big)}{\sqrt{\lambda_n}},$$

where the coefficient A, λ_n, μ_n and the functions $y_n(x)$ remain as before. If the inequality $\lambda_n<0$ holds for several first values $n=1,\dots,m$, then the expressions $\sqrt{\lambda_n}$ must be replaced by $\sqrt{|\lambda_n|}$ and the sines by the hyperbolic sines in the corresponding terms of the series.

6°. Domain: $0\le x\le l$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w&=f_0(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_x w-s_1w&=g_1(t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\ \partial_x w+s_2w&=g_2(t) &&\text{at}\quad x=l &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(x,t)$ is determined by the formula in Item 5° with

$$G(x,\xi,t)=\exp\left[\frac{b(\xi-x)}{2a^2}-\frac{kt}{2}\right]\sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)\sin\big(t\sqrt{\lambda_n}\big)}{B_n\sqrt{\lambda_n}}.$$

Here,

$$y_n(x)=\cos(\mu_nx)+\frac{2a^2s_1+b}{2a^2\mu_n}\sin(\mu_nx),\qquad \lambda_n=a^2\mu_n^2+\frac{b^2}{4a^2}-c-\frac{k^2}{4},$$
$$B_n=\frac{2a^2s_2-b}{4a^2\mu_n^2}\frac{4a^4\mu_n^2+(2a^2s_1+b)^2}{4a^4\mu_n^2+(2a^2s_2-b)^2}+\frac{2a^2s_1+b}{4a^2\mu_n^2}+\frac{l}{2}+\frac{l(2a^2s_1+b)^2}{8a^4\mu_n^2},$$

where the μ_n are positive roots of the transcendental equation

$$\frac{\tan(\mu l)}{\mu}=\frac{4a^4(s_1+s_2)}{4a^4\mu^2-(2a^2s_1+b)(2a^2s_2-b)}.$$

6.4.2 Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + h(x)w + \Phi(x,t)$

1. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a^2\Bigl(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\Bigr).$

This equation describes vibration of a circular membrane in a resisting medium characterized by a velocity-proportional resistance coefficient.

1°. Domain: $0 \le r \le R$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= g(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x = R && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\bigl[A_n\cos(\lambda_n t) + B_n\sin(\lambda_n t)\bigr]J_0\biggl(\frac{\mu_n r}{R}\biggr), \qquad \lambda_n = \sqrt{\frac{a^2\mu_n^2}{R^2} - \frac{k^2}{4}}.$$

Here,

$$A_n = \frac{2}{R^2 J_1^2(\mu_n)}\int_0^R f(r)J_0\biggl(\frac{\mu_n r}{R}\biggr)r\,dr, \quad B_n = \frac{A_n k}{2\lambda_n} + \frac{2}{\lambda_n R^2 J_1^2(\mu_n)}\int_0^R g(r)J_0\biggl(\frac{\mu_n r}{R}\biggr)r\,dr,$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$.

2°. For the solution of the second and third boundary value problems, see equation 6.4.2.2 (Items 3° and 4° with $b = 0$).

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a^2\Bigl(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\Bigr) - bw + \Phi(r,t).$

1°. The substitution $w(r,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)u(r,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\biggl(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\biggr) - \bigl(b - \tfrac{1}{4}k^2\bigr)u + \exp\bigl(\tfrac{1}{2}kt\bigr)\Phi(r,t),$$

which is discussed in Section 6.2.5.

2°. Domain: $0 \le r \le R$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g(t) && \text{at} \quad r = R && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R \bigl[f_1(\xi)+kf_0(\xi)\bigr]G(r,\xi,t)\,d\xi$$
$$- a^2\int_0^t g(\tau)\left[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\right]_{\xi=R} d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

Here,

$$G(r,\xi,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\frac{2\xi}{R^2J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}},$$
$$\lambda_n = \frac{a^2\mu_n^2}{R^2} + b - \frac{k^2}{4},$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

3°. Domain: $0 \le r \le R$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) &&\text{at } t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r) &&\text{at } t=0 &&\text{(initial condition)},\\ \partial_r w &= g(t) &&\text{at } r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi + \int_0^R \bigl[f_1(\xi)+kf_0(\xi)\bigr]G(r,\xi,t)\,d\xi$$
$$+ a^2\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

Here,

$$G(r,\xi,t)=\exp\bigl(-\tfrac{1}{2}kt\bigr)\left[\frac{2\xi\sin\bigl(t\sqrt{\lambda_0}\,\bigr)}{R^2\sqrt{\lambda_0}}+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\xi}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}\right],$$

where $\lambda_0 = b-\frac{1}{4}k^2$; $\lambda_n = a^2\mu_n^2R^{-2}+b-\frac{1}{4}k^2$; the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

4°. Domain: $0 \le r \le R$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) &&\text{at } t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r) &&\text{at } t=0 &&\text{(initial condition)},\\ \partial_r w + sw &= g(t) &&\text{at } r=R &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,t)$ is given by the formula in Item 3° with

$$G(r,\xi,t)=\frac{2}{R^2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\frac{\mu_n^2\xi}{(s^2R^2+\mu_n^2)J_0^2(\mu_n)}J_0\biggl(\frac{\mu_n r}{R}\biggr)J_0\biggl(\frac{\mu_n\xi}{R}\biggr)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}.$$

Here, $\lambda_n=a^2\mu_n^2R^{-2}+b-\frac{1}{4}k^2$ and the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu)-sRJ_0(\mu)=0.$$

The numerical values of the first six roots μ_n can be found in Abramowitz and Stegun (1964) and Carslaw and Jaeger (1984).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2}+k\frac{\partial w}{\partial t}=a^2\biggl(\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial w}{\partial r}\biggr)-bw+\Phi(r,t).$

1°. The substitution $w(r,t)=\exp\bigl(-\frac{1}{2}kt\bigr)u(r,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\biggl(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\biggr)-\bigl(b-\tfrac{1}{4}k^2\bigr)u+\exp\bigl(\tfrac{1}{2}kt\bigr)\Phi(r,t),$$

which is discussed in Section 6.2.6.

2°. Domain: $0\le r\le R$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,t)=\frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi \int_0^R\bigl[f_1(\xi)+kf_0(\xi)\bigr]G(r,\xi,t)\,d\xi\\ -a^2\int_0^t g(\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\xi,t-\tau)\biggr]_{\xi=R}d\tau+\int_0^t\!\!\int_0^R\Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,\end{aligned}$$

where

$$G(r,\xi,t)=\frac{2\xi}{Rr}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sin\biggl(\frac{n\pi r}{R}\biggr)\sin\biggl(\frac{n\pi\xi}{R}\biggr)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}},\quad \lambda_n=\frac{a^2\pi^2n^2}{R^2}+b-\frac{k^2}{4}.$$

3°. Domain: $0\le r\le R$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w&=g(t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f_0(\xi)G(r,\xi,t)\,d\xi \int_0^R \bigl[f_1(\xi)+kf_0(\xi)\bigr]G(r,\xi,t)\,d\xi$$
$$+\,a^2\int_0^t g(\tau)G(r,R,t-\tau)\,d\tau + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(r,\xi,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)\Biggl[\frac{3\xi^2\sin\bigl(t\sqrt{\lambda_0}\,\bigr)}{R^3\sqrt{\lambda_0}}$$
$$+\frac{2\xi}{Rr}\sum_{n=1}^{\infty}\frac{\mu_n^2+1}{\mu_n^2\sqrt{\lambda_n}}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n\xi}{R}\right)\sin\bigl(t\sqrt{\lambda_n}\,\bigr)\Biggr].$$

Here, $\lambda_0 = b-\frac{1}{4}k^2$; $\lambda_n = a^2\mu_n^2R^{-2}+b-\frac{1}{4}k^2$; and the μ_n are positive roots of the transcendental equation $\tan\mu-\mu=0$. The numerical values of the first five roots μ_n are specified in Section 3.2.3 (see the second boundary value problem for $0\le r\le R$).

4°. Domain: $0\le r\le R$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + sw &= g(t) && \text{at} \quad r=R && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,t)$ is given by the formula in Item 3° with

$$G(r,\xi,t) = \frac{2\xi}{Rr}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\frac{\mu_n^2+(sR-1)^2}{\mu_n^2+sR(sR-1)}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n\xi}{R}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}.$$

Here, $\lambda_n = a^2\mu_n^2R^{-2}+b-\frac{1}{4}k^2$ and the μ_n are positive roots of the transcendental equation $\mu\cot\mu+sR-1=0$. The numerical values of the first six roots μ_n can be found in Carslaw and Jaeger (1984).

4. $\displaystyle \boldsymbol{\frac{\partial^2 w}{\partial t^2}+k\frac{\partial w}{\partial t} = a^2\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right)-bw+\Phi(x,t).}$

1°. The substitution $w(r,t)=\exp\bigl(-\frac{1}{2}kt\bigr)u(r,t)$ leads to an equation of the form 6.3.1.2:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)-\bigl(b-\tfrac{1}{4}k^2\bigr)u+\exp\bigl(\tfrac{1}{2}kt\bigr)\Phi(x,t).$$

2°. Domain: $0\le x\le l$. First boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g(t) && \text{at} \quad x=l && \text{(boundary condition)},\\ w &\ne \infty && \text{at} \quad x=0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l \bigl[f_1(\xi) + kf_0(\xi)\bigr]G(x,\xi,t)\,d\xi$$
$$- a^2 l\int_0^t g(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,t-\tau)\right]_{\xi=l} d\tau + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t) = \frac{1}{l}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\mu_n\sqrt{\frac{x}{l}}\right)J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}.$$

Here, $\lambda_n=\frac{1}{4}a^2\mu_n^2 l^{-1}+b-\frac{1}{4}k^2$; the μ_n are positive zeros of the Bessel function, $J_0(\mu)=0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

3°. Domain: $0 \le x \le l$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w &= g(t) && \text{at} \quad x = l && \text{(boundary condition)},\\ w &\ne \infty && \text{at} \quad x = 0 && \text{(boundedness condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \frac{\partial}{\partial t}\int_0^l f_0(\xi)G(x,\xi,t)\,d\xi + \int_0^l \bigl[f_1(\xi) + kf_0(\xi)\bigr]\,d\xi$$
$$+ a^2 l\int_0^t g(\tau)G(x,l,t-\tau)\,d\tau + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(r,\xi,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)\left[\frac{\sin\bigl(t\sqrt{\lambda_0}\,\bigr)}{l\sqrt{\lambda_0}}\right.$$
$$\left. + \frac{1}{l}\sum_{n=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\mu_n\sqrt{\frac{x}{l}}\right)J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)\frac{\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\sqrt{\lambda_n}}\right].$$

Here, $\lambda_0 = b - \frac{1}{4}k^2$; $\lambda_n = \frac{1}{4}a^2\mu_n^2 l^{-1} + b - \frac{1}{4}k^2$; the μ_n are positive zeros of the first-order Bessel function, $J_1(\mu) = 0$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

4°. Domain: $0 \le x \le l$. Third boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_x w + kw &= g(t) && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

The solution $w(x,t)$ is given by the formula in Item 3° with

$$G(r,\xi,t)=\frac{1}{l}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\frac{\mu_n^2}{(4k^2l+\mu_n^2)J_0^2(\mu_n)}J_0\biggl(\mu_n\sqrt{\frac{x}{l}}\biggr)J_0\biggl(\mu_n\sqrt{\frac{\xi}{l}}\biggr)\frac{\sin\bigl(t\sqrt{\lambda_n}\bigr)}{\sqrt{\lambda_n}}.$$

Here, $\lambda_n=\frac{1}{4}a^2\mu_n^2l^{-1}+b-\frac{1}{4}k^2$, and the μ_n are positive roots of the transcendental equation

$$\mu J_1(\mu)-2k\sqrt{l}\,J_0(\mu)=0.$$

The numerical values of the first six roots μ_n can be found in Abramowitz and Stegun (1964) and Carslaw and Jaeger (1984).

5. $\displaystyle\frac{\partial^2 w}{\partial t^2}+k\frac{\partial w}{\partial t}=(ax^m+b)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}amx^{m-1}\frac{\partial w}{\partial x}+cw.$

The substitution $z=\displaystyle\int\frac{dx}{\sqrt{ax^m+b}}$ leads to a constant coefficient equation of the form 6.4.1.2: $\partial_{tt}w+k\partial_t w=\partial_{zz}w+cw$.

6.4.3 Other Equations

1. $\displaystyle\frac{\partial^2 w}{\partial t^2}+\frac{k-1}{t}\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}.$

Darboux equation. Domain: $-\infty<x<\infty$. Cauchy problem.
Initial conditions are prescribed:

$$w=f(x)\quad\text{at}\quad t=0,$$
$$\partial_t w=0\quad\text{at}\quad t=0.$$

Solution:

$$w(x,t)=\frac{\Gamma\bigl(\frac{k}{2}\bigr)}{\sqrt{\pi}\,\Gamma\bigl(\frac{k}{2}-\frac{1}{2}\bigr)}\int_{-1}^{1}f(x+t\xi)(1-\xi^2)^{\frac{k-3}{2}}\,d\xi\qquad(k>1).$$

⊙ *Literature*: R. Courant and D. Hilbert (1989).

2. $\displaystyle\frac{\partial^2 w}{\partial t^2}+\frac{2a}{t}\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}-b^2w.$

Domain: $-\infty<x<\infty$. Cauchy problem.
Initial conditions are prescribed:

$$w=f(x)\quad\text{at}\quad t=0,$$
$$t^{2a}\partial_t w=g(x)\quad\text{at}\quad t=0.$$

Solution for $0<2a<1$:

$$w(x,t)=\frac{\Gamma(2a)}{\Gamma^2(a)}\int_0^1 f\bigl(x+t(2\xi-1)\bigr)\bar{J}_{a-1}\bigl(2bt\sqrt{\xi(1-\xi)}\,\bigr)\xi^{a-1}(1-\xi)^{a-1}d\xi$$
$$+\frac{\Gamma(2-2a)t^{1-2a}}{(1-2a)\Gamma^2(1-a)}\int_0^1 g\bigl(x+t(2\xi-1)\bigr)\bar{J}_{-a}\bigl(2bt\sqrt{\xi(1-\xi)}\,\bigr)\xi^{-a}(1-\xi)^{-a}d\xi,$$

where

$$\bar{J}_\nu(z) = 2^\nu \Gamma(1+\nu) z^{-\nu} J_\nu(z), \quad \Gamma(\nu) = \int_0^\infty e^{-s} s^{\nu-1}\,ds.$$

⊙ *Literature*: M. M. Smirnov (1975).

3. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{2a}{t}\dfrac{\partial w}{\partial t} = t^m \dfrac{\partial^2 w}{\partial x^2}.$

Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$t^{2a}\partial_t w = g(x) \quad \text{at} \quad t = 0.$$

Solution for $0 \le 2a < 1$ and $m > 0$:

$$w(x,t) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \int_0^1 f\biggl(x + \frac{2}{2+m} t^{\frac{2+m}{2}}(2\xi - 1)\biggr) \xi^{\beta-1}(1-\xi)^{\beta-1}\,d\xi$$
$$+ \frac{\Gamma(2-2\beta) t^{1-2a}}{(1-2a)\Gamma^2(1-\beta)} \int_0^1 g\biggl(x + \frac{2}{2+m} t^{\frac{2+m}{2}}(2\xi - 1)\biggr) \xi^{-\beta}(1-\xi)^{-\beta}\,d\xi,$$

where

$$\beta = \frac{m+4a}{2(m+2)}, \quad \Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\,ds.$$

⊙ *Literature*: M. M. Smirnov (1975).

4. $t^2 \dfrac{\partial^2 w}{\partial t^2} + kt \dfrac{\partial w}{\partial t} = a^2 \dfrac{\partial^2 w}{\partial x^2} + b \dfrac{\partial w}{\partial x} + cw.$

The substitution $t = Ae^\tau$ $(A \neq 0)$ leads to a constant coefficient equation of the form 6.4.1.3:

$$\frac{\partial^2 w}{\partial \tau^2} + (k-1)\frac{\partial w}{\partial \tau} = a^2 \frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw.$$

5. $t^2 \dfrac{\partial^2 w}{\partial t^2} + kt \dfrac{\partial w}{\partial t} = a^2 x^2 \dfrac{\partial^2 w}{\partial x^2} + bx \dfrac{\partial w}{\partial x} + cw.$

The transformation

$$t = Ae^\tau, \quad x = Be^\xi \qquad (A \neq 0,\ B \neq 0)$$

leads to a constant coefficient equation of the form 6.4.1.3:

$$\frac{\partial^2 w}{\partial \tau^2} + (k-1)\frac{\partial w}{\partial \tau} = a^2 \frac{\partial^2 w}{\partial \xi^2} + (b - a^2)\frac{\partial w}{\partial \xi} + cw.$$

6. $\boldsymbol{t^m \dfrac{\partial^2 w}{\partial t^2} + a t^{m-1} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}, \qquad 0 < m < 2.}$

Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0,$$
$$t^a \partial_t w = g(x) \quad \text{at} \quad t = 0.$$

1°. Solution for $\frac{1}{2}m < a < 1$:

$$w(x,t) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \int_0^1 f\biggl(x + \frac{2}{2-m} t^{\frac{2-m}{2}}(2\xi - 1)\biggr) \xi^{\beta-1}(1-\xi)^{\beta-1}\,d\xi$$
$$+ \frac{\Gamma(2-2\beta)}{(1-a)\Gamma^2(1-\beta)} t^{1-a} \int_0^1 g\biggl(x + \frac{2}{2-m} t^{\frac{2-m}{2}}(2\xi - 1)\biggr) \xi^{-\beta}(1-\xi)^{-\beta}\,d\xi,$$

where

$$\beta = \frac{2a-m}{2(2-m)}, \quad \Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\,ds.$$

2°. Solution for $a = \frac{1}{2}m$:

$$w(x,t) = \frac{f(y)+f(z)}{2} + \frac{1}{2}\int_z^y g(\xi)\,d\xi,$$
$$y = x - \frac{2}{2-m} t^{\frac{2-m}{2}}, \quad z = x + \frac{2}{2-m} t^{\frac{2-m}{2}}.$$

⊙ *Literature*: M. M. Smirnov (1975).

7. $\boldsymbol{(t^m + k)\dfrac{\partial^2 w}{\partial t^2} + \tfrac{1}{2} m t^{m-1} \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + cw.}$

The substitution $\tau = \displaystyle\int \frac{dt}{\sqrt{t^m + k}}$ leads to the equation $\partial_{\tau\tau} w = a\partial_{xx} w + b\partial_x w + cw$, which is discussed in Section 6.1.5.

6.5 Equations Containing Arbitrary Functions

6.5.1 Equations of the Form $s(x)\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\Bigl[p(x)\dfrac{\partial w}{\partial x}\Bigr] - q(x)w + \Phi(x,t)$

It is assumed that the functions s, p, p'_x, and q are continuous and the inequalities $s > 0$, $p > 0$ hold for $x_1 \le x \le x_2$.

► **General relations to solve linear nonhomogeneous boundary value problems.**

The solution of the equation in question under the general initial conditions

$$\begin{aligned} w &= f_0(x) \quad \text{at} \quad t=0,\\ \partial_t w &= f_1(x) \quad \text{at} \quad t=0 \end{aligned} \tag{1}$$

and the arbitrary linear nonhomogeneous boundary conditions

$$\begin{aligned} a_1\partial_x w + b_1 w &= g_1(t) \quad \text{at} \quad x=x_1,\\ a_2\partial_x w + b_2 w &= g_2(t) \quad \text{at} \quad x=x_2 \end{aligned} \tag{2}$$

can be represented as the sum

$$\begin{aligned} w(x,t) &= \int_0^t \int_{x_1}^{x_2} \Phi(\xi,\tau)\mathcal{G}(x,\xi,t-\tau)\,d\xi\,d\tau\\ &+ \frac{\partial}{\partial t}\int_{x_1}^{x_2} s(\xi)f_0(\xi)\mathcal{G}(x,\xi,t)\,d\xi + \int_{x_1}^{x_2} s(\xi)f_1(\xi)\mathcal{G}(x,\xi,t)\,d\xi\\ &+ p(x_1)\int_0^t g_1(\tau)\Lambda_1(x,t-\tau)\,d\tau + p(x_2)\int_0^t g_2(\tau)\Lambda_2(x,t-\tau)\,d\tau. \end{aligned} \tag{3}$$

Here, the modified Green's function is determined by

$$\mathcal{G}(x,\xi,t) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\|y_n\|^2\sqrt{\lambda_n}}, \qquad \|y_n\|^2 = \int_{x_1}^{x_2} s(x)y_n^2(x)\,dx, \tag{4}$$

where the λ_n and $y_n(x)$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the second-order linear ordinary differential equation

$$\begin{aligned} &[p(x)y'_x]'_x + [\lambda s(x) - q(x)]y = 0,\\ &a_1 y'_x + b_1 y = 0 \quad \text{at} \quad x=x_1,\\ &a_2 y'_x + b_2 y = 0 \quad \text{at} \quad x=x_2. \end{aligned} \tag{5}$$

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ that occur in the integrands of the last two terms in solution (3) are expressed in terms of the Green's function of (4). The corresponding formulas will be specified below in studying specific boundary value problems.

General properties of the Sturm–Liouville problem (5):

1°. There are finitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$; hence the number of negative eigenvalues is finite.

2°. Any two eigenfunctions $y_n(x)$ and $y_m(x)$ for $n \neq m$ are orthogonal to each other with weight $s(x)$ on the interval $x_1 \le x \le x_2$; specifically,

$$\int_{x_1}^{x_2} s(x)y_n(x)y_m(x)\,dx = 0 \quad \text{at} \quad n \neq m.$$

3°. If the conditions

$$q(x) \ge 0, \quad a_1b_1 \le 0, \quad a_2b_2 \ge 0 \tag{6}$$

are satisfied, then there are no negative eigenvalues. If $q \equiv 0$ and $b_1 = b_2 = 0$, the least eigenvalue is $\lambda_1 = 0$ and the corresponding eigenfunction is $\varphi_1 = \text{const}$. In the other cases where conditions (6) are satisfied, all eigenvalues are positive.

Remark 6.2. More detailed information about the properties of the Sturm–Liouville problem (5) can be found in Section 3.8.9. Asymptotic and approximate formulas for eigenvalues and eigenfunctions are also presented there.

► **First boundary value problem (case $a_1 = a_2 = 0$, $b_1 = b_2 = 1$).**

The solution of the first boundary value problem for the equation in question with the initial conditions (1) and the boundary conditions

$$w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by relations (3) and (4) in which

$$\Lambda_1(x,t) = \frac{\partial}{\partial\xi}\mathcal{G}(x,\xi,t)\Big|_{\xi=x_1}, \quad \Lambda_2(x,t) = -\frac{\partial}{\partial\xi}\mathcal{G}(x,\xi,t)\Big|_{\xi=x_2}.$$

► **Second boundary value problem (case $a_1 = a_2 = 1$, $b_1 = b_2 = 0$).**

The solution of the second boundary value problem for the equation in question with the initial conditions (1) and the boundary conditions

$$\partial_x w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$\partial_x w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by relations (3) and (4) with

$$\Lambda_1(x,t) = -\mathcal{G}(x,x_1,t), \quad \Lambda_2(x,t) = \mathcal{G}(x,x_2,t).$$

► **Third boundary value problem (case $a_1 = a_2 = 1$, $b_1 \neq 0$, $b_2 \neq 0$).**

The solution of the third boundary value problem for the equation in question with the initial conditions (1) and the boundary conditions (2) with $a_1 = a_2 = 1$ is given by relations (3) and (4) in which

$$\Lambda_1(x,t) = -\mathcal{G}(x,x_1,t), \quad \Lambda_2(x,t) = \mathcal{G}(x,x_2,t).$$

► **Mixed boundary value problem (case $a_1 = b_2 = 0$, $a_2 = b_1 = 1$).**

The solution of the mixed boundary value problem for the equation in question with the initial conditions (1) and the boundary conditions

$$w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$\partial_x w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by relations (3) and (4) with

$$\Lambda_1(x,t) = \frac{\partial}{\partial\xi}\mathcal{G}(x,\xi,t)\Big|_{\xi=x_1}, \quad \Lambda_2(x,t) = \mathcal{G}(x,x_2,t).$$

► **Mixed boundary value problem (case $a_1 = b_2 = 1$, $a_2 = b_1 = 0$).**

The solution of the mixed boundary value problem with the initial conditions (1) and the boundary conditions

$$\partial_x w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$w = g_2(t) \quad \text{at} \quad x = x_2$$

is given by relations (3) and (4) with

$$\Lambda_1(x,t) = -\mathcal{G}(x, x_1, t), \quad \Lambda_2(x,t) = -\frac{\partial}{\partial \xi}\mathcal{G}(x,\xi,t)\Big|_{\xi=x_2}.$$

⊙ *Literature for Section* 6.5.1: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), V. A. Marchenko (1986), V. S. Vladimirov (1988), A. D. Polyanin (2000a).

6.5.2 Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} + a(t)\dfrac{\partial w}{\partial t} = b(t)\Big\{\dfrac{\partial}{\partial x}\Big[p(x)\dfrac{\partial w}{\partial x}\Big] - q(x)w\Big\} + \Phi(x,t)$

It is assumed that the functions p, p'_x, and q are continuous and $p > 0$ for $x_1 \le x \le x_2$.

► **General relations to solve linear nonhomogeneous boundary value problems.**

The solution of the equation in question under the general initial conditions

$$w = f_0(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0 \qquad (1)$$

and the arbitrary linear nonhomogeneous boundary conditions

$$s_1 \partial_x w + k_1 w = g_1(t) \quad \text{at} \quad x = x_1,$$
$$s_2 \partial_x w + k_2 w = g_2(t) \quad \text{at} \quad x = x_2 \qquad (2)$$

can be represented as the sum

$$w(x,t) = \int_0^t \int_{x_1}^{x_2} \Phi(\xi,\tau) G(x,\xi,t,\tau)\, d\xi\, d\tau$$
$$- \int_{x_1}^{x_2} f_0(\xi)\left[\frac{\partial}{\partial \tau} G(x,\xi,t,\tau)\right]_{\tau=0} d\xi + \int_{x_1}^{x_2} \big[f_1(\xi) + a(0) f_0(\xi)\big] G(x,\xi,t,0)\, d\xi$$
$$+ p(x_1) \int_0^t g_1(\tau) b(\tau) \Lambda_1(x,t,\tau)\, d\tau + p(x_2) \int_0^t g_2(\tau) b(\tau) \Lambda_2(x,t,\tau)\, d\tau. \qquad (3)$$

Here, the modified Green's function is determined by

$$G(x,\xi,t,\tau) = \sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{\|y_n\|^2} U_n(t,\tau), \qquad \|y_n\|^2 = \int_{x_1}^{x_2} y_n^2(x)\, dx, \qquad (4)$$

where the λ_n and $y_n(x)$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the following second-order linear ordinary differential equation with homogeneous boundary conditions:

$$\begin{aligned}&[p(x)y'_x]'_x + [\lambda - q(x)]y = 0,\\ &s_1 y'_x + k_1 y = 0 \quad \text{at} \quad x = x_1,\\ &s_2 y'_x + k_2 y = 0 \quad \text{at} \quad x = x_2.\end{aligned} \tag{5}$$

The functions $U_n = U_n(t,\tau)$ are determined by solving the Cauchy problem for the linear ordinary differential equation

$$\begin{gathered}U''_n + a(t)U'_n + \lambda_n b(t)U_n = 0,\\ U_n\big|_{t=\tau} = 0, \quad U'_n\big|_{t=\tau} = 1.\end{gathered} \tag{6}$$

The prime denotes the derivative with respect to t, and τ is a free parameter occurring in the initial conditions.

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ that occur in the integrands of the last two terms in solution (3) are expressed in terms of the Green's function of (4). The corresponding formulas will be specified below when studying specific boundary value problems.

The properties of the Sturm–Liouville problem (5) are detailed in Section 3.8.9. Asymptotic and approximate formulas for eigenvalues and eigenfunctions are also presented there.

► First, second, third, and mixed boundary value problems.

1°. *First boundary value problem.* The solution of the equation in question with the initial conditions (1) and boundary conditions (2) for $s_1 = s_2 = 0$ and $k_1 = k_2 = 1$ is given by relations (3) and (4), where

$$\Lambda_1(x,t,\tau) = \frac{\partial}{\partial\xi}G(x,\xi,t,\tau)\Big|_{\xi=x_1}, \quad \Lambda_2(x,t,\tau) = -\frac{\partial}{\partial\xi}G(x,\xi,t,\tau)\Big|_{\xi=x_2}.$$

2°. *Second boundary value problem.* The solution of the equation with the initial conditions (1) and boundary conditions (2) for $s_1 = s_2 = 1$ and $k_1 = k_2 = 0$ is given by relations (3) and (4) with

$$\Lambda_1(x,t,\tau) = -G(x,x_1,t,\tau), \quad \Lambda_2(x,t,\tau) = G(x,x_2,t,\tau).$$

3°. *Third boundary value problem.* The solution of the equation with the initial conditions (1) and boundary conditions (2) for $s_1 = s_2 = 1$ and $k_1 k_2 \neq 0$ is given by relations (3) and (4) in which

$$\Lambda_1(x,t,\tau) = -G(x,x_1,t,\tau), \quad \Lambda_2(x,t,\tau) = G(x,x_2,t,\tau).$$

4°. *Mixed boundary value problem.* The solution of the equation with the initial conditions (1) and boundary conditions (2) for $s_1 = k_2 = 0$ and $s_2 = k_1 = 1$ is given by relations (3) and (4) with

$$\Lambda_1(x,t,\tau) = \frac{\partial}{\partial\xi}G(x,\xi,t,\tau)\Big|_{\xi=x_1}, \quad \Lambda_2(x,t,\tau) = G(x,x_2,t,\tau).$$

5°. *Mixed boundary value problem.* The solution of the equation with the initial conditions (1) and boundary conditions (2) for $s_1 = k_2 = 1$ and $s_2 = k_1 = 0$ is given by relations (3) and (4) with

$$\Lambda_1(x,t,\tau) = -G(x,x_1,t,\tau), \quad \Lambda_2(x,t,\tau) = -\frac{\partial}{\partial\xi}G(x,\xi,t,\tau)\Big|_{\xi=x_2}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), A. V. Bitsadze and D. F. Kalinichenko (1985), A. D. Polyanin (2000a).

6.5.3 Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of the equation of Section 6.5.1 with $s(x) = 1/f(x)$, $p(x) = 1$, and $q = \Phi = 0$.

1°. Particular solutions:

$$w = C_1 xt + C_2 t + C_3 x + C_4,$$
$$w = C_1 t^2 + C_2 xt + C_3 t + C_4 x + 2C_1 \int_a^x \frac{x-\xi}{f(\xi)}\,d\xi + C_5,$$
$$w = C_1 t^3 + C_2 xt + C_3 t + C_4 x + 6C_1 t \int_a^x \frac{x-\xi}{f(\xi)}\,d\xi + C_5,$$
$$w = (C_1 x + C_2)t^2 + C_3 xt + C_4 t + C_5 x + 2\int_a^x (x-\xi)\frac{(C_1\xi + C_2)}{f(\xi)}\,d\xi + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants and a is an arbitrary real number.

2°. Separable particular solution:

$$w = (C_1 e^{\lambda t} + C_2 e^{-\lambda t})H(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $H = H(x)$ is determined by the ordinary differential equation $f(x)H''_{xx} - \lambda^2 H = 0$.

3°. Separable particular solution:

$$w = [C_1 \sin(\lambda t) + C_2 \cos(\lambda t)]Z(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $Z = Z(x)$ is determined by the ordinary differential equation $f(x)Z''_{xx} + \lambda^2 Z = 0$.

4°. Particular solutions with even powers of t:

$$w = \sum_{k=0}^{n} \varphi_k(x)t^{2k},$$

where the functions $\varphi_k = \varphi_k(x)$ are defined by the recurrence relations

$$\varphi_n(x) = A_n x + B_n,$$
$$\varphi_{k-1}(x) = A_k x + B_k + 2k(2k-1)\int_a^x (x-\xi)\frac{\varphi_k(\xi)}{f(\xi)}\,d\xi,$$

where A_k, B_k are arbitrary constants ($k = n, \ldots, 1$).

5°. Particular solutions with odd powers of t:

$$w = \sum_{k=0}^{n} \psi_k(x) t^{2k+1},$$

where the functions $\psi_k = \psi_k(x)$ are defined by the recurrence relations

$$\psi_n(x) = A_n x + B_n,$$
$$\psi_{k-1}(x) = A_k x + B_k + 2k(2k+1) \int_a^x (x-\xi) \frac{\psi_k(\xi)}{f(\xi)}\, d\xi,$$

where A_k, B_k are arbitrary constants ($k = n, \ldots, 1$).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\Big[f(x)\dfrac{\partial w}{\partial x}\Big]$.

This is a special case of the equation of Section 6.5.1 with $s(x) = 1$, $p(x) = f(x)$, and $q = \Phi = 0$.

1°. Particular solutions:

$$w = C_1 t^2 + C_2 t + 2 \int \frac{C_1 x + C_3}{f(x)}\, dx + C_4,$$
$$w = C_1 t^3 + C_2 t + 6t \int \frac{C_1 x + C_3}{f(x)}\, dx + C_4,$$
$$w = [C_1 \Phi(x) + C_2] t + C_3 \Phi(x) + C_4, \quad \Phi(x) = \int \frac{dx}{f(x)},$$
$$w = [C_1 \Phi(x) + C_2] t^2 + C_3 \Phi(x) + C_4 + 2 \int \left\{ \frac{1}{f(x)} \int [C_1 \Phi(x) + C_2]\, dx \right\} dx,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

2°. Separable particular solution:

$$w = (C_1 e^{\lambda t} + C_2 e^{-\lambda t}) H(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $H = H(x)$ is determined by the ordinary differential equation $[f(x) H'_x]'_x - \lambda^2 H = 0$.

3°. Separable particular solution:

$$w = [C_1 \sin(\lambda t) + C_2 \cos(\lambda t)] Z(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $Z = Z(x)$ is determined by the ordinary differential equation $[f(x) Z'_x]'_x + \lambda^2 Z = 0$.

4°. Particular solutions with even powers of t:

$$w = \sum_{k=0}^{n} \zeta_k(x) t^{2k},$$

where the functions $\zeta_k = \zeta_k(x)$ are defined by the recurrence relations

$$\zeta_n(x) = A_n\Phi(x) + B_n, \qquad \Phi(x) = \int \frac{dx}{f(x)},$$
$$\zeta_{k-1}(x) = A_k\Phi(x) + B_k + 2k(2k-1)\int \frac{1}{f(x)}\Bigl\{\int \zeta_k(x)\,dx\Bigr\}\,dx,$$

where A_k, B_k are arbitrary constants ($k = n, \dots, 1$).

5°. Particular solutions with odd powers of t:

$$w = \sum_{k=0}^{n} \eta_k(x) t^{2k+1},$$

where the functions $\eta_k = \eta_k(x)$ are defined by the recurrence relations

$$\eta_n(x) = A_n\Phi(x) + B_n, \qquad \Phi(x) = \int \frac{dx}{f(x)},$$
$$\eta_{k-1}(x) = A_k\Phi(x) + B_k + 2k(2k+1)\int \frac{1}{f(x)}\Bigl\{\int \eta_k(x)\,dx\Bigr\}\,dx,$$

where A_k, B_k are arbitrary constants ($k = n, \dots, 1$).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x)\frac{\partial w}{\partial x} + \Phi(x,t), \qquad 0 < f(x) < \infty.$

This equation can be rewritten in the form of the equation from Section 6.5.1 with $q(x) \equiv 0$:

$$s(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right] + s(x)\Phi(x,t),$$

where

$$s(x) = \frac{1}{f(x)}\exp\left[\int \frac{g(x)}{f(x)} dx\right], \quad p(x) = \exp\left[\int \frac{g(x)}{f(x)} dx\right].$$

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x)\frac{\partial w}{\partial x} + h(x)w + \Phi(x,t).$

This equation can be rewritten in the form of the equation from Section 6.5.1:

$$s(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right] - q(x)w + s(x)\Phi(x,t),$$

where

$$s(x) = \frac{1}{f(x)}\exp\left[\int \frac{g(x)}{f(x)}\,dx\right], \quad p(x) = \exp\left[\int \frac{g(x)}{f(x)}\,dx\right],$$
$$q(x) = -\frac{h(x)}{f(x)}\exp\left[\int \frac{g(x)}{f(x)}\,dx\right].$$

5. $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + \big[h_1(x) + h_2(t)\big]w.$

1°. There are separable solutions in the product form $w(x,t) = \varphi(x)\psi(t)$, where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ satisfy the ordinary differential equations (λ is an arbitrary constant):

$$f(x)\varphi''_{xx} + g(x)\varphi'_x + \big[\lambda + h_1(x)\big]\varphi = 0, \qquad \psi''_{tt} + \big[\lambda - h_2(t)\big]\psi = 0.$$

2°. For the solution of various boundary value problems for the original equation, see Sections 15.1.1 and 15.1.3.

6. $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + \tfrac{1}{2}f'(x)\dfrac{\partial w}{\partial x} + bw.$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{f(x)}}$ leads to the constant coefficient equation $\partial_{tt} w = \partial_{zz} w + bw$, which is discussed in Section 6.1.3.

7. $\dfrac{\partial^2 w}{\partial t^2} = f^2\dfrac{\partial^2 w}{\partial x^2} + f(f'_x + 2g)\dfrac{\partial w}{\partial x} + (fg'_x + g^2)w, \quad f = f(x),\ g = g(x).$

The transformation

$$w(x,t) = u(\xi,t)\exp\biggl(-\int \frac{g}{f}\,dx\biggr), \quad \xi = \int \frac{dx}{f(x)}$$

leads to the wave equation $\partial_{tt} u = \partial_{\xi\xi} u$, which is discussed in Section 6.1.1.

8. $\dfrac{\partial^2 w}{\partial t^2} + a\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + \tfrac{1}{2}f'(x)\dfrac{\partial w}{\partial x} + bw.$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{f(x)}}$ leads to a constant coefficient equation of the form 6.4.1.2: $\partial_{tt} w + a\partial_t w = \partial_{zz} w + bw$.

9. $f(t)\dfrac{\partial^2 w}{\partial t^2} + \tfrac{1}{2}f'(t)\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + cw.$

The substitution $\tau = \displaystyle\int \frac{dt}{\sqrt{f(t)}}$ leads to the equation $\partial_{\tau\tau} w = a\partial_{xx} w + b\partial_x w + cw$, which is discussed in Section 6.1.5.

10. $f(t)\dfrac{\partial^2 w}{\partial t^2} + \tfrac{1}{2}f'(t)\dfrac{\partial w}{\partial t} = g(x)\dfrac{\partial^2 w}{\partial x^2} + \tfrac{1}{2}g'(x)\dfrac{\partial w}{\partial x} + cw.$

The transformation $\tau = \displaystyle\int \frac{dt}{\sqrt{f(t)}}$, $z = \displaystyle\int \frac{dx}{\sqrt{g(x)}}$ leads to the constant coefficient equation $\partial_{\tau\tau} w = \partial_{zz} w + cw$, which is discussed in Section 6.1.3.

Chapter 7

Second-Order Hyperbolic Equations with Two Space Variables

7.1 Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w$

7.1.1 Problems in Cartesian Coordinates

The wave equation with two space variables in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right).$$

► **Particular solutions and some relations.**

1°. Particular solutions:

$$w(x,y,t) = A\exp\Bigl(k_1x + k_2y \pm at\sqrt{k_1^2+k_2^2}\Bigr),$$
$$w(x,y,t) = A\sin(k_1x + C_1)\sin(k_2y + C_2)\sin\Bigl(at\sqrt{k_1^2+k_2^2}\Bigr),$$
$$w(x,y,t) = A\sin(k_1x + C_1)\sin(k_2y + C_2)\cos\Bigl(at\sqrt{k_1^2+k_2^2}\Bigr),$$
$$w(x,y,t) = A\sinh(k_1x + C_1)\sinh(k_2y + C_2)\sinh\Bigl(at\sqrt{k_1^2+k_2^2}\Bigr),$$
$$w(x,y,t) = A\sinh(k_1x + C_1)\sinh(k_2y + C_2)\cosh\Bigl(at\sqrt{k_1^2+k_2^2}\Bigr),$$
$$w(x,y,t) = \varphi(x\sin\beta + y\cos\beta + at) + \psi(x\sin\beta + y\cos\beta - at),$$

where A, C_1, C_2, k_1, k_2, and β are arbitrary constants, and $\varphi(z)$ and $\psi(z)$ are arbitrary functions.

2°. Particular solutions that are expressed in terms of solutions to simpler equations:

$$w(x,y,t)=\big[A\cos(ky)+B\sin(ky)\big]u(x,t) \quad \text{for} \quad \partial_{tt}u=a^2\partial_{xx}u-a^2k^2u, \qquad (1)$$

$$w(x,y,t)=\big[A\cosh(ky)+B\sinh(ky)\big]u(x,t) \quad \text{for} \quad \partial_{tt}u=a^2\partial_{xx}u+a^2k^2u, \qquad (2)$$

$$w(x,y,t)=\big[A\cos(kt)+B\sin(kt)\big]u(x,y) \quad \text{for} \quad \partial_{xx}u+\partial_{yy}u=-(k/a)^2u, \qquad (3)$$

$$w(x,y,t)=\big[A\cosh(kt)+B\sinh(kt)\big]u(x,y) \quad \text{for} \quad \partial_{xx}u+\partial_{yy}u=(k/a)^2u, \qquad (4)$$

$$w(x,y,t)=\exp\left(\frac{at\pm y}{2b}\right)u(x,\tau),\quad \tau=\frac{at\mp y}{2} \quad \text{for} \quad \partial_{\tau}u=b\partial_{xx}u. \qquad (5)$$

For particular solutions of equations (1) and (2) for the function $u(x,t)$, see the Klein–Gordon equation 6.1.3. For particular solutions of equations (3) and (4) for the function $u(x,y)$, see Section 9.3.2. For particular solutions of the heat equation (5) for the function $u(x,\tau)$, see Section 3.1.1.

3°. Fundamental solution:

$$\mathscr{E}(x,y,t)=\frac{\vartheta(at-r)}{2\pi a\sqrt{a^2t^2-r^2}},\qquad \vartheta(z)=\begin{cases}1 & \text{for } z>0,\\ 0 & \text{for } z\le 0,\end{cases}$$

where $r=\sqrt{x^2+y^2}$.

4°. Infinite series solutions that contain arbitrary functions of the space variables:

$$w(x,y,t)=f(x,y)+\sum_{n=1}^{\infty}\frac{(at)^{2n}}{(2n)!}\Delta^n f(x,y),\qquad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

$$w(x,y,t)=tg(x,y)+t\sum_{n=1}^{\infty}\frac{(at)^{2n}}{(2n+1)!}\Delta^n g(x,y),$$

where $f(x,y)$ and $g(x,y)$ are any infinitely differentiable functions. The first solution satisfies the initial conditions $w(x,y,0)=f(x,y)$, $\partial_t w(x,y,0)=0$ and the second solution satisfies the initial conditions $w(x,y,0)=0$, $\partial_t w(x,y,0)=g(x,y)$. The sums are finite if $f(x,y)$ and $g(x,y)$ are bivariate polynomials.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

5°. A wide class of solutions to the wave equation with two space variables is described by the formulas

$$w(x,y,t)=\operatorname{Re}F(\theta)\quad\text{and}\quad w(x,y,t)=\operatorname{Im}F(\theta). \qquad (6)$$

Here, $F(\theta)$ is an arbitrary analytic function of the complex argument θ connected with the variables (x,y,t) by the implicit relation

$$at-(x-x_0)\theta+(y-y_0)\sqrt{1-\theta^2}=G(\theta), \qquad (7)$$

where $G(\theta)$ is any analytic function and x_0, y_0 are arbitrary constants. Solutions of the forms (6), (7) find wide application in the theory of diffraction. If the argument θ obtained by solving (7) with a prescribed $G(\theta)$ is real in some domain D, then one should set $\operatorname{Re}F(\theta)=F(\theta)$ in relation (6) everywhere in D.

⊙ *Literature*: V. I. Smirnov (1974, Vol. 3, Pt. 2).

6°. Suppose $w = w(x, y, t)$ is a solution of the wave equation. Then the functions

$$\begin{aligned}
w_1 &= Aw(\pm\lambda x + C_1,\ \pm\lambda y + C_2,\ \pm\lambda t + C_3),\\
w_2 &= Aw\biggl(\frac{x - vt}{\sqrt{1 - (v/a)^2}},\ y,\ \frac{t - va^{-2}x}{\sqrt{1 - (v/a)^2}}\biggr),\\
w_3 &= \frac{A}{\sqrt{|r^2 - a^2t^2|}}\, w\biggl(\frac{x}{r^2 - a^2t^2},\ \frac{y}{r^2 - a^2t^2},\ \frac{t}{r^2 - a^2t^2}\biggr),\\
w_4 &= \frac{A}{\sqrt{\Xi}}\, w\biggl(\frac{x + k_1(a^2t^2 - r^2)}{\Xi},\ \frac{y + k_2(a^2t^2 - r^2)}{\Xi},\ \frac{at + k_3(a^2t^2 - r^2)}{a\Xi}\biggr),\\
r^2 &= x^2 + y^2, \quad \Xi = 1 - 2(k_1x + k_2y - ak_3t) + (k_1^2 + k_2^2 - k_3^2)(r^2 - a^2t^2),
\end{aligned}$$

where A, C_1, C_2, C_3, k_1, k_2, k_3, v, and λ are arbitrary constants, are also solutions of the equation. The signs at λ in the expression of w_1 can be taken independently of one another. The function w_2 results from the invariance of the wave equation under the *Lorentz transformation*.

More detailed information about particular solutions and transformations of the wave equation with two space variables can be found in the references cited below.

⊙ *Literature*: E. Kalnins and W. Miller, Jr. (1975, 1976), W. Miller, Jr. (1977).

► Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$w = f(x, y) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x, y) \quad \text{at} \quad t = 0.$$

Solution (*Poisson's formula*):

$$w(x, y, t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint_{C_{at}} \frac{f(\xi, \eta)\, d\xi\, d\eta}{\sqrt{a^2t^2 - (\xi - x)^2 - (\eta - y)^2}} + \frac{1}{2\pi a}\iint_{C_{at}} \frac{g(\xi, \eta)\, d\xi\, d\eta}{\sqrt{a^2t^2 - (\xi - x)^2 - (\eta - y)^2}},$$

where the integration is performed over the interior of the circle of radius at with center at (x, y).

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970), A. N. Tikhonov and A. A. Samarskii (1990).

► Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. First boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x, y) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_t w &= f_1(x, y) && \text{at} \quad t = 0 && \text{(initial condition)},\\
w &= g_1(y, t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\
w &= g_2(y, t) && \text{at} \quad x = l_1 && \text{(boundary condition)},\\
w &= g_3(x, t) && \text{at} \quad y = 0 && \text{(boundary condition)},\\
w &= g_4(x, t) && \text{at} \quad y = l_2 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ \int_0^{l_1}\!\!\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau \\
&- a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1} d\eta\,d\tau \\
&+ a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\
&- a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=l_2} d\xi\,d\tau,
\end{aligned}
$$

where

$$
G(x,y,\xi,\eta,t) = \frac{4}{al_1l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\lambda_{nm}}\sin(p_nx)\sin(q_my)\sin(p_n\xi)\sin(q_m\eta)\sin(a\lambda_{nm}t),
$$
$$
p_n = \frac{n\pi}{l_1},\quad q_m = \frac{m\pi}{l_2},\quad \lambda_{nm} = \sqrt{p_n^2+q_m^2}.
$$

The problem of vibration of a rectangular membrane with sides l_1 and l_2 rigidly fixed in its contour is characterized by homogeneous boundary conditions, $g_s \equiv 0$ ($s = 1,\ 2,\ 3,\ 4$).

⊙ *Literature*: M. M. Smirnov (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Second boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t} \int_0^{l_1}\int_0^{l_2} f_0(\xi,\eta) G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ \int_0^{l_1}\int_0^{l_2} f_1(\xi,\eta) G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&- a^2 \int_0^{t}\int_0^{l_2} g_1(\eta,\tau) G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau \\
&+ a^2 \int_0^{t}\int_0^{l_2} g_2(\eta,\tau) G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau \\
&- a^2 \int_0^{t}\int_0^{l_1} g_3(\xi,\tau) G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau \\
&+ a^2 \int_0^{t}\int_0^{l_1} g_4(\xi,\tau) G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,
\end{aligned}
$$

where

$$
G(x,y,\xi,\eta,t) = \frac{t}{l_1 l_2} + \frac{2}{a l_1 l_2}\Biggl[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{A_{nm}}{\sqrt{p_n^2+q_m^2}} \cos(p_n x)\cos(q_m y) \times \cos(p_n \xi)\cos(q_m \eta)\sin(a\lambda_{nm} t)\Biggr],
$$

$$
p_n = \frac{n\pi}{l_1}, \quad q_m = \frac{m\pi}{l_2}, \quad A_{nm} = \begin{cases} 0 & \text{for } n=m=0, \\ 1 & \text{for } nm=0\ (n\neq m), \\ 2 & \text{for } nm\neq 0. \end{cases}
$$

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Third boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)}, \\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)}, \\
\partial_x w - k_1 w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)}, \\
\partial_x w + k_2 w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)}, \\
\partial_y w - k_3 w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)}, \\
\partial_y w + k_4 w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t) = \frac{4}{a}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{E_{nm}\sqrt{\mu_n^2+\nu_m^2}}\sin(\mu_n x+\varepsilon_n)\sin(\nu_m y+\sigma_m)$$
$$\times\sin(\mu_n\xi+\varepsilon_n)\sin(\nu_m\eta+\sigma_m)\sin\bigl(at\sqrt{\mu_n^2+\nu_m^2}\,\bigr),$$
$$\varepsilon_n = \arctan\frac{\mu_n}{l_1},\qquad \sigma_m = \arctan\frac{\nu_m}{l_2},$$
$$E_{nm} = \left[l_1+\frac{(k_1k_2+\mu_n^2)(k_1+k_2)}{(k_1^2+\mu_n^2)(k_2^2+\mu_n^2)}\right]\left[l_2+\frac{(k_3k_4+\nu_m^2)(k_3+k_4)}{(k_3^2+\nu_m^2)(k_4^2+\nu_m^2)}\right],$$

where the μ_n and ν_m are positive roots of the transcendental equations

$$\mu^2-k_1k_2=(k_1+k_2)\mu\cot(l_1\mu),\qquad \nu^2-k_3k_4=(k_3+k_4)\nu\cot(l_2\nu).$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Mixed boundary value problems.

1°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&+\int_0^{l_1}\!\!\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&+a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0}d\eta\,d\tau\\
&-a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1}d\eta\,d\tau\\
&-a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau\\
&+a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{2}{al_1l_2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m}{\lambda_{nm}}\sin(p_nx)\cos(q_my)\sin(p_n\xi)\cos(q_m\eta)\sin(a\lambda_{nm}t),$$

$$p_n = \frac{n\pi}{l_1},\quad q_m = \frac{m\pi}{l_2},\quad \lambda_{nm} = \sqrt{p_n^2+q_m^2},\quad A_m = \begin{cases}1 & \text{for } m=0,\\ 2 & \text{for } m\neq 0.\end{cases}$$

2°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad+\int_0^{l_1}\!\!\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0}d\eta\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{4}{al_1l_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\lambda_{nm}}\sin(p_nx)\sin(q_my)\sin(p_n\xi)\sin(q_m\eta)\sin(a\lambda_{nm}t),$$

$$p_n = \frac{\pi(2n+1)}{2l_1},\quad q_m = \frac{\pi(2m+1)}{2l_2},\quad \lambda_{nm} = \sqrt{p_n^2+q_m^2}.$$

7.1.2 Problems in Polar Coordinates

The wave equation with two space variables in the polar coordinate system has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}\right),\qquad r=\sqrt{x^2+y^2}.$$

One-dimensional solutions $w = w(r,t)$ that are independent of the angular coordinate φ are considered in Section 6.2.1.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g(\varphi,t) && \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+\int_0^{2\pi}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &- a^2R\int_0^t\!\!\int_0^{2\pi} g(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R}d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t) = \frac{1}{\pi aR^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{\mu_{nm}[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin(\mu_{nm}at),$$

$$A_0 = 1, \quad A_n = 2 \quad (n = 1,\,2,\,\ldots),$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

The problem of vibration of a circular membrane of radius R rigidly fixed in its contour is characterized by the homogeneous boundary condition, $g(\varphi,t) \equiv 0$.

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970), A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_r w &= g(\varphi,t) && \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+\int_0^{2\pi}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+ a^2R\int_0^t\!\!\int_0^{2\pi} g(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t) = \frac{t}{\pi R^2} + \frac{1}{\pi a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\sin(\mu_{nm}at),$$
$$A_0 = 1, \quad A_n = 2 \quad (n = 1,\, 2,\, \ldots),$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▸ Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Third boundary value problem.

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ \partial_r w + kw &= g(\varphi,t) \quad &&\text{at} \quad r = R \quad &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2+k^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\sin(\mu_{nm}at),$$
$$A_0=1, \quad A_n=2 \quad (n=1,\, 2,\, \ldots).$$

Here, the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

▸ Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. First boundary value problem.

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) \quad &&\text{at} \quad t = 0 \quad &&\text{(initial condition)},\\ w &= g_1(\varphi,t) \quad &&\text{at} \quad r = R_1 \quad &&\text{(boundary condition)},\\ w &= g_2(\varphi,t) \quad &&\text{at} \quad r = R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad + \int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad + a^2R_1\int_0^t\int_0^{2\pi} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R_1} d\eta\,d\tau\\ &\quad - a^2R_2\int_0^t\int_0^{2\pi} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R_2} d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{\pi}{2a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_nB_{nm}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin(\mu_{nm}at),$$

$$A_n=\begin{cases}1/2 & \text{for } n=0,\\ 1 & \text{for } n\neq 0,\end{cases}\qquad B_{nm}=\frac{\mu_{nm}J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)},$$

$$Z_n(\mu_{nm}r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2)-Y_n(\mu R_1)J_n(\mu R_2)=0.$$

► **Domain: $R_1\le r\le R_2,\ 0\le\varphi\le 2\pi$. Second boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w&=g_1(\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w&=g_2(\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,t)&=\frac{\partial}{\partial t}\int_0^{2\pi}\int_{R_1}^{R_2}f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&+\int_0^{2\pi}\int_{R_1}^{R_2}f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&-a^2R_1\int_0^t\int_0^{2\pi}g_1(\eta,\tau)G(r,\varphi,R_1,\eta,t-\tau)\,d\eta\,d\tau\\
&+a^2R_2\int_0^t\int_0^{2\pi}g_2(\eta,\tau)G(r,\varphi,R_2,\eta,t-\tau)\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(r,\varphi,\xi,\eta,t)&=\frac{t}{\pi(R_2^2-R_1^2)}\\
&\quad+\frac{1}{\pi a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin(\mu_{nm}at)}{(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)},\\
Z_n(\mu_{nm}r)&=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),
\end{aligned}$$

where $A_0=1$ and $A_n=2$ for $n=1,\ 2,\ \dots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w - k_1 w &= g_1(\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w + k_2 w &= g_2(\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t) = \frac{1}{\pi a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}}{B_{nm}} Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin(\mu_{nm}at),$$

$$B_{nm} = (k_2^2R_2^2 + \mu_{nm}^2R_2^2 - n^2)Z_n^2(\mu_{nm}R_2) - (k_1^2R_1^2 + \mu_{nm}^2R_1^2 - n^2)Z_n^2(\mu_{nm}R_1),$$

$$\begin{aligned}
Z_n(\mu_{nm}r) &= \bigl[\mu_{nm}J_n'(\mu_{nm}R_1) - k_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r)\\
&\quad - \bigl[\mu_{nm}Y_n'(\mu_{nm}R_1) - k_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r).
\end{aligned}$$

Here, $A_0 = 1$ and $A_n = 2$ for $n = 1,\ 2,\ \dots$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation

$$\begin{aligned}
&\bigl[\mu J_n'(\mu R_1) - k_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2) + k_2Y_n(\mu R_2)\bigr]\\
&\qquad = \bigl[\mu Y_n'(\mu R_1) - k_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2) + k_2J_n(\mu R_2)\bigr].
\end{aligned}$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. First boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
w &= g_2(r,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{\varphi_0}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad + \int_0^{\varphi_0}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad - a^2R\int_0^t\!\!\int_0^{\varphi_0} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau\\
&\quad + a^2\int_0^t\!\!\int_0^R g_2(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\
&\quad - a^2\int_0^t\!\!\int_0^R g_3(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=\varphi_0} d\xi\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{4}{aR^2\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{\mu_{nm}[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\times\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\sin(\mu_{nm}at),$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

► **Domain: $0\le r\le R$, $0\le\varphi\le\varphi_0$. Second boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w&=g_1(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w&=g_2(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w&=g_3(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,t)&=\frac{\partial}{\partial t}\int_0^{\varphi_0}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&+\int_0^{\varphi_0}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&+a^2R\int_0^t\!\!\int_0^{\varphi_0} g_1(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau\\
&-a^2\int_0^t\!\!\int_0^R g_2(\xi,\tau)G(r,\varphi,\xi,0,t-\tau)\,d\xi\,d\tau\\
&+a^2\int_0^t\!\!\int_0^R g_3(\xi,\tau)G(r,\varphi,\xi,\varphi_0,t-\tau)\,d\xi\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{2t}{R^2\varphi_0}+\frac{4\varphi_0}{a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_{nm}J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{(R^2\varphi_0^2\mu_{nm}^2-n^2\pi^2)\big[J_{n\pi/\varphi_0}(\mu_{nm}R)\big]^2}\times\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\eta}{\varphi_0}\right)\sin(\mu_{nm}at),$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J'_{n\pi/\varphi_0}(\mu R)=0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Mixed boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
\partial_r w + kw &= g(\varphi,t) &&\text{at} \quad r=R &&\text{(boundary condition)},\\
\partial_\varphi w &= 0 &&\text{at} \quad \varphi=0 &&\text{(boundary condition)},\\
\partial_\varphi w &= 0 &&\text{at} \quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{\varphi_0}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad+\int_0^{\varphi_0}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad+a^2R\int_0^t\!\!\int_0^{\varphi_0} g(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\varphi,\xi,\eta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}J_{s_n}(\mu_{nm}r)J_{s_n}(\mu_{nm}\xi)\cos(s_n\varphi)\cos(s_n\eta)\sin(\mu_{nm}at),
$$

$$
s_n = \frac{n\pi}{\varphi_0}, \qquad A_{nm} = \frac{4\mu_{nm}}{a\varphi_0(\mu_{nm}^2R^2 + k^2R^2 - s_n^2)\big[J_{s_n}(\mu_{nm}R)\big]^2},
$$

where the $J_{s_n}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$
\mu J'_{s_n}(\mu R) + kJ_{s_n}(\mu R) = 0.
$$

7.1.3 Axisymmetric Problems

In the axisymmetric case the wave equation in the cylindrical system of coordinates has the form

$$
\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right), \qquad r = \sqrt{x^2+y^2}.
$$

One-dimensional problems with axial symmetry that have solutions $w = w(r,t)$ are considered in Section 6.2.1.

In the solution of the problems considered below, the modified Green's function $\mathcal{G}(r,z,\xi,\eta,t) = 2\pi\xi G(r,z,\xi,\eta,t)$ is used for convenience.

► Domain: $0 \le r \le R$, $0 \le z \le l$. First boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) = \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
+\int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
-a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau\\
+a^2\int_0^t\int_0^R g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\
-a^2\int_0^t\int_0^R g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) = \frac{4\xi}{R^2 l}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\\
\times\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin\left(at\sqrt{\lambda_{nm}}\right)}{a\sqrt{\lambda_{nm}}},
\end{aligned}
$$

$$
\lambda_{nm} = \frac{\mu_n^2}{R^2}+\frac{\pi^2 m^2}{l^2},
$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu)=0$. The numerical values of the first ten λ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

► Domain: $0 \le r \le R$, $0 \le z \le l$. Second boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$w(r,z,t)=\frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta+\int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta$$
$$+a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R,\eta,t-\tau)\,d\eta\,d\tau$$
$$-a^2\int_0^t\int_0^R g_2(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau$$
$$+a^2\int_0^t\int_0^R g_3(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi\,d\tau.$$

Here,

$$\mathcal{G}(r,z,\xi,\eta,t)=\frac{2t\xi}{R^2 l}+\frac{2\xi}{R^2 l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_{nm}}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)$$
$$\times\cos\left(\frac{m\pi z}{l}\right)\cos\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(at\sqrt{\lambda_{nm}}\,\bigr)}{a\sqrt{\lambda_{nm}}},$$

$$\lambda_{nm}=\frac{\mu_n^2}{R^2}+\frac{\pi^2 m^2}{l^2},\qquad A_{nm}=\begin{cases}0 & \text{for } m=0,\ n=0,\\ 1 & \text{for } m=0,\ n>0,\\ 2 & \text{for } m>0,\end{cases}$$

where the μ_n are zeros of the first-order Bessel function, $J_1(\mu)=0$ $(\mu_0=0)$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

► Domain: $0\le r\le R$, $0\le z\le l$. Third boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w + k_1 w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w - k_2 w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w + k_3 w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\mathcal{G}(r,z,\xi,\eta,t)=\frac{2\xi}{R^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_n J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2}\frac{\sin\bigl(at\sqrt{\lambda_{nm}}\,\bigr)}{a\sqrt{\lambda_{nm}}},$$

$$A_n=\frac{\mu_n^2}{(k_1^2R^2+\mu_n^2)J_0^2(\mu_n)},\quad \lambda_{nm}=\frac{\mu_n^2}{R^2}+\beta_m^2,\quad \varphi_m(z)=\cos(\beta_m z)+\frac{k_2}{\beta_m}\sin(\beta_m z),$$

$$\|\varphi_m\|^2=\frac{k_3}{2\beta_m^2}\frac{\beta_m^2+k_2^2}{\beta_m^2+k_3^2}+\frac{k_2}{2\beta_m^2}+\frac{l}{2}\left(1+\frac{k_2^2}{\beta_m^2}\right).$$

Here, the μ_n and β_m are positive roots of the transcendental equations

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0, \qquad \frac{\tan(\beta l)}{\beta} = \frac{k_2 + k_3}{\beta^2 - k_2 k_3}.$$

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,z) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ \partial_t w &= f_1(r,z) \quad &\text{at}\quad t=0 \quad &\text{(initial condition)},\\ w &= g_1(z,t) \quad &\text{at}\quad r=R \quad &\text{(boundary condition)},\\ \partial_z w &= g_2(r,t) \quad &\text{at}\quad z=0 \quad &\text{(boundary condition)},\\ \partial_z w &= g_3(r,t) \quad &\text{at}\quad z=l \quad &\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) = {} & \frac{\partial}{\partial t}\int_0^l\!\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta + \int_0^l\!\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\ & - a^2\int_0^t\!\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau\\ & - a^2\int_0^t\!\int_0^R g_2(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau\\ & + a^2\int_0^t\!\int_0^R g_3(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} \mathcal{G}(r,z,\xi,\eta,t) &= \frac{2\xi}{R^2 l}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m}{J_1^2(\mu_n)}J_0\!\left(\frac{\mu_n r}{R}\right)J_0\!\left(\frac{\mu_n \xi}{R}\right)\\ &\qquad\times\cos\!\left(\frac{m\pi z}{l}\right)\cos\!\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(at\sqrt{\lambda_{nm}}\bigr)}{a\sqrt{\lambda_{nm}}},\\ \lambda_{nm} &= \frac{\mu_n^2}{R^2} + \frac{\pi^2 m^2}{l^2}, \quad A_m = \begin{cases} 1 & \text{for } m=0,\\ 2 & \text{for } m>0, \end{cases} \end{aligned}$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten λ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

2°. A circular cylinder of finite length is considered. The following conditions are pre-

scribed:

$$
\begin{aligned}
w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) = {} & \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta + \int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&+ a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\
&+ a^2\int_0^t\int_0^R g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\
&- a^2\int_0^t\int_0^R g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{4\xi}{R^2 l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\\
&\qquad\times\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin\left(at\sqrt{\lambda_{nm}}\right)}{a\sqrt{\lambda_{nm}}},\\
\lambda_{nm} &= \frac{\mu_n^2}{R^2}+\frac{\pi^2 m^2}{l^2},
\end{aligned}
$$

where the μ_n are zeros of the first-order Bessel function, $J_1(\mu)=0$ $(\mu_0=0)$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. First boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$w(r,z,t)=\frac{\partial}{\partial t}\int_0^l\int_{R_1}^{R_2} f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta+\int_0^l\int_{R_1}^{R_2} f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta$$
$$+a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R_1} d\eta\,d\tau$$
$$-a^2\int_0^t\int_0^l g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R_2} d\eta\,d\tau$$
$$+a^2\int_0^t\int_{R_1}^{R_2} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau$$
$$-a^2\int_0^t\int_{R_1}^{R_2} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau.$$

Here,

$$\mathcal{G}(r,z,\xi,\eta,t)=\frac{\pi^2\xi}{R_1^2 l}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n\Psi_n(r)\Psi_n(\xi)\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin\left(at\sqrt{\lambda_{nm}}\,\right)}{a\sqrt{\lambda_{nm}}},$$
$$A_n=\frac{\mu_n^2 J_0^2(s\mu_n)}{J_0^2(\mu_n)-J_0^2(s\mu_n)},\quad \Psi_n(r)=Y_0(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right)-J_0(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right),$$
$$s=\frac{R_2}{R_1},\quad \lambda_{nm}=\frac{\mu_n^2}{R_1^2}+\frac{\pi^2 m^2}{l^2},$$

where $J_0(\mu)$ and $Y_0(\mu)$ are Bessel functions, and the μ_n are positive roots of the transcendental equation

$$J_0(\mu)Y_0(s\mu)-J_0(s\mu)Y_0(\mu)=0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,z) && \text{at } t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,z) && \text{at } t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(z,t) && \text{at } r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(z,t) && \text{at } r=R_2 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,t) && \text{at } z=0 && \text{(boundary condition)},\\ \partial_z w &= g_4(r,t) && \text{at } z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,z,t)=\frac{\partial}{\partial t}\int_0^l\int_{R_1}^{R_2} f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta+\int_0^l\int_{R_1}^{R_2} f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta$$
$$-a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R_1,\eta,t-\tau)\,d\eta\,d\tau+a^2\int_0^t\int_0^l g_2(\eta,\tau)\mathcal{G}(r,z,R_2,\eta,t-\tau)\,d\eta\,d\tau$$
$$-a^2\int_0^t\int_{R_1}^{R_2} g_3(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau+a^2\int_0^t\int_{R_1}^{R_2} g_4(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi\,d\tau.$$

Here,

$$\mathcal{G}(r,z,\xi,\eta,t) = \frac{2t\xi}{(R_2^2-R_1^2)l} + \frac{4\xi}{\pi a(R_2^2-R_1^2)}\sum_{m=1}^{\infty}\frac{1}{m}\cos\left(\frac{m\pi z}{l}\right)\cos\left(\frac{m\pi\eta}{l}\right)\sin\left(\frac{m\pi at}{l}\right)$$
$$+\frac{\pi^2\xi}{2R_1^2 l}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m\mu_n^2 J_1^2(s\mu_n)}{J_1^2(\mu_n)-J_1^2(s\mu_n)}\Psi_n(r)\Psi_n(\xi)\cos\left(\frac{m\pi z}{l}\right)\cos\left(\frac{m\pi\eta}{l}\right)\frac{\sin\left(at\sqrt{\lambda_{nm}}\,\right)}{a\sqrt{\lambda_{nm}}},$$

where

$$\Psi_n(r) = Y_1(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right) - J_1(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right), \qquad s = \frac{R_2}{R_1},$$
$$A_m = \begin{cases} 1 & \text{for } m=0, \\ 2 & \text{for } m>1, \end{cases} \qquad \lambda_{nm} = \frac{\mu_n^2}{R_1^2} + \frac{\pi^2 m^2}{l^2};$$

$J_k(\mu)$ and $Y_k(\mu)$ are Bessel functions ($k = 0,\ 1$); and the μ_n are positive roots of the transcendental equation

$$J_1(\mu)Y_1(s\mu) - J_1(s\mu)Y_1(\mu) = 0.$$

7.2 Nonhomogeneous Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2\Delta_2 w + \Phi(x,y,t)$

7.2.1 Problems in Cartesian Coordinates

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t=0,$$
$$\partial_t w = g(x,y) \quad \text{at} \quad t=0.$$

Solution:

$$w(x,y,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{\rho\le at}\frac{f(\xi,\eta)\,d\xi\,d\eta}{\sqrt{a^2t^2-\rho^2}} + \frac{1}{2\pi a}\iint\limits_{\rho\le at}\frac{g(\xi,\eta)\,d\xi\,d\eta}{\sqrt{a^2t^2-\rho^2}}$$
$$+\frac{1}{2\pi a}\int_0^t\left[\ \iint\limits_{\rho\le a(t-\tau)}\frac{\Phi(\xi,\eta,\tau)\,d\xi\,d\eta}{\sqrt{a^2(t-\tau)^2-\rho^2}}\right]d\tau, \qquad \rho^2 = (\xi-x)^2+(\eta-y)^2.$$

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970).

► Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. First boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is given by the formula in Section 7.1.1 (see the first boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term

$$
\int_0^t \int_0^{l_1} \int_0^{l_2} \Phi(\xi,\eta,\tau) G(x,y,\xi,\eta,t-\tau)\, d\eta\, d\xi\, d\tau, \qquad (1)
$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Second boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_2(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is given by the formula in Section 7.1.1 (see the second boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term (1); the Green's function is also taken from Section 7.1.1.

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Third boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_2(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w + k_2 w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w - k_3 w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w + k_4 w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is given by the formula in Section 7.1.1 (see the third boundary value problem for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term (1); the Green's function is also taken from Section 7.1.1.

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Mixed boundary value problems.**

1°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_2(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(x,y,t)$ is given by the formula in Section 7.1.1 (see Item 1° for the mixed boundary value problems for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term (1).

2°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_2(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(x,y,t)$ is given by the formula in Section 7.1.1 (see Item 2° for the mixed boundary value problems for $0 \le x \le l_1$, $0 \le y \le l_2$) with the additional term (1).

7.2.2 Problems in Polar Coordinates

A nonhomogeneous wave equation in the polar coordinate system has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \qquad r = \sqrt{x^2+y^2}.$$

One-dimensional boundary value problems independent of the angular coordinate φ are considered in Section 6.2.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,t)$ is given by the formula in Section 7.1.2 (see the first boundary value problem for $0\le r\le R$, $0\le\varphi\le 2\pi$) with the additional term

$$\int_0^t\int_0^{2\pi}\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau, \tag{1}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0\le r\le R$, $0\le\varphi\le 2\pi$. Second boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,t)$ is given by the formula in Section 7.1.2 (see the second boundary value problem for $0\le r\le R$, $0\le\varphi\le 2\pi$) with the additional term (1); the Green's function is also taken from Section 7.1.2.

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0\le r\le R$, $0\le\varphi\le 2\pi$. Third boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + kw &= g(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,t)$ is given by the formula in Section 7.1.2 (see the third boundary value problem for $0\le r\le R$, $0\le\varphi\le 2\pi$) with the additional term (1); the Green's function is also taken from Section 7.1.2.

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1\le r\le R_2$, $0\le\varphi\le 2\pi$. First boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ w &= g_2(\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r, \varphi, t)$ is given by the formula in Section 7.1.2 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$) with the additional term

$$\int_0^t \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi, \eta, \tau) G(r, \varphi, \xi, \eta, t - \tau) \xi \, d\xi \, d\eta \, d\tau, \tag{2}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

▶ Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Second boundary value problem.

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r, \varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r, \varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w &= g_1(\varphi, t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ \partial_r w &= g_2(\varphi, t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r, \varphi, t)$ is given by the formula in Section 7.1.2 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$) with the additional term (2); the Green's function is also taken from Section 7.1.2.

▶ Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Third boundary value problem.

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r, \varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r, \varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w - k_1 w &= g_1(\varphi, t) && \text{at} \quad r = R_1 && \text{(boundary condition)}, \\ \partial_r w + k_2 w &= g_2(\varphi, t) && \text{at} \quad r = R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r, \varphi, t)$ is given by the formula in Section 7.1.2 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$) with the additional term (2).

▶ Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. First boundary value problem.

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r, \varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r, \varphi) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ w &= g_1(\varphi, t) && \text{at} \quad r = R && \text{(boundary condition)}, \\ w &= g_2(r, t) && \text{at} \quad \varphi = 0 && \text{(boundary condition)}, \\ w &= g_3(r, t) && \text{at} \quad \varphi = \varphi_0 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r, \varphi, t)$ is given by the formula in Section 7.1.2 (see the first boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$) with the additional term

$$\int_0^t \int_0^{\varphi_0} \int_0^R \Phi(\xi, \eta, \tau) G(r, \varphi, \xi, \eta, t - \tau) \xi \, d\xi \, d\eta \, d\tau, \tag{3}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Second boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w &= g_2(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
r^{-1}\partial_\varphi w &= g_3(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,\varphi,t)$ is given by the formula in Section 7.1.2 (see the second boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$) with the additional term (3); the Green's function is also taken from Section 7.1.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Mixed boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w + kw &= g(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_\varphi w &= 0 &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
\partial_\varphi w &= 0 &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,\varphi,t)$ is given by the formula in Section 7.1.2 (see the mixed boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$) with the additional term (3); the Green's function is also taken from Section 7.1.2.

7.2.3 Axisymmetric Problems

In the axisymmetric case, a nonhomogeneous wave equation in the cylindrical system of coordinates has the form

$$
\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) + \Phi(r,z,t), \qquad r = \sqrt{x^2+y^2}.
$$

► **Domain: $0 \le r \le R$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r, z, t)$ is given by the formula in Section 7.1.3 (see the first boundary value problem for $0 \le r \le R$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^R \Phi(\xi, \eta, \tau)\mathcal{G}(r, z, \xi, \eta, t - \tau)\, d\xi\, d\eta\, d\tau, \tag{1}$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

► Domain: $0 \le r \le R$, $0 \le z \le l$. Second boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\
\partial_r w &= g_1(z, t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\
\partial_z w &= g_2(r, t) &&\text{at}\quad z = 0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r, t) &&\text{at}\quad z = l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r, z, t)$ is given by the formula in Section 7.1.3 (see the second boundary value problem for $0 \le r \le R$, $0 \le z \le l$) with the additional term (1); the Green's function is also taken from Section 7.1.3.

► Domain: $0 \le r \le R$, $0 \le z \le l$. Third boundary value problem.

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\
\partial_r w + k_1 w &= g_1(z, t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\
\partial_z w - k_2 w &= g_2(r, t) &&\text{at}\quad z = 0 &&\text{(boundary condition)},\\
\partial_z w + k_3 w &= g_3(r, t) &&\text{at}\quad z = l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r, z, t)$ is given by the formula in Section 7.1.3 (see the third boundary value problem for $0 \le r \le R$, $0 \le z \le l$) with the additional term (1).

► Domain: $0 \le r \le R$, $0 \le z \le l$. Mixed boundary value problems.

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r, z) &&\text{at}\quad t = 0 &&\text{(initial condition)},\\
w &= g_1(z, t) &&\text{at}\quad r = R &&\text{(boundary condition)},\\
\partial_z w &= g_2(r, t) &&\text{at}\quad z = 0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r, t) &&\text{at}\quad z = l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r, z, t)$ is given by the formula in Section 7.1.3 (see Item 1° for the mixed boundary value problems for $0 \le r \le R$, $0 \le z \le l$) with the additional term (1).

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,z,t)$ is given by the formula in Section 7.1.3 (see Item 2° for the mixed boundary value problems for $0 \le r \le R$, $0 \le z \le l$) with the additional term (1); the Green's function is also taken from Section 7.1.3.

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. First boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,z,t)$ is given by the formula in Section 7.1.3 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le z \le l$) with the additional term

$$
\int_0^t \int_0^l \int_{R_1}^{R_2} \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau, \tag{2}
$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

► Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Second boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
\partial_z w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,z,t)$ is given by the formula in Section 7.1.3 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le z \le l$) with the additional term (2); the Green's function is also taken from Section 7.1.3.

7.3 Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_2 w - bw + \Phi(x,y,t)$

7.3.1 Problems in Cartesian Coordinates

The *two-dimensional nonhomogeneous Klein–Gordon equation* with two space variables in the rectangular Cartesian coordinate system is written as

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - bw + \Phi(x,y,t).$$

► **Fundamental solutions.**

1°. Case $b = -\sigma^2 < 0$:

$$\mathscr{E}(x,y,t) = \frac{\vartheta(at-r)}{2\pi a^2}\frac{\cosh\bigl(\sigma\sqrt{t^2 - r^2/a^2}\,\bigr)}{\sqrt{t^2 - r^2/a^2}}, \qquad r = \sqrt{x^2+y^2},$$

where $\vartheta(z)$ is the Heaviside unit step function.

2°. Case $b = \sigma^2 > 0$:

$$\mathscr{E}(x,y,t) = \frac{\vartheta(at-r)}{2\pi a^2}\frac{\cos\bigl(\sigma\sqrt{t^2 - r^2/a^2}\,\bigr)}{\sqrt{t^2 - r^2/a^2}}, \qquad r = \sqrt{x^2+y^2}.$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x,y) \quad \text{at} \quad t = 0.$$

1°. Solution for $b = -a^2c^2 < 0$:

$$\begin{aligned} w(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{\rho\le at} f(\xi,\eta)\frac{\cosh\bigl(c\sqrt{a^2t^2-\rho^2}\,\bigr)}{\sqrt{a^2t^2-\rho^2}}\,d\xi\,d\eta \\ &+ \frac{1}{2\pi a}\iint\limits_{\rho\le at} g(\xi,\eta)\frac{\cosh\bigl(c\sqrt{a^2t^2-\rho^2}\,\bigr)}{\sqrt{a^2t^2-\rho^2}}\,d\xi\,d\eta \\ &+ \frac{1}{2\pi a}\int_0^t d\tau \iint\limits_{\rho\le a(t-\tau)} \Phi(\xi,\eta,\tau)\frac{\cosh\bigl(c\sqrt{a^2(t-\tau)^2-\rho^2}\,\bigr)}{\sqrt{a^2(t-\tau)^2-\rho^2}}\,d\xi\,d\eta, \end{aligned}$$

where $\rho = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

2°. Solution for $b = a^2c^2 > 0$:

$$
\begin{aligned}
w(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{\rho\le at} f(\xi,\eta)\frac{\cos\bigl(c\sqrt{a^2t^2-\rho^2}\,\bigr)}{\sqrt{a^2t^2-\rho^2}}\,d\xi\,d\eta \\
&+ \frac{1}{2\pi a}\iint\limits_{\rho\le at} g(\xi,\eta)\frac{\cos\bigl(c\sqrt{a^2t^2-\rho^2}\,\bigr)}{\sqrt{a^2t^2-\rho^2}}\,d\xi\,d\eta \\
&+ \frac{1}{2\pi a}\int_0^t d\tau \iint\limits_{\rho\le a(t-\tau)} \Phi(\xi,\eta,\tau)\frac{\cos\bigl(c\sqrt{a^2(t-\tau)^2-\rho^2}\,\bigr)}{\sqrt{a^2(t-\tau)^2-\rho^2}}\,d\xi\,d\eta,
\end{aligned}
$$

where $\rho = \sqrt{(x-\xi)^2+(y-\eta)^2}$.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. First boundary value problem.

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ \int_0^{l_1}\!\!\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau \\
&- a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1} d\eta\,d\tau \\
&+ a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\
&- a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=l_2} d\xi\,d\tau \\
&+ \int_0^{t}\!\!\int_0^{l_1}\!\!\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,
\end{aligned}
$$

where

$$G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\lambda_{nm}}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)\sin(\lambda_{nm}t),$$

$$p_n = \frac{n\pi}{l_1},\quad q_m = \frac{m\pi}{l_2},\quad \lambda_{nm} = \sqrt{a^2p_n^2 + a^2q_m^2 + b}.$$

► **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2$. Second boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y) && \text{at} && t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} && t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,t) && \text{at} && x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} && x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} && y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} && y=l_2 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) =& \frac{\partial}{\partial t}\int_0^{l_1}\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi + \int_0^{l_1}\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&- a^2\int_0^t\int_0^{l_2} g_1(\eta,\tau)G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau\\
&+ a^2\int_0^t\int_0^{l_2} g_2(\eta,\tau)G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau\\
&- a^2\int_0^t\int_0^{l_1} g_3(\xi,\tau)G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau\\
&+ a^2\int_0^t\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau\\
&+ \int_0^t\int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{\sin(t\sqrt{b})}{l_1 l_2\sqrt{b}} + \frac{2}{l_1 l_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_{nm}}{\lambda_{nm}}\cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)\sin(\lambda_{nm}t),$$

$$p_n = \frac{n\pi}{l_1},\quad q_m = \frac{m\pi}{l_2},\quad \lambda_{nm} = \sqrt{a^2p_n^2 + a^2q_m^2 + b},\quad A_{nm} = \begin{cases} 0 & \text{for } n=m=0,\\ 1 & \text{for } nm=0\ (n\neq m),\\ 2 & \text{for } nm\neq 0.\end{cases}$$

▶ **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Third boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w - k_1 w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w + k_2 w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w - k_3 w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w + k_4 w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\begin{gathered}
G(x,y,\xi,\eta,t) = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{E_{nm}\sqrt{a^2\mu_n^2+a^2\nu_m^2+b}}\sin(\mu_n x+\varepsilon_n)\sin(\nu_m y+\sigma_m)\\
\times\sin(\mu_n\xi+\varepsilon_n)\sin(\nu_m\eta+\sigma_m)\sin\bigl(t\sqrt{a^2\mu_n^2+a^2\nu_m^2+b}\,\bigr),\\
E_{nm} = \left[l_1+\frac{(k_1k_2+\mu_n^2)(k_1+k_2)}{(k_1^2+\mu_n^2)(k_2^2+\mu_n^2)}\right]\left[l_2+\frac{(k_3k_4+\nu_m^2)(k_3+k_4)}{(k_3^2+\nu_m^2)(k_4^2+\nu_m^2)}\right],\\
\varepsilon_n = \arctan\frac{\mu_n}{l_1},\qquad \sigma_m = \arctan\frac{\nu_m}{l_2}.
\end{gathered}
$$

Here, the μ_n and ν_m are positive roots of the transcendental equations

$$
\begin{aligned}
\mu^2 - k_1k_2 &= (k_1+k_2)\mu\cot(l_1\mu),\\
\nu^2 - k_3k_4 &= (k_3+k_4)\nu\cot(l_2\nu).
\end{aligned}
$$

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Mixed boundary value problems.**

1°. A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ \int_0^{l_1}\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ a^2\int_0^{t}\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau \\
&- a^2\int_0^{t}\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1} d\eta\,d\tau \\
&- a^2\int_0^{t}\int_0^{l_1} g_3(\xi,\tau)G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau \\
&+ a^2\int_0^{t}\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau \\
&+ \int_0^{t}\int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,y,\xi,\eta,t) = \frac{2}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m}{\lambda_{nm}}\sin(p_n x)\cos(q_m y)\sin(p_n\xi)\cos(q_m\eta)\sin(\lambda_{nm}t),$$

$$p_n = \frac{n\pi}{l_1},\quad q_m = \frac{m\pi}{l_2},\quad \lambda_{nm} = \sqrt{a^2p_n^2 + a^2q_m^2 + b},\quad A_m = \begin{cases}1 & \text{for } m=0,\\ 2 & \text{for } m\neq 0.\end{cases}$$

2°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y) \quad &&\text{at} \quad t=0 \quad &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,t) &&\text{at} \quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,t) &&\text{at} \quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,t) &&\text{at} \quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,t) &&\text{at} \quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ \int_0^{l_1}\int_0^{l_2} f_1(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ a^2\int_0^{t}\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau \\
&+ a^2\int_0^{t}\int_0^{l_2} g_2(\eta,\tau)G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau
\end{aligned}$$

$$+ a^2 \int_0^t \int_0^{l_1} g_3(\xi,\tau) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta,t-\tau) \right]_{\eta=0} d\xi \, d\tau$$
$$+ a^2 \int_0^t \int_0^{l_1} g_4(\xi,\tau) G(x,y,\xi,l_2,t-\tau) \, d\xi \, d\tau$$
$$+ \int_0^t \int_0^{l_1} \int_0^{l_2} \Phi(\xi,\eta,\tau) G(x,y,\xi,\eta,t-\tau) \, d\eta \, d\xi \, d\tau,$$

where

$$G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\lambda_{nm}} \sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta) \sin(\lambda_{nm} t),$$
$$p_n = \frac{\pi(2n+1)}{2l_1}, \quad q_m = \frac{\pi(2m+1)}{2l_2}, \quad \lambda_{nm} = \sqrt{a^2 p_n^2 + a^2 q_m^2 + b}.$$

7.3.2 Problems in Polar Coordinates

A nonhomogeneous Klein–Gordon equation with two space variables in the polar coordinate system has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} \right) - bw + \Phi(r,\varphi,t), \qquad r = \sqrt{x^2+y^2}.$$

One-dimensional solutions $w = w(r,t)$ independent of the angular coordinate φ are considered in Section 6.2.5.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)}, \\ \partial_t w &= f_1(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)}, \\ w &= g(\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(r,\varphi,t) = \frac{\partial}{\partial t} \int_0^{2\pi} \int_0^R f_0(\xi,\eta) G(r,\varphi,\xi,\eta,t) \xi \, d\xi \, d\eta$$
$$+ \int_0^{2\pi} \int_0^R f_1(\xi,\eta) G(r,\varphi,\xi,\eta,t) \xi \, d\xi \, d\eta$$
$$- a^2 R \int_0^t \int_0^{2\pi} g(\eta,\tau) \left[\frac{\partial}{\partial \xi} G(r,\varphi,\xi,\eta,t-\tau) \right]_{\xi=R} d\eta \, d\tau$$
$$+ \int_0^t \int_0^{2\pi} \int_0^R \Phi(\xi,\eta,\tau) G(r,\varphi,\xi,\eta,t-\tau) \xi \, d\xi \, d\eta \, d\tau.$$

Here,*

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}},$$

$$A_0=1,\quad A_n=2\quad (n=1,\,2,\,\ldots),\quad \lambda_{nm}=\sqrt{a^2\mu_{nm}^2+b},$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R)=0$.

► **Domain: $0\le r\le R$, $0\le\varphi\le 2\pi$. Second boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad+\int_0^{2\pi}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad+a^2R\int_0^t\!\!\int_0^{2\pi} g(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau\\ &\quad+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,\xi,\eta,t) &= \frac{\sin\bigl(t\sqrt{b}\bigr)}{\pi R^2\sqrt{b}}\\ &\quad+\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\,\bigr)}{\sqrt{a^2\mu_{nm}^2+b}}, \end{aligned}$$

where $A_0=1$ and $A_n=2$ for $n=1,\,2,\,\ldots$; the $J_n(\xi)$ are Bessel functions; and the μ_m are positive roots of the transcendental equation $J_n'(\mu R)=0$.

► **Domain: $0\le r\le R$, $0\le\varphi\le 2\pi$. Third boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w + kw &= g(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}. \end{aligned}$$

*The ratios $\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\bigr)\big/\sqrt{a^2\mu_{nm}^2+b}$ in the expressions of the Green's functions specified in Section 7.3.2 must be replaced with $\sinh\bigl(t\sqrt{|a^2\mu_{nm}^2+b|}\,\bigr)\big/\sqrt{|a^2\mu_{nm}^2+b|}$ if $a^2\mu_{nm}^2+b<0$.

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2}{B_{nm}}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\,\bigr)}{\sqrt{a^2\mu_{nm}^2+b}},$$

$$A_0=1,\quad A_n=2\quad (n=1,\,2,\,\dots),\quad B_{nm}=(\mu_{nm}^2R^2+k^2R^2-n^2)[J_n(\mu_{nm}R)]^2.$$

Here, the $J_n(\xi)$ are Bessel functions and the μ_m are positive roots of the transcendental equation

$$\mu J_n'(\mu R)+kJ_n(\mu R)=0.$$

► **Domain: $R_1\le r\le R_2$, $0\le\varphi\le 2\pi$. First boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_1(\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
w&=g_2(\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,t)&=\frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta+\int_0^{2\pi}\!\!\int_{R_1}^{R_2} f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&\quad+a^2R_1\int_0^t\!\!\int_0^{2\pi} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R_1}d\eta\,d\tau\\
&\quad-a^2R_2\int_0^t\!\!\int_0^{2\pi} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R_2}d\eta\,d\tau\\
&\quad+\int_0^t\!\!\int_0^{2\pi}\!\!\int_{R_1}^{R_2}\Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_nB_{nm}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\,\bigr)}{\sqrt{a^2\mu_{nm}^2+b}},$$

$$A_n=\begin{cases}1/2 & \text{for } n=0,\\ 1 & \text{for } n\neq 0,\end{cases}\qquad B_{nm}=\frac{\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)},$$

$$Z_n(\mu_{nm}r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2)-Y_n(\mu R_1)J_n(\mu R_2)=0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) = \frac{\partial}{\partial t}\int_0^{2\pi}\int_{R_1}^{R_2} & f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &+ \int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &- a^2R_1\int_0^t\int_0^{2\pi} g_1(\eta,\tau)G(r,\varphi,R_1,\eta,t-\tau)\,d\eta\,d\tau \\ &+ a^2R_2\int_0^t\int_0^{2\pi} g_2(\eta,\tau)G(r,\varphi,R_2,\eta,t-\tau)\,d\eta\,d\tau \\ &+ \int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,\xi,\eta,t) = & \frac{\sin\bigl(t\sqrt{b}\,\bigr)}{\pi(R_2^2-R_1^2)\sqrt{b}} \\ &+ \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\,\bigr)}{\bigl[(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)\bigr]\sqrt{a^2\mu_{nm}^2+b}}, \end{aligned}$$

where

$$Z_n(\mu_{nm}r) = J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r), \qquad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n>0, \end{cases}$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2) - Y_n'(\mu R_1)J_n'(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w - k_1 w &= g_1(\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w + k_2 w &= g_2(\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2}{B_{nm}\lambda_{nm}}Z_{nm}(r)Z_{nm}(\xi)\cos[n(\varphi-\eta)]\sin(\lambda_{nm}t),$$

$$B_{nm}=(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1),$$

$$Z_{nm}(r)=\big[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\big]Y_n(\mu_{nm}r)$$
$$-\big[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\big]J_n(\mu_{nm}r).$$

Here, $A_0=1$ and $A_n=2$ for $n=1,\,2,\,\dots;$ $\lambda_{nm}=\sqrt{a^2\mu_{nm}^2+b}$; the $J_n(r)$ and $Y_n(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation

$$\big[\mu J_n'(\mu R_1)-k_1J_n(\mu R_1)\big]\big[\mu Y_n'(\mu R_2)+k_2Y_n(\mu R_2)\big]$$
$$=\big[\mu Y_n'(\mu R_1)-k_1Y_n(\mu R_1)\big]\big[\mu J_n'(\mu R_2)+k_2J_n(\mu R_2)\big].$$

► **Domain: $0\le r\le R$, $0\le\varphi\le\varphi_0$. First boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w&=f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_1(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w&=g_2(r,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
w&=g_3(r,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,t)&=\frac{\partial}{\partial t}\int_0^{\varphi_0}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta+\int_0^{\varphi_0}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&-a^2R\int_0^t\!\!\int_0^{\varphi_0} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R}d\eta\,d\tau\\
&+a^2\int_0^t\!\!\int_0^R g_2(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau\\
&-a^2\int_0^t\!\!\int_0^R g_3(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\tau\\
&+\int_0^t\!\!\int_0^{\varphi_0}\!\!\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{4}{R^2\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}},$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$, and $\lambda_{nm}=\sqrt{a^2\mu_{nm}^2+b}$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Second boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
r^{-1}\partial_\varphi w &= g_2(r,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\
r^{-1}\partial_\varphi w &= g_3(r,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,t) = {}& \frac{\partial}{\partial t}\int_0^{\varphi_0}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta + \int_0^{\varphi_0}\!\!\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\
&+ a^2R\int_0^t\!\!\int_0^{\varphi_0} g_1(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau\\
&- a^2\int_0^t\!\!\int_0^R g_2(\xi,\tau)G(r,\varphi,\xi,0,t-\tau)\,d\xi\,d\tau\\
&+ a^2\int_0^t\!\!\int_0^R g_3(\xi,\tau)G(r,\varphi,\xi,\varphi_0,t-\tau)\,d\xi\,d\tau\\
&+ \int_0^t\!\!\int_0^{\varphi_0}\!\!\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,\xi,\eta,t) = {}& \frac{2\sin\bigl(t\sqrt{b}\bigr)}{R^2\varphi_0\sqrt{b}} + 4\varphi_0\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_{nm}^2 J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{(R^2\varphi_0^2\mu_{nm}^2 - n^2\pi^2)\bigl[J_{n\pi/\varphi_0}(\mu_{nm}R)\bigr]^2}\\
&\times\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\eta}{\varphi_0}\right)\frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\bigr)}{\sqrt{a^2\mu_{nm}^2+b}},
\end{aligned}
$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J'_{n\pi/\varphi_0}(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Mixed boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w + kw &= g(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
\partial_\varphi w &= 0 && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\
\partial_\varphi w &= 0 && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{\varphi_0}\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &\quad + \int_0^{\varphi_0}\int_0^R f_1(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &\quad + a^2R\int_0^t\int_0^{\varphi_0} g(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau \\ &\quad + \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{gathered} G(r,\varphi,\xi,\eta,t) = 4\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_{nm}^2}{B_{nm}\sqrt{a^2\mu_{nm}^2+b}}J_{s_n}(\mu_{nm}r)J_{s_n}(\mu_{nm}\xi) \\ \times\cos(s_n\varphi)\cos(s_n\eta)\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b}\,\bigr), \\ s_n = \frac{n\pi}{\varphi_0}, \qquad B_{nm} = \varphi_0(\mu_{nm}^2R^2 + k^2R^2 - s_n^2)\bigl[J_{s_n}(\mu_{nm}R)\bigr]^2, \end{gathered}$$

where the $J_{s_n}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J'_{s_n}(\mu R) + kJ_{s_n}(\mu R) = 0.$$

7.3.3 Axisymmetric Problems

In the axisymmetric case, a nonhomogeneous Klein–Gordon equation in the cylindrical system of coordinates has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) - bw + \Phi(r,z,t), \qquad r = \sqrt{x^2+y^2}.$$

In the solutions of the problems considered below, the modified Green's function $\mathcal{G}(r,z,\xi,\eta,t) = 2\pi\xi G(r,z,\xi,\eta,t)$ is used for convenience.

► **Domain: $0 \le r \le R$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\ w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)}, \\ w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)}, \\ w &= g_3(r,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ \int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&- a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau \\
&+ a^2\int_0^t\int_0^R g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\
&- a^2\int_0^t\int_0^R g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau \\
&+ \int_0^t\int_0^l\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$\mathcal{G}(r,z,\xi,\eta,t)=\frac{4\xi}{R^2 l}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{J_1^2(\mu_n)}J_0\!\left(\frac{\mu_n r}{R}\right)J_0\!\left(\frac{\mu_n\xi}{R}\right)\sin\!\left(\frac{m\pi z}{l}\right)\sin\!\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\bigr)}{\sqrt{\lambda_{nm}}},$$

$$\lambda_{nm}=\frac{a^2\mu_n^2}{R^2}+\frac{a^2\pi^2 m^2}{l^2}+b,$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ \int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R,\eta,t-\tau)\,d\eta\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&- a^2\int_0^t\int_0^R g_2(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau \\
&+ a^2\int_0^t\int_0^R g_3(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi(\,d\tau \\
&+ \int_0^t\int_0^l\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) =& \frac{2\xi\sin\bigl(t\sqrt{b}\bigr)}{R^2 l\sqrt{b}} \\
&+\frac{2\xi}{R^2 l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_{nm}}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\cos\left(\frac{m\pi z}{l}\right)\cos\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}},
\end{aligned}
$$

$$
\lambda_{nm} = \frac{a^2\mu_n^2}{R^2} + \frac{a^2\pi^2 m^2}{l^2} + b, \qquad A_{nm} = \begin{cases} 0 & \text{for } m=0,\, n=0, \\ 1 & \text{for } m=0,\, n>0, \\ 2 & \text{for } m>0, \end{cases}
$$

where the μ_n are zeros of the first-order Bessel function, $J_1(\mu)=0$ $(\mu_0=0)$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

► **Domain: $0\le r\le R$, $0\le z\le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w + k_1 w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w - k_2 w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w + k_3 w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\mathcal{G}(r,z,\xi,\eta,t) = \frac{2\xi}{R^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_n^2}{B_n}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2}\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}},
$$

$$
B_n = (k_1^2R^2+\mu_n^2)J_0^2(\mu_n), \quad \lambda_{nm} = \frac{a^2\mu_n^2}{R^2} + a^2\beta_m^2 + b, \quad \varphi_m(z) = \cos(\beta_m z) + \frac{k_2}{\beta_m}\sin(\beta_m z),
$$

$$
\|\varphi_m\|^2 = \frac{k_3}{2\beta_m^2}\frac{\beta_m^2+k_2^2}{\beta_m^2+k_3^2} + \frac{k_2}{2\beta_m^2} + \frac{l}{2}\left(1+\frac{k_2^2}{\beta_m^2}\right).
$$

Here, the μ_n and β_m are positive roots of the transcendental equations

$$
\mu J_1(\mu) - k_1 R J_0(\mu) = 0, \qquad \frac{\tan(\beta l)}{\beta} = \frac{k_2+k_3}{\beta^2 - k_2 k_3}.
$$

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(r,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
w &= g_1(z,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)},\\
\partial_z w &= g_2(r,t) \quad \text{at} \quad z=0 \quad \text{(boundary condition)},\\
\partial_z w &= g_3(r,t) \quad \text{at} \quad z=l \quad \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&\quad + \int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&\quad - a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau\\
&\quad - a^2\int_0^t\int_0^R g_2(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau\\
&\quad + a^2\int_0^t\int_0^R g_3(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi\,d\tau\\
&\quad + \int_0^t\int_0^l\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{2\xi}{R^2 l}\sum_{n=1}^\infty\sum_{m=0}^\infty \frac{A_m}{J_1^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right)\\
&\qquad\qquad \times \cos\left(\frac{m\pi z}{l}\right)\cos\left(\frac{m\pi\eta}{l}\right)\frac{\sin\left(t\sqrt{\lambda_{nm}}\,\right)}{\sqrt{\lambda_{nm}}},\\
\lambda_{nm} &= \frac{a^2\mu_n^2}{R^2} + \frac{a^2\pi^2 m^2}{l^2} + b, \quad A_m = \begin{cases} 1 & \text{for } m=0,\\ 2 & \text{for } m>0,\end{cases}
\end{aligned}$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

2°. A circular cylinder of finite length is considered. The following conditions are pre-

scribed:

$$
\begin{aligned}
w &= f_0(r,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(r,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_r w &= g_1(z,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)},\\
w &= g_2(r,t) \quad \text{at} \quad z=0 \quad \text{(boundary condition)},\\
w &= g_3(r,t) \quad \text{at} \quad z=l \quad \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta + \int_0^l\int_0^R f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta\\
&+ a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R,\eta,t-\tau)\,d\eta\,d\tau\\
&+ a^2\int_0^t\int_0^R g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\
&- a^2\int_0^t\int_0^R g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau\\
&+ \int_0^t\int_0^l\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{4\xi}{R^2 l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)\\
&\qquad\times\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\bigr)}{\sqrt{\lambda_{nm}}},\\
\lambda_{nm} &= \frac{a^2\mu_n^2}{R^2} + \frac{a^2\pi^2 m^2}{l^2} + b,
\end{aligned}
$$

where the μ_n are zeros of the first-order Bessel function, $J_1(\mu)=0$ $(\mu_0=0)$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

► Domain: $R_1 \le r \le R_2$, $0\le z\le l$. First boundary value problem.

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(r,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
w &= g_1(z,t) \quad \text{at} \quad r=R_1 \quad \text{(boundary condition)},\\
w &= g_2(z,t) \quad \text{at} \quad r=R_2 \quad \text{(boundary condition)},\\
w &= g_3(r,t) \quad \text{at} \quad z=0 \quad \text{(boundary condition)},\\
w &= g_4(r,t) \quad \text{at} \quad z=l \quad \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_{R_1}^{R_2} f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ \int_0^l\int_{R_1}^{R_2} f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R_1} d\eta\,d\tau \\
&- a^2\int_0^t\int_0^l g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R_2} d\eta\,d\tau \\
&+ a^2\int_0^t\int_{R_1}^{R_2} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\
&- a^2\int_0^t\int_{R_1}^{R_2} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau \\
&+ \int_0^t\int_0^l\int_{R_1}^{R_2} \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{\pi^2\xi}{R_1^2 l}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_n^2 J_0^2(s\mu_n)}{J_0^2(\mu_n)-J_0^2(s\mu_n)}\Psi_n(r)\Psi_n(\xi) \\
&\quad\times\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\bigr)}{\sqrt{\lambda_{nm}}},
\end{aligned}
$$

$$
\Psi_n(r) = Y_0(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right) - J_0(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right),\quad s=\frac{R_2}{R_1},\quad \lambda_{nm}=\frac{a^2\mu_n^2}{R_1^2}+\frac{a^2\pi^2 m^2}{l^2}+b,
$$

where $J_0(\mu)$ and $Y_0(\mu)$ are Bessel functions, and the μ_n are positive roots of the transcendental equation

$$
J_0(\mu)Y_0(s\mu) - J_0(s\mu)Y_0(\mu) = 0.
$$

► **Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_{R_1}^{R_2} f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ \int_0^l\int_{R_1}^{R_2} f_1(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&- a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R_1,\eta,t-\tau)\,d\eta\,d\tau \\
&+ a^2\int_0^t\int_0^l g_2(\eta,\tau)\mathcal{G}(r,z,R_2,\eta,t-\tau)\,d\eta\,d\tau \\
&- a^2\int_0^t\int_{R_1}^{R_2} g_3(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau \\
&+ a^2\int_0^t\int_{R_1}^{R_2} g_4(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi\,d\tau \\
&+ \int_0^t\int_0^l\int_{R_1}^{R_2} \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{2\xi\sin\bigl(t\sqrt{b}\,\bigr)}{(R_2^2-R_1^2)l\sqrt{b}} + \frac{4\xi}{(R_2^2-R_1^2)l}\sum_{m=1}^{\infty}\cos\Bigl(\frac{m\pi z}{l}\Bigr)\cos\Bigl(\frac{m\pi\eta}{l}\Bigr)\frac{\sin\bigl(t\sqrt{\beta_m}\,\bigr)}{\sqrt{\beta_m}} \\
&+ \frac{\pi^2\xi}{2R_1^2 l}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m\mu_n^2 J_1^2(s\mu_n)}{J_1^2(\mu_n)-J_1^2(s\mu_n)}\Psi_n(r)\Psi_n(\xi)\cos\Bigl(\frac{m\pi z}{l}\Bigr)\cos\Bigl(\frac{m\pi\eta}{l}\Bigr)\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}},
\end{aligned}
$$

where

$$
\Psi_n(r) = Y_1(\mu_n)J_0\Bigl(\frac{\mu_n r}{R_1}\Bigr) - J_1(\mu_n)Y_0\Bigl(\frac{\mu_n r}{R_1}\Bigr), \qquad s = \frac{R_2}{R_1},
$$

$$
A_m = \begin{cases} 1 & \text{for } m = 0, \\ 2 & \text{for } m > 1, \end{cases} \qquad \beta_m = \frac{a^2\pi^2 m^2}{l^2} + b, \qquad \lambda_{nm} = \frac{a^2\mu_n^2}{R_1^2} + \frac{a^2\pi^2 m^2}{l^2} + b;
$$

$J_k(\mu)$ and $Y_k(\mu)$ are Bessel functions ($k = 0,\ 1$); and the μ_n are positive roots of the transcendental equation

$$
J_1(\mu)Y_1(s\mu) - J_1(s\mu)Y_1(\mu) = 0.
$$

7.4 Telegraph Equation $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a^2\Delta_2 w - bw + \Phi(x,y,t)$

7.4.1 Problems in Cartesian Coordinates

A two-dimensional nonhomogeneous telegraph equation in the rectangular Cartesian coordinate system is written as

$$
\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\Bigl(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Bigr) - bw + \Phi(x,y,t).
$$

► **Reduction to the two-dimensional Klein–Gordon equation.**

The substitution $w(x,y,t) = \exp\bigl(-\frac{1}{2}kt\bigr)u(x,y,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \bigl(b - \tfrac{1}{4}k^2\bigr)u + \exp\bigl(\tfrac{1}{2}kt\bigr)\Phi(x,y,t),$$

which is discussed in Section 7.3.1.

► **Fundamental solutions.**

1°. Case $b - \frac{1}{4}k^2 = \sigma^2 > 0$:

$$\mathscr{E}(x,y,t) = \vartheta(at - r)\exp\bigl(-\tfrac{1}{2}kt\bigr)\frac{\cos\bigl(\sigma\sqrt{t^2 - r^2/a^2}\,\bigr)}{2\pi a^2\sqrt{t^2 - r^2/a^2}},$$

where $r = \sqrt{x^2 + y^2}$ and $\vartheta(z)$ is the Heaviside unit step function.

2°. Case $b - \frac{1}{4}k^2 = -\sigma^2 < 0$:

$$\mathscr{E}(x,y,t) = \vartheta(at - r)\exp\bigl(-\tfrac{1}{2}kt\bigr)\frac{\cosh\bigl(\sigma\sqrt{t^2 - r^2/a^2}\,\bigr)}{2\pi a^2\sqrt{t^2 - r^2/a^2}}.$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x,y) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned} w(x,y,t) &= \exp\bigl(-\tfrac{1}{2}kt\bigr)\frac{\partial}{\partial t}\iint\limits_{\rho\le at} f(\xi,\eta)H(x,y,\xi,\eta,t)\,d\xi\,d\eta \\ &\quad + \exp\bigl(-\tfrac{1}{2}kt\bigr)\iint\limits_{\rho\le at}\bigl[g(\xi,\eta) + \tfrac{1}{2}kf(\xi,\eta)\bigr]H(x,y,\xi,\eta,t)\,d\xi\,d\eta \\ &\quad + \int_0^t d\tau \iint\limits_{\rho\le a(t-\tau)} \exp\bigl[-\tfrac{1}{2}k(t-\tau)\bigr]\Phi(\xi,\eta,\tau)H(x,y,\xi,\eta,t-\tau)\,d\xi\,d\eta. \end{aligned}$$

Here,

$$H(x,y,\xi,\eta,t) = \begin{cases} \dfrac{\cos\bigl(\sigma\sqrt{t^2 - \rho^2/a^2}\,\bigr)}{2\pi a^2\sqrt{t^2 - \rho^2/a^2}} & \text{for } b - \frac{1}{4}k^2 = \sigma^2 > 0, \\ \dfrac{\cosh\bigl(\sigma\sqrt{t^2 - \rho^2/a^2}\,\bigr)}{2\pi a^2\sqrt{t^2 - \rho^2/a^2}} & \text{for } b - \frac{1}{4}k^2 = -\sigma^2 < 0, \end{cases}$$

where $\rho = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. First boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&+ \int_0^{l_1}\!\!\int_0^{l_2} \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&+ a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0} d\eta\,d\tau\\
&- a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1} d\eta\,d\tau\\
&+ a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau\\
&- a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=l_2} d\xi\,d\tau\\
&+ \int_0^{t}\!\!\int_0^{l_1}\!\!\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,
\end{aligned}
$$

where

$$
G(x,y,\xi,\eta,t)=\frac{4}{l_1l_2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\lambda_{nm}}\sin(p_nx)\sin(q_my)\sin(p_n\xi)\sin(q_m\eta)\sin(\lambda_{nm}t),
$$

$$
p_n=\frac{n\pi}{l_1},\quad q_m=\frac{m\pi}{l_2},\quad \lambda_{nm}=\sqrt{a^2p_n^2+a^2q_m^2+b-\tfrac{1}{4}k^2}.
$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Second boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,t) &= \frac{\partial}{\partial t}\int_0^{l_1}\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&+ \int_0^{l_1}\int_0^{l_2}\big[f_1(\xi,\eta)+kf_0(\xi,\eta)\big]G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\
&- a^2\int_0^{t}\int_0^{l_2} g_1(\eta,\tau)G(x,y,0,\eta,t-\tau)\,d\eta\,d\tau \\
&+ a^2\int_0^{t}\int_0^{l_2} g_2(\eta,\tau)G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau \\
&- a^2\int_0^{t}\int_0^{l_1} g_3(\xi,\tau)G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau \\
&+ a^2\int_0^{t}\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau \\
&+ \int_0^{t}\int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(x,y,\xi,\eta,t) = \exp\big(-\tfrac{1}{2}kt\big)\Bigg[&\frac{\sin\big(\lambda_{00}t\big)}{l_1 l_2\lambda_{00}} \\
&+ \frac{2}{l_1 l_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_{nm}}{\lambda_{nm}}\cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)\sin(\lambda_{nm}t)\Bigg],
\end{aligned}
$$

where

$$
p_n=\frac{n\pi}{l_1},\quad q_m=\frac{m\pi}{l_2},\quad \lambda_{nm}=\sqrt{a^2p_n^2+a^2q_m^2+b-\tfrac{1}{4}k^2},\quad A_{nm}=\begin{cases}0 & \text{for } n=m=0,\\ 1 & \text{for } nm=0\ (n\neq m),\\ 2 & \text{for } nm\neq 0.\end{cases}
$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Third boundary value problem.**

A rectangle is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w - s_1 w &= g_1(y,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w + s_2 w &= g_2(y,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w - s_3 w &= g_3(x,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w + s_4 w &= g_4(x,t) && \text{at} \quad y=l_2 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t) = 4\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{E_{nm}\lambda_{mn}}\sin(\mu_n x+\varepsilon_n)\sin(\nu_m y+\sigma_m)$$
$$\times\sin(\mu_n\xi+\varepsilon_n)\sin(\nu_m\eta+\sigma_m)\sin\bigl(\lambda_{mn}t\bigr).$$

Here,

$$\lambda_{mn}=\sqrt{a^2\mu_n^2+a^2\nu_m^2+b-\tfrac{1}{4}k^2},\quad \varepsilon_n=\arctan\frac{\mu_n}{l_1},\quad \sigma_m=\arctan\frac{\nu_m}{l_2},$$
$$E_{nm}=\left[l_1+\frac{(s_1s_2+\mu_n^2)(s_1+s_2)}{(s_1^2+\mu_n^2)(s_2^2+\mu_n^2)}\right]\left[l_2+\frac{(s_3s_4+\nu_m^2)(s_3+s_4)}{(s_3^2+\nu_m^2)(s_4^2+\nu_m^2)}\right];$$

the μ_n and ν_m are positive roots of the transcendental equations

$$\mu^2-s_1s_2=(s_1+s_2)\mu\cot(l_1\mu),\quad \nu^2-s_3s_4=(s_3+s_4)\nu\cot(l_2\nu).$$

► **Domain: $0\le x\le l_1$, $0\le y\le l_2$. Mixed boundary value problems.**

1°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w&=g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w&=g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w&=g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t)&=\frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad+\int_0^{l_1}\!\!\int_0^{l_2}\bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0}d\eta\,d\tau\\
&\quad-a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=l_1}d\eta\,d\tau\\
&\quad-a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)G(x,y,\xi,0,t-\tau)\,d\xi\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau\\
&\quad+\int_0^{t}\!\!\int_0^{l_1}\!\!\int_0^{l_2}\Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,y,\xi,\eta,t)=\frac{2}{l_1l_2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m}{\lambda_{nm}}\sin(p_nx)\cos(q_my)\sin(p_n\xi)\cos(q_m\eta)\sin(\lambda_{nm}t),$$

$$p_n=\frac{n\pi}{l_1},\quad q_m=\frac{m\pi}{l_2},\quad \lambda_{nm}=\sqrt{a^2p_n^2+a^2q_m^2+b-\tfrac{1}{4}k^2},\quad A_m=\begin{cases}1 & \text{for } m=0,\\ 2 & \text{for } m\neq 0.\end{cases}$$

2°. A rectangle is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(x,y) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_1(y,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w&=g_2(y,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w&=g_3(x,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w&=g_4(x,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,t)&=\frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f_0(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad+\int_0^{l_1}\!\!\int_0^{l_2}\bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(x,y,\xi,\eta,t)\,d\eta\,d\xi\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta,t-\tau)\right]_{\xi=0}d\eta\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2} g_2(\eta,\tau)G(x,y,l_1,\eta,t-\tau)\,d\eta\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_1} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau\\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_1} g_4(\xi,\tau)G(x,y,\xi,l_2,t-\tau)\,d\xi\,d\tau\\
&\quad+\int_0^{t}\!\!\int_0^{l_1}\!\!\int_0^{l_2}\Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau,
\end{aligned}$$

where

$$G(x,y,\xi,\eta,t)=\frac{4}{l_1l_2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\lambda_{nm}}\sin(p_nx)\sin(q_my)\sin(p_n\xi)\sin(q_m\eta)\sin(\lambda_{nm}t),$$

$$p_n=\frac{\pi(2n+1)}{2l_1},\quad q_m=\frac{\pi(2m+1)}{2l_2},\quad \lambda_{nm}=\sqrt{a^2p_n^2+a^2q_m^2+b-\tfrac{1}{4}k^2}.$$

7.4.2 Problems in Polar Coordinates

A two-dimensional nonhomogeneous telegraph equation in the polar coordinate system has the form

$$\frac{\partial^2 w}{\partial t^2}+k\frac{\partial w}{\partial t}=a^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}\right)-bw+\Phi(r,\varphi,t),\qquad r=\sqrt{x^2+y^2}.$$

For one-dimensional solutions $w=w(r,t)$, see equation 6.4.2.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ w &= g(\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+\int_0^{2\pi}\!\!\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &-a^2R\int_0^t\!\!\int_0^{2\pi} g(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R}d\eta\,d\tau\\ &+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi R^2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_nJ_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{[J_n'(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}},$$
$$\lambda_{nm}=a^2\mu_{nm}^2+b-\tfrac{1}{4}k^2,\quad A_0=1,\quad A_n=2\quad (n=1,\,2,\,\ldots),$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R)=0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_r w &= g(\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+\int_0^{2\pi}\!\!\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &+a^2R\int_0^t\!\!\int_0^{2\pi} g(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau\\ &+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\exp\bigl(-\tfrac{1}{2}kt\bigr)\Biggl[\frac{\sin\bigl(t\sqrt{b-k^2/4}\,\bigr)}{\pi R^2\sqrt{b-k^2/4}} + \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}}\Biggr],$$

$$\lambda_{nm}=a^2\mu_{nm}^2+b-\tfrac{1}{4}k^2,\quad A_0=1,\quad A_n=2\quad (n=1,\,2,\,\ldots),$$

where the $J_n(\xi)$ are Bessel functions and the μ_m are positive roots of the transcendental equation $J_n'(\mu R)=0$.

▶ Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Third boundary value problem.

A circle is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w + sw &= g(\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{(\mu_{nm}^2R^2+s^2R^2-n^2)[J_n(\mu_{nm}R)]^2\sqrt{\lambda_{nm}}},$$

$$\lambda_{nm}=a^2\mu_{nm}^2+b-\tfrac{1}{4}k^2,\quad A_0=1,\quad A_n=2\quad (n=1,\,2,\,\ldots).$$

Here, the $J_n(\xi)$ are Bessel functions and the μ_m are positive roots of the transcendental equation

$$\mu J_n'(\mu R)+sJ_n(\mu R)=0.$$

▶ Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. First boundary value problem.

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ w &= g_2(\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$
\begin{aligned}
w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\
&\quad+\int_0^{2\pi}\!\!\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\
&\quad+a^2R_1\int_0^t\!\!\int_0^{2\pi} g_1(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\biggr]_{\xi=R_1} d\eta\,d\tau \\
&\quad-a^2R_2\int_0^t\!\!\int_0^{2\pi} g_2(\eta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\biggr]_{\xi=R_2} d\eta\,d\tau \\
&\quad+\int_0^t\!\!\int_0^{2\pi}\!\!\int_{R_1}^{R_2}\Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\varphi,\xi,\eta,t)=\frac{\pi}{2}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_nB_{nm}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\bigr)}{\sqrt{\lambda_{nm}}},
$$

$$
A_n=\begin{cases}1/2 & \text{for } n=0,\\ 1 & \text{for } n\neq 0,\end{cases}\qquad B_{nm}=\frac{\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)},
$$

$$
Z_n(\mu_{nm}r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),\qquad \lambda_{nm}=a^2\mu_{nm}^2+b-\tfrac{1}{4}k^2,
$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$
J_n(\mu R_1)Y_n(\mu R_2)-Y_n(\mu R_1)J_n(\mu R_2)=0.
$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\
&\quad+\int_0^{2\pi}\!\!\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\
&\quad-a^2R_1\int_0^t\!\!\int_0^{2\pi} g_1(\eta,\tau)G(r,\varphi,R_1,\eta,t-\tau)\,d\eta\,d\tau
\end{aligned}
$$

$$+\,a^2R_2\int_0^t\int_0^{2\pi} g_2(\eta,\tau)G(r,\varphi,R_2,\eta,t-\tau)\,d\eta\,d\tau$$
$$+\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau.$$

Here,

$$G(r,\varphi,\xi,\eta,t)=\exp\bigl(-\tfrac{1}{2}kt\bigr)\Biggl[\frac{\sin\bigl(t\sqrt{b-k^2/4}\,\bigr)}{\pi(R_2^2-R_1^2)\sqrt{b-k^2/4}}$$
$$+\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b-k^2/4}\,\bigr)}{\bigl[(\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1)\bigr]\sqrt{a^2\mu_{nm}^2+b-k^2/4}}\Biggr],$$

where

$$Z_n(\mu_{nm}r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r),\qquad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.$$

► **Domain: $R_1\le r\le R_2,\ 0\le\varphi\le 2\pi$. Third boundary value problem.**

An annular domain is considered. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(r,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w-s_1w&=g_1(\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w+s_2w&=g_2(\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,\xi,\eta,t)=\frac{1}{\pi}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2}{B_{nm}\lambda_{nm}}Z_n(\mu_{nm}r)Z_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin(\lambda_{nm}t).$$

Here,

$$A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}\qquad \lambda_{nm}=\sqrt{a^2\mu_{nm}^2+b-\tfrac{1}{4}k^2},$$
$$B_{nm}=(s_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_n^2(\mu_{nm}R_2)-(s_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_n^2(\mu_{nm}R_1),$$
$$Z_n(\mu_{nm}r)=\bigl[\mu_{nm}J_n'(\mu_{nm}R_1)-s_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r)$$
$$-\bigl[\mu_{nm}Y_n'(\mu_{nm}R_1)-s_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$\bigl[\mu J_n'(\mu R_1)-s_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2)+s_2Y_n(\mu R_2)\bigr]$$
$$=\bigl[\mu Y_n'(\mu R_1)-s_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2)+s_2J_n(\mu R_2)\bigr].$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. First boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ w &= g_3(r,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{\varphi_0}\!\!\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad+\int_0^{\varphi_0}\!\!\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta\\ &\quad-a^2R\int_0^t\!\!\int_0^{\varphi_0} g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\xi=R}d\eta\,d\tau\\ &\quad+a^2\int_0^t\!\!\int_0^R g_2(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau\\ &\quad-a^2\int_0^t\!\!\int_0^R g_3(\xi,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\tau\\ &\quad+\int_0^t\!\!\int_0^{\varphi_0}\!\!\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,\xi,\eta,t) &= \frac{4}{R^2\varphi_0}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\\ &\qquad\times\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b-k^2/4}\,\bigr)}{\sqrt{a^2\mu_{nm}^2+b-k^2/4}}, \end{aligned}$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Second boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_2(r,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\ r^{-1}\partial_\varphi w &= g_3(r,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{\varphi_0}\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &+ \int_0^{\varphi_0}\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &+ a^2R\int_0^t\int_0^{\varphi_0} g_1(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau \\ &- a^2\int_0^t\int_0^R g_2(\xi,\tau)G(r,\varphi,\xi,0,t-\tau)\,d\xi\,d\tau \\ &+ a^2\int_0^t\int_0^R g_3(\xi,\tau)G(r,\varphi,\xi,\varphi_0,t-\tau)\,d\xi\,d\tau \\ &+ \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,\xi,\eta,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr)\Biggl[\frac{2\sin\bigl(t\sqrt{b-k^2/4}\,\bigr)}{R^2\varphi_0\sqrt{b-k^2/4}} + 4\varphi_0\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_{nm}^2 J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{(R^2\varphi_0^2\mu_{nm}^2-n^2\pi^2)J_{n\pi/\varphi_0}^2(\mu_{nm}R)} \\ \times\cos\Bigl(\frac{n\pi\varphi}{\varphi_0}\Bigr)\cos\Bigl(\frac{n\pi\eta}{\varphi_0}\Bigr)\frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2+b-k^2/4}\,\bigr)}{\sqrt{a^2\mu_{nm}^2+b-k^2/4}}\Biggr], \end{aligned}$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J'_{n\pi/\varphi_0}(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$. Mixed boundary value problem.**

A circular sector is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi) && \text{at}\quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi) && \text{at}\quad t=0 && \text{(initial condition)},\\ \partial_r w + \beta w &= g(\varphi,t) && \text{at}\quad r=R && \text{(boundary condition)},\\ \partial_\varphi w &= 0 && \text{at}\quad \varphi=0 && \text{(boundary condition)},\\ \partial_\varphi w &= 0 && \text{at}\quad \varphi=\varphi_0 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{\varphi_0}\int_0^R f_0(\xi,\eta)G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &+ \int_0^{\varphi_0}\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]G(r,\varphi,\xi,\eta,t)\xi\,d\xi\,d\eta \\ &+ a^2R\int_0^t\int_0^{\varphi_0} g(\eta,\tau)G(r,\varphi,R,\eta,t-\tau)\,d\eta\,d\tau \\ &+ \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\tau)G(r,\varphi,\xi,\eta,t-\tau)\xi\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,\xi,\eta,t) = \exp\bigl(-\tfrac{1}{2}kt\bigr) \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_{s_n}(\mu_{nm} r) J_{s_n}(\mu_{nm}\xi)\cos(s_n\varphi)\cos(s_n\eta)\sin\bigl(\lambda_{nm} t\bigr),$$

$$s_n = \frac{n\pi}{\varphi_0},\quad A_{nm} = \frac{4\mu_{nm}^2}{\varphi_0(\mu_{nm}^2 R^2 + \beta^2 R^2 - s_n^2)\bigl[J_{s_n}(\mu_{nm} R)\bigr]^2 \lambda_{nm}},\quad \lambda_{nm} = \sqrt{a^2\mu_{nm}^2 + b - \tfrac{1}{4}k^2},$$

where the $J_{s_n}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J'_{s_n}(\mu R) + \beta J_{s_n}(\mu R) = 0.$$

7.4.3 Axisymmetric Problems

In the axisymmetric case, a nonhomogeneous telegraph equation in the cylindrical coordinate system has the form

$$\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) - bw + \Phi(r,z,t), \qquad r = \sqrt{x^2+y^2}.$$

In the solutions of the problems considered below, the modified Green's function $\mathcal{G}(r,z,\xi,\eta,t) = 2\pi\xi G(r,z,\xi,\eta,t)$ is used for convenience.

► **Domain: $0 \le r \le R$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ w &= g_2(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ w &= g_3(r,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\!\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\ &+ \int_0^l\!\int_0^R \bigl[f_1(\xi,\eta) + kf_0(\xi,\eta)\bigr]\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\ &- a^2\int_0^t\!\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau \\ &+ a^2\int_0^t\!\int_0^R g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\ &- a^2\int_0^t\!\int_0^R g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau \\ &+ \int_0^t\!\int_0^l\!\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\mathcal{G}(r, z, \xi, \eta, t) = \frac{4\xi e^{-kt/2}}{R^2 l} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{J_1^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \times \sin\left(\frac{m\pi z}{l}\right) \sin\left(\frac{m\pi\eta}{l}\right) \frac{\sin(\lambda_{nm} t)}{\lambda_{nm}},$$

$$\lambda_{nm} = \sqrt{\frac{a^2\mu_n^2}{R^2} + \frac{a^2\pi^2 m^2}{l^2} + b - \frac{k^2}{4}},$$

where the μ_n are positive zeros of the Bessel function, $J_0(\mu) = 0$. The numerical values of the first ten μ_n are specified in Section 3.2.1 (see the first boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(r, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_r w &= g_1(z, t) && \text{at} \quad r = R && \text{(boundary condition)},\\ \partial_z w &= g_2(r, t) && \text{at} \quad z = 0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r, t) && \text{at} \quad z = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r, z, t) &= \frac{\partial}{\partial t} \int_0^l \int_0^R f_0(\xi, \eta) \mathcal{G}(r, z, \xi, \eta, t)\, d\xi\, d\eta \\ &+ \int_0^l \int_0^R \big[f_1(\xi, \eta) + k f_0(\xi, \eta)\big] \mathcal{G}(r, z, \xi, \eta, t)\, d\xi\, d\eta \\ &+ a^2 \int_0^t \int_0^l g_1(\eta, \tau) \mathcal{G}(r, z, R, \eta, t - \tau)\, d\eta\, d\tau \\ &- a^2 \int_0^t \int_0^R g_2(\xi, \tau) \mathcal{G}(r, z, \xi, 0, t - \tau)\, d\xi\, d\tau \\ &+ a^2 \int_0^t \int_0^R g_3(\xi, \tau) \mathcal{G}(r, z, \xi, l, t - \tau)\, d\xi\, d\tau \\ &+ \int_0^t \int_0^l \int_0^R \Phi(\xi, \eta, \tau) \mathcal{G}(r, z, \xi, \eta, t - \tau)\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Here,

$$\mathcal{G}(r, z, \xi, \eta, t) = 2\xi \exp\left(-\tfrac{1}{2}kt\right) \left[\frac{\sin\left(t\sqrt{c}\right)}{R^2 l \sqrt{c}} + \frac{1}{R^2 l} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{A_{nm}}{J_0^2(\mu_n)} J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \times \cos\left(\frac{m\pi z}{l}\right) \cos\left(\frac{m\pi\eta}{l}\right) \frac{\sin\left(t\sqrt{\lambda_{nm}}\right)}{\sqrt{\lambda_{nm}}}\right],$$

where

$$c = b - \frac{k^2}{4}, \quad \lambda_{nm} = \frac{a^2\mu_n^2}{R^2} + \frac{a^2\pi^2m^2}{l^2} + b - \frac{k^2}{4}, \quad A_{nm} = \begin{cases} 0 & \text{for } m = 0, n = 0, \\ 1 & \text{for } m = 0, n > 0, \\ 2 & \text{for } m > 0, \end{cases}$$

and the μ_n are zeros of the first-order Bessel function, $J_1(\mu) = 0$ $(\mu_0 = 0)$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0 \le r \le R$).

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w + s_1 w &= g_1(z,t) && \text{at} \quad r = R && \text{(boundary condition)}, \\ \partial_z w - s_2 w &= g_2(r,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\ \partial_z w + s_3 w &= g_3(r,t) && \text{at} \quad z = l && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\mathcal{G}(r,z,\xi,\eta,t) = \frac{2\xi}{R^2} \exp\bigl(-\tfrac{1}{2}kt\bigr) \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n J_0\left(\frac{\mu_n r}{R}\right) J_0\left(\frac{\mu_n \xi}{R}\right) \frac{\varphi_m(z)\varphi_m(\eta)}{\|\varphi_m\|^2} \frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}}.$$

Here,

$$A_n = \frac{\mu_n^2}{(s_1^2R^2 + \mu_n^2)J_0^2(\mu_n)}, \quad \lambda_{nm} = \frac{a^2\mu_n^2}{R^2} + a^2\beta_m^2 + b - \frac{k^2}{4},$$

$$\varphi_m(z) = \cos(\beta_m z) + \frac{s_2}{\beta_m}\sin(\beta_m z), \quad \|\varphi_m\|^2 = \frac{s_3}{2\beta_m^2}\frac{\beta_m^2 + s_2^2}{\beta_m^2 + s_3^2} + \frac{s_2}{2\beta_m^2} + \frac{l}{2}\left(1 + \frac{s_2^2}{\beta_m^2}\right);$$

the μ_n and β_m are positive roots of the transcendental equations

$$\mu J_1(\mu) - s_1 R J_0(\mu) = 0, \qquad \frac{\tan(\beta l)}{\beta} = \frac{s_2 + s_3}{\beta^2 - s_2 s_3}.$$

► **Domain: $0 \le r \le R$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ w &= g_1(z,t) && \text{at} \quad r = R && \text{(boundary condition)}, \\ \partial_z w &= g_2(r,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\ \partial_z w &= g_3(r,t) && \text{at} \quad z = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ \int_0^l\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&- a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R} d\eta\,d\tau \\
&- a^2\int_0^t\int_0^R g_2(\xi,\tau)\mathcal{G}(r,z,\xi,0,t-\tau)\,d\xi\,d\tau \\
&+ a^2\int_0^t\int_0^R g_3(\xi,\tau)\mathcal{G}(r,z,\xi,l,t-\tau)\,d\xi\,d\tau \\
&+ \int_0^t\int_0^l\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{2\xi e^{-kt/2}}{R^2 l}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m}{J_1^2(\mu_n)}J_0\!\left(\frac{\mu_n r}{R}\right)J_0\!\left(\frac{\mu_n\xi}{R}\right) \\
&\qquad\times\cos\!\left(\frac{m\pi z}{l}\right)\cos\!\left(\frac{m\pi\eta}{l}\right)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}}, \\
\lambda_{nm} &= \sqrt{\frac{a^2\mu_n^2}{R^2}+\frac{a^2\pi^2m^2}{l^2}+b-\frac{k^2}{4}},\quad A_m=\begin{cases}1 & \text{for } m=0,\\ 2 & \text{for } m>0,\end{cases}
\end{aligned}
$$

where the μ_n are zeros of the Bessel function, $J_0(\mu) = 0$.

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_3(r,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^R f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ \int_0^l\int_0^R \bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&+ a^2\int_0^t\int_0^l g_1(\eta,\tau)\mathcal{G}(r,z,R,\eta,t-\tau)\,d\eta\,d\tau \\
&+ a^2\int_0^t\int_0^R g_2(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0} d\xi\,d\tau \\
&- a^2\int_0^t\int_0^R g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l} d\xi\,d\tau \\
&+ \int_0^t\int_0^l\int_0^R \Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{4\xi e^{-kt/2}}{R^2 l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{J_0^2(\mu_n)}J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right) \\
&\qquad\times\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}}, \\
\lambda_{nm} &= \sqrt{\frac{a^2\mu_n^2}{R^2}+\frac{a^2\pi^2m^2}{l^2}+b-\frac{k^2}{4}},
\end{aligned}
$$

where the μ_n are zeros of the first-order Bessel function, $J_1(\mu)=0$ $(\mu_0=0)$. The numerical values of the first ten roots μ_n are specified in Section 3.2.1 (see the second boundary value problem for $0\le r\le R$).

► **Domain: $R_1 \le r \le R_2$, $0 \le z \le l$. First boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_4(r,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_{R_1}^{R_2} f_0(\xi,\eta)\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&\quad+\int_0^l\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta)+kf_0(\xi,\eta)\bigr]\mathcal{G}(r,z,\xi,\eta,t)\,d\xi\,d\eta \\
&\quad+a^2\int_0^t\int_0^l g_1(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R_1}d\eta\,d\tau \\
&\quad-a^2\int_0^t\int_0^l g_2(\eta,\tau)\left[\frac{\partial}{\partial\xi}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\xi=R_2}d\eta\,d\tau \\
&\quad+a^2\int_0^t\int_{R_1}^{R_2} g_3(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=0}d\xi\,d\tau \\
&\quad-a^2\int_0^t\int_{R_1}^{R_2} g_4(\xi,\tau)\left[\frac{\partial}{\partial\eta}\mathcal{G}(r,z,\xi,\eta,t-\tau)\right]_{\eta=l}d\xi\,d\tau \\
&\quad+\int_0^t\int_0^l\int_{R_1}^{R_2}\Phi(\xi,\eta,\tau)\mathcal{G}(r,z,\xi,\eta,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathcal{G}(r,z,\xi,\eta,t) &= \frac{\pi^2\xi}{R_1^2 l}e^{-kt/2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_n^2 J_0^2(s\mu_n)\Psi_n(r)\Psi_n(\xi)}{J_0^2(\mu_n)-J_0^2(s\mu_n)} \\
&\qquad\times\sin\left(\frac{m\pi z}{l}\right)\sin\left(\frac{m\pi\eta}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_{nm}}\,\bigr)}{\sqrt{\lambda_{nm}}}, \\
\Psi_n(r) &= Y_0(\mu_n)J_0\left(\frac{\mu_n r}{R_1}\right)-J_0(\mu_n)Y_0\left(\frac{\mu_n r}{R_1}\right),\qquad s=\frac{R_2}{R_1}, \\
\lambda_{nm} &= \frac{a^2\mu_n^2}{R_1^2}+\frac{a^2\pi^2 m^2}{l^2}+b-\frac{k^2}{4},
\end{aligned}
$$

where $J_0(\mu)$ and $Y_0(\mu)$ are Bessel functions, and the μ_n are positive roots of the transcendental equation

$$J_0(\mu)Y_0(s\mu)-J_0(s\mu)Y_0(\mu)=0.$$

7.5 Other Equations with Two Space Variables

1. $\displaystyle \frac{\partial^2 w}{\partial t^2}+k\frac{\partial w}{\partial t}=a^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+b_1\frac{\partial w}{\partial x}+b_2\frac{\partial w}{\partial y}+cw.$

The transformation

$$w(x,y,t)=u(x,y,\tau)\exp\left(-\frac{1}{2}kt-\frac{b_1x+b_2y}{2a^2}\right),\qquad \tau=at$$

leads to the equation from Section 7.1.3:

$$\frac{\partial^2 u}{\partial\tau^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\beta u,\qquad \beta=\frac{c}{a^2}+\frac{k^2}{4a^2}-\frac{1}{4a^4}(b_1^2+b_2^2).$$

2. $\displaystyle t^m \frac{\partial^2 w}{\partial t^2} + \frac{m}{2} t^{m-1} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$

Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x, y) \quad \text{at} \quad t = 0,$$
$$t^{m/2} \partial_t w = g(x, y) \quad \text{at} \quad t = 0.$$

Solution for $1 \le m < 2$:

$$w(x, y, t) = \frac{1}{2\pi} t^{m/2} \frac{\partial}{\partial t} \iint\limits_{C_t} \frac{f(\xi, \eta)\, d\xi\, d\eta}{\sqrt{k_m^2 t^{2-m} - \rho^2}} + \frac{1}{2\pi} \iint\limits_{C_t} \frac{g(\xi, \eta)\, d\xi\, d\eta}{\sqrt{k_m^2 t^{2-m} - \rho^2}},$$
$$k_m = \frac{2}{2-m}, \qquad \rho = \sqrt{(x-\xi)^2 + (y-\eta)^2},$$

where $C_t = \{\rho^2 \le k_m^2 t^{2-m}\}$ is the circle with center at (x, y) and radius $k_m t^{1/k_m}$.

⊙ *Literature*: M. M. Smirnov (1975).

Chapter 8

Second-Order Hyperbolic Equations with Three or More Space Variables

8.1 Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w$

8.1.1 Problems in Cartesian Coordinates

The wave equation with three space variables in the rectangular Cartesian coordinate system has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

This equation is of fundamental importance in sound propagation theory, the propagation of electromagnetic fields theory, and a number of other areas of physics and mechanics.

► **Particular solutions and their properties.**

1°. Particular solutions:

$$w(x,y,z,t) = A\exp\Big(k_1x+k_2y+k_3z \pm at\sqrt{k_1^2+k_2^2+k_3^2}\,\Big),$$
$$w(x,y,z,t) = A\sin(k_1x+C_1)\sin(k_2y+C_2)\sin(k_3z+C_3)\sin\Big(at\sqrt{k_1^2+k_2^2+k_3^2}\,\Big),$$
$$w(x,y,z,t) = A\sin(k_1x+C_1)\sin(k_2y+C_2)\sin(k_3z+C_3)\cos\Big(at\sqrt{k_1^2+k_2^2+k_3^2}\,\Big),$$
$$w(x,y,z,t) = A\sinh(k_1x+C_1)\sinh(k_2y+C_2)\sinh(k_3z+C_3)\sinh\Big(at\sqrt{k_1^2+k_2^2+k_3^2}\,\Big),$$
$$w(x,y,z,t) = A\sinh(k_1x+C_1)\sinh(k_2y+C_2)\sinh(k_3z+C_3)\cosh\Big(at\sqrt{k_1^2+k_2^2+k_3^2}\,\Big),$$

where A, C_1, C_2, C_3, k_1, k_2, and k_3 are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}(x,y,z,t) = \frac{1}{2\pi a}\delta(a^2t^2 - r^2), \qquad r = \sqrt{x^2+y^2+z^2},$$

where $\delta(\xi)$ is the Dirac delta function.

⊙ *Literature*: V. S. Vladimirov (1988).

3°. Infinite series solutions containing arbitrary functions of space variables:

$$w(x,y,z,t) = f(x,y,z) + \sum_{n=1}^{\infty} \frac{(at)^{2n}}{(2n)!} \Delta^n f(x,y,z), \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$
$$w(x,y,z,t) = tg(x,y,z) + t\sum_{n=1}^{\infty} \frac{(at)^{2n}}{(2n+1)!} \Delta^n g(x,y,z),$$

where $f(x,y,z)$ and $g(x,y,z)$ are any infinitely differentiable functions. The first solution satisfies the initial conditions $w(x,y,z,0) = f(x,y,z)$, $\partial_t w(x,y,z,0) = 0$, while the second solution satisfies the initial conditions $w(x,y,z,0) = 0$, $\partial_t w(x,y,z,0) = g(x,y,z)$. The sums are finite if $f(x,y,z)$ and $g(x,y,z)$ are polynomials in x, y, z.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

4°. Suppose $w = w(x,y,z,t)$ is a solution of the wave equation. Then the functions

$$w_1 = Aw(\pm\lambda x + C_1,\ \pm\lambda y + C_2,\ \pm\lambda z + C_3,\ \pm\lambda t + C_4),$$
$$w_2 = Aw\left(\frac{x - vt}{\sqrt{1-(v/a)^2}},\ y,\ z,\ \frac{t - va^{-2}x}{\sqrt{1-(v/a)^2}}\right),$$
$$w_3 = \frac{A}{r^2 - a^2t^2} w\left(\frac{x}{r^2 - a^2t^2}, \frac{y}{r^2 - a^2t^2}, \frac{z}{r^2 - a^2t^2}, \frac{t}{r^2 - a^2t^2}\right),$$

where A, C_n, v, and λ are arbitrary constants, are also solutions of the equation. The signs at λ in the expression of w_1 can be taken independently of one another. The function w_2 is a consequence of the invariance of the wave equation under the *Lorentz transformation*.

⊙ *Literature*: G. N. Polozhii (1964), W. Miller, Jr. (1977), A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x,y,z) \quad \text{at} \quad t = 0.$$

Solution (*Kirchhoff's formula*):

$$w(x,y,z,t) = \frac{1}{4\pi a}\frac{\partial}{\partial t} \iint_{S_{at}} \frac{f(\xi,\eta,\zeta)}{r}\,dS + \frac{1}{4\pi a} \iint_{S_{at}} \frac{g(\xi,\eta,\zeta)}{r}\,dS,$$
$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2},$$

where the integration is performed over the surface of the sphere of radius at with center at (x,y,z).

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970), A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3$. First boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_6(x,y,t) &&\text{at}\quad z=l_3 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t) = \frac{\partial}{\partial t}\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} & f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
+\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} & f_1(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} & g_1(\eta,\zeta,\tau)\Big[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\Big]_{\xi=0}\,d\eta\,d\zeta\,d\tau\\
-a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} & g_2(\eta,\zeta,\tau)\Big[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\Big]_{\xi=l_1}\,d\eta\,d\zeta\,d\tau\\
+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} & g_3(\xi,\zeta,\tau)\Big[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\Big]_{\eta=0}\,d\xi\,d\zeta\,d\tau\\
-a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} & g_4(\xi,\zeta,\tau)\Big[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\Big]_{\eta=l_2}\,d\xi\,d\zeta\,d\tau\\
+a^2\int_0^{t}\!\!\int_0^{l_2}\!\!\int_0^{l_1} & g_5(\xi,\eta,\tau)\Big[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\Big]_{\zeta=0}\,d\xi\,d\eta\,d\tau\\
-a^2\int_0^{t}\!\!\int_0^{l_2}\!\!\int_0^{l_1} & g_6(\xi,\eta,\tau)\Big[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\Big]_{\zeta=l_3}\,d\xi\,d\eta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{al_1l_2l_3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\lambda_{nmk}}&\sin(\alpha_n x)\sin(\beta_m y)\sin(\gamma_k z)\\
&\times\sin(\alpha_n\xi)\sin(\beta_m\eta)\sin(\gamma_k\zeta)\sin(a\lambda_{nmk}t),
\end{aligned}$$

where

$$\alpha_n = \frac{n\pi}{l_1},\quad \beta_m = \frac{m\pi}{l_2},\quad \gamma_k = \frac{k\pi}{l_3},$$
$$\lambda_{nmk} = \sqrt{\alpha_n^2+\beta_m^2+\gamma_k^2}.$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Second boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) = {} & \frac{\partial}{\partial t}\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_1(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&-a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_1(\eta,\zeta,\tau)G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_2(\eta,\zeta,\tau)G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&-a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_3(\xi,\zeta,\tau)G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&+a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_4(\xi,\zeta,\tau)G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&-a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau)G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{t}{l_1l_2l_3} + \frac{1}{al_1l_2l_3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_nA_mA_k}{\lambda_{nmk}}\cos(\alpha_n x)\cos(\beta_m y)\cos(\gamma_k z)\\
\times\cos(\alpha_n\xi)\cos(\beta_m\eta)\cos(\gamma_k\zeta)\sin(a\lambda_{nmk}t),
\end{aligned}
$$

$$
\alpha_n=\frac{n\pi}{l_1},\quad \beta_m=\frac{m\pi}{l_2},\quad \gamma_k=\frac{k\pi}{l_3},\quad \lambda_{nmk}=\sqrt{\alpha_n^2+\beta_m^2+\gamma_k^2},\quad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0.\end{cases}
$$

The summation here is performed over the indices satisfying the condition $n+m+k>0$; the term corresponding to $n=m=k=0$ is singled out.

▶ **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3$. Third boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w - s_1 w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w + s_2 w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w - s_3 w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w + s_4 w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
\partial_z w - s_5 w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w + s_6 w &= g_6(x,y,t) &&\text{at}\quad z=l_3 &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t) = \frac{8}{a}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{E_{nmk}\sqrt{\alpha_n^2+\beta_m^2+\gamma_k^2}}\sin(\alpha_n x+\varepsilon_n)\sin(\beta_m y+\sigma_m)$$
$$\times\sin(\gamma_k z+\nu_k)\sin(\alpha_n\xi+\varepsilon_n)\sin(\beta_m\eta+\sigma_m)\sin(\gamma_k\zeta+\nu_k)\sin\Big(at\sqrt{\alpha_n^2+\beta_m^2+\gamma_k^2}\Big)$$

with

$$\varepsilon_n = \arctan\frac{\alpha_n}{l_1},\quad \sigma_m = \arctan\frac{\beta_m}{l_2},\quad \nu_k = \arctan\frac{\gamma_k}{l_3},$$
$$E_{nmk} = \left[l_1+\frac{(s_1s_2+\alpha_n^2)(s_1+s_2)}{(s_1^2+\alpha_n^2)(s_2^2+\alpha_n^2)}\right]\left[l_2+\frac{(s_3s_4+\beta_m^2)(s_3+s_4)}{(s_3^2+\beta_m^2)(s_4^2+\beta_m^2)}\right]\left[l_3+\frac{(s_5s_6+\gamma_k^2)(s_5+s_6)}{(s_5^2+\gamma_k^2)(s_6^2+\gamma_k^2)}\right].$$

Here, the α_n, β_m, and γ_k are positive roots of the transcendental equations

$$\alpha^2 - s_1s_2 = (s_1+s_2)\alpha\cot(l_1\alpha),\qquad \beta^2 - s_3s_4 = (s_3+s_4)\beta\cot(l_2\beta),$$
$$\gamma^2 - s_5s_6 = (s_5+s_6)\gamma\cot(l_3\gamma).$$

▶ **Domain: $0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3$. Mixed boundary value problems.**

1°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) &&\text{at}\quad z=l_3 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f_1(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ a^2\int_0^{t}\int_0^{l_3}\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau \\
&- a^2\int_0^{t}\int_0^{l_3}\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\,d\zeta\,d\tau \\
&- a^2\int_0^{t}\int_0^{l_3}\int_0^{l_1} g_3(\xi,\zeta,\tau)G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
&+ a^2\int_0^{t}\int_0^{l_3}\int_0^{l_1} g_4(\xi,\zeta,\tau)G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
&- a^2\int_0^{t}\int_0^{l_2}\int_0^{l_1} g_5(\xi,\eta,\tau)G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau \\
&+ a^2\int_0^{t}\int_0^{l_2}\int_0^{l_1} g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{2}{al_1l_2l_3}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_mA_k}{\lambda_{nmk}}\sin(\alpha_n x)\cos(\beta_m y)\cos(\gamma_k z) \\
\times \sin(\alpha_n\xi)\cos(\beta_m\eta)\cos(\gamma_k\zeta)\sin(a\lambda_{nmk}t),
\end{aligned}
$$

where

$$
\alpha_n = \frac{n\pi}{l_1},\qquad \beta_m = \frac{m\pi}{l_2},\qquad \gamma_k = \frac{k\pi}{l_3},
$$

$$
\lambda_{nmk} = \sqrt{\alpha_n^2+\beta_m^2+\gamma_k^2},\quad A_m = \begin{cases} 1 & \text{for } m=0,\\ 2 & \text{for } m>0.\end{cases}
$$

2°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y,z) && \text{at} && t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} && t=0 && \text{(initial condition)},\\
w &= g_1(y,z,t) && \text{at} && x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) && \text{at} && x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,z,t) && \text{at} && y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at} && y=l_2 && \text{(boundary condition)},\\
w &= g_5(x,y,t) && \text{at} && z=0 && \text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) && \text{at} && z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t} \int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f_0(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\
&\quad + \int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f_1(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\
&\quad + a^2 \int_0^{t}\int_0^{l_3}\int_0^{l_2} g_1(\eta,\zeta,\tau) \left[\frac{\partial}{\partial \xi} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\, d\zeta\, d\tau \\
&\quad + a^2 \int_0^{t}\int_0^{l_3}\int_0^{l_2} g_2(\eta,\zeta,\tau) G(x,y,z,l_1,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau \\
&\quad + a^2 \int_0^{t}\int_0^{l_3}\int_0^{l_1} g_3(\xi,\zeta,\tau) \left[\frac{\partial}{\partial \eta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\, d\zeta\, d\tau \\
&\quad + a^2 \int_0^{t}\int_0^{l_3}\int_0^{l_1} g_4(\xi,\zeta,\tau) G(x,y,z,\xi,l_2,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
&\quad + a^2 \int_0^{t}\int_0^{l_2}\int_0^{l_1} g_5(\xi,\eta,\tau) \left[\frac{\partial}{\partial \zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\, d\eta\, d\tau \\
&\quad + a^2 \int_0^{t}\int_0^{l_2}\int_0^{l_1} g_6(\xi,\eta,\tau) G(x,y,z,\xi,\eta,l_3,t-\tau)\, d\xi\, d\eta\, d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{a l_1 l_2 l_3} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\lambda_{nmk}} \sin(\alpha_n x)\sin(\beta_m y)\sin(\gamma_k z) \\
\times \sin(\alpha_n \xi)\sin(\beta_m \eta)\sin(\gamma_k \zeta)\sin(a\lambda_{nmk} t),
\end{aligned}
$$

where

$$
\alpha_n = \frac{\pi(2n+1)}{2l_1}, \quad \beta_m = \frac{\pi(2m+1)}{2l_2}, \quad \gamma_k = \frac{\pi(2k+1)}{2l_3}, \quad \lambda_{nmk} = \sqrt{\alpha_n^2 + \beta_m^2 + \gamma_k^2}.
$$

8.1.2 Problems in Cylindrical Coordinates

The three-dimensional wave equation in the cylindrical coordinate system is written as

$$
\frac{\partial^2 w}{\partial t^2} = a^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}\right], \qquad r = \sqrt{x^2+y^2}.
$$

One-dimensional problems with axial symmetry that have solutions $w = w(r,t)$ are considered in Section 6.2.1. Two-dimensional problems whose solutions have the form $w = w(r,\varphi,t)$ or $w = w(r,z,t)$ are discussed in Sections 7.1.2 and 7.1.3.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &+ \int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &- a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\ &+ a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau\\ &- a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{gathered} G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{2}{\pi a R^2 l}\sum_{n=0}^\infty\sum_{m=1}^\infty\sum_{k=1}^\infty \frac{A_n}{[J_n'(\mu_{nm}R)]^2\sqrt{\lambda_{nmk}}}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\\ \times\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\left(at\sqrt{\lambda_{nmk}}\right),\\ \lambda_{nmk} = \mu_{nm}^2 + \frac{k^2\pi^2}{l^2},\qquad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases} \end{gathered}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\ \partial_z w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&- a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau \\
&+ a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{t}{\pi R^2 l}+\frac{2}{\pi^2 aR^2}\sum_{k=1}^{\infty}\frac{1}{k}\cos\left(\frac{k\pi x}{l}\right)\cos\left(\frac{k\pi\xi}{l}\right)\sin\left(\frac{ak\pi t}{l}\right)
$$
$$
+\frac{1}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_nA_k\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\cos\left(\frac{k\pi x}{l}\right)\cos\left(\frac{k\pi\xi}{l}\right)\frac{\sin(\lambda_{nmk}t)}{\lambda_{nmk}},
$$
$$
\lambda_{nmk}=a\sqrt{\mu_{nm}^2+\frac{k^2\pi^2}{l^2}},\qquad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}
$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w + k_1 w &= g(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w - k_2 w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w + k_3 w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2}{(\mu_{nm}^2R^2+k_1^2R^2-n^2)[J_n(\mu_{nm}R)]^2\|h_s\|^2\lambda_{nms}}
$$
$$
\times J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]h_s(z)h_s(\zeta)\sin(\lambda_{nms}t),
$$
$$
\lambda_{nms} = a\sqrt{\mu_{nm}^2+\beta_s^2},\qquad h_s(z)=\cos(\beta_s z)+\frac{k_2}{\beta_s}\sin(\beta_s z),
$$
$$
\|h_s\|^2 = \frac{k_3}{2\beta_s^2}\frac{\beta_s^2+k_2^2}{\beta_s^2+k_3^2}+\frac{k_2}{2\beta_s^2}+\frac{l}{2}\left(1+\frac{k_2^2}{\beta_s^2}\right).
$$

Here, $A_0 = 1$ and $A_n = 2$ for $n = 1, 2, \ldots$; the $J_n(\xi)$ are Bessel functions; and the μ_{nm} and β_s are positive roots of the transcendental equations

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0, \qquad \frac{\tan(\beta l)}{\beta} = \frac{k_2 + k_3}{\beta^2 - k_2 k_3}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta) G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &+ \int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta) G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &- a^2 R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\ &- a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau) G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\ &+ a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau) G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Here,

$$\begin{gathered} G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi a R^2 l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_n A_k}{[J_n'(\mu_{nm}R)]^2\sqrt{\lambda_{nmk}}} J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)\\ \times\cos[n(\varphi-\eta)]\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\sin\bigl(at\sqrt{\lambda_{nmk}}\,\bigr),\\ \lambda_{nmk} = \mu_{nm}^2 + \frac{k^2\pi^2}{l^2}, \quad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n>0, \end{cases} \end{gathered}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

2°. A circular cylinder of finite length is considered. The following conditions are pre-

scribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)},\\
w &= g_2(r,\varphi,t) \quad \text{at} \quad z=0 \quad \text{(boundary condition)},\\
w &= g_3(r,\varphi,t) \quad \text{at} \quad z=l \quad \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&\quad+\int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&\quad+a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&\quad+a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}d\xi\,d\eta\,d\tau\\
&\quad-a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{2}{\pi^2 aR^2}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\left(\frac{k\pi at}{l}\right)\\
&\quad+\frac{2}{\pi al}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2\sqrt{\lambda_{nmk}}}\\
&\qquad\times\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\bigl(at\sqrt{\lambda_{nmk}}\,\bigr),\\
\lambda_{nmk} &= \mu_{nm}^2+\frac{k^2\pi^2}{l^2},\quad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}
\end{aligned}
$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R)=0$.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. First boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1} d\eta\,d\zeta\,d\tau\\
&-a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2} d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau\\
&-a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{gathered}
G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{\pi}{2l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)}Z_{nm}(r)Z_{nm}(\xi)\\
\times\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(at\sqrt{\lambda_{nmk}}\right)}{a\sqrt{\lambda_{nmk}}},\\
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nmk}=\mu_{nm}^2+\frac{k^2\pi^2}{l^2},\\
Z_{nm}(r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),
\end{gathered}
$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2)-Y_n(\mu R_1)J_n(\mu R_2)=0.$$

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&\quad+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&\quad-a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&\quad+a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&\quad-a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&\quad+a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{t}{\pi(R_2^2-R_1^2)l}+\frac{2}{\pi^2a(R_2^2-R_1^2)}\sum_{k=1}^{\infty}\frac{1}{k}\cos\Big(\frac{k\pi z}{l}\Big)\cos\Big(\frac{k\pi\zeta}{l}\Big)\sin\Big(\frac{k\pi at}{l}\Big)\\
&\quad+\frac{1}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_nA_k\mu_{nm}^2Z_{nm}(r)Z_{nm}(\xi)}{(\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1)}\\
&\quad\times\cos[n(\varphi-\eta)]\cos\Big(\frac{k\pi z}{l}\Big)\cos\Big(\frac{k\pi\zeta}{l}\Big)\frac{\sin\big(at\sqrt{\lambda_{nmk}}\,\big)}{a\sqrt{\lambda_{nmk}}},
\end{aligned}
$$

where

$$
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nmk}=\mu_{nm}^2+\frac{k^2\pi^2}{l^2},
$$

$$
Z_{nm}(r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r);
$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$
J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.
$$

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. Third boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at } t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at } t=0 && \text{(initial condition)},\\
\partial_r w - k_1 w &= g_1(\varphi,z,t) && \text{at } r=R_1 && \text{(boundary condition)},\\
\partial_r w + k_2 w &= g_2(\varphi,z,t) && \text{at } r=R_2 && \text{(boundary condition)},\\
\partial_z w - k_3 w &= g_3(r,\varphi,t) && \text{at } z=0 && \text{(boundary condition)},\\
\partial_z w + k_4 w &= g_4(r,\varphi,t) && \text{at } z=l && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{1}{\pi a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2}{\|h_s\|^2\sqrt{\mu_{nm}^2+\lambda_s^2}}\\
&\quad\times\frac{Z_{nm}(r)Z_{nm}(\xi)\cos[n(\varphi-\eta)]h_s(z)h_s(\zeta)\sin\bigl(at\sqrt{\mu_{nm}^2+\lambda_s^2}\,\bigr)}{(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1)}.
\end{aligned}
$$

Here,

$$
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad
\begin{aligned}Z_{nm}(r) &= \bigl[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r)\\ &\quad-\bigl[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r),\end{aligned}
$$

$$
h_s(z)=\cos(\lambda_s z)+\frac{k_3}{\lambda_s}\sin(\lambda_s z),\qquad \|h_s\|^2=\frac{k_4}{2\lambda_s^2}\,\frac{\lambda_s^2+k_3^2}{\lambda_s^2+k_4^2}+\frac{k_3}{2\lambda_s^2}+\frac{l}{2}\left(1+\frac{k_3^2}{\lambda_s^2}\right),
$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions; the μ_{nm} are positive roots of the transcendental equation

$$
\begin{aligned}
&\bigl[\mu J_n'(\mu R_1)-k_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2)+k_2Y_n(\mu R_2)\bigr]\\
&\qquad=\bigl[\mu Y_n'(\mu R_1)-k_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2)+k_2J_n(\mu R_2)\bigr];
\end{aligned}
$$

and the λ_s are positive roots of the transcendental equation $\dfrac{\tan(\lambda l)}{\lambda}=\dfrac{k_3+k_4}{\lambda^2-k_3k_4}$.

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. Mixed boundary value problems.**

1°. A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at } t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at } t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at } r=R_1 && \text{(boundary condition)},\\
w &= g_2(\varphi,z,t) && \text{at } r=R_2 && \text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) && \text{at } z=0 && \text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) && \text{at } z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&+ a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1} d\eta\,d\zeta\,d\tau \\
&- a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2} d\eta\,d\zeta\,d\tau \\
&- a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau \\
&+ a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{gathered}
G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{\pi}{4l}\sum_{n=0}^\infty\sum_{m=1}^\infty\sum_{k=0}^\infty \frac{A_nA_k\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)}Z_{nm}(r)Z_{nm}(\xi) \\
\times\cos[n(\varphi-\eta)]\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\bigl(at\sqrt{\lambda_{nmk}}\,\bigr)}{a\sqrt{\lambda_{nmk}}}, \\
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nmk}=\mu_{nm}^2+\frac{k^2\pi^2}{l^2}, \\
Z_{nm}(r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),
\end{gathered}
$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

2°. A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&- a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&+ a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&+ a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau \\
&- a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{2}{\pi^2a(R_2^2-R_1^2)}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\left(\frac{k\pi at}{l}\right) \\
&+\frac{2}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n\mu_{nm}^2Z_{nm}(r)Z_{nm}(\xi)}{(\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1)} \\
&\qquad\times\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(at\sqrt{\lambda_{nmk}}\right)}{a\sqrt{\lambda_{nmk}}},
\end{aligned}
$$

where

$$
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nmk}=\mu_{nm}^2+\frac{k^2\pi^2}{l^2},
$$

$$
Z_{nm}(r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r);
$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$
J_n'(\mu R_1)Y_n'(\mu R_2)-Y_n'(\mu R_1)J_n'(\mu R_2)=0.
$$

► **Domain: $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z\le l$. First boundary value problem.**

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
w &= g_3(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\
w &= g_4(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_5(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{\varphi_0}\int_0^R f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^l\int_0^{\varphi_0}\int_0^R f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta \\
&- a^2R\int_0^t\int_0^l\int_0^{\varphi_0} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau \\
&+ a^2\int_0^t\int_0^l\int_0^R g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau \\
&- a^2\int_0^t\int_0^l\int_0^R g_3(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\,d\zeta\,d\tau \\
&+ a^2\int_0^t\int_0^{\varphi_0}\int_0^R g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} \xi\,d\xi\,d\eta\,d\tau \\
&- a^2\int_0^t\int_0^{\varphi_0}\int_0^R g_5(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} \xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{8}{R^2 l\varphi_0}\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{k=1}^\infty \frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \\
&\quad\times\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\bigl(at\sqrt{\mu_{nm}^2+k^2\pi^2/l^2}\,\bigr)}{a\sqrt{\mu_{nm}^2+k^2\pi^2/l^2}},
\end{aligned}
$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

► **Domain: $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z\le l$. Mixed boundary value problem.**

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
w &= g_3(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_5(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{\varphi_0}\int_0^R f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{\varphi_0}\int_0^R f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&-a^2R\int_0^t\int_0^l\int_0^{\varphi_0} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^t\int_0^l\int_0^R g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0}d\xi\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^l\int_0^R g_3(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^{\varphi_0}\int_0^R g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\int_0^{\varphi_0}\int_0^R g_5(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{4}{R^2 l\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_k J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\\
&\quad\times\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(at\sqrt{\mu_{nm}^2+k^2\pi^2/l^2}\right)}{a\sqrt{\mu_{nm}^2+k^2\pi^2/l^2}},
\end{aligned}
$$

where $A_0=1$ and $A_k=2$ for $k\ge 1$; the $J_{n\pi/\varphi_0}(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

8.1.3 Problems in Spherical Coordinates

The three-dimensional wave equation in the spherical coordinate system is represented as

$$
\frac{\partial^2 w}{\partial t^2}=a^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}\right],
$$
$$
r=\sqrt{x^2+y^2+z^2}.
$$

One-dimensional problems with central symmetry that have solutions $w=w(r,t)$ are considered in Section 6.2.3.

► **Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. First boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
\partial_t w &= f_1(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\
w &= g(\theta,\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$w(r,\theta,\varphi,t)=\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta$$
$$+\int_0^{2\pi}\int_0^{\pi}\int_0^{R} f_1(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta$$
$$-a^2R^2\int_0^{t}\int_0^{2\pi}\int_0^{\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}\sin\eta\,d\eta\,d\zeta\,d\tau,$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{2\pi aR^2\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\sin(\lambda_{nm}at),$$

$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{(2n+1)(n-k)!}{(n+k)!\left[J'_{n+1/2}(\lambda_{nm}R)\right]^2\lambda_{nm}}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^k(\mu)=(1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_n(\mu),\qquad P_n(\mu)=\frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation $J_{n+1/2}(\lambda R)=0$.

► **Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. Second boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w &= g(\theta,\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$w(r,\theta,\varphi,t)=\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta$$
$$+\int_0^{2\pi}\int_0^{\pi}\int_0^{R} f_1(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta$$
$$+a^2R^2\int_0^{t}\int_0^{2\pi}\int_0^{\pi} g(\eta,\zeta,\tau)G(r,\theta,\varphi,R,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau,$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{3t}{4\pi R^3}+\frac{1}{2\pi a\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\sin(\lambda_{nm}at),$$

$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{\lambda_{nm}(2n+1)(n-k)!}{(n+k)!\left[R^2\lambda_{nm}^2-n(n+1)\right]\left[J_{n+1/2}(\lambda_{nm}R)\right]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the paragraph above), and the λ_{nm} are positive roots of the transcendental equation

$$2\lambda R J'_{n+1/2}(\lambda R) - J_{n+1/2}(\lambda R) = 0.$$

⊙ *Literature*: M. M. Smirnov (1975).

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + kw &= g(\theta,\varphi,t) && \text{at} \quad r=R && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = \frac{1}{2\pi a\sqrt{r\xi}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=0}^{n} A_s B_{nms} J_{n+1/2}(\lambda_{nm} r) J_{n+1/2}(\lambda_{nm}\xi) \times P_n^s(\cos\theta) P_n^s(\cos\eta) \cos[s(\varphi-\zeta)] \sin(\lambda_{nm} a t),$$

$$A_s = \begin{cases} 1 & \text{for } s=0, \\ 2 & \text{for } s\neq 0, \end{cases} \qquad B_{nms} = \frac{\lambda_{nm}(2n+1)(n-s)!}{(n+s)!\big[R^2\lambda_{nm}^2 + (kR+n)(kR-n-1)\big]\big[J_{n+1/2}(\lambda_{nm}R)\big]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda R J'_{n+1/2}(\lambda R) + \big(kR - \tfrac{1}{2}\big) J_{n+1/2}(\lambda R) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= g_1(\theta,\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ w &= g_2(\theta,\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta) G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta \\ &+ \int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta) G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta \\ &+ a^2R_1^2 \int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1} \sin\eta\, d\eta\, d\zeta\, d\tau \\ &- a^2R_2^2 \int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2} \sin\eta\, d\eta\, d\zeta\, d\tau, \end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = \frac{\pi}{8a\sqrt{r\xi}} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n} A_k B_{nmk} Z_{n+1/2}(\lambda_{nm} r) Z_{n+1/2}(\lambda_{nm}\xi) \times P_n^k(\cos\theta) P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\sin(\lambda_{nm} a t).$$

Here,

$$Z_{n+1/2}(\lambda_{nm} r) = J_{n+1/2}(\lambda_{nm}R_1) Y_{n+1/2}(\lambda_{nm} r) - Y_{n+1/2}(\lambda_{nm}R_1) J_{n+1/2}(\lambda_{nm} r),$$

$$A_k = \begin{cases} 1 & \text{for } k=0, \\ 2 & \text{for } k\neq 0, \end{cases} \qquad B_{nmk} = \frac{\lambda_{nm}(2n+1)(n-k)!\, J^2_{n+1/2}(\lambda_{nm}R_2)}{(n+k)!\big[J^2_{n+1/2}(\lambda_{nm}R_1) - J^2_{n+1/2}(\lambda_{nm}R_2)\big]},$$

where the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^k(\mu) = (1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_n(\mu), \qquad P_n(\mu) = \frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation $Z_{n+1/2}(\lambda R_2) = 0$.

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\theta,\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\ \partial_r w &= g_2(\theta,\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta) G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta \\ &+ \int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta) G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta \\ &- a^2 R_1^2 \int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_1(\eta,\zeta,\tau) G(r,\theta,\varphi,R_1,\eta,\zeta,t-\tau)\, \sin\eta\, d\eta\, d\zeta\, d\tau \\ &+ a^2 R_2^2 \int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_2(\eta,\zeta,\tau) G(r,\theta,\varphi,R_2,\eta,\zeta,t-\tau)\, \sin\eta\, d\eta\, d\zeta\, d\tau, \end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{3t}{4\pi(R_2^3-R_1^3)}+\frac{1}{4\pi a\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}\frac{A_k}{B_{nmk}}Z_{n+1/2}(\lambda_{nm}r)Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\sin(\lambda_{nm}at),$$

$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{\lambda_{nm}(n+k)!}{(2n+1)(n-k)!}\int_{R_1}^{R_2} rZ_{n+1/2}^2(\lambda_{nm}r)\,dr,$$

$$Z_{n+1/2}(\lambda_{nm}r)=\left[\lambda_{nm}J'_{n+1/2}(\lambda_{nm}R_1)-\frac{1}{2R_1}J_{n+1/2}(\lambda_{nm}R_1)\right]Y_{n+1/2}(\lambda_{nm}r)$$
$$-\left[\lambda_{nm}Y'_{n+1/2}(\lambda_{nm}R_1)-\frac{1}{2R_1}Y_{n+1/2}(\lambda_{nm}R_1)\right]J_{n+1/2}(\lambda_{nm}r).$$

Here, the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the paragraph above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2)-\frac{1}{2R_2}Z_{n+1/2}(\lambda R_2)=0.$$

► **Domain: $R_1\le r\le R_2,\ 0\le\theta\le\pi,\ 0\le\varphi\le 2\pi$. Third boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w-k_1w&=g_1(\theta,\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ \partial_r w+k_2w&=g_2(\theta,\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{4\pi a\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n}\frac{A_s}{B_{nms}}Z_{n+1/2}(\lambda_{nm}r)Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\sin(\lambda_{nm}at).$$

Here,

$$A_s=\begin{cases}1 & \text{for } s=0,\\ 2 & \text{for } s\neq 0,\end{cases}\qquad B_{nms}=\frac{\lambda_{nm}(n+s)!}{(2n+1)(n-s)!}\int_{R_1}^{R_2} rZ_{n+1/2}^2(\lambda_{nm}r)\,dr,$$

$$Z_{n+1/2}(\lambda r)=\left[\lambda J'_{n+1/2}(\lambda R_1)-\left(k_1+\frac{1}{2R_1}\right)J_{n+1/2}(\lambda R_1)\right]Y_{n+1/2}(\lambda r)$$
$$-\left[\lambda Y'_{n+1/2}(\lambda R_1)-\left(k_1+\frac{1}{2R_1}\right)Y_{n+1/2}(\lambda R_1)\right]J_{n+1/2}(\lambda r),$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2)+\left(k_2-\frac{1}{2R_2}\right)Z_{n+1/2}(\lambda R_2)=0.$$

8.2 Nonhomogeneous Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2\Delta_3 w + \Phi(x,y,z,t)$

8.2.1 Problems in Cartesian Coordinates

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0,$$
$$\partial_t w = g(x,y,z) \quad \text{at} \quad t=0.$$

Solution:

$$w(x,y,z,t) = \frac{1}{4\pi a}\frac{\partial}{\partial t}\iint\limits_{r=at}\frac{f(\xi,\eta,\zeta)}{r}\,dS + \frac{1}{4\pi a}\iint\limits_{r=at}\frac{g(\xi,\eta,\zeta)}{r}\,dS$$
$$+\frac{1}{4\pi a^2}\iiint\limits_{r\le at}\frac{1}{r}\Phi\Bigl(\xi,\eta,\zeta,t-\frac{r}{a}\Bigr)\,d\xi\,d\eta\,d\zeta,$$
$$r = \sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2},$$

where the integration is performed over the surface of the sphere ($r=at$) and the volume of the sphere ($r\le at$) with center at (x,y,z).

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970).

► **Domain: $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$. Different boundary value problems.**

1°. The solution $w(x,y,z,t)$ of the first boundary value problem for a parallelepiped is given by the formula in Section 8.1.1 (see the first boundary value problem for $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$) with the additional term

$$\int_0^t\int_0^{l_1}\int_0^{l_2}\int_0^{l_3}\Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\zeta\,d\eta\,d\xi\,d\tau,$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(x,y,z,t)$ of the second boundary value problem for a parallelepiped is given by the formula from Section 8.1.1 (see the second boundary value problem for $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$) with the additional term specified in Item 1° above; the Green's function is also taken from Section 8.1.1.

3°. The solution $w(x,y,z,t)$ of the third boundary value problem for a parallelepiped is given by the formula from Section 8.1.1 (see the third boundary value problem for $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$) with the additional term specified in Item 1° above; the Green's function is also taken from Section 8.1.1.

4°. The solutions of mixed boundary value problems for a parallelepiped are given by the formulas from Section 8.1.1 (see the mixed boundary value problems for $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$) to which one should add the term specified in Item 1° above; the Green's function is also taken from Section 8.1.1.

8.2.2 Problems in Cylindrical Coordinates

A three-dimensional nonhomogeneous wave equation in the cylindrical coordinate system is written as

$$\frac{\partial^2 w}{\partial t^2} = a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}\right] + \Phi(r, \varphi, z, t).$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(r, \varphi, z, t)$ of the first boundary value problem for a circular cylinder of finite length is given by the formula from Section 8.1.2 (see the first boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^{2\pi} \int_0^R \Phi(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\xi \, d\xi \, d\eta \, d\zeta \, d\tau, \qquad (1)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r, \varphi, z, t)$ of the second boundary value problem for a circular cylinder of finite length is given by the formula from Section 8.1.2 (see the second boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term (1); the Green's function is also taken from Section 8.1.2.

3°. The solution $w(r, \varphi, z, t)$ of the third boundary value problem for a circular cylinder of finite length is the sum of the solution specified in Section 8.1.2 (see the third boundary value problem for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) and expression (1); the Green's function is also taken from Section 8.1.2.

4°. The solutions $w(r, \varphi, z, t)$ of mixed boundary value problems for a circular cylinder of finite length are given by the formulas from Section 8.1.2 (see the mixed boundary value problems for $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with additional terms of the form (1); the Green's function is also taken from Section 8.1.2.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(r, \varphi, z, t)$ of the first boundary value problem for a hollow cylinder of finite dimensions is given by the formula from Section 8.1.2 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\xi \, d\xi \, d\eta \, d\zeta \, d\tau, \qquad (2)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r, \varphi, z, t)$ of the second boundary value problem for a hollow cylinder of finite dimensions is given by the formula from Section 8.1.2 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with the additional term (2); the Green's function is also taken from Section 8.1.2.

3°. The solution $w(r,\varphi,z,t)$ of the third boundary value problem for a hollow cylinder of finite dimensions is the sum of the solution specified in Section 8.1.2 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) and expression (2); the Green's function is also taken from Section 8.1.2.

4°. The solutions $w(r,\varphi,z,t)$ of mixed boundary value problems for a hollow cylinder of finite dimensions are given by the formulas from Section 8.1.2 (see the mixed boundary value problems for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$) with additional terms of the form (2); the Green's function is also taken from Section 8.1.2.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. Different boundary value problems.**

1°. The solution $w(r,\varphi,z,t)$ of the first boundary value problem for a cylindrical sector of finite thickness is given by the formula from Section 8.1.2 (see the first boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$) with the additional term

$$\int_0^t \int_0^l \int_0^{\varphi_0} \int_0^R \Phi(\xi,\eta,\zeta,\tau) G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi \, d\xi \, d\eta \, d\zeta \, d\tau, \qquad (3)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\varphi,z,t)$ of a mixed boundary value problem for a cylindrical sector of finite thickness is given by the formula from Section 8.1.2 (see the mixed boundary value problem for $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$) with the additional term (3); the Green's function is also taken from Section 8.1.2.

8.2.3 Problems in Spherical Coordinates

A three-dimensional nonhomogeneous wave equation in the spherical coordinate system is represented as

$$\frac{\partial^2 w}{\partial t^2} = a^2 \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial w}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial w}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 w}{\partial \varphi^2} \right] + \Phi(r,\theta,\varphi,t).$$

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Boundary value problem.**

1°. The solution $w(r,\theta,\varphi,t)$ of the first boundary value problem for a sphere is given by the formula from Section 8.1.3 (see the first boundary value problem for $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) with the additional term

$$\int_0^t \int_0^{2\pi} \int_0^{\pi} \int_0^R \Phi(\xi,\eta,\zeta,\tau) G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2 \sin\eta \, d\xi \, d\eta \, d\zeta \, d\tau, \qquad (1)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\theta,\varphi,t)$ of the second boundary value problem for a sphere is given by the formula from Section 8.1.3 (see the second boundary value problem for $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) with the additional term (1); the Green's function is also taken from Section 8.1.3.

3°. The solution $w(r,\theta,\varphi,t)$ of the third boundary value problem for a sphere is the sum of the solution specified in Section 8.1.3 (see the third boundary value problem for $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) and expression (1); the Green's function is also taken from Section 8.1.3.

▶ **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Boundary value problems.**

1°. The solution $w(r,\theta,\varphi,t)$ of the first boundary value problem for a spherical layer is given by the formula from Section 8.1.3 (see the first boundary value problem for $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) with the additional term

$$\int_0^t \int_0^{2\pi} \int_0^{\pi} \int_{R_1}^{R_2} \Phi(\xi,\eta,\zeta,\tau) G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau) \xi^2 \sin\eta \, d\xi \, d\eta \, d\zeta \, d\tau, \qquad (2)$$

which allows for the equation's nonhomogeneity; this term is the solution of the nonhomogeneous equation with homogeneous initial and boundary conditions.

2°. The solution $w(r,\theta,\varphi,t)$ of the second boundary value problem for a spherical layer is given by the formula from Section 8.1.3 (see the second boundary value problem for $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) with the additional term (2); the Green's function is also taken from Section 8.1.3.

3°. The solution $w(r,\theta,\varphi,t)$ of the third boundary value problem for a spherical layer is the sum of the solution specified in Section 8.1.3 (see the third boundary value problem for $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$) and expression (2); the Green's function is also taken from Section 8.1.3.

8.3 Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w - bw + \Phi(x,y,z,t)$

8.3.1 Problems in Cartesian Coordinates

A *three-dimensional nonhomogeneous Klein–Gordon equation* in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - bw + \Phi(x,y,z,t).$$

▶ **Fundamental solutions.**

1°. For $b = -c^2 < 0$,

$$\mathscr{E}(x,y,z,t) = \frac{1}{4\pi a^2} \left[\frac{\delta(t - r/a)}{r} - \frac{c}{a} \frac{I_1\bigl(c\sqrt{t^2 - r^2/a^2}\,\bigr)}{\sqrt{t^2 - r^2/a^2}} \vartheta(t - r/a) \right],$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $\delta(\xi)$ is the Dirac delta function, $\vartheta(\xi)$ is the Heaviside unit step function, and $I_1(z)$ is the modified Bessel function.

2°. For $b = c^2 > 0$,

$$\mathscr{E}(x,y,z,t) = \frac{1}{4\pi a^2}\left[\frac{\delta(t-r/a)}{r} - \frac{c}{a}\frac{J_1\bigl(c\sqrt{t^2-r^2/a^2}\,\bigr)}{\sqrt{t^2-r^2/a^2}}\vartheta(t-r/a)\right],$$

where $J_1(z)$ is the Bessel function.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(x,y,z) \quad \text{at} \quad t = 0.$$

Let $a = 1$ and $\Phi(x,y,z,t) \equiv 0$.

1°. Solution for $b = -c^2 < 0$:

$$w(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{t}\frac{\partial}{\partial t}\int_0^t r^2 I_0\bigl(c\sqrt{t^2-r^2}\,\bigr)T_r\bigl[f(x,y,z)\bigr]\,dr\right] + \frac{1}{t}\frac{\partial}{\partial t}\int_0^t r^2 I_0\bigl(c\sqrt{t^2-r^2}\,\bigr)T_r\bigl[g(x,y,z)\bigr]\,dr.$$

Here, $I_0(z)$ is the modified Bessel function and $T_r\bigl[h(x,y,z)\bigr]$ is the average of $h(x,y,z)$ over the spherical surface with center at (x,y,z) and radius r:

$$T_r\bigl[h(x,y,z)\bigr] = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} h(x+r\sin\theta\cos\varphi,\ y+r\sin\theta\sin\varphi,\ z+r\cos\theta)\sin\theta\,d\theta\,d\varphi.$$

2°. Solution for $b = c^2 > 0$:

$$w(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{t}\frac{\partial}{\partial t}\int_0^t r^2 J_0\bigl(c\sqrt{t^2-r^2}\,\bigr)T_r\bigl[f(x,y,z)\bigr]\,dr\right] + \frac{1}{t}\frac{\partial}{\partial t}\int_0^t r^2 J_0\bigl(c\sqrt{t^2-r^2}\,\bigr)T_r\bigl[g(x,y,z)\bigr]\,dr,$$

where $J_0(z)$ is the Bessel function.

⊙ *Literature*: V. I. Smirnov (1974, Vol. 2).

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. First boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+ \int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} f_1(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+ a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial \xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau\\
&- a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial \xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1} d\eta\,d\zeta\,d\tau\\
&+ a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\left[\frac{\partial}{\partial \eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau\\
&- a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} g_4(\xi,\zeta,\tau)\left[\frac{\partial}{\partial \eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=l_2} d\xi\,d\zeta\,d\tau\\
&+ a^2\int_0^{t}\!\!\int_0^{l_2}\!\!\int_0^{l_1} g_5(\xi,\eta,\tau)\left[\frac{\partial}{\partial \zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau\\
&- a^2\int_0^{t}\!\!\int_0^{l_2}\!\!\int_0^{l_1} g_6(\xi,\eta,\tau)\left[\frac{\partial}{\partial \zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l_3} d\xi\,d\eta\,d\tau\\
&+ \int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{l_1l_2l_3}\Biggl[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\sqrt{\lambda_{nmk}}}\sin(\alpha_n x)\sin(\beta_m y)\sin(\gamma_k z)\\
\times \sin(\alpha_n \xi)\sin(\beta_m \eta)\sin(\gamma_k \zeta)\sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr)\Biggr],
\end{aligned}
$$

where

$$
\alpha_n = \frac{n\pi}{l_1},\quad \beta_m = \frac{m\pi}{l_2},\quad \gamma_k = \frac{k\pi}{l_3},\quad \lambda_{nmk} = a^2(\alpha_n^2+\beta_m^2+\gamma_k^2)+b.
$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Second boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_0(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+ \int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_1(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&- a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_1(\eta,\zeta,\tau) G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&+ a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_2(\eta,\zeta,\tau) G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&- a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_3(\xi,\zeta,\tau) G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&+ a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_4(\xi,\zeta,\tau) G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&- a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau) G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+ a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_6(\xi,\eta,\tau) G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+ \int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau) G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{\sin\bigl(t\sqrt{b}\,\bigr)}{l_1 l_2 l_3\sqrt{b}} + \frac{1}{l_1 l_2 l_3}\Biggl[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_n A_m A_k}{\sqrt{\lambda_{nmk}}}\cos(\alpha_n x)\cos(\beta_m y)\\
\times\cos(\gamma_k z)\cos(\alpha_n \xi)\cos(\beta_m \eta)\cos(\gamma_k \zeta)\sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr)\Biggr],
\end{aligned}$$

$$\alpha_n = \frac{n\pi}{l_1},\quad \beta_m = \frac{m\pi}{l_2},\quad \gamma_k = \frac{k\pi}{l_3},\quad \lambda_{nmk} = a^2(\alpha_n^2+\beta_m^2+\gamma_k^2)+b,\quad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0.\end{cases}$$

The summation is performed over the indices satisfying the condition $n+m+k>0$; the term corresponding to $n=m=k=0$ is singled out.

► **Domain: $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$. Third boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w - s_1 w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w + s_2 w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w - s_3 w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w + s_4 w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
\partial_z w - s_5 w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w + s_6 w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\begin{aligned}
G(x,y,\xi,\eta,t) &= 8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{E_{nmk}\sqrt{\lambda_{nmk}}}\sin(\alpha_n x+\varepsilon_n)\sin(\beta_m y+\sigma_m)\sin(\gamma_k z+\nu_k)\\
&\quad\times\sin(\alpha_n\xi+\varepsilon_n)\sin(\beta_m\eta+\sigma_m)\sin(\gamma_k\zeta+\nu_k)\sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr),
\end{aligned}$$

$$\varepsilon_n=\arctan\frac{\alpha_n}{l_1},\quad \sigma_m=\arctan\frac{\beta_m}{l_2},\quad \nu_k=\arctan\frac{\gamma_k}{l_3},\quad \lambda_{nmk}=a^2(\alpha_n^2+\beta_m^2+\gamma_k^2)+b,$$

$$E_{nmk}=\left[l_1+\frac{(s_1s_2+\alpha_n^2)(s_1+s_2)}{(s_1^2+\alpha_n^2)(s_2^2+\alpha_n^2)}\right]\left[l_2+\frac{(s_3s_4+\beta_m^2)(s_3+s_4)}{(s_3^2+\beta_m^2)(s_4^2+\beta_m^2)}\right]\left[l_3+\frac{(s_5s_6+\gamma_k^2)(s_5+s_6)}{(s_5^2+\gamma_k^2)(s_6^2+\gamma_k^2)}\right].$$

Here, the α_n, β_m, and γ_k are positive roots of the transcendental equations

$$\alpha^2-s_1s_2=(s_1+s_2)\alpha\cot(l_1\alpha),\quad \beta^2-s_3s_4=(s_3+s_4)\beta\cot(l_2\beta),$$
$$\gamma^2-s_5s_6=(s_5+s_6)\gamma\cot(l_3\gamma).$$

► **Domain: $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$. Mixed boundary value problems.**

1°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,y,z,t) = {} & \frac{\partial}{\partial t}\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
& + \int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_1(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
& + a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0}d\eta\,d\zeta\,d\tau \\
& - a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1}d\eta\,d\zeta\,d\tau \\
& - a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_3(\xi,\zeta,\tau)G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
& + a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_4(\xi,\zeta,\tau)G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
& - a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau)G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau \\
& + a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau \\
& + \int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = {} & \frac{2}{l_1 l_2 l_3}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_m A_k}{\sqrt{\lambda_{nmk}}}\sin(\alpha_n x)\cos(\beta_m y)\cos(\gamma_k z) \\
& \times \sin(\alpha_n\xi)\cos(\beta_m\eta)\cos(\gamma_k\zeta)\sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr),
\end{aligned}$$

where

$$A_m = \begin{cases} 1 & \text{for } m=0,\\ 2 & \text{for } m>0,\end{cases}\qquad A_k = \begin{cases} 1 & \text{for } k=0,\\ 2 & \text{for } k>0,\end{cases}$$

$$\alpha_n = \frac{n\pi}{l_1},\quad \beta_m = \frac{m\pi}{l_2},\quad \gamma_k = \frac{k\pi}{l_3},\quad \lambda_{nmk} = a^2(\alpha_n^2 + \beta_m^2 + \gamma_k^2) + b.$$

2°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) &&\text{at}\quad z=l_3 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&\quad+\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} f_1(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0}d\eta\,d\zeta\,d\tau \\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} g_2(\eta,\zeta,\tau)G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0}d\xi\,d\zeta\,d\tau \\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} g_4(\xi,\zeta,\tau)G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau \\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2}\!\!\int_0^{l_1} g_5(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}d\xi\,d\eta\,d\tau \\
&\quad+a^2\int_0^{t}\!\!\int_0^{l_2}\!\!\int_0^{l_1} g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau \\
&\quad+\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{l_1l_2l_3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\sqrt{\lambda_{nmk}}}\sin(\alpha_n x)\sin(\beta_m y)\sin(\gamma_k z)\\
\times\sin(\alpha_n\xi)\sin(\beta_m\eta)\sin(\gamma_k\zeta)\sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr),
\end{aligned}
$$

where

$$
\alpha_n=\frac{\pi(2n+1)}{2l_1},\quad \beta_m=\frac{\pi(2m+1)}{2l_2},\quad \gamma_k=\frac{\pi(2k+1)}{2l_3},
$$
$$
\lambda_{nmk}=a^2(\alpha_n^2+\beta_m^2+\gamma_k^2)+b.
$$

8.3.2 Problems in Cylindrical Coordinates

A *nonhomogeneous Klein–Gordon equation* in the cylindrical coordinate system is written as

$$
\frac{\partial^2 w}{\partial t^2}=a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial^2 w}{\partial z^2}\right]-bw+\Phi(r,\varphi,z,t),\qquad r=\sqrt{x^2+y^2}.
$$

One-dimensional problems with axial symmetry that have solutions $w=w(r,t)$ are treated in Section 6.2.5. Two-dimensional problems whose solutions have the form $w=w(r,\varphi,t)$ or $w=w(r,z,t)$ are considered in Sections 7.3.2 and 7.3.3.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+ \int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&- a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\
&+ a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\,d\eta\,d\tau\\
&- a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\,d\eta\,d\tau\\
&+ \int_0^t\int_0^l\int_0^{2\pi}\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{2}{\pi R^2 l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2\sqrt{\lambda_{nmk}}}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\\
&\quad\times\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr),
\end{aligned}$$

where

$$\lambda_{nmk} = a^2\mu_{nm}^2 + \frac{a^2k^2\pi^2}{l^2} + b, \qquad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n>0, \end{cases}$$

the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument), and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &+\int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\ &+a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\ &-a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\ &+a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau\\ &+\int_0^t\int_0^l\int_0^{2\pi}\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{\sin\bigl(t\sqrt{b}\bigr)}{\pi R^2 l\sqrt{b}}+\frac{2}{\pi R^2 l}\sum_{k=1}^{\infty}\frac{1}{\sqrt{\beta_k}}\cos\Bigl(\frac{k\pi x}{l}\Bigr)\cos\Bigl(\frac{k\pi\xi}{l}\Bigr)\sin\bigl(t\sqrt{\beta_k}\bigr)$$
$$+\frac{1}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_nA_k\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\cos\Bigl(\frac{k\pi x}{l}\Bigr)\cos\Bigl(\frac{k\pi\xi}{l}\Bigr)\frac{\sin(\lambda_{nmk}t)}{\lambda_{nmk}},$$

$$\beta_k=\frac{a^2k^2\pi^2}{l^2}+b,\qquad \lambda_{nmk}=\sqrt{a^2\mu_{nm}^2+\frac{a^2k^2\pi^2}{l^2}+b},\qquad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R)=0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_r w + k_1 w &= g(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\ \partial_z w - k_2 w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\ \partial_z w + k_3 w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2}{(\mu_{nm}^2R^2+k_1^2R^2-n^2)[J_n(\mu_{nm}R)]^2\|h_s\|^2\lambda_{nms}} \times J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]h_s(z)h_s(\zeta)\sin(\lambda_{nms}t).$$

Here,

$$A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}\qquad \lambda_{nms}=\sqrt{a^2\mu_{nm}^2+a^2\beta_s^2+b},$$

$$h_s(z)=\cos(\beta_s z)+\frac{k_2}{\beta_s}\sin(\beta_s z),\quad \|h_s\|^2=\frac{k_3}{2\beta_s^2}\frac{\beta_s^2+k_2^2}{\beta_s^2+k_3^2}+\frac{k_2}{2\beta_s^2}+\frac{l}{2}\left(1+\frac{k_2^2}{\beta_s^2}\right),$$

the $J_n(\xi)$ are Bessel functions, and the μ_{nm} and β_s are positive roots of the transcendental equations

$$\mu J_n'(\mu R)+k_1J_n(\mu R)=0,\qquad \frac{\tan(\beta l)}{\beta}=\frac{k_2+k_3}{\beta^2-k_2k_3}.$$

► **Domain: $0\le r\le R$, $0\le\varphi\le 2\pi$, $0\le z\le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_0^R \xi f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&-a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{2\pi}\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi R^2 l} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{A_n A_k}{[J_n'(\mu_{nm} R)]^2 \sqrt{\lambda_{nmk}}} J_n(\mu_{nm} r) J_n(\mu_{nm} \xi) \times \cos[n(\varphi-\eta)] \cos\left(\frac{k\pi z}{l}\right) \cos\left(\frac{k\pi \zeta}{l}\right) \sin\bigl(t\sqrt{\lambda_{nmk}}\,\bigr),$$

$$\lambda_{nmk} = a^2 \mu_{nm}^2 + \frac{a^2 k^2 \pi^2}{l^2} + b, \quad A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n > 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_r w &= g_1(\varphi,z,t) && \text{at} \quad r = R && \text{(boundary condition)}, \\ w &= g_2(r,\varphi,t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\ w &= g_3(r,\varphi,t) && \text{at} \quad z = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \frac{\partial}{\partial t} \int_0^l \int_0^{2\pi} \int_0^R \xi f_0(\xi,\eta,\zeta) G(r,\varphi,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\ &+ \int_0^l \int_0^{2\pi} \int_0^R \xi f_1(\xi,\eta,\zeta) G(r,\varphi,z,\xi,\eta,\zeta,t)\, d\xi\, d\eta\, d\zeta \\ &+ a^2 R \int_0^t \int_0^l \int_0^{2\pi} g_1(\eta,\zeta,\tau) G(r,\varphi,z,R,\eta,\zeta,t-\tau)\, d\eta\, d\zeta\, d\tau \\ &+ a^2 \int_0^t \int_0^{2\pi} \int_0^R \xi g_2(\xi,\eta,\tau) \left[\frac{\partial}{\partial \zeta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0} d\xi\, d\eta\, d\tau \\ &- a^2 \int_0^t \int_0^{2\pi} \int_0^R \xi g_3(\xi,\eta,\tau) \left[\frac{\partial}{\partial \zeta} G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l} d\xi\, d\eta\, d\tau \\ &+ \int_0^t \int_0^l \int_0^{2\pi} \int_0^R \xi \Phi(\xi,\eta,\zeta,\tau) G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{2}{\pi R^2 l}\sum_{k=1}^{\infty}\frac{1}{\sqrt{\beta_k}}\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\left(t\sqrt{\beta_k}\right)$$
$$+\frac{2}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n\mu_{nm}^2}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2\sqrt{\lambda_{nmk}}}J_n(\mu_{nm}r)$$
$$\times J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\sin\left(t\sqrt{\lambda_{nmk}}\right),$$

$$\beta_k = \frac{a^2k^2\pi^2}{l^2}+b,\quad \lambda_{nmk} = a^2\mu_{nm}^2+\frac{a^2k^2\pi^2}{l^2}+b,\quad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. First boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(r,\varphi,z,t) = \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta$$
$$+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta$$
$$+a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}d\eta\,d\zeta\,d\tau$$
$$-a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}d\eta\,d\zeta\,d\tau$$
$$+a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau$$
$$-a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau$$
$$+\int_0^t\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{\pi}{2l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)}Z_{nm}(r)Z_{nm}(\xi)$$
$$\times\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{\lambda_{nmk}}\right)}{\sqrt{\lambda_{nmk}}},$$

where

$$A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nmk} = a^2\mu_{nm}^2 + \frac{a^2k^2\pi^2}{l^2} + b,$$
$$Z_{nm}(r) = J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r);$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - Y_n(\mu R_1)J_n(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2,\ 0\le\varphi\le 2\pi,\ 0\le z\le l$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&- a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&+ a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&- a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+ a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{\sin\left(t\sqrt{b}\,\right)}{\pi(R_2^2-R_1^2)l\sqrt{b}} + \frac{2}{\pi(R_2^2-R_1^2)l}\sum_{k=1}^{\infty}\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{\beta_k}\,\right)}{\sqrt{\beta_k}}$$
$$+\frac{1}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_nA_k\mu_{nm}^2 Z_{nm}(r)Z_{nm}(\xi)}{(\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1)}$$
$$\times\cos[n(\varphi-\eta)]\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{\lambda_{nmk}}\,\right)}{\sqrt{\lambda_{nmk}}},$$

where

$$A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases} \qquad \beta_k = \frac{a^2k^2\pi^2}{l^2} + b, \qquad \lambda_{nmk} = a^2\mu_{nm}^2 + \frac{a^2k^2\pi^2}{l^2} + b,$$
$$Z_{nm}(r) = J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r) - Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r);$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2) - Y_n'(\mu R_1)J_n'(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Third boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w - k_1 w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w + k_2 w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
\partial_z w - k_3 w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w + k_4 w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2}{\|h_s\|^2\sqrt{a^2\mu_{nm}^2+a^2\lambda_s^2+b}}$$
$$\times\frac{Z_{nm}(r)Z_{nm}(\xi)\cos[n(\varphi-\eta)]h_s(z)h_s(\zeta)\sin\left(t\sqrt{a^2\mu_{nm}^2+a^2\lambda_s^2+b}\,\right)}{(k_2^2R_2^2+\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(k_1^2R_1^2+\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1)}.$$

Here,

$$Z_{nm}(r) = \left[\mu_{nm}J_n'(\mu_{nm}R_1)-k_1J_n(\mu_{nm}R_1)\right]Y_n(\mu_{nm}r)$$
$$-\left[\mu_{nm}Y_n'(\mu_{nm}R_1)-k_1Y_n(\mu_{nm}R_1)\right]J_n(\mu_{nm}r),$$
$$A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases} \quad h_s(z) = \cos(\lambda_s z)+\frac{k_3}{\lambda_s}\sin(\lambda_s z), \quad \|h_s\|^2 = \frac{k_4}{2\lambda_s^2}\frac{\lambda_s^2+k_3^2}{\lambda_s^2+k_4^2}+\frac{k_3}{2\lambda_s^2}+\frac{l}{2}\left(1+\frac{k_3^2}{\lambda_s^2}\right),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$\big[\mu J_n'(\mu R_1) - k_1 J_n(\mu R_1)\big]\big[\mu Y_n'(\mu R_2) + k_2 Y_n(\mu R_2)\big] = \big[\mu Y_n'(\mu R_1) - k_1 Y_n(\mu R_1)\big]\big[\mu J_n'(\mu R_2) + k_2 J_n(\mu R_2)\big],$$

and the λ_s are positive roots of the transcendental equation $\dfrac{\tan(\lambda l)}{\lambda} = \dfrac{k_3 + k_4}{\lambda^2 - k_3 k_4}$.

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. Mixed boundary value problems.**

1°. A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\!\int_0^{2\pi}\!\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\!\int_0^{2\pi}\!\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+a^2R_1\int_0^t\!\int_0^l\!\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1} d\eta\,d\zeta\,d\tau\\
&-a^2R_2\int_0^t\!\int_0^l\!\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2} d\eta\,d\zeta\,d\tau\\
&-a^2\int_0^t\!\int_0^{2\pi}\!\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\!\int_0^{2\pi}\!\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\!\int_0^l\!\int_0^{2\pi}\!\int_{R_1}^{R_2} \Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{\pi}{4l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_nA_k\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)}Z_{nm}(r)Z_{nm}(\xi)\\
&\quad\times\cos[n(\varphi-\eta)]\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\big(t\sqrt{\lambda_{nmk}}\,\big)}{\sqrt{\lambda_{nmk}}},
\end{aligned}$$

where

$$A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0, \end{cases} \qquad \lambda_{nmk} = a^2 \mu_{nm}^2 + \frac{a^2 k^2 \pi^2}{l^2} + b,$$

$$Z_{nm}(r) = J_n(\mu_{nm} R_1) Y_n(\mu_{nm} r) - Y_n(\mu_{nm} R_1) J_n(\mu_{nm} r);$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1) Y_n(\mu R_2) - Y_n(\mu R_1) J_n(\mu R_2) = 0.$$

2°. A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r, \varphi, z) &&\text{at} \quad t = 0 &&\text{(initial condition)}, \\ \partial_t w &= f_1(r, \varphi, z) &&\text{at} \quad t = 0 &&\text{(initial condition)}, \\ \partial_r w &= g_1(\varphi, z, t) &&\text{at} \quad r = R_1 &&\text{(boundary condition)}, \\ \partial_r w &= g_2(\varphi, z, t) &&\text{at} \quad r = R_2 &&\text{(boundary condition)}, \\ w &= g_3(r, \varphi, t) &&\text{at} \quad z = 0 &&\text{(boundary condition)}, \\ w &= g_4(r, \varphi, t) &&\text{at} \quad z = l &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r, \varphi, z, t) &= \frac{\partial}{\partial t} \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} f_0(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\ &+ \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} f_1(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\ &- a^2 R_1 \int_0^t \int_0^l \int_0^{2\pi} g_1(\eta, \zeta, \tau) G(r, \varphi, z, R_1, \eta, \zeta, t - \tau) \, d\eta \, d\zeta \, d\tau \\ &+ a^2 R_2 \int_0^t \int_0^l \int_0^{2\pi} g_2(\eta, \zeta, \tau) G(r, \varphi, z, R_2, \eta, \zeta, t - \tau) \, d\eta \, d\zeta \, d\tau \\ &+ a^2 \int_0^t \int_0^{2\pi} \int_{R_1}^{R_2} g_3(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\zeta=0} \xi \, d\xi \, d\eta \, d\tau \\ &- a^2 \int_0^t \int_0^{2\pi} \int_{R_1}^{R_2} g_4(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\zeta=l} \xi \, d\xi \, d\eta \, d\tau \\ &+ \int_0^t \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \xi \, d\xi \, d\eta \, d\zeta \, d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r, \varphi, z, \xi, \eta, \zeta, t) = &\frac{2}{\pi (R_2^2 - R_1^2) l} \sum_{k=1}^{\infty} \sin\left(\frac{k\pi z}{l}\right) \sin\left(\frac{k\pi \zeta}{l}\right) \frac{\sin\left(t\sqrt{\beta_k}\right)}{\sqrt{\beta_k}} \\ &+ \frac{2}{\pi l} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{A_n \mu_{nm}^2 Z_{nm}(r) Z_{nm}(\xi)}{(\mu_{nm}^2 R_2^2 - n^2) Z_{nm}^2(R_2) - (\mu_{nm}^2 R_1^2 - n^2) Z_{nm}^2(R_1)} \\ &\times \cos[n(\varphi - \eta)] \sin\left(\frac{k\pi z}{l}\right) \sin\left(\frac{k\pi \zeta}{l}\right) \frac{\sin\left(t\sqrt{\lambda_{nmk}}\right)}{\sqrt{\lambda_{nmk}}}, \end{aligned}$$

where

$$A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0, \end{cases} \quad \beta_k = \frac{a^2 k^2 \pi^2}{l^2} + b, \quad \lambda_{nmk} = a^2 \mu_{nm}^2 + \frac{a^2 k^2 \pi^2}{l^2} + b,$$
$$Z_{nm}(r) = J_n'(\mu_{nm} R_1) Y_n(\mu_{nm} r) - Y_n'(\mu_{nm} R_1) J_n(\mu_{nm} r);$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1) Y_n'(\mu R_2) - Y_n'(\mu R_1) J_n'(\mu R_2) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. First boundary value problem.**

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ w &= g_1(\varphi, z, t) && \text{at} \quad r = R && \text{(boundary condition)}, \\ w &= g_2(r, z, t) && \text{at} \quad \varphi = 0 && \text{(boundary condition)}, \\ w &= g_3(r, z, t) && \text{at} \quad \varphi = \varphi_0 && \text{(boundary condition)}, \\ w &= g_4(r, \varphi, t) && \text{at} \quad z = 0 && \text{(boundary condition)}, \\ w &= g_5(r, \varphi, t) && \text{at} \quad z = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r, \varphi, z, t) = {} & \frac{\partial}{\partial t} \int_0^l \int_0^{\varphi_0} \int_0^R f_0(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\ & + \int_0^l \int_0^{\varphi_0} \int_0^R f_1(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\ & - a^2 R \int_0^t \int_0^l \int_0^{\varphi_0} g_1(\eta, \zeta, \tau) \left[\frac{\partial}{\partial \xi} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\xi = R} d\eta \, d\zeta \, d\tau \\ & + a^2 \int_0^t \int_0^l \int_0^R g_2(\xi, \zeta, \tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\eta = 0} d\xi \, d\zeta \, d\tau \\ & - a^2 \int_0^t \int_0^l \int_0^R g_3(\xi, \zeta, \tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\eta = \varphi_0} d\xi \, d\zeta \, d\tau \\ & + a^2 \int_0^t \int_0^{\varphi_0} \int_0^R g_4(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\zeta = 0} \xi \, d\xi \, d\eta \, d\tau \\ & - a^2 \int_0^t \int_0^{\varphi_0} \int_0^R g_5(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \right]_{\zeta = l} \xi \, d\xi \, d\eta \, d\tau \\ & + \int_0^t \int_0^l \int_0^{\varphi_0} \int_0^R \Phi(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \xi \, d\xi \, d\eta \, d\zeta \, d\tau. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{8}{R^2 l\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)$$
$$\times\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\sin\left(\frac{k\pi z}{l}\right)\sin\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{a^2\mu_{nm}^2+a^2k^2\pi^2l^{-2}+b}\right)}{\sqrt{a^2\mu_{nm}^2+a^2k^2\pi^2l^{-2}+b}},$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

► **Domain: $0\le r\le R$, $0\le\varphi\le\varphi_0$, $0\le z\le l$. Mixed boundary value problem.**

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,z,t) &&\text{at}\quad \varphi=0 &&\text{(boundary condition)},\\
w &= g_3(r,z,t) &&\text{at}\quad \varphi=\varphi_0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_5(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{\varphi_0}\int_0^R f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{\varphi_0}\int_0^R f_1(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&-a^2R\int_0^t\int_0^l\int_0^{\varphi_0} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^t\int_0^l\int_0^R g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0}d\xi\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^l\int_0^R g_3(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0}d\xi\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^{\varphi_0}\int_0^R g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\int_0^{\varphi_0}\int_0^R g_5(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{\varphi_0}\int_0^R \Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{4}{R^2 l\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_k J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)$$
$$\times\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\cos\left(\frac{k\pi z}{l}\right)\cos\left(\frac{k\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{a^2\mu_{nm}^2+a^2k^2\pi^2l^{-2}+b}\,\right)}{\sqrt{a^2\mu_{nm}^2+a^2k^2\pi^2l^{-2}+b}},$$

where $A_0=1$ and $A_k=2$ for $k\ge 1$; the $J_{n\pi/\varphi_0}(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R)=0$.

8.3.3 Problems in Spherical Coordinates

A *nonhomogeneous Klein–Gordon equation* in the spherical coordinate system is written as

$$\frac{\partial^2 w}{\partial t^2}=a^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}\right]-bw+\Phi(r,\theta,\varphi,t),$$
$$r=\sqrt{x^2+y^2+z^2}.$$

One-dimensional problems with central symmetry that have solutions of the form $w=w(r,t)$ are treated in Section 6.2.6.

► Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. First boundary value problem.

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g(\theta,\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t)=&\frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{R} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{R} f_1(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &-a^2R^2\int_0^{t}\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}\sin\eta\,d\eta\,d\zeta\,d\tau\\ &+\int_0^{t}\!\!\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{R}\Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau.\end{aligned}$$

Here,

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{2\pi R^2\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}A_kB_{nmk}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\sin\left(t\sqrt{a^2\lambda_{nm}^2+b}\,\right),$$

where

$$A_k = \begin{cases} 1 & \text{for } k=0, \\ 2 & \text{for } k\neq 0, \end{cases} \qquad B_{nmk} = \frac{(2n+1)(n-k)!}{(n+k)!\,[J'_{n+1/2}(\lambda_{nm}R)]^2\sqrt{a^2\lambda_{nm}^2+b}};$$

the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^k(\mu) = (1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_n(\mu), \qquad P_n(\mu) = \frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation $J_{n+1/2}(\lambda R)=0$.

► **Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. Second boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) && \text{at } t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) && \text{at } t=0 && \text{(initial condition)},\\ \partial_r w &= g(\theta,\varphi,t) && \text{at } r=R && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{R} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{R} f_1(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+a^2R^2\int_0^{t}\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g(\eta,\zeta,\tau)G(r,\theta,\varphi,R,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau\\ &+\int_0^{t}\!\!\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{R}\Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\theta,\varphi,\xi,\eta,\zeta,t) &= \frac{3\sin\bigl(t\sqrt{b}\bigr)}{4\pi R^3\sqrt{b}} + \frac{1}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}\frac{A_kB_{nmk}}{\sqrt{a^2\lambda_{nm}^2+b}}J_{n+1/2}(\lambda_{nm}r)\\ &\quad\times J_{n+1/2}(\lambda_{nm}\xi)P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b}\bigr), \end{aligned}$$

where

$$A_k = \begin{cases} 1 & \text{for } k=0, \\ 2 & \text{for } k\neq 0, \end{cases} \qquad B_{nmk} = \frac{\lambda_{nm}^2(2n+1)(n-k)!}{(n+k)!\bigl[R^2\lambda_{nm}^2-n(n+1)\bigr]\bigl[J_{n+1/2}(\lambda_{nm}R)\bigr]^2};$$

the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the paragraph above), and the λ_{nm} are positive roots of the transcendental equation

$$2\lambda RJ'_{n+1/2}(\lambda R) - J_{n+1/2}(\lambda R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w + kw &= g(\theta,\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = \frac{1}{2\pi\sqrt{r\xi}} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n} \frac{A_s B_{nms}}{\sqrt{a^2\lambda_{nm}^2+b}} J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b}\,\bigr).$$

Here,

$$A_s=\begin{cases}1 & \text{for } s=0,\\ 2 & \text{for } s\neq 0,\end{cases}\qquad B_{nms}=\frac{\lambda_{nm}^2(2n+1)(n-s)!}{(n+s)!\bigl[R^2\lambda_{nm}^2+(kR+n)(kR-n-1)\bigr]\bigl[J_{n+1/2}(\lambda_{nm}R)\bigr]^2};$$

the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda R J'_{n+1/2}(\lambda R) + \bigl(kR-\tfrac{1}{2}\bigr)J_{n+1/2}(\lambda R)=0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w &= g_1(\theta,\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ w &= g_2(\theta,\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+a^2R_1^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}\sin\eta\,d\eta\,d\zeta\,d\tau\\ &-a^2R_2^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}\sin\eta\,d\eta\,d\zeta\,d\tau\\ &+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau, \end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = \frac{\pi}{8\sqrt{r\xi}} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n} \frac{A_k B_{nmk}}{\sqrt{a^2\lambda_{nm}^2 + b}} Z_{n+1/2}(\lambda_{nm} r) Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta) P_n^k(\cos\eta) \cos[k(\varphi-\zeta)] \sin\bigl(t\sqrt{a^2\lambda_{nm}^2 + b}\bigr).$$

Here,

$$Z_{n+1/2}(\lambda_{nm} r) = J_{n+1/2}(\lambda_{nm} R_1) Y_{n+1/2}(\lambda_{nm} r) - Y_{n+1/2}(\lambda_{nm} R_1) J_{n+1/2}(\lambda_{nm} r),$$
$$A_k = \begin{cases} 1 & \text{for } k=0, \\ 2 & \text{for } k\neq 0, \end{cases} \qquad B_{nmk} = \frac{\lambda_{nm}(2n+1)(n-k)!\, J_{n+1/2}^2(\lambda_{nm} R_2)}{(n+k)!\,[J_{n+1/2}^2(\lambda_{nm} R_1) - J_{n+1/2}^2(\lambda_{nm} R_2)]},$$

where the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^k(\mu) = (1-\mu^2)^{k/2} \frac{d^k}{d\mu^k} P_n(\mu), \qquad P_n(\mu) = \frac{1}{n!\,2^n} \frac{d^n}{d\mu^n} (\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation $Z_{n+1/2}(\lambda R_2) = 0$.

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)}, \\ \partial_t w &= f_1(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)}, \\ \partial_r w &= g_1(\theta,\varphi,t) \quad \text{at} \quad r=R_1 \quad &&\text{(boundary condition)}, \\ \partial_r w &= g_2(\theta,\varphi,t) \quad \text{at} \quad r=R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) &= \frac{\partial}{\partial t} \int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta) G(r,\theta,\varphi,\xi,\eta,\zeta,t) \xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta \\ &+ \int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2} f_1(\xi,\eta,\zeta) G(r,\theta,\varphi,\xi,\eta,\zeta,t) \xi^2 \sin\eta \, d\xi\, d\eta\, d\zeta \\ &- a^2 R_1^2 \int_0^t\int_0^{2\pi}\int_0^{\pi} g_1(\eta,\zeta,\tau) G(r,\theta,\varphi,R_1,\eta,\zeta,t-\tau) \sin\eta\, d\eta\, d\zeta\, d\tau \\ &+ a^2 R_2^2 \int_0^t\int_0^{2\pi}\int_0^{\pi} g_2(\eta,\zeta,\tau) G(r,\theta,\varphi,R_2,\eta,\zeta,t-\tau) \sin\eta\, d\eta\, d\zeta\, d\tau \\ &+ \int_0^t\int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2} \Phi(\xi,\eta,\zeta,\tau) G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau) \xi^2 \sin\eta\, d\xi\, d\eta\, d\zeta\, d\tau, \end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{3\sin\bigl(t\sqrt{b}\,\bigr)}{4\pi(R_2^3-R_1^3)\sqrt{b}}+\frac{1}{4\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n}\frac{A_k}{B_{nmk}}Z_{n+1/2}(\lambda_{nm}r)$$
$$\times Z_{n+1/2}(\lambda_{nm}\xi)P_n^k(\cos\theta)P_n^k(\cos\eta)\cos[k(\varphi-\zeta)]\frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b}\,\bigr)}{\sqrt{a^2\lambda_{nm}^2+b}}.$$

Here,

$$A_k=\begin{cases}1 & \text{for } k=0,\\ 2 & \text{for } k\neq 0,\end{cases}\qquad B_{nmk}=\frac{(n+k)!}{(2n+1)(n-k)!}\int_{R_1}^{R_2} rZ_{n+1/2}^2(\lambda_{nm}r)\,dr,$$
$$Z_{n+1/2}(\lambda_{nm}r)=\left[\lambda_{nm}J'_{n+1/2}(\lambda_{nm}R_1)-\frac{1}{2R_1}J_{n+1/2}(\lambda_{nm}R_1)\right]Y_{n+1/2}(\lambda_{nm}r)$$
$$-\left[\lambda_{nm}Y'_{n+1/2}(\lambda_{nm}R_1)-\frac{1}{2R_1}Y_{n+1/2}(\lambda_{nm}R_1)\right]J_{n+1/2}(\lambda_{nm}r),$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see the paragraph above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2)-\frac{1}{2R_2}Z_{n+1/2}(\lambda R_2)=0.$$

► **Domain: $R_1\le r\le R_2,\ 0\le\theta\le\pi,\ 0\le\varphi\le 2\pi$. Third boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_r w-k_1w&=g_1(\theta,\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\ \partial_r w+k_2w&=g_2(\theta,\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.\end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{1}{4\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n}\frac{A_s}{B_{nms}}Z_{n+1/2}(\lambda_{nm}r)Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b}\,\bigr)}{\sqrt{a^2\lambda_{nm}^2+b}}.$$

Here,

$$A_s=\begin{cases}1 & \text{for } s=0,\\ 2 & \text{for } s\neq 0,\end{cases}\qquad B_{nms}=\frac{(n+s)!}{(2n+1)(n-s)!}\int_{R_1}^{R_2} rZ_{n+1/2}^2(\lambda_{nm}r)\,dr,$$
$$Z_{n+1/2}(\lambda r)=\left[\lambda J'_{n+1/2}(\lambda R_1)-\left(k_1+\frac{1}{2R_1}\right)J_{n+1/2}(\lambda R_1)\right]Y_{n+1/2}(\lambda r)$$
$$-\left[\lambda Y'_{n+1/2}(\lambda R_1)-\left(k_1+\frac{1}{2R_1}\right)Y_{n+1/2}(\lambda R_1)\right]J_{n+1/2}(\lambda r),$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2) + \left(k_2 - \frac{1}{2R_2}\right) Z_{n+1/2}(\lambda R_2) = 0.$$

8.4 Telegraph Equation $\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\Delta_3 w - bw + \Phi(x,y,z,t)$

8.4.1 Problems in Cartesian Coordinates

A *three-dimensional nonhomogeneous telegraph equation* in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - bw + \Phi(x,y,z,t).$$

► **Reduction to the three-dimensional Klein–Gordon equation.**

The substitution $w(x,y,z,t) = \exp\left(-\frac{1}{2}kt\right)u(x,y,z,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \left(b - \tfrac{1}{4}k^2\right)u + \exp\left(\tfrac{1}{2}kt\right)\Phi(x,y,z,t),$$

which is discussed in Section 8.3.1.

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. First boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_6(x,y,t) &&\text{at}\quad z=l_3 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} \bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=0} d\eta\,d\zeta\,d\tau \\
&- a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=l_1} d\eta\,d\zeta\,d\tau \\
&+ a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_3(\xi,\zeta,\tau)\biggl[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\eta=0} d\xi\,d\zeta\,d\tau \\
&- a^2\int_0^{t}\!\int_0^{l_3}\!\int_0^{l_1} g_4(\xi,\zeta,\tau)\biggl[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\eta=l_2} d\xi\,d\zeta\,d\tau \\
&+ a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0} d\xi\,d\eta\,d\tau \\
&- a^2\int_0^{t}\!\int_0^{l_2}\!\int_0^{l_1} g_6(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=l_3} d\xi\,d\eta\,d\tau \\
&+ \int_0^{t}\!\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{l_1l_2l_3}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{\sqrt{\lambda_{nms}}}\sin(\alpha_n x)\sin(\beta_m y)\sin(\gamma_s z) \\
\times\sin(\alpha_n\xi)\sin(\beta_m\eta)\sin(\gamma_s\zeta)\sin\bigl(t\sqrt{\lambda_{nms}}\,\bigr),
\end{aligned}
$$

where

$$
\alpha_n = \frac{n\pi}{l_1},\quad \beta_m = \frac{m\pi}{l_2},\quad \gamma_s = \frac{s\pi}{l_3},\quad \lambda_{nms} = a^2(\alpha_n^2+\beta_m^2+\gamma_s^2)+b-\tfrac{1}{4}k^2.
$$

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Second boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w &= g_1(y,z,t) &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) &&\text{at}\quad x=l_1 &&\text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) &&\text{at}\quad y=0 &&\text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) &&\text{at}\quad y=l_2 &&\text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) &&\text{at}\quad z=l_3 &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&-a^2\int_0^t\!\int_0^{l_3}\!\int_0^{l_2} g_1(\eta,\zeta,\tau)G(x,y,z,0,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^t\!\int_0^{l_3}\!\int_0^{l_2} g_2(\eta,\zeta,\tau)G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&-a^2\int_0^t\!\int_0^{l_3}\!\int_0^{l_1} g_3(\xi,\zeta,\tau)G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&+a^2\int_0^t\!\int_0^{l_3}\!\int_0^{l_1} g_4(\xi,\zeta,\tau)G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&-a^2\int_0^t\!\int_0^{l_2}\!\int_0^{l_1} g_5(\xi,\eta,\tau)G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\!\int_0^{l_2}\!\int_0^{l_1} g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\!\int_0^{l_3}\!\int_0^{l_2}\!\int_0^{l_1}\Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
G(x,y,z,\xi,\eta,\zeta,t) = \frac{e^{-kt/2}}{l_1l_2l_3}\Biggl[\frac{\sin\bigl(t\sqrt{c}\bigr)}{\sqrt{c}} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\frac{A_nA_mA_s}{\sqrt{\lambda_{nms}}}\cos(\alpha_n x)\cos(\beta_m y)\\
\times\cos(\gamma_s z)\cos(\alpha_n\xi)\cos(\beta_m\eta)\cos(\gamma_s\zeta)\sin\bigl(t\sqrt{\lambda_{nms}}\bigr)\Biggr],
\end{aligned}
$$

$$
A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}\qquad \alpha_n = \frac{n\pi}{l_1},\quad \beta_m = \frac{m\pi}{l_2},\quad \gamma_s = \frac{s\pi}{l_3},
$$

$$
c = b - \tfrac14 k^2,\quad \lambda_{nms} = a^2(\alpha_n^2+\beta_m^2+\gamma_s^2) + b - \tfrac14 k^2.
$$

The summation is performed over the indices satisfying the condition $n+m+s>0$; the term corresponding to $n=m=s=0$ is singled out.

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Third boundary value problem.**

A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_x w - s_1 w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w + s_2 w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
\partial_y w - s_3 w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w + s_4 w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
\partial_z w - s_5 w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w + s_6 w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

The solution $w(x,y,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta,t) = 8\exp\bigl(-\tfrac{1}{2}kt\bigr)\Biggl[\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{E_{npq}\sqrt{\lambda_{npq}}}\sin(\alpha_n x+\varepsilon_n)\sin(\beta_p y+\sigma_p)$$
$$\times\sin(\gamma_q z+\nu_q)\sin(\alpha_n\xi+\varepsilon_n)\sin(\beta_p\eta+\sigma_p)\sin(\gamma_q\zeta+\nu_q)\sin\bigl(t\sqrt{\lambda_{npq}}\bigr)\Biggr].$$

Here,

$$\varepsilon_n=\arctan\frac{\alpha_n}{l_1},\quad \sigma_p=\arctan\frac{\beta_p}{l_2},\quad \nu_q=\arctan\frac{\gamma_q}{l_3},\quad \lambda_{npq}=a^2(\alpha_n^2+\beta_p^2+\gamma_q^2)+b-\tfrac{1}{4}k^2,$$
$$E_{npq}=\left[l_1+\frac{(s_1s_2+\alpha_n^2)(s_1+s_2)}{(s_1^2+\alpha_n^2)(s_2^2+\alpha_n^2)}\right]\left[l_2+\frac{(s_3s_4+\beta_p^2)(s_3+s_4)}{(s_3^2+\beta_p^2)(s_4^2+\beta_p^2)}\right]\left[l_3+\frac{(s_5s_6+\gamma_q^2)(s_5+s_6)}{(s_5^2+\gamma_q^2)(s_6^2+\gamma_q^2)}\right],$$

where the α_n, β_p, and γ_q are positive roots of the transcendental equations

$$\alpha^2-s_1s_2=(s_1+s_2)\alpha\cot(l_1\alpha),$$
$$\beta^2-s_3s_4=(s_3+s_4)\beta\cot(l_2\beta),$$
$$\gamma^2-s_5s_6=(s_5+s_6)\gamma\cot(l_3\gamma).$$

► **Domain: $0\le x\le l_1$, $0\le y\le l_2$, $0\le z\le l_3$. Mixed boundary value problems.**

1°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(x,y,z) && \text{at } t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at } t=0 && \text{(initial condition)},\\
w &= g_1(y,z,t) && \text{at } x=0 && \text{(boundary condition)},\\
w &= g_2(y,z,t) && \text{at } x=l_1 && \text{(boundary condition)},\\
\partial_y w &= g_3(x,z,t) && \text{at } y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at } y=l_2 && \text{(boundary condition)},\\
\partial_z w &= g_5(x,y,t) && \text{at } z=0 && \text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) && \text{at } z=l_3 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(x,y,z,t)=\frac{\partial}{\partial t}\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta$$
$$+\int_0^{l_3}\!\!\int_0^{l_2}\!\!\int_0^{l_1}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta$$
$$+a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0}d\eta\,d\zeta\,d\tau$$
$$-a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_2} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=l_1}d\eta\,d\zeta\,d\tau$$
$$-a^2\int_0^{t}\!\!\int_0^{l_3}\!\!\int_0^{l_1} g_3(\xi,\zeta,\tau)G(x,y,z,\xi,0,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau$$

$$
\begin{aligned}
&+ a^2\int_0^t\int_0^{l_3}\int_0^{l_1} g_4(\xi,\zeta,\tau)G(x,y,z,\xi,l_2,\zeta,t-\tau)\,d\xi\,d\zeta\,d\tau\\
&- a^2\int_0^t\int_0^{l_2}\int_0^{l_1} g_5(\xi,\eta,\tau)G(x,y,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+ a^2\int_0^t\int_0^{l_2}\int_0^{l_1} g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,l_3,t-\tau)\,d\xi\,d\eta\,d\tau\\
&+ \int_0^t\int_0^{l_3}\int_0^{l_2}\int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau,
\end{aligned}
$$

where

$$
G(x,y,z,\xi,\eta,\zeta,t)=\frac{2}{l_1l_2l_3}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\frac{A_mA_s}{\sqrt{\lambda_{nms}}}\sin(\alpha_n x)\cos(\beta_m y)\cos(\gamma_s z)
$$
$$
\times\sin(\alpha_n\xi)\cos(\beta_m\eta)\cos(\gamma_s\zeta)\sin\bigl(t\sqrt{\lambda_{nms}}\,\bigr),
$$

$$
A_m=\begin{cases}1 & \text{for } m=0,\\ 2 & \text{for } m>0,\end{cases}\qquad \alpha_n=\frac{n\pi}{l_1},\quad \beta_m=\frac{m\pi}{l_2},\quad \gamma_s=\frac{s\pi}{l_3},
$$
$$
\lambda_{nms}=a^2(\alpha_n^2+\beta_m^2+\gamma_s^2)+b-\tfrac{1}{4}k^2.
$$

2°. A rectangular parallelepiped is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(x,y,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(y,z,t) && \text{at} \quad x=0 && \text{(boundary condition)},\\
\partial_x w &= g_2(y,z,t) && \text{at} \quad x=l_1 && \text{(boundary condition)},\\
w &= g_3(x,z,t) && \text{at} \quad y=0 && \text{(boundary condition)},\\
\partial_y w &= g_4(x,z,t) && \text{at} \quad y=l_2 && \text{(boundary condition)},\\
w &= g_5(x,y,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_6(x,y,t) && \text{at} \quad z=l_3 && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z,t) &= \frac{\partial}{\partial t}\int_0^{l_3}\int_0^{l_2}\int_0^{l_1} f_0(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+ \int_0^{l_3}\int_0^{l_2}\int_0^{l_1}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(x,y,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+ a^2\int_0^t\int_0^{l_3}\int_0^{l_2} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=0} d\eta\,d\zeta\,d\tau\\
&+ a^2\int_0^t\int_0^{l_3}\int_0^{l_2} g_2(\eta,\zeta,\tau)G(x,y,z,l_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ a^2 \int_0^t \int_0^{l_3} \int_0^{l_1} g_3(\xi,\zeta,\tau) \left[\frac{\partial}{\partial \eta} G(x,y,z,\xi,\eta,\zeta,t-\tau) \right]_{\eta=0} d\xi\, d\zeta\, d\tau \\
&+ a^2 \int_0^t \int_0^{l_3} \int_0^{l_1} g_4(\xi,\zeta,\tau) G(x,y,z,\xi,l_2,\zeta,t-\tau)\, d\xi\, d\zeta\, d\tau \\
&+ a^2 \int_0^t \int_0^{l_2} \int_0^{l_1} g_5(\xi,\eta,\tau) \left[\frac{\partial}{\partial \zeta} G(x,y,z,\xi,\eta,\zeta,t-\tau) \right]_{\zeta=0} d\xi\, d\eta\, d\tau \\
&+ a^2 \int_0^t \int_0^{l_2} \int_0^{l_1} g_6(\xi,\eta,\tau) G(x,y,z,\xi,\eta,l_3,t-\tau)\, d\xi\, d\eta\, d\tau \\
&+ \int_0^t \int_0^{l_3} \int_0^{l_2} \int_0^{l_1} \Phi(\xi,\eta,\zeta,\tau) G(x,y,z,\xi,\eta,\zeta,t-\tau)\, d\xi\, d\eta\, d\zeta\, d\tau,
\end{aligned}
$$

where

$$
G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{l_1 l_2 l_3} \exp\left(-\tfrac{1}{2}kt\right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{\sqrt{\lambda_{nms}}} \sin(\alpha_n x) \sin(\beta_m y) \sin(\gamma_s z)
$$
$$
\times \sin(\alpha_n \xi) \sin(\beta_m \eta) \sin(\gamma_s \zeta) \sin\left(t\sqrt{\lambda_{nms}}\,\right),
$$
$$
\alpha_n = \frac{\pi(2n+1)}{2l_1}, \quad \beta_m = \frac{\pi(2m+1)}{2l_2}, \quad \gamma_s = \frac{\pi(2s+1)}{2l_3}, \quad \lambda_{nms} = a^2(\alpha_n^2 + \beta_m^2 + \gamma_s^2) + b - \tfrac{1}{4}k^2.
$$

8.4.2 Problems in Cylindrical Coordinates

A three-dimensional nonhomogeneous telegraph equation in the cylindrical coordinate system is written as

$$
\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2} \right] - bw + \Phi(r,\varphi,z,t), \qquad r = \sqrt{x^2+y^2}.
$$

One-dimensional problems with axial symmetry that have solutions $w = w(r,t)$ are treated in Section 6.4.2. Two-dimensional problems whose solutions have the form $w = w(r,\varphi,t)$ or $w = w(r,z,t)$ are considered in Sections 7.4.2 and 7.4.3.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. First boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &\text{at}\quad & t = 0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &\text{at}\quad & t = 0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &\text{at}\quad & r = R &&\text{(boundary condition)},\\
w &= g_2(r,\varphi,t) &\text{at}\quad & z = 0 &&\text{(boundary condition)},\\
w &= g_3(r,\varphi,t) &\text{at}\quad & z = l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^l\int_0^{2\pi}\int_0^R \xi\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&- a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=R} d\eta\,d\zeta\,d\tau \\
&+ a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0} d\xi\,d\eta\,d\tau \\
&- a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=l} d\xi\,d\eta\,d\tau \\
&+ \int_0^t\int_0^l\int_0^{2\pi}\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{2e^{-kt/2}}{\pi R^2 l}\biggl[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi) \\
\times\cos[n(\varphi-\eta)]\sin\Bigl(\frac{s\pi z}{l}\Bigr)\sin\Bigl(\frac{s\pi\zeta}{l}\Bigr)\frac{\sin\bigl(t\sqrt{\lambda_{nms}}\,\bigr)}{\sqrt{\lambda_{nms}}}\biggr],
\end{aligned}$$

where

$$\lambda_{nms} = a^2\mu_{nm}^2 + \frac{a^2s^2\pi^2}{l^2} + b - \tfrac{1}{4}k^2, \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n>0, \end{cases}$$

the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument), and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Second boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ \int_0^l\int_0^{2\pi}\int_0^R \xi\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\
&+ a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau \\
&- a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau \\
&+ a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau \\
&+ \int_0^t\int_0^l\int_0^{2\pi}\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\exp\bigl(-\tfrac{1}{2}kt\bigr)\biggl[\frac{\sin\bigl(t\sqrt{c}\bigr)}{\pi R^2 l\sqrt{c}}+\frac{2}{\pi R^2 l}\sum_{s=1}^{\infty}\frac{1}{\sqrt{\beta_s}}\cos\Bigl(\frac{s\pi x}{l}\Bigr)\cos\Bigl(\frac{s\pi\xi}{l}\Bigr)\sin\bigl(t\sqrt{\beta_s}\bigr)$$
$$+\frac{1}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{\infty}\frac{A_nA_s\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)]\cos\Bigl(\frac{s\pi x}{l}\Bigr)\cos\Bigl(\frac{s\pi\xi}{l}\Bigr)\frac{\sin(\lambda_{nms}t)}{\lambda_{nms}}\biggr],$$
$$c=b-\frac{k^2}{4},\quad \beta_s=\frac{a^2s^2\pi^2}{l^2}+b-\frac{k^2}{4},\quad \lambda_{nms}=\sqrt{a^2\mu_{nm}^2+\frac{a^2s^2\pi^2}{l^2}+b-\frac{k^2}{4}},\quad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R)=0$.

► **Domain: $0\le r\le R$, $0\le\varphi\le 2\pi$, $0\le z\le l$. Third boundary value problem.**

A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w + s_1 w &= g(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
\partial_z w - s_2 w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w + s_3 w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t)=\frac{1}{\pi}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{A_n\mu_{nm}^2J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2+s_1^2R^2-n^2)[J_n(\mu_{nm}R)]^2}$$
$$\times\cos[n(\varphi-\eta)]\frac{h_p(z)h_p(\zeta)}{\|h_p\|^2}\frac{\sin\bigl(t\sqrt{\lambda_{nmp}}\bigr)}{\sqrt{\lambda_{nmp}}}.$$

Here, the $J_n(\xi)$ are Bessel functions,

$$A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n>0, \end{cases} \qquad \lambda_{nmp} = a^2\mu_{nm}^2 + a^2\beta_p^2 + b - \tfrac{1}{4}k^2,$$

$$h_p(z) = \cos(\beta_p z) + \frac{s_2}{\beta_p}\sin(\beta_p z), \qquad \|h_p\|^2 = \frac{s_3}{2\beta_p^2}\frac{\beta_p^2+s_2^2}{\beta_p^2+s_3^2} + \frac{s_2}{2\beta_p^2} + \frac{l}{2}\left(1+\frac{s_2^2}{\beta_p^2}\right);$$

the μ_{nm} and β_p are positive roots of the transcendental equations

$$\mu J_n'(\mu R) + s_1 J_n(\mu R) = 0, \qquad \frac{\tan(\beta l)}{\beta} = \frac{s_2+s_3}{\beta^2 - s_2 s_3}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Mixed boundary value problems.**

1°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\ \partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)}, \\ w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)}, \\ \partial_z w &= g_2(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)}, \\ \partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\!\int_0^{2\pi}\!\int_0^R \xi f_0(\xi,\eta,\zeta) G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\ &+ \int_0^l\!\int_0^{2\pi}\!\int_0^R \xi\big[f_1(\xi,\eta,\zeta) + kf_0(\xi,\eta,\zeta)\big] G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta \\ &- a^2 R\int_0^t\!\int_0^l\!\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau \\ &- a^2\int_0^t\!\int_0^{2\pi}\!\int_0^R \xi g_2(\xi,\eta,\tau) G(r,\varphi,z,\xi,\eta,0,t-\tau)\,d\xi\,d\eta\,d\tau \\ &+ a^2\int_0^t\!\int_0^{2\pi}\!\int_0^R \xi g_3(\xi,\eta,\tau) G(r,\varphi,z,\xi,\eta,l,t-\tau)\,d\xi\,d\eta\,d\tau \\ &+ \int_0^t\!\int_0^l\!\int_0^{2\pi}\!\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau) G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{1}{\pi R^2 l}\exp\left(-\tfrac{1}{2}kt\right)\Bigg[\sum_{n=0}^\infty\sum_{m=1}^\infty\sum_{s=0}^\infty \frac{A_n A_s}{[J_n'(\mu_{nm}R)]^2\sqrt{\lambda_{nms}}} J_n(\mu_{nm}r) \\ &\quad\times J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\cos\left(\frac{s\pi z}{l}\right)\cos\left(\frac{s\pi\zeta}{l}\right)\sin\left(t\sqrt{\lambda_{nms}}\right)\Bigg], \end{aligned}$$

$$\lambda_{nms} = a^2\mu_{nm}^2 + \frac{a^2 s^2\pi^2}{l^2} + b - \tfrac{1}{4}k^2, \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n>0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

2°. A circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) &&\text{at}\quad r=R &&\text{(boundary condition)},\\
w &= g_2(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_3(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_0^R \xi f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_0^R \xi\big[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\big]G(r,\varphi,z,\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\\
&+a^2R\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_2(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=0}d\xi\,d\eta\,d\tau\\
&-a^2\int_0^t\int_0^{2\pi}\int_0^R \xi g_3(\xi,\eta,\tau)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\zeta=l}d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{2\pi}\int_0^R \xi\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{2}{\pi R^2 l}\exp\left(-\tfrac{1}{2}kt\right)\sum_{s=1}^{\infty}\frac{1}{\sqrt{\beta_s}}\sin\left(\frac{s\pi z}{l}\right)\sin\left(\frac{s\pi\zeta}{l}\right)\sin\left(t\sqrt{\beta_s}\right)\\
&+\frac{2}{\pi l}\exp\left(-\tfrac{1}{2}kt\right)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2}{(\mu_{nm}^2R^2-n^2)[J_n(\mu_{nm}R)]^2}J_n(\mu_{nm}r)\\
&\times J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin\left(\frac{s\pi z}{l}\right)\sin\left(\frac{s\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{\lambda_{nms}}\right)}{\sqrt{\lambda_{nms}}},
\end{aligned}
$$

$$
\beta_s=\frac{a^2s^2\pi^2}{l^2}+b-\tfrac{1}{4}k^2,\qquad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n>0,\end{cases}
$$

$$
\lambda_{nms}=a^2\mu_{nm}^2+\frac{a^2s^2\pi^2}{l^2}+b-\tfrac{1}{4}k^2,
$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. First boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=R_1}d\eta\,d\zeta\,d\tau\\
&-a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\biggl[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\xi=R_2}d\eta\,d\zeta\,d\tau\\
&+a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=0}\xi\,d\xi\,d\eta\,d\tau\\
&-a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)\biggl[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\biggr]_{\zeta=l}\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{\pi}{2l}\exp\bigl(-\tfrac12 kt\bigr)\Biggl[\sum_{n=0}^\infty\sum_{m=1}^\infty\sum_{s=1}^\infty \frac{A_n\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)}Z_{nm}(r)\\
\times Z_{nm}(\xi)\cos[n(\varphi-\eta)]\sin\biggl(\frac{s\pi z}{l}\biggr)\sin\biggl(\frac{s\pi\zeta}{l}\biggr)\frac{\sin\bigl(t\sqrt{\lambda_{nms}}\,\bigr)}{\sqrt{\lambda_{nms}}}\Biggr],
\end{aligned}
$$

$$
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nms}=a^2\mu_{nm}^2+\frac{a^2s^2\pi^2}{l^2}+b-\tfrac14k^2,
$$

$$
Z_{nm}(r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),
$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$
J_n(\mu R_1)Y_n(\mu R_2)-Y_n(\mu R_1)J_n(\mu R_2)=0.
$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\varphi,z,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(\varphi,z,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) &&\text{at}\quad z=0 &&\text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) &&\text{at}\quad z=l &&\text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) = {}& \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&-a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)G(r,\varphi,z,R_1,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&+a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)G(r,\varphi,z,R_2,\eta,\zeta,t-\tau)\,d\eta\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) = {}& \frac{e^{-kt/2}}{\pi(R_2^2-R_1^2)l}\left[\frac{\sin\bigl(t\sqrt{c}\bigr)}{\sqrt{c}}+2\sum_{s=1}^{\infty}\cos\left(\frac{s\pi z}{l}\right)\cos\left(\frac{s\pi\zeta}{l}\right)\frac{\sin\bigl(t\sqrt{\beta_s}\bigr)}{\sqrt{\beta_s}}\right]\\
&+\frac{e^{-kt/2}}{\pi l}\Biggl[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{\infty}\frac{A_nA_s\mu_{nm}^2Z_{nm}(r)Z_{nm}(\xi)}{(\mu_{nm}^2R_2^2-n^2)Z_{nm}^2(R_2)-(\mu_{nm}^2R_1^2-n^2)Z_{nm}^2(R_1)}\\
&\qquad\times\cos[n(\varphi-\eta)]\cos\left(\frac{s\pi z}{l}\right)\cos\left(\frac{s\pi\zeta}{l}\right)\frac{\sin\bigl(t\sqrt{\lambda_{nms}}\bigr)}{\sqrt{\lambda_{nms}}}\Biggr],
\end{aligned}
$$

where

$$
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\quad c=b-\tfrac14k^2,\quad \beta_s=\frac{a^2s^2\pi^2}{l^2}+b-\tfrac14k^2,\quad \lambda_{nms}=a^2\mu_{nm}^2+\frac{a^2s^2\pi^2}{l^2}+b-\tfrac14k^2,
$$

$$
Z_{nm}(r)=J_n'(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n'(\mu_{nm}R_1)J_n(\mu_{nm}r);
$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$J_n'(\mu R_1)Y_n'(\mu R_2) - Y_n'(\mu R_1)J_n'(\mu R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le l$. Third boundary value problem.**

A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_r w - s_1 w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
\partial_r w + s_2 w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
\partial_z w - s_3 w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w + s_4 w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}$$

The solution $w(r,\varphi,z,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi}\exp\bigl(-\tfrac{1}{2}kt\bigr)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{A_n\mu_{nm}^2}{\|h_p\|^2\sqrt{a^2\mu_{nm}^2 + a^2\lambda_p^2 + b - k^2/4}}$$
$$\times\frac{Z_{nm}(r)Z_{nm}(\xi)\cos[n(\varphi-\eta)]h_p(z)h_p(\zeta)\sin\bigl(t\sqrt{a^2\mu_{nm}^2 + a^2\lambda_p^2 + b - k^2/4}\,\bigr)}{(s_2^2R_2^2 + \mu_{nm}^2R_2^2 - n^2)Z_{nm}^2(R_2) - (s_1^2R_1^2 + \mu_{nm}^2R_1^2 - n^2)Z_{nm}^2(R_1)}.$$

Here,

$$Z_{nm}(r) = \bigl[\mu_{nm}J_n'(\mu_{nm}R_1) - s_1J_n(\mu_{nm}R_1)\bigr]Y_n(\mu_{nm}r) - \bigl[\mu_{nm}Y_n'(\mu_{nm}R_1) - s_1Y_n(\mu_{nm}R_1)\bigr]J_n(\mu_{nm}r),$$

$$A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases} \qquad h_p(z) = \cos(\lambda_p z) + \frac{s_3}{\lambda_p}\sin(\lambda_p z),$$

$$\|h_p\|^2 = \frac{s_4}{2\lambda_p^2}\frac{\lambda_p^2 + s_3^2}{\lambda_p^2 + s_4^2} + \frac{s_3}{2\lambda_p^2} + \frac{l}{2}\left(1 + \frac{s_3^2}{\lambda_p^2}\right),$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, the μ_{nm} are positive roots of the transcendental equation

$$\bigl[\mu J_n'(\mu R_1) - s_1J_n(\mu R_1)\bigr]\bigl[\mu Y_n'(\mu R_2) + s_2Y_n(\mu R_2)\bigr] = \bigl[\mu Y_n'(\mu R_1) - s_1Y_n(\mu R_1)\bigr]\bigl[\mu J_n'(\mu R_2) + s_2J_n(\mu R_2)\bigr],$$

and the λ_p are positive roots of the transcendental equation $\dfrac{\tan(\lambda l)}{\lambda} = \dfrac{s_3 + s_4}{\lambda^2 - s_3s_4}$.

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le l$. Mixed boundary value problems.**

1°. A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(\varphi,z,t) && \text{at} \quad r=R_2 && \text{(boundary condition)},\\
\partial_z w &= g_3(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+a^2R_1\int_0^t\int_0^l\int_0^{2\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1} d\eta\,d\zeta\,d\tau\\
&-a^2R_2\int_0^t\int_0^l\int_0^{2\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2} d\eta\,d\zeta\,d\tau\\
&-a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_3(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+a^2\int_0^t\int_0^{2\pi}\int_{R_1}^{R_2} g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\int_0^l\int_0^{2\pi}\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{\pi}{4l}\exp\bigl(-\tfrac12 kt\bigr)\Biggl[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{\infty}\frac{A_nA_s\mu_{nm}^2J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_1)-J_n^2(\mu_{nm}R_2)}Z_{nm}(r)\\
&\qquad\times Z_{nm}(\xi)\cos[n(\varphi-\eta)]\cos\Bigl(\frac{s\pi z}{l}\Bigr)\cos\Bigl(\frac{s\pi\zeta}{l}\Bigr)\frac{\sin\bigl(t\sqrt{\lambda_{nms}}\,\bigr)}{\sqrt{\lambda_{nms}}}\Biggr],
\end{aligned}
$$

$$
A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}\qquad \lambda_{nms}=a^2\mu_{nm}^2+\frac{a^2s^2\pi^2}{l^2}+b-\tfrac14k^2,
$$

$$
Z_{nm}(r)=J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)-Y_n(\mu_{nm}R_1)J_n(\mu_{nm}r),
$$

where the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$
J_n(\mu R_1)Y_n(\mu R_2)-Y_n(\mu R_1)J_n(\mu R_2)=0.
$$

2°. A hollow circular cylinder of finite length is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_t w &= f_1(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_r w &= g_1(\varphi, z, t) && \text{at} \quad r = R_1 && \text{(boundary condition)},\\
\partial_r w &= g_2(\varphi, z, t) && \text{at} \quad r = R_2 && \text{(boundary condition)},\\
w &= g_3(r, \varphi, t) && \text{at} \quad z = 0 && \text{(boundary condition)},\\
w &= g_4(r, \varphi, t) && \text{at} \quad z = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r, \varphi, z, t) &= \frac{\partial}{\partial t} \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} f_0(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\
&+ \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} \big[f_1(\xi, \eta, \zeta) + k f_0(\xi, \eta, \zeta)\big] G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\
&- a^2 R_1 \int_0^t \int_0^l \int_0^{2\pi} g_1(\eta, \zeta, \tau) G(r, \varphi, z, R_1, \eta, \zeta, t - \tau) \, d\eta \, d\zeta \, d\tau \\
&+ a^2 R_2 \int_0^t \int_0^l \int_0^{2\pi} g_2(\eta, \zeta, \tau) G(r, \varphi, z, R_2, \eta, \zeta, t - \tau) \, d\eta \, d\zeta \, d\tau \\
&+ a^2 \int_0^t \int_0^{2\pi} \int_{R_1}^{R_2} g_3(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\zeta=0} \xi \, d\xi \, d\eta \, d\tau \\
&- a^2 \int_0^t \int_0^{2\pi} \int_{R_1}^{R_2} g_4(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\zeta=l} \xi \, d\xi \, d\eta \, d\tau \\
&+ \int_0^t \int_0^l \int_0^{2\pi} \int_{R_1}^{R_2} \Phi(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \xi \, d\xi \, d\eta \, d\zeta \, d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r, \varphi, z, \xi, \eta, \zeta, t) &= \frac{2e^{-kt/2}}{\pi(R_2^2 - R_1^2) l} \sum_{s=1}^{\infty} \sin\left(\frac{s\pi z}{l}\right) \sin\left(\frac{s\pi \zeta}{l}\right) \frac{\sin\bigl(t\sqrt{\beta_s}\,\bigr)}{\sqrt{\beta_s}} \\
&+ \frac{2e^{-kt/2}}{\pi l} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{A_n \mu_{nm}^2 Z_{nm}(r) Z_{nm}(\xi)}{(\mu_{nm}^2 R_2^2 - n^2) Z_{nm}^2(R_2) - (\mu_{nm}^2 R_1^2 - n^2) Z_{nm}^2(R_1)} \\
&\qquad \times \cos[n(\varphi - \eta)] \sin\left(\frac{s\pi z}{l}\right) \sin\left(\frac{s\pi \zeta}{l}\right) \frac{\sin\bigl(t\sqrt{\lambda_{nms}}\,\bigr)}{\sqrt{\lambda_{nms}}},
\end{aligned}
$$

where

$$
A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0, \end{cases} \quad \beta_s = \frac{a^2 s^2 \pi^2}{l^2} + b - \tfrac{1}{4} k^2, \quad \lambda_{nms} = a^2 \mu_{nm}^2 + \frac{a^2 s^2 \pi^2}{l^2} + b - \tfrac{1}{4} k^2,
$$

$$
Z_{nm}(r) = J_n'(\mu_{nm} R_1) Y_n(\mu_{nm} r) - Y_n'(\mu_{nm} R_1) J_n(\mu_{nm} r);
$$

the $J_n(r)$ and $Y_n(r)$ are Bessel functions, and the μ_{nm} are positive roots of the transcendental equation

$$
J_n'(\mu R_1) Y_n'(\mu R_2) - Y_n'(\mu R_1) J_n'(\mu R_2) = 0.
$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. First boundary value problem.**

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
\partial_t w &= f_1(r, \varphi, z) && \text{at} \quad t = 0 && \text{(initial condition)},\\
w &= g_1(\varphi, z, t) && \text{at} \quad r = R && \text{(boundary condition)},\\
w &= g_2(r, z, t) && \text{at} \quad \varphi = 0 && \text{(boundary condition)},\\
w &= g_3(r, z, t) && \text{at} \quad \varphi = \varphi_0 && \text{(boundary condition)},\\
w &= g_4(r, \varphi, t) && \text{at} \quad z = 0 && \text{(boundary condition)},\\
w &= g_5(r, \varphi, t) && \text{at} \quad z = l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r, \varphi, z, t) = {} & \frac{\partial}{\partial t} \int_0^l \int_0^{\varphi_0} \int_0^R f_0(\xi, \eta, \zeta) G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\
& + \int_0^l \int_0^{\varphi_0} \int_0^R \bigl[f_1(\xi, \eta, \zeta) + k f_0(\xi, \eta, \zeta)\bigr] G(r, \varphi, z, \xi, \eta, \zeta, t) \xi \, d\xi \, d\eta \, d\zeta \\
& - a^2 R \int_0^t \int_0^l \int_0^{\varphi_0} g_1(\eta, \zeta, \tau) \left[\frac{\partial}{\partial \xi} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\xi = R} d\eta \, d\zeta \, d\tau \\
& + a^2 \int_0^t \int_0^l \int_0^R g_2(\xi, \zeta, \tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\eta = 0} d\xi \, d\zeta \, d\tau \\
& - a^2 \int_0^t \int_0^l \int_0^R g_3(\xi, \zeta, \tau) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\eta = \varphi_0} d\xi \, d\zeta \, d\tau \\
& + a^2 \int_0^t \int_0^{\varphi_0} \int_0^R g_4(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\zeta = 0} \xi \, d\xi \, d\eta \, d\tau \\
& - a^2 \int_0^t \int_0^{\varphi_0} \int_0^R g_5(\xi, \eta, \tau) \left[\frac{\partial}{\partial \zeta} G(r, \varphi, z, \xi, \eta, \zeta, t - \tau)\right]_{\zeta = l} \xi \, d\xi \, d\eta \, d\tau \\
& + \int_0^t \int_0^l \int_0^{\varphi_0} \int_0^R \Phi(\xi, \eta, \zeta, \tau) G(r, \varphi, z, \xi, \eta, \zeta, t - \tau) \xi \, d\xi \, d\eta \, d\zeta \, d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r, \varphi, z, \xi, \eta, \zeta, t) = {} & \frac{8 e^{-kt/2}}{R^2 l \varphi_0} \Biggl[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{J_{n\pi/\varphi_0}(\mu_{nm} r) J_{n\pi/\varphi_0}(\mu_{nm} \xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm} R)]^2} \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \\
& \times \sin\left(\frac{n\pi\eta}{\varphi_0}\right) \sin\left(\frac{s\pi z}{l}\right) \sin\left(\frac{s\pi\zeta}{l}\right) \frac{\sin\bigl(t\sqrt{a^2\mu_{nm}^2 + a^2 s^2 \pi^2 l^{-2} + b - k^2/4}\,\bigr)}{\sqrt{a^2\mu_{nm}^2 + a^2 s^2 \pi^2 l^{-2} + b - k^2/4}} \Biggr],
\end{aligned}
$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le l$. Mixed boundary value problem.**

A cylindrical sector of finite thickness is considered. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\varphi,z) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\varphi,z,t) && \text{at} \quad r=R && \text{(boundary condition)},\\
w &= g_2(r,z,t) && \text{at} \quad \varphi=0 && \text{(boundary condition)},\\
w &= g_3(r,z,t) && \text{at} \quad \varphi=\varphi_0 && \text{(boundary condition)},\\
\partial_z w &= g_4(r,\varphi,t) && \text{at} \quad z=0 && \text{(boundary condition)},\\
\partial_z w &= g_5(r,\varphi,t) && \text{at} \quad z=l && \text{(boundary condition)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(r,\varphi,z,t) &= \frac{\partial}{\partial t}\int_0^l\!\int_0^{\varphi_0}\!\int_0^R f_0(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&+\int_0^l\!\int_0^{\varphi_0}\!\int_0^R \big[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\big]G(r,\varphi,z,\xi,\eta,\zeta,t)\xi\,d\xi\,d\eta\,d\zeta\\
&- a^2R\int_0^t\!\int_0^l\!\int_0^{\varphi_0} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R} d\eta\,d\zeta\,d\tau\\
&+ a^2\int_0^t\!\int_0^l\!\int_0^R g_2(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=0} d\xi\,d\zeta\,d\tau\\
&- a^2\int_0^t\!\int_0^l\!\int_0^R g_3(\xi,\zeta,\tau)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\right]_{\eta=\varphi_0} d\xi\,d\zeta\,d\tau\\
&- a^2\int_0^t\!\int_0^{\varphi_0}\!\int_0^R g_4(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,0,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+ a^2\int_0^t\!\int_0^{\varphi_0}\!\int_0^R g_5(\xi,\eta,\tau)G(r,\varphi,z,\xi,\eta,l,t-\tau)\xi\,d\xi\,d\eta\,d\tau\\
&+\int_0^t\!\int_0^l\!\int_0^{\varphi_0}\!\int_0^R \Phi(\xi,\eta,\zeta,\tau)G(r,\varphi,z,\xi,\eta,\zeta,t-\tau)\xi\,d\xi\,d\eta\,d\zeta\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(r,\varphi,z,\xi,\eta,\zeta,t) &= \frac{4e^{-kt/2}}{R^2 l\varphi_0}\Bigg[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{\infty}\frac{A_s J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2}\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\\
&\times\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\cos\left(\frac{s\pi z}{l}\right)\cos\left(\frac{s\pi\zeta}{l}\right)\frac{\sin\left(t\sqrt{a^2\mu_{nm}^2+a^2s^2\pi^2l^{-2}+b-k^2/4}\right)}{\sqrt{a^2\mu_{nm}^2+a^2s^2\pi^2l^{-2}+b-k^2/4}}\Bigg],
\end{aligned}
$$

where $A_0 = 1$ and $A_s = 2$ for $s \ge 1$; the $J_{n\pi/\varphi_0}(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

8.4.3 Problems in Spherical Coordinates

A three-dimensional nonhomogeneous telegraph equation in the spherical coordinate system is written as

$$\frac{\partial^2 w}{\partial t^2}+k\frac{\partial w}{\partial t}=a^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}\right]$$
$$-bw+\Phi(r,\theta,\varphi,t).$$

► **Domain: $0\le r\le R$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. First boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w&=f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ \partial_t w&=f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\ w&=g(\theta,\varphi,t) &&\text{at}\quad r=R &&\text{(boundary condition)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t)=&\frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^R f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^R\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &-a^2R^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R}\sin\eta\,d\eta\,d\zeta\,d\tau\\ &+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^R\Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau,\end{aligned}$$

where

$$\begin{aligned} G(r,\theta,\varphi,\xi,\eta,\zeta,t)&=\frac{1}{2\pi R^2\sqrt{r\xi}}\exp\bigl(-\tfrac{1}{2}kt\bigr)\Biggl[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n}A_sB_{nms}J_{n+1/2}(\lambda_{nm}r)\\ &\times J_{n+1/2}(\lambda_{nm}\xi)P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b-k^2/4}\bigr)}{\sqrt{a^2\lambda_{nm}^2+b-k^2/4}}\Biggr],\end{aligned}$$

$$A_s=\begin{cases}1 & \text{for } s=0,\\ 2 & \text{for } s\ne 0,\end{cases}\qquad B_{nms}=\frac{(2n+1)(n-s)!}{(n+s)!\bigl[J'_{n+1/2}(\lambda_{nm}R)\bigr]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^s(\mu)=(1-\mu^2)^{s/2}\frac{d^s}{d\mu^s}P_n(\mu),\qquad P_n(\mu)=\frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation $J_{n+1/2}(\lambda R)=0$.

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_r w &= g(\theta,\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(r,\theta,\varphi,t) =& \frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}\int_0^R f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+\int_0^{2\pi}\int_0^{\pi}\int_0^R \bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\ &+a^2R^2\int_0^t\int_0^{2\pi}\int_0^{\pi} g(\eta,\zeta,\tau)G(r,\theta,\varphi,R,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau\\ &+\int_0^t\int_0^{2\pi}\int_0^{\pi}\int_0^R \Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau, \end{aligned}$$

where

$$\begin{aligned} G(r,\theta,\varphi,\xi,\eta,\zeta,t) =& \frac{3e^{-kt/2}}{4\pi R^3}\frac{\sin\bigl(t\sqrt{c}\bigr)}{\sqrt{c}} + \frac{e^{-kt/2}}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n} A_s B_{nms} J_{n+1/2}(\lambda_{nm} r)\\ &\times J_{n+1/2}(\lambda_{nm}\xi)P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+c}\bigr)}{\sqrt{a^2\lambda_{nm}^2+c}}, \end{aligned}$$

$$A_s = \begin{cases} 1 & \text{if } s=0,\\ 2 & \text{if } s\neq 0,\end{cases}\quad B_{nms} = \frac{\lambda_{nm}^2(2n+1)(n-s)!}{(n+s)!\bigl[R^2\lambda_{nm}^2 - n(n+1)\bigr]\bigl[J_{n+1/2}(\lambda_{nm}R)\bigr]^2},\quad c = b - \tfrac14 k^2.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see the paragraph above), and the λ_{nm} are positive roots of the transcendental equation

$$2\lambda R J'_{n+1/2}(\lambda R) - J_{n+1/2}(\lambda R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical domain is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad \text{(initial condition)},\\ \partial_r w + sw &= g(\theta,\varphi,t) \quad \text{at} \quad r=R \quad \text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{e^{-kt/2}}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{l=0}^{n}A_l B_{nml}J_{n+1/2}(\lambda_{nm}r)J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^l(\cos\theta)P_n^l(\cos\eta)\cos[l(\varphi-\zeta)]\frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b-k^2/4}\,\bigr)}{\sqrt{a^2\lambda_{nm}^2+b-k^2/4}},$$

$$A_l=\begin{cases}1 & \text{if } l=0,\\ 2 & \text{if } l\neq 0,\end{cases}\qquad B_{nml}=\frac{\lambda_{nm}^2(2n+1)(n-l)!}{(n+l)!\bigl[R^2\lambda_{nm}^2+(sR+n)(sR-n-1)\bigr]\bigl[J_{n+1/2}(\lambda_{nm}R)\bigr]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^l(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda R J'_{n+1/2}(\lambda R)+\bigl(sR-\tfrac{1}{2}\bigr)J_{n+1/2}(\lambda R)=0.$$

► **Domain: $R_1\le r\le R_2$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$. First boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(r,\theta,\varphi) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_1(\theta,\varphi,t) && \text{at} \quad r=R_1 && \text{(boundary condition)},\\
w &= g_2(\theta,\varphi,t) && \text{at} \quad r=R_2 && \text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\theta,\varphi,t)={}&\frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\
&+\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2}\bigl[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\bigr]G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\
&+a^2R_1^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_1(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_1}\sin\eta\,d\eta\,d\zeta\,d\tau\\
&-a^2R_2^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_2(\eta,\zeta,\tau)\left[\frac{\partial}{\partial\xi}G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\right]_{\xi=R_2}\sin\eta\,d\eta\,d\zeta\,d\tau\\
&+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau,
\end{aligned}$$

where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t)=\frac{\pi e^{-kt/2}}{8\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n}A_sB_{nms}Z_{n+1/2}(\lambda_{nm}r)Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2+b-k^2/4}\,\bigr)}{\sqrt{a^2\lambda_{nm}^2+b-k^2/4}},$$

$$Z_{n+1/2}(\lambda_{nm}r)=J_{n+1/2}(\lambda_{nm}R_1)Y_{n+1/2}(\lambda_{nm}r)-Y_{n+1/2}(\lambda_{nm}R_1)J_{n+1/2}(\lambda_{nm}r),$$

$$A_s=\begin{cases}1 & \text{for } s=0,\\ 2 & \text{for } s\neq 0,\end{cases}\qquad B_{nms}=\frac{\lambda_{nm}(2n+1)(n-s)!\,J_{n+1/2}^2(\lambda_{nm}R_2)}{(n+s)!\bigl[J_{n+1/2}^2(\lambda_{nm}R_1)-J_{n+1/2}^2(\lambda_{nm}R_2)\bigr]}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^s(\mu) = (1-\mu^2)^{s/2}\frac{d^s}{d\mu^s}P_n(\mu), \qquad P_n(\mu) = \frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation $Z_{n+1/2}(\lambda R_2) = 0$.

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned}
w &= f_0(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(r,\theta,\varphi) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_r w &= g_1(\theta,\varphi,t) &&\text{at}\quad r=R_1 &&\text{(boundary condition)},\\
\partial_r w &= g_2(\theta,\varphi,t) &&\text{at}\quad r=R_2 &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(r,\theta,\varphi,t) &= \frac{\partial}{\partial t}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2} f_0(\xi,\eta,\zeta)G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\
&+\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2}\big[f_1(\xi,\eta,\zeta)+kf_0(\xi,\eta,\zeta)\big]G(r,\theta,\varphi,\xi,\eta,\zeta,t)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\\
&-a^2R_1^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_1(\eta,\zeta,\tau)G(r,\theta,\varphi,R_1,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau\\
&+a^2R_2^2\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi} g_2(\eta,\zeta,\tau)G(r,\theta,\varphi,R_2,\eta,\zeta,t-\tau)\sin\eta\,d\eta\,d\zeta\,d\tau\\
&+\int_0^t\!\!\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{R_1}^{R_2}\Phi(\xi,\eta,\zeta,\tau)G(r,\theta,\varphi,\xi,\eta,\zeta,t-\tau)\xi^2\sin\eta\,d\xi\,d\eta\,d\zeta\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(r,\theta,\varphi,\xi,\eta,\zeta,t) &= \frac{3e^{-kt/2}\sin\big(t\sqrt{c}\big)}{4\pi(R_2^3-R_1^3)\sqrt{c}} + \frac{e^{-kt/2}}{4\pi\sqrt{r\xi}}\Bigg[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n}\frac{A_s}{B_{nms}}Z_{n+1/2}(\lambda_{nm}r)\\
&\times Z_{n+1/2}(\lambda_{nm}\xi)P_n^s(\cos\theta)P_n^s(\cos\eta)\cos[s(\varphi-\zeta)]\frac{\sin\big(t\sqrt{a^2\lambda_{nm}^2+c}\big)}{\sqrt{a^2\lambda_{nm}^2+c}}\Bigg].
\end{aligned}$$

Here,

$$A_s = \begin{cases}1 & \text{for } s=0,\\ 2 & \text{for } s\ne 0,\end{cases}\qquad B_{nms} = \frac{(n+s)!}{(2n+1)(n-s)!}\int_{R_1}^{R_2} rZ_{n+1/2}^2(\lambda_{nm}r)\,dr,\qquad c = b-\tfrac14 k^2,$$

$$\begin{aligned}
Z_{n+1/2}(\lambda_{nm}r) &= \left[\lambda_{nm}J'_{n+1/2}(\lambda_{nm}R_1) - \frac{1}{2R_1}J_{n+1/2}(\lambda_{nm}R_1)\right]Y_{n+1/2}(\lambda_{nm}r)\\
&-\left[\lambda_{nm}Y'_{n+1/2}(\lambda_{nm}R_1) - \frac{1}{2R_1}Y_{n+1/2}(\lambda_{nm}R_1)\right]J_{n+1/2}(\lambda_{nm}r),
\end{aligned}$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions (see the paragraph above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2) - \frac{1}{2R_2} Z_{n+1/2}(\lambda R_2) = 0.$$

► **Domain: $R_1 \le r \le R_2,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical layer is considered. The following conditions are prescribed:

$$\begin{aligned} w &= f_0(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)},\\ \partial_t w &= f_1(r,\theta,\varphi) \quad \text{at} \quad t=0 \quad &&\text{(initial condition)},\\ \partial_r w - s_1 w &= g_1(\theta,\varphi,t) \quad \text{at} \quad r=R_1 \quad &&\text{(boundary condition)},\\ \partial_r w + s_2 w &= g_2(\theta,\varphi,t) \quad \text{at} \quad r=R_2 \quad &&\text{(boundary condition)}. \end{aligned}$$

The solution $w(r,\theta,\varphi,t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(r,\theta,\varphi,\xi,\eta,\zeta,t) = \frac{e^{-kt/2}}{4\pi\sqrt{r\xi}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{l=0}^{n} \frac{A_l}{B_{nml}} Z_{n+1/2}(\lambda_{nm} r) Z_{n+1/2}(\lambda_{nm}\xi) \\ \times P_n^l(\cos\theta) P_n^l(\cos\eta) \cos[l(\varphi-\zeta)] \frac{\sin\bigl(t\sqrt{a^2\lambda_{nm}^2 + c}\,\bigr)}{\sqrt{a^2\lambda_{nm}^2 + c}}.$$

Here,

$$A_l = \begin{cases} 1 & \text{for } l=0,\\ 2 & \text{for } l\ne 0,\end{cases} \quad B_{nml} = \frac{(n+l)!}{(2n+1)(n-l)!} \int_{R_1}^{R_2} r Z_{n+1/2}^2(\lambda_{nm} r)\,dr, \quad c = b - \tfrac{1}{4}k^2,$$

$$Z_{n+1/2}(\lambda r) = \left[\lambda J'_{n+1/2}(\lambda R_1) - \left(s_1 + \frac{1}{2R_1}\right) J_{n+1/2}(\lambda R_1)\right] Y_{n+1/2}(\lambda r) \\ - \left[\lambda Y'_{n+1/2}(\lambda R_1) - \left(s_1 + \frac{1}{2R_1}\right) Y_{n+1/2}(\lambda R_1)\right] J_{n+1/2}(\lambda r),$$

where the $J_{n+1/2}(r)$ and $Y_{n+1/2}(r)$ are Bessel functions, the $P_n^l(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation

$$\lambda Z'_{n+1/2}(\lambda R_2) + \left(s_2 - \frac{1}{2R_2}\right) Z_{n+1/2}(\lambda R_2) = 0.$$

8.5 Other Equations with Three Space Variables

8.5.1 Equations Containing Arbitrary Parameters

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\Big(ax^n \dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(by^m \dfrac{\partial w}{\partial y}\Big) + \dfrac{\partial}{\partial z}\Big(cz^k \dfrac{\partial w}{\partial z}\Big).$

This equation admits separable solutions. In addition, for $n \neq 2$, $m \neq 2$, and $k \neq 2$, there are particular solutions of the form

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right],$$

where $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi}, \qquad A = 2\left(\frac{1}{2-n} + \frac{1}{2-m} + \frac{1}{2-k}\right) - 1.$$

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a^2\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\Big) + b_1\dfrac{\partial w}{\partial x} + b_2\dfrac{\partial w}{\partial y} + b_3\dfrac{\partial w}{\partial z} + cw.$

The transformation

$$w(x,y,z,t) = u(x,y,z,\tau)\exp\left(-\frac{1}{2}kt - \frac{b_1x + b_2y + b_3z}{2a^2}\right), \quad \tau = at$$

leads to the equation in Section 8.3.1:

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \beta u, \qquad \beta = \frac{c}{a^2} + \frac{k^2}{4a^2} - \frac{1}{4a^4}\big(b_1^2 + b_2^2 + b_3^2\big).$$

8.5.2 Equation of the Form $\rho(x,y,z)\dfrac{\partial^2 w}{\partial t^2} = \mathrm{div}\big[a(x,y,z)\nabla w\big] - q(x,y,z)w + \Phi(x,y,z,t)$

Such equations are encountered when studying vibration of finite volumes. The equation is written using the notation

$$\mathrm{div}\big[a(\mathbf{r})\nabla w\big] = \frac{\partial}{\partial x}\left[a(\mathbf{r})\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[a(\mathbf{r})\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[a(\mathbf{r})\frac{\partial w}{\partial z}\right], \qquad \mathbf{r} = \{x, y, z\}.$$

The problems for the equation in question are considered below for the interior of a bounded domain V with smooth surface S. In what follows, it is assumed that $\rho(\mathbf{r}) > 0$, $a(\mathbf{r}) > 0$, and $q(\mathbf{r}) \geq 0$.

► **First boundary value problem.**

The solution of the equation in question with the initial conditions

$$w = f_0(\mathbf{r}) \quad \text{at} \quad t = 0, \qquad \partial_t w = f_1(\mathbf{r}) \quad \text{at} \quad t = 0 \tag{1}$$

and the nonhomogeneous boundary conditions of the first kind

$$w = g(\mathbf{r}, t) \quad \text{for} \quad \mathbf{r} \in S \tag{2}$$

can be written as the sum

$$\begin{aligned} w(\mathbf{r}, t) = {} & \frac{\partial}{\partial t} \int_V f_0(\boldsymbol{\xi}) \rho(\boldsymbol{\xi}) \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t)\, dV_\xi + \int_V f_1(\boldsymbol{\xi}) \rho(\boldsymbol{\xi}) \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t)\, dV_\xi \\ & - \int_0^t \int_S g(\boldsymbol{\xi}, \tau) a(\boldsymbol{\xi}) \left[\frac{\partial}{\partial N_\xi} \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t - \tau) \right] dS_\xi\, d\tau \\ & + \int_0^t \int_V \Phi(\boldsymbol{\xi}, \tau) \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t - \tau)\, dV_\xi\, d\tau. \end{aligned} \tag{3}$$

Here, the modified Green's function is expressed as

$$\begin{aligned} \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n} \|u_n\|^2} u_n(\mathbf{r}) u_n(\boldsymbol{\xi}) \sin\left(\sqrt{\lambda_n}\, t\right), \\ \|u_n\|^2 &= \int_V \rho(\mathbf{r}) u_n^2(\mathbf{r})\, dV, \qquad \boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}, \end{aligned} \tag{4}$$

where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation with homogeneous boundary conditions of the first kind

$$\operatorname{div}\left[a(\mathbf{r}) \nabla u\right] - q(\mathbf{r}) u + \lambda \rho(\mathbf{r}) u = 0, \tag{5}$$

$$u = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{6}$$

The integration in solution (3) is performed with respect to ξ_1, ξ_2, and ξ_3; $\frac{\partial}{\partial N_\xi}$ is the derivative along the outward normal to the surface S with respect to ξ_1, ξ_2, and ξ_3.

General properties of the Sturm–Liouville problem (5)–(6):

1°. There are finitely many eigenvalues. All eigenvalues are real and can be ordered so that $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$; therefore the number of negative eigenvalues is finite.

2°. If $\rho(\mathbf{r}) > 0$, $a(\mathbf{r}) > 0$, and $q(\mathbf{r}) \ge 0$, then all eigenvalues are positive, $\lambda_n > 0$.

3°. An eigenfunction is determined up to a constant multiplier. Two eigenfunctions $u_n(\mathbf{r})$ and $u_m(\mathbf{r})$ corresponding to different eigenvalues λ_n and λ_m are orthogonal with weight $\rho(\mathbf{r})$ in the domain V, that is,

$$\int_V \rho(\mathbf{r}) u_n(\mathbf{r}) u_m(\mathbf{r})\, dV = 0 \quad \text{for} \quad n \ne m.$$

4°. An arbitrary function $F(\mathbf{r})$ twice continuously differentiable and satisfying the boundary condition of the Sturm–Liouville problem ($F = 0$ for $\mathbf{r} \in S$) can be expanded into an absolutely and uniformly convergent series in the eigenfunctions:

$$F(\mathbf{r}) = \sum_{n=1}^{\infty} F_n u_n(\mathbf{r}), \qquad F_n = \frac{1}{\|u_n\|^2} \int_V F(\mathbf{r}) \rho(\mathbf{r}) u_n(\mathbf{r})\, dV,$$

where $\|u_n\|^2$ is defined in (4).

Remark 8.1. In a three-dimensional problem, finitely many linearly independent eigenfunctions $u_n^{(1)}, \ldots, u_n^{(m)}$ generally correspond to each eigenvalue λ_n. These functions can always be replaced by their linear combinations

$$\bar{u}_n^{(k)} = A_{k,1}u_n^{(1)} + \cdots + A_{k,k-1}u_n^{(k-1)} + u_n^{(k)}, \qquad k = 1, \ldots, m,$$

so that $\bar{u}_n^{(1)}, \ldots, \bar{u}_n^{(m)}$ are now orthogonal pairwise. For this reason, without loss of generality, all eigenfunctions can be assumed orthogonal.

► **Second boundary value problem.**

The solution of the equation with the initial conditions (1) and nonhomogeneous boundary conditions of the second kind,

$$\frac{\partial w}{\partial N} = g(\mathbf{r}, t) \quad \text{for} \quad \mathbf{r} \in S,$$

can be represented as the sum

$$\begin{aligned} w(\mathbf{r}, t) &= \frac{\partial}{\partial t}\int_V f_0(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t)\, dV_\xi + \int_V f_1(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t)\, dV_\xi \\ &+ \int_0^t \int_S g(\boldsymbol{\xi}, \tau)a(\boldsymbol{\xi})\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t-\tau)\, dS_\xi\, d\tau \\ &+ \int_0^t \int_V \Phi(\boldsymbol{\xi}, \tau)\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}, t-\tau)\, dV_\xi\, d\tau. \end{aligned} \tag{7}$$

Here, the modified Green's function $\mathcal{G}$ is given by relation (4), the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and corresponding eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation (5) with homogeneous boundary conditions of the second kind,

$$\frac{\partial u}{\partial N} = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{8}$$

For $q(\mathbf{r}) > 0$, the general properties of the eigenvalue problem (5), (8) are the same as those of the first boundary value problem (all λ_n are positive).

► **Third boundary value problem.**

The solution of the equation with the initial conditions (1) and nonhomogeneous boundary conditions of the third kind,

$$\frac{\partial w}{\partial N} + k(\mathbf{r})w = g(\mathbf{r}, t) \quad \text{for} \quad \mathbf{r} \in S,$$

is determined by relations (7) and (4), where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation (5) with homogeneous boundary conditions of the third kind,

$$\frac{\partial u}{\partial N} + k(\mathbf{r})u = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{9}$$

If $q(\mathbf{r}) \geq 0$ and $k(\mathbf{r}) > 0$, the general properties of the eigenvalue problem (5), (9) are the same as those of the first boundary value problem (see the first paragraph of this section).

Suppose $k(\mathbf{r}) = k = \text{const}$. Denote the Green's functions of the second and third boundary value problems by $G_2(\mathbf{r}, \boldsymbol{\xi}, t)$ and $G_3(\mathbf{r}, \boldsymbol{\xi}, t, k)$, respectively. If $q(\mathbf{r}) > 0$, the limit relation $G_2(\mathbf{r}, \boldsymbol{\xi}, t) = \lim\limits_{k\to 0} G_3(\mathbf{r}, \boldsymbol{\xi}, t, k)$ holds.

⊙ *Literature for Section* 8.5.2: V. S. Vladimirov (1988), A. D. Polyanin (2000a).

8.6 Equations with n Space Variables

Throughout this section the following notation is used:

$$\Delta_n w = \sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2}, \quad \mathbf{x} = \{x_1, \dots, x_n\}, \quad \mathbf{y} = \{y_1, \dots, y_n\}, \quad |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

8.6.1 Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w$

► **Fundamental solution:**

$$\mathscr{E}(\mathbf{x}, t) = \begin{cases} \dfrac{(-1)^{\frac{n-2}{2}}}{2a\pi^{\frac{n+1}{2}}} \Gamma\left(\dfrac{n-1}{2}\right) \dfrac{\vartheta(at - |\mathbf{x}|)}{(a^2t^2 - |\mathbf{x}|^2)^{\frac{n-1}{2}}} & \text{if } n \geq 2 \text{ is even;} \\ \dfrac{1}{2\pi a}\left(\dfrac{1}{2\pi a^2 t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \delta(a^2t^2 - |\mathbf{x}|^2) & \text{if } n \geq 3 \text{ is odd;} \end{cases}$$

where $\vartheta(z)$ is the Heaviside unit step function and $\delta(z)$ is the Dirac delta function.

⊙ *Literature*: V. S. Vladimirov (1988).

► **Properties of solutions.**

Suppose $w(x_1, \dots, x_n, t)$ is a solution of the wave equation. Then the functions

$$\begin{aligned} w_1 &= Aw(\pm\lambda x_1 + C_1,\ \dots,\ \pm\lambda x_n + C_n,\ \pm\lambda t + C_{n+1}), \\ w_2 &= Aw\left(\frac{x_1 - vt}{\sqrt{1-(v/a)^2}},\ x_2,\ \dots,\ x_n,\ \frac{t - va^{-2}x_1}{\sqrt{1-(v/a)^2}}\right), \\ w_3 &= A|r^2 - a^2t^2|^{-\frac{n-1}{2}} w\left(\frac{x_1}{r^2 - a^2t^2}, \dots, \frac{x_n}{r^2 - a^2t^2}, \frac{t}{r^2 - a^2t^2}\right), \quad r = |\mathbf{x}|, \end{aligned}$$

are also solutions of this equation everywhere they are defined; A, C_1, …, C_{n+1}, v, and λ are arbitrary constants. The signs at λ in the expression of w_1 can be taken independently of one another.

▶ **Domain: $-\infty < x_k < \infty$; $k = 1, \dots, n$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(\mathbf{x}) \quad \text{at} \quad t = 0.$$

Solution:

$$w(\mathbf{x}, t) = \frac{1}{a^{n-1}(n-2)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^{at} \big(a^2t^2 - r^2\big)^{\frac{n-3}{2}} r T_r[f(\mathbf{x})]\, dr$$
$$+ \frac{1}{a^{n-1}(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^{at} \big(a^2t^2 - r^2\big)^{\frac{n-3}{2}} r T_r[g(\mathbf{x})]\, dr.$$

Here, $T_r[f(\mathbf{x})]$ is the average of f over the surface of the sphere of radius r with center at $\mathbf{x}$:

$$T_r[f(\mathbf{x})] \equiv \frac{1}{\sigma_n r^{n-1}} \int\limits_{|\mathbf{x}-\mathbf{y}|=r} f(\mathbf{y})\, dS_y, \qquad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where $\sigma_n r^{n-1}$ is the area of the surface of an n-dimensional sphere of radius r, dS_y is the area element of this surface, and $|\mathbf{x} - \mathbf{y}|^2 = (x_1 - y_1)^2 + \dots + (x_n - y_n)^2$.

For odd n, the solution can be alternatively represented as

$$w(\mathbf{x}, t) = \frac{1}{1 \times 3 \dots (n-2)} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \big(t^{n-2} T_{at}[f(\mathbf{x})]\big)$$
$$+ \frac{1}{1 \times 3 \dots (n-2)} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \big(t^{n-2} T_{at}[g(\mathbf{x})]\big).$$

For even n, the solution can be alternatively represented as

$$w(\mathbf{x}, t) = \frac{1}{2 \times 4 \dots (n-2) a^{n-1}} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \int_0^{at} T_r[f(\mathbf{x})] \frac{r^{n-1}\, dr}{\sqrt{a^2t^2 - r^2}}$$
$$+ \frac{1}{2 \times 4 \dots (n-2) a^{n-1}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \int_0^{at} T_r[g(\mathbf{x})] \frac{r^{n-1}\, dr}{\sqrt{a^2t^2 - r^2}}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), R. Courant and D. Hilbert (1989), D. Zwillinger (1998).

▶ **Domain: $0 \le x_k \le l_k$; $k = 1, \dots, n$. Boundary value problems.**

For solutions of the first, second, third, and mixed boundary value problems with nonhomogeneous conditions of general form, see Section 8.6.2 for $\Phi \equiv 0$.

8.6.2 Nonhomogeneous Wave Equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w + \Phi(x_1, \dots, x_n, t)$$

► **Domain: $-\infty < x_k < \infty$; $k = 1, \dots, n$. Cauchy problem.**

Initial conditions are prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(\mathbf{x}) \quad \text{at} \quad t = 0.$$

Solution:

$$\begin{aligned} w(\mathbf{x}, t) &= \frac{1}{a^{n-1}(n-2)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^{at} \left(a^2 t^2 - r^2\right)^{\frac{n-3}{2}} r T_r[f(\mathbf{x})]\, dr \\ &+ \frac{1}{a^{n-1}(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^{at} \left(a^2 t^2 - r^2\right)^{\frac{n-3}{2}} r T_r[g(\mathbf{x})]\, dr \\ &+ \frac{1}{a^{n-1}(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^{at} d\tau \int_0^{a\tau} \left(a^2 \tau^2 - r^2\right)^{\frac{n-3}{2}} r T_r[\Phi(\mathbf{x}, t-\tau)]\, dr. \end{aligned}$$

Here, $T_r[f(\mathbf{x})]$ is the average of f over the spherical surface of radius r with center at $\mathbf{x}$:

$$T_r[f(\mathbf{x})] \equiv \frac{1}{\sigma_n r^{n-1}} \int\limits_{|\mathbf{x}-\mathbf{y}|=r} f(\mathbf{y})\, dS_y, \qquad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where $\sigma_n r^{n-1}$ is the area of the surface of an n-dimensional sphere of radius r and dS_y is the area element of this surface.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), R. Courant and D. Hilbert (1989).

► **Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\ \dots,\ n\}$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)}, \\ w &= g_k(\mathbf{x}, t) && \text{at} \quad x_k = 0 && \text{(boundary conditions)}, \\ w &= h_k(\mathbf{x}, t) && \text{at} \quad x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(\mathbf{x}, t) &= \int_0^t \int_V \Phi(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t-\tau)\, dV_y\, d\tau \\ &+ \frac{\partial}{\partial t} \int_V f_0(\mathbf{y}) G(\mathbf{x}, \mathbf{y}, t)\, dV_y + \int_V f_1(\mathbf{y}) G(\mathbf{x}, \mathbf{y}, t)\, dV_y \\ &+ a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y}, \tau) \frac{\partial}{\partial y_k} G(\mathbf{x}, \mathbf{y}, t-\tau) \right]_{y_k=0} dS_y^{(k)}\, d\tau \\ &- a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[h_k(\mathbf{y}, \tau) \frac{\partial}{\partial y_k} G(\mathbf{x}, \mathbf{y}, t-\tau) \right]_{y_k=l_k} dS_y^{(k)}\, d\tau, \end{aligned}$$

where

$$\mathbf{x}=\{x_1,\dots,x_n\},\quad \mathbf{y}=\{y_1,\dots,y_n\},\quad dV_y=dy_1\,dy_2\dots dy_n,$$
$$dS_y^{(k)}=dy_1\dots dy_{k-1}\,dy_{k+1}\dots dy_n,$$
$$S^{(k)}=\{0\le y_m\le l_m \text{ for } m=1,\,\dots,\,k-1,\,k+1,\,\dots,\,n\}.$$

Green's function:

$$G(\mathbf{x},\mathbf{y},t)=\frac{2^n}{l_1l_2\dots l_n}\sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty}\dots\sum_{s_n=1}^{\infty}\sin(\lambda_{s_1}x_1)\sin(\lambda_{s_2}x_2)\dots\sin(\lambda_{s_n}x_n)$$
$$\times\sin(\lambda_{s_1}y_1)\sin(\lambda_{s_2}y_2)\dots\sin(\lambda_{s_n}y_n)\frac{\sin\bigl(at\sqrt{\lambda_{s_1}^2+\dots+\lambda_{s_n}^2}\,\bigr)}{a\sqrt{\lambda_{s_1}^2+\dots+\lambda_{s_n}^2}},$$

where

$$\lambda_{s_1}=\frac{s_1\pi}{l_1},\quad \lambda_{s_2}=\frac{s_2\pi}{l_2},\quad \dots,\quad \lambda_{s_n}=\frac{s_n\pi}{l_n}.$$

► **Domain: $V=\{0\le x_k\le l_k;\ k=1,\,\dots,\,n\}$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_{x_k} w&=g_k(\mathbf{x},t) &&\text{at}\quad x_k=0 &&\text{(boundary conditions)},\\
\partial_{x_k} w&=h_k(\mathbf{x},t) &&\text{at}\quad x_k=l_k &&\text{(boundary conditions)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(\mathbf{x},t)&=\int_0^t\int_V\Phi(\mathbf{y},\tau)G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau\\
&+\int_V f_0(\mathbf{y})G(\mathbf{x},\mathbf{y},t)\,dV_y+\int_V f_1(\mathbf{y})G(\mathbf{x},\mathbf{y},t)\,dV_y\\
&-a^2\sum_{k=1}^n\int_0^t\int_{S^{(k)}}\bigl[g_k(\mathbf{y},\tau)G(\mathbf{x},\mathbf{y},t-\tau)\bigr]_{y_k=0}\,dS_y^{(k)}\,d\tau\\
&+a^2\sum_{k=1}^n\int_0^t\int_{S^{(k)}}\bigl[h_k(\mathbf{y},\tau)G(\mathbf{x},\mathbf{y},t-\tau)\bigr]_{y_k=l_k}\,dS_y^{(k)}\,d\tau.
\end{aligned}$$

Here,

$$G(\mathbf{x},\mathbf{y},t)=\frac{t}{l_1l_2\dots l_n}+\frac{1}{l_1l_2\dots l_n}\Biggl[\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\dots\sum_{s_n=0}^{\infty}\frac{A_{s_1}A_{s_2}\dots A_{s_n}}{a\sqrt{\lambda_{s_1}^2+\dots+\lambda_{s_n}^2}}\sin\bigl(at\sqrt{\lambda_{s_1}^2+\dots+\lambda_{s_n}^2}\,\bigr)$$
$$\times\cos(\lambda_{s_1}x_1)\cos(\lambda_{s_2}x_2)\dots\cos(\lambda_{s_n}x_n)\cos(\lambda_{s_1}y_1)\cos(\lambda_{s_2}y_2)\dots\cos(\lambda_{s_n}y_n)\Biggr],$$

where

$$\lambda_{s_1}=\frac{s_1\pi}{l_1},\quad \lambda_{s_2}=\frac{s_2\pi}{l_2},\quad \dots,\quad \lambda_{s_n}=\frac{s_n\pi}{l_n};\quad A_{s_m}=\begin{cases}1 & \text{if}\quad s_m=0,\\ 2 & \text{if}\quad s_m\ne 0,\end{cases}\qquad m=1,\,2,\,\dots,\,n.$$

The summation is performed over the indices satisfying the condition $s_1 + \cdots + s_n > 0$; the term corresponding to $s_1 = \cdots = s_n = 0$ is singled out.

► **Domain: $V = \{0 \le x_k \le l_k;\ k = 1, \ldots, n\}$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_{x_k} w - b_k w &= g_k(\mathbf{x}, t) && \text{at} \quad x_k = 0 && \text{(boundary conditions)},\\ \partial_{x_k} w + c_k w &= h_k(\mathbf{x}, t) && \text{at} \quad x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

The solution $w(\mathbf{x}, t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}, t) = 2^n \sum_{s_1=1}^{\infty} \sum_{s_2=1}^{\infty} \cdots \sum_{s_n=1}^{\infty} & \frac{\sin\bigl(at\sqrt{\lambda_{s_1}^2 + \lambda_{s_2}^2 + \cdots + \lambda_{s_n}^2}\,\bigr)}{aE_{s_1}E_{s_2}\ldots E_{s_n}\sqrt{\lambda_{s_1}^2 + \lambda_{s_2}^2 + \cdots + \lambda_{s_n}^2}}\\ & \times \sin(\lambda_{s_1}x_1 + \varphi_{s_1})\sin(\lambda_{s_2}x_2 + \varphi_{s_2})\ldots\sin(\lambda_{s_n}x_n + \varphi_{s_n})\\ & \times \sin(\lambda_{s_1}y_1 + \varphi_{s_1})\sin(\lambda_{s_2}y_2 + \varphi_{s_2})\ldots\sin(\lambda_{s_n}y_n + \varphi_{s_n}). \end{aligned}$$

Here,

$$\varphi_{s_m} = \arctan\frac{\lambda_{s_m}}{l_m}, \quad E_{s_m} = l_m + \frac{(b_m c_m + \lambda_{s_m}^2)(b_m + c_m)}{(b_m^2 + \lambda_{s_m}^2)(c_m^2 + \lambda_{s_m}^2)}, \qquad m = 1,\ 2,\ \ldots,\ n;$$

the λ_{s_m} are positive roots of the transcendental equations

$$\frac{1}{b_m + c_m}\left(\lambda - \frac{b_m c_m}{\lambda}\right) = \cot(l_m\lambda), \qquad m = 1,\ 2,\ \ldots,\ n.$$

► **Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\ \ldots,\ n\}$. Mixed boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= f_1(\mathbf{x}) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= g_k(\mathbf{x}, t) && \text{at} \quad x_k = 0 && \text{(boundary conditions)},\\ \partial_{x_k} w &= h_k(\mathbf{x}, t) && \text{at} \quad x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(\mathbf{x}, t) = & \int_0^t \int_V \Phi(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t - \tau)\, dV_y\, d\tau\\ & + \frac{\partial}{\partial t}\int_V f_0(\mathbf{y}) G(\mathbf{x}, \mathbf{y}, t)\, dV_y + \int_V f_1(\mathbf{y}) G(\mathbf{x}, \mathbf{y}, t)\, dV_y\\ & + a^2 \sum_{k=1}^{n} \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y}, \tau)\frac{\partial}{\partial y_k} G(\mathbf{x}, \mathbf{y}, t - \tau)\right]_{y_k=0} dS_y^{(k)}\, d\tau\\ & + a^2 \sum_{k=1}^{n} \int_0^t \int_{S^{(k)}} \Bigl[h_k(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t - \tau)\Bigr]_{y_k=l_k} dS_y^{(k)}\, d\tau, \end{aligned}$$

where

$$G(\mathbf{x},\mathbf{y},t)=\frac{2^n}{l_1 l_2\dots l_n}\sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty}\dots\sum_{s_n=1}^{\infty}\sin(\lambda_{s_1}x_1)\sin(\lambda_{s_2}x_2)\dots\sin(\lambda_{s_n}x_n)$$

$$\times\sin(\lambda_{s_1}y_1)\sin(\lambda_{s_2}y_2)\dots\sin(\lambda_{s_n}y_n)\frac{\sin\left(at\sqrt{\lambda_{s_1}^2+\dots+\lambda_{s_n}^2}\,\right)}{a\sqrt{\lambda_{s_1}^2+\dots+\lambda_{s_n}^2}},$$

$$\lambda_{s_1}=\frac{\pi(2s_1+1)}{2l_1},\quad \lambda_{s_2}=\frac{\pi(2s_2+1)}{2l_2},\quad \dots,\quad \lambda_{s_n}=\frac{\pi(2s_n+1)}{2l_n}.$$

8.6.3 Equations of the Form $\dfrac{\partial^2 w}{\partial t^2}=a^2\Delta_n w-bw+\Phi(x_1,\dots,x_n,t)$

▶ **Domain: $-\infty<x_k<\infty$; $k=1,\dots,n$. Cauchy problem.**

Initial conditions are prescribed:

$$w=f(\mathbf{x})\quad\text{at}\quad t=0,$$
$$\partial_t w=g(\mathbf{x})\quad\text{at}\quad t=0,$$

where $\mathbf{x}=\{x_1,\dots,x_n\}$.

1°. Let $b=-c^2<0$ and $\Phi\equiv 0$. The solution is sought by the descent method in the form

$$w(\mathbf{x},t)=\frac{1}{\exp(cx_{n+1})}u(\mathbf{x},x_{n+1},t),\tag{1}$$

where u is the solution of the Cauchy problem for the auxiliary $(n+1)$-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}=\Delta_{n+1}u\tag{2}$$

with the initial conditions

$$\begin{aligned}u&=\exp(cx_{n+1})f(\mathbf{x})\quad\text{at}\quad t=0,\\ \partial_t u&=\exp(cx_{n+1})g(\mathbf{x})\quad\text{at}\quad t=0.\end{aligned}\tag{3}$$

For the solution of the Cauchy problem (2), (3), see Section 8.6.1.

2°. Let $b=c^2>0$ and $\Phi\equiv 0$. In this case the function $\exp(cx_{n+1})$ in (1) and (3) must be replaced by $\cos(cx_{n+1})$.

⊙ *Literature*: R. Courant and D. Hilbert (1989).

▶ **Domain: $V=\{0\le x_k\le l_k;\ k=1,\ \dots,\ n\}$. First boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}w&=f_0(\mathbf{x})&&\text{at}\quad t=0&&\text{(initial condition)},\\ \partial_t w&=f_1(\mathbf{x})&&\text{at}\quad t=0&&\text{(initial condition)},\\ w&=g_k(\mathbf{x},t)&&\text{at}\quad x_k=0&&\text{(boundary conditions)},\\ w&=h_k(\mathbf{x},t)&&\text{at}\quad x_k=l_k&&\text{(boundary conditions)}.\end{aligned}$$

Solution:

$$\begin{aligned} w(\mathbf{x},t) &= \int_0^t \int_V \Phi(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau \\ &\quad + \frac{\partial}{\partial t}\int_V f_0(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V f_1(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y \\ &\quad + a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau) \right]_{y_k=0} dS_y^{(k)}\,d\tau \\ &\quad - a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[h_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau) \right]_{y_k=l_k} dS_y^{(k)}\,d\tau, \end{aligned}$$

where

$$\mathbf{x} = \{x_1,\dots,x_n\}, \quad \mathbf{y} = \{y_1,\dots,y_n\}, \quad dV_y = dy_1\,dy_2\dots dy_n,$$
$$dS_y^{(k)} = dy_1\dots dy_{k-1}\,dy_{k+1}\dots dy_n,$$
$$S^{(k)} = \{0 \le y_m \le l_m \text{ for } m = 1,\,\dots,\,k-1,\,k+1,\,\dots,\,n\}.$$

Green's function:

$$\begin{aligned} G(\mathbf{x},\mathbf{y},t) &= \frac{2^n}{l_1 l_2 \dots l_n} \sum_{s_1=1}^\infty \sum_{s_2=1}^\infty \dots \sum_{s_n=1}^\infty \sin(\lambda_{s_1}x_1)\sin(\lambda_{s_2}x_2)\dots\sin(\lambda_{s_n}x_n) \\ &\quad \times \sin(\lambda_{s_1}y_1)\sin(\lambda_{s_2}y_2)\dots\sin(\lambda_{s_n}y_n) \frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b}\,\bigr)}{\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b}}, \end{aligned}$$

where

$$\lambda_{s_1} = \frac{s_1\pi}{l_1}, \quad \lambda_{s_2} = \frac{s_2\pi}{l_2}, \quad \dots, \quad \lambda_{s_n} = \frac{s_n\pi}{l_n}.$$

► **Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\,\dots,\,n\}$. Second boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(\mathbf{x}) && \text{at } t = 0 && \text{(initial condition)}, \\ \partial_t w &= f_1(\mathbf{x}) && \text{at } t = 0 && \text{(initial condition)}, \\ \partial_{x_k} w &= g_k(\mathbf{x},t) && \text{at } x_k = 0 && \text{(boundary conditions)}, \\ \partial_{x_k} w &= h_k(\mathbf{x},t) && \text{at } x_k = l_k && \text{(boundary conditions)}. \end{aligned}$$

Solution:

$$\begin{aligned} w(\mathbf{x},t) &= \int_0^t \int_V \Phi(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau \\ &\quad + \int_V f_0(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V f_1(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y \\ &\quad - a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \bigl[g_k(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau) \bigr]_{y_k=0}\, dS_y^{(k)}\,d\tau \\ &\quad + a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \bigl[h_k(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau) \bigr]_{y_k=l_k}\, dS_y^{(k)}\,d\tau. \end{aligned}$$

Here,

$$G(\mathbf{x},\mathbf{y},t)=\frac{1}{l_1l_2\dots l_n}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\dots\sum_{s_n=0}^{\infty}A_{s_1}A_{s_2}\dots A_{s_n}\cos(\lambda_{s_1}x_1)\cos(\lambda_{s_2}x_2)\dots\cos(\lambda_{s_n}x_n)$$
$$\times\cos(\lambda_{s_1}y_1)\cos(\lambda_{s_2}y_2)\dots\cos(\lambda_{s_n}y_n)\frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b}\bigr)}{\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b}},$$

where

$$\lambda_{s_1}=\frac{s_1\pi}{l_1},\quad \lambda_{s_2}=\frac{s_2\pi}{l_2},\quad \dots,\quad \lambda_{s_n}=\frac{s_n\pi}{l_n};\quad A_{s_m}=\begin{cases}1 & \text{for } s_m=0,\\ 2 & \text{for } s_m\neq 0,\end{cases}\qquad m=1,\,2,\,\dots,\,n.$$

► **Domain: $V=\{0\le x_k\le l_k;\ k=1,\,\dots,\,n\}$. Third boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_{x_k}w-b_kw&=g_k(\mathbf{x},t) &&\text{at}\quad x_k=0 &&\text{(boundary conditions)},\\
\partial_{x_k}w+c_kw&=h_k(\mathbf{x},t) &&\text{at}\quad x_k=l_k &&\text{(boundary conditions)}.
\end{aligned}$$

The solution $w(\mathbf{x},t)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(\mathbf{x},\mathbf{y},t)=2^n\sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty}\dots\sum_{s_n=1}^{\infty}\frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\lambda_{s_2}^2+\dots+\lambda_{s_n}^2)+b}\bigr)}{E_{s_1}E_{s_2}\dots E_{s_n}\sqrt{a^2(\lambda_{s_1}^2+\lambda_{s_2}^2+\dots+\lambda_{s_n}^2)+b}}$$
$$\times\sin(\lambda_{s_1}x_1+\varphi_{s_1})\sin(\lambda_{s_2}x_2+\varphi_{s_2})\dots\sin(\lambda_{s_n}x_n+\varphi_{s_n})$$
$$\times\sin(\lambda_{s_1}y_1+\varphi_{s_1})\sin(\lambda_{s_2}y_2+\varphi_{s_2})\dots\sin(\lambda_{s_n}y_n+\varphi_{s_n}).$$

Here,

$$\varphi_{s_m}=\arctan\frac{\lambda_{s_m}}{l_m},\quad E_{s_m}=l_m+\frac{(b_mc_m+\lambda_{s_m}^2)(b_m+c_m)}{(b_m^2+\lambda_{s_m}^2)(c_m^2+\lambda_{s_m}^2)},\qquad m=1,\,2,\,\dots,\,n;$$

the λ_{s_m} are positive roots of the transcendental equations

$$\frac{1}{b_m+c_m}\left(\lambda-\frac{b_mc_m}{\lambda}\right)=\cot(l_m\lambda),\qquad m=1,\,2,\,\dots,\,n.$$

► **Domain: $V=\{0\le x_k\le l_k;\ k=1,\,\dots,\,n\}$. Mixed boundary value problem.**

The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_k(\mathbf{x},t) &&\text{at}\quad x_k=0 &&\text{(boundary conditions)},\\
\partial_{x_k}w&=h_k(\mathbf{x},t) &&\text{at}\quad x_k=l_k &&\text{(boundary conditions)}.
\end{aligned}$$

Solution:

$$
\begin{aligned}
w(\mathbf{x},t) &= \int_0^t \int_V \Phi(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau \\
&+ \frac{\partial}{\partial t}\int_V f_0(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V f_1(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y \\
&+ a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y},\tau) \frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau)\right]_{y_k=0} dS_y^{(k)}\,d\tau \\
&+ a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \Bigl[h_k(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\Bigr]_{y_k=l_k} dS_y^{(k)}\,d\tau.
\end{aligned}
$$

Here,

$$
\begin{aligned}
G(\mathbf{x},\mathbf{y},t) = \frac{2^n}{l_1 l_2 \dots l_n} \sum_{s_1=1}^\infty \sum_{s_2=1}^\infty \dots \sum_{s_n=1}^\infty \sin(\lambda_{s_1}x_1)\sin(\lambda_{s_2}x_2)\dots\sin(\lambda_{s_n}x_n) \\
\times \sin(\lambda_{s_1}y_1)\sin(\lambda_{s_2}y_2)\dots\sin(\lambda_{s_n}y_n) \frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b}\,\bigr)}{\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b}},
\end{aligned}
$$

where

$$
\lambda_{s_1} = \frac{\pi(2s_1+1)}{2l_1}, \quad \lambda_{s_2} = \frac{\pi(2s_2+1)}{2l_2}, \quad \dots, \quad \lambda_{s_n} = \frac{\pi(2s_n+1)}{2l_n}.
$$

8.6.4 Equations Containing the First Time Derivative

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} + \beta\frac{\partial w}{\partial t} = a^2\Delta_n w - bw + \Phi(x_1,\dots,x_n,t).$

Nonhomogeneous telegraph equation with n space variables.

1°. The substitution $w = \exp\bigl(-\frac{1}{2}\beta t\bigr)u$ leads to the equation

$$
\frac{\partial^2 u}{\partial t^2} = a^2\Delta_n u - \bigl(b - \tfrac{1}{4}\beta^2\bigr)u + \exp\bigl(\tfrac{1}{2}\beta t\bigr)\Phi(x_1,\dots,x_n,t),
$$

which is considered in Section 8.6.3.

2°. Domain: $V=\{0\le x_k \le l_k;\ k=1,\,\dots,\,n\}$. First boundary value problem. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(\mathbf{x}) && \text{at} \quad t=0 && \text{(initial condition)},\\
\partial_t w &= f_1(\mathbf{x}) && \text{at} \quad t=0 && \text{(initial condition)},\\
w &= g_k(\mathbf{x},t) && \text{at} \quad x_k=0 && \text{(boundary conditions)},\\
w &= h_k(\mathbf{x},t) && \text{at} \quad x_k=l_k && \text{(boundary conditions)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(\mathbf{x},t) &= \int_0^t \int_V \Phi(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau \\
&\quad + \frac{\partial}{\partial t}\int_V f_0(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V \bigl[f_1(\mathbf{y}) + \beta f_0(\mathbf{y})\bigr] G(\mathbf{x},\mathbf{y},t)\,dV_y \\
&\quad + a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[g_k(\mathbf{y},\tau)\frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau)\right]_{y_k=0} dS_y^{(k)}\,d\tau \\
&\quad - a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \left[h_k(\mathbf{y},\tau)\frac{\partial}{\partial y_k} G(\mathbf{x},\mathbf{y},t-\tau)\right]_{y_k=l_k} dS_y^{(k)}\,d\tau,
\end{aligned}
$$

where

$$
\mathbf{x} = \{x_1,\dots,x_n\}, \quad \mathbf{y} = \{y_1,\dots,y_n\}, \quad dV_y = dy_1\,dy_2\dots dy_n,
$$
$$
dS_y^{(k)} = dy_1\dots dy_{k-1}\,dy_{k+1}\dots dy_n,
$$
$$
S^{(k)} = \{0 \le y_m \le l_m \ \text{ for } \ m = 1,\,\dots,\,k-1,\,k+1,\,\dots,\,n\}.
$$

Green's function:

$$
\begin{aligned}
G(\mathbf{x},\mathbf{y},t) &= \frac{2^n e^{-\beta t/2}}{l_1 l_2 \dots l_n} \sum_{s_1=1}^\infty \sum_{s_2=1}^\infty \dots \sum_{s_n=1}^\infty \sin(\lambda_{s_1}x_1)\sin(\lambda_{s_2}x_2)\dots\sin(\lambda_{s_n}x_n) \\
&\quad \times \sin(\lambda_{s_1}y_1)\sin(\lambda_{s_2}y_2)\dots\sin(\lambda_{s_n}y_n) \frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2 + \dots + \lambda_{s_n}^2) + b - \beta^2/4}\,\bigr)}{\sqrt{a^2(\lambda_{s_1}^2 + \dots + \lambda_{s_n}^2) + b - \beta^2/4}},
\end{aligned}
$$

where

$$
\lambda_{s_1} = \frac{s_1\pi}{l_1}, \quad \lambda_{s_2} = \frac{s_2\pi}{l_2}, \quad \dots, \quad \lambda_{s_n} = \frac{s_n\pi}{l_n}.
$$

3°. Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\,\dots,\,n\}$. Second boundary value problem. The following conditions are prescribed:

$$
\begin{aligned}
w &= f_0(\mathbf{x}) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
\partial_t w &= f_1(\mathbf{x}) &&\text{at} \quad t=0 &&\text{(initial condition)},\\
\partial_{x_k} w &= g_k(\mathbf{x},t) &&\text{at} \quad x_k=0 &&\text{(boundary conditions)},\\
\partial_{x_k} w &= h_k(\mathbf{x},t) &&\text{at} \quad x_k=l_k &&\text{(boundary conditions)}.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(\mathbf{x},t) &= \int_0^t \int_V \Phi(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau \\
&\quad + \int_V f_0(\mathbf{y}) G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V \bigl[f_1(\mathbf{y}) + \beta f_0(\mathbf{y})\bigr] G(\mathbf{x},\mathbf{y},t)\,dV_y \\
&\quad - a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \bigl[g_k(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\bigr]_{y_k=0}\,dS_y^{(k)}\,d\tau \\
&\quad + a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \bigl[h_k(\mathbf{y},\tau) G(\mathbf{x},\mathbf{y},t-\tau)\bigr]_{y_k=l_k} dS_y^{(k)}\,d\tau.
\end{aligned}
$$

Here,

$$G(\mathbf{x},\mathbf{y},t)=\frac{e^{-\beta t/2}}{l_1 l_2\dots l_n}\biggl[\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\dots\sum_{s_n=0}^{\infty}A_{s_1}A_{s_2}\dots A_{s_n}\cos(\lambda_{s_1}x_1)\cos(\lambda_{s_2}x_2)\dots\cos(\lambda_{s_n}x_n)$$
$$\times\cos(\lambda_{s_1}y_1)\cos(\lambda_{s_2}y_2)\dots\cos(\lambda_{s_n}y_n)\frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b-\beta^2/4}\bigr)}{\sqrt{a^2(\lambda_{s_1}^2+\dots+\lambda_{s_n}^2)+b-\beta^2/4}}\biggr],$$

where

$$\lambda_{s_1}=\frac{s_1\pi}{l_1},\quad \lambda_{s_2}=\frac{s_2\pi}{l_2},\quad \dots,\quad \lambda_{s_n}=\frac{s_n\pi}{l_n};\quad A_{s_m}=\begin{cases}1 & \text{for } s_m=0,\\ 2 & \text{for } s_m\neq 0,\end{cases}\qquad m=1,\,2,\,\dots,\,n.$$

4°. Domain: $V=\{0\le x_k\le l_k;\ k=1,\,\dots,\,n\}$. Third boundary value problem. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_{x_k}w-b_k w&=g_k(\mathbf{x},t) &&\text{at}\quad x_k=0 &&\text{(boundary conditions)},\\
\partial_{x_k}w+c_k w&=h_k(\mathbf{x},t) &&\text{at}\quad x_k=l_k &&\text{(boundary conditions)}.
\end{aligned}$$

The solution $w(\mathbf{x},t)$ is given by the formula in Item 3° with

$$G(\mathbf{x},\mathbf{y},t)=2^n e^{-\beta t/2}\sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty}\dots\sum_{s_n=1}^{\infty}\frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\lambda_{s_2}^2+\dots+\lambda_{s_n}^2)+b-\beta^2/4}\bigr)}{E_{s_1}E_{s_2}\dots E_{s_n}\sqrt{a^2(\lambda_{s_1}^2+\lambda_{s_2}^2+\dots+\lambda_{s_n}^2)+b-\beta^2/4}}$$
$$\times\sin(\lambda_{s_1}x_1+\varphi_{s_1})\sin(\lambda_{s_2}x_2+\varphi_{s_2})\dots\sin(\lambda_{s_n}x_n+\varphi_{s_n})$$
$$\times\sin(\lambda_{s_1}y_1+\varphi_{s_1})\sin(\lambda_{s_2}y_2+\varphi_{s_2})\dots\sin(\lambda_{s_n}y_n+\varphi_{s_n}).$$

Here,

$$\varphi_{s_m}=\arctan\frac{\lambda_{s_m}}{l_m},\quad E_{s_m}=l_m+\frac{(b_m c_m+\lambda_{s_m}^2)(b_m+c_m)}{(b_m^2+\lambda_{s_m}^2)(c_m^2+\lambda_{s_m}^2)},\qquad m=1,\,2,\,\dots,\,n;$$

the λ_{s_m} are positive roots of the transcendental equation

$$\frac{1}{b_m+c_m}\left(\lambda-\frac{b_m c_m}{\lambda}\right)=\cot(l_m\lambda),\qquad m=1,\,2,\,\dots,\,n.$$

5°. Domain: $V=\{0\le x_k\le l_k;\ k=1,\,\dots,\,n\}$. Mixed boundary value problem. The following conditions are prescribed:

$$\begin{aligned}
w&=f_0(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=f_1(\mathbf{x}) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
w&=g_k(\mathbf{x},t) &&\text{at}\quad x_k=0 &&\text{(boundary conditions)},\\
\partial_{x_k}w&=h_k(\mathbf{x},t) &&\text{at}\quad x_k=l_k &&\text{(boundary conditions)}.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(\mathbf{x},t) &= \int_0^t \int_V \Phi(\mathbf{y},\tau)G(\mathbf{x},\mathbf{y},t-\tau)\,dV_y\,d\tau \\
&\quad + \frac{\partial}{\partial t}\int_V f_0(\mathbf{y})G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V \bigl[f_1(\mathbf{y}) + \beta f_0(\mathbf{y})\bigr]G(\mathbf{x},\mathbf{y},t)\,dV_y \\
&\quad + a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \Bigl[g_k(\mathbf{y},\tau)\frac{\partial}{\partial y_k}G(\mathbf{x},\mathbf{y},t-\tau)\Bigr]_{y_k=0} dS_y^{(k)}\,d\tau \\
&\quad + a^2 \sum_{k=1}^n \int_0^t \int_{S^{(k)}} \Bigl[h_k(\mathbf{y},\tau)G(\mathbf{x},\mathbf{y},t-\tau)\Bigr]_{y_k=l_k} dS_y^{(k)}\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
G(\mathbf{x},\mathbf{y},t) &= \frac{2^n e^{-\beta t/2}}{l_1 l_2 \ldots l_n} \sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty}\ldots\sum_{s_n=1}^{\infty} \sin(\lambda_{s_1}x_1)\sin(\lambda_{s_2}x_2)\ldots\sin(\lambda_{s_n}x_n) \\
&\quad \times \sin(\lambda_{s_1}y_1)\sin(\lambda_{s_2}y_2)\ldots\sin(\lambda_{s_n}y_n)\frac{\sin\bigl(t\sqrt{a^2(\lambda_{s_1}^2+\cdots+\lambda_{s_n}^2)+b-\beta^2/4}\,\bigr)}{\sqrt{a^2(\lambda_{s_1}^2+\cdots+\lambda_{s_n}^2)+b-\beta^2/4}}, \\
\lambda_{s_1} &= \frac{\pi(2s_1+1)}{2l_1}, \quad \lambda_{s_2} = \frac{\pi(2s_2+1)}{2l_2}, \quad \ldots, \quad \lambda_{s_n} = \frac{\pi(2s_n+1)}{2l_n}.
\end{aligned}$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} + \beta\frac{\partial w}{\partial t} = a^2\Delta_n w + \sum_{k=1}^n b_k \frac{\partial w}{\partial x_k} + cw.$

The transformation

$$w(x_1,\ldots,x_n,t) = u(x_1,\ldots,x_n,\tau)\exp\biggl(-\frac{1}{2}\beta t - \frac{1}{2a^2}\sum_{k=1}^n b_k x_k\biggr), \qquad \tau = at$$

leads to the equation

$$\frac{\partial^2 u}{\partial \tau^2} = \Delta_n u + \lambda u, \qquad \lambda = \frac{c}{a^2} + \frac{\beta^2}{4a^2} - \frac{1}{4a^4}\sum_{k=1}^n b_k^2,$$

which is considered in Section 8.6.3.

⊙ *Literature*: R. Courant and D. Hilbert (1989).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} + \frac{n-1}{t}\frac{\partial w}{\partial t} = \Delta_n w.$

Darboux equation. Cauchy problem.
Initial conditions are prescribed:

$$\begin{aligned}
w &= f(\mathbf{x}) \quad \text{at} \quad t = 0, \\
\partial_t w &= 0 \quad\quad \text{at} \quad t = 0.
\end{aligned}$$

Solution:

$$w(\mathbf{x},t) = \frac{1}{\sigma_n t^{n-1}} \int\limits_{|\mathbf{x}-\mathbf{y}|=t} f(\mathbf{y})\,dS_y, \qquad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where dS_y is the area element of the surface of an n-dimensional sphere of radius t (i.e., the solution w is the average of the function f over the sphere for radius t with center at **x**).

⊙ *Literature*: R. Courant and D. Hilbert (1989).

Chapter 9

Second-Order Elliptic Equations with Two Space Variables

9.1 Laplace Equation $\Delta_2 w = 0$

The *Laplace equation* is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. For example, in heat and mass transfer theory, this equation describes steady-state temperature distribution in the absence of heat sources and sinks in the domain under study.

A regular solution of the Laplace equation is called a harmonic function. The first boundary value problem for the Laplace equation is often referred to as the *Dirichlet problem*, and the second boundary value problem as the *Neumann problem*.

Extremum principle: Given a domain D, a harmonic function w in D that is not identically constant in D cannot attain its maximum or minimum value at any interior point of D.

9.1.1 Problems in Cartesian Coordinate System

The Laplace equation with two space variables in the rectangular Cartesian system of coordinates is written as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

► **Particular solutions and a method for their construction.**

1°. Particular solutions:

$$w(x,y) = Ax + By + C,$$
$$w(x,y) = A(x^2 - y^2) + Bxy,$$
$$w(x,y) = A(x^3 - 3xy^2) + B(3x^2y - y^3),$$
$$w(x,y) = \frac{Ax + By}{x^2 + y^2} + C,$$

$$w(x,y) = \exp(\pm\mu x)(A\cos\mu y + B\sin\mu y),$$
$$w(x,y) = (A\cos\mu x + B\sin\mu x)\exp(\pm\mu y),$$
$$w(x,y) = (A\sinh\mu x + B\cosh\mu x)(C\cos\mu y + D\sin\mu y),$$
$$w(x,y) = (A\cos\mu x + B\sin\mu x)(C\sinh\mu y + D\cosh\mu y),$$
$$w(x,y) = A\ln\bigl[(x-x_0)^2 + (y-y_0)^2\bigr] + B,$$

where A, B, C, D, x_0, y_0, and μ are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}(x,y) = \frac{1}{2\pi}\ln\frac{1}{r}, \qquad r = \sqrt{x^2+y^2}.$$

3°. If $w(x,y)$ is a solution of the Laplace equation, then the functions

$$w_1 = Aw(\pm\lambda x + C_1, \pm\lambda y + C_2),$$
$$w_2 = Aw(x\cos\beta + y\sin\beta,\ -x\sin\beta + y\cos\beta),$$
$$w_3 = Aw\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right),$$

are also solutions everywhere they are defined; A, C_1, C_2, β, and λ are arbitrary constants. The signs at λ in w_1 are taken independently of each other.

4°. A fairly general method for constructing particular solutions involves the following. Let $f(z) = u(x,y) + iv(x,y)$ be any analytic function of the complex variable $z = x + iy$ (u and v are real functions of the real variables x and y; $i^2 = -1$). Then the real and imaginary parts of f both satisfy the two-dimensional Laplace equation,

$$\Delta_2 u = 0, \qquad \Delta_2 v = 0.$$

Recall that the *Cauchy–Riemann conditions*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

are necessary and sufficient conditions for the function f to be analytic. Thus, by specifying analytic functions $f(z)$ and taking their real and imaginary parts, one obtains various solutions of the two-dimensional Laplace equation.

⊙ *Literature*: M. A. Lavrent'ev and B. V. Shabat (1973), A. G. Sveshnikov and A. N. Tikhonov (1974), A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Specific features of stating boundary value problems for the Laplace equation.**

1°. For outer boundary value problems on the plane, it is (usually) required to set the additional condition that the solution of the Laplace equation must be bounded at infinity.

2°. The solution of the second boundary value problem is determined up to an arbitrary additive term.

3°. Let the second boundary value problem in a closed bounded domain D with piecewise smooth boundary Σ be characterized by the boundary condition*

$$\frac{\partial w}{\partial N} = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in \Sigma,$$

where $\frac{\partial w}{\partial N}$ is the derivative along the (outward) normal to Σ. The necessary and sufficient condition of solvability of the problem has the form

$$\int_\Sigma f(\mathbf{r})\, d\Sigma = 0. \tag{2}$$

Remark 9.1. The same solvability condition occurs for the outer second boundary value problem if the domain is infinite but has a finite boundary.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► Domain: $-\infty < x < \infty$, $0 \le y < \infty$. First boundary value problem.

A half-plane is considered. A boundary condition is prescribed:

$$w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\xi)\, d\xi}{(x-\xi)^2 + y^2} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x + y\tan\theta)\, d\theta.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $-\infty < x < \infty$, $0 \le y < \infty$. Second boundary value problem.

A half-plane is considered. A boundary condition is prescribed:

$$\partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi) \ln\sqrt{(x-\xi)^2 + y^2}\, d\xi + C,$$

where C is an arbitrary constant.

⊙ *Literature*: V. S. Vladimirov (1988).

*More rigorously, Σ must satisfy the Lyapunov condition [see Babich, Kapilevich, Mikhlin et al. (1964) and Tikhonov and Samarskii (1990)].

► **Domain: $0 \le x < \infty,\ 0 \le y < \infty$. First boundary value problem.**

A quadrant of the plane is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \frac{4}{\pi} xy \int_0^\infty \frac{f_1(\eta)\eta\, d\eta}{[x^2 + (y-\eta)^2][x^2 + (y+\eta)^2]} \\ & + \frac{4}{\pi} xy \int_0^\infty \frac{f_2(\xi)\xi\, d\xi}{[(x-\xi)^2 + y^2][(x+\xi)^2 + y^2]}. \end{aligned}$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

► **Domain: $-\infty < x < \infty,\ 0 \le y \le a$. First boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(x) \quad \text{at} \quad y = 0, \qquad w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \frac{1}{2a} \sin\left(\frac{\pi y}{a}\right) \int_{-\infty}^\infty \frac{f_1(\xi)\, d\xi}{\cosh[\pi(x-\xi)/a] - \cos(\pi y/a)} \\ & + \frac{1}{2a} \sin\left(\frac{\pi y}{a}\right) \int_{-\infty}^\infty \frac{f_2(\xi)\, d\xi}{\cosh[\pi(x-\xi)/a] + \cos(\pi y/a)}. \end{aligned}$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $-\infty < x < \infty,\ 0 \le y \le a$. Second boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$\partial_y w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \frac{1}{2\pi} \int_{-\infty}^\infty f_1(\xi) \ln\bigl\{\cosh[\pi(x-\xi)/a] - \cos(\pi y/a)\bigr\}\, d\xi \\ & - \frac{1}{2\pi} \int_{-\infty}^\infty f_2(\xi) \ln\bigl\{\cosh[\pi(x-\xi)/a] + \cos(\pi y/a)\bigr\}\, d\xi + C, \end{aligned}$$

where C is an arbitrary constant.

► Domain: $0 \le x < \infty$, $0 \le y \le a$. First boundary value problem.

A semiinfinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0, \qquad w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \frac{2}{a} \sum_{n=1}^{\infty} \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \int_0^a f_1(\eta) \sin\left(\frac{n\pi \eta}{a}\right) d\eta \\ & + \frac{1}{2a} \sin\left(\frac{\pi y}{a}\right) \int_0^{\infty} \left\{ \frac{1}{\cosh[\pi(x-\xi)/a] - \cos(\pi y/a)} - \frac{1}{\cosh[\pi(x+\xi)/a] - \cos(\pi y/a)} \right\} f_2(\xi)\, d\xi \\ & + \frac{1}{2a} \sin\left(\frac{\pi y}{a}\right) \int_0^{\infty} \left\{ \frac{1}{\cosh[\pi(x-\xi)/a] + \cos(\pi y/a)} - \frac{1}{\cosh[\pi(x+\xi)/a] + \cos(\pi y/a)} \right\} f_3(\xi)\, d\xi. \end{aligned}$$

Example 9.1. Consider the first boundary value problem for the Laplace equation in a semiinfinite strip with $f_1(y) = 1$ and $f_2(x) = f_3(x) = 0$.

Using the general formula and carrying out transformations, we obtain the solution

$$w(x,y) = \frac{2}{\pi} \arctan\left[\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right].$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► Domain: $0 \le x \le a$, $0 \le y \le b$. First boundary value problem.

A rectangle is considered. Boundary conditions are prescribed:

$$\begin{aligned} w &= f_1(y) \quad \text{at} \quad x = 0, \qquad & w &= f_2(y) \quad \text{at} \quad x = a, \\ w &= f_3(x) \quad \text{at} \quad y = 0, \qquad & w &= f_4(x) \quad \text{at} \quad y = b. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y) = {} & \sum_{n=1}^{\infty} A_n \sinh\left[\frac{n\pi}{b}(a-x)\right] \sin\left(\frac{n\pi}{b} y\right) + \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{b} x\right) \sin\left(\frac{n\pi}{b} y\right) \\ & + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left[\frac{n\pi}{a}(b-y)\right] + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right), \end{aligned}$$

where the coefficients A_n, B_n, C_n, and D_n are expressed as

$$\begin{aligned} A_n &= \frac{2}{\lambda_n} \int_0^b f_1(\xi) \sin\left(\frac{n\pi\xi}{b}\right) d\xi, \quad & B_n &= \frac{2}{\lambda_n} \int_0^b f_2(\xi) \sin\left(\frac{n\pi\xi}{b}\right) d\xi, \\ C_n &= \frac{2}{\mu_n} \int_0^a f_3(\xi) \sin\left(\frac{n\pi\xi}{a}\right) d\xi, \quad & D_n &= \frac{2}{\mu_n} \int_0^a f_4(\xi) \sin\left(\frac{n\pi\xi}{a}\right) d\xi, \end{aligned}$$

$$\lambda_n = b \sinh\left(\frac{n\pi a}{b}\right), \quad \mu_n = a \sinh\left(\frac{n\pi b}{a}\right).$$

⊙ *Literature*: M. M. Smirnov (1975), H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le x \le a$, $0 \le y \le b$. Second boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y) \quad \text{at} \quad x=0, \qquad \partial_x w = f_2(y) \quad \text{at} \quad x=a,$$
$$\partial_y w = f_3(x) \quad \text{at} \quad y=0, \qquad \partial_y w = f_4(x) \quad \text{at} \quad y=b.$$

Solution:

$$\begin{aligned} w(x,y) = &-\frac{A_0}{4a}(x-a)^2 + \frac{B_0}{4a}x^2 - \frac{C_0}{4b}(x-b)^2 + \frac{D_0}{4b}y^2 + K \\ &- b\sum_{n=1}^{\infty}\frac{A_n}{\lambda_n}\cosh\left[\frac{n\pi}{b}(a-x)\right]\cos\left(\frac{n\pi}{b}y\right) + b\sum_{n=1}^{\infty}\frac{B_n}{\lambda_n}\cosh\left(\frac{n\pi}{b}x\right)\cos\left(\frac{n\pi}{b}y\right) \\ &- a\sum_{n=1}^{\infty}\frac{C_n}{\mu_n}\cos\left(\frac{n\pi}{a}x\right)\cosh\left[\frac{n\pi}{a}(b-y)\right] + a\sum_{n=1}^{\infty}\frac{D_n}{\mu_n}\cos\left(\frac{n\pi}{a}x\right)\cosh\left(\frac{n\pi}{a}y\right), \end{aligned}$$

where K is an arbitrary constant, and the coefficients A_n, B_n, C_n, D_n, λ_n, and μ_n are expressed as

$$A_n = \frac{2}{b}\int_0^b f_1(\xi)\cos\left(\frac{n\pi\xi}{b}\right)d\xi, \quad B_n = \frac{2}{b}\int_0^b f_2(\xi)\cos\left(\frac{n\pi\xi}{b}\right)d\xi,$$
$$C_n = \frac{2}{a}\int_0^a f_3(\xi)\cos\left(\frac{n\pi\xi}{a}\right)d\xi, \quad D_n = \frac{2}{a}\int_0^a f_4(\xi)\cos\left(\frac{n\pi\xi}{a}\right)d\xi,$$
$$\lambda_n = n\pi\sinh\left(\frac{n\pi a}{b}\right), \quad \mu_n = n\pi\sinh\left(\frac{n\pi b}{a}\right).$$

The solvability condition for the problem in question has the form (see Section 9.1.1, condition (2)):

$$\int_0^b f_1(y)\,dy + \int_0^b f_2(y)\,dy - \int_0^a f_3(x)\,dx - \int_0^a f_4(x)\,dx = 0.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$. Third boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = f_1(y) \quad \text{at} \quad x=0, \qquad \partial_x w + k_2 w = f_2(y) \quad \text{at} \quad x=a,$$
$$\partial_y w - k_3 w = f_3(x) \quad \text{at} \quad y=0, \qquad \partial_y w + k_4 w = f_4(x) \quad \text{at} \quad y=b.$$

For the solution, see Section 9.2.2 (the third boundary value problem for $0 \le x \le a$, $0 \le y \le b$) with $\Phi \equiv 0$.

► **Domain: $0 \le x \le a$, $0 \le y \le b$. Mixed boundary value problems.**

1°. A rectangle is considered. Boundary conditions are prescribed:

$$\partial_x w = f(y) \quad \text{at} \quad x=0, \qquad \partial_x w = g(y) \quad \text{at} \quad x=a,$$
$$w = h(x) \quad \text{at} \quad y=0, \qquad w = s(x) \quad \text{at} \quad y=b.$$

Solution:

$$w(x,y) = -\frac{b}{\pi}\sum_{n=1}^{\infty}\frac{f_n}{n\lambda_n}\cosh\left[\frac{\pi n}{b}(a-x)\right]\sin\left(\frac{\pi n y}{b}\right) + \frac{b}{\pi}\sum_{n=1}^{\infty}\frac{g_n}{n\lambda_n}\cosh\left(\frac{\pi n x}{b}\right)\sin\left(\frac{\pi n y}{b}\right)$$
$$+\sum_{n=1}^{\infty}\frac{h_n}{\mu_n}\cos\left(\frac{\pi n x}{a}\right)\sinh\left[\frac{\pi n}{a}(b-y)\right] + \sum_{n=1}^{\infty}\frac{s_n}{\mu_n}\cos\left(\frac{\pi n x}{a}\right)\sinh\left(\frac{\pi n y}{a}\right)$$
$$+\frac{b-y}{ab}\int_0^a h(x)\,dx + \frac{y}{ab}\int_0^a s(x)\,dx,$$

where

$$f_n = \frac{2}{b}\int_0^b f(\xi)\sin\left(\frac{\pi n\xi}{b}\right)d\xi, \quad g_n = \frac{2}{b}\int_0^b g(\xi)\sin\left(\frac{\pi n\xi}{b}\right)d\xi,$$
$$h_n = \frac{2}{a}\int_0^a h(\xi)\cos\left(\frac{\pi n\xi}{a}\right)d\xi, \quad s_n = \frac{2}{a}\int_0^a s(\xi)\cos\left(\frac{\pi n\xi}{a}\right)d\xi,$$
$$\lambda_n = \sinh\left(\frac{\pi n a}{b}\right), \quad \mu_n = \sinh\left(\frac{\pi n b}{a}\right).$$

⊙ *Literature*: M. M. Smirnov (1975).

2°. A rectangle is considered. Boundary conditions are prescribed:

$$w = f(y) \quad \text{at} \quad x=0, \qquad \partial_x w = g(y) \quad \text{at} \quad x=a,$$
$$w = h(x) \quad \text{at} \quad y=0, \qquad \partial_y w = s(x) \quad \text{at} \quad y=b,$$

where $f(0) = h(0)$.

Solution:

$$w(x,y) = \sum_{n=0}^{\infty}\frac{f_n}{\cosh\lambda_n}\cosh\left(\lambda_n\frac{a-x}{a}\right)\sin\left(\lambda_n\frac{y}{a}\right) + a\sum_{n=0}^{\infty}\frac{g_n}{\lambda_n\cosh\lambda_n}\sinh\left(\lambda_n\frac{x}{a}\right)\sin\left(\lambda_n\frac{y}{a}\right)$$
$$+\sum_{n=0}^{\infty}\frac{h_n}{\cosh\mu_n}\sin\left(\mu_n\frac{x}{b}\right)\cosh\left(\mu_n\frac{b-y}{b}\right) + b\sum_{n=0}^{\infty}\frac{s_n}{\mu_n\cosh\mu_n}\sin\left(\mu_n\frac{x}{b}\right)\sinh\left(\mu_n\frac{y}{b}\right),$$

where

$$f_n = \frac{2}{b}\int_0^b f(\xi)\sin\left[\frac{\pi(2n+1)}{2b}\xi\right]d\xi, \quad g_n = \frac{2}{b}\int_0^b g(\xi)\sin\left[\frac{\pi(2n+1)}{2b}\xi\right]d\xi,$$
$$h_n = \frac{2}{a}\int_0^a h(\xi)\sin\left[\frac{\pi(2n+1)}{2a}\xi\right]d\xi, \quad s_n = \frac{2}{a}\int_0^a s(\xi)\sin\left[\frac{\pi(2n+1)}{2a}\xi\right]d\xi,$$
$$\lambda_n = \frac{\pi(2n+1)a}{2b}, \quad \mu_n = \frac{\pi(2n+1)b}{2a}.$$

⊙ *Literature*: M. M. Smirnov (1975).

9.1.2 Problems in Polar Coordinate System

The two-dimensional Laplace equation in the polar coordinate system is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} = 0, \qquad r = \sqrt{x^2+y^2}.$$

▶ **Particular solutions:**

$$w(r) = A\ln r + B,$$
$$w(r,\varphi) = \left(Ar^m + \frac{B}{r^m}\right)(C\cos m\varphi + D\sin m\varphi),$$

where $m = 1,\ 2,\ \ldots;\ A$, B, C, and D are arbitrary constants.

▶ **Domain: $0 \le r \le R$ or $R \le r < \infty$. First boundary value problem.**

The condition

$$w = f(\varphi) \quad \text{at} \quad r = R$$

is set at the boundary of the circle; $f(\varphi)$ is a given function.

1°. Solution of the inner problem ($r \le R$):

$$w(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{R^2 - r^2}{r^2 - 2Rr\cos(\varphi - \psi) + R^2}\,d\psi.$$

This formula is conventionally referred to as the *Poisson integral*.

Solution of the outer problem in series form:

$$w(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\varphi + b_n\sin n\varphi),$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\cos(n\psi)\,d\psi, \qquad n = 0,\ 1,\ 2,\ \ldots,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\sin(n\psi)\,d\psi, \qquad n = 1,\ 2,\ 3,\ \ldots$$

2°. Bounded solution of the outer problem ($r \ge R$):

$$w(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{r^2 - R^2}{r^2 - 2Rr\cos(\varphi - \psi) + R^2}\,d\psi.$$

Bounded solution of the outer problem in series form:

$$w(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n (a_n\cos n\varphi + b_n\sin n\varphi),$$

where the coefficients a_0, a_n, and b_n are defined by the same relations as in the inner problem.

In hydrodynamics and other applications, outer problems are sometimes encountered in which one has to consider unbounded solutions for $r \to \infty$.

Example 9.2. The potential flow of an ideal (inviscid) incompressible fluid about a circular cylinder of radius R with a constant incident velocity U at infinity is characterized by the following boundary conditions for the stream function:

$$w = 0 \quad \text{at} \quad r = R, \qquad w \to Ur\sin\varphi \quad \text{as} \quad r \to \infty.$$

Solution:

$$w(r,\varphi) = U\left(r - \frac{R^2}{r}\right)\sin\varphi.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), A. N. Tikhonov and A. A. Samarskii (1990).

► Domain: $0 \le r \le R$ or $R \le r < \infty$. Second boundary value problem.

The condition

$$\partial_r w = f(\varphi) \quad \text{at} \quad r = R$$

is set at the boundary of the circle. The function $f(\varphi)$ must satisfy the solvability condition

$$\int_0^{2\pi} f(\varphi)\,d\varphi = 0.$$

1°. Solution of the inner problem ($r \le R$):

$$w(r,\varphi) = \frac{R}{2\pi}\int_0^{2\pi} f(\psi)\ln\frac{r^2 - 2Rr\cos(\varphi-\psi) + R^2}{R^2}\,d\psi + C,$$

where C is an arbitrary constant; this formula is known as the Dini integral.

Series solution of the inner problem:

$$w(r,\varphi) = \sum_{n=1}^{\infty}\frac{R}{n}\left(\frac{r}{R}\right)^n(a_n\cos n\varphi + b_n\sin n\varphi) + C,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\cos(n\psi)\,d\psi, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\sin(n\psi)\,d\psi,$$

where C is an arbitrary constant.

2°. Solution of the outer problem ($r \ge R$):

$$w(r,\varphi) = -\frac{R}{2\pi}\int_0^{2\pi} f(\psi)\ln\frac{r^2 - 2Rr\cos(\varphi-\psi) + R^2}{r^2}\,d\psi + C,$$

where C is an arbitrary constant.

Series solution of the outer problem:

$$w(r,\varphi) = -\sum_{n=1}^{\infty}\frac{R}{n}\left(\frac{R}{r}\right)^n(a_n\cos n\varphi + b_n\sin n\varphi) + C,$$

where the coefficients a_n and b_n are defined by the same relations as in the inner problem, and C is an arbitrary constant.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

▶ **Domain: $0 \le r \le R$ or $R \le r < \infty$. Third boundary value problem.**

The condition

$$\partial_r w + kw = f(\varphi) \quad \text{at} \quad r = R$$

is set at the circle boundary; $f(\varphi)$ is a given function.

1°. Solution of the inner problem ($r \le R$):

$$w(r,\varphi) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} \frac{R}{kR+n}\left(\frac{r}{R}\right)^n (a_n \cos n\varphi + b_n \sin n\varphi),$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\cos(n\psi)\,d\psi, \qquad n = 0,\ 1,\ 2,\ \ldots,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\sin(n\psi)\,d\psi, \qquad n = 1,\ 2,\ 3,\ \ldots$$

2°. Solution of the outer problem ($r \ge R$):

$$w(r,\varphi) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} \frac{R}{kR-n}\left(\frac{R}{r}\right)^n (a_n \cos n\varphi + b_n \sin n\varphi),$$

where the coefficients a_0, a_n, and b_n are defined by the same relations as in the inner problem.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

▶ **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

An annular domain is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi) \quad \text{at} \quad r = R_1, \qquad w = f_2(\varphi) \quad \text{at} \quad r = R_2.$$

Solution:

$$w(r,\varphi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi) + \sum_{n=1}^{\infty} \frac{1}{r^n}(C_n \cos n\varphi + D_n \sin n\varphi),$$

where the coefficients A_0, B_0, A_n, B_n, C_n, and D_n are expressed as

$$A_0 = \frac{1}{2}\frac{a_0^{(1)}\ln R_2 - a_0^{(2)}\ln R_1}{\ln R_2 - \ln R_1}, \qquad B_0 = \frac{1}{2}\frac{a_0^{(2)} - a_0^{(1)}}{\ln R_2 - \ln R_1},$$

$$A_n = \frac{R_2^n a_n^{(2)} - R_1^n a_n^{(1)}}{R_2^{2n} - R_1^{2n}}, \qquad B_n = \frac{R_2^n b_n^{(2)} - R_1^n b_n^{(1)}}{R_2^{2n} - R_1^{2n}},$$

$$C_n = (R_1 R_2)^n \frac{R_2^n a_n^{(1)} - R_1^n a_n^{(2)}}{R_2^{2n} - R_1^{2n}}, \qquad D_n = (R_1 R_2)^n \frac{R_2^n b_n^{(1)} - R_1^n b_n^{(2)}}{R_2^{2n} - R_1^{2n}}.$$

Here, the $a_n^{(i)}$ and $b_n^{(i)}$ $(i = 1,\ 2)$ are the coefficients of the Fourier series expansions of the functions $f_1(\varphi)$ and $f_2(\varphi)$:

$$a_n^{(i)} = \frac{1}{\pi}\int_0^{2\pi} f_i(\psi)\cos(n\psi)\,d\psi, \qquad n = 0,\ 1,\ 2,\ \dots,$$
$$b_n^{(i)} = \frac{1}{\pi}\int_0^{2\pi} f_i(\psi)\sin(n\psi)\,d\psi, \qquad n = 1,\ 2,\ 3,\ \dots$$

⊙ *Literature*: M. M. Smirnov (1975).

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

An annular domain is considered. Boundary conditions are prescribed:

$$\partial_r w = f_1(\varphi) \quad \text{at} \quad r = R_1, \qquad \partial_r w = f_2(\varphi) \quad \text{at} \quad r = R_2.$$

Solution:

$$w(r,\varphi) = B\ln r + \sum_{n=1}^{\infty} r^n (A_n\cos n\varphi + B_n\sin n\varphi) + \sum_{n=1}^{\infty}\frac{1}{r^n}(C_n\cos n\varphi + D_n\sin n\varphi) + K.$$

The coefficients B, A_n, B_n, C_n, and D_n are expressed as

$$B = \frac{1}{2}R_1 a_0^{(1)}, \quad A_n = \frac{R_2^{n+1}a_n^{(2)} - R_1^{n+1}a_n^{(1)}}{n(R_2^{2n} - R_1^{2n})}, \quad B_n = \frac{R_2^{n+1}b_n^{(2)} - R_1^{n+1}b_n^{(1)}}{n(R_2^{2n} - R_1^{2n})},$$
$$C_n = (R_1R_2)^{n+1}\frac{R_1^{n-1}a_n^{(2)} - R_2^{n-1}a_n^{(1)}}{n(R_2^{2n} - R_1^{2n})}, \quad D_n = (R_1R_2)^{n+1}\frac{R_1^{n-1}b_n^{(2)} - R_2^{n-1}b_n^{(1)}}{n(R_2^{2n} - R_1^{2n})},$$

where the constants $a_n^{(i)}$ and $b_n^{(i)}$ $(i = 1,\ 2)$ are defined by the same relations as in the first boundary value problem; K is an arbitrary constant.

Remark 9.2. Note that the condition $a_0^{(1)}R_1 = a_0^{(2)}R_2$ must hold; this relation is a consequence of the solvability condition for the problem,

$$\int_{r=R_1} f_1\,dS - \int_{r=R_2} f_2\,dS = 0.$$

► **Domain: $R_1 \le r \le R_2$. Mixed boundary value problem.**

An annular domain is considered. Boundary conditions are prescribed:

$$\partial_r w = f_1(\varphi) \quad \text{at} \quad r = R_1, \qquad w = f_2(\varphi) \quad \text{at} \quad r = R_2.$$

Solution:

$$w(r,\varphi) = \tfrac{1}{2}a_0^{(2)} + \tfrac{1}{2}a_0^{(1)}R_1\ln\frac{r}{R_2}$$
$$+ \sum_{n=1}^{\infty} r^n(A_n\cos n\varphi + B_n\sin n\varphi) + \sum_{n=1}^{\infty}\frac{1}{r^n}(C_n\cos n\varphi + D_n\sin n\varphi).$$

Here, the coefficients A_n, B_n, C_n, and D_n are expressed as

$$A_n = \frac{nR_2^n a_n^{(2)} + R_1^{n+1} a_n^{(1)}}{n(R_2^{2n} + R_1^{2n})}, \quad B_n = \frac{nR_2^n b_n^{(2)} + R_1^{n+1} b_n^{(1)}}{n(R_2^{2n} + R_1^{2n})},$$
$$C_n = R_1^{n+1} R_2^n \frac{nR_1^{n-1} a_n^{(2)} - R_2^n a_n^{(1)}}{n(R_2^{2n} + R_1^{2n})}, \quad D_n = R_1^{n+1} R_2^n \frac{nR_1^{n-1} b_n^{(2)} - R_2^n b_n^{(1)}}{n(R_2^{2n} + R_1^{2n})},$$

where the constants $a_n^{(i)}$ and $b_n^{(i)}$ $(i = 1,\, 2)$ are defined by the same formulas as in the first boundary value problem.

⊙ *Literature*: M. M. Smirnov (1975).

9.1.3 Other Coordinate Systems. Conformal Mappings Method

► **Parabolic, elliptic, and bipolar coordinate systems.**

In a number of applications, it is convenient to solve the Laplace equation in other orthogonal systems of coordinates. Some of those commonly encountered are displayed in Table 9.1. In all the coordinate systems presented, the Laplace equation $\Delta_2 w = 0$ is reduced to the equation considered in Section 9.1.1 in detail (particular solutions and solutions to boundary value problems are given there).

TABLE 9.1
Two-dimensional Laplace operator in some curvilinear orthogonal systems of coordinates

Coordinates	Transformation ($c > 0$)	Laplace operator, $\Delta_2 w$
Parabolic coordinates $u,\, v$	$x = cuv,\ y = \frac{1}{2}c(v^2 - u^2)$ $-\infty < u < \infty,\ 0 \le v < \infty$	$\dfrac{1}{c^2(u^2+v^2)}\left(\dfrac{\partial^2 w}{\partial u^2} + \dfrac{\partial^2 w}{\partial v^2}\right)$
Elliptic coordinates $\xi,\, \eta$	$x = c\cosh\xi\cos\eta,\ y = c\sinh\xi\sin\eta$ $0 \le \xi < \infty,\ 0 \le \eta < 2\pi$	$\dfrac{1}{c^2(\sinh^2\xi + \sin^2\eta)}\left(\dfrac{\partial^2 w}{\partial \xi^2} + \dfrac{\partial^2 w}{\partial \eta^2}\right)$
Bipolar coordinates $\sigma,\, \tau$	$x = \dfrac{c\sinh\tau}{\cosh\tau - \cos\sigma},\ y = \dfrac{c\sin\sigma}{\cosh\tau - \cos\sigma}$ $0 \le \sigma < 2\pi,\ -\infty < \tau < \infty$	$\dfrac{1}{c^2}(\cosh\tau - \cos\sigma)^2\left(\dfrac{\partial^2 w}{\partial \sigma^2} + \dfrac{\partial^2 w}{\partial \tau^2}\right)$

The orthogonal transformations presented in Table 9.1 can be written in the language of complex variables as follows:

$$x + iy = -\tfrac{1}{2}ic(u + iv)^2 \qquad \text{(parabolic coordinates)},$$
$$x + iy = c\cosh(\xi + i\eta) \qquad \text{(elliptic coordinates)},$$
$$x + iy = ic\cot\left[\tfrac{1}{2}(\sigma + i\tau)\right] \qquad \text{(bipolar coordinates)}.$$

The real parts, as well as the imaginary parts, in both sides of these relations must be equated to each other ($i^2 = -1$).

Example 9.3. Plane hydrodynamic problems of potential flows of ideal (inviscid) incompressible fluid are reduced to the Laplace equation for the stream function. In particular, the motion of an elliptic cylinder with semiaxes a and b at a velocity U in the direction parallel to the major semiaxis ($a > b$) in ideal fluid is described by the stream function

$$w(\xi,\eta) = -Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{\xi}\sin\eta, \qquad c^2 = a^2 - b^2,$$

where ξ and η are the elliptic coordinates.

⊙ *Literature*: H. Lamb (1945), J. Happel and H. Brenner (1965), G. Korn and T. Korn (2000).

▶ Domain of arbitrary shape. Method of conformal mappings.

1°. Let $\zeta = \zeta(z)$ be an analytic function that defines a conformal mapping from the complex plane $z = x + iy$ into a complex plane $\zeta = u + iv$, where $u = u(x,y)$ and $v = v(x,y)$ are new independent variables. With reference to the fact that the real and imaginary parts of an analytic function satisfy the Cauchy–Riemann conditions, we have $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, and hence

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = |\zeta'(z)|^2\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right).$$

Therefore, the Laplace equation in the xy-plane transforms under a conformal mapping into the Laplace equation in the uv-plane.

2°. Any simply connected domain D in the xy-plane with a piecewise smooth boundary can be mapped, with appropriate conformal mappings, onto the upper half-plane or into a unit circle in the uv-plane. Consequently, a first and a second boundary value problem for the Laplace equation in D can be reduced, respectively, to a first and a second boundary value problem for the upper half-space or a circle; such problems are considered in Sections 9.1.1 and 9.1.2.

Section 9.2.4 presents conformal mappings of some domains onto the upper half-plane or a unit circle. Moreover, examples of solving specific boundary value problems for the Poisson equation by the conformal mappings method are given there; the Green's functions for a semicircle and a quadrant of a circle are obtained.

A large number of conformal mappings of various domains can be found, for example, in the references cited below.

⊙ *Literature*: V. I. Lavrik and V. N. Savenkov (1970), M. A. Lavrent'ev and B. V. Shabat (1973), V. I. Ivanov and M. K. Trubetskov (1994).

▶ Reduction of the two-dimensional Neumann problem to the Dirichlet problem.

Let the position of any point (x_*, y_*) located on the boundary Σ of a domain D be specified by a parameter s, so that $x_* = x_*(s)$ and $y_* = y_*(s)$. Then a function of two variables, $f(x,y)$, is determined on Σ by the parameter s as well, $f(x,y)|_\Sigma = f(x_*(s), y_*(s)) = f_*(s)$.

The solution of the two-dimensional Neumann problem for the Laplace equation $\Delta_2 w = 0$ in D with the boundary condition of the second kind

$$\frac{\partial w}{\partial N} = f_*(s) \quad \text{for} \quad \mathbf{r} \in \Sigma$$

can be expressed in terms of the solution of the two-dimensional Dirichlet problem for the Laplace equation $\Delta_2 u = 0$ in D with the boundary condition of the first kind

$$u = F_*(s) \quad \text{for} \quad \mathbf{r} \in \Sigma,$$

where $F_*(s) = \int f_*(s)\, ds$, as follows:

$$w(x,y) = \int_{x_0}^{x} \frac{\partial u}{\partial y}(t, y_0)\, dt - \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt + C.$$

Here, (x_0, y_0) are the coordinates of any point in D, and C is an arbitrary constant.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

9.2 Poisson Equation $\Delta_2 w = -\Phi(\mathbf{x})$

9.2.1 Preliminary Remarks. Solution Structure

Just as the Laplace equation, the Poisson equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. For example, it describes steady-state temperature distribution in the presence of heat sources or sinks in the domain under study.

The Laplace equation is a special case of the Poisson equation with $\Phi \equiv 0$.

In what follows, we consider a finite domain S with a sufficiently smooth boundary L. Let $\mathbf{r} \in S$ and $\boldsymbol{\rho} \in S$, where $\mathbf{r} = \{x, y\}$, $\boldsymbol{\rho} = \{\xi, \eta\}$, $|\mathbf{r} - \boldsymbol{\rho}|^2 = (x-\xi)^2 + (y-\eta)^2$.

► **First boundary value problem.**

The solution of the first boundary value problem for the Poisson equation

$$\Delta_2 w = -\Phi(\mathbf{r}) \tag{1}$$

in the domain S with the nonhomogeneous boundary condition

$$w = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in L$$

can be represented as

$$w(\mathbf{r}) = \int_S \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dS_\rho - \int_L f(\boldsymbol{\rho}) \frac{\partial G}{\partial N_\rho}\, dL_\rho. \tag{2}$$

Here, $G(\mathbf{r}, \boldsymbol{\rho})$ is the Green's function of the first boundary value problem, $\frac{\partial G}{\partial N_\rho}$ is the derivative of the Green's function with respect to ξ, η along the outward normal $\mathbf{N}$ to the boundary L. The integration is performed with respect to ξ, η, with $dS_\rho = d\xi\, d\eta$.

The Green's function $G = G(\mathbf{r}, \boldsymbol{\rho})$ of the first boundary value problem is determined by the following conditions.

1°. The function G satisfies the Laplace equation in x, y in the domain S everywhere except for the point (ξ, η), at which G has a singularity of the form $\frac{1}{2\pi} \ln \frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$.

2°. With respect to x, y, the function G satisfies the homogeneous boundary condition of the first kind at the domain boundary, i.e., the condition $G|_L = 0$.

The Green's function can be represented in the form

$$G(\mathbf{r}, \boldsymbol{\rho}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{r} - \boldsymbol{\rho}|} + u, \tag{3}$$

where the auxiliary function $u = u(\mathbf{r}, \boldsymbol{\rho})$ is determined by solving the first boundary value problem for the Laplace equation $\Delta_2 u = 0$ with the boundary condition $u|_L = -\frac{1}{2\pi} \ln \frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$; in this problem, $\boldsymbol{\rho}$ is treated as a two-dimensional free parameter.

The Green's function is symmetric with respect to its arguments: $G(\mathbf{r}, \boldsymbol{\rho}) = G(\boldsymbol{\rho}, \mathbf{r})$.

Remark 9.3. When using the polar coordinate system, one should set

$$\mathbf{r} = \{r, \varphi\}, \quad \boldsymbol{\rho} = \{\xi, \eta\}, \quad |\mathbf{r} - \boldsymbol{\rho}|^2 = r^2 + \xi^2 - 2r\xi \cos(\varphi - \eta), \quad dS_\rho = \xi \, d\xi \, d\eta$$

in relations (2) and (3).

► Second boundary value problem.

The second boundary value problem for the Poisson equation (1) is characterized by the boundary condition

$$\frac{\partial w}{\partial N} = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in L.$$

The necessary solvability condition for this problem is

$$\int_S \Phi(\mathbf{r}) \, dS + \int_L f(\mathbf{r}) \, dL = 0. \tag{4}$$

The solution of the second boundary value problem, provided that condition (4) is satisfied, can be represented as

$$w(\mathbf{r}) = \int_S \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho}) \, dS_\rho + \int_L f(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho}) \, dL_\rho + C, \tag{5}$$

where C is an arbitrary constant.

The Green's function $G = G(\mathbf{r}, \boldsymbol{\rho})$ of the second boundary value problem is determined by the following conditions:

1°. The function G satisfies the Laplace equation in x, y in the domain S everywhere except for the point (ξ, η), at which G has a singularity of the form $\frac{1}{2\pi} \ln \frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$.

2°. With respect to x, y, the function G satisfies the homogeneous boundary condition of the second kind at the domain boundary:

$$\left.\frac{\partial G}{\partial N}\right|_L = \frac{1}{L_0},$$

where L_0 is the length of the boundary of S.

The Green's function is unique up to an additive constant.

Remark 9.4. The Green's function cannot be determined by condition 1° and the homogeneous boundary condition $\left.\frac{\partial G}{\partial N}\right|_L = 0$. The point is that the problem is unsolvable for G in this case, because, on representing G in the form (3), for u we obtain a problem with a nonhomogeneous boundary condition of the second kind for which the solvability condition (4) now is not satisfied.

► **Third boundary value problem.**

The solution of the third boundary value problem for the Poisson equation (1) in the domain S with the nonhomogeneous boundary condition

$$\frac{\partial w}{\partial N} + kw = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in L$$

is given by formula (5) with $C = 0$, where $G = G(\mathbf{r}, \boldsymbol{\rho})$ is the Green's function of the third boundary value problem and is determined by the following conditions:

1°. The function G satisfies the Laplace equation in x, y in the domain S everywhere except for the point (ξ, η), at which G has a singularity of the form $\frac{1}{2\pi} \ln \frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$.

2°. With respect to x, y, the function G satisfies the homogeneous boundary condition of the third kind at the domain boundary, i.e., the condition $\left[\frac{\partial G}{\partial N} + kG\right]_L = 0$.

The Green's function can be represented in the form (3); the auxiliary function u is identified by solving the corresponding third boundary value problem for the Laplace equation $\Delta_2 u = 0$.

The Green's function is symmetric with respect to its arguments: $G(\mathbf{r}, \boldsymbol{\rho}) = G(\boldsymbol{\rho}, \mathbf{r})$.

⊙ *Literature for Section* 9.2.1: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970).

9.2.2 Problems in Cartesian Coordinate System

The two-dimensional Poisson equation in the rectangular Cartesian coordinate system has the form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \Phi(x, y) = 0.$$

► **Particular solutions of the Poisson equation with a special right-hand side.**

1°. If $\Phi(x, y) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij} \exp(b_i x + c_j y)$, the equation has solutions of the form

$$w(x, y) = -\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{b_i^2 + c_j^2} \exp(b_i x + c_j y).$$

2°. If $\Phi(x, y) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij} \sin(b_i x + p_i) \sin(c_j y + q_j)$, the equation admits solutions of the form

$$w(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{b_i^2 + c_j^2} \sin(b_i x + p_i) \sin(c_j y + q_j).$$

► **Domain:** $-\infty < x < \infty$, $-\infty < y < \infty$.

Solution:

$$w(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\xi, \eta) \ln \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \, d\xi \, d\eta.$$

► **Domain: $-\infty < x < \infty,\ 0 \le y < \infty$. First boundary value problem.**

A half-plane is considered. A boundary condition is prescribed:

$$w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\xi)\,d\xi}{(x-\xi)^2 + y^2} + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta) \ln \frac{\sqrt{(x-\xi)^2 + (y+\eta)^2}}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\,d\xi\,d\eta.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $-\infty < x < \infty, 0 \le y < \infty$. Second boundary value problem.**

A half-plane is considered. A boundary condition is prescribed:

$$\partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi) \ln \sqrt{(x-\xi)^2 + y^2}\,d\xi$$
$$+ \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta)\left[\ln \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} + \ln \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2}}\right] d\xi\,d\eta + C,$$

where C is an arbitrary constant.

⊙ *Literature*: V. S. Vladimirov (1988).

► **Domain: $-\infty < x < \infty,\ 0 \le y \le a$. First boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(x) \quad \text{at} \quad y = 0, \qquad w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$\begin{aligned} w(x,y) &= \frac{1}{2a}\sin\left(\frac{\pi y}{a}\right)\int_{-\infty}^{\infty} \frac{f_1(\xi)\,d\xi}{\cosh[\pi(x-\xi)/a] - \cos(\pi y/a)} \\ &+ \frac{1}{2a}\sin\left(\frac{\pi y}{a}\right)\int_{-\infty}^{\infty} \frac{f_2(\xi)\,d\xi}{\cosh[\pi(x-\xi)/a] + \cos(\pi y/a)} \\ &+ \frac{1}{4\pi}\int_0^{a}\int_{-\infty}^{\infty} \Phi(\xi,\eta) \ln \frac{\cosh[\pi(x-\xi)/a] - \cos[\pi(y+\eta)/a]}{\cosh[\pi(x-\xi)/a] - \cos[\pi(y-\eta)/a]}\,d\xi\,d\eta. \end{aligned}$$

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

▶ **Domain: $-\infty < x < \infty$, $0 \le y \le a$. Second boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$\partial_y w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = -\int_{-\infty}^{\infty} f_1(\xi) G(x,y,\xi,0)\, d\xi + \int_{-\infty}^{\infty} f_2(\xi) G(x,y,\xi,a)\, d\xi + \int_0^a \int_{-\infty}^{\infty} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\xi\, d\eta + C.$$

Here,

$$G(x,y,\xi,\eta) = \frac{1}{4\pi} \ln \frac{1}{\cosh[\pi(x-\xi)/a] - \cos[\pi(y-\eta)/a]} + \frac{1}{4\pi} \ln \frac{1}{\cosh[\pi(x-\xi)/a] - \cos[\pi(y+\eta)/a]},$$

where C is an arbitrary constant.

▶ **Domain: $-\infty < x < \infty$, $0 \le y \le a$. Third boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$\partial_y w - k_1 w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w + k_2 w = f_2(x) \quad \text{at} \quad y = a.$$

The solution $w(x,y)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\varphi_n(y)\varphi_n(\eta)}{\|\varphi_n\|^2 \mu_n} \exp\bigl(-\mu_n |x-\xi|\bigr),$$

$$\varphi_n(y) = \mu_n \cos(\mu_n y) + k_1 \sin(\mu_n y), \quad \|\varphi_n\|^2 = \frac{1}{2}(\mu_n^2 + k_1^2)\left[a + \frac{(k_1+k_2)(\mu_n^2 + k_1 k_2)}{(\mu_n^2 + k_1^2)(\mu_n^2 + k_2^2)}\right].$$

Here, the μ_n are positive roots of the transcendental equation $\tan(\mu a) = \dfrac{(k_1+k_2)\mu}{\mu^2 - k_1 k_2}$.

▶ **Domain: $-\infty < x < \infty$, $0 \le y \le a$. Mixed boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f_1(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=0} d\xi + \int_{-\infty}^{\infty} f_2(\xi) G(x,y,\xi,a)\, d\xi + \int_0^a \int_{-\infty}^{\infty} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\xi\, d\eta,$$

where

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=0}^{\infty}\frac{1}{\mu_n}\exp\bigl(-\mu_n|x-\xi|\bigr)\sin(\mu_n y)\sin(\mu_n\eta), \quad \mu_n = \frac{\pi(2n+1)}{2a}.$$

► **Domain: $0 \le x < \infty$, $0 \le y \le a$. First boundary value problem.**

A semiinfinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x=0, \qquad w = f_2(x) \quad \text{at} \quad y=0, \qquad w = f_3(x) \quad \text{at} \quad y=a.$$

Solution:

$$\begin{aligned} w(x,y) &= \int_0^a f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=0} d\eta + \int_0^\infty f_2(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi \\ &\quad - \int_0^\infty f_3(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=a} d\xi + \int_0^a\int_0^\infty \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta, \end{aligned}$$

where

$$\begin{aligned} G(x,y,\xi,\eta) &= \frac{1}{4\pi}\ln\frac{\cosh[\pi(x-\xi)/a] - \cos[\pi(y+\eta)/a]}{\cosh[\pi(x-\xi)/a] - \cos[\pi(y-\eta)/a]} \\ &\quad - \frac{1}{4\pi}\ln\frac{\cosh[\pi(x+\xi)/a] - \cos[\pi(y+\eta)/a]}{\cosh[\pi(x+\xi)/a] - \cos[\pi(y-\eta)/a]}. \end{aligned}$$

Alternatively, the Green's function can be represented in the series form

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{q_n}\bigl[\exp\bigl(-q_n|x-\xi|\bigr) - \exp\bigl(-q_n|x+\xi|\bigr)\bigr]\sin(q_n y)\sin(q_n\eta), \quad q_n = \frac{\pi n}{a}.$$

⊙ *Literature*: N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand (1955), A. G. Butkovskiy (1979).

► **Domain: $0 \le x < \infty$, $0 \le y \le a$. Third boundary value problem.**

A semiinfinite strip is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = f_1(y) \ \text{at} \ x=0, \quad \partial_y w - k_2 w = f_2(x) \ \text{at} \ y=0, \quad \partial_y w + k_3 w = f_3(x) \ \text{at} \ y=a.$$

Solution:

$$\begin{aligned} w(x,y) &= \int_0^a\int_0^\infty \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta - \int_0^a f_1(\eta)G(x,y,0,\eta)\,d\eta \\ &\quad - \int_0^\infty f_2(\xi)G(x,y,\xi,0)\,d\xi + \int_0^\infty f_3(\xi)G(x,y,\xi,a)\,d\xi, \end{aligned}$$

where

$$G(x,y,\xi,\eta) = \sum_{n=1}^{\infty}\frac{\varphi_n(y)\varphi_n(\eta)}{\|\varphi_n\|^2\mu_n(\mu_n+k_1)}H_n(x,\xi),$$

$$\varphi_n(y) = \mu_n\cos(\mu_n y) + k_2\sin(\mu_n y), \quad \|\varphi_n\|^2 = \frac{1}{2}(\mu_n^2+k_2^2)\left[a + \frac{(k_2+k_3)(\mu_n^2+k_2k_3)}{(\mu_n^2+k_2^2)(\mu_n^2+k_3^2)}\right],$$

$$H_n(x,\xi) = \begin{cases} \exp(-\mu_n x)\bigl[\mu_n\cosh(\mu_n\xi) + k_1\sinh(\mu_n\xi)\bigr] & \text{for } x > \xi, \\ \exp(-\mu_n \xi)\bigl[\mu_n\cosh(\mu_n x) + k_1\sinh(\mu_n x)\bigr] & \text{for } \xi > x. \end{cases}$$

Here, the μ_n are positive roots of the transcendental equation $\tan(\mu a) = \dfrac{(k_2+k_3)\mu}{\mu^2 - k_2 k_3}$.

► **Domain: $0 \le x < \infty$, $0 \le y \le a$. Mixed boundary value problems.**

1°. A semiinfinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \int_0^a f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=0} d\eta - \int_0^\infty f_2(\xi)G(x,y,\xi,0)\,d\xi \\ & + \int_0^\infty f_3(\xi)G(x,y,\xi,a)\,d\xi + \int_0^a\int_0^\infty \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta, \end{aligned}$$

where

$$G(x,y,\xi,\eta) = \frac{1}{2a}\sum_{n=0}^{\infty}\frac{\varepsilon_n}{q_n}\bigl[\exp\bigl(-q_n|x-\xi|\bigr) - \exp\bigl(-q_n|x+\xi|\bigr)\bigr]\cos(q_n y)\cos(q_n\eta),$$

$$q_n = \frac{\pi n}{a}, \qquad \varepsilon = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \ne 0. \end{cases}$$

2°. A semiinfinite strip is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0, \qquad w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$\begin{aligned} w(x,y) = {} & -\int_0^a f_1(\eta)G(x,y,0,\eta)\,d\eta + \int_0^\infty f_2(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi \\ & - \int_0^\infty f_3(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=a} d\xi + \int_0^a\int_0^\infty \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta, \end{aligned}$$

where

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{q_n}\bigl[\exp\bigl(-q_n|x-\xi|\bigr) + \exp\bigl(-q_n|x+\xi|\bigr)\bigr]\sin(q_n y)\sin(q_n\eta), \quad q_n = \frac{\pi n}{a}.$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. First boundary value problem.**

A quadrant of the plane is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \frac{4}{\pi}xy\int_0^\infty \frac{f_1(\eta)\eta\,d\eta}{[x^2+(y-\eta)^2][x^2+(y+\eta)^2]} + \frac{4}{\pi}xy\int_0^\infty \frac{f_2(\xi)\xi\,d\xi}{[(x-\xi)^2+y^2][(x+\xi)^2+y^2]} \\ & + \frac{1}{2\pi}\int_0^\infty\int_0^\infty \Phi(\xi,\eta)\ln\frac{\sqrt{(x-\xi)^2+(y+\eta)^2}\sqrt{(x+\xi)^2+(y-\eta)^2}}{\sqrt{(x-\xi)^2+(y-\eta)^2}\sqrt{(x+\xi)^2+(y+\eta)^2}}\,d\xi\,d\eta. \end{aligned}$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le a,\ 0 \le y \le b$. First boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(y) \quad \text{at} \quad x = a,$$
$$w = f_3(x) \quad \text{at} \quad y = 0, \qquad w = f_4(x) \quad \text{at} \quad y = b.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \int_0^a \int_0^b \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\eta\,d\xi \\ & + \int_0^b f_1(\eta) \left[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\right]_{\xi=0} d\eta - \int_0^b f_2(\eta) \left[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\right]_{\xi=a} d\eta \\ & + \int_0^a f_3(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=0} d\xi - \int_0^a f_4(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=b} d\xi. \end{aligned}$$

Two forms of representation of the Green's function:

$$G(x,y,\xi,\eta) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(p_n x)\sin(p_n \xi)}{p_n \sinh(p_n b)} H_n(y,\eta) = \frac{2}{b} \sum_{m=1}^{\infty} \frac{\sin(q_m y)\sin(q_m \eta)}{q_m \sinh(q_m a)} Q_m(x,\xi),$$

where

$$p_n = \frac{\pi n}{a}, \quad H_n(y,\eta) = \begin{cases} \sinh(p_n \eta)\sinh[p_n(b-y)] & \text{for } b \ge y > \eta \ge 0, \\ \sinh(p_n y)\sinh[p_n(b-\eta)] & \text{for } b \ge \eta > y \ge 0, \end{cases}$$

$$q_m = \frac{\pi m}{b}, \quad Q_m(x,\xi) = \begin{cases} \sinh(q_m \xi)\sinh[q_m(a-x)] & \text{for } a \ge x > \xi \ge 0, \\ \sinh(q_m x)\sinh[q_m(a-\xi)] & \text{for } a \ge \xi > x \ge 0. \end{cases}$$

The Green's function can be written in the form of a double series:

$$G(x,y,\xi,\eta) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(p_n x)\sin(q_m y)\sin(p_n \xi)\sin(q_m \eta)}{p_n^2 + q_m^2}, \quad p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}.$$

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $0 \le x \le a,\ 0 \le y \le b$. Third boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = f_1(y) \quad \text{at} \quad x = 0, \qquad \partial_x w + k_2 w = f_2(y) \quad \text{at} \quad x = a,$$
$$\partial_y w - k_3 w = f_3(x) \quad \text{at} \quad y = 0, \qquad \partial_y w + k_4 w = f_4(x) \quad \text{at} \quad y = b.$$

Solution:

$$\begin{aligned} w(x,y) = {} & \int_0^a \int_0^b \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\eta\,d\xi \\ & - \int_0^b f_1(\eta) G(x,y,0,\eta)\,d\eta + \int_0^b f_2(\eta) G(x,y,a,\eta)\,d\eta \\ & - \int_0^a f_3(\xi) G(x,y,\xi,0)\,d\xi + \int_0^a f_4(\xi) G(x,y,\xi,b)\,d\xi. \end{aligned}$$

Here,

$$G(x,y,\xi,\eta)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)\psi_m(y)\psi_m(\eta)}{\|\varphi_n\|^2\|\psi_m\|^2(\mu_n^2+\lambda_m^2)},$$

$$\varphi_n(x)=\cos(\mu_n x)+\frac{k_1}{\mu_n}\sin(\mu_n x),\qquad \|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{a}{2}\left(1+\frac{k_1^2}{\mu_n^2}\right),$$

$$\psi_m(y)=\cos(\lambda_m y)+\frac{k_3}{\lambda_m}\sin(\lambda_m y),\quad \|\psi_m\|^2=\frac{k_4}{2\lambda_m^2}\frac{\lambda_m^2+k_3^2}{\lambda_m^2+k_4^2}+\frac{k_3}{2\lambda_m^2}+\frac{b}{2}\left(1+\frac{k_3^2}{\lambda_m^2}\right),$$

where the μ_n and λ_m are positive roots of the transcendental equations

$$\frac{\tan(\mu a)}{\mu}=\frac{k_1+k_2}{\mu^2-k_1k_2},\qquad \frac{\tan(\lambda b)}{\lambda}=\frac{k_3+k_4}{\lambda^2-k_3k_4}.$$

► Domain: $0\le x\le a,\ 0\le y\le b$. Mixed boundary value problem.

A rectangle is considered. Boundary conditions are prescribed:

$$w=f_1(y)\quad\text{at}\quad x=0,\qquad \partial_x w=f_2(y)\quad\text{at}\quad x=a,$$
$$w=f_3(x)\quad\text{at}\quad y=0,\qquad \partial_y w=f_4(x)\quad\text{at}\quad y=b.$$

Solution:

$$\begin{aligned}w(x,y)&=\int_0^a\int_0^b\Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi\\&+\int_0^b f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=0}d\eta+\int_0^b f_2(\eta)G(x,y,a,\eta)\,d\eta\\&+\int_0^a f_3(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0}d\xi+\int_0^a f_4(\xi)G(x,y,\xi,b)\,d\xi.\end{aligned}$$

Two forms of representation of the Green's function:

$$G(x,y,\xi,\eta)=\frac{2}{a}\sum_{n=0}^{\infty}\frac{\sin(p_n x)\sin(p_n\xi)}{p_n\cosh(p_n b)}H_n(y,\eta)=\frac{2}{b}\sum_{m=0}^{\infty}\frac{\sin(q_m y)\sin(q_m\eta)}{q_m\cosh(q_m a)}Q_m(x,\xi),$$

where

$$p_n=\frac{\pi(2n+1)}{a},\quad H_n(y,\eta)=\begin{cases}\sinh(p_n\eta)\cosh[p_n(b-y)] & \text{for } b\ge y>\eta\ge 0,\\ \sinh(p_n y)\cosh[p_n(b-\eta)] & \text{for } b\ge\eta>y\ge 0,\end{cases}$$

$$q_m=\frac{\pi(2m+1)}{b},\quad Q_m(x,\xi)=\begin{cases}\sinh(q_m\xi)\cosh[q_m(a-x)] & \text{for } a\ge x>\xi\ge 0,\\ \sinh(q_m x)\cosh[q_m(a-\xi)] & \text{for } a\ge\xi>x\ge 0.\end{cases}$$

The Green's function can be written in the form of a double series:

$$G(x,y,\xi,\eta)=\frac{4}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)}{p_n^2+q_m^2},$$
$$p_n=\frac{\pi(2n+1)}{2a},\quad q_m=\frac{\pi(2m+1)}{2b}.$$

9.2.3 Problems in Polar Coordinate System

The two-dimensional Poisson equation in the polar coordinate system is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \Phi(r,\varphi) = 0, \qquad r = \sqrt{x^2+y^2}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A circle is considered. A boundary condition is prescribed:

$$w = f(\varphi) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\frac{R^2 - r^2}{r^2 - 2Rr\cos(\varphi-\eta) + R^2}\,d\eta + \int_0^{2\pi}\int_0^R \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where

$$G(r,\varphi,\xi,\eta) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{r}-\mathbf{r}_0|} - \frac{1}{2\pi}\ln\frac{R}{r_0|(R/r_0)^2\mathbf{r}_0 - \mathbf{r}|},$$

$$\mathbf{r} = \{x,y\}, \qquad x = r\cos\varphi, \qquad y = r\sin\varphi,$$

$$\mathbf{r}_0 = \{x_0,y_0\}, \quad x_0 = \xi\cos\eta, \quad y_0 = \xi\sin\eta.$$

The magnitude of a vector difference is calculated as

$$|a\mathbf{r} - b\mathbf{r}_0|^2 = a^2r^2 - 2abr\xi\cos(\varphi-\eta) + b^2\xi^2$$

where a and b are any scalars. Thus, we obtain

$$G(r,\varphi,\xi,\eta) = \frac{1}{4\pi}\ln\frac{r^2\xi^2 - 2R^2r\xi\cos(\varphi-\eta) + R^4}{R^2[r^2 - 2r\xi\cos(\varphi-\eta) + \xi^2]}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), A. G. Butkovskiy (1979).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A circle is considered. A boundary condition is prescribed:

$$\partial_r w + kw = f(\varphi) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi) = R\int_0^{2\pi} f(\eta)G(r,\varphi,R,\eta)\,d\eta + \int_0^{2\pi}\int_0^R \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where

$$G(r,\varphi,\xi,\eta) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_nJ_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2 + k^2R^2 - n^2)[J_n(\mu_{nm}R)]^2}\cos[n(\varphi-\eta)],$$

$$A_0 = 1, \quad A_n = 2 \quad (n = 1,\,2,\,\ldots).$$

Here, the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

► **Domain: $R \le r < \infty$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

The exterior of a circle is considered. A boundary condition is prescribed:

$$w = f(\varphi) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\frac{r^2 - R^2}{r^2 - 2Rr\cos(\varphi - \eta) + R^2}\,d\eta + \int_0^{2\pi}\!\!\int_R^{\infty} \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where the Green's function $G(r,\varphi,\xi,\eta)$ is defined by the formula presented in the first paragraph for $0 \le r \le R$, $0 \le \varphi \le 2\pi$.

⊙ *Literature*: A. G. Butkovskiy (1979).

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

An annular domain is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi) \quad \text{at} \quad r = R_1, \qquad w = f_2(\varphi) \quad \text{at} \quad r = R_2.$$

Solution:

$$w(r,\varphi) = R_1\int_0^{2\pi} f_1(\eta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta)\right]_{\xi=R_1} d\eta - R_2\int_0^{2\pi} f_2(\eta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta)\right]_{\xi=R_2} d\eta$$
$$+ \int_0^{2\pi}\!\!\int_{R_1}^{R_2} \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta.$$

Here,

$$G(r,\varphi,\xi,\eta) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\ln\frac{1}{r_n} - \ln\frac{R_1}{\xi r_n^*}\right),$$

where

$$r_n^2 = r^2 + \rho_n^2 - 2r\rho_n\cos(\varphi - \eta), \quad (r_n^*)^2 = r^2 + (\rho_n^*)^2 - 2r\rho_n^*\cos(\varphi - \eta),$$
$$\rho_n = \begin{cases} (R_1/R_2)^{2k}\xi & \text{for } n = 2k, \\ (R_2/R_1)^{2k+2}\xi & \text{for } n = 2k+1, \end{cases} \qquad \rho_n^* = \frac{R_1^2}{\rho_n}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \pi$. First boundary value problem.**

A semicircle is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi) \quad \text{at} \quad r = R, \qquad w = f_2(r) \quad \text{at} \quad \varphi = 0, \qquad w = f_3(r) \quad \text{at} \quad \varphi = \pi.$$

Solution:

$$w(r,\varphi)=-R\int_0^{\pi} f_1(\eta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta)\right]_{\xi=R} d\eta+\int_0^R f_2(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=0} d\xi$$
$$-\int_0^R f_3(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=\pi} d\xi+\int_0^{\pi}\int_0^R \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where

$$G(r,\varphi,\xi,\eta)=\frac{1}{4\pi}\ln\frac{r^2\xi^2-2R^2r\xi\cos(\varphi-\eta)+R^4}{R^2[r^2-2r\xi\cos(\varphi-\eta)+\xi^2]}-\frac{1}{4\pi}\ln\frac{r^2\xi^2-2R^2r\xi\cos(\varphi+\eta)+R^4}{R^2[r^2-2r\xi\cos(\varphi+\eta)+\xi^2]}.$$

See also Example 9.5 in Section 9.2.4.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ Domain: $0 \le r \le R$, $0 \le \varphi \le \pi/2$. First boundary value problem.

A quadrant of a circle is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi) \quad \text{at} \quad r = R, \qquad w = f_2(r) \quad \text{at} \quad \varphi = 0, \qquad w = f_3(r) \quad \text{at} \quad \varphi = \pi/2.$$

Solution:

$$w(r,\varphi)=-R\int_0^{\pi/2} f_1(\eta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta)\right]_{\xi=R} d\eta+\int_0^R f_2(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=0} d\xi$$
$$-\int_0^R f_3(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=\pi/2} d\xi+\int_0^{\pi/2}\int_0^R \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where

$$G(r,\varphi,\xi,\eta)=G_1(r,\varphi,\xi,\eta)-G_1(r,\varphi,\xi,2\pi-\eta)-G_1(r,\varphi,\xi,\pi-\eta)+G_1(r,\varphi,\xi,\pi+\eta),$$
$$G_1(r,\varphi,\xi,\eta)=\frac{1}{4\pi}\ln\frac{r^2\xi^2-2R^2r\xi\cos(\varphi-\eta)+R^4}{R^2[r^2-2r\xi\cos(\varphi-\eta)+\xi^2]}.$$

See also Example 9.6 in Section 9.2.4.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ Domain: $0 \le r \le R$, $0 \le \varphi \le \beta$. First boundary value problem.

A circular sector is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi) \quad \text{at} \quad r = R, \qquad w = f_2(r) \quad \text{at} \quad \varphi = 0, \qquad w = f_3(r) \quad \text{at} \quad \varphi = \beta.$$

Solution:

$$w(r,\varphi)=-R\int_0^\beta f_1(\eta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,\xi,\eta)\right]_{\xi=R}d\eta+\int_0^R f_2(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=0}d\xi$$
$$-\int_0^R f_3(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=\beta}d\xi+\int_0^\beta\int_0^R\Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta.$$

1°. For $\beta=\pi/n$, where n is a positive integer, the Green's function is expressed as

$$G(r,\varphi,\xi,\eta)=\sum_{k=0}^{n-1}\bigl[G_1(r,\varphi,\xi,2k\beta+\eta)-G_1(r,\varphi,\xi,2k\beta-\eta)\bigr],$$
$$G_1(r,\varphi,\xi,\eta)=\frac{1}{4\pi}\ln\frac{r^2\xi^2-2R^2r\xi\cos(\varphi-\eta)+R^4}{R^2[r^2-2r\xi\cos(\varphi-\eta)+\xi^2]}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

2°. For arbitrary β, the Green's function is given by

$$G(r,\varphi,\xi,\eta)=\frac{1}{2\pi}\ln\frac{\bigl|z^{\pi/\beta}-\bar\zeta^{\pi/\beta}\bigr|\bigl|R^{2\pi/\beta}-(\bar\zeta z)^{\pi/\beta}\bigr|}{\bigl|z^{\pi/\beta}-\zeta^{\pi/\beta}\bigr|\bigl|R^{2\pi/\beta}-(\zeta z)^{\pi/\beta}\bigr|},$$

where $z=re^{i\varphi}$, $\zeta=\xi e^{i\eta}$, $\bar\zeta=\xi e^{-i\eta}$, and $i^2=-1$.

► **Domain: $0\le r<\infty$, $0\le\varphi\le\beta$. First boundary value problem.**

A wedge domain is considered. Boundary conditions are prescribed:

$$w=f_1(r)\quad\text{at}\quad\varphi=0,\qquad w=f_2(r)\quad\text{at}\quad\varphi=\beta.$$

Solution:

$$w(r,\varphi)=\int_0^\infty f_1(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=0}d\xi-\int_0^\infty f_2(\xi)\frac{1}{\xi}\left[\frac{\partial}{\partial\eta}G(r,\varphi,\xi,\eta)\right]_{\eta=\beta}d\xi$$
$$+\int_0^\beta\int_0^\infty\Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where

$$G(r,\varphi,\xi,\eta)=\frac{1}{4\pi}\ln\frac{r^{2\pi/\beta}-2(r\xi)^{\pi/\beta}\cos[\pi(\varphi+\eta)/\beta]+\xi^{2\pi/\beta}}{r^{2\pi/\beta}-2(r\xi)^{\pi/\beta}\cos[\pi(\varphi-\eta)/\beta]+\xi^{2\pi/\beta}}.$$

Alternatively, the Green's function can be represented in the complex form

$$G(r,\varphi,\xi,\eta)=\frac{1}{2\pi}\ln\frac{\bigl|z^{\pi/\beta}-\bar\zeta^{\pi/\beta}\bigr|}{\bigl|z^{\pi/\beta}-\zeta^{\pi/\beta}\bigr|},\qquad z=re^{i\varphi},\quad\zeta=\xi e^{i\eta},\quad\bar\zeta=\xi e^{-i\eta},\quad i^2=-1.$$

9.2.4 Arbitrary Shape Domain. Conformal Mappings Method

► Description of the method. Tables of conformal mappings.

Any simply connected domain D in the xy-plane with a piecewise smooth boundary can be mapped in a mutually unique way, with an appropriate conformal mapping, onto the upper half-plane or into a unit circle in a uv-plane. Under a conformal mapping, a Poisson equation in the xy-plane transforms into a Poisson equation in the uv-plane; what is changed is the function Φ, as well as the function f in the boundary condition. Consequently, a first and a second boundary value problem for the plane domain D can be reduced, respectively, to a first and a second boundary value problem for the upper half-plane or a unit circle. The latter problems are considered above (see Sections 9.2.2 and 9.2.3).

A large number of conformal mappings (mappings defined by analytic functions) of various domains onto the upper half-plane or a unit circle can be found, for example, in Lavrik and Savenkov (1970), Lavrent'ev and Shabat (1973), and Ivanov and Trubetskov (1994).

Table 9.2 presents conformal mappings of some domains D in the complex plane z onto the upper half-plane $\operatorname{Im}\omega \geq 0$ in the complex plane ω. In the relations involving square roots, it is assumed that $\sqrt{\zeta} = \sqrt{|\zeta|}\,\bigl[\cos\bigl(\tfrac{1}{2}\varphi\bigr) + i\sin\bigl(\tfrac{1}{2}\varphi\bigr)\bigr]$, where $\varphi = \arg\zeta$ (i.e., the first branch of $\sqrt{\zeta}$ is taken).

Table 9.3 presents conformal mappings of some domains D in the complex plane z onto the unit circle $|\omega| \leq 1$ in the complex plane ω.

► General formula for the Green's function. Example boundary value problems.

Let a function $\omega = \omega(z)$ define a conformal mapping of a domain D in the complex plane z onto the upper half-plane in the complex plane ω. Then the Green's function of the first boundary value problem in D for the Poisson (Laplace) equation is expressed as

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\ln\left|\frac{\omega(z) - \bar{\omega}(\zeta)}{\omega(z) - \omega(\zeta)}\right|, \qquad z = x + iy, \;\; \zeta = \xi + i\eta, \tag{1}$$

where $\omega(z) = u(x,y) + iv(x,y)$ and $\bar{\omega}(z) = u(x,y) - iv(x,y)$.

The solution of the first boundary value problem for the Poisson equation is determined by the above Green's function in accordance with formula (2) specified in Section 9.2.1.

Example 9.4. Consider the first boundary value problem for the Poisson equation in the strip $-\infty < x < \infty$, $0 \leq y \leq a$. The function that maps this strip onto the upper half-plane has the form $\omega(z) = \exp(\pi z/a)$ (see the second row of Table 9.2). Substituting this expression into relation (1) and performing elementary transformations, we obtain the Green's function

$$G(x,y,\xi,\eta) = \frac{1}{4\pi}\ln\frac{\cosh[\pi(x-\xi)/a] - \cos[\pi(y+\eta)/a]}{\cosh[\pi(x-\xi)/a] - \cos[\pi(y-\eta)/a]}.$$

Example 9.5. Consider the first boundary value problem for the Poisson equation in a semicircle of radius a such that $D = \{x^2 + y^2 \leq a^2,\ y \geq 0\}$. The domain D is conformally mapped onto the upper half-plane by the function $\omega(z) = -(z/a + a/z)$ (see the sixth row in Table 9.2). Substituting this expression into (1), we arrive at the Green's function

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\ln\frac{|z-\bar{\zeta}|\,|a^2 - z\bar{\zeta}|}{|z-\zeta|\,|a^2 - z\zeta|}, \qquad z = x + iy, \quad \zeta = \xi + i\eta.$$

TABLE 9.2
Conformal mapping of some domains D in the z-plane onto the upper half-plane $\operatorname{Im}\omega \geq 0$ in the ω-plane. Notation: $z = x + iy$ and $\omega = u + iv$

No	Domain D in the z-plane	Transformation
1	First quadrant: $0 \leq x < \infty, 0 \leq y < \infty$	$\omega = a^2 z^2 + b$, a, b are real numbers
2	Infinite strip of width a: $-\infty < x < \infty, 0 \leq y \leq a$	$\omega = \exp(\pi z/a)$
3	Semiinfinite strip of width a: $0 \leq x < \infty, 0 \leq y \leq a$	$\omega = \cosh(\pi z/a)$
4	Plane with the cut in the real axis	$\omega = \sqrt{z}$
5	Interior of an infinite sector with angle β: $0 \leq \arg z \leq \beta, 0 \leq \lvert z\rvert < \infty$ $(0 < \beta \leq 2\pi)$	$\omega = z^{\pi/\beta}$
6	Upper half of a circle of radius a: $x^2 + y^2 \leq a^2, y \geq 0$	$\omega = -\dfrac{z}{a} - \dfrac{a}{z}$
7	Quadrant of a circle of radius a: $x^2 + y^2 \leq a^2, x \geq 0, y \geq 0$	$\omega = -\dfrac{z^2}{a^2} - \dfrac{a^2}{z^2}$
8	Sector of a circle of radius a with angle β: $x^2 + y^2 \leq a^2, 0 \leq \arg z \leq \beta$	$\omega = -\left(\dfrac{z}{a}\right)^{\pi/\beta} - \left(\dfrac{a}{z}\right)^{\pi/\beta}$
9	Upper half-plane with a circular domain or radius a removed: $y \geq 0, x^2 + y^2 \geq a^2$	$\omega = \dfrac{z}{a} + \dfrac{a}{z}$
10	Exterior of a parabola: $y^2 - 2px \geq 0$	$\omega = \sqrt{z - \frac{1}{2}p} - i\sqrt{\frac{1}{2}p}$
11	Interior of a parabola: $y^2 - 2px \leq 0$	$\omega = i\cosh\Bigl(\pi\sqrt{\frac{1}{2}z/p - \frac{1}{4}}\Bigr)$

Example 9.6. Consider the first boundary value problem for the Poisson equation in a quadrant of a circle of radius a, so that $D = \{x^2 + y^2 \leq a^2,\ x \geq 0,\ y \geq 0\}$. The conformal mapping of the domain D onto the upper half-plane is performed with the function $\omega(z) = -(z/a)^2 - (a/z)^2$ (see the seventh row of Table 9.2). Substituting this expression into (1) yields

$$G(x, y, \xi, \eta) = \frac{1}{2\pi} \ln \frac{|z^2 - \bar{\zeta}^2|\,|a^4 - z^2\bar{\zeta}^2|}{|z^2 - \zeta^2|\,|a^4 - z^2\zeta^2|}, \qquad z = x + iy, \quad \zeta = \xi + i\eta.$$

Let a function $\omega = \omega(z)$ define a conformal mapping of a domain D in the complex plane z onto the unit circle $|\omega| \leq 1$ in the complex plane ω. Then the Green's function of the first boundary value problem in D for the Laplace equation is given by

$$G(x, y, \xi, \eta) = \frac{1}{2\pi} \ln\left|\frac{1 - \bar{\omega}(\zeta)\omega(z)}{\omega(z) - \omega(\zeta)}\right|, \qquad z = x + iy,\ \ \zeta = \xi + i\eta. \tag{2}$$

⊙ *References for Section* 9.2.4: N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand (1955), A. G. Sveshnikov and A. N. Tikhonov (1974).

TABLE 9.3
Conformal mapping of some domains D in the z-plane onto the unit circle $|\omega| \le 1$.
Notation: $z = x + iy$, $\omega = u + iv$, $z_0 = x_0 + iy_0$, and $\bar{z}_0 = x_0 - iy_0$

No	Domain D in z-plane	Transformation
1	Upper half-plane: $-\infty < x < \infty,\ 0 \le y < \infty$	$\omega = e^{i\lambda}\dfrac{z - z_0}{z - \bar{z}_0}$, λ is a real number
2	A circle of unit radius: $x^2 + y^2 \le 1$	$\omega = e^{i\lambda}\dfrac{z - z_0}{1 - \bar{z}_0 z}$, λ is a real number
3	Exterior of a circle of radius a: $x^2 + y^2 \ge a^2$	$\omega = \dfrac{a}{z}$
4	Infinite strip of width a: $-\infty < x < \infty,\ 0 \le y \le a$	$\omega = \dfrac{\exp(\pi z/a) - \exp(\pi z_0/a)}{\exp(\pi z/a) - \exp(\pi \bar{z}_0/a)}$
5	Semicircle of radius a: $x^2 + y^2 \le a^2,\ x \ge 0$	$\omega = i\dfrac{z^2 + 2az - a^2}{z^2 - 2az - a^2}$
6	Sector of a unit circle with angle β: $|z| \le 1,\ 0 \le \arg z \le \beta$	$\omega = \dfrac{(1 + z^{\pi/\beta})^2 - i(1 - z^{\pi/\beta})^2}{(1 + z^{\pi/\beta})^2 + i(1 - z^{\pi/\beta})^2}$
7	Exterior of an ellipse with semiaxes a and b: $(x/a)^2 + (y/b)^2 \ge 1$	$z = \dfrac{1}{2}\left[(a - b)\omega + \dfrac{a + b}{\omega}\right]$

9.3 Helmholtz Equation $\Delta_2 w + \lambda w = -\Phi(\mathrm{x})$

Many problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic, etc.) lead to the two-dimensional Helmholtz equation. For $\lambda < 0$, this equation describes mass transfer processes with volume chemical reactions of the first order. Moreover, any elliptic equation with constant coefficients can be reduced to the Helmholtz equation.

9.3.1 General Remarks, Results, and Formulas

► **Some definitions.**

The Helmholtz equation is called homogeneous if $\Phi = 0$ and nonhomogeneous if $\Phi \ne 0$. A homogeneous boundary value problem is a boundary value problem for the homogeneous Helmholtz equation with homogeneous boundary conditions; a particular solution of a homogeneous boundary value problem is $w = 0$.

The values λ_n of the parameter λ for which there are nontrivial solutions (solutions other than identical zero) of the homogeneous boundary value problem are called eigenvalues and the corresponding solutions, $w = w_n$, are called eigenfunctions of the boundary value problem.

In what follows, the first, second, and third boundary value problems for the two-dimensional Helmholtz equation in a finite two-dimensional domain S with boundary L are considered. For the third boundary value problem with the boundary condition

$$\frac{\partial w}{\partial N} + kw = 0 \quad \text{for} \quad \mathbf{r} \in L,$$

it is assumed that $k > 0$. Here, $\frac{\partial w}{\partial N}$ is the derivative along the outward normal to the contour L, and $\mathbf{r} = \{x, y\}$.

► **Properties of eigenvalues and eigenfunctions.**

1°. There are infinitely many eigenvalues $\{\lambda_n\}$; the set of eigenvalues forms a discrete spectrum for the given boundary value problem.

2°. All eigenvalues are positive, except for the eigenvalue $\lambda_0 = 0$ existing in the second boundary value problem (the corresponding eigenfunction is $w_0 = \text{const}$). We number the eigenvalues in order of increasing magnitude so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$.

3°. The eigenvalues tend to infinity as the number n increases. The following asymptotic estimate holds:

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{S_2}{4\pi},$$

where S_2 is the area of the two-dimensional domain under study.

4°. The eigenfunctions $w_n = w_n(x, y)$ are defined up to a constant multiplier. Any two eigenfunctions corresponding to different eigenvalues, $\lambda_n \neq \lambda_m$, are orthogonal:

$$\int_S w_n w_m \, dS = 0.$$

5°. Any twice continuously differentiable function $f = f(\mathbf{r})$ that satisfies the boundary conditions of a boundary value problem can be expanded into a uniformly convergent series in the eigenfunctions of the boundary value problem:

$$f = \sum_{n=1}^{\infty} f_n w_n, \qquad \text{where} \quad f_n = \frac{1}{\|w_n\|^2} \int_S f w_n \, dS, \quad \|w_n\|^2 = \int_S w_n^2 \, dS.$$

If f is square summable, then the series converges in mean.

6°. The eigenvalues of the first boundary value problem do not increase if the domain is extended.

Remark 9.5. In a two-dimensional problem, finitely many linearly independent eigenfunctions $w_n^{(1)}$, $w_n^{(2)}$, …, $w_n^{(p)}$ generally correspond to each eigenvalue λ_n. These functions can always be replaced by their linear combinations,

$$\bar{w}_n^{(j)} = c_{j,1} w_n^{(1)} + \cdots + c_{j,j-1} w_n^{(j-1)} + w_n^{(j)}, \qquad j = 1,\, 2,\, \ldots,\, p,$$

so that the new eigenfunctions $\bar{w}_n^{(1)}$, $\bar{w}_n^{(2)}$, …, $\bar{w}_n^{(p)}$ are pairwise orthogonal. Therefore, without loss of generality, we assume that all the eigenfunctions are orthogonal.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► **Nonhomogeneous Helmholtz equation with homogeneous boundary conditions.**

Three cases are possible.

1°. If the equation parameter λ is not equal to any one of the eigenvalues, then there exists the series solution

$$w = \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n - \lambda} w_n, \quad \text{where} \quad A_n = \frac{1}{\|w_n\|^2} \int_S \Phi w_n \, dS, \quad \|w_n\|^2 = \int_S w_n^2 \, dS.$$

2°. If λ is equal to some eigenvalue, $\lambda = \lambda_m$, then the solution of the nonhomogeneous problem exists only if the function Φ is orthogonal to w_m, i.e.,

$$\int_S \Phi w_m \, dS = 0.$$

In this case the system is expressed as

$$w = \sum_{n=1}^{m-1} \frac{A_n}{\lambda_n - \lambda_m} w_n + \sum_{n=m+1}^{\infty} \frac{A_n}{\lambda_n - \lambda_m} w_n + C w_m, \quad A_n = \frac{1}{\|w_n\|^2} \int_S \Phi w_n \, dS,$$

where $\|w_n\|^2 = \displaystyle\int_S w_n^2 \, dS$, and C is an arbitrary constant.

3°. If $\lambda = \lambda_m$ and $\displaystyle\int_S \Phi w_m \, dS \neq 0$, then the boundary value problem for the nonhomogeneous equation does not have solutions.

Remark 9.6. If p_n mutually orthogonal eigenfunctions $w_n^{(j)}$ $(j = 1, 2, \ldots, p_n)$ correspond to each eigenvalue λ_n, then, for $\lambda \neq \lambda_n$, the solution is written as

$$w = \sum_{n=1}^{\infty} \sum_{j=1}^{p_n} \frac{A_n^{(j)}}{\lambda_n - \lambda} w_n^{(j)}, \quad \text{where} \quad A_n^{(j)} = \frac{1}{\|w_n^{(j)}\|^2} \int_S \Phi w_n^{(j)} \, dS, \quad \|w_n^{(j)}\|^2 = \int_S \bigl[w_n^{(j)}\bigr]^2 \, dS.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► **Solution of nonhomogeneous boundary value problem of general form.**

1°. The solution of the first boundary value problem for the Helmholtz equation with the boundary condition

$$w = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in L$$

can be represented in the form

$$w(\mathbf{r}) = \int_S \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho}) \, dS_\rho - \int_L f(\boldsymbol{\rho}) \frac{\partial}{\partial N_\rho} G(\mathbf{r}, \boldsymbol{\rho}) \, dL_\rho. \tag{1}$$

Here, $\mathbf{r} = \{x, y\}$ and $\boldsymbol{\rho} = \{\xi, \eta\}$ $(\mathbf{r} \in S, \boldsymbol{\rho} \in S)$; $\frac{\partial}{\partial N_\rho}$ denotes the derivative along the outward normal to the contour L with respect to the variables ξ and η. The Green's function is given by the series

$$G(\mathbf{r}, \boldsymbol{\rho}) = \sum_{n=1}^{\infty} \frac{w_n(\mathbf{r}) w_n(\boldsymbol{\rho})}{\|w_n\|^2 (\lambda_n - \lambda)}, \qquad \lambda \neq \lambda_n, \tag{2}$$

where the w_n and λ_n are the eigenfunctions and eigenvalues of the homogeneous first boundary value problem.

2°. The solution of the second boundary value problem with the boundary condition

$$\frac{\partial w}{\partial N} = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in L$$

can be written as

$$w(\mathbf{r}) = \int_S \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dS_\rho + \int_L f(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dL_\rho. \tag{3}$$

Here, the Green's function is given by the series

$$G(\mathbf{r}, \boldsymbol{\rho}) = -\frac{1}{S_2 \lambda} + \sum_{n=1}^{\infty} \frac{w_n(\mathbf{r}) w_n(\boldsymbol{\rho})}{\|w_n\|^2 (\lambda_n - \lambda)}, \qquad \lambda \neq \lambda_n, \tag{4}$$

where S_2 is the area of the two-dimensional domain under consideration, and the λ_n and w_n are the positive eigenvalues and the corresponding eigenfunctions of the homogeneous second boundary value problem. For clarity, the term corresponding to the zero eigenvalue $\lambda_0 = 0$ ($w_0 = \text{const}$) is singled out in (4).

3°. The solution of the third boundary value problem for the Helmholtz equation with the boundary condition

$$\frac{\partial w}{\partial N} + kw = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in L$$

is given by formula (3), where the Green's function is defined by series (2), which involves the eigenfunctions w_n and eigenvalues λ_n of the homogeneous third boundary value problem.

► Boundary conditions at infinity in the case of an infinite domain.

In what follows, the function Φ is assumed to be finite or sufficiently rapidly decaying as $r \to \infty$.

1°. For $\lambda < 0$, in the case of an infinite domain, the vanishing condition of the solution at infinity is set,

$$w \to 0 \quad \text{as} \quad r \to \infty.$$

2°. For $\lambda > 0$, if the domain is unbounded, the radiation conditions (*Sommerfeld conditions*) at infinity are used. In two-dimensional problems, these conditions are written as

$$\lim_{r\to\infty} \sqrt{r}\, w = \text{const}, \qquad \lim_{r\to\infty} \sqrt{r} \left(\frac{\partial w}{\partial r} + i\sqrt{\lambda}\, w \right) = 0, \tag{5}$$

where $i^2 = -1$.

To identify a single solution, the principle of limit absorption and the principle of limit amplitude are also used.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

9.3.2 Problems in Cartesian Coordinate System

A two-dimensional nonhomogeneous Helmholtz equation in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \lambda w = -\Phi(x, y).$$

► **Particular solutions and some relations.**

1°. Particular solutions of the homogeneous equation ($\Phi \equiv 0$):

$$\begin{aligned}
&w = (Ax + B)(C\cos\mu y + D\sin\mu y), && \lambda = \mu^2,\\
&w = (Ax + B)(C\cosh\mu y + D\sinh\mu y), && \lambda = -\mu^2,\\
&w = (A\cos\mu x + B\sin\mu x)(Cy + D), && \lambda = \mu^2,\\
&w = (A\cosh\mu x + B\sinh\mu x)(Cy + D), && \lambda = -\mu^2,\\
&w = (A\cos\mu_1 x + B\sin\mu_1 x)(C\cos\mu_2 y + D\sin\mu_2 y), && \lambda = \mu_1^2 + \mu_2^2,\\
&w = (A\cos\mu_1 x + B\sin\mu_1 x)(C\cosh\mu_2 y + D\sinh\mu_2 y), && \lambda = \mu_1^2 - \mu_2^2,\\
&w = (A\cosh\mu_1 x + B\sinh\mu_1 x)(C\cos\mu_2 y + D\sin\mu_2 y), && \lambda = -\mu_1^2 + \mu_2^2,\\
&w = (A\cosh\mu_1 x + B\sinh\mu_1 x)(C\cosh\mu_2 y + D\sinh\mu_2 y), && \lambda = -\mu_1^2 - \mu_2^2,
\end{aligned}$$

where A, B, C, and D are arbitrary constants.

2°. Fundamental solutions:

$$\begin{aligned}
\mathscr{E}(x, y) &= \frac{1}{2\pi} K_0(sr) && \text{if} \quad \lambda = -s^2 < 0,\\
\mathscr{E}(x, y) &= \frac{i}{4} H_0^{(1)}(kr) && \text{if} \quad \lambda = k^2 > 0,\\
\mathscr{E}(x, y) &= -\frac{i}{4} H_0^{(2)}(kr) && \text{if} \quad \lambda = k^2 > 0,
\end{aligned}$$

where $r = \sqrt{x^2 + y^2}$, $K_0(z)$ is the modified Bessel function of the second kind, $H_0^{(1)}(z)$ and $H_0^{(2)}(z)$ are the Hankel functions of the first and second kind of order 0, and $i^2 = -1$. The leading term of the asymptotic expansion of the fundamental solutions, as $r \to 0$, is given by $\frac{1}{2\pi} \ln \frac{1}{r}$.

3°. Suppose $w = w(x, y)$ is a solution of the homogeneous Helmholtz equation. Then the functions

$$\begin{aligned}
w_1 &= w(x + C_1, \pm y + C_2),\\
w_2 &= w(-x + C_1, \pm y + C_2),\\
w_3 &= w(x\cos\theta + y\sin\theta + C_1,\ -x\sin\theta + y\cos\theta + C_2),
\end{aligned}$$

where C_1, C_2, and θ are arbitrary constants, are also solutions of the equation.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

▶ **Domain:** $-\infty < x < \infty$, $-\infty < y < \infty$.

1°. Solution for $\lambda = -s^2 < 0$:

$$w(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta)K_0(s\varrho)\,d\xi\,d\eta, \qquad \varrho = \sqrt{(x-\xi)^2+(y-\eta)^2}.$$

2°. Solution for $\lambda = k^2 > 0$:

$$w(x,y) = -\frac{i}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta)H_0^{(2)}(k\varrho)\,d\xi\,d\eta, \qquad \varrho = \sqrt{(x-\xi)^2+(y-\eta)^2}.$$

The radiation conditions (Sommerfeld conditions) at infinity were used to obtain this solution (see Section 9.3.1, condition (5)).

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), A. N. Tikhonov and A. A. Samarskii (1990).

▶ **Domain:** $-\infty < x < \infty$, $0 \le y < \infty$. **First boundary value problem.**

A half-plane is considered. A boundary condition is prescribed:

$$w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi + \int_0^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta.$$

1°. The Green's function for $\lambda = -s^2 < 0$:

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\big[K_0(s\varrho_1) - K_0(s\varrho_2)\big],$$
$$\varrho_1 = \sqrt{(x-\xi)^2+(y-\eta)^2}, \quad \varrho_2 = \sqrt{(x-\xi)^2+(y+\eta)^2}.$$

2°. The Green's function for $\lambda = k^2 > 0$:

$$G(x,y,\xi,\eta) = -\frac{i}{4}\big[H_0^{(2)}(k\varrho_1) - H_0^{(2)}(k\varrho_2)\big].$$

The radiation conditions at infinity were used to obtain this relation (see Section 9.3.1, condition (5)).

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ **Domain:** $-\infty < x < \infty$, $0 \le y < \infty$. **Second boundary value problem.**

A half-plane is considered. A boundary condition is prescribed:

$$\partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = -\int_{-\infty}^{\infty} f(\xi) G(x,y,\xi,0)\, d\xi + \int_0^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\xi\, d\eta.$$

1°. The Green's function for $\lambda = -s^2 < 0$:

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\bigl[K_0(s\varrho_1) + K_0(s\varrho_2)\bigr],$$
$$\varrho_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad \varrho_2 = \sqrt{(x-\xi)^2 + (y+\eta)^2}.$$

2°. The Green's function for $\lambda = k^2 > 0$:

$$G(x,y,\xi,\eta) = -\frac{i}{4}\bigl[H_0^{(2)}(k\varrho_1) + H_0^{(2)}(k\varrho_2)\bigr].$$

The radiation conditions at infinity were used to obtain this relation (see Section 9.3.1, condition (5)).

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. First boundary value problem.**

A quadrant of the plane is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \int_0^{\infty} f_1(\eta)\left[\frac{\partial}{\partial\xi} G(x,y,\xi,\eta)\right]_{\xi=0} d\eta + \int_0^{\infty} f_2(\xi)\left[\frac{\partial}{\partial\eta} G(x,y,\xi,\eta)\right]_{\eta=0} d\xi$$
$$+ \int_0^{\infty}\int_0^{\infty} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\xi\, d\eta.$$

1°. The Green's function for $\lambda = -s^2 < 0$:

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\bigl[K_0(s\varrho_1) - K_0(s\varrho_2) - K_0(s\varrho_3) + K_0(s\varrho_4)\bigr],$$
$$\varrho_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad \varrho_2 = \sqrt{(x-\xi)^2 + (y+\eta)^2},$$
$$\varrho_3 = \sqrt{(x+\xi)^2 + (y-\eta)^2}, \quad \varrho_4 = \sqrt{(x+\xi)^2 + (y+\eta)^2}.$$

2°. The Green's function for $\lambda = k^2 > 0$:

$$G(x,y,\xi,\eta) = -\frac{i}{4}\bigl[H_0^{(2)}(k\varrho_1) - H_0^{(2)}(k\varrho_2) - H_0^{(2)}(k\varrho_3) + H_0^{(2)}(k\varrho_4)\bigr].$$

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$. Second boundary value problem.**

A quadrant of the plane is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y) \quad \text{at} \quad x = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = -\int_0^\infty f_1(\eta) G(x,y,0,\eta)\,d\eta - \int_0^\infty f_2(\xi) G(x,y,\xi,0)\,d\xi + \int_0^\infty \int_0^\infty \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\xi\,d\eta.$$

1°. The Green's function for $\lambda = -s^2 < 0$:

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\bigl[K_0(s\varrho_1) + K_0(s\varrho_2) + K_0(s\varrho_3) + K_0(s\varrho_4)\bigr],$$
$$\varrho_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad \varrho_2 = \sqrt{(x-\xi)^2 + (y+\eta)^2},$$
$$\varrho_3 = \sqrt{(x+\xi)^2 + (y-\eta)^2}, \quad \varrho_4 = \sqrt{(x+\xi)^2 + (y+\eta)^2}.$$

2°. The Green's function for $\lambda = k^2 > 0$:

$$G(x,y,\xi,\eta) = -\frac{i}{4}\bigl[H_0^{(2)}(k\varrho_1) + H_0^{(2)}(k\varrho_2) + H_0^{(2)}(k\varrho_3) + H_0^{(2)}(k\varrho_4)\bigr].$$

► **Domain: $-\infty < x < \infty$, $0 \le y \le a$. First boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(x) \quad \text{at} \quad y = 0, \qquad w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = \int_{-\infty}^\infty f_1(\xi)\left[\frac{\partial}{\partial\eta} G(x,y,\xi,\eta)\right]_{\eta=0} d\xi - \int_{-\infty}^\infty f_2(\xi)\left[\frac{\partial}{\partial\eta} G(x,y,\xi,\eta)\right]_{\eta=a} d\xi + \int_0^a \int_{-\infty}^\infty \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\xi\,d\eta.$$

Green's function:

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=1}^\infty \frac{1}{\beta_n}\exp\bigl(-\beta_n|x-\xi|\bigr)\sin(q_n y)\sin(q_n\eta), \quad q_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{q_n^2 - \lambda}.$$

Alternatively, the Green's function for $\lambda = -s^2 < 0$ can be represented as

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\bigl[K_0(s\varrho_{1n}) - K_0(s\varrho_{2n})\bigr],$$
$$\varrho_{n1} = \sqrt{(x-\xi)^2 + (y-\eta-2na)^2}, \quad \varrho_{n2} = \sqrt{(x-\xi)^2 + (y+\eta+2na)^2}.$$

► **Domain: $-\infty < x < \infty$, $0 \le y \le a$. Second boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$\partial_y w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = -\int_{-\infty}^{\infty} f_1(\xi) G(x,y,\xi,0)\,d\xi + \int_{-\infty}^{\infty} f_2(\xi) G(x,y,\xi,a)\,d\xi + \int_0^a \int_{-\infty}^{\infty} \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\xi\,d\eta.$$

Green's function:

$$G(x,y,\xi,\eta) = \frac{1}{2a} \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\beta_n} \exp\bigl(-\beta_n |x-\xi|\bigr) \cos(q_n y) \cos(q_n \eta),$$

$$q_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{q_n^2 - \lambda}, \quad \varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

Alternatively, the Green's function for $\lambda = -s^2 < 0$ can be represented as

$$G(x,y,\xi,\eta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \bigl[K_0(s\varrho_{1n}) + K_0(s\varrho_{2n})\bigr],$$

$$\varrho_{n1} = \sqrt{(x-\xi)^2 + (y-\eta_{n1})^2}, \quad \eta_{n1} = 2na + \eta,$$
$$\varrho_{n2} = \sqrt{(x-\xi)^2 + (y-\eta_{n2})^2}, \quad \eta_{n2} = 2na - \eta.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $-\infty < x < \infty$, $0 \le y \le a$. Third boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$\partial_y w - k_1 w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w + k_2 w = f_2(x) \quad \text{at} \quad y = a.$$

The solution $w(x,y)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\varphi_n(y)\varphi_n(\eta)}{\|\varphi_n\|^2 \beta_n} \exp\bigl(-\beta_n |x-\xi|\bigr), \qquad \beta_n = \sqrt{\mu_n^2 - \lambda},$$

$$\varphi_n(y) = \mu_n \cos(\mu_n y) + k_1 \sin(\mu_n y), \quad \|\varphi_n\|^2 = \frac{1}{2}(\mu_n^2 + k_1^2)\left[a + \frac{(k_1+k_2)(\mu_n^2 + k_1 k_2)}{(\mu_n^2 + k_1^2)(\mu_n^2 + k_2^2)}\right].$$

Here, the μ_n are positive roots of the transcendental equation $\tan(\mu a) = \dfrac{(k_1+k_2)\mu}{\mu^2 - k_1 k_2}$.

► **Domain: $-\infty < x < \infty,\ 0 \le y \le a$. Mixed boundary value problem.**

An infinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f_1(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi + \int_{-\infty}^{\infty} f_2(\xi)G(x,y,\xi,a)\,d\xi$$
$$+ \int_0^a\int_{-\infty}^{\infty} \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta,$$

where

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=0}^{\infty}\frac{1}{\beta_n}\exp\bigl(-\beta_n|x-\xi|\bigr)\sin(q_n y)\sin(q_n\eta),$$
$$q_n = \frac{\pi(2n+1)}{2a}, \qquad \beta_n = \sqrt{q_n^2-\lambda}.$$

► **Domain: $0 \le x < \infty,\ 0 \le y \le a$. First boundary value problem.**

A semiinfinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0, \qquad w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = \int_0^a\int_0^{\infty} \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta + \int_0^a f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=0} d\eta$$
$$+ \int_0^{\infty} f_2(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi - \int_0^{\infty} f_3(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=a} d\xi,$$

where

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{\beta_n}\bigl[\exp\bigl(-\beta_n|x-\xi|\bigr) - \exp\bigl(-\beta_n|x+\xi|\bigr)\bigr]\sin(q_n y)\sin(q_n\eta),$$
$$q_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{q_n^2-\lambda}.$$

► **Domain: $0 \le x < \infty,\ 0 \le y \le a$. Second boundary value problem.**

A semiinfinite strip is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y) \quad \text{at} \quad x = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = \int_0^a\int_0^{\infty} \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta - \int_0^a f_1(\eta)G(x,y,0,\eta)\,d\eta$$
$$- \int_0^{\infty} f_2(\xi)G(x,y,\xi,0)\,d\xi + \int_0^{\infty} f_3(\xi)G(x,y,\xi,a)\,d\xi,$$

where

$$G(x,y,\xi,\eta) = \frac{1}{2a}\sum_{n=0}^{\infty}\frac{\varepsilon_n}{\beta_n}\bigl[\exp\bigl(-\beta_n|x-\xi|\bigr) + \exp\bigl(-\beta_n|x+\xi|\bigr)\bigr]\cos(q_n y)\cos(q_n\eta),$$

$$q_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{q_n^2 - \lambda}, \quad \varepsilon = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

► **Domain: $0 \le x < \infty,\ 0 \le y \le a$. Third boundary value problem.**

A semiinfinite strip is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = f_1(y) \text{ at } x=0, \quad \partial_y w - k_2 w = f_2(x) \text{ at } y=0, \quad \partial_y w + k_3 w = f_3(x) \text{ at } y=a.$$

The solution $w(x,y)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,\xi,\eta) = \sum_{n=1}^{\infty}\frac{\varphi_n(y)\varphi_n(\eta)}{\|\varphi_n\|^2\beta_n(\beta_n+k_1)}H_n(x,\xi), \qquad \beta_n = \sqrt{\mu_n^2 - \lambda},$$

$$\varphi_n(y) = \mu_n\cos(\mu_n y) + k_2\sin(\mu_n y), \quad \|\varphi_n\|^2 = \frac{1}{2}(\mu_n^2 + k_2^2)\left[a + \frac{(k_2+k_3)(\mu_n^2 + k_2 k_3)}{(\mu_n^2 + k_2^2)(\mu_n^2 + k_3^2)}\right],$$

$$H_n(x,\xi) = \begin{cases} \exp(-\beta_n x)\bigl[\beta_n\cosh(\beta_n\xi) + k_1\sinh(\beta_n\xi)\bigr] & \text{for } x > \xi, \\ \exp(-\beta_n\xi)\bigl[\beta_n\cosh(\beta_n x) + k_1\sinh(\beta_n x)\bigr] & \text{for } \xi > x. \end{cases}$$

Here, the μ_n are positive roots of the transcendental equation $\tan(\mu a) = \dfrac{(k_2+k_3)\mu}{\mu^2 - k_2 k_3}$.

► **Domain: $0 \le x < \infty,\ 0 \le y \le a$. Mixed boundary value problems.**

1°. A semiinfinite strip is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad \partial_y w = f_2(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = \int_0^a f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=0} d\eta - \int_0^\infty f_2(\xi)G(x,y,\xi,0)\,d\xi$$
$$+ \int_0^\infty f_3(\xi)G(x,y,\xi,a)\,d\xi + \int_0^a\int_0^\infty \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta,$$

where

$$G(x,y,\xi,\eta) = \frac{1}{2a}\sum_{n=0}^{\infty}\frac{\varepsilon_n}{\beta_n}\bigl[\exp\bigl(-\beta_n|x-\xi|\bigr) - \exp\bigl(-\beta_n|x+\xi|\bigr)\bigr]\cos(q_n y)\cos(q_n\eta),$$

$$q_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{q_n^2 - \lambda}, \quad \varepsilon = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

2°. A semiinfinite strip is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0, \qquad w = f_3(x) \quad \text{at} \quad y = a.$$

Solution:

$$w(x,y) = -\int_0^a f_1(\eta) G(x,y,0,\eta)\,d\eta + \int_0^\infty f_2(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=0} d\xi$$
$$- \int_0^\infty f_3(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=a} d\xi + \int_0^a \int_0^\infty \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\xi\,d\eta,$$

where

$$G(x,y,\xi,\eta) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{\beta_n} \left[\exp\bigl(-\beta_n |x-\xi|\bigr) + \exp\bigl(-\beta_n |x+\xi|\bigr)\right] \sin(q_n y) \sin(q_n \eta),$$
$$q_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{q_n^2 - \lambda}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$. First boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(y) \quad \text{at} \quad x = a,$$
$$w = f_3(x) \quad \text{at} \quad y = 0, \qquad w = f_4(x) \quad \text{at} \quad y = b.$$

1°. Eigenvalues of the one-dimensional problem (it is convenient to label them with a double subscript):

$$\lambda_{nm} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right); \qquad n = 1,\, 2,\, \ldots; \quad m = 1,\, 2,\, \ldots$$

Eigenfunctions and the norm squared:

$$w_{nm} = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right), \qquad \|w_{nm}\|^2 = \frac{ab}{4}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

2°. Solution for $\lambda \neq \lambda_{nm}$:

$$w(x,y) = \int_0^a \int_0^b \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\eta\,d\xi$$
$$+ \int_0^b f_1(\eta) \left[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\right]_{\xi=0} d\eta - \int_0^b f_2(\eta) \left[\frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\right]_{\xi=a} d\eta$$
$$+ \int_0^a f_3(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=0} d\xi - \int_0^a f_4(\xi) \left[\frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\right]_{\eta=b} d\xi.$$

Two forms of representation of the Green's function:

$$G(x,y,\xi,\eta) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin(p_n x)\sin(p_n\xi)}{\beta_n \sinh(\beta_n b)} H_n(y,\eta) = \frac{2}{b}\sum_{m=1}^{\infty} \frac{\sin(q_m y)\sin(q_m\eta)}{\mu_m \sinh(\mu_m a)} Q_m(x,\xi),$$

where

$$p_n = \frac{\pi n}{a}, \quad \beta_n = \sqrt{p_n^2 - \lambda}, \quad q_m = \frac{\pi m}{b}, \quad \mu_m = \sqrt{q_m^2 - \lambda},$$

$$H_n(y,\eta) = \begin{cases} \sinh(\beta_n \eta)\sinh[\beta_n(b-y)] & \text{for } b \ge y > \eta \ge 0, \\ \sinh(\beta_n y)\sinh[\beta_n(b-\eta)] & \text{for } b \ge \eta > y \ge 0, \end{cases}$$

$$Q_m(x,\xi) = \begin{cases} \sinh(\mu_m \xi)\sinh[\mu_m(a-x)] & \text{for } a \ge x > \xi \ge 0, \\ \sinh(\mu_m x)\sinh[\mu_m(a-\xi)] & \text{for } a \ge \xi > x \ge 0. \end{cases}$$

Alternatively, the Green's function can be written as the double series

$$G(x,y,\xi,\eta) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)}{p_n^2 + q_m^2 - \lambda}, \quad p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$. Second boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y) \ \text{at} \ x = 0, \qquad \partial_x w = f_2(y) \ \text{at} \ x = a,$$
$$\partial_y w = f_3(x) \ \text{at} \ y = 0, \qquad \partial_y w = f_4(x) \ \text{at} \ y = b.$$

1°. Eigenvalues of the homogeneous problem:

$$\lambda_{nm} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right); \qquad n = 0,\ 1,\ 2,\ \ldots; \quad m = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions and the norm squared:

$$w_{nm} = \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right), \quad \|w_{nm}\|^2 = \frac{ab}{4}(1+\delta_{n0})(1+\delta_{m0}), \quad \delta_{n0} = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{if } n \ne 0. \end{cases}$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

2°. Solution for $\lambda \ne \lambda_{nm}$:

$$\begin{aligned} w(x,y) = {} & \int_0^a \int_0^b \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\eta\, d\xi \\ & - \int_0^b f_1(\eta) G(x,y,0,\eta)\, d\eta + \int_0^b f_2(\eta) G(x,y,a,\eta)\, d\eta \\ & - \int_0^a f_3(\xi) G(x,y,\xi,0)\, d\xi + \int_0^a f_4(\xi) G(x,y,\xi,b)\, d\xi. \end{aligned}$$

Two forms of representation of the Green's function:

$$G(x,y,\xi,\eta) = \frac{1}{a}\sum_{n=0}^{\infty}\frac{\varepsilon_n\cos(p_n x)\cos(p_n\xi)}{\beta_n\sinh(\beta_n b)}H_n(y,\eta)$$
$$= \frac{1}{b}\sum_{m=0}^{\infty}\frac{\varepsilon_m\cos(q_m y)\cos(q_m\eta)}{\mu_m\sinh(\mu_m a)}Q_m(x,\xi),$$

where

$$p_n = \frac{\pi n}{a}, \quad H_n(y,\eta) = \begin{cases}\cosh(\beta_n\eta)\cosh[\beta_n(b-y)] & \text{for } y>\eta,\\ \cosh(\beta_n y)\cosh[\beta_n(b-\eta)] & \text{for } \eta>y,\end{cases}$$

$$q_m = \frac{\pi m}{b}, \quad Q_m(x,\xi) = \begin{cases}\cosh(\mu_m\xi)\cosh[\mu_m(a-x)] & \text{for } x>\xi,\\ \cosh(\mu_m x)\cosh[\mu_m(a-\xi)] & \text{for } \xi>x,\end{cases}$$

$$\beta_n = \sqrt{p_n^2-\lambda}, \quad \mu_m = \sqrt{q_m^2-\lambda}, \quad \varepsilon_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0.\end{cases}$$

The Green's function can also be written as the double series

$$G(x,y,\xi,\eta) = \frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)}{p_n^2+q_m^2-\lambda},$$
$$p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}.$$

◆ *Only the eigenvalues and eigenfunctions of homogeneous boundary value problems for the homogeneous Helmholtz equation (with $\Phi\equiv 0$) are presented below. The solutions of the corresponding nonhomogeneous boundary value problems (with $\Phi\not\equiv 0$) can be constructed by the relations specified in Section 9.3.1.*

► **Domain: $0\le x\le a$, $0\le y\le b$. Third boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = 0 \quad \text{at} \quad x=0, \qquad \partial_x w + k_2 w = 0 \quad \text{at} \quad x=a,$$
$$\partial_y w - k_3 w = 0 \quad \text{at} \quad y=0, \qquad \partial_y w + k_4 w = 0 \quad \text{at} \quad y=b.$$

Eigenvalues:

$$\lambda_{nm} = \mu_n^2 + \nu_m^2,$$

where the μ_n and ν_m are positive roots of the transcendental equations

$$\tan(\mu a) = \frac{(k_1+k_2)\mu}{\mu^2-k_1k_2}, \qquad \tan(\nu b) = \frac{(k_3+k_4)\nu}{\nu^2-k_3k_4}.$$

Eigenfunctions:

$$w_{nm} = (\mu_n\cos\mu_n x + k_1\sin\mu_n x)(\nu_m\cos\nu_m y + k_3\sin\nu_m y).$$

The square of the norm of an eigenfunction:

$$\|w_{nm}\|^2 = \frac{1}{4}(\mu_n^2 + k_1^2)(\nu_m^2 + k_3^2)\left[a + \frac{(k_1 + k_2)(\mu_n^2 + k_1 k_2)}{(\mu_n^2 + k_1^2)(\mu_n^2 + k_2^2)}\right]\left[b + \frac{(k_3 + k_4)(\nu_m^2 + k_3 k_4)}{(\nu_m^2 + k_3^2)(\nu_m^2 + k_4^2)}\right].$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le a$, $0 \le y \le b$. Mixed boundary value problems.

1°. A rectangle is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = a,$$
$$\partial_y w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = 0 \quad \text{at} \quad y = b.$$

Eigenvalues:

$$\lambda_{nm} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right); \qquad n = 1,\ 2,\ 3,\ \ldots\ ; \quad m = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions and the norm squared:

$$w_{nm} = \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right), \quad \|w_{nm}\|^2 = \frac{ab}{4}(1 + \delta_{m0}), \quad \delta_{m0} = \begin{cases} 1 & \text{for } m = 0, \\ 0 & \text{for } m \neq 0. \end{cases}$$

2°. A rectangle is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = 0 \quad \text{at} \quad x = a,$$
$$w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = 0 \quad \text{at} \quad y = b.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\pi^2}{4}\left[\frac{(2n+1)^2}{a^2} + \frac{(2m+1)^2}{b^2}\right]; \qquad n = 0,\ 1,\ 2,\ \ldots\ ; \quad m = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions and the norm squared:

$$w_{nm} = \sin\left[\frac{\pi(2n+1)x}{2a}\right]\sin\left[\frac{\pi(2m+1)y}{2b}\right], \qquad \|w_{nm}\|^2 = \frac{ab}{4}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► First boundary value problem for a triangular domain.

The sides of the triangle are defined by the equations

$$x = 0, \quad y = 0, \quad y = a - x.$$

The unknown quantity is zero for these sides.

Eigenvalues:

$$\lambda_{nm} = \frac{\pi^2}{a^2}\big[(n+m)^2 + m^2\big]; \qquad n = 1,\, 2,\, \dots\,; \quad m = 1,\, 2,\, \dots$$

Eigenfunctions:

$$w_{nm} = \sin\left[\frac{\pi}{a}(n+m)x\right]\sin\left(\frac{\pi}{a}my\right) - (-1)^n \sin\left(\frac{\pi}{a}mx\right)\sin\left[\frac{\pi}{a}(n+m)y\right].$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► **Second boundary value problem for a triangular domain.**

The sides of the triangle are defined by the equations

$$x = 0, \quad y = 0, \quad y = a - x.$$

The normal derivative of the unknown quantity for these sides is zero.

Eigenvalues:

$$\lambda_{nm} = \frac{\pi^2}{a^2}\big[(n+m)^2 + m^2\big]; \qquad n = 0,\, 1,\, \dots\,; \quad m = 0,\, 1,\, \dots$$

Eigenfunctions:

$$w_{nm} = \cos\left[\frac{\pi}{a}(n+m)x\right]\cos\left(\frac{\pi}{a}my\right) - (-1)^n \cos\left(\frac{\pi}{a}mx\right)\cos\left[\frac{\pi}{a}(n+m)y\right].$$

9.3.3 Problems in Polar Coordinate System

A two-dimensional nonhomogeneous Helmholtz equation in the polar coordinate system is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \lambda w = -\Phi(r,\varphi), \qquad r = \sqrt{x^2+y^2}.$$

► **Particular solutions of the homogeneous equation ($\Phi \equiv 0$):**

$$\begin{aligned}
w &= [AJ_0(\mu r) + BY_0(\mu r)](C\varphi + D), \quad \lambda = \mu^2,\\
w &= [AI_0(\mu r) + BK_0(\mu r)](C\varphi + D), \quad \lambda = -\mu^2,\\
w &= [AJ_m(\mu r) + BY_m(\mu r)](C\cos m\varphi + D\sin m\varphi), \quad \lambda = \mu^2,\\
w &= [AI_m(\mu r) + BK_m(\mu r)](C\cos m\varphi + D\sin m\varphi), \quad \lambda = -\mu^2,
\end{aligned}$$

where $m = 1,\, 2,\, \dots\,;$ A, B, C, D are arbitrary constants; the $J_m(\mu)$ and $Y_m(\mu)$ are Bessel functions; and the $I_m(\mu)$ and $K_m(\mu)$ are modified Bessel functions.

◆ *Only the eigenvalues and eigenfunctions of homogeneous boundary value problems for the homogeneous Helmholtz equation (with $\Phi \equiv 0$) are presented below. The solutions of the corresponding nonhomogeneous boundary value problems (with $\Phi \not\equiv 0$) can be constructed by the relations specified in Section 9.3.1.*

► **Domain: $0 \le r \le R$. First boundary value problem.**

A circle is considered. A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2}; \qquad n = 0,\ 1,\ 2,\ \dots\ ;\ m = 1,\ 2,\ 3,\ \dots$$

Here, the μ_{nm} are positive zeros of the Bessel functions, $J_n(\mu) = 0$.

Eigenfunctions:

$$w_{nm}^{(1)} = J_n\bigl(r\sqrt{\lambda_{nm}}\,\bigr)\cos n\varphi, \quad w_{nm}^{(2)} = J_n\bigl(r\sqrt{\lambda_{nm}}\,\bigr)\sin n\varphi.$$

Eigenfunctions possessing the axial symmetry property: $w_{0m}^{(1)} = J_0\bigl(r\sqrt{\lambda_{0m}}\,\bigr)$.

The square of the norm of an eigenfunction is given by

$$\|w_{nm}^{(k)}\|^2 = \tfrac{1}{2}\pi R^2(1+\delta_{n0})[J_n'(\mu_{nm})]^2, \qquad k = 1,\ 2; \qquad \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases}$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$. Second boundary value problem.**

A circle is considered. A boundary condition is prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r = R.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2},$$

where the μ_{nm} are roots of the quadratic equation $J_n'(\mu) = 0$.

Eigenfunctions:

$$w_{nm}^{(1)} = J_n(r\sqrt{\lambda_{nm}}\,)\cos n\varphi, \quad w_{nm}^{(2)} = J_n(r\sqrt{\lambda_{nm}}\,)\sin n\varphi.$$

Here, $n = 0,\ 1,\ 2,\ \dots$; for $n \ne 0$, the parameter m assumes the values $m = 1,\ 2,\ 3,\ \dots$; for $n = 0$, a root $\mu_{00} = 0$ (the corresponding eigenfunction is $w_{00} = 1$).

Eigenfunctions possessing the axial symmetry property: $w_{0m}^{(1)} = J_0\bigl(r\sqrt{\lambda_{0m}}\,\bigr)$.

The square of the norm of an eigenfunction is given by

$$\|w_{nm}^{(k)}\|^2 = \frac{\pi^2 R^2(1+\delta_{n0})}{2\mu_{nm}^2}(\mu_{nm}^2 - n^2)[J_n(\mu_{nm})]^2, \quad \|w_{00}\|^2 = \pi R^2,$$

where $k = 1, 2;\ \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases}$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le r \le R$. Third boundary value problem.

A circle is considered. A boundary condition is prescribed:

$$\partial_r w + kw = 0 \quad \text{at} \quad r = R.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2}; \qquad n = 0,\ 1,\ 2,\ \ldots\ ;\ m = 1,\ 2,\ 3,\ \ldots$$

Here, the μ_{nm} is the mth root of the transcendental equation $\mu J_n'(\mu) + kRJ_n(\mu) = 0$.

Eigenfunctions:

$$w_{nm}^{(1)} = J_n\bigl(r\sqrt{\lambda_{nm}}\,\bigr)\cos n\varphi, \quad w_{nm}^{(2)} = J_n\bigl(r\sqrt{\lambda_{nm}}\,\bigr)\sin n\varphi.$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}^{(1)}\|^2 = \|w_{nm}^{(2)}\|^2 = \frac{\pi R^2(1+\delta_{n0})}{2\mu_{nm}^2}(k^2R^2+\mu_{nm}^2-n^2)[J_n(\mu_{nm})]^2, \quad \delta_{ij} = \begin{cases} 1 & \text{for } i=j, \\ 0 & \text{for } i\ne j. \end{cases}$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $R_1 \le r \le R_2$. First boundary value problem.

An annular domain is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R_1, \qquad w = 0 \quad \text{at} \quad r = R_2.$$

Eigenvalues:

$$\lambda_{nm} = \mu_{nm}^2; \qquad n = 0,\ 1,\ 2,\ \ldots\ ; \quad m = 1,\ 2,\ 3,\ \ldots$$

Here, the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - J_n(\mu R_2)Y_n(\mu R_1) = 0.$$

Eigenfunctions:

$$w_{nm}^{(1)} = [J_n(\mu_{nm}r)Y_n(\mu_{nm}R_1) - J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\cos n\varphi,$$
$$w_{nm}^{(2)} = [J_n(\mu_{nm}r)Y_n(\mu_{nm}R_1) - J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\sin n\varphi.$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}^{(1)}\|^2 = \|w_{nm}^{(2)}\|^2 = \frac{2(1+\delta_{n0})}{\pi\mu_{nm}^2}\,\frac{J_n^2(\mu_{nm}R_1) - J_n^2(\mu_{nm}R_2)}{J_n^2(\mu_{nm}R_2)}, \quad \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases}$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$. Second boundary value problem.**

An annular domain is considered. Boundary conditions are prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r = R_1, \qquad \partial_r w = 0 \quad \text{at} \quad r = R_2.$$

Eigenvalues:

$$\lambda_{nm} = \mu_{nm}^2; \qquad n = 0,\, 1,\, 2,\, \dots\,; \quad m = 0,\, 1,\, 2,\, \dots$$

Here, the μ_{nm} are roots of the quadratic equation

$$J_n'(\mu R_1) Y_n'(\mu R_2) - J_n'(\mu R_2) Y_n'(\mu R_1) = 0.$$

If $n = 0$, there is a root $\mu_{00} = 0$ and the corresponding eigenfunction is $w_{00}^{(1)} = 1$.

Eigenfunctions:

$$w_{nm}^{(1)} = \left[J_n(\mu_{nm} r) Y_n'(\mu_{nm} R_1) - J_n'(\mu_{nm} R_1) Y_n(\mu_{nm} r)\right] \cos n\varphi,$$
$$w_{nm}^{(2)} = \left[J_n(\mu_{nm} r) Y_n'(\mu_{nm} R_1) - J_n'(\mu_{nm} R_1) Y_n(\mu_{nm} r)\right] \sin n\varphi.$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}^{(1)}\|^2 = \|w_{nm}^{(2)}\|^2 = \frac{2(1+\delta_{n0})}{\pi \mu_{nm}^2} \left\{ \left(1 - \frac{n^2}{R_2^2 \mu_{nm}^2}\right) \left[\frac{J_n'(\mu_{nm} R_1)}{J_n'(\mu_{nm} R_2)}\right]^2 - \left(1 - \frac{n^2}{R_1^2 \mu_{nm}^2}\right) \right\},$$
$$\|w_{00}^{(1)}\|^2 = \pi (R_2^2 - R_1^2); \qquad \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases}$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$. Third boundary value problem.**

An annular domain is considered. Boundary conditions are prescribed:

$$\partial_r w - k w = 0 \quad \text{at} \quad r = R_1, \qquad \partial_r w + k w = 0 \quad \text{at} \quad r = R_2.$$

Eigenvalues:

$$\lambda_{nm} = \mu_{nm}^2; \qquad n = 0,\, 1,\, 2,\, \dots\,; \quad m = 1,\, 2,\, 3,\, \dots\,;$$

where the μ_{nm} are positive roots of the transcendental equation

$$A_1(\mu R_1) B_2(\mu R_2) - A_2(\mu R_2) B_1(\mu R_1) = 0.$$

Here, we use the notation

$$A_1(\mu R) = J_n'(\mu R) - \frac{k}{\mu} J_n(\mu R), \quad B_1(\mu R) = Y_n'(\mu R) - \frac{k}{\mu} Y_n(\mu R),$$
$$A_2(\mu R) = J_n'(\mu R) + \frac{k}{\mu} J_n(\mu R), \quad B_2(\mu R) = Y_n'(\mu R) + \frac{k}{\mu} Y_n(\mu R).$$

Eigenfunctions:

$$w_{nm}^{(1)} = [B_1(\mu_{nm}R_1)J_n(\mu_{nm}r) - A_1(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\cos n\varphi,$$
$$w_{nm}^{(2)} = [B_1(\mu_{nm}R_1)J_n(\mu_{nm}r) - A_1(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\sin n\varphi.$$

The square of the norm of an eigenfunction is given by $(s = 1,\ 2)$

$$\|w_{nm}^{(s)}\|^2 = \tfrac{1}{2}\pi\varepsilon_n R_2^2\biggl\{\bigl[F'_{nm}(R_2)\bigr]^2 + \biggl(1 - \frac{n^2}{R_2^2\mu_{nm}^2}\biggr)F_{nm}^2(R_2)\biggr\}$$
$$- \tfrac{1}{2}\pi\varepsilon_n R_1^2\biggl\{\bigl[F'_{nm}(R_1)\bigr]^2 + \biggl(1 - \frac{n^2}{R_1^2\mu_{nm}^2}\biggr)F_{nm}^2(R_1)\biggr\},$$
$$F_{nm}(r) = B_1(\mu_{nm}R_1)J_n(\mu_{nm}r) - A_1(\mu_{nm}R_1)Y_n(\mu_{nm}r), \quad \varepsilon_{ij} = \begin{cases} 2 & \text{for } i = j, \\ 1 & \text{for } i \neq j. \end{cases}$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le r \le R,\ 0 \le \varphi \le \alpha$. First boundary value problem.

A circular sector is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R, \qquad w = 0 \quad \text{at} \quad \varphi = 0, \qquad w = 0 \quad \text{at} \quad \varphi = \alpha.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2}; \qquad n = 1,\ 2,\ 3,\ \dots; \quad m = 1,\ 2,\ 3,\ \dots$$

Here, the μ_{nm} are positive zeros of the Bessel functions, $J_{\frac{n\pi}{\alpha}}(\mu) = 0$.

Eigenfunctions:

$$w_{nm} = J_{\frac{n\pi}{\alpha}}\biggl(\mu_{nm}\frac{r}{R}\biggr)\sin\biggl(\frac{n\pi}{\alpha}\varphi\biggr).$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}\|^2 = \frac{\alpha R^2}{4}\Bigl[J'_{\frac{n\pi}{\alpha}}(\mu_{nm})\Bigr]^2.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le r \le R,\ 0 \le \varphi \le \alpha$. Second boundary value problem.

A circular sector is considered. Boundary conditions are prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r = R, \qquad \partial_\varphi w = 0 \quad \text{at} \quad \varphi = 0, \qquad \partial_\varphi w = 0 \quad \text{at} \quad \varphi = \alpha.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2}; \qquad n = 0,\ 1,\ 2,\ \dots; \quad m = 0,\ 1,\ 2,\ \dots$$

Here, the μ_{nm} are roots of the quadratic equation $J'_{\frac{n\pi}{\alpha}}(\mu) = 0$.

Eigenfunctions:

$$w_{nm} = J_{\frac{n\pi}{\alpha}}\Bigl(\mu_{nm}\frac{r}{R}\Bigr)\cos\Bigl(\frac{n\pi}{\alpha}\varphi\Bigr), \qquad w_{00} = 1.$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}\|^2 = \frac{\alpha R^2}{4}(1+\delta_{n0})\biggl(1-\frac{n^2}{\mu_{nm}^2}\biggr)\Bigl[J_{\frac{n\pi}{\alpha}}(\mu_{nm})\Bigr]^2, \qquad \|w_{00}\|^2 = \frac{\alpha R^2}{2}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le r \le R$, $0 \le \varphi \le \alpha$. Third boundary value problem.

A circular sector is considered. Boundary conditions are prescribed:

$$\partial_r w + k_1 w = 0 \quad \text{at} \quad r = R, \qquad \partial_\varphi w - k_2 w = 0 \quad \text{at} \quad \varphi = 0, \qquad \partial_\varphi w + k_3 w = 0 \quad \text{at} \quad \varphi = \alpha.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2}; \qquad n = 1,\ 2,\ 3,\ \dots; \quad m = 1,\ 2,\ 3,\ \dots$$

Here, the μ_{nm} are positive roots of the transcendental equation $\mu J'_{\nu_n}(\mu) + k_1 R J_{\nu_n}(\mu) = 0$; the ν_n are positive roots of the transcendental equation $\tan(\alpha\nu) = \dfrac{(k_2+k_3)\nu}{\nu^2 - k_2 k_3}$.

Eigenfunctions:

$$w_{nm} = J_{\nu_n}\Bigl(\mu_{nm}\frac{r}{R}\Bigr)\frac{\nu_n\cos(\nu_n\varphi) + k_2\sin(\nu_n\varphi)}{\sqrt{\nu_n^2 + k_2^2}}.$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}\|^2 = \frac{R^2}{4}\biggl[\alpha + \frac{(k_2+k_3)(\nu_n^2 + k_2k_3)}{(\nu_n^2+k_2^2)(\nu_n^2+k_3^2)}\biggr]\biggl(1 + \frac{k_1^2R^2 - \nu_n^2}{\mu_{nm}^2}\biggr)J_{\nu_n}^2(\mu_{nm}).$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le \alpha$. First boundary value problem.

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R_1, \qquad w = 0 \quad \text{at} \quad r = R_2,$$
$$w = 0 \quad \text{at} \quad \varphi = 0, \qquad w = 0 \quad \text{at} \quad \varphi = \alpha.$$

Eigenvalues:

$$\lambda_{nm} = \mu_{nm}^2,$$

where the μ_{nm} are positive roots of the transcendental equation

$$J_{\nu_n}(\mu R_1)Y_{\nu_n}(\mu R_2) - J_{\nu_n}(\mu R_2)Y_{\nu_n}(\mu R_1) = 0, \qquad \nu_n = \frac{n\pi}{\alpha}.$$

Eigenfunctions:

$$w_{nm} = \bigl[J_{\nu_n}(\mu_{nm} r)Y_{\nu_n}(\mu_{nm} R_1) - J_{\nu_n}(\mu_{nm} R_1)Y_{\nu_n}(\mu_{nm} r)\bigr]\sin(\nu_n\varphi).$$

The square of the norm of an eigenfunction is given by

$$\|w_{nm}\|^2 = \frac{\alpha}{\pi^2\mu_{nm}^2}\,\frac{\bigl[J_{\nu_n}(\mu_{nm}R_1)\bigr]^2 - \bigl[J_{\nu_n}(\mu_{nm}R_2)\bigr]^2}{\bigl[J_{\nu_n}(\mu_{nm}R_2)\bigr]^2}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

9.3.4 Other Orthogonal Coordinate Systems. Elliptic Domain

In the first and second paragraphs below, two other orthogonal systems of coordinates are described in which the homogeneous Helmholtz equation admits separation of variables.

► **Parabolic coordinate system.**

In the parabolic coordinates that are introduced by the relations

$$x = \tfrac{1}{2}(\xi^2 - \eta^2), \quad y = \xi\eta \qquad (0 \le \xi < \infty,\ -\infty < \eta < \infty),$$

the Helmholtz equation has the form

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + \lambda(\xi^2 + \eta^2)w = 0.$$

Setting $w = f(\xi)g(\eta)$, we arrive at the following linear ordinary differential equations for $f = f(\xi)$ and $g = g(\eta)$:

$$f'' + (\lambda\xi^2 + k)f = 0, \quad g'' + (\lambda\eta^2 - k)g = 0,$$

where k is the separation constant. The general solutions of these equations are given by

$$f(\xi) = A_1 D_{\mu-1/2}(\sigma\xi) + A_2 D_{\mu-1/2}(-\sigma\xi), \quad g(\eta) = B_1 D_{-\mu-1/2}(\sigma\eta) + B_2 D_{-\mu-1/2}(-\sigma\eta),$$
$$\mu = \tfrac{1}{2}k(-\lambda)^{-1/2}, \quad \sigma = (-4\lambda)^{1/4}.$$

Here, A_1, B_1, A_2, and B_2 are arbitrary constants, and $D_\nu(z)$ is the parabolic cylinder function,

$$D_\nu(z) = 2^{1/2}\exp\bigl(-\tfrac{1}{4}z^2\bigr)\left[\frac{\Gamma\bigl(\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}-\frac{\nu}{2}\bigr)}\Phi\bigl(-\tfrac{\nu}{2}, \tfrac{1}{2}; \tfrac{1}{2}z^2\bigr) + 2^{-1/2}\frac{\Gamma\bigl(-\frac{1}{2}\bigr)}{\Gamma\bigl(-\frac{\nu}{2}\bigr)} z\Phi\bigl(\tfrac{1}{2}-\tfrac{\nu}{2}, \tfrac{3}{2}; \tfrac{1}{2}z^2\bigr)\right].$$

For $\nu = n = 0,\ 1,\ 2,\ \ldots$, we have

$$D_n(z) = 2^{-n/2}\exp\bigl(-\tfrac{1}{4}z^2\bigr)H_n\bigl(2^{-1/2}z\bigr), \quad \text{where} \quad H_n(z) = (-1)^n\exp\bigl(z^2\bigr)\frac{d^n}{z^n}\exp\bigl(-z^2\bigr).$$

⊙ *Literature*: M. Abramowitz and I. Stegun (1964), W. Miller, Jr. (1977).

► Elliptic coordinate system.

In the elliptic coordinates that are introduced by the relations

$$x = a\cosh u\cos v, \quad y = a\sinh u\sin v \qquad (0 \le u < \infty,\ 0 \le v < 2\pi,\ a > 0),$$

the Helmholtz equation is expressed as

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} + a^2\lambda(\cosh^2 u - \cos^2 v)w = 0.$$

Setting $w = F(u)G(v)$, we arrive at the following linear ordinary differential equations for $F = F(u)$ and $G = G(v)$:

$$F'' + \left(\tfrac{1}{2}a^2\lambda\cosh 2u - k\right)F = 0, \quad G'' - \left(\tfrac{1}{2}a^2\lambda\cos 2v - k\right)G = 0,$$

where k is the separation constant. The solutions of these equations periodic in v are given by

$$F(u) = \begin{cases} \mathrm{Ce}_n(u,q), \\ \mathrm{Se}_n(u,q), \end{cases} \qquad G(v) = \begin{cases} \mathrm{ce}_n(v,q), \\ \mathrm{se}_n(v,q), \end{cases} \qquad q = \tfrac{1}{4}a^2\lambda,$$

where $\mathrm{Ce}_n(u,q)$ and $\mathrm{Se}_n(u,q)$ are the modified Mathieu functions, while $\mathrm{ce}_n(v,q)$ and $\mathrm{se}_n(v,q)$ are the Mathieu functions; to each value of q there is a corresponding $k = k_n(q)$.

⊙ *Literature*: M. Abramowitz and I. Stegun (1964), W. Miller, Jr. (1977).

► Domain: $(x/a)^2 + (y/b)^2 \le 1$. First boundary value problem.

The unknown quantity is zero at the boundary of the elliptic domain:

$$w = 0 \quad \text{if} \quad (x/a)^2 + (y/b)^2 = 1 \qquad (a \ge b).$$

The first three eigenvalues and eigenfunctions are given by the approximate relations

$$\lambda_1 = \frac{\gamma_{10}^2}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \qquad w_1(\mathcal{R}) = J_0(\gamma_{10}\mathcal{R}),$$
$$\lambda_2^{(\mathrm{c})} = \frac{\gamma_{11}^2}{4}\left(\frac{3}{a^2} + \frac{1}{b^2}\right), \quad w_2^{(\mathrm{c})}(\mathcal{R},\varphi) = J_1(\gamma_{11}\mathcal{R})\cos\varphi,$$
$$\lambda_2^{(\mathrm{s})} = \frac{\gamma_{11}^2}{4}\left(\frac{1}{a^2} + \frac{3}{b^2}\right), \quad w_2^{(\mathrm{s})}(\mathcal{R},\varphi) = J_1(\gamma_{11}\mathcal{R})\sin\varphi,$$

where $\gamma_{10} = 2.4048$ and $\gamma_{11} = 3.8317$ are the first roots of the Bessel functions J_0 and J_1, i.e., $J_0(\gamma_{10}) = 0$ and $J_1(\gamma_{11}) = 0$; $\mathcal{R} = \sqrt{(x/a)^2 + (y/b)^2}$.

The above relations were obtained using the generalized (nonorthogonal) polar coordinates $\mathcal{R}, \varphi$ defined by

$$x = a\mathcal{R}\cos\varphi, \quad y = b\mathcal{R}\sin\varphi \qquad (0 \le \mathcal{R} \le 1,\ 0 \le \varphi \le 2\pi)$$

and the variational method.

For $\varepsilon = \sqrt{1-(b/a)^2} \le 0.9$, the above formulas provide an accuracy of 1% for λ_1 and 2% for $\lambda_2^{(\mathrm{c})}$ and $\lambda_2^{(\mathrm{s})}$. For $\varepsilon \le 0.5$, the errors in calculating λ_1 and $\lambda_2^{(\mathrm{c})}$ do not exceed 0.01%, and the maximum error in determining $\lambda_2^{(\mathrm{s})}$ is 0.12%. In the limit case $\varepsilon = 0$ that corresponds to a circular domain, the above formulas are exact.

⊙ *Literature*: L. D. Akulenko and S. V. Nesterov (2000).

9.4 Other Equations

9.4.1 Stationary Schrödinger Equation $\Delta_2 w = f(x,y)w$

1. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = a(x^2+y^2)w.$

The transformation

$$z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to the Helmholtz equation

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \zeta^2} - aw = 0,$$

which is discussed in Section 9.3.2.

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = a(x^2+y^2)^2 w.$

The transformation

$$z = \tfrac{1}{3}x^3 - xy^2, \quad \zeta = x^2 y - \tfrac{1}{3}y^3$$

leads to the Helmholtz equation

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \zeta^2} - aw = 0,$$

which is discussed in Section 9.3.2.

3. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = a(x^2+y^2)^k w.$

This is a special case of equation 9.4.1.7 for $f(u) = au^k$. Table 9.4 presents transformations that reduce this equation to the Helmholtz equation that is discussed in Section 9.3.2; the sixth row involves the imaginary unit, $i^2 = -1$.

4. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = ae^{\beta x} w.$

The transformation

$$u(x,y) = \exp\bigl(\tfrac{1}{2}\beta x\bigr)\cos\bigl(\tfrac{1}{2}\beta y\bigr), \quad v(x,y) = \exp\bigl(\tfrac{1}{2}\beta x\bigr)\sin\bigl(\tfrac{1}{2}\beta y\bigr)$$

leads to the Helmholtz equation

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 4a\beta^{-2} w,$$

which is discussed in Section 9.3.2.

TABLE 9.4
Transformations reducing equation 9.4.1.3 to the Helmholtz equation $\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = bw$

No	Exponent k	Transformation	Factor b
1	$k=1$	$\xi = \frac{1}{2}(x^2 - y^2),\ \eta = xy$	$b = a$
2	$k=2$	$\xi = \frac{1}{3}x^3 - xy^2,\ \eta = x^2 y - \frac{1}{3}y^3$	$b = a$
3	$k=-1$	$\xi = \frac{1}{2}\ln(x^2+y^2),\ \eta = \arctan\dfrac{y}{x}$	$b = a$
4	$k=-2$	$\xi = -\dfrac{x}{x^2+y^2},\ \eta = \dfrac{y}{x^2+y^2}$	$b = a$
5	$k = -\frac{1}{2}$	$x = \frac{1}{2}(\xi^2 - \eta^2),\ y = \xi\eta$	$b = 2a$
6	$k = \pm 3,\ \pm 4,\ \ldots$	$\xi = \dfrac{(x+iy)^{k+1} + (x-iy)^{k+1}}{2(k+1)},\ \eta = \dfrac{(x+iy)^{k+1} - (x-iy)^{k+1}}{2(k+1)i}$	$b = a$
7	k is any ($k \neq -1$)	$\xi = \dfrac{\rho^{k+1}\cos[(k+1)\varphi]}{k+1},\ \eta = \dfrac{\rho^{k+1}\sin[(k+1)\varphi]}{k+1}$, $x = \rho\cos\varphi,\ y = \rho\sin\varphi$	$b = a$

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ke^{ax+by}w.$

The transformation

$$\xi = ax + by, \qquad \eta = bx - ay$$

leads to an equation of the form 9.4.1.4:

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = \frac{k}{a^2+b^2} e^{\xi} w.$$

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(ax+by)w.$

This is a special case of equation 9.4.1.9 for $g(u) = 0$. Particular solutions:

$$w(x,y) = \bigl\{C_1 \cos[k(bx - ay)] + C_2 \sin[k(bx - ay)]\bigr\}\varphi(ax+by),$$

where C_1, C_2, and k are arbitrary constants, and the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$\varphi''_{\xi\xi} - \left[\frac{1}{a^2+b^2} f(\xi) + k^2\right]\varphi = 0.$$

7. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(x^2+y^2)w.$

1°. This equation admits separation of variables in the polar coordinates ρ, φ ($x = \rho\cos\varphi$, $y = \rho\sin\varphi$). Particular solution:

$$w(x,y) = \bigl[C_1 \cos(k\varphi) + C_2 \sin(k\varphi)\bigr]U(\rho),$$

where C_1, C_2, and k are arbitrary constants, and the function $U = U(\rho)$ is determined by the ordinary differential equation

$$\rho(\rho U'_\rho)'_\rho - \big[k^2 + \rho^2 f(\rho^2)\big]U = 0.$$

2°. The transformation

$$z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to a similar equation

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \zeta^2} = F(z^2 + \zeta^2)w, \qquad F(u) = \frac{f(2\sqrt{u}\,)}{2\sqrt{u}}.$$

In the special case $f(u) = 2a$, we have $F(u) = a/\sqrt{u}$. For $f(u) = bu^3$, we obtain an equation of the form 9.4.1.1 with $F(u) = 4bu$.

8. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = [f(x) + g(y)]w.$

A particular separable solution:

$$w(x, y) = \varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the second-order ordinary differential equations

$$\varphi''_{xx} - [f(x) - C]\varphi = 0, \qquad \psi''_{yy} - [g(y) + C]\psi = 0,$$

where C is an arbitrary constant.

9. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = [f(ax + by) + g(bx - ay)]w.$

The transformation

$$\xi = ax + by, \quad \eta = bx - ay$$

leads to an equation of the form 9.4.1.8:

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = \left[\frac{f(\xi)}{a^2 + b^2} + \frac{g(\eta)}{a^2 + b^2}\right]w.$$

10. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = (x^2 + y^2)[f(x^2 - y^2) + g(xy)]w.$

The transformation

$$z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to an equation of the form 9.4.1.8:

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \zeta^2} = [f(2z) + g(\zeta)]w.$$

9.4.2 Convective Heat and Mass Transfer Equations

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha \dfrac{\partial w}{\partial x}$.

This is a convective heat and mass transfer equation. It describes a stationary temperature (concentration) field in a continuous medium moving with a constant velocity along the x-axis. In particular, it models convective-molecular heat transfer from a heated flat plate in a flow of a thermal-transfer ideal fluid moving along the plate. This occurs, for example, if a liquid-metal coolant flows past a flat plate or if a plate is in a seepage flow through a granular medium.

In the sequel, it is assumed that the equation is written in dimensionless variables x, y related to the characteristic length (for a flat plate of length $2h$, the characteristic length is taken to be h).

1°. The substitution $w(x,y) = \exp\bigl(\tfrac{1}{2}\alpha x\bigr) U(x,y)$ brings the original equation to the Helmholtz equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{1}{4}\alpha^2 U.$$

Particular solutions of this equation in Cartesian and polar coordinates can be found in Sections 9.3.2 and 9.3.3.

2°. In the elliptic coordinates

$$x = \cosh\zeta\cos\eta, \qquad y = \sinh\zeta\sin\eta$$

a wide class of particular solutions (vanishing as $\zeta \to \infty$) can be indicated; this class of solutions of the original equation is represented in series form as

$$w = \exp\bigl(\tfrac{1}{2}\alpha x\bigr) \sum_{m=0}^{\infty} A_m \operatorname{ce}_m(\eta, -q) \operatorname{Fek}_m(\zeta, -q), \qquad q = -\tfrac{1}{16}\alpha^2,$$

where the A_m are arbitrary constants, the $\operatorname{ce}_m(\eta, -q)$ are the Mathieu functions, and the $\operatorname{Fek}_m(\zeta, -q)$ are the modified Mathieu functions [e.g., see McLachlan (1947) and Bateman and Erdélyi (1955)].

3°. Consider the first boundary value problem in the upper half-plane ($-\infty < x < \infty$, $0 \le y < \infty$). We assume that the surface of a plate of finite length is maintained at a constant temperature w_0 and the medium has a temperature $w_\infty = \text{const}$ far away from the plate:

$$\begin{aligned} w &= w_0 && \text{for} \quad y = 0,\ |x| < 1, \\ \partial_y w &= 0 && \text{for} \quad y = 0,\ |x| > 1, \\ w &\to w_\infty && \text{for} \quad x^2 + y^2 \to \infty. \end{aligned}$$

The solution of this problem in the elliptic coordinates ζ, η (see Item 2°) has the form

$$w(\eta,\zeta) = w_\infty + (w_0 - w_\infty) \exp\bigl(\tfrac{1}{2}\alpha\cos\eta\cosh\zeta\bigr) \sum_{m=0}^{\infty} D_m \operatorname{ce}_m(\eta, -q) \frac{\operatorname{Fek}_m(\zeta, -q)}{\operatorname{Fer}_m(0, -q)},$$

where

$$D_{2n} = 2\frac{\mathrm{ce}_{2n}(0,-q)}{\mathrm{ce}_{2n}(0,q)}A_0^{(2n)}, \quad D_{2n+1} = -\frac{1}{2}\frac{\mathrm{ce}_{2n+1}(0,-q)}{\mathrm{ce}_{2n+1}(0,q)}\alpha B_1^{(2n+1)}, \quad q = -\frac{1}{16}\alpha^2.$$

Here, the $A_0^{(2n)}$ and $B_1^{(2n+1)}$ are the coefficients in the series expansions of the Mathieu functions; these can be found in McLachlan (1947).

4°. Consider the second boundary value problem in the upper half-plane ($-\infty < x < \infty$, $0 \le y < \infty$). We assume that a thermal flux is prescribed on the surface of a plate of finite length and the medium has a constant temperature far away from the plate:

$$\begin{aligned} \partial_y w &= f(x) &&\text{for}\quad y=0,\ |x|<1,\\ \partial_y w &= 0 &&\text{for}\quad y=0,\ |x|>1,\\ w &\to w_\infty &&\text{as}\quad x^2+y^2\to\infty. \end{aligned}$$

The solution of this problem in the Cartesian coordinates has the form

$$w(x,y) = w_\infty - \frac{1}{\pi}\int_{-1}^{1} f(\xi)\exp\bigl[\tfrac{1}{2}\alpha(x-\xi)\bigr]K_0\Bigl(\tfrac{1}{2}\alpha\sqrt{(x-\xi)^2+y^2}\Bigr)\,d\xi,$$

where $K_0(z)$ is the modified Bessel function of the second kind.

⊙ *Literature*: P. V. Cherpakov (1975), A. A. Borzykh and G. P. Cherepanov (1978).

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \alpha\frac{\partial w}{\partial x} + \beta\frac{\partial w}{\partial y} + \gamma w.$

This equation describes a stationary temperature field in a medium moving with a constant velocity, provided there is volume release heat (or absorption) proportional to temperature.

The substitution

$$w(x,y) = \exp\bigl[\tfrac{1}{2}(\alpha x+\beta y)\bigr]U(x,y)$$

brings the original equation to the Helmholtz equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \bigl(\gamma + \tfrac{1}{4}\alpha^2 + \tfrac{1}{4}\beta^2\bigr)U,$$

which is discussed in Sections 9.3.1 through 9.3.3.

3. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \mathrm{Pe}\,(1-y^2)\frac{\partial w}{\partial x}.$

The Graetz–Nusselt equation. It governs steady-state heat exchange in a laminar fluid flow with a parabolic velocity profile in a plane channel. The equation is written in terms of the dimensionless Cartesian coordinates x, y related to the channel half-width h; $\mathrm{Pe} = Uh/a$ is the Peclet number and U is the fluid velocity at the channel axis ($y = 0$). The walls of the channel correspond to $y = \pm 1$.

1°. Particular solutions:

$$w(y) = A + By,$$
$$w(x,y) = 12Ax + A\,\mathsf{Pe}\,(6y^2 - y^4) + B,$$
$$w(x,y) = \sum_{n=1}^{m} A_n \exp\biggl(-\frac{\lambda_n^2}{\mathsf{Pe}}x\biggr) f_n(y).$$

Here, A, B, A_n, and λ_n are arbitrary constants, and the functions f_n are defined by

$$f_n(y) = \exp\bigl(-\tfrac{1}{2}\lambda_n y^2\bigr)\Phi\bigl(\alpha_n,\ \tfrac{1}{2};\ \lambda_n y^2\bigr), \quad \alpha_n = \tfrac{1}{4} - \tfrac{1}{4}\lambda_n - \tfrac{1}{4}\lambda_n^3\,\mathsf{Pe}^{-2}, \tag{1}$$

where $\Phi(\alpha,\beta;\xi) = 1 + \sum\limits_{k=1}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{\beta(\beta+1)\ldots(\beta+k-1)}\frac{\xi^k}{k!}$ is the degenerate hypergeometric function.

2°. Let the walls of the channel be maintained at a constant temperature, $w = 0$ for $x < 0$ and $w = w_0$ for $x > 0$. Due to the symmetry of the problem about the x-axis, it suffices to consider only half of the domain, $0 \le y \le 1$. The boundary conditions are written as

$$y = 0, \qquad \frac{\partial w}{\partial y} = 0; \qquad y = 1, \qquad w = \begin{cases} 0 & \text{for } x < 0, \\ w_0 & \text{for } x > 0; \end{cases}$$
$$x \to -\infty, \quad w \to 0; \qquad x \to \infty, \quad w \to w_0.$$

The solution of the original equation under these boundary conditions is sought in the form

$$w(x,y) = w_0 \sum_{n=1}^{\infty} B_n \exp\biggl(\frac{\mu_n^2}{\mathsf{Pe}}x\biggr) g_n(y) \qquad \text{for} \quad x < 0,$$
$$w(x,y) = w_0\biggl[1 - \sum_{n=1}^{\infty} A_n \exp\biggl(-\frac{\lambda_n^2}{\mathsf{Pe}}x\biggr) f_n(y)\biggr] \quad \text{for} \quad x > 0.$$

The series coefficients must satisfy the matching conditions at the boundary:

$$w(x,y)\big|_{x\to 0,\,x<0} - w(x,y)\big|_{x\to 0,\,x>0} = 0,$$
$$\partial_x w(x,y)\big|_{x\to 0,\,x<0} - \partial_x w(x,y)\big|_{x\to 0,\,x>0} = 0.$$

For $x > 0$, the function $f_n(y)$ is defined by relation (1), where the eigenvalues λ_n are roots of the transcendental equation

$$\Phi\bigl(\alpha_n,\ \tfrac{1}{2};\ \lambda_n\bigr) = 0, \quad \text{where} \quad \alpha_n = \tfrac{1}{4} - \tfrac{1}{4}\lambda_n - \tfrac{1}{4}\lambda_n^3\,\mathsf{Pe}^{-2}.$$

For $\mathsf{Pe} \to \infty$, it is convenient to use the following approximate relation to identify the λ_n:

$$\lambda_n = 4(n-1) + 1.68 \qquad (n = 1,\ 2,\ 3,\ \ldots). \tag{2}$$

The error of this formula does not exceed 0.2%. The corresponding numerical values of the coefficients A_n are rather well approximated by the relations

$$A_1 = 1.2, \quad A_n = 2.27\,(-1)^{n-1}\lambda_n^{-7/6} \qquad \text{for} \quad n = 2,\ 3,\ 4,\ \ldots,$$

whose maximum error is less than 0.1%, provided that the λ_n are calculated by (2).

For $\mathsf{Pe} \to 0$, the following asymptotic relations hold:

$$\lambda_n = \sqrt{\pi\left(n - \tfrac{1}{2}\right)\mathsf{Pe}}, \quad A_n = \frac{4(-1)^{n-1}}{\pi^2(2n-1)^2}, \quad f_n(y) = \cos\left[\pi\left(n - \tfrac{1}{2}\right)y\right] \quad n = 1,\ 2,\ 3,\ \ldots$$

No results for $x < 0$ are given here, because they are of secondary importance in applications.

3°. Let a constant thermal flux be prescribed at the walls for $x > 0$ and let, for $x < 0$, the walls be insulated from heat and the temperature vanishes as $x \to -\infty$. Then the boundary conditions have the form

$$y = 0, \quad \frac{\partial w}{\partial y} = 0; \qquad y = 1, \quad \frac{\partial w}{\partial y} = \begin{cases} 0 & \text{for } x < 0, \\ q & \text{for } x > 0; \end{cases} \qquad x \to -\infty, \quad w \to 0.$$

In the domain of thermal stabilization, the asymptotic behavior of the solution (as $x \to \infty$) is as follows:

$$w(x, y) = q\left(\frac{3}{2}\frac{x}{\mathsf{Pe}} + \frac{3}{4}y^2 - \frac{1}{8}y^4 + \frac{9}{4\,\mathsf{Pe}^2} - \frac{39}{280}\right).$$

⊙ *Literature*: L. Graetz (1883), W. Nusselt (1910), C. A. Deavours (1974), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

4. $\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{\partial^2 w}{\partial z^2} = \mathsf{Pe}\,(1 - r^2)\dfrac{\partial w}{\partial z}$.

This equation governs steady-state heat exchange in a laminar fluid flow with parabolic (Poiseuille's) velocity profile in a circular tube. The equation is written in terms of the dimensionless cylindrical coordinates x, y related to the tube radius R; $\mathsf{Pe} = UR/a$ is the Peclet number and U is the fluid velocity at the tube axis (at $r = 0$). The walls of the tube correspond to $r = 1$.

1°. Particular solutions:

$$\begin{aligned} w(r) &= A + B\ln r, \\ w(r, z) &= 16Az + A\,\mathsf{Pe}\,(4r^2 - r^4) + B, \\ w(r, z) &= \sum_{n=1}^{m} A_n \exp\left(-\frac{\lambda_n^2}{\mathsf{Pe}}z\right) f_n(r). \end{aligned}$$

Here, A, B, A_n, and λ_n are arbitrary constants, and the functions f_n are defined by

$$f_n(r) = \exp\left(-\tfrac{1}{2}\lambda_n r^2\right)\Phi\left(\alpha_n,\ 1;\ \lambda_n r^2\right), \quad \alpha_n = \tfrac{1}{2} - \tfrac{1}{4}\lambda_n - \tfrac{1}{4}\lambda_n^3\,\mathsf{Pe}^{-2}, \qquad (1)$$

where $\Phi(\alpha, \beta; \xi)$ is the degenerate hypergeometric function (see equation 9.4.2.3, Item 1°).

2°. Let the tube wall be maintained at a constant temperature such that $w = 0$ for $z < 0$ and $w = w_0$ for $z > 0$. The boundary conditions are written as

$$r = 0, \qquad \frac{\partial w}{\partial r} = 0; \qquad r = 1, \qquad w = \begin{cases} 0 & \text{for } z < 0, \\ w_0 & \text{for } z > 0; \end{cases}$$
$$z \to -\infty, \quad w \to 0; \qquad z \to \infty, \quad w \to w_0.$$

The solution of the original equation under these boundary conditions is sought in the form

$$w(r,z) = w_0 \sum_{n=1}^{\infty} B_n \exp\biggl(\frac{\mu_n^2}{\mathsf{Pe}} z\biggr) g_n(r) \qquad \text{for} \quad z < 0,$$
$$w(r,z) = w_0 \biggl[1 - \sum_{n=1}^{\infty} A_n \exp\biggl(-\frac{\lambda_n^2}{\mathsf{Pe}} z\biggr) f_n(r)\biggr] \quad \text{for} \quad z > 0.$$

The series coefficients must satisfy the matching conditions at the boundary,

$$w(r,z)\big|_{z\to 0,\, z<0} - w(r,z)\big|_{z\to 0,\, z>0} = 0,$$
$$\partial_z w(r,z)\big|_{z\to 0,\, z<0} - \partial_z w(r,z)\big|_{z\to 0,\, z>0} = 0.$$

For $z > 0$, the functions $f_n(r)$ are defined by relations (1), where the eigenvalues λ_n are roots of the transcendental equation

$$\Phi\bigl(\alpha_n,\ 1;\ \lambda_n\bigr) = 0, \quad \text{where} \quad \alpha_n = \tfrac{1}{2} - \tfrac{1}{4}\lambda_n - \tfrac{1}{4}\lambda_n^3\, \mathsf{Pe}^{-2}.$$

For $\mathsf{Pe} \to \infty$, it is convenient to use the following approximate relation to identify the λ_n:

$$\lambda_n = 4\,(n-1) + 2.7 \qquad (n = 1,\ 2,\ 3,\ \dots). \tag{2}$$

The error of this formula does not exceed 0.3%. The corresponding numerical values of the coefficients A_n are rather well approximated by the relations

$$A_n = 2.85\,(-1)^{n-1}\lambda_n^{-2/3} \qquad \text{for} \quad n = 1,\ 2,\ 3,\ \dots,$$

whose maximum error is 0.5%,

No results for $z < 0$ are given here, since they are of secondary importance in applications.

3°. Let a constant thermal flux be prescribed at the wall for $z > 0$ and let, for $z < 0$, the tube surface be insulated from heat and the temperature vanishes as $z \to -\infty$. Then the boundary conditions have the form

$$r = 0, \quad \frac{\partial w}{\partial r} = 0; \qquad r = 1, \quad \frac{\partial w}{\partial r} = \begin{cases} 0 & \text{for } z < 0, \\ q & \text{for } z > 0; \end{cases} \qquad z \to -\infty, \quad w \to 0.$$

In the domain of thermal stabilization, the asymptotic behavior of the solution (as $z \to \infty$) is as follows:

$$w(r,z) = q\biggl(4\frac{z}{\mathsf{Pe}} + r^2 - \frac{1}{4}r^4 + \frac{8}{\mathsf{Pe}^2} - \frac{7}{24}\biggr).$$

⊙ *Literature*: C. A. Deavours (1974), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(y)\dfrac{\partial w}{\partial x}.$

This equation describes steady-state heat exchange in a laminar fluid flow with an arbitrary velocity profile $f = f(y)$ in a plane channel.

1°. Particular solutions:

$$w(x,y) = Ax + A\int_{y_0}^{y}(y-\xi)f(\xi)\,d\xi + By + C, \quad (1)$$

$$w(x,y) = B + \sum_{n=1}^{m} A_n \exp(-\beta_n x)u_n(y). \quad (2)$$

Here, A, B, C, y_0, A_n, and β_n are arbitrary constants, and the functions $u_n = u_n(y)$ are determined by the second-order linear ordinary differential equation

$$\frac{d^2 u_n}{dy^2} + \big[\beta_n f(y) + \beta_n^2\big]u_n = 0.$$

2°. Solution (1) describes the temperature distribution far away from the inlet section of the tube, in the domain of thermal stabilization, provided that a constant thermal flux is prescribed at the channel walls.

6. $a\Big(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\Big) = v_1(x,y)\dfrac{\partial w}{\partial x} + v_2(x,y)\dfrac{\partial w}{\partial y}.$

This is an equation of steady-state convective heat and mass transfer in the Cartesian coordinate system. Here, $v_1 = v_1(x,y)$ and $v_2 = v_2(x,y)$ are the components of the fluid velocity that are assumed to be known from the solution of the hydrodynamic problem.

1°. In plane problems of convective heat exchange in liquid metals modeled by an ideal fluid, as well as in describing seepage (filtration) streams employing the model of potential flows, the fluid velocity components $v_1(x,y)$ and $v_2(x,y)$ can be expressed in terms of the potential $\varphi = \varphi(x,y)$ and stream function $\psi = \psi(x,y)$ as follows:

$$v_1 = \frac{\partial\varphi}{\partial x} = -\frac{\partial\psi}{\partial y}, \qquad v_2 = \frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial x}. \quad (1)$$

The function φ is determined by solving the Laplace equation $\Delta\varphi = 0$. In specific problems, the potential φ and stream function ψ may be identified by invoking the complex variable theory [e.g., see Lavrent'ev and Shabat (1973) and Sedov (1980)].

By passing in the convective heat exchange equation from x, y to the new variables φ, ψ (Boussinesq transformation) and taking into account (1), we arrive a simpler equation with constant coefficients of the form 9.4.2.1:

$$\frac{\partial^2 w}{\partial\varphi^2} + \frac{\partial^2 w}{\partial\psi^2} = \frac{1}{a}\frac{\partial w}{\partial\varphi}. \quad (2)$$

The Boussinesq transformation brings any plane contour in a potential flow to a cut in the φ-axis, simultaneously with the reduction of the original equation to the form (2). Consequently, the heat transfer problem of a potential flow about this contour is reduced to the heat exchange problem of a longitudinal flow of an ideal fluid past a flat plate (see equation 9.4.2.1, Items 3° and 4°).

2°. Asymptotic analyses of plane problems on heat/mass exchange of bodies of various shape with laminar translational and shear flows of a viscous (and ideal) incompressible fluid for large and small Peclet numbers were carried out in the references cited below. In the thermal boundary layer approximation, the solution of the heat exchange problem for a flat plate in a longitudinal translational flow of a viscous incompressible fluid at large Reynolds numbers is presented in 3.9.1.4, Item 3°.

⊙ *Literature*: V. G. Levich (1962), P. V. Cherpakov (1975), A. A. Borzykh and G. P. Cherepanov (1978), Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

7. $\displaystyle \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial w}{\partial r}\Big)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial w}{\partial\theta}\Big)=\cos\theta\,\frac{\partial w}{\partial r}-\frac{\sin\theta}{r}\frac{\partial w}{\partial\theta}.$

This is a special case of equation 9.4.1.8 with $a=1$, $v_r=\cos\theta$, and $v_\theta=-\sin\theta$. This equation is obtained from the equation $\partial_{xx}w+\partial_{yy}w+\partial_{zz}w=\partial_x w$ by the passage to the spherical coordinate system in the axisymmetric case.

The general solution satisfying the decay condition ($w\to 0$ as $r\to\infty$) is expressed as

$$w(r,\theta)=\Big(\frac{\pi}{r}\Big)^{1/2}\exp\Big(\frac{r\cos\theta}{2}\Big)\sum_{n=0}^{\infty}A_nK_{n+\frac12}\Big(\frac{r}{2}\Big)P_n(\cos\theta),$$

where the A_n are arbitrary constants. The Legendre polynomials $P_n(\xi)$ and the modified Bessel functions $K_{n+\frac12}(z)$ are given by

$$P_n(\xi)=\frac{1}{n!\,2^n}\frac{d^n}{d\xi^n}(\xi^2-1)^n,\quad K_{n+\frac12}\Big(\frac{r}{2}\Big)=\Big(\frac{\pi}{r}\Big)^{1/2}\exp\Big(-\frac{r}{2}\Big)\sum_{m=0}^{n}\frac{(n+m)!}{(n-m)!\,m!\,r^m}.$$

⊙ *Literature*: P. L. Rimmer (1968).

8. $\displaystyle a\bigg[\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial w}{\partial r}\Big)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial w}{\partial\theta}\Big)\bigg]=v_r\frac{\partial w}{\partial r}+\frac{v_\theta}{r}\frac{\partial w}{\partial\theta}.$

This equation is often encountered in axisymmetric problems of convective heat and mass exchange of solid particles, drops, and bubbles with a flow of a viscous incompressible fluid. The fluid velocity components $v_r=v_r(r,\theta)$ and $v_\theta=v_\theta(r,\theta)$ can be expressed in terms of the stream function $\psi=\psi(r,\theta)$ as

$$v_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\qquad v_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.\tag{1}$$

Asymptotic analyses for a wide class of axisymmetric problems on heat/mass exchange of solid particles, drops, and bubbles of various shape with a laminar translational or straining flow of a viscous incompressible fluid at large and small Peclet numbers $\mathsf{Pe}=UR/a$ are performed in the books cited below. The Peclet number is written in terms of the characteristic velocity U (e.g., the unperturbed fluid velocity far away from the particle in the case of translation flow), the characteristic size of the particle R (e.g., the radius for a spherical particle), and the thermal conductivity or diffusion coefficient a.

The following boundary conditions are usually specified:

$$w = w_0 \quad \text{at} \quad r = R, \qquad w \to w_\infty \quad \text{as} \quad r \to \infty, \tag{2}$$

where R is the particle radius, w_0 the temperature at the particle surface, and w_∞ the temperature far away from the particle (w_0 and w_∞ are constant).

Convective mass transfer problems are characterized by large Peclet numbers. To solve such problems, the diffusion boundary layer approximation is often used; in this case, the left-hand side of the equation takes into account only the diffusion mass transfer in the normal direction to the particle surface (the tangential mass transfer is neglected). The convective terms on the right-hand side are partially preserved—the fluid velocity components are approximated by their leading terms of the asymptotic expansion near the phase surface. Presented below are some important results obtained by solving the original equation under the boundary conditions (2) in the diffusion boundary layer approximation.

Example 9.7. For the translational Stokes flow of a viscous incompressible fluid about a spherical bubble, the stream function is expressed as

$$\psi(r,\theta) = \tfrac{1}{2}Ur(r-R)\sin^2\theta.$$

Here, U is the unperturbed fluid velocity in the incident flow, R the bubble radius (the value $\theta = \pi$ corresponds to the front critical point at the bubble surface).

In this case, the solution of the convective heat/mass transfer equation with the boundary conditions (2) for $\mathsf{Pe} = UR/a \gg 1$ in the diffusion boundary layer approximation is given by

$$w(r,\theta) = w_0 + (w_\infty - w_0)\operatorname{erf}\xi, \quad \xi = \sqrt{\tfrac{3}{8}\,\mathsf{Pe}}\left(\frac{r}{R} - 1\right)\frac{1-\cos\theta}{\sqrt{2-\cos\theta}},$$

where $\operatorname{erf}\xi$ is the error function.

Example 9.8. For the translational Stokes flow of a viscous incompressible fluid about a solid spherical particle, the stream function is expressed as

$$\psi(r,\theta) = \frac{1}{4}U(r-R)^2\left(2 + \frac{R}{r}\right)\sin^2\theta.$$

Here, the notation is the same as in the case of a bubble above.

For a solid particle, the solution of the convective heat/mass transfer equation with the boundary conditions (2) for $\mathsf{Pe} = UR/a \gg 1$ in the diffusion boundary layer approximation is given by

$$w(r,\theta) = w_0 + (w_\infty - w_0)\bigl[\Gamma\bigl(\tfrac{1}{3}\bigr)\bigr]^{-1}\gamma\bigl(\tfrac{1}{3},\xi\bigr), \qquad \xi = \frac{\mathsf{Pe}\,(r-R)^3\sin^3\theta}{3R^3\bigl(\pi - \theta + \frac{1}{2}\sin 2\theta\bigr)},$$

where $\Gamma(\beta)$ is the gamma function and $\gamma(\beta,\xi) = \int_0^\xi e^{-z}z^{\beta-1}\,dz$ is the incomplete gamma function.

⊙ *Literature*: V. G. Levich (1962), Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

9.4.3 Equations of Heat and Mass Transfer in Anisotropic Media

1. $\dfrac{\partial}{\partial x}\Big(ax^n\dfrac{\partial w}{\partial x}\Big)+\dfrac{\partial}{\partial y}\Big(by^m\dfrac{\partial w}{\partial y}\Big)=0.$

This is a two-dimensional equation of the heat and mass transfer theory in a inhomogeneous anisotropic medium. Here, $a_1(x)=ax^n$ and $a_2(y)=by^m$ are the principal thermal diffusivities.

1°. Particular solutions (A, B, C are arbitrary constants):

$$w(x,y)=Ax^{1-n}+By^{1-m}+C,$$
$$w(x,y)=A\left[\frac{x^{2-n}}{a(2-n)}-\frac{y^{2-m}}{b(2-m)}\right]+B,$$
$$w(x,y)=Ax^{1-n}y^{1-m}+B.$$

2°. For $n\neq 2$ and $m\neq 2$, there are particular solutions of the form

$$w=w(\xi),\qquad \xi=\big[b(2-m)^2x^{2-n}+a(2-n)^2y^{2-m}\big]^{1/2}.$$

The function $w=w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{A}{\xi}w'_{\xi}=0,\qquad A=\frac{4-nm}{(2-n)(2-m)}.\qquad (1)$$

The general solution of equation (1) is given by

$$w(\xi)=\begin{cases}C_1\xi^{1-A}+C_2 & \text{for } A\neq 1,\\ C_1\ln\xi+C_2 & \text{for } A=1,\end{cases}$$

where C_1 and C_2 are arbitrary constants.

3°. There are multiplicative separable particular solutions in the form

$$w(x,y)=\varphi(x)\psi(y),\qquad (2)$$

where $\varphi(x)$ and $\psi(y)$ are determined by the following second-order linear ordinary differential equations (A_1 is an arbitrary constant):

$$(ax^n\varphi'_x)'_x=-A_1\varphi,\qquad (3)$$
$$(by^m\psi'_y)'_y=A_1\psi.\qquad (4)$$

The solution of equation (3) is given by

$$\varphi(x)=\begin{cases}x^{\frac{1-n}{2}}\Big[C_1J_\nu\Big(\beta x^{\frac{2-n}{2}}\Big)+C_2Y_\nu\Big(\beta x^{\frac{2-n}{2}}\Big)\Big] & \text{for } A_1>0,\\ x^{\frac{1-n}{2}}\Big[C_1I_\nu\Big(\beta x^{\frac{2-n}{2}}\Big)+C_2K_\nu\Big(\beta x^{\frac{2-n}{2}}\Big)\Big] & \text{for } A_1<0,\end{cases}$$
$$\nu=\frac{|1-n|}{2-n},\qquad \beta=\frac{2}{2-n}\sqrt{\frac{|A_1|}{a}},$$

where C_1 and C_2 are arbitrary constants, $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, and $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions.

The solution of equation (4) is expressed as

$$\psi(y) = \begin{cases} y^{\frac{1-m}{2}}\Big[C_1 J_\sigma\Big(\mu y^{\frac{2-m}{2}}\Big) + C_2 Y_\sigma\Big(\mu y^{\frac{2-m}{2}}\Big)\Big] & \text{for } A_1 < 0, \\ y^{\frac{1-m}{2}}\Big[C_1 I_\sigma\Big(\mu y^{\frac{2-m}{2}}\Big) + C_2 K_\sigma\Big(\mu y^{\frac{2-m}{2}}\Big)\Big] & \text{for } A_1 > 0, \end{cases}$$

$$\sigma = \frac{|1-m|}{2-m}, \qquad \mu = \frac{2}{2-m}\sqrt{\frac{|A_1|}{b}},$$

where C_1 and C_2 are arbitrary constants.

The sum of solutions of the form (2) corresponding to different values of the parameter A_1 is also a solution of the original equation; the solutions of some boundary value problems may be obtained by separation of variables.

4°. See equation 9.4.3.3, Item 4°, for $c = 0$.

2. $\dfrac{\partial}{\partial x}\Big(ax^n\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(by^m\dfrac{\partial w}{\partial y}\Big) = c.$

This is a two-dimensional equation of the heat and mass transfer theory with constant volume release of heat in an inhomogeneous anisotropic medium. Here, $a_1(x) = ax^n$ and $a_2(y) = by^m$ are the principal thermal diffusivities.

1°. For $n \neq 2$ and $m \neq 2$, there are particular solutions of the form

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\big]^{1/2}. \tag{1}$$

The function $w = w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = B, \tag{2}$$

where

$$A = \frac{4-nm}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}. \tag{3}$$

The general solution of equation (2) is given by

$$w(\xi) = \begin{cases} C_1\xi^{1-A} + C_2 + \frac{B}{2(A+1)}\xi^2 & \text{for } A \neq \pm 1, \\ C_1 \ln\xi + C_2 + \frac{1}{4}B\xi^2 & \text{for } A = 1, \\ C_1\xi^2 + C_2 + \frac{1}{2}B\xi^2\ln\xi & \text{for } A = -1, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution

$$w(x,y) = U(x,y) + \frac{c}{a(2-n)}x^{2-n}$$

leads to a homogeneous equation of the form 9.4.3.1:

$$\frac{\partial}{\partial x}\Big(ax^n\frac{\partial U}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(by^m\frac{\partial U}{\partial y}\Big) = 0.$$

3. $\dfrac{\partial}{\partial x}\Big(ax^n\dfrac{\partial w}{\partial x}\Big)+\dfrac{\partial}{\partial y}\Big(by^m\dfrac{\partial w}{\partial y}\Big)=cw.$

This is a two-dimensional equation of the heat and mass transfer theory with a linear source in an inhomogeneous anisotropic medium.

1°. For $n\neq 2$ and $m\neq 2$, there are particular solutions of the form

$$w=w(\xi),\qquad \xi=\big[b(2-m)^2x^{2-n}+a(2-n)^2y^{2-m}\big]^{1/2}.$$

The function $w=w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{A}{\xi}w'_\xi=Bw,\tag{1}$$

where

$$A=\frac{4-nm}{(2-n)(2-m)},\quad B=\frac{4c}{ab(2-n)^2(2-m)^2}.$$

The general solution of equation (1) is given by

$$w(\xi)=\xi^{\frac{1-A}{2}}\Big[C_1J_\nu\big(\xi\sqrt{|B|}\,\big)+C_2Y_\nu\big(\xi\sqrt{|B|}\,\big)\Big]\quad\text{for } B<0,$$
$$w(\xi)=\xi^{\frac{1-A}{2}}\Big[C_1I_\nu\big(\xi\sqrt{B}\,\big)+C_2K_\nu\big(\xi\sqrt{B}\,\big)\Big]\qquad\text{for } B>0,$$

where $\nu=\frac{1}{2}|1-A|$; C_1 and C_2 are arbitrary constants; $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions; and $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions.

2°. There are multiplicative separable particular solutions of the form

$$w(x,y)=\varphi(x)\psi(y),$$

where $\varphi(x)$ and $\psi(y)$ are determined by the following second-order linear ordinary differential equations (A_1 is an arbitrary constant):

$$(ax^n\varphi'_x)'_x=A_1\varphi,\qquad (by^m\psi'_y)'_y=(c-A_1)\psi.\tag{2}$$

The solutions of equations (2) are expressed in terms of the Bessel functions (or modified Bessel functions); see equation 9.4.3.1, Item 3°.

3°. There are additively separable particular solutions of the form

$$w(x,y)=f(x)+g(y),$$

where $f(x)$ and $g(y)$ are determined by the following second-order linear ordinary differential equations (A_2 is an arbitrary constant):

$$(ax^nf'_x)'_x-cf=A_2,\qquad (by^mg'_y)'_y-cg=-A_2.\tag{3}$$

The solutions of equations (3) are expressed in terms of the Bessel functions (or modified Bessel functions).

4°. The transformation (specified by A. I. Zhurov, private communication, 2001)

$$x^{\frac{2-n}{2}} = Ar\cos\theta, \quad y^{\frac{2-m}{2}} = Br\sin\theta,$$

where $A^2 = a(2-n)^2$ and $B^2 = b(2-m)^2$, leads to the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{4-nm}{(2-n)(2-m)}\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{2}{r^2}\frac{(nm-n-m)\cos 2\theta + (n-m)}{(2-n)(2-m)\sin 2\theta}\frac{\partial w}{\partial \theta} = 4cw,$$

which admits separable solutions of the form $w(r,\theta) = F_1(r)F_2(\theta)$.

4. $\dfrac{\partial}{\partial x}\left[a(x+k)^n\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[b(y+s)^m\dfrac{\partial w}{\partial y}\right] = c.$

The transformation $\zeta = x + k$, $\eta = y + s$ leads to an equation of the form 9.4.3.2:

$$\frac{\partial}{\partial \zeta}\left(a\zeta^n\frac{\partial w}{\partial \zeta}\right) + \frac{\partial}{\partial \eta}\left(b\eta^m\frac{\partial w}{\partial \eta}\right) = c.$$

5. $\dfrac{\partial}{\partial x}\left[a(x+k)^n\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[b(y+s)^m\dfrac{\partial w}{\partial y}\right] = cw.$

The transformation $\zeta = x + k$, $\eta = y + s$ leads to an equation of the form 9.4.3.3:

$$\frac{\partial}{\partial \zeta}\left(a\zeta^n\frac{\partial w}{\partial \zeta}\right) + \frac{\partial}{\partial \eta}\left(b\eta^m\frac{\partial w}{\partial \eta}\right) = cw.$$

6. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) = 0.$

This is a two-dimensional equation of the heat and mass transfer theory in an inhomogeneous anisotropic medium. Here, $a_1(x) = ae^{\beta x}$ and $a_2(y) = be^{\mu y}$ are the principal thermal diffusivities.

1°. Particular solutions (A, B, C are arbitrary constants):

$$w(x,y) = Ae^{-\beta x} + Be^{-\mu y} + C,$$
$$w(x,y) = \frac{A}{a\beta^2}(\beta x + 1)e^{-\beta x} - \frac{A}{b\mu^2}(\mu y + 1)e^{-\mu y} + B,$$
$$w(x,y) = Ae^{-\beta x - \mu y} + B.$$

2°. There are multiplicative separable particular solutions of the form

$$w(x,y) = \varphi(x)\psi(y), \tag{1}$$

where $\varphi(x)$ and $\psi(y)$ are determined by the following second-order linear ordinary differential equations (A_1 is an arbitrary constant):

$$(ae^{\beta x}\varphi'_x)'_x = -A_1\varphi, \tag{2}$$
$$(be^{\mu y}\psi'_y)'_y = A_1\psi. \tag{3}$$

The solution of equation (2) is given by

$$\varphi(x) = \begin{cases} e^{-\beta x/2}\big[C_1 J_1\big(ke^{-\beta x/2}\big) + C_2 Y_1\big(ke^{-\beta x/2}\big)\big] & \text{for } A_1 > 0, \\ e^{-\beta x/2}\big[C_1 I_1\big(ke^{-\beta x/2}\big) + C_2 K_1\big(ke^{-\beta x/2}\big)\big] & \text{for } A_1 < 0, \end{cases}$$

where $k = -(2/\beta)\sqrt{|A_1|/a}$; C_1 and C_2 are arbitrary constants; $J_1(z)$ and $Y_1(z)$ are Bessel functions; and $I_1(z)$ and $K_1(z)$ are modified Bessel functions.

The solution of equation (3) is given by

$$\psi(y) = \begin{cases} e^{-\mu y/2}\big[C_1 J_1\big(se^{-\mu y/2}\big) + C_2 Y_1\big(se^{-\mu y/2}\big)\big] & \text{for } A_1 < 0, \\ e^{-\mu y/2}\big[C_1 I_1\big(se^{-\mu y/2}\big) + C_2 K_1\big(se^{-\mu y/2}\big)\big] & \text{for } A_1 > 0, \end{cases}$$

where $s = -(2/\mu)\sqrt{|A_1|/b}$; C_1 and C_2 are arbitrary constants.

The sum of solutions of the form (1) corresponding to different values of the parameter A_1 is also a solution of the original equation.

3°. See equation 9.4.3.8, Item 3°, for $c = 0$.

7. $\dfrac{\partial}{\partial x}\Big(ae^{\beta x}\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(be^{\mu y}\dfrac{\partial w}{\partial y}\Big) = c.$

This is a two-dimensional equation of the heat and mass transfer theory with constant volume release of heat in an inhomogeneous anisotropic medium. Here, $a_1(x) = ae^{\beta x}$ and $a_2(y) = be^{\mu y}$ are the principal thermal diffusivities.

The substitution

$$w(x,y) = U(x,y) - \frac{c}{a\beta^2}(\beta x + 1)e^{-\beta x}$$

leads to a homogeneous equation of the form 9.4.3.6:

$$\frac{\partial}{\partial x}\Big(ae^{\beta x}\frac{\partial U}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(be^{\mu y}\frac{\partial U}{\partial y}\Big) = 0.$$

8. $\dfrac{\partial}{\partial x}\Big(ae^{\beta x}\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(be^{\mu y}\dfrac{\partial w}{\partial y}\Big) = cw.$

This is a two-dimensional equation of the heat and mass transfer theory with a linear source in an inhomogeneous anisotropic medium.

1°. For $\beta\mu \neq 0$, there are particular solutions of the form

$$w = w(\xi), \qquad \xi = \big(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\big)^{1/2}.$$

The function $w = w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} - \frac{1}{\xi}w'_{\xi} = Bw, \qquad B = \frac{4c}{ab\beta^2\mu^2}.$$

For the solution of this equation, see 9.4.3.3 (Item 1° for $A = -1$).

2°. The original equation admits multiplicative (and additively) separable solutions. See equation 7.4.3.12 with $f(x) = ae^{\beta x}$ and $g(y) = be^{\mu y}$.

3°. The transformation (specified by A. I. Zhurov, private communication, 2001)

$$e^{-\beta x/2} = Ar\cos\theta, \quad e^{-\mu y/2} = Br\sin\theta,$$

where $A^2 = a\beta^2$ and $B^2 = b\mu^2$, leads to the equation

$$\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{2}{r^2}\cot 2\theta\,\frac{\partial w}{\partial \theta} = 4cw,$$

which admits separable solutions of the form $w(r,\theta) = F_1(r)F_2(\theta)$.

9. $\displaystyle \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta y}\frac{\partial w}{\partial x}\right) = cw.$

1°. For $n \neq 2$ and $\beta \neq 0$, there are particular solutions of the form

$$w = w(r), \qquad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\beta y}}{b\beta^2}.$$

The function $w = w(r)$ is determined by the ordinary differential equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{n}{2-n}\frac{1}{r}\frac{\partial w}{\partial r} = 4cw.$$

For the solution of this equation, see 9.4.3.3 (Item 1°).

2°. The original equation admits multiplicative (and additively) separable solutions. See equation 7.4.3.12 with $f(x) = ax^n$ and $g(y) = be^{\beta y}$.

3°. The transformation (specified by A. I. Zhurov, private communication, 2001)

$$x^{1-\frac{1}{2}n} = Ar\cos\theta, \quad e^{-\frac{1}{2}\beta y} = Br\sin\theta,$$

where $A^2 = a(2-n)^2$ and $B^2 = b\beta^2$, leads to the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{n}{2-n}\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{2}{r^2}\frac{(1-n)\cos 2\theta + 1}{(2-n)\sin 2\theta}\frac{\partial w}{\partial \theta} = 4cw,$$

which admits separable solutions of the form $w(r,\theta) = F_1(r)F_2(\theta)$.

10. $\displaystyle \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial^2 w}{\partial y^2} = 0.$

1°. Particular solutions:

$$w = C_1y^2 + C_2y - 2\int\frac{C_1x + C_3}{f(x)}\,dx + C_4,$$

$$w = C_1y^3 + C_2y - 6y\int\frac{C_1x + C_3}{f(x)}\,dx + C_4,$$

$$w = [C_1\Phi(x) + C_2]y + C_3\Phi(x) + C_4, \quad \Phi(x) = \int\frac{dx}{f(x)},$$

$$w = [C_1\Phi(x) + C_2]y^2 + C_3\Phi(x) + C_4 - 2\int\left\{\frac{1}{f(x)}\int[C_1\Phi(x) + C_2]\,dx\right\}dx,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

2°. Separable particular solution:

$$w = (C_1 e^{\lambda y} + C_2 e^{-\lambda y}) H(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $H = H(x)$ is determined by the ordinary differential equation $[f(x)H'_x]'_x + \lambda^2 H = 0$.

3°. Separable particular solution:

$$w = [C_1 \sin(\lambda y) + C_2 \cos(\lambda y)] Z(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $Z = Z(x)$ is determined by the ordinary differential equation $[f(x)Z'_x]'_x - \lambda^2 Z = 0$.

4°. Particular solutions with even powers of y:

$$w = \sum_{k=0}^{n} \zeta_k(x) y^{2k},$$

where the functions $\zeta_k = \zeta_k(x)$ are defined by the recurrence relations

$$\zeta_n(x) = A_n \Phi(x) + B_n, \qquad \Phi(x) = \int \frac{dx}{f(x)},$$
$$\zeta_{k-1}(x) = A_k \Phi(x) + B_k - 2k(2k-1) \int \frac{1}{f(x)} \left\{ \int \zeta_k(x)\, dx \right\} dx,$$

where A_k and B_k are arbitrary constants ($k = n, \ldots, 1$).

5°. Particular solutions with odd powers of y:

$$w = \sum_{k=0}^{n} \eta_k(x) y^{2k+1},$$

where the functions $\eta_k = \eta_k(x)$ are defined by the recurrence relations

$$\eta_n(x) = A_n \Phi(x) + B_n, \qquad \Phi(x) = \int \frac{dx}{f(x)},$$
$$\eta_{k-1}(x) = A_k \Phi(x) + B_k - 2k(2k+1) \int \frac{1}{f(x)} \left\{ \int \eta_k(x)\, dx \right\} dx,$$

where A_k and B_k are arbitrary constants ($k = n, \ldots, 1$).

11. $\displaystyle \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] = 0.$

This is a two-dimensional sourceless equation of the heat and mass transfer theory in an inhomogeneous anisotropic medium. The functions $f = f(x)$ and $g = g(y)$ are the principal thermal diffusivities.

1°. Particular solutions:

$$w(x,y) = A_1 \int \frac{dx}{f(x)} + B_1 \int \frac{dy}{g(y)} + C_1,$$
$$w(x,y) = A_2 \int \frac{x\,dx}{f(x)} - A_2 \int \frac{y\,dy}{g(y)} + B_2,$$
$$w(x,y) = A_3 \int \frac{dx}{f(x)} \int \frac{dy}{g(y)} + B_3,$$

where A_1, A_2, A_3, B_1, B_2, B_3, and C_1 are arbitrary constants. A linear combination of these solutions is also a solution of the original equation.

2°. There are multiplicative separable particular solutions of the form

$$w(x,y) = \varphi(x)\psi(y), \tag{1}$$

where $\varphi(x)$ and $\psi(y)$ are determined by the following second-order linear ordinary differential equations (A is an arbitrary constant):

$$\begin{aligned} (f\varphi'_x)'_x &= A\varphi, & f &= f(x), \\ (g\psi'_y)'_y &= -A\psi, & g &= g(y). \end{aligned} \tag{2}$$

The sum of solutions of the form (1) corresponding to different values of the parameter A in (2) is also a solution of the original equation (the solutions of some boundary value problems may be obtained by separation of variables).

12. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] = \beta w.$

This is a two-dimensional equation of the heat and mass transfer theory with a linear source in an inhomogeneous anisotropic medium. The functions $f = f(x)$ and $g = g(y)$ are the principal thermal diffusivities.

1°. There are multiplicative separable particular solutions of the form

$$w(x,y) = \varphi(x)\psi(y), \tag{1}$$

where $\varphi(x)$ and $\psi(y)$ are determined by the following second-order linear ordinary differential equations (A is an arbitrary constant):

$$\begin{aligned} (f\varphi'_x)'_x &= A\varphi, & f &= f(x), \\ (g\psi'_y)'_y &= (\beta - A)\psi, & g &= g(y). \end{aligned} \tag{2}$$

The sum of solutions of the form (1) corresponding to different values of the parameter A in (2) is also a solution of the original equation; the solutions of some boundary value problems may be obtained by separation of variables.

2°. There are additively separable particular solutions of the form

$$w(x,y) = \Phi(x) + \Psi(y),$$

where $\Phi(x)$ and $\Psi(y)$ are determined by the following second-order linear ordinary differential equations (C is an arbitrary constant):

$$(f\Phi'_x)'_x - \beta\Phi = C, \qquad f = f(x),$$
$$(g\Psi'_y)'_y - \beta\Psi = -C, \quad g = g(y).$$

In the special case $\beta = 0$, the solutions of these equations can be represented as

$$\Phi(x) = C\int\frac{x\,dx}{f(x)} + A_1\int\frac{dx}{f(x)} + B_1,$$
$$\Psi(y) = -C\int\frac{y\,dy}{g(y)} + A_2\int\frac{dy}{g(y)} + B_2,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

9.4.4 Other Equations Arising in Applications

1. $y\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0.$

Tricomi equation. It is used to describe near-sonic flows of gas.

1°. Particular solutions:

$$w = Axy + Bx + Cy + D,$$
$$w = A(3x^2 - y^3) + B(x^3 - xy^3) + C(6yx^2 - y^4),$$

where A, B, C, and D are arbitrary constants.

2°. Particular solutions with even powers of x:

$$w = \sum_{k=0}^{n}\varphi_k(y)x^{2k},$$

where the functions $\varphi_k = \varphi_k(y)$ are defined by the recurrence relations

$$\varphi_n(y) = A_n y + B_n, \quad \varphi_{k-1}(y) = A_k y + B_k - 2k(2k-1)\int_0^y (y-t)t\varphi_k(t)\,dt,$$

where A_k and B_k are arbitrary constants ($k = n, \dots, 1$).

3°. Particular solutions with odd powers of x:

$$w = \sum_{k=0}^{n}\psi_k(y)x^{2k+1},$$

where the functions $\psi_k = \psi_k(y)$ are defined by the recurrence relations

$$\psi_n(y) = A_n y + B_n, \quad \psi_{k-1}(y) = A_k y + B_k - 2k(2k+1)\int_0^y (y-t)t\psi_k(t)\,dt,$$

where A_k and B_k are arbitrary constants ($k = n, \dots, 1$).

4°. Separable particular solutions:

$$w(x,y) = \left[A\sinh(3\lambda x) + B\cosh(3\lambda x)\right]\sqrt{y}\,\left[CJ_{1/3}(2\lambda y^{3/2}) + DY_{1/3}(2\lambda y^{3/2})\right],$$
$$w(x,y) = \left[A\sin(3\lambda x) + B\cos(3\lambda x)\right]\sqrt{y}\,\left[CI_{1/3}(2\lambda y^{3/2}) + DK_{1/3}(2\lambda y^{3/2})\right],$$

where A, B, C, D, and λ are arbitrary constants, $J_{1/3}(z)$ and $Y_{1/3}(z)$ are Bessel functions, and $I_{1/3}(z)$ and $K_{1/3}(z)$ are modified Bessel functions.

5°. For $y > 0$, see also equation 7.4.4.2 with $n = 1$. For $y < 0$, the change of variable $y = -t$ leads to an equation of the form 6.3.3.11 with $m = 1$.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

2. $\displaystyle y^n\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$

1°. Particular solutions:

$$w = Axy + Bx + Cy + D,$$
$$w = Ax^2 - \frac{2A}{(n+1)(n+2)}y^{n+2},$$
$$w = Ax^3 - \frac{6A}{(n+1)(n+2)}xy^{n+2},$$
$$w = Ayx^2 - \frac{2A}{(n+2)(n+3)}y^{n+3},$$

where A, B, C, and D are arbitrary constants.

2°. Particular solutions with even powers of x:

$$w = \sum_{k=0}^{m}\varphi_k(y)x^{2k},$$

where the functions $\varphi_k = \varphi_k(y)$ are defined by the recurrence relations

$$\varphi_m(y) = A_m y + B_m, \quad \varphi_{k-1}(y) = A_k y + B_k - 2k(2k-1)\int_a^y (y-t)t^n\varphi_k(t)\,dt,$$

where A_k and B_k are arbitrary constants ($k = m, \dots, 1$), and a is any number.

3°. Particular solutions with odd powers of x:

$$w = \sum_{k=0}^{m}\psi_k(y)x^{2k+1},$$

where the functions $\psi_k = \psi_k(y)$ are defined by the recurrence relations

$$\psi_m(y) = A_m y + B_m, \quad \psi_{k-1}(y) = A_k y + B_k - 2k(2k+1)\int_a^y (y-t)t^n\psi_k(t)\,dt,$$

where A_k and B_k are arbitrary constants ($k = m, \dots, 1$), and a is any number.

4°. Separable particular solutions:

$$w(x,y)=\bigl[A\sinh(\lambda qx)+B\cosh(\lambda qx)\bigr]\sqrt{y}\,\bigl[CJ_{\frac{1}{2q}}(\lambda y^q)+DY_{\frac{1}{2q}}(\lambda y^q)\bigr],\quad q=\tfrac{1}{2}(n+2),$$
$$w(x,y)=\bigl[A\sin(\lambda qx)+B\cos(\lambda qx)\bigr]\sqrt{y}\,\bigl[CI_{\frac{1}{2q}}(\lambda y^q)+DK_{\frac{1}{2q}}(\lambda y^q)\bigr],$$

where A, B, C, D, and λ are arbitrary constants, $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, and $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions.

5°. Fundamental solutions (for $y>0$):

$$w_1(x,y,x_0,y_0)=k_1(r_1^2)^{-\beta}F(\beta,\,\beta,\,2\beta;\,1-\xi),\quad \beta=\frac{n}{2(n+2)},\quad \xi=\frac{r_2^2}{r_1^2},$$
$$w_2(x,y,x_0,y_0)=k_2(r_1^2)^{-\beta}(1-\xi)^{1-2\beta}F(1-\beta,\,1-\beta,\,2-2\beta;\,1-\xi).$$

Here, $F(a,b,c;\xi)$ is the hypergeometric function and

$$r_1^2=(x-x_0)^2+\frac{4}{(n+2)^2}\left(y^{\frac{n+2}{2}}+y_0^{\frac{n+2}{2}}\right),\quad k_1=\frac{1}{4\pi}\left(\frac{4}{n+2}\right)^{2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)},$$
$$r_2^2=(x-x_0)^2+\frac{4}{(n+2)^2}\left(y^{\frac{n+2}{2}}-y_0^{\frac{n+2}{2}}\right),\quad k_2=\frac{1}{4\pi}\left(\frac{4}{n+2}\right)^{2\beta}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)},$$

where $\Gamma(\beta)$ is the gamma function; x_0 and y_0 are arbitrary constants.

The fundamental solutions satisfy the conditions

$$\partial_y w_1\big|_{y=0}=0,\quad w_2\big|_{y=0}=0\qquad (x \text{ and } x_0 \text{ are any}, \ y_0>0).$$

The solutions of some boundary value problems can be found in the first book cited below.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), A. D. Polyanin (2001).

3. $\dfrac{\partial^2 w}{\partial r^2}+\dfrac{\alpha}{r}\dfrac{\partial w}{\partial r}+\dfrac{\partial^2 w}{\partial z^2}=0.$

Elliptic analogue of the Euler–Poisson–Darboux equation.

1°. For $\alpha=1$, see Sections 10.1.2 and 10.2.3 with $w=w(r,z)$. For $\alpha\neq 1$, the transformation $x=(1-\alpha)z$, $y=r^{1-\alpha}$ leads to an equation of the form 9.4.4.2:

$$y^{\frac{2\alpha}{1-\alpha}}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0.$$

2°. Suppose $w_\alpha=w_\alpha(r,z)$ is a solution of the equation in question for a fixed value of the parameter α. Then the functions $\widetilde{w}_\alpha$ defined by the relations

$$\widetilde{w}_\alpha=\frac{\partial w_\alpha}{\partial z},$$
$$\widetilde{w}_\alpha=r\frac{\partial w_\alpha}{\partial r}+z\frac{\partial w_\alpha}{\partial z},$$
$$\widetilde{w}_\alpha=2rz\frac{\partial w_\alpha}{\partial r}+(z^2-r^2)\frac{\partial w_\alpha}{\partial z}+\alpha z w_\alpha$$

are also solutions of this equation.

3°. Suppose $w_\alpha = w_\alpha(r,z)$ is a solution of the equation in question for a fixed value of the parameter α. Using this w_α, one can construct solutions of the equation with other values of the parameter by the formulas

$$
\begin{aligned}
w_{2-\alpha} &= r^{\alpha-1} w_\alpha,\\
w_{\alpha-2} &= r\frac{\partial w_\alpha}{\partial r} + (\alpha-1)w_\alpha,\\
w_{\alpha-2} &= rz\frac{\partial w_\alpha}{\partial r} - r^2\frac{\partial w_\alpha}{\partial z} + (\alpha-1)zw_\alpha,\\
w_{\alpha-2} &= r(r^2-z^2)\frac{\partial w_\alpha}{\partial r} + 2r^2z\frac{\partial w_\alpha}{\partial z} + \left[r^2-(\alpha-1)z^2\right] w_\alpha,\\
w_{\alpha+2} &= \frac{1}{r}\frac{\partial w_\alpha}{\partial r},\\
w_{\alpha+2} &= \frac{z}{r}\frac{\partial w_\alpha}{\partial r} - \frac{\partial w_\alpha}{\partial z},\\
w_{\alpha+2} &= \frac{r^2-z^2}{r}\frac{\partial w_\alpha}{\partial r} + 2z\frac{\partial w_\alpha}{\partial z} + \alpha w_\alpha.
\end{aligned}
$$

⊙ *Literature*: A. V. Aksenov (2001).

4. $\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial^2 w}{\partial y^2} = 0.$

1°. Particular solutions:

$$
\begin{aligned}
w &= C_1xy + C_2y + C_3x + C_4,\\
w &= C_1y^2 + C_2xy + C_3y + C_4x - 2C_1\int_a^x (x-t)f(t)\,dt + C_5,\\
w &= C_1y^3 + C_2xy + C_3y + C_4x - 6C_1y\int_a^x (x-t)f(t)\,dt + C_5,\\
w &= (C_1x+C_2)y^2 + C_3xy + C_4y + C_5x - 2\int_a^x (x-t)(C_1t+C_2)f(t)\,dt + C_6,
\end{aligned}
$$

where $C_1, \ldots, C_6$ are arbitrary constants and a is any number.

2°. Separable particular solution:

$$w = (C_1e^{\lambda y} + C_2e^{-\lambda y})H(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $H = H(x)$ is determined by the ordinary differential equation $H''_{xx} + \lambda^2 f(x)H = 0$.

3°. Separable particular solution:

$$w = [C_1\sin(\lambda y) + C_2\cos(\lambda y)]Z(x),$$

where C_1, C_2, and λ are arbitrary constants, and the function $Z = Z(x)$ is determined by the ordinary differential equation $Z''_{xx} - \lambda^2 f(x)Z = 0$.

4°. Particular solutions with even powers of y:

$$w=\sum_{k=0}^{n}\varphi_k(x)y^{2k},$$

where the functions $\varphi_k=\varphi_k(x)$ are defined by the recurrence relations

$$\varphi_n(x)=A_nx+B_n,\quad \varphi_{k-1}(x)=A_kx+B_k-2k(2k-1)\int_a^x(x-t)f(t)\varphi_k(t)\,dt,$$

where A_k and B_k are arbitrary constants ($k=n,\dots,1$), and a is any number.

5°. Particular solutions with odd powers of y:

$$w=\sum_{k=0}^{n}\psi_k(x)y^{2k+1},$$

where the functions $\psi_k=\psi_k(x)$ are defined by the recurrence relations

$$\psi_n(x)=A_nx+B_n,\quad \psi_{k-1}(x)=A_kx+B_k-2k(2k+1)\int_a^x(x-t)f(t)\psi_k(t)\,dt,$$

where A_k and B_k are arbitrary constants ($k=n,\dots,1$), and a is any number.

5. $\dfrac{\partial}{\partial x}\Big[f_1(x)\dfrac{\partial w}{\partial x}\Big]+\dfrac{\partial}{\partial y}\Big[f_2(y)\dfrac{\partial w}{\partial y}\Big]+\lambda\big[g_1(x)+g_2(y)\big]w=0.$

This equation is encountered in the theory of vibration of inhomogeneous membranes. Its separable solutions are sought in the form $w(x,y)=\varphi(x)\psi(y)$.

The article cited below presents an algorithm for accelerated convergence of solutions to eigenvalue boundary value problems for this equation.

⊙ *Literature*: L. D. Akulenko and S. V. Nesterov (1999).

9.4.5 Equations of the Form $a(x)\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+b(x)\dfrac{\partial w}{\partial x}+c(x)w=-\Phi(x,y)$

► **Statements of boundary value problems. Relations for the Green's function.**

Consider two-dimensional boundary value problems for the equation

$$a(x)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+b(x)\frac{\partial w}{\partial x}+c(x)w=-\Phi(x,y)\tag{1}$$

with general boundary conditions in x,

$$\begin{aligned}s_1\partial_x w-k_1w&=f_1(y)\quad\text{at}\quad x=x_1,\\ s_2\partial_x w+k_2w&=f_2(y)\quad\text{at}\quad x=x_2,\end{aligned}\tag{2}$$

and different boundary conditions in y. We assume that the coefficients of equation (1) and the boundary conditions (2) meet the requirement

$$a(x),\ b(x),\ c(x) \text{ are continuous functions; } \ a(x)>0,\ \ |s_1|+|k_1|>0,\ \ |s_2|+|k_2|>0,$$

where $x_1 \le x \le x_2$.

In the general case, the Green's function can be represented as

$$G(x,y,\xi,\eta)=\rho(\xi)\sum_{n=1}^{\infty}\frac{u_n(x)u_n(\xi)}{\|u_n\|^2}\Psi_n(y,\eta;\lambda_n). \tag{3}$$

Here,

$$\rho(x)=\frac{1}{a(x)}\exp\biggl[\int\frac{b(x)}{a(x)}\,dx\biggr],\quad \|u_n\|^2=\int_{x_1}^{x_2}\rho(x)u_n^2(x)\,dx, \tag{4}$$

and the λ_n and $u_n(x)$ are the eigenvalues and eigenfunctions of the homogeneous boundary value problem for the ordinary differential equation

$$a(x)u''_{xx}+b(x)u'_x+[\lambda+c(x)]u=0, \tag{5}$$

$$s_1u'_x-k_1u=0 \quad\text{at}\quad x=x_1, \tag{6}$$

$$s_2u'_x+k_2u=0 \quad\text{at}\quad x=x_2. \tag{7}$$

The functions Ψ_n for various boundary conditions in y are specified in Table 9.5.

Equation (5) can be rewritten in self-adjoint form as

$$[p(x)u'_x]'_x+[\lambda\rho(x)-q(x)]u=0, \tag{8}$$

where the functions $p(x)$ and $q(x)$ are given by

$$p(x)=\exp\biggl[\int\frac{b(x)}{a(x)}\,dx\biggr],\ \ q(x)=-\frac{c(x)}{a(x)}\exp\biggl[\int\frac{b(x)}{a(x)}\,dx\biggr],$$

and $\rho(x)$ is defined in (4).

The eigenvalue problem (8), (6), (7) possesses the following properties:

1°. All eigenvalues $\lambda_1,\ \lambda_2,\ \ldots$ are real and $\lambda_n\to\infty$ as $n\to\infty$.

2°. The system of eigenfunctions $\{u_1(x),u_2(x),\ldots\}$ is orthogonal on the interval $x_1\le x\le x_2$ with weight $\rho(x)$, that is,

$$\int_{x_1}^{x_2}\rho(x)u_n(x)u_m(x)\,dx=0 \quad\text{for}\quad n\ne m.$$

3°. If the conditions

$$q(x)\ge 0,\quad s_1k_1\ge 0,\quad s_2k_2\ge 0 \tag{9}$$

are satisfied, there are no negative eigenvalues. If $q\equiv 0$ and $k_1=k_2=0$, then the least eigenvalue is $\lambda_0=0$ and the corresponding eigenfunction is $u_0=\text{const}$; in this case, the summation in (3) must start with $n=0$. In the other cases, if conditions (9) are satisfied, all eigenvalues are positive; for example, the first inequality in (9) holds if $c(x)\le 0$.

TABLE 9.5
The functions Ψ_n in (3) for various boundary conditions. Notation: $\beta_n = \sqrt{\lambda_n}$

Domain	Boundary conditions	Function $\Psi_n(y,\eta;\lambda_n)$
$-\infty<y<\infty$	$\lvert w\rvert<\infty$ as $y\to\pm\infty$	$\dfrac{1}{2\beta_n}e^{-\beta_n\lvert y-\eta\rvert}$
$0\le y<\infty$	$w=0$ at $y=0$	$\dfrac{1}{\beta_n}\begin{cases} e^{-\beta_n y}\sinh(\beta_n\eta) & \text{if } y>\eta, \\ e^{-\beta_n\eta}\sinh(\beta_n y) & \text{if } \eta>y \end{cases}$
$0\le y<\infty$	$\partial_y w=0$ at $y=0$	$\dfrac{1}{\beta_n}\begin{cases} e^{-\beta_n y}\cosh(\beta_n\eta) & \text{if } y>\eta, \\ e^{-\beta_n\eta}\cosh(\beta_n y) & \text{if } \eta>y \end{cases}$
$0\le y<\infty$	$\partial_y w-k_3w=0$ at $y=0$	$\dfrac{1}{\beta_n(\beta_n+k_3)}\begin{cases} e^{-\beta_n y}[\beta_n\cosh(\beta_n\eta)+k_3\sinh(\beta_n\eta)] & \text{if } y>\eta, \\ e^{-\beta_n\eta}[\beta_n\cosh(\beta_n y)+k_3\sinh(\beta_n y)] & \text{if } \eta>y \end{cases}$
$0\le y\le h$	$w=0$ at $y=0$, $w=0$ at $y=h$	$\dfrac{1}{\beta_n\sinh(\beta_n h)}\begin{cases} \sinh(\beta_n\eta)\sinh[\beta_n(h-y)] & \text{if } y>\eta, \\ \sinh(\beta_n y)\sinh[\beta_n(h-\eta)] & \text{if } \eta>y \end{cases}$
$0\le y\le h$	$\partial_y w=0$ at $y=0$, $\partial_y w=0$ at $y=h$	$\dfrac{1}{\beta_n\sinh(\beta_n h)}\begin{cases} \cosh(\beta_n\eta)\cosh[\beta_n(h-y)] & \text{if } y>\eta, \\ \cosh(\beta_n y)\cosh[\beta_n(h-\eta)] & \text{if } \eta>y \end{cases}$
$0\le y\le h$	$w=0$ at $y=0$, $\partial_y w=0$ at $y=h$	$\dfrac{1}{\beta_n\cosh(\beta_n h)}\begin{cases} \sinh(\beta_n\eta)\cosh[\beta_n(h-y)] & \text{if } y>\eta, \\ \sinh(\beta_n y)\cosh[\beta_n(h-\eta)] & \text{if } \eta>y \end{cases}$

Section 3.8.9 presents some relations for estimating the eigenvalues λ_n and eigenfunctions $u_n(x)$.

The Green's function of the two-dimensional third boundary value problem (1)–(2) augmented by the boundary conditions

$$\frac{\partial w}{\partial y} - k_3 w = 0 \quad \text{at} \quad y = 0,$$

$$\frac{\partial w}{\partial y} + k_4 w = 0 \quad \text{at} \quad y = h$$

is given by relation (3) with

$$\Psi_n(y,\eta;\lambda_n)=\begin{cases} \dfrac{\big[\beta_n\cosh(\beta_n\eta)+k_3\sinh(\beta_n\eta)\big]\big\{\beta_n\cosh[\beta_n(h-y)]+k_4\sinh[\beta_n(h-y)]\big\}}{\beta_n\big[\beta_n(k_3+k_4)\cosh(\beta_n h)+(\beta_n^2+k_3k_4)\sinh(\beta_n h)\big]} & \text{if } y>\eta, \\ \dfrac{\big[\beta_n\cosh(\beta_n y)+k_3\sinh(\beta_n y)\big]\big\{\beta_n\cosh[\beta_n(h-\eta)]+k_4\sinh[\beta_n(h-\eta)]\big\}}{\beta_n\big[\beta_n(k_3+k_4)\cosh(\beta_n h)+(\beta_n^2+k_3k_4)\sinh(\beta_n h)\big]} & \text{if } y<\eta. \end{cases}$$

► **Representation of solutions to boundary value problems using the Green's function.**

1°. The solution* of the first boundary value problem for equation (1) with the boundary conditions

$$w = f_1(y) \quad \text{at} \quad x = x_1, \qquad w = f_2(y) \quad \text{at} \quad x = x_2,$$
$$w = f_3(x) \quad \text{at} \quad y = 0, \qquad w = f_4(x) \quad \text{at} \quad y = h$$

is expressed in terms of the Green's function as

$$\begin{aligned} w(x,y) &= a(x_1)\int_0^h f_1(\eta)\Big[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\Big]_{\xi=x_1} d\eta \\ &\quad - a(x_2)\int_0^h f_2(\eta)\Big[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\Big]_{\xi=x_2} d\eta \\ &\quad + \int_{x_1}^{x_2} f_3(\xi)\Big[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\Big]_{\eta=0} d\xi - \int_{x_1}^{x_2} f_4(\xi)\Big[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\Big]_{\eta=h} d\xi \\ &\quad + \int_{x_1}^{x_2}\int_0^h \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi. \end{aligned}$$

2°. The solution of the second boundary value problem for equation (1) with boundary conditions

$$\partial_x w = f_1(y) \quad \text{at} \quad x = x_1, \qquad \partial_x w = f_2(y) \quad \text{at} \quad x = x_2,$$
$$\partial_y w = f_3(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x) \quad \text{at} \quad y = h$$

is expressed in terms of the Green's function as

$$\begin{aligned} w(x,y) &= -a(x_1)\int_0^h f_1(\eta)G(x,y,x_1,\eta)\,d\eta + a(x_2)\int_0^h f_2(\eta)G(x,y,x_2,\eta)\,d\eta \\ &\quad - \int_{x_1}^{x_2} f_3(\xi)G(x,y,\xi,0)\,d\xi + \int_{x_1}^{x_2} f_4(\xi)G(x,y,\xi,h)\,d\xi \\ &\quad + \int_{x_1}^{x_2}\int_0^h \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi. \end{aligned}$$

3°. The solution of the third boundary value problem for equation (1) in terms of the Green's function is represented in the same way as the solution of the second boundary value problem (the Green's function is now different).

*For unbounded domains, the condition of boundedness of the solution as $y \to \pm\infty$ is set; in Table 9.5, this condition is omitted.

Chapter 10

Second-Order Elliptic Equations with Three or More Space Variables

10.1 Laplace Equation $\Delta_3 w = 0$

The *three-dimensional Laplace equation* is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. For example, in heat and mass transfer theory, this equation describes stationary temperature distribution in the absence of heat sources and sinks in the domain under study.

A regular solution of the Laplace equation is called a harmonic function. The first boundary value problem for the Laplace equation is often referred to as the *Dirichlet problem*, and the second boundary value problem as the *Neumann problem*.

Extremum principle: Given a domain D, a harmonic function w in D that is not identically constant in D cannot attain its maximum or minimum value at any interior point of D.

10.1.1 Problems in Cartesian Coordinates

The three-dimensional Laplace equation in the rectangular Cartesian system of coordinates is written as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.$$

► **Particular solutions and some relations.**

1°. Particular solutions:

$$\begin{aligned}
w(x,y,z) &= Ax + By + Cz + D,\\
w(x,y,z) &= Ax^2 + By^2 - (A+B)z^2 + Cxy + Dxz + Eyz,\\
w(x,y,z) &= \cos(\mu_1 x + \mu_2 y)\exp(\pm\mu z),\\
w(x,y,z) &= \sin(\mu_1 x + \mu_2 y)\exp(\pm\mu z),
\end{aligned}$$

$$
\begin{aligned}
w(x,y,z) &= \exp(\mu_1 x + \mu_2 y)\cos(\mu z + A),\\
w(x,y,z) &= \exp(\pm\mu x)\cos(\mu_1 y + A)\cos(\mu_2 z + B),\\
w(x,y,z) &= \cosh(\mu_1 x)\cosh(\mu_2 y)\cos(\mu z + B),\\
w(x,y,z) &= \cosh(\mu_1 x)\sinh(\mu_2 y)\cos(\mu z + B),\\
w(x,y,z) &= \cosh(\mu x)\cos(\mu_1 y + A)\cos(\mu_2 z + B),\\
w(x,y,z) &= \sinh(\mu_1 x)\sinh(\mu_2 y)\sin(\mu z + B),\\
w(x,y,z) &= \sinh(\mu x)\sin(\mu_1 y + A)\sin(\mu_2 z + B),
\end{aligned}
$$

where A, B, C, D, E, μ_1, and μ_2 are arbitrary constants, and $\mu = \sqrt{\mu_1^2 + \mu_2^2}$.

2°. Fundamental solution:

$$
\mathscr{E}(x,y,z) = \frac{1}{4\pi\sqrt{x^2+y^2+z^2}}.
$$

3°. Suppose $w = w(x,y,z)$ is a solution of the Laplace equation. Then the functions

$$
\begin{aligned}
&w_1 = Aw(\pm\lambda x + C_1,\ \pm\lambda y + C_2,\ \pm\lambda x + C_3),\\
&w_2 = \frac{A}{r} w\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right), \qquad r = \sqrt{x^2+y^2+z^2},\\
&w_3 = \frac{A}{\sqrt{\Xi}} w\left(\frac{x-ar^2}{\Xi}, \frac{y-br^2}{\Xi}, \frac{z-cr^2}{\Xi}\right), \qquad \Xi = 1-2(ax+by+cz)+(a^2+b^2+c^2)r^2,
\end{aligned}
$$

where A, C_n, a, b, c, and λ are arbitrary constants, are also solutions of this equation. The signs at λ in the expression of w_1 can be taken independently of one another.

⊙ *Literature*: W. Miller, Jr. (1977), R. Courant and D. Hilbert (1989).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z < \infty$. First boundary value problem.**

A half-space is considered. A boundary condition is prescribed:

$$
w = f(x,y) \quad \text{at} \quad z = 0.
$$

Solution:

$$
w(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z f(\xi,\eta)\,d\xi\,d\eta}{\left[(x-\xi)^2 + (y-\eta)^2 + z^2\right]^{3/2}}.
$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z < \infty$. Second boundary value problem.**

A half-space is considered. A boundary condition is prescribed:

$$
\partial_z w = f(x,y) \quad \text{at} \quad z = 0.
$$

Solution:

$$w(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\xi,\eta)\,d\xi\,d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}} + C,$$

where C is an arbitrary constant.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. First boundary value problem.**

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$\begin{aligned} &w = f_1(y,z) \quad \text{at} \quad x=0, \qquad && w = f_2(y,z) \quad \text{at} \quad x=a,\\ &w = f_3(x,z) \quad \text{at} \quad y=0, \qquad && w = f_4(x,z) \quad \text{at} \quad y=b,\\ &w = f_5(x,y) \quad \text{at} \quad z=0, \qquad && w = f_6(x,y) \quad \text{at} \quad z=c. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z) =& \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{f_{nm}^2\sinh(\lambda_{nm}^1 x)+f_{nm}^1\sinh[\lambda_{nm}^1(a-x)]}{\sinh(\lambda_{nm}^1 a)}\sin\Big(\frac{\pi n y}{b}\Big)\sin\Big(\frac{\pi m z}{c}\Big)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{f_{nm}^4\sinh(\lambda_{nm}^2 y)+f_{nm}^3\sinh[\lambda_{nm}^2(b-y)]}{\sinh(\lambda_{nm}^2 b)}\sin\Big(\frac{\pi n x}{a}\Big)\sin\Big(\frac{\pi m z}{c}\Big)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{f_{nm}^6\sinh(\lambda_{nm}^3 z)+f_{nm}^5\sinh[\lambda_{nm}^3(c-z)]}{\sinh(\lambda_{nm}^3 c)}\sin\Big(\frac{\pi n x}{a}\Big)\sin\Big(\frac{\pi m y}{b}\Big), \end{aligned}$$

where the constant coefficients are given by

$$\lambda_{nm}^1 = \pi\sqrt{\frac{n^2}{b^2}+\frac{m^2}{c^2}}, \quad \lambda_{nm}^2 = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{c^2}}, \quad \lambda_{nm}^3 = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}},$$

$$f_{nm}^i = \begin{cases} \dfrac{4}{bc}\displaystyle\int_0^b\int_0^c f_i(y,z)\sin\Big(\frac{\pi n y}{b}\Big)\sin\Big(\frac{\pi m z}{c}\Big)\,dy\,dz & \text{for } i=1,\,2;\\ \dfrac{4}{ac}\displaystyle\int_0^a\int_0^c f_i(x,z)\sin\Big(\frac{\pi n x}{a}\Big)\sin\Big(\frac{\pi m z}{c}\Big)\,dx\,dz & \text{for } i=3,\,4;\\ \dfrac{4}{ab}\displaystyle\int_0^a\int_0^b f_i(x,y)\sin\Big(\frac{\pi n x}{a}\Big)\sin\Big(\frac{\pi m y}{b}\Big)\,dx\,dy & \text{for } i=5,\,6. \end{cases}$$

Example 10.1. The planes $x=0$ and $x=a$ have constant temperatures w_1 and w_2, respectively. The other planes are maintained at zero temperature ($f_3=f_4=f_5=f_6=0$).

Solution:

$$w = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{w_2\sinh(\mu_{nm}x)+w_1\sinh[\mu_{nm}(a-x)]}{(2n-1)(2m-1)\sinh(\mu_{nm}a)}\sin(p_n y)\sin(q_m z),$$

$$p_n = \frac{\pi(2n-1)}{b}, \quad q_m = \frac{\pi(2m-1)}{c}, \quad \mu_{nm} = \sqrt{p_n^2+q_m^2}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), H. S. Carslaw and J. C. Jaeger (1984).

◆ *For the solution of other boundary value problems for the three-dimensional Laplace equation in the Cartesian coordinate system, see Section 10.2.2 for* $\Phi \equiv 0$.

10.1.2 Problems in Cylindrical Coordinates

The three-dimensional Laplace equation in the cylindrical coordinate system is written as

$$\Delta_3 w \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}+\frac{\partial^2 w}{\partial z^2}=0, \qquad r=\sqrt{x^2+y^2}.$$

► **Particular solutions:**

$$\begin{aligned}
w(r,\varphi,z) &= \left(Ar^m+\frac{B}{r^m}\right)(C\cos m\varphi+D\sin m\varphi)(\alpha+\beta z),\\
w(r,\varphi,z) &= J_m(\mu r)(A\cos m\varphi+B\sin m\varphi)(C\cosh\mu z+D\sinh\mu z),\\
w(r,\varphi,z) &= Y_m(\mu r)(A\cos m\varphi+B\sin m\varphi)(C\cosh\mu z+D\sinh\mu z),\\
w(r,\varphi,z) &= I_m(\mu r)(A\cos m\varphi+B\sin m\varphi)(C\cos\mu z+D\sin\mu z),\\
w(r,\varphi,z) &= K_m(\mu r)(A\cos m\varphi+B\sin m\varphi)(C\cos\mu z+D\sin\mu z),
\end{aligned}$$

where $m=0,\,1,\,2,\,\dots;$ A, B, C, D, α, β, and μ are arbitrary constants; the $J_m(\xi)$ and $Y_m(\xi)$ are Bessel functions; and the $I_m(\xi)$ and $K_m(\xi)$ are modified Bessel functions.

► **Domain: $0\le r\le a,\ 0\le\varphi\le 2\pi,\ -\infty<z<\infty$. First boundary value problem.**

An infinite circular cylinder is considered. A boundary condition is prescribed:

$$w=f(\varphi,z) \quad \text{at} \quad r=a.$$

Solution:

$$w(r,\varphi,z)=-\frac{1}{a}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{J_n(\lambda_{nm}r)}{J_n'(\lambda_{nm}a)}\int_{-\infty}^{\infty}\bigl[A_n(\xi)\cos n\varphi+B_n(\xi)\sin n\varphi\bigr]\exp\bigl(-\lambda_{nm}|\xi-z|\bigr)\,d\xi,$$

where the $J_n(r)$ are the Bessel functions and λ_{mn} are positive roots of the transcendental equation $J_n(a\lambda)=0$. The functions $A_n(z)$ and $B_n(z)$ are the coefficients of the Fourier series expansion of $f(\varphi,z)$,

$$A_n(z)=\frac{\varepsilon_n}{\pi}\int_0^{2\pi}f(\varphi,z)\cos(n\varphi)\,d\varphi, \quad B_n(z)=\frac{1}{\pi}\int_0^{2\pi}f(\varphi,z)\sin(n\varphi)\,d\varphi,$$

where $\varepsilon_0=1/2$ and $\varepsilon_n=1$ for $n=1,\,2,\,\dots$

If the surface temperature is independent of φ, i.e., $f(z,\varphi)=f(z)$, then the solution takes the form

$$w(r,\varphi,z)=\frac{1}{a}\sum_{m=1}^{\infty}\frac{J_0(\lambda_m r)}{J_1(\lambda_m a)}\int_0^{\infty}\bigl[f(z+\zeta)+f(z-\zeta)\bigr]\exp\bigl(-\lambda_m\zeta\bigr)\,d\zeta,$$

where the λ_m are positive roots of the transcendental equation $J_0(a\lambda)=0$.

⊙ *Literature*: H. S. Carslaw and J. C. Jaeger (1984).

► **Domain: $0 \le r \le a,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le b$. First boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$w = f(\varphi, z) \quad \text{at} \quad r = a, \qquad w = g_1(r,\varphi) \quad \text{at} \quad z = 0, \qquad w = g_2(r,\varphi) \quad \text{at} \quad z = b.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{I_n\left(\dfrac{\pi m r}{b}\right)}{I_n\left(\dfrac{\pi m a}{b}\right)} (A_{nm}\cos n\varphi + B_{nm}\sin n\varphi)\sin\left(\frac{\pi m z}{b}\right) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{\mu_{mn} r}{a}\right)(C_{nm}^{(1)}\cos n\varphi + D_{nm}^{(1)}\sin n\varphi)\frac{\sinh\left(\dfrac{\mu_{mn}(b-z)}{a}\right)}{\sinh\left(\dfrac{\mu_{mn} b}{a}\right)} \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{\mu_{mn} r}{a}\right)(C_{nm}^{(2)}\cos n\varphi + D_{nm}^{(2)}\sin n\varphi)\frac{\sinh\left(\dfrac{\mu_{mn} z}{a}\right)}{\sinh\left(\dfrac{\mu_{mn} b}{a}\right)}, \end{aligned}$$

where the $J_n(r)$ are Bessel functions, the $I_n(r)$ are modified Bessel functions, and μ_{mn} is the mth root of the equation $J_n(\mu) = 0$. The coefficients A_{nm}, B_{nm}, $C_{nm}^{(i)}$, and $D_{nm}^{(i)}$ are defined by

$$\begin{aligned} A_{nm} &= \frac{\varepsilon_n}{\pi b}\int_0^{2\pi}\int_0^b f(\varphi,z)\cos(n\varphi)\sin\left(\frac{\pi m z}{b}\right)d\varphi\,dz, \\ B_{nm} &= \frac{2}{\pi b}\int_0^{2\pi}\int_0^b f(\varphi,z)\sin(n\varphi)\sin\left(\frac{\pi m z}{b}\right)d\varphi\,dz, \\ C_{nm}^{(i)} &= \frac{\varepsilon_n}{\pi a^2[J_n'(\mu_{mn})]^2}\int_0^{2\pi}\int_0^a g_i(r,\varphi)\cos(n\varphi)J_n\left(\frac{\mu_{mn} r}{a}\right)r\,dr\,d\varphi, \\ D_{nm}^{(i)} &= \frac{2}{\pi a^2[J_n'(\mu_{mn})]^2}\int_0^{2\pi}\int_0^a g_i(r,\varphi)\sin(n\varphi)J_n\left(\frac{\mu_{mn} r}{a}\right)r\,dr\,d\varphi, \\ \varepsilon_n &= \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \ne 0, \end{cases} \qquad i = 1,\ 2. \end{aligned}$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

◆ *For the solution of other boundary value problems for the three-dimensional Laplace equation in the cylindrical coordinate system, see Section 10.2.3 for* $\Phi \equiv 0$.

10.1.3 Problems in Spherical Coordinates

The three-dimensional Laplace equation in the spherical coordinate system is written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} = 0, \qquad r = \sqrt{x^2+y^2+z^2}.$$

► **Particular solutions:**

$$w(r) = A + \frac{B}{r},$$
$$w(r,\theta) = \left(Ar^n + \frac{B}{r^{n+1}}\right)P_n(\cos\theta),$$
$$w(r,\theta,\varphi) = \left(Ar^n + \frac{B}{r^{n+1}}\right)P_n^m(\cos\theta)(C\cos m\varphi + D\sin m\varphi),$$

where $n = 0, 1, 2, \ldots;\ m = 0, 1, 2, \ldots, n;\ A, B, C, D$ are arbitrary constants; the $P_n(\xi)$ are the Legendre polynomials; and the $P_n^m(\xi)$ are associated Legendre functions, which are expressed as

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n, \quad P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x).$$

► **Domain: $0 \le r \le R$ or $R \le r < \infty$. First boundary value problem.**

A boundary condition at the sphere surface is prescribed:

$$w = f(\theta,\varphi) \quad \text{at} \quad r = R.$$

1°. Solution of the inner problem (for $r \le R$):

$$w(r,\theta,\varphi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\frac{R^2-r^2}{(r^2-2Rr\cos\gamma+R^2)^{3/2}}\sin\theta_0\,d\theta_0\,d\varphi_0,$$
$$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi-\varphi_0).$$

This formula is conventionally called the *Poisson integral for a sphere*.

Series solution:

$$w(r,\theta,\varphi) = \sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n Y_n(\theta,\varphi), \quad Y_n(\theta,\varphi) = \sum_{m=0}^{n}(A_{nm}\cos m\varphi + B_{nm}\sin m\varphi)P_n^m(\cos\theta),$$

where

$$A_{00} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\sin\theta\,d\theta\,d\varphi,$$
$$A_{nm} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)P_n^m(\cos\theta)\cos m\varphi\sin\theta\,d\theta\,d\varphi,$$
$$B_{nm} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)P_n^m(\cos\theta)\sin m\varphi\sin\theta\,d\theta\,d\varphi.$$

2°. Solution of the outer problem (for $r \ge R$):

$$w(r,\theta,\varphi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\frac{r^2-R^2}{(r^2-2Rr\cos\gamma+R^2)^{3/2}}\sin\theta_0\,d\theta_0\,d\varphi_0,$$

where $\cos\gamma$ is expressed in the same way as in the inner problem.

Series solution:

$$w(r,\theta,\varphi) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} Y_n(\theta,\varphi),$$
$$Y_n(\theta,\varphi) = \sum_{m=0}^{n} (A_{nm}\cos m\varphi + B_{nm}\sin m\varphi) P_n^m(\cos\theta),$$

where the coefficients A_{nm} and B_{nm} are defined by the same relations as in the inner problem.

⊙ *Literature*: G. N. Polozhii (1964), V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), A. N. Tikhonov and A. A. Samarskii (1990).

► Domain: $0 \le r \le R$ or $R \le r < \infty$. Second boundary value problem.

A boundary condition at the sphere surface is prescribed:

$$\frac{\partial w}{\partial r} = f(\theta,\varphi) \quad \text{at} \quad r = R.$$

The function $f(\theta,\varphi)$ must satisfy the solvability condition

$$\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\sin\theta\, d\theta\, d\varphi = 0.$$

1°. Solution of the inner problem (for $r \le R$):

$$w(r,\theta,\varphi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\left[\frac{1}{R}\ln(R + r_1 - r\cos\gamma) - \frac{2}{r_1}\right]\sin\theta_0\, d\theta_0\, d\varphi_0,$$
$$r_1 = \sqrt{r^2 - 2Rr\cos\gamma + R^2}, \quad \cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0).$$

Series solution:

$$w(r,\theta,\varphi) = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{R}{n}\left(\frac{r}{R}\right)^{n} (A_{nm}\cos m\varphi + B_{nm}\sin m\varphi) P_n^m(\cos\theta) + C,$$

where the coefficients A_{nm} and B_{nm} are expressed in the same way as in the inner first boundary value problem (see the paragraph above), and C is an arbitrary constant.

2°. Solution of the outer problem (for $r \ge R$):

$$w(r,\theta,\varphi) = -\frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\left[\frac{1}{R}\ln\frac{R + r_1 - r\cos\gamma}{r(1-\cos\gamma)} - \frac{2}{r_1}\right]\sin\theta_0\, d\theta_0\, d\varphi_0,$$
$$r_1 = \sqrt{r^2 - 2Rr\cos\gamma + R^2}, \quad \cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0).$$

Series solution:

$$w(r,\theta,\varphi) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{R}{n+1}\left(\frac{R}{r}\right)^{n+1} (A_{nm}\cos m\varphi + B_{nm}\sin m\varphi) P_n^m(\cos\theta) + C,$$

where the coefficients A_{nm} and B_{nm} are expressed in the same way as in the inner first boundary value problem, and C is an arbitrary constant.

3°. Outer boundary value problems where unbounded solutions as $r \to \infty$ are sought are also encountered in applications.

Example 10.2. A potential translational flow of an ideal incompressible fluid about a sphere of radius R is governed by the Laplace equation with the boundary conditions:

$$\partial_r w = 0 \quad \text{at} \quad r = R, \qquad |w - Ur\cos\theta| \to 0 \quad \text{as} \quad r \to \infty,$$

where w is the potential, U the unperturbed flow velocity at infinity; the fluid velocity is expressed in terms of the potential as $\mathbf{v} = \nabla\varphi$.

Solution:

$$w = Ur\left(1 + \frac{R^3}{2r^3}\right)\cos\theta.$$

This solution is a special case of the second formula from the first paragraph for $n = 1$.

⊙ *Literature*: G. N. Polozhii (1964), V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), L. G. Loitsyanskii (1996).

◆ *For the solution of other boundary value problems for the three-dimensional Laplace equation in the spherical coordinate system, see Section 10.2.4 for* $\Phi \equiv 0$.

10.1.4 Other Orthogonal Curvilinear Systems of Coordinates

The three-dimensional Laplace equation admits separation of variables in the eleven orthogonal coordinate systems that are listed in Table 10.1.

For the general ellipsoidal and conical coordinate systems, the functions f, g, and h are determined by Lamé equations that involve the Jacobian elliptic function $\operatorname{sn} z = \operatorname{sn}(z, k)$. The solutions of these equations under some conditions can be represented in the form of finite series called Lamé polynomials. For details about the Lamé equation and its solutions, see Whittaker and Watson (1963), Bateman and Erdélyi (1955), Arscott (1964), and Miller, Jr. (1977).

There are also coordinate systems that allow the so-called $\mathcal{R}$-separation of variables of the three-dimensional Laplace equation. Such solutions in the new coordinate system, μ, ν, ρ, can be represented in the form $w = \sqrt{\mathcal{R}(\mu,\nu,\rho)}\, f(\mu)g(\nu)h(\rho)$. Coordinates that allow the $\mathcal{R}$-separation of variables are listed in Table 10.2.

Only the bicylindrical and toroidal coordinate systems are fairly widely used in applications. In three subsequent coordinate systems, the functions $f = f(\mu)$ and $g = g(\rho)$ are determined by identical equations. With the change of variables $\mu = \operatorname{sn}^2(\alpha, k)$, $\rho = \operatorname{sn}^2(\beta, k)$, where $k = a^{-1/2}$, these equations are reduced to Lamé equations (α and β are the new independent variables).

⊙ *Literature for Section* 10.1.4: M. Bôcher (1894), N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand (1955), F. M. Morse and H. Feshbach (1953, Vols. 1–2), P. Moon and D. Spencer (1988), A. Makarov, J. Smorodinsky, K. Valiev, and P. Winternitz (1967), W. Miller, Jr. (1977).

10.2 Poisson Equation $\Delta_3 w + \Phi(\mathbf{x}) = 0$

10.2.1 Preliminary Remarks. Solution Structure

Like the three-dimensional Laplace equation, the three-dimensional Poisson equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics,

TABLE 10.1
Orthogonal coordinates $\bar{x}, \bar{y}, \bar{z}$ that allow separable solutions of the form $w = f(\bar{x})g(\bar{y})h(\bar{z})$ for the three-dimensional Laplace equation $\Delta_3 w = 0$

Coordinates	Transformations	Particular solutions (or equations for f, g, h)
Cartesian x, y, z	$x = x$, $y = y$, $z = z$	$w = \cos(k_1 x + s_1)\cos(k_2 y + s_2)\cosh(k_3 z + s_3)$, where $k_1^2 + k_2^2 = k_3^2$; see also Section 10.1.1 (particular solutions)
Cylindrical r, φ, z	$x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$	$w = [AJ_n(kr) + BY_n(kr)]\cos(n\varphi + c)\exp(\pm kz)$, $J_n(z)$ and $Y_n(z)$ are the Bessel functions; see also Section 10.1.2 (particular solutions)
Parabolic cylindrical ξ, η, z	$x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$, $z = z$	$w = D_{\mu-1/2}(\pm\sigma\xi)D_{-\mu-1/2}(\pm\sigma\eta)\cos(kz + s)$, where $\sigma = \sqrt{2k}$, $D_\mu(z)$ is the parabolic cylinder function
Elliptic cylindrical u, v, z	$x = a\cosh u\cos v$, $y = a\sinh u\sin v$, $z = z$	$w = \begin{cases} \mathrm{Ce}_n(u, -q)\,\mathrm{ce}_n(v, -q)\cos(kz + s), \\ \mathrm{Se}_n(u, -q)\,\mathrm{se}_n(v, -q)\cos(kz + s), \end{cases}$ Ce_n and Se_n are the modified Mathieu functions, ce_n and se_n are the Mathieu functions, $q = \frac{1}{4}a^2k^2$
Spherical r, θ, φ	$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$	$w = (Ar^n + Br^{-n-1})P_n^k(\cos\theta)\cos(k\varphi + s)$, $P_n^k(\xi)$ are the associated Legendre functions, see also Section 10.1.3 (particular solutions)
Prolate spheroidal u, v, φ	$x = a\sinh u\sin v\cos\varphi$, $y = a\sinh u\sin v\sin\varphi$, $z = a\cosh u\cos v$	$w = P_n^k(\cosh u)P_n^k(\cos v)\cos(k\varphi + s)$, $P_n^k(\xi)$ are the associated Legendre functions
Oblate spheroidal u, v, φ	$x = a\cosh u\sin v\cos\varphi$, $y = a\cosh u\sin v\sin\varphi$, $z = a\sinh u\cos v$	$w = P_n^k(-i\sinh u)P_n^k(\cos v)\cos(k\varphi + s)$, $P_n^k(\xi)$ are the associated Legendre functions
Parabolic ξ, η, φ	$x = a\xi\eta\cos\varphi$, $y = a\xi\eta\sin\varphi$, $z = \frac{1}{2}a(\xi^2 - \eta^2)$	$w = I_{\pm k}(\beta\xi)J_{\pm k}(\beta\eta)\cos(k\varphi + s)$, $J_k(z)$ are the Bessel functions, $I_k(z)$ are the modified Bessel functions
Paraboloidal u, v, φ	$x = 2a\cosh u\cos v\sinh\varphi$, $y = 2a\sinh u\sin v\cosh\varphi$, $z = \frac{1}{2}a(\cosh 2u$ $+ \cos 2v - \cosh 2\varphi)$	$w = \begin{cases} \mathrm{Ce}_n(u, -b)\,\mathrm{ce}_n(v, -b)\,\mathrm{Ce}_n(\varphi + i\pi/2, -b), \\ \mathrm{Se}_n(u, -b)\,\mathrm{se}_n(v, -b)\,\mathrm{Se}_n(\varphi + i\pi/2, -b), \end{cases}$ $b = \frac{1}{2}a\beta$; ce_n and se_n are the Mathieu functions, Ce_n and Se_n are the modified Mathieu functions
General ellipsoidal μ, ν, ρ	$x = \sqrt{\frac{(\mu-a)(\nu-a)(\rho-a)}{a(a-1)}}$, $y = \sqrt{\frac{(\mu-1)(\nu-1)(\rho-1)}{1-a}}$, $z = \sqrt{\frac{\mu\nu\rho}{a}}$	$f''_{\xi\xi} + (\beta_2 + \beta_1\,\mathrm{sn}^2\,\xi)f = 0$ (Lamé equation), $g''_{\eta\eta} + (\beta_2 + \beta_1\,\mathrm{sn}^2\,\eta)g = 0$ (Lamé equation), $h''_{\zeta\zeta} + (\beta_2 + \beta_1\,\mathrm{sn}^2\,\zeta)g = 0$ (Lamé equation), $\mu = \mathrm{sn}^2(\xi, k)$, $\nu = \mathrm{sn}^2(\eta, k)$, $\rho = \mathrm{sn}^2(\zeta, k)$, $k = a^{-1/2}$
Conical ρ, μ, ν	$x = \rho\sqrt{\frac{(a\mu-1)(a\nu-1)}{1-a}}$, $y = \rho\sqrt{\frac{a(\mu-1)(\nu-1)}{a-1}}$, $z = \rho\sqrt{a\mu\nu}$	$f(\rho) = A\rho^n + B\rho^{-n-1}$, $n = 0, 1, \ldots,$ $g''_{\xi\xi} + [\beta - n(n+1)k^2\,\mathrm{sn}^2\,\xi]g = 0$ (Lamé equation), $h''_{\eta\eta} + [\beta - n(n+1)k^2\,\mathrm{sn}^2\,\eta]h = 0$ (Lamé equation), where $\mu = \mathrm{sn}^2(\xi, k)$, $\nu = \mathrm{sn}^2(\eta, k)$, $k = a^{1/2}$, g and h are expressed in terms of the Lamé polynomials

TABLE 10.2
Coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ that allow $\mathcal{R}$-separated solutions of the form $w=\sqrt{\mathcal{R}(\bar{x},\bar{y},\bar{z})}\,f(\bar{x})g(\bar{y})h(\bar{z})$ for the three-dimensional Laplace equation $\Delta_3 w=0$

New coordinates, function $\mathcal{R}$	Transformations of coordinates	Functions f, g, h (equations for f, g, h)
Bicylindrical coordinates α, β, φ, $\mathcal{R}=\cosh\beta-\cos\alpha$	$x=c\mathcal{R}^{-1}\sin\alpha\cos\varphi$, $y=c\mathcal{R}^{-1}\sin\alpha\sin\varphi$, $z=c\mathcal{R}^{-1}\sinh\beta$; $0\le\alpha\le\pi$, β is any, $0\le\varphi<2\pi$	$f(\alpha)=A_1P_n^m(\cos\alpha)+A_2Q_n^m(\cos\alpha)$, $g(\beta)=B_1\cosh\big[(n+\frac{1}{2})\beta\big]+B_2\sinh\big[(n+\frac{1}{2})\beta\big]$, $h(\varphi)=C_1\cos(m\varphi)+C_2\sin(m\varphi)$, $n=0,\,1,\,2,\,\ldots;\ m=0,\,1,\,2,\,\ldots$
Toroidal coordinates α, β, φ, $\mathcal{R}=\cosh\alpha-\cos\beta$	$x=c\mathcal{R}^{-1}\sinh\alpha\cos\varphi$, $y=c\mathcal{R}^{-1}\sinh\alpha\sin\varphi$, $z=c\mathcal{R}^{-1}\sin\beta$; $\alpha\ge0$, $-\pi\le\beta\le\pi$, $0\le\varphi<2\pi$	$f(\alpha)=A_1P_{n-1/2}^m(\cosh\alpha)+A_2Q_{n-1/2}^m(\cosh\alpha)$, $g(\beta)=B_1\cos(n\beta)+B_2\sin(n\beta)$, $h(\varphi)=C_1\cos(m\varphi)+C_2\sin(m\varphi)$, $n=0,\,1,\,2,\,\ldots;\ m=0,\,1,\,2,\,\ldots$
Coordinates μ, ρ, φ, $\mathcal{R}=\sqrt{\frac{(\mu-a)(a-\rho)}{a(a-1)}}-\sqrt{\frac{(\mu-1)(1-\rho)}{a-1}}$	$x=\mathcal{R}^{-1}\cos\varphi$, $y=\mathcal{R}^{-1}\sin\varphi$, $z=\mathcal{R}^{-1}\sqrt{-\mu\rho/a}$; $\mu>a>1$, $\rho<0$, $0\le\varphi<2\pi$	$\sqrt{U(\mu)}\,[\sqrt{U(\mu)}\,f']'+\big[(\frac{1}{4}-n^2)\mu-\lambda\big]f=0$, $\sqrt{U(\rho)}\,[\sqrt{U(\rho)}\,g']'+\big[(\frac{1}{4}-n^2)\rho-\lambda\big]g=0$, $h(\varphi)=C_1\cos(n\varphi)+C_2\sin(n\varphi)$, $U(t)=4t(t-1)(t-a)$
Coordinates μ, ρ, φ, $\mathcal{R}=\sqrt{\frac{\mu\rho}{a}}+\sqrt{\frac{(\mu-1)(\rho-1)}{a-1}}$	$x=\mathcal{R}^{-1}\cos\varphi$, $y=\mathcal{R}^{-1}\sin\varphi$, $z=\mathcal{R}^{-1}\sqrt{\frac{(\mu-a)(a-\rho)}{a(a-1)}}$; $1<\rho<a<\mu$, $0\le\varphi<2\pi$	$\sqrt{U(\mu)}\,[\sqrt{U(\mu)}\,f']'+\big[(\frac{1}{4}-n^2)\mu-\lambda\big]f=0$, $\sqrt{U(\rho)}\,[\sqrt{U(\rho)}\,g']'+\big[(\frac{1}{4}-n^2)\rho-\lambda\big]g=0$, $h(\varphi)=C_1\cos(n\varphi)+C_2\sin(n\varphi)$, $U(t)=4t(t-1)(t-a)$
Coordinates μ, ρ, φ, $\mathcal{R}=2\,\mathrm{Re}\sqrt{\frac{i(\mu-a)(\rho-a)}{a(a-b)}}$, $a=\bar{b}=\alpha+i\beta$, α, β are real numbers	$x=\mathcal{R}^{-1}\cos\varphi$, $y=\mathcal{R}^{-1}\sin\varphi$, $z=\mathcal{R}^{-1}\sqrt{-\mu\rho/(ab)}$; $\mu>0$, $\rho<0$, $0\le\varphi<2\pi$	$\sqrt{U(\mu)}\,[\sqrt{U(\mu)}\,f']'+\big[(\frac{1}{4}-n^2)\mu-\lambda\big]f=0$, $\sqrt{U(\rho)}\,[\sqrt{U(\rho)}\,g']'+\big[(\frac{1}{4}-n^2)\rho-\lambda\big]g=0$, $h(\varphi)=C_1\cos(n\varphi)+C_2\sin(n\varphi)$, $U(t)=4t(t-a)(t-b)$
Coordinates μ, ν, ρ, $\mathcal{R}=1+\sqrt{\frac{\mu\nu\rho}{ab}}$	$x=\mathcal{R}^{-1}\sqrt{\frac{(\mu-a)(\nu-a)(\rho-a)}{(b-a)(a-1)a}}$, $y=\mathcal{R}^{-1}\sqrt{\frac{(\mu-b)(\nu-b)(\rho-b)}{(a-b)(b-1)b}}$, $z=\mathcal{R}^{-1}\sqrt{\frac{(\mu-1)(\nu-1)(\rho-1)}{(a-1)(b-1)}}$; $0<\rho<1<\nu<b<\mu<a$	$\sqrt{U(\mu)}\,[\sqrt{U(\mu)}\,f']'-(3\mu^2+\lambda_1\mu+\lambda_2)f=0$, $\sqrt{U(\nu)}\,[\sqrt{U(\nu)}\,g']'-(3\nu^2+\lambda_1\nu+\lambda_2)g=0$, $\sqrt{U(\rho)}\,[\sqrt{U(\rho)}\,h']'-(3\rho^2+\lambda_1\rho+\lambda_2)h=0$, $U(t)=16t(t-1)(t-a)(t-b)$
Coordinates μ, ν, ρ, $\mathcal{R}=2\,\mathrm{Re}\sqrt{\frac{(\mu-a)(\nu-a)(\rho-a)}{ia(a-1)(a-b)}}$, $a=\bar{b}=\alpha+i\beta$, α, β are real numbers	$x=\mathcal{R}^{-1}\sqrt{\frac{(\mu-1)(\nu-1)(\rho-1)}{(a-1)(b-1)}}$, $y=\mathcal{R}^{-1}\sqrt{-\frac{\mu\nu\rho}{ab}}$, $z=\mathcal{R}^{-1}$; $\rho<0<\mu<1<\nu$	$\sqrt{U(\mu)}\,[\sqrt{U(\mu)}\,f']'-(3\mu^2+\lambda_1\mu+\lambda_2)f=0$, $\sqrt{U(\nu)}\,[\sqrt{U(\nu)}\,g']'-(3\nu^2+\lambda_1\nu+\lambda_2)g=0$, $\sqrt{U(\rho)}\,[\sqrt{U(\rho)}\,h']'-(3\rho^2+\lambda_1\rho+\lambda_2)h=0$, $U(t)=16t(t-1)(t-a)(t-b)$

and other areas of mechanics and physics. In particular, the Poisson equation describes stationary temperature distribution in the presence of thermal sources or sinks in the domain under consideration.

The Laplace equation is a special case of the Poisson equation with $\Phi\equiv0$.

Throughout this section, we consider a three-dimensional bounded domain V with a sufficiently smooth boundary S. We assume that $\mathbf{r}\in V$ and $\boldsymbol{\rho}\in V$, where $\mathbf{r}=\{x,y,z\}$ and $\boldsymbol{\rho}=\{\xi,\eta,\zeta\}$.

► **First boundary value problem.**

The solution of the first boundary value problem for the Poisson equation

$$\Delta_3 w + \Phi(\mathbf{r}) = 0 \tag{1}$$

in a domain V with the nonhomogeneous boundary condition

$$w = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S$$

can be represented in the form

$$w(\mathbf{r}) = \int_V \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dV_\rho - \int_S f(\boldsymbol{\rho}) \frac{\partial G}{\partial N_\rho}\, dS_\rho. \tag{2}$$

Here, $G(\mathbf{r}, \boldsymbol{\rho})$ is the Green's function of the first boundary value problem, $\frac{\partial G}{\partial N_\rho}$ is the derivative of the Green's function with respect to ξ, η, ζ along the outward normal $\mathbf{N}$ to the boundary S of the domain V. The integration is everywhere with respect to ξ, η, ζ.

The volume elements in solution (2) for basic coordinate systems are presented in Table 10.3. In addition, the expressions of the gradients are given, which enable one to find the derivative along the normal in accordance with the formula $\frac{\partial G}{\partial N_\rho} = (\mathbf{N} \cdot \nabla_\rho G)$.

TABLE 10.3
The volume elements and distances occurring in relations (2) and (5) in some coordinate systems. In all cases, $\boldsymbol{\rho} = \{\xi, \eta, \zeta\}$

Coordinate system	Volume element, dV_ρ	Gradient, $\nabla_\rho u$ ($\lvert\mathbf{i}_\xi\rvert = \lvert\mathbf{i}_\eta\rvert = \lvert\mathbf{i}_\zeta\rvert = 1$)	Distance, $d = \lvert\mathbf{r} - \boldsymbol{\rho}\rvert$
Cartesian $\mathbf{r} = \{x, y, z\}$	$d\xi\, d\eta\, d\zeta$	$\mathbf{i}_\xi \frac{\partial u}{\partial \xi} + \mathbf{i}_\eta \frac{\partial u}{\partial \eta} + \mathbf{i}_\zeta \frac{\partial u}{\partial \zeta}$	$d = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$
Cylindrical $\mathbf{r} = \{r, \varphi, z\}$	$\xi\, d\xi\, d\eta\, d\zeta$	$\mathbf{i}_\xi \frac{\partial u}{\partial \xi} + \mathbf{i}_\eta \frac{1}{\xi} \frac{\partial u}{\partial \eta} + \mathbf{i}_\zeta \frac{\partial u}{\partial \zeta}$	$d = \sqrt{r^2 + \xi^2 - 2r\xi\cos(\varphi - \eta) + (z-\zeta)^2}$
Spherical $\mathbf{r} = \{r, \theta, \varphi\}$	$\xi^2 \sin\eta\, d\xi\, d\eta\, d\zeta$	$\mathbf{i}_\xi \frac{\partial u}{\partial \xi} + \mathbf{i}_\eta \frac{1}{\xi} \frac{\partial u}{\partial \eta} + \mathbf{i}_\zeta \frac{1}{\xi \sin\eta} \frac{\partial u}{\partial \zeta}$	$d = \sqrt{r^2 + \xi^2 - 2r\xi\cos\gamma}$, where $\cos\gamma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos(\varphi - \zeta)$

The Green's function $G = G(\mathbf{r}, \boldsymbol{\rho})$ of the first boundary value problem is determined by the following conditions:

1°. The function G satisfies the Laplace equation with respect to x, y, z in the domain V everywhere except for the point (ξ, η, ζ), at which it can have a singularity of the form $\frac{1}{4\pi} \frac{1}{\lvert\mathbf{r} - \boldsymbol{\rho}\rvert}$.

2°. The function G, with respect to x, y, z, satisfies the homogeneous boundary condition of the first kind at the boundary, i.e., the condition $G|_S = 0$.

The Green's function can be represented as

$$G(\mathbf{r}, \boldsymbol{\rho}) = \frac{1}{4\pi} \frac{1}{\lvert\mathbf{r} - \boldsymbol{\rho}\rvert} + u, \tag{3}$$

where the auxiliary function $u = u(\mathbf{r}, \boldsymbol{\rho})$ is determined by solving the first boundary value problem for the Laplace equation $\Delta_3 u = 0$ with the boundary condition $u|_S = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$; the vector quantity $\boldsymbol{\rho}$ in this problem is treated as a three-dimensional free parameter.

The Green's function possesses the symmetry property with respect to the arguments: $G(\mathbf{r}, \boldsymbol{\rho}) = G(\boldsymbol{\rho}, \mathbf{r})$.

The construction of Green's functions is discussed in Section 10.3.2 for $\lambda = 0$.

Remark 10.1. For outer first boundary value problems for the Laplace equation, the following condition is usually set at infinity: $|w| < A/|\mathbf{r}|$ ($|\mathbf{r}| \to \infty,\ A = \text{const}$).

► Second boundary value problem.

The second boundary value problem for the Poisson equation (1) is characterized by the boundary condition

$$\frac{\partial w}{\partial N} = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S.$$

Necessary condition solvability of the inner problem:

$$\int_V \Phi(\mathbf{r})\, d\mathbf{r} + \int_S f(\mathbf{r})\, dS = 0. \tag{4}$$

The solution of the second boundary value problem can be written as

$$w(\mathbf{r}) = \int_V \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dV_\rho + \int_S f(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dS_\rho + C, \tag{5}$$

where C is an arbitrary constant, provided that the solvability condition is met.

The Green's function $G = G(\mathbf{r}, \boldsymbol{\rho})$ of the second boundary value problem is determined by the following conditions:

1°. The function G satisfies the Laplace equation with respect to x, y, z in the domain V everywhere except for the point (ξ, η, ζ) at which it has a singularity of the form $\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$.

2°. The function G, with respect to x, y, z, satisfies the homogeneous condition of the second kind at the boundary, i.e., the condition

$$\left.\frac{\partial G}{\partial N}\right|_S = \frac{1}{S_0},$$

where S_0 is the area of the surface S.

The Green's function is unique up to an additive constant.

Remark 10.2. The Green's function cannot be identified with condition 1° and the homogeneous boundary condition $\frac{\partial G}{\partial N}\big|_S = 0$; this problem for G has no solution, because, on representing G in the form (3), for u we obtain a problem with a nonhomogeneous boundary condition of the second kind, for which the solvability condition (2) is not met.

Remark 10.3. Condition (4) is not extended to the outer second boundary value problem (for infinite domain).

► **Third boundary value problem.**

The solution of the third boundary value problem for the Poisson equation (1) in a bounded domain V with the nonhomogeneous boundary condition

$$\frac{\partial w}{\partial N} + kw = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S$$

is given by relation (5) with $C = 0$, where $G = G(\mathbf{r}, \boldsymbol{\rho})$ is the Green's function of the third boundary value problem; the Green's function is determined by the following conditions:

1°. The function G satisfies the Laplace equation with respect to x, y, z in V everywhere except for the point (ξ, η, ζ) at which it has a singularity of the form $\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\rho}|}$.

2°. The function G, with respect to x, y, z, satisfies the homogeneous boundary condition of the third kind at the boundary, i.e., the condition $\left[\frac{\partial G}{\partial N} + kG\right]_S = 0$.

The Green's function can be represented in the form (3), where the auxiliary function u is determined by solving the corresponding third boundary value problem for the Laplace equation $\Delta_3 u = 0$.

The construction of Green's functions is discussed in Section 10.3.2 for $\lambda = 0$.

⊙ *Literature for Section* 10.2.1: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970).

10.2.2 Problems in Cartesian Coordinates

The three-dimensional Poisson equation in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \Phi(x, y, z) = 0.$$

► **Domain:** $-\infty < x < \infty,\ -\infty < y < \infty,\ -\infty < z < \infty$.

Solution:

$$w(x, y, z) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Phi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}.$$

⊙ *Literature*: R. Courant and D. Hilbert (1989).

► **Domain:** $-\infty < x < \infty, -\infty < y < \infty, 0 \le z < \infty$.

First boundary value problem.

A half-space is considered. A boundary condition is prescribed:

$$w = f(x, y) \quad \text{at} \quad z = 0.$$

Solution:

$$w(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z f(\xi, \eta)\, d\xi\, d\eta}{\left[(x-\xi)^2 + (y-\eta)^2 + z^2\right]^{3/2}}$$
$$+ \frac{1}{4\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{R_-} - \frac{1}{R_+}\right)\Phi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta,$$

where

$$R_- = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}, \quad R_+ = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}.$$

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z < \infty$. Third boundary value problem.**

A half-space is considered. A boundary condition is prescribed:

$$\partial_z w - kw = f(x,y) \quad \text{at} \quad z = 0.$$

Solution:

$$w(x,y,z) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta) G(x,y,z,\xi,\eta,0)\, d\xi\, d\eta + \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta,$$

where

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi}\Biggl[\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} + \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}} - 2k\int_0^{\infty} \frac{\exp(-ks)\, ds}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta+s)^2}}\Biggr].$$

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. First boundary value problem.**

A dihedral angle is considered. Boundary conditions are prescribed:

$$w = f_1(x,z) \quad \text{at} \quad y = 0, \qquad w = f_2(x,y) \quad \text{at} \quad z = 0.$$

Solution:

$$w(x,y,z) = \int_0^{\infty}\int_{-\infty}^{\infty} f_1(\xi,\zeta)\Bigl[\frac{\partial}{\partial\eta} G(x,y,z,\xi,\eta,\zeta)\Bigr]_{\eta=0} d\xi\, d\zeta + \int_0^{\infty}\int_{-\infty}^{\infty} f_2(\xi,\eta)\Bigl[\frac{\partial}{\partial\zeta} G(x,y,z,\xi,\eta,\zeta)\Bigr]_{\zeta=0} d\xi\, d\eta + \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta,$$

where

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi}\Biggl[\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} - \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}} - \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2}} + \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2+(z+\zeta)^2}}\Biggr].$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), A. G. Butkovskiy (1979).

► **Domain: $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. First boundary value problem.**

An octant is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad w = f_2(x,z) \quad \text{at} \quad y = 0, \qquad w = f_3(x,y) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(x,y,z) &= \int_0^\infty\!\!\int_0^\infty f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\eta\,d\zeta \\ &+ \int_0^\infty\!\!\int_0^\infty f_2(\xi,\zeta)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\xi\,d\zeta \\ &+ \int_0^\infty\!\!\int_0^\infty f_3(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\xi\,d\eta \\ &+ \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \end{aligned}$$

where

$$\begin{aligned} G(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi}\Bigg[&\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} - \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}} \\ &- \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2}} + \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2+(z+\zeta)^2}} \\ &- \frac{1}{\sqrt{(x+\xi)^2+(y-\eta)^2+(z-\zeta)^2}} + \frac{1}{\sqrt{(x+\xi)^2+(y-\eta)^2+(z+\zeta)^2}} \\ &+ \frac{1}{\sqrt{(x+\xi)^2+(y+\eta)^2+(z-\zeta)^2}} - \frac{1}{\sqrt{(x+\xi)^2+(y+\eta)^2+(z+\zeta)^2}}\Bigg]. \end{aligned}$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), A. G. Butkovskiy (1979).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le a$. First boundary value problem.**

An infinite layer is considered. Boundary conditions are prescribed:

$$w = f_1(x,y) \quad \text{at} \quad z = 0, \qquad w = f_2(x,y) \quad \text{at} \quad z = a.$$

Solution:

$$\begin{aligned} w(x,y,z) &= \int_{-\infty}^\infty\!\!\int_{-\infty}^\infty f_1(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\xi\,d\eta \\ &- \int_{-\infty}^\infty\!\!\int_{-\infty}^\infty f_2(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=a} d\xi\,d\eta \\ &+ \int_0^a\!\!\int_{-\infty}^\infty\!\!\int_{-\infty}^\infty \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{r_{n1}}-\frac{1}{r_{n2}}\right),$$

where

$$r_{n1}=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta-2na)^2},$$
$$r_{n2}=\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta-2na)^2}.$$

⊙ *Literature*: A. G. Butkovskiy (1979), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $-\infty<x<\infty$, $-\infty<y<\infty$, $0\le z\le a$. Mixed boundary value problem.**

An infinite layer is considered. Boundary conditions are prescribed:

$$w=f_1(x,y)\quad\text{at}\quad z=0,\qquad \partial_z w=f_2(x,y)\quad\text{at}\quad z=a.$$

Solution:

$$\begin{aligned}w(x,y,z)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_1(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0}d\xi\,d\eta\\&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_2(\xi,\eta)G(x,y,z,\xi,\eta,a)\,d\xi\,d\eta\\&+\int_0^a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta.\end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{r_{n1}}-\frac{1}{r_{n2}}\right),$$

where

$$r_{n1}=\sqrt{(x-\xi)^2+(y-\eta)^2+[z-(-1)^n\zeta-2na]^2},$$
$$r_{n2}=\sqrt{(x-\xi)^2+(y-\eta)^2+[z+(-1)^n\zeta-2na]^2}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0\le x<\infty$, $-\infty<y<\infty$, $0\le z\le a$. First boundary value problem.**

A semiinfinite layer is considered. Boundary conditions are prescribed:

$$w=f_1(y,z)\quad\text{at}\quad x=0,\qquad w=f_2(x,y)\quad\text{at}\quad z=0,\qquad w=f_3(x,y)\quad\text{at}\quad z=a.$$

Solution:

$$
\begin{aligned}
w(x,y,z) &= \int_0^a \int_{-\infty}^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\eta\,d\zeta \\
&+ \int_{-\infty}^{\infty}\int_0^{\infty} f_2(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\xi\,d\eta \\
&- \int_{-\infty}^{\infty}\int_0^{\infty} f_3(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=a} d\xi\,d\eta \\
&+ \int_0^a \int_{-\infty}^{\infty}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta.
\end{aligned}
$$

Green's function:

$$
G(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{r_{n1}} - \frac{1}{r_{n2}} - \frac{1}{r_{n3}} + \frac{1}{r_{n4}}\right),
$$

where

$$
\begin{aligned}
r_{n1} &= \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-2na)^2}, \\
r_{n2} &= \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2na)^2}, \\
r_{n3} &= \sqrt{(x+\xi)^2 + (y-\eta)^2 + (z-\zeta-2na)^2}, \\
r_{n4} &= \sqrt{(x+\xi)^2 + (y-\eta)^2 + (z+\zeta-2na)^2}.
\end{aligned}
$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le a$, $0 \le y \le b$, $-\infty < z < \infty$. First boundary value problem.

An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$
\begin{aligned}
&w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad w = f_2(y,z) \quad \text{at} \quad x = a, \\
&w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad w = f_4(x,z) \quad \text{at} \quad y = b.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z) &= \int_0^b \int_{-\infty}^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\,d\eta \\
&- \int_0^b \int_{-\infty}^{\infty} f_2(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=a} d\zeta\,d\eta \\
&+ \int_0^a \int_{-\infty}^{\infty} f_3(\xi,\zeta)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\zeta\,d\xi \\
&- \int_0^a \int_{-\infty}^{\infty} f_4(\xi,\zeta)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=b} d\zeta\,d\xi \\
&+ \int_0^a \int_0^b \int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi.
\end{aligned}
$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{2}{ab}\biggl[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\beta_{nm}}\sin(p_n x)\sin(q_m y)$$
$$\times\sin(p_n\xi)\sin(q_m\eta)\exp(-\beta_{nm}|z-\zeta|)\biggr],$$
$$p_n=\frac{n\pi}{a},\quad q_m=\frac{m\pi}{b},\quad \beta_{nm}=\sqrt{p_n^2+q_m^2}.$$

Alternatively, the Green's function can be represented as

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{r_{nm}^{(1)}}-\frac{1}{r_{nm}^{(2)}}-\frac{1}{r_{nm}^{(3)}}+\frac{1}{r_{nm}^{(4)}}\right),$$

where

$$r_{nm}^{(1)}=\sqrt{(x-\xi-2na)^2+(y-\eta-2mb)^2+(z-\zeta)^2},$$
$$r_{nm}^{(2)}=\sqrt{(x+\xi-2na)^2+(y-\eta-2mb)^2+(z-\zeta)^2},$$
$$r_{nm}^{(3)}=\sqrt{(x-\xi-2na)^2+(y+\eta-2mb)^2+(z-\zeta)^2},$$
$$r_{nm}^{(4)}=\sqrt{(x+\xi-2na)^2+(y+\eta-2mb)^2+(z-\zeta)^2}.$$

► **Domain: $0\le x\le a$, $0\le y\le b$, $-\infty<z<\infty$. Third boundary value problem.**

An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w-k_1w=f_1(y,z)\quad\text{at}\quad x=0,\qquad \partial_x w+k_2w=f_2(y,z)\quad\text{at}\quad x=a,$$
$$\partial_y w-k_3w=f_3(x,z)\quad\text{at}\quad y=0,\qquad \partial_y w+k_4w=f_4(x,z)\quad\text{at}\quad y=b.$$

Solution:

$$w(x,y,z)=-\int_0^b\int_{-\infty}^{\infty}f_1(\eta,\zeta)G(x,y,z,0,\eta,\zeta)\,d\zeta\,d\eta$$
$$+\int_0^b\int_{-\infty}^{\infty}f_2(\eta,\zeta)G(x,y,z,a,\eta,\zeta)\,d\zeta\,d\eta$$
$$-\int_0^a\int_{-\infty}^{\infty}f_3(\xi,\zeta)G(x,y,z,\xi,0,\zeta)\,d\zeta\,d\xi$$
$$+\int_0^a\int_{-\infty}^{\infty}f_4(\xi,\zeta)G(x,y,z,\xi,b,\zeta)\,d\zeta\,d\xi$$
$$+\int_0^a\int_0^b\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{u_{nm}(x,y)u_{nm}(\xi,\eta)}{\|u_{nm}\|^2\beta_{nm}}\exp(-\beta_{nm}|z-\zeta|),$$

where

$$w_{nm}(x,y) = (\mu_n \cos\mu_n x + k_1 \sin\mu_n x)(\nu_m \cos\nu_m y + k_3 \sin\nu_m y), \quad \beta_{nm} = \sqrt{\mu_n^2 + \nu_m^2},$$
$$\|w_{nm}\|^2 = \frac{1}{4}(\mu_n^2 + k_1^2)(\nu_m^2 + k_3^2)\left[a + \frac{(k_1+k_2)(\mu_n^2+k_1k_2)}{(\mu_n^2+k_1^2)(\mu_n^2+k_2^2)}\right]\left[b + \frac{(k_3+k_4)(\nu_m^2+k_3k_4)}{(\nu_m^2+k_3^2)(\nu_m^2+k_4^2)}\right].$$

Here, the μ_n and ν_m are positive roots of the transcendental equations

$$\tan(\mu a) = \frac{(k_1+k_2)\mu}{\mu^2 - k_1k_2}, \qquad \tan(\nu b) = \frac{(k_3+k_4)\nu}{\nu^2 - k_3k_4}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $-\infty < z < \infty$. Mixed boundary value problems.**

1°. An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad \partial_x w = f_2(y,z) \quad \text{at} \quad x = a,$$
$$w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x,z) \quad \text{at} \quad y = b.$$

Solution:

$$\begin{aligned} w(x,y,z) = {} & \int_0^b \int_{-\infty}^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\, d\eta \\ & + \int_0^b \int_{-\infty}^{\infty} f_2(\eta,\zeta) G(x,y,z,a,\eta,\zeta)\, d\zeta\, d\eta \\ & + \int_0^a \int_{-\infty}^{\infty} f_3(\xi,\zeta)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\zeta\, d\xi \\ & + \int_0^a \int_{-\infty}^{\infty} f_4(\xi,\zeta) G(x,y,z,\xi,b,\zeta)\, d\zeta\, d\xi \\ & + \int_0^a \int_0^b \int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\zeta\, d\eta\, d\xi. \end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{2}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\beta_{nm}}\sin(p_n x)\sin(q_m y)\sin(p_n \xi)\sin(q_m \eta)\exp(-\beta_{nm}|z-\zeta|),$$

where

$$p_n = \frac{(2n+1)\pi}{2a}, \quad q_m = \frac{(2m+1)\pi}{2b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2}.$$

2°. An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad w = f_2(y,z) \quad \text{at} \quad x = a,$$
$$\partial_y w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x,z) \quad \text{at} \quad y = b.$$

Solution:

$$\begin{aligned} w(x,y,z) &= \int_0^b\int_{-\infty}^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\,d\eta \\ &- \int_0^b\int_{-\infty}^{\infty} f_2(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=a} d\zeta\,d\eta \\ &- \int_0^a\int_{-\infty}^{\infty} f_3(\xi,\zeta)G(x,y,z,\xi,0,\zeta)\,d\zeta\,d\xi \\ &+ \int_0^a\int_{-\infty}^{\infty} f_4(\xi,\zeta)G(x,y,z,\xi,b,\zeta)\,d\zeta\,d\xi \\ &+ \int_0^a\int_0^b\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{ab}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{A_m}{\beta_{nm}}\sin(p_n x)\cos(q_m y)\sin(p_n\xi)\cos(q_m\eta)\exp(-\beta_{nm}|z-\zeta|),$$

where

$$p_n = \frac{n\pi}{a},\quad q_m = \frac{m\pi}{b},\quad \beta_{nm} = \sqrt{p_n^2+q_m^2},\quad A_m = \begin{cases}1 & \text{for } m=0,\\ 2 & \text{for } m\neq 0.\end{cases}$$

◆ *Below, in Section 10.2.2, we present only Green's functions; the complete solution is constructed with the formulas given in Section 10.2.1.*

► **Domain: $0 \le x \le a$, $0 \le y \le x$, $-\infty < z < \infty$. First boundary value problem.**

An infinite cylindrical domain of triangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad\text{at}\quad x=0,\qquad w = f_2(x,z)\quad\text{at}\quad y=0,\qquad w=f_3(x,z)\quad\text{at}\quad y=x.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = H(x,y,z,\xi,\eta,\zeta) - H(x,y,z,\eta,\xi,\zeta),$$

where

$$H(x,y,z,\xi,\eta,\zeta) = \frac{2}{a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\beta_{nm}}\sin(p_n x)\sin(p_m y)\sin(p_n\xi)\sin(p_m\eta)\exp(-\beta_{nm}|z-\zeta|),$$

$$p_n = \frac{n\pi}{a},\quad p_m = \frac{m\pi}{a},\quad \beta_{nm} = \sqrt{p_n^2+p_m^2}.$$

An alternative representation of the Green's function can be obtained by setting

$$H(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{r_{nm}^{(1)}} - \frac{1}{r_{nm}^{(2)}} - \frac{1}{r_{nm}^{(3)}} + \frac{1}{r_{nm}^{(4)}}\right),$$

where

$$r_{nm}^{(1)} = \sqrt{(x-\xi-2na)^2 + (y-\eta-2ma)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(2)} = \sqrt{(x+\xi-2na)^2 + (y-\eta-2ma)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(3)} = \sqrt{(x-\xi-2na)^2 + (y+\eta-2ma)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(4)} = \sqrt{(x+\xi-2na)^2 + (y+\eta-2ma)^2 + (z-\zeta)^2}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

▶ Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z < \infty$. First boundary value problem.

A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x=0, \qquad w = f_2(y,z) \quad \text{at} \quad x=a,$$
$$w = f_3(x,z) \quad \text{at} \quad y=0, \qquad w = f_4(x,z) \quad \text{at} \quad y=b,$$
$$w = f_5(x,y) \quad \text{at} \quad z=0.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\beta_{nm}}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)H_{nm}(z,\zeta),$$
$$p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2},$$
$$H_{nm}(z,\zeta) = \begin{cases} \exp(-\beta_{nm} z)\sinh(\beta_{nm}\zeta) & \text{for } z > \zeta \ge 0, \\ \exp(-\beta_{nm}\zeta)\sinh(\beta_{nm} z) & \text{for } \zeta > z \ge 0. \end{cases}$$

An alternative representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{r_{nm}^{(1)}} - \frac{1}{r_{nm}^{(2)}} - \frac{1}{r_{nm}^{(3)}} + \frac{1}{r_{nm}^{(4)}} - \frac{1}{r_{nm}^{(5)}} + \frac{1}{r_{nm}^{(6)}} + \frac{1}{r_{nm}^{(7)}} - \frac{1}{r_{nm}^{(8)}}\right),$$

where

$$r_{nm}^{(1)} = \sqrt{(x-\xi-2na)^2 + (y-\eta-2mb)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(2)} = \sqrt{(x+\xi-2na)^2 + (y-\eta-2mb)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(3)} = \sqrt{(x-\xi-2na)^2 + (y+\eta-2mb)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(4)} = \sqrt{(x+\xi-2na)^2 + (y+\eta-2mb)^2 + (z-\zeta)^2},$$
$$r_{nm}^{(5)} = \sqrt{(x-\xi-2na)^2 + (y-\eta-2mb)^2 + (z+\zeta)^2},$$
$$r_{nm}^{(6)} = \sqrt{(x+\xi-2na)^2 + (y-\eta-2mb)^2 + (z+\zeta)^2},$$
$$r_{nm}^{(7)} = \sqrt{(x-\xi-2na)^2 + (y+\eta-2mb)^2 + (z+\zeta)^2},$$
$$r_{nm}^{(8)} = \sqrt{(x+\xi-2na)^2 + (y+\eta-2mb)^2 + (z+\zeta)^2}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z < \infty$. Third boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = f_1(y,z) \quad \text{at} \quad x=0, \qquad \partial_x w + k_2 w = f_2(y,z) \quad \text{at} \quad x=a,$$
$$\partial_y w - k_3 w = f_3(x,z) \quad \text{at} \quad y=0, \qquad \partial_y w + k_4 w = f_4(x,z) \quad \text{at} \quad y=b,$$
$$\partial_z w - k_5 w = f_5(x,y) \quad \text{at} \quad z=0.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{u_{nm}(x,y)u_{nm}(\xi,\eta)}{\|u_{nm}\|^2} H_{nm}(z,\zeta),$$

where

$$w_{nm}(x,y) = (\mu_n \cos\mu_n x + k_1 \sin\mu_n x)(\nu_m \cos\nu_m y + k_3 \sin\nu_m y),$$
$$\|w_{nm}\|^2 = \frac{1}{4}(\mu_n^2 + k_1^2)(\nu_m^2 + k_3^2)\left[a + \frac{(k_1+k_2)(\mu_n^2+k_1k_2)}{(\mu_n^2+k_1^2)(\mu_n^2+k_2^2)}\right]\left[b + \frac{(k_3+k_4)(\nu_m^2+k_3k_4)}{(\nu_m^2+k_3^2)(\nu_m^2+k_4^2)}\right],$$
$$H_{nm}(z,\zeta) = \begin{cases} \dfrac{\exp(-\beta_{nm}z)\left[\beta_{nm}\cosh(\beta_{nm}\zeta) + k_5\sinh(\beta_{nm}\zeta)\right]}{\beta_{nm}(\beta_{nm}+k_5)} & \text{if } z > \zeta, \\ \dfrac{\exp(-\beta_{nm}\zeta)\left[\beta_{nm}\cosh(\beta_{nm}z) + k_5\sinh(\beta_{nm}z)\right]}{\beta_{nm}(\beta_{nm}+k_5)} & \text{if } \zeta > z, \end{cases} \qquad \beta_{nm} = \sqrt{\mu_n^2 + \nu_m^2}.$$

Here, the μ_n and ν_m are positive roots of the transcendental equations

$$\tan(\mu a) = \frac{(k_1+k_2)\mu}{\mu^2 - k_1k_2}, \qquad \tan(\nu b) = \frac{(k_3+k_4)\nu}{\nu^2 - k_3k_4}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z < \infty$. Mixed boundary value problems.**

1°. A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x=0, \qquad w = f_2(y,z) \quad \text{at} \quad x=a,$$
$$w = f_3(x,z) \quad \text{at} \quad y=0, \qquad w = f_4(x,z) \quad \text{at} \quad y=b,$$
$$\partial_z w = f_5(x,y) \quad \text{at} \quad z=0.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\beta_{nm}} \sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta) H_{nm}(z,\zeta),$$
$$p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2},$$
$$H_{nm}(z,\zeta) = \begin{cases} \exp(-\beta_{nm}z)\cosh(\beta_{nm}\zeta) & \text{for } z > \zeta \ge 0, \\ \exp(-\beta_{nm}\zeta)\cosh(\beta_{nm}z) & \text{for } \zeta > z \ge 0. \end{cases}$$

2°. A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y,z) \quad \text{at} \quad x=0, \qquad \partial_x w = f_2(y,z) \quad \text{at} \quad x=a,$$
$$\partial_y w = f_3(x,z) \quad \text{at} \quad y=0, \qquad \partial_y w = f_4(x,z) \quad \text{at} \quad y=b,$$
$$w = f_5(x,y) \quad \text{at} \quad z=0.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_n A_m}{\beta_{nm}}\cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)H_{nm}(z,\zeta),$$

$$p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2}, \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n \neq 0, \end{cases}$$

$$H_{nm}(z,\zeta) = \begin{cases} \exp(-\beta_{nm}z)\sinh(\beta_{nm}\zeta) & \text{for } z > \zeta \geq 0, \\ \exp(-\beta_{nm}\zeta)\sinh(\beta_{nm}z) & \text{for } \zeta > z \geq 0. \end{cases}$$

► **Domain: $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. First boundary value problem.**

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x=0, \qquad w = f_2(y,z) \quad \text{at} \quad x=a,$$
$$w = f_3(x,z) \quad \text{at} \quad y=0, \qquad w = f_4(x,z) \quad \text{at} \quad y=b,$$
$$w = f_5(x,y) \quad \text{at} \quad z=0, \qquad w = f_6(x,y) \quad \text{at} \quad z=c.$$

1°. Representation of the Green's function in the form of a double series:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \begin{cases} \dfrac{\sinh(\beta_{nm}\zeta)\sinh[\beta_{nm}(c-z)]}{\beta_{nm}\sinh(\beta_{nm}c)} & \text{for } c \geq z > \zeta \geq 0, \\ \dfrac{\sinh(\beta_{nm}z)\sinh[\beta_{nm}(c-\zeta)]}{\beta_{nm}\sinh(\beta_{nm}c)} & \text{for } c \geq \zeta > z \geq 0, \end{cases}$$

$$p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2}.$$

This relation can be used to obtain two other representations of the Green's function by means of the following cyclic permutations:

$$\begin{array}{ccc} & (x,\xi,a) & \\ \nearrow & & \searrow \\ (z,\zeta,c) & \longleftarrow & (y,\eta,b) \end{array}$$

2°. Representation of the Green's function in the form of a triple series:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{8}{abc}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin(p_n x)\sin(q_m y)\sin(s_k z)\sin(p_n\xi)\sin(q_m\eta)\sin(s_k\zeta)}{p_n^2 + q_m^2 + s_k^2},$$

$$p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}, \quad s_k = \frac{\pi k}{c}.$$

3°. An alternative representation of the Green's function in the form of a triple series:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\biggl(\frac{1}{r^{(1)}_{nmk}}-\frac{1}{r^{(2)}_{nmk}}-\frac{1}{r^{(3)}_{nmk}}+\frac{1}{r^{(4)}_{nmk}}-\frac{1}{r^{(5)}_{nmk}}+\frac{1}{r^{(6)}_{nmk}}+\frac{1}{r^{(7)}_{nmk}}-\frac{1}{r^{(8)}_{nmk}}\biggr),$$

where

$$\begin{aligned}
r^{(1)}_{nmk}&=\sqrt{(x-\xi-2na)^2+(y-\eta-2mb)^2+(z-\zeta-2kc)^2},\\
r^{(2)}_{nmk}&=\sqrt{(x+\xi-2na)^2+(y-\eta-2mb)^2+(z-\zeta-2kc)^2},\\
r^{(3)}_{nmk}&=\sqrt{(x-\xi-2na)^2+(y+\eta-2mb)^2+(z-\zeta-2kc)^2},\\
r^{(4)}_{nmk}&=\sqrt{(x+\xi-2na)^2+(y+\eta-2mb)^2+(z-\zeta-2kc)^2},\\
r^{(5)}_{nmk}&=\sqrt{(x-\xi-2na)^2+(y-\eta-2mb)^2+(z+\zeta-2kc)^2},\\
r^{(6)}_{nmk}&=\sqrt{(x+\xi-2na)^2+(y-\eta-2mb)^2+(z+\zeta-2kc)^2},\\
r^{(7)}_{nmk}&=\sqrt{(x-\xi-2na)^2+(y+\eta-2mb)^2+(z+\zeta-2kc)^2},\\
r^{(8)}_{nmk}&=\sqrt{(x+\xi-2na)^2+(y+\eta-2mb)^2+(z+\zeta-2kc)^2}.
\end{aligned}$$

► **Domain: $0\le x\le a$, $0\le y\le b$, $0\le z\le c$. Third boundary value problem.**

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$\begin{aligned}
&\partial_x w-k_1w=f_1(y,z)\quad\text{at}\quad x=0, &&\partial_x w+k_2w=f_2(y,z)\quad\text{at}\quad x=a,\\
&\partial_y w-k_3w=f_3(x,z)\quad\text{at}\quad y=0, &&\partial_y w+k_4w=f_4(x,z)\quad\text{at}\quad y=b,\\
&\partial_z w-k_5w=f_5(x,y)\quad\text{at}\quad z=0, &&\partial_z w+k_6w=f_6(x,y)\quad\text{at}\quad z=c.
\end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)\psi_m(y)\psi_m(\eta)\chi_s(z)\chi_s(\zeta)}{\|\varphi_n\|^2\|\psi_m\|^2\|\chi_s(\zeta)\|^2(\mu_n^2+\lambda_m^2+\nu_s^2)},$$

$$\varphi_n(x)=\cos(\mu_n x)+\frac{k_1}{\mu_n}\sin(\mu_n x),\qquad \|\varphi_n\|^2=\frac{k_2}{2\mu_n^2}\frac{\mu_n^2+k_1^2}{\mu_n^2+k_2^2}+\frac{k_1}{2\mu_n^2}+\frac{a}{2}\biggl(1+\frac{k_1^2}{\mu_n^2}\biggr),$$

$$\psi_m(y)=\cos(\lambda_m y)+\frac{k_3}{\lambda_m}\sin(\lambda_m y),\qquad \|\psi_m\|^2=\frac{k_4}{2\lambda_m^2}\frac{\lambda_m^2+k_3^2}{\lambda_m^2+k_4^2}+\frac{k_3}{2\lambda_m^2}+\frac{b}{2}\biggl(1+\frac{k_3^2}{\lambda_m^2}\biggr),$$

$$\chi_s(z)=\cos(\nu_s z)+\frac{k_5}{\nu_s}\sin(\nu_s z),\qquad \|\chi_s\|^2=\frac{k_6}{2\nu_s^2}\frac{\nu_s^2+k_5^2}{\nu_s^2+k_6^2}+\frac{k_5}{2\nu_s^2}+\frac{c}{2}\biggl(1+\frac{k_5^2}{\nu_s^2}\biggr),$$

where the μ_n, λ_m, and ν_s are positive roots of the transcendental equations

$$\frac{\tan(\mu a)}{\mu}=\frac{k_1+k_2}{\mu^2-k_1k_2},\qquad \frac{\tan(\lambda b)}{\lambda}=\frac{k_3+k_4}{\lambda^2-k_3k_4},\qquad \frac{\tan(\nu c)}{\nu}=\frac{k_5+k_6}{\nu^2-k_5k_6}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Mixed boundary value problem.**

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$\begin{aligned} w &= f_1(y,z) \quad \text{at} \quad x=0, & \partial_x w &= f_2(y,z) \quad \text{at} \quad x=a,\\ w &= f_3(x,z) \quad \text{at} \quad y=0, & \partial_y w &= f_4(x,z) \quad \text{at} \quad y=b,\\ w &= f_5(x,y) \quad \text{at} \quad z=0, & \partial_z w &= f_6(x,y) \quad \text{at} \quad z=c. \end{aligned}$$

Solution:

$$\begin{aligned} w(x,y,z) =& \int_0^a\int_0^b\int_0^c \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi\\ &+\int_0^b\int_0^c f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\,d\eta + \int_0^b\int_0^c f_2(\eta,\zeta)G(x,y,z,a,\eta,\zeta)\,d\zeta\,d\eta\\ &+\int_0^a\int_0^c f_3(\xi,\zeta)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\zeta\,d\xi + \int_0^a\int_0^c f_4(\xi,\zeta)G(x,y,z,\xi,b,\zeta)\,d\zeta\,d\xi\\ &+\int_0^a\int_0^b f_5(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\eta\,d\xi + \int_0^a\int_0^b f_6(\xi,\eta)G(x,y,z,\xi,\eta,c)\,d\eta\,d\xi. \end{aligned}$$

1°. A double-series representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{4}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \begin{cases} \dfrac{\sinh(\beta_{nm}\zeta)\cosh[\beta_{nm}(c-z)]}{\beta_{nm}\cosh(\beta_{nm}c)} & \text{for } c \ge z > \zeta \ge 0,\\ \dfrac{\sinh(\beta_{nm}z)\cosh[\beta_{nm}(c-\zeta)]}{\beta_{nm}\cosh(\beta_{nm}c)} & \text{for } c \ge \zeta > z \ge 0, \end{cases}$$

$$p_n = \frac{\pi(2n+1)}{2a}, \quad q_m = \frac{\pi(2m+1)}{2b}, \quad \beta_{nm} = \sqrt{p_n^2+q_m^2}.$$

This relation can be used to obtain two other representations of the Green's function by means of the following cyclic permutations:

$$\begin{array}{ccc} & (x,\xi,a) & \\ \nearrow & & \searrow \\ (z,\zeta,c) & \longleftarrow & (y,\eta,b) \end{array}$$

2°. A triple series representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{8}{abc}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{\sin(p_n x)\sin(q_m y)\sin(s_k z)\sin(p_n\xi)\sin(q_m\eta)\sin(s_k\zeta)}{p_n^2+q_m^2+s_k^2},$$

$$p_n = \frac{\pi(2n+1)}{2a}, \quad q_m = \frac{\pi(2m+1)}{2b}, \quad s_k = \frac{\pi(2k+1)}{2c}.$$

10.2.3 Problems in Cylindrical Coordinates

The three-dimensional Poisson equation in the cylindrical coordinate system is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{\partial^2 w}{\partial z^2} = -\Phi(r,\varphi,z), \qquad r = \sqrt{x^2+y^2}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. First boundary value problem.**

An infinite circular cylinder is considered. A boundary condition is prescribed:

$$w = f(\varphi, z) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi,z) = -R\int_0^{2\pi}\int_{-\infty}^{\infty} f(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\xi=R} d\zeta\, d\eta + \int_0^R\int_0^{2\pi}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\, d\zeta\, d\eta\, d\xi.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\left[J_n'(\mu_{nm}R)\right]^2\mu_{nm}}\cos[n(\varphi-\eta)]\exp\bigl(-\mu_{nm}|z-\zeta|\bigr),$$

where $A_0 = 1$ and $A_n = 2$ for $n \neq 0$; the $J_n(\xi)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Third boundary value problem.**

An infinite circular cylinder is considered. A boundary condition is prescribed:

$$\partial_r w + kw = f(\varphi, z) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi,z) = R\int_0^{2\pi}\int_{-\infty}^{\infty} f(\eta,\zeta)G(r,\varphi,z,R,\eta,\zeta)\, d\zeta\, d\eta + \int_0^R\int_0^{2\pi}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\, d\zeta\, d\eta\, d\xi.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]}{(\mu_{nm}^2R^2 + k^2R^2 - n^2)J_n^2(\mu_{nm}R)}\exp\bigl(-\mu_{nm}|z-\zeta|\bigr),$$

where $A_0 = 1$ and $A_n = 2$ for $n \neq 0$; the $J_n(\xi)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad w = f_2(r, \varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) &= -R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\xi=R} d\zeta\,d\eta \\ &\quad + \int_0^{2\pi}\int_0^{R} f_2(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\zeta=0} \xi\,d\xi\,d\eta \\ &\quad + \int_0^{R}\int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\bigl[J_n'(\mu_{nm}R)\bigr]^2\mu_{nm}}\cos[n(\varphi-\eta)]F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \exp(-\mu_{nm}|z-\zeta|) - \exp(-\mu_{nm}|z+\zeta|), \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases}$$

where the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Third boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$\partial_r w + k_1 w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad \partial_z w - k_2 w = f_2(r, \varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) &= R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)G(r,\varphi,z,R,\eta,\zeta)\,d\zeta\,d\eta \\ &\quad - \int_0^{2\pi}\int_0^{R} f_2(\xi,\eta)G(r,\varphi,z,R,\eta,0)\xi\,d\xi\,d\eta \\ &\quad + \int_0^{R}\int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]}{(\mu_{nm}^2R^2 + k_1^2R^2 - n^2)J_n^2(\mu_{nm}R)}F_{nm}(z,\zeta),$$

$$A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases} \quad F_{nm}(z,\zeta) = \begin{cases} \dfrac{\exp(-\mu_{nm}z)\bigl[\mu_{nm}\cosh(\mu_{nm}\zeta) + k_2\sinh(\mu_{nm}\zeta)\bigr]}{\mu_{nm}(\mu_{nm}+k_2)} & \text{for } z > \zeta, \\ \dfrac{\exp(-\mu_{nm}\zeta)\bigl[\mu_{nm}\cosh(\mu_{nm}z) + k_2\sinh(\mu_{nm}z)\bigr]}{\mu_{nm}(\mu_{nm}+k_2)} & \text{for } \zeta > z, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Mixed boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad \partial_z w = f_2(r, \varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$w(r,\varphi,z) = -R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\xi=R} d\zeta\,d\eta$$
$$-\int_0^{2\pi}\int_0^{R} f_2(\xi,\eta)G(r,\varphi,z,\xi,\eta,0)\xi\,d\xi\,d\eta$$
$$+\int_0^{R}\int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\bigl[J_n'(\mu_{nm}R)\bigr]^2\mu_{nm}}\cos[n(\varphi-\eta)]F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \exp(-\mu_{nm}|z-\zeta|) + \exp(-\mu_{nm}|z+\zeta|), \quad A_n = \begin{cases}1 & \text{for } n = 0,\\ 2 & \text{for } n \ne 0,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are roots of the transcendental equation $J_n(\mu R) = 0$.

◆ *Below, in Section 10.2.3, we present only Green's functions; the complete solution is constructed with the formulas given in Section 10.2.1. See also Section 10.3.2 for $\lambda = 0$.*

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. First boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad w = f_2(r, \varphi) \quad \text{at} \quad z = 0, \qquad w = f_3(r, \varphi) \quad \text{at} \quad z = a.$$

A double series representation of the Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\bigl[J_n'(\mu_{nm}R)\bigr]^2\mu_{nm}\sinh(\mu_{nm}a)}\cos[n(\varphi-\eta)]F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \begin{cases}\sinh(\mu_{nm}\zeta)\sinh[\mu_{nm}(a-z)] & \text{for } a \ge z > \zeta \ge 0,\\ \sinh(\mu_{nm}z)\sinh[\mu_{nm}(a-\zeta)] & \text{for } a \ge \zeta > z \ge 0,\end{cases} \qquad A_n = \begin{cases}1 & \text{for } n = 0,\\ 2 & \text{for } n \ne 0,\end{cases}$$

where the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument) and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

A triple series representation of the Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{2a}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{A_n}{[J_n'(\mu_{nm}R)]^2\bigl[(a\mu_{nm})^2 + (\pi k)^2\bigr]}J_n(\mu_{nm}r)$$
$$\times J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\sin\left(\frac{k\pi z}{a}\right)\sin\left(\frac{k\pi\zeta}{a}\right).$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. Third boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$\partial_r w + k_1 w = f_1(\varphi, z) \quad \text{at} \quad r = R,$$
$$\partial_z w - k_2 w = f_2(r, \varphi) \quad \text{at} \quad z = 0,$$
$$\partial_z w + k_3 w = f_3(r, \varphi) \quad \text{at} \quad z = a.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty} \frac{A_n \mu_{nm}^2 J_n(\mu_{nm} r) J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)] h_s(z) h_s(\zeta)}{(\mu_{nm}^2 R^2 + k_1^2 R^2 - n^2)(\mu_{nm}^2+\lambda_s^2)[J_n(\mu_{nm}R)]^2 \|h_s\|^2},$$

$$h_s(z) = \cos(\lambda_s z) + \frac{k_2}{\lambda_s}\sin(\lambda_s z), \qquad \|h_s\|^2 = \frac{k_3}{2\lambda_s^2}\frac{\lambda_s^2+k_2^2}{\lambda_s^2+k_3^2} + \frac{k_2}{2\lambda_s^2} + \frac{a}{2}\left(1+\frac{k_2^2}{\lambda_s^2}\right).$$

Here, $A_0 = 1$ and $A_n = 2$ for $n \neq 0$; the $J_n(\xi)$ are Bessel functions; and the μ_{nm} and λ_s are positive roots of the transcendental equations

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0, \qquad \frac{\tan(\lambda a)}{\lambda} = \frac{k_2 + k_3}{\lambda^2 - k_2 k_3}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. Mixed boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad \partial_z w = f_2(r,\varphi) \quad \text{at} \quad z = 0, \qquad \partial_z w = f_3(r,\varphi) \quad \text{at} \quad z = a.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{a}{\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{A_n A_k}{[J_n'(\mu_{nm}R)]^2\left[(a\mu_{nm})^2 + (\pi k)^2\right]} J_n(\mu_{nm} r)$$
$$\times J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]\cos\left(\frac{k\pi z}{a}\right)\cos\left(\frac{k\pi\zeta}{a}\right),$$

where $A_0 = 1$ and $A_n = 2$ for $n \neq 0$; the $J_n(\xi)$ are Bessel functions (the prime denotes a derivative with respect to the argument); and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le a$. First boundary value problem.**

A cylindrical sector of finite thickness is considered. Boundary conditions are prescribed:

$$w = f_1(r, z) \quad \text{at} \quad \varphi = 0, \qquad w = f_2(r, z) \quad \text{at} \quad \varphi = \varphi_0, \qquad w = f_3(\varphi, z) \quad \text{at} \quad r = R,$$
$$w = f_4(r, \varphi) \quad \text{at} \quad z = 0, \qquad w = f_5(r, \varphi) \quad \text{at} \quad z = a.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{8a}{R^2\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{J_{n\pi/\varphi_0}(\mu_{nm} r) J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J_{n\pi/\varphi_0}'(\mu_{nm}R)]^2\left[(a\mu_{nm})^2 + (\pi k)^2\right]}$$
$$\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\sin\left(\frac{k\pi z}{a}\right)\sin\left(\frac{k\pi\zeta}{a}\right),$$

where the $J_{n\pi/\varphi_0}(r)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le a$. Mixed boundary value problem.**

A cylindrical sector of finite thickness is considered. Boundary conditions are prescribed:

$$w=f_1(r,z) \quad \text{at} \quad \varphi=0, \qquad w=f_2(r,z) \quad \text{at} \quad \varphi=\varphi_0, \qquad w=f_3(\varphi,z) \quad \text{at} \quad r=R,$$
$$\partial_z w=f_4(r,\varphi) \quad \text{at} \quad z=0, \qquad \partial_z w=f_5(r,\varphi) \quad \text{at} \quad z=a.$$

Green's function:

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{4a}{R^2\varphi_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{A_k J_{n\pi/\varphi_0}(\mu_{nm}r)J_{n\pi/\varphi_0}(\mu_{nm}\xi)}{[J'_{n\pi/\varphi_0}(\mu_{nm}R)]^2\big[(a\mu_{nm})^2+(\pi k)^2\big]}$$
$$\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\eta}{\varphi_0}\right)\cos\left(\frac{k\pi z}{a}\right)\cos\left(\frac{k\pi\zeta}{a}\right),$$

where $A_0 = 1$ and $A_k = 2$ for $k \ne 0$; the $J_{n\pi/\varphi_0}(r)$ are Bessel functions; and the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu R) = 0$.

10.2.4 Problems in Spherical Coordinates

The three-dimensional Poisson equation in the spherical coordinate system is written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} = -\Phi(r,\theta,\varphi),$$

where $r = \sqrt{x^2+y^2+z^2}$.

◆ *Only Green's functions are presented below; the complete solutions can be constructed with the formulas given in Section 10.2.1.*

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical domain is considered. A boundary condition is prescribed:

$$w = f(\varphi,\theta) \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\theta,\varphi,\xi,\eta,\zeta) = \frac{1}{4\pi\sqrt{r^2-2r\xi\cos\gamma+\xi^2}} - \frac{1}{4\pi\sqrt{r^2\xi^2-2R^2r\xi\cos\gamma+R^4}},$$
$$\cos\gamma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos(\varphi-\zeta).$$

An alternative representation of the Green's function:

$$G(\mathbf{r},\mathbf{r}_0) = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} - \frac{1}{4\pi}\frac{R}{r_0|(R/r_0)^2\mathbf{r}_0-\mathbf{r}|}, \qquad r_0 = |\mathbf{r}_0|,$$

where

$$\mathbf{r} = \{x, y, z\}, \qquad x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$
$$\mathbf{r}_0 = \{x_0, y_0, z_0\}, \quad x_0 = \xi\sin\eta\cos\zeta, \quad y_0 = \xi\sin\eta\sin\zeta, \quad z_0 = \xi\cos\eta.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

A spherical domain is considered. A boundary condition is prescribed:

$$\partial_r w = f(\varphi, \theta) \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\theta,\varphi,\xi,\eta,\zeta) = \frac{1}{4\pi}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{R}{|\mathbf{r}_0|\,|\mathbf{r}_1|} + \frac{1}{R}\ln\frac{2R^2}{R^2 + |\mathbf{r}_0|\,|\mathbf{r}_1| - (\mathbf{r}\cdot\mathbf{r}_0)}\right\},$$

where

$$|\mathbf{r}-\mathbf{r}_0| = \sqrt{r^2 - 2r\xi\cos\gamma + \xi^2}, \quad |\mathbf{r}_0|\,|\mathbf{r}_1| = \sqrt{r^2\xi^2 - 2R^2 r\xi\cos\gamma + R^4},$$
$$|\mathbf{r}_0| = \xi, \quad (\mathbf{r}\cdot\mathbf{r}_0) = r\xi\cos\gamma, \quad \cos\gamma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos(\varphi-\zeta).$$

For a solution of the second boundary value problem to exist the solvability condition must be satisfied (see Section 10.2.1).

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970).

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Third boundary value problem.**

A spherical domain is considered. A boundary condition is prescribed:

$$\partial_r w + kw = f(\theta, \varphi) \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\theta,\varphi,\xi,\eta,\zeta) = \frac{1}{2\pi\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=0}^{n} A_s B_{nms} J_{n+1/2}(\lambda_{nm} r) J_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^s(\cos\theta) P_n^s(\cos\eta)\cos[s(\varphi-\zeta)],$$

$$A_s = \begin{cases} 1 & \text{for } s=0, \\ 2 & \text{for } s\neq 0, \end{cases} \qquad B_{nms} = \frac{(2n+1)(n-s)!}{(n+s)!\left[R^2\lambda_{nm}^2 + (kR+n)(kR-n-1)\right]\left[J_{n+1/2}(\lambda_{nm}R)\right]^2}.$$

Here, the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^s(\mu)$ are associated Legendre functions, which are expressed in terms of the Legendre polynomials $P_n(\mu)$ as

$$P_n^s(\mu) = (1-\mu^2)^{s/2}\frac{d^s}{d\mu^s}P_n(\mu), \qquad P_n(\mu) = \frac{1}{n!\,2^n}\frac{d^n}{d\mu^n}(\mu^2-1)^n,$$

and the λ_{nm} are positive roots of the transcendental equation

$$\lambda R J'_{n+1/2}(\lambda R) + \left(kR - \tfrac{1}{2}\right)J_{n+1/2}(\lambda R) = 0.$$

► **Domain: $R \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

Three-dimensional space with a spherical cavity is considered. A boundary condition is prescribed:

$$w = f(\varphi, \theta) \quad \text{at} \quad r = R.$$

The Green's function of the outer first boundary value problem is given by the same relation as that for the inner first boundary value problem (see the first paragraph in this section), except that $r \ge R$ and $\xi \ge R$.

► **Domain: $R \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Second boundary value problem.**

Three-dimensional space with a spherical cavity is considered. A boundary condition is prescribed:

$$\partial_r w = f(\varphi, \theta) \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\theta,\varphi,\xi,\eta,\zeta) = \frac{1}{4\pi}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{R}{|\mathbf{r}_0|\,|\mathbf{r}_1|} + \frac{1}{R}\ln\frac{(1-\cos\gamma)|\mathbf{r}|\,|\mathbf{r}_0|}{R^2 + |\mathbf{r}_0|\,|\mathbf{r}_1| - (\mathbf{r}\cdot\mathbf{r}_0)}\right\},$$

where

$$|\mathbf{r}| = r, \qquad |\mathbf{r}_0| = \xi, \qquad |\mathbf{r}-\mathbf{r}_0| = \sqrt{r^2 - 2r\xi\cos\gamma + \xi^2},$$
$$|\mathbf{r}_0|\,|\mathbf{r}_1| = \sqrt{r^2\xi^2 - 2R^2 r\xi\cos\gamma + R^4}, \quad (\mathbf{r}\cdot\mathbf{r}_0) = r\xi\cos\gamma,$$
$$\cos\gamma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos(\varphi-\zeta).$$

⊙ *Literature*: N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov (1970).

► **Domain: $R_1 \le r \le R_2$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A spherical layer is considered. Boundary conditions are prescribed:

$$w = f_1(\theta, \varphi) \quad \text{at} \quad r = R_1, \qquad w = f_2(\theta,\varphi) \quad \text{at} \quad r = R_2.$$

Green's function:

$$G(r,\theta,\varphi,\xi,\eta,\zeta) = \frac{\pi}{8\sqrt{r\xi}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{n} A_k B_{nmk} Z_{n+1/2}(\lambda_{nm} r) Z_{n+1/2}(\lambda_{nm}\xi)$$
$$\times P_n^k(\cos\theta) P_n^k(\cos\eta)\cos[k(\varphi-\zeta)],$$

where

$$Z_{n+1/2}(\lambda_{nm} r) = J_{n+1/2}(\lambda_{nm}R_1) Y_{n+1/2}(\lambda_{nm} r) - Y_{n+1/2}(\lambda_{nm}R_1) J_{n+1/2}(\lambda_{nm} r),$$
$$A_k = \begin{cases} 1 & \text{for } k = 0, \\ 2 & \text{for } k \ne 0, \end{cases} \qquad B_{nmk} = \frac{(2n+1)(n-k)!\, J^2_{n+1/2}(\lambda_{nm}R_2)}{(n+k)!\big[J^2_{n+1/2}(\lambda_{nm}R_1) - J^2_{n+1/2}(\lambda_{nm}R_2)\big]};$$

the $J_{n+1/2}(r)$ are Bessel functions, the $P_n^k(\mu)$ are associated Legendre functions (see above), and the λ_{nm} are positive roots of the transcendental equation $Z_{n+1/2}(\lambda R_2) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi/2$, $0 \le \varphi \le 2\pi$. First boundary value problem.**

A hemisphere is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, \theta) \quad \text{at} \quad r = R, \qquad w = f_2(r, \varphi) \quad \text{at} \quad \theta = \pi/2.$$

Green's function in the spherical coordinate system:

$$G(r, \theta, \varphi, \xi, \eta, \zeta) = G_{\rm s}(r, \theta, \varphi, \xi, \eta, \zeta) - G_{\rm s}(r, \theta, \varphi, \xi, \pi - \eta, \zeta),$$
$$G_{\rm s}(r, \theta, \varphi, \xi, \eta, \zeta) = \frac{1}{4\pi\sqrt{r^2 - 2r\xi\cos\gamma + \xi^2}} - \frac{1}{4\pi\sqrt{r^2\xi^2 - 2R^2 r\xi\cos\gamma + R^4}},$$
$$\cos\gamma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos(\varphi - \zeta).$$

where $G_{\rm s}(r, \theta, \varphi, \xi, \eta, \zeta)$ is the Green's functions for a sphere.

Green's function in the Cartesian coordinate system:

$$G(x, y, z, x_0, y_0, z_0) = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} - \frac{R}{|\mathbf{r}_0|\,|\mathbf{r} - \mathbf{r}_0^*|}\right) - \frac{1}{4\pi}\left(\frac{1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{R}{|\mathbf{r}_0|\,|\mathbf{r} - \mathbf{r}_1^*|}\right),$$
$$\mathbf{r} = \{x, y, z\}, \;\; \mathbf{r}_0 = \{x_0, y_0, z_0\}, \;\; \mathbf{r}_1 = \{x_0, y_0, -z_0\}, \;\; \mathbf{r}_k^* = (R/r_0)^2 \mathbf{r}_k, \;\; k = 0,\, 1.$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \theta \le \pi/2$, $0 \le \varphi \le \pi$. First boundary value problem.**

A quarter of a sphere is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, \theta) \quad \text{at} \quad r = R, \qquad w = f_2(r, \varphi) \quad \text{at} \quad \theta = \pi/2,$$
$$w = f_3(r, \theta) \quad \text{at} \quad \varphi = 0, \qquad w = f_4(r, \theta) \quad \text{at} \quad \varphi = \pi.$$

Green's function in the spherical coordinate system:

$$G(r, \theta, \varphi, \xi, \eta, \zeta) = G_{\rm s}(r, \theta, \varphi, \xi, \eta, \zeta) - G_{\rm s}(r, \theta, \varphi, \xi, \pi - \eta, \zeta)$$
$$+ G_{\rm s}(r, \theta, \varphi, \xi, \pi - \eta, 2\pi - \zeta) - G_{\rm s}(r, \theta, \varphi, \xi, \eta, 2\pi - \zeta),$$

where $G_{\rm s}(r, \theta, \varphi, \xi, \eta, \zeta)$ is the Green's function for a sphere; see the previous paragraph.

Green's function in the Cartesian coordinate system:

$$G(x, y, z, x_0, y_0, z_0) = \frac{1}{4\pi} \sum_{n,k=0}^{1} (-1)^{n+k} \left(\frac{1}{|\mathbf{r} - \mathbf{r}_{nk}|} - \frac{R}{|\mathbf{r}_0|\,|\mathbf{r} - \mathbf{r}_{nk}^*|}\right),$$
$$\mathbf{r} = \{x, y, z\}, \;\; \mathbf{r}_0 = \{x_0, y_0, z_0\}, \;\; \mathbf{r}_{nk} = \{x_0, (-1)^n y_0, (-1)^k z_0\}, \;\; \mathbf{r}_{nk}^* = (R/r_0)^2 \mathbf{r}_{nk},$$

where $r_0 = |\mathbf{r}_0|$; $n = 0,\, 1$; $k = 0,\, 1$.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

10.3 Helmholtz Equation $\Delta_3 w + \lambda w = -\Phi(\mathbf{x})$

A variety of problems related to steady-state oscillations (mechanical, acoustic, thermal, electromagnetic, etc.) lead to the three-dimensional Helmholtz equation with $\lambda > 0$. This equation governs mass transfer phenomena with volume chemical reaction of the first order for $\lambda < 0$. Any elliptic equation with constant coefficients can be reduced to the Helmholtz equation.

The Helmholtz equation is called homogeneous if $\Phi = 0$ and nonhomogeneous if $\Phi \neq 0$.

10.3.1 Homogeneous Helmholtz Equation. Eigenvalue problems

► Some definitions. Eigenvalue problems

A homogeneous boundary value problem is a boundary value problem for a homogeneous equation with homogeneous boundary conditions; $w = 0$ is a particular solution of a homogeneous boundary value problem.

The values λ_n of the parameter λ for which there are nontrivial solutions (i.e., not identically zero solutions) of a homogeneous boundary value problem are called eigenvalues. The corresponding solutions, $w = w_n$, are called eigenfunctions of this boundary value problem.

In what follows, we consider simultaneously the first, second, and third boundary value problems for the three-dimensional homogeneous Helmholtz equation

$$\Delta_3 w + \lambda w = 0 \tag{1}$$

in a finite three-dimensional domain V with a sufficiently smooth surface S. Let the homogeneous boundary condition have the form

$$s\frac{\partial w}{\partial N} + kw = 0 \quad \text{for} \quad \mathbf{r} \in S, \tag{2}$$

where $\frac{\partial w}{\partial N}$ is the derivative along the outward normal to the surface S, $sk \geq 0$, and $\mathbf{r} = \{x, y, z\}$. By appropriately choosing the constants s and k, one can obtain boundary conditions of the first ($s = 0,\ k = 1$), second ($s = 1,\ k = 0$), and third ($s = 1,\ k > 0$) kind.

► General properties of eigenvalues and eigenfunctions.

1°. There are infinitely many eigenvalues $\{\lambda_n\}$; they form the discrete spectrum of the boundary value problem (1)–(2).

2°. All eigenvalues are positive, except for one eigenvalue $\lambda_0 = 0$ of the second boundary value problem (the corresponding eigenfunction is $w_0 = \text{const}$). The eigenvalues are assumed to be ordered so that $\lambda_{\leq} < \lambda_2 \leq \lambda_3 \leq \cdots$.

3°. The eigenvalues tend to infinity as the number n increases. The following asymptotic estimate holds:

$$\lim_{n\to\infty} \frac{n}{\lambda_n^{3/2}} = \frac{V_3}{6\pi^2},$$

where V_3 is the volume of the domain under consideration.

4°. The eigenfunctions are defined up to a constant factor, and they can be chosen to be real. Any two eigenfunctions, w_n and w_m, that correspond to distinct eigenvalues $\lambda_n \neq \lambda_m$ are orthogonal; that is,

$$\int_V w_n w_m \, dV = 0.$$

Distinct eigenfunctions corresponding to coinciding eigenvalues $\lambda_n = \lambda_m$ can be chosen to be orthogonal.

5°. Any twice continuously differentiable function $f = f(\mathbf{r})$ that satisfies the boundary conditions of a boundary value problem can be expanded into a uniformly convergent series in the eigenfunctions of this boundary value problem; specifically,

$$f = \sum_{n=1}^{\infty} a_n w_n, \qquad \text{where} \quad a_n = \frac{1}{\|w_n\|^2} \int_V f w_n \, dV, \quad \|w_n\|^2 = \int_V w_n^2 \, dV.$$

If f is square integrable, then the series is convergent in mean.

6°. The eigenvalues of the first boundary value problem decrease if the domain is extended. The nonzero eigenvalues of the second boundary value problem decrease if the domain is extended.

7°. Let λ_n and $w_n = w_n(\mathbf{x})$ be the eigenvalues and eigenfunctions of problem (1)–(2) in a domain V with the first or second boundary conditions ($s = 0$, $k = 1$, or $k = 0$, $s = 1$). Then the eigenvalues and eigenfunctions λ'_n and w'_n of the similar problem in the domain V' obtained from V by dilation with ratio k (the shape of the domain remains unchanged) are determined by the formulas

$$\lambda'_n = \frac{\lambda_n}{k^2}, \qquad w'_n = w_n\left(\frac{\mathbf{x}}{k}\right).$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1984).

10.3.2 Nonhomogeneous Helmholtz Equation. General Remarks, Results, and Formulas

► Nonhomogeneous Helmholtz equation with homogeneous boundary conditions.

Three cases are possible.

1°. If λ is not equal to any one of the eigenvalues, then the solution of the problem is given by

$$w = \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n - \lambda} w_n, \quad \text{where} \quad A_n = \frac{1}{\|w_n\|^2} \int_V \Phi w_n \, dV, \quad \|w_n\|^2 = \int_V w_n^2 \, dV.$$

2°. If λ coincides with one of the eigenvalues, $\lambda = \lambda_m$, then the condition of the orthogonality of the function Φ to the eigenfunction w_m,

$$\int_V \Phi w_m \, dV = 0,$$

is a necessary condition for a solution of the nonhomogeneous problem to exist. The solution is then given by

$$w=\sum_{n=1}^{m-1}\frac{A_n}{\lambda_n-\lambda_m}w_n+\sum_{n=m+1}^{\infty}\frac{A_n}{\lambda_n-\lambda_m}w_n+Cw_m,\quad A_n=\frac{1}{\|w_n\|^2}\int_V \Phi w_n\,dV,$$

where C is an arbitrary constant and $\|w_n\|^2=\int_V w_n^2\,dV$.

3°. If $\lambda=\lambda_m$ and $\int_V \Phi w_m\,dV\neq 0$, then the boundary value problem for the nonhomogeneous equation has no solution.

Remark 10.4. If to each eigenvalue λ_n there are corresponding p_n mutually orthogonal eigenfunctions $w_n^{(s)}$ $(s=1,\dots,p_n)$, then the solution is written as

$$w=\sum_{n=1}^{\infty}\sum_{s=1}^{p_n}\frac{A_n^{(s)}}{\lambda_n-\lambda}w_n^{(s)},\quad\text{where}\quad A_n^{(s)}=\frac{1}{\|w_n^{(s)}\|^2}\int_V \Phi w_n^{(s)}\,dV,\quad \|w_n^{(s)}\|^2=\int_V\bigl[w_n^{(s)}\bigr]^2\,dV,$$

provided that $\lambda\neq\lambda_n$.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1984).

► Solution of nonhomogeneous boundary value problems of general form.

1°. The solution of the first boundary value problem for the Helmholtz equation with the boundary condition

$$w=f(\mathbf{r})\quad\text{for}\quad \mathbf{r}\in S$$

can be represented in the form

$$w(\mathbf{r})=\int_V \Phi(\boldsymbol{\rho})G(\mathbf{r},\boldsymbol{\rho})\,dV_\rho-\int_S f(\boldsymbol{\rho})\frac{\partial}{\partial N_\rho}G(\mathbf{r},\boldsymbol{\rho})\,dS_\rho. \tag{1}$$

Here, $\mathbf{r}=\{x,y,z\}$, $\boldsymbol{\rho}=\{\xi,\eta,\zeta\}$ ($\mathbf{r}\in V$, $\boldsymbol{\rho}\in V$); $\frac{\partial}{\partial N_\rho}$ denotes the derivative along the outward normal to the surface S with respect to ξ,η,ζ. The Green's function is given by the series

$$G(\mathbf{r},\boldsymbol{\rho})=\sum_{n=1}^{\infty}\frac{w_n(\mathbf{r})w_n(\boldsymbol{\rho})}{\|w_n\|^2(\lambda_n-\lambda)},\qquad \lambda\neq\lambda_n, \tag{2}$$

where the w_n and λ_n are the eigenfunctions and eigenvalues of the homogeneous first boundary value problem.

2°. The solution of the second boundary value problem with the boundary condition

$$\frac{\partial w}{\partial N}=f(\mathbf{r})\quad\text{for}\quad \mathbf{r}\in S$$

can be represented in the form

$$w(\mathbf{r})=\int_V \Phi(\boldsymbol{\rho})G(\mathbf{r},\boldsymbol{\rho})\,dV_\rho+\int_S f(\boldsymbol{\rho})G(\mathbf{r},\boldsymbol{\rho})\,dS_\rho. \tag{3}$$

Here, the Green's function is given by the series

$$G(\mathbf{r}, \boldsymbol{\rho}) = -\frac{1}{V_3\lambda} + \sum_{n=1}^{\infty} \frac{w_n(\mathbf{r})w_n(\boldsymbol{\rho})}{\|w_n\|^2(\lambda_n - \lambda)}, \tag{4}$$

where V_3 is the volume of the three-dimensional domain under consideration, and the λ_n and w_n are the positive eigenvalues and corresponding eigenfunctions of the homogeneous second boundary value problem. For clarity, the term corresponding to the zero eigenvalue $\lambda_0 = 0$ ($w_0 = \text{const}$) is singled out in (4). It is assumed that $\lambda \neq 0$ and $\lambda \neq \lambda_n$.

3°. The solution of the third boundary value problem for the Helmholtz equation with the boundary condition

$$\frac{\partial w}{\partial N} + kw = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S$$

is given by relation (3) in which the Green's function is defined by series (2) with the eigenfunctions w_n and eigenvalues λ_n of the homogeneous third boundary value problem.

4°. Let nonhomogeneous boundary conditions of various types be set on different portions S_i of the surface $S = \sum\limits_{i=1}^{m} S_i$,

$$\Gamma_i[w] = f_i(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S_i.$$

Then the solution of the corresponding mixed boundary value problem can be written as

$$w(\mathbf{r}) = \int_V \Phi(\boldsymbol{\rho})G(\mathbf{r}, \boldsymbol{\rho})\, dV_\rho + \sum_{i=1}^{m} \int_{S_i} f_i(\boldsymbol{\rho})\Lambda_i(\mathbf{r}, \boldsymbol{\rho})\, dS_\rho^{(i)},$$

where

$$\Lambda_i(\mathbf{r}, \boldsymbol{\rho}) = \begin{cases} -\dfrac{\partial}{\partial N_\rho} G(\mathbf{r}, \boldsymbol{\rho}) & \text{if a first-kind boundary condition is set on } S_i, \\ G(\mathbf{r}, \boldsymbol{\rho}) & \text{if a second- or third-kind boundary condition is set on } S_i. \end{cases}$$

The Green's function is expressed by series (2) that involves the eigenfunctions w_n and eigenvalues λ_n of the homogeneous mixed boundary value problem.

► **Boundary conditions at infinity in the case of an unbounded domain.**

Below it is assumed that the function Φ is finite or sufficiently rapidly decaying as $r \to \infty$.

1°. If $\lambda < 0$ and the domain is unbounded, the additional condition that the solution must vanish at infinity is set:

$$w \to 0 \quad \text{as} \quad r \to \infty. \tag{5}$$

2°. If $\lambda > 0$, the radiation conditions (*Sommerfeld conditions*) are often used at infinity. In three-dimensional problems, these conditions are expressed as

$$\lim_{r\to\infty} rw = \text{const}, \qquad \lim_{r\to\infty} r\left(\frac{\partial w}{\partial r} + i\sqrt{\lambda}\, w\right) = 0, \tag{6}$$

where $i^2 = -1$.

The principle of limit absorption and the principle of limit amplitude are also employed to separate a single solution.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► Green's function for an infinite cylindrical domain of arbitrary cross-section.

Consider the three-dimensional Helmholtz equation

$$\Delta_3 w + \lambda w = -\Phi(\mathbf{r}) \tag{7}$$

inside an infinite cylindrical domain $V = \{(x,y) \in D,\ -\infty < z < \infty\}$ with arbitrary cross-section D. On the surface of this domain, let $S = \{(x,y) \in L,\ -\infty < z < \infty\}$, where L is the boundary of D, the homogeneous boundary condition of general form

$$s\frac{\partial w}{\partial N} + kw = 0 \quad \text{for} \quad \mathbf{r} \in S \tag{8}$$

be set, with $sk \geq 0$. By appropriately choosing the constants s and k in (8), one can obtain boundary conditions of the first $(s = 0,\ k = 1)$, second $(s = 1,\ k = 0)$, and third $(sk \neq 0)$ kind.

The Green's function of the first or third boundary value problem can be represented in the form*

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{u_n(x,y)u_n(\xi,\eta)}{\|u_n\|^2\sqrt{\mu_n - \lambda}} e^{-\sqrt{\mu_n-\lambda}\,|z-\zeta|},$$
$$\|u_n\|^2 = \int_D u_n^2(x,y)\,dx\,dy, \tag{9}$$

where the μ_n and u_n are the eigenvalues and eigenfunctions of the corresponding two-dimensional boundary value problem in D,

$$\Delta_2 u + \mu u = 0 \qquad \text{for } (x,y) \in D,$$
$$s\frac{\partial u}{\partial N} + ku = 0 \qquad \text{for } (x,y) \in L. \tag{10}$$

Recall that all μ_n are positive.

In the second boundary value problem, the zero eigenvalue $\mu_0 = 0$ appears, and hence the summation in (9) must start with $n = 0$. In this case, $u_0 = 1$ and $\|u_0\|^2 = D_2$, where D_2 is the area of the cross-section D.

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980), A. N. Tikhonov and A. A. Samarskii (1990).

---

*Here and later in this section, the cross-section D is assumed to have finite dimensions.

► **Green's function for a semiinfinite cylindrical domain.**

1°. The Green's function of the three-dimensional first boundary value problem for equation (7) in a semiinfinite cylindrical domain $V = \{(x, y) \in D,\ 0 \le z < \infty\}$ with arbitrary cross-section D is given by

$$G(x, y, z, \xi, \eta, \zeta) = \sum_{n=1}^{\infty} \frac{u_n(x, y) u_n(\xi, \eta)}{\|u_n\|^2} H_n(z, \zeta), \tag{11}$$

where

$$\begin{aligned} H_n(z, \zeta) &= \frac{1}{2\beta_n} \big[\exp(-\beta_n |z - \zeta|\,) - \exp(-\beta_n |z + \zeta|\,)\big] \\ &= \begin{cases} \dfrac{1}{\beta_n} \exp(-\beta_n z) \sinh(\beta_n \zeta) & \text{for } z > \zeta \ge 0, \\ \dfrac{1}{\beta_n} \exp(-\beta_n \zeta) \sinh(\beta_n z) & \text{for } \zeta > z \ge 0, \end{cases} \qquad \beta_n = \sqrt{\mu_n - \lambda}. \end{aligned} \tag{12}$$

Relations (11) and (12) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional first boundary value problem (10) with $s = 0$ and $k = 1$.

2°. The Green's function of the three-dimensional second boundary value problem for equation (7) in a semiinfinite cylindrical domain $V = \{(x, y) \in D,\ 0 \le z < \infty\}$ with arbitrary cross-section D is given by

$$G(x, y, z, \xi, \eta, \zeta) = \frac{1}{D_2} H_0(z, \zeta) + \sum_{n=1}^{\infty} \frac{u_n(x, y) u_n(\xi, \eta)}{\|u_n\|^2} H_n(z, \zeta), \tag{13}$$

where

$$\begin{aligned} H_n(z, \zeta) &= \frac{1}{2\beta_n} \big[\exp(-\beta_n |z - \zeta|\,) + \exp(-\beta_n |z + \zeta|\,)\big] \\ &= \begin{cases} \dfrac{1}{\beta_n} \exp(-\beta_n z) \cosh(\beta_n \zeta) & \text{for } z > \zeta \ge 0, \\ \dfrac{1}{\beta_n} \exp(-\beta_n \zeta) \cosh(\beta_n z) & \text{for } \zeta > z \ge 0, \end{cases} \qquad \beta_n = \sqrt{\mu_n - \lambda}. \end{aligned} \tag{14}$$

Relations (13) and (14) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional second boundary value problem (10) with $s = 1$ and $k = 0$. Note that in (13) the term corresponding to the zero eigenvalue $\mu_0 = 0$ is specially singled out; D_2 is the area of the cross-section D.

3°. The Green's function of the three-dimensional third boundary value problem for equation (7) with the boundary conditions

$$\frac{\partial w}{\partial z} - k_1 w = 0 \quad \text{for} \quad z = 0, \qquad \frac{\partial w}{\partial N} + k_2 w = 0 \quad \text{for} \quad \mathbf{r} \in S$$

in a semiinfinite cylindrical domain $V=\{(x,y)\in D,\ 0\le z<\infty\}$ with arbitrary cross-section D and lateral surface S is given by relation (11) with

$$H_n(z,\zeta)=\begin{cases}\dfrac{\exp(-\beta_n z)\big[\beta_n\cosh(\beta_n\zeta)+k_1\sinh(\beta_n\zeta)\big]}{\beta_n(\beta_n+k_1)} & \text{if } z>\zeta\ge 0,\\[2mm] \dfrac{\exp(-\beta_n\zeta)\big[\beta_n\cosh(\beta_n z)+k_1\sinh(\beta_n z)\big]}{\beta_n(\beta_n+k_1)} & \text{if } \zeta>z\ge 0,\end{cases} \tag{15}$$
$$\beta_n=\sqrt{\mu_n-\lambda}.$$

Relations (11) and (15) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional third boundary value problem (10) with $s=1$ and $k=k_2$.

4°. The Green's function of the three-dimensional mixed boundary value problem for equation (7) with a second-kind boundary condition at the end face and a first-kind boundary condition at the lateral surface is given by relations (11) and (14), where the μ_n and u_n are the eigenvalues and eigenfunctions of the two-dimensional first boundary value problem (10) with $s=0$ and $k=1$.

The Green's functions of other mixed boundary value problems can be constructed likewise.

► Green's function for a cylindrical domain of finite dimensions.

1°. The Green's function of the three-dimensional first boundary value problem for equation (7) in a cylindrical domain of finite dimensions $V=\{(x,y)\in D,\ 0\le z\le a\}$ with arbitrary cross-section D is given by relation (11) with

$$H_n(z,\zeta)=\begin{cases}\dfrac{\sinh(\beta_n\zeta)\sinh[\beta_n(a-z)]}{\beta_n\sinh(\beta_n a)} & \text{if } a\ge z>\zeta\ge 0,\\[2mm] \dfrac{\sinh(\beta_n z)\sinh[\beta_n(a-\zeta)]}{\beta_n\sinh(\beta_n a)} & \text{if } a\ge\zeta>z\ge 0,\end{cases}\qquad \beta_n=\sqrt{\mu_n-\lambda}. \tag{16}$$

Relations (11) and (16) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional first boundary value problem (10) with $s=0$ and $k=1$.

Another representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{2}{a}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{u_n(x,y)u_n(\xi,\eta)\sin(q_m z)\sin(q_m\zeta)}{\|u_n\|^2(\mu_n+q_m^2-\lambda)},\qquad q_m=\frac{\pi m}{a}.$$

It is a consequence of formula (2).

2°. The Green's function of the three-dimensional second boundary value problem for equation (7) in a cylindrical domain of finite dimensions $V=\{(x,y)\in D,\ 0\le z\le a\}$ with arbitrary cross-section D is given by relation (13) with

$$H_n(z,\zeta)=\begin{cases}\dfrac{\cosh(\beta_n\zeta)\cosh[\beta_n(a-z)]}{\beta_n\sinh(\beta_n a)} & \text{if } a\ge z>\zeta\ge 0,\\[2mm] \dfrac{\cosh(\beta_n z)\cosh[\beta_n(a-\zeta)]}{\beta_n\sinh(\beta_n a)} & \text{if } a\ge\zeta>z\ge 0,\end{cases}\qquad \beta_n=\sqrt{\mu_n-\lambda}. \tag{17}$$

Relations (13) and (17) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional second boundary value problem (10) with $s = 1$ and $k = 0$.

Another representation of the Green's function:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varepsilon_m u_n(x, y) u_n(\xi, \eta) \cos(q_m z) \cos(q_m \zeta)}{\|u_n\|^2 (\mu_n + q_m^2 - \lambda)},$$

$$q_m = \frac{\pi m}{a}, \quad \varepsilon_m = \begin{cases} 1 & \text{for } m = 0, \\ 2 & \text{for } m \neq 0, \end{cases} \quad \mu_0 = 0, \quad u_0 = 1.$$

It is a consequence of formula (4).

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

3°. The Green's function of the three-dimensional third boundary value problem for equation (7) with the boundary conditions

$$\frac{\partial w}{\partial z} - k_1 w = 0 \quad \text{at} \quad z = 0, \qquad \frac{\partial w}{\partial z} + k_2 w = 0 \quad \text{at} \quad z = a, \qquad \frac{\partial w}{\partial N} + k_3 w = 0 \quad \text{for} \quad \mathbf{r} \in S$$

in a cylindrical domain of finite dimensions $V = \{(x, y) \in D,\ 0 \leq z \leq a\}$ with arbitrary cross-section D and lateral surface S is given by relation (11) with

$$H_n(z,\zeta) = \begin{cases} \dfrac{[\beta_n \cosh(\beta_n \zeta) + k_1 \sinh(\beta_n \zeta)]\{\beta_n \cosh[\beta_n(a-z)] + k_2 \sinh[\beta_n(a-z)]\}}{\beta_n[\beta_n(k_1+k_2)\cosh(\beta_n a) + (\beta_n^2 + k_1 k_2)\sinh(\beta_n a)]} & \text{if } z > \zeta, \\ \dfrac{[\beta_n \cosh(\beta_n z) + k_1 \sinh(\beta_n z)]\{\beta_n \cosh[\beta_n(a-\zeta)] + k_2 \sinh[\beta_n(a-\zeta)]\}}{\beta_n[\beta_n(k_1+k_2)\cosh(\beta_n a) + (\beta_n^2 + k_1 k_2)\sinh(\beta_n a)]} & \text{if } z < \zeta, \end{cases} \tag{18}$$

$$\beta_n = \sqrt{\mu_n - \lambda} \qquad (0 \leq z \leq a,\ 0 \leq \zeta \leq a).$$

Relations (11) and (18) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional third boundary value problem (10) with $s = 1$ and $k = k_3$.

4°. The Green's function of the three-dimensional mixed boundary value problem for equation (7) with second-kind boundary conditions at the end faces and a first-kind boundary condition at the lateral surface is given by relations (11) and (17), where the μ_n and u_n are the eigenvalues and eigenfunctions of the two-dimensional first boundary value problem (10) with $s = 0$ and $k = 1$.

The Green's function of the three-dimensional mixed boundary value problem for equation (7) with the boundary conditions

$$w = 0 \quad \text{for} \quad z = 0, \qquad \partial_z w = 0 \quad \text{for} \quad z = a, \qquad w = 0 \quad \text{for} \quad \mathbf{r} \in S$$

in a cylindrical domain of finite dimensions $V = \{(x, y) \in D,\ 0 \leq z \leq a\}$ with arbitrary cross-section D and lateral surface S is given by relation (11) with

$$H_n(z, \zeta) = \begin{cases} \dfrac{\sinh(\beta_n \zeta) \cosh[\beta_n(a - z)]}{\beta_n \cosh(\beta_n a)} & \text{if } a \geq z > \zeta \geq 0, \\ \dfrac{\sinh(\beta_n z) \cosh[\beta_n(a - \zeta)]}{\beta_n \cosh(\beta_n a)} & \text{if } a \geq \zeta > z \geq 0, \end{cases} \qquad \beta_n = \sqrt{\mu_n - \lambda}. \tag{19}$$

Relations (11) and (19) involve the eigenfunctions u_n and eigenvalues μ_n of the two-dimensional first boundary value problem (10) with $s = 0$ and $k = 1$.

The Green's functions of other mixed boundary value problems can be constructed likewise.

10.3.3 Problems in Cartesian Coordinates

The three-dimensional nonhomogeneous Helmholtz equation in the rectangular Cartesian system of coordinates has the form

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\lambda w=-\Phi(x,y,z).$$

► **Particular solutions of the homogeneous equation ($\Phi \equiv 0$):**

$w=(A_1\cos kx+A_2\sin kx)(B_1\cos my+B_2\sin my)(C_1z+C_2),\quad \lambda=k^2+m^2;$

$w=(A_1\cos kx+A_2\sin kx)(B_1\cosh my+B_2\sinh my)(C_1z+C_2),\quad \lambda=k^2-m^2;$

$w=(A_1\cos kx+A_2\sin kx)(B_1\cos my+B_2\sin my)\cos nz,\quad \lambda=k^2+m^2+n^2;$

$w=(A_1\cos kx+A_2\sin kx)(B_1\cos my+B_2\sin my)\sin nz,\quad \lambda=k^2+m^2+n^2;$

$w=(A_1\cosh kx+A_2\sinh kx)(B_1\cos my+B_2\sin my)\cos nz,\quad \lambda=-k^2+m^2+n^2;$

$w=(A_1\cosh kx+A_2\sinh kx)(B_1\cos my+B_2\sin my)\sin nz,\quad \lambda=-k^2+m^2+n^2;$

$w=(A_1\cosh kx+A_2\sinh kx)(B_1\cosh my+B_2\sinh my)\cos nz,\quad \lambda=-k^2-m^2+n^2;$

$w=(A_1\cosh kx+A_2\sinh kx)(B_1\cosh my+B_2\sinh my)\sin nz,\quad \lambda=-k^2-m^2+n^2;$

$w=(A_1\cosh kx+A_2\sinh kx)(B_1\cosh my+B_2\sinh my)\cosh nz,\quad \lambda=-k^2-m^2-n^2,$

$w=(A_1\cosh kx+A_2\sinh kx)(B_1\cosh my+B_2\sinh my)\sinh nz,\quad \lambda=-k^2-m^2-n^2,$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants.

Fundamental solutions:

$$\mathscr{E}(x,y,z)=\frac{1}{4\pi r}\exp(-kr),\qquad \lambda=-k^2<0,$$
$$\mathscr{E}(x,y,z)=\frac{1}{4\pi r}\exp(\mp ikr),\qquad \lambda=k^2>0,$$

where $r=\sqrt{x^2+y^2+z^2}$, $k>0$, $i^2=-1$.

► **Domain: $-\infty<x<\infty,\ -\infty<y<\infty,\ -\infty<z<\infty$.**

1°. Solution for $\lambda=-k^2<0$:

$$w(x,y,z)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)\frac{\exp\bigl[-k\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}\,\bigr]}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\xi\,d\eta\,d\zeta.$$

2°. Solution for $\lambda=k^2>0$:

$$w(x,y,z)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)\frac{\exp\bigl[-ik\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}\,\bigr]}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\xi\,d\eta\,d\zeta.$$

This solution was obtained taking into account the radiation condition at infinity (see Section 10.3.2, condition (6)).

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z < \infty$. First boundary value problem.**

A half-space is considered. A boundary condition is prescribed:

$$w = f(x, y) \quad \text{at} \quad z = 0.$$

Solution:

$$w(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\left[\frac{\partial}{\partial \zeta} G(x, y, z, \xi, \eta, \zeta)\right]_{\zeta=0} d\xi\, d\eta + \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi, \eta, \zeta) G(x, y, z, \xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta.$$

Green's function for $\lambda = -k^2 < 0$:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{\exp(-k\mathcal{R}_1)}{4\pi\mathcal{R}_1} - \frac{\exp(-k\mathcal{R}_2)}{4\pi\mathcal{R}_2},$$
$$\mathcal{R}_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}, \quad \mathcal{R}_2 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}.$$

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z < \infty$. Second boundary value problem.**

A half-space is considered. A boundary condition is prescribed:

$$\partial_z w = f(x, y) \quad \text{at} \quad z = 0.$$

Solution:

$$w(x, y, z) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta) G(x, y, z, \xi, \eta, 0)\, d\xi\, d\eta + \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi, \eta, \zeta) G(x, y, z, \xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta.$$

Green's function for $\lambda = -k^2 < 0$:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{\exp(-k\mathcal{R}_1)}{4\pi\mathcal{R}_1} + \frac{\exp(-k\mathcal{R}_2)}{4\pi\mathcal{R}_2},$$
$$\mathcal{R}_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}, \quad \mathcal{R}_2 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. First boundary value problem.**

A dihedral angle is considered. Boundary conditions are prescribed:

$$w = f_1(x,z) \quad \text{at} \quad y=0, \qquad w = f_2(x,y) \quad \text{at} \quad z=0.$$

Solution:

$$\begin{aligned} w(x,y,z) &= \int_0^\infty \int_{-\infty}^\infty f_1(\xi,\zeta)\left[\frac{\partial}{\partial\eta}G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\xi\,d\zeta \\ &+ \int_0^\infty \int_{-\infty}^\infty f_2(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\xi\,d\eta \\ &+ \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}$$

Green's function for $\lambda = -k^2 < 0$:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{\exp(-k\mathcal{R}_1)}{4\pi\mathcal{R}_1} - \frac{\exp(-k\mathcal{R}_2)}{4\pi\mathcal{R}_2} - \frac{\exp(-k\mathcal{R}_3)}{4\pi\mathcal{R}_3} + \frac{\exp(-k\mathcal{R}_4)}{4\pi\mathcal{R}_4},$$
$$\mathcal{R}_1 = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}, \quad \mathcal{R}_2 = \sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2},$$
$$\mathcal{R}_3 = \sqrt{(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2}, \quad \mathcal{R}_4 = \sqrt{(x-\xi)^2+(y+\eta)^2+(z+\zeta)^2}.$$

► **Domain: $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. Second boundary value problem.**

A dihedral angle is considered. Boundary conditions are prescribed:

$$\partial_y w = f_1(x,z) \quad \text{at} \quad y=0, \qquad \partial_z w = f_2(x,y) \quad \text{at} \quad z=0.$$

Solution:

$$\begin{aligned} w(x,y,z) &= -\int_0^\infty \int_{-\infty}^\infty f_1(\xi,\zeta)G(x,y,z,\xi,0,\zeta)\,d\xi\,d\zeta \\ &- \int_0^\infty \int_{-\infty}^\infty f_2(\xi,\eta)G(x,y,z,\xi,\eta,0)\,d\xi\,d\eta \\ &+ \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}$$

Green's function for $\lambda = -k^2 < 0$:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{\exp(-k\mathcal{R}_1)}{4\pi\mathcal{R}_1} + \frac{\exp(-k\mathcal{R}_2)}{4\pi\mathcal{R}_2} + \frac{\exp(-k\mathcal{R}_3)}{4\pi\mathcal{R}_3} + \frac{\exp(-k\mathcal{R}_4)}{4\pi\mathcal{R}_4},$$
$$\mathcal{R}_1 = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}, \quad \mathcal{R}_2 = \sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2},$$
$$\mathcal{R}_3 = \sqrt{(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2}, \quad \mathcal{R}_4 = \sqrt{(x-\xi)^2+(y+\eta)^2+(z+\zeta)^2}.$$

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le a$. First boundary value problem.**

An infinite layer is considered. Boundary conditions are prescribed:

$$w = f_1(x,y) \quad \text{at} \quad z = 0, \qquad w = f_2(x,y) \quad \text{at} \quad z = a.$$

Solution:

$$\begin{aligned} w(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\xi\,d\eta \\ &\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=a} d\xi\,d\eta \\ &\quad + \int_0^a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}$$

Green's function for $\lambda = -k^2 < 0$:

$$G(x,y,z,\xi,\eta,\zeta) = \sum_{n=-\infty}^{\infty}\left[\frac{\exp(-k\mathcal{R}_{n1})}{4\pi\mathcal{R}_{n1}} - \frac{\exp(-k\mathcal{R}_{2n})}{4\pi\mathcal{R}_{2n}}\right],$$

$$\mathcal{R}_{1n} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-2na)^2},$$
$$\mathcal{R}_{2n} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2na)^2}.$$

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \le z \le a$. Second boundary value problem.**

An infinite layer is considered. Boundary conditions are prescribed:

$$\partial_z w = f_1(x,y) \quad \text{at} \quad z = 0, \qquad \partial_z w = f_2(x,y) \quad \text{at} \quad z = a.$$

Solution:

$$\begin{aligned} w(x,y,z) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\xi,\eta)G(x,y,z,\xi,\eta,0)\,d\xi\,d\eta \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(\xi,\eta)G(x,y,z,\xi,\eta,a)\,d\xi\,d\eta \\ &\quad + \int_0^a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}$$

Green's function for $\lambda = -k^2 < 0$:

$$G(x,y,z,\xi,\eta,\zeta) = \sum_{n=-\infty}^{\infty}\left[\frac{\exp(-k\mathcal{R}_{n1})}{4\pi\mathcal{R}_{n1}} + \frac{\exp(-k\mathcal{R}_{2n})}{4\pi\mathcal{R}_{2n}}\right],$$

$$\mathcal{R}_{1n} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-2na)^2},$$
$$\mathcal{R}_{2n} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2na)^2}.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le a$, $0 \le y \le b$, $-\infty < z < \infty$. First boundary value problem.

An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad w = f_2(y,z) \quad \text{at} \quad x = a,$$
$$w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad w = f_4(x,z) \quad \text{at} \quad y = b.$$

Solution:

$$\begin{aligned} w(x,y,z) &= \int_0^b \int_{-\infty}^{\infty} f_1(\eta,\zeta) \left[\frac{\partial}{\partial\xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\, d\eta \\ &- \int_0^b \int_{-\infty}^{\infty} f_2(\eta,\zeta) \left[\frac{\partial}{\partial\xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=a} d\zeta\, d\eta \\ &+ \int_0^a \int_{-\infty}^{\infty} f_3(\xi,\zeta) \left[\frac{\partial}{\partial\eta} G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\zeta\, d\xi \\ &- \int_0^a \int_{-\infty}^{\infty} f_4(\xi,\zeta) \left[\frac{\partial}{\partial\eta} G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=b} d\zeta\, d\xi \\ &+ \int_0^a \int_0^b \int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\zeta\, d\eta\, d\xi. \end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{2}{ab}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\beta_{nm}} \sin(p_n x)\sin(q_m y) \times \sin(p_n \xi)\sin(q_m \eta)\exp(-\beta_{nm}|z-\zeta|)\right],$$

$$p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2 - \lambda}.$$

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► Domain: $0 \le x \le a$, $0 \le y \le b$, $-\infty < z < \infty$. Second boundary value problem.

An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad \partial_x w = f_2(y,z) \quad \text{at} \quad x = a,$$
$$\partial_y w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x,z) \quad \text{at} \quad y = b.$$

Solution:

$$\begin{aligned} w(x,y,z) = &- \int_0^b \int_{-\infty}^{\infty} f_1(\eta,\zeta) G(x,y,z,0,\eta,\zeta)\, d\zeta\, d\eta + \int_0^b \int_{-\infty}^{\infty} f_2(\eta,\zeta) G(x,y,z,a,\eta,\zeta)\, d\zeta\, d\eta \\ &- \int_0^a \int_{-\infty}^{\infty} f_3(\xi,\zeta) G(x,y,z,\xi,0,\zeta)\, d\zeta\, d\xi + \int_0^a \int_{-\infty}^{\infty} f_4(\xi,\zeta) G(x,y,z,\xi,b,\zeta)\, d\zeta\, d\xi \\ &+ \int_0^a \int_0^b \int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\zeta\, d\eta\, d\xi. \end{aligned}$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{2ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_n A_m}{\beta_{nm}}\cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)\exp(-\beta_{nm}|z-\zeta|),$$

$$p_n=\frac{n\pi}{a},\quad q_m=\frac{m\pi}{b},\quad \beta_{nm}=\sqrt{p_n^2+q_m^2-\lambda},\quad A_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0.\end{cases}$$

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $0\le x\le a$, $0\le y\le b$, $-\infty<z<\infty$. Third boundary value problem.**

An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1 w = f_1(y,z) \quad\text{at}\quad x=0, \qquad \partial_x w + k_2 w = f_2(y,z) \quad\text{at}\quad x=a,$$
$$\partial_y w - k_3 w = f_3(x,z) \quad\text{at}\quad y=0, \qquad \partial_y w + k_4 w = f_4(x,z) \quad\text{at}\quad y=b.$$

The solution $w(x,y,z)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{u_{nm}(x,y)u_{nm}(\xi,\eta)}{\|u_{nm}\|^2\beta_{nm}}\exp(-\beta_{nm}|z-\zeta|).$$

Here,

$$w_{nm}(x,y)=(\mu_n\cos\mu_n x+k_1\sin\mu_n x)(\nu_m\cos\nu_m y+k_3\sin\nu_m y),\quad \beta_{nm}=\sqrt{\mu_n^2+\nu_m^2-\lambda},$$
$$\|w_{nm}\|^2=\frac{1}{4}(\mu_n^2+k_1^2)(\nu_m^2+k_3^2)\left[a+\frac{(k_1+k_2)(\mu_n^2+k_1k_2)}{(\mu_n^2+k_1^2)(\mu_n^2+k_2^2)}\right]\left[b+\frac{(k_3+k_4)(\nu_m^2+k_3k_4)}{(\nu_m^2+k_3^2)(\nu_m^2+k_4^2)}\right],$$

where the μ_n and ν_m are positive roots of the transcendental equations

$$\tan(\mu a)=\frac{(k_1+k_2)\mu}{\mu^2-k_1k_2},\qquad \tan(\nu b)=\frac{(k_3+k_4)\nu}{\nu^2-k_3k_4}.$$

► **Domain: $0\le x\le a$, $0\le y\le b$, $-\infty<z<\infty$. Mixed boundary value problems.**

1°. An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w=f_1(y,z) \quad\text{at}\quad x=0, \qquad \partial_x w=f_2(y,z) \quad\text{at}\quad x=a,$$
$$w=f_3(x,z) \quad\text{at}\quad y=0, \qquad \partial_y w=f_4(x,z) \quad\text{at}\quad y=b.$$

Solution:

$$
\begin{aligned}
w(x,y,z) &= \int_0^b \int_{-\infty}^{\infty} f_1(\eta,\zeta) \left[\frac{\partial}{\partial \xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\, d\eta \\
&+ \int_0^b \int_{-\infty}^{\infty} f_2(\eta,\zeta) G(x,y,z,a,\eta,\zeta)\, d\zeta\, d\eta \\
&+ \int_0^a \int_{-\infty}^{\infty} f_3(\xi,\zeta) \left[\frac{\partial}{\partial \eta} G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\zeta\, d\xi \\
&+ \int_0^a \int_{-\infty}^{\infty} f_4(\xi,\zeta) G(x,y,z,\xi,b,\zeta)\, d\zeta\, d\xi \\
&+ \int_0^a \int_0^b \int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\zeta\, d\eta\, d\xi.
\end{aligned}
$$

Green's function:

$$
G(x,y,z,\xi,\eta,\zeta) = \frac{2}{ab} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\beta_{nm}} \sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta) \exp(-\beta_{nm}|z-\zeta|),
$$

$$
p_n = \frac{(2n+1)\pi}{2a}, \quad q_m = \frac{(2m+1)\pi}{2b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2 - \lambda}.
$$

2°. An infinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$
\begin{aligned}
w &= f_1(y,z) \quad \text{at} \quad x=0, & w &= f_2(y,z) \quad \text{at} \quad x=a, \\
\partial_y w &= f_3(x,z) \quad \text{at} \quad y=0, & \partial_y w &= f_4(x,z) \quad \text{at} \quad y=b.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z) &= \int_0^b \int_{-\infty}^{\infty} f_1(\eta,\zeta) \left[\frac{\partial}{\partial \xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\, d\eta \\
&- \int_0^b \int_{-\infty}^{\infty} f_2(\eta,\zeta) \left[\frac{\partial}{\partial \xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=a} d\zeta\, d\eta \\
&- \int_0^a \int_{-\infty}^{\infty} f_3(\xi,\zeta) G(x,y,z,\xi,0,\zeta)\, d\zeta\, d\xi \\
&+ \int_0^a \int_{-\infty}^{\infty} f_4(\xi,\zeta) G(x,y,z,\xi,b,\zeta)\, d\zeta\, d\xi \\
&+ \int_0^a \int_0^b \int_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\zeta\, d\eta\, d\xi.
\end{aligned}
$$

Green's function:

$$
G(x,y,z,\xi,\eta,\zeta) = \frac{1}{ab} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{A_m}{\beta_{nm}} \sin(p_n x) \cos(q_m y) \sin(p_n \xi) \cos(q_m \eta) \exp(-\beta_{nm}|z-\zeta|),
$$

$$
p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2 - \lambda}, \quad A_m = \begin{cases} 1 & \text{for } m=0, \\ 2 & \text{for } m \neq 0. \end{cases}
$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$
\begin{aligned}
&w = f_1(y,z) \quad \text{at} \quad x = 0, && w = f_2(y,z) \quad \text{at} \quad x = a,\\
&w = f_3(x,z) \quad \text{at} \quad y = 0, && w = f_4(x,z) \quad \text{at} \quad y = b,\\
&w = f_5(x,y) \quad \text{at} \quad z = 0.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,y,z) &= \int_0^b \int_0^\infty f_1(\eta,\zeta) \left[\frac{\partial}{\partial \xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=0} d\zeta\, d\eta\\
&\quad - \int_0^b \int_0^\infty f_2(\eta,\zeta) \left[\frac{\partial}{\partial \xi} G(x,y,z,\xi,\eta,\zeta)\right]_{\xi=a} d\zeta\, d\eta\\
&\quad + \int_0^a \int_0^\infty f_3(\xi,\zeta) \left[\frac{\partial}{\partial \eta} G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=0} d\zeta\, d\xi\\
&\quad - \int_0^a \int_0^\infty f_4(\xi,\zeta) \left[\frac{\partial}{\partial \eta} G(x,y,z,\xi,\eta,\zeta)\right]_{\eta=b} d\zeta\, d\xi\\
&\quad + \int_0^a \int_0^b f_5(\xi,\eta) \left[\frac{\partial}{\partial \zeta} G(x,y,z,\xi,\eta,\zeta)\right]_{\zeta=0} d\eta\, d\xi\\
&\quad + \int_0^a \int_0^b \int_0^\infty \Phi(\xi,\eta,\zeta) G(x,y,z,\xi,\eta,\zeta)\, d\zeta\, d\eta\, d\xi.
\end{aligned}
$$

Green's function:

$$
G(x,y,z,\xi,\eta,\zeta) = \frac{4}{ab} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{\beta_{nm}} \sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta) H_{nm}(z,\zeta),
$$

$$
p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2 - \lambda},
$$

$$
H_{nm}(z,\zeta) = \begin{cases} \exp(-\beta_{nm} z) \sinh(\beta_{nm} \zeta) & \text{for } z > \zeta \ge 0,\\ \exp(-\beta_{nm} \zeta) \sinh(\beta_{nm} z) & \text{for } \zeta > z \ge 0. \end{cases}
$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z < \infty$. Second boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$
\begin{aligned}
&\partial_x w = f_1(y,z) \quad \text{at} \quad x = 0, && \partial_x w = f_2(y,z) \quad \text{at} \quad x = a,\\
&\partial_y w = f_3(x,z) \quad \text{at} \quad y = 0, && \partial_y w = f_4(x,z) \quad \text{at} \quad y = b,\\
&\partial_z w = f_5(x,y) \quad \text{at} \quad z = 0.
\end{aligned}
$$

Solution:

$$w(x,y,z) = \int_0^a\int_0^b\int_0^\infty \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi$$
$$-\int_0^b\int_0^\infty f_1(\eta,\zeta)G(x,y,z,0,\eta,\zeta)\,d\zeta\,d\eta + \int_0^b\int_0^\infty f_2(\eta,\zeta)G(x,y,z,a,\eta,\zeta)\,d\zeta\,d\eta$$
$$-\int_0^a\int_0^\infty f_3(\xi,\zeta)G(x,y,z,\xi,0,\zeta)\,d\zeta\,d\xi + \int_0^a\int_0^\infty f_4(\xi,\zeta)G(x,y,z,\xi,b,\zeta)\,d\zeta\,d\xi$$
$$-\int_0^a\int_0^b f_5(\xi,\eta)G(x,y,z,\xi,\eta,0)\,d\eta\,d\xi.$$

Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{ab}\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{A_nA_m}{\beta_{nm}}\cos(p_nx)\cos(q_my)\cos(p_n\xi)\cos(q_m\eta)H_{nm}(z,\zeta),$$

$$p_n = \frac{n\pi}{a},\quad q_m = \frac{m\pi}{b},\quad \beta_{nm} = \sqrt{p_n^2+q_m^2-\lambda},\quad A_n = \begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}$$

$$H_{nm}(z,\zeta) = \begin{cases}\exp(-\beta_{nm}z)\cosh(\beta_{nm}\zeta) & \text{for } z>\zeta\ge 0,\\ \exp(-\beta_{nm}\zeta)\cosh(\beta_{nm}z) & \text{for } \zeta>z\ge 0.\end{cases}$$

► **Domain: $0\le x\le a$, $0\le y\le b$, $0\le z<\infty$. Third boundary value problem.**

A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w - k_1w = f_1(y,z) \quad\text{at}\quad x=0, \qquad \partial_x w + k_2w = f_2(y,z) \quad\text{at}\quad x=a,$$
$$\partial_y w - k_3w = f_3(x,z) \quad\text{at}\quad y=0, \qquad \partial_y w + k_4w = f_4(x,z) \quad\text{at}\quad y=b,$$
$$\partial_z w - k_5w = f_5(x,y) \quad\text{at}\quad z=0.$$

The solution $w(x,y,z)$ is determined by the formula in the previous paragraph (for the second boundary value problem) where

$$G(x,y,z,\xi,\eta,\zeta) = \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{u_{nm}(x,y)u_{nm}(\xi,\eta)}{\|u_{nm}\|^2}H_{nm}(z,\zeta).$$

Here,

$$w_{nm}(x,y) = (\mu_n\cos\mu_nx + k_1\sin\mu_nx)(\nu_m\cos\nu_my + k_3\sin\nu_my),$$
$$\|w_{nm}\|^2 = \frac{1}{4}(\mu_n^2+k_1^2)(\nu_m^2+k_3^2)\left[a+\frac{(k_1+k_2)(\mu_n^2+k_1k_2)}{(\mu_n^2+k_1^2)(\mu_n^2+k_2^2)}\right]\left[b+\frac{(k_3+k_4)(\nu_m^2+k_3k_4)}{(\nu_m^2+k_3^2)(\nu_m^2+k_4^2)}\right],$$
$$H_{nm}(z,\zeta) = \begin{cases}\dfrac{\exp(-\beta_{nm}z)[\beta_{nm}\cosh(\beta_{nm}\zeta)+k_5\sinh(\beta_{nm}\zeta)]}{\beta_{nm}(\beta_{nm}+k_5)} & \text{for } z>\zeta,\\[2ex] \dfrac{\exp(-\beta_{nm}\zeta)[\beta_{nm}\cosh(\beta_{nm}z)+k_5\sinh(\beta_{nm}z)]}{\beta_{nm}(\beta_{nm}+k_5)} & \text{for } \zeta>z,\end{cases}$$
$$\beta_{nm} = \sqrt{\mu_n^2+\nu_m^2-\lambda},$$

where the μ_n and ν_m are positive roots of the transcendental equations

$$\tan(\mu a) = \frac{(k_1 + k_2)\mu}{\mu^2 - k_1 k_2}, \qquad \tan(\nu b) = \frac{(k_3 + k_4)\nu}{\nu^2 - k_3 k_4}.$$

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z < \infty$. Mixed boundary value problems.**

1°. A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$w = f_1(y, z) \quad \text{at} \quad x = 0, \qquad w = f_2(y, z) \quad \text{at} \quad x = a,$$
$$w = f_3(x, z) \quad \text{at} \quad y = 0, \qquad w = f_4(x, z) \quad \text{at} \quad y = b,$$
$$\partial_z w = f_5(x, y) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(x, y, z) = &\int_0^b \int_0^\infty f_1(\eta, \zeta) \left[\frac{\partial}{\partial \xi} G(x, y, z, \xi, \eta, \zeta) \right]_{\xi=0} d\zeta\, d\eta \\ &- \int_0^b \int_0^\infty f_2(\eta, \zeta) \left[\frac{\partial}{\partial \xi} G(x, y, z, \xi, \eta, \zeta) \right]_{\xi=a} d\zeta\, d\eta \\ &+ \int_0^a \int_0^\infty f_3(\xi, \zeta) \left[\frac{\partial}{\partial \eta} G(x, y, z, \xi, \eta, \zeta) \right]_{\eta=0} d\zeta\, d\xi \\ &- \int_0^a \int_0^\infty f_4(\xi, \zeta) \left[\frac{\partial}{\partial \eta} G(x, y, z, \xi, \eta, \zeta) \right]_{\eta=b} d\zeta\, d\xi \\ &- \int_0^a \int_0^b f_5(\xi, \eta) G(x, y, z, \xi, \eta, 0)\, d\eta\, d\xi \\ &+ \int_0^a \int_0^b \int_0^\infty \Phi(\xi, \eta, \zeta) G(x, y, z, \xi, \eta, \zeta)\, d\zeta\, d\eta\, d\xi. \end{aligned}$$

Green's function:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{4}{ab} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{\beta_{nm}} \sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta) H_{nm}(z, \zeta),$$
$$p_n = \frac{n\pi}{a}, \quad q_m = \frac{m\pi}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2 - \lambda},$$
$$H_{nm}(z, \zeta) = \begin{cases} \exp(-\beta_{nm} z) \cosh(\beta_{nm} \zeta) & \text{for } z > \zeta \ge 0, \\ \exp(-\beta_{nm} \zeta) \cosh(\beta_{nm} z) & \text{for } \zeta > z \ge 0. \end{cases}$$

2°. A semiinfinite cylindrical domain of a rectangular cross-section is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y, z) \quad \text{at} \quad x = 0, \qquad \partial_x w = f_2(y, z) \quad \text{at} \quad x = a,$$
$$\partial_y w = f_3(x, z) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x, z) \quad \text{at} \quad y = b,$$
$$w = f_5(x, y) \quad \text{at} \quad z = 0.$$

Solution:

$$
\begin{aligned}
w(x,y,z) = &\int_0^a\int_0^b\int_0^\infty \Phi(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi \\
&-\int_0^b\int_0^\infty f_1(\eta,\zeta)G(x,y,z,0,\eta,\zeta)\,d\zeta\,d\eta + \int_0^b\int_0^\infty f_2(\eta,\zeta)G(x,y,z,a,\eta,\zeta)\,d\zeta\,d\eta \\
&-\int_0^a\int_0^\infty f_3(\xi,\zeta)G(x,y,z,\xi,0,\zeta)\,d\zeta\,d\xi + \int_0^a\int_0^\infty f_4(\xi,\zeta)G(x,y,z,\xi,b,\zeta)\,d\zeta\,d\xi \\
&+\int_0^a\int_0^b f_5(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(x,y,z,\xi,\eta,\xi)\right]_{\zeta=0} d\eta\,d\xi.
\end{aligned}
$$

Green's function:

$$
G(x,y,z,\xi,\eta,\zeta) = \frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_n A_m}{\beta_{nm}}\cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)H_{nm}(z,\zeta).
$$

Here,

$$
p_n = \frac{n\pi}{a},\quad q_m = \frac{m\pi}{b},\quad \beta_{nm} = \sqrt{p_n^2+q_m^2-\lambda},\quad A_n = \begin{cases} 1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0,\end{cases}
$$

$$
H_{nm}(z,\zeta) = \begin{cases} \exp(-\beta_{nm}z)\sinh(\beta_{nm}\zeta) & \text{for } z>\zeta\ge 0,\\ \exp(-\beta_{nm}\zeta)\sinh(\beta_{nm}z) & \text{for } \zeta>z\ge 0.\end{cases}
$$

◆ *Only the eigenvalues and eigenfunctions of homogeneous boundary value problems for the homogeneous Helmholtz equation (with $\Phi \equiv 0$) are presented below. The solutions of the corresponding nonhomogeneous boundary value problems (with $\Phi \not\equiv 0$) can be constructed by the relations specified in Section 10.3.2.*

► **Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. First boundary value problem.**

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$
\begin{aligned}
&w = f_1(y,z) \quad\text{at}\quad x=0, &&w = f_2(y,z) \quad\text{at}\quad x=a,\\
&w = f_3(x,z) \quad\text{at}\quad y=0, &&w = f_4(x,z) \quad\text{at}\quad y=b,\\
&w = f_5(x,y) \quad\text{at}\quad z=0, &&w = f_6(x,y) \quad\text{at}\quad z=c.
\end{aligned}
$$

1°. Eigenvalues of the homogeneous problem:

$$
\lambda_{nmk} = \pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{k^2}{c^2}\right);\qquad n,\ m,\ k = 1,\ 2,\ 3,\ \ldots
$$

Eigenfunctions and the norm squared:

$$
w_{nmk} = \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{b}\right)\sin\left(\frac{\pi k z}{c}\right),\qquad \|w_{nmk}\|^2 = \frac{abc}{8}.
$$

2°. A double-series representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)H_{nm}(z,\zeta),$$

$$H_{nm}(z,\zeta) = \begin{cases} \dfrac{\sinh(\beta_{nm}\zeta)\sinh[\beta_{nm}(c-z)]}{\beta_{nm}\sinh(\beta_{nm}c)} & \text{for } c \ge z > \zeta \ge 0, \\ \dfrac{\sinh(\beta_{nm}z)\sinh[\beta_{nm}(c-\zeta)]}{\beta_{nm}\sinh(\beta_{nm}c)} & \text{for } c \ge \zeta > z \ge 0, \end{cases}$$

$$p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}, \quad \beta_{nm} = \sqrt{p_n^2 + q_m^2 - \lambda}.$$

This relation can be used to obtain two other representations of the Green's function with the aid of the cyclic permutations of triples:

$$\begin{array}{ccc} & (x,\xi,a) & \\ \nearrow & & \searrow \\ (z,\zeta,c) & \longleftarrow & (y,\eta,b) \end{array}$$

A triple series representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{8}{abc}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin(p_n x)\sin(q_m y)\sin(s_k z)\sin(p_n\xi)\sin(q_m\eta)\sin(s_k\zeta)}{p_n^2+q_m^2+s_k^2-\lambda},$$

$$p_n = \frac{\pi n}{a}, \quad q_m = \frac{\pi m}{b}, \quad s_k = \frac{\pi k}{c}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Second boundary value problem.

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$\begin{aligned} &\partial_x w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad &&\partial_x w = f_2(y,z) \quad \text{at} \quad x = a, \\ &\partial_y w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad &&\partial_y w = f_4(x,z) \quad \text{at} \quad y = b, \\ &\partial_z w = f_5(x,y) \quad \text{at} \quad z = 0, \qquad &&\partial_z w = f_6(x,y) \quad \text{at} \quad z = c. \end{aligned}$$

1°. Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2}\right); \qquad n,\, m,\, k = 0,\, 1,\, 2,\, \dots$$

Eigenfunctions:

$$w_{nmk} = \cos\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi m y}{b}\right)\cos\left(\frac{\pi k z}{c}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}\|^2 = \frac{abc}{8}(1+\delta_{n0})(1+\delta_{m0})(1+\delta_{k0}), \qquad \delta_{n0} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \ne 0. \end{cases}$$

2°. A double series representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varepsilon_n\varepsilon_m\cos(p_nx)\cos(q_my)\cos(p_n\xi)\cos(q_m\eta)H_{nm}(z,\zeta),$$

$$H_{nm}(z,\zeta)=\begin{cases}\dfrac{\cosh(\beta_{nm}\zeta)\cosh[\beta_{nm}(c-z)]}{\beta_{nm}\sinh(\beta_{nm}c)} & \text{for } c\ge z>\zeta\ge 0,\\[2mm] \dfrac{\cosh(\beta_{nm}z)\cosh[\beta_{nm}(c-\zeta)]}{\beta_{nm}\sinh(\beta_{nm}c)} & \text{for } c\ge \zeta>z\ge 0,\end{cases}$$

$$p_n=\frac{\pi n}{a},\quad q_m=\frac{\pi m}{b},\quad \beta_{nm}=\sqrt{p_n^2+q_m^2-\lambda},\quad \varepsilon_n=\begin{cases}1 & \text{for } n=0,\\ 2 & \text{for } n\neq 0.\end{cases}$$

This relation can be used to obtain two other representations of the Green's function with the aid of the cyclic permutations:

$$\begin{array}{ccc} & (x,\xi,a) & \\ \nearrow & & \searrow \\ (z,\zeta,c) & \longleftarrow & (y,\eta,b)\end{array}$$

A triple series representation of the Green's function:

$$G(x,y,z,\xi,\eta,\zeta)=\frac{1}{abc}\Biggl[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\varepsilon_k}{p_n^2+q_m^2+s_k^2-\lambda}\cos(p_nx)\cos(q_my)$$
$$\times\cos(s_kz)\cos(p_n\xi)\cos(q_m\eta)\cos(s_k\zeta)\Biggr],$$

$$p_n=\frac{\pi n}{a},\quad q_m=\frac{\pi m}{b},\quad s_k=\frac{\pi k}{c}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0\le x\le a$, $0\le y\le b$, $0\le z\le c$. Third boundary value problem.

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$\begin{aligned}
&\partial_x w-k_1w=f_1(y,z) \text{ at } x=0, && \partial_x w+k_2w=f_2(y,z) \text{ at } x=a,\\
&\partial_y w-k_3w=f_3(x,z) \text{ at } y=0, && \partial_y w+k_4w=f_4(x,z) \text{ at } y=b,\\
&\partial_z w-k_5w=f_5(x,y) \text{ at } z=0, && \partial_z w+k_6w=f_6(x,y) \text{ at } z=c.
\end{aligned}$$

Eigenvalues of the homogeneous problem:

$$\lambda_{nml}=\mu_n^2+\nu_m^2+\sigma_l^2;\qquad n,\,m,\,l=1,\,2,\,3,\,\dots$$

Here, the μ_n, ν_m, and σ_l are positive roots of the transcendental equations

$$\tan(\mu a)=\frac{(k_1+k_2)\mu}{\mu^2-k_1k_2},\quad \tan(\nu b)=\frac{(k_3+k_4)\nu}{\nu^2-k_3k_4},\quad \tan(\sigma c)=\frac{(k_5+k_6)\sigma}{\sigma^2-k_5k_6}.$$

Eigenfunctions:

$$w_{nml} = \frac{1}{A_n B_m C_l}(\mu_n \cos\mu_n x + k_1 \sin\mu_n x)(\nu_m \cos\nu_m y + k_3 \sin\nu_m y)(\sigma_l \cos\sigma_l z + k_5 \sin\sigma_l z),$$

$$A_n = \sqrt{\mu_n^2 + k_1^2}, \quad B_m = \sqrt{\nu_m^2 + k_3^2}, \quad C_l = \sqrt{\sigma_l^2 + k_5^2}.$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nml}\|^2 = \frac{1}{8}\left[a + \frac{(k_1+k_2)(\mu_n^2+k_1k_2)}{(\mu_n^2+k_1^2)(\mu_n^2+k_2^2)}\right]\left[b + \frac{(k_3+k_4)(\nu_m^2+k_3k_4)}{(\nu_m^2+k_3^2)(\nu_m^2+k_4^2)}\right]\left[c + \frac{(k_5+k_6)(\sigma_l^2+k_5k_6)}{(\sigma_l^2+k_5^2)(\sigma_l^2+k_6^2)}\right].$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► Domain: $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Mixed boundary value problems.

1°. A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad w = f_2(y,z) \quad \text{at} \quad x = a,$$
$$w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad w = f_4(x,z) \quad \text{at} \quad y = b,$$
$$\partial_z w = f_5(x,y) \quad \text{at} \quad z = 0, \qquad \partial_z w = f_6(x,y) \quad \text{at} \quad z = c.$$

Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2}\right); \qquad n,\ m = 1,\ 2,\ 3,\ \ldots; \quad k = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions:

$$w_{nmk} = \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{b}\right)\cos\left(\frac{\pi k z}{c}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}\|^2 = \frac{abc}{8}(1 + \delta_{k0}), \qquad \delta_{k0} = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

2°. A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \qquad w = f_2(y,z) \quad \text{at} \quad x = a,$$
$$\partial_y w = f_3(x,z) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x,z) \quad \text{at} \quad y = b,$$
$$\partial_z w = f_5(x,y) \quad \text{at} \quad z = 0, \qquad \partial_z w = f_6(x,y) \quad \text{at} \quad z = c.$$

Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2}\right); \qquad n = 1,\ 2,\ 3,\ \ldots; \quad m,\ k = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions:

$$w_{nmk} = \sin\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi m y}{b}\right)\cos\left(\frac{\pi k z}{c}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}\|^2 = \frac{abc}{8}(1 + \delta_{m0})(1 + \delta_{k0}), \qquad \delta_{m0} = \begin{cases} 1 & \text{for } m = 0, \\ 0 & \text{for } m \neq 0. \end{cases}$$

► **Domain: $0 \le x \le a$, $0 \le y \le x$, $0 \le z \le c$. First boundary value problem.**

A right prism whose base is an isosceles right-angled triangle is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \quad w = f_2(x,z) \quad \text{at} \quad y = 0, \quad w = f_3(x,z) \quad \text{at} \quad y = x,$$
$$w = f_4(x,y) \quad \text{at} \quad z = 0, \quad w = f_5(x,y) \quad \text{at} \quad z = c.$$

Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \frac{\pi^2}{a^2}\left[(n+m)^2 + m^2\right] + \frac{\pi^2 k^2}{c^2}; \qquad n, m, k = 1,\ 2,\ 3,\ \ldots$$

Eigenfunctions:

$$w_{nmk} = \left\{\sin\left[\frac{\pi}{a}(n+m)x\right]\sin\left(\frac{\pi m y}{a}\right) - (-1)^n \sin\left(\frac{\pi m x}{a}\right)\sin\left[\frac{\pi}{a}(n+m)y\right]\right\}\sin\left(\frac{\pi k z}{c}\right).$$

► **Domain: $0 \le x \le a$, $0 \le y \le x$, $0 \le z \le c$. Second boundary value problem.**

A right prism whose base is an isosceles right-angled triangle is considered. Boundary conditions are prescribed:

$$\partial_x w = f_1(y,z) \quad \text{at} \quad x = 0, \quad \partial_y w = f_2(x,z) \quad \text{at} \quad y = 0, \quad \partial_N w = f_3(x,z) \quad \text{at} \quad y = x,$$
$$\partial_z w = f_4(x,y) \quad \text{at} \quad z = 0, \quad \partial_z w = f_5(x,y) \quad \text{at} \quad z = c,$$

where $\partial_N w = \mathbf{N} \cdot \nabla w = \frac{1}{\sqrt{2}}(\partial_x w + \partial_y w)$.

Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \frac{\pi^2}{a^2}\left[(n+m)^2 + m^2\right] + \frac{\pi^2 k^2}{c^2}; \qquad n, m, k = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions:

$$w_{nmk} = \left\{\cos\left[\frac{\pi}{a}(n+m)x\right]\cos\left(\frac{\pi m y}{a}\right) - (-1)^n \cos\left(\frac{\pi m x}{a}\right)\cos\left[\frac{\pi}{a}(n+m)y\right]\right\}\cos\left(\frac{\pi k z}{c}\right).$$

► **Domain: $0 \le x \le a$, $0 \le y \le x$, $0 \le z \le c$. Mixed boundary value problems.**

1°. A right prism whose base is an isosceles right-angled triangle is considered. Boundary conditions are prescribed:

$$w = f_1(y,z) \quad \text{at} \quad x = 0, \quad w = f_2(x,z) \quad \text{at} \quad y = 0, \quad w = f_3(x,z) \quad \text{at} \quad y = x,$$
$$\partial_z w = f_4(x,y) \quad \text{at} \quad z = 0, \quad \partial_z w = f_5(x,y) \quad \text{at} \quad z = c.$$

Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \frac{\pi^2}{a^2}\left[(n+m)^2 + m^2\right] + \frac{\pi^2 k^2}{c^2}; \qquad n, m = 1,\ 2,\ 3,\ \ldots; \quad k = 0,\ 1,\ 2,\ \ldots$$

Eigenfunctions:

$$w_{nmk}=\left\{\sin\left[\frac{\pi}{a}(n+m)x\right]\sin\left(\frac{\pi my}{a}\right)-(-1)^n\sin\left(\frac{\pi mx}{a}\right)\sin\left[\frac{\pi}{a}(n+m)y\right]\right\}\cos\left(\frac{\pi kz}{c}\right).$$

2°. A right prism whose base is an isosceles right-angled triangle is considered. Boundary conditions are prescribed:

$$\partial_x w=f_1(y,z) \quad\text{at}\quad x=0, \qquad \partial_y w=f_2(x,z) \quad\text{at}\quad y=0, \qquad \partial_N w=f_3(x,z) \quad\text{at}\quad y=x,$$
$$w=f_4(x,y) \quad\text{at}\quad z=0, \qquad w=f_5(x,y) \quad\text{at}\quad z=c,$$

where $\partial_N w = \mathbf{N}\cdot\nabla w = \frac{1}{\sqrt{2}}(\partial_x w + \partial_y w)$.

Eigenvalues of the homogeneous problem:

$$\lambda_{nmk} = \frac{\pi^2}{a^2}\left[(n+m)^2 + m^2\right] + \frac{\pi^2 k^2}{c^2}; \qquad n,m = 0,\ 1,\ 2,\ \dots; \quad k = 1,\ 2,\ 3,\ \dots$$

Eigenfunctions:

$$w_{nmk}=\left\{\cos\left[\frac{\pi}{a}(n+m)x\right]\cos\left(\frac{\pi my}{a}\right)-(-1)^n\cos\left(\frac{\pi mx}{a}\right)\cos\left[\frac{\pi}{a}(n+m)y\right]\right\}\sin\left(\frac{\pi kz}{c}\right).$$

10.3.4 Problems in Cylindrical Coordinates

The three-dimensional nonhomogeneous Helmholtz equation in the cylindrical coordinate system is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{\partial^2 w}{\partial z^2} + \lambda w = -\Phi(r,\varphi,z), \qquad r = \sqrt{x^2+y^2}.$$

► **Particular solutions of the homogeneous equation ($\Phi \equiv 0$):**

$$\begin{aligned}
&w = \left[AJ_0\big(r\sqrt{\lambda}\big) + BY_0\big(r\sqrt{\lambda}\big)\right](C_1\varphi + D_1)(C_2 z + D_2),\\
&w = J_m\big(r\sqrt{\lambda-\mu^2}\big)(A\cos m\varphi + B\sin m\varphi)(C\cos\mu z + D\sin\mu z), && \lambda > \mu^2,\\
&w = Y_m\big(r\sqrt{\lambda-\mu^2}\big)(A\cos m\varphi + B\sin m\varphi)(C\cos\mu z + D\sin\mu z), && \lambda > \mu^2,\\
&w = J_m\big(r\sqrt{\lambda+\mu^2}\big)(A\cos m\varphi + B\sin m\varphi)(C\cosh\mu z + D\sinh\mu z), && \lambda > -\mu^2,\\
&w = Y_m\big(r\sqrt{\lambda+\mu^2}\big)(A\cos m\varphi + B\sin m\varphi)(C\cosh\mu z + D\sinh\mu z), && \lambda > -\mu^2,\\
&w = I_m\big(r\sqrt{\mu^2-\lambda}\big)(A\cos m\varphi + B\sin m\varphi)(C\cos\mu z + D\sin\mu z), && \lambda < \mu^2,\\
&w = Y_m\big(r\sqrt{\mu^2-\lambda}\big)(A\cos m\varphi + B\sin m\varphi)(C\cos\mu z + D\sin\mu z), && \lambda < \mu^2,
\end{aligned}$$

where $m = 0,\ 1,\ 2,\ \dots;$ $A,\ B,\ C,\ D,\ C_1,\ C_2,\ D_1,\ D_2$, and μ are arbitrary constants; the $J_m(\xi)$ and $Y_m(\xi)$ are Bessel functions; and the $I_m(\xi)$ and $K_m(\xi)$ are modified Bessel functions.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. First boundary value problem.**

An infinite circular cylinder is considered. A boundary condition is prescribed:

$$w = f(\varphi, z) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi,z) = -R\int_0^{2\pi}\int_{-\infty}^{\infty} f(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\xi=R} d\zeta\, d\eta + \int_0^R\int_0^{2\pi}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\, d\zeta\, d\eta\, d\xi.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\left[J_n'(\mu_{nm}R)\right]^2\beta_{nm}}\cos[n(\varphi-\eta)]\exp\bigl(-\beta_{nm}|z-\zeta|\bigr),$$

$$\beta_{nm} = \sqrt{\mu_{nm}^2 - \lambda}, \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Second boundary value problem.**

An infinite circular cylinder is considered. A boundary condition is prescribed:

$$\partial_r w = f(\varphi, z) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi,z) = R\int_0^{2\pi}\int_{-\infty}^{\infty} f(\eta,\zeta)G(r,\varphi,z,R,\eta,\zeta)\, d\zeta\, d\eta + \int_0^R\int_0^{2\pi}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\, d\zeta\, d\eta\, d\xi.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{\exp\bigl(-\sqrt{-\lambda}\,|z-\zeta|\bigr)}{2\pi R^2\sqrt{-\lambda}} + \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]}{(\mu_{nm}^2R^2 - n^2)J_n^2(\mu_{nm}R)\beta_{nm}}\exp\bigl(-\beta_{nm}|z-\zeta|\bigr),$$

$$\beta_{nm} = \sqrt{\mu_{nm}^2 - \lambda}, \quad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. Third boundary value problem.**

An infinite circular cylinder is considered. A boundary condition is prescribed:

$$\partial_r w + kw = f(\varphi, z) \quad \text{at} \quad r = R.$$

Solution:

$$w(r,\varphi,z) = R\int_0^{2\pi}\int_{-\infty}^{\infty} f(\eta,\zeta)G(r,\varphi,z,R,\eta,\zeta)\,d\zeta\,d\eta + \int_0^R\int_0^{2\pi}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi.$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]}{(\mu_{nm}^2R^2+k^2R^2-n^2)J_n^2(\mu_{nm}R)\beta_{nm}}\exp\bigl(-\beta_{nm}|z-\zeta|\bigr),$$

$$\beta_{nm} = \sqrt{\mu_{nm}^2-\lambda}, \qquad A_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n\neq 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + kJ_n(\mu R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. First boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad w = f_2(r, \varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) = {}& -R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\xi=R} d\zeta\,d\eta \\ &+ \int_0^{2\pi}\int_0^R f_2(\xi,\eta)\left[\frac{\partial}{\partial\zeta}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\zeta=0}\xi\,d\xi\,d\eta \\ &+ \int_0^R\int_0^{2\pi}\int_0^{\infty}\Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{A_nJ_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\bigl[J_n'(\mu_{nm}R)\bigr]^2\beta_{nm}}\cos[n(\varphi-\eta)]F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \exp(-\beta_{nm}|z-\zeta|) - \exp(-\beta_{nm}|z+\zeta|),$$

$$\beta_{nm} = \sqrt{\mu_{nm}^2-\lambda}, \qquad A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Second boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$\partial_r w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad \partial_z w = f_2(r, \varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) &= R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)G(r,\varphi,z,R,\eta,\zeta)\,d\zeta\,d\eta \\ &\quad - \int_0^{2\pi}\int_0^{R} f_2(\xi,\eta)G(r,\varphi,z,R,\eta,0)\xi\,d\xi\,d\eta \\ &\quad + \int_0^{R}\int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Here,

$$\begin{aligned} G(r,\varphi,z,\xi,\eta,\zeta) &= \frac{\exp\bigl(-\sqrt{-\lambda}\,|z-\zeta|\bigr) + \exp\bigl(-\sqrt{-\lambda}\,|z+\zeta|\bigr)}{2\pi R^2\sqrt{-\lambda}} \\ &\quad + \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]}{(\mu_{nm}^2R^2 - n^2)J_n^2(\mu_{nm}R)\beta_{nm}} F_{nm}(z,\zeta), \\ F_{nm}(z,\zeta) &= \exp(-\beta_{nm}|z-\zeta|) + \exp(-\beta_{nm}|z+\zeta|), \\ \beta_{nm} &= \sqrt{\mu_{nm}^2 - \lambda}, \qquad A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \ne 0, \end{cases} \end{aligned}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n'(\mu R) = 0$.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Third boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$\partial_r w + k_1 w = f(\varphi, z) \quad \text{at} \quad r = R, \qquad \partial_z w - k_2 w = f_2(r, \varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) &= R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)G(r,\varphi,z,R,\eta,\zeta)\,d\zeta\,d\eta \\ &\quad - \int_0^{2\pi}\int_0^{R} f_2(\xi,\eta)G(r,\varphi,z,\xi,\eta,0)\xi\,d\xi\,d\eta \\ &\quad + \int_0^{R}\int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)\cos[n(\varphi-\eta)]}{(\mu_{nm}^2R^2 + k_1^2R^2 - n^2)J_n^2(\mu_{nm}R)} F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \begin{cases} \dfrac{\exp(-\beta_{nm}z)[\beta_{nm}\cosh(\beta_{nm}\zeta) + k_2\sinh(\beta_{nm}\zeta)]}{\beta_{nm}(\beta_{nm}+k_2)} & \text{for } z > \zeta, \\ \dfrac{\exp(-\beta_{nm}\zeta)[\beta_{nm}\cosh(\beta_{nm}z) + k_2\sinh(\beta_{nm}z)]}{\beta_{nm}(\beta_{nm}+k_2)} & \text{for } \zeta > z, \end{cases}$$

$$\beta_{nm} = \sqrt{\mu_{nm}^2 - \lambda}, \qquad A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation

$$\mu J_n'(\mu R) + k_1 J_n(\mu R) = 0.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z < \infty$. Mixed boundary value problem.**

A semiinfinite circular cylinder is considered. Boundary conditions are prescribed:

$$w = f_1(\varphi, z) \quad \text{at} \quad r = R, \qquad \partial_z w = f_2(r,\varphi) \quad \text{at} \quad z = 0.$$

Solution:

$$\begin{aligned} w(r,\varphi,z) = {} & -R\int_0^{2\pi}\int_0^{\infty} f_1(\eta,\zeta)\left[\frac{\partial}{\partial\xi}G(r,\varphi,z,\xi,\eta,\zeta)\right]_{\xi=R} d\zeta\,d\eta \\ & - \int_0^{2\pi}\int_0^{R} f_2(\xi,\eta)G(r,\varphi,z,\xi,\eta,0)\xi\,d\xi\,d\eta \\ & + \int_0^{R}\int_0^{2\pi}\int_0^{\infty} \Phi(\xi,\eta,\zeta)G(r,\varphi,z,\xi,\eta,\zeta)\xi\,d\zeta\,d\eta\,d\xi. \end{aligned}$$

Here,

$$G(r,\varphi,z,\xi,\eta,\zeta) = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_n J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{\left[J_n'(\mu_{nm}R)\right]^2\beta_{nm}} \cos[n(\varphi-\eta)]F_{nm}(z,\zeta),$$

$$F_{nm}(z,\zeta) = \exp(-\beta_{nm}|z-\zeta|) + \exp(-\beta_{nm}|z+\zeta|),$$

$$\beta_{nm} = \sqrt{\mu_{nm}^2 - \lambda}, \qquad A_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0, \end{cases}$$

where the $J_n(\xi)$ are Bessel functions and the μ_{nm} are positive roots of the transcendental equation $J_n(\mu R) = 0$.

◆ *Only the eigenvalues and eigenfunctions of homogeneous boundary value problems for the homogeneous Helmholtz equation (with $\Phi \equiv 0$) are presented below. The solutions of the corresponding nonhomogeneous boundary value problems (with $\Phi \not\equiv 0$) can be constructed by the relations specified in Section 10.3.2.*

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. First boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R, \qquad w = 0 \quad \text{at} \quad z = 0, \qquad w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \frac{\mu_{nm}^2}{R^2}; \qquad n = 0,\ 1,\ \dots; \quad m,\ k = 1,\ 2,\ \dots$$

Here, the μ_{nm} are positive zeros of the Bessel functions, $J_n(\mu) = 0$.

Eigenfunctions:

$$w_{nmk}^{(1)} = J_n\left(\mu_{nm}\frac{r}{R}\right)\cos(n\varphi)\sin\left(\frac{\pi k z}{a}\right),$$
$$w_{nmk}^{(2)} = J_n\left(\mu_{nm}\frac{r}{R}\right)\sin(n\varphi)\sin\left(\frac{\pi k z}{a}\right).$$

Eigenfunctions possessing the axial symmetry property:

$$w_{0mk}^{(1)} = J_0\left(\mu_{0m}\frac{r}{R}\right)\sin\left(\frac{\pi k z}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}^{(1)}\|^2 = \|w_{nmk}^{(2)}\|^2 = \frac{\pi R^2 a}{4}(1+\delta_{n0})\bigl[J_n'(\mu_{nm})\bigr]^2, \qquad \delta_{nm} = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{for } n \ne m. \end{cases}$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. Second boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r = R, \qquad \partial_z w = 0 \quad \text{at} \quad z = 0, \qquad \partial_z w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{000} = 0, \quad \lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \frac{\mu_{nm}^2}{R^2}; \qquad n = 0,\ 1,\ \dots; \quad k,\ m = 0,\ 1,\ \dots$$

Here, the μ_{nm} are roots of the quadratic equation $J_n'(\mu) = 0$.

Eigenfunctions:

$$w_{nmk}^{(1)} = J_n\left(\mu_{nm}\frac{r}{R}\right)\cos(n\varphi)\cos\left(\frac{\pi k z}{a}\right), \qquad w_{000}^{(1)} = 1,$$
$$w_{nmk}^{(2)} = J_n\left(\mu_{nm}\frac{r}{R}\right)\sin(n\varphi)\cos\left(\frac{\pi k z}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2 = \frac{\pi R^2 a}{4\mu^2_{nm}}(1+\delta_{n0})(\mu^2_{nm} - n^2)\big[J_n(\mu_{nm})\big]^2, \quad \|w^{(1)}_{000}\|^2 = \pi R^2 a,$$

where δ_{n0} is the Kronecker delta.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. Third boundary value problem.**

A circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$\partial_r w + k_1 w = 0 \quad \text{at} \quad r = R, \qquad \partial_z w - k_2 w = 0 \quad \text{at} \quad z = 0, \qquad \partial_z w + k_3 w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nml} = \nu_l^2 + \frac{\mu^2_{nm}}{R^2},$$

where the ν_l and μ_{nm} are positive roots of the transcendental equations

$$\tan(\nu a) = \frac{(k_2 + k_3)\nu}{\nu^2 - k_2 k_3}, \qquad \mu J'_n(\mu) + R k_1 J_n(\mu) = 0.$$

Eigenfunctions:

$$w^{(1)}_{nml} = J_n\left(\mu_{nm}\frac{r}{R}\right)\cos(n\varphi)\,\frac{\nu_l \cos \nu_l z + k_2 \sin \nu_l z}{\sqrt{\nu_l^2 + k_2^2}},$$

$$w^{(2)}_{nml} = J_n\left(\mu_{nm}\frac{r}{R}\right)\sin(n\varphi)\,\frac{\nu_l \cos \nu_l z + k_2 \sin \nu_l z}{\sqrt{\nu_l^2 + k_2^2}}.$$

The square of the norm of an eigenfunction is defined as

$$\|w^{(i)}_{nml}\|^2 = \frac{\pi R^2}{4\mu^2_{nm}}(1+\delta_{n0})(R^2 k_1^2 + \mu^2_{nm} - n^2)\big[J_n(\mu_{nm})\big]^2\left[a + \frac{(k_2+k_3)(\nu_l^2 + k_2 k_3)}{(\nu_l^2 + k_2^2)(\nu_l^2 + k_3^2)}\right],$$

where δ_{n0} is the Kronecker delta.

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. First boundary value problem.**

A hollow circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R_1, \qquad w = 0 \quad \text{at} \quad r = R_2,$$
$$w = 0 \quad \text{at} \quad z = 0, \qquad w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \mu^2_{nm}; \qquad n = 0,\,1,\,2,\,\dots\,; \quad m,\,k = 1,\,2,\,3,\,\dots$$

Here, the μ_{nm} are positive roots of the transcendental equation

$$J_n(\mu R_1)Y_n(\mu R_2) - J_n(\mu R_2)Y_n(\mu R_1) = 0.$$

Eigenfunctions:

$$w^{(1)}_{nmk} = [J_n(\mu_{nm}r)Y_n(\mu_{nm}R_1) - J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\cos(n\varphi)\sin\left(\frac{\pi kz}{a}\right),$$
$$w^{(2)}_{nmk} = [J_n(\mu_{nm}r)Y_n(\mu_{nm}R_1) - J_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\sin(n\varphi)\sin\left(\frac{\pi kz}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2 = \frac{a}{\pi\mu^2_{nm}}(1+\delta_{n0})\frac{\big[J_n(\mu_{nm}R_1)\big]^2 - \big[J_n(\mu_{nm}R_2)\big]^2}{\big[J_n(\mu_{nm}R_2)\big]^2}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, $0 \le z \le a$. Second boundary value problem.**

A hollow circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r = R_1, \qquad \partial_r w = 0 \quad \text{at} \quad r = R_2,$$
$$\partial_z w = 0 \quad \text{at} \quad z = 0, \qquad \partial_z w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \mu^2_{nm}; \qquad n,\ m,\ k = 0,\ 1,\ 2,\ \ldots$$

Here, the μ_{nm} are roots of the quadratic equation

$$J'_n(\mu R_1)Y'_n(\mu R_2) - J'_n(\mu R_2)Y'_n(\mu R_1) = 0.$$

Eigenfunctions:

$$w^{(1)}_{nmk} = [J_n(\mu_{nm}r)Y'_n(\mu_{nm}R_1) - J'_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\cos(n\varphi)\cos\left(\frac{\pi kz}{a}\right),$$
$$w^{(2)}_{nmk} = [J_n(\mu_{nm}r)Y'_n(\mu_{nm}R_1) - J'_n(\mu_{nm}R_1)Y_n(\mu_{nm}r)]\sin(n\varphi)\cos\left(\frac{\pi kz}{a}\right).$$

To the zero eigenvalue $\lambda_{000} = 0$ there is a corresponding eigenfunction $w^{(1)}_{000} = 1$.

The square of the norm of an eigenfunction is defined as

$$\|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2 = \frac{a(1+\delta_{n0})(1+\delta_{k0})}{\pi\mu^2_{nm}}\left\{\left(1-\frac{n^2}{R_2^2\mu^2_{nm}}\right)\left[\frac{J'_n(\mu_{nm}R_1)}{J'_n(\mu_{nm}R_2)}\right]^2 - \left(1-\frac{n^2}{R_1^2\mu^2_{nm}}\right)\right\},$$

where δ_{n0} is the Kronecker delta.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Domain: $R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le a$. Mixed boundary value problems.**

1°. A hollow circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R_1, \qquad w = 0 \quad \text{at} \quad r = R_2,$$
$$\partial_z w = 0 \quad \text{at} \quad z = 0, \qquad \partial_z w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \mu_{nm}^2; \qquad n,\ k = 0,\ 1,\ 2,\ \dots; \quad m = 1,\ 2,\ 3,\ \dots$$

Here, the μ_{nm} are roots of the quadratic equation

$$J_n(\mu R_1) Y_n(\mu R_2) - J_n(\mu R_2) Y_n(\mu R_1) = 0.$$

Eigenfunctions:

$$w_{nmk}^{(1)} = [J_n(\mu_{nm} r) Y_n(\mu_{nm} R_1) - J_n(\mu_{nm} R_1) Y_n(\mu_{nm} r)] \cos(n\varphi) \cos\left(\frac{\pi k z}{a}\right),$$
$$w_{nmk}^{(2)} = [J_n(\mu_{nm} r) Y_n(\mu_{nm} R_1) - J_n(\mu_{nm} R_1) Y_n(\mu_{nm} r)] \sin(n\varphi) \cos\left(\frac{\pi k z}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}^{(1)}\|^2 = \|w_{nmk}^{(2)}\|^2 = \frac{a\varepsilon_n \varepsilon_k}{\pi \mu_{nm}^2} \frac{[J_n(\mu_{nm} R_1)]^2 - [J_n(\mu_{nm} R_2)]^2}{[J_n(\mu_{nm} R_2)]^2}, \qquad \varepsilon_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n \neq 0. \end{cases}$$

2°. A hollow circular cylinder of finite length is considered. Boundary conditions are prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r = R_1, \qquad \partial_r w = 0 \quad \text{at} \quad r = R_2,$$
$$w = 0 \quad \text{at} \quad z = 0, \qquad w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \mu_{nm}^2; \qquad n = 0,\ 1,\ 2,\ \dots; \quad m,\ k = 1,\ 2,\ 3,\ \dots$$

Here, the μ_{nm} are roots of the quadratic equation

$$J_n'(\mu R_1) Y_n'(\mu R_2) - J_n'(\mu R_2) Y_n'(\mu R_1) = 0.$$

Eigenfunctions:

$$w_{nmk}^{(1)} = [J_n(\mu_{nm} r) Y_n'(\mu_{nm} R_1) - J_n'(\mu_{nm} R_1) Y_n(\mu_{nm} r)] \cos(n\varphi) \sin\left(\frac{\pi k z}{a}\right),$$
$$w_{nmk}^{(2)} = [J_n(\mu_{nm} r) Y_n'(\mu_{nm} R_1) - J_n'(\mu_{nm} R_1) Y_n(\mu_{nm} r)] \sin(n\varphi) \sin\left(\frac{\pi k z}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}^{(1)}\|^2 = \|w_{nmk}^{(2)}\|^2 = \frac{a\varepsilon_n}{\pi \mu_{nm}^2} \left\{ \left(1 - \frac{n^2}{R_2^2 \mu_{nm}^2}\right) \left[\frac{J_n'(\mu_{nm} R_1)}{J_n'(\mu_{nm} R_2)}\right]^2 - \left(1 - \frac{n^2}{R_1^2 \mu_{nm}^2}\right) \right\},$$

where ε_n is defined in Item 1°.

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le a$. First boundary value problem.**

A cylindrical sector of finite thickness is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad \varphi = 0, \qquad w = 0 \quad \text{at} \quad \varphi = \varphi_0, \qquad w = 0 \quad \text{at} \quad r = R,$$
$$w = 0 \quad \text{at} \quad z = 0, \qquad w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \frac{\mu_{nm}^2}{R^2}; \qquad n,\, m,\, k = 1,\, 2,\, 3,\, \ldots$$

Here, the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu) = 0$.

Eigenfunctions:

$$w_{nmk} = J_{n\pi/\varphi_0}\left(\frac{\mu_{nm} r}{R}\right) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{k\pi z}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}\|^2 = \tfrac{1}{8} a R^2 \varphi_0 \big[J'_{n\pi/\varphi_0}(\mu_{nm})\big]^2.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le \varphi_0$, $0 \le z \le a$. Mixed boundary value problem.**

A cylindrical sector of finite thickness is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad \varphi = 0, \qquad w = 0 \quad \text{at} \quad \varphi = \varphi_0, \qquad w = 0 \quad \text{at} \quad r = R,$$
$$\partial_z w = 0 \quad \text{at} \quad z = 0, \qquad \partial_z w = 0 \quad \text{at} \quad z = a.$$

Eigenvalues:

$$\lambda_{nmk} = \frac{\pi^2 k^2}{a^2} + \frac{\mu_{nm}^2}{R^2}; \qquad n,\, m = 1,\, 2,\, 3,\, \ldots; \quad k = 0,\, 1,\, 2,\, \ldots$$

Here, the μ_{nm} are positive roots of the transcendental equation $J_{n\pi/\varphi_0}(\mu) = 0$.

Eigenfunctions:

$$w_{nmk} = J_{n\pi/\varphi_0}\left(\frac{\mu_{nm} r}{R}\right) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{k\pi z}{a}\right).$$

The square of the norm of an eigenfunction is defined as

$$\|w_{nmk}\|^2 = \tfrac{1}{8} a R^2 \varphi_0 (1 + \delta_{k0}) \big[J'_{n\pi/\varphi_0}(\mu_{nm})\big]^2, \qquad \delta_{k0} = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

10.3.5 Problems in Spherical Coordinates

The three-dimensional homogeneous Helmholtz equation in the spherical coordinate system is written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial w}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial w}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2} + \lambda w = 0, \qquad r = \sqrt{x^2 + y^2 + z^2}.$$

► **Particular solutions:**

$$w = \frac{1}{r}(A \sin \mu r + B \cos \mu r), \qquad \lambda = \mu^2,$$
$$w = \frac{1}{r}(A \sinh \mu r + B \cosh \mu r), \qquad \lambda = -\mu^2,$$
$$w = \frac{1}{\sqrt{r}} J_{n+1/2}(\mu r) P_n^m(\cos\theta)(A \cos m\varphi + B \sin m\varphi), \quad \lambda = \mu^2,$$
$$w = \frac{1}{\sqrt{r}} Y_{n+1/2}(\mu r) P_n^m(\cos\theta)(A \cos m\varphi + B \sin m\varphi), \quad \lambda = \mu^2,$$
$$w = \frac{1}{\sqrt{r}} I_{n+1/2}(\mu r) P_n^m(\cos\theta)(A \cos m\varphi + B \sin m\varphi), \quad \lambda = -\mu^2,$$
$$w = \frac{1}{\sqrt{r}} K_{n+1/2}(\mu r) P_n^m(\cos\theta)(A \cos m\varphi + B \sin m\varphi), \quad \lambda = -\mu^2,$$

where n, $m = 0, 1, 2, \ldots;$ A and B are arbitrary constants; $J_\nu(\xi)$ and $Y_\nu(\xi)$ are Bessel functions; $I_\nu(\xi)$ and $K_\nu(\xi)$ are modified Bessel functions; and the $P_n^m(\xi)$ are associated Legendre functions, which are expressed in terms of the Legendre polynomials $P_n(\xi)$ as

$$P_n^m(\xi) = (1-\xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi), \qquad P_n(\xi) = \frac{1}{n!\,2^n} \frac{d^n}{d\xi^n}(\xi^2-1)^n.$$

► **Domain: $0 \le r \le R$. First boundary value problem.**

1°. A spherical domain is considered. A homogeneous boundary condition is prescribed,

$$w = 0 \quad \text{at} \quad r = R.$$

Eigenvalues:

$$\lambda_{nk} = \frac{\mu_{nk}^2}{R^2}; \qquad n = 0, 1, 2, \ldots; \quad k = 1, 2, 3, \ldots$$

Here, the μ_{nk} are positive zeros of the Bessel functions, $J_{n+1/2}(\mu) = 0$. Note that the $J_{n+1/2}(\mu)$ can be expressed in terms of elementary functions, see Bateman and Erdélyi (1953, Vol. 2).

Eigenfunctions:

$$w_{nmk}^{(1)} = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right) P_n^m(\cos\theta) \cos m\varphi, \qquad m = 0, 1, 2, \ldots;$$
$$w_{nmk}^{(2)} = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right) P_n^m(\cos\theta) \sin m\varphi, \qquad m = 1, 2, 3, \ldots$$

Here, the $P_n^m(\xi)$ are associated Legendre functions.

Eigenfunctions possessing central symmetry (i.e., independent of θ and φ):

$$w_{00k}^{(1)} = J_{1/2}\left(\mu_{0k}\frac{r}{R}\right).$$

Eigenfunctions possessing axial symmetry (i.e., independent of φ):

$$w^{(1)}_{n0k} = J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right)P_n(\cos\theta).$$

The square of the norm of an eigenfunction:

$$\|w^{(1)}_{nmk}\|^2 = \frac{\pi R^2(1+\delta_{m0})(n+m)!}{(2n+1)(n-m)!}\left[J'_{n+1/2}(\mu_{nk})\right]^2, \qquad \delta_{m0} = \begin{cases} 1 & \text{for } m=0,\\ 0 & \text{for } m\neq 0,\end{cases}$$

$$\|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2, \qquad m=1,\,2,\,3,\,\ldots$$

2°. A spherical domain is considered. A nonhomogeneous boundary condition is prescribed,

$$w = f(\theta,\varphi) \quad \text{at} \quad r=R.$$

Solution:

$$w(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}\frac{\Psi_n(r\sqrt{\lambda}\,)}{\Psi_n(R\sqrt{\lambda}\,)}Y^m_n(\theta,\varphi), \qquad \Psi_n(x) = \frac{1}{\sqrt{x}}J_{n+1/2}(x),$$

where

$$f_{nm} = \frac{1}{\|Y^m_n\|}\int_0^{2\pi}\!\!\int_0^{\pi} f(\theta,\varphi)Y^m_n(\theta,\varphi)\sin\theta\,d\theta\,d\varphi, \qquad \|Y^m_n\| = \frac{2\pi\varepsilon_m}{2n+1}\frac{(n+m)!}{(n-m)!},$$

$$Y^m_n(\theta,\varphi) = \begin{cases} P_n(\cos\theta) & \text{for } m=0,\\ P^m_n(\cos\theta)\sin m\varphi & \text{for } m=1,\,2,\,\ldots,\\ P^{|m|}_n(\cos\theta)\cos m\varphi & \text{for } m=-1,\,-2,\,\ldots,\end{cases} \qquad \varepsilon_m = \begin{cases} 2 & \text{for } m=0,\\ 1 & \text{for } m\neq 0.\end{cases}$$

The solution was written out under the assumption that $J_{n+1/2}(R\sqrt{\lambda}\,)\neq 0$, where $n = 0,\,1,\,2,\,\ldots$

⊙ *Literature*: M. M. Smirnov (1975), A. N. Tikhonov and A. A. Samarskii (1990).

◆ *Only the eigenvalues and eigenfunctions of homogeneous boundary value problems for the homogeneous Helmholtz equation (with* $\Phi\equiv 0$*) are presented below. The solutions of the corresponding nonhomogeneous boundary value problems (with* $\Phi\not\equiv 0$*) can be constructed by the relations specified in Section 10.3.2.*

► **Domain: $0\le r\le R$. Second boundary value problem.**

A spherical domain is considered. A boundary condition is prescribed:

$$\partial_r w = 0 \quad \text{at} \quad r=R.$$

Eigenvalues:

$$\lambda_{00}=0, \quad \lambda_{nk} = \frac{\mu^2_{nk}}{R^2}; \qquad n=0,\,1,\,2,\,\ldots; \quad k=1,\,2,\,3,\,\ldots$$

Here, the μ_{nk} are roots of the quadratic equation

$$2\mu J'_{n+1/2}(\mu) - J_{n+1/2}(\mu) = 0.$$

Eigenfunctions:

$$w^{(1)}_{000} = 1, \quad w^{(1)}_{nmk} = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right) P^m_n(\cos\theta)\cos m\varphi, \qquad m = 0,\,1,\,2,\,\dots\,;$$

$$w^{(2)}_{nmk} = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right) P^m_n(\cos\theta)\sin m\varphi, \qquad m = 1,\,2,\,3,\,\dots$$

The square of the norm of an eigenfunction:

$$\|w^{(1)}_{000}\|^2 = \tfrac{4}{3}\pi R^3, \quad \|w^{(1)}_{nmk}\|^2 = \frac{\pi R^2 \varepsilon_m (n+m)!}{(2n+1)(n-m)!}\left[1 - \frac{n(n+1)}{\mu^2_{nk}}\right] J^2_{n+1/2}(\mu_{nk}),$$

$$\|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2, \qquad m = 1,\,2,\,3,\,\dots,$$

where $\varepsilon_m = \begin{cases} 2 & \text{for } m = 0, \\ 1 & \text{for } m \neq 0. \end{cases}$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► Domain: $0 \le r \le R$. Third boundary value problem.

A spherical domain is considered. A boundary condition is prescribed:

$$\partial_r w + sw = 0 \quad \text{at} \quad r = R.$$

Eigenvalues:

$$\lambda_{nk} = \frac{\mu^2_{nk}}{R^2}; \qquad n = 0,\,1,\,2,\,\dots\,; \quad k = 1,\,2,\,3,\,\dots$$

Here, the μ_{nk} are positive roots of the transcendental equation

$$2\mu J'_{n+1/2}(\mu) - (1 - 2Rs) J_{n+1/2}(\mu) = 0.$$

Eigenfunctions:

$$w^{(1)}_{nmk} = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right) P^m_n(\cos\theta)\cos m\varphi, \qquad m = 0,\,1,\,2,\,\dots\,;$$

$$w^{(2)}_{nmk} = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\mu_{nk}\frac{r}{R}\right) P^m_n(\cos\theta)\sin m\varphi, \qquad m = 1,\,2,\,3,\,\dots$$

Here, the $P^m_n(\xi)$ are associated Legendre functions.

The square of the norm of an eigenfunction:

$$\|w^{(1)}_{nmk}\|^2 = \frac{\pi R^2 \varepsilon_m (n+m)!}{(2n+1)(n-m)!}\left[1 + \frac{(Rs+n)(Rs-n-1)}{\mu^2_{nk}}\right] J^2_{n+1/2}(\mu_{nk}),$$

$$\varepsilon_m = \begin{cases} 2 & \text{for } m = 0, \\ 1 & \text{for } m \neq 0, \end{cases} \quad \|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2, \quad m = 1,\,2,\,3,\,\dots$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

► **Domain: $R \le r < \infty$. First boundary value problem.**

A spherical cavity is considered and the dependent variable is prescribed at its surface:

$$w = f(\theta,\varphi) \quad \text{at} \quad r = R,$$

and the radiation conditions are prescribed at infinity (see Section 10.3.2, condition (6)).

Solution for $\lambda = k^2 > 0$:

$$w(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}\frac{\Xi_n(kr)}{\Xi_n(kR)}Y_n^m(\theta,\varphi), \qquad \Xi_n(\rho) = \frac{1}{\sqrt{\rho}}H^{(2)}_{n+1/2}(\rho),$$

$$f_{nm} = \frac{1}{\|Y_n^m\|}\int_0^{2\pi}\!\!\int_0^{\pi} f(\theta,\varphi)Y_n^m(\theta,\varphi)\sin\theta\,d\theta\,d\varphi, \qquad \|Y_n^m\| = \frac{2\pi\varepsilon_m}{2n+1}\frac{(n+m)!}{(n-m)!},$$

$$Y_n^m(\theta,\varphi) = \begin{cases} P_n(\cos\theta) & \text{for } m = 0,\\ P_n^m(\cos\theta)\sin m\varphi & \text{for } m = 1,\,2,\,\dots,\\ P_n^{|m|}(\cos\theta)\cos m\varphi & \text{for } m = -1,\,-2,\,\dots, \end{cases} \qquad \varepsilon_m = \begin{cases} 2 & \text{for } m = 0,\\ 1 & \text{for } m \neq 0, \end{cases}$$

where $H^{(2)}_{n+1/2}(\rho)$ is the Hankel function of the second kind.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► **Domain: $R_1 \le r \le R_2$. First boundary value problem.**

A spherical layer is considered. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = R_1, \qquad w = 0 \quad \text{at} \quad r = R_2.$$

Eigenvalues:

$$\lambda_{nk} = \mu_{nk}^2; \qquad n = 0,\,1,\,2,\,\dots\,; \quad k = 1,\,2,\,3,\,\dots$$

Here, the μ_{nk} are positive roots of the transcendental equation

$$J_{n+1/2}(\mu R_1)Y_{n+1/2}(\mu R_2) - J_{n+1/2}(\mu R_2)Y_{n+1/2}(\mu R_1) = 0.$$

Eigenfunctions:

$$w^{(1)}_{nmk} = \frac{1}{\sqrt{r}}Z_{n+1/2}(\mu_{nk}r)P_n^m(\cos\theta)\cos m\varphi, \qquad m = 0,\,1,\,2,\,\dots\,;$$
$$w^{(2)}_{nmk} = \frac{1}{\sqrt{r}}Z_{n+1/2}(\mu_{nk}r)P_n^m(\cos\theta)\sin m\varphi, \qquad m = 1,\,2,\,3,\,\dots$$

Here, the $P_n^m(\xi)$ are associated Legendre functions and

$$Z_{n+1/2}(\mu r) = J_{n+1/2}(\mu R_1)Y_{n+1/2}(\mu r) - Y_{n+1/2}(\mu R_1)J_{n+1/2}(\mu r).$$

The square of the norm of an eigenfunction:

$$\|w^{(1)}_{nmk}\|^2 = \frac{4\varepsilon_m(n+m)!}{\pi(2n+1)(n-m)!}\,\frac{J^2_{n+1/2}(\mu_{nk}R_1) - J^2_{n+1/2}(\mu_{nk}R_2)}{\mu_{nk}^2 J^2_{n+1/2}(\mu_{nk}R_2)}, \qquad \varepsilon_m = \begin{cases} 2 & \text{if } m=0,\\ 1 & \text{if } m\neq 0, \end{cases}$$

$$\|w^{(1)}_{nmk}\|^2 = \|w^{(2)}_{nmk}\|^2, \qquad m = 1,\,2,\,3,\,\dots$$

10.3.6 Other Orthogonal Curvilinear Coordinates

The homogenous three-dimensional Helmholtz equation admits separation of variables in the eleven orthogonal systems of coordinates listed in Table 10.4.

For the parabolic cylindrical system of coordinates, the multipliers f and g are expressed in terms of the parabolic cylinder functions as

$$f(\xi)=A_1 D_{\mu-1/2}(\sigma\xi)+A_2 D_{\mu-1/2}(-\sigma\xi), \quad g(\eta)=B_1 D_{-\mu-1/2}(\sigma\eta)+B_2 D_{-\mu-1/2}(-\sigma\eta),$$
$$\mu=\tfrac{1}{2}\beta(k^2-\lambda)^{-1/2}, \quad \sigma=\bigl[4(k^2-\lambda)\bigr]^{1/4},$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

For the elliptic cylindrical system of coordinates, the functions f and g are determined by the modified Mathieu equation and Mathieu equation, respectively, so that

$$f(u)=\begin{cases}\mathrm{Ce}_n(u,q),\\ \mathrm{Se}_n(u,q),\end{cases} \qquad g(v)=\begin{cases}\mathrm{ce}_n(v,q),\\ \mathrm{se}_n(v,q),\end{cases} \qquad q=\tfrac{1}{4}a^2(\lambda-k^2),$$

where $\mathrm{Ce}_n(u,q)$ and $\mathrm{Se}_n(u,q)$ are the modified Mathieu functions, and $\mathrm{ce}_n(v,q)$ and $\mathrm{se}_n(v,q)$ are the Mathieu functions; to each value of the parameter q there are certain corresponding eigenvalues $\beta=\beta_n(q)$ [see Abramowitz and Stegun (1964)].

In the prolate and oblate spheroidal systems of coordinates, the equations for f and g are different forms of the spheroidal wave equation, whose bounded solutions are given by

$$f(u)=\mathrm{Ps}_n^{|k|}(\cosh u, a^2\lambda), \qquad g(u)=\mathrm{Ps}_n^{|k|}(\cos v, a^2\lambda) \qquad \text{for prolate spheroid,}$$
$$f(u)=\mathrm{Ps}_n^{|k|}(-i\sinh u, a^2\lambda), \quad g(u)=\mathrm{Ps}_n^{|k|}(\cos v, -a^2\lambda) \qquad \text{for oblate spheroid,}$$
$$k \text{ is an integer}, \quad n=0,\,1,\,2,\,\ldots, \quad -n\le k\le n,$$

where $\mathrm{Ps}_n^k(z,a)$ are the spheroidal wave functions; see Bateman and Erdélyi (1955, Vol. 3), Arscott (1964), and Meixner and Schäfke (1965). The separation of variables for the Helmholtz equation in modified prolate and oblate spheroidal systems of coordinates, as well as the spheroidal wave functions, are discussed in Abramowitz and Stegun (1964).

In the parabolic coordinate system, the solutions of the equations for f and g are expressed in terms of the degenerate hypergeometric functions [see Miller, Jr. (1977)] as follows:

$$f(\xi)=\xi^k\exp\bigl(\pm\tfrac{1}{2}\omega\xi^2\bigr)\Phi\biggl(-\frac{\beta}{4\omega}+\frac{k+1}{2},\,k+1;\,\mp\omega\xi^2\biggr), \quad \omega=\sqrt{-\lambda},$$
$$g(\eta)=\eta^k\exp\bigl(\pm\tfrac{1}{2}\omega\eta^2\bigr)\Phi\biggl(\frac{\beta}{4\omega}+\frac{k+1}{2},\,k+1;\,\mp\omega\eta^2\biggr).$$

In the case of the paraboloidal coordinate system, the equations for f, g, and h are reduced to the Whittaker–Hill equation

$$G''_{\theta\theta}+\bigl(\mu+\tfrac{1}{8}b^2+bc\cos 2\theta-\tfrac{1}{8}b^2\cos 4\theta\bigr)G=0.$$

Denote by $\mathrm{gc}_n(\theta;b,c)$ and $\mathrm{gs}_n(\theta;b,c)$, respectively, the even and odd 2π-periodic solutions of the Whittaker–Hill equation, which is a generalization of the Mathieu equation. The

TABLE 10.4
Orthogonal coordinates $\bar{x}, \bar{y}, \bar{z}$ that allow separable solutions of the form $w = f(\bar{x})g(\bar{y})h(\bar{z})$ for the three-dimensional Helmholtz equation $\Delta_3 w + \lambda w = 0$

Coordinates	Transformations	Particular solutions (or equations for f, g, h)
Cartesian x, y, z	$x = x$, $y = y$, $z = z$	$w = \cos(k_1 x + s_1)\cos(k_2 y + s_2)\cos(k_3 z + s_3)$, where $k_1^2 + k_2^2 + k_3^2 = \lambda$; see also Section 10.3.3 (particular solutions)
Cylindrical r, φ, z	$x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$	$w = [AJ_n(\beta r) + BY_n(\beta r)]\cos(n\varphi + c)\cos(kz + s)$, where J_n and Y_n are the Bessel functions, $k^2 + \beta^2 = \lambda$; see also Section 10.3.4
Parabolic cylindrical ξ, η, z	$x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$, $z = z$	$w = f(\xi)g(\eta)\cos(kz + s)$, $f'' + [(\lambda - k^2)\xi^2 + \beta]f = 0$, $g'' + [(\lambda - k^2)\eta^2 - \beta]g = 0$
Elliptic cylindrical u, v, z	$x = a\cosh u\cos v$, $y = a\sinh u\sin v$, $z = z$	$w = f(u)g(v)\cos(kz + s)$, $f'' + [\frac{1}{2}a^2(\lambda - k^2)\cosh 2u - \beta]f = 0$, $g'' - [\frac{1}{2}a^2(\lambda - k^2)\cos 2v - \beta]g = 0$
Spherical r, θ, φ	$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$	$w = r^{-1/2}J_{n+1/2}(\beta r)P_n^m(\cos\theta)\cos(m\varphi + s)$, $w = r^{-1/2}Y_{n+1/2}(\beta r)P_n^m(\cos\theta)\cos(m\varphi + s)$, where $\lambda = \beta^2$; see also Section 10.3.5
Prolate spheroidal u, v, φ	$x = a\sinh u\sin v\cos\varphi$, $y = a\sinh u\sin v\sin\varphi$, $z = a\cosh u\cos v$	$w = f(u)g(v)\cos(k\varphi + s)$, $f'' + f'\coth u + (-\beta + a^2\lambda\sinh^2 u - k^2/\sinh^2 u)f = 0$, $g'' + g'\cot v + (\beta + a^2\lambda\sin^2 v - k^2/\sin^2 v)g = 0$
Oblate spheroidal u, v, φ	$x = a\cosh u\sin v\cos\varphi$, $y = a\cosh u\sin v\sin\varphi$, $z = a\sinh u\cos v$	$w = f(u)g(v)\cos(k\varphi + s)$, $f'' + f'\tanh u + (-\beta + a^2\lambda\cosh^2 u + k^2/\cosh^2 u)f = 0$, $g'' + g'\cot v + (\beta - a^2\lambda\sin^2 v - k^2/\sin^2 v)g = 0$
Parabolic ξ, η, φ	$x = \xi\eta\cos\varphi$, $y = \xi\eta\sin\varphi$, $z = \frac{1}{2}(\xi^2 - \eta^2)$	$w = f(\xi)g(\eta)\cos(k\varphi + s)$, $\xi^2 f'' + \xi f' + (\lambda\xi^4 - \beta\xi^2 - k^2)f = 0$, $\eta^2 g'' + \eta g' + (\lambda\eta^4 + \beta\eta^2 - k^2)g = 0$
Paraboloidal u, v, φ	$x = 2a\cosh u\cos v\sinh\varphi$, $y = 2a\sinh u\sin v\cosh\varphi$, $z = \frac{1}{2}a(\cosh 2u + \cos 2v - \cosh 2\varphi)$	$f'' + (-k - a\beta\cosh 2u + \frac{1}{2}a^2\lambda\cosh 4u)f = 0$, $g'' + (k + a\beta\cos 2v - \frac{1}{2}a^2\lambda\cos 4v)g = 0$, $h'' + (-k + a\beta\cosh 2\varphi - \frac{1}{2}a^2\lambda\cosh 4\varphi)h = 0$
General ellipsoidal μ, ν, ρ	$x = \sqrt{\frac{(\mu-a)(\nu-a)(\rho-a)}{a(a-1)}}$, $y = \sqrt{\frac{(\mu-1)(\nu-1)(\rho-1)}{1-a}}$, $z = \sqrt{\frac{\mu\nu\rho}{a}}$	$4\sqrt{\varphi(\mu)}\,[\sqrt{\varphi(\mu)}\,f']' + (\lambda\mu^2 + \beta_1\mu + \beta_2)f = 0$, $4\sqrt{\varphi(\nu)}\,[\sqrt{\varphi(\nu)}\,g']' + (\lambda\nu^2 + \beta_1\nu + \beta_2)g = 0$, $4\sqrt{\varphi(\rho)}\,[\sqrt{\varphi(\rho)}\,h']' + (\lambda\rho^2 + \beta_1\rho + \beta_2)h = 0$, $\varphi(t) = t(t-1)(t-a)$
Conical ρ, μ, ν	$x = \rho\sqrt{\frac{(a\mu-1)(a\nu-1)}{1-a}}$, $y = \rho\sqrt{\frac{a(\mu-1)(\nu-1)}{a-1}}$, $z = \rho\sqrt{a\mu\nu}$	$w = \rho^{-1/2}J_{\pm(n+1/2)}(\rho\sqrt{\lambda})g(\xi)h(\eta)$, $g'' + [\beta - n(n+1)k^2\,\mathrm{sn}^2\,\xi]g = 0$, $h'' + [\beta - n(n+1)k^2\,\mathrm{sn}^2\,\eta]h = 0$, where $\mu = \mathrm{sn}^2(\xi, k)$, $\nu = \mathrm{sn}^2(\eta, k)$, $k = \sqrt{a}$

subscript $n = 0, 1, 2, \ldots$ labels the discrete eigenvalues $\mu = \mu_n$. Each of the solutions gc_n and gs_n can be represented in the form of an infinite convergent trigonometric series in $\cos n\theta$ and $\sin n\theta$, respectively; see Urvin and Arscott (1970). The functions f, g, and h can be expressed in terms of the periodic solutions of the Whittaker–Hill equation as follows [Miller, Jr. (1977)]:

$$f(u) = \begin{cases} \mathrm{gc}_n\bigl(iu; 2a\omega, \frac{1}{2}\beta/\omega\bigr), \\ \mathrm{gs}_n\bigl(iu; 2a\omega, \frac{1}{2}\beta/\omega\bigr), \end{cases} \qquad g(v) = \begin{cases} \mathrm{gc}_n\bigl(v; 2a\omega, \frac{1}{2}\beta/\omega\bigr), \\ \mathrm{gs}_n\bigl(v; 2a\omega, \frac{1}{2}\beta/\omega\bigr), \end{cases}$$

$$h(\varphi) = \begin{cases} \mathrm{gc}_n\bigl(i\varphi + \frac{\pi}{2}; 2a\omega, \frac{1}{2}\beta/\omega\bigr), \\ \mathrm{gs}_n\bigl(i\varphi + \frac{\pi}{2}; 2a\omega, \frac{1}{2}\beta/\omega\bigr), \end{cases}$$

where $\omega = \sqrt{\lambda}$ and $k = \mu_n - \frac{1}{2}a^2\lambda$.

For the general ellipsoidal coordinates, the functions f, g, and h are expressed in terms of the ellipsoidal wave functions; for details, see Arscott (1964) and Miller, Jr. (1977).

For the conical coordinate system, the functions g and h are determined by the Lamé equations that involve the Jacobian elliptic function $\operatorname{sn} z = \operatorname{sn}(z, k)$.

The unambiguity conditions for the transformation yield $n = 0, 1, 2, \ldots$ It is known that, for any positive integer n, there exist exactly $2n+1$ solutions corresponding to $2n+1$ different eigenvalues β. These solutions can be represented the form of finite series known as Lamé polynomials. For more details about the Lamé equation and its solutions, see Whittaker and Watson (1963), Arscott (1964), Bateman and Erdélyi (1955), and Miller, Jr. (1977).

Unlike the Laplace equation, there are no nontrivial transformations for the three-dimensional Helmholtz equation that allow the $\mathcal{R}$-separation of variables.

⊙ *Literature for Section* 10.3.5: F. M. Morse and H. Feshbach (1953, Vols. 1–2), P. Moon and D. Spencer (1988), A. Makarov, J. Smorodinsky, K. Valiev, and P. Winternitz (1967), W. Miller, Jr. (1977).

10.4 Other Equations with Three Space Variables

10.4.1 Equations Containing Arbitrary Functions

1. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \Bigl(\lambda + \frac{a}{r}\Bigr) w = 0, \qquad r^2 = x^2 + y^2 + z^2.$

Schrödinger's equation. It governs the motion of an electron in the Coulomb field of a nucleus ($a > 0$).

The desired solutions must satisfy the normalizing condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |w(x,y,z)|^2 \, dx\, dy\, dz = 1.$$

Eigenvalues:

$$\lambda_n = -\frac{a^2}{4n^2}; \qquad n = 1, 2, 3, \ldots$$

Normalized eigenfunctions (in the spherical coordinate system r, θ, φ):

$$w_{nmk}=\left(\frac{2}{n}\right)^{3/2}\sqrt{\frac{(2k+1)(k-m)!\,(n-k-1)!}{4\pi\varepsilon_m n(n+k)!\,(m+k)!}}\left(\frac{ar}{n}\right)^k\exp\left(-\frac{ar}{2n}\right)L_{n-k-1}^{2k+1}\left(\frac{ar}{n}\right)Y_k^{(m)}(\theta,\varphi),$$
$$n=1,\,2,\,3,\,\dots;\qquad m=0,\,\pm1,\,\pm2,\,\dots,\,\pm k;\qquad k=0,\,1,\,2\,\dots,\,n-1;$$

where

$$\varepsilon_m=\begin{cases}2 & \text{for } m=0,\\ 1 & \text{for } m\neq 0,\end{cases}\qquad Y_k^{(m)}(\theta,\varphi)=\begin{cases}P_k(\cos\theta) & \text{for } m=0,\\ P_k^m(\cos\theta)\sin m\varphi & \text{for } m=1,\,2,\,\dots,\\ P_k^{|m|}(\cos\theta)\cos m\varphi & \text{for } m=-1,\,-2,\,\dots,\end{cases}$$
$$L_k^s(x)=\frac{1}{k!}x^{-s}e^x\frac{d^k}{dx^k}\big(x^{k+s}e^{-x}\big),\quad P_k^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_k(x),$$
$$P_k(x)=\frac{1}{k!\,2^k}\frac{d^k}{dx^k}(x^2-1)^k.$$

These relations involve the generalized Laguerre polynomials $L_k^s(x)$ and the associated Legendre functions $P_n^m(\xi)$; the $P_n(\xi)$ are the Legendre polynomials.

⊙ *Literature*: G. Korn and T. Korn (2000), A. N. Tikhonov and A. A. Samarskii (1990).

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}=ay\dfrac{\partial w}{\partial x}.$

This equation is encountered in problems of convective heat and mass transfer in a simple shear flow.

Fundamental solution:

$$\mathscr{E}(x,y,z,\xi,\eta,\zeta)=\frac{1}{(4\pi)^{3/2}}\int_0^\infty\frac{1}{\sqrt{t^3(1+\frac{1}{12}a^2t^2)}}$$
$$\times\exp\left\{-\frac{[x-\xi-\frac{1}{2}at(y+\eta)]^2}{4t(1+\frac{1}{12}a^2t^2)}-\frac{(y-\eta)^2+(z-\zeta)^2}{4t}\right\}dt.$$

⊙ *Literature*: E. A. Novikov (1958), D. E. Elrick (1962).

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}+a_1x\dfrac{\partial w}{\partial x}+a_2y\dfrac{\partial w}{\partial y}+a_3z\dfrac{\partial w}{\partial z}=0.$

This equation is encountered in problems of convective heat and mass transfer in a straining flow.

Fundamental solution:

$$\mathscr{E}(x,y,z,\xi,\eta,\zeta)=\int_0^\infty F(x,\xi,t;a_1)F(y,\eta,t;a_2)F(z,\zeta,t;a_3)\,dt,$$
$$F(x,\xi,t;a)=\left[\frac{2\pi}{a}\big(e^{2at}-1\big)\right]^{-1/2}\exp\left[-\frac{a(xe^{at}-\xi)^2}{2(e^{2at}-1)}\right].$$

4. $\dfrac{\partial^2 w}{\partial x_1^2}+\dfrac{\partial^2 w}{\partial x_2^2}+\dfrac{\partial^2 w}{\partial x_3^2}=\sum_{n,k=1}^{3} a_{nk}x_n\dfrac{\partial w}{\partial x_k}.$

This equation is encountered in problems of convective heat and mass transfer in an arbitrary linear shear flow.

The solution that corresponds to a source of unit power at the origin of coordinates is given by

$$w(x_1,x_2,x_3)=\frac{1}{(4\pi)^{3/2}}\int_0^\infty \exp\biggl[-\sum_{n,k=1}^{3}\frac{b_{nk}(t)x_nx_k}{4D(t)}\biggr]\frac{dt}{\sqrt{D(t)}}.$$

Here, $D=D(t)$ is the determinant of the matrix $\mathbf{B}=\{B_{nk}\}$; the $b_{nk}=b_{nk}(t)$ are the cofactors of the entries $B_{nk}=B_{nk}(t)$; the B_{nk} are determined by solving the following system of ordinary differential equations with constant coefficients:

$$\frac{dB_{nk}}{dt}=\delta_{nk}+\sum_{m=1}^{3}a_{nm}B_{km}+\sum_{m=1}^{3}a_{km}B_{nm},$$
$$B_{nk}\to\delta_{nk}t \quad \text{as} \quad t\to 0 \quad \text{(initial conditions)},$$

where $\delta_{nn}=1$ $(n=1,2,3)$ and $\delta_{nk}=0$ if $n\neq k$.

⊙ *Literature*: G. K. Batchelor (1979).

5. $\dfrac{\partial}{\partial x}\Bigl[f_1(x)\dfrac{\partial w}{\partial x}\Bigr]+\dfrac{\partial}{\partial y}\Bigl[f_2(y)\dfrac{\partial w}{\partial y}\Bigr]+\dfrac{\partial}{\partial z}\Bigl[f_3(z)\dfrac{\partial w}{\partial z}\Bigr]=\beta w.$

This is a three-dimensional linear equation of heat and mass transfer theory with a source in an inhomogeneous anisotropic medium. Here, $f_1=f_1(x)$, $f_2=f_2(y)$, and $f_3=f_3(z)$ are the principal thermal diffusivities.

1°. The equation admits multiplicatively separable solutions, $w=\varphi_1(x)\varphi_2(y)\varphi_3(z)$.

2°. There are also additively separable solutions, $w=\psi_1(x)+\psi_2(y)+\psi_3(z)$.

3°. If $f_1=ax^n$, $f_2=by^m$, and $f_3=cz^k$ $(n\neq 2,\ m\neq 2,\ k\neq 2)$, there are particular solutions of the form

$$w=w(\xi),\qquad \xi^2=4\biggl[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\biggr],$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$\frac{d^2w}{d\xi^2}+\frac{A}{\xi}\frac{dw}{d\xi}=\beta w,\qquad A=2\biggl(\frac{1}{2-n}+\frac{1}{2-m}+\frac{1}{2-k}\biggr)-1,$$

whose solutions are expressed in terms of the Bessel functions.

10.4.2 Equations of the Form $\operatorname{div}\,[a(x,y,z)\nabla w] - q(x,y,z)w = -\Phi(x,y,z)$

Equations of this sort are often encountered in heat and mass transfer theory. For brevity, the equation is written using the notation

$$\operatorname{div}[a(\mathbf{r})\nabla w] = \frac{\partial}{\partial x}\left[a(\mathbf{r})\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[a(\mathbf{r})\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[a(\mathbf{r})\frac{\partial w}{\partial z}\right], \qquad \mathbf{r} = \{x, y, z\}.$$

In what follows, the problems for the equation in question will be considered in a bounded domain V with a sufficiently smooth surface S. It is assumed that $a(\mathbf{r}) > 0$ and $q(\mathbf{r}) \geq 0$.

► **First boundary value problem.**

The following boundary condition of the first kind is imposed:

$$w = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S.$$

Solution:

$$w(\mathbf{r}) = \int_V \Phi(\boldsymbol{\rho})G(\mathbf{r}, \boldsymbol{\rho})\,dV_\rho - \int_S f(\boldsymbol{\rho})a(\boldsymbol{\rho})\frac{\partial}{\partial N_\rho}G(\mathbf{r}, \boldsymbol{\rho})\,dS_\rho. \tag{1}$$

Here, the Green's function is given by

$$G(\mathbf{r}, \boldsymbol{\rho}) = \sum_{n=1}^{\infty} \frac{u_n(\mathbf{r})u_n(\boldsymbol{\rho})}{\|u_n\|^2\lambda_n}, \qquad \|u_n\|^2 = \int_V u_n^2(\mathbf{r})\,dV, \quad \boldsymbol{\rho} = \{\xi, \eta, \zeta\}, \tag{2}$$

where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and eigenfunctions of the Sturm–Liouville problem for the following second-order elliptic equation with a homogeneous boundary condition of the first kind:

$$\operatorname{div}\big[a(\mathbf{r})\nabla u\big] - q(\mathbf{r})u + \lambda u = 0, \tag{3}$$

$$u = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{4}$$

The integration in (1) is performed with respect to ξ, η, ζ; $\frac{\partial}{\partial N_\rho}$ denotes the derivative along the outward normal to the surface S with respect to ξ, η, ζ.

General properties of the Sturm–Liouville problem (3)–(4):

1°. There are countably many eigenvalues. All eigenvalues are real and can be ordered so that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$; therefore the number of negative eigenvalues is finite.

2°. If $a(\mathbf{r}) > 0$ and $q(\mathbf{r}) \geq 0$, all eigenvalues are positive, $\lambda_n > 0$.

3°. The eigenfunctions are defined up to a constant factor, and they can be chosen to be real. Two arbitrary eigenfunctions, $u_n(\mathbf{r})$ and $u_m(\mathbf{r})$, corresponding to distinct eigenvalues $\lambda_n \neq \lambda_m$ are orthogonal to each other in V,

$$\int_V u_n(\mathbf{r})u_m(\mathbf{r})\,dV = 0.$$

Distinct eigenfunctions corresponding to coinciding eigenvalues $\lambda_n = \lambda_m$ can be chosen to be orthogonal.

4°. An arbitrary function $F(\mathbf{r})$ that is twice continuously differentiable and satisfies the boundary condition of the Sturm–Liouville problem ($F = 0$ for $\mathbf{r} \in S$) can be expanded into an absolutely and uniformly convergent series in the eigenfunctions; specifically,

$$F(\mathbf{r}) = \sum_{n=1}^{\infty} F_n u_n(\mathbf{r}), \quad F_n = \frac{1}{\|u_n\|^2} \int_V F(\mathbf{r}) u_n(\mathbf{r})\, dV,$$

where the squared norm $\|u_n\|^2$ is defined in (2).

5°. An increase in a and/or a decay in q results in an increase in the eigenvalues. The eigenvalues decrease if the domain is extended.

6°. Constraint-strengthening modifications of the eigenvalue problem (including extra conditions on w) result in nondecreasing of the eigenvalues.

► Second boundary value problem.

A boundary condition of the second kind is imposed,

$$\frac{\partial w}{\partial N} = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S.$$

It is assumed that $q(\mathbf{r}) > 0$.

Solution:

$$w(\mathbf{r}) = \int_V \Phi(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dV_\rho + \int_S f(\boldsymbol{\rho}) a(\boldsymbol{\rho}) G(\mathbf{r}, \boldsymbol{\rho})\, dS_\rho. \tag{5}$$

Here, the Green's function is defined by relation (2), where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation (3) with the following homogeneous boundary condition of the second kind:

$$\frac{\partial u}{\partial N} = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{6}$$

If $q(\mathbf{r}) > 0$, the general properties of the eigenvalue problem (3), (6) are the same as those of the first boundary value problem (see the paragraph above).

► Third boundary value problem.

The following boundary condition of the third kind is set:

$$\frac{\partial w}{\partial N} + k(\mathbf{r}) w = f(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S.$$

The solution of the third boundary value problem is given by relations (5) and (2), where the λ_n and $u_n(\mathbf{r})$ are the eigenvalues and eigenfunctions of the Sturm–Liouville problem for the second-order elliptic equation (3) with the following homogeneous boundary condition of the third kind:

$$\frac{\partial u}{\partial N} + k(\mathbf{r}) u = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{7}$$

If $q(\mathbf{r}) \geq 0$ and $k(\mathbf{r}) > 0$, the general properties of the eigenvalue problem (3), (7) are the same as those of the first boundary value problem (see the first paragraph).

Let $k(\mathbf{r}) = k = \text{const}$. Denote the Green's functions of the second and third boundary value problems by $G_2(\mathbf{r}, \boldsymbol{\rho})$ and $G_3(\mathbf{r}, \boldsymbol{\rho}, k)$, respectively. For $q(\mathbf{r}) > 0$, the following limit relation holds:

$$G_2(\mathbf{r}, \boldsymbol{\rho}) = \lim_{k \to 0} G_3(\mathbf{r}, \boldsymbol{\rho}, k).$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1984), G. A. Korn and T. M. Korn (2000).

10.5 Equations with n Space Variables

10.5.1 Laplace Equation $\Delta_n w = 0$

The n-dimensional Laplace equation in the rectangular Cartesian system of coordinates $x_1, \ldots, x_n$ has the form

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial^2 w}{\partial x_n^2} = 0.$$

For $n = 2$ and $n = 3$, see Sections 9.1.1 and 10.1.1.

A regular solution of the Laplace equation is called a harmonic function.

In what follows we use the notation: $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$.

► **Particular solutions.**

1°. Fundamental solution:

$$\mathscr{E}(\mathbf{x}) = -\frac{1}{(n-2)\sigma_n |\mathbf{x}|^{n-2}}, \quad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \qquad (n \geq 3).$$

2°. Solution containing arbitrary functions of $n-1$ variables:

$$w(x_1, \ldots, x_n) = \sum_{k=0}^{\infty} (-1)^k \left[\frac{x_n^{2k}}{(2k)!} \Delta^k f(x_1, \ldots, x_{n-1}) + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^k g(x_1, \ldots, x_{n-1}) \right],$$

where $f(x_1, \ldots, x_{n-1})$ and $g(x_1, \ldots, x_{n-1})$ are arbitrary infinitely differentiable functions.

3°. Let $w(x_1, \ldots, x_n)$ be a harmonic function. Then the functions

$$w_1 = A w(\pm\lambda x_1 + C_1, \ldots, \pm\lambda x_n + C_n),$$
$$w_2 = \frac{A}{|\mathbf{x}|^{n-2}} w\left(\frac{x_1}{|\mathbf{x}|^2}, \ldots, \frac{x_n}{|\mathbf{x}|^2} \right),$$

are also harmonic functions everywhere they are defined; A, $C_1, \ldots, C_n$, and λ are arbitrary constants. The signs at λ in the expression of w_1 can be taken independently of one another.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985), R. Courant and D. Hilbert (1989).

► **Domain: $-\infty < x_1 < \infty,\ \ldots,\ -\infty < x_{n-1} < \infty,\ 0 \le x_n < \infty$.**

The first boundary value problem for an n-dimensional half-space is considered. A boundary condition is prescribed:

$$w = f(x_1, \ldots, x_{n-1}) \quad \text{at} \quad x_n = 0.$$

Solution:

$$w(x_1,\ldots,x_n)=\frac{\Gamma(n/2)}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{k=1}^{n-1}(y_k-x_k)^2+x_n^2\right]^{-n/2} x_n f(y_1,\ldots,y_{n-1})\,dy_1\ldots dy_{n-1},$$

where $\Gamma(z)$ is the gamma function.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Domain: $|\mathbf{x}| \le 1$. First boundary value problem.**

A sphere of unit radius in the n-dimensional space is considered. A boundary condition is prescribed:

$$w = f(\mathbf{x}) \quad \text{for} \quad |\mathbf{x}| = 1.$$

Solution (*Poisson integral*):

$$w(\mathbf{x}) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int\limits_{|\mathbf{y}|=1} \frac{1-|\mathbf{x}|^2}{|\mathbf{y}-x|^n} f(\mathbf{y})\,dS_{\mathbf{y}}.$$

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

10.5.2 Other Equations

1. $\Delta_n w = -\Phi(x_1, \ldots, x_n)$.

This is the *Poisson equation* in n independent variables. For $n = 2$ and $n = 3$, see Sections 9.2 and 10.2.

1°. Solution:

$$w(x_1,\ldots,x_n) = \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \int_{\mathbb{R}^n} \frac{\Phi(y_1,\ldots,y_n)\,dy_1\,\ldots\,dy_n}{\left[(x_1-y_1)^2+\cdots+(x_n-y_n)^2\right]^{\frac{n-2}{2}}}.$$

⊙ *Literature*: S. G. Krein (1972).

2°. Domain: $0 \le x_k \le a_k$; $k = 1, \ldots, n$. First boundary value problem.
A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w = f_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n) \quad \text{at} \quad x_k = 0,$$
$$w = g_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n) \quad \text{at} \quad x_k = a_k.$$

Green's function:

$$G(x_1,\dots,x_n,y_1,\dots,y_n)=\frac{2^n}{a_1\dots a_n}\sum_{k_1=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\frac{\sin(p_{k_1}x_1)\sin(p_{k_1}y_1)\dots\sin(p_{k_n}x_n)\sin(p_{k_n}y_n)}{p_{k_1}^2+\dots+p_{k_n}^2},$$

$$p_{k_1}=\frac{\pi k_1}{a_1},\quad p_{k_2}=\frac{\pi k_2}{a_2},\quad \dots,\quad p_{k_n}=\frac{\pi k_n}{a_n}.$$

3°. Domain: $0\le x_k\le a_k$; $k=1,\,\dots,\,n$. Mixed boundary value problem.
A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w=f_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\quad\text{at}\quad x_k=0,$$
$$\partial_{x_k}w=g_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\quad\text{at}\quad x_k=a_k.$$

Green's function:

$$G(x_1,\dots,x_n,y_1,\dots,y_n)=\frac{2^n}{a_1\dots a_n}\sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty}\frac{\sin(p_{k_1}x_1)\sin(p_{k_1}y_1)\dots\sin(p_{k_n}x_n)\sin(p_{k_n}y_n)}{p_{k_1}^2+\dots+p_{k_n}^2},$$

$$p_{k_1}=\frac{\pi(2k_1+1)}{2a_1},\quad p_{k_2}=\frac{\pi(2k_2+1)}{2a_2},\quad \dots,\quad p_{k_n}=\frac{\pi(2k_n+1)}{2a_n}.$$

2. $\boldsymbol{\Delta_n w+\lambda w=0}$.

This is the *Helmholtz equation* in n independent variables. For $n=2$ and $n=3$, see Sections 9.3 and 10.3.

1°. Fundamental solution for $\lambda=k^2>0$:

$$\mathscr{E}(\mathbf{x},\mathbf{y})=\frac{k^{\frac{n-2}{2}}}{4(2\pi)^{\frac{n-2}{2}}}r^{-\frac{n-2}{2}}Y_{\frac{n-2}{2}}(kr),\quad r=|\mathbf{x}-\mathbf{y}|\qquad\text{for even } n,$$

$$\mathscr{E}(\mathbf{x},\mathbf{y})=\frac{k^{\frac{n-2}{2}}}{4(2\pi)^{\frac{n-2}{2}}\sin\left(\frac{1}{2}\pi n\right)}r^{-\frac{n-2}{2}}J_{-\frac{n-2}{2}}(kr)\qquad\text{for odd } n,$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

2°. Domain: $0\le x_k\le a_k$; $k=1,\,\dots,\,n$. First boundary value problem.
A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w=f_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\quad\text{at}\quad x_k=0,$$
$$w=g_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\quad\text{at}\quad x_k=a_k.$$

Green's function:

$$G(x_1,\dots,x_n,y_1,\dots,y_n)=\frac{2^n}{a_1a_2\dots a_n}\sum_{k_1=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\frac{\sin(p_{k_1}x_1)\sin(p_{k_1}y_1)\dots\sin(p_{k_n}x_n)\sin(p_{k_n}y_n)}{p_{k_1}^2+\dots+p_{k_n}^2-\lambda},$$

$$p_{k_1}=\frac{\pi k_1}{a_1},\quad p_{k_2}=\frac{\pi k_2}{a_2},\quad \dots,\quad p_{k_n}=\frac{\pi k_n}{a_n}.$$

3°. Domain: $0\le x_k\le a_k$; $k=1,\,\dots,\,n$. Second boundary value problem.

A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$\partial_{x_k} w = f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \quad \text{at} \quad x_k = 0,$$
$$\partial_{x_k} w = g_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \quad \text{at} \quad x_k = a_k.$$

Green's function:

$$G(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} A_{k_1 k_2 \dots k_n} \frac{\cos(p_{k_1} x_1)\cos(p_{k_1} y_1)\dots\cos(p_{k_n} x_n)\cos(p_{k_n} y_n)}{p_{k_1}^2 + \dots + p_{k_n}^2 - \lambda},$$

$$A_{k_1 k_2 \dots k_n} = \frac{\varepsilon_{k_1}\varepsilon_{k_2}\dots\varepsilon_{k_n}}{a_1 a_2 \dots a_n}, \quad p_{k_1} = \frac{\pi k_1}{a_1}, \quad p_{k_2} = \frac{\pi k_2}{a_2}, \quad \dots, \quad p_{k_n} = \frac{\pi k_n}{a_n}, \quad \varepsilon_m = \begin{cases} 1 & \text{for } m=0, \\ 2 & \text{for } m\neq 0. \end{cases}$$

4°. Domain: $0 \le x_k \le a_k$; $k = 1, \dots, n$. Mixed boundary value problem.
A rectangular parallelepiped is considered. Boundary conditions are prescribed:

$$w = f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \quad \text{at} \quad x_k = 0,$$
$$\partial_{x_k} w = g_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \quad \text{at} \quad x_k = a_k.$$

Green's function:

$$G(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{2^n}{a_1 a_2 \dots a_n} \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{\sin(p_{k_1} x_1)\sin(p_{k_1} y_1)\dots\sin(p_{k_n} x_n)\sin(p_{k_n} y_n)}{p_{k_1}^2 + \dots + p_{k_n}^2 - \lambda},$$

$$p_{k_1} = \frac{\pi(2k_1+1)}{2a_1}, \quad p_{k_2} = \frac{\pi(2k_2+1)}{2a_2}, \quad \dots, \quad p_{k_n} = \frac{\pi(2k_n+1)}{2a_n}.$$

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

3. $\displaystyle \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = 0.$

It is assumed that for any real numbers $y_1, \dots, y_n$ the relation $\left| \sum_{i,j=1}^{n} a_{ij} y_i y_j \right| \ge k \sum_{i=1}^{n} y_i^2$ holds, where k is some positive constant.

Fundamental solution:

$$\mathscr{E}(x_1, \dots, x_n, y_1, \dots, y_n) = \begin{cases} \dfrac{\Gamma(n/2)}{2(n-2)\pi^{n/2}\sqrt{A}} \left[\displaystyle\sum_{i,j=1}^{n} b_{ij}(x_i - y_i)(x_j - y_j) \right]^{-\frac{n-2}{2}} & \text{if } n \ge 3, \\ \dfrac{1}{2\pi\sqrt{A}} \ln \left[\displaystyle\sum_{i,j=1}^{2} b_{ij}(x_i - y_i)(x_j - y_j) \right]^{-1/2} & \text{if } n = 2, \end{cases}$$

where A is the determinant of the matrix $\mathbf{A} = \{a_{ij}\}$ and the b_{ij} are the entries of the inverse of $\mathbf{A}$.

⊙ *Literature*: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964).

4. $\displaystyle\sum_{i=1}^{n-1}\frac{\partial^2 w}{\partial x_i^2}+\frac{\partial}{\partial x_n}\left(x_n^\beta\frac{\partial w}{\partial x_n}\right)+\lambda w=0.$

Domain: $a_i \le x_i \le b_i$ $(i=1,\dots,n-1)$, $0 \le x_n \le c$.

1°. Case $0<\beta<1$. First boundary value problem. The condition $w=0$ is set on the entire boundary of the domain.

Eigenvalues and eigenfunctions:

$$\lambda_{k_1,\dots,k_{n-1},m}=\sum_{i=1}^{n-1}\frac{k_i^2\pi^2}{(b_i-a_i)^2}+\frac{(2-\beta)^2\gamma_{\nu m}}{4c^{2-\beta}},$$
$$w_{k_1,\dots,k_{n-1},m}=x_n^{\frac{1-\beta}{2}}J_\nu\left(\gamma_{\nu m}\left(\frac{x_n}{a}\right)^{\frac{2-\beta}{2}}\right)\prod_{i=1}^{n-1}\sin\frac{k_i\pi(x_i-a_i)}{b_i-a_i},$$

where $\gamma_{\nu m}$ is the mth positive root of the equation $J_\nu(\gamma)=0$,

$$k_1,\,\dots,\,k_{n-1}=1,\,2,\,\dots\,;\qquad m=1,\,2,\,\dots\,;\qquad \nu=\frac{1-\beta}{2-\beta}.$$

2°. Case $1\le\beta<2$. Boundary conditions: the solution must be bounded at $x_n=0$, and the condition $w=0$ must hold on the rest of the boundary of the domain.

The eigenvalues and eigenfunctions of this problem are given by the relations of Item 1° with $\nu=(\beta-1)/(2-\beta)$.

⊙ *Literature*: M. M. Smirnov (1975).

Chapter 11

Higher-Order Partial Differential Equations

11.1 Third-Order Partial Differential Equations

11.1.1 One-Dimensional Equations Containing the First Derivative in t

1. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} = \Phi(x, t).$

Linearized Korteweg–de Vries equation.

1°. Particular solutions of the homogeneous equation with $\Phi(x,t) = 0$:

$$
\begin{aligned}
w(x,t) &= a(x^3 - 6t) + bx^2 + cx + k,\\
w(x,t) &= a(x^5 - 60x^2 t) + b(x^4 - 24xt),\\
w(x,t) &= a\sin(\lambda x + \lambda^3 t) + b\cos(\lambda x + \lambda^3 t) + c,\\
w(x,t) &= a\sinh(\lambda x - \lambda^3 t) + b\cosh(\lambda x - \lambda^3 t) + c,\\
w(x,t) &= \exp\bigl(-\lambda^3 t\bigr)\bigl[a\exp\bigl(\lambda x\bigr) + b\exp\bigl(-\tfrac{1}{2}\lambda x\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}\lambda x + c\bigr)\bigr],
\end{aligned}
$$

where a, b, c, k, and λ are arbitrary constants.

2°. Solution given by a formal series in powers of t ($\Phi(x,t) = 0$):

$$
w(x,t) = f(x) + \sum_{k=1}^{\infty} (-1)^k \frac{t^k}{k!} \frac{d^{3k} f(x)}{dx^{3k}},
$$

where $f(x)$ is an arbitrary infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = f(x)$. If a polynomial of degree n is taken for the function $f(x)$, then the solution is a polynomial in x of degree n as well.

3°. Solution given by a formal series in powers of x ($\Phi(x,t)=0$):

$$w(x,t)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{(3k)!}\frac{d^kf(t)}{dt^k}+\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+1}}{(3k+1)!}\frac{d^kg(t)}{dt^k}+\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+2}}{(3k+2)!}\frac{d^kh(t)}{dt^k},$$

where $f(t)$, $g(t)$, and $h(t)$ are arbitrary infinitely differentiable functions and $d^0f(t)/dt^0=f(t)$. This solution satisfies the boundary conditions $w(0,t)=f(t)$, $\partial_x w(0,t)=g(t)$, and $\partial_{xx}w(0,t)=h(t)$. If polynomials of degree $\le n$ are taken for the functions $f(t)$, $g(t)$, and $h(t)$, then the solution is a polynomial in t of degree $\le n$ as well.

4°. Fundamental solution:

$$\mathscr{E}(x,t)=\frac{1}{\pi}\int_0^\infty\cos(t\xi^3+x\xi)\,d\xi=\frac{1}{(3t)^{1/3}}\operatorname{Ai}(z),\qquad z=\frac{x}{(3t)^{1/3}},$$

where $\operatorname{Ai}(z)=\frac{1}{\pi}\int_0^\infty\cos\bigl(\frac{1}{3}\xi^3+z\xi\bigr)\,d\xi$ is the Airy function.

5°. Domain: $-\infty<x<\infty$. Cauchy problem.
An initial condition is prescribed:

$$w=f(x)\quad\text{at}\quad t=0.$$

Solution:

$$w(x,t)=\int_{-\infty}^{\infty}\mathscr{E}(x-y,t)f(y)\,dy+\int_0^t\int_{-\infty}^{\infty}\mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau.$$

6°. Domain: $0\le x<\infty$. First boundary value problem.
Initial and boundary conditions are prescribed:

$$w=0\quad\text{at}\quad t=0,\qquad w=f(t)\quad\text{at}\quad x=0,\qquad w\to0\quad\text{as}\quad x\to\infty.$$

Solution:

$$w(x,t)=3\int_0^t\operatorname{Ai}''\biggl(\frac{x}{(t-\tau)^{1/3}}\biggr)\frac{f(\tau)}{t-\tau}\,d\tau,$$

where $\operatorname{Ai}''(z)$ is the second derivative of the Airy function.

7°. Domain: $-\infty<x\le0$. First boundary value problem.
Initial and boundary conditions are prescribed:

$$w=0\quad\text{at}\quad t=0,\qquad w=f(t)\quad\text{at}\quad x=0,\qquad w\to0\quad\text{as}\quad x\to-\infty.$$

Solution:

$$w(x,t)=-\frac{3}{2}\int_0^t\operatorname{Ai}''\biggl(\frac{x}{(t-\tau)^{1/3}}\biggr)\frac{f(\tau)}{t-\tau}\,d\tau.$$

⊙ *Literature*: A. V. Faminskii (1999), N. A. Kudryashov and D. I. Sinelshchikov (2014).

2. $\dfrac{\partial w}{\partial t}+a\dfrac{\partial^3 w}{\partial x^3}-b\dfrac{\partial^2 w}{\partial x^2}=\Phi(x,t).$

Linearized Burgers–Korteweg–de Vries equation.

1°. Fundamental solution:

$$\mathscr{E}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[(-bu^2+iau^3)t]e^{ixu}\,du=\frac{1}{\pi}\int_0^{\infty}\exp(-bu^2)\cos(au^3t+xu)\,du.$$

2°. Domain: $-\infty<x<\infty$. Cauchy problem.

An initial condition is prescribed:

$$w=f(x)\quad\text{at}\quad t=0.$$

Solution:

$$w(x,t)=\int_{-\infty}^{\infty}\mathscr{E}(x-y,t)f(y)\,dy+\int_0^t\int_{-\infty}^{\infty}\mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau.$$

3. $\dfrac{\partial w}{\partial t}=ax^6\dfrac{\partial^3 w}{\partial x^3}.$

The transformation

$$u(z,\tau)=wx^{-2},\quad z=1/x,\quad \tau=at$$

leads to a constant coefficient equation of the form 11.1.1.1:

$$\frac{\partial u}{\partial\tau}=-\frac{\partial^3 u}{\partial z^3}.$$

4. $\dfrac{\partial w}{\partial t}=k(t)\dfrac{\partial^3 w}{\partial x^3}+\big[xf(t)+g(t)\big]\dfrac{\partial w}{\partial x}+h(t)w.$

The transformation

$$w(x,t)=u(z,\tau)\exp\left[\int h(t)\,dt\right],\quad z=xF(t)+\int g(t)F(t)\,dt,\quad \tau=\int k(t)F^3(t)\,dt,$$

where $F(t)=\exp\left[\int f(t)\,dt\right]$, leads to the linearized Korteweg–de Vries equation of the form 11.1.1.1:

$$\frac{\partial u}{\partial\tau}=\frac{\partial^3 u}{\partial z^3}.$$

5. $\dfrac{\partial w}{\partial t}=(ax^2+bx+c)^3\dfrac{\partial^3 w}{\partial x^3}.$

This is a special case of equation 11.7.4.5 with $k=1$ and $n=3$. The transformation

$$w(x,t)=u(z,t)(ax^2+bx+c),\qquad z=\int\frac{dx}{ax^2+bx+c}$$

leads to the constant coefficient equation

$$\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial z^3}+(4ac-b^2)\frac{\partial u}{\partial z}.$$

11.1.2 One-Dimensional Equations Containing the Second Derivative in t

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3}$.

1°. Particular solutions:

$$\begin{aligned}
w(x,t) &= A(x^3 + 3at^2) + Bx^2 + C_1 x + C_2 t + C_3,\\
w(x,t) &= A(x^5 + 30ax^2t^2) + B(x^4 + 12axt^2),\\
w(x,t) &= e^{\beta x}\bigl[A\exp\bigl(-\sqrt{a\beta^3}\,t\bigr) + B\exp\bigl(\sqrt{a\beta^3}\,t\bigr)\bigr],\\
w(x,t) &= e^{-\beta x}\bigl[A\cos\bigl(\sqrt{a\beta^3}\,t\bigr) + B\sin\bigl(\sqrt{a\beta^3}\,t\bigr)\bigr],
\end{aligned}$$

where A, B, C_n, and β are arbitrary constants.

2°. Solution given by a formal series in powers of t:

$$w(x,t) = \sum_{k=0}^{\infty} \frac{a^k t^{2k}}{(2k)!}\frac{d^{3k} f(x)}{dx^{3k}} + \sum_{k=0}^{\infty} \frac{a^k t^{2k+1}}{(2k+1)!}\frac{d^{3k} g(x)}{dx^{3k}},$$

where $f(x)$ and $g(x)$ are arbitrary infinitely differentiable functions and $d^0 f(x)/dx^0 = f(x)$. This solution satisfies the initial conditions $w(x,0) = f(x)$ and $\partial_t w(x,0) = g(x)$. If polynomials of degree $\le n$ are taken for the functions $f(x)$ and $g(x)$, then the solution is a polynomial in x of degree $\le n$ as well.

3°. Solution given by a formal series in powers of x:

$$\begin{aligned}
w(x,t) = {}&\sum_{k=0}^{\infty} \frac{x^{3k}}{a^k (3k)!}\frac{d^{2k} f(t)}{dt^{2k}} + \sum_{k=0}^{\infty} \frac{x^{3k+1}}{a^k (3k+1)!}\frac{d^{2k} g(t)}{dt^{2k}}\\
&+ \sum_{k=0}^{\infty} \frac{x^{3k+2}}{a^k (3k+2)!}\frac{d^{2k} h(t)}{dt^{2k}},
\end{aligned}$$

where $f(t)$, $g(t)$, and $h(t)$ are arbitrary infinitely differentiable functions and $d^0 f(t)/dt^0 = f(t)$. This solution satisfies the boundary conditions $w(0,t) = f(t)$, $\partial_x w(0,t) = g(t)$, and $\partial_{xx} w(0,t) = h(t)$. If polynomials of degree $\le n$ are taken for the functions $f(t)$, $g(t)$, and $h(t)$, then the solution is a polynomial in t of degree $\le n$ as well.

2. $\dfrac{\partial^2 w}{\partial t^2} = ax^6\dfrac{\partial^3 w}{\partial x^3}$.

The transformation $z = 1/x$, $u = wx^{-2}$ leads to the constant coefficient equation of the form 11.1.2.1:

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial^3 u}{\partial z^3}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = (ax^2 + bx + c)^3\dfrac{\partial^3 w}{\partial x^3}$.

This is a special case of equation 11.7.4.5 with $k = 2$ and $n = 3$. The transformation

$$w(x,t) = u(z,t)(ax^2 + bx + c), \qquad z = \int \frac{dx}{ax^2 + bx + c}$$

leads to the constant coefficient equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial z^3} + (4ac - b^2)\frac{\partial u}{\partial z}.$$

11.1.3 One-Dimensional Equations Containing a Mixed Derivative and the First Derivative in t

1. $\dfrac{\partial w}{\partial t} - a\dfrac{\partial^2 w}{\partial x^2} - b\dfrac{\partial^3 w}{\partial t \partial x^2} = 0.$

Equation of filtration of a compressible fluid in a cracked porous medium (e.g., see Barenblatt, Zheltov, and Kochina (1960), Barenblatt (1963)). This equation also describes one-dimensional unsteady motions of second-grade non-Newtonian fluids (e.g., see Puri (1984), Christov (2010)).

1°. Particular solutions:

$$w(x,t) = Ax^4 + (12aAt + B)x^2 + 12a^2At^2 + 2a(12A + B)t + C,$$
$$w(x,t) = \exp\left(\frac{a\beta^2 t}{1 - b\beta^2}\right)\Big[A\exp(-\beta x) + B\exp(\beta x)\Big],$$
$$w(x,t) = \exp\left(-\frac{a\beta^2 t}{1 + b\beta^2}\right)\Big[A\cos(\beta x) + B\sin(\beta x)\Big],$$

where A, B, C, and β are arbitrary constants. The last solution is periodic in x.

2°. Solutions periodic in t:

$$w(x,t) = e^{-\lambda x}\big[A\cos(\omega t - \mu x) + B\sin(\omega t - \mu x)\big],$$
$$\lambda = \pm\left[\frac{\omega\sqrt{a^2 + b^2} + b\omega^2}{2(a^2 + b^2\omega^2)}\right]^{1/2}, \qquad \mu = \pm\left[\frac{\omega\sqrt{a^2 + b^2} - b\omega^2}{2(a^2 + b^2\omega^2)}\right]^{1/2}.$$

Here one simultaneously takes only the upper or only the lower signs; A and B are arbitrary constants.

For $\lambda > 0$, these formulas give the solution of the Stokes second problem for a second-grade fluid with the boundary conditions

$$w = A\cos(\omega t) + B\sin(\omega t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty.$$

This is a problem without initial conditions, $-\infty < t < \infty$; see also Item 12°.

3°. Fundamental solution:

$$\begin{aligned}\mathscr{E}_\mathrm{e}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1 + b\xi^2}\exp\left(-\frac{a\xi^2 t}{1 + b\xi^2} + ix\xi\right)d\xi \\ &= \frac{1}{\pi}\int_0^{\infty} \exp\left(-\frac{a\xi^2 t}{1 + b\xi^2}\right)\frac{\cos(x\xi)}{1 + b\xi^2}\,d\xi.\end{aligned} \tag{1}$$

4°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0. \tag{2}$$

One also assumes that $w \to 0$ and $\partial_x w \to 0$ as $|x| \to \infty$.

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)[f(y) - bf''(y)]\,dy,$$

where the function $\mathscr{E}_{\rm e}(x,t)$ is defined in (1).

5°. Domain: $0 \le x < \infty$. First boundary value problem with initial condition (2).

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad x = 0.$$

Solution:

$$w(x,t) = \int_0^{\infty} \bigl[\mathscr{E}_{\rm e}(x-y,t) - \mathscr{E}_{\rm e}(x+y,t)\bigr][f(y) - bf''(y)]\,dy,$$

where the function $\mathscr{E}_{\rm e}(x,t)$ is defined in (1).

6°. Domain: $0 \le x < \infty$. Second boundary value problem with initial condition (2).

A boundary condition is prescribed:

$$\partial_x w = 0 \quad \text{at} \quad x = 0.$$

Solution:

$$w(x,t) = \int_0^{\infty} \bigl[\mathscr{E}_{\rm e}(x-y,t) + \mathscr{E}_{\rm e}(x+y,t)\bigr][f(y) - bf''(y)]\,dy.$$

7°. Domain: $0 \le x \le l$. First boundary value problem with initial condition (2).

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Solution:

$$w(x,t) = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) \exp\left(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\right),$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\beta_n x)\,dx, \qquad \beta_n = \frac{\pi n}{l}.$$

8°. Domain: $0 \le x \le l$. Second boundary value problem with initial condition (2).

Boundary conditions are prescribed:

$$\partial_x w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = 0 \quad \text{at} \quad x = l.$$

Solution:

$$w(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\beta_n x) \exp\left(-\frac{a\beta_n^2 t}{1+b\beta_n^2}\right),$$

$$A_0 = \frac{1}{l}\int_0^l f(x)\,dx, \qquad A_n = \frac{2}{l}\int_0^l f(x)\cos(\beta_n x)\,dx, \qquad \beta_n = \frac{\pi n}{l}.$$

9°. Domain: $0 \le x \le l$. Mixed boundary value problem with initial condition (2).
Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = 0 \quad \text{at} \quad x = l.$$

Solution:

$$w(x,t) = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) \exp\left(-\frac{a\beta_n^2 t}{1+b\beta_n^2}\right),$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\beta_n x)\,dx, \qquad \beta_n = \frac{\pi(2n+1)}{2l}.$$

10°. Domain: $0 \le x \le l$. Mixed boundary value problem with initial condition (2).
Boundary conditions are prescribed:

$$\partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Solution:

$$w(x,t) = \sum_{n=1}^{\infty} A_n \cos(\beta_n x) \exp\left(-\frac{a\beta_n^2 t}{1+b\beta_n^2}\right),$$

$$A_n = \frac{2}{l}\int_0^l f(x)\cos(\beta_n x)\,dx, \qquad \beta_n = \frac{\pi(2n+1)}{2l}.$$

11°. Domain: $0 \le x < \infty$. First boundary value problem. The following conditions are prescribed:

$$\begin{aligned} &w = 0 && \text{at} \quad t = 0 && \text{(initial condition)},\\ &w = f(t) && \text{at} \quad x = 0 && \text{(boundary condition)},\\ &w \to 0 && \text{as} \quad x \to \infty && \text{(boundary condition)}. \end{aligned}$$

It is assumed that the initial and boundary conditions are consistent; i.e., $f(0) = 0$.
Solution:

$$w(x,t) = \frac{2}{\pi}\int_0^{\infty} U(\xi,t)\sin(x\xi)\,d\xi,$$

$$U(\xi,t) = \frac{\xi}{1+b\xi^2}\int_0^t \varphi(\tau)\exp\left[-\frac{a\xi^2(t-\tau)}{1+b\xi^2}\right]d\tau, \qquad \varphi(\tau) = af(\tau) + bf'(\tau).$$

An alternative representation of the solution:

$$w(x,t) = \int_0^t \varphi(\tau) G(x,t-\tau)\,d\tau, \qquad \varphi(\tau) = af(\tau) + bf'(\tau),$$

$$G(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{\xi\sin(x\xi)}{1+b\xi^2}\exp\left(-\frac{a\xi^2 t}{1+b\xi^2}\right)d\xi.$$

12°. Domain: $0 \le x < \infty$. Problems without initial conditions ($-\infty < t < \infty$).

In applications, problems are encountered in which the process is studied at a time instant fairly remote from the initial instant; in this case, the initial conditions essentially do not affect the distribution of the desired variable at the observation instant. In such problems, no initial condition is stated, and the boundary conditions are assumed to be prescribed for all preceding time instants, $-\infty < t$. However, in addition, the boundedness condition in the entire domain is imposed on the solution.

As an example, consider the first boundary value problem for the half-space $0 \le x < \infty$ with the boundary conditions

$$w = f(t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty.$$

The estimate $|f(t)| < C\exp(-\lambda|t|)$ with $C > 0$ and $\lambda > a/b$ is assumed to hold as $t \to -\infty$.

Solution:

$$w(x,t) = \int_{-\infty}^{t} \varphi(\tau) G(x, t-\tau)\, d\tau, \qquad \varphi(\tau) = af(\tau) + bf'(\tau),$$

$$G(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\xi\sin(x\xi)}{1+b\xi^2}\exp\biggl(-\frac{a\xi^2 t}{1+b\xi^2}\biggr)\, d\xi.$$

13°. *The Stokes first problem* (a special case of Item 12°). The problem of unidirectional plane flow of a second-grade fluid in a half-plane caused by an impulsive motion of a plate is characterized by the boundary conditions

$$w = U_0\vartheta(t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty,$$

where $\vartheta(t)$ is the Heaviside unit step function and $-\infty < t < \infty$.

Three representations of the solution:

$$\begin{aligned} w(x,t) &= U_0\vartheta(t)\biggl[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(x\xi)}{\xi(1+b\xi^2)}\exp\biggl(-\frac{a\xi^2 t}{1+b\xi^2}\biggr)\, d\xi\biggr] \\ &= U_0\vartheta(t)\biggl[1 - \frac{1}{\pi}\int_0^{1/b} \exp(-at\eta)\sin\biggl(x\sqrt{\frac{\eta}{1-b\eta}}\biggr)\frac{d\eta}{\eta}\biggr] \\ &= U_0\vartheta(t)e^{-at/b}\int_0^\infty e^{-\zeta}\operatorname{erfc}\biggl(\frac{x}{2\sqrt{b\zeta}}\biggr) I_0\biggl(2\sqrt{\frac{at\zeta}{b}}\biggr)\, d\zeta, \end{aligned}$$

where $\operatorname{erfc} z$ is the complementary error function and $I_0(y)$ is the modified Bessel function of the first kind of order zero.

14°. *A modified Stokes second problem* (a special case of Item 12°). The transient version of the Stokes second problem for a second-grade fluid is characterized by the boundary conditions

$$w = U_0\vartheta(t)\cos(\omega t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty,$$

where $\vartheta(t)$ is the Heaviside unit step function and $-\infty < t < \infty$.

Solution:

$$w(x,t)=U_0\vartheta(t)\frac{2}{\pi}\int_0^\infty \frac{\xi\sin(x\xi)}{a^2\xi^4+\omega^2(1+b\xi^2)^2}\biggl\{-\frac{a^2\xi^2}{1+b\xi^2}\exp\biggl(-\frac{a\xi^2 t}{1+b\xi^2}\biggr) + a\omega\sin(\omega t)+\bigl[a^2\xi^2+b\omega^2(1+b\xi^2)\bigr]\cos(\omega t)\biggr\}\,d\xi.$$

15°. Domain: $0\le x<\infty$. *Transient filtration of a fluid in an adit* (w is the pressure).

The following conditions are prescribed:

$$w=w_0 \quad\text{at}\quad t=0,\qquad w=w_1+(w_0-w_1)e^{-at/b}\quad\text{at}\quad x=0,$$

where w_1 is the pressure in the stratum to the left of the boundary $x=0$.

Solution:

$$w(x,t)=w_1+(w_0-w_1)\exp\biggl(-\frac{at}{b}\biggr) + (w_0-w_1)\frac{2}{\pi}\int_0^\infty \frac{\sin(x\xi)}{\xi}\biggl[\exp\biggl(-\frac{a\xi^2t}{1+b\xi^2}\biggr)-\exp\biggl(-\frac{at}{b}\biggr)\biggr]\,d\xi.$$

⊙ *Literature for Items* 13°–15°: G. I. Barenblatt, Yu. P. Zheltov, and I. N. Kochina (1960), P. Puri (1984), R. Bandelli, K. R. Rajagopal, and G. P. Galdi (1995), C. Fetecău and J. Zierep (2001), I. C. Christov (2010), I. C. Christov and C. I. Christov (2010), I. C. Christov and P. M. Jordan (2012).

2. $\boldsymbol{\dfrac{\partial w}{\partial t}-a\dfrac{\partial^2 w}{\partial x^2}-b\dfrac{\partial^3 w}{\partial t\partial x^2}=\Phi(x,t).}$

1°. Domain: $-\infty<x<\infty$. Cauchy problem.

An initial condition is prescribed:

$$w=f(x)\quad\text{at}\quad t=0.$$

Solution:

$$w(x,t)=\int_0^t\int_{-\infty}^\infty \mathscr{E}_{\rm e}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau+\int_{-\infty}^\infty \mathscr{E}_{\rm e}(x-y,t)[f(y)-bf''(y)]\,dy,$$

where

$$\mathscr{E}_{\rm e}(x,t)=\frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{1+b\xi^2}\exp\biggl(-\frac{a\xi^2t}{1+b\xi^2}+ix\xi\biggr)\,d\xi =\frac{1}{\pi}\int_0^\infty \exp\biggl(-\frac{a\xi^2t}{1+b\xi^2}\biggr)\frac{\cos(x\xi)}{1+b\xi^2}\,d\xi.$$

2°. In Items 3°–6°, we consider problems on an interval $0\le x\le l$ with the general initial condition

$$w=f(x)\quad\text{at}\quad t=0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t)=\int_0^t\int_0^l G(x,\xi,t-\tau)\Phi(\xi,\tau)\,d\xi\,d\tau+\int_0^l G(x,\xi,t)[f(\xi)-bf''(\xi)]\,d\xi.$$

The function $f(x)$ is assumed to be consistent with the homogeneous boundary conditions for w at $x=0$ and $x=l$.

3°. Domain: $0 \le x \le l$. First boundary value problem.
Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} \frac{\sin(\beta_n x)\sin(\beta_n \xi)}{1 + b\beta_n^2} \exp\biggl(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\biggr), \qquad \beta_n = \frac{\pi n}{l}.$$

4°. Domain: $0 \le x \le l$. Second boundary value problem.
Boundary conditions are prescribed:

$$\partial_x w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \frac{\cos(\beta_n x)\cos(\beta_n \xi)}{1 + b\beta_n^2} \exp\biggl(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\biggr), \qquad \beta_n = \frac{\pi n}{l}.$$

5°. Domain: $0 \le x \le l$. Mixed boundary value problem.
Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=0}^{\infty} \frac{\sin(\beta_n x)\sin(\beta_n \xi)}{1 + b\beta_n^2} \exp\biggl(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\biggr), \qquad \beta_n = \frac{\pi(2n+1)}{2l}.$$

6°. Domain: $0 \le x \le l$. Mixed boundary value problem.
Boundary conditions are prescribed:

$$\partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=0}^{\infty} \frac{\cos(\beta_n x)\cos(\beta_n \xi)}{1 + b\beta_n^2} \exp\biggl(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\biggr), \qquad \beta_n = \frac{\pi(2n+1)}{2l}.$$

3. $\displaystyle \boldsymbol{\frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} - b\frac{\partial^3 w}{\partial t \partial x^2} + cw = \Phi(x, t).}$

The substitution $w = e^{-ct}u$ leads to an equation of the form 11.1.3.2:

$$\frac{\partial u}{\partial t} - (a - bc)\frac{\partial^2 u}{\partial x^2} - b\frac{\partial^3 u}{\partial t \partial x^2} = e^{ct}\Phi(x, t).$$

11.1.4 One-Dimensional Equations Containing a Mixed Derivative and the Second Derivative in t

1. $\dfrac{\partial^2 w}{\partial t^2} - a\dfrac{\partial^2 w}{\partial x^2} - b\dfrac{\partial^3 w}{\partial t \partial x^2} = 0.$

This equation describes one-dimensional unsteady motions of viscous compressible barotropic fluids.

1°. Particular solutions:

$$w(x,t) = (A_1 t + A_2)x^2 + (B_1 t + B_2)x + \tfrac{1}{3}aA_1 t^3 + (aA_2 + bA_1)t^2 + C_1 t + C_2;$$
$$w(x,t) = e^{\beta t}\left[A_1 \exp\left(-\frac{\beta x}{\sqrt{a+b\beta}}\right) + A_2 \exp\left(\frac{\beta x}{\sqrt{a+b\beta}}\right)\right], \qquad a + b\beta > 0;$$
$$w(x,t) = e^{\beta t}\left[A_1 \cos\left(\frac{\beta x}{\sqrt{|a+b\beta|}}\right) + A_2 \sin\left(\frac{\beta x}{\sqrt{|a+b\beta|}}\right)\right], \qquad a + b\beta < 0,$$

where A_n, B_n, C_n, and β are arbitrary constants. The last solution is periodic in x.

2°. Solutions periodic in t:

$$w(x,t) = e^{-\lambda x}\big[A\cos(\omega t - \mu x) + B\sin(\omega t - \mu x)\big],$$
$$\lambda = \pm\omega\left[\frac{\sqrt{a^2 + b^2\omega^2} - a}{2(a^2 + b^2\omega^2)}\right]^{1/2}, \qquad \mu = \pm\omega\left[\frac{\sqrt{a^2 + b^2\omega^2} + a}{2(a^2 + b^2\omega^2)}\right]^{1/2}.$$

Here one simultaneously takes only the upper or only the lower signs; A and B are arbitrary constants.

3°. Domain: $0 \le x \le l$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t = 0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x = 0 && \text{(boundary condition)},\\ w &= 0 && \text{at} \quad x = l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t) = \sum_{n=1}^{\infty}\big[A_n\psi_{n1}(t) + B_n\psi_{n2}(t)\big]\sin(\beta_n x), \qquad \beta_n = \frac{\pi n}{l},$$

where

$$\psi_{n1}(t) = \frac{\lambda_{n1}\exp(-\lambda_{n2}t) - \lambda_{n2}\exp(-\lambda_{n1}t)}{\lambda_{n1} - \lambda_{n2}}, \qquad \psi_{n2}(t) = \frac{\exp(-\lambda_{n2}t) - \exp(-\lambda_{n1}t)}{\lambda_{n1} - \lambda_{n2}},$$
$$\lambda_{n1} = \frac{b\beta_n^2 + \beta_n\sqrt{b^2\beta_n^2 - 4a}}{2}, \qquad \lambda_{n2} = \frac{b\beta_n^2 - \beta_n\sqrt{b^2\beta_n^2 - 4a}}{2},$$
$$A_n = \frac{2}{l}\int_0^l f(x)\sin(\beta_n x)\,dx, \qquad B_n = \frac{2}{l}\int_0^l g(x)\sin(\beta_n x)\,dx.$$

Remark 11.1. This solution holds for $b^2\beta_n^2-4a>0$ as well as for $b^2\beta_n^2-4a<0$. In the latter case, one should write $\sqrt{b^2\beta_n^2-4a}=i\sqrt{4a-b^2\beta_n^2}$, where $i^2=-1$, and transform the exponentials occurring in the functions $\psi_{n1}(t)$ and $\psi_{n2}(t)$ by using the Moivre formulas.

4°. Domain: $0\le x\le l$. Second boundary value problem.

The following conditions are prescribed:

$$\begin{aligned}
w&=f(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_t w&=g(x) &&\text{at}\quad t=0 &&\text{(initial condition)},\\
\partial_x w&=0 &&\text{at}\quad x=0 &&\text{(boundary condition)},\\
\partial_x w&=0 &&\text{at}\quad x=l &&\text{(boundary condition)}.
\end{aligned}$$

Solution:

$$w(x,t)=A_0+B_0t+\sum_{n=1}^{\infty}\bigl[A_n\psi_{n1}(t)+B_n\psi_{n2}(t)\bigr]\cos(\beta_n x),\qquad \beta_n=\frac{\pi n}{l},$$

where

$$\psi_{n1}(t)=\frac{\lambda_{n1}\exp(-\lambda_{n2}t)-\lambda_{n2}\exp(-\lambda_{n1}t)}{\lambda_{n1}-\lambda_{n2}},\qquad \psi_{n2}(t)=\frac{\exp(-\lambda_{n2}t)-\exp(-\lambda_{n1}t)}{\lambda_{n1}-\lambda_{n2}},$$

$$\lambda_{n1}=\frac{b\beta_n^2+\beta_n\sqrt{b^2\beta_n^2-4a}}{2},\qquad \lambda_{n2}=\frac{b\beta_n^2-\beta_n\sqrt{b^2\beta_n^2-4a}}{2},$$

$$A_0=\frac{1}{l}\int_0^l f(x)\,dx,\qquad B_0=\frac{1}{l}\int_0^l g(x)\,dx,$$

$$A_n=\frac{2}{l}\int_0^l f(x)\cos(\beta_n x)\,dx,\qquad B_n=\frac{2}{l}\int_0^l g(x)\cos(\beta_n x)\,dx,\qquad n=1,\,2,\,\dots.$$

2. $\displaystyle c\frac{\partial^2 w}{\partial t^2}+\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^3 w}{\partial t\partial x^2}.$

This equation describes one-dimensional unsteady motions of a viscoelastic incompressible Oldroyd-B fluid ($a>0$, $b>0$, and $c>0$).

1°. Solutions periodic in x:

$$w(x,t)=e^{-\gamma t}\bigl[A\cos(\beta x)+B\sin(\beta x)\bigr],$$

$$\gamma_{1,2}=\frac{1}{2c}\Bigl[b\beta^2+1\pm\sqrt{(b\beta^2+1)^2-4ac\beta^2}\Bigr],$$

where A, B, and β are arbitrary constants.

2°. Solutions periodic in t:

$$w(x,t)=e^{-\lambda x}\bigl[A\cos(\omega t-\mu x)+B\sin(\omega t-\mu x)\bigr],$$

$$\lambda=\pm\biggl[\frac{\omega\sqrt{(bc\omega^2+a)^2+(ac-b)^2\omega^2}-(ac-b)\omega^2}{2(a^2+b^2\omega^2)}\biggr]^{1/2},$$

$$\mu=\pm\biggl[\frac{\omega\sqrt{(bc\omega^2+a)^2+(ac-b)^2\omega^2}+(ac-b)\omega^2}{2(a^2+b^2\omega^2)}\biggr]^{1/2}.$$

Here one simultaneously takes only the upper or only the lower signs; A and B are arbitrary constants.

3°. *The Stokes first problem.* The problem of unidirectional plane flow of an *Oldroyd-B fluid* in a half-plane due to the impulsive motion of the plate is characterized by the boundary conditions

$$w = U_0 \vartheta(t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty,$$

where $\vartheta(t)$ is the Heaviside unit step function and $-\infty < t < \infty$.

Solution:

$$w(x,t) = U_0 \vartheta(t) \biggl\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp\Bigl(-xg(\eta)[\cos h(\eta) - \sin h(\eta)] \Bigr) \times \sin\Bigl(\frac{t\eta}{c} - xg(\eta)[\cos h(\eta) + \sin h(\eta)] \Bigr) \frac{d\eta}{\eta} \biggr\},$$

where

$$g(\eta) = \left(\frac{\eta}{2ac} \right)^{1/2} \left(\frac{1+\eta^2}{1+k^2\eta^2} \right)^{1/4}, \qquad h(\eta) = \frac{1}{2}\bigl[\tan^{-1}\eta - \tan^{-1}(k\eta)\bigr], \qquad k = \frac{b}{ac}.$$

4°. For the special case of $b = ac$, the equation admits the factorization

$$(c\partial_t + 1)(\partial_t - a\partial_{xx})[w] = 0.$$

Hence its general solution has the form

$$w = f(x) \exp(-t/c) + u(x,t),$$

where $f(x)$ is an arbitrary function and the function $u = u(x,t)$ is an arbitrary solution of the heat equation $\partial_t u - a\partial_{xx} u = 0$.

The solution of the Stokes first problem (see Item 3°) with $b = ac$ can be expressed via the complementary error function

$$w(x,t) = U_0 \vartheta(t) \operatorname{erfc}\left(\frac{x}{2\sqrt{at}} \right).$$

⊙ *Literature for Item* 3°: R. I. Tanner (1962), I. C. Christov (2010).

3. $\displaystyle \varepsilon \frac{\partial^2 w}{\partial t^2} + \sigma \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b \frac{\partial^3 w}{\partial t \partial x^2} + kw.$

1°. Solutions periodic in x:

$$w(x,t) = e^{-\gamma t}\bigl[A\cos(\beta x) + B\sin(\beta x)\bigr],$$
$$\gamma_{1,2} = \frac{1}{2\varepsilon}\Bigl[b\beta^2 + \sigma \pm \sqrt{(b\beta^2+\sigma)^2 + 4\varepsilon(k - a\beta^2)}\Bigr], \qquad \varepsilon \neq 0,$$

where A, B, and β are arbitrary constants.

2°. Solutions periodic in t:

$$w(x,t) = e^{-\lambda x}\big[A\cos(\omega t - \mu x) + B\sin(\omega t - \mu x)\big],$$

$$\lambda = \pm\left[\frac{\sqrt{\omega^2(b\varepsilon\omega^2 + a\sigma + bk)^2 + [(a\varepsilon - b\sigma)\omega^2 + ak]^2} - (a\varepsilon - b\sigma)\omega^2 - ak}{2(a^2 + b^2\omega^2)}\right]^{1/2},$$

$$\mu = \pm\left[\frac{\sqrt{\omega^2(b\varepsilon\omega^2 + a\sigma + bk)^2 + [(a\varepsilon - b\sigma)\omega^2 + ak]^2} + (a\varepsilon - b\sigma)\omega^2 + ak}{2(a^2 + b^2\omega^2)}\right]^{1/2}.$$

Here one simultaneously takes only the upper or only the lower signs; A and B are arbitrary constants.

3°. The substitution $w = e^{-\lambda t}u$, where λ is a root of the quadratic equation $\varepsilon\lambda^2 - \sigma\lambda - k = 0$, leads to an equation of the form 11.1.4.2:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + (\sigma - 2\lambda)\frac{\partial u}{\partial t} = (a - b\lambda)\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^3 u}{\partial t\partial x^2}.$$

11.1.5 Two- and Three-Dimensional Equations

1. $\displaystyle \frac{\partial w}{\partial t} - a\Delta w - b\frac{\partial}{\partial t}\Delta w = 0.$

Equation of filtration of a fluid in a cracked porous medium (e.g., see Barenblatt, Zheltov, and Kochina (1960), Barenblatt (1963)).

1°. Fundamental solution in two-dimensional case:

$$\mathscr{E}_{\rm e}(\mathbf{x},t) = \frac{\vartheta(t)}{2\pi}\int_0^\infty \frac{\rho J_0(r\rho)}{1 + b\rho^2}\exp\left(-\frac{a\rho^2 t}{1 + b\rho^2}\right)d\rho,$$

where $r = |\mathbf{x}| = \sqrt{x^2 + y^2}$ and $J_0(z)$ is the Bessel function.

2°. Fundamental solution in three-dimensional case:

$$\mathscr{E}_{\rm e}(\mathbf{x},t) = \frac{\vartheta(t)}{2\pi^2 r}\int_0^\infty \frac{\rho\sin(r\rho)}{1 + b\rho^2}\exp\left(-\frac{a\rho^2 t}{1 + b\rho^2}\right)d\rho,$$

where $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$.

3°. Domain: $\mathbf{x} \in \mathbb{R}^n$. Cauchy problem.

An initial condition is prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0.$$

Solution:

$$w = \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x} - \mathbf{y}, t)\big[f(\mathbf{y}) - b\Delta_y f(\mathbf{y})\big]\,dV_y, \qquad dV_y = dy_1 \ldots dy_n,$$

where Δ_y is the Laplace operator in the integration variables $y_1, \ldots, y_n$. See Items 1° and 2° for the function $\mathscr{E}_{\rm e}(\mathbf{x},t)$ with $n = 2$ and $n = 3$.

4°. Consider an open bounded domain V with boundary S. The following conditions are prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},$$
$$\alpha w + \beta \frac{\partial w}{\partial n} = 0 \quad \text{at} \quad \mathbf{x} \in S \quad \text{(boundary condition)},$$

where $\alpha = \alpha(\mathbf{x}) \geq 0$, $\beta = \beta(\mathbf{x}) \geq 0$, $\alpha + \beta > 0$ (all functions are assumed to be continuous in V), $\mathbf{x} = (x, y, z)$, and $\partial w/\partial n$ is the outward normal derivative on S. The boundary conditions of the first and second kind correspond to the special values $\beta = 0$ ($\alpha = 1$) and $\alpha = 0$ ($\beta = 1$), respectively.

Solution:

$$w(\mathbf{x}, t) = \sum_{k=1}^{\infty} A_k u_k(\mathbf{x}) \exp\left(-\frac{a\lambda_k t}{1 + b\lambda_k}\right),$$
$$A_k = \frac{1}{\|u_k\|^2} \int_V f(\mathbf{x}) u_k(\mathbf{x})\, dv, \qquad \|u_k\|^2 = \int_V u_k^2(\mathbf{x})\, dv,$$

where λ_k and $u_k(\mathbf{x})$ are the eigenvalues and eigenfunctions of the auxiliary problem

$$\Delta u + \lambda u = 0, \qquad \mathbf{x} \in V,$$
$$\alpha u + \beta \frac{\partial u}{\partial n} = 0, \qquad \mathbf{x} \in S.$$

Let us list the main properties of the eigenvalues λ_k and eigenfunctions $u_k(\mathbf{x})$ (e.g., see Miranda (1970) and Vladimirov (1988)).

1. All eigenvalues are nonnegative. They can be numbered in ascending order,

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lambda_k \to \infty \quad \text{as} \quad k \to \infty,$$

where each λ_k is repeated a number of times equal to its multiplicity.

2. The eigenfunctions $u_k = u_k(\mathbf{x})$ can be chosen to be real and orthogonal,

$$\int_V u_k u_m\, dv = 0 \quad \text{if} \quad k \neq m.$$

3. Every function $f(\mathbf{x})$ twice continuously differentiable in V, continuously differentiable in $V \cup S$, and satisfying the same boundary condition as w and the condition $\Delta f \in L_2(V)$ can be expanded in a Fourier series in the orthonormal eigenfunction system,

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} A_k u_k(\mathbf{x}), \qquad A_k = \frac{1}{\|u_k\|^2} \int_V f(\mathbf{x}) u_k(\mathbf{x})\, dv.$$

4. The system of eigenfunctions is complete in $L_2(V)$.

5. A necessary and sufficient condition that $\lambda = 0$ is an eigenvalue is that $\alpha = 0$; then $\lambda_1 = 0$ is a simple eigenvalue, and the corresponding eigenfunction is $u_1 = \text{const}$.

2. $\dfrac{\partial w}{\partial t} - a\Delta w - b\dfrac{\partial}{\partial t}\Delta w = \Phi(\mathbf{x}, t).$

Equation of filtration of a fluid in a cracked porous medium.

Domain: $\mathbf{x} \in \mathbb{R}^n$. Cauchy problem.

An initial condition is prescribed:

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0.$$

Solution:

$$w = \int_0^t \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x} - \mathbf{y}, t - \tau)\Phi(\mathbf{y}, \tau)\, dV_y\, d\tau + \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x} - \mathbf{y}, t)\big[f(\mathbf{y}) - b\Delta_y f(\mathbf{y})\big]\, dV_y,$$

where Δ_y is the Laplace operator in the integration variables $y_1, \dots, y_n$.

For the function $\mathscr{E}_{\rm e}(\mathbf{x}, t)$ with $n = 2$ and $n = 3$, see Items 1° and 2° in Eq. 11.1.5.1.

3. $\dfrac{\partial^2 w}{\partial t^2} - a\Delta w - b\dfrac{\partial}{\partial t}\Delta w = 0.$

This equation arises when decomposing unsteady equations describing 3D motions of viscous compressible barotropic fluids (see Section 12.12.2).

Consider a finite open domain V with boundary S. The following conditions are prescribed:

$$\begin{aligned} w &= f(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_t w &= g(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \alpha w + \beta\frac{\partial w}{\partial n} &= 0 \quad \text{at} \quad \mathbf{x} \in S \quad \text{(boundary condition)}, \end{aligned}$$

where $\alpha = \alpha(\mathbf{x}) \ge 0$, $\beta = \beta(\mathbf{x}) \ge 0$, $\alpha + \beta > 0$ (all functions are assumed to be continuous in V), $\mathbf{x} = (x, y, z)$, and $\partial w/\partial n$ is the outward normal derivative on S. The boundary conditions of the first and second kind correspond to the special values $\beta = 0$ ($\alpha = 1$) and $\alpha = 0$ ($\beta = 1$), respectively.

Solution:

$$w(x, t) = \sum_{k=1}^{\infty} \big[A_k \psi_{k1}(t) + B_k \psi_{k2}(t)\big] u_k(\mathbf{x}),$$

where

$$\psi_{k1}(t) = \frac{\gamma_{k1}\exp(-\gamma_{k2}t) - \gamma_{k2}\exp(-\gamma_{k1}t)}{\gamma_{k1} - \gamma_{k2}}, \qquad \psi_{k2}(t) = \frac{\exp(-\gamma_{k2}t) - \exp(-\gamma_{k1}t)}{\gamma_{k1} - \gamma_{k2}},$$

$$\gamma_{k1} = \frac{b\lambda_k + \sqrt{b^2\lambda_k^2 - 4a\lambda_k}}{2}, \qquad \gamma_{k2} = \frac{b\lambda_k - \sqrt{b^2\lambda_k^2 - 4a\lambda_k}}{2},$$

$$A_k = \frac{1}{\|u_k\|^2}\int_V f(\mathbf{x}) u_k(\mathbf{x})\, dv, \qquad B_k = \frac{1}{\|u_k\|^2}\int_V g(\mathbf{x}) u_k(\mathbf{x})\, dv,$$

and the eigenvalues λ_k and eigenfunctions $u_k(\mathbf{x})$ are the same as in the solution of equation 11.1.5.1, Item 4°.

11.2 Fourth-Order One-Dimensional Nonstationary Equations

11.2.1 Equation of the Form $\dfrac{\partial w}{\partial t} + a^2 \dfrac{\partial^4 w}{\partial x^4} = \Phi(x,t)$

► **Particular solutions and the formal series solution.**

1°. Particular solutions of the homogeneous equation with $\Phi(x,t)=0$:

$$w(x) = Ax^3 + Bx^2 + Cx + D,$$
$$w(x,t) = A(x^5 - 120a^2xt) + B(x^4 - 24a^2t),$$
$$w(x,t) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\exp(-\lambda^4a^2t),$$

where A, B, C, D, and λ are arbitrary constants.

2°. Formal series solution with $\Phi(x,t)=0$:

$$w(x,t) = f(x) + \sum_{k=1}^{\infty}(-1)^k \frac{a^{2k}t^k}{k!}\frac{d^{4k}f(x)}{dx^{4k}},$$

where $f(x)$ is an arbitrary infinitely differentiable function. This solution satisfies the initial condition $w(x,0)=f(x)$. If the function $f(x)$ is a polynomial of degree n, then the solution is a polynomial in x of degree n as well.

► **Fundamental solution and the Cauchy problem.**

1°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{\pi}\int_0^\infty \exp(-a^2\xi^4t)\cos(x\xi)\,d\xi.$$

2°. Domain: $-\infty < x < \infty$. Cauchy problem.
An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty}\mathscr{E}(x-y,t)f(y)\,dy + \int_0^t\int_{-\infty}^{\infty}\mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau.$$

► **Solutions of boundary value problems in terms of the Green's function.**

1°. Consider problems on an interval $0 \le x \le l$ with the general initial condition

$$w = f(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t)=\int_0^l f(\xi)G(x,\xi,t)\,d\xi+\int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau. \tag{1}$$

2°. The Green's functions can be evaluated from the formula

$$G(x,\xi,t)=\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2}\exp(-\lambda_n^4a^2t), \tag{2}$$

where the λ_n and $\varphi_n(x)$ are determined by solving the self-adjoint eigenvalue problem for the fourth-order ordinary differential equation

$$\varphi''''-\lambda^4\varphi=0$$

subject to appropriate boundary conditions; the prime denotes differentiation with respect to x. The norms of eigenfunctions can be calculated by the formula

$$\|\varphi_n\|^2=\int_0^l\varphi_n^2(x)\,dx=\frac{l}{4}\varphi_n^2(l)+\frac{l}{4\lambda_n^4}\bigl[\varphi_n''(l)\bigr]^2-\frac{l}{2\lambda_n^4}\varphi_n'(l)\varphi_n'''(l). \tag{3}$$

Relations (2) and (3) are written under the assumption that $\lambda=0$ is not an eigenvalue.

► **Green's functions for various boundary value problems.**

Items 1° through 8° below present the Green's functions for various types of boundary conditions. The solutions of these boundary value problems are represented via the Green's functions by formula (1).

1°. Domain: $0\le x\le l$. The function and its first derivative are prescribed at the boundaries:

$$w=\partial_x w=0\quad\text{at}\quad x=0,\qquad w=\partial_x w=0\quad\text{at}\quad x=l.$$

Green's function:

$$G(x,\xi,t)=\frac{4}{l}\sum_{n=1}^{\infty}\frac{\lambda_n^4}{\bigl[\varphi_n''(l)\bigr]^2}\varphi_n(x)\varphi_n(\xi)\exp(-\lambda_n^4a^2t),$$

where

$$\begin{aligned}\varphi_n(x)=\bigl[\sinh(\lambda_nl)-\sin(\lambda_nl)\bigr]\bigl[\cosh(\lambda_nx)-\cos(\lambda_nx)\bigr]\\-\bigl[\cosh(\lambda_nl)-\cos(\lambda_nl)\bigr]\bigl[\sinh(\lambda_nx)-\sin(\lambda_nx)\bigr];\end{aligned}$$

the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l)=1$. The numerical values of the roots can be calculated from the formulas

$$\lambda_n=\frac{\mu_n}{l},\quad\text{where}\quad\mu_1=4.730,\quad\mu_2=7.859,\quad\mu_n=\frac{\pi}{2}(2n+1)\quad\text{for}\quad n\ge 3. \tag{4}$$

2°. Domain: $0 \le x \le l$. The function and its second derivative are prescribed at the boundaries:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} \sin(\lambda_n x) \sin(\lambda_n \xi) \exp(-\lambda_n^4 a^2 t), \qquad \lambda_n = \frac{\pi n}{l}.$$

3°. Domain: $0 \le x \le l$. The first and third derivatives are prescribed at the boundaries:

$$\partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \cos(\lambda_n x) \cos(\lambda_n \xi) \exp(-\lambda_n^4 a^2 t), \qquad \lambda_n = \frac{\pi n}{l}.$$

4°. Domain: $0 \le x \le l$. The second and third derivatives are prescribed at the boundaries:

$$w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{1}{l} + \frac{3}{l^3}(2x - l)(2\xi - l) + \frac{4}{l} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \exp(-\lambda_n^4 a^2 t),$$

where

$$\begin{aligned} \varphi_n(x) = {} & \big[\sinh(\lambda_n l) - \sin(\lambda_n l)\big]\big[\cosh(\lambda_n x) + \cos(\lambda_n x)\big] \\ & - \big[\cosh(\lambda_n l) - \cos(\lambda_n l)\big]\big[\sinh(\lambda_n x) + \sin(\lambda_n x)\big] \end{aligned}$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$. For the numerical values of the roots, see the end of Item 1°.

5°. Domain: $0 \le x \le l$. Mixed conditions are prescribed at the boundaries (case 1):

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} \lambda_n^4 \frac{\varphi_n(x)\varphi_n(\xi)}{|\varphi_n'(l)\varphi_n'''(l)|} \exp(-\lambda_n^4 a^2 t),$$

where

$$\begin{aligned} \varphi_n(x) = {} & \big[\sinh(\lambda_n l) - \sin(\lambda_n l)\big]\big[\cosh(\lambda_n x) - \cos(\lambda_n x)\big] \\ & - \big[\cosh(\lambda_n l) - \cos(\lambda_n l)\big]\big[\sinh(\lambda_n x) - \sin(\lambda_n x)\big], \end{aligned}$$

and the λ_n are positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$.

6°. Domain: $0 \le x \le l$. Mixed conditions are prescribed at the boundaries (case 2):

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{l} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \exp(-\lambda_n^4 a^2 t),$$

where

$$\varphi_n(x) = \big[\sinh(\lambda_n l) + \sin(\lambda_n l)\big]\big[\cosh(\lambda_n x) - \cos(\lambda_n x)\big] - \big[\cosh(\lambda_n l) + \cos(\lambda_n l)\big]\big[\sinh(\lambda_n x) - \sin(\lambda_n x)\big]$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = -1$.

7°. Domain: $0 \le x \le l$. Mixed conditions are prescribed at the boundaries (case 3):

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=0}^{\infty} \sin(\lambda_n x)\sin(\lambda_n \xi)\exp(-\lambda_n^4 a^2 t), \qquad \lambda_n = \frac{\pi(2n+1)}{2l}.$$

8°. Domain: $0 \le x \le l$. Mixed conditions are prescribed at the boundaries (case 4):

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{l} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \exp(-\lambda_n^4 a^2 t),$$

where

$$\varphi_n(x) = \sin(\lambda_n l)\sinh(\lambda_n x) + \sinh(\lambda_n l)\sin(\lambda_n x);$$

the λ_n are positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$.

11.2.2 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2} + a^2 \dfrac{\partial^4 w}{\partial x^4} = 0$

This equation is encountered when studying *transverse vibrations of elastic bars*.

► **Particular solutions and the formal series solution.**

1°. Particular solutions:

$$w(x,t) = (Ax^3 + Bx^2 + Cx + D)t + A_1x^3 + B_1x^2 + C_1x + D_1,$$
$$w(x,t) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\sin(\lambda^2 at),$$
$$w(x,t) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\cos(\lambda^2 at),$$
$$w(x,t) = \exp(-\lambda x)\big[A\sin(\lambda x) + B\cos(\lambda x)\big]\big[C\exp(-2\lambda^2 at) + D\exp(2\lambda^2 at)\big],$$
$$w(x,t) = \exp(\lambda x)\big[A\sin(\lambda x) + B\cos(\lambda x)\big]\big[C\exp(-2\lambda^2 at) + D\exp(2\lambda^2 at)\big],$$

where A, B, C, D, A_1, B_1, C_1, D_1, and λ are arbitrary constants.

2°. Formal series solution:

$$w(x,t) = \sum_{k=0}^{\infty}(-1)^k \frac{a^{2k}t^{2k}}{(2k)!}\frac{d^{4k}f(x)}{dx^{4k}} + \sum_{k=0}^{\infty}(-1)^k \frac{a^{2k}t^{2k+1}}{(2k+1)!}\frac{d^{4k}g(x)}{dx^{4k}},$$

where $f(x)$ and $g(x)$ are arbitrary infinitely differentiable functions and $d^0f(x)/dx^0 = f(x)$. This solution satisfies the initial conditions $w(x,0) = f(x)$ and $\partial_t w(x,0) = g(x)$. If the functions $f(x)$ and $g(x)$ are polynomials of degree $\le n$, then the solution is a polynomial in x of degree $\le n$ as well.

► **Fundamental solution and the Cauchy problem.**

1°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(a\xi^2 t)}{a\xi^2}e^{ix\xi}\,d\xi = \frac{1}{\pi}\int_0^{\infty}\frac{\sin(a\xi^2 t)}{a\xi^2}\cos(x\xi)\,d\xi$$
$$= \frac{x}{2a}\left[S\left(\frac{x^2}{4at}\right) - C\left(\frac{x^2}{4at}\right) + \sqrt{\pi a}\,\sin\left(\frac{x^2}{4at} + \frac{\pi}{4}\right)\right],$$

where $C(z)$ and $S(z)$ are the Fresnel integrals.

2°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = ag''(x) \quad \text{at} \quad t = 0.$$

Boussinesq solution:

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\big(x - 2\xi\sqrt{at}\,\big)\big(\cos\xi^2 + \sin\xi^2\big)\,d\xi$$
$$+ \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} g\big(x - 2\xi\sqrt{at}\,\big)\big(\cos\xi^2 - \sin\xi^2\big)\,d\xi.$$

⊙ *Literature*: I. Sneddon (1951).

► **Boundary value problems.**

1°. Domain: $0 \le x < \infty$. The problem on free vibrations of a semi-infinite bar. The following conditions are prescribed:

$$\begin{aligned} &w = 0 \quad \text{at} \quad t = 0, &&\partial_t w = 0 \quad \text{at} \quad t = 0 &&\text{(initial conditions)},\\ &w = f(t) \quad \text{at} \quad x = 0, &&\partial_{xx} w = 0 \quad \text{at} \quad x = 0 &&\text{(boundary conditions)}. \end{aligned}$$

Boussinesq solution:

$$w(x,t) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{2at}}^{\infty} f\left(t - \frac{x^2}{2a\xi^2}\right)\left(\sin\frac{\xi^2}{2} + \cos\frac{\xi^2}{2}\right) d\xi.$$

⊙ *Literature*: I. Sneddon (1951).

2°. For solutions of various boundary value problems in the domain $0 \le x \le l$, see Section 11.2.3 for $\Phi \equiv 0$.

11.2.3 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2} + a^2 \dfrac{\partial^4 w}{\partial x^4} = \Phi(x,t)$

This equation is encountered when studying forced (transverse) vibrations of elastic bars.

► **Solutions of boundary value problems in terms of the Green's function.**

1°. In this section we consider boundary value problems on an interval $0 \le x \le l$ with the general initial conditions

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$\begin{aligned} w(x,t) = \frac{\partial}{\partial t} \int_0^l f(\xi) G(x,\xi,t)\, d\xi + \int_0^l g(\xi) G(x,\xi,t)\, d\xi \\ + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau. \end{aligned} \tag{1}$$

2°. The Green's functions can be evaluated from the formula

$$G(x,\xi,t) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2 \|\varphi_n\|^2} \sin(\lambda_n^2 a t), \tag{2}$$

where the λ_n and $\varphi_n(x)$ are determined by solving the self-adjoint eigenvalue problem for the fourth-order ordinary differential equation

$$\varphi'''' - \lambda^4 \varphi = 0$$

subject to appropriate boundary conditions; the prime denotes differentiation with respect to x. The norms of eigenfunctions can be calculated by Krylov's formula [see Krylov (1949)]:

$$\|\varphi_n\|^2 = \int_0^l \varphi_n^2(x)\, dx = \frac{l}{4}\varphi_n^2(l) + \frac{l}{4\lambda_n^4}\left[\varphi_n''(l)\right]^2 - \frac{l}{2\lambda_n^4}\varphi_n'(l)\varphi_n'''(l). \tag{3}$$

Relations (2) and (3) are written under the assumption that $\lambda = 0$ is not an eigenvalue.

► **Green's functions for various boundary value problems.**

Items 1° through 8° below present the Green's functions for various types of boundary conditions. The solutions of these boundary value problems are represented via the Green's functions by formula (1).

1°. Domain: $0 \le x \le l$. Both ends of the bar are clamped.
Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{al} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{\bigl[\varphi_n''(l)\bigr]^2} \varphi_n(x)\varphi_n(\xi) \sin(\lambda_n^2 at),$$

where

$$\begin{aligned}\varphi_n(x) = \bigl[\sinh(\lambda_n l) - \sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x) - \cos(\lambda_n x)\bigr]\\ - \bigl[\cosh(\lambda_n l) - \cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x) - \sin(\lambda_n x)\bigr]\end{aligned}$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$. The numerical values of the roots can be calculated from the formulas

$$\lambda_n = \frac{\mu_n}{l}, \quad \text{where} \quad \mu_1 = 4.730, \quad \mu_2 = 7.859, \quad \mu_n = \frac{\pi}{2}(2n+1) \quad \text{for} \quad n \ge 3.$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

2°. Domain: $0 \le x \le l$. Both ends of the bar are hinged.
Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2l}{a\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\lambda_n x)\sin(\lambda_n \xi)\sin(\lambda_n^2 at), \qquad \lambda_n = \frac{\pi n}{l}.$$

⊙ *Literature*: A. N. Krylov (1949), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

3°. Domain: $0 \le x \le l$. Both ends of the bar are free.
Boundary conditions are prescribed:

$$w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{t}{l} + \frac{3t}{l^3}(2x - l)(2\xi - l) + \frac{4}{al} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2 \varphi_n^2(l)} \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = \big[\sinh(\lambda_n l) - \sin(\lambda_n l)\big]\big[\cosh(\lambda_n x) + \cos(\lambda_n x)\big] \\ - \big[\cosh(\lambda_n l) - \cos(\lambda_n l)\big]\big[\sinh(\lambda_n x) + \sin(\lambda_n x)\big]$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$. For the numerical values of the roots, see the end of Item 1°.

The first two terms in the expression for the Green's function correspond to the zero eigenvalue $\lambda_0 = 0$, to which there correspond two orthogonal eigenfunctions $w_0^{(1)} = 1$ and $w_0^{(2)} = 2x - l$ with $\|w_0^{(1)}\|^2 = l$ and $\|w_0^{(2)}\|^2 = \frac{1}{3}l^3$.

⊙ *Literature*: A. N. Krylov (1949).

4°. Domain: $0 \le x \le l$. One end of the bar is clamped and the other is hinged.
Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{al} \sum_{n=1}^{\infty} \lambda_n^2 \frac{\varphi_n(x)\varphi_n(\xi)}{|\varphi_n'(l)\varphi_n'''(l)|} \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = \big[\sinh(\lambda_n l) - \sin(\lambda_n l)\big]\big[\cosh(\lambda_n x) - \cos(\lambda_n x)\big] \\ - \big[\cosh(\lambda_n l) - \cos(\lambda_n l)\big]\big[\sinh(\lambda_n x) - \sin(\lambda_n x)\big]$$

and the λ_n are positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$.

5°. Domain: $0 \le x \le l$. One end of the bar is clamped and the other is free.
Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{al} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2 \varphi_n^2(l)} \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = \big[\sinh(\lambda_n l) + \sin(\lambda_n l)\big]\big[\cosh(\lambda_n x) - \cos(\lambda_n x)\big] \\ - \big[\cosh(\lambda_n l) + \cos(\lambda_n l)\big]\big[\sinh(\lambda_n x) - \sin(\lambda_n x)\big]$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = -1$.

6°. Domain: $0 \le x \le l$. One end of the bar is hinged and the other is free.
Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{al} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2 \varphi_n^2(l)} \sin(\lambda_n^2 a t),$$

where

$$\varphi_n(x) = \sin(\lambda_n l) \sinh(\lambda_n x) + \sinh(\lambda_n l) \sin(\lambda_n x)$$

and the λ_n are positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$.

7°. Domain: $0 \le x \le l$. The first and third derivatives are prescribed at the ends:

$$\partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{t}{l} + \frac{2}{al} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \cos(\lambda_n x) \cos(\lambda_n \xi) \sin(\lambda_n^2 a t), \qquad \lambda_n = \frac{\pi n}{l}.$$

8°. Domain: $0 \le x \le l$. Mixed boundary conditions are prescribed at the ends:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{al} \sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} \sin(\lambda_n x) \sin(\lambda_n \xi) \sin(\lambda_n^2 a t), \qquad \lambda_n = \frac{\pi(2n+1)}{2l}.$$

11.2.4 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} + kw = \Phi(x, t)$

► **Particular solutions.**

Particular solutions of the homogeneous equation ($\Phi \equiv 0$):

$$w(x, t) = (Ax^3 + Bx^2 + Cx + D) \sin\bigl(t\sqrt{k}\,\bigr),$$
$$w(x, t) = (Ax^3 + Bx^2 + Cx + D) \cos\bigl(t\sqrt{k}\,\bigr),$$
$$w(x, t) = \bigl[A \sin(\lambda x) + B \cos(\lambda x) + C \sinh(\lambda x) + D \cosh(\lambda x)\bigr] \sin\bigl(t\sqrt{a^2\lambda^4 + k}\,\bigr),$$
$$w(x, t) = \bigl[A \sin(\lambda x) + B \cos(\lambda x) + C \sinh(\lambda x) + D \cosh(\lambda x)\bigr] \cos\bigl(t\sqrt{a^2\lambda^4 + k}\,\bigr),$$

where A, B, C, D, and λ are arbitrary constants.

► **Solution of boundary value problems in terms of the Green's function.**

1°. Consider boundary value problems on the interval $0 \le x \le l$ with the general initial conditions

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$\begin{aligned} w(x,t) = \frac{\partial}{\partial t}\int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^l g(\xi)G(x,\xi,t)\,d\xi \\ + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau. \end{aligned} \tag{1}$$

2°. The Green's functions can be evaluated from the formula

$$G(x,\xi,t) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2} \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}}, \tag{2}$$

where the λ_n and $\varphi_n(x)$ are determined by solving the self-adjoint eigenvalue problem for the fourth-order ordinary differential equation $\varphi'''' - \lambda^4\varphi = 0$ subject to appropriate boundary conditions. The norms of eigenfunctions can be calculated by formula (3) in Section 11.2.3.

► **Green's functions for various boundary value problems.**

Items 1° through 8° below present the Green's functions for various types of boundary conditions. The solutions of these boundary value problems are represented via the Green's functions by formula (1).

1°. Domain: $0 \le x \le l$. The function and its first derivative are prescribed at the ends:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t) = \frac{4}{l}\sum_{n=1}^{\infty} \lambda_n^4 \frac{\varphi_n(x)\varphi_n(\xi)}{\bigl[\varphi_n''(l)\bigr]^2} \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}}, \qquad \varphi_n''(x) = \frac{d^2\varphi_n}{dx^2},$$

where

$$\begin{aligned} \varphi_n(x) = \bigl[\sinh(\lambda_n l) - \sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x) - \cos(\lambda_n x)\bigr] \\ - \bigl[\cosh(\lambda_n l) - \cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x) - \sin(\lambda_n x)\bigr] \end{aligned}$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$.

2°. Domain: $0 \le x \le l$. The function and its second derivative are prescribed at the ends:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} \sin(\lambda_n x) \sin(\lambda_n \xi) \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}}, \qquad \lambda_n = \frac{\pi n}{l}.$$

3°. Domain: $0 \le x \le l$. The first and third derivatives are prescribed at the ends:

$$\partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{\sin\bigl(t\sqrt{k}\,\bigr)}{l\sqrt{k}} + \frac{2}{l} \sum_{n=1}^{\infty} \cos(\lambda_n x) \cos(\lambda_n \xi) \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}}, \qquad \lambda_n = \frac{\pi n}{l}.$$

4°. Domain: $0 \le x \le l$. The second and third derivatives are prescribed at the ends:

$$w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \left[1 + \frac{3}{l^2}(2x - l)(2\xi - l)\right] \frac{\sin\bigl(t\sqrt{k}\,\bigr)}{l\sqrt{k}} + \frac{4}{l} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}},$$

where

$$\begin{aligned} \varphi_n(x) = {} & \bigl[\sinh(\lambda_n l) - \sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x) + \cos(\lambda_n x)\bigr] \\ & - \bigl[\cosh(\lambda_n l) - \cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x) + \sin(\lambda_n x)\bigr] \end{aligned}$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$. For the numerical values of the roots, see formulas (4) in Section 11.2.1.

5°. Domain: $0 \le x \le l$. Mixed boundary conditions are prescribed at the ends (case 1):

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} \lambda_n^4 \frac{\varphi_n(x)\varphi_n(\xi)}{|\varphi_n'(l)\varphi_n'''(l)|} \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}},$$

where

$$\begin{aligned} \varphi_n(x) = {} & \bigl[\sinh(\lambda_n l) - \sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x) - \cos(\lambda_n x)\bigr] \\ & - \bigl[\cosh(\lambda_n l) - \cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x) - \sin(\lambda_n x)\bigr] \end{aligned}$$

and the λ_n are positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$.

6°. Domain: $0 \le x \le l$. Mixed boundary conditions are prescribed at the ends (case 2):

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{l} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}},$$

where

$$\varphi_n(x) = \bigl[\sinh(\lambda_n l) + \sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x) - \cos(\lambda_n x)\bigr] - \bigl[\cosh(\lambda_n l) + \cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x) - \sin(\lambda_n x)\bigr]$$

and the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = -1$.

7°. Domain: $0 \le x \le l$. Mixed boundary conditions are prescribed at the ends (case 3):

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=0}^{\infty} \sin(\lambda_n x)\sin(\lambda_n \xi) \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}}, \qquad \lambda_n = \frac{\pi(2n+1)}{2l}.$$

8°. Domain: $0 \le x \le l$. Mixed boundary conditions are prescribed at the ends (case 4):

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{l} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \frac{\sin\bigl(t\sqrt{a^2\lambda_n^4 + k}\,\bigr)}{\sqrt{a^2\lambda_n^4 + k}},$$

where

$$\varphi_n(x) = \sin(\lambda_n l)\sinh(\lambda_n x) + \sinh(\lambda_n l)\sin(\lambda_n x);$$

the λ_n are positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$.

11.2.5 Other Equations without Mixed Derivatives

► **Equations containing the first derivative with respect to t.**

1. $\dfrac{\partial w}{\partial t} + a^2 \dfrac{\partial^4 w}{\partial x^4} + kw = \Phi(x, t).$

The change of variable $w(x, t) = e^{-kt} u(x, t)$ leads to the equation

$$\frac{\partial u}{\partial t} + a^2 \frac{\partial^4 u}{\partial x^4} = e^{kt}\Phi(x, t),$$

which is discussed in Section 11.2.1.

2. $\dfrac{\partial w}{\partial t} = ax^8 \dfrac{\partial^4 w}{\partial x^4}.$

This is a special case of equation 11.7.4.3 with $k = 1$ and $n = 4$.

3. $\dfrac{\partial w}{\partial t} = k(t) \dfrac{\partial^4 w}{\partial x^4} + [xf(t) + g(t)] \dfrac{\partial w}{\partial x} + h(t)w.$

This is a special case of equation 11.7.4.1 with $n = 4$. The transformation

$$w(x,t) = u(z,\tau) \exp\left[\int h(t)\,dt\right], \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int k(t)F^4(t)\,dt,$$

where $F(t) = \exp\left[\int f(t)\,dt\right]$, leads to the constant coefficient equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^4 u}{\partial z^4},$$

which is discussed in Section 11.2.1.

4. $\dfrac{\partial w}{\partial t} = (ax^2 + bx + c)^4 \dfrac{\partial^4 w}{\partial x^4}.$

This is a special case of equation 11.7.4.5 with $k = 1$ and $n = 4$.

► Equations containing the second derivative with respect to t.

5. $\dfrac{\partial^2 w}{\partial t^2} - a^2 \dfrac{\partial^4 w}{\partial x^4} = 0.$

1°. General solution:

$$w = w_1 + w_2,$$

where w_1 and w_2 are arbitrary solutions of two second-order equations

$$\frac{\partial w_1}{\partial t} - a\frac{\partial^2 w_1}{\partial x^2} = 0, \qquad \frac{\partial w_2}{\partial t} + a\frac{\partial^2 w_2}{\partial x^2} = 0.$$

2°. Formal series solution:

$$w(x,t) = \sum_{k=0}^{\infty} \frac{a^{2k} t^{2k}}{(2k)!} \frac{d^{4k} f(x)}{dx^{4k}} + \sum_{k=0}^{\infty} \frac{a^{2k} t^{2k+1}}{(2k+1)!} \frac{d^{4k} g(x)}{dx^{4k}},$$

where $f(x)$ and $g(x)$ are arbitrary infinitely differentiable functions and $d^0 f(x)/dx^0 = f(x)$. This solution satisfies the initial conditions $w(x,0) = f(x)$ and $\partial_t w(x,0) = g(x)$. If the functions $f(x)$ and $g(x)$ are polynomials of degree $\le n$, then the solution is a polynomial in x of degree $\le n$ as well.

6. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} + a^2 \dfrac{\partial^4 w}{\partial x^4} = \Phi(x,t).$

For $\Phi(x,t) \equiv 0$, this equation describes transverse vibrations of an elastic bar in a resisting medium with resistance coefficient proportional to the velocity.

The change of variable $w(x,t) = \exp\left(-\tfrac{1}{2}kt\right)u(x,t)$ leads to the equation

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} - \tfrac{1}{4}k^2 u = \exp\left(\tfrac{1}{2}kt\right)\Phi(x,t),$$

which is discussed in Section 11.2.4.

7. $\dfrac{\partial^2 w}{\partial t^2} = ax^8 \dfrac{\partial^4 w}{\partial x^4}.$

This is a special case of equation 11.7.4.3 with $k = 2$ and $n = 4$.

8. $\dfrac{\partial^2 w}{\partial t^2} = (ax^2 + bx + c)^4 \dfrac{\partial^4 w}{\partial x^4}.$

This is a special case of equation 11.7.4.5 with $k = 2$ and $n = 4$.

► **Equation containing the fourth derivative with respect to t.**

9. $\dfrac{\partial^4 w}{\partial t^4} - \dfrac{\partial^4 w}{\partial x^4} = 0.$

1°. Particular solution:

$$w(x,t) = f(x-t) + g(x+t),$$

where $f(z_1)$ and $g(z_2)$ are arbitrary functions.

2°. General solution:

$$w = w_1 + w_2,$$

where w_1 and w_2 are arbitrary solutions of two second-order equations

$$\frac{\partial^2 w_1}{\partial t^2} - \frac{\partial^2 w_1}{\partial x^2} = 0, \qquad \frac{\partial^2 w_2}{\partial t^2} + \frac{\partial^2 w_2}{\partial x^2} = 0.$$

3°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\pi}\biggl\{t \ln\sqrt{x^2+t^2} - x \arctan\frac{x}{t} - \frac{1}{2}(t+x)\ln|t+x| \\ - \frac{1}{2}(t-x)\ln|t-x| + \frac{1}{8}|t+x| + \frac{1}{8}|t-x|\biggr\}.$$

4°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = 0 \quad \text{at} \quad t = 0, \qquad \partial_t w = 0 \quad \text{at} \quad t = 0, \qquad \partial_{tt} w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}(x-\xi, t) f(\xi)\, d\xi.$$

⊙ *Literature*: G. E. Shilov (1965).

11.2.6 Equations Containing Second Derivative in x and Mixed Derivatives

1. $\dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^4 w}{\partial t^2 \partial x^2} = 0.$

One-dimensional wave equation with strong dispersion. This equation describes the dynamics of interior one-dimensional wave motions in an exponentially stratified fluid as well as one-dimensional longitudinal vibrations of a rigid Rayleigh bar of constant cross-section. This is a special case of equation 11.2.6.4 with $a(x) = b(x) = c(x) = 1$ and $\Phi(x,t) = 0$.

1°. Particular solutions:

$$w = \big[C_1 \exp(-\lambda t) + C_2 \exp(\lambda t)\big]\left[C_3 \exp\left(-\frac{\lambda x}{\sqrt{1+\lambda^2}}\right) + C_4 \exp\left(\frac{\lambda x}{\sqrt{1+\lambda^2}}\right)\right],$$
$$w = \big[C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\big]\left[C_3 \cos\left(\frac{\lambda t}{\sqrt{1+\lambda^2}}\right) + C_4 \sin\left(\frac{\lambda t}{\sqrt{1+\lambda^2}}\right)\right],$$
$$w = \big[C_1 \cos(\lambda t) + C_2 \sin(\lambda t)\big]\left[C_3 \exp\left(-\frac{\lambda x}{\sqrt{\lambda^2-1}}\right) + C_4 \exp\left(\frac{\lambda x}{\sqrt{\lambda^2-1}}\right)\right], \quad |\lambda| > 1,$$

where C_1, C_2, C_3, C_4, and λ are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}_{\rm e}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i|x|\xi}}{\xi\sqrt{1+\xi^2}} \sin\left(\frac{t\xi}{\sqrt{1+\xi^2}}\right) d\xi,$$
$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos(|x|\xi)}{\xi\sqrt{1+\xi^2}} \sin\left(\frac{t\xi}{\sqrt{1+\xi^2}}\right) d\xi, \qquad i^2 = -1.$$

Here the absolute value is only written to emphasize that the function $\mathscr{E}_{\rm e}(x,t)$ is even in the variable x.

The following asymptotic formulas hold:

(i) For $x = 0$ and $t \to \infty$,

$$\mathscr{E}_{\rm e}(0,t) = \frac{1}{2}\int_0^t J_0(\xi)\,d\xi = \frac{1}{2} + O(t^{-1/2}).$$

(ii) For $|x|/t \in [\delta, 1-\delta]$, where $\delta \in (0, 1/2)$, and $t \to \infty$,

$$\mathscr{E}_{\rm e}(x,t) = \frac{1}{2} + \frac{\sin[(t^{2/3} - |x|^{2/3})^{3/2} - \pi/4]}{\sqrt{6\pi}(t^{2/3} - |x|^{2/3})^{3/4}} + O(t^{-1}).$$

(iii) For $|x| = t$ and $t \to \infty$,

$$\mathscr{E}_{\rm e}(x,t) = \frac{1}{6} + O(t^{-1}).$$

(iv) For $|x|/t \in [1+\delta, \infty)$, where $\delta > 0$, and $|x| \to \infty$,

$$\mathscr{E}_{\rm e}(x,t) = \frac{\exp[-(|x|^{2/3} - t^{2/3})^{3/2}]}{2\sqrt{6\pi}(|x|^{2/3} - t^{2/3})^{3/4}}\big[1 + O(|x|^{-1/2})\big].$$

(v) For $t \to 0$,

$$\mathscr{E}_{\rm e}(x,t) = \tfrac{1}{2} t e^{-|x|}\bigl[1 + O(t^2)\bigr].$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f_0(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)[f_0(y) - f_0''(y)]\,dy + \int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)[f_1(y) - f_1''(y)]\,dy.$$

4°. Domain: $0 \le x < \infty$. First boundary value problem.
The following conditions are prescribed:

$$w = f_0(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0,$$
$$w = g(t) \quad \text{at} \quad x = 0.$$

Solution:

$$w(x,t) = \frac{\partial}{\partial t}\int_0^{\infty} G_1(x,y,t)[f_0(y) - f_0''(y)]\,dy + \int_0^{\infty} G_1(x,y,t)[f_1(y) - f_1''(y)]\,dy - 2g(t)\biggl[\frac{\partial^2}{\partial t^2}\mathscr{E}_{\rm e}(x,t)\biggr]_{t=0} - 2\int_0^t B[\mathscr{E}_{\rm e}](x,t-\tau)g(\tau)\,d\tau,$$

$$G_1(x,y,t) = \mathscr{E}_{\rm e}(x-y,t) - \mathscr{E}_{\rm e}(x+y,t), \qquad B[u](x,t) = u(x,t) + \frac{\partial^2}{\partial t^2}u(x,t).$$

5°. Domain: $0 \le x < \infty$. A boundary value problem.
The following conditions are prescribed:

$$w = f_0(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0,$$
$$\partial_x w + \partial_{ttx} w = g(t) \quad \text{at} \quad x = 0.$$

Solution:

$$w(x,t) = \frac{\partial}{\partial t}\int_0^{\infty} G_2(x,y,t)[f_0(y) - f_0''(y)]\,dy + \int_0^{\infty} G_2(x,y,t)[f_1(y) - f_1''(y)]\,dy - \int_0^t G_2(x,0,t-\tau)g(\tau)\,d\tau, \qquad G_2(x,y,t) = \mathscr{E}_{\rm e}(x-y,t) + \mathscr{E}_{\rm e}(x+y,t).$$

6°. Domain: $0 \le x \le l$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x=0 && \text{(boundary condition)},\\ w &= 0 && \text{at} \quad x=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t)=\sum_{n=1}^{\infty}\Big[a_n\cos(\lambda_n t)+\frac{b_n}{\lambda_n}\sin(\lambda_n t)\Big]\sin(\beta_n x),\quad \beta_n=\frac{\pi n}{l},\quad \lambda_n=\frac{\beta_n}{\sqrt{1+\beta_n^2}},$$
$$a_n=\frac{2}{l}\int_0^l f_0(\xi)\sin(\beta_n\xi)\,d\xi,\qquad b_n=\frac{2}{l}\int_0^l f_1(\xi)\sin(\beta_n\xi)\,d\xi.$$

An alternative representation of the solution:

$$w(x,t)=\frac{\partial}{\partial t}\int_0^l G(x,\xi,t)f_0(\xi)\,d\xi+\int_0^l G(x,\xi,t)f_1(\xi)\,d\xi,$$
$$G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\sin(\beta_n x)\sin(\beta_n\xi)\sin(\lambda_n t).$$

7°. Domain: $0\le x\le l$. Second boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_x w &= 0 && \text{at} \quad x=0 && \text{(boundary condition)},\\ \partial_x w &= 0 && \text{at} \quad x=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t)=a_0+b_0t+\sum_{n=1}^{\infty}\Big[a_n\cos(\lambda_n t)+\frac{b_n}{\lambda_n}\sin(\lambda_n t)\Big]\cos(\beta_n x),$$
$$\beta_n=\frac{\pi n}{l},\quad \lambda_n=\frac{\beta_n}{\sqrt{1+\beta_n^2}},\quad a_0=\frac{1}{l}\int_0^l f_0(\xi)\,d\xi,\quad b_n=\frac{1}{l}\int_0^l f_1(\xi)\,d\xi,$$
$$a_n=\frac{2}{l}\int_0^l f_0(\xi)\sin(\beta_n\xi)\,d\xi,\qquad b_n=\frac{2}{l}\int_0^l f_1(\xi)\sin(\beta_n\xi)\,d\xi.$$

8°. Domain: $0\le x\le l$. Mixed boundary value problem.
The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w &= 0 && \text{at} \quad x=0 && \text{(boundary condition)},\\ \partial_x w &= 0 && \text{at} \quad x=l && \text{(boundary condition)}. \end{aligned}$$

Solution:

$$w(x,t)=\sum_{n=0}^{\infty}\Big[a_n\cos(\lambda_n t)+\frac{b_n}{\lambda_n}\sin(\lambda_n t)\Big]\sin(\beta_n x),$$

$$\beta_n=\frac{\pi(2n+1)}{2l},\qquad \lambda_n=\frac{\beta_n}{\sqrt{1+\beta_n^2}},$$

$$a_n=\frac{2}{l}\int_0^l f_0(\xi)\sin(\beta_n\xi)\,d\xi,\qquad b_n=\frac{2}{l}\int_0^l f_1(\xi)\sin(\beta_n\xi)\,d\xi.$$

⊙ *Literature for equation 11.2.6.1*: S. A. Gabov and B. B. Orazov (1986), S. A. Gabov and A. G. Sveshnikov (1990), I. A. Fedotov, A. D. Polyanin, and M. Yu. Shatalov (2007), I. A. Fedotov, Y. Gai, A. D. Polyanin, and M. Yu. Shatalov (2008).

2. $\dfrac{\partial^2 w}{\partial t^2}-\dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial^4 w}{\partial t^2\partial x^2}=\Phi(x,t).$

This is a special case of equation 11.2.6.4 with $a(x)=b(x)=c(x)=1$. For $\Phi(x,t)=0$, see equation 11.2.6.1.

1°. Domain: $-\infty<x<\infty$. Cauchy problem.
Initial conditions are prescribed:

$$w=f_0(x)\quad\text{at}\quad t=0,$$
$$\partial_t w=f_1(x)\quad\text{at}\quad t=0.$$

The solution of the Cauchy problem is given by the formula in Item 3° of equation 11.2.6.1 with the additional term

$$\int_0^t\int_{-\infty}^{\infty}\mathscr{E}_{\rm e}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau.$$

2°. Domain: $0\le x<\infty$. First boundary value problem.
The following conditions are prescribed:

$$w=f_0(x)\quad\text{at}\quad t=0,$$
$$\partial_t w=f_1(x)\quad\text{at}\quad t=0,$$
$$w=g(t)\quad\text{at}\quad x=0.$$

The solution of the first boundary value problem is given by the formula in Item 4° of equation 11.2.6.1 with the additional term

$$\int_0^t\int_0^{\infty}G_1(x,y,t-\tau)\Phi(y,\tau)\,dy\,d\tau.$$

3°. Domain: $0\le x<\infty$. A boundary value problem.
The following conditions are prescribed:

$$w=f_0(x)\quad\text{at}\quad t=0,$$
$$\partial_t w=f_1(x)\quad\text{at}\quad t=0,$$
$$\partial_x w+\partial_{ttx}w=g(t)\quad\text{at}\quad x=0.$$

The solution of the boundary value problem is given by the formula in Item 5° of equation 11.2.6.1 with the additional term

$$\int_0^t \int_0^\infty G_2(x, y, t-\tau)\Phi(y, \tau)\, dy\, d\tau.$$

4°. Domain: $0 \le x \le l$. First boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 \quad \text{(initial condition)},\\ w &= 0 && \text{at} \quad x = 0 \quad \text{(boundary condition)},\\ w &= 0 && \text{at} \quad x = l \quad \text{(boundary condition)}. \end{aligned}$$

The solution of the boundary value problem is given by the formula in Item 6° of equation 11.2.6.1 with the additional term

$$\sum_{n=1}^{\infty} \frac{\sin(\beta_n x)}{\beta_n\sqrt{1+\beta_n^2}} \int_0^t \varphi_n(\tau) \sin\left[\frac{\beta_n(t-\tau)}{\sqrt{1+\beta_n^2}}\right] d\tau,$$

where

$$\beta_n = \frac{\pi n}{l}, \quad \varphi_n(\tau) = \frac{2}{l}\int_0^l \Phi(\xi, \tau)\sin(\beta_n \xi)\, d\xi.$$

⊙ *Literature*: S. A. Gabov and A. G. Sveshnikov (1990).

3. $\dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^4 w}{\partial t^2 \partial x^2} + \sigma^2 w = \Phi(x, t), \qquad \sigma^2 \neq 1.$

One-dimensional wave equation with strong dispersion taking into account the rotation of an exponentially stratified fluid.

1°. Particular solutions of the homogeneous equation with $\Phi \equiv 0$:

$$w = \big[C_1 \exp(-\lambda t) + C_2 \exp(\lambda t)\big]\left[C_3 \exp\left(-x\sqrt{\frac{\lambda^2+\sigma^2}{\lambda^2+1}}\right) + C_4 \exp\left(x\sqrt{\frac{\lambda^2+\sigma^2}{\lambda^2+1}}\right)\right],$$

$$w = \big[C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\big]\left[C_3 \cos\left(t\sqrt{\frac{\lambda^2+\sigma^2}{\lambda^2+1}}\right) + C_4 \sin\left(t\sqrt{\frac{\lambda^2+\sigma^2}{\lambda^2+1}}\right)\right],$$

$$w = \big[C_1 \cos(\lambda t) + C_2 \sin(\lambda t)\big]\left[C_3 \exp\left(-x\sqrt{\frac{\lambda^2-\sigma^2}{\lambda^2-1}}\right) + C_4 \exp\left(x\sqrt{\frac{\lambda^2-\sigma^2}{\lambda^2-1}}\right)\right],$$

where C_1, C_2, C_3, C_4, and λ are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}_{\rm e}(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\mu x}}{\sqrt{1+\mu^2}\sqrt{\sigma^2+\mu^2}} \sin\left(t\frac{\sqrt{\sigma^2+\mu^2}}{\sqrt{1+\mu^2}}\right) d\mu, \qquad i^2 = -1.$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$w = f_0(x) \quad \text{at} \quad t = 0,$$
$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_0^t \int_{-\infty}^{\infty} \mathscr{E}(x-y, t-\tau)\Phi(y,\tau)\,dy\,d\tau$$
$$+ \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}(x-y,t)[f_0(y) - f_0''(y)]\,dy + \int_{-\infty}^{\infty} \mathscr{E}(x-y,t)[f_1(y) - f_1''(y)]\,dy.$$

⊙ *Literature*: S. A. Gabov and A. G. Sveshnikov (1990).

4. $\displaystyle a(x)\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\Big[b(x)\frac{\partial w}{\partial x} + c(x)\frac{\partial^3 w}{\partial^2 t\,\partial x}\Big] = \Phi(x,t).$

This equation describes one-dimensional longitudinal vibrations of a rigid Rayleigh bar of variable cross-section and takes into account the inertial effects of the transverse motion. The coefficients of the equation are positive functions and can be expressed as follows via geometric and physical variables:

$$a(x) = \rho(x)S(x), \quad b(x) = S(x)E(x), \quad c(x) = \rho(x)\nu^2(x)I(x),$$

where $\rho(x)$ is the bar mass density, $S(x)$ is the transverse cross-section area, $E(x)$ is the Young modulus, $\nu(x)$ is the Poisson ratio, and $I(x)$ is the polar moment of inertia.

► Initial and boundary conditions.

This equation is considered with the general boundary conditions

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0 \tag{1}$$

and with two types of boundary conditions given below.

(i) Geometric boundary conditions (rigid clamping of the bar ends):

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l. \tag{2}$$

(ii) Dynamic boundary conditions (the bar ends are fixed with the use of springs and lumped masses):

$$\begin{aligned} b(x)\frac{\partial w}{\partial x} + c(x)\frac{\partial^3 w}{\partial^2 t\,\partial x} - K_1 w - M_1\frac{\partial^2 w}{\partial t^2} = 0 \quad \text{at} \quad x = 0,\\ b(x)\frac{\partial w}{\partial x} + c(x)\frac{\partial^3 w}{\partial^2 t\,\partial x} + K_2 w + M_2\frac{\partial^2 w}{\partial t^2} = 0 \quad \text{at} \quad x = l, \end{aligned} \tag{3}$$

where $K_n \ge 0$ and $M_n \ge 0$. The boundary conditions (3) express the balance of elastic deformation forces of the Rayleigh bar (the first two terms), elastic deformation forces of the springs (the terms proportional to K_n), and the inertia forces exerted on the bar by the accelerated lumped masses (the terms proportional to M_n).

► **Free vibrations of the Rayleigh bar. The Sturm–Liouville problem.**

Free vibrations are determined by the relations

$$w(x,t) = [A\cos(\omega t) + B\sin(\omega t)]\varphi(x), \qquad \Phi(x,t) = 0.$$

By substituting these relations into the equation and the dynamic boundary conditions (3), we arrive at the nonclassical Sturm–Liouville boundary value problem

$$[p(x,\omega)\varphi'_x]'_x + \omega^2 a(x)\varphi = 0, \tag{4}$$

$$p(0,\omega)\varphi'_x - (K_1 - \omega^2 M_1)\varphi = 0 \quad \text{at} \quad x = 0, \tag{5}$$

$$p(l,\omega)\varphi'_x + (K_2 - \omega^2 M_2)\varphi = 0 \quad \text{at} \quad x = l, \tag{6}$$

where

$$p(x,\omega) = b(x) - \omega^2 c(x). \tag{7}$$

We point out that the spectral parameter ω occurs in Eq. (4) as well as in the boundary conditions (5)–(6) in a complicated way. Without loss of generality, we can assume that $\omega > 0$.

In the general case, the eigenvalues ω_n of the Sturm–Liouville problem (4)–(7) are bounded and sit on the interval

$$0 \le |\omega_n| < \omega_\infty = \inf_{0\le x\le l} \sqrt{b(x)/c(x)},$$

which is ensured by the positivity of the coefficient $p(x,\omega_n)$ in Eq. (4).

► **Two types of orthogonality of eigenfunctions.**

Let ω_m and ω_n be two distinct eigenvalues of the Sturm–Liouville problem (4)–(7), and let $\varphi_m(x)$ and $\varphi_n(x)$ be the corresponding eigenfunctions.

1°. *First type of orthogonality* is the mass orthogonality of eigenfunctions for the case of the dynamic boundary conditions (3),

$$\begin{aligned}\int_0^l \big[a(x)\varphi_m(x)\varphi_n(x) + c(x)\varphi'_m(x)\varphi'_n(x)\big]\,dx \\ + M_1\varphi_m(0)\varphi_n(0) + M_2\varphi_m(l)\varphi_n(l) = 0, \qquad m \ne n.\end{aligned} \tag{8}$$

The corresponding squared mass norm of the eigenfunction is given by

$$\|\varphi_n\|_1^2 = \int_0^l \big[a(x)\varphi_n^2(x) + c(x)\varphi_n'^2(x)\big]\,dx + M_1\varphi_n^2(0) + M_2\varphi_n^2(l). \tag{9}$$

For the geometric boundary conditions (2), one should set $M_1 = M_2 = 0$ in formulas (8) and (9).

2°. *Second type of orthogonality* is the stiffness orthogonality of eigenfunctions for the case of the dynamic boundary conditions (3),

$$\int_0^l b(x)\varphi_m'(x)\varphi_n'(x)\,dx + K_1\varphi_m(0)\varphi_n(0) + K_2\varphi_m(l)\varphi_n(l) = 0, \qquad m \neq n. \tag{10}$$

The corresponding squared stiffness norm of the eigenfunction is given by

$$\|\varphi_n\|_2^2 = \int_0^l b(x)[\varphi_n'(x)]^2\,dx + K_1\varphi_n^2(0) + K_2\varphi_n^2(l) = \omega_n^2\|\varphi_n\|_1^2. \tag{11}$$

For the geometric boundary conditions (2), one should set $K_1 = K_2 = 0$ in formulas (10) and (11).

► **Two representations of solutions of the general initial-boundary value problem via Green's functions.**

1°. The solution of the problem on longitudinal vibrations of a Rayleigh bar with the general initial conditions (1) and the dynamic boundary conditions (3) can be represented in the form

$$\begin{aligned} w(x,t) &= \frac{\partial}{\partial t}\int_0^l \left[a(y)f(y)G_1(x,y,t) + c(y)f'(y)\frac{\partial}{\partial y}G_1(x,y,t)\right]dy \\ &\quad + \int_0^l \left[a(y)g(y)G_1(x,y,t) + c(y)g'(y)\frac{\partial}{\partial y}G_1(x,y,t)\right]dy \\ &\quad + \int_0^t\int_0^l \Phi(y,\tau)G_1(x,y,t-\tau)\,dy\,d\tau \\ &\quad + M_1\left[f(0)\frac{\partial}{\partial t}G_1(x,0,t) + g(0)G_1(x,0,t)\right] \\ &\quad + M_2\left[f(l)\frac{\partial}{\partial t}G_1(x,l,t) + g(l)G_1(x,l,t)\right], \end{aligned} \tag{12}$$

where $G_1(x,y,t)$ is the mass Green's function

$$G_1(x,y,t) = \sum_{n=1}^{\infty} \frac{1}{\omega_n\|\varphi_n\|_1^2}\varphi_n(x)\varphi_n(y)\sin(\omega_n t). \tag{13}$$

For the geometric boundary conditions (2), one should set $M_1 = M_2 = 0$ in the above formula for $w(x,t)$.

Remark 11.2. The solution (12) can be reduced to the form

$$\begin{aligned} w(x,t) &= \frac{\partial}{\partial t}\int_0^l G_1(x,y,t)\bigl\{a(y)f(y) - [c(y)f'(y)]'\bigr\}\,dy \\ &\quad + \int_0^l G_1(x,y,t)\bigl\{a(y)g(y) - [c(y)g'(y)]'\bigr\}\,dy + \cdots, \end{aligned}$$

where we should add the last three lines in (12).

2°. An alternative representation of the solution of the problem on longitudinal vibrations of a Rayleigh bar:

$$w(x,t) = \int_0^l b(y)\left[f'(y)\frac{\partial^2}{\partial t\partial y}G_2(x,y,t) + g'(y)\frac{\partial}{\partial y}G_2(x,y,t)\right]dy$$
$$- \int_0^t\int_0^l \Phi(y,\tau)\frac{\partial^2}{\partial t^2}G_2(x,y,t-\tau)\,dy\,d\tau$$
$$+ K_1\left[f(0)\frac{\partial}{\partial t}G_2(x,0,t) + g(0)G_2(x,0,t)\right]$$
$$+ K_2\left[f(l)\frac{\partial}{\partial t}G_2(x,l,t) + g(l)G_2(x,l,t)\right],$$

where $G_2(x,y,t)$ is the stiffness Green's function

$$G_2(x,y,t) = \sum_{n=1}^{\infty}\frac{1}{\omega_n\|\varphi_n\|_2^2}\varphi_n(x)\varphi_n(y)\sin(\omega_n t).$$

For the geometric boundary conditions (2), one should set $K_1 = K_2 = 0$ in the above formula for $w(x,t)$.

► **Homogeneous bar of constant cross-section.**

Consider longitudinal vibrations of a homogeneous Rayleigh bar of constant cross-section. In this case, the coefficients of the original equation are constants, and the general solution of Eq. (4) is given by the formula

$$\varphi(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x), \qquad \lambda = \omega\sqrt{\frac{a}{b-\omega^2 c}}, \tag{14}$$

where C_1 and C_2 are arbitrary constants. The vibration frequencies are bounded, $|\omega| < \omega_\infty = \sqrt{b/c}$. In the degenerate case ($c = 0$), we have the usual model of vibrations of a thin bar described by a second-order wave equation with $\lambda = \omega/v$, where $v = \sqrt{b/a}$ is the phase speed of propagation of perturbations in the bar.

For the dynamic boundary conditions (3) in the absence of lumped masses, $M_1 = M_2 = 0$, the dispersion equation for the eigenvalues ω_n has the form

$$(p^2\lambda^2 - K_1K_2)\sin(\lambda l) - (K_1+K_2)p\lambda\cos(\lambda l) = 0,$$

where $p = b - \omega^2 c$ and λ is defined in (14).

For the geometric boundary conditions (2), one should set $K_1 = K_2 = 0$ in the dispersion equation, which results in the following eigenvalues and eigenfunctions:

$$\omega_n = \sqrt{\frac{b(\pi n)^2}{al^2 + c(\pi n)^2}}, \qquad \varphi_n(x) = \sin\left(\frac{\pi n x}{l}\right), \qquad n = 1,\ 2,\ \dots \tag{15}$$

Formulas (12), (13), and (15) permit one to obtain the solution of the problem on the longitudinal vibrations of a cylindrical Rayleigh bar with the general initial conditions (1) and the homogeneous boundary conditions (2). See also equation 11.2.6.1.

► **Uniform conical bar.**

Consider a homogeneous conical bar of circular cross-section with constant physical parameters ρ, E, and ν. The cross-sectional area of the cone and the moment of inertia with respect to the axis are given by the formulas

$$S(x) = \pi(x - x_0)^2, \quad I(x) = \tfrac{1}{2}\pi(x - x_0)^4,$$

where x_0 is the coordinate of the cone vertex ($x_0 < 0$). The coefficients of the original equation have the form

$$a(x) = \pi\rho(x - x_0)^2, \quad b(x) = \pi E(x - x_0)^2, \quad c(x) = \tfrac{1}{2}\pi\rho\nu^2(x - x_0)^4.$$

In this case, the general solution of Eq. (4) can be represented in the form

$$\varphi(x) = C_1 \frac{P_\sigma(\mu(x - x_0))}{x - x_0} + C_2 \frac{Q_\sigma(\mu(x - x_0))}{x - x_0},$$

$$\mu = \nu\omega\sqrt{\frac{\rho}{2E}}, \quad \sigma = -\frac{1}{2} + \sqrt{\frac{9}{4} + \frac{2}{\nu^2}},$$

where C_1 and C_2 are arbitrary constants and $P_\sigma(z)$ and $Q_\sigma(z)$ are the Legendre functions of the first and second kind, respectively. One can show that $0 < \mu(x - x_0) \le \mu(l - x_0) < 1$.

⊙ *Literature for equation 11.2.6.4*: I. A. Fedotov, A. D. Polyanin, and M. Yu. Shatalov (2007), I. A. Fedotov, Y. Gai, A. D. Polyanin, and M. Yu. Shatalov (2008).

11.2.7 Equations Containing Fourth Derivative in x and Mixed Derivatives

► **Equations containing the second derivative with respect to t.**

1. $\displaystyle \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^2 w = 0.$

1°. General solution (two representations):

$$w(x,t) = tu_1(x,t) + u_2(x,t),$$
$$w(x,t) = xu_1(x,t) + u_2(x,t),$$

where $u_k = u_k(x,t)$ is an arbitrary function satisfying the heat equation $\partial_t u_k - \partial_{xx} u_k = 0$; $k = 1,\ 2$.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{\sqrt{t}}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4t}\right).$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = 0 \quad \text{at} \quad t = 0, \qquad \partial_t w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \frac{\sqrt{t}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x - \xi)^2}{4t}\right] f(\xi)\, d\xi.$$

⊙ *Literature*: G. E. Shilov (1965).

2. $\dfrac{\partial^2 w}{\partial t^2} - a\dfrac{\partial^2 w}{\partial x^2} - b\dfrac{\partial^4 w}{\partial^2 t\partial x^2} + c\dfrac{\partial^4 w}{\partial x^4} = \Phi(x,t).$

Rayleigh–Bishop equation with $a > 0$, $b > 0$, and $c > 0$. This equation describes one-dimensional longitudinal vibrations of a thick short circular bar of constant cross-section. This is a special case of equation 11.2.7.3.

1°. Particular solutions for the homogeneous equation with $\Phi(x,t) = 0$:

$$w = [A\exp(-\beta x) + B\exp(\beta x)][C\cos(\lambda t) + \sin(\lambda t)], \qquad \lambda = \beta\sqrt{\frac{a - c\beta^2}{b\beta^2 - 1}},$$

$$w = [A\exp(-\beta x) + B\exp(\beta x)][C\exp(-\lambda t) + D\exp(\lambda t)], \qquad \lambda = \beta\sqrt{\frac{a - c\beta^2}{1 - b\beta^2}},$$

$$w = [A\cos(\beta x) + B\sin(\beta x)][C\cos(\lambda t) + \sin(\lambda t)], \qquad \lambda = \beta\sqrt{\frac{a + c\beta^2}{1 + b\beta^2}},$$

where A, B, C, D, and β are arbitrary constants. (It is assumed in all cases that the radicands are positive.)

2°. Fundamental solution:

$$\begin{aligned}\mathscr{E}_{\rm e}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-i|x|\xi)}{\xi\sqrt{(1+b\xi^2)(a+c\xi^2)}}\sin\biggl(t\xi\sqrt{\frac{a+c\xi^2}{1+b\xi^2}}\biggr)d\xi \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{\cos(|x|\xi)}{\xi\sqrt{(1+b\xi^2)(a+c\xi^2)}}\sin\biggl(t\xi\sqrt{\frac{a+c\xi^2}{1+b\xi^2}}\biggr)d\xi.\end{aligned}$$

Here the absolute value is only written to emphasize that the function $\mathscr{E}_{\rm e}(x,t)$ is even in the variable x.

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x) \quad\text{at}\quad t = 0,$$
$$\partial_t w = g(x) \quad\text{at}\quad t = 0.$$

Solution:

$$\begin{aligned}w(x,t) &= \int_0^t\int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau \\ &\quad+ \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)[f(y) - f''(y)]\,dy + \int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)[g(y) - g''(y)]\,dy.\end{aligned}$$

4°. Domain: $0 \le x < \infty$. First boundary value problem.
The following conditions are prescribed:

$$w = f(x) \quad\text{at}\quad t = 0,$$
$$\partial_t w = g(x) \quad\text{at}\quad t = 0,$$
$$w = 0 \quad\text{at}\quad x = 0.$$

Solution:

$$w(x,t) = \int_0^t \int_0^\infty G(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau + \frac{\partial}{\partial t}\int_0^\infty G(x-y,t)[f(y)-f''(y)]\,dy$$
$$+ \int_0^\infty G(x-y,t)[g(y)-g''(y)]\,dy, \qquad G(x,y,t) = \mathscr{E}_{\rm e}(x-y,t) - \mathscr{E}_{\rm e}(x+y,t).$$

5°. Domain: $0 \le x \le l$. The boundary value problem corresponding to the rigid clamping of the bar ends.

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w = \partial_{xx} w &= 0 && \text{at} \quad x=0 && \text{(boundary conditions)},\\ w = \partial_{xx} w &= 0 && \text{at} \quad x=l && \text{(boundary conditions)}. \end{aligned}$$

It is assumed that $f(0) = f(l) = g(0) = g(l) = 0$.

Solution:

$$\begin{aligned} w(x,t) &= \int_0^t \int_0^l G(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau \\ &\quad + \frac{\partial}{\partial t}\int_0^l G(x-y,t)[f(y)-f''(y)]\,dy + \int_0^l G(x-y,t)[g(y)-g''(y)]\,dy, \end{aligned} \tag{1}$$

where

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^\infty \frac{1}{(1+b\beta_n^2)\lambda_n}\sin(\beta_n x)\sin(\beta_n y)\sin(\lambda_n t),$$
$$\beta_n = \frac{\pi n}{l}, \qquad \lambda_n = \beta_n\sqrt{\frac{a+c\beta_n^2}{1+b\beta_n^2}}.$$

6°. Domain: $0 \le x \le l$. A boundary value problem.

The following conditions are prescribed:

$$\begin{aligned} w &= f(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ \partial_t w &= g(x) && \text{at} \quad t=0 && \text{(initial condition)},\\ w = \partial_{xx} w &= 0 && \text{at} \quad x=0 && \text{(boundary conditions)},\\ w_x = \partial_{xxx} w &= 0 && \text{at} \quad x=l && \text{(boundary conditions)}. \end{aligned}$$

The solution is determined by formula (1), where

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^\infty \frac{1}{(1+b\beta_n^2)\lambda_n}\sin(\beta_n x)\sin(\beta_n y)\sin(\lambda_n t),$$
$$\beta_n = \frac{\pi(2n-1)}{2l}, \qquad \lambda_n = \beta_n\sqrt{\frac{a+c\beta_n^2}{1+b\beta_n^2}}.$$

7°. Domain: $0 \le x \le l$. A boundary value problem.

The following conditions are prescribed:

$$
\begin{aligned}
w = f(x) &\quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\
\partial_t w = g(x) &\quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\
w_x = \partial_{xxx} w = 0 &\quad \text{at} \quad x = 0 \quad \text{(boundary conditions)},\\
w = \partial_{xx} w = 0 &\quad \text{at} \quad x = l \quad \text{(boundary conditions)}.
\end{aligned}
$$

The solution is determined by formula (1), where

$$
G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{(1+b\beta_n^2)\lambda_n}\cos(\beta_n x)\cos(\beta_n y)\sin(\lambda_n t),
$$

$$
\beta_n = \frac{\pi(2n-1)}{2l}, \qquad \lambda_n = \beta_n\sqrt{\frac{a+c\beta_n^2}{1+b\beta_n^2}}.
$$

3. $\displaystyle a(x)\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[b(x)\frac{\partial w}{\partial x}\right] - \frac{\partial}{\partial x}\left[c(x)\frac{\partial^3 w}{\partial^2 t\,\partial x}\right] + \frac{\partial^2}{\partial x^2}\left[d(x)\frac{\partial^2 w}{\partial x^2}\right] = \Phi(x,t).$

This equation describes one-dimensional longitudinal vibrations of a thick short bar that is a body of revolution around the x-axis. The Rayleigh–Bishop model is used, which takes into account the lateral displacements as well as the transverse shear stresses. The coefficients of the equation are positive functions and can be expressed as follows via geometric and physical variables:

$$
a(x) = \rho S(x), \quad b(x) = ES(x), \quad c(x) = \rho\nu^2 I(x), \quad d(x) = \mu\nu^2 I(x),
$$

where ρ is the bar mass density, E is the Young modulus, ν is the Poisson ratio, $\mu = \frac{E}{2(1+\nu)}$ is the shear modulus, $S(x)$ is the transverse cross-section area, and $I(x)$ is the polar moment of inertia.

▶ **Initial and boundary conditions.**

This equation is considered with the general initial conditions

$$
w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0 \tag{1}
$$

and the homogeneous boundary conditions

$$
w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, \qquad w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = l, \tag{2}
$$

which correspond to the rigid clamping of the bar ends.

▶ **Free vibrations of a Rayleigh bar. The Sturm–Liouville problem.**

Free vibrations are determined by the relations

$$
w(x,t) = [A\cos(\omega t) + B\sin(\omega t)]\varphi(x), \qquad \Phi(x,t) = 0. \tag{3}
$$

By substituting these relations into the equation and the boundary conditions (2), we arrive at the nonclassical Sturm–Liouville eigenvalue problem

$$[d(x)\varphi'']'' + \{[\omega^2 c(x) - b(x)]\varphi'\}' = \omega^2 a(x)\varphi, \tag{4}$$

$$\varphi = \varphi'' = 0 \quad \text{at} \quad x = 0, \tag{5}$$

$$\varphi = \varphi'' = 0 \quad \text{at} \quad x = l, \tag{6}$$

where the primes stand for derivatives with respect to x. We point out that the spectral parameter ω occurs in Eq. (4) in a complicated way.

► **Two types of orthogonality of eigenfunctions.**

Let ω_m and ω_n be two distinct eigenvalues of the Sturm–Liouville problem (4)–(6), and let $\varphi_m(x)$ and $\varphi_n(x)$ be the corresponding eigenfunctions.

1°. First type of orthogonality of eigenfunctions:

$$\int_0^l \bigl[a(x)\varphi_m(x)\varphi_n(x) + c(x)\varphi'_m(x)\varphi'_n(x)\bigr]\,dx = 0, \qquad m \neq n.$$

The corresponding squared norm of the eigenfunction is given by

$$\|\varphi_n\|_1^2 = \int_0^l \bigl[a(x)\varphi_n^2(x) + c(x)\varphi_n'^2(x)\bigr]\,dx.$$

2°. Second type of orthogonality of eigenfunctions:

$$\int_0^l \bigl[b(x)\varphi'_m(x)\varphi'_n(x) + d(x)\varphi''_m(x)\varphi''_n(x)\bigr]\,dx = 0, \qquad m \neq n.$$

The corresponding squared norm of the eigenfunction is given by

$$\|\varphi_n\|_2^2 = \int_0^l \bigl\{b(x)[\varphi'_n(x)]^2 + d(x)[\varphi''_n(x)]^2\bigr\}\,dx = \omega_n^2\|\varphi_n\|_1^2.$$

► **Representation of the solution of a general initial-boundary value problem via the Green's function.**

1°. The solution of the problem on the longitudinal vibrations of a Rayleigh–Bishop bar with the general initial conditions (1) and the homogeneous boundary conditions (2) can be represented in the form

$$\begin{aligned} w(x,t) = {} & \frac{\partial}{\partial t}\int_0^l \left[a(y)f(y)G(x,y,t) + c(y)f'(y)\frac{\partial}{\partial y}G(x,y,t)\right]dy \\ & + \int_0^l \left[a(y)g(y)G(x,y,t) + c(y)g'(y)\frac{\partial}{\partial y}G(x,y,t)\right]dy \\ & + \int_0^t\!\int_0^l \Phi(y,\tau)G(x,y,t-\tau)\,dy\,d\tau, \end{aligned} \tag{7}$$

where $G(x,y,t)$ is the Green's function

$$G(x,y,t) = \sum_{n=1}^{\infty} \frac{1}{\omega_n\|\varphi_n\|_1^2}\varphi_n(x)\varphi_n(y)\sin(\omega_n t). \tag{8}$$

► **Homogeneous cylindrical bar of constant cross-section.**

Let us look at longitudinal vibrations of a homogeneous cylindrical bar of constant cross-section. In this case, the coefficients of the original equation are constants, and the general solution of the corresponding Sturm–Liouville problem (4)–(6) results in the following eigenvalues and eigenfunctions:

$$\omega_n = \frac{\pi n}{l}\sqrt{\frac{bl^2 + d(\pi n)^2}{al^2 + c(\pi n)^2}}, \qquad \varphi_n(x) = \sin\left(\frac{\pi n x}{l}\right), \qquad n = 1,\ 2,\ \ldots \tag{9}$$

Formulas (7)–(9) permit one to obtain the solution of the problem on the longitudinal vibrations of a cylindrical Rayleigh–Bishop bar with the general initial conditions (1) and homogeneous boundary conditions (2).

⊙ *Literature for equation 11.2.7.3*: I. A. Fedotov, A. D. Polyanin, M. Yu. Shatalov, and H. M. Tenkam (2010).

► **Equations containing the fourth derivative with respect to t.**

4. $\displaystyle \frac{\partial^4 w}{\partial t^4} - 2\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^4 w}{\partial x^4} = 0.$

General solution (three representations):

$$w(x,t) = f_1(t-x) + f_2(t+x) + t\big[g_1(t-x) + g_2(t+x)\big],$$
$$w(x,t) = f_1(t-x) + f_2(t+x) + x\big[g_1(t-x) + g_2(t+x)\big],$$
$$w(x,t) = f_1(t-x) + f_2(t+x) + (t+x)g_1(t-x) + (t-x)g_2(t+x),$$

where $f_1(y)$, $f_2(z)$, $g_1(y)$, and $g_2(z)$ are arbitrary functions.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

5. $\displaystyle \Big(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial^2}{\partial t^2} - b^2\frac{\partial^2}{\partial x^2}\Big)w = 0.$

1°. General solution with $a \neq b$:

$$w = f_1(x-at) + f_2(x+at) + f_3(x-bt) + f_4(x+bt), \tag{1}$$

where $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$, and $f_4(z_4)$ are arbitrary functions.

2°. General solution with $a = b$:

$$w = f_1(x-at) + f_2(x+at) + tf_3(x-at) + tf_4(x+at) \tag{2}$$

where $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$, and $f_4(z_4)$ are arbitrary functions.

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w(x,0) = \varphi_0(x), \quad w_t(x,0) = \varphi_1(x), \quad w_{tt}(x,0) = \varphi_2(x), \quad w_{ttt}(x,0) = \varphi_3(x). \tag{3}$$

For $a \neq b$, the solution of the problem is described by formula (1), where

$$f_1(x) = \frac{1}{2(a^2-b^2)}\left[-b^2\varphi_0(x) + \frac{b^2}{a}\Phi_1(x) + \Phi_2(x) - \frac{1}{2a}\Phi_3(x)\right],$$
$$f_2(x) = \frac{1}{2(a^2-b^2)}\left[-b^2\varphi_0(x) - \frac{b^2}{a}\Phi_1(x) + \Phi_2(x) + \frac{1}{2a}\Phi_3(x)\right],$$
$$f_3(x) = \frac{1}{2(b^2-a^2)}\left[-a^2\varphi_0(x) + \frac{a^2}{b}\Phi_1(x) + \Phi_2(x) - \frac{1}{2b}\Phi_3(x)\right],$$
$$f_4(x) = \frac{1}{2(b^2-a^2)}\left[-a^2\varphi_0(x) - \frac{a^2}{b}\Phi_1(x) + \Phi_2(x) + \frac{1}{2b}\Phi_3(x)\right],$$
$$\Phi_n(x) = \int_{x_0}^{x} \varphi_n(\xi)(x-\xi)^{n-1}\,d\xi, \qquad n = 1,\,2,\,3,$$

and x_0 is an arbitrary constant.

For $a = b$, the solution of the problem is described by formula (2), where

$$f_1(x) = \frac{1}{2}\varphi_0(x) - \frac{3}{4a}\int_{x_0}^{x}\varphi_1(\xi)\,d\xi + \frac{1}{8a^3}\int_{x_0}^{x}\varphi_3(\xi)(x-\xi)^2\,d\xi,$$
$$f_2(x) = \frac{1}{2}\varphi_0(x) + \frac{3}{4a}\int_{x_0}^{x}\varphi_1(\xi)\,d\xi - \frac{1}{8a^3}\int_{x_0}^{x}\varphi_3(\xi)(x-\xi)^2\,d\xi,$$
$$f_3(x) = \frac{1}{4}a\varphi_0'(x) - \frac{1}{4}\varphi_1(x) - \frac{1}{4a}\int_{x_0}^{x}\varphi_2(\xi)\,d\xi + \frac{1}{4a^2}\int_{x_0}^{x}\varphi_3(\xi)(x-\xi)\,d\xi,$$
$$f_4(x) = -\frac{1}{4}a\varphi_0'(x) - \frac{1}{4}\varphi_1(x) + \frac{1}{4a}\int_{x_0}^{x}\varphi_2(\xi)\,d\xi + \frac{1}{4a^2}\int_{x_0}^{x}\varphi_3(\xi)(x-\xi)\,d\xi.$$

11.3 Two-Dimensional Nonstationary Fourth-Order Equations

11.3.1 Equation of the Form $\frac{\partial w}{\partial t} + a^2\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4}\right) = \Phi(x, y, t)$

► **Solutions of boundary value problems in terms of the Green's function.**

In this section we consider boundary value problems in a rectangular domain $0 \le x \le l_1$, $0 \le y \le l_2$ with the general initial condition

$$w = f(x, y) \quad \text{at} \quad t = 0 \tag{1}$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$\begin{aligned} w(x,y,t) = &\int_0^{l_1}\!\!\int_0^{l_2} f(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\ &+ \int_0^t\!\!\int_0^{l_1}\!\!\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,\xi,\eta,t-\tau)\,d\eta\,d\xi\,d\tau. \end{aligned} \tag{2}$$

► **Green's functions for various boundary value problems ($0 \le x \le l_1,\ 0 \le y \le l_2$).**

Items 1° through 5° below present the Green's functions for various types of boundary conditions. The solutions of these boundary value problems are represented via the Green's functions by formula (2).

1°. The function and its first derivatives are prescribed on the sides of the rectangle:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_y w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{16}{l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{p_n^4 q_m^4 \varphi_n(x)\psi_m(y)\varphi_n(\xi)\psi_m(\eta) \exp\bigl[-(p_n^4 + q_m^4)a^2 t\bigr]}{\bigl[\varphi_n''(l_1)\psi_m''(l_2)\bigr]^2},$$
$$\varphi_n''(x) = \frac{d^2\varphi_n}{dx^2}, \quad \psi_m''(y) = \frac{d^2\psi_m}{dy^2}.$$

Here

$$\varphi_n(x) = \bigl[\sinh(p_n l_1) - \sin(p_n l_1)\bigr]\bigl[\cosh(p_n x) - \cos(p_n x)\bigr] - \bigl[\cosh(p_n l_1) - \cos(p_n l_1)\bigr]\bigl[\sinh(p_n x) - \sin(p_n x)\bigr],$$
$$\psi_m(y) = \bigl[\sinh(q_m l_2) - \sin(q_m l_2)\bigr]\bigl[\cosh(q_m y) - \cos(q_m y)\bigr] - \bigl[\cosh(q_m l_2) - \cos(q_m l_2)\bigr]\bigl[\sinh(q_m y) - \sin(q_m y)\bigr],$$

where the p_n and q_m are positive roots of the transcendental equations

$$\cosh(pl_1)\cos(pl_1) = 1, \quad \cosh(ql_2)\cos(ql_2) = 1 \qquad (q_m = p_m l_1/l_2).$$

2°. The function and its second derivatives are prescribed on the sides of the rectangle:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(p_n x)\sin(q_m y)\sin(p_n \xi)\sin(q_m \eta)\exp\bigl[-(p_n^4 + q_m^4)a^2 t\bigr],$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}.$$

3°. The first and third derivatives are prescribed on the sides of the rectangle:

$$w_x = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad w_x = \partial_{xxx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w_y = \partial_{yyy} w = 0 \quad \text{at} \quad y = 0, \qquad w_y = \partial_{yyy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{1}{l_1 l_2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon_n \varepsilon_m \cos(p_n x)\sin(q_m y) \times \cos(p_n \xi)\cos(q_m \eta)\exp\bigl[-(p_n^4 + q_m^4)a^2 t\bigr],$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

4°. The second and third derivatives are prescribed on the sides of the rectangle:

$$w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad w_{xx} = \partial_{xxx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w_{yy} = \partial_{yyy} w = 0 \quad \text{at} \quad y = 0, \qquad w_{yy} = \partial_{yyy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x, y, \xi, \eta, t) = G_1(x, \xi, t) G_2(y, \eta, t),$$
$$G_1(x, \xi, t) = \frac{1}{l_1} + \frac{3}{l_1^3}(2x - l_1)(2\xi - l_1) + \frac{4}{l_1} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l_1)} \exp(-p_n^4 a^2 t),$$
$$G_2(y, \eta, t) = \frac{1}{l_2} + \frac{3}{l_2^3}(2y - l_2)(2\eta - l_2) + \frac{4}{l_2} \sum_{m=1}^{\infty} \frac{\psi_m(y)\psi_m(\eta)}{\psi_m^2(l_2)} \exp(-q_m^4 a^2 t).$$

Here

$$\begin{aligned}\varphi_n(x) &= \big[\sinh(p_n l_1) - \sin(p_n l_1)\big]\big[\cosh(p_n x) + \cos(p_n x)\big] \\ &\quad - \big[\cosh(p_n l_1) - \cos(p_n l_1)\big]\big[\sinh(p_n x) + \sin(p_n x)\big], \\ \psi_m(y) &= \big[\sinh(q_m l_2) - \sin(q_m l_2)\big]\big[\cosh(q_m y) + \cos(q_m y)\big] \\ &\quad - \big[\cosh(q_m l_2) - \cos(q_m l_2)\big]\big[\sinh(q_m y) + \sin(q_m y)\big],\end{aligned}$$

where the p_n and q_m are positive roots of the transcendental equations

$$\cosh(p l_1)\cos(p l_1) = 1, \quad \cosh(q l_2)\cos(q l_2) = 1.$$

5°. Mixed boundary conditions are prescribed on the sides of the rectangle:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x, y, \xi, \eta, t) = \frac{4}{l_1 l_2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin(p_n x)\sin(q_m y)\sin(p_n \xi)\sin(q_m \eta) \exp\big[-(p_n^4 + q_m^4)a^2 t\big],$$
$$p_n = \frac{\pi(2n+1)}{2l_1}, \quad q_m = \frac{\pi(2m+1)}{2l_2}.$$

11.3.2 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta\Delta w = 0$

► **Preliminary remarks. Particular solutions.**

1°. This equation describes two-dimensional free transverse vibrations of a thin elastic plate; the unknown w is the deflection (transverse displacement) of the plate midplane points relative to the original plane position. Here $\Delta\Delta = \Delta^2$ and Δ is the Laplace operator defined as

$$\Delta = \begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & \text{in the Cartesian coordinate system,} \\ \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \varphi^2} & \text{in the polar coordinate system.} \end{cases}$$

2°. Particular solutions:

$$w(x,y,t) = \big[A_1 \sin(k_1 x) + B_1 \cos(k_1 x)\big]\big[A_2 \sin(k_2 y) + B_2 \cos(k_2 y)\big] \sin\big[(k_1^2 + k_2^2)at\big],$$
$$w(x,y,t) = \big[A_1 \sin(k_1 x) + B_1 \cos(k_1 x)\big]\big[A_2 \sin(k_2 y) + B_2 \cos(k_2 y)\big] \cos\big[(k_1^2 + k_2^2)at\big],$$
$$w(x,y,t) = \big[A_1 \sinh(k_1 x) + B_1 \cosh(k_1 x)\big] \sinh(k_2 y) \sin\big[(k_1^2 + k_2^2)at\big],$$
$$w(x,y,t) = \big[A_1 \sinh(k_1 x) + B_1 \cosh(k_1 x)\big] \cosh(k_2 y) \sin\big[(k_1^2 + k_2^2)at\big],$$
$$w(x,y,t) = \big[A_1 \sinh(k_1 x) + B_1 \cosh(k_1 x)\big] \sinh(k_2 y) \cos\big[(k_1^2 + k_2^2)at\big],$$
$$w(x,y,t) = \big[A_1 \sinh(k_1 x) + B_1 \cosh(k_1 x)\big] \cosh(k_2 y) \cos\big[(k_1^2 + k_2^2)at\big],$$
$$w(r,\varphi,t) = \big[A_1 J_n(kr) + A_2 Y_n(kr) + A_3 I_n(kr) + A_4 K_n(kr)\big] \cos(n\varphi) \sin(k^2 at),$$
$$w(r,\varphi,t) = \big[A_1 J_n(kr) + A_2 Y_n(kr) + A_3 I_n(kr) + A_4 K_n(kr)\big] \sin(n\varphi) \cos(k^2 at),$$

where A_1, A_2, A_3, A_4, B_1, B_2, k, k_1, k_2 are arbitrary constants, $J_n(\xi)$ and $Y_n(\xi)$ are Bessel functions of the first and second kind, $I_n(\xi)$ and $K_n(\xi)$ are modified Bessel functions of the first and second kind, $r = \sqrt{x^2 + y^2}$, and $n = 0, 1, 2, \ldots$

► Cauchy problem.

1°. Domain: $-\infty < x < \infty, -\infty < y < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x,y) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x,y) \quad \text{at} \quad t = 0.$$

Poisson solution:

$$w(x,y,t) = \frac{1}{\pi} \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} f\big(x + 2\xi\sqrt{at},\, y + 2\eta\sqrt{at}\,\big) \sin\big(\xi^2 + \eta^2\big)\, d\xi\, d\eta$$
$$+ \frac{1}{\pi} \int_0^t d\tau \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} g\big(x + 2\xi\sqrt{a\tau},\, y + 2\eta\sqrt{a\tau}\,\big) \sin\big(\xi^2 + \eta^2\big)\, d\xi\, d\eta.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{1}{4\pi a} \int_0^t \sin\left[\frac{(x-\xi)^2 + (y-\eta)^2}{4a\tau}\right] \frac{d\tau}{\tau}.$$

⊙ *Literature*: A. N. Krylov (1949), I. Sneddon (1951), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

2°. Domain: $0 \le r < \infty$, $0 \le \varphi \le 2\pi$. Cauchy problem.
Initial conditions for the symmetric case in the polar coordinate system:

$$w = f(r) \quad \text{at} \quad t = 0, \qquad \partial_t w = 0 \quad \text{at} \quad t = 0.$$

Solution:

$$w(r,t) = \frac{1}{2at} \int_0^{\infty} \xi f(\xi) J_0\left(\frac{\xi r}{2at}\right) \sin\left(\frac{\xi^2 + r^2}{4at}\right) d\xi,$$

where $J_0(z)$ is the zeroth Bessel function.

⊙ *Literature*: I. Sneddon (1951), B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

► **Solutions of boundary value problems in terms of the Green's function.**

Consider boundary value problems in the rectangular domain $0 \le x \le l_1$, $0 \le y \le l_2$ with the general initial conditions

$$w = f(x,y) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x,y) \quad \text{at} \quad t = 0 \tag{1}$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$\begin{aligned} w(x,y,t) = \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi \\ + \int_0^{l_1}\!\!\int_0^{l_2} g(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi. \end{aligned} \tag{2}$$

► **Green's functions for various boundary value problems ($0 \le x \le l_1$, $0 \le y \le l_2$).**

Items 1° through 3° below present the Green's functions for various types of boundary conditions in Cartesian coordinates. The solutions of these boundary value problems are represented via the Green's functions by formula (2).

1°. Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. All sides of the plate are hinged.
Boundary conditions are prescribed:

$$\begin{aligned} &w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad &w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1, \\ &w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad &w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2. \end{aligned}$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{4}{al_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)\frac{\sin(\lambda_{nm} at)}{\lambda_{nm}},$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = p_n^2 + q_m^2.$$

2°. Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. The 1st and 3rd derivatives are prescribed on the sides:

$$\begin{aligned} &\partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad &\partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l_1, \\ &\partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = 0, \qquad &\partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = l_2. \end{aligned}$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{1}{al_1 l_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon_n\varepsilon_m \cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)\frac{\sin(\lambda_{nm} at)}{\lambda_{nm}},$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = p_n^2 + q_m^2, \quad \varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

For $n = m = 0$, the ratio $\sin(\lambda_{nm} at)/\lambda_{nm}$ must be replaced by at.

3°. Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Mixed boundary conditions are set on the sides:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$\partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x, y, \xi, \eta, t) = \frac{2}{a l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \varepsilon_m \sin(p_n x) \cos(q_m y) \sin(p_n \xi) \cos(q_m \eta) \frac{\sin(\lambda_{nm} a t)}{\lambda_{nm}},$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = p_n^2 + q_m^2, \quad \varepsilon_m = \begin{cases} 1 & \text{for } m = 0, \\ 2 & \text{for } m \neq 0. \end{cases}$$

► **Solution of the boundary value problem on transverse vibrations of a circular plate.**

Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Initial and boundary conditions for symmetric transverse vibrations of a circular plate of radius R with clamped contour in the polar coordinate system:

$$w = f(r) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(t) \quad \text{at} \quad t = 0;$$
$$w = 0 \quad \text{at} \quad r = R, \qquad \partial_r w = 0 \quad \text{at} \quad r = R.$$

Solution:

$$w(r, t) = \sum_{n=1}^{\infty} \bigl[A_n \cos(a k_n^2 t) + B_n \sin(a k_n^2 t)\bigr] \Psi_n(r),$$
$$\Psi_n(r) = I_0(k_n R) J_0(k_n r) - J_0(k_n R) I_0(k_n r),$$

where the k_n are positive roots of the transcendental equation (the prime denotes the derivative)

$$J_0(kR) I_0'(kR) - I_0(kR) J_0'(kR) = 0,$$

and the coefficients A_n and B_n are given by

$$A_n = \frac{1}{\|\Psi_n\|^2} \int_0^R f(r) \Psi_n(r) r\, dr, \quad B_n = \frac{1}{a k_n^2 \|\Psi_n\|^2} \int_0^R g(r) \Psi_n(r) r\, dr,$$
$$\|\Psi_n\|^2 = \tfrac{1}{4} R^6 \bigl[\Psi_n''(R)\bigr]^2 = R^2 J_0^2(k_n R) I_0^2(k_n R).$$

⊙ *Literature*: B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (1980).

11.3.3 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta\Delta w + k w = \Phi(x, y, t)$

► **Solutions of boundary value problems in terms of the Green's function.**

Consider boundary value problems in the rectangular domain $0 \le x \le l_1$, $0 \le y \le l_2$ with the general initial conditions

$$w = f(x, y) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x, y) \quad \text{at} \quad t = 0 \tag{1}$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w = \frac{\partial}{\partial t}\int_0^{l_1}\int_0^{l_2} f(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi + \int_0^{l_1}\int_0^{l_2} g(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi$$
$$+ \int_0^{t}\int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\eta\,d\xi\,d\tau. \qquad (2)$$

► **Green's functions for various boundary value problems ($0 \le x \le l_1,\ 0 \le y \le l_2$).**

Items 1° through 3° below present the Green's functions for various types of boundary conditions in Cartesian coordinates. The solutions of these boundary value problems are represented via the Green's functions by formula (2).

1°. The function and its second derivatives are prescribed on the sides of the rectangle:

$$w = \partial_{xx}w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx}w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy}w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_{yy}w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}},$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = \sqrt{a^2(p_n^2+q_m^2)^2 + k}.$$

2°. The first and third derivatives are prescribed on the sides of the rectangle:

$$\partial_x w = \partial_{xxx}w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx}w = 0 \quad \text{at} \quad x = l_1,$$
$$\partial_y w = \partial_{yyy}w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = \partial_{yyy}w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{1}{l_1 l_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon_n\varepsilon_m \cos(p_n x)\cos(q_m y)\cos(p_n\xi)\cos(q_m\eta)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}},$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = \sqrt{a^2(p_n^2+q_m^2)^2 + k}, \quad \varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

3°. Mixed boundary conditions are prescribed on the sides of the rectangle:

$$w = \partial_{xx}w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx}w = 0 \quad \text{at} \quad x = l_1,$$
$$\partial_y w = \partial_{yyy}w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = \partial_{yyy}w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{2}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \varepsilon_m \sin(p_n x)\cos(q_m y)\sin(p_n\xi)\cos(q_m\eta)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}},$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = \sqrt{a^2(p_n^2+q_m^2)^2 + k}, \quad \varepsilon_m = \begin{cases} 1 & \text{for } m = 0, \\ 2 & \text{for } m \neq 0. \end{cases}$$

11.3.4 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\Big(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4}\Big) + kw = \Phi(x,y,t)$

► **Solutions of boundary value problems in terms of the Green's function.**

Consider boundary value problems in the rectangular domain $0 \le x \le l_1$, $0 \le y \le l_2$ with the general initial conditions

$$w = f(x,y) \quad \text{at} \quad t=0, \qquad \partial_t w = g(x,y) \quad \text{at} \quad t=0 \tag{1}$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w = \frac{\partial}{\partial t}\int_0^{l_1}\!\!\int_0^{l_2} f(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi + \int_0^{l_1}\!\!\int_0^{l_2} g(\xi,\eta)G(x,y,\xi,\eta,t)\,d\eta\,d\xi$$
$$+ \int_0^t\!\!\int_0^{l_1}\!\!\int_0^{l_2} \Phi(\xi,\eta,\tau)G(x,y,z,\xi,\eta,\zeta,t-\tau)\,d\eta\,d\xi\,d\tau. \tag{2}$$

► **Green's functions for various boundary value problems ($0 \le x \le l_1$, $0 \le y \le l_2$).**

Items 1° through 3° below present the Green's functions for various types of boundary conditions in Cartesian coordinates. The solutions of these boundary value problems are represented via the Green's functions by formula (2).

1°. The function and its first derivatives are prescribed on the sides of the rectangle:

$$w = \partial_x w = 0 \quad \text{at} \quad x=0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x=l_1,$$
$$w = \partial_y w = 0 \quad \text{at} \quad y=0, \qquad w = \partial_y w = 0 \quad \text{at} \quad y=l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{16}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{p_n^4 q_m^4}{\big[\varphi_n''(l_1)\psi_m''(l_2)\big]^2}\varphi_n(x)\psi_m(y)\varphi_n(\xi)\psi_m(\eta)\frac{\sin(\lambda_{nm}t)}{\lambda_{nm}},$$
$$\lambda_{nm} = \sqrt{a^2(p_n^4+q_m^4)+k}, \quad \varphi_n''(x) = \frac{d^2\varphi_n}{dx^2}, \quad \psi_m''(y) = \frac{d^2\psi_m}{dy^2}.$$

Here

$$\varphi_n(x) = \big[\sinh(p_n l_1) - \sin(p_n l_1)\big]\big[\cosh(p_n x) - \cos(p_n x)\big]$$
$$- \big[\cosh(p_n l_1) - \cos(p_n l_1)\big]\big[\sinh(p_n x) - \sin(p_n x)\big],$$
$$\psi_m(y) = \big[\sinh(q_m l_2) - \sin(q_m l_2)\big]\big[\cosh(q_m y) - \cos(q_m y)\big]$$
$$- \big[\cosh(q_m l_2) - \cos(q_m l_2)\big]\big[\sinh(q_m y) - \sin(q_m y)\big],$$

where the p_n and q_m are positive roots of the transcendental equations

$$\cosh(pl_1)\cos(pl_1) = 1, \quad \cosh(ql_2)\cos(ql_2) = 1.$$

2°. The function and its second derivatives are prescribed on the sides of the rectangle:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x=0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x=l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y=0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y=l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta) \frac{\sin(\lambda_{nm} t)}{\lambda_{nm}},$$

$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = \sqrt{a^2(p_n^4 + q_m^4) + k}.$$

3°. The first and third derivatives are prescribed on the sides of the rectangle:

$$\partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = 0, \qquad \partial_x w = \partial_{xxx} w = 0 \quad \text{at} \quad x = l_1,$$
$$\partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = \partial_{yyy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta,t) = \frac{1}{l_1 l_2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon_n \varepsilon_m \cos(p_n x) \cos(q_m y) \cos(p_n \xi) \cos(q_m \eta) \frac{\sin(\lambda_{nm} t)}{\lambda_{nm}},$$

$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}, \quad \lambda_{nm} = \sqrt{a^2(p_n^4 + q_m^4) + k}, \quad \varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n \neq 0. \end{cases}$$

11.3.5 Other Two-Dimensional Nonstationary Fourth-Order Equations

1. $$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + a^2 \frac{\partial^2 w}{\partial x^2} + b^2 \frac{\partial^2 w}{\partial y^2} = 0.$$

Two-dimensional equation of gravitational–gyroscopic waves in the Boussinesq approximation. In what follows, we assume that $a \geq 0$ and $b \geq 0$.

1°. Particular solutions:

$$w = (c_1 e^{-k_1 x} + c_2 e^{k_1 x})(c_3 e^{-k_2 y} + c_4 e^{k_2 y}) \cos(\lambda t + c_5), \qquad \lambda = \sqrt{\frac{a^2 k_1^2 + b^2 k_2^2}{k_1^2 + k_2^2}};$$

$$w = (c_1 e^{-k_1 x} + c_2 e^{k_1 x}) \cos(k_2 y + c_3) \cos(\lambda t + c_4), \qquad \lambda = \sqrt{\frac{a^2 k_1^2 - b^2 k_2^2}{k_1^2 - k_2^2}};$$

$$w = \cos(k_1 x + c_1)(c_2 e^{-k_2 y} + c_3 e^{k_2 y}) \cos(\lambda t + c_4), \qquad \lambda = \sqrt{\frac{a^2 k_1^2 - b^2 k_2^2}{k_1^2 - k_2^2}};$$

$$w = \cos(k_1 x + c_1) \cos(k_2 y + c_2) \cos(\lambda t + c_3), \qquad \lambda = \sqrt{\frac{a^2 k_1^2 + b^2 k_2^2}{k_1^2 + k_2^2}};$$

$$w = (c_1 e^{-k_1 x} + c_2 e^{k_1 x}) \cos(k_2 y + c_3)(c_4 e^{-\lambda t} + c_5 e^{\lambda t}), \qquad \lambda = \sqrt{\frac{b^2 k_2^2 - a^2 k_1^2}{k_1^2 - k_2^2}};$$

$$w = \cos(k_1 x + c_1)(c_2 e^{-k_2 y} + c_3 e^{k_2 y})(c_4 e^{-\lambda t} + c_5 e^{\lambda t}), \qquad \lambda = \sqrt{\frac{a^2 k_1^2 - b^2 k_2^2}{k_2^2 - k_1^2}},$$

where c_1, c_2, c_3, k_1, and k_2 are arbitrary constants and the radicands are assumed to be positive.

2°. Singular particular solution:

$$w_2(\mathbf{r},t)=\frac{1}{2\pi}t\bigl(\ln|\mathbf{r}|+\ln t+\mathcal{C}-1\bigr)-\frac{1}{2\pi}\int_0^t \mathrm{Ci}\biggl(\xi\frac{|\mathbf{r}|_*}{|\mathbf{r}|}\biggr)d\xi,$$
$$\mathrm{Ci}(\xi)=-\int_\xi^\infty\frac{\cos z}{z}\,dz=\mathcal{C}+\ln\xi+\int_0^\xi\frac{\cos z-1}{z}\,dz, \tag{1}$$

where $|\mathbf{r}|=(x^2+y^2)^{1/2}$, $|\mathbf{r}|_*=(b^2x^2+a^2y^2)^{1/2}$, $\mathrm{Ci}(\xi)$ is the cosine integral, and $\mathcal{C}=0.5772\ldots$ is the Euler constant.

The singular solution (1) satisfies the initial conditions

$$w_2=0,\quad \frac{\partial w_2}{\partial t}=\frac{1}{2\pi}\ln|\mathbf{r}|\quad\text{at}\quad t=0. \tag{2}$$

3°. Fundamental solution:

$$\mathscr{E}_\mathrm{e}(\mathbf{r},t)=(I-aJ_{at}*)(I-bJ_{bt}*)w_2(\mathbf{r},t),$$
$$(I-aJ_{at}*)f(t)\equiv f(t)-a\int_0^t J_1(a(t-\tau))f(\tau)\,d\tau, \tag{3}$$

where I is the identity operator and $J_1(\tau)$ is the Bessel function.

The fundamental solution satisfies the initial conditions

$$\mathscr{E}_\mathrm{e}=0,\quad \frac{\partial\mathscr{E}_\mathrm{e}}{\partial t}=-\frac{1}{2\pi}\ln|\mathbf{r}|\quad\text{at}\quad t=0.$$

4°. Set $\lambda=\lambda(\mathbf{r})=|\mathbf{r}|_*/|\mathbf{r}|$. For $|x|\ge\delta>0$, $|y|\ge\delta>0$, and $a>b>0$, the following asymptotic formula holds for the fundamental solution as $t\to\infty$:

$$\begin{aligned}\mathscr{E}_\mathrm{e}(\mathbf{r},t)&=\frac{1}{2\pi}\sqrt{\frac{2}{\pi|a^2-b^2|}}\biggl[\frac{\sin(at-\pi/4)}{\sqrt{at}}+\frac{\sin(bt-\pi/4)}{\sqrt{bt}}\biggr]\ln|\mathbf{r}|\\&+\frac{1}{2\pi}\sqrt{\frac{1}{2\pi|a^2-b^2|}}\biggl[\frac{\sin(at-\pi/4)}{\sqrt{at}}\ln\biggl|1-\frac{\lambda^2(\mathbf{r})}{a^2}\biggr|+\frac{\sin(bt-\pi/4)}{\sqrt{bt}}\ln\biggl|1-\frac{\lambda^2(\mathbf{r})}{b^2}\biggr|\biggr]\\&-\frac{1}{2\pi}\sqrt{\frac{\pi}{2|a^2-b^2|}}\frac{\cos(bt-\pi/4)}{\sqrt{bt}}+\frac{1}{2\pi abt}-\frac{\sin(\lambda(\mathbf{r})t)}{2\pi t\sqrt{a^2-\lambda^2(\mathbf{r})}\sqrt{b^2-\lambda^2(\mathbf{r})}},\end{aligned}$$

where terms of the order of $t^{-3/2}$ are dropped. For $b>a>0$, one should exchange a and b in the asymptotic formula.

5°. Cauchy problem ($t\ge0$, $\mathbf{r}\in\mathbb{R}^2$). Initial conditions are prescribed:

$$w=f(\mathbf{r})\quad\text{at}\quad t=0,$$
$$\partial_t w=g(\mathbf{r})\quad\text{at}\quad t=0.$$

Solution:

$$w(\mathbf{r},t)=\frac{\partial}{\partial t}\int_{\mathbb{R}^2}\mathscr{E}_\mathrm{e}(\mathbf{r}-\mathbf{r}',t)\Delta_2'f(\mathbf{r}')\,dV'+\int_{\mathbb{R}^2}\mathscr{E}_\mathrm{e}(\mathbf{r}-\mathbf{r}',t)\Delta_2'g(\mathbf{r}')\,dV',$$

where Δ_2' is the two-dimensional Laplace operator in the integration variables $(x',\,y')$ and $dV'=dx'\,dy'$.

⊙ *Literature*: S. A. Gabov and A. G. Sveshnikov (1990).

2. $\dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)+a^2\dfrac{\partial^2 w}{\partial x^2}+b^2\dfrac{\partial^2 w}{\partial y^2}=\Phi(x,y,t).$

For $\Phi(x,y,t)=0$, see equation 11.3.5.1.

Cauchy problem ($t\ge 0$, $\mathbf{r}\in\mathbb{R}^2$). Initial conditions:

$$w=f(\mathbf{r})\quad\text{at}\quad t=0,$$
$$\partial_t w=g(\mathbf{r})\quad\text{at}\quad t=0,$$

where $\mathbf{r}=(x,y)$.

Solution:

$$w(\mathbf{r},t)=\frac{\partial}{\partial t}\int_{\mathbb{R}^2}\mathscr{E}_{\rm e}(\mathbf{r}-\mathbf{r}',t)\Delta_2' f(\mathbf{r}')\,dV'+\int_{\mathbb{R}^2}\mathscr{E}_{\rm e}(\mathbf{r}-\mathbf{r}',t)\Delta_2' g(\mathbf{r}')\,dV'$$
$$+\int_0^t\int_{\mathbb{R}^2}\mathscr{E}_{\rm e}(\mathbf{r}-\mathbf{r}',t-\tau)\Phi(\mathbf{r}',\tau)\,dV'd\tau,$$

where $\mathscr{E}(\mathbf{r},t)$ is the fundamental solution given in Item 3° of equation 11.3.5.1, Δ_2' is the two-dimensional Laplace operator in the integration variables $(x',\ y')$, and $dV'=dx'\,dy'$.

⊙ *Literature*: S. A. Gabov and A. G. Sveshnikov (1990).

3. $\dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)-\dfrac{\partial^2 w}{\partial t^2}+\dfrac{\partial^2 w}{\partial x^2}=0.$

This equation describes interior waves propagating in an infinite flat channel bounded by horizontal rigid walls and filled with an exponentially stratified fluid.

1°. Particular solutions:

$$w=(A_1e^{-\alpha x}+A_2e^{\alpha x})(B_1e^{-\beta y}+B_2e^{\beta y})(C_1e^{-\lambda t}+C_2e^{\lambda t}),\quad \lambda=\frac{\alpha}{\sqrt{1-\alpha^2-\beta^2}},$$
$$w=(A_1e^{-\alpha x}+A_2e^{\alpha x})(B_1e^{-\beta y}+B_2e^{\beta y})\cos(\lambda t+C_1),\quad \lambda=\frac{\alpha}{\sqrt{\alpha^2+\beta^2-1}},$$
$$w=(A_1e^{-\alpha x}+A_2e^{\alpha x})\cos(\beta y+C_1)\cos(\lambda t+C_2),\quad \lambda=\frac{\alpha}{\sqrt{\alpha^2-\beta^2-1}},$$
$$w=\cos(\alpha x+C_1)(B_1e^{-\beta y}+B_2e^{\beta y})\cos(\lambda t+C_2),\quad \lambda=\frac{\alpha}{\sqrt{1+\alpha^2-\beta^2}},$$
$$w=\cos(\alpha x+C_1)\cos(\beta y+C_2)\cos(\lambda t+C_3),\quad \lambda=\frac{\alpha}{\sqrt{1+\alpha^2+\beta^2}},$$

where A_n, B_n, C_n, α, and β are arbitrary constants.

2°. Domain: $S=\{-\infty<x<\infty,\,0\le y\le\pi\}$. First boundary value problem.

The following conditions are prescribed:

$$w=f_0(x,y)\quad\text{at}\quad t=0,$$
$$\partial_t w=f_1(x,y)\quad\text{at}\quad t=0,$$
$$w=0\quad\text{at}\quad y=0,$$
$$w=0\quad\text{at}\quad y=\pi.$$

Solution:

$$w(\mathbf{r},t) = \frac{\partial}{\partial t}\int_{S'} G(\mathbf{r},\mathbf{r}',t)\mathrm{M}[f_0(\mathbf{r}')]\,dV' + \int_{S'} G(\mathbf{r},\mathbf{r}',t)\mathrm{M}[f_1(\mathbf{r}')]\,dV',$$

where

$$\mathbf{r} = (x,y), \quad \mathbf{r}' = (x',y'), \quad \mathrm{M}[f(\mathbf{r}')] \equiv \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} - f, \quad dV' = dx'\,dy',$$

$$G(\mathbf{r},\mathbf{r}',t) = -\sum_{n=1}^{\infty} \frac{\sin(ny)\sin(ny')}{\pi^2\sqrt{1+n^2}} \int_{-\infty}^{\infty} \frac{\exp\bigl(i\mu|x-x'|\sqrt{1+n^2}\,\bigr)}{\mu\sqrt{1+\mu^2}} \sin\biggl(\frac{t\mu}{\sqrt{1+\mu^2}}\biggr)\,d\mu.$$

⊙ *Literature*: S. A. Gabov and A. G. Sveshnikov (1990).

11.4 Three- and n-Dimensional Nonstationary Fourth-Order Equations

11.4.1 Equation of the Form $\dfrac{\partial^2 w}{\partial t^2} + a^2\Delta\Delta w = 0$

► **Three-dimensional case.**

1°. Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w = f(x,y,z) \quad \text{at} \quad t=0, \qquad \partial_t w = 0 \quad \text{at} \quad t=0.$$

Solution:

$$w(x,y,z,t) = \frac{1}{\bigl(2\sqrt{\pi a t}\,\bigr)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\xi, y+\eta, z+\zeta) \times \cos\biggl(\frac{\xi^2+\eta^2+\zeta^2}{4at} - \frac{3\pi}{4}\biggr)\,d\xi\,d\eta\,d\zeta.$$

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

2°. Domain: $0 \le x \le l_1$, $0 \le y \le l_2$, $0 \le z \le l_3$. Boundary value problem.
Initial conditions:

$$w = f(x,y,z) \quad \text{at} \quad t=0, \qquad \partial_t w = g(x,y,z) \quad \text{at} \quad t=0.$$

Boundary conditions:

$$\begin{aligned}
&w = \partial_{xx} w = 0 \quad \text{at} \quad x=0, \qquad && w = \partial_{xx} w = 0 \quad \text{at} \quad x=l_1,\\
&w = \partial_{yy} w = 0 \quad \text{at} \quad y=0, \qquad && w = \partial_{yy} w = 0 \quad \text{at} \quad y=l_2,\\
&w = \partial_{zz} w = 0 \quad \text{at} \quad z=0, \qquad && w = \partial_{zz} w = 0 \quad \text{at} \quad z=l_3.
\end{aligned}$$

Solution:

$$w(x,y,z,t) = \frac{\partial}{\partial t}\int_0^{l_1}\int_0^{l_2}\int_0^{l_3} f(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\zeta\,d\eta\,d\xi + \int_0^{l_1}\int_0^{l_2}\int_0^{l_3} g(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta,t)\,d\zeta\,d\eta\,d\xi,$$

where

$$G(x,y,z,\xi,\eta,\zeta,t) = \frac{8}{al_1l_2l_3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\lambda_{nmk}}\sin(p_nx)\sin(q_my)\sin(s_kz) \times \sin(p_n\xi)\sin(q_m\eta)\sin(s_k\zeta)\sin(\lambda_{nmk}at),$$

$$p_n = \frac{\pi n}{l_1},\quad q_m = \frac{\pi m}{l_2},\quad s_k = \frac{\pi k}{l_3},\quad \lambda_{nmk} = p_n^2 + q_m^2 + s_k^2.$$

► **n-dimensional case.**

1°. Domain: $\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1,\ \dots,\ n\}$. Cauchy problem.

Initial conditions are prescribed:

$$w = f(\mathbf{x}) \quad\text{at}\quad t=0, \qquad \partial_t w = 0 \quad\text{at}\quad t=0,$$

where $\mathbf{x} = \{x_1,\dots,x_n\}$.

Solution:

$$w(\mathbf{x},t) = \frac{1}{\left(2\sqrt{\pi at}\,\right)^n}\int_{\mathbb{R}^n} f(\mathbf{y})\cos\left(\frac{|\mathbf{x}-\mathbf{y}|}{4at} - \frac{\pi n}{4}\right)dV_y,$$

where $\mathbf{y} = \{y_1,\dots,y_n\}$ and $dV_y = dy_1\dots dy_n$.

⊙ *Literature*: V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

2°. Domain: $V = \{0 \le x_k \le l_k;\ k = 1,\ 2,\ \dots\ n\}$. Boundary value problem.

Initial conditions:

$$w = f(\mathbf{x}) \quad\text{at}\quad t=0, \qquad \partial_t w = g(\mathbf{x}) \quad\text{at}\quad t=0.$$

Boundary conditions:

$$w = \partial_{x_kx_k}w = 0 \quad\text{at}\quad x_k=0, \qquad w = \partial_{x_kx_k}w = 0 \quad\text{at}\quad x_k = l_k.$$

Solution:

$$w(\mathbf{x},t) = \frac{\partial}{\partial t}\int_V f(\mathbf{y})G(\mathbf{x},\mathbf{y},t)\,dV_y + \int_V g(\mathbf{y})G(\mathbf{x},\mathbf{y},t)\,dV_y,$$

where

$$G(\mathbf{x},\mathbf{y},t) = \frac{2^n}{al_1l_2\dots l_n}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\frac{1}{\lambda_k}\sin(p_{k_1}x_1)\sin(p_{k_2}x_2)\dots\sin(p_{k_n}x_n) \times \sin(p_{k_1}y_1)\sin(p_{k_2}y_2)\dots\sin(p_{k_n}y_n)\sin\bigl(\lambda_k at\bigr),$$

$$p_{k_1} = \frac{\pi k_1}{l_1},\quad p_{k_2} = \frac{\pi k_2}{l_2},\quad \dots,\quad p_{k_n} = \frac{\pi k_n}{l_n},\quad \lambda_k = p_{k_1}^2 + p_{k_2}^2 + \dots + p_{k_n}^2.$$

11.4.2 Equations Containing Mixed Derivatives

1. $\dfrac{\partial^2 w}{\partial t^2} - a\Delta w - b\dfrac{\partial^2}{\partial t^2}\Delta w = 0.$

Let $a > 0$ and $b > 0$. Consider a bounded open domain V with boundary S. The following conditions are prescribed:

$$\begin{aligned} w &= f(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \partial_t w &= g(\mathbf{x}) \quad \text{at} \quad t = 0 \quad \text{(initial condition)},\\ \alpha w + \beta\frac{\partial w}{\partial n} &= 0 \quad \text{at} \quad \mathbf{x} \in S \quad \text{(boundary condition)}, \end{aligned}$$

where $\alpha = \alpha(\mathbf{x}) \ge 0$, $\beta = \beta(\mathbf{x}) \ge 0$, $\alpha + \beta > 0$ (all functions are assumed to be continuous in V), $\mathbf{x} = (x, y, z)$, and $\partial w/\partial n$ is the outward normal derivative on S. The boundary conditions of the first and second kind correspond to the special values $\beta = 0$ ($\alpha = 1$) and $\alpha = 0$ ($\beta = 1$), respectively.

Solution:

$$w(x,t) = \sum_{k=1}^{\infty} \bigl[A_k\psi_{k1}(t) + B_k\psi_{k2}(t)\bigr]u_k(\mathbf{x}),$$

where

$$\psi_{k1}(t) = \sqrt{\frac{1+b\lambda_n}{a\lambda_n}}\sin\biggl(t\sqrt{\frac{a\lambda_n}{1+b\lambda_n}}\biggr), \qquad \psi_{k2}(t) = \cos\biggl(t\sqrt{\frac{a\lambda_n}{1+b\lambda_n}}\biggr),$$
$$A_k = \frac{1}{\|u_k\|^2}\int_V f(\mathbf{x})u_k(\mathbf{x})\,dv, \qquad B_k = \frac{1}{\|u_k\|^2}\int_V g(\mathbf{x})u_k(\mathbf{x})\,dv,$$

and λ_k and $u_k(\mathbf{x})$ are the eigenvalues and eigenfunctions of the auxiliary problem

$$\begin{aligned} \Delta u + \lambda u &= 0, \qquad \mathbf{x} \in V,\\ \alpha u + \beta\frac{\partial u}{\partial n} &= 0, \qquad \mathbf{x} \in S. \end{aligned}$$

For the main properties of the eigenvalues λ_k and eigenfunctions $u_k(\mathbf{x})$, see Item 4° of equation 11.1.5.1.

2. $\Delta\Bigl(\dfrac{\partial}{\partial t} - \nu\Delta\Bigr)w = \Phi(\mathbf{x}, t).$

Here $\mathbf{x} = (x, y, z)$ and Δ is the three-dimensional Laplace operator. Equations of this kind occur in the hydrodynamics of a viscous incompressible fluid (see Section 12.7.3).

1°. The general solution of the homogeneous equation with $\Phi \equiv 0$:

$$w = w_1 + w_2,$$

where w_1 and w_2 are arbitrary solutions of the Laplace and the heat equations

$$\Delta w_1 = 0, \qquad \frac{\partial w_2}{\partial t} - \nu\Delta w_2 = 0.$$

2°. A particular solution of the nonhomogeneous equation for an arbitrary domain V_* lying inside the rectangular parallelepiped $V = \{0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$ can be represented in the form of the series

$$w_p(\mathbf{x}, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \varphi_{nmk}(t) u_{nmk}(\mathbf{x}),$$

where

$$\varphi_{nmk}(t) = -\frac{1}{\lambda_{nmk}} \int_0^t \exp\bigl[-\nu\lambda_{nmk}(t-\tau)\bigr] A_{nmk}(\tau)\,d\tau,$$
$$u_{nmk}(\mathbf{x}) = \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{b}\right)\sin\left(\frac{\pi k z}{c}\right),$$
$$\lambda_{nmk} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2}\right), \quad A_{nmk}(t) = \frac{8}{abc}\int_V f(\mathbf{x}, t) u_{nmk}(\mathbf{x})\,dV.$$

3. $\boldsymbol{\Delta\Bigl(\sigma\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} - \nu\Delta\Bigr)w = \Phi(\mathbf{x}, t).}$

Here $\mathbf{x} = (x, y, z)$ and Δ is the three-dimensional Laplace operator. Equations of this kind arise when decomposing the equations of a viscoelastic incompressible Maxwell fluid (see Section 12.9.3).

1°. The general solution of the homogeneous equation with $\Phi \equiv 0$:

$$w = w_1 + w_2,$$

where w_1 and w_2 are arbitrary solutions of the Laplace and the telegraph equations

$$\Delta w_1 = 0, \qquad \sigma\frac{\partial^2 w_2}{\partial t^2} + \frac{\partial w_2}{\partial t} - \nu\Delta w_2 = 0.$$

2°. A particular solution of the nonhomogeneous equation for an arbitrary domain V_* lying inside the rectangular parallelepiped $V = \{0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$ can be represented in the form of the series

$$w_p(\mathbf{x}, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \varphi_{nmk}(t) u_{nmk}(\mathbf{x}),$$

where

$$\varphi_{nmk}(t)=\begin{cases}-\dfrac{2}{s_{nmk}\lambda_{nmk}}\displaystyle\int_0^t A_{nmk}(\tau)\exp\biggl[-\frac{1}{2\sigma}(t-\tau)\biggr]\sin\biggl[\frac{s_{nmk}}{2\sigma}(t-\tau)\biggr]\,d\tau \\ \qquad\qquad\qquad\text{if } 4\nu\sigma\lambda_{nmk}-1=s_{nmk}^2>0,\\ -\dfrac{2}{s_{nmk}\lambda_{nmk}}\displaystyle\int_0^t A_{nmk}(\tau)\exp\biggl[-\frac{1}{2\sigma}(t-\tau)\biggr]\sinh\biggl[\frac{s_{nmk}}{2\sigma}(t-\tau)\biggr]\,d\tau \\ \qquad\qquad\qquad\text{if } 4\nu\sigma\lambda_{nmk}-1=-s_{nmk}^2<0,\\ -\dfrac{1}{\sigma\lambda_{nmk}}\displaystyle\int_0^t A_{nmk}(\tau)\exp\biggl[-\frac{1}{2\sigma}(t-\tau)\biggr](t-\tau)\,d\tau \\ \qquad\qquad\qquad\text{if } 4\nu\sigma\lambda_{nmk}-1=0,\end{cases}$$

$$u_{nmk}(\mathbf{x})=\sin\Bigl(\frac{\pi nx}{a}\Bigr)\sin\Bigl(\frac{\pi my}{b}\Bigr)\sin\Bigl(\frac{\pi kz}{c}\Bigr),$$

$$\lambda_{nmk}=\pi^2\Bigl(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{k^2}{c^2}\Bigr),\qquad A_{nmk}(t)=\frac{8}{abc}\int_V f(\mathbf{x},t)u_{nmk}(\mathbf{x})\,dV.$$

4. $\Bigl(\dfrac{\partial^2}{\partial t^2}-c_1^2\Delta\Bigr)\Bigl(\dfrac{\partial^2}{\partial t^2}-c_2^2\Delta\Bigr)w=\Phi(\mathbf{x},t).$

Here $\mathbf{x}=(x,y,z)$ and Δ is the three-dimensional Laplace operator. Similar equations occur in elasticity theory (see Section 12.6.3).

1°. The general solution of the homogeneous equation with $\Phi\equiv 0$:

$$w=w_1+w_2,$$

where w_1 and w_2 are arbitrary solutions of two independent wave equations

$$\frac{\partial^2 w_1}{\partial t^2}-c_1^2\Delta w_1=0,\qquad \frac{\partial^2 w_2}{\partial t^2}-c_2^2\Delta w_2=0.$$

2°. If $c_1^2\neq c_2^2$, then a particular solution of the nonhomogeneous equation for an arbitrary domain V_* lying inside the rectangular parallelepiped $V=\{0\le x\le a_1, 0\le y\le a_2, 0\le z\le a_3\}$ can be represented in the form of the series

$$w_p(\mathbf{x},t)=\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{k=1}^\infty \varphi_{nmk}(t)u_{nmk}(\mathbf{x}),$$

where

$$\varphi_{nmk}(t)=\int_0^t\frac{c_1\sin\bigl[c_2\lambda_{nmk}^{1/2}(t-\tau)\bigr]-c_2\sin\bigl[c_1\lambda_{nmk}^{1/2}(t-\tau)\bigr]}{c_1c_2\lambda_{nmk}^{3/2}(c_1^2-c_2^2)}A_{nmk}(\tau)\,d\tau,$$

$$u_{nmk}(\mathbf{x})=\sin\Bigl(\frac{\pi nx}{a_1}\Bigr)\sin\Bigl(\frac{\pi my}{a_2}\Bigr)\sin\Bigl(\frac{\pi kz}{a_3}\Bigr),$$

$$\lambda_{nmk}=\pi^2\Bigl(\frac{n^2}{a_1^2}+\frac{m^2}{a_2^2}+\frac{k^2}{a_3^2}\Bigr),\qquad A_{nmk}(t)=\frac{8}{a_1a_2a_3}\int_V f(\mathbf{x},t)u_{nmk}(\mathbf{x})\,dV.$$

5. $\dfrac{\partial^2}{\partial t^2}\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}\Big)+a^2\Big(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\Big)+b^2\dfrac{\partial^2 w}{\partial z^2}=0.$

Equation of gravitational–gyroscopic waves in the Boussinesq approximation. In the special case of $a=0$, it is known as the *Sobolev equation* and describes small vibrations of a homogeneous inviscid rotating fluid. It is assumed in what follows that $a\ge 0$ and $b\ge 0$.

1°. Particular solutions:

$$w=\exp(k_1x+k_2y+k_3z)[c_1\cos(\lambda t)+c_2\sin(\lambda t)],$$
$$w=\cos(k_1x+c_1)\cos(k_2y+c_2)\cos(k_3z+c_3)\cos(\lambda t+c_4),$$

where c_1, c_2, c_3, k_1, k_2, and k_3 are arbitrary constants, and

$$\lambda=\sqrt{\frac{a^2(k_1^2+k_2^2)+b^2k_3^2}{k_1^2+k_2^2+k_3^2}}.$$

2°. Singular particular solution:

$$w_3(\mathbf{r},t)=-\frac{1}{4\pi|\mathbf{r}|_*}\int_0^{t|\mathbf{r}|_*/|\mathbf{r}|}J_0(\xi)\,d\xi,$$
$$|\mathbf{r}|=(x^2+y^2+z^2)^{1/2},\qquad |\mathbf{r}|_*=[b^2(x^2+y^2)+a^2z^2]^{1/2},\tag{1}$$

where $J_0(\xi)$ is the Bessel function. Note that $\min\{a,b\}\le|\mathbf{r}|_*/|\mathbf{r}|\le\max\{a,b\}$.

The singular solution (1) satisfies the initial conditions

$$w_3=0,\qquad \frac{\partial w_3}{\partial t}=-\frac{1}{4\pi|\mathbf{r}|}\quad\text{at}\quad t=0.\tag{2}$$

3°. Fundamental solution:

$$\mathscr{E}_{\rm e}(\mathbf{r},t)=-\frac{1}{4\pi|\mathbf{r}|}\int_0^t J_0(a(t-\tau))J_0\Big(\frac{|\mathbf{r}|_*}{|\mathbf{r}|}\tau\Big)\,d\tau.\tag{3}$$

The fundamental solution satisfies the initial conditions (2) with w_3 replaced by $\mathscr{E}_{\rm e}$.

On the z-axis, which corresponds to $x=y=0$ and $|\mathbf{r}|_*/|\mathbf{r}|=a$, formula (3) implies that

$$\mathscr{E}_{\rm e}(\mathbf{r},t)\big|_{x=y=0}=-\frac{\sin(at)}{4\pi a|z|}.$$

4°. Let $a\ne b$. Set $\lambda=\lambda(\mathbf{r})=|\mathbf{r}|_*/|\mathbf{r}|$. An alternative representation for the fundamental solution with $\lambda\ne a$ is given by

$$\mathscr{E}_{\rm e}(\mathbf{r},t)=-\frac{1}{2\pi^2a|\mathbf{r}|}\int_{\lambda/a}^1\frac{\sin(at\mu)\,d\mu}{\sqrt{1-\mu^2}\sqrt{\mu^2-(\lambda/a)^2}}.$$

Let $|\lambda-a|\ge\delta>0$, where δ is a fixed sufficiently small number. (Thus, a narrow domain near the z-axis is eliminated.) Then the leading term of the asymptotic expansion of the fundamental solution as $at\to\infty$ has the form

$$\mathscr{E}_{\rm e}(\mathbf{r},t)=-\frac{1}{2\pi^2}\sqrt{\frac{\pi a}{2\lambda|a^2-\lambda^2|}}\,\frac{\sin(\lambda t-\pi/4)}{|\mathbf{r}|\sqrt{at}}$$
$$-\frac{1}{2\pi^2}\sqrt{\frac{\pi}{2|a^2-\lambda^2|}}\,\frac{\sin(at-\pi/4)}{|\mathbf{r}|\sqrt{at}}+\frac{1}{|\mathbf{r}|}O\Big(\frac{1}{at}\Big).$$

5°. *The Cauchy problem* ($t \geq 0$, $\mathbf{r} \in \mathbb{R}^3$). The initial conditions:

$$w = f(\mathbf{r}) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(\mathbf{r}) \quad \text{at} \quad t = 0.$$

Solution:

$$w(\mathbf{r}, t) = \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \mathscr{E}_{\rm e}(\mathbf{r} - \mathbf{r}', t)\Delta_3' f(\mathbf{r}')\, dV' + \int_{\mathbb{R}^3} \mathscr{E}_{\rm e}(\mathbf{r} - \mathbf{r}', t)\Delta_3' g(\mathbf{r}')\, dV',$$

where Δ_3' is the three-dimensional Laplace operator in the integration variables (x', y', z') and $dV' = dx'\, dy'\, dz'$.

⊙ *Literature*: S. Ya. Sekerzh–Zenkovich (1979), S. A. Gabov and A. G. Sveshnikov (1990).

6. $\displaystyle \frac{\partial^2}{\partial t^2}\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) + a^2\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big) + b^2\frac{\partial^2 w}{\partial z^2} = \Phi(x, y, z, t).$

The Cauchy problem ($t \geq 0$, $\mathbf{r} \in \mathbb{R}^3$). The initial conditions:

$$w = f(\mathbf{r}) \quad \text{at} \quad t = 0,$$
$$\partial_t w = g(\mathbf{r}) \quad \text{at} \quad t = 0,$$

where $\mathbf{r} = (x, y, z)$.

Solution:

$$w(\mathbf{r}, t) = \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \mathscr{E}_{\rm e}(\mathbf{r} - \mathbf{r}', t)\Delta_3' f(\mathbf{r}')\, dV' + \int_{\mathbb{R}^3} \mathscr{E}_{\rm e}(\mathbf{r} - \mathbf{r}', t)\Delta_3' g(\mathbf{r}')\, dV'$$
$$+ \int_0^t \int_{\mathbb{R}^3} \mathscr{E}_{\rm e}(\mathbf{r} - \mathbf{r}', t - \tau)\Phi(\mathbf{r}', \tau)\, dV' d\tau,$$

where $\mathscr{E}_{\rm e}(\mathbf{r}, t)$ is the fundamental solution given in Item 3° of the preceding equation, Δ_3' is the three-dimensional Laplace operator in the integration variables x', y', z', and $dV' = dx'\, dy'\, dz'$.

⊙ *Literature*: S. A. Gabov and A. G. Sveshnikov (1990).

11.5 Fourth-Order Stationary Equations

11.5.1 Biharmonic Equation $\Delta\Delta w = 0$

The biharmonic equation is encountered in plane problems of elasticity (w is the Airy stress function). It is also used to describe slow flows of viscous incompressible fluids (in this case w is the stream function).

All solutions of the Laplace equation $\Delta w = 0$ (see Sections 9.1 and 10.1) are also solutions of the biharmonic equation.

► **Two-dimensional equation. Particular solutions. Fundamental solution.**

1°. In the rectangular Cartesian system of coordinates, the biharmonic operator has the form

$$\Delta\Delta \equiv \Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

2°. Particular solutions:

$$w(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3 + ax^2 + bxy + cy^2 + \alpha x + \beta y + \gamma,$$
$$w(x,y) = (A\cosh\beta x + B\sinh\beta x + Cx\cosh\beta x + Dx\sinh\beta x)(a\cos\beta y + b\sin\beta y),$$
$$w(x,y) = (A\cos\beta x + B\sin\beta x + Cx\cos\beta x + Dx\sin\beta x)(a\cosh\beta y + b\sinh\beta y),$$
$$w(x,y) = Ar^2\ln r + Br^2 + C\ln r + D, \quad r = \sqrt{(x-a)^2 + (y-b)^2},$$
$$w(x,y) = (Ax + By + C)(D\cosh\beta x + E\sinh\beta x)(a\cos\beta y + b\sin\beta y),$$
$$w(x,y) = (Ax + By + C)(D\cosh\beta y + E\sinh\beta y)(a\cos\beta x + b\sin\beta x),$$
$$w(x,y) = (x^2 + y^2)(D\cosh\beta x + E\sinh\beta x)(a\cos\beta y + b\sin\beta y),$$
$$w(x,y) = (x^2 + y^2)(D\cosh\beta y + E\sinh\beta y)(a\cos\beta x + b\sin\beta x),$$

where A, B, C, D, E, a, b, c, α, β, and γ are arbitrary constants.

3°. Particular solutions of the biharmonic equation in some orthogonal curvilinear coordinate systems are listed in Table 11.1.

TABLE 11.1
Particular solutions of the biharmonic equation in some orthogonal curvilinear coordinate systems; A, B, C, D, a, b, and λ are arbitrary constants

Transformation	Particular solutions
Polar coordinates r, φ: $x = r\cos\varphi,\ y = r\sin\varphi$	$w = (Ar^{2+\lambda} + Br^{2-\lambda} + Cr^{\lambda} + Dr^{-\lambda})\cos(\lambda\varphi)$, $w = (Ar^{2+\lambda} + Br^{2-\lambda} + Cr^{\lambda} + Dr^{-\lambda})\sin(\lambda\varphi)$, $w = Ar^2\ln r + Br^2 + C\ln r + D$ (at $\lambda = 0$)
Bipolar coordinates ξ, η: $x = \dfrac{c\sinh\xi}{\cosh\xi - \cos\eta},\ y = \dfrac{c\sin\eta}{\cosh\xi - \cos\eta}$	$w = \dfrac{a\cos\lambda\eta + b\sin\lambda\eta}{\cosh\xi - \cos\eta}\big[A\cosh(\lambda+1)\xi + B\sinh(\lambda+1)\xi + C\cosh(\lambda-1)\xi + D\sinh(\lambda-1)\xi\big]$
Degenerate bipolar coordinates u, v: $x = \dfrac{u}{u^2+v^2},\ y = -\dfrac{v}{u^2+v^2}$	$w = \dfrac{a\cos(\lambda v) + b\sin(\lambda v)}{(u^2+v^2)^2}\big[A\cosh(\lambda u) + B\sinh(\lambda u) + Cu\cosh(\lambda u) + Du\sinh(\lambda u)\big]$

⊙ *Literature*: N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand (1955).

4°. Fundamental solution:

$$\mathscr{E}_{\rm e}(x,y) = \frac{1}{8\pi}r^2\ln r, \qquad r = \sqrt{x^2+y^2}. \tag{1}$$

► **Two-dimensional equation. Various representations of the general solution.**

1°. Various representations of the general solution of the biharmonic equation in terms of harmonic functions:

$$w(x,y) = xu_1(x,y) + u_2(x,y),$$
$$w(x,y) = yu_1(x,y) + u_2(x,y),$$
$$w(x,y) = (x^2+y^2)u_1(x,y) + u_2(x,y),$$

where u_1 and u_2 are arbitrary functions satisfying the Laplace equation $\Delta u_k = 0$ $(k=1,\ 2)$.

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

2°. Complex form of representation of the general solution:

$$w(x,y) = \operatorname{Re}\bigl[\overline{z} f(z) + g(z)\bigr],$$

where $f(z)$ and $g(z)$ are arbitrary analytic functions of the complex variable $z = x + iy$; $\overline{z} = x - iy$, $i^2 = -1$. The symbol $\operatorname{Re}[A]$ stands for the real part of a complex number A.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Two-dimensional boundary value problems for the upper half-plane.**

1°. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. The desired function and its derivative along the normal are prescribed on the boundary:

$$w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi, y)\, d\xi, \qquad G(x,y) = \frac{1}{\pi}\frac{y^2}{x^2+y^2}.$$

⊙ *Literature*: G. E. Shilov (1965).

2°. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. The derivatives of the desired function are prescribed on the boundary:

$$\partial_x w = f(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = g(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\left[\arctan\left(\frac{x-\xi}{y}\right) + \frac{y(x-\xi)}{(x-\xi)^2+y^2}\right] d\xi$$
$$+ \frac{y^2}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi)\, d\xi}{(x-\xi)^2+y^2} + C,$$

where C is an arbitrary constant.

Example 11.1. Consider the problem of a slow (Stokes) inflow of a viscous fluid into a half-plane through a slit of width $2a$ at a constant velocity U that makes an angle β with the normal to the boundary (the angle is measured off from the normal direction counterclockwise).

With the stream function w introduced by the relations $v_x = -\frac{\partial w}{\partial y}$ and $v_y = \frac{\partial w}{\partial x}$ (v_x and v_y are the fluid velocity components), the problem is reduced to the special case of the previous problem with

$$f(x) = \begin{cases} U\cos\beta & \text{for } |x| < a, \\ 0 & \text{for } |x| > a, \end{cases} \qquad g(x) = \begin{cases} U\sin\beta & \text{for } |x| < a, \\ 0 & \text{for } |x| > a. \end{cases}$$

Dean's solution:

$$\begin{aligned} w(x,y) &= \frac{U}{\pi}\big[(x-a)\cos\beta + y\sin\beta\big]\arctan\left(\frac{y}{x-a}\right) \\ &\quad - \frac{U}{\pi}\big[(x+a)\cos\beta + y\sin\beta\big]\arctan\left(\frac{y}{x+a}\right) + C. \end{aligned}$$

⊙ *Literature*: I. Sneddon (1951).

► Two-dimensional boundary value problem for a disk.

Domain: $0 \le r \le a$, $0 \le \varphi \le 2\pi$. Boundary conditions in the polar coordinate system:

$$w = f(\varphi) \quad \text{at} \quad r = a, \qquad \partial_r w = g(\varphi) \quad \text{at} \quad r = a.$$

Solution:

$$\begin{aligned} w(r,\varphi) = \frac{1}{2\pi a}(r^2 - a^2)^2 \bigg[&\int_0^{2\pi} \frac{[a - r\cos(\eta - \varphi)]f(\eta)\,d\eta}{[r^2 + a^2 - 2ar\cos(\eta - \varphi)]^2} \\ &- \frac{1}{2}\int_0^{2\pi} \frac{g(\eta)\,d\eta}{r^2 + a^2 - 2ar\cos(\eta - \varphi)} \bigg]. \end{aligned}$$

⊙ *Literature*: A. N. Tikhonov and A. A. Samarskii (1990).

► Three-dimensional equation. Particular solutions.

In the rectangular Cartesian coordinate system, the three-dimensional biharmonic operator is expressed as

$$\Delta\Delta \equiv \Delta^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + 2\frac{\partial^4}{\partial x^2 \partial z^2} + 2\frac{\partial^4}{\partial y^2 \partial z^2}.$$

1°. Particular solutions in the Cartesian coordinate system:

$$w(x,y,z) = Ar^2 + Br + C + \frac{D}{r}, \quad r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2},$$
$$w(x,y,z) = \bigl[Ax\sin(\beta x + C_1) + B\sin(\beta x + C_2)\bigr]\sin(\mu y)\exp\bigl(\pm z\sqrt{\beta^2+\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\sin(\beta x + C_1) + B\sin(\beta x + C_2)\bigr]\cos(\mu y)\exp\bigl(\pm z\sqrt{\beta^2+\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\sin(\beta x + C_1) + B\sin(\beta x + C_2)\bigr]\sinh(\mu y)\exp\bigl(\pm z\sqrt{\beta^2-\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\sin(\beta x + C_1) + B\sin(\beta x + C_2)\bigr]\cosh(\mu y)\exp\bigl(\pm z\sqrt{\beta^2-\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\sinh(\beta x + C_1) + B\sinh(\beta x + C_2)\bigr]\sinh(\mu y)\sin\bigl(z\sqrt{\beta^2+\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\cosh(\beta x + C_1) + B\cosh(\beta x + C_2)\bigr]\sinh(\mu y)\sin\bigl(z\sqrt{\beta^2+\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\sinh(\beta x + C_1) + B\sinh(\beta x + C_2)\bigr]\cosh(\mu y)\cos\bigl(z\sqrt{\beta^2+\mu^2}\,\bigr),$$
$$w(x,y,z) = \bigl[Ax\cosh(\beta x + C_1) + B\cosh(\beta x + C_2)\bigr]\cosh(\mu y)\cos\bigl(z\sqrt{\beta^2+\mu^2}\,\bigr),$$

where A, B, C, C_1, C_2, D, β, and μ are arbitrary constants.

2°. Particular solutions in the cylindrical coordinate system $\bigl(r = \sqrt{x^2+y^2}\,\bigr)$:

$$w(r,\varphi,z) = J_n(\mu r)(Ar\cos\varphi + Br\sin\varphi + C)(a_1\cos n\varphi + b_1\sin n\varphi)\exp(\pm\mu z),$$
$$w(r,\varphi,z) = Y_n(\mu r)(Ar\cos\varphi + Br\sin\varphi + C)(a_1\cos n\varphi + b_1\sin n\varphi)\exp(\pm\mu z),$$
$$w(r,\varphi,z) = I_n(\mu r)(Ar\cos\varphi + Br\sin\varphi + C)(a_1\cos n\varphi + b_1\sin n\varphi)\sin(\mu z + \beta),$$
$$w(r,\varphi,z) = K_n(\mu r)(Ar\cos\varphi + Br\sin\varphi + C)(a_1\cos n\varphi + b_1\sin n\varphi)\sin(\mu z + \beta),$$
$$w(r,\varphi,z) = J_n(\mu r)\cos(n\varphi + \beta)(a_1\cosh\mu z + b_1\sinh\mu z + a_2 z\cosh\mu z + b_2 z\sinh\mu z),$$
$$w(r,\varphi,z) = Y_n(\mu r)\cos(n\varphi + \beta)(a_1\cosh\mu z + b_1\sinh\mu z + a_2 z\cosh\mu z + b_2 z\sinh\mu z),$$
$$w(r,\varphi,z) = I_n(\mu r)\cos(n\varphi + \beta)(a_1\cos\mu z + b_1\sin\mu z + a_2 z\cos\mu z + b_2 z\sin\mu z),$$
$$w(r,\varphi,z) = K_n(\mu r)\cos(n\varphi + \beta)(a_1\cos\mu z + b_1\sin\mu z + a_2 z\cos\mu z + b_2 z\sin\mu z),$$

where $n = 0,\,1,\,2,\,\ldots;$ A, B, C, a_1, a_2, b_1, b_2, β, and μ are arbitrary constants; the $J_n(\xi)$ and $Y_n(\xi)$ are Bessel functions; and the $I_n(\xi)$ and $K_n(\xi)$ are modified Bessel functions.

3°. Particular solutions in the spherical coordinate system $\bigl(r = \sqrt{x^2+y^2+z^2}\,\bigr)$:

$$w(r) = Ar^2 + Br + C + Dr^{-1},$$
$$w(r,\theta) = \bigl(Ar^{n+2} + Br^n + Cr^{1-n} + Dr^{-1-n}\bigr)P_n(\cos\theta),$$
$$w(r,\theta,\varphi) = \bigl(Ar^{n+2} + Br^n + Cr^{1-n} + Dr^{-1-n}\bigr)P_n^m(\cos\theta)(a\cos m\varphi + b\sin m\varphi),$$

where $n = 0,\,1,\,2,\,\ldots;$ $m = 0,\,1,\,2,\,\ldots,\,n;$ A, B, C, D, a, and b are arbitrary constants; the $P_n(\xi)$ are the Legendre polynomials; and the $P_n^m(\xi)$ are the associated Legendre functions defined by

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n, \quad P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x).$$

► Three-dimensional equation. Fundamental solution. General solution.

1°. Fundamental solution:

$$\mathscr{E}_{\rm e}(x,y,z) = -\frac{1}{8\pi}\sqrt{x^2+y^2+z^2}. \qquad (2)$$

2°. Various representations of the general solution of the biharmonic equation in terms of harmonic functions:

$$w(x,y,z) = xu_1(x,y,z) + u_2(x,y,z),$$
$$w(x,y,z) = (x^2+y^2+z^2)u_1(x,y,z) + u_2(x,y,z),$$

where u_1 and u_2 are arbitrary functions satisfying the three-dimensional Laplace equation $\Delta_3 u_k = 0$ ($k = 1,\ 2$). The coefficient x of u_1 in the first formula can be replaced by y or z.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► n-dimensional equation.

1°. Particular solutions:

$$w(\mathbf{x}) = \sum_{i,j,k=1}^{n} A_{ijk}x_ix_jx_k + \sum_{i,j=1}^{n} B_{ij}x_ix_j + \sum_{i=1}^{n} C_ix_i + D,$$

$$w(\mathbf{x}) = Ar^2 + B + Cr^{4-n} + Dr^{2-n}, \quad r^2 = \sum_{k=1}^{n}(x_k-\alpha_k)^2,$$

$$w(\mathbf{x}) = (A + Br^{2-n})\biggl(\sum_{i=1}^{n} C_ix_i + D\biggr), \quad r^2 = \sum_{k=1}^{n}(x_k-\alpha_k)^2,$$

$$w(\mathbf{x}) = \exp\bigl(\pm x_n\sqrt{\lambda_n}\,\bigr)\biggl(\sum_{i=1}^{n} A_ix_i + B\biggr)\prod_{k=1}^{n-1}\sin(\alpha_kx_k+\beta_k), \quad \lambda_n = \sum_{k=1}^{n-1}\alpha_k^2,$$

$$w(\mathbf{x}) = \biggl(\sum_{i=1}^{n} A_ix_i + B\biggr)\biggl[\prod_{k=1}^{m-1}\sin(\alpha_kx_k+\beta_k)\biggr]\biggl[\prod_{k=m}^{n}\sinh(\gamma_kx_k)\biggr], \quad \sum_{k=1}^{m-1}\alpha_k^2 = \sum_{k=m}^{n}\gamma_k^2,$$

where the A_{ijk}, B_{ij}, A_i, C_i, A, B, C, D, α_k, β_k, and γ_k are arbitrary constants.

2°. Fundamental solution:

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \begin{cases} \dfrac{\Gamma(n/2)|\mathbf{x}|^{4-n}}{4\pi^{n/2}(n-2)(n-4)} & \text{for } n = 3,\ 5,\ 6,\ 7,\ \ldots; \\ -\dfrac{1}{8\pi^2}\ln|\mathbf{x}| & \text{for } n = 4. \end{cases}$$

For $n = 2$, see formula (1) above.

⊙ *Literature*: G. E. Shilov (1965).

3°. Various representations of solutions of the biharmonic equation in terms of harmonic functions:

$$w(\mathbf{x}) = x_s u_1(\mathbf{x}) + u_2(\mathbf{x}), \qquad s = 1,\, 2,\, \dots,\, n;$$

$$w(\mathbf{x}) = |\mathbf{x}|^2 u_1(\mathbf{x}) + u_2(\mathbf{x}), \qquad |\mathbf{x}|^2 = \sum_{k=1}^{n} x_k^2,$$

where u_1 and u_2 are arbitrary functions satisfying the n-dimensional Laplace equation $\Delta_n u_m = 0$ $(m = 1,\, 2)$.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

11.5.2 Equation of the Form $\Delta\Delta w = \Phi$

Nonhomogeneous biharmonic equation. It is encountered in problems of elasticity and hydrodynamics.

► **Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Particular solution.**

Particular solution with $\Phi = \Phi(x,y)$:

$$w(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta)\mathscr{E}_{\rm e}(x-\xi, y-\eta)\, d\xi\, d\eta, \qquad \mathscr{E}_{\rm e}(x,y) = \frac{1}{8\pi}(x^2+y^2)\ln\sqrt{x^2+y^2}.$$

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

► **Boundary value problems in the Cartesian coordinates.**

1°. The upper half-plane is considered $(-\infty < x < \infty, 0 \le y < \infty)$. The derivatives are prescribed on the boundary:

$$\partial_x w = f(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = g(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\left[\arctan\left(\frac{x-\xi}{y}\right) + \frac{y(x-\xi)}{(x-\xi)^2+y^2}\right] d\xi + \frac{y^2}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi)\, d\xi}{(x-\xi)^2+y^2}$$
$$+ \frac{1}{8\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty}\left[\frac{1}{2}(R_+^2 - R_-^2) - R_-^2 \ln\frac{R_+}{R_-}\right]\Phi(\xi,\eta)\, d\eta + C,$$

where C is an arbitrary constant and

$$R_+^2 = (x-\xi)^2 + (y+\eta)^2, \qquad R_-^2 = (x-\xi)^2 + (y-\eta)^2.$$

⊙ *Literature*: I. Sneddon (1951).

2°. A rectangle is considered ($0 \le x \le l_1$, $0 \le y \le l_2$). The sides of the plate are hinged. Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2.$$

Solution:

$$w(x,y) = \int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\eta\, d\xi,$$

where

$$G(x,y,\xi,\eta) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(p_n^2+q_m^2)^2}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta),$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}.$$

► **Domain: $0 \le r \le 1$, $0 \le \varphi \le 2\pi$. Various boundary value problems for the disk.**

1°. Consider boundary value problems in the circle $0 \le r \le 1$, $0 \le \varphi \le 2\pi$ with various homogeneous boundary conditions in the polar coordinate system, $\Phi = \Phi(r,\varphi)$. The solution can be represented in terms of the Green's function as

$$w(r,\varphi) = \int_0^{1}\int_0^{2\pi} \Phi(\xi,\eta) G(r,\varphi,\xi,\eta)\, d\eta\, d\xi.$$

2°. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = 1, \qquad \partial_r w = 0 \quad \text{at} \quad r = 1.$$

These boundary conditions correspond to the rigid clamping of the circular plate. (The deflection w and the rotation angle $\partial_r w$ of the normal element vanish on the support contour.)

Green's function:

$$G_0(r,\varphi,\xi,\eta) = \frac{1}{16\pi}\xi\big[r^2+\xi^2-2\xi r\cos(\varphi-\eta)\big]\left[\frac{1+r^2\xi^2-2\xi r\cos(\varphi-\eta)}{r^2+\xi^2-2\xi r\cos(\varphi-\eta)} - 1 + \ln\left(\frac{r^2+\xi^2-2\xi r\cos(\varphi-\eta)}{1+r^2\xi^2-2\xi r\cos(\varphi-\eta)}\right)\right].$$

Here the Green's function is equipped with the zero subscript, which will prove useful in the subsequent exposition.

⊙ *Literature*: J. H. Michell (1902).

3°. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = 1, \qquad \partial_{rr} w = 0 \quad \text{at} \quad r = 1.$$

Green's function:

$$G(r,\varphi,\xi,\eta) = G_0(r,\varphi,\xi,\eta) + \frac{\sqrt{\xi}\,(\xi^2-1)(r^2-1)}{8\pi\sqrt{r}}\Big[e^{i\frac{\varphi-\eta}{2}}\tanh^{-1}\big(\sqrt{\xi r}\,e^{-i\frac{\varphi-\eta}{2}}\big) + e^{-i\frac{\varphi-\eta}{2}}\tanh^{-1}\big(\sqrt{\xi r}\,e^{i\frac{\varphi-\eta}{2}}\big) - \sqrt{r\xi}\,\Big],$$

where $G_0(r,\varphi,\xi,\eta)$ is the Green's function in Item 2°.

4°. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = 1, \qquad \partial_{rr} w + \frac{\nu}{r}\,\partial_r w = 0 \quad \text{at} \quad r = 1.$$

These boundary conditions correspond to a hinged circular plate. (The deflection w and the bending torque vanish on the support contour; ν is the Poisson ratio.)

Green's function:

$$G(r,\varphi,\xi,\eta) = G_0(r,\varphi,\xi,\eta) + \frac{\xi(\xi^2-1)(r^2-1)}{8\pi(\nu+1)}\biggl[F\biggl(1,\frac{\nu+1}{2};\frac{\nu+3}{2};\xi r e^{-i(\varphi-\eta)}\biggr) + F\biggl(1,\frac{\nu+1}{2};\frac{\nu+3}{2};\xi r e^{i(\varphi-\eta)}\biggr) - 1\biggr],$$

where $G_0(r,\varphi,\xi,\eta)$ is the Green's function in Item 2° and $F(a,b;c;z)$ is the hypergeometric function.

5°. Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad r = 1, \qquad \partial_{rr} w + \Bigl(\frac{\nu}{r} + \gamma\Bigr)\,\partial_r w = 0 \quad \text{at} \quad r = 1.$$

These boundary conditions correspond to an elastically supported circular plate. (The deflection w on the support contour is zero, and the bending torque is proportional to the angle of rotation of the normal element; γ characterizes the elasticity of the support contour.)

The Green's function in this case can be obtained from the Green's function given in Item 4° by the replacement of the parameter ν with $\nu+\gamma$.

6°. Singular solution with $\Phi(r,\varphi) = \delta(r-\xi)\delta(\varphi-\eta)$:

$$w = G_0(r,\varphi,\xi,\eta) - \frac{\xi(\xi^2-1)(r^2-1)}{16\pi}\ln\bigl[1 + r^2\xi^2 - 2\xi r\cos(\varphi-\eta)\bigr],$$

where $\delta(r)$ is the Dirac delta function and ξ and η are arbitrary constants. This solution satisfies the boundary conditions

$$w = 0 \quad \text{at} \quad r = 1, \qquad \partial_{rr} w - \partial_r w = \frac{\xi(1-\xi^2)}{4\pi} \quad \text{at} \quad r = 1.$$

7°. Solution with $\Phi(r,\varphi) = \alpha + \beta r\cos\varphi$:

$$w = \frac{r^2-1}{192}\biggl[(3\alpha + \beta r\cos\varphi)(r^2-1) - \frac{12\alpha}{1+\nu+\gamma} - \frac{4r\beta\cos\varphi}{3+\nu+\gamma}\biggr].$$

This solution satisfies the boundary conditions in Item 5°.

⊙ The results of Items 3°–7° were obtained by S. A. Lychev (private communication, 2014).

► Domain: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Particular solution.

Particular solution with $\Phi = \Phi(x,y,z)$:

$$w(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)\mathscr{E}_\text{e}(x-\xi,y-\eta,z-\zeta)\,d\xi\,d\eta\,d\zeta,$$

$$\mathscr{E}_\text{e}(x,y,z) = -\frac{1}{8\pi}\sqrt{x^2+y^2+z^2}.$$

⊙ *Literature*: O. A. Ladyzhenskaya (1969).

11.5.3 Equation of the Form $\Delta\Delta w - \lambda w = \Phi(x, y)$

▶ **Homogeneous equation ($\Phi \equiv 0$).**

1°. This equation describes the shapes of two-dimensional free transverse vibrations of a thin elastic plate; the function w defines the deflection (transverse displacement) of the plate midplane points relative to the original plane position, and $k = \lambda^{1/4}$ is the frequency parameter. Here $\Delta\Delta = \Delta^2$ is the biharmonic operator and Δ is the Laplace operator defined as

$$\Delta = \begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & \text{in the Cartesian coordinate system,} \\ \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \varphi^2} & \text{in the polar coordinate system.} \end{cases}$$

2°. Particular solutions (A, B, C, and D are arbitrary constants):

$$w(x,y) = \big[A\sin(k_1x) + B\cos(k_1x)\big]\big[C\sin(k_2y) + D\cos(k_2y)\big], \qquad \lambda = (k_1^2 + k_2^2)^2,$$
$$w(x,y) = \big[A\sin(k_1x) + B\cos(k_1x)\big]\big[C\exp(-k_2y) + D\exp(k_2y)\big], \qquad \lambda = (k_1^2 - k_2^2)^2,$$
$$w(x,y) = \big[A\exp(-k_1x) + B\exp(k_1x)\big]\big[C\sin(k_2y) + D\cos(k_2y)\big], \qquad \lambda = (k_1^2 - k_2^2)^2,$$
$$w(x,y) = \big[A\exp(-k_1x) + B\exp(k_1x)\big]\big[C\exp(-k_2y) + \exp(k_2y)\big], \qquad \lambda = (k_1^2 + k_2^2)^2,$$
$$w(r,\varphi) = \big[AJ_n(kr) + BY_n(kr) + CI_n(kr) + DK_n(kr)\big]\cos(n\varphi), \qquad \lambda = k^4 > 0,$$
$$w(r,\varphi) = \big[AJ_n(kr) + BY_n(kr) + CI_n(kr) + DK_n(kr)\big]\sin(n\varphi), \qquad \lambda = k^4 > 0,$$

where the $J_n(\xi)$ and $Y_n(\xi)$ are Bessel functions of the first and second kind, the $I_n(\xi)$ and $K_n(\xi)$ are modified Bessel functions of the first and second kind, $r = \sqrt{x^2 + y^2}$, and $n = 0,\ 1,\ 2,\ \dots$

3°. General solution with $\lambda > 0$:

$$w(x,y) = u_1(x,y) + u_2(x,y),$$

where u_1 and u_2 are arbitrary functions satisfying the Helmholtz equations

$$\Delta u_1 + \sqrt{\lambda}\, u_1 = 0, \qquad \Delta u_2 - \sqrt{\lambda}\, u_2 = 0.$$

For the solutions of these equations, see Section 9.3.

▶ **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Boundary value problem.**

A rectangle is considered. Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2.$$

Solution:

$$w(x,y) = \int_0^{l_1}\!\int_0^{l_2} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\eta\, d\xi,$$

where

$$G(x,y,\xi,\eta,t) = \frac{4}{l_1 l_2} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin(p_n x)\sin(q_m y)\sin(p_n \xi)\sin(q_m \eta)}{(p_n^2 + q_m^2)^2 - \lambda},$$
$$p_n = \frac{\pi n}{l_1}, \qquad q_m = \frac{\pi m}{l_2}.$$

► **Domain: $0 \le r \le R$, $0 \le \varphi \le 2\pi$. Eigenvalue problem with $\Phi \equiv 0$.**

The unknown and its normal derivative are zero on the boundary of a circular domain:

$$w = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Eigenvalues:

$$\lambda_{nm} = \frac{\beta_{nm}^4}{R^4}, \qquad n = 0,\, 1,\, 2,\, \dots,\; m = 1,\, 2,\, 3,\, \dots,$$

where the β_{nm} are positive roots of the transcendental equation

$$J_n(\beta) I_n'(\beta) - I_n(\beta) J_n'(\beta) = 0.$$

Numerical values of some roots:

$$\begin{aligned}
&\beta_{01} = 3.196, \quad \beta_{02} = 6.306, \quad \beta_{03} = 9.439, \quad \beta_{04} = 12.58;\\
&\beta_{11} = 4.611, \quad \beta_{12} = 7.799, \quad \beta_{13} = 10.96, \quad \beta_{14} = 14.11;\\
&\beta_{21} = 5.906, \quad \beta_{22} = 9.197, \quad \beta_{23} = 12.40, \quad \beta_{24} = 15.58,\\
&\beta_{31} = 7.144, \quad \beta_{32} = 10.54, \quad \beta_{33} = 13.79, \quad \beta_{34} = 17.01.
\end{aligned}$$

Eigenvalues:

$$\begin{aligned}
w_{nm}^{(\mathrm{c})}(r,\varphi) &= \Big[I_n(\beta_{nm}) J_n\Big(\beta_{nm}\frac{r}{R}\Big) - J_n(\beta_{nm}) I_n\Big(\beta_{nm}\frac{r}{R}\Big)\Big]\cos(n\varphi),\\
w_{nm}^{(\mathrm{s})}(r,\varphi) &= \Big[I_n(\beta_{nm}) J_n\Big(\beta_{nm}\frac{r}{R}\Big) - J_n(\beta_{nm}) I_n\Big(\beta_{nm}\frac{r}{R}\Big)\Big]\sin(n\varphi).
\end{aligned}$$

⊙ *Literature*: V. V. Bolotin (1978).

► **Domain: $(x/a)^2 + (y/b)^2 \le 1$. Eigenvalue problem with $\Phi \equiv 0$.**

The unknown and its normal derivative are zero on the boundary of an elliptic domain:

$$w = \frac{\partial w}{\partial N} = 0 \quad \text{on} \quad (x/a)^2 + (y/b)^2 = 1 \qquad (a \ge b).$$

Eigenvalues and eigenfunctions (approximate formulas):

$$\begin{aligned}
\lambda_0 &= \frac{\beta_0^4}{8}\Big(\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}\Big), \qquad w_0(\rho) = I_0(\beta_0) J_0(\beta_0\rho) - J_0(\beta_0) I_0(\beta_0\rho),\\
\lambda_{11}^{(\mathrm{c})} &= \frac{\beta_{11}^4}{8}\Big(\frac{5}{a^4} + \frac{1}{b^4} + \frac{2}{a^2b^2}\Big), \quad w_{11}^{(\mathrm{c})}(\rho,\varphi) = f(\rho)\cos\varphi,\\
\lambda_{11}^{(\mathrm{s})} &= \frac{\beta_{11}^4}{8}\Big(\frac{1}{a^4} + \frac{5}{b^4} + \frac{2}{a^2b^2}\Big), \quad w_{11}^{(\mathrm{s})}(\rho,\varphi) = f(\rho)\sin\varphi,
\end{aligned}$$

where

$$\rho = \sqrt{(x/a)^2 + (y/b)^2}, \qquad f(\rho) = \big[I_1(\beta_{11}) J_1(\beta_{11}\rho) - J_1(\beta_{11}) I_1(\beta_{11}\rho)\big],$$

$J_n(z)$ and $I_n(z)$ are the Bessel and the modified Bessel functions, and $\beta_0 = 3.196$ and $\beta_{11} = 4.611$ are the least roots of the transcendental equations

$$J_0(\beta)I_1(\beta) + J_1(\beta)I_0(\beta) = 0,$$
$$J_1(\beta)I_1'(\beta) - J_1'(\beta)I_1(\beta) = 0.$$

The above formulas were obtained with the aid of generalized (nonorthogonal) polar coordinates ρ, φ defined by

$$x = a\rho\cos\varphi, \quad y = b\rho\sin\varphi \qquad (0 \le \rho \le 1,\ 0 \le \varphi \le 2\pi)$$

and the variational method.

The maximum error in the eigenvalue λ_1 for $\varepsilon = \sqrt{1-(b/a)^2} \le 0.86$ is less than 1%. The errors in $\lambda_{11}^{(\mathrm{c})}$ and $\lambda_{11}^{(\mathrm{s})}$ for $\varepsilon \le 0.6$ do not exceed 2%. In the limit case $\varepsilon = 0$ that corresponds to a circular domain, the formulas provide exact results.

⊙ *Literature*: L. D. Akulenko, S. V. Nesterov, and A. L. Popov (2001).

11.5.4 Equation of the Form $\dfrac{\partial^4 w}{\partial x^4} + \dfrac{\partial^4 w}{\partial y^4} = \Phi(x, y)$

► Homogeneous equation. Particular solutions. General solution.

1°. Particular solutions with $\Phi \equiv 0$:

$$w(x,y) = \bigl[A\sin(\lambda x+\beta) + B\exp(-\lambda x) + C\exp(\lambda x)\bigr]\exp\bigl(\tfrac{1}{\sqrt{2}}\lambda y\bigr)\sin\bigl(\tfrac{1}{\sqrt{2}}\lambda y\bigr),$$
$$w(x,y) = \bigl[A\sin(\lambda x+\beta) + B\exp(-\lambda x) + C\exp(\lambda x)\bigr]\exp\bigl(\tfrac{1}{\sqrt{2}}\lambda y\bigr)\cos\bigl(\tfrac{1}{\sqrt{2}}\lambda y\bigr),$$
$$w(x,y) = \bigl[A\sin(\lambda x+\beta) + B\exp(-\lambda x) + C\exp(\lambda x)\bigr]\exp\bigl(-\tfrac{1}{\sqrt{2}}\lambda y\bigr)\sin\bigl(\tfrac{1}{\sqrt{2}}\lambda y\bigr),$$
$$w(x,y) = \bigl[A\sin(\lambda x+\beta) + B\exp(-\lambda x) + C\exp(\lambda x)\bigr]\exp\bigl(-\tfrac{1}{\sqrt{2}}\lambda y\bigr)\cos\bigl(\tfrac{1}{\sqrt{2}}\lambda y\bigr),$$

where A, B, C, β, and λ are arbitrary constants.

2°. General solution:

$$w(x,y) = \mathrm{Re}\bigl[f(z_1) + g(z_2)\bigr].$$

Here $f(z_1)$ and $g(z_2)$ are arbitrary analytic functions of the complex variables $z_1 = x - \frac{1}{\sqrt{2}}(1+i)y$ and $z_2 = x + \frac{1}{\sqrt{2}}(1+i)y$. The symbol $\mathrm{Re}[A]$ stands for the real part of a complex number A.

⊙ *Literature*: A. V. Bitsadze end D. F. Kalinichenko (1985).

► Homogeneous equation. Boundary value problem.

The upper half-space is considered ($-\infty < x < \infty$, $0 \le y < \infty$).

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi, y)\, d\xi,$$

where

$$G(x,y) = \frac{1}{\pi\sqrt{2}} \left[\arctan\left(1 - \frac{x\sqrt{2}}{y}\right) + \arctan\left(1 + \frac{x\sqrt{2}}{y}\right) \right].$$

⊙ *Literature*: G. E. Shilov (1965).

► Nonhomogeneous equation. Boundary value problems in a rectangle.

Consider problems in the rectangular domain $0 \le x \le l_1$, $0 \le y \le l_2$ with various homogeneous boundary conditions. The solution can be expressed in terms of the Green's function as

$$w(x,y) = \int_0^{l_1} \int_0^{l_2} \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\eta\, d\xi.$$

Given below are the Green's functions for two types of boundary conditions.

1°. The function and its first derivatives are prescribed on the sides of the rectangle:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_y w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta) = \frac{16}{l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{p_n^4 q_m^4 \varphi_n(x) \psi_m(y) \varphi_n(\xi) \psi_m(\eta)}{(p_n^4 + q_m^4) \big[\varphi_n''(l_1) \psi_m''(l_2)\big]^2}.$$

Here

$$\varphi_n(x) = \big[\sinh(p_n l_1) - \sin(p_n l_1)\big] \big[\cosh(p_n x) - \cos(p_n x)\big]$$
$$- \big[\cosh(p_n l_1) - \cos(p_n l_1)\big] \big[\sinh(p_n x) - \sin(p_n x)\big],$$
$$\psi_m(y) = \big[\sinh(q_m l_2) - \sin(q_m l_2)\big] \big[\cosh(q_m y) - \cos(q_m y)\big]$$
$$- \big[\cosh(q_m l_2) - \cos(q_m l_2)\big] \big[\sinh(q_m y) - \sin(q_m y)\big],$$

where p_n and q_m are positive roots of the transcendental equations

$$\cosh(p l_1) \cos(p l_1) = 1, \quad \cosh(q l_2) \cos(q l_2) = 1.$$

2°. The function and its second derivatives are prescribed on the sides of the rectangle:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta) = \frac{4}{l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{p_n^4 + q_m^4} \sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta),$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}.$$

11.5.5 Equation of the Form $\dfrac{\partial^4 w}{\partial x^4} + \dfrac{\partial^4 w}{\partial y^4} + kw = \Phi(x, y)$

► **Particular solutions of the homogeneous equation ($\Phi \equiv 0$):**

$$w(x,y) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\exp(\beta y)\sin(\beta y),$$
$$w(x,y) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\exp(\beta y)\cos(\beta y),$$
$$w(x,y) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\exp(-\beta y)\sin(\beta y),$$
$$w(x,y) = \big[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cosh(\lambda x)\big]\exp(-\beta y)\cos(\beta y),$$

where $\beta = \frac{1}{\sqrt{2}}(\lambda^4 + k)^{1/4}$; A, B, C, D, and λ are arbitrary constants.

► **Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. Boundary value problems.**

1°. Consider problems in the rectangular domain with various homogeneous boundary conditions. The solution can be expressed in terms of the Green's function as

$$w(x,y) = \int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta) G(x,y,\xi,\eta)\,d\eta\,d\xi.$$

Given below are the Green's functions for two types of boundary conditions.

2°. The function and its first derivatives are prescribed on the sides of the rectangle:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_y w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta) = \frac{16}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{p_n^4 q_m^4 \varphi_n(x)\psi_m(y)\varphi_n(\xi)\psi_m(\eta)}{(p_n^4 + q_m^4 + k)\big[\varphi_n''(l_1)\psi_m''(l_2)\big]^2}.$$

Here

$$\varphi_n(x) = \big[\sinh(p_n l_1) - \sin(p_n l_1)\big]\big[\cosh(p_n x) - \cos(p_n x)\big] - \big[\cosh(p_n l_1) - \cos(p_n l_1)\big]\big[\sinh(p_n x) - \sin(p_n x)\big],$$
$$\psi_m(y) = \big[\sinh(q_m l_2) - \sin(q_m l_2)\big]\big[\cosh(q_m y) - \cos(q_m y)\big] - \big[\cosh(q_m l_2) - \cos(q_m l_2)\big]\big[\sinh(q_m y) - \sin(q_m y)\big],$$

where p_n and q_m are positive roots of the transcendental equations

$$\cosh(pl_1)\cos(pl_1) = 1, \quad \cosh(ql_2)\cos(ql_2) = 1.$$

3°. The function and its second derivatives are prescribed on the sides of the rectangle:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2.$$

Green's function:

$$G(x,y,\xi,\eta) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{p_n^4 + q_m^4 + k}\sin(p_n x)\sin(q_m y)\sin(p_n \xi)\sin(q_m \eta),$$
$$p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}.$$

11.5.6 Stokes Equation (Axisymmetric Flows of Viscous Fluids)

► **Stokes equation for the stream function in the spherical coordinate system.**

1°. The Stokes equation for the stream function in the axisymmetric case is written as

$$E^2(E^2 w) = 0, \qquad E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

It describes slow axisymmetric flows of viscous incompressible fluids; w is the stream function, and (r,θ) are the spherical coordinates. The fluid velocity components are related to the stream function by $v_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial w}{\partial\theta}$ and $v_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial w}{\partial r}$.

2°. General solution (A_n, B_n, C_n, D_n, $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, and $\widetilde{D}_n$ are arbitrary constants):

$$\begin{aligned} w(r,\theta) &= \sum_{n=0}^{\infty}\bigl(A_n r^n + B_n r^{1-n} + C_n r^{n+2} + D_n r^{3-n}\bigr)\mathcal{J}_n(\cos\theta) \\ &\quad + \sum_{n=2}^{\infty}(\widetilde{A}_n r^n + \widetilde{B}_n r^{1-n} + \widetilde{C}_n r^{n+2} + \widetilde{D}_n r^{3-n}\bigr)\mathcal{H}_n(\cos\theta), \end{aligned} \qquad (1)$$

where $\mathcal{J}_n(\zeta)$ and $\mathcal{H}_n(\zeta)$ are the Gegenbauer functions of the first and second kind, respectively. These are linearly related to the Legendre functions $P_n(\zeta)$ and $Q_n(\zeta)$ by

$$\mathcal{J}_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1}, \quad \mathcal{H}_n(\zeta) = \frac{Q_{n-2}(\zeta) - Q_n(\zeta)}{2n-1} \quad (n \ge 2).$$

The Gegenbauer functions of the first kind are represented in the form of finite power series as

$$\begin{aligned} \mathcal{J}_n(\zeta) &= -\frac{1}{(n-1)!}\left(\frac{d}{d\zeta}\right)^{n-2}\left(\frac{\zeta^2-1}{2}\right)^{n-1} \\ &= \frac{1\cdot 3\dots(2n-3)}{1\cdot 2\dots n}\left[\zeta^n - \frac{n(n-1)}{2(2n-3)}\zeta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-3)(2n-5)}\zeta^{n-4} - \cdots\right]. \end{aligned}$$

In particular,

$$\mathcal{J}_0(\zeta) = 1, \quad \mathcal{J}_1(\zeta) = -\zeta, \quad \mathcal{J}_2(\zeta) = \tfrac{1}{2}(1-\zeta^2), \quad \mathcal{J}_3(\zeta) = \tfrac{1}{2}\zeta(1-\zeta^2),$$
$$\mathcal{J}_4(\zeta) = \tfrac{1}{8}(1-\zeta^2)(5\zeta^2-1), \quad \mathcal{J}_5(\zeta) = \tfrac{1}{8}\zeta(1-\zeta^2)(7\zeta^2-3).$$

The Gegenbauer functions of the second kind are defined as

$$\mathcal{H}_0(\zeta) = -\zeta, \quad \mathcal{H}_1(\zeta) = -1, \quad \mathcal{H}_n(\zeta) = \frac{1}{2}\mathcal{J}_n(\zeta)\ln\frac{1+\zeta}{1-\zeta} + \mathcal{K}_n(\zeta) \quad \text{at} \quad n \ge 2,$$

where the functions $\mathcal{K}_n(\zeta)$ are expressed in terms of the Gegenbauer functions of the first kind as

$$\mathcal{K}_n(\zeta) = -\sum_{k}^{\frac{1}{2}n \le k \le \frac{1}{2}n+\frac{1}{2}} \frac{(2n-4k+1)}{(2k-1)(n-k)}\left[1 - \frac{(2k-1)(n-k)}{n(n-1)}\right]\mathcal{J}_{n-2k+1}(\zeta);$$

the series start with $\mathcal{J}_0$ or $\mathcal{J}_1$ depending on whether n is odd or even. In particular,

$$\mathcal{K}_2(\zeta) = \tfrac{1}{2}\zeta, \quad \mathcal{K}_3(\zeta) = \tfrac{1}{6}(3\,\zeta^2 - 2), \quad \mathcal{K}_4(\zeta) = \tfrac{1}{24}\zeta(15\,\zeta^2 - 13),$$
$$\mathcal{K}_5(\zeta) = \tfrac{1}{120}(105\,\zeta^4 - 115\,\zeta^2 + 16).$$

For $n \geq 2$, the Gegenbauer functions of the second kind take infinite values at the points $\zeta = \pm 1$, which correspond to $\theta = 0$ and $\theta = \pi$. Therefore, if physically there are no singularities in the problem, then the constants in (1) labeled with a tilde must be zero. In an overwhelming majority of problems on the flow about particles, drops, or bubbles, the stream function in the spherical coordinates is given by formula (1) with

$$A_1 = A_0 = B_1 = B_0 = C_1 = C_0 = D_1 = D_0 = 0; \ \widetilde{A}_n = \widetilde{B}_n = \widetilde{C}_n = \widetilde{D}_n = 0 \ \text{ for } n = 2,\ 3,\ \ldots$$

Example 11.2. In the problem on the translational Stokes flow about a solid spherical particle, the following boundary conditions are imposed on the stream function w:

$$w = 0 \quad \text{at} \quad r = R, \qquad \partial_r w = 0 \quad \text{at} \quad r = R, \qquad w \to \tfrac{1}{2}Ur^2 \sin^2\theta \quad \text{as} \quad r \to \infty,$$

where R is the radius of the particle and U is the unperturbed fluid velocity at infinity.

Stokes solution:

$$w(r,\theta) = \frac{1}{4}U(r-R)^2\left(2 + \frac{R}{r}\right)\sin^2\theta.$$

Here the only remaining terms are those in the first sum in (1) with $n = 2$.

Example 11.3. In the problem on the axisymmetric straining Stokes flow about a solid spherical particle, the following boundary conditions are imposed on the stream function w:

$$w = 0 \quad \text{at} \quad r = R, \qquad \partial_r w = 0 \quad \text{at} \quad r = R, \qquad w \to \tfrac{1}{2}Er^3 \sin^2\theta\cos\theta \quad \text{as} \quad r \to \infty,$$

where R is the particle radius and E is the shear modulus.

Solution:

$$w(r,\theta) = \frac{1}{2}ER^3\left(\frac{r^3}{R^3} - \frac{5}{2} + \frac{3}{2}\frac{R^2}{r^2}\right)\sin^2\theta\cos\theta.$$

Here the only remaining terms are those in the first sum in (1) with $n = 3$.

Example 11.4. Solving the problem of the translational Stokes flow about a spherical drop (or bubble) is reduced to solving the Stokes equation outside and inside the drop. The boundary condition at infinity is given in Example 11.2. Transmission boundary conditions are set on the drop surface; these conditions can be found in the references cited below and are not written out here.

Hadamard–Rybczynski solution:

$$w(r,\theta) = \frac{1}{4}Ur^2\left(2 - \frac{3\beta+2}{\beta+1}\frac{R}{r} + \frac{\beta}{\beta+1}\frac{R^3}{r^3}\right)\sin^2\theta \quad \text{for} \quad r > R,$$
$$w(r,\theta) = \frac{U}{4(\beta+1)}r^2\left(\frac{r^2}{R^2} - 1\right)\sin^2\theta \qquad \text{for} \quad r < R,$$

where R is the drop radius, U is the unperturbed fluid velocity at infinity, and β is the ratio of dynamic viscosities of the fluids inside and outside the drop. (The value $\beta = 0$ corresponds to a gas bubble; $\beta = \infty$, to a solid particle.)

Example 11.5. Solving the problem of the axisymmetric straining Stokes flow about a spherical drop (or bubble) is reduced to solving the Stokes equation outside and inside the drop. The boundary condition at infinity is specified in Example 11.3. Transmission boundary conditions are set at the drop surface; these conditions can be found in the references cited below and are not written out here.

Taylor solution:

$$w(r,\theta)=\frac{1}{2}ER^3\left(\frac{r^3}{R^3}-\frac{1}{2}\frac{5\beta+2}{\beta+1}+\frac{3}{2}\frac{\beta}{\beta+1}\frac{R^2}{r^2}\right)\sin^2\theta\cos\theta \quad \text{for} \quad r>R,$$
$$w(r,\theta)=\frac{3}{4}\frac{ER^3}{\beta+1}\frac{r^3}{R^3}\left(\frac{r^2}{R^2}-1\right)\sin^2\theta\cos\theta \quad \text{for} \quad r<R,$$

where R is the drop radius, E is the shear modulus, and β is the ratio of dynamic viscosities of the fluids inside and outside the drop. (The value $\beta=0$ corresponds to a gas bubble; $\beta=\infty$, to a solid particle.)

⊙ *Literature*: G. I. Taylor (1932), V. G. Levich (1962), J. Happel and H. Brenner (1965), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

► Stokes equation in the bipolar coordinate system.

1°. When studying axisymmetric problems of a flow about two spherical particles (drops, bubbles), one uses the bipolar coordinates ξ, η; these are related to the cylindrical coordinates $\rho=r\cos\theta$, $z=r\sin\theta$ by

$$\rho=\frac{a\sin\xi}{\cosh\eta-\cos\xi},\qquad z=\frac{a\sinh\eta}{\cosh\eta-\cos\xi}.$$

2°. The general solution of the equation $E^2(E^2w)=0$ in the bipolar coordinate system has the form

$$w(\xi,\eta)=\frac{1}{(\cosh\eta-\cos\xi)^{3/2}}\left[\sum_{n=0}^{\infty}\mathcal{J}_{n+1}(\cos\xi)f_n(\eta)+\sum_{n=0}^{\infty}\mathcal{H}_{n+1}(\cos\xi)g_n(\eta)\right],$$
$$f_n(\eta)=A_n\cosh\left[(n-\tfrac{1}{2})\eta\right]+B_n\sinh\left[(n-\tfrac{1}{2})\eta\right]+C_n\cosh\left[(n+\tfrac{3}{2})\eta\right]+D_n\sinh\left[(n+\tfrac{3}{2})\eta\right],$$
$$g_n(\eta)=\widetilde{A}_n\cosh\left[(n-\tfrac{1}{2})\eta\right]+\widetilde{B}_n\sinh\left[(n-\tfrac{1}{2})\eta\right]+\widetilde{C}_n\cosh\left[(n+\tfrac{3}{2})\eta\right]+\widetilde{D}_n\sinh\left[(n+\tfrac{3}{2})\eta\right],$$

where A_n, B_n, C_n, D_n, $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, and $\widetilde{D}_n$ are arbitrary constants and $\mathcal{J}_n(\zeta)$ and $\mathcal{H}_n(\zeta)$ are the Gegenbauer functions.

⊙ *Literature*: J. Happel and H. Brenner (1965).

► Stokes equation in the oblate spheroidal coordinate system.

1°. When studying axisymmetric problems of flows about spheroidal particles, one uses the oblate spheroidal coordinates ξ, η; these are related to the cylindrical coordinates $\rho=r\cos\theta$, $z=r\sin\theta$ by

$$\rho=c\cosh\xi\sin\eta,\qquad z=c\sinh\xi\cos\eta.$$

2°. The solution of the equation $E^2(E^2w)=0$ that describes the flow of a fluid about a oblate spheroid in the direction parallel to the spheroid axis is expressed as

$$w=\frac{1}{2}Uc^2\cosh^2\xi\sin^2\eta\left\{1-\frac{[\lambda/(\lambda^2+1)]-[(\lambda_0^2-1)/(\lambda_0^2+1)]\operatorname{arccot}\lambda}{[\lambda_0/(\lambda_0^2+1)]-[(\lambda_0^2-1)/(\lambda_0^2+1)]\operatorname{arccot}\lambda_0}\right\},$$
$$\lambda=\sinh\xi,\qquad \lambda_0=\sinh\xi_0.$$

Here w is the stream function, U is the fluid velocity at infinity, and c and λ_0 are the constants related to the spheroid semiaxes a and b ($a>b$) by $c=\sqrt{a^2-b^2}$ and $\lambda_0=b/c$.

⊙ *Literature*: J. Happel and H. Brenner (1965).

11.6 Higher-Order Linear Equations with Constant Coefficients

◆ *Throughout Section 11.6, the following notation is used*:

$$\mathbf{x}=\{x_1,\dots,x_n\},\quad \mathbf{y}=\{y_1,\dots,y_n\},\quad \boldsymbol{\omega}=\{\omega_1,\dots,\omega_n\},\quad \boldsymbol{\xi}=\{\xi_1,\dots,\xi_n\},$$
$$|\mathbf{x}|=\sqrt{x_1^2+\cdots+x_n^2},\quad |\boldsymbol{\omega}|=\sqrt{\omega_1^2+\cdots+\omega_n^2},\quad \boldsymbol{\omega}\cdot\mathbf{x}=\omega_1x_1+\cdots+\omega_nx_n.$$

11.6.1 Fundamental Solutions. Cauchy Problem

► **Domain: $\mathbb{R}^n=\{-\infty<x_k<\infty;\ k=1,\dots,n\}$. Fundamental solution.**

Let P be a constant coefficient linear differential operator such that

$$P\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)\equiv\sum_{s=0}^{M}a_{s_1,\dots,s_n}\frac{\partial^s}{\partial x_1^{s_1}\dots\partial x_n^{s_n}},\qquad s=s_1+\cdots+s_n,$$

where $s_1,\dots,s_n$ are nonnegative integers, $a_{s_1,\dots,s_n}$ are some constants, and M is the order of the operator. A generalized function (distribution) $\mathscr{E}_{\rm e}(\mathbf{x})=\mathscr{E}_{\rm e}(x_1,\dots,x_n)$ that satisfies the equation

$$P\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)\mathscr{E}_{\rm e}(\mathbf{x})=\delta(\mathbf{x}),$$

where $\delta(\mathbf{x})=\delta(x_1)\dots\delta(x_n)$ is the Dirac delta function in the n-dimensional Euclidian space, is called a *fundamental solution* corresponding to the operator P.

Any constant coefficient linear differential operator has a fundamental solution $\mathscr{E}_{\rm e}(\mathbf{x})$. The fundamental solution is not unique; it is defined up to an additive term $w_0(\mathbf{x})$ that is an arbitrary solution of the homogeneous equation $P\bigl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\bigr)w_0(\mathbf{x})=0$.

A solution of the nonhomogeneous equation

$$P\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)w=\Phi(\mathbf{x})$$

with an arbitrary right-hand side has the form

$$w(\mathbf{x})=\mathscr{E}_{\rm e}(\mathbf{x})*\Phi(\mathbf{x}),\qquad \mathscr{E}_{\rm e}(\mathbf{x})*\Phi(\mathbf{x})=\int_{\mathbb{R}^n}\mathscr{E}_{\rm e}(\mathbf{x}-y)\Phi(\mathbf{y})\,dV_y.$$

Here $dV_y = dy_1 \ldots dy_n$ and the convolution $\mathscr{E}_\mathrm{e} * \Phi$ is assumed to make sense.

⊙ *Literature*: G. E. Shilov (1965), S. G. Krein (1972), L. Hörmander (1983), V. S. Vladimirov (1988).

▶ **Domain:** $0 \le t < \infty,\ -\infty < x_k < \infty;\ k = 1, \ldots, n$. **Cauchy problem.**

Now let $P\bigl(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\bigr)$ be a constant coefficient linear differential operator of order m with respect to t. Then a distribution $\mathscr{E}(t, \mathbf{x}) = \mathscr{E}(t, x_1, \ldots, x_n)$ that is a solution of the homogeneous equation

$$P\biggl(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\biggr)\mathscr{E}(t, \mathbf{x}) = 0 \tag{1}$$

and satisfies the initial conditions*

$$\mathscr{E}\big|_{t=0} = 0, \quad \frac{\partial \mathscr{E}}{\partial t}\bigg|_{t=0} = 0, \quad \ldots, \quad \frac{\partial^{m-2} \mathscr{E}}{\partial t^{m-2}}\bigg|_{t=0} = 0, \quad \frac{\partial^{m-1} \mathscr{E}}{\partial t^{m-1}}\bigg|_{t=0} = \delta(\mathbf{x}) \tag{2}$$

is called the *fundamental solution of the Cauchy problem* corresponding to the operator P.

The solution of the Cauchy problem for the linear differential equation

$$P\biggl(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\biggr) w = 0 \tag{3}$$

with the special initial conditions

$$w\big|_{t=0} = 0, \quad \frac{\partial w}{\partial t}\bigg|_{t=0} = 0, \quad \ldots, \quad \frac{\partial^{m-2} w}{\partial t^{m-2}}\bigg|_{t=0} = 0, \quad \frac{\partial^{m-1} w}{\partial t^{m-1}}\bigg|_{t=0} = f(\mathbf{x}) \tag{4}$$

is given by

$$w(t, \mathbf{x}) = \mathscr{E}(t, \mathbf{x}) * f(\mathbf{x}), \qquad \mathscr{E}(t, \mathbf{x}) * f(\mathbf{x}) \equiv \int_{\mathbb{R}^n} \mathscr{E}(t, \mathbf{x} - y) f(\mathbf{y})\, dV_y.$$

⊙ *Literature*: S. G. Krein (1972).

▶ **Solution of the Cauchy problem for general initial conditions.**

If the general initial conditions

$$w\big|_{t=0} = f_0(\mathbf{x}), \quad \frac{\partial w}{\partial t}\bigg|_{t=0} = f_1(\mathbf{x}), \quad \ldots, \quad \frac{\partial^{m-1} w}{\partial t^{m-1}}\bigg|_{t=0} = f_{m-1}(\mathbf{x}) \tag{5}$$

are set, then the solution of Eq. (3) is sought in the form

$$w(t, \mathbf{x}) = \mathscr{E}(t, \mathbf{x}) * \varphi_0(\mathbf{x}) + \frac{\partial \mathscr{E}(t, \mathbf{x})}{\partial t} * \varphi_1(\mathbf{x}) + \cdots + \frac{\partial^{m-1} \mathscr{E}(t, \mathbf{x})}{\partial t^{m-1}} * \varphi_{m-1}(\mathbf{x}). \tag{6}$$

*The number of initial conditions can be less than m (see Section 11.6.4).

Each term in (6) satisfies Eq. (3), and the functions φ_{m-1}, φ_{m-2}, ..., φ_0 are determined successively from the linear system

$$
\begin{aligned}
&f_0(\mathbf{x}) = \varphi_{m-1}(\mathbf{x}),\\
&f_1(\mathbf{x}) = \varphi_{m-2}(\mathbf{x}) + \frac{\partial^m \mathscr{E}(0,\mathbf{x})}{\partial t^m} * \varphi_{m-1}(\mathbf{x}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&f_k(\mathbf{x}) = \varphi_{m-k-1}(\mathbf{x}) + \frac{\partial^m \mathscr{E}(0,\mathbf{x})}{\partial t^m} * \varphi_{m-k}(\mathbf{x}) + \dots + \frac{\partial^{m+k-1} \mathscr{E}(0,\mathbf{x})}{\partial t^{m+k-1}} * \varphi_{m-1}(\mathbf{x}),\\
&\qquad k = 2, \dots, m-1.
\end{aligned}
$$

This system of equations is obtained by successively differentiating relation (6) followed by substituting $t = 0$ and taking into account the initial conditions (2) and (5).

⊙ *Literature*: G. E. Shilov (1965).

► **Nonhomogeneous constant coefficient linear equations of special form.**

Consider a nonhomogeneous constant coefficient linear equation of the special form

$$
\frac{\partial^m w}{\partial t^m} + \sum_{k=0}^{m-1} P_k\biggl(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\biggr)\frac{\partial^k w}{\partial t^k} = \Phi(t,\mathbf{x}). \qquad (7)
$$

Let $\mathscr{E}(t,\mathbf{x}) = \mathscr{E}(t, x_1, \dots, x_n)$ be the fundamental solution of the Cauchy problem for the homogeneous equation (7) ($\Phi \equiv 0$) with the initial conditions (2). Then the function

$$
\mathscr{E}_{\rm e}(t,\mathbf{x}) = \vartheta(t)\mathscr{E}(t,\mathbf{x}),
$$

where $\vartheta(t)$ is the Heaviside unit step function ($\vartheta = 0$ for $t \le 0$ and $\vartheta = 1$ for $t > 0$), is a fundamental solution of Eq. (7) with $\Phi(\mathbf{x}, t) = \delta(t)\delta(\mathbf{x})$.

The solution of the Cauchy problem for the linear differential equation (7) with the special initial conditions (4) is given by

$$
w(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^n} \mathscr{E}(t-\tau, \mathbf{x}-y)\Phi(\tau,\mathbf{y})\,dV_y\,d\tau + \int_{\mathbb{R}^n} \mathscr{E}(t,\mathbf{x}-y) f(\mathbf{y})\,dV_y.
$$

11.6.2 Elliptic Operators and Elliptic Equations

► **Homogeneous elliptic differential operator.**

A constant coefficient linear homogeneous differential operator of order k has the form

$$
P_k\biggl(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\biggr) \equiv \sum a_{s_1,\dots,s_n}\biggl(\frac{\partial}{\partial x_1}\biggr)^{s_1} \dots \biggl(\frac{\partial}{\partial x_n}\biggr)^{s_n}, \qquad \sum_{i=1}^n s_i = k,
$$

where $s_1, \dots, s_n$ are nonnegative integers. From now on, we adopt the notation

$$
\biggl(\frac{\partial}{\partial x_1}\biggr)^{s_1} \dots \biggl(\frac{\partial}{\partial x_n}\biggr)^{s_n} \equiv \frac{\partial^{s_1+\dots+s_n}}{\partial x_1^{s_1} \dots \partial x_n^{s_n}}.
$$

A linear homogeneous differential operator of order k possesses the property

$$P_k\biggl(b\frac{\partial}{\partial x_1},\dots,b\frac{\partial}{\partial x_n}\biggr)=b^kP_k\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr),\qquad b\neq 0 \text{ is an arbitrary constant.}$$

A linear homogeneous differential operator P_k is said to be elliptic if the replacement of the symbols $\frac{\partial}{\partial x_1}$, …, $\frac{\partial}{\partial x_n}$ in P_k with the variables ω_1, …, ω_n gives a polynomial $P_k(\omega_1,\dots,\omega_n)$ that does not vanish for real $\boldsymbol{\omega}\neq\mathbf{0}$, i.e., if

$$P_k(\omega_1,\dots,\omega_n)\equiv\sum a_{s_1,\dots,s_n}\omega_1^{s_1}\dots\omega_n^{s_n}\neq 0\qquad\text{if}\quad \boldsymbol{\omega}\in\mathbb{R}^n,\quad |\boldsymbol{\omega}|\neq\mathbf{0}.$$

A linear differential equation

$$P_k\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)w\equiv\sum a_{s_1,\dots,s_n}\frac{\partial^{s_1+\dots+s_n}w}{\partial x_1^{s_1}\dots\partial x_n^{s_n}}=0,\qquad \sum_{i=1}^n s_i=k,\qquad (1)$$

is said to be elliptic if the linear homogeneous differential operator P_k is elliptic.

► **Elliptic differential operator of the general form.**

In general, a constant coefficient linear differential operator of order k has the form

$$\mathscr{P}_k\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)=P_k\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)+\sum_{i=0}^{k-1}P_i\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr),$$

where P_k is the principal part of the operator and P_j ($j=0,\ 1\ \dots,\ k$) is a linear homogeneous differential operator of order j. The operator $\mathscr{P}_k$ is said to be elliptic if its principal part P_k is elliptic.

A linear differential equation

$$\mathscr{P}_k\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)w=0\qquad (2)$$

is said to be elliptic if the linear differential operator $\mathscr{P}_k$ is elliptic.

Remark 11.3. A linear elliptic operator and a linear elliptic differential equation can only be of even order $k=2m$, where m is a positive integer.

▶ **Fundamental solution of a homogeneous elliptic equation.**

The fundamental solution of the homogeneous elliptic equation (1) with $k = 2m$ is given by

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \frac{(-1)^{\frac{n-1}{2}}}{4(2\pi)^{n-1}(2m-n)!}\int_{\Omega_n} |\boldsymbol{\omega}\cdot\mathbf{x}|^{2m-n}\frac{d\Omega_n}{P_{2m}(\boldsymbol{\omega})} \qquad \text{if } n \text{ is odd and } 2m \ge n,$$

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \frac{(-1)^{\frac{n-2}{2}}}{(2\pi)^{n}(2m-n)!}\int_{\Omega_n} |\boldsymbol{\omega}\cdot\mathbf{x}|^{2m-n}\ln|\boldsymbol{\omega}\cdot\mathbf{x}|\frac{d\Omega_n}{P_{2m}(\boldsymbol{\omega})} \qquad \text{if } n \text{ is even and } 2m \ge n,$$

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{\frac{n-1}{2}}}\int_{\Omega_n} \delta^{(n-2m-1)}(\boldsymbol{\omega}\cdot\mathbf{x})\frac{d\Omega_n}{P_{2m}(\boldsymbol{\omega})} \qquad \text{if } n \text{ is odd and } 2m < n,$$

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \frac{(-1)^{\frac{n}{2}}(n-2m-1)!}{(2\pi)^{n}}\int_{\Omega_n} |\boldsymbol{\omega}\cdot\mathbf{x}|^{2m-n}\frac{d\Omega_n}{P_{2m}(\boldsymbol{\omega})} \qquad \text{if } n \text{ is even and } 2m < n.$$

Here the integration is over the surface of the n-dimensional sphere Ω_n of unit radius defined by the equation $|\boldsymbol{\omega}| = 1$, $\boldsymbol{\omega}\cdot\mathbf{x} = \omega_1 x_1 + \cdots + \omega_n x_n$, and $P_{2m}(\boldsymbol{\omega}) = P_{2m}(\omega_1, \ldots, \omega_n)$.

The fundamental solution is an ordinary function analytic at any point $\mathbf{x} \neq 0$; this function is described in a neighborhood of the origin (as $|\mathbf{x}| \to 0$) by the relations

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \begin{cases} b_{n,m}|\mathbf{x}|^{2m-n} & \text{if } n \text{ is odd or } n \text{ is even and } n > 2m, \\ c_{n,m}|\mathbf{x}|^{2m-n}\ln|\mathbf{x}| & \text{if } n \text{ is even and } n \le 2m. \end{cases}$$

Here $b_{n,m}$ and $c_{n,m}$ are some nonzero constants. If $2m > n$, then the fundamental solution has continuous derivatives of order $\le 2m-n-1$ at the origin.

▶ **Fundamental solution of the general elliptic equation.**

The fundamental solution of the general elliptic equation (2) with $k = 2m$ is determined by the relation

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \int_{\Omega_n} Z_\omega(\boldsymbol{\omega}\cdot\mathbf{x},\, -n)\, d\Omega_n, \tag{3}$$

where

$$Z_\omega(\xi,\lambda) = \frac{1}{\sigma_n \pi^{\frac{n-1}{2}}\Gamma\big(\frac{\lambda+1}{2}\big)}\int_{-\infty}^{\infty} G(\xi-\eta,\,\boldsymbol{\omega})|\eta|^\lambda\, d\eta, \qquad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Here the function $G(\xi,\,\boldsymbol{\omega})$ is the fundamental solution of the constant coefficient linear ordinary differential equation

$$\mathscr{P}_{2m}\left(\omega_1\frac{d}{d\xi},\, \ldots,\, \omega_n\frac{d}{d\xi}\right)G(\xi,\,\boldsymbol{\omega}) = \delta(\xi).$$

If n is odd, then the fundamental solution (3) can be represented as

$$\mathscr{E}_{\rm e}(\mathbf{x}) = A_n\int_{\Omega_n}\left[\frac{\partial^{n-1}}{\partial\xi^{n-1}}G(\xi,\,\boldsymbol{\omega})\right]d\Omega_n, \qquad A_n = \frac{(-1)^{\frac{n-1}{2}}}{1\times 3\times\ldots\times(n-2)\,\sigma_n(2\pi)^{\frac{n-1}{2}}}.$$

⊙ *Literature*: I. M. Gelfand and G. E. Shilov (1959), S. G. Krein (1972).

11.6.3 Hyperbolic Operators and Hyperbolic Equations

Let $P\big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$ be a constant coefficient linear homogeneous differential operator of order m with respect to t. The operator P is said to be hyperbolic if, for any real numbers $\omega_1, \dots, \omega_n$ such that $\sum\limits_{s=1}^{n} \omega_s^2 = 1$, the mth-order algebraic equation

$$P(\lambda, \omega_1, \dots, \omega_n) = 0$$

for λ has m distinct real roots.

Fundamental solution of the Cauchy problem for $m \ge n-1$:

$$\mathscr{E}(t,\mathbf{x}) = \frac{(-1)^{\frac{n+1}{2}}}{2(2\pi)^{n-1}(m-n-1)!} \int\limits_{H=0} (\boldsymbol{\xi}\cdot\mathbf{x}+t)^{m-n-1} \frac{[\operatorname{sign}(\boldsymbol{\xi}\cdot\mathbf{x}+t)]^{m-1}}{|\nabla H| \operatorname{sign}(\boldsymbol{\xi}\cdot\nabla H)}\, d\sigma_H$$

if n is odd,

$$\mathscr{E}(t,\mathbf{x}) = \frac{2(-1)^{\frac{n}{2}}}{(2\pi)^{n}(m-n-1)!} \int\limits_{H=0} \frac{(\boldsymbol{\xi}\cdot\mathbf{x}+t)^{m-n-1}}{|\nabla H| \operatorname{sign}(\boldsymbol{\xi}\cdot\nabla H)} \ln\left|\frac{\boldsymbol{\xi}\cdot\mathbf{x}+t}{\boldsymbol{\xi}\cdot\mathbf{x}}\right| d\sigma_H$$

if n is even.

Here and in the following, we use the notation

$$H = P(1, \xi_1, \dots, \xi_n),$$

$$|\nabla H| = \left[\left(\frac{\partial H}{\partial \xi_1}\right)^2 + \dots + \left(\frac{\partial H}{\partial \xi_n}\right)^2\right]^{1/2}, \qquad \boldsymbol{\xi}\cdot\nabla H = \xi_1\frac{\partial H}{\partial \xi_1} + \dots + \xi_n\frac{\partial H}{\partial \xi_n},$$

and $d\sigma_H$ is the element of the surface $H=0$.

Fundamental solution of the Cauchy problem for $m < n-1$:

$$\mathscr{E}(t,\mathbf{x}) = \frac{(-1)^{\frac{n+1}{2}}}{(2\pi)^{n-1}} \int\limits_{H=0} \frac{\delta^{(n-m)}(\boldsymbol{\xi}\cdot\mathbf{x}+t)}{|\nabla H| \operatorname{sign}(\boldsymbol{\xi}\cdot\nabla H)}\, d\sigma_H \qquad \text{if } n \text{ is odd},$$

$$\mathscr{E}(t,\mathbf{x}) = \frac{(-1)^{\frac{n}{2}}(n-m)!}{(2\pi)^{n}} \int\limits_{H=0} \frac{(\boldsymbol{\xi}\cdot\mathbf{x}+t)^{m-n-1}}{|\nabla H| \operatorname{sign}(\boldsymbol{\xi}\cdot\nabla H)}\, d\sigma_H \qquad \text{if } n \text{ is even}.$$

⊙ *Literature*: I. M. Gelfand and G. E. Shilov (1959), S. G. Krein (1972).

11.6.4 Regular Equations. Number of Initial Conditions in the Cauchy Problem

► **Equations with two independent variables ($0 \le t < \infty$, $-\infty < x < \infty$).**

1°. Consider the constant coefficient linear differential equation

$$\frac{\partial^m w}{\partial t^m} = \sum_{k=0}^{m-1} p_k\left(i\frac{\partial}{\partial x}\right)\frac{\partial^k w}{\partial t^k}, \tag{1}$$

where $p_k(z)$ is a polynomial and $i^2 = -1$. Let $r = r(\sigma)$ be the number of roots (taking into account their multiplicities) of the characteristic equation

$$\lambda^m - \sum_{k=0}^{m-1} p_k(\sigma)\lambda^k = 0 \tag{2}$$

whose real parts are nonpositive (or bounded above) for a given σ. If r is the same (up to a set of measure zero) for all $\sigma \in (-\infty, \infty)$, then Eq. (1) is said to be regular with regularity index r.

Classical equations such as the heat, wave, and Laplace equations are regular.

2°. In the Cauchy problem for a regular equation (1), one should set r initial conditions of the form

$$w\big|_{t=0} = f_0(x), \quad \frac{\partial w}{\partial t}\bigg|_{t=0} = f_1(x), \quad \dots, \quad \frac{\partial^{r-1} w}{\partial t^{r-1}}\bigg|_{t=0} = f_{r-1}(x). \tag{3}$$

We point out that the regularity index r can generally differ from the order m of the equation with respect to t. In particular, for the two-dimensional Laplace equation $\partial_{tt} w = -\partial_{xx} w$, we have $r = 1$ and $m = 2$; here y is replaced by t and the first boundary value problem in the upper half-plane $t \geq 0$ is considered. For the heat equation $\partial_t w = \partial_{xx} w$ and wave equation $\partial_{tt} w = \partial_{xx} w$, we have $r = m = 1$ and $r = m = 2$, respectively.

Example 11.6. The regularity indices for some fourth-order equations are as follows:

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^4 w}{\partial x^4} = 0 \quad (r = 1,\ m = 2); \qquad \frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0 \quad (r = 2,\ m = 2);$$

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)^2 w = 0 \quad (r = 2,\ m = 4); \qquad \frac{\partial^4 w}{\partial t^4} - a^2 \frac{\partial^4 w}{\partial x^4} = 0 \quad (r = 3,\ m = 4).$$

3°. The special solution $\mathscr{E} = \mathscr{E}(t, x)$ that satisfies the initial conditions

$$\mathscr{E}\big|_{t=0} = 0, \quad \frac{\partial \mathscr{E}}{\partial t}\bigg|_{t=0} = 0, \quad \dots, \quad \frac{\partial^{r-2} \mathscr{E}}{\partial t^{r-2}}\bigg|_{t=0} = 0, \quad \frac{\partial^{r-1} \mathscr{E}}{\partial t^{r-1}}\bigg|_{t=0} = \delta(x) \tag{4}$$

is called the fundamental solution.

The fundamental solution can be found by applying the Fourier transform in the space variable to Eq. (1) (with $w = \mathscr{E}$) and the initial conditions (4).

Example 11.7. Consider the *polyharmonic equation*

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)^n w = 0. \tag{5}$$

Taking into account the representation $\frac{\partial^2}{\partial x^2} = -\left(i\frac{\partial}{\partial x}\right)^2$, we rewrite the characteristic equation (2) in the form

$$(\lambda^2 - \sigma^2)^n = 0.$$

It has only one solution, $\lambda = -|\sigma|$, whose real part is nonpositive. Considering the multiplicity of the root, we find that the regularity index r is equal to n.

In Eq. (5) with $w = \mathscr{E}$ and the initial conditions (4) with $r = n$, we make the Fourier transform with respect to the space variable,

$$U(t,\sigma) = \int_{-\infty}^{\infty} e^{i\sigma x} \mathscr{E}(t,x)\,dx.$$

As a result, we arrive at the ordinary differential equation

$$\left(\frac{d^2}{dt^2} - \sigma^2\right)^n U = 0 \tag{6}$$

with the initial conditions

$$U\big|_{t=0} = 0, \quad U'_t\big|_{t=0} = 0, \quad \dots, \quad U_t^{(n-2)}\big|_{t=0} = 0, \quad U_t^{(n-1)}\big|_{t=0} = 1. \tag{7}$$

The bounded solution of problem (6), (7) is given by

$$U(t,\sigma) = \frac{t^{n-1}}{(n-1)!} e^{-|\sigma| t}.$$

By applying the inverse Fourier transform, we obtain the fundamental solution of the polyharmonic equation in the form

$$\begin{aligned}
\mathscr{E}(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\sigma x} U(t,\sigma)\,d\sigma = \frac{1}{2\pi}\frac{t^{n-1}}{(n-1)!}\int_{-\infty}^{\infty} e^{-i\sigma x - |\sigma| t}\,d\sigma \\
&= \frac{1}{2\pi}\frac{t^{n-1}}{(n-1)!}\left(\int_0^{\infty} e^{-i\sigma x - \sigma t}\,d\sigma + \int_{-\infty}^{0} e^{-i\sigma x + \sigma t}\,d\sigma\right) \\
&= \frac{1}{2\pi}\frac{t^{n-1}}{(n-1)!}\left(\frac{1}{t+ix} + \frac{1}{t-ix}\right) = \frac{1}{\pi}\frac{t^{n-1}}{(n-1)!}\frac{t}{t^2+x^2}.
\end{aligned}$$

4°. For general initial conditions of the form (3), the solution of Eq. (1) is determined on the basis of the fundamental solution from the relation

$$w(t,x) = \mathscr{E}(t,x) * \varphi_0(x) + \frac{\partial \mathscr{E}(t,x)}{\partial t} * \varphi_1(x) + \dots + \frac{\partial^{r-1}\mathscr{E}(t,x)}{\partial t^{r-1}} * \varphi_{r-1}(x). \tag{8}$$

Each term in (8) satisfies Eq. (1), and the functions φ_{r-1}, φ_{r-2}, $\dots$, φ_0 are calculated successively by solving the linear system

$$\begin{aligned}
&f_0(x) = \varphi_{r-1}(x), \\
&f_1(x) = \varphi_{r-2}(x) + \frac{\partial^r \mathscr{E}(0,x)}{\partial t^r} * \varphi_{r-1}(x), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&f_k(x) = \varphi_{r-k-1}(x) + \frac{\partial^r \mathscr{E}(0,x)}{\partial t^r} * \varphi_{r-k}(x) + \dots + \frac{\partial^{r+k-1}\mathscr{E}(0,x)}{\partial t^{r+k-1}} * \varphi_{r-1}(x), \\
&\qquad k = 2, \dots, r-1.
\end{aligned}$$

This system of equations is obtained by successively differentiating relation (8) followed by substituting $t = 0$ and by taking into account the initial conditions (3) and (4).

In the special case of $f_0(x) = f_1(x) = \dots = f_{r-2}(x) = 0$, one should set $\varphi_0(x) = f_{r-1}(x)$ and $\varphi_1(x) = \dots = \varphi_{r-1}(x) = 0$ in (8).

⊙ *Literature*: G. E. Shilov (1965).

► Equations with many independent variables ($0 \le t < \infty$, $\mathbf{x} \in \mathbb{R}^n$).

Solving the Cauchy problem for the constant coefficient linear differential equation

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) w = 0 \tag{9}$$

with arbitrarily many space variables $x_1, \ldots, x_n$ can be reduced to solving the Cauchy problem for an equation with one space variable ξ. Take the auxiliary linear differential operator

$$P_\omega\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi}\right) \equiv P\left(\frac{\partial}{\partial t}, \omega_1 \frac{\partial}{\partial \xi}, \ldots, \omega_n \frac{\partial}{\partial \xi}\right)$$

that depends on two independent variables t and ξ so that the Cauchy problem for the equation

$$P_\omega\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi}\right) v = 0 \tag{10}$$

is well posed. Then the fundamental solution of the Cauchy problem for the original equation (9) is given by

$$\mathscr{E}(t, \mathbf{x}) = \int_{\Omega_n} v_\omega(t,\, \boldsymbol{\omega} \cdot \mathbf{x},\, -n)\, d\Omega_n.$$

Here

$$v_\omega(t, \xi, \lambda) = \frac{1}{\sigma_n \pi^{\frac{n-1}{2}} \Gamma\bigl(\frac{\lambda+1}{2}\bigr)} \int_{-\infty}^{\infty} G_\omega(t, \xi - \eta) |\eta|^\lambda \, d\eta, \qquad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where $G_\omega(t, \xi)$ is the fundamental solution of the Cauchy problem for the auxiliary equation (10).

If the number of space variables is odd, one can use the simpler formula

$$\mathscr{E}(t, \mathbf{x}) = \frac{(-1)^{\frac{n-1}{2}} \bigl(\frac{n-1}{2}\bigr)!}{\sigma_n \pi^{\frac{n-1}{2}} (n-1)!} \int_{\Omega_n} \left[\frac{d^{n-1}}{d\xi^{n-1}} G_\omega(t, \xi)\right] d\Omega_n, \qquad \xi = \boldsymbol{\omega} \cdot \mathbf{x}.$$

Remark 11.4. The above relations hold for all equations for which the Cauchy problem is well posed.

⊙ *Literature*: I. M. Gelfand and G. E. Shilov (1959), S. G. Krein (1972).

► Stationary homogeneous regular equations ($\mathbf{x} \in \mathbb{R}^n$).

A linear differential operator $P_k\bigl(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\bigr)$ is said to be regular if it is homogeneous and if the gradient of the function $P_k(\omega_1, \ldots, \omega_n)$ on the set defined by the equation

$$P_k(\omega_1, \ldots, \omega_n) = 0$$

is everywhere nonzero whenever $|\boldsymbol{\omega}| \neq 0$.

The fundamental solution of the linear regular PDE $P_k\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0$ generated by the linear regular differential operator P_k is expressed as

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \int_{\Omega_n} \frac{\varphi_{nk}(\boldsymbol{\omega}\cdot\mathbf{x})}{P_k(\boldsymbol{\omega})}\,d\Omega_n, \tag{11}$$

where the function $\varphi_{nk}(z)$ is defined by

$$\varphi_{nk}(z) = \frac{(-1)^{\frac{n-2}{2}}}{(2\pi)^n(k-n)!}z^{k-n}\ln|z| \qquad \text{if } n \text{ is even and } k \ge n,$$
$$\varphi_{nk}(z) = \frac{(-1)^{\frac{n+2k}{2}}(n-k-1)!}{(2\pi)^n}z^{k-n} \qquad \text{if } n \text{ is even and } k < n,$$
$$\varphi_{nk}(z) = \frac{(-1)^{\frac{n-1}{2}}}{4(2\pi)^{n-1}(k-n)!}z^{k-n}\operatorname{sign} z \qquad \text{if } n \text{ is odd and } k \ge n,$$
$$\varphi_{nk}(z) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}}\delta^{(n-k-1)}(z) \qquad \text{if } n \text{ is odd and } k < n.$$

The integral in (11) is understood in the sense of its regularized value; i.e.,

$$\mathscr{E}_{\rm e}(\mathbf{x}) = \lim_{\varepsilon\to 0} I_\varepsilon(\mathbf{x}), \qquad I_\varepsilon(\mathbf{x}) = \int_{\Omega_n^{(\varepsilon)}} \frac{\varphi_{nk}(\boldsymbol{\omega}\cdot\mathbf{x})}{P_k(\boldsymbol{\omega})}\,d\Omega_n^{(\varepsilon)},$$

where $\Omega_n^{(\varepsilon)}$ is the set of points on the sphere of unit radius for which $|P_k(\boldsymbol{\omega})| > \varepsilon$.

⊙ *Literature*: I. M. Gelfand and G. E. Shilov (1959), S. G. Krein (1972).

11.6.5 Some Equations with Two Independent Variables Containing the First Derivative in t

1. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^{2n} w}{\partial x^{2n}} = \Phi(x,t), \qquad n = 1,\,2,\,\dots.$

1°. Particular solutions of the homogeneous equation with $\Phi \equiv 0$:

$$w(x,t) = b\left[\frac{x^{2n}}{(2n)!} - at\right] + \sum_{k=0}^{2n-1} c_k x^k,$$
$$w(x,t) = b\left[\frac{2x^{2n+2}}{(2n+2)!} - ax^2t\right] + c\left[\frac{x^{2n+1}}{(2n+1)!} - axt\right],$$
$$w(x,t) = \exp(-a\lambda^{2n})\bigl[b\exp(-\lambda x) + c\exp(\lambda x)\bigr],$$
$$w(x,t) = \exp\bigl[a(-1)^{n+1}\lambda^{2n}\bigr]\bigl[b\cos(\lambda x) + c\sin(\lambda x)\bigr],$$
$$w(x,t) = \exp\bigl(-a\lambda^{2n}t\bigr)\bigl\{b\exp\bigl(-\lambda x\bigr) + c\exp\bigl(\lambda x\bigr)$$
$$+ \sum_{k=1}^{n-1}\exp(\lambda x\cos\beta_k)[b_k\cos(\lambda x\sin\beta_k) + c_k\sin(\lambda x\sin\beta_k)]\bigr\}, \qquad \beta_k = \frac{\pi k}{n},$$

where b, b_k, c, c_k, and λ are arbitrary constants.

2°. Formal series solution for $\Phi \equiv 0$:

$$w(x,t) = f(x) + \sum_{k=1}^{\infty} (-1)^k \frac{a^k t^k}{k!} \frac{d^{2nk} f(x)}{dx^{2nk}},$$

where $f(x)$ is an arbitrary infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = f(x)$. If the function $f(x)$ is a polynomial of degree m, then the solution is a polynomial in x of degree m as well.

3°. Let $a = (-1)^n \alpha^2$, where $\alpha > 0$.

Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\bigl(-\alpha^2 t \xi^{2n} + ix\xi\bigr)\, d\xi = \frac{1}{\pi} \int_0^{\infty} \exp\bigl(-\alpha^2 t \xi^{2n}\bigr) \cos(x\xi)\, d\xi.$$

By using the self-similar variables

$$\zeta = \xi \bigl(2n\alpha^2 t\bigr)^{\frac{1}{2n}}, \qquad z = x\bigl(2n\alpha^2 t\bigr)^{-\frac{1}{2n}},$$

one can reduce the fundamental solution to the form

$$\mathscr{E}(x,t) = \frac{1}{\pi} \bigl(2n\alpha^2 t\bigr)^{-\frac{1}{2n}} I_{2n}(z), \qquad I_{2n}(z) = \int_0^{\infty} \exp\Bigl(-\frac{1}{2n}\zeta^{2n}\Bigr) \cos(z\zeta)\, d\zeta.$$

The function $I_{2n}(z)$ is known as the *hyper-Airy function of even index*. These functions are bounded, integrable, and infinitely differentiable.

The function $I_{2n}(z)$ can be represented in the form of the series

$$I_{2n}(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(2n)^{\nu_k - 1}}{(2k)!} \Gamma(\nu_k) z^{2k}, \qquad \nu_k = \frac{2k+1}{2n},$$

where $\Gamma(\nu)$ is the gamma function.

The leading term of the asymptotic expansion for $I_{2n}(z)$, $n > 1$, as $z \to \pm\infty$ has the form

$$I_{2n}(z) \doteq \left[\frac{2}{\pi(2n-1)}\right]^{1/2} |z|^{-\frac{n-1}{2n-1}} \exp\left[-\frac{2n-1}{2n} \cos\Bigl(\pi \frac{n-1}{2n-1}\Bigr) |z|^{\frac{2n}{2n-1}}\right] \times \cos\left[\frac{2n-1}{2n} \sin\Bigl(\pi \frac{n-1}{2n-1}\Bigr) |z|^{\frac{2n}{2n-1}} - \frac{\pi}{2} \frac{n-1}{2n-1}\right].$$

4°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution with $a = (-1)^n \alpha^2$:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}(x-y,t) f(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \mathscr{E}(x-y,t-\tau) \Phi(y,\tau)\, dy\, d\tau,$$

where the function $\mathscr{E}(x,t)$ is defined in Item 3°.

⊙ *Literature*: A. Winter (1957), R. B. Paris and A. D. Wood (1987), A. Karlsson and S. Ritke (1998), G. Kristensson, A. Karlsson, and S. Rikte (2002), N. A. Kudryashov and D. I. Sinelshchikov (2014).

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} = \Phi(x,t), \qquad n = 1,\ 2,\ \ldots.$

1°. Particular solutions of the homogeneous equation with $\Phi \equiv 0$:

$$
\begin{aligned}
w(x,t) &= b\left[\frac{x^{2n+1}}{(2n+1)!} - at\right] + \sum_{k=0}^{2n} c_k x^k,\\
w(x,t) &= b\left[\frac{2x^{2n+3}}{(2n+3)!} - ax^2 t\right] + c\left[\frac{x^{2n+2}}{(2n+2)!} - axt\right],\\
w(x,t) &= b\exp(\lambda x - a\lambda^{2n+1}t) + c\exp(-\lambda x + a\lambda^{2n+1}t),\\
w(x,t) &= b\sin\bigl[\lambda x + (-1)^{n+1}a\lambda^{2n+1}t\bigr] + c\cos\bigl[\lambda x + (-1)^{n+1}a\lambda^{2n+1}t\bigr],\\
w(x,t) &= \exp\bigl(-a\lambda^{2n+1}t\bigr)\bigl\{b\exp\bigl(\lambda x\bigr)\\
&\quad + \sum_{k=1}^{n}\exp(\lambda x\cos\beta_k)[b_k\cos(\lambda x\sin\beta_k) + c_k\sin(\lambda x\sin\beta_k)]\bigr\}, \quad \beta_k = \frac{2\pi k}{2n+1},
\end{aligned}
$$

where b, b_k, c, c_k, and λ are arbitrary constants.

2°. Formal series solution for $\Phi \equiv 0$:

$$
w(x,t) = f(x) + \sum_{k=1}^{\infty}(-1)^k\frac{a^k t^k}{k!}\frac{d^{2nk+k}f(x)}{dx^{2nk+k}},
$$

where $f(x)$ is an arbitrary infinitely differentiable function. This solution satisfies the initial condition $w(x,0) = f(x)$. If the function $f(x)$ is a polynomial of degree m, then solution is a polynomial in x of degree m as well.

3°. Let $a = (-1)^{n+1}\alpha^2$, where $\alpha > 0$.

Fundamental solution:

$$
\mathscr{E}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\bigl[i(\alpha^2 t\xi^{2n+1} + x\xi)\bigr]\,d\xi = \frac{1}{\pi}\int_0^{\infty}\cos(\alpha^2 t\xi^{2n+1} + x\xi)\,d\xi.
$$

By using the self-similar variables

$$
\zeta = \xi\bigl[(2n+1)\alpha^2 t\bigr]^{\frac{1}{2n+1}}, \qquad z = x\bigl[(2n+1)\alpha^2 t\bigr]^{-\frac{1}{2n+1}},
$$

one can reduce the fundamental solution to the form

$$
\begin{aligned}
\mathscr{E}(x,t) &= \frac{1}{\pi}\bigl[(2n+1)\alpha^2 t\bigr]^{-\frac{1}{2n+1}} I_{2n+1}(z),\\
I_{2n+1}(z) &= \int_0^{\infty}\cos\Bigl(\frac{1}{2n+1}\zeta^{2n+1} + z\zeta\Bigr)\,d\zeta\\
&= \int_0^{\infty}\frac{2n\zeta^{2n-1}}{(z+\zeta^{2n})^2}\sin\Bigl(\frac{1}{2n+1}\zeta^{2n+1} + z\zeta\Bigr)\,d\zeta.
\end{aligned}
$$

The family of integrals $I_{2n+1}(z)$ is called the *hyper-Airy function of odd index*. The leading term of the asymptotic expansion for $I_{2n+1}(z)$ as $z \to \infty$ has the form

$$I_{2n+1}(z) \doteq \frac{1}{\sqrt{\pi n}} z^{-\frac{2n-1}{4n}} \exp\left[-\frac{2n}{2n+1}\cos\left(\frac{\pi}{2}\frac{n-1}{n}\right) z^{\frac{2n+1}{2n}}\right] \times \cos\left[\frac{2n}{2n+1}\sin\left(\frac{\pi}{2}\frac{n-1}{n}\right) z^{\frac{2n+1}{2n}} - \frac{\pi}{2}\frac{n-1}{2n}\right].$$

For $I_{2n+1}(z)$ as $z \to -\infty$ we have

$$I_{2n+1}(z) \doteq \frac{1}{\sqrt{\pi n}} |z|^{-\frac{2n-1}{4n}} \cos\left(\frac{2n}{2n+1}|z|^{\frac{2n+1}{2n}} - \frac{\pi}{4}\right).$$

4°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution with $a = (-1)^{n+1}\alpha^2$:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}(x-y,t) f(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau,$$

where the function $\mathscr{E}_{\rm e}(x,t)$ is defined in Item 3°.

⊙ *Literature*: A. Winter (1957), R. B. Paris and A. D. Wood (1987), A. Karlsson and S. Ritke (1998), G. Kristensson, A. Karlsson, and S. Rikte (2002), N. A. Kudryashov and D. I. Sinelshchikov (2014).

3. $\dfrac{\partial w}{\partial t} + \displaystyle\sum_{k=0}^{n} a_k \frac{\partial^{2k} w}{\partial x^{2k}} = \Phi(x,t).$

1°. Particular solutions of the homogeneous equation with $\Phi(x,t) = 0$:

$$w = \exp(-\lambda t)[A\exp(-\beta x) + B\exp(\beta x)], \qquad \lambda = \sum_{k=0}^{n} a_k \beta^{2k},$$

$$w = \exp(-\lambda t)[A\cos(\beta x) + B\sin(\beta x)], \qquad \lambda = \sum_{k=0}^{n} (-1)^k a_k \beta^{2k},$$

where A, B, and β are arbitrary constants.

2°. The condition

$$P(\xi) = \sum_{k=0}^{n} (-1)^k a_k \xi^{2k} \ge 0 \qquad (1)$$

is assumed to be met for all real ξ. The condition is satisfied, say, if $a_k = (-1)^k b_k$ with $b_k \ge 0$ for all k.

Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\bigl[-tP(\xi) + ix\xi\bigr]\,d\xi = \frac{1}{\pi} \int_0^{\infty} \exp\bigl[-tP(\xi)\bigr] \cos(x\xi)\,d\xi. \qquad (2)$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}(x-y,t) f(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau,$$

where the function $\mathscr{E}(x,t)$ is defined in Item 2°.

Remark 11.5. One can weaken condition (1) by replacing it with the condition $(-1)^n a_n > 0$.

4. $\displaystyle \frac{\partial w}{\partial t} + \sum_{k=0}^{n} a_k \frac{\partial^{2k+1} w}{\partial x^{2k+1}} = \Phi(x,t).$

1°. Particular solutions of the homogeneous equation with $\Phi(x,t) = 0$:

$$w = A\exp(\beta x - \lambda t) + B\exp(-\beta x + \lambda t), \qquad \lambda = \sum_{k=0}^{n} a_k \beta^{2k+1},$$

$$w = A\cos(\beta x + \lambda t) + B\sin(\beta x + \lambda t), \qquad \lambda = \sum_{k=0}^{n} (-1)^{k+1} a_k \beta^{2k+1},$$

where A, B, and β are arbitrary constants.

2°. The condition

$$P(\xi) = \sum_{k=0}^{n} (-1)^{k+1} a_k \xi^{2k+1} \geq 0$$

is assumed to be met for real $\xi \geq 0$. The condition is satisfied, say, if $a_k = (-1)^{k+1} b_k$ with $b_k \geq 0$ for all k.

Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\bigl[itP(\xi) + ix\xi\bigr]\,d\xi = \frac{1}{\pi} \int_0^{\infty} \cos\bigl[tP(\xi) + x\xi\bigr]\,d\xi.$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}(x-y,t) f(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau,$$

where the function $\mathscr{E}(x,t)$ is defined in Item 2°.

5. $\displaystyle \frac{\partial w}{\partial t}+\sum_{k=0}^{n} a_k\frac{\partial^k w}{\partial x^k}=\Phi(x,t).$

For $n=2m$ and $a_1=a_3=\cdots=a_{2m-1}=0$, see equation 11.6.5.3; for $n=2m+1$ and $a_0=a_2=\cdots=a_{2m}=0$, see equation 11.6.5.4.

1°. Particular solutions of the homogeneous equation with $\Phi(x,t)=0$:

$$
\begin{aligned}
w&=\exp(\beta x-\lambda t), \qquad \lambda=\sum_{k=0}^{n} a_k\beta^k,\\
w&=\exp\bigl[-P(\beta)t\bigr]\cos\bigl[\beta x+Q(\beta)t\bigr],\\
w&=\exp\bigl[-P(\beta)t\bigr]\sin\bigl[\beta x+Q(\beta)t\bigr],
\end{aligned}
$$

where β is an arbitrary constant and

$$
P(\beta)=\sum_{k=0}^{2k\le n}(-1)^k a_{2k}\beta^{2k}, \qquad Q(\beta)=\sum_{k=0}^{2k+1\le n}(-1)^{k+1}a_{2k+1}\beta^{2k+1}.
$$

2°. The condition

$$
P(\xi)=\sum_{k=0}^{2k\le n}(-1)^k a_{2k}\xi^{2k}>0
$$

is assumed to be met for real $\xi>0$.

Fundamental solution:

$$
\begin{aligned}
\mathscr{E}(x,t)&=\frac{1}{\pi}\int_0^\infty \exp\bigl[-tP(\xi)\bigr]\cos\bigl[tQ(\xi)+x\xi\bigr]\,d\xi,\\
Q(\xi)&=\sum_{k=0}^{2k+1\le n}(-1)^{k+1}a_{2k+1}\xi^{2k+1}.
\end{aligned}
$$

3°. Domain: $-\infty<x<\infty$. Cauchy problem.

An initial condition is prescribed:

$$
w=f(x) \quad \text{at} \quad t=0.
$$

Solution:

$$
w(x,t)=\int_{-\infty}^{\infty}\mathscr{E}(x-y,t)f(y)\,dy+\int_0^t\int_{-\infty}^{\infty}\mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau,
$$

where the function $\mathscr{E}(x,t)$ is defined in Item 2°.

⊙ *Literature*: S. G. Krein (1972), V. S. Vladimirov, V. P. Mikhailov, A. A. Vasharin et al. (1974).

6. $\dfrac{\partial w}{\partial t} - b\dfrac{\partial^3 w}{\partial t \partial x^2} + \sum_{k=0}^{n} a_k \dfrac{\partial^{2k} w}{\partial x^{2k}} = \Phi(x,t).$

1°. Particular solutions of the homogeneous equation with $\Phi(x,t) = 0$:

$$w = \exp(-\lambda t)[A\exp(-\beta x) + B\exp(\beta x)], \qquad \lambda = \frac{1}{1-b\beta^2}\sum_{k=0}^{n} a_k \beta^{2k},$$
$$w = \exp(-\lambda t)[A\cos(\beta x) + B\sin(\beta x)], \qquad \lambda = \frac{1}{1+b\beta^2}\sum_{k=0}^{n} (-1)^k a_k \beta^{2k},$$

where A, B, and β are arbitrary constants.

2°. Let $b > 0$, and let the condition $P(\xi) = \sum_{k=0}^{n}(-1)^k a_k \xi^{2k} \ge 0$ be satisfied for all real ξ. The condition is satisfied, say, if $a_k = (-1)^k c_k$ with $c_k \ge 0$ for all k.

Fundamental solution:

$$\mathscr{E}_\mathrm{e}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+b\xi^2}\exp\biggl[-\frac{P(\xi)}{1+b\xi^2}t + ix\xi\biggr]d\xi$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{1+b\xi^2}\exp\biggl[-\frac{P(\xi)}{1+b\xi^2}t\biggr]\cos(|x|\xi)\,d\xi.$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_0^t \int_{-\infty}^{\infty} \mathscr{E}_\mathrm{e}(x-y, t-\tau)\Phi(y,\tau)\,dy\,d\tau + \int_{-\infty}^{\infty} \mathscr{E}_\mathrm{e}(x-y,t)\bigl[f(y) - bf''(y)\bigr]\,dy.$$

11.6.6 Some Equations with Two Independent Variables Containing the Second Derivative in t

1. $\dfrac{\partial^2 w}{\partial t^2} + a\dfrac{\partial^{2n} w}{\partial x^{2n}} = \Phi(x,t).$

1°. Particular solutions of the homogeneous equation with $\Phi(x,t) = 0$:

$$w = [A\exp(-\beta x) + B\exp(\beta x)][C\cos(\lambda t) + D\sin(\lambda t)], \qquad \lambda = \sqrt{a\beta^{2n}},$$
$$w = [A\exp(-\beta x) + B\exp(\beta x)][C\exp(-\lambda t) + D\exp(\lambda t)], \qquad \lambda = \sqrt{-a\beta^{2n}},$$
$$w = [A\cos(\beta x) + B\sin(\beta x)][C\cos(\lambda t) + D\sin(\lambda t)], \qquad \lambda = \sqrt{(-1)^n a\beta^{2n}},$$
$$w = [A\cos(\beta x) + B\sin(\beta x)][C\exp(-\lambda t) + D\exp(\lambda t)], \qquad \lambda = \sqrt{(-1)^{n+1} a\beta^{2n}},$$

where A, B, C, D, and β are arbitrary constants. (In all cases, the sign of the coefficient a is chosen to ensure that the radicands are positive).

2°. General solution with $\Phi(x,t)=0$ and $a=-\alpha^2<0$:

$$w = w_1 + w_2,$$

where w_1 and w_2 are arbitrary solutions of two simpler nth-order equations

$$\frac{\partial w_1}{\partial t} + \alpha\frac{\partial^n w_1}{\partial x^n} = 0, \qquad \frac{\partial w_2}{\partial t} - \alpha\frac{\partial^n w_2}{\partial x^n} = 0.$$

See equation 11.6.5.1 or 11.6.5.2 for information on the solution of these equations depending on whether n is odd or even.

3°. Formal series solution for $\Phi(x,t)=0$:

$$w(x,t) = \sum_{k=0}^{\infty}(-1)^k \frac{a^k t^{2k}}{(2k)!}\frac{d^{2nk}f(x)}{dx^{2nk}} + \sum_{k=0}^{\infty}(-1)^k \frac{a^k t^{2k+1}}{(2k+1)!}\frac{d^{2nk}g(x)}{dx^{2nk}},$$

where $f(x)$ and $g(x)$ are arbitrary infinitely differentiable functions and $d^0 f(x)/dx^0 = f(x)$. This solution satisfies the initial conditions $w(x,0)=f(x)$ and $\partial_t w(x,0) = g(x)$. If the functions $f(x)$ and $g(x)$ are polynomials of degree $\le m$, then the solution is a polynomial in x of degree $\le m$ as well.

4°. Let $a=(-1)^n\alpha^2$, where $\alpha>0$.

Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(\alpha t\xi^n)}{\alpha\xi^n}\exp(ix\xi)\,d\xi = \frac{1}{\pi}\int_0^{\infty}\frac{\sin(\alpha t\xi^n)}{\alpha\xi^n}\cos(x\xi)\,d\xi.$$

By using the self-similar variables

$$\zeta = \xi(\alpha t)^{1/n}, \qquad z = x(\alpha t)^{-1/n},$$

one can reduce the fundamental solution to the form

$$\mathscr{E}(x,t) = \frac{t^{(n-1)/n}}{\pi\alpha^{1/n}}I(z), \qquad I(z) = \int_0^{\infty}\frac{\sin(\zeta^n)}{\zeta^n}\cos(z\zeta)\,d\zeta.$$

5°. Domain: $-\infty<x<\infty$. Cauchy problem with $a=(-1)^n\alpha^2$, where $\alpha>0$.

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t=0,$$
$$\partial_t w = g(x) \quad \text{at} \quad t=0.$$

Solution:

$$w(x,t) = \int_0^t\int_{-\infty}^{\infty}\mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau + \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\mathscr{E}(x-y,t)f(y)\,dy + \int_{-\infty}^{\infty}\mathscr{E}(x-y,t)g(y)\,dy.$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} + \sum_{k=0}^{n} a_k \frac{\partial^{2k} w}{\partial x^{2k}} = \Phi(x,t).$

1°. Particular solutions of the homogeneous equation for $\Phi(x,t)=0$:

$$
\begin{aligned}
w &= [A\exp(-\beta x) + B\exp(\beta x)]\sin(\lambda t + C), &\quad \lambda &= \Big(\sum_{k=0}^{n} a_k \beta^{2k}\Big)^{1/2},\\
w &= [A\exp(-\beta x) + B\exp(\beta x)]\exp(\pm\lambda t), &\quad \lambda &= \Big(-\sum_{k=0}^{n} a_k \beta^{2k}\Big)^{1/2},\\
w &= \sin(\beta x + A)\sin(\lambda t + B), &\quad \lambda &= \Big(\sum_{k=0}^{n} (-1)^k a_k \beta^{2k}\Big)^{1/2},\\
w &= \sin(\beta x + A)[B\exp(-\lambda t) + C\exp(\lambda t)], &\quad \lambda &= \Big(-\sum_{k=0}^{n} (-1)^k a_k \beta^{2k}\Big)^{1/2},
\end{aligned}
$$

where A, B, C, and β are arbitrary constants. (In all cases, the radicands are assumed to be positive.)

2°. Let $b>0$, and let the condition $P(\xi)=\sum_{k=0}^{n}(-1)^k a_k \xi^{2k} \ge 0$ be satisfied for all real ξ. The condition is satisfied if, say, $a_k = (-1)^k c_k$ with $c_k \ge 0$ for all k.

Fundamental solution:

$$
\begin{aligned}
\mathscr{E}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{P(\xi)}} \sin\big[t\sqrt{P(\xi)}\,\big] \exp(ix\xi)\,d\xi\\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{\sqrt{P(\xi)}} \sin\big[t\sqrt{P(\xi)}\,\big] \cos(|x|\xi)\,d\xi.
\end{aligned}
$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$
\begin{aligned}
w &= f(x) \quad \text{at} \quad t=0,\\
\partial_t w &= g(x) \quad \text{at} \quad t=0.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
w(x,t) &= \int_0^t \int_{-\infty}^{\infty} \mathscr{E}(x-y,t-\tau)\Phi(y,\tau)\,dy\,d\tau\\
&\quad + \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}(x-y,t) f(y)\,dy + \int_{-\infty}^{\infty} \mathscr{E}(x-y,t) g(y)\,dy.
\end{aligned}
$$

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} - b\frac{\partial^4 w}{\partial t^2 \partial x^2} + \sum_{k=0}^{n} a_k \frac{\partial^{2k} w}{\partial x^{2k}} = \Phi(x,t).$

1°. Particular solutions of the homogeneous equation for $\Phi(x,t) = 0$:

$$w = [A\exp(-\beta x) + B\exp(\beta x)]\sin(\lambda t + C), \quad \lambda = \left(\frac{1}{1-b\beta^2}\sum_{k=0}^{n} a_k \beta^{2k}\right)^{1/2},$$

$$w = [A\exp(-\beta x) + B\exp(\beta x)]\exp(\pm\lambda t), \quad \lambda = \left(\frac{1}{b\beta^2 - 1}\sum_{k=0}^{n} a_k \beta^{2k}\right)^{1/2},$$

$$w = \sin(\beta x + A)\sin(\lambda t + B), \quad \lambda = \left(\frac{1}{1+b\beta^2}\sum_{k=0}^{n}(-1)^k a_k \beta^{2k}\right)^{1/2},$$

$$w = \sin(\beta x + A)[B\exp(-\lambda t) + C\exp(\lambda t)], \quad \lambda = \left(-\frac{1}{1+b\beta^2}\sum_{k=0}^{n}(-1)^k a_k \beta^{2k}\right)^{1/2},$$

where A, B, C, and β are arbitrary constants.

2°. Let $b > 0$, and let the condition

$$P(\xi) = \sum_{k=0}^{n}(-1)^k a_k \xi^{2k} \ge 0$$

be satisfied for all real ξ. The condition is satisfied if, say, $a_k = (-1)^k c_k$ with $c_k \ge 0$ for all k.

Fundamental solution:

$$\begin{aligned}
\mathscr{E}_{\rm e}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{(1+b\xi^2)P(\xi)}} \sin\left(t\sqrt{\frac{P(\xi)}{1+b\xi^2}}\right)\exp(ix\xi)\,d\xi \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{\sqrt{(1+b\xi^2)P(\xi)}} \sin\left(t\sqrt{\frac{P(\xi)}{1+b\xi^2}}\right)\cos(|x|\xi)\,d\xi.
\end{aligned}$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$\begin{aligned}
w &= f(x) \quad \text{at} \quad t = 0, \\
\partial_t w &= g(x) \quad \text{at} \quad t = 0.
\end{aligned}$$

Solution:

$$\begin{aligned}
w(x,t) &= \int_0^t \int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y, t-\tau)\Phi(y,\tau)\,dy\,d\tau \\
&\quad + \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)\big[f(y) - bf''(y)\big]\,dy + \int_{-\infty}^{\infty} \mathscr{E}_{\rm e}(x-y,t)\big[g(y) - bg''(y)\big]\,dy.
\end{aligned}$$

11.6.7 Other Equations with Two Independent Variables

1. $\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)^n w = 0, \qquad n = 1,\ 2,\ \ldots$

1°. General solution (two representations):

$$w(x,t) = \sum_{k=0}^{n-1} t^k f_k(x+at),$$
$$w(x,t) = \sum_{k=0}^{n-1} x^k f_k(x+at),$$

where the $f_k = f_k(z)$ are arbitrary functions.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{t^{n-1}}{(n-1)!}\delta(x+at).$$

2. $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^n w = 0, \qquad n = 1,\ 2,\ \ldots$

1°. General solution (two representations):

$$w(x,t) = \sum_{k=0}^{n-1} t^k u_k(x,t),$$
$$w(x,t) = \sum_{k=0}^{n-1} x^k u_k(x,t),$$

where the $u_k = u_k(x,t)$ are arbitrary solutions of the heat equations $\partial_t u_k - \partial_{xx} u_k = 0$.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{1}{2\sqrt{\pi}\,(n-1)!} t^{n-3/2} \exp\left(-\frac{x^2}{4t}\right).$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.
Initial conditions are prescribed:

$$w\big|_{t=0} = 0, \quad \frac{\partial w}{\partial t}\bigg|_{t=0} = 0, \quad \ldots, \quad \frac{\partial^{n-2} w}{\partial t^{n-2}}\bigg|_{t=0} = 0, \quad \frac{\partial^{n-1} w}{\partial t^{n-1}}\bigg|_{t=0} = f(x).$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} f(\xi)\mathscr{E}(x-\xi,t)\,d\xi.$$

⊙ *Literature*: G. E. Shilov (1965).

3. $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^n w = 0, \qquad n = 1,\ 2,\ \ldots$

1°. General solution (two representations):

$$w(x,t) = \sum_{k=0}^{n-1} t^k \big[f_k(x+t) + g_k(x-t)\big],$$

$$w(x,t) = \sum_{k=0}^{n-1} x^k \big[f_k(x+t) + g_k(x-t)\big],$$

where $f_k = f_k(y)$ and $g_k = g_k(z)$ are arbitrary functions.

2°. Fundamental solution:

$$\mathscr{E}(x,t) = \frac{(-1)^{n-1}}{4^n (n-1)!}\left[\operatorname{sign}(x-t)\sum_{k=0}^{n-1}\frac{(2t)^k (x-t)^{2n-k-2}}{k!(n-k-1)!} + (-1)^n \operatorname{sign}(x+t)\sum_{k=0}^{n-1}\frac{(-2t)^k (x+t)^{2n-k-2}}{k!(n-k-1)!}\right].$$

⊙ *Literature*: G. E. Shilov (1965).

4. $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n w = 0, \qquad n = 1,\ 2,\ \ldots$

This is the *polyharmonic equation* of order n with two independent variables.

1°. General solution (two representations):

$$w(x,y) = \sum_{k=0}^{n-1} r^{2k} u_k(x,y), \qquad r = \sqrt{x^2+y^2},$$

$$w(x,y) = \sum_{k=0}^{n-1} x^k u_k(x,y),$$

where the $u_k(x,y)$ are arbitrary harmonic functions ($\Delta u_k = 0$). In the second relation, x^k can be replaced by y^k.

2°. Domain: $-\infty < x < \infty$, $-\infty < y < \infty$. Fundamental solution:

$$\mathscr{E}_{\rm e}(x,y) = \frac{1}{\pi 2^{2n-1}\big[(n-1)!\big]^2} r^{2n-2} \ln r, \qquad r = \sqrt{x^2+y^2}.$$

3°. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. Boundary value problem.
Boundary conditions are prescribed:

$$w\big|_{y=0} = 0, \quad \frac{\partial w}{\partial y}\bigg|_{y=0} = 0, \quad \ldots, \quad \frac{\partial^{n-2} w}{\partial y^{n-2}}\bigg|_{y=0} = 0, \quad \frac{\partial^{n-1} w}{\partial y^{n-1}}\bigg|_{y=0} = f(x).$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi, y)\, d\xi, \qquad G(x,y) = \frac{1}{\pi(n-1)!}\frac{y^n}{x^2+y^2}.$$

See also Example 11.7 in Section 11.6.4.

⊙ *Literature*: G. E. Shilov (1965), L. D. Faddeev (1998).

5. $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^n w=\Phi(x,y), \qquad n=1,\,2,\,\ldots$

This is a *nonhomogeneous polyharmonic equation* of order n with two independent variables.

Particular solution:

$$w(x,y)=\frac{1}{k_n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta)[(x-\xi)^2+(y-\eta)^2]^{n-1}\ln[(x-\xi)^2+(y-\eta)^2]\,d\xi\,d\eta,$$
$$k_n=\pi 2^{2n}[(n-1)!]^2.$$

The general solution is given by the sum of any particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation (see equation 11.6.7.4, Item 1°).

6. $\sum_{k=0}^{m} a_k\Delta^k w=0, \qquad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$

Particular solutions:

$$w(x,y)=\sum_{n=1}^{m}u_n(x,y),$$

where the u_n are solutions of the Helmholtz equations $\Delta u_n-\lambda_n u_n=0$ and the λ_n are roots of the characteristic equation $\sum\limits_{k=0}^{m}a_k\lambda^k=0$.

⊙ *Literature*: A. V. Bitsadze and D. F. Kalinichenko (1985).

11.6.8 Equations with Three and More Independent Variables

1. $\Delta_n^m w=0, \qquad \Delta_n=\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}.$

This is the *polyharmonic equation* of order m with n independent variables. For $m=1$, see Sections 9.1 and 10.1. For $m=2$, see Section 11.5.1. For $n=2$, see equation 11.6.7.4.

1°. Particular solutions:

$$w(\mathbf{x})=\sum_{s=0}^{m-1}x_j^s u_s(\mathbf{x}), \qquad j=1,\,2,\,\ldots,\,n,$$

where the $u_s(\mathbf{x})$ are arbitrary harmonic functions ($\Delta_n u_s=0$).

2°. Fundamental solution for $m\ge 1$ and $n\ge 3$:

$$\mathscr{E}_{\rm e}(\mathbf{x})=\begin{cases} b_{n,m}|\mathbf{x}|^{2m-n} & \text{if } n \text{ is odd or } n \text{ is even and } n>2m,\\ c_{n,m}|\mathbf{x}|^{2m-n}\ln|\mathbf{x}| & \text{if } n \text{ is even and } n\le 2m.\end{cases}$$

Here

$$b_{n,m} = \frac{a_{n,m}}{(2-n)(4-n)\dots(2m-n)}, \qquad a_{n,m} = \frac{\Gamma(n/2)}{2^m(m-1)!\,\pi^{n/2}},$$
$$c_{n,m} = \frac{a_{n,m}}{(2-n)(4-n)\dots(2m_0-2-n)(2m_0+2-n)(2m_0+4-n)\dots(2m-n)},$$

where $m_0 = n/2$. The expression of the coefficient $c_{n,m}$ can be obtained formally from the expression of $b_{n,m}$ by removing the zero factor $(2m_0 - n)$ from the denominator.

⊙ *Literature*: G. E. Shilov (1965).

2. $\dfrac{\partial w}{\partial t} + \sum\limits_{s=1}^{m} a_s \Delta_n^s w = \Phi(\mathbf{x}, t).$

Here $\mathbf{x} = (x_1, \dots, x_n)$ and $\Delta_n = \sum_{s=1}^{n} \frac{\partial^2}{\partial x_s^2}$.

Let an initial-boundary value problem for this equation be stated in a bounded domain V. Particular solutions of the nonhomogeneous equation can be found as follows.

Consider the auxiliary problem

$$\Delta u + \lambda u = 0 \tag{1}$$

for the Helmholtz equation in a bounded domain V_1 containing V (moreover, the boundaries of V and V_1 do not have common points) with the homogeneous boundary condition of the first kind

$$u|_{\mathbf{x}\in S_1} = 0. \tag{2}$$

Let λ_k and $u_k(\mathbf{x})$ be the eigenvalues and eigenfunctions of problem (1)–(2). All eigenvalues are positive. We arrange them in the nondescending order $\lambda_1 \le \lambda_2 \le \dots$. It is well known that the eigenfunctions are orthogonal,

$$\int_{V_1} u_i(\mathbf{x}) u_j(\mathbf{x})\, dv_1 = 0 \qquad \text{if} \qquad \lambda_i \ne \lambda_j,$$

and can be chosen to be real. The eigenfunctions corresponding to coinciding eigenvalues $\lambda_i = \lambda_j$ can be chosen to be orthogonal as well. The system of eigenfunctions u_k is complete in $L_2(V_1)$.

Assume that the function $\Phi(\mathbf{x}, t)$ is defined in V_1. Let us expand it in the eigenfunctions u_k,

$$\Phi(\mathbf{x}, t) = \sum_{k=1}^{\infty} A_k(t) u_k(\mathbf{x}),$$
$$A_k(t) = \frac{1}{\|u_k\|^2} \int_{V_1} \Phi(\mathbf{x}, t) u_k(\mathbf{x})\, dv_1, \qquad \|u_k\|^2 = \int_{V_1} u_k^2(\mathbf{x})\, dv_1. \tag{3}$$

We seek a particular solution of the original equation in the form

$$w = \sum_{k=1}^{\infty} f_k(t) u_k(\mathbf{x}). \tag{4}$$

Since the eigenfunctions u_k satisfy Eq. (1), we have $\Delta^s u_k = (-\lambda_k)^s u_k$. In view of the representation (3), for the functions $f_k = f_k(t)$ we obtain the first-order linear ODEs

$$f_k' + \beta_k f = A_k(t), \qquad \beta_k = \sum_{s=1}^{m} a_s(-\lambda_k)^s.$$

The general solutions of these equations have the form

$$f_k(t) = C_k e^{-\beta_k t} + e^{-\beta_k t}\int_0^t e^{\beta_k \tau} A_k(\tau)\,d\tau, \tag{5}$$

where the C_k are arbitrary constants. Formulas (4)–(5) give a particular solution of the original inhomogeneous equation.

We point out that for V_1 one can take a domain of simple geometric shape (for example, an n-dimensional cube or sphere) for which the eigenvalues and eigenfunctions are known.

Remark 11.6. Instead of the boundary condition (2) of the first kind, one can use homogeneous boundary conditions of the second or third kind.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} + \sum_{s=1}^{m} a_s \Delta_n^s w = \Phi(\mathbf{x}, t).$

Here $\mathbf{x} = (x_1, \dots, x_n)$ and $\Delta_n = \sum_{s=1}^{n} \frac{\partial^2}{\partial x_s^2}$.

Let λ_k and $u_k(\mathbf{x})$ be the eigenvalues and eigenfunctions of the auxiliary problem for the homogeneous Helmholtz equation described in 11.6.8.2 (see Eqs. (1)–(2)).

Let us expand $\Phi(\mathbf{x}, t)$ in a series in the eigenfunctions $u_k(\mathbf{x})$. We seek a particular solution of the equation in question in the form

$$w = \sum_{k=1}^{\infty} f_k(t) u_k(\mathbf{x}).$$

As result, for the functions $f_k = f_k(t)$ we obtain the second-order linear ODEs

$$f_k'' + \beta_k f = A_k(t), \qquad \beta_k = \sum_{s=1}^{m} a_s(-\lambda_k)^s, \qquad A_k(t) = \frac{1}{\|u_k\|^2}\int_{V_1} \Phi(\mathbf{x}, t) u_k(\mathbf{x})\,dv_1.$$

The general solution of these equations has the form

$$f_k = \begin{cases} C_1\cos(b_k t) + C_2\sin(b_k t) + \dfrac{1}{b_k}\displaystyle\int_0^t A_k(\tau)\sin[b_k(t-\tau)]\,d\tau & \text{if } \beta_k = b_k^2 > 0, \\ C_1\cosh(b_k t) + C_2\sinh(b_k t) + \dfrac{1}{b_k}\displaystyle\int_0^t A_k(\tau)\sinh[b_k(t-\tau)]\,d\tau & \text{if } \beta_k = -b_k^2 < 0, \\ C_1 t + C_2 + \displaystyle\int_0^t A_k(\tau)(t-\tau)\,d\tau & \text{if } \beta_k = 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

4. $\displaystyle\sum_{k=0}^{n} a_k \frac{\partial^n w}{\partial t^k \partial x^{n-k}} = 0.$

1°. Let the algebraic equation

$$\sum_{k=0}^{n} a_k \lambda^k = 0 \tag{1}$$

have m distinct real roots $\lambda_1, \ldots, \lambda_m$ (which corresponds to the homogeneous hyperbolic equation). Then the function

$$w = \sum_{k=1}^{m} f_k(x + \lambda_k t),$$

where the $f_k(z_k)$ are arbitrary functions, is a solution of the equation.

2°. Let λ_s be a real root of Eq. (1) of multiplicity r. Then the function

$$w = \sum_{k=1}^{r} x^k g_k(x + \lambda_s t),$$

where $g_k(z)$ are arbitrary functions, is a solution of the equation.

3°. Let $\lambda_{p,1} = b_p + ic_p$ and $\lambda_{p,2} = b_p - ic_p$ be complex conjugate roots of Eq. (1) ($p = 1, \ldots, q$). Then the original equation has solutions w_p that satisfy second-order elliptic equations of the form

$$\frac{\partial^2 w_p}{\partial t^2} - 2b_p \frac{\partial^2 w_p}{\partial t \partial x} + (b_p^2 + c_p^2) \frac{\partial^2 w_p}{\partial x^2} = 0. \tag{2}$$

Each equation (2) can be reduced to the Laplace equation

$$\frac{\partial^2 U_p}{\partial \xi_p^2} + \frac{\partial^2 U_p}{\partial \eta_p^2} = 0$$

by the transformation

$$w_p = U_p(\xi_p, \eta_p), \qquad \xi_p = (b_p^2 + c_p^2)t + b_p x, \qquad \eta_p = c_p x.$$

5. $\displaystyle\sum_{k=0}^{m} a_k L^k[w] = 0.$

Here L is any constant coefficient linear differential operator with arbitrarily many independent variables $x_1, \ldots, x_n$.

Particular solutions:

$$w(x_1, \ldots, x_n) = \sum_{s=1}^{m} C_s u_s(x_1, \ldots, x_n),$$

where the u_s are solutions of the equations $L[u_s] - \lambda_s u_s = 0$, the λ_s are roots of the characteristic equation $\sum\limits_{k=0}^{m} a_k \lambda^k = 0$, and the C_s are arbitrary constants.

6. $\displaystyle\sum_{k=0}^{m} a_k L^k M^{m-k}[w] = 0.$

Here L and M are any constant coefficient linear differential operators with arbitrarily many independent variables $x_1, \ldots, x_n$.

Solution:

$$w(x_1, \ldots, x_n) = \sum_{s=1}^{m} C_s u_s(x_1, \ldots, x_n),$$

where the u_s are solutions of the equations $L[u_s] - \lambda_s M[u_s] = 0$, the λ_s are roots of the characteristic equation $\sum\limits_{k=0}^{m} a_k \lambda^k = 0$, and the C_s are arbitrary constants.

7. $L_m \ldots L_2 L_1 w = 0.$

Here $L_1, \ldots L_m$ are any linear differential operator with constant coefficients and arbitrarily many independent variables $x_1, \ldots, x_n$.

Solution:

$$w = \sum_{k=1}^{m} w_k,$$

where $L_k w_k = 0$ and $k = 1, \ldots, m$. (If $L_i \neq L_j$ for all i and j, then this is the general solution.)

11.7 Higher-Order Linear Equations with Variable Coefficients

11.7.1 Equations Containing the First Time Derivative

► **Statement of the problem for an equation with two independent variables.**

Consider the linear nonhomogeneous partial differential equation

$$\frac{\partial w}{\partial t} - L_{x,t}[w] = \Phi(x,t), \tag{1}$$

where $L_{x,t}$ is a general linear differential operator of order n with respect to the space variable x,

$$L_{x,t}[w] \equiv \sum_{k=0}^{n} a_k(x,t) \frac{\partial^k w}{\partial x^k}, \tag{2}$$

whose coefficients $a_k = a_k(x,t)$ are sufficiently smooth functions of both arguments for $t \geq 0$ and $x_1 \leq x \leq x_2$. The subscripts x and t indicate that the operator $L_{x,t}$ depends on the variables x and t.

We set the initial condition

$$w = f(x) \quad \text{at} \quad t = 0 \tag{3}$$

and the general nonhomogeneous boundary conditions

$$\Gamma_m^{(1)}[w] \equiv \sum_{k=0}^{n-1} b_{mk}^{(1)}(t) \frac{\partial^k w}{\partial x^k} = g_m^{(1)}(t) \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s),$$
$$\Gamma_m^{(2)}[w] \equiv \sum_{k=0}^{n-1} b_{mk}^{(2)}(t) \frac{\partial^k w}{\partial x^k} = g_m^{(2)}(t) \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n), \tag{4}$$

where $s \geq 1$ and $n \geq s+1$. We assume that both sets of the boundary forms $\Gamma_m^{(1)}[w]$ $(m = 1, \dots, s)$ and $\Gamma_m^{(2)}[w]$ $(m = s+1, \dots, n)$ are linearly independent, which means that for any nonzero $\psi_m = \psi_m(t)$ the following relations hold:

$$\sum_{m=1}^{s} \psi_m(t) \Gamma_m^{(1)}[w] \not\equiv 0, \qquad \sum_{m=s+1}^{n} \psi_m(t) \Gamma_m^{(2)}[w] \not\equiv 0.$$

In what follows, we deal with the nonstationary boundary value problem (1)–(4). It is assumed that there exist solutions of the problems considered in what follows.

► Homogeneous boundary conditions. The Green's function.

The solution of Eq. (1) with the initial condition (3) and the homogeneous boundary conditions

$$\Gamma_m^{(1)}[w] = 0 \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s),$$
$$\Gamma_m^{(2)}[w] = 0 \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n) \tag{5}$$

can be written as

$$w(x,t) = \int_{x_1}^{x_2} f(y) G(x,y,t,0)\, dy + \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau) G(x,y,t,\tau)\, dy\, d\tau. \tag{6}$$

Here $G(x,y,t,\tau)$ is the Green's function, which satisfies, for $t > \tau \geq 0$, the homogeneous equation

$$\frac{\partial G}{\partial t} - L_{x,t}[G] = 0 \tag{7}$$

with the special nonhomogeneous initial condition

$$G = \delta(x-y) \quad \text{at} \quad t = \tau \tag{8}$$

and the homogeneous boundary conditions

$$\Gamma_m^{(1)}[G] = 0 \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s),$$
$$\Gamma_m^{(2)}[G] = 0 \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n). \tag{9}$$

The quantities y and τ appear in problem (7)–(9) as free parameters $(x_1 \leq y \leq x_2)$, and $\delta(x)$ is the Dirac delta function.

We point out that the Green's function G is independent of the functions $\Phi(x,t)$, $f(x)$, $g_m^{(1)}(t)$, and $g_m^{(2)}(t)$ that characterize various inhomogeneities of the boundary value problem. If the coefficients a_k, $b_{mk}^{(1)}$, and $b_{mk}^{(2)}$ determining the differential operator (2) and boundary conditions (4) are independent of time t, then the Green's function depends only on three arguments, $G(x, y, t, \tau) = G(x, y, t - \tau)$.

⊙ *Literature*: Mathematical Encyclopedia (1977, Vol. 1).

► Nonhomogeneous boundary conditions. Preliminary transformations.

To solve the problem with nonhomogeneous boundary conditions (1), (3), (4), we choose a sufficiently smooth "test function" $\varphi = \varphi(x, t)$ that satisfies the same boundary conditions as the unknown function; thus,

$$\begin{aligned} \Gamma_m^{(1)}[\varphi] &= g_m^{(1)}(t) \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s), \\ \Gamma_m^{(2)}[\varphi] &= g_m^{(2)}(t) \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n). \end{aligned} \tag{10}$$

Otherwise the choice of the "test function" φ is arbitrary and is not linked to the solution of the equation in question; there are infinitely many such functions.

Let us pass from $w = w(x, t)$ to the new unknown $u = u(x, t)$ by the relation

$$w(x, t) = u(x, t) + \varphi(x, t). \tag{11}$$

By substituting (11) into (1), (3), and (4), we arrive at the problem for an equation with a modified right-hand side,

$$\frac{\partial u}{\partial t} - L_{x,t}[u] = \overline{\Phi}(x, t), \qquad \overline{\Phi}(x, t) = \Phi(x, t) - \frac{\partial \varphi}{\partial t} + L_{x,t}[\varphi], \tag{12}$$

subject to the nonhomogeneous initial condition

$$u = f(x) - \varphi(x, 0) \quad \text{at} \quad t = 0 \tag{13}$$

and the homogeneous boundary conditions

$$\begin{aligned} \Gamma_m^{(1)}[u] &= 0 \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s), \\ \Gamma_m^{(2)}[u] &= 0 \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n). \end{aligned} \tag{14}$$

The solution of problem (12)–(14) can be found using the Green's function by formula (6) in which one should replace w by u, $\Phi(x, t)$ by $\overline{\Phi}(x, t)$, and $f(x)$ by $\overline{f}(x) = f(x) - \varphi(x, 0)$. Taking into account relation (11), for w we obtain the representation

$$\begin{aligned} w(x, t) &= \int_{x_1}^{x_2} f(y) G(x, y, t, 0)\, dy + \int_0^t \int_{x_1}^{x_2} \Phi(y, \tau) G(x, y, t, \tau)\, dy\, d\tau + \varphi(x, t) \\ &\quad - \int_{x_1}^{x_2} \varphi(y, 0) G(x, y, t, 0)\, dy - \int_0^t \int_{x_1}^{x_2} \frac{\partial \varphi}{\partial \tau}(y, \tau) G(x, y, t, \tau)\, dy\, d\tau \\ &\quad + \int_0^t \int_{x_1}^{x_2} G(x, y, t, \tau) L_{y,\tau}[\varphi(y, \tau)]\, dy\, d\tau. \end{aligned} \tag{15}$$

Changing the order of integration and integrating by parts with respect to τ, we find, with reference to the initial condition (8) for the Green's function,

$$\int_0^t \frac{\partial \varphi}{\partial \tau} G\, d\tau = \varphi(y, t)\delta(x - y) - \varphi(y, 0) G(x, y, t, 0) - \int_0^t \varphi(y, \tau) \frac{\partial G}{\partial \tau}(x, y, t, \tau)\, d\tau. \tag{16}$$

We transform the inner integral of the last term in (15) using the Lagrange–Green's formula [see Kamke (1977)] to obtain

$$\int_{x_1}^{x_2} G L_{y,\tau}[\varphi]\,dy = \int_{x_1}^{x_2} \varphi L^*_{y,\tau}[G]\,dy + \mathscr{L}[\varphi, G]\big|_{y=x_1}^{y=x_2}, \tag{17}$$

$$L^*_{y,\tau}[G] \equiv \sum_{k=0}^{n} (-1)^k \frac{\partial^k}{\partial y^k}\bigl[a_k(y,\tau)G\bigr],\quad \mathscr{L}[\varphi, G] \equiv \sum_{r=0}^{n-1} \sum_{p+q=r} (-1)^p \frac{\partial^q \varphi}{\partial y^q} \frac{\partial^p}{\partial y^p}\bigl[a_{r+1}(y,\tau)G\bigr],$$

where $L^*_{x,t}[w]$ is the differential form adjoint to $L_{x,t}[w]$ in (2), $\varphi = \varphi(y,\tau)$, and p and q are non-negative integers.

Using relations (16) and (17), we rewrite solution (15) in the form

$$\begin{aligned} w(x,t) = {} & \int_{x_1}^{x_2} f(y) G(x,y,t,0)\,dy + \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau) G(x,y,t,\tau)\,dy\,d\tau \\ & + \int_0^t \mathscr{L}[\varphi, G]\big|_{y=x_1}^{y=x_2}\,d\tau. \end{aligned} \tag{18}$$

This formula was derived taking into account the fact that the Green's function with respect to y and τ satisfies the adjoint equation*

$$\frac{\partial G}{\partial \tau} + L^*_{y,\tau}[G] = 0.$$

For subsequent analysis, it is convenient to represent the bilinear differential form $\mathscr{L}[\varphi, G]$ as

$$\mathscr{L}[\varphi, G] = \sum_{k=0}^{n-1} \frac{\partial^k \varphi}{\partial y^k} \Psi_k[G], \qquad \Psi_k[G] = \sum_{s=0}^{n-k-1} (-1)^s \frac{\partial^s}{\partial y^s}\bigl[a_{s+k+1}(y,\tau)G\bigr]. \tag{19}$$

Note that in the special case where the operator (2) is binomial,

$$L_{x,t}[w] = a_n \frac{\partial^n w}{\partial x^n} + a_0(x,t) w, \qquad a_n = \text{const},$$

the differential forms in (19) are written as

$$\mathscr{L}[\varphi, G] = a_n \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{\partial^k \varphi}{\partial y^k} \frac{\partial^{n-k-1} G}{\partial y^{n-k-1}}, \qquad \Psi_k[G] = a_n (-1)^{n-k-1} \frac{\partial^{n-k-1} G}{\partial y^{n-k-1}}.$$

► Nonhomogeneous boundary conditions of the special form.

Consider the following nonhomogeneous boundary conditions of special form that are often encountered in applications:

$$\begin{aligned} \frac{\partial^{k_m} w}{\partial x^{k_m}} &= g^{(1)}_{k_m}(t) \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s), \\ \frac{\partial^{k_m} w}{\partial x^{k_m}} &= g^{(2)}_{k_m}(t) \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n). \end{aligned} \tag{20}$$

*This equation can be derived by considering the case of homogeneous initial and boundary conditions and using arbitrariness in the choice of the test function $\varphi = \varphi(x,t)$; it should be taken into account that the solution itself must be independent of the specific form of φ, because φ does not occur in the original statement of the problem. By appropriately selecting the test function, one can also derive the boundary conditions (21).

Without loss of generality, we assume that the following inequalities hold:

$$n-1 \geq k_1 > k_2 > \cdots > k_s, \qquad n-1 \geq k_{s+1} > k_{s+2} > \cdots > k_n.$$

The Green's function satisfies the corresponding homogeneous boundary conditions that can be obtained from (20) by replacing w by G and by setting $g_{k_m}^{(1)}(t) = g_{k_m}^{(2)}(t) = 0$.

The homogeneous boundary conditions adjoint to (20), which must be satisfied by the Green's function with respect to y and τ, have the form

$$\begin{aligned} &\Psi_{k_\beta}[G] = 0 \quad \text{at} \quad x = x_1 \qquad (k_\beta \neq k_m,\ \beta = s+1,\dots,n;\ m = 1,\dots,s), \\ &\Psi_{k_\beta}[G] = 0 \quad \text{at} \quad x = x_2 \qquad (k_\beta \neq k_m,\ \beta = 1,\dots,s;\ m = s+1,\dots,n). \end{aligned} \tag{21}$$

These conditions involve the linear differential forms $\Psi_k[G]$ defined in (19). For each endpoint of the interval in question, the set $\{k_\beta\}$ of indices in the boundary operators (21) together with the set $\{k_m\}$ of the orders of derivatives in the boundary conditions (20) make up a complete set of nonnegative integers from 0 to $n-1$.

Taking into account the fact that the test function φ must satisfy the boundary conditions (20) and the Green's function G must satisfy conditions (21), we rewrite the solution (18) to obtain

$$\begin{aligned} w(x,t) &= \int_{x_1}^{x_2} f(y) G(x,y,t,0)\,dy + \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau) G(x,y,t,\tau)\,dy\,d\tau \\ &\quad - \sum_{m=1}^{s} \int_0^t g_{k_m}^{(1)}(\tau) \Psi_{k_m}[G]\big|_{y=x_1} d\tau + \sum_{m=s+1}^{n} \int_0^t g_{k_m}^{(2)}(\tau) \Psi_{k_m}[G]\big|_{y=x_2} d\tau, \end{aligned} \tag{22}$$

where the $\Psi_{k_m}[G]$ are differential operators with respect to y defined in (19).

If the Green's function is known, then formula (22) can be used to obtain the solution of the nonhomogeneous boundary value problem (1), (3), (20) readily for arbitrary $\Phi(x,t)$, $f(x)$, $g_{k_m}^{(1)}(t)$ ($m = 1,\dots,s$), and $g_{k_m}^{(2)}(t)$ ($m = s+1,\dots,n$).

► General nonhomogeneous boundary conditions.

On solving (4) for the highest derivatives, we reduce the boundary conditions (4) to the canonical form

$$\begin{aligned} &\frac{\partial^{k_m} w}{\partial x^{k_m}} + \sum_{i=0}^{k_m-1} c_{mi}^{(1)}(t) \frac{\partial^i w}{\partial x^i} = h_{k_m}^{(1)}(t) \quad \text{at} \quad x = x_1 \qquad (m = 1,\dots,s), \\ &\frac{\partial^{k_m} w}{\partial x^{k_m}} + \sum_{i=0}^{k_m-1} c_{mi}^{(2)}(t) \frac{\partial^i w}{\partial x^i} = h_{k_m}^{(2)}(t) \quad \text{at} \quad x = x_2 \qquad (m = s+1,\dots,n), \end{aligned} \tag{23}$$

where the leading terms in distinct boundary conditions are distinct,

$$n-1 \geq k_1 > k_2 > \cdots > k_s, \qquad n-1 \geq k_{s+1} > k_{s+2} > \cdots > k_n.$$

The sums in (23) do not contain the derivatives of orders $k_1, \ldots, k_s$ (for $x = x_1$) and $k_{s+1}, \ldots, k_n$ (for $x = x_2$); thus,

$$c^{(1)}_{mi}(t) = 0 \quad \text{at} \quad i = k_j \qquad (j = 1, \ldots, s),$$
$$c^{(2)}_{mi}(t) = 0 \quad \text{at} \quad i = k_j \qquad (j = s+1, \ldots, n).$$

It can be shown that the solution of problem (1), (3), (23) is given by

$$\begin{aligned} w(x,t) &= \int_{x_1}^{x_2} f(y)G(x,y,t,0)\,dy + \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau)G(x,y,t,\tau)\,dy\,d\tau \\ &- \sum_{m=1}^{s} \int_0^t h^{(1)}_{k_m}(\tau)\Psi_{k_m}[G]\big|_{y=x_1} d\tau + \sum_{m=s+1}^{n} \int_0^t h^{(2)}_{k_m}(\tau)\Psi_{k_m}[G]\big|_{y=x_2} d\tau, \end{aligned} \tag{24}$$

where the $\Psi_{k_m}[G]$ are the differential operators with respect to y defined in (19). Relation (24) is similar to (22) but contains the Green's function satisfying the more complicated boundary conditions that can be obtained from (23) by substituting G for w and by setting $h^{(1)}_{k_m}(t) = h^{(2)}_{k_m}(t) = 0$.

⊙ *Literature for Section 11.7.1*: A. G. Butkovskiy (1979, 1982), A. D. Polyanin (2000a, 2002).

11.7.2 Equations Containing the Second Time Derivative

► Homogeneous initial and boundary conditions.

Consider the linear nonhomogeneous differential equation

$$\frac{\partial^2 w}{\partial t^2} + \psi(x,t)\frac{\partial w}{\partial t} - \sum_{k=0}^{n} a_k(x,t)\frac{\partial^k w}{\partial x^k} = \Phi(x,t). \tag{1}$$

We set the homogeneous initial conditions

$$\begin{aligned} w &= 0 \quad \text{at} \quad t = 0, \\ \partial_t w &= 0 \quad \text{at} \quad t = 0 \end{aligned} \tag{2}$$

and the homogeneous boundary conditions

$$\begin{aligned} \Gamma^{(1)}_m[w] &= 0 \quad \text{at} \quad x = x_1 \qquad (m = 1, \ldots, s), \\ \Gamma^{(2)}_m[w] &= 0 \quad \text{at} \quad x = x_2 \qquad (m = s+1, \ldots, n), \end{aligned} \tag{3}$$

where the boundary operators $\Gamma^{(1)}_m[w]$ and $\Gamma^{(2)}_m[w]$ are defined in Section 11.7.1, see formulas (4).

The solution of problem (1)–(3) can be represented in the form*

$$w(x,t) = \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau)G(x,y,t,\tau)\,dy\,d\tau. \tag{4}$$

*Problem (1)–(3) is assumed to be well posed.

Here $G = G(x, y, t, \tau)$ is the Green's function; for $t > \tau \geq 0$, it satisfies the homogeneous equation

$$\frac{\partial^2 G}{\partial t^2} + \psi(x,t)\frac{\partial G}{\partial t} - \sum_{k=0}^{n} a_k(x,t)\frac{\partial^k G}{\partial x^k} = 0 \tag{5}$$

with the special semihomogeneous initial conditions

$$\begin{aligned} G &= 0 && \text{at} \quad t = \tau, \\ \partial_t G &= \delta(x-y) && \text{at} \quad t = \tau \end{aligned} \tag{6}$$

and the corresponding homogeneous boundary conditions

$$\begin{aligned} \Gamma_m^{(1)}[G] &= 0 \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s), \\ \Gamma_m^{(2)}[G] &= 0 \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n). \end{aligned} \tag{7}$$

The quantities y and τ appear in problem (5)–(7) as free parameters ($x_1 \leq y \leq x_2$), and $\delta(x)$ is the Dirac delta function.

One can verify formula (4) by a straightforward substitution into the equation and the initial and boundary conditions (1)–(3) and by taking into account properties (5)–(7) of the Green's function.

► Nonhomogeneous initial and boundary conditions.

Consider the linear nonhomogeneous differential equation (1) with the general nonhomogeneous initial conditions

$$\begin{aligned} w &= f_0(x) && \text{at} \quad t = 0, \\ \partial_t w &= f_1(x) && \text{at} \quad t = 0 \end{aligned} \tag{8}$$

and the nonhomogeneous boundary conditions reduced to the canonical form (the reduction of boundary conditions to the canonical form is described at the end of Section 9.7.1):

$$\begin{aligned} \frac{\partial^{k_m} w}{\partial x^{k_m}} + \sum_{i=0}^{k_m-1} c_{mi}^{(1)}(t)\frac{\partial^i w}{\partial x^i} &= h_{k_m}^{(1)}(t) \quad \text{at} \quad x = x_1 \qquad (m = 1, \dots, s), \\ \frac{\partial^{k_m} w}{\partial x^{k_m}} + \sum_{i=0}^{k_m-1} c_{mi}^{(2)}(t)\frac{\partial^i w}{\partial x^i} &= h_{k_m}^{(2)}(t) \quad \text{at} \quad x = x_2 \qquad (m = s+1, \dots, n). \end{aligned} \tag{9}$$

Introducing a test function $\varphi = \varphi(x,t)$ that satisfies the nonhomogeneous initial and boundary conditions (8), (9) and using the same line of reasoning as in Section 11.7.1 for a simpler equation, we arrive at the solution of problem (1), (8), (9) in the form

$$\begin{aligned} w(x,t) &= \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau) G(x,y,t,\tau)\,dy\,d\tau - \int_{x_1}^{x_2} f_0(y)\frac{\partial}{\partial \tau}\Big[G(x,y,t,\tau)\Big]_{\tau=0} dy \\ &\quad + \int_{x_1}^{x_2} \big[f_1(y) + f_0(y)\psi(y,0)\big] G(x,y,t,0)\,dy \\ &\quad - \sum_{m=1}^{s} \int_0^t h_{k_m}^{(1)}(\tau)\Psi_{k_m}[G]\big|_{y=x_1} d\tau + \sum_{m=s+1}^{n} \int_0^t h_{k_m}^{(2)}(\tau)\Psi_{k_m}[G]\big|_{y=x_2} d\tau, \end{aligned} \tag{10}$$

where the $\Psi_{k_m}[G]$ are differential operators with respect to y defined in Section 11.7.1; see relations (19).

Remark 11.7. If the coefficients of Eq. (1) and those of the boundary conditions (9) are time independent, i.e., if

$$\psi = \psi(x), \quad a_k = a_k(x), \quad c_{mi}^{(1)} = \text{const}, \quad c_{mi}^{(2)} = \text{const},$$

then in solution (10) one should set

$$G(x,y,t,\tau) = \widetilde{G}(x,y,t-\tau), \quad \frac{\partial}{\partial\tau}G(x,y,t,\tau)\Big|_{\tau=0} = -\frac{\partial}{\partial t}\widetilde{G}(x,y,t).$$

⊙ *Literature for Section 11.7.2*: A. G. Butkovskiy (1979, 1982), A. D. Polyanin (2000a, 2002).

11.7.3 Nonstationary Problems with Many Space Variables

► **Equations with the first partial derivative with respect to t.**

Consider the following linear differential operator with respect to variables $x_1, \dots, x_n$:

$$\mathfrak{L}_{\mathbf{x},t}[w] \equiv \sum A_{k_1,\dots,k_n}(x_1,\dots,x_n,t)\frac{\partial^{k_1+\cdots+k_n} w}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}. \tag{1}$$

The coefficients $A_{k_1,\dots,k_n}$ of the operator are assumed to be sufficiently smooth functions of $x_1,\dots,x_n$ and t (and also bounded if necessary). The coefficients of the highest derivatives are assumed to be everywhere nonzero.

1°. *Cauchy problem* ($t \ge 0$, $\mathbf{x} \in \mathbb{R}^n$). The solution of the Cauchy problem for the linear nonhomogeneous parabolic differential equation with variable coefficients

$$\frac{\partial w}{\partial t} - \mathfrak{L}_{\mathbf{x},t}[w] = \Phi(\mathbf{x},t) \tag{2}$$

under the initial conditions

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0 \tag{3}$$

is given by

$$w(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y},\tau)\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)\,dV_y\,d\tau + \int_{\mathbb{R}^n} f(\mathbf{y})\mathscr{E}(\mathbf{x},\mathbf{y},t,0)\,dV_y. \tag{4}$$

Here $dV_y = dy_1 \dots dy_n$ and $\mathscr{E} = \mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)$ is the fundamental solution of the Cauchy problem, which satisfies the equation

$$\frac{\partial\mathscr{E}}{\partial t} - \mathfrak{L}_{\mathbf{x},t}[\mathscr{E}] = 0 \tag{5}$$

for $t > \tau \ge 0$ and the special initial condition

$$\mathscr{E}\Big|_{t=\tau} = \delta(\mathbf{x} - y). \tag{6}$$

The variables $\mathbf{y}$ and τ appear in problem (5), (6) as free parameters ($\mathbf{y} \in \mathbb{R}^n$), and $\delta(\mathbf{x})$ is the n-dimensional Dirac delta function.

If the coefficients $A_{k_1,\dots,k_n}$ of operator (1) are independent of time t, then the fundamental solution only depends on three arguments, $\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau) = \mathscr{E}(\mathbf{x},\mathbf{y},t-\tau)$. If the coefficients of operator (1) are constants, then $\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau) = \mathscr{E}(\mathbf{x}-\mathbf{y},t-\tau)$.

2°. *Boundary value problems* ($t \ge 0$, $\mathbf{x} \in \mathcal{D}$). The solutions of linear boundary value problems in a spatial domain $\mathcal{D}$ for Eq. (2) with initial condition (3) and homogeneous boundary conditions for $\mathbf{x} \in \partial\mathcal{D}$ (these conditions are not written out here) are given by formula (4) in which the domain of integration $\mathbb{R}^n$ should be replaced by $\mathcal{D}$. Here by $\mathscr{E}$ we mean the Green's function that must satisfy, apart from Eq. (5) and the boundary condition (6), the same homogeneous boundary conditions for $\mathbf{x} \in \partial\mathcal{D}$ as the original equation (2). For boundary value problems, the parameter $\mathbf{y}$ belongs to the same domain as $\mathbf{x}$; i.e., $\mathbf{y} \in \mathcal{D}$.

⊙ *Literature*: Mathematical Encyclopedia (1977, Vol. 1).

► Equations with the second partial derivative with respect to t.

1°. *Cauchy problem* ($t \ge 0$, $\mathbf{x} \in \mathbb{R}^n$). The solution of the Cauchy problem for the linear nonhomogeneous differential equation with variable coefficients

$$\frac{\partial^2 w}{\partial t^2} - \mathfrak{L}_{\mathbf{x},t}[w] = \Phi(\mathbf{x}, t) \tag{7}$$

under the initial conditions

$$\begin{aligned} w &= f(\mathbf{x}) \quad \text{at} \quad t = 0, \\ \partial_t w &= g(\mathbf{x}) \quad \text{at} \quad t = 0 \end{aligned} \tag{8}$$

is given by

$$\begin{aligned} w(\mathbf{x}, t) = &\int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \tau)\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)\, dV_y\, d\tau \\ &- \int_{\mathbb{R}^n} f(\mathbf{y}) \left[\frac{\partial}{\partial \tau}\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)\right]_{\tau=0} dV_y + \int_{\mathbb{R}^n} g(\mathbf{y})\mathscr{E}(\mathbf{x}, \mathbf{y}, t, 0)\, dV_y. \end{aligned}$$

Here $\mathscr{E} = \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)$ is the fundamental solution of the Cauchy problem

$$\begin{gathered} \frac{\partial^2 \mathscr{E}}{\partial t^2} - \mathfrak{L}_{\mathbf{x},t}[\mathscr{E}] = 0, \\ \mathscr{E}\big|_{t=\tau} = 0, \qquad \frac{\partial \mathscr{E}}{\partial t}\bigg|_{t=\tau} = \delta(\mathbf{x} - y), \end{gathered}$$

where $\mathbf{y}$ and τ play the role of parameters.

If the coefficients $A_{k_1,\dots,k_n}$ of operator (1) are independent of time t, then the fundamental solution only depends on three arguments, $\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau) = \mathscr{E}(\mathbf{x}, \mathbf{y}, t - \tau)$, and the relation $\frac{\partial}{\partial \tau}\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)\big|_{\tau=0} = -\frac{\partial}{\partial t}\mathscr{E}(\mathbf{x}, \mathbf{y}, t)$ holds. If the coefficients of operator (1) are constant, then $\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau) = \mathscr{E}(\mathbf{x} - \mathbf{y}, t - \tau)$.

2°. The solution of the Cauchy problem for the more complicated linear nonhomogeneous differential equation with variable coefficients

$$\frac{\partial^2 w}{\partial t^2} + \psi(\mathbf{x}, t)\frac{\partial w}{\partial t} - \mathfrak{L}_{\mathbf{x},t}[w] = \Phi(\mathbf{x}, t)$$

with initial conditions (8) is expressed as

$$w(\mathbf{x},t)=\int_0^t\int_{\mathbb{R}^n}\Phi(\mathbf{y},\tau)\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)\,dV_y\,d\tau-\int_{\mathbb{R}^n}f(\mathbf{y})\left[\frac{\partial}{\partial\tau}\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)\right]_{\tau=0}dV_y \\ +\int_{\mathbb{R}^n}\big[g(\mathbf{y})+\psi(\mathbf{y},0)f(\mathbf{y})\big]\mathscr{E}(\mathbf{x},\mathbf{y},t,0)\,dV_y. \tag{9}$$

Here $\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)$ is the corresponding fundamental solution of the Cauchy problem

$$\frac{\partial^2\mathscr{E}}{\partial t^2}+\psi(\mathbf{x},t)\frac{\partial\mathscr{E}}{\partial t}-\mathfrak{L}_{\mathbf{x},t}[\mathscr{E}]=0,$$

$$\mathscr{E}\big|_{t=\tau}=0,\qquad \frac{\partial\mathscr{E}}{\partial t}\bigg|_{t=\tau}=\delta(\mathbf{x}-y).$$

3°. *Boundary value problems* ($t\ge 0$, $\mathbf{x}\in\mathcal{D}$). The solutions of linear boundary value problems in a spatial domain $\mathcal{D}$ for Eq. (7) with initial condition (8) and homogeneous boundary conditions for $\mathbf{x}\in\partial\mathcal{D}$ (these conditions are not written out here) are given by formula (9) in which the domain of integration $\mathbb{R}^n$ should be replaced by $\mathcal{D}$. Here by $\mathscr{E}$ we mean the Green's function that must satisfy, apart from Eq. (7) and the initial conditions (8), the same homogeneous boundary conditions as the solutions of the original equation (7).

⊙ *Literature for Section 11.7.3*: A. G. Butkovskiy (1979, 1982), A. D. Polyanin (2000a, 2002).

11.7.4 Some Special Equations with Variable Coefficients

1. $\displaystyle \frac{\partial w}{\partial t}=k(t)\frac{\partial^n w}{\partial x^n}+\big[xf(t)+g(t)\big]\frac{\partial w}{\partial x}+h(t)w.$

The transformation

$$w(x,t)=u(z,\tau)\exp\left[\int h(t)\,dt\right],\quad z=xF(t)+\int g(t)F(t)\,dt,\quad \tau=\int k(t)F^n(t)\,dt,$$

where $F(t)=\exp\left[\int f(t)\,dt\right]$, leads to the simpler constant coefficient equation

$$\frac{\partial u}{\partial\tau}=\frac{\partial^n u}{\partial z^n}.$$

Concerning the solutions of this equation, see equation 11.6.5.1 (for even n) and equation 11.6.5.2 (for odd n).

2. $\displaystyle \frac{\partial w}{\partial t}+\sum_{k=0}^{n}a_k(t)\frac{\partial^k w}{\partial x^k}=\Phi(x,t).$

1°. Particular solutions of the homogeneous equation with $\Phi(x,t)=0$:

$$w=\exp\big[\beta x-\lambda(t)\big],\qquad \lambda(t)=\int_{t_0}^t\left[\sum_{k=0}^n a_k(\tau)\beta^k\right]d\tau,$$

$$w=\exp\left[-\int_{t_0}^t p(\tau)\,d\tau\right]\cos\left[\beta x+\int_{t_0}^t q(\tau)\,d\tau\right],$$

$$w=\exp\left[-\int_{t_0}^t p(\tau)\,d\tau\right]\sin\left[\beta x+\int_{t_0}^t q(\tau)\,d\tau\right],$$

where β and t_0 are arbitrary constants and

$$p(t) = \sum_{k=0}^{2k\le n} (-1)^k a_{2k}(t)\beta^{2k}, \qquad q(t) = \sum_{k=0}^{2k+1\le n} (-1)^{k+1} a_{2k+1}(t)\beta^{2k+1}.$$

2°. Let the condition

$$P(\xi,t) = \sum_{k=0}^{2k\le n} (-1)^k a_{2k}(t)\xi^{2k} > 0$$

be satisfied for $\xi > 0$ and $t > 0$.

Fundamental solution:

$$\mathscr{E}(x,t,\tau) = \frac{1}{\pi}\int_0^\infty \exp\left[-\int_\tau^t P(\xi,\mu)\,d\mu\right]\cos\left[\int_\tau^t Q(\xi,\mu)\,d\mu + x\xi\right]d\xi,$$

$$Q(\xi,t) = \sum_{k=0}^{2k+1\le n} (-1)^{k+1} a_{2k+1}(t)\xi^{2k+1}.$$

3°. Domain: $-\infty < x < \infty$. Cauchy problem.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} \mathscr{E}(x-y,t,0)f(y)\,dy + \int_0^t\int_{-\infty}^{\infty} \mathscr{E}(x-y,t,\tau)\Phi(y,\tau)\,dy\,d\tau,$$

where the function $\mathscr{E}(x,t,\tau)$ is defined in Item 2°.

3. $\displaystyle \frac{\partial^k w}{\partial t^k} = ax^{2n}\frac{\partial^n w}{\partial x^n}.$

The transformation $z = 1/x$, $u = wx^{1-n}$ leads to the constant coefficient equation

$$\frac{\partial^k u}{\partial t^k} = a(-1)^n\frac{\partial^n u}{\partial z^n}.$$

4. $\displaystyle \frac{\partial^k w}{\partial t^k} = \sum_{m=0}^{n} a_m x^m\frac{\partial^m w}{\partial x^m}.$

The change of variable $z = \ln|x|$ leads to a constant coefficient equation.

5. $\displaystyle \frac{\partial^k w}{\partial t^k} = (ax^2 + bx + c)^n\frac{\partial^n w}{\partial x^n}.$

The transformation

$$w(x,t) = u(z,t)|ax^2 + bx + c|^{\frac{n-1}{2}}, \qquad z = \int\frac{dx}{ax^2 + bx + c}$$

leads to a constant coefficient equation.

6. $\left(\dfrac{\partial}{\partial t} - L_x\right)^n w = 0, \qquad n = 1,\ 2,\ \ldots$

Here L_x is a linear differential operator of any order with respect to the space variable x whose coefficients can depend on x.

1°. General solution:

$$w(x,t) = \sum_{k=0}^{n-1} t^k u_k(x,t),$$

where the $u_k = u_k(x,t)$ are arbitrary functions that satisfy the original equation with $n = 1$: $(\partial_t - L_x)u_k = 0$.

2°. Fundamental solution:

$$\mathscr{E}_n(x,t) = \frac{t^{n-1}}{(n-1)!}\mathscr{E}_1(x,t),$$

where $\mathscr{E}_1(x,t)$ is the fundamental solution of the equation with $n = 1$.

Remark 11.8. The linear differential operator L_x can involve arbitrarily many space variables.

7. $\left(\dfrac{\partial^2}{\partial t^2} - L_x\right)^n w = 0, \qquad n = 1,\ 2,\ \ldots$

Here L_x is a linear differential operator of any order with respect to the space variable x whose coefficients can depend on x.

1°. General solution:

$$w(x,t) = \sum_{k=0}^{n-1} t^k u_k(x,t),$$

where the $u_k = u_k(x,t)$ are arbitrary functions that satisfy the original equation with $n = 1$: $(\partial_{tt} - L_x)u_k = 0$.

2°. Suppose that the Cauchy problem for the special case of the equation with $n = 1$ is well posed if only one initial condition is set at $t = 0$; this means that the constant coefficient differential operator L_x is such that the equation with $n = 1$ is regular with regularity index $r = 1$. Then the fundamental solution of the original equation can be found by the formula

$$\mathscr{E}_n(x,t) = \frac{t^{n-1}}{(n-1)!}\mathscr{E}_1(x,t),$$

where $\mathscr{E}_1(x,t)$ is the fundamental solution for $n = 1$.

Remark 11.9. The linear differential operator L_x can involve arbitrarily many space variables.

8. $\displaystyle\sum_{k=0}^{m} a_k L^k[w] = 0.$

Here L is any linear differential operator that depends on arbitrarily many independent variables $x_1, \ldots, x_n$.

Solution:

$$w(x_1, \dots, x_n) = \sum_{s=1}^{m} C_s u_s(x_1, \dots, x_n),$$

where the u_s are solutions of the simpler equations $L[u_s] - \lambda_s u_s = 0$, the λ_s are roots of the characteristic equation $\sum\limits_{k=0}^{m} a_k \lambda^k = 0$, and the C_s are arbitrary constants.

Chapter 12

Systems of Linear Partial Differential Equations

12.1 Preliminary Remarks. Some Notation and Helpful Relations

Let f and $\mathbf{u} = u\,\mathbf{e}_1 + v\,\mathbf{e}_2 + w\,\mathbf{e}_3$ be arbitrary sufficiently smooth scalar and vector functions, and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the unit coordinate vectors corresponding to the Cartesian coordinates x, y, and z. Then, by definition, we have

$$\begin{aligned}
&\nabla f = f_x\mathbf{e}_1 + f_y\mathbf{e}_2 + f_z\mathbf{e}_3,\\
&\operatorname{div}\mathbf{u} = \nabla\cdot\mathbf{u} = u_x + v_y + w_z,\\
&\operatorname{curl}\mathbf{u} = \nabla\times\mathbf{u} = (w_y - v_z)\mathbf{e}_1 + (u_z - w_x)\mathbf{e}_2 + (v_x - u_y)\mathbf{e}_3,\\
&\Delta f = f_{xx} + f_{yy} + f_{zz},
\end{aligned}$$

where the subscripts x, y, and z stand for the derivatives. The following differential relations hold:

$$\begin{aligned}
&\operatorname{curl}\nabla f = 0, \quad \operatorname{div}\operatorname{curl}\,\mathbf{u} = 0, \quad \operatorname{div}\nabla f = \Delta f,\\
&\operatorname{curl}\,\operatorname{curl}\mathbf{u} = \nabla\operatorname{div}\mathbf{u} - \Delta\mathbf{u}, \quad \Delta(\mathbf{x}f) = \mathbf{x}\Delta f + 2\nabla f, \quad \mathbf{x} = (x, y, z),\\
&\operatorname{curl}[\Delta(\mathbf{x}f)] = \operatorname{curl}(\mathbf{x}\Delta f), \quad \Delta(\mathbf{x}\cdot\mathbf{u}) = \mathbf{x}\cdot(\Delta\mathbf{u}) + 2\operatorname{div}\mathbf{u},
\end{aligned}$$

which are often used in what follows in Sections 12.5–12.19.

12.2 Systems of Two First-Order Equations

1. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + f_1(t)u + g_1(t)w, \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial w}{\partial x} + f_2(t)u + g_2(t)w.$

Variable coefficient first-order linear system.

General solution:

$$\begin{aligned}
u &= \varphi_1(t)U(x + at) + \varphi_2(t)W(x + at),\\
w &= \psi_1(t)U(x + at) + \psi_2(t)W(x + at),
\end{aligned}$$

where $U = U(z)$ and $W = W(z)$ are arbitrary functions and the pairs of functions $\varphi_1 = \varphi_1(t)$, $\psi_1 = \psi_1(t)$ and $\varphi_2 = \varphi_2(t)$, $\psi_2 = \psi_2(t)$ are linearly independent (fundamental) solutions of the system of linear ordinary differential equations

$$\varphi'_t = f_1(t)\varphi + g_1(t)\psi,$$
$$\psi'_t = f_2(t)\varphi + g_2(t)\psi.$$

2. $\displaystyle \frac{\partial u}{\partial t} = a(t)\frac{\partial u}{\partial x} + f_1(t)u + g_1(t)w + h_1(t),$
$\displaystyle \frac{\partial w}{\partial t} = a(t)\frac{\partial w}{\partial x} + f_2(t)u + g_2(t)w + h_2(t).$

General solution:

$$u = \varphi_1(t)U(z) + \varphi_2(t)W(z) + u_0(t), \qquad z = x + \int a(t)\,dt,$$
$$w = \psi_1(t)U(z) + \psi_2(t)W(z) + w_0(t),$$

where $U = U(z)$ and $W = W(z)$ are arbitrary functions, the pairs of functions $\varphi_1 = \varphi_1(t)$, $\psi_1 = \psi_1(t)$ and $\varphi_2 = \varphi_2(t)$, $\psi_2 = \psi_2(t)$ are linearly independent (fundamental) solutions of the system of linear ordinary differential equations

$$\varphi'_t = f_1(t)\varphi + g_1(t)\psi,$$
$$\psi'_t = f_2(t)\varphi + g_2(t)\psi,$$

and $u_0 = u_0(t)$, $w_0 = w_0(t)$ is a solution of the system of linear ordinary differential equations

$$u'_0 = f_1(t)u_0 + g_1(t)w_0 + h_1(t),$$
$$w'_0 = f_2(t)u_0 + g_2(t)w_0 + h_2(t).$$

3. $\displaystyle C\frac{\partial u}{\partial t} + \frac{\partial w}{\partial x} + Au = 0, \qquad L\frac{\partial w}{\partial t} + \frac{\partial u}{\partial x} + Rw = 0.$

This system of equations occurs in the theory of propagation of quasistationary electric oscillations in cables. (Here u is the voltage, w is the current, L is the self-induction, R is the ohmic resistance, C is the capacitance, and A is the creepage.)

1°. Let us introduce an auxiliary function ψ by the formulas

$$u = \frac{\partial \psi}{\partial x}, \qquad w = -C\frac{\partial \psi}{\partial t} - A\psi.$$

Then the first equation in the system is satisfied identically, and the second equation is reduced to the second-order linear equation

$$CL\frac{\partial^2 \psi}{\partial t^2} + (AL + CR)\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} + AR\psi = 0. \tag{1}$$

The substitution

$$\psi = e^{-\mu t}\theta(x,t), \qquad \mu = \frac{AL + CR}{2CL} \tag{2}$$

reduces Eq. (1) to the simpler form

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{1}{CL}\frac{\partial^2 \theta}{\partial x^2} + \beta^2\theta, \qquad \beta = \frac{AL - CR}{2CL}.$$

For the solutions of this equation, see Section 3.1.3.

2°. With $AL = CR$, the general solution of the system is given by the formulas

$$u = e^{-\mu t}[\varphi_1(x - at) + \varphi_2(x + at)], \qquad a = (CL)^{-1/2},$$
$$w = be^{-\mu t}[\varphi_1(x - at) - \varphi_2(x + at)], \qquad b = (C/L)^{1/2},$$

where $\varphi_1(z_1)$ and $\varphi_2(z_2)$ are arbitrary functions and the coefficient μ is defined in (2).

⊙ *Literature*: V. I. Smirnov (1974, Vol. 2, pp. 571–577).

4. $\displaystyle \boldsymbol{a_1\frac{\partial u}{\partial t} + b_1\frac{\partial u}{\partial x} + c_1\frac{\partial w}{\partial t} + d_1\frac{\partial w}{\partial x} + p_1 u + q_1 w = 0,}$
$\displaystyle \boldsymbol{a_2\frac{\partial u}{\partial t} + b_2\frac{\partial u}{\partial x} + c_2\frac{\partial w}{\partial t} + d_2\frac{\partial w}{\partial x} + p_2 u + q_2 w = 0.}$

It is a system of constant coefficient first-order homogeneous linear PDEs.

1°. Set

$$u = c_1\frac{\partial \psi}{\partial t} + d_1\frac{\partial \psi}{\partial x} + q_1\psi, \qquad w = -a_1\frac{\partial \psi}{\partial t} - b_1\frac{\partial \psi}{\partial x} - p_1\psi. \tag{1}$$

By substituting (1) into the second equation of the system, we arrive at a second-order constant coefficient linear PDE for ψ,

$$\begin{aligned}(a_2c_1 - a_1c_2)\frac{\partial^2\psi}{\partial t^2} &+ (a_2d_1 - a_1d_2 + b_2c_1 - b_1c_2)\frac{\partial^2\psi}{\partial x\partial t} + (b_2d_1 - b_1d_2)\frac{\partial^2\psi}{\partial x^2}\\ &+ (a_2q_1 - a_1q_2 + c_1p_2 - c_2p_1)\frac{\partial\psi}{\partial t} + (b_2q_1 - b_1q_2 + d_1p_2 - d_2p_1)\frac{\partial\psi}{\partial x}\\ &+ (p_2q_1 - q_2p_1)\psi = 0.\end{aligned} \tag{2}$$

The substitution of the expressions (1) into the first equation of the system results in an identity.

Remark 12.1. A similar technique of reducing two linear equations to one equation can be used if, instead of the second equation, we deal with any other linear or nonlinear PDE of arbitrary order. (The first equation remains the same.)

If we set

$$u = c_2\frac{\partial \psi}{\partial t} + d_2\frac{\partial \psi}{\partial x} + q_2\psi, \qquad w = -a_2\frac{\partial \psi}{\partial t} - b_2\frac{\partial \psi}{\partial x} - p_2\psi,$$

then the first equation of the system gives Eq. (2), and the second equation gives an identity.

2°. Consider the nondegenerate case $\Delta = a_1d_1 - b_1c_1 \neq 0$. The transformation

$$U = \exp(\beta x + \lambda t)(a_1u + c_1w), \qquad u = \frac{1}{\Delta}\exp(-\beta x - \lambda t)(d_1U - c_1W),$$
$$W = \exp(\beta x + \lambda t)(b_1u + d_1w), \qquad w = \frac{1}{\Delta}\exp(-\beta x - \lambda t)(a_1W - b_1U),$$

where β and λ are a solution of the linear algebraic system

$$a_1\lambda + b_1\beta = p_1,$$
$$c_1\lambda + d_1\beta = q_1,$$

leads to the system

$$\frac{\partial U}{\partial t} + \frac{\partial W}{\partial x} = 0, \tag{3}$$

$$A\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} + C\frac{\partial W}{\partial t} + D\frac{\partial W}{\partial x} + PU + QW = 0. \tag{4}$$

Here Eq. (4) uses the notation

$$A = d_1a_2 - b_1c_2, \quad B = d_1b_2 - b_1d_2, \quad C = a_1c_2 - c_1a_2, \quad D = a_1d_2 - c_1b_2,$$
$$P = -A\lambda - B\beta + d_1p_2 - b_1q_2, \quad Q = -C\lambda - D\beta - c_1p_2 + a_1q_2.$$

Let us differentiate Eq. (4) with respect to x and then eliminate the derivative W_x with the help of the first equation. As a result, we obtain a second-order PDE for U,

$$C\frac{\partial^2 U}{\partial t^2} + (D - A)\frac{\partial^2 U}{\partial x \partial t} - B\frac{\partial^2 U}{\partial x^2} + Q\frac{\partial U}{\partial t} - P\frac{\partial U}{\partial x} = 0. \tag{5}$$

In a similar way, one can derive exactly the same second-order PDE for W.

We can satisfy Eq. (3) identically by setting $U = \partial\Psi/\partial x$ and $W = -\partial\Psi/\partial t$, where Ψ is a new unknown function (an analog of the stream function in hydrodynamic problems). Then Eq. (4) leads to the second-order constant coefficient linear partial differential equation considered in Section 14.1.1 (see Eq. 14.1.1.8).

3°. Consider the degenerate case $\Delta = a_1d_1 - b_1c_1 = 0$.

3.1. Let $a_1 \neq 0$ and $c_1 \neq 0$. Then one can set $b_1/a_1 = d_1/c_1 = k$. By passing from u and w to the new unknown functions u and $v = a_1u + c_1w$, we transform the first equation of the system to the form

$$\frac{\partial v}{\partial t} + k\frac{\partial v}{\partial x} + \frac{q_1}{c_1}v + \left(p_1 - \frac{a_1}{c_1}q_1\right)u = 0. \tag{6}$$

If $c_1p_1 - a_1q_1 \neq 0$, then, by eliminating u from (6) and from the transformed second equation of the system, we obtain a second-order constant coefficient linear partial differential equation for the function v. If $c_1p_1 - a_1g_1 = 0$, then Eq. (6) permits finding the function v immediately (because the equation is independent of u).

3.2. Let $a_1 = c_1 = 0$. By passing from u and w to the new unknown functions u and $v = b_1u + d_1w$, we reduce the first equation of the system to the form

$$\frac{\partial v}{\partial x} + \frac{q_1}{d_1}v + \left(p_1 - \frac{b_1}{d_1}q_1\right)u = 0. \tag{7}$$

If $d_1p_1 - b_1q_1 \neq 0$, then, by eliminating u from (7) and from the transformed second equation of the system, we obtain a second-order constant coefficient linear partial differential equation for the function v. If $d_1p_1 - b_1q_1 = 0$, then Eq. (7) permits finding the function v immediately (because the equation is independent of u).

3.3. Let $a_1 = b_1 = 0$ and $c_1 \neq 0$. For $p_1 \neq 0$, the simple elimination of u from the system leads to a second-order constant coefficient linear partial differential equation for the function w. For $p_1 = 0$, the first equation of the system permits finding the function w immediately.

3.4. Let $c_1 = d_1 = 0$ and $a_1 \neq 0$. For $q_1 \neq 0$, the simple elimination of w from the system leads to a second-order constant coefficient linear partial differential equation for the function u. For $q_1 = 0$, the first equation of the system permits finding the function u immediately.

12.3 Systems of Two Second-Order Equations

12.3.1 Systems of Parabolic Equations

1. $\displaystyle \frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b_1 u + c_1 w, \quad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b_2 u + c_2 w.$

Constant coefficient second-order linear system of parabolic type.

Solution:

$$u = \frac{b_1 - \lambda_2}{b_2(\lambda_1 - \lambda_2)} e^{\lambda_1 t}\theta_1 - \frac{b_1 - \lambda_1}{b_2(\lambda_1 - \lambda_2)} e^{\lambda_2 t}\theta_2,$$
$$w = \frac{1}{\lambda_1 - \lambda_2}\bigl(e^{\lambda_1 t}\theta_1 - e^{\lambda_2 t}\theta_2\bigr),$$

where λ_1 and λ_2 are roots of the quadratic equation

$$\lambda^2 - (b_1 + c_2)\lambda + b_1 c_2 - b_2 c_1 = 0$$

and the functions $\theta_n = \theta_n(x,t)$ satisfy the independent linear heat equations

$$\frac{\partial \theta_1}{\partial t} = a\frac{\partial^2 \theta_1}{\partial x^2}, \qquad \frac{\partial \theta_2}{\partial t} = a\frac{\partial^2 \theta_2}{\partial x^2}.$$

2. $\displaystyle \frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f_1(t)u + g_1(t)w, \quad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f_2(t)u + g_2(t)w.$

Variable coefficient second-order linear system of parabolic type.

Solution:

$$u = \varphi_1(t)U(x,t) + \varphi_2(t)W(x,t),$$
$$w = \psi_1(t)U(x,t) + \psi_2(t)W(x,t),$$

where the pairs of functions $\varphi_1 = \varphi_1(t)$, $\psi_1 = \psi_1(t)$ and $\varphi_2 = \varphi_2(t)$, $\psi_2 = \psi_2(t)$ are linearly independent (fundamental) solutions of the system of linear ordinary differential equations

$$\varphi'_t = f_1(t)\varphi + g_1(t)\psi,$$
$$\psi'_t = f_2(t)\varphi + g_2(t)\psi,$$

and the functions $U = U(x,t)$ and $W = W(x,t)$ satisfy the independent linear heat equations

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2}, \qquad \frac{\partial W}{\partial t} = a\frac{\partial^2 W}{\partial x^2}.$$

⊙ *Literature for Section 12.3.1*: A. D. Polyanin and A. V. Manzhirov (2007, pp. 1341–1342).

12.3.2 Systems of Hyperbolic or Elliptic Equations

1. $\dfrac{\partial^2 u}{\partial t^2} = k\dfrac{\partial^2 u}{\partial x^2} + a_1 u + b_1 w, \qquad \dfrac{\partial^2 w}{\partial t^2} = k\dfrac{\partial^2 w}{\partial x^2} + a_2 u + b_2 w.$

Constant coefficient second-order linear system of hyperbolic type.

Solution:

$$u = \frac{a_1 - \lambda_2}{a_2(\lambda_1 - \lambda_2)}\theta_1 - \frac{a_1 - \lambda_1}{a_2(\lambda_1 - \lambda_2)}\theta_2, \qquad w = \frac{1}{\lambda_1 - \lambda_2}\big(\theta_1 - \theta_2\big),$$

where λ_1 and λ_2 are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0$$

and the functions $\theta_n = \theta_n(x,t)$ satisfy the independent linear Klein–Gordon equations

$$\frac{\partial^2\theta_1}{\partial t^2} = k\frac{\partial^2\theta_1}{\partial x^2} + \lambda_1\theta_1, \qquad \frac{\partial^2\theta_2}{\partial t^2} = k\frac{\partial^2\theta_2}{\partial x^2} + \lambda_2\theta_2.$$

2. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = a_1 u + b_1 w, \qquad \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a_2 u + b_2 w.$

Constant coefficient second-order linear system of elliptic type.

Solution:

$$u = \frac{a_1 - \lambda_2}{a_2(\lambda_1 - \lambda_2)}\theta_1 - \frac{a_1 - \lambda_1}{a_2(\lambda_1 - \lambda_2)}\theta_2, \qquad w = \frac{1}{\lambda_1 - \lambda_2}\big(\theta_1 - \theta_2\big),$$

where λ_1 and λ_2 are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0$$

and the functions $\theta_n = \theta_n(x,y)$ satisfy the independent linear Helmholtz equations

$$\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_1}{\partial y^2} = \lambda_1\theta_1, \qquad \frac{\partial^2\theta_2}{\partial x^2} + \frac{\partial^2\theta_2}{\partial y^2} = \lambda_2\theta_2.$$

⊙ *Literature for Section 12.3.2*: A. D. Polyanin and A. V. Manzhirov (2007, p. 1342).

12.4 Systems of Two Higher-Order Equations

1. $\dfrac{\partial u}{\partial t} = L[u] + f_1(t)u + g_1(t)w, \qquad \dfrac{\partial w}{\partial t} = L[w] + f_2(t)u + g_2(t)w.$

Here L is an arbitrary linear differential operator with respect to the coordinates $x_1, \dots, x_n$ (of any order in derivatives) whose coefficients may depend on $x_1, \dots, x_n, t$. It is assumed that $L[\mathrm{const}] = 0$.

Solution:

$$u = \varphi_1(t)U(x_1, \dots, x_n, t) + \varphi_2(t)W(x_1, \dots, x_n, t),$$
$$w = \psi_1(t)U(x_1, \dots, x_n, t) + \psi_2(t)W(x_1, \dots, x_n, t),$$

where the two pairs of functions $\varphi_1 = \varphi_1(t)$, $\psi_1 = \psi_1(t)$ and $\varphi_2 = \varphi_2(t)$, $\psi_2 = \psi_2(t)$ are linearly independent (fundamental) solutions of the system of first-order linear ordinary differential equations

$$\varphi'_t = f_1(t)\varphi + g_1(t)\psi,$$
$$\psi'_t = f_2(t)\varphi + g_2(t)\psi$$

and the functions $U = U(x_1, \dots, x_n, t)$ and $W = W(x_1, \dots, x_n, t)$ satisfy the independent linear equations

$$\frac{\partial U}{\partial t} = L[U], \qquad \frac{\partial W}{\partial t} = L[W].$$

⊙ *Literature*: A. D. Polyanin and A. V. Manzhirov (2007, p. 1374).

2. $\boldsymbol{\dfrac{\partial^2 u}{\partial t^2} = L[u] + a_1 u + b_1 w, \qquad \dfrac{\partial^2 w}{\partial t^2} = L[w] + a_2 u + b_2 w.}$

Here L is an arbitrary linear differential operator with respect to the coordinates $x_1, \dots, x_n$ (of any order in derivatives).

Solution:

$$u = \frac{a_1 - \lambda_2}{a_2(\lambda_1 - \lambda_2)}\theta_1 - \frac{a_1 - \lambda_1}{a_2(\lambda_1 - \lambda_2)}\theta_2, \qquad w = \frac{1}{\lambda_1 - \lambda_2}(\theta_1 - \theta_2),$$

where λ_1 and λ_2 are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0$$

and the functions $\theta_n = \theta_n(x_1, \dots, x_n, t)$ satisfy the independent linear equations

$$\frac{\partial^2 \theta_1}{\partial t^2} = L[\theta_1] + \lambda_1\theta_1, \qquad \frac{\partial^2 \theta_2}{\partial t^2} = L[\theta_2] + \lambda_2\theta_2.$$

⊙ *Literature*: A. D. Polyanin and A. V. Manzhirov (2007, p. 1374).

3. $\boldsymbol{L[u] = a_1 u + b_1 w, \qquad L[w] = a_2 u + b_2 w.}$

Here L is an arbitrary linear differential operator with respect to the variables $x_1, \dots, x_n, t$ (of any order in derivatives).

Solution:

$$u = \frac{a_1 - \lambda_2}{a_2(\lambda_1 - \lambda_2)}\theta_1 - \frac{a_1 - \lambda_1}{a_2(\lambda_1 - \lambda_2)}\theta_2, \qquad w = \frac{1}{\lambda_1 - \lambda_2}(\theta_1 - \theta_2),$$

where λ_1 and λ_2 are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0$$

and the functions $\theta_n = \theta_n(x_1, \dots, x_n, t)$ satisfy the independent linear equations

$$L[\theta_1] = \lambda_1\theta_1, \qquad L[\theta_2] = \lambda_2\theta_2.$$

4. $\boldsymbol{L_1[u] + L_2[w] = 0, \quad M[u, w] = 0.}$

Here L_1 and L_2 are arbitrary constant coefficient linear differential operators with respect to the variables x and t (of any order in derivatives) and M is an arbitrary linear or nonlinear differential operator with respect to the variables x and t whose coefficients may depend on the independent variables.

One can identically satisfy the first equation by setting

$$u = L_2[\psi], \qquad w = -L_1[\psi], \tag{1}$$

where ψ is the new unknown function (an analog of the stream function in hydrodynamic problems). By substituting (1) into the second equation of the system, we obtain a equation for ψ,

$$M\bigl[L_2[\psi], -L_1[\psi]\bigr] = 0.$$

12.5 Simplest Systems Containing Vector Functions and Operators div and curl

12.5.1 Equation $\operatorname{curl} \mathbf{u} = \mathbf{A}(\mathbf{x})$

► **Homogeneous equation.**

Let $\mathbf{u} = (u, v, w)$ be a vector field, where u, v, w are its components depending on the rectangular Cartesian coordinates x, y, z. Then the homogeneous equation

$$\operatorname{curl} \mathbf{u} = \mathbf{0} \tag{1}$$

is equivalent to the system

$$w_y - v_z = 0, \qquad u_z - w_x = 0, \qquad v_x - u_y = 0 \tag{2}$$

of three coupled first-order linear equations for the components of $\mathbf{u}$.

The general solution of the vector equation (1) or system (2) is

$$\mathbf{u} = \nabla\varphi \qquad \text{or} \qquad u = \varphi_x, \quad v = \varphi_y, \quad w = \varphi_z,$$

where $\varphi = \varphi(x, y, z)$ is an arbitrary function.

► **Nonhomogeneous equation.**

Consider the nonhomogeneous vector equation

$$\operatorname{curl} \mathbf{u} = \mathbf{A}(\mathbf{x}), \tag{3}$$

where $\mathbf{A}(\mathbf{x}) = \bigl(a(\mathbf{x}), b(\mathbf{x}), c(\mathbf{x})\bigr)$ is a given vector function and $\mathbf{x} = (x, y, z)$.

The solvability condition for this equation has the form

$$\operatorname{div} \mathbf{A}(\mathbf{x}) \equiv a_x + b_y + c_z = 0. \tag{4}$$

The vector equation (3) is equivalent to the system

$$w_y - v_z = a(\mathbf{x}), \quad u_z - w_x = b(\mathbf{x}), \quad v_x - u_y = c(\mathbf{x}) \tag{5}$$

of three coupled first-order linear equations for the components of **u**.

Let condition (4) be satisfied. We seek a solution of system (5) in the form

$$u = \varphi_x, \quad v = \varphi_y + \eta, \quad w = \varphi_z + \zeta. \tag{6}$$

By substituting (6) into (5), we have

$$\eta_x = c(\mathbf{x}), \quad \zeta_x = -b(\mathbf{x}), \quad \zeta_y - \eta_z = a(\mathbf{x}). \tag{7}$$

The first two equations can be integrated in an elementary way. The resulting solutions are substituted into the third equation, and then we use condition (4). Finally, after simple transformations, we obtain

$$\eta = \int_{x_0}^{x} c(\bar{x}, y, z)\, d\bar{x}, \quad \zeta = -\int_{x_0}^{x} b(\bar{x}, y, z)\, d\bar{x} + \int_{y_0}^{y} a(x_0, \bar{y}, z)\, d\bar{y}, \tag{8}$$

where x_0 and y_0 are arbitrary constants lying in the domain of Eq. (3).

Formulas (6) and (8), where $\varphi = \varphi(x, y, z)$ is an arbitrary function, give the general solution of the nonhomogeneous vector equation (3) (or system (5)) under the solvability condition (4).

12.5.2 Simple Systems of Equations Containing Operators div and curl

► Homogeneous system.

Consider the homogeneous system of equations

$$\operatorname{div} \mathbf{u} = 0, \quad \operatorname{curl} \mathbf{u} = \mathbf{0}.$$

It is an *overdetermined system*, because it consists of four scalar equations for the three components of the vector **u**.

The general solution of the system has the form

$$\mathbf{u} = \nabla\varphi, \tag{9}$$

where $\varphi = \varphi(\mathbf{x})$ is an arbitrary *harmonic function* satisfying the Laplace equation

$$\Delta\varphi = 0.$$

► **Nonhomogeneous system I.**

Consider an overdetermined nonhomogeneous system of equations consisting of two coupled equations,

$$\operatorname{div}\mathbf{u} = f(\mathbf{x}), \qquad \operatorname{curl}\mathbf{u} = \mathbf{0}. \tag{10}$$

The general solution of system (10) has the form (9), where the function $\varphi = \varphi(\mathbf{x})$ satisfies the Poisson equation

$$\Delta\varphi = f(\mathbf{x}).$$

A particular solution of the equation in a bounded domain V has the form

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{f(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dV',$$

where $dV' = dx'\,dy'\,dz'$.

► **Nonhomogeneous system II.**

Consider an overdetermined nonhomogeneous system of equations consisting of two coupled equations,

$$\operatorname{div}\mathbf{u} = 0, \qquad \operatorname{curl}\mathbf{u} = \mathbf{A}(\mathbf{x}). \tag{11}$$

We assume that the vector function $\mathbf{A}(\mathbf{x})$ satisfies the solvability condition (4).

1°. We seek a solution of system (11) under condition (4) in the form (6), (8). As a result, for the function φ we obtain the Poisson equation

$$\Delta\varphi + g(\mathbf{x}) = 0,$$

where

$$g(\mathbf{x}) = \int_{x_0}^{x} c_y(\bar{x},y,z)\,d\bar{x} - \int_{x_0}^{x} b_z(\bar{x},y,z)\,d\bar{x} + \int_{y_0}^{y} a_z(x_0,\bar{y},z)\,d\bar{y}. \tag{12}$$

2°. A particular solution of system (11) under condition (4) in a bounded domain V has the form

$$\mathbf{u} = \operatorname{curl}\boldsymbol{\psi}, \qquad \boldsymbol{\psi}(\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{\mathbf{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dV'.$$

► **Nonhomogeneous system III (Helmholtz problem on the reconstruction of a vector field from its divergence and curl).**

Consider the overdetermined nonhomogeneous system of equations

$$\operatorname{div}\mathbf{u} = f(\mathbf{x}), \qquad \operatorname{curl}\mathbf{u} = \mathbf{A}(\mathbf{x}), \tag{13}$$

which generalizes systems (10) and (11). We assume that the solvability condition (4) is satisfied.

1°. We seek a solution of problem (13) in the form (6), (8). For the function φ, we obtain the Poisson equation

$$\Delta\varphi + g(\mathbf{x}) = f(\mathbf{x}), \tag{14}$$

where the function $g(\mathbf{x})$ is determined by formula (12).

2°. The solution of problem (13) can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\psi} + \nabla\theta,$$
$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{f(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dV', \qquad \boldsymbol{\psi}(\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{\mathbf{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dV',$$

where the function θ satisfies the Laplace equation, $\Delta\theta = 0$. The surface V may be unbounded if both integrals converge and decay as $|\mathbf{x}| \to \infty$ at least at the rate of $|\mathbf{x}|^{-(1+\varepsilon)}$, where $\varepsilon > 0$.

3°. Equations (13) are often supplemented with the boundary condition

$$(\mathbf{n}\cdot\mathbf{u}) = h(\mathbf{x}) \quad \text{at} \quad \mathbf{x} \in S,$$

where $\mathbf{n}$ is the normal vector on the boundary S of the domain V and $h(\mathbf{x})$ is a given function. For this problem to be solvable, the solvability condition (4) should be satisfied and, in the case of a bounded domain V, the additional condition $\int_V f(\mathbf{x})\,dV = \int_S h(\mathbf{x})\,dS$ should hold.

12.5.3 Two Representations of Vector Functions

► **Stokes–Helmholtz representation of a vector function.**

Each sufficiently smooth vector function $\mathbf{f} = \mathbf{f}(\mathbf{x})$ can be represented as the sum of a potential and a solenoidal vector (the *Stokes–Helmholtz decomposition*):

$$\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}. \tag{15}$$

In what follows, we assume that the vector $\mathbf{f}$ is given and we need to determine the scalar and vector functions γ and $\boldsymbol{\omega}$.

1°. The scalar function γ satisfies the Poisson equation

$$\Delta\gamma = \operatorname{div}\mathbf{f}, \tag{16}$$

which can be derived by applying the operator div to relation (15). It follows from (16) that the function γ is not uniquely determined. Assume that we have constructed a solution γ of Eq. (16). Then we have an equation of the form (3) for the vector $\boldsymbol{\omega}$,

$$\operatorname{curl}\boldsymbol{\omega} = \mathbf{f} - \nabla\gamma, \tag{17}$$

whose right-hand side is known and satisfies the solvability condition. The vector $\boldsymbol{\omega}$ can be found by formulas (6) and (8), where u, v, and w should be replaced by ω_1, ω_2, and ω_3, respectively, and $a(\mathbf{x})$, $b(\mathbf{x})$, and $c(\mathbf{x})$ are the components of the vector $\mathbf{f}-\nabla\gamma$. Formulas (6) include the arbitrary function φ, and hence the vector $\boldsymbol{\omega}$ is determined nonuniquely.

2°. The scalar and vector functions γ and $\boldsymbol{\omega}$ in (15), for example, can be sought in the form

$$\gamma = \operatorname{div} \mathbf{U}, \qquad \boldsymbol{\omega} = -\operatorname{curl} \mathbf{U}. \tag{18}$$

By substituting (18) into (15) and by taking into account the identity

$$\operatorname{curl}\operatorname{curl}\ \mathbf{U} = \nabla \operatorname{div} \mathbf{U} - \Delta \mathbf{U},$$

we obtain Poisson's vector equation for the *Lamé vector potential* **U**,

$$\Delta \mathbf{U} = \mathbf{f}. \tag{19}$$

The solutions of Eq. (19) permit using formulas (18) to find the functions γ and $\boldsymbol{\omega}$ determining the decomposition (15). A particular solution of Eq. (19) has the form

$$\mathbf{U} = -\frac{1}{4\pi} \int_V \frac{\mathbf{f}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, dV'.$$

The Stokes–Helmholtz representation (15) of a vector function contains four scalar functions γ, ω_1, ω_2, and ω_3.

► **Representation of a vector function via a potential and two stream functions.**

The vector function $\mathbf{f} = (f_1, f_2, f_3)$ can also be represented in the form

$$\begin{gathered} \mathbf{f} = \nabla\gamma + \mathbf{h}, \qquad \mathbf{h} = (h_1, h_2, h_3), \\ h_1 = \theta^{(1)}_y, \quad h_2 = -\theta^{(1)}_x + \theta^{(2)}_z, \quad h_3 = -\theta^{(2)}_y. \end{gathered} \tag{20}$$

Here the potential γ is the same as in (15) and the stream functions $\theta^{(1)}$ and $\theta^{(2)}$ can be expressed via the components of the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ occurring in the decomposition (15) as follows:

$$\theta^{(1)} = \omega_3 - \frac{\partial}{\partial z} \int_a^y \omega_2(x, \bar{y}, z)\, d\bar{y}, \qquad \theta^{(2)} = \omega_1 - \frac{\partial}{\partial x} \int_a^y \omega_2(x, \bar{y}, z)\, d\bar{y}, \tag{21}$$

where a is an arbitrary constant.

The representation (20)–(21) of the mass force contains three scalar functions γ, $\theta^{(1)}$, and $\theta^{(2)}$ (one less than the Stokes–Helmholtz representation (15)), and the vector **h** satisfies the incompressibility condition $\operatorname{div} \mathbf{h} = 0$.

Remark 12.2. The conditions under which the vector function $\mathbf{f} = \mathbf{f}(\mathbf{x})$ in a finite or infinite domain V admits the Stokes–Helmholtz representation (15), as well as related questions (concerning the determination and uniqueness of γ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, the smoothness of the boundary of V, etc.) are discussed in the literature cited below.

⊙ *Literature for Section 12.5*: G. G. Stokes (1849), H. Helmholtz (1858), P. M. Morse and H. Feshbach (1953, Section 1.5), M. E. Gurtin (1962), D. A. W. Pecknold (1971), A. C. Eringen and E. S. Suhubi (1975), L. Morino (1986), G. A. Korn and T. M. Korn (2000), G. B. Arfken and H. J. Weber (2005).

12.6 Elasticity Equations

12.6.1 Elasticity Equations in Various Coordinate Systems

► Vector forms of the elasticity equations.

The *linear elasticity equations* (in displacements) have the form

$$\rho \mathbf{u}_{tt} = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \rho \mathbf{f}, \tag{1}$$

where $\mathbf{u}$ is the *displacement vector*, t is time, ρ is the material density, μ is the *shear modulus* (or the *modulus of elasticity of the second kind*), λ is the *Lamé coefficient*, Δ is the Laplace operator, and $\mathbf{f}$ is the mass force.

The vector equation (1) is a short representation of the *elasticity equations*, which describe the motion of a linearly elastic isotropic body (medium) in the case of small strains.

The vector elasticity equation (1) is sometimes represented as follows:

$$\mathbf{u}_{tt} = c_2^2 \Delta \mathbf{u} + (c_1^2 - c_2^2)\nabla \operatorname{div} \mathbf{u} + \mathbf{f}, \tag{2}$$

where c_1 and c_2 are the *longitudinal* and *transverse wave velocities*, respectively, which are determined by the formulas

$$c_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \qquad c_2 = \left(\frac{\mu}{\rho}\right)^{1/2}. \tag{3}$$

Remark 12.3. The longitudinal waves, which propagate at the velocity c_1, are also known as *primary waves* (or *p-waves*), because these waves are recorded first in geophysics, while the transverse waves, which propagate at the velocity c_2, are also known as *secondary waves* (or *s-waves*).

In view of the identity $\nabla \operatorname{div} \mathbf{u} \equiv \Delta \mathbf{u} + \operatorname{curl}\,\operatorname{curl} \mathbf{u}$ and the formula $\operatorname{curl} \mathbf{u} \equiv \nabla \times \mathbf{u}$, Eqs. (1) and (2) can be represented in the form

$$\begin{aligned}
\rho \mathbf{u}_{tt} &= (\lambda + 2\mu)\nabla \operatorname{div} \mathbf{u} - \mu \operatorname{curl}\,\operatorname{curl} \mathbf{u} + \rho \mathbf{f},\\
\mathbf{u}_{tt} &= c_1^2 \nabla \operatorname{div} \mathbf{u} - c_2^2 \operatorname{curl}\,\operatorname{curl} \mathbf{u} + \mathbf{f}.
\end{aligned}$$

► Elasticity equations in rectangular Cartesian coordinates.

In the rectangular Cartesian coordinates (x, y, z) the elasticity equations (1) have the form

$$\begin{aligned}
\rho \frac{\partial^2 u_1}{\partial t^2} &= \mu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) + (\lambda + \mu)\frac{\partial}{\partial x} \operatorname{div} \mathbf{u} + \rho f_1,\\
\rho \frac{\partial^2 u_2}{\partial t^2} &= \mu\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}\right) + (\lambda + \mu)\frac{\partial}{\partial y} \operatorname{div} \mathbf{u} + \rho f_2,\\
\rho \frac{\partial^2 u_3}{\partial t^2} &= \mu\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial^2 u_3}{\partial z^2}\right) + (\lambda + \mu)\frac{\partial}{\partial z} \operatorname{div} \mathbf{u} + \rho f_3,\\
\operatorname{div} \mathbf{u} &\equiv \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z},
\end{aligned} \tag{4}$$

where u_1, u_2, u_3 are the displacement vector components and $f_1 = f_1(x,y,z,t)$, $f_2 = f_2(x,y,z,t)$, $f_3 = f_3(x,y,z,t)$ are the mass force components.

Equations (4) can be represented in the form

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial}{\partial x_j}S_{ij} + \rho f_i, \qquad j = 1,\,2,\,3,$$

where we have used the notation $x_1 = x$, $x_2 = y$, $x_3 = z$ and the stress tensor components S_{ij} are expressed via the displacement vector components as follows:

$$S_{ii} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_i}{\partial x_i}, \quad \operatorname{div}\mathbf{u} = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i};$$

$$S_{ij} = S_{ji} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for} \quad i \neq j.$$

► **Elasticity equations in the cylindrical coordinates.**

In the cylindrical coordinates $(r,\,\varphi,\,z)$ the elasticity equations (1) have the form

$$\rho\frac{\partial^2 u_r}{\partial t^2} = \mu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}\right) + (\lambda+\mu)\frac{\partial}{\partial r}\operatorname{div}\mathbf{u} + \rho f_r,$$

$$\rho\frac{\partial^2 u_\varphi}{\partial t^2} = \mu\left(\Delta u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}\right) + (\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial\varphi}\operatorname{div}\mathbf{u} + \rho f_\varphi,$$

$$\rho\frac{\partial^2 u_z}{\partial t^2} = \mu\Delta u_z + (\lambda+\mu)\frac{\partial}{\partial z}\operatorname{div}\mathbf{u} + \rho f_z,$$

$$\operatorname{div}\mathbf{u} \equiv \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r},$$

where u_r, u_φ, and u_z are the displacement vector components, $f_r = f_r(r,\varphi,z,t)$, $f_\varphi = f_\varphi(r,\varphi,z,t)$, and $f_z = f_z(r,\varphi,z,t)$ are the mass force components, and the Laplace operator Δ is given by the formula

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}.$$

► **Elasticity equations in the spherical coordinates.**

In the spherical coordinates $(R,\,\theta,\,\varphi)$ the elasticity equations (1) have the form

$$\rho\frac{\partial^2 u_R}{\partial t^2} = (\lambda+\mu)\frac{\partial}{\partial R}\operatorname{div}\mathbf{u} + \mu\Delta u_R - \frac{2\mu}{R^2}\left(\frac{\partial u_\theta}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + u_R + \cot\theta\, u_\theta\right) + \rho f_R,$$

$$\rho\frac{\partial^2 u_\theta}{\partial t^2} = (\lambda+\mu)\frac{1}{R}\frac{\partial}{\partial\theta}\operatorname{div}\mathbf{u} + \mu\Delta u_\theta + \frac{\mu}{R^2\sin^2\theta}\left(2\sin^2\theta\,\frac{\partial u_R}{\partial\theta} - 2\cos\theta\,\frac{\partial u_\varphi}{\partial\varphi} - u_\theta\right) + \rho f_\theta,$$

$$\rho\frac{\partial^2 u_\varphi}{\partial t^2} = (\lambda+\mu)\frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}\operatorname{div}\mathbf{u} + \mu\Delta u_\varphi + \frac{\mu}{R^2\sin^2\theta}\left(2\sin\theta\,\frac{\partial u_R}{\partial\varphi} + 2\cos\theta\,\frac{\partial u_\theta}{\partial\varphi} - u_\varphi\right) + \rho f_\varphi,$$

$$\operatorname{div}\mathbf{u} \equiv \frac{1}{R^2}\frac{\partial}{\partial R}\big(R^2 u_R\big) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta u_\theta\big) + \frac{1}{R\sin\theta}\frac{\partial u_\varphi}{\partial\varphi},$$

where u_R, u_θ, and u_φ are the displacement vector components and the Laplace operator Δ is given by the formula

$$\Delta \equiv \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

► **Invariant transformations.**

The elasticity equations (1) are invariant under the following transformations:

$$\mathbf{u} = \widetilde{\mathbf{u}} + \mathbf{a}_1 x + \mathbf{a}_2 y + \mathbf{a}_3 y + \mathbf{a}_4 t + \mathbf{b},$$
$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl}\boldsymbol{\Psi}^\circ,$$
$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\boldsymbol{\Phi},$$

where $\mathbf{a}_n$ and $\mathbf{b}$ are arbitrary constant vectors and the vector function $\boldsymbol{\Psi}^\circ$ and the scalar function Φ are arbitrary solutions of the wave equations

$$\boldsymbol{\Psi}^\circ_{tt} - c_2^2\Delta\boldsymbol{\Psi}^\circ = \mathbf{0},$$
$$\Phi_{tt} - c_1^2\Delta\Phi = 0.$$

12.6.2 Various Forms of Decompositions of Homogeneous Elasticity Equations with f = 0

► **Decomposition based on two stream functions.**

Any solution of the homogeneous elasticity equations (4) with $\mathbf{f} = \mathbf{0}$ can also be represented as

$$u_1 = \varphi_x + \psi^{(1)}_y, \quad u_2 = \varphi_y - \psi^{(1)}_x + \psi^{(2)}_z, \quad u_3 = \varphi_z - \psi^{(2)}_y, \tag{5}$$

where the functions $\varphi = \varphi(x,y,z)$, $\psi^{(1)} = \psi^{(1)}(x,y,z)$, and $\psi^{(2)} = \psi^{(2)}(x,y,z)$ are solutions of the wave equations

$$\varphi_{tt} - c_1^2\Delta\varphi = 0, \tag{6}$$

$$\psi^{(1)}_{tt} - c_2^2\Delta\psi^{(1)} = 0, \quad \psi^{(2)}_{tt} - c_2^2\Delta\psi^{(2)} = 0. \tag{7}$$

Remark 12.4. For $\varphi = 0$, formulas (5) satisfy the incompressibility condition

$$\operatorname{div}\mathbf{u} = (u_1)_x + (u_2)_y + (u_3)_z = 0,$$

and hence the functions $\psi^{(1)}$ and $\psi^{(2)}$ can be interpreted as stream functions.

► **Using the Stokes–Helmholtz representation of the displacement vector.**

Every solution of Eqs. (1) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \tag{8}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the vector wave equation

$$\boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} = \mathbf{0} \tag{9}$$

and the function φ is a solution of the scalar wave equation (6).

► **Cauchy–Kovalevskaya solution.**

Any solution of Eqs. (1) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \Box_1[\mathbf{w}] + (c_1^2 - c_2^2)\nabla \operatorname{div}\mathbf{w}, \tag{10}$$

where the vector function $\mathbf{w}$ satisfies the equation

$$\Box_2\Box_1[\mathbf{w}] = 0. \tag{11}$$

Here and in the following, the *d'Alembert operators* $\Box_1$ and $\Box_2$ are given by

$$\Box_1 \equiv \partial_t^2 - c_1^2\Delta, \qquad \Box_2 \equiv \partial_t^2 - c_2^2\Delta. \tag{12}$$

If $\mathbf{u}_t \not\equiv \mathbf{0}$, then the general solution of Eq. (10) can be represented as the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$\Box_1[\mathbf{w}_1] = \mathbf{0}, \qquad \Box_2[\mathbf{w}_2] = \mathbf{0}.$$

► **Chadwick–Trowbridge solution (toroidal–poloidal decomposition).**

Any solution of Eqs. (1) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{x}\xi) + \operatorname{curl}\operatorname{curl}(\mathbf{x}\eta), \qquad \mathbf{x} = (x, y, z),$$

where φ, ξ, and η are scalar functions satisfying the wave equations

$$\Box_1\varphi = 0, \qquad \Box_2\xi = 0, \qquad \Box_2\eta = 0. \tag{13}$$

► **Two other representations of solutions via three scalar functions.**

The solutions Eqs. (1) with $\mathbf{f} = \mathbf{0}$ can also be represented in the following two forms:

$$\begin{aligned} \mathbf{u} &= \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{b}\eta), \qquad && \mathbf{a}\cdot\mathbf{b} \neq 0,\\ \mathbf{u} &= \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{x}\eta), \qquad && |\mathbf{a}| \neq 0, \end{aligned}$$

where φ, ξ, and η are scalar functions satisfying the wave equations (13).

▶ **Steady-state elasticity equations. The biharmonic equation for the displacement vector.**

For the steady-state elasticity equations (1) with $\mathbf{u}_t = \mathbf{0}$, the displacement field $\mathbf{u}$ satisfies the biharmonic equation

$$\Delta\Delta\mathbf{u} = \mathbf{0}. \tag{14}$$

Remark 12.5. Equation (13) remains valid for the steady-state nonhomogeneous elasticity equations (1) if the mass force satisfies the conditions $\operatorname{div}\mathbf{f} = 0$ and $\operatorname{curl}\mathbf{f} = \mathbf{0}$. (This was proved by Cauchy in 1828.)

▶ **Steady-state elasticity equations. Slobodyanskii's solution.**

The solution of the steady-state elasticity equations (1) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \frac{2(2\mu+\lambda)}{\mu+\lambda}\mathbf{w} + (\mathbf{x}\cdot\nabla)\mathbf{w} - \mathbf{x}\operatorname{div}\mathbf{w},$$

where the vector function $\mathbf{w}$ satisfies the Laplace equation

$$\Delta\mathbf{w} = \mathbf{0}.$$

12.6.3 Various Forms of Decompositions for Nonhomogeneous Elasticity Equations

▶ **Lamé decomposition of the elasticity equations.**

Assume that the mass force in the elasticity equations (1) is represented as the sum of potential and solenoidal components (this is the Stokes–Helmholtz representation of the mass force; see Section 12.5.3),

$$\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}.$$

In this case, the displacement can be represented by formula (8), where the scalar function φ and the vector function $\boldsymbol{\Psi}$ satisfy the nonhomogeneous wave equations

$$\varphi_{tt} - c_1^2\Delta\varphi = \gamma,$$
$$\boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} = \boldsymbol{\omega}.$$

▶ **Cauchy–Kovalevskaya decomposition not requiring the force to split into components.**

Any solution of Eqs. (1) can be represented in the form (10), where the vector function $\mathbf{w}$ satisfies the equation

$$\Box_2\Box_1[\mathbf{w}] = \mathbf{f}.$$

Here the d'Alembert operators $\Box_1$ and $\Box_2$ are defined in (12).

► **Integro-differential representation of solutions.**

A fairly broad class of solutions of the elasticity vector equations (1) can be represented in the form

$$\mathbf{u} = \boldsymbol{\vartheta} + \frac{(c_1^2 - c_2^2)}{4\pi c_1^2}\nabla \operatorname{div} \int_{r \le c_1 t} \boldsymbol{\vartheta}\Big(\mathbf{x}', t - \frac{r}{c_1}\Big)\frac{dv'}{r}, \qquad r = |\mathbf{x}' - \mathbf{x}|,$$

where the integration is over the ball of radius $c_1 t$ centered at $\mathbf{x}$ and the vector function $\boldsymbol{\vartheta}$ satisfies the nonhomogeneous wave equation

$$\boldsymbol{\vartheta}_{tt} - c_2^2 \Delta \boldsymbol{\vartheta} = \mathbf{f}.$$

► **Papkovich–Neuber decomposition for the steady-state elasticity equations.**

Let $\mathbf{u}_t = \mathbf{0}$. Then the solution of the elasticity equations (1) can be represented in the form

$$\mathbf{u} = \mathbf{v} + \nabla\Big[\Phi - \frac{\lambda + \mu}{2(\lambda + 2\mu)}\mathbf{x} \cdot \mathbf{v}\Big], \tag{15}$$

where the vector function $\mathbf{v}$ and the scalar function Φ satisfy the Poisson equations

$$\mu \Delta \mathbf{v} = -\rho \mathbf{f}, \qquad \Delta \Phi = -\frac{\rho(\lambda + \mu)}{2\mu(\lambda + 2\mu)}\mathbf{x} \cdot \mathbf{f}. \tag{16}$$

Remark 12.6. The Papkovich–Neuber solution (15)–(16) admits a generalization and can be represented in the form

$$\mathbf{u} = \mathbf{v} + \nabla\big[\Phi - \mathbf{x} \cdot (\mathrm{Q}\mathbf{v})\big], \qquad \mathrm{Q} = \frac{\Delta \mathrm{H} + \mu + \lambda}{2(\lambda + 2\mu)},$$

where H is an arbitrary constant coefficient linear differential operator, the vector function $\mathbf{v}$ is a solution of the first equation in (16), and the scalar function Φ satisfies the Poisson equation

$$\Delta \Phi + \frac{\rho}{\mu}\mathbf{x} \cdot (\mathrm{Q}\mathbf{f}) + \frac{\rho}{\mu(\lambda + 2\mu)}\mathrm{H}[\operatorname{div} \mathbf{f}] = 0.$$

12.6.4 Cauchy Problem and Its Solution. Fundamental Solution Matrix

The 3D Cauchy problem for the elasticity equations (1) in the infinite domain R_3 is determined by the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}), \qquad \mathbf{u}_t|_{t=0} = \mathbf{u}_1(\mathbf{x}), \tag{17}$$

where $\mathbf{u}_0(\mathbf{x})$ and $\mathbf{u}_1(\mathbf{x})$ are given vector functions.

The main role in the construction of the solution of the Cauchy problem is played by the fundamental solutions of Eq. (1), which represent the displacements of an infinite elastic space subjected to a lumped force applied at a point $\mathbf{x}^0 \equiv (x_1^0, x_2^0, x_3^0)$ with magnitude $\delta(t)$, where $\delta(t)$ is the Dirac delta function.

If the force acts in the direction of the x_k-axis ($k = 1,\ 2,\ 3$), then the displacement components $u_j^{(k)}(x,t) = \Gamma_{kj}$ ($k, j = 1,\ 2,\ 3$) are given by the formulas

$$\Gamma_{kj}(\mathbf{x}-\mathbf{x}^0,t) = \frac{1}{4\pi\rho}\biggl[\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\frac{\delta(\zeta_1)}{c_1^2 r} + \biggl(\delta_{kj} - \frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\biggr)\frac{\delta(\zeta_2)}{c_2^2 r}$$
$$+\frac{r}{c_1}\theta(\zeta_1) - \frac{r}{c_2}\theta(\zeta_2) + \zeta_1\theta(\zeta_1) - \zeta_2\theta(\zeta_2)\biggr],$$

where c_1 and c_2 are the characteristic velocities (3) and we have used the notation

$$r = |\mathbf{x}-\mathbf{x}^0|, \quad \zeta_1 = t - \frac{r}{c_1}, \qquad \zeta_2 = t - \frac{r}{c_2},$$
$$\theta(\zeta) = \begin{cases} 1 & \text{if } \zeta \ge 0, \\ 0 & \text{if } \zeta < 0, \end{cases} \qquad \delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \ne j. \end{cases}$$

The solution of the Cauchy problem (1) with the initial conditions (17) can be represented via the symmetric fundamental solution matrix $\mathbf{\Gamma}(\mathbf{x}-\mathbf{x}^0,t) = \|\Gamma_{kj}(\mathbf{x}-\mathbf{x}^0,t)\|$ by the *Volterra formula*

$$\mathbf{u}(\mathbf{x}^0,t) = \frac{\partial}{\partial t}\int_{r\le c_1 t}\mathbf{\Gamma}(\mathbf{x}^0-\mathbf{x},t)\mathbf{u}_0(\mathbf{x})\,d\mathbf{x} + \int_{r\le c_1 t}\mathbf{\Gamma}(\mathbf{x}^0-\mathbf{x},t)\mathbf{u}_1(\mathbf{x})\,d\mathbf{x}$$
$$+\int_0^t\int_{r\le c_1(t-\tau)}\mathbf{\Gamma}(\mathbf{x}^0-\mathbf{x},t-\tau)\mathbf{f}(\mathbf{x},\tau)\,d\mathbf{x}\,d\tau,$$

where we have used the notation $d\mathbf{x} = dx_1\,dx_2\,dx_3$.

⊙ *Literature for Section 12.6*: C.-L.-M.-H. Navier (1821), A.-L. Cauchy (1828), S. Kovalevski (1885), C. Somigliana (1889), B. Galerkin (1930), P. F. Papkovich (1932), U. Neuber (1934), M. G. Slobodianskii (1959), P. Chadwick and E. A. Trowbridge (1967), M. E. Gurtin (1972), W. Nowacki (1975), A. C. Eringen and E. S. Suhubi (1975), Mathematical Encyclopedia (1979, pp. 149–154), D. S. Chandrasechariah (1988), A. D. Polyanin and S. A. Lychev (2014a).

12.7 Stokes Equations for Viscous Incompressible Fluids

12.7.1 Stokes Equations in Various Coordinate Systems

► **Vector form of the Stokes equations.**

The closed system of equations for slow (creeping) flows of a *viscous incompressible Newtonian fluid* has the form

$$\mathbf{u}_t = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}, \tag{1}$$
$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity, t is time, ν is the kinematic viscosity of the fluid, p is the ratio of the pressure to the fluid density, $\mathbf{f} = (f_1, f_2, f_3)$ is the mass force (e.g., the gravitational force), ∇ is the gradient operator, and Δ is the Laplace operator.

The vector equation (1) is a concise representation of the *Stokes equations*, and Eq. (2) is the *continuity equation*. In what follows, for brevity, system (1)–(2) will be referred to as the Stokes equations (without distinguishing the continuity equation), and *the ratio p of the pressure to the fluid density will be referred to as the pressure*.

To derive the Stokes equations, one linearizes the Navier–Stokes equations using the order-of-magnitude relations

$$|\mathbf{u}| \sim \varepsilon, \quad |p - p_0| \sim \varepsilon, \quad |\mathbf{f}| \sim \varepsilon,$$

where ε is a small parameter.

We apply the operator div to Eq. (1), take into account (2), and obtain an equation for the pressure,

$$\Delta p = \operatorname{div} \mathbf{f}.$$

► Invariant transformations.

The Stokes equations (1)–(2) are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl} \mathbf{\Psi}^\circ, \qquad p = \widetilde{p},$$

where $\mathbf{\Psi}^\circ$ is an arbitrary solution of the heat equation $\mathbf{\Psi}^\circ_t - \nu\Delta\mathbf{\Psi}^\circ = \mathbf{0}$.

The Stokes equations (1)–(2) are also invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi, \qquad p = \widetilde{p} - \Phi_t + p_0(t),$$

where $p_0(t)$ is an arbitrary function and $\Phi = \Phi(\mathbf{x}, t)$ is an arbitrary solution of the Laplace equation $\Delta\Phi = 0$.

► Problems for the Stokes equations.

Cauchy problem. One seeks a solution of Eqs. (1)–(2) in $\mathbb{R}_3$ for $t > 0$ with the initial condition

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}), \tag{3}$$

where $\mathbf{u}_0(\mathbf{x})$ is a given vector function.

Interior initial-boundary value problem. To describe the motion of a fluid in an immovable bounded domain V, one supplements the initial condition (3) with the no-slip condition on the boundary S of the domain,

$$\mathbf{u}|_{\mathbf{x}\in S} = \mathbf{0}. \tag{4}$$

Exterior initial-boundary value problems. In the problem on the flow, homogeneous at infinity with velocity $\mathbf{a}$, past an immovable bounded body, one supplements the initial condition (3) with the boundary conditions

$$\mathbf{u}|_{\mathbf{x}\in S} = \mathbf{0}, \qquad \mathbf{u}|_{|x|\to\infty} \to \mathbf{a}, \tag{5}$$

where S is the surface of the body.

In the problem on the motion of a bounded body at a constant velocity **a** in an immovable fluid, the initial condition (3) is supplemented with the boundary conditions

$$\mathbf{u}|_{\mathbf{x}\in S} = \mathbf{a}, \qquad \mathbf{u}\big|_{|x|\to\infty} \to \mathbf{0}. \tag{6}$$

We point out that no conditions on the pressure p are needed in the above-stated problems for the Stokes equations.

► Stokes equations in the rectangular Cartesian coordinates.

In the rectangular Cartesian coordinates (x, y, z), the Stokes equations (1)–(2) have the form

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) + f_1,\\
\frac{\partial u_2}{\partial t} &= -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}\right) + f_2,\\
\frac{\partial u_3}{\partial t} &= -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial^2 u_3}{\partial z^2}\right) + f_3,\\
\frac{\partial u_1}{\partial x} &+ \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0,
\end{aligned} \tag{7}$$

where u_1, u_2, u_3 are the fluid velocity components.

Remark 12.7. For $f_1 = f_2 = f_3 = 0$, the scaling transformation

$$x = b\bar{x}, \quad y = b\bar{y}, \quad z = b\bar{z}, \quad t = \frac{b^2}{\nu}\bar{t}, \quad u_1 = a\bar{u}_1, \quad u_2 = a\bar{u}_2, \quad u_3 = a\bar{u}_3, \quad p = \frac{a\nu}{b}\bar{p},$$

where a and b are arbitrary (nonzero) constants, reduces system (7) to the same form with $\nu = 1$.

► Stokes equations in the cylindrical coordinates.

The cylindrical coordinates (r, φ, z) are related to the rectangular Cartesian coordinates (x, y, z) by

$$\begin{gathered}
r = \sqrt{x^2 + y^2}, \quad \tan\varphi = y/x, \quad z = z \quad (\sin\varphi = y/r);\\
x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z,
\end{gathered}$$

where $0 \le \varphi \le 2\pi$.

The fluid velocity components in the cylindrical coordinates are expressed in terms of those in the rectangular Cartesian coordinates as follows:

$$u_r = u_1\cos\varphi + u_2\sin\varphi, \quad u_\varphi = u_2\cos\varphi - u_1\sin\varphi, \quad u_z = u_3.$$

Remark 12.8. For $u_z = 0$, the cylindrical coordinates r and φ are also used as the polar coordinates on the xy-plane.

The Stokes equations (1)–(2) in the cylindrical coordinates become

$$\begin{aligned}
\frac{\partial u_r}{\partial t} &= -\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\right) + f_r,\\
\frac{\partial u_\varphi}{\partial t} &= -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \nu\left(\Delta u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi}\right) + f_\varphi,\\
\frac{\partial u_z}{\partial t} &= -\frac{\partial p}{\partial z} + \nu\Delta u_z + f_z,\\
\frac{\partial u_r}{\partial r} &+ \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0,
\end{aligned} \tag{8}$$

where u_r, u_φ, u_z are the fluid velocity components and the Laplace operator Δ is given by the formula

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

► **Stokes equations in the spherical coordinates.**

The spherical coordinates (R, θ, φ) are related to the rectangular Cartesian coordinates (x, y, z) by

$$R = \sqrt{x^2+y^2+z^2}, \quad \theta = \arccos\frac{z}{R}, \quad \tan\varphi = \frac{y}{x} \quad \left(\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}\right);$$
$$x = R\sin\theta\cos\varphi, \quad y = R\sin\theta\sin\varphi, \quad z = R\cos\theta,$$

where $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$.

The Stokes equations (1)–(2) in the spherical coordinates become

$$\begin{aligned}
\frac{\partial u_R}{\partial t} &= -\frac{\partial p}{\partial R} + \nu\Delta u_R - \frac{2\nu}{R^2}\left(\frac{\partial u_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + u_R + \cot\theta\, u_\theta\right) + f_R,\\
\frac{\partial u_\theta}{\partial t} &= -\frac{1}{R}\frac{\partial p}{\partial \theta} + \nu\Delta u_\theta + \frac{\nu}{R^2\sin^2\theta}\left(2\sin^2\theta\frac{\partial u_R}{\partial \theta} - 2\cos\theta\frac{\partial u_\varphi}{\partial \varphi} - u_\theta\right) + f_\theta,\\
\frac{\partial u_\varphi}{\partial t} &= -\frac{1}{R\sin\theta}\frac{\partial p}{\partial \varphi} + \nu\Delta u_\varphi + \frac{\nu}{R^2\sin^2\theta}\left(2\sin\theta\frac{\partial u_R}{\partial \varphi} + 2\cos\theta\frac{\partial u_\theta}{\partial \varphi} - u_\varphi\right) + f_\varphi,\\
&\frac{\partial}{\partial R}\left(R^2\sin\theta u_R\right) + \frac{\partial}{\partial \theta}\left(R\sin\theta u_\theta\right) + \frac{\partial}{\partial \varphi}\left(R u_\varphi\right) = 0,
\end{aligned}$$

where u_R, u_θ, u_φ are the fluid velocity components and the Laplace operator Δ is given by the formula

$$\Delta \equiv \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}.$$

12.7.2 Various Forms of Decompositions for the Stokes Equations with $\mathbf{f} = \mathbf{0}$

► **Decomposition based on two stream functions.**

Each solution of Eqs. (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = p_0 - \varphi_t,\\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\psi^{(2)}_y,\end{gathered} \tag{9}$$

where $p_0 = p_0(t)$ is an arbitrary function, $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are arbitrary solutions of the heat equation

$$\psi_t - \nu\Delta\psi = 0, \tag{10}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the Laplace equation

$$\Delta\varphi = 0. \tag{11}$$

Remark 12.9. The functions $\psi^{(1)}$ and $\psi^{(2)}$ in (9) can be interpreted as two stream functions permitting one to eliminate the continuity equation from the 3D incompressible fluid equations (where the fluid velocity components are denoted by v_1, v_2, v_3). In the special case of $\psi^{(2)} = 0$, we obtain the usual representation of the fluid velocity components for 2D planar flows with $v_3 = 0$ and with one stream function.

Remark 12.10. The Laplace equation (11) does not include time t explicitly, but the arbitrary constants C_n occurring in the solution of this steady-state equation may depend on t arbitrarily.

► **Using the Stokes–Helmholtz representation of the fluid velocity.**

Each solution of Eqs. (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad p = p_0 - \varphi_t, \tag{12}$$

where $p_0 = p_0(t)$ is an arbitrary function, $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is an arbitrary solution of the vector heat equation

$$\boldsymbol{\Psi}_t - \nu\Delta\boldsymbol{\Psi} = \mathbf{0}, \tag{13}$$

and the function φ is a solution of the Laplace equation (11).

► **Special types of decomposition using representations of the vector u via three scalar functions.**

Table 12.1 describes various decompositions of the Stokes equations (1)–(2) with $\mathbf{f} = \mathbf{0}$ on the basis of a representation of solutions by formulas of the form (12),

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}_n, \quad p = p_0 - \varphi_t, \tag{14}$$

where $p_0 = p_0(t)$ is an arbitrary function and the vector function $\boldsymbol{\Psi}_n$ ($n = 1, \ldots, 7$) can be expressed in a special way via two scalar functions $\psi^{(1)}$ and $\psi^{(2)}$. To ensure completeness

Table 12.1: Various decompositions of the Stokes equations (1)–(2) with $\mathbf{f} = \mathbf{0}$ based on the representation (14) of the solutions

n	Form of vector function $\mathbf{\Psi}_n$	Order of decomposition	Remarks
1	$\mathbf{e}_1\psi^{(2)} + \mathbf{e}_3\psi^{(1)}$	First	Is equivalent to the representation (9)
2	$\mathbf{a}\psi^{(1)} + \mathbf{b}\psi^{(2)}$	First	$\mathbf{a}\cdot\mathbf{b} \neq 0$
3	$\mathbf{a}\psi^{(1)} + \operatorname{curl}(\mathbf{b}\psi^{(2)})$	Second	$\mathbf{a}\cdot\mathbf{b} \neq 0$
4	$\mathbf{a}\psi^{(1)} + \mathbf{x}\psi^{(2)}$	First	$\lvert\mathbf{a}\rvert \neq 0$
5	$\mathbf{a}\psi^{(1)} + \operatorname{curl}(\mathbf{x}\psi^{(2)})$	Second	$\lvert\mathbf{a}\rvert \neq 0$
6	$\mathbf{x}\psi^{(1)} + \operatorname{curl}(\mathbf{a}\psi^{(2)})$	Second	$\lvert\mathbf{a}\rvert \neq 0$
7	$\mathbf{x}\psi^{(1)} + \operatorname{curl}(\mathbf{x}\psi^{(2)})$	Second	No remark

and clarity, Table 12.1 also includes the decomposition determined by formulas (9) and Eqs. (10)–(11). We use the following notation: $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors in a Cartesian coordinate system, and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors.

For all solutions of the Stokes equations (1)–(2) based on formula (14) and the seven distinct representations of the vector function $\mathbf{\Psi}_n$ in Table 12.1, the functions $\psi^{(1)} = \psi^{(1)}(\mathbf{x},t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x},t)$ are arbitrary solutions of the heat equation (10), and the function φ satisfies the Laplace equation (11).

Note that all above-described special representations of solutions determined by formula (14) and Table 12.1 contain one unknown function less than the Stokes–Helmholtz representation (12).

► **Steady-state Stokes equations. Solution of Slobodyanskii type.**

The solution of the steady-state Stokes equations (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = 2\mathbf{w} + (\mathbf{x}\cdot\nabla)\mathbf{w} - \mathbf{x}\operatorname{div}\mathbf{w}, \qquad p = p_0 - 2\nu\rho\operatorname{div}\mathbf{w},$$

where p_0 is an arbitrary constant and the vector function $\mathbf{w}$ satisfies the Laplace equation

$$\Delta\mathbf{w} = \mathbf{0}.$$

12.7.3 Various Forms of Decompositions for the Stokes Equations with f $\neq$ 0

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force in the Stokes equations (1)–(2) be written as the sum of potential and solenoidal components (see Section 12.5.3)

$$\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}. \tag{15}$$

Then the solution of system (1)–(2) can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\mathbf{\Psi}, \qquad p = p_0 - \varphi_t + \gamma, \tag{16}$$

where $p_0 = p_0(t)$ is an arbitrary function and the functions φ and $\boldsymbol{\Psi}$ satisfy the independent equations

$$\begin{gathered}\boldsymbol{\Psi}_t - \nu\Delta\boldsymbol{\Psi} = \boldsymbol{\omega},\\ \Delta\varphi = 0.\end{gathered} \tag{17}$$

► **Decomposition not requiring the mass force to split into components.**

The solution of the Stokes equations (1)–(2) can be represented in the form

$$\mathbf{u} = -\operatorname{curl}\operatorname{curl}\mathbf{U} \equiv \Delta\mathbf{U} - \nabla\operatorname{div}\mathbf{U}, \qquad p = p_0(t) + \operatorname{div}(\mathbf{U}_t - \nu\Delta\mathbf{U}),$$

where $p_0(t)$ is an arbitrary function and the vector function $\mathbf{U}$ satisfies the equation

$$\Delta(\partial_t - \nu\Delta)[\mathbf{U}] = \mathbf{f}.$$

► **Papkovich–Neuber decomposition for the steady-state Stokes equations.**

Let $\mathbf{u}_t = \mathbf{0}$. Then the solution of the Stokes equations (1)–(2) can be represented in the form

$$\mathbf{u} = \mathbf{v} + \nabla\Big(\Phi - \frac{1}{2}\mathbf{x}\cdot\mathbf{v}\Big), \qquad p = p_0 - \nu\rho\operatorname{div}\mathbf{v}, \tag{17a}$$

where p_0 is an arbitrary constant and the vector function $\mathbf{v}$ and the scalar function Φ satisfy the Poisson equations

$$\nu\Delta\mathbf{v} = -\mathbf{f}, \qquad \nu\Delta\Phi = -\frac{1}{2}\mathbf{x}\cdot\mathbf{f}. \tag{17b}$$

Remark 12.11. The solution (17a)–(17b) admits a generalization and can be represented in the form

$$\mathbf{u} = \mathbf{v} + \nabla\big[\Phi + \mathbf{x}\cdot(\mathrm{Q}\mathbf{v})\big], \qquad p = p_0 - \nu\rho\operatorname{div}\mathbf{v}, \qquad \mathrm{Q} = \Delta\mathrm{H} - \frac{1}{2},$$

where H is an arbitrary constant coefficient linear differential operator, the vector function $\mathbf{v}$ is a solution of the first equation in (17b), and the scalar function Φ satisfies the Poisson equation

$$\nu\Delta\Phi = \mathbf{x}\cdot(\mathrm{Q}\mathbf{f}) + 2\mathrm{H}[\operatorname{div}\mathbf{f}].$$

► **Incomplete asymmetric decomposition of the Stokes equations.**

Each solution of system (1)–(2) with $\mathbf{f} = (f_1, f_2, f_3)$ can be represented in the form

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, \qquad p = -\varphi_t + \nu\Delta\varphi + F,\\ F = F(\mathbf{x}, t) = F(x, y, z, t) = \int_0^x f_1(x_1, y, z, t)\,dx_1,\end{gathered} \tag{18}$$

where $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors of the Cartesian coordinate y- and z-axes, the scalar functions $v_2 = v_2(\mathbf{x}, t)$ and $v_3 = v_3(\mathbf{x}, t)$ satisfy the two independent linear nonhomogeneous heat equations

$$(v_2)_t - \nu\Delta v_2 = f_2 - F_y, \qquad (v_3)_t - \nu\Delta v_3 = f_3 - F_z, \tag{19}$$

and the function φ is related to v_2 and v_3 by the equation

$$\Delta\varphi + (v_2)_y + (v_3)_z = 0. \tag{20}$$

► **Incomplete symmetric decomposition of the Stokes equations.**

Every solution of system (1)–(2) can also be represented in the symmetric form

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = -\varphi_t + \nu\Delta\varphi + G, \tag{21}$$

where the vector function $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the independent linear nonhomogeneous equation

$$\mathbf{v}_t - \Delta\mathbf{v} = \mathbf{f} - \nabla G \tag{22}$$

and the functions φ and $\mathbf{v}$ are related by the single equation

$$\Delta\varphi + \operatorname{div}\mathbf{v} = 0. \tag{23}$$

Formulas (21) and Eqs. (22) include an arbitrary scalar function $G = G(\mathbf{x}, t)$.

By setting $v_1 = 0$ and $G = F$ in (21)–(23), we obtain an asymmetric representation of the solutions (18)–(20).

12.7.4 General Solution of the Steady-State Homogeneous Stokes Equations

The general solution of the steady-state homogeneous Stokes equations (1)–(2) with $\mathbf{u}_t \equiv 0$ and $\mathbf{f} = \mathbf{0}$ is given by the formulas

$$\mathbf{u} = \sum_{n=-\infty}^{\infty}\Big[\nabla\varphi_n + \nabla\times(\mathbf{x}\psi_n)\Big] + \frac{1}{\nu}\sum_{n=-\infty,\,n\neq-1}^{\infty}\bigg[\frac{(n+3)}{2(n+1)(2n+3)}|\mathbf{x}|^2\nabla p_n - \frac{n}{(n+1)(2n+3)}\mathbf{x}p_n\bigg],$$

$$p = \sum p_n, \qquad \mathbf{x} = (x, y, z), \qquad |\mathbf{x}| = R = \sqrt{x^2 + y^2 + z^2},$$

where φ_n, ψ_n, and p_n are arbitrary spherical harmonics of order n, which satisfy the Laplace equations

$$\Delta\varphi_n = 0, \qquad \Delta\psi_n = 0, \qquad \Delta p_n = 0$$

and have the form

$$\varphi_n = |\mathbf{x}|^n \sum_{m=0}^{n} P_n^m(\cos\theta)\big[a_{nm}\cos(m\varphi) + \widetilde{a}_{nm}\sin(m\varphi)\big],$$

$$\psi_n = |\mathbf{x}|^n \sum_{m=0}^{n} P_n^m(\cos\theta)\big[b_{nm}\cos(m\varphi) + \widetilde{b}_{nm}\sin(m\varphi)\big],$$

$$p_n = |\mathbf{x}|^n \sum_{m=0}^{n} P_n^m(\cos\theta)\big[c_{nm}\cos(m\varphi) + \widetilde{c}_{nm}\sin(m\varphi)\big],$$

where a_{nm}, $\widetilde{a}_{nm}$, b_{nm}, $\widetilde{b}_{nm}$, c_{nm}, and $\widetilde{c}_{nm}$ are arbitrary constants and the P_n^m are the associated Legendre polynomials,

$$P_n^m(\zeta) = (1-\zeta^2)^{m/2}\frac{d^m}{d\zeta^m}P_n(\zeta), \qquad P_n(\zeta) = \frac{1}{n!2^n}\frac{d^n}{d\zeta^n}(\zeta^2-1)^n,$$

$$n = 1,\ 2,\ 3,\ \ldots, \qquad m = 0,\ 1,\ 2,\ \ldots$$

12.7.5 Solution of the Steady-State Nonhomogeneous Stokes Equations

Consider the auxiliary problems

$$\begin{aligned}&\nu\Delta\mathbf{u}_k - \nabla p_k = \delta(\mathbf{x}-\mathbf{y}), \quad \operatorname{div}\mathbf{u}_k = 0, \quad k = 1,\,2,\,3,\\ &|\mathbf{u}_k| \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \qquad |p_k| \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty,\end{aligned} \tag{24}$$

where $\delta(\mathbf{x})$ is the Dirac delta function, and $\mathbf{y}$ plays the role of a parameter.

The solution of problems (24) is given by the *Lorentz formulas*

$$\begin{aligned}&\mathbf{u}_k(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{3} u_{kj}(\mathbf{x},\mathbf{y})\mathbf{e}_j, \quad u_{kj}(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi\nu}\left[\frac{\delta_{kj}}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_k-y_k)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\right],\\ &p_k(\mathbf{x},\mathbf{y}) = -\frac{x_k-y_k}{4\pi|\mathbf{x}-\mathbf{y}|^3},\end{aligned} \tag{25}$$

where $\mathbf{e}_j$ is the unit vector of the jth coordinate axis and δ_{kj} is the Kronecker delta.

The solution of the steady-state nonhomogeneous Stokes equations (1) and (2) in a domain V with sufficiently smooth boundary ∂V can be expressed via the functions (25) and the mass force $\mathbf{f} = (f_1, f_2, f_3)$ as follows:

$$\mathbf{u}(\mathbf{x}) = \int_V \sum_{k=1}^{3} \mathbf{u}_k(\mathbf{x},\mathbf{y}) f_k(\mathbf{y})\, dV_y, \quad p(\mathbf{x}) = \int_V \sum_{k=1}^{3} p_k(\mathbf{x},\mathbf{y}) f_k(\mathbf{y})\, dV_y, \tag{26}$$

where $dV_y = dy_1\,dy_2\,dy_3$. Formulas (26) are the volume potentials for the steady-state Stokes equations.

12.7.6 Solution of the Cauchy Problem

Consider the Cauchy problem for the Stokes equations (1)–(2) with the homogeneous initial condition

$$\mathbf{u} = \mathbf{0} \quad \text{at} \quad t = 0. \tag{27}$$

The solution of problem (1)–(2), (27) for $x \in \mathbb{R}^3$ and $t > 0$ has the form

$$\mathbf{u} = (u_1, u_2, u_3), \quad u_k = u_k(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^3} \mathbf{w}_k(\mathbf{x}-\mathbf{y}, t-\tau) \cdot \mathbf{f}(\mathbf{y},\tau)\, dy\, d\tau, \tag{28}$$

$$p = p(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^{3} q_k(\mathbf{x}-\mathbf{y}, t-\tau) f_k(\mathbf{y},\tau)\, dy\, d\tau, \quad dy = dy_1\,dy_2\,dy_3, \tag{29}$$

where the $f_k(\mathbf{x},t)$ are the components of the mass force $\mathbf{f}$ and the functions $\mathbf{w}_k$ and q_k are given by the formulas

$$\mathbf{w}_k(\mathbf{x},t) = \Gamma(\mathbf{x},t)\mathbf{e}_k + \frac{1}{4\pi}\nabla\frac{\partial}{\partial x_k}\int_{\mathbb{R}^3} \frac{\Gamma(\mathbf{x}-\mathbf{y},t)}{|\mathbf{y}|}\, dy, \tag{30}$$

$$q_k(\mathbf{x},t) = -\frac{\partial}{\partial x_k}\frac{1}{4\pi|\mathbf{x}|}\delta(t), \quad \Gamma(\mathbf{x},t) = \frac{1}{(4\nu\pi t)^{3/2}} \exp\biggl(-\frac{|\mathbf{x}|^2}{4\nu t}\biggr). \tag{31}$$

Here $\mathbf{e}_k$ is the unit vector of the x_k-axis and $\delta(t)$ is the Dirac delta function.

The vectors $\mathbf{w}_k$ in (28) and (30) can be represented in the form

$$\mathbf{w}_k(\mathbf{x},t) = \operatorname{curl}\operatorname{curl}\mathbf{U}_k \equiv -\Delta\mathbf{U}_k + \nabla\operatorname{div}\mathbf{U}_k,$$

where

$$\mathbf{U}_k(\mathbf{x},t) = \mathbf{e}_k\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\Gamma(\mathbf{x}-\mathbf{y},t)}{|\mathbf{y}|}\,dy = \frac{1}{2\pi^{3/2}|\mathbf{x}|}\theta\biggl(\frac{|\mathbf{x}|}{2\sqrt{\nu t}}\biggr)\mathbf{e}_k, \quad \theta(z) = \int_0^z e^{-\xi^2}\,d\xi.$$

⊙ *Literature for Section 12.7*: H. Lamb (1945), R. Berker (1963), J. Happel and H. Brenner (1965), O. A. Ladyzhenskaya (1969), G. K. Batchelor (1970), H. Schlichting (1981), L. G. Loitsyanskiy (1996), A. D. Polyanin and A. V. Vyazmin (2012, 2013a), A. D. Polyanin and A. I. Zhurov (2013), A. D. Polyanin and S. A. Lychev (2014a).

12.8 Oseen Equations for Viscous Incompressible Fluids

12.8.1 Vector Form of Oseen Equations. Some Remarks

The *Oseen equations* for a viscous incompressible Newtonian fluid have the form

$$\begin{gathered}\mathbf{u}_t + (\mathbf{a}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f},\\ \operatorname{div}\mathbf{u} = 0,\end{gathered} \tag{1}$$

where $(\mathbf{a}\cdot\nabla)\mathbf{u} = a_1\mathbf{u}_x + a_2\mathbf{u}_y + a_3\mathbf{u}_z$, a_1, a_2, and a_3 are some constants and the remaining notation is the same as in the Stokes equations (see Eqs. (1)–(2) in Section 12.7).

The Oseen equations (1) describe the evolution of small perturbations of velocity components for a viscous incompressible fluid near the unperturbed flow with constant velocity vector $\mathbf{a} = (a_1, a_2, a_3)$ and constant pressure p_0. To derive the Oseen equations, one linearizes the Navier–Stokes equations on the basis of the order-of-magnitude relations

$$|\mathbf{u}-\mathbf{a}| \sim \varepsilon, \quad |p-p_0| \sim \varepsilon, \quad |\mathbf{f}| \sim \varepsilon,$$

where ε is a small parameter.

In the special case of $a_1 = a_2 = a_3 = 0$, the Oseen equations become the Stokes equations.

In the time-dependent case of $\mathbf{u}_t \not\equiv \mathbf{0}$, the transition in the Oseen equations (1) from x, y, z, and t to the new independent variables $\bar{x} = x - a_1 t$, $\bar{y} = y - a_2 t$, $\bar{z} = z - a_3 t$, and $\bar{t} = t$ results in the Stokes equations (see Eqs. (1)–(2) in Section 12.7).

The Oseen equations (1) are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl}\mathbf{\Psi}^\circ, \quad p = \widetilde{p},$$

where $\mathbf{\Psi}^\circ$ is an arbitrary solution of the equation $\mathbf{\Psi}_t^\circ + (\mathbf{a}\cdot\nabla)\Psi^\circ - \nu\Delta\Psi^\circ = \mathbf{0}$.

From (1), we obtain an equation for the pressure,

$$\Delta p = \operatorname{div}\mathbf{f}.$$

12.8.2 Various Forms of Decompositions for the Oseen Equations with $\mathbf{f} = \mathbf{0}$

► **Decomposition based on two stream functions.**

Any solution of the Oseen equations (1) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = p_0 - \varphi_t - (\mathbf{a}\cdot\nabla)\varphi, \\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi_y^{(1)}, \quad v_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad v_3 = -\psi_y^{(2)},\end{gathered} \tag{2}$$

where $p_0 = p_0(t)$ is an arbitrary function, $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are some solutions of the convective heat equation

$$\psi_t + (\mathbf{a}\cdot\nabla)\psi - \nu\Delta\psi = 0, \tag{3}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the Laplace equation $\Delta\varphi = 0$.

► **Using the Stokes–Helmholtz representation of the fluid velocity.**

Every solution of the Oseen equations (1) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad p = p_0 - \varphi_t - (\mathbf{a}\cdot\nabla)\varphi, \tag{4}$$

where $p_0 = p_0(t)$ is an arbitrary function, $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the convective heat vector equation

$$\boldsymbol{\Psi}_t + (\mathbf{a}\cdot\nabla)\boldsymbol{\Psi} - \nu\Delta\boldsymbol{\Psi} = \mathbf{0},$$

and the function φ is a solution of the Laplace equation $\Delta\varphi = 0$.

► **Decomposition of the steady-state Oseen equations.**

Let the vector $\mathbf{a}$ be directed along the x-axis; i.e., $\mathbf{a} = a\mathbf{e}_1$. Then the solution of the steady-state Oseen equations (1) can be represented in the form

$$\mathbf{u} = \nabla\varphi + \frac{1}{2k}\nabla\theta - \mathbf{e}_1\theta + \mathbf{e}_2\psi_z - \mathbf{e}_3\psi_y, \quad p = p_0 - a\rho\varphi_x, \quad k = \frac{a}{2\nu}, \tag{5}$$

where p_0 is an arbitrary constant and the functions φ, θ, and ψ satisfy the equations

$$\Delta\varphi = 0, \quad (\Delta - 2k\partial_x)\theta = 0, \quad (\Delta - 2k\partial_x)\psi = 0.$$

The transformations

$$\theta = e^{kx}\widetilde{\theta}, \quad \psi = e^{kx}\widetilde{\psi}$$

reduce the last two equations to the Helmholtz equations

$$\Delta\widetilde{\theta} - k^2\widetilde{\theta} = 0, \quad \Delta\widetilde{\psi} - k^2\widetilde{\psi} = 0.$$

12.8.3 Various Forms of Decompositions for the Oseen Equations with f $\neq$ 0

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force in the Oseen equations (1) be represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of system (1) can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\mathbf{\Psi}, \qquad p = p_0 - \varphi_t - (\mathbf{a}\cdot\nabla)\varphi + \gamma, \tag{6}$$

where $p_0 = p_0(t)$ is an arbitrary function, the vector function $\mathbf{\Psi}$ satisfies the equation

$$\mathbf{\Psi}_t + (\mathbf{a}\cdot\nabla)\mathbf{\Psi} - \nu\Delta\mathbf{\Psi} = \boldsymbol{\omega},$$

and the function φ is a solution of the Laplace equation $\Delta\varphi = 0$.

► **Decomposition not requiring the mass force to split into components.**

The solution of the Oseen equations (1) can be represented in the form

$$\mathbf{u} = -\operatorname{curl}\operatorname{curl}\mathbf{U} \equiv \Delta\mathbf{U} - \nabla\operatorname{div}\mathbf{U}, \qquad p = p_0(t) + \operatorname{div}[\mathbf{U}_t + (\mathbf{a}\cdot\nabla)\mathbf{U} - \nu\Delta\mathbf{U}],$$

where $p_0(t)$ is an arbitrary function and the vector function $\mathbf{U}$ satisfies the equation

$$\Delta[\mathbf{U}_t + (\mathbf{a}\cdot\nabla)\mathbf{U} - \nu\Delta\mathbf{U}] = \mathbf{f}.$$

► **Incomplete symmetric decomposition of the Oseen equations.**

Any solution of the Oseen equations (1) can also be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = -\varphi_t - (\mathbf{a}\cdot\nabla)\varphi + \nu\Delta\varphi + G, \tag{7}$$

where the vector function $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the independent linear nonhomogeneous equation

$$\mathbf{v}_t + (\mathbf{a}\cdot\nabla)\mathbf{v} - \Delta\mathbf{v} = \mathbf{f} - \nabla G \tag{8}$$

and the function φ satisfies Eq. (23) in Section 12.7. Formulas (7) and Eqs. (8) contain an arbitrary scalar function $G = G(\mathbf{x}, t)$.

By setting $v_1 = 0$ and $G_x = f_1$ in (7)–(8), we obtain an asymmetric representation of solutions of the Oseen equations (1).

12.8.4 Oseen Equations with Variable Coefficients

The coefficients of the Oseen equations (1) may arbitrarily depend on time, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $a_3 = a_3(t)$. This corresponds to the linearization of the Navier–Stokes equations in a neighborhood of an exact solution on the basis of the order-of-magnitude relations

$$|\mathbf{u} - \mathbf{a}| \sim \varepsilon, \quad |p - p_0 + \mathbf{a}'_t\cdot\mathbf{x}| \sim \varepsilon, \quad |\mathbf{f}| \sim \varepsilon,$$

where ε is a small parameter, $\mathbf{a} = (a_1, a_2, a_3)$, and $\mathbf{x} = (x, y, z)$.

The passage from x, y, z, t to the new independent variables $\bar{x} = x - \int a_1\,dt$, $\bar{y} = y - \int a_2\,dt$, $\bar{z} = z - \int a_3\,dt$, $\bar{t} = t$ in the Oseen equations with variable coefficients (1) gives the Stokes equations (see Eqs. (1)–(2) in Section 12.7).

The above-described decompositions of the Oseen equations with constant coefficients remain valid for the Oseen equations with variable coefficients.

⊙ *Literature for Section 12.8*: C. W. Oseen (1927), H. Lamb (1945), G. K. Batchelor (1970), A. D. Polyanin and A. V. Vyazmin (2012, 2013a), A. D. Polyanin and S. A. Lychev (2014a).

12.9 Maxwell Equations for Viscoelastic Incompressible Fluids

12.9.1 Vector Form of the Maxwell Equations

Slow motions of incompressible *Maxwell viscoelastic fluids* are described by the coupled linear equations

$$\begin{gathered}\tau\mathbf{u}_{tt} + \mathbf{u}_t = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f},\\ \operatorname{div}\mathbf{u} = 0,\end{gathered} \tag{1}$$

where τ is the relaxation time and all remaining notation is as in the Stokes equations (see Eqs. (1)–(2) in Section 12.7).

Some linearized hyperbolic modifications of the Navier–Stokes equations can be reduced to Eqs. (1) as well.

In the special case of $\tau = 0$, Eqs. (1) become the Stokes equations.

The Maxwell equations (1) are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl}\boldsymbol{\Psi}^\circ, \qquad p = \widetilde{p},$$

where $\boldsymbol{\Psi}^\circ$ is an arbitrary solution of the telegraph equation $\tau\Psi^\circ_{tt} + \boldsymbol{\Psi}^\circ_t - \nu\Delta\boldsymbol{\Psi}^\circ = \mathbf{0}$.

The Maxwell equations (1) are also invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi, \qquad p = \widetilde{p} - \tau\Phi_{tt} - \Phi_t + p_0(t),$$

where $p_0(t)$ is an arbitrary function and Φ is an arbitrary solution of the Laplace equation $\Delta\Phi = 0$.

From (1), one obtains an equation for the pressure,

$$\Delta p = \operatorname{div}\mathbf{f}.$$

12.9.2 Various Forms of Decompositions for the Maxwell Equations with $\mathbf{f} = \mathbf{0}$

► **Decomposition based on two stream functions.**

Each solution of the Maxwell equations (1) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = p_0 - \tau\varphi_{tt} - \varphi_t,\\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\psi^{(2)}_y,\end{gathered} \tag{2}$$

where $p_0 = p_0(t)$ is an arbitrary function, $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are some solutions of the telegraph equation

$$\tau\psi_{tt} + \psi_t - \nu\Delta\psi = 0, \tag{3}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the Laplace equation $\Delta\varphi = 0$.

► **Using the Stokes–Helmholtz representation of the fluid velocity.**

Any solution of the Maxwell equations (1) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad p = p_0 - \tau\varphi_{tt} - \varphi_t, \tag{4}$$

where $p_0 = p_0(t)$ is an arbitrary function, $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the vector telegraph equation

$$\tau\boldsymbol{\Psi}_t + \boldsymbol{\Psi}_t - \nu\Delta\boldsymbol{\Psi} = \mathbf{0},$$

and the function φ is a solution of the Laplace equation $\Delta\varphi = 0$.

► **Special types of decompositions using representations of the vector u via three scalar functions.**

Other representations of solutions of the Maxwell equations (1) with $\mathbf{f} = \mathbf{0}$ can be represented with the use of formulas of the form (4)

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}_n, \qquad p = p_0 - \tau\varphi_{tt} - \varphi_t \tag{5}$$

and Table 12.1, where the functions $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are arbitrary solutions of the telegraph equation (3) and the function φ satisfies the Laplace equation $\Delta\varphi = 0$.

12.9.3 Various Forms of Decompositions for the Maxwell Equations with f $\neq$ 0

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force in the Maxwell equations (1) be written in the form of the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of system (1) can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad p = p_0 - \tau\varphi_{tt} - \varphi_t + \gamma, \tag{6}$$

where $p_0 = p_0(t)$ is an arbitrary function, the vector function $\boldsymbol{\Psi}$ satisfies the equation

$$\tau\boldsymbol{\Psi}_{tt} + \boldsymbol{\Psi}_t - \nu\Delta\boldsymbol{\Psi} = \boldsymbol{\omega},$$

and the function φ is a solution of the Laplace equation $\Delta\varphi = 0$.

► **Decomposition not requiring the mass force to split into components.**

The solution of the Maxwell equations (1) can be represented in the form

$$\mathbf{u} = -\operatorname{curl}\operatorname{curl}\mathbf{U} \equiv \Delta\mathbf{U} - \nabla\operatorname{div}\mathbf{U}, \quad p = p_0(t) + \operatorname{div}(\tau\mathbf{U}_{tt} + \mathbf{U}_t - \nu\Delta\mathbf{U}),$$

where $p_0(t)$ is an arbitrary function and the vector function $\mathbf{U}$ satisfies the equation

$$\Delta(\tau\partial_{tt} + \partial_t - \nu\Delta)[\mathbf{U}] = \mathbf{f}.$$

► **Incomplete symmetric decomposition of the Maxwell equations.**

Each solution of the Maxwell equations (1) can also be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = -\tau\varphi_{tt} - \varphi_t + \nu\Delta\varphi + G, \tag{7}$$

where the vector function $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the independent linear nonhomogeneous equation

$$\tau\mathbf{v}_{tt} + \mathbf{v}_t - \Delta\mathbf{v} = \mathbf{f} - \nabla G \tag{8}$$

and the function φ satisfies Eq. (23) in Section 12.7. Formulas (7) and Eqs. (8) contain an arbitrary scalar function $G = G(\mathbf{x}, t)$.

By setting $v_1 = 0$ and $G_x = f_1$ in (7)–(8), we obtain an asymmetric representation of solutions of the Maxwell equations (1).

⊙ *Literature for Section 12.9*: W. L. Wilkinson (1960), K.-C. Lee and D. A. Finlayson (1986), D. D. Joseph (1990), K. R. Rajagopal (1993), R. Racke and J. Saal (2012), A. D. Polyanin and A. I. Zhurov (2013), A. D. Polyanin and A. V. Vyazmin (2013b), A. D. Polyanin and S. A. Lychev (2014a).

12.10 Equations of Viscoelastic Incompressible Fluids (General Model)

12.10.1 Linearized Equations of Viscoelastic Incompressible Fluids. Some Models of Viscoelastic Fluids

► **Vector form of linearized equations for the general model of viscoelastic incompressible fluids.**

In the general case, slow motions of incompressible *viscoelastic fluids* are described by the coupled linear equations

$$\begin{gathered} \mathrm{L}[\mathbf{u}] = -\nabla p + \mathbf{f}, \\ \operatorname{div}\mathbf{u} = 0, \end{gathered} \tag{1}$$

where L is a linear operator whose properties are described in what follows.

Equations (1) of viscoelastic incompressible fluids are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl}\boldsymbol{\Psi}^\circ, \quad p = \widetilde{p},$$

where $\boldsymbol{\Psi}^\circ$ is an arbitrary solution of the equation $\mathrm{L}[\Psi^\circ] = \mathbf{0}$.

Equations (1) of viscoelastic incompressible fluids are also invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi, \qquad p = \widetilde{p} - \mathrm{L}[\Phi] + p_0(t),$$

where $p_0(t)$ is an arbitrary function and Φ is an arbitrary solution of the Laplace equation $\Delta\Phi = 0$.

From (1), we obtain an equation for the pressure,

$$\Delta p = \operatorname{div}\mathbf{f}.$$

► **Some models of viscoelastic incompressible fluids.**

In general, the linearized rheological equation of state describing slow motions of any isotropic viscoelastic incompressible fluid can be written as

$$\mathrm{M}[\sigma_{ij}] = -p\delta_{ij} + \mathrm{K}[e_{ij}], \tag{2}$$

where the σ_{ij} are the stress tensor components, the $e_{ij} = \frac{1}{2}\bigl(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\bigr)$ are the strain rate tensor components ($x_1 = x$, $x_2 = y$, $x_3 = z$), M and K are linear operators in t, and δ_{ij} is the Kronecker delta. The linear operator L in Eqs. (1) can be expressed in terms of M and K as follows:

$$\mathrm{L}[u] = \mathrm{M}[u_t] - \tfrac{1}{2}\mathrm{K}[\Delta u]. \tag{3}$$

Usually, M and K are differential operators, but they can also be integral, integro-differential, or differential-difference operators. For example, the linearized equations of incompressible *Oldroyd viscoelastic fluids* are determined by the operators

$$\mathrm{M}[\sigma] = \sigma + a_1\sigma_t, \qquad \mathrm{K}[e] = 2\nu e + a_2 e_t,$$

where ν is the kinematic viscosity, and a_1 and a_2 are constants. (Maxwell fluids correspond to the value $a_2 = 0$ for $a_1 = \tau$.)

Table 12.2 lists examples of specific linear operators M and K that determine the equation of state (2) and are used in some models of incompressible viscoelastic fluids.

Remark 12.12. The fractional derivative of order q, $0 < q < 1$ (see row 7 in Table 12.2), is defined as

$$f_t^{[q]}(t) \equiv \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t \frac{f(s)\,ds}{(t-s)^q},$$

where $\Gamma(z)$ is the gamma function. The function $\sigma|_{t+\tau}$ in the last row of the table is evaluated at $t+\tau$ (with a shift in time).

12.10.2 Various Forms of Decompositions for Equations of Viscoelastic Incompressible Fluids with $\mathbf{f} = \mathbf{0}$

► **Decomposition based on two stream functions.**

Each solution of Eqs. (1) of viscoelastic fluids with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\begin{gathered} \mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = p_0 - \mathrm{L}[\varphi], \\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi_y^{(1)}, \quad v_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad v_3 = -\psi_y^{(2)}, \end{gathered} \tag{2}$$

Table 12.2: Linear operators used in various models of incompressible viscoelastic fluids

No.	Operator $\mathrm{M}[\sigma]$	Operator $\mathrm{K}[e]$	Rheologic model
1	σ	$2\nu e$	Newtonian
2	$\sigma+\tau\sigma_t$	$2\nu e$	Maxwell
3	$\sigma+a\sigma_t$	$2\nu e+be_t$	Oldroyd
4	$\sigma+a_1\sigma_t+a_2\sigma_{tt}$	$2\nu e+be_t$	Burgers
5	$\sigma+a_1\sigma_t+a_2\sigma_{tt}$	$2\nu e+b_1e_t+b_2e_{tt}$	Burgers, generalized
6	$\sigma+\sum_{n=1}^{m} a_n\sigma_t^{(n)}$	$2\nu e+\sum_{n=1}^{k} b_n e_t^{(n)}$	General differential
7	$\sigma+a\sigma_t^{[q]}$	$2\nu e+be_t^{[r]}$	With fractional derivatives of orders q and r
8	σ	$2\nu e+a\int_0^t e^{-\lambda(t-s)}e\big\|_{t=s}\,ds$	Integro-differential Oldroyd
9	σ	$2\nu e+\int_0^t F(t-s)e\big\|_{t=s}\,ds$	Integro-differential with difference kernel
10	$\sigma\big\|_{t+\tau}$	$2\nu e$	Difference-differential

where $p_0=p_0(t)$ is an arbitrary function, $\psi^{(1)}=\psi^{(1)}(\mathbf{x},t)$ and $\psi^{(2)}=\psi^{(2)}(\mathbf{x},t)$ are some solutions of the equation

$$\mathrm{L}[\psi]=0, \tag{3}$$

and the function $\varphi=\varphi(\mathbf{x},t)$ is a solution of the Laplace equation $\Delta\varphi=0$.

► **Using the Stokes–Helmholtz representation of the fluid velocity.**

Each solution of Eqs. (1) of viscoelastic fluids for $\mathbf{f}=\mathbf{0}$ can be represented by the formulas

$$\mathbf{u}=\nabla\varphi+\operatorname{curl}\mathbf{\Psi},\qquad p=p_0-\mathrm{L}[\varphi], \tag{4}$$

where $p_0=p_0(t)$ is an arbitrary function, $\mathbf{\Psi}=(\Psi_1,\Psi_2,\Psi_3)$ is a solution of the vector equation

$$\mathrm{L}[\mathbf{\Psi}]=\mathbf{0}, \tag{5}$$

and the function φ is a solution of the Laplace equation $\Delta\varphi=0$.

12.10.3 Various Forms of Decompositions for Equations of Viscoelastic Incompressible Fluids with $\mathbf{f}\neq 0$

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force in the equations (1) of viscoelastic fluids be represented as the sum $\mathbf{f}=\nabla\gamma+\operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of system (1) can be wtitten as

$$\mathbf{u}=\nabla\varphi+\operatorname{curl}\mathbf{\Psi},\qquad p=p_0-\mathrm{L}[\varphi]+\gamma, \tag{6}$$

where $p_0=p_0(t)$ is an arbitrary function, the vector function $\mathbf{\Psi}$ satisfies the equation

$$\mathrm{L}[\mathbf{\Psi}]=\boldsymbol{\omega},$$

and the function φ is a solution of the Laplace equation $\Delta\varphi=0$.

► **Decomposition not requiring the mass force to split into components (the first approach).**

The solution of Eqs. (1) of viscoelastic fluids can be represented in the form

$$\mathbf{u} = -\operatorname{curl}\operatorname{curl}\mathbf{U} \equiv \Delta\mathbf{U} - \nabla\operatorname{div}\mathbf{U}, \qquad p = p_0(t) + \mathrm{L}[\operatorname{div}\mathbf{U}],$$

where $p_0(t)$ is an arbitrary function and the vector function $\mathbf{U}$ satisfies the equation

$$\Delta\mathrm{L}[\mathbf{U}] = \mathbf{f}.$$

► **Decomposition not requiring the mass force to split into components (the second approach).**

The solution of Eqs. (1) of viscoelastic fluids can be represented in the form

$$\mathbf{u} = \Delta\mathbf{U} + \nabla\bigl\{\varphi - (\mathrm{LH}+1)[\operatorname{div}\mathbf{U}]\bigr\}, \qquad p = p_0(t) - \rho\mathrm{L}\bigl\{\varphi - (\mathrm{LH}+1)[\operatorname{div}\mathbf{U}]\bigr\},$$

where $p_0(t)$ is an arbitrary function, H is an arbitrary constant coefficient linear differential operator, and the vector and scalar functions $\mathbf{U}$ and Φ satisfy the equations

$$\Delta\mathrm{L}[\mathbf{U}] = \mathbf{f}, \qquad \Delta\Phi = \mathrm{H}[\operatorname{div}\mathbf{f}].$$

► **Incomplete symmetric decomposition.**

Each solution of Eqs. (1) of viscoelastic fluids can also be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = -\mathrm{L}[\varphi] + G, \tag{7}$$

where the vector function $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the independent linear nonhomogeneous equation

$$\mathrm{L}[\mathbf{v}] = \mathbf{f} - \nabla G \tag{8}$$

and the function φ satisfies Eq. (23) in Section 12.7. Formulas (7) and Eqs. (8) contain an arbitrary scalar function $G = G(\mathbf{x}, t)$.

By setting $v_1 = 0$ and $G_x = f_1$ in (7)–(8), we obtain asymmetric representation of solutions of Eqs. (1).

⊙ *Literature for Section 12.10*: J. G. Oldroyd (1956), W. L. Wilkinson (1960), K.-C. Lee and D. A. Finlayson (1986), D. D. Joseph (1990), K. R. Rajagopal (1993), I. C. Christov (2010), D. Tong (2010), A. D. Polyanin and A. I. Zhurov (2013), A. D. Polyanin and A. V. Vyazmin (2013b), A. D. Polyanin and S. A. Lychev (2014a).

12.11 Linearized Equations for Inviscid Compressible Barotropic Fluids

12.11.1 Vector Form of Equations without Mass Forces. Some Remarks

The equations describing slow motions of an inviscid compressible barotropic fluid without mass forces have the form

$$\rho_0\mathbf{u}_t + \nabla p = \mathbf{0}, \tag{1}$$

$$p_t + \rho_0 c^2 \operatorname{div}\mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}$ is the fluid velocity, ρ_0 is the unperturbed density, p is the pressure, and c is the speed of sound (which is a constant).

System (1)–(2) can be reduced to the vector equation

$$\mathbf{u}_{tt} - c^2\nabla \operatorname{div} \mathbf{u} = \mathbf{0}$$

for the velocity and the scalar wave equation

$$p_{tt} - c^2\Delta p = 0$$

for the pressure.

12.11.2 Decompositions of Equations for Inviscid Compressible Barotropic Fluid

► Decomposition based on two stream functions.

Each solution of Eqs. (1)–(2) can be represented in the form

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = -\varphi_t,\\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\rho_0\psi^{(2)}_y,\end{gathered} \tag{3}$$

where $\psi^{(1)} = \psi^{(1)}(\mathbf{x})$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x})$ are arbitrary functions depending only on the spatial variables and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the wave equation

$$\varphi_{tt} - c^2\Delta\varphi = 0. \tag{4}$$

► Using the Stokes–Helmholtz representation of the fluid velocity.

Each solution of Eqs. (1)–(2) can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl} \mathbf{\Psi}, \quad p = -\rho_0\varphi_t, \tag{5}$$

where $\mathbf{\Psi} = \mathbf{\Psi}(\mathbf{x})$ is an arbitrary vector function depending only on the spatial variables and the function φ is a solution of the scalar equation (4).

Remark 12.13. See also Section 12.12 with $\nu = \varkappa = 0$.

12.11.3 Vector Form of Equations with Mass Forces

The density and pressure distribution in an immovable compressible barotropic fluid with the equation of state $p = p(\rho)$ in the presence of a stationary potential mass force $\mathbf{f} = \mathbf{f}(\mathbf{x}) = (f_1, f_2, f_3)$ (e.g., the force of gravity) is described by the equations

$$\mathbf{u} = \mathbf{0}, \qquad \nabla p_0 = \rho_0\mathbf{f}, \qquad p_0 = p_0(\rho_0). \tag{6}$$

The linearization of the full equations of an inviscid compressible barotropic fluid in a neighborhood of the steady-state solution (6) on the basis of the relations

$$\mathbf{u} = \varepsilon\mathbf{v}, \quad p = p_0 + \varepsilon\bar{p}, \quad \rho = \rho_0 + \varepsilon\bar{\rho},$$

where ε is a small parameter, leads to the equations

$$\mathbf{v}_t + \nabla q = \mathbf{0}, \qquad q = \bar{p}/\rho_0, \tag{7}$$

$$q_t + c^2 \operatorname{div} \mathbf{v} + \mathbf{f}\cdot\mathbf{v} = 0. \tag{8}$$

Here $c = \sqrt{p'_\rho}\,|_{\rho=\rho_0}$ is the local speed of sound, which depends on the spatial variables in this case.

By eliminating the function q from (7)–(8), we arrive at the equation

$$\mathbf{v}_{tt} - \nabla(c^2 \operatorname{div} \mathbf{v} + \mathbf{f}\cdot\mathbf{v}) = \mathbf{0} \tag{9}$$

for the fluid velocity.

The solution of system (7)–(8) has the form

$$\mathbf{v} = \nabla\varphi + \mathbf{w}(\mathbf{x}), \qquad q = q_0 - \varphi_t, \tag{10}$$

where q_0 is an arbitrary constant, the function $\varphi = \varphi(\mathbf{x}, t)$ satisfies the wave type equation

$$\varphi_{tt} = c^2\Delta\varphi + \mathbf{f}\cdot\nabla\varphi, \tag{11}$$

and the stationary vector function $\mathbf{w} = \mathbf{w}(\mathbf{x})$ is a solution of the linear first-order PDE

$$c^2 \operatorname{div} \mathbf{w} + \mathbf{f}\cdot\mathbf{w} = 0. \tag{12}$$

The underdetermined equation (12) can readily be integrated. (Two arbitrary components of the vector **v** are specified arbitrarily, and the remaining component satisfies a linear first-order ODE in which the first two spatial coordinates occur as parameters.) For example, the general solution of Eq. (12) can be represented in the form

$$\mathbf{w} = (w_1, w_2, w_3), \qquad w_1 = E\Big(A - \int FE^{-1}\,dx\Big), \qquad w_2 = w_2(\mathbf{x}), \qquad w_3 = w_3(\mathbf{x}),$$
$$F = (w_2)_y + (w_3)_z + c^{-2}(f_2 w_2 + f_3 w_3), \qquad E = \exp\Big(-\int f_1 c^{-2}\,dx\Big),$$

where $w_1(\mathbf{x})$, $w_2(\mathbf{x})$, and $A = A(y,z)$ are arbitrary functions and $\mathbf{x} = (x, y, z)$.

⊙ *Literature for Section 12.11*: H. Lamb (1945), L. V. Ovsyannikov (1981).

12.12 Stokes Equations for Viscous Compressible Barotropic Fluids

12.12.1 Linearized Equations of Viscous Compressible Barotropic Fluids

The equations describing slow motions of a viscous compressible barotropic fluid have the form

$$\mathbf{u}_t = -\nabla p + \nu\Delta\mathbf{u} + (\nu + \varkappa)\nabla \operatorname{div} \mathbf{u} + \mathbf{f}, \tag{1}$$

$$p_t + c^2 \operatorname{div} \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity, t is time, $\nu = \mu/\rho_0$, $\varkappa = \lambda/\rho_0$, μ and λ are the dynamic viscosities, ρ_0 is the unperturbed density, p is the ratio of the pressure to the unperturbed density (i.e., it is introduced in a different way from that in Section 12.11), c is the speed of sound, and $\mathbf{f} = (f_1, f_2, f_3)$ is the mass force.

Equations (1)–(2) describe the evolution of small perturbations of velocity and pressure near the stationary solution by linearizing the full nonlinear equations of motion of a viscous compressible barotropic fluid on the basis of the order-of-magnitude relations

$$|\mathbf{u}| \sim \varepsilon, \quad |p - p_0| \sim \varepsilon, \quad |\rho - \rho_0| \sim \varepsilon, \quad |\mathbf{f}| \sim \varepsilon,$$

where ε is a small parameter and $p = p(\rho)$.

In the limit case of $c^2 \to \infty$, Eqs. (1)–(2) become the Stokes equations for the viscous incompressible fluid (see Eqs. (1)–(2) in Section 12.7).

12.12.2 Decompositions of Equations of Viscous Compressible Barotropic Fluid with f = 0

► **Decomposition based on two stream functions.**

Each solution of Eqs. (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = -\varphi_t + (2\nu + \varkappa)\Delta\varphi, \\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\psi^{(2)}_y,\end{gathered} \tag{3}$$

where $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are some solutions of the heat equation

$$\psi_t - \nu\Delta\psi = 0 \tag{4}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the third-order equation

$$\varphi_{tt} - (2\nu + \varkappa)\Delta\varphi_t - c^2\Delta\varphi = 0. \tag{5}$$

► **Using the Stokes–Helmholtz representation of the fluid velocity.**

Each solution of Eqs. (1)–(2) for $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad p = -\varphi_t + (2\nu + \varkappa)\Delta\varphi, \tag{6}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the vector heat equation

$$\boldsymbol{\Psi}_t - \nu\Delta\boldsymbol{\Psi} = \mathbf{0} \tag{7}$$

and the function φ is a solution of the scalar equation (5).

► **Special types of decompositions using representations of the vector u via two scalar functions.**

Other representations of solutions for equations (1)–(2) of a viscous compressible barotropic fluid with $\mathbf{f} = \mathbf{0}$ can be obtained with the use of formulas of the form (6)

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}_n, \qquad p = -\varphi_t + (2\nu + \varkappa)\Delta\varphi \tag{8}$$

and Table 12.1, where the functions $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are arbitrary solutions of the heat equation (4) and the function φ satisfies Eq. (5).

12.12.3 Decompositions of Equations of a Viscous Compressible Barotropic Fluid with f $\neq$ 0

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force in Eqs. (1)–(2) be represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of system (1)–(2) can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad p = -\varphi_t + (2\nu + \varkappa)\Delta\varphi + \gamma, \tag{9}$$

where the vector function $\boldsymbol{\Psi}$ satisfies the equation

$$\boldsymbol{\Psi}_t - \nu\Delta\boldsymbol{\Psi} = \boldsymbol{\omega}$$

and the function φ is a solution of the equation

$$\varphi_{tt} - (2\nu + \varkappa)\Delta\varphi_t - c^2\Delta\varphi = \gamma_t.$$

► **Decomposition not requiring the force to split into components.**

The solution of system (1)–(2) can be represented in the form

$$\begin{gathered}\mathbf{u} = [\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta][\mathbf{w}] + \nabla\bigl\{\varphi + [c^2 + (\nu + \varkappa)\partial_t][\operatorname{div}\mathbf{w}]\bigr\},\\ p = -[\partial_t - (2\nu + \varkappa)\Delta][\varphi] - c^2(\partial_t - \nu\Delta)[\operatorname{div}\mathbf{w}],\end{gathered} \tag{10}$$

where the vector and scalar functions $\mathbf{w}$ and φ satisfy the independent equations

$$(\partial_t - \nu\Delta)[\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta][\mathbf{w}] = \mathbf{f}, \tag{11}$$

$$[\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta][\varphi] = 0. \tag{12}$$

The representation of the solution in the form (10)–(12) corresponds to a decomposition of the third order.

Remark 12.14. The general solution of the homogeneous equation (11) with $\mathbf{f} = \mathbf{0}$ can be represented as the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$(\partial_t - \nu\Delta)[\mathbf{w}_1] = \mathbf{0}, \qquad [\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta][\mathbf{w}_2] = \mathbf{0}.$$

12.12.4 Reduction to One Vector Equation and Its Decompositions

► **Reduction to one vector equation.**

Let us reduce system (1)–(2) of coupled equations to a single vector equation. To this end, we set

$$\mathbf{u} = \mathbf{w}_t, \qquad p = -c^2\operatorname{div}\mathbf{w}, \tag{13}$$

where $\mathbf{w}$ is the new unknown vector function. By substituting (13) into (1), we obtain the equation

$$\mathbf{w}_{tt} - \nu\Delta\mathbf{w}_t - \nabla[c^2\operatorname{div}\mathbf{w} + (\nu + \varkappa)\operatorname{div}\mathbf{w}_t] = \mathbf{f}. \tag{14}$$

Equation (2) is satisfied by the expressions (13) identically.

► **Decomposition based on the Stokes–Helmholtz representation of the mass force.**

Let the mass force in Eqs. (14) be represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$. Then the solution of system (14) admits the representation

$$\mathbf{w} = \nabla\widetilde{\varphi} + \operatorname{curl}\widetilde{\boldsymbol{\Psi}},$$

where the vector function $\widetilde{\boldsymbol{\Psi}}$ satisfies the equation

$$\widetilde{\boldsymbol{\Psi}}_t - \nu\Delta\widetilde{\boldsymbol{\Psi}} = \boldsymbol{\omega}$$

and the function $\widetilde{\varphi}$ is a solution of the equation

$$\widetilde{\varphi}_{tt} - (2\nu + \varkappa)\Delta\widetilde{\varphi}_t - c^2\Delta\widetilde{\varphi} = \gamma.$$

► **Decomposition not requiring the force to split into components.**

The solution of system (14) can be represented in the form

$$\mathbf{w} = \big[\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta\big][\widetilde{\mathbf{v}}] + \nabla\big[c^2 + (\nu + \varkappa)\partial_t\big][\operatorname{div}\widetilde{\mathbf{v}}], \qquad (15)$$

where the vector function $\widetilde{\mathbf{v}}$ satisfies the equation

$$\big(\partial_t^2 - \nu\partial_t\Delta\big)\big[\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta\big][\widetilde{\mathbf{v}}] = \mathbf{f}. \qquad (16)$$

One can reduce the order of this equation by integrating with respect to t.

Let us differentiate (15) with respect to t and take into account the first relation in (13). As a result, we obtain the following representation of the velocity:

$$\mathbf{u} = \big[\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta\big][\boldsymbol{\zeta}] + \nabla\big[c^2 + (\nu + \varkappa)\partial_t\big][\operatorname{div}\boldsymbol{\zeta}], \qquad \boldsymbol{\zeta} = \widetilde{\mathbf{v}}_t,$$

where the vector function $\boldsymbol{\zeta}$ satisfies the equation

$$\big(\partial_t - \nu\partial\Delta\big)\big[\partial_t^2 - (2\nu + \varkappa)\partial_t\Delta - c^2\Delta\big][\boldsymbol{\zeta}] = \mathbf{f}.$$

12.12.5 Independent Equations for u and p

Let us differentiate Eq. (1) with respect to t and then eliminate p_t with the use of Eq. (2). As a result, we obtain the independent equation

$$\mathbf{u}_{tt} - \nu\Delta\mathbf{u}_t = \nabla\big[c^2\operatorname{div}\mathbf{u} + (\nu + \varkappa)\operatorname{div}\mathbf{u}_t\big] + \mathbf{f}_t \qquad (17)$$

for the fluid velocity.

Let us apply the operator div to Eq. (1) and then eliminate $\operatorname{div}\mathbf{u}$ with the use of Eq. (2). As a result, we obtain the independent equation

$$p_{tt} - (2\nu + \varkappa)\Delta p_t - c^2\Delta p = -c^2\operatorname{div}\mathbf{f} \qquad (18)$$

for the pressure.

The solutions of Eqs. (17) and (18) do not give a solution of the original system (1)–(2) in general, because (17) and (18) were obtained by an application of various differential operators to the original model, which makes the class of solutions broader. One can dispense with the spurious solutions by using the original equations (1)–(2), which can be interpreted as differential constraints for Eqs. (17) and (18).

⊙ *Literature for Section 12.12*: P. B. Mucha and W. M. Zajaczkowski (2002), N. A. Gusev (2011), I. I. Lipatov and A. D. Polyanin (2013), A. D. Polyanin and S. A. Lychev (2014a,b).

12.13 Oseen Equations for Viscous Compressible Barotropic Fluids

12.13.1 Vector Form of Equations. Some Remarks

The *Oseen equations for viscous compressible barotropic fluids* have the form

$$\mathbf{u}_t + (\mathbf{a}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + (\nu+\varkappa)\nabla\operatorname{div}\mathbf{u} + \mathbf{f}, \tag{1}$$

$$p_t + c^2\operatorname{div}\mathbf{u} = 0, \tag{2}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and the remaining notation is as in Section 12.12.

Equations (1)–(2) describe the evolution of small perturbations of velocity and pressure near a stationary solution with the use of the linearization of the full nonlinear equations of motion of viscous compressible barotropic fluids on the basis of the order-of-magnitude relations

$$|\mathbf{u}-\mathbf{a}| \sim \varepsilon, \quad |p-p_0| \sim \varepsilon, \quad |\rho-\rho_0| \sim \varepsilon, \quad |\mathbf{f}| \sim \varepsilon,$$

where ε is a small parameter and $p = p(\rho)$.

12.13.2 Decompositions of Equations with $\mathbf{f} = 0$

► **Decomposition based on two stream functions.**

Each solution of Eqs. (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = -\varphi_t - (\mathbf{a}\cdot\nabla)\varphi + (2\nu+\varkappa)\Delta\varphi,\\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\psi^{(2)}_y,\end{gathered} \tag{3}$$

where $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are some solutions of the heat-type equation

$$\psi_t + (\mathbf{a}\cdot\nabla)\psi - \nu\Delta\psi = 0 \tag{4}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the third-order equation

$$\varphi_{tt} + (\mathbf{a}\cdot\nabla)\varphi_t - (2\nu+\varkappa)\Delta\varphi_t - c^2\Delta\varphi = 0. \tag{5}$$

► **Using the Stokes–Helmholtz representation of the fluid velocity.**

Each solution of Eqs. (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad p = -\varphi_t - (\mathbf{a}\cdot\nabla)\varphi + (2\nu+\varkappa)\Delta\varphi, \tag{6}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the vector heat-type equation

$$\boldsymbol{\Psi}_t + (\mathbf{a}\cdot\nabla)\boldsymbol{\Psi} - \nu\Delta\boldsymbol{\Psi} = \mathbf{0}, \tag{7}$$

and the function φ is a solution of the scalar equation (5).

12.13.3 Decomposition of Equations with f $\neq$ 0

Let the mass force in Eqs. (1)–(2) be represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of system (1)–(2) can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad p = -\varphi_t - (\mathbf{a}\cdot\nabla)\varphi + (2\nu + \varkappa)\Delta\varphi + \gamma, \tag{8}$$

where the vector function $\boldsymbol{\Psi}$ satisfies the equation

$$\boldsymbol{\Psi}_t + (\mathbf{a}\cdot\nabla)\boldsymbol{\Psi} - \nu\Delta\boldsymbol{\Psi} = \boldsymbol{\omega},$$

and the function φ is a solution of the equation

$$\varphi_{tt} + (\mathbf{a}\cdot\nabla)\varphi_t - (2\nu + \varkappa)\Delta\varphi_t - c^2\Delta\varphi = \gamma_t.$$

⊙ *Literature for Section 12.13*: I. I. Lipatov and A. D. Polyanin (2013), A. D. Polyanin and S. A. Lychev (2014a,b).

12.14 Equations of Thermoelasticity

12.14.1 Vector Form of Thermoelasticity Equations

Coupled thermoelasticity equations have the form

$$\rho\mathbf{u}_{tt} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\operatorname{div}\mathbf{u} - \alpha\nabla T + \rho\mathbf{f}, \tag{1}$$

$$T_t = a\Delta T - \beta(\operatorname{div}\mathbf{u})_t, \tag{2}$$

where $\mathbf{u}$ is the *displacement vector*, t is time, ρ is the material density, μ is the *shear modulus* (or the *modulus of elasticity of the second kind*), λ is the *Lamé coefficient*, T is temperature, α and β are the thermomechanical moduli, a is the thermal diffusivity, and $\mathbf{f}$ is the mass force.

We also use the following modified representation of the vector equation (1):

$$\mathbf{u}_{tt} = c_2^2\Delta\mathbf{u} + (c_1^2 - c_2^2)\nabla\operatorname{div}\mathbf{u} - (\alpha/\rho)\nabla T + \mathbf{f}, \tag{3}$$

where $c_1 = \sqrt{(2\mu + \lambda)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the longitudinal and transverse wave velocities.

12.14.2 Decompositions of Thermoelasticity Equations with f $=$ 0

► **Decomposition based on two stream functions.**

Each solution of the thermoelasticity equations (2)–(3) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\begin{gathered}\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt}\right), \\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi_y^{(1)}, \quad v_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad v_3 = -\psi_y^{(2)},\end{gathered} \tag{4}$$

where $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are some solutions of the wave equations

$$\psi_{tt} - c_2^2\Delta\psi = 0 \tag{5}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the fourth-order equation

$$\varphi_{ttt} - [c_1^2 + (\alpha\beta)/\rho]\Delta\varphi_t - a\Delta\varphi_{tt} + ac_1^2\Delta\Delta\varphi = 0. \tag{6}$$

► **Using the Stokes–Helmholtz representation of the displacement vector.**

Each solution of the thermoelasticity equations (2)–(3) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt}\right), \tag{7}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the vector wave equation

$$\boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} = \mathbf{0} \tag{8}$$

and the function φ is a solution of the fourth-order equation (6).

► **Toroidal–poloidal decomposition.**

Each solution of the thermoelasticity equations (2)–(3) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{x}\xi) + \operatorname{curl}\operatorname{curl}(\mathbf{x}\eta), \quad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt}\right), \quad \mathbf{x} = (x, y, z),$$

where ξ and η are scalar functions satisfying the wave equations

$$\xi_{tt} - c_2^2\Delta\xi = 0, \quad \eta_{tt} - c_2^2\Delta\eta = 0 \tag{9}$$

and the function φ is a solution of the fourth-order equation (6).

► **Two other representations of solutions via three scalar functions.**

The solutions of Eqs. (2)–(3) with $\mathbf{f} = \mathbf{0}$ can also be represented in the following two forms:

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{b}\eta), \qquad \mathbf{a}\cdot\mathbf{b} \neq 0,$$
$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{x}\eta), \qquad |\mathbf{a}| \neq 0,$$

where the functions ξ and η are solutions of the wave equations (9) and the function φ is a solution of the fourth-order equation (6).

12.14.3 Decompositions of Thermoelasticity Equations with $\mathbf{f} \neq \mathbf{0}$

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force be represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of the thermoelasticity equations (2)–(3) admits the representation

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt} + \gamma\right),$$

where the functions $\boldsymbol{\Psi}$ and φ satisfy the independent equations

$$\boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} = \boldsymbol{\omega},$$
$$\varphi_{ttt} - [c_1^2 + (\alpha\beta)/\rho]\Delta\varphi_t - a\Delta\varphi_{tt} + ac_1^2\Delta\Delta\varphi = \gamma_t - a\Delta\gamma.$$

► **Decomposition not requiring the force to split into components.**

The solution of system (2)–(3) can be represented by the formulas

$$\begin{aligned}\mathbf{u} &= [(\partial_t - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t\Delta][\mathbf{w}]\\ &\quad + \nabla\bigl\{\varphi + [(\alpha\beta/\rho)\partial_t + (c_1^2 - c_2^2)(\partial_t - a\Delta)][\operatorname{div}\mathbf{w}]\bigr\},\\ T &= -\frac{\rho}{\alpha}(\partial_t^2 - c_1^2\Delta)[\varphi] - \beta\partial_t(\partial_t^2 - c_2^2\Delta)[\operatorname{div}\mathbf{w}],\end{aligned}$$

where the vector function **w** and the scalar function φ satisfy the equations

$$(\partial_t^2 - c_2^2\Delta)[(\partial_t - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t\Delta][\mathbf{w}] = \mathbf{f}, \tag{10}$$

$$[(\partial_t - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t\Delta][\varphi] = 0. \tag{11}$$

If $\mathbf{u}_t \not\equiv \mathbf{0}$, then the general solution of equation (10) with $\mathbf{f} = \mathbf{0}$ can be represented as the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the simpler two equations

$$\begin{aligned}(\partial_t^2 - c_2^2\Delta)[\mathbf{w}_1] &= \mathbf{0},\\ [(\partial_t - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t\Delta][\mathbf{w}_2] &= \mathbf{0}.\end{aligned}$$

⊙ *Literature for Section 12.14*: A. D. Polyanin and S. A. Lychev (2014a,b).

12.15 Nondissipative Thermoelasticity Equations (the Green–Naghdi Model)

12.15.1 Vector Form of the Nondissipative Thermoelasticity Equations

The coupled nondissipative thermoelasticity equations of the Green–Naghdi model have the form

$$\rho\mathbf{u}_{tt} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\operatorname{div}\mathbf{u} - \alpha\nabla T + \rho\mathbf{f}, \tag{1}$$

$$T_{tt} = a\Delta T - \beta(\operatorname{div}\mathbf{u})_{tt}. \tag{2}$$

Equations (1)–(2) differ from the ordinary thermoelasticity equations (see Eqs. (1)–(2) in Section 12.14) in the second equation, which for the Green–Naghdi model contains second time derivatives.

Below we use a modified form of the vector equation (1),

$$\mathbf{u}_{tt} = c_2^2\Delta\mathbf{u} + (c_1^2 - c_2^2)\nabla\operatorname{div}\mathbf{u} - (\alpha/\rho)\nabla T + \mathbf{f}. \tag{3}$$

12.15.2 Decompositions of the Nondissipative Thermoelasticity Equations with $\mathbf{f} = \mathbf{0}$

► **Decomposition based on two stream functions.**

Each solution of the nondissipative thermoelasticity equations (2)–(3) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\begin{gathered} \mathbf{u} = \nabla\varphi + \mathbf{v}, \quad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt}\right), \\ \mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi_y^{(1)}, \quad v_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad v_3 = -\psi_y^{(2)}, \end{gathered} \tag{4}$$

where $\psi^{(1)} = \psi^{(1)}(\mathbf{x}, t)$ and $\psi^{(2)} = \psi^{(2)}(\mathbf{x}, t)$ are some solutions of the wave equation

$$\psi_{tt} - c_2^2\Delta\psi = 0, \tag{5}$$

and the function $\varphi = \varphi(\mathbf{x}, t)$ is a solution of the fourth-order equation

$$\varphi_{tttt} - [a + c_1^2 + (\alpha\beta)/\rho]\Delta\varphi_{tt} + ac_1^2\Delta\Delta\varphi = 0. \tag{6}$$

Equation (6) can be represented in the form

$$(\partial_t^2 - k_1\Delta)(\partial_t^2 - k_2\Delta)[\varphi] = 0, \tag{7}$$

where k_1 and k_2 are the roots of the quadratic equation

$$k^2 - [a + c_1^2 + (\alpha\beta)/\rho]k + ac_1^2 = 0. \tag{8}$$

For $\varphi_t \not\equiv 0$, the general solution of the fourth-order equation (7) can be represented as the sum

$$\varphi = \varphi_1 + \varphi_2,$$

where φ_1 and φ_2 are arbitrary solutions of the wave equations

$$(\partial_t^2 - k_1\Delta)\varphi_1 = 0, \quad (\partial_t^2 - k_2\Delta)\varphi_2 = 0.$$

► **Using the Stokes–Helmholtz representation of the displacement vector.**

Each solution of the nondissipative thermoelasticity equations (2)–(3) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt}\right), \tag{9}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the wave vector equation

$$\boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} = \mathbf{0}, \tag{10}$$

and the function φ is a solution of the fourth-order equation (7).

► **Toroidal–poloidal decomposition.**

Each solution of the nondissipative thermoelasticity equations (2)–(3) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{x}\xi) + \operatorname{curl}\operatorname{curl}(\mathbf{x}\eta), \qquad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt}\right), \qquad \mathbf{x} = (x, y, z),$$

where ξ and η are scalar functions satisfying the wave equations

$$\xi_{tt} - c_2^2\Delta\xi = 0, \qquad \eta_{tt} - c_2^2\Delta\eta = 0 \tag{11}$$

and the function φ is a solution of the fourth-order equation (7).

► **Two other representations of solutions via three scalar functions.**

The solutions of Eqs. (2)–(3) with $\mathbf{f} = \mathbf{0}$ can also be represented in the following two forms:

$$\begin{aligned} \mathbf{u} &= \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{b}\eta), \qquad && \mathbf{a}\cdot\mathbf{b} \neq 0, \\ \mathbf{u} &= \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{x}\eta), \qquad && |\mathbf{a}| \neq 0, \end{aligned}$$

where the functions ξ and η are solutions of the wave equations (5) and the function φ is a solution of the fourth-order equation (7).

12.15.3 Decompositions of Thermoelasticity Equations with $\mathbf{f} \neq \mathbf{0}$

► **Using the Stokes–Helmholtz representation of the mass force.**

Let the mass force be represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of the nondissipative thermoelasticity equations (2)–(3) admits the representation

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad T = \frac{\rho}{\alpha}\left(c_1^2\Delta\varphi - \varphi_{tt} + \gamma\right),$$

where the functions $\boldsymbol{\Psi}$ and φ satisfy the independent equations

$$\begin{gathered} \boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} = \boldsymbol{\omega}, \\ \varphi_{tttt} - [a + c_1^2 + (\alpha\beta)/\rho]\Delta\varphi_{tt} + ac_1^2\Delta\Delta\varphi = \gamma_{tt} - a\Delta\gamma. \end{gathered}$$

► **Decomposition not requiring the force to split into components.**

The solution of system (2)–(3) can be represented by the formulas

$$\begin{aligned} \mathbf{u} &= [(\partial_t^2 - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t^2\Delta][\mathbf{w}] \\ &\quad + \nabla\bigl\{\varphi + [(\alpha\beta/\rho)\partial_t^2 + (c_1^2 - c_2^2)(\partial_t^2 - a\Delta)][\operatorname{div}\mathbf{w}]\bigr\}, \\ T &= -\frac{\rho}{\alpha}(\partial_t^2 - c_1^2\Delta)[\varphi] - \beta\partial_t^2(\partial_t^2 - c_2^2\Delta)[\operatorname{div}\mathbf{w}], \end{aligned}$$

where the vector function $\mathbf{w}$ and the scalar function φ satisfy the equations

$$\begin{aligned} (\partial_t^2 - c_2^2\Delta)[(\partial_t^2 - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t^2\Delta][\mathbf{w}] &= \mathbf{f}, \\ [(\partial_t - a\Delta)(\partial_t^2 - c_1^2\Delta) - (\alpha\beta/\rho)\partial_t^2\Delta][\varphi] &= 0. \end{aligned} \tag{12}$$

If $\mathbf{u}_t \not\equiv \mathbf{0}$, then the general solution of the first equation (12) with $\mathbf{f}=\mathbf{0}$ can be represented as the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3,$$

where $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ are arbitrary solutions of the three wave equations

$$(\partial_t^2 - k_1\Delta)[\mathbf{w}_1] = \mathbf{0}, \quad (\partial_t^2 - k_2\Delta)[\mathbf{w}_2] = \mathbf{0}, \quad (\partial_t^2 - c_2^2\Delta)[\mathbf{w}_3] = \mathbf{0}.$$

Here k_1 and k_2 are the roots of the quadratic equation (8).

⊙ *Literature for Section 12.15*: A. E. Green and P. M. Naghdi (1993), A. D. Polyanin and S. A. Lychev (2014a,b), S. A. Lychev and A. D. Polyanin (2015).

12.16 Viscoelasticity Equations

12.16.1 Vector Form of Viscoelasticity Equations

Coupled viscoelasticity equations have the form

$$\rho\mathbf{u}_{tt} - \mu_0\Delta\mathbf{u} - \mu_1\Delta\mathbf{u}_t - \nabla[(\lambda_0+\mu_0)\operatorname{div}\mathbf{u} + (\lambda_1+\mu_1)\operatorname{div}\mathbf{u}_t] = \rho\mathbf{f}, \tag{1}$$

where λ_0 and μ_0 are the elastic Lamé moduli, λ_1 and μ_1 are the viscosity coefficients, and the remaining notation is the same as in the elasticity equations (see Eqs. (1) in Section 12.6).

In what follows, we use a modified form of the vector equation (1),

$$\mathbf{u}_{tt} - c_2^2\Delta\mathbf{u} - \nu\Delta\mathbf{u}_t - \nabla[(c_1^2-c_2^2)\operatorname{div}\mathbf{u} + (\nu+\varkappa)\operatorname{div}\mathbf{u}_t] = \mathbf{f}, \tag{2}$$

where $c_1 = \sqrt{(2\mu_0+\lambda_0)/\rho}$, $c_2 = \sqrt{\mu_0/\rho}$, $\nu = \mu_1/\rho$, and $\varkappa = \lambda_1/\rho$.

12.16.2 Decompositions of Viscoelasticity Equations with $\mathbf{f} = \mathbf{0}$

► **Decomposition based on two stream functions.**

Any solution of the homogeneous viscoelasticity equations (2) with $\mathbf{f} = \mathbf{0}$ can also be represented as

$$u_1 = \varphi_x + \psi_y^{(1)}, \quad u_2 = \varphi_y - \psi_x^{(1)} + \psi_z^{(2)}, \quad u_3 = \varphi_z - \psi_y^{(2)}, \tag{3}$$

where the functions $\varphi = \varphi(x,y,z)$, $\psi^{(1)} = \psi^{(1)}(x,y,z)$, and $\psi^{(2)} = \psi^{(2)}(x,y,z)$ are solutions of the third-order equations

$$\varphi_{tt} - c_1^2\Delta\varphi - (2\nu+\varkappa)\Delta\varphi_t = 0, \tag{4}$$

$$\psi_{tt}^{(1)} - c_2^2\Delta\psi^{(1)} - \nu\Delta\psi_t^{(1)} = 0, \quad \psi_{tt}^{(2)} - c_2^2\Delta\psi^{(2)} - \nu\Delta\psi_t^{(2)} = 0. \tag{5}$$

► **Using the Stokes–Helmholtz representation of the displacement vector.**

Each solution of the viscoelasticity equations (2) with $\mathbf{f} = \mathbf{0}$ can be represented by the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \tag{6}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is a solution of the third-order vector equation

$$\boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} - \nu\Delta\boldsymbol{\Psi}_t = \mathbf{0} \tag{7}$$

and the function φ is a solution of the scalar equation (4).

► **Solution of the Cauchy–Kovalevskaya type.**

Each solution of the viscoelasticity equations (2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = [\partial_t^2 - c_1^2\Delta - (2\nu + \varkappa)\partial_t\Delta][\mathbf{w}] + \nabla[c_1^2 - c_2^2 + (\nu + \varkappa)\partial_t][\operatorname{div}\mathbf{w}], \tag{8}$$

where the vector function $\mathbf{w}$ satisfies the equation

$$(\partial_t^2 - c_2^2\Delta - \nu\partial_t\Delta)[\partial_t^2 - c_1^2\Delta - (2\nu + \varkappa)\partial_t\Delta][\mathbf{w}] = 0. \tag{9}$$

If $\mathbf{u}_t \not\equiv \mathbf{0}$, then the general solution of equation (9) can be represented as

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$(\partial_t^2 - c_2^2\Delta - \nu\partial_t\Delta)[\mathbf{w}_1] = \mathbf{0}, \qquad [\partial_t^2 - c_1^2\Delta - (2\nu + \varkappa)\partial_t\Delta][\mathbf{w}_2] = \mathbf{0}.$$

► **Chadwick–Trowbridge solution (toroidal–poloidal decomposition).**

Each solution of the viscoelasticity equations (2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{x}\xi) + \operatorname{curl}\operatorname{curl}(\mathbf{x}\eta), \qquad \mathbf{x} = (x, y, z),$$

where φ, ξ, and η are scalar functions satisfying the equations

$$\varphi_{tt} - c_1^2\Delta\varphi - (2\nu + \varkappa)\Delta\varphi_t = 0, \tag{10}$$

$$\xi_{tt} - c_2^2\Delta\xi - \nu\Delta\xi_t = 0, \qquad \eta_{tt} - c_2^2\Delta\eta - \nu\Delta\eta_t = 0. \tag{11}$$

► **Two other representations of solutions via three scalar functions.**

The solutions of Eqs. (2) with $\mathbf{f} = \mathbf{0}$ can also be represented in the following two forms:

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{b}\eta), \qquad \mathbf{a}\cdot\mathbf{b} \neq 0,$$
$$\mathbf{u} = \nabla\varphi + \operatorname{curl}(\mathbf{a}\xi + \mathbf{x}\eta), \qquad |\mathbf{a}| \neq 0,$$

where φ, ξ, and η are scalar functions satisfying Eqs. (10)–(11).

12.16.3 Various Forms of Decompositions for Viscoelasticity Equations with $\mathbf{f} \neq 0$

► **Lamé type decomposition of the elasticity equations.**

Assume that the mass force in the viscoelasticity equations (2) is represented as the sum of potential and solenoidal components (it is the Stokes–Helmholtz representation of the mass force; see Section 12.5.3),

$$\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}.$$

In this case, the displacement can be represented by formula (6), where the scalar function φ and the vector function $\boldsymbol{\Psi}$ satisfy the nonhomogeneous third-order equations

$$\begin{aligned} \varphi_{tt} - c_1^2\Delta\varphi - (2\nu + \varkappa)\Delta\varphi_t &= \gamma, \\ \boldsymbol{\Psi}_{tt} - c_2^2\Delta\boldsymbol{\Psi} - \nu\Delta\boldsymbol{\Psi}_t &= \boldsymbol{\omega}. \end{aligned}$$

► **Decomposition of Cauchy–Kovalevskaya type not requiring the force to split into components.**

Each solution of the viscoelasticity equations (2) can be represented in the form (8), where the vector function $\mathbf{w}$ satisfies the equation

$$(\partial_t^2 - c_2^2\Delta - \nu\partial_t\Delta)[\partial_t^2 - c_1^2\Delta - (2\nu + \varkappa)\partial_t\Delta][\mathbf{w}] = \mathbf{f}.$$

⊙ *Literature for Section 12.16*: M. E. Gurtin and E. Sternberg (1962), A. D. Polyanin and S. A. Lychev (2014a,b).

12.17 Maxwell Equations (Electromagnetic Field Equations)

12.17.1 Maxwell Equations in a Medium and Constitutive Relations

The Maxwell equations in a medium are written as

$$\begin{aligned} \operatorname{curl}\mathbf{E} &= -\mathbf{B}_t, \\ \operatorname{curl}\mathbf{H} &= \mathbf{j} + \mathbf{D}_t, \\ \operatorname{div}\mathbf{D} &= \rho, \\ \operatorname{div}\mathbf{B} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{E}$ is the electric field intensity, $\mathbf{H}$ is the magnetic field intensity, $\mathbf{D}$ is the electric induction, $\mathbf{B}$ is the magnetic induction, t is time, $\mathbf{j}$ is the current density, and ρ is the charge density.

The first and fourth Maxwell equations are satisfied identically if one sets

$$\mathbf{B} = \operatorname{curl}\boldsymbol{\Psi}, \qquad \mathbf{E} = -\nabla\Phi - \boldsymbol{\Psi}_t,$$

where the scalar potential Φ and vector potential $\boldsymbol{\Psi}$ are chosen arbitrarily.

The second and third equations in (1) imply the continuity equation

$$\rho_t + \operatorname{div} \mathbf{j} = 0,$$

which expresses the charge conservation law.

According to experimental data, the vector fields **E**, **H**, **D**, **B**, and **j** are not independent and should be supplemented with the material equations (constitutive relations) of the medium. For isotropic linear media, these equations are

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{j} = \lambda \mathbf{E}, \tag{2}$$

where ε and μ are the dielectric permittivity and the magnetic permeability of the medium and λ is the conductance of the medium. For a perfect dielectric, i.e., for a medium that does not conduct current at all, one should set $\lambda = 0$.

12.17.2 Some Transformations and Solutions of the Maxwell Equations

1°. By substituting (2) into the Maxwell equations (1), we obtain

$$\begin{aligned} \operatorname{curl} \mathbf{E} &= -\mu \mathbf{H}_t, \\ \operatorname{curl} \mathbf{H} &= \lambda \mathbf{E} + \varepsilon \mathbf{E}_t, \\ \operatorname{div} \mathbf{E} &= \rho/\varepsilon, \\ \operatorname{div} \mathbf{H} &= 0. \end{aligned} \tag{3}$$

Here the vectors **E** and **H** are the unknowns, and the electric charge density is assumed to be a given function, $\rho = \rho(\mathbf{x}, t)$. System (3) is overdetermined, because it consists of 8 scalar equations for the 6 unknown components of the vector functions **E** and **H**.

Remark 12.15. It follows from the second and third equations in (3) that the charge density distribution cannot be arbitrary and must have the special form

$$\rho = f_1(\mathbf{x}) \exp(-\lambda t/\varepsilon), \tag{4}$$

where $f_1(\mathbf{x})$ is an arbitrary function.

The general solution of the Maxwell equations (3) with $\rho = 0$ is

$$\mathbf{H} = -\operatorname{curl}\operatorname{curl} \boldsymbol{\Psi}, \qquad \mathbf{E} = \mu \operatorname{curl} \boldsymbol{\Psi}_t,$$

where the vector function $\boldsymbol{\Psi}$ satisfies the equation

$$\mu(\varepsilon \boldsymbol{\Psi}_{tt} + \lambda \boldsymbol{\Psi}_t) - \Delta \boldsymbol{\Psi} = \mathbf{0}.$$

2°. By successively eliminating the vectors **H** and **E** from system (3), we obtain two independent overdetermined subsystems:

For the magnetic field intensity,

$$\begin{aligned} \mu(\varepsilon \mathbf{H}_{tt} + \lambda \mathbf{H}_t) - \Delta \mathbf{H} &= \mathbf{0}, \\ \operatorname{div} \mathbf{H} &= 0. \end{aligned} \tag{5}$$

For the electric field intensity,

$$\mu(\varepsilon\mathbf{E}_{tt} + \lambda\mathbf{E}_t) - \Delta\mathbf{E} + \nabla\operatorname{div}\mathbf{E} = \mathbf{0},$$
$$\operatorname{div}\mathbf{E} = \rho/\varepsilon. \tag{6}$$

The general solution of Eqs. (5) for the magnetic field intensity can be expressed via two scalar functions,

$$\mathbf{H} = (H_1, H_2, H_3), \quad H_1 = \psi_y^{(1)}, \quad H_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad H_3 = -\psi_y^{(2)}, \tag{7}$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are arbitrary solutions of the equation

$$\mathrm{L}[\psi] = 0, \quad \mathrm{L} = \mu(\varepsilon\partial_t^2 + \lambda\partial_t) - \Delta. \tag{8}$$

The general solution of Eqs. (5) can also be expressed via the vector $\boldsymbol{\psi}$ (i.e., via three scalar functions)

$$\mathbf{H} = \operatorname{curl}\boldsymbol{\psi}, \quad \mathrm{L}[\boldsymbol{\psi}] = \mathbf{0}, \tag{9}$$

where the operator L is defined in (8).

3°. For the overdetermined system (6) for the electric field intensity to be solvable, the consistency conditions (4) should be satisfied. If this is the case, then the solution can be represented in the form

$$\mathbf{E} = \nabla\varphi + \operatorname{curl}\boldsymbol{\psi}, \quad \varphi = \varphi_0(\mathbf{x}) + \varphi_1(\mathbf{x})\exp(-\lambda t/\varepsilon), \tag{10}$$

where the functions $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ satisfied the Laplace and Poisson equations

$$\Delta\varphi_0 = 0, \quad \Delta\varphi_1 = f_1(\mathbf{x})/\varepsilon,$$

the vector $\boldsymbol{\psi}$ satisfies the equation $\mathrm{L}[\boldsymbol{\psi}] = \mathbf{0}$, and the operator L is defined in (8).

⊙ *Literature for Section 12.17*: V. I. Smirnov (1974, Vol. 2, pp. 375–377), L. D. Landau and E. M. Lifshitz (1980), M. Schwartz (1987), D. J. Griffiths (1999), F. Melia (2001), W. Benenson, J. W. Harris, H. Stocker, and H. Lutz (2002), R. Fitzpatrick (2008), A. D. Polyanin and A. I. Chernoutsan (2011).

12.18 Vector Equations of General Form

12.18.1 Vector Equations Containing Operators div and ∇

► Class of vector equations considered.

Consider the vector equations

$$\mathrm{L}[\mathbf{u}] + \nabla\mathrm{K}[\operatorname{div}\mathbf{u}] = \mathbf{f}, \tag{1}$$

which are systems of three coupled scalar equations for $\mathbf{u} = (u_1, u_2, u_3)$. Here we assume that L and K are arbitrary constant coefficient linear differential operators.

The vector equation (1) is a generalization of the elasticity and viscoelasticity equations considered in Sections 12.6 and 12.16 (see also equation (17) for the fluid velocity in Section 12.12). The more complex system considered below in Section 12.19 also can be reduced to Eq. (1).

► **Invariant transformations.**

1°. The vector equation (1) is invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl} \boldsymbol{\Psi}^{\circ},$$

where the vector function $\boldsymbol{\Psi}^{\circ}$ is an arbitrary solution of the equation $\mathrm{L}[\boldsymbol{\Psi}^{\circ}] = \mathbf{0}$.

2°. The vector equation (1) is invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi,$$

where the scalar function Φ is an arbitrary solution of the equation

$$(\mathrm{L} + \mathrm{K}_1\Delta)[\Phi] = 0.$$

12.18.2 Decompositions of the Homogeneous Vector Equation

1°. Each solution of the homogeneous equation (1) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad \mathbf{v} = (v_1, v_2, v_3),$$
$$v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\psi^{(2)}_y. \tag{2}$$

Since $\operatorname{div}\mathbf{v} = 0$, we have three independent equations for the unknown functions φ, $\psi^{(1)}$, and $\psi^{(2)}$,

$$(\mathrm{L} + \Delta\mathrm{K})[\varphi] = 0, \tag{3}$$
$$\mathrm{L}[\psi^{(1)}] = 0, \quad \mathrm{L}[\psi^{(2)}] = 0. \tag{4}$$

2°. Each solution of the homogeneous equation (1) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \tag{5}$$

where the scalar function φ and the vector function $\boldsymbol{\Psi}$ satisfy the independent equations

$$(\mathrm{L} + \Delta\mathrm{K})[\varphi] = 0, \quad \mathrm{L}[\boldsymbol{\Psi}] = \mathbf{0}. \tag{6}$$

The representation (5) of the solution contains one scalar function more than the representation (2).

3°. Other representations of solutions of the vector equation (1) with $\mathbf{f} = \mathbf{0}$ have the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}_n, \qquad n = 1, \dots, 7, \tag{7}$$

where the scalar function φ satisfies Eq. (2) and the vector function $\boldsymbol{\Psi}_n$ can be expressed via the two scalar functions $\psi^{(1)}$ and $\psi^{(2)}$ by the formulas indicated in Table 12.1. It is assumed in rows 4–7 of Table 12.1 that the operator L in system (1) has the special form

$$\mathrm{L}[u] = \mathrm{L}_1[u] + \mathrm{L}_2[\Delta u],$$

where L_1 and L_2 are linear differential operators in time t. The functions $\psi^{(1)}$ and $\psi^{(2)}$ are described by independent (and identical) equations (4).

4°. The solution of system (1) with $\mathbf{f} = \mathbf{0}$ can also be represented in the following form (see the proof in Section 19.3.1):

$$\mathbf{u} = \bigl(\mathrm{L} + \Delta\mathrm{K}\bigr)[\mathbf{w}] - \nabla\mathrm{K}[\operatorname{div}\mathbf{w}], \tag{8}$$

where the vector function $\mathbf{w}$ satisfies the equation

$$\mathrm{L}\bigl(\mathrm{L} + \Delta\mathrm{K}\bigr)[\mathbf{w}] = \mathbf{0}. \tag{9}$$

The vector equation (9) consists of three independent scalar equations.

The general solution of the homogeneous equation (9) for $\Delta\mathrm{K} \neq \mathrm{const}\,\mathrm{L}$ can be represented as the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$\mathrm{L}[\mathbf{w}_1] = \mathbf{0}, \qquad \bigl(\mathrm{L} + \Delta\mathrm{K}\bigr)[\mathbf{w}_2] = \mathbf{0}.$$

12.18.3 Decompositions of the Nonhomogeneous Vector Equation

1°. For the vector function $\mathbf{f}$ in Eq. (1), we use the representation

$$\mathbf{f} = \nabla\gamma + \mathbf{h}, \qquad \mathbf{h} = (\theta_y^{(1)}, -\theta_x^{(1)} + \theta_z^{(2)}, -\theta_y^{(2)}),$$

where the formulas for the functions γ, $\theta^{(1)}$, and $\theta^{(2)}$ are given in Section 12.5.3. Then the solution of Eq. (1) admits the representation (2), where the scalar function φ and the stream functions $\psi^{(1)}$ and $\psi^{(2)}$ satisfy the independent equations

$$(\mathrm{L} + \Delta\mathrm{K})[\varphi] = \gamma, \quad \mathrm{L}[\psi^{(1)}] = \theta^{(1)}, \quad \mathrm{L}[\psi^{(21)}] = \theta^{(2)}.$$

2°. Assume that the mass force $\mathbf{f}$ in the vector equation (1) is represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of vector equation (1) admits the representation (5), where the scalar function φ and the vector function $\boldsymbol{\Psi}$ satisfy the independent equations

$$(\mathrm{L} + \Delta\mathrm{K})[\varphi] = \gamma, \quad \mathrm{L}[\boldsymbol{\Psi}] = \boldsymbol{\omega}. \tag{10}$$

3°. The solution of the vector equation (1) can be represented without decomposing the vector $\mathbf{f}$ into components in the form (8), where the vector function $\mathbf{w}$ satisfies the equation

$$\mathrm{L}\bigl(\mathrm{L} + \Delta\mathrm{K}\bigr)[\mathbf{w}] = \mathbf{f}. \tag{11}$$

The vector equation (11) consists of three independent scalar equations.

4°. The solution of the vector equation (1) can also be represented in the form

$$\mathbf{u} = \bigl(\mathrm{L} + \Delta\mathrm{K}\bigr)[\mathbf{w}] + \nabla\bigl\{\varphi - \mathrm{K}[\operatorname{div}\mathbf{w}] - \mathrm{L}(\mathbf{Q}\cdot\mathbf{w})\bigr\}, \tag{11a}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3)$, Q_n are arbitrary constants or arbitrary constant coefficient linear differential operators with respect to x, y, z, and t, the vector function $\mathbf{w}$ satisfies Eq. (11), and the function φ is a solution of the equation

$$\bigl(\mathrm{L} + \Delta\mathrm{K}\bigr)[\varphi] = \mathbf{Q}\cdot\mathbf{f}. \tag{11b}$$

Note two special cases of the operator $\mathbf{\Omega}$,

$$\mathbf{Q} = \mathbf{a}, \qquad \mathbf{Q}\cdot\mathbf{w} = \mathbf{a}\cdot\mathbf{w},$$
$$\mathbf{Q} = \nabla, \qquad \mathbf{Q}\cdot\mathbf{w} = \operatorname{div}\mathbf{w},$$

where **a** is an arbitrary constant vector. (With $\mathbf{a}=\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$, we have $\mathbf{Q}\cdot\mathbf{w}=w_1+w_2+w_3$.)

The representation of the solution of the vector equation (1) in the form (8), (11) is a special case of the representation (11a)–(11b) with $\mathbf{Q} = \mathbf{0}$ and $\varphi = 0$.

12.18.4 Vector Equations Containing More General Operators

► Class of vector equations considered.

Consider the vector equation

$$\mathrm{L}[\mathbf{u}] + \mathbf{\Lambda}\mathrm{K}[\mathbf{\Omega}\cdot\mathbf{u}] = \mathbf{f}, \tag{12}$$

where $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$; $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$; L, K, Λ_n, and Ω_n are constant coefficient linear differential operators with respect to t, x, y, and z. In the special case $\mathbf{\Lambda} = \mathbf{\Omega} = \nabla$, system (12) becomes system (1).

► Decompositions of the homogeneous vector equation.

1°. We seek the solutions of the vector equation (12) with $\mathbf{f} = \mathbf{0}$ in the form

$$\begin{gathered}\mathbf{u} = \mathbf{\Lambda}\varphi + \mathbf{v}, \quad \mathbf{v} = (v_1, v_2, v_3),\\ v_1 = \Omega_2\psi^{(1)}, \quad v_2 = -\Omega_1\psi^{(1)} + \Omega_3\psi^{(2)}, \quad v_3 = -\Omega_2\psi^{(2)}.\end{gathered} \tag{13}$$

Since $\mathbf{\Omega}\cdot\mathbf{v} = 0$, we obtain two independent identical homogeneous equations for the functions $\psi^{(1)}$ and $\psi^{(2)}$,

$$\mathrm{L}\,[\psi^{(1)}] = 0, \quad \mathrm{L}\,[\psi^{(2)}] = 0, \tag{14}$$

and an independent equation for the function φ,

$$\mathrm{L}[\varphi] + \mathrm{K}[\mathbf{\Omega}\cdot\mathbf{\Lambda}\varphi] = 0. \tag{15}$$

Remark 12.16. The functions $\psi^{(1)}$ and $\psi^{(2)}$ in (13) are generalized counterparts of the stream functions in hydrodynamics; see Remark 12.9 in Section 12.7.

2°. See also the next paragraph with $\mathbf{f} = \mathbf{0}$.

► Decompositions of the nonhomogeneous vector equation.

The solution of the vector equation (12) can be represented in the form

$$\mathbf{u} = \big(\mathrm{L} + \mathrm{K}\mathbf{\Omega}\cdot\mathbf{\Lambda}\big)[\mathbf{w}] - \mathbf{\Lambda}\mathrm{K}[\mathbf{\Omega}\cdot\mathbf{w}], \tag{16}$$

where the vector function **w** satisfies the equation

$$\mathrm{L}\big(\mathrm{L} + \mathrm{K}\mathbf{\Omega}\cdot\mathbf{\Lambda}\big)[\mathbf{w}] = \mathbf{f}. \tag{17}$$

The vector equation (17) consists of three independent scalar equations; i.e., we have obtained complete decomposition of the original system (12) in this case. In the special case of $\mathbf{\Lambda} = \mathbf{\Omega} = \nabla$, formulas (16) and Eq. (17) become (8) and (11), respectively.

The general solution of the homogeneous equation (17) with $\mathbf{f} = \mathbf{0}$ and $\mathrm{K}\mathbf{\Omega}\cdot\mathbf{\Lambda} \neq \mathrm{const}\,\mathrm{L}$ can be represented in the form of the sum

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$\mathrm{L}[\mathbf{w}_1] = \mathbf{0}, \qquad \big(\mathrm{L} + \mathrm{K}\mathbf{\Omega}\cdot\mathbf{\Lambda}\big)[\mathbf{w}_2] = \mathbf{0}.$$

⊙ *Literature for Section 12.19*: A. D. Polyanin and A. I. Zhurov (2013), A. D. Polyanin and S. A. Lychev (2014a,b), S. A. Lychev and A. D. Polyanin (2015).

12.19 General Systems Involving Vector and Scalar Equations: Part I

12.19.1 Systems Containing Operators div and ∇

► **Class of systems considered.**

Consider the systems consisting of one vector and one scalar equation of the form

$$\mathrm{L}[\mathbf{u}] + \nabla\big(\sigma p + \mathrm{K}_1[\mathrm{div}\,\mathbf{u}]\big) = \mathbf{f}, \tag{1}$$

$$\mathrm{M}_1[p] + \mathrm{M}_2[\mathrm{div}\,\mathbf{u}] = f_4, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and p are the unknown functions, L, K_1, M_1, and M_2 are arbitrary constant coefficient linear differential operators with respect to t, x, y, and z, σ is a constant, and $\mathbf{f} = (f_1, f_2, f_3)$ and f_4 are given functions.

Systems of coupled equations of the form (1)–(2) often occur in continuum mechanics and physics; see Sections 12.6–12.16 and 12.17.2. In the degenerate case of $\sigma = 0$, Eq. (1) is independent of p and coincides with the vector equation for $\mathbf{u}$ considered in Section 12.18. (In this case, the function p is found from Eq. (2) after $\mathbf{u}$ has been determined.) In what follows, we consider the nondegenerate case of $\sigma \neq 0$.

► **Invariant transformations.**

1°. The coupled equations (1)–(2) are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \mathrm{curl}\,\mathbf{\Psi}^\circ, \qquad p = \widetilde{p},$$

where $\mathbf{\Psi}^\circ$ is an arbitrary solution of the equation $\mathrm{L}[\mathbf{\Psi}^\circ] = \mathbf{0}$.

2°. The coupled equations (1)–(2) are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi, \qquad p = \widetilde{p} - \frac{1}{\sigma}(\mathrm{L} + \mathrm{K}_1\Delta)[\Phi],$$

where the function Φ is an arbitrary solution of the equation

$$(\mathrm{M}_1\mathrm{L} + \mathrm{M}_1\mathrm{K}_1\Delta - \sigma\mathrm{M}_2\Delta)[\Phi] = 0.$$

12.19.2 Decompositions of Systems with Homogeneous Vector Equation

1°. Each solution of system (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\begin{aligned} &\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = -\frac{1}{\sigma}\Big(\mathrm{L}\,[\varphi] + \mathrm{K}_1\,[\Delta\varphi]\Big), \\ &\mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi_y^{(1)}, \quad v_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad v_3 = -\psi_y^{(2)}, \end{aligned} \tag{3}$$

where the functions $\psi^{(1)}$, $\psi^{(2)}$, and φ satisfy the independent equations

$$\mathrm{L}\,[\psi^{(1)}] = 0, \quad \mathrm{L}\,[\psi^{(2)}] = 0, \tag{4}$$

$$\mathrm{M}_1\mathrm{L}\,[\varphi] + (\mathrm{M}_1\mathrm{K}_1 - \sigma\mathrm{M}_2)\,[\Delta\varphi] = -\sigma f_4. \tag{5}$$

The solution representation (3) contains one unknown function less than the original system (1)–(2).

Remark 12.17. If $\mathrm{M}_1 = 0$, then an arbitrary function $p_0(t)$ can be added on the right-hand side of the second formula in (3) for p.

2°. Each solution of system (1)–(2) with $\mathbf{f} = \mathbf{0}$ can be represented in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \quad p = -\frac{1}{\sigma}\Big(\mathrm{L}\,[\varphi] + \mathrm{K}_1\,[\Delta\varphi]\Big), \tag{6}$$

where the vector function $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ satisfies the equation

$$\mathrm{L}[\boldsymbol{\Psi}] = \mathbf{0} \tag{7}$$

and the scalar function φ is determined from Eq. (5).

3°. See also Item 3° in Section 12.19.3 with $\mathbf{f} = \mathbf{0}$.

12.19.3 Decompositions of Systems with Nonhomogeneous Vector Equation

1°. For the vector function $\mathbf{f}$ in Eq. (1) we use the representation

$$\mathbf{f} = \nabla\gamma + \mathbf{h}, \qquad \mathbf{h} = (\theta_y^{(1)}, -\theta_x^{(1)} + \theta_z^{(2)}, -\theta_y^{(2)}),$$

where the formulas for finding the functions γ, $\theta^{(1)}$, and $\theta^{(2)}$ are given in Section 12.5.3. Then the solution of system (1)–(2) admits the representation

$$\begin{aligned} &\mathbf{u} = \nabla\varphi + \mathbf{v}, \quad p = \frac{1}{\sigma}\Big(\gamma - \mathrm{L}\,[\varphi] - \mathrm{K}_1\,[\Delta\varphi]\Big), \\ &\mathbf{v} = (v_1, v_2, v_3), \quad v_1 = \psi_y^{(1)}, \quad v_2 = -\psi_x^{(1)} + \psi_z^{(2)}, \quad v_3 = -\psi_y^{(2)}, \end{aligned}$$

where the stream functions $\psi^{(1)}$ and $\psi^{(2)}$ satisfy the independent equations

$$\mathrm{L}[\psi^{(1)}] = \theta^{(1)}, \quad \mathrm{L}[\psi^{(21)}] = \theta^{(2)}$$

and the scalar function φ is described by the independent equation

$$-\mathrm{M}_1\mathrm{L}\,[\varphi] + (\sigma\mathrm{M}_2 - \mathrm{M}_1\mathrm{K}_1)\,[\Delta\varphi] = \sigma f_4 - \mathrm{M}_1[\gamma]. \tag{8}$$

2°. Assume that the mass force $\mathbf{f}$ in the vector equation (1) is represented as the sum $\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}$ (see Section 12.5.3). Then the solution of system (1)–(2) admits the representation (6), where the vector function $\boldsymbol{\Psi}$ satisfies the equation

$$\mathrm{L}[\boldsymbol{\Psi}] = \boldsymbol{\omega} \tag{9}$$

and the scalar function φ is described by the independent equation (8).

3°. The solution of system (1)–(2) can also be represented without splitting the vector $\mathbf{f}$ into components as follows (for the proof, see Section 19.3.2):

$$\begin{aligned} \mathbf{u} &= (\mathrm{M}_1\mathrm{L} + \Delta\mathrm{M}_1\mathrm{K}_1 - \sigma\Delta\mathrm{M}_2)[\mathbf{w}] + \nabla\bigl\{\varphi + (\sigma\mathrm{M}_2 - \mathrm{M}_1\mathrm{K}_1)[\operatorname{div}\mathbf{w}]\bigr\},\\ p &= -\frac{1}{\sigma}\bigl(\mathrm{L} + \Delta\mathrm{K}_1\bigr)[\varphi] - \mathrm{M}_2\mathrm{L}[\operatorname{div}\mathbf{w}], \end{aligned} \tag{10}$$

where the vector and scalar functions $\mathbf{w}$ and φ satisfy the independent equations

$$\mathrm{L}(\mathrm{M}_1\mathrm{L} + \Delta\mathrm{M}_1\mathrm{K}_1 - \sigma\Delta\mathrm{M}_2)[\mathbf{w}] = \mathbf{f}, \tag{11}$$

$$(\mathrm{M}_1\mathrm{L} + \Delta\mathrm{M}_1\mathrm{K}_1 - \sigma\Delta\mathrm{M}_2)[\varphi] = -\sigma f_4. \tag{12}$$

Remark 12.18. The solution of the homogeneous equation (11) can be represented in the form of the sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$\mathrm{L}[\mathbf{w}_1] = \mathbf{0}, \qquad (\mathrm{M}_1\mathrm{L} + \Delta\mathrm{M}_1\mathrm{K}_1 - \sigma\Delta\mathrm{M}_2)[\mathbf{w}_2] = \mathbf{0}.$$

Remark 12.19. For $f_4 = 0$, one can set $\varphi = 0$ in formulas (10) and Eq. (12) without loss of generality.

12.19.4 Equations for u and p. Reduction to One Vector Equation

► **Independent equations for u and p.**

1°. Let $\mathrm{M}_1 \neq 0$. We apply the operator M_1 to and then eliminate $\mathrm{M}_1[p]$ from Eq. (1) with the use of Eq. (2). As a result, we obtain an equation for $\mathbf{u}$,

$$\mathrm{M}_1\mathrm{L}[\mathbf{u}] + \nabla\bigl\{(\mathrm{K}_1\mathrm{M}_1 - \sigma\mathrm{M}_2)[\operatorname{div}\mathbf{u}]\bigr\} = \mathrm{M}_1[\mathbf{f}] - \sigma\nabla f_4. \tag{13}$$

2°. By applying the operator div to Eq. (1), after elementary transformations we obtain

$$(\mathrm{L} + \Delta\mathrm{K}_1)[\operatorname{div}\mathbf{u}] + \sigma\Delta p = \operatorname{div}\mathbf{f}. \tag{14}$$

Let us apply the operator $(\mathrm{L} + \Delta\mathrm{K}_1)$ to Eq. (2) and the operator $-\mathrm{M}_2$ to Eq. (14) and then add the resulting relations. Then we obtain an equation for p,

$$\bigl(\mathrm{L}\mathrm{M}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\bigr)[p] = (\mathrm{L} + \Delta\mathrm{K}_1)[f_4] - \mathrm{M}_2[\operatorname{div}\mathbf{f}]. \tag{15}$$

The solutions of the independent equations (13) and (15) do not in general give a solution of the original system (1)–(2), because (13) and (15) were obtained by an application of various differential operators to the original equations, which results in a broader set of solutions. The spurious solutions can be discarded with the use of the original equations (1)–(2), which can be interpreted as differential constraints for Eqs. (13) and (15).

► **Reduction of the system with $f_4 = 0$ to one vector equation.**

Let us reduce system (1)–(2) to one vector equation. To this end, set

$$\mathbf{u} = \mathrm{M}_1[\boldsymbol{\xi}], \quad p = -\mathrm{M}_2[\operatorname{div}\boldsymbol{\xi}], \tag{16}$$

where $\boldsymbol{\xi}$ is the new unknown vector function. By substituting (16) into (1), we obtain the equation

$$\mathrm{LM}_1[\boldsymbol{\xi}] + \nabla\big(\mathrm{K}_1\mathrm{M}_1 - \sigma\mathrm{M}_2\big)[\operatorname{div}\boldsymbol{\xi}] = \mathbf{f}. \tag{17}$$

Equation (2) with $f_4 = 0$ is identically satisfied after the substitution of the expressions (16).

Vector equations of the form (17) are considered in Section 12.18. By using the results of this section, we arrive at the following representation of the solution of the vector equation (17):

$$\boldsymbol{\xi} = \big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\boldsymbol{\eta}] + \nabla\big\{(\sigma\mathrm{M}_2 - \mathrm{K}_1\mathrm{M}_1\big)[\operatorname{div}\boldsymbol{\eta}]\big\}, \tag{18}$$

where the vector function $\boldsymbol{\eta}$ satisfies the equation

$$\mathrm{LM}_1\big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\boldsymbol{\eta}] = \mathbf{f}, \tag{19}$$

which consists of three independent scalar equations for the components of the vector $\boldsymbol{\eta}$.

We point out that the above-described full decomposition of the original system consisting of the four equations (1)–(2) for $f_4 = 0$ provides a description of the solution via the three components of the vector $\boldsymbol{\eta}$. (The solution was expressed via four functions in Section 12.19.3, Item 3°.)

Let us represent the solution (18) and Eq. (19) in terms of the original unknown $\mathbf{u} = \mathrm{M}_1[\boldsymbol{\xi}]$ (see formula (16)). To this end, we apply the operator M_1 to (18) and introduce the new unknown $\widetilde{\boldsymbol{\eta}} = \mathrm{M}_1[\boldsymbol{\eta}]$. As a result, we obtain

$$\mathbf{u} = \big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\widetilde{\boldsymbol{\eta}}] + \nabla\big\{(\sigma\mathrm{M}_2 - \mathrm{K}_1\mathrm{M}_1\big)[\operatorname{div}\widetilde{\boldsymbol{\eta}}]\big\}. \tag{20}$$

Equation (19) in terms of $\widetilde{\boldsymbol{\eta}} = \mathrm{M}_1[\boldsymbol{\eta}]$ can be reduced to the form

$$\mathrm{L}\big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\widetilde{\boldsymbol{\eta}}] = \mathbf{f}. \tag{21}$$

Formula (20), up to notation ($\widetilde{\boldsymbol{\eta}}$ should be replaced by $\mathbf{w}$), coincides with the first formula in (10) with $\varphi = 0$; Eqs. (21) and (11) coincide in this case. We see that for $f_4 = 0$ one can set $\varphi = 0$ in formulas (10) without loss of generality.

12.19.5 Systems Containing More General Operators

Consider the system

$$\mathrm{L}[\mathbf{u}] + \boldsymbol{\Lambda}\big(\sigma p + \mathrm{K}_1[\boldsymbol{\Omega}\cdot\mathbf{u}]\big) = \mathbf{0}, \tag{22}$$

$$\mathrm{M}_1[p] + \mathrm{M}_2[\boldsymbol{\Omega}\cdot\mathbf{u}] = f_4, \tag{23}$$

where $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$; $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$; L, K_1, M_1, M_2, Λ_n, and Ω_n are constant coefficient linear differential operators with respect to t, x, y, and z; and σ is a constant. In the special case of $\boldsymbol{\Lambda} = \boldsymbol{\Omega} = \nabla$, system (22)–(23) becomes system (1)–(2) with $\mathbf{f} = 0$.

In the degenerate case of $\sigma = 0$, Eq. (22) was considered in Section 12.18.4. In what follows, we assume that $\sigma \neq 0$.

The solution of system (22)–(23) can be represented in the form

$$\begin{gathered}\mathbf{u} = \boldsymbol{\Lambda}\varphi + \mathbf{v}, \quad \mathbf{v} = (v_1, v_2, v_3), \quad p = -\frac{1}{\sigma}\Big(\mathrm{L}\,[\varphi] + \mathrm{K}_1\,[\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\varphi]\Big),\\ v_1 = \Omega_2\psi^{(1)}, \quad v_2 = -\Omega_1\psi^{(1)} + \Omega_3\psi^{(2)}, \quad v_3 = -\Omega_2\psi^{(2)},\end{gathered} \tag{24}$$

where the functions $\psi^{(1)}$ and $\psi^{(2)}$ are arbitrary solutions of the two identical independent equations

$$\mathrm{L}\,[\psi^{(1)}] = 0, \quad \mathrm{L}\,[\psi^{(2)}] = 0,$$

and the function φ is described by the equation

$$\mathrm{M}_1\mathrm{L}[\varphi] + (\mathrm{M}_1\mathrm{K}_1 - \sigma\mathrm{M}_2)[\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\varphi] = -\sigma f_4.$$

⊙ *Literature for Section 12.19*: A. D. Polyanin and A. I. Zhurov (2013), A. D. Polyanin and S. A. Lychev (2014a,b), S. A. Lychev and A. D. Polyanin (2015).

12.20 General Systems Involving Vector and Scalar Equations: Part II

12.20.1 Class of Systems Considered

Consider a system of coupled equations of the form

$$\mathrm{L}[\mathbf{u}] + \nabla\mathrm{K}[\mathbf{u}, p] = \mathbf{f}, \tag{1}$$

$$\mathrm{M}[\mathbf{u}, p] = f_4, \tag{2}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$ and $p = p(\mathbf{x}, t)$ are the unknown functions, $\mathbf{x} = (x, y, z)$, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1, f_2, f_3)$ and $f_4 = f_4(\mathbf{x}, t)$ are given functions, and L, K, and M are constant coefficient linear differential operators with respect to x, y, z, and t.

Remark 12.20. The coefficients of the operator L may depend on time t, and the coefficients of the operators K and M may depend on all independent variables x, y, z, and t.

12.20.2 Asymmetric Decomposition

Each solution of system (1)–(2) can be represented in the form

$$\mathbf{u} = \nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, \quad p = p, \tag{3}$$

where $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors of the y- and z-axes of the Cartesian coordinate system, the two scalar functions $v_2 = v_2(\mathbf{x}, t)$ and $v_3 = v_3(\mathbf{x}, t)$ satisfy two independent linear equations of the same type,

$$\mathrm{L}\,[v_2] = f_2 - \partial_y F, \quad \mathrm{L}\,[v_3] = f_3 - \partial_z F, \tag{4}$$

$$F = F(\mathbf{x}, t) = F(x, y, z, t) = \int_0^x f_1(x_1, y, z, t)\,dx_1, \tag{5}$$

and the functions $\varphi = \varphi(\mathbf{x}, t)$ and $p = p(\mathbf{x}, t)$ are determined from the system of equations

$$\mathrm{L}\,[\varphi] + \mathrm{K}\,[\nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, p] = F, \tag{6}$$

$$\mathrm{M}\,[\nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, p] = f_4. \tag{7}$$

To avoid introducing excessive new notation, in (3) and in the following we write $p = p$ instead of $p = \widetilde{p}$, where $\widetilde{p}$ is the new unknown function.

System (4)–(7) consists of two independent linear equations (4) and a subsystem of two coupled equations (6)–(7) and is substantially simpler than the original system of four coupled linear equations (1)–(2).

12.20.3 Symmetric Decomposition

Each solution of system (1)–(2) can also be represented in the symmetric form

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = p, \tag{8}$$

where the vector function $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the independent linear nonhomogeneous equation

$$\mathrm{L}[\mathbf{v}] = \mathbf{f} - \nabla G \tag{9}$$

and the functions φ and p are described by the system of equations

$$\mathrm{L}\,[\varphi] + \mathrm{K}\,[\nabla\varphi + \mathbf{v}, p] = G, \tag{10}$$

$$\mathrm{M}\,[\nabla\varphi + \mathbf{v}, p] = f_4. \tag{11}$$

Equations (9)–(11) contain the arbitrary scalar function $G = G(\mathbf{x}, t)$.

Remark 12.21. The representation of the components of the vector (8) contains one extra (additional) function as compared with the representation (3). This permits one to simplify Eqs. (9)–(11) by imposing an additional condition on the components v_1, v_2, v_3 of the vector function $\mathbf{v}$ and by choosing an appropriate function G. In particular, without loss in generality, one can set $f_1 = G_x$ and $v_1 = 0$ in (8)–(11), which gives the representation (3)–(7) with $F = G$.

Remark 12.22. All coupled equations of continuum mechanics and physics, as well as the more general equations considered in Sections 12.6–12.19, are special cases of system (1)–(2).

⊙ *Literature for Section 12.20*: A. D. Polyanin and A. I. Zhurov (2013), I. I. Lipatov and A. D. Polyanin (2013), A. D. Polyanin and S. A. Lychev (2014a,b).

Part II

Analytical Methods

Chapter 13

Methods for First-Order Linear PDEs

13.1 Linear PDEs with Two Independent Variables

13.1.1 Special First-Order Linear PDEs with Two Independent Variables

► **Physical interpretation and the characteristic equation.**

Consider a first-order linear homogeneous partial differential equation with two independent variables of the special form

$$f(x,y)\frac{\partial w}{\partial x} + g(x,y)\frac{\partial w}{\partial y} = 0. \tag{13.1.1.1}$$

Equation (13.1.1.1) describes a steady-state distribution of the concentration of a substance in a plane flow (without regard to diffusion). Moreover, it is assumed that the fluid velocity components along the x- and y-axes are specified by the functions f and g.

The transformation

$$w = \Psi(u),$$

where $\Psi(u)$ is an arbitrary function ($\Psi \not\equiv \text{const}$), preserves the form of Eq. (13.1.1.1).

The first-order ordinary differential equation

$$\frac{dx}{f(x,y)} = \frac{dy}{g(x,y)} \tag{13.1.1.2}$$

is called the *characteristic equation* corresponding to the partial differential equation (13.1.1.1). The integral curves of Eq. (13.1.1.2) are called *characteristics*.

Remark 13.1. Suppose that the variables x and y belong to a domain V. Let the functions $f(x,y)$ and $g(x,y)$ be continuously differentiable with respect to both x and y in V, and let $f^2(x,y) + g^2(x,y) \neq 0$ in V. Then there exists a unique characteristic through each point of V.

► Formula for the general solution. General solutions of some first-order PDEs.

Let the general solution of the characteristic equation (13.1.1.2) be given by

$$\Xi(x, y) = C, \tag{13.1.1.3}$$

where C is an arbitrary constant. Then the general solution of Eq. (13.1.1.1) has the form

$$w = \Phi(\Xi), \tag{13.1.1.4}$$

where $\Phi = \Phi(\Xi)$ is an arbitrary function. The left-hand side $\Xi(x, y)$ of Eq. (13.1.1.3) is called a *first integral* of the partial differential equation (13.1.1.1).

Remark 13.2. In formulas (13.1.1.3) and (13.1.1.4), one can take an arbitrary nonconstant particular solution of Eq. (13.1.1.1) for Ξ.

Example 13.1. Consider the linear constant coefficient equation

$$\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = 0. \tag{13.1.1.5}$$

The characteristic equation for this equation is

$$\frac{dx}{1} = \frac{dy}{a}.$$

The general solution of the characteristic equation is $y - ax = C$. Thus, the general solution of the original first-order PDE (13.1.1.5) can be expressed via an arbitrary function Φ of one variable as

$$w = \Phi(y - ax). \tag{13.1.1.6}$$

It is a traveling wave solution.

Table 13.1 lists general solutions of some linear first-order partial differential equations in two independent variables of the form (13.1.1.1).

TABLE 13.1
General solutions of some first-order partial differential equations of the form (13.1.1.1); $\Phi(u)$ is an arbitrary function. The subscripts x and y indicate the corresponding partial derivatives

No.	Equations	General solutions	Notation
1	$w_x + [f(x)y + g(x)]w_y = 0$	$w = \Phi\Big(e^{-F}y - \int e^{-F}g(x)\,dx\Big)$	$F = \int f(x)\,dx$
2	$w_x + [f(x)y + g(x)y^k]w_y = 0$	$w = \Phi\Big(e^{-F}y^{1-k} - (1-k)\int e^{-F}g(x)\,dx\Big)$	$F = (1-k)\int f(x)\,dx$
3	$w_x + [f(x)e^{\lambda y} + g(x)]w_y = 0$	$w = \Phi\Big(e^{-\lambda y}E + \lambda\int f(x)E\,dx\Big)$	$E = \exp\Big(\lambda\int g(x)\,dx\Big)$
4	$f(x)w_x + g(y)w_y = 0$	$w = \Phi\Big(\int \frac{dx}{f(x)} - \int \frac{dy}{g(y)}\Big)$	

► Classical Cauchy problem.

Find a solution $w = w(x, y)$ of Eq. (13.1.1.1) satisfying the condition

$$w = s(y) \quad \text{at} \quad x = x_0, \tag{13.1.1.7}$$

where $s(y)$ is a given function.

The solution of the Cauchy problem (also called the *initial value problem*) can be obtained from the general solution (13.1.1.4). Substituting the initial data (13.1.1.7) into Eq. (13.1.1.4), we have

$$s(y) = \Phi\bigl(\Xi(x_0, y)\bigr).$$

This relation serves to determine the function Φ.

Example 13.2. Find a solution of the Cauchy problem for Eq. (13.1.1.5) with the initial condition

$$w = y^k \quad \text{at} \quad x = 1. \tag{13.1.1.8}$$

Substituting the initial data (13.1.1.8) into the general solution (13.1.1.6) yields

$$y^k = \Phi(y - a).$$

This allows us to conclude that $\Phi(u) = (u + a)^k$. Substituting this expression into Eq. (13.1.1.6), we find the solution of the Cauchy problem in the form

$$w = (y - ax + a)^k.$$

► Physical interpretation of the classical Cauchy problem.

Let x and y be spatial coordinates, and let w be the concentration of a substance. It is assumed that the concentration distribution is described by the steady-state transport equation (13.1.1.1) and the concentration profile (13.1.1.7) is specified at the input cross-section $x = x_0$. The concentration $w = w(x, y)$ is to be determined in the flow after the input cross-section (for $x \ge x_0$).

Another, nonsteady-state interpretation of the Cauchy problem is possible. Let x be time, let y be the spatial coordinate, and let w be the concentration ($f \equiv 1$). It is assumed that the concentration distribution is described by the nonsteady-state transport equation (13.1.1.1) and that the concentration profile (13.1.1.7) is prescribed at the initial instant $x = x_0$. The concentration $w = w(x, y)$ is to be determined for all subsequent instants of time ($x \ge x_0$).

► Generalized Cauchy problem.

Find a solution $w = w(x, y)$ of Eq. (13.1.1.1) with the initial conditions

$$x = s_1(\xi), \quad y = s_2(\xi), \quad w = s_3(\xi), \tag{13.1.1.9}$$

where ξ is a parameter ($\alpha \le \xi \le \beta$), $s_1 = s_1(\xi)$ and $s_2 = s_2(\xi)$ are given functions, and $|s_1'| + |s_2'| \ne 0$.

A geometric interpretation of this problem is as follows: find an integral surface of Eq. (13.1.1.1) passing through the parametrically defined line (13.1.1.9).

The solution of the generalized Cauchy problem can be obtained from the general solution (13.1.1.4) by substituting the initial data (13.1.1.9) into it.

The classical Cauchy problem (13.1.1.1), (13.1.1.7) can be represented as the generalized Cauchy problem (13.1.1.1), (13.1.1.9) if one rewrites the initial condition (13.1.1.7) in the parametric form

$$x = x_0, \quad y = \xi, \quad w = s(\xi). \tag{13.1.1.10}$$

Remark 13.3. In the statement of the Cauchy problem (13.1.1.1), (13.1.1.9), the plane curve $x = s_1(\xi)$, $y = s_2(\xi)$ is assumed not to be tangent to any characteristic at any point; that is,

$$f(s_1, s_2)s_2' - g(s_1, s_2)s_1' \neq 0.$$

Remark 13.4. If the curve $x = s_1(\xi)$, $y = s_2(\xi)$ is a characteristic, then
(a) The Cauchy problem has no solution if $s_3(\xi) \not\equiv$ const.
(b) The Cauchy problem has infinitely many solutions if $s_3(\xi) \equiv$ const.

13.1.2 General First-Order Linear PDE with Two Independent Variables

▶ The representation of the general solution via particular solutions.

In the general case, a first-order linear nonhomogeneous equation with two independent variables has the form

$$f(x, y)\frac{\partial w}{\partial x} + g(x, y)\frac{\partial w}{\partial y} = h_1(x, y)w + h_0(x, y). \tag{13.1.2.1}$$

The special case $h_0 = h_1 = 0$ is discussed in Section 13.1.1.

The general solution of the linear nonhomogeneous equation (13.1.2.1) can be represented as the sum of any particular solution of this equation and the general solution of the corresponding homogeneous equation (with $h_0 \equiv 0$). In what follows, we give a more detailed statement about the representation of the general solution via particular solutions.

The general solution of Eq. (13.1.2.1) can be represented in the form

$$w = w_2 + w_1\Phi(w_0),$$

where $w_0 = w_0(x, y)$ is any nonconstant particular solution of the "truncated" homogeneous equation (13.1.2.1) with $h_0 = h_1 = 0$; $w_1 = w_1(x, y)$ is a nontrivial particular solution of the truncated homogeneous equation (13.1.2.1) with $h_0 = 0$; $w_2 = w_2(x, y)$ is a particular solution of the nonhomogeneous equation (13.1.2.1); and $\Phi(w_0)$ is an arbitrary function.

▶ Solution by using the characteristic system.

Given two distinct (functionally independent) integrals,

$$u_1(x, y, w) = C_1, \qquad u_2(x, y, w) = C_2 \tag{13.1.2.2}$$

of the *characteristic system*

$$\frac{dx}{f(x, y)} = \frac{dy}{g(x, y)} = \frac{dw}{h_1(x, y)w + h_0(x, y)}, \tag{13.1.2.3}$$

the general solution of the nonhomogeneous equation (13.1.2.1) is defined by

$$\Phi(u_1, u_2) = 0, \tag{13.1.2.4}$$

where Φ is an arbitrary function of two variables. With Eq. (13.1.2.4) solved for u_1 or u_2, we often specify the general solution in the form

$$u_k = \Psi(u_{3-k}),$$

where $k = 1,\ 2$ and $\Psi(u)$ is an arbitrary function of one variable.

Remark 13.5. In the general case, the integrals (13.1.2.2) can be represented in the form

$$\Xi(x,y) = C_1, \qquad p_1(x,y)w + p_0(x,y) = C_2,$$

where $\Xi(x,y)$ is a first integral of the partial differential equation (13.1.1.1) (or Eq. (13.1.2.1) with $h_0 = h_1 = 0$) and $p_0(x,y)$ and $p_1(x,y)$ are some functions.

Remark 13.6. The degenerate case $h_0 = h_1 = 0$ in (13.1.2.3) corresponds to the simplest integral $w = C_2$ in (13.1.2.2).

Example 13.3. Consider the equation

$$\frac{\partial w}{\partial x} + ax\frac{\partial w}{\partial y} = bw.$$

The corresponding characteristic system

$$\frac{dx}{1} = \frac{dy}{ax} = \frac{dw}{bw},$$

has two independent integrals

$$y - \tfrac{1}{2}ax^2 = C_1, \qquad we^{-bx} = C_2.$$

Therefore, the general solution of the equation in question is expressed in terms of an arbitrary function of two variables as $\Phi\bigl(y - \frac{1}{2}ax^2, we^{-bx}\bigr) = 0$. By solving this equation for the second argument, we obtain the solution in explicit form,

$$w = e^{bx}\Psi(y - \tfrac{1}{2}ax^2), \qquad \text{where} \quad \Psi(u) \text{ is an arbitrary function.}$$

► **Solution based on a change of variables.**

Let a particular solution $u = u(x,y)$ (first integral) of the corresponding truncated homogeneous equation

$$f(x,y)\frac{\partial u}{\partial x} + g(x,y)\frac{\partial u}{\partial y} = 0 \qquad (u \not\equiv \text{const}) \tag{13.1.2.5}$$

be known. By switching from x and y to the new variables x and $u = u(x,y)$ in equation (13.1.2.1), we obtain

$$\bar{f}(x,u)\frac{\partial w}{\partial x} = \bar{h}_1(x,u)w + \bar{h}_0(x,u),$$

where $\bar{f}(x,u) = f(x,y)$, $\bar{h}_1(x,u) = h_1(x,y)$, and $\bar{h}_0(x,u) = h_0(x,y)$ are the coefficients of the original equation (13.1.2.1) rewritten in terms of x and u.

Equation (13.1.2.5) can be treated as an ordinary differential equation for $w = w(x)$ with parameter u. The general solution of Eq. (13.1.2.5) has the form

$$w = E\biggl[\int \frac{\bar{h}_0(x,u)}{\bar{f}(x,u)}\frac{dx}{E} + \Phi(u)\biggr], \qquad E = \exp\biggl[\int \frac{\bar{h}_1(x,u)}{\bar{f}(x,u)}\,dx\biggr],$$

where Φ is an arbitrary function; in the integration, u is considered a parameter. To find a general integral of Eq. (13.1.2.1), one should compute the integrals in the last relation and then return to the original variables x and y.

► Classical and generalized Cauchy problems. Solution methods.

1°. The classical and generalized Cauchy problems for Eq. (13.1.2.1) are stated in the same manner as for the "truncated" equation (with $h_0 = h_1 = 0$); see Section 13.1.1.

2°. The solution of the Cauchy problem can be obtained from the expression for the general solution, into which the initial data should be substituted.

3°. An alternative (and more convenient) approach to the solution of the Cauchy problem (13.1.1.1), (13.1.1.9) consists of several stages. First, two independent integrals in (13.1.2.2) of the characteristic system (13.1.2.3) are determined. Then the initial data (13.1.1.9) are substituted into the integrals (13.1.2.2),

$$u_1\bigl(s_1(\xi), s_2(\xi), s_3(\xi)\bigr) = C_1, \qquad u_2\bigl(s_1(\xi), s_2(\xi), s_3(\xi)\bigr) = C_2, \tag{13.1.2.6}$$

to determine the constants of integration C_1 and C_2. By eliminating C_1 and C_2 from Eqs. (13.1.2.2) and (13.1.2.6), we have

$$\begin{aligned} u_1(x,y,w) &= u_1\bigl(s_1(\xi), s_2(\xi), s_3(\xi)\bigr),\\ u_2(x,y,w) &= u_2\bigl(s_1(\xi), s_2(\xi), s_3(\xi)\bigr). \end{aligned} \tag{13.1.2.7}$$

These relations are a parametric form of the solution of the Cauchy problem (13.1.1.1), (13.1.1.9). By eliminating the parameter ξ from (13.1.2.7), one can obtain the solution in explicit form.

Example 13.4. Consider the classical Cauchy problem for the equation

$$\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = b \tag{13.1.2.8}$$

subject to the initial condition

$$w = ky^2 \quad \text{at} \quad x = 0. \tag{13.1.2.9}$$

The characteristic system

$$\frac{dx}{1} = \frac{dy}{a} = \frac{dw}{b} \tag{13.1.2.10}$$

corresponding to Eq. (13.1.2.8) has two independent integrals

$$y - ax = C_1, \qquad w - bx = C_2. \tag{13.1.2.11}$$

Let us rewrite the initial condition (13.1.2.9) for the classical Cauchy problem in terms of the generalized Cauchy problem:

$$x = 0, \quad y = \xi, \quad w = k\xi^2.$$

By substituting these initial conditions into the integrals (13.1.2.11), we obtain

$$\xi = C_1, \qquad k\xi^2 = C_2. \tag{13.1.2.12}$$

By eliminating C_1 and C_2 from Eqs. (13.1.2.11) and (13.1.2.12), we have

$$y - ax = \xi, \qquad w - bx = k\xi^2.$$

On eliminating the parameter ξ from these algebraic equations, we obtain the solution of the Cauchy problem (13.1.2.8)–(13.1.2.9) in explicit form,

$$w = bx + k(y - ax)^2.$$

Example 13.5. Consider the generalized Cauchy problem for Eq. (13.1.2.8) with the initial conditions

$$x = \xi, \quad y = \xi, \quad w = \xi^2. \tag{13.1.2.13}$$

The corresponding characteristic system (13.1.2.10) has two independent integrals (13.1.2.11). Substituting the initial data (13.1.2.13) into the integrals (13.1.2.11) yields

$$(1 - a)\xi = C_1, \qquad \xi^2 - b\xi = C_2. \tag{13.1.2.14}$$

On eliminating the parameters C_1, C_2, and ξ from Eqs. (13.1.2.11) and (13.1.2.14), we obtain the solution of the generalized Cauchy problem (13.1.2.8), (13.1.2.13),

$$w = bx - b\frac{y - ax}{1 - a} + \left(\frac{y - ax}{1 - a}\right)^2.$$

This is the solution for $a \neq 1$. For $a = 1$, the straight line $y = x = \xi$ (where the initial data are specified) is a characteristic, and the generalized Cauchy problem in question has no solutions.

Remark 13.7. For the existence and uniqueness theorem for the solution of the Cauchy problem, see the end of Section 13.2.

13.2 First-Order Linear PDEs with Three or More Independent Variables

13.2.1 Characteristic System. General Solution

► Linear homogeneous equations. Characteristic system. General solution.

Consider a linear homogeneous equation with n independent variables of the form

$$\sum_{i=1}^{n} f_i(x_1, \dots, x_n)\frac{\partial w}{\partial x_i} = 0. \tag{13.2.1.1}$$

If $n - 1$ functionally independent integrals

$$u_1(x_1, \dots, x_n) = C_1, \quad u_2(x_1, \dots, x_n) = C_2, \quad \dots, \quad u_{n-1}(x_1, \dots, x_n) = C_{n-1} \tag{13.2.1.2}$$

(a *basis of integrals*) of the characteristic system

$$\frac{dx_1}{f_1(x_1, \dots, x_n)} = \frac{dx_2}{f_2(x_1, \dots, x_n)} = \dots = \frac{dx_n}{f_n(x_1, \dots, x_n)} \tag{13.2.1.3}$$

are known, then the general solution of Eq. (13.2.1.1) is defined by

$$w = \Phi(u_1, u_2, \dots, u_{n-1}),$$

where Φ is an arbitrary function of $n-1$ variables.

► Reducing the number of independent variables.

Assume that one integral $u(x_1, \dots, x_n) = C$ of system (13.2.1.3) is known. By passing from $x_1, \dots, x_{n-1}, x_n$ to the new variables $x_1, \dots, x_{n-1}, u$, we arrive at a first-order linear homogeneous partial differential equation with one independent variable fewer than the original equation,

$$\sum_{i=1}^{n-1} \widetilde{f}_i(x_1, \dots, x_{n-1}, u) \frac{\partial w}{\partial x_i} = 0,$$

where u plays the role of a parameter.

► Linear nonhomogeneous equations. Characteristic system. General solution.

Consider a linear nonhomogeneous equation with n independent variables of the general form

$$\sum_{i=1}^{n} f_i(x_1, \dots, x_n) \frac{\partial w}{\partial x_i} = g(x_1, \dots, x_n)w + h(x_1, \dots, x_n). \tag{13.2.1.4}$$

Given n functionally independent integrals

$$u_1(x_1, \dots, x_n, w) = C_1, \quad u_2(x_1, \dots, x_n, w) = C_2, \quad \dots, \quad u_n(x_1, \dots, x_n, w) = C_n \tag{13.2.1.5}$$

(a *basis of integrals*) of the characteristic system

$$\frac{dx_1}{f_1(x_1, \dots, x_n)} = \dots = \frac{dx_n}{f_n(x_1, \dots, x_n)} = \frac{dw}{g(x_1, \dots, x_n)w + h(x_1, \dots, x_n)}, \tag{13.2.1.6}$$

the general solution of Eq. (13.2.1.4) is defined by

$$\Phi(u_1, u_2, \dots, u_n) = 0,$$

where Φ is an arbitrary function of n variables.

► Some formulas for the solution.

1°. Let an integral basis $u_k = u_k(x_1, \dots, x_n)$ ($k = 1, \dots, n-1$) of the corresponding "truncated" homogeneous equation (13.2.1.1) be known. By passing from $x_1, x_2, \dots, x_n$ to the new variables $x_1, u_1, \dots, u_{n-1}$, we arrive at the linear equation

$$\widetilde{f}_1(x, \mathbf{u}) \frac{\partial w}{\partial x} = \widetilde{g}(x, \mathbf{u})w + \widetilde{h}(x, \mathbf{u}), \qquad x = x_1,$$

which can be treated as a first-order linear ordinary differential equation for $w = w(x)$ with the parameter vector $\mathbf{u} = (u_1, u_2, \dots, u_{n-1})$. By solving this equation, we obtain

$$w = E\biggl[\Phi(\mathbf{u}) + \int \frac{\widetilde{h}(x,\mathbf{u})}{\widetilde{f}_1(x,\mathbf{u})}\frac{dx}{E}\biggr], \qquad E = \exp\biggl[\int \frac{\widetilde{g}(x,\mathbf{u})}{\widetilde{f}_1(x,\mathbf{u})}\,dx\biggr],$$

where Φ is an arbitrary function. When computing the two integrals, the components of the vector $\mathbf{u}$ are treated as parameters. To determine the general integral of Eq. (13.2.1.4), one should return to the original variables $x_1, \dots, x_n$ after the integration has been performed.

2°. Let $g \equiv 0$. Given a basis $\mathbf{u} = (u_1, \dots, u_{n-1})$ of integrals of the corresponding homogeneous equation (with $h \equiv 0$) and a particular solution $w_0 = w_0(x_1, \dots, x_n)$ of the original nonhomogeneous equation, the general solution is defined by

$$w = w_0 + \Phi(\mathbf{u}),$$

where Φ is an arbitrary function.

13.2.2 Cauchy Problems

► Classical and generalized Cauchy problems (initial value problems).

Consider two statements of the Cauchy problem.

1°. *Generalized Cauchy problem.* Find a solution $w = w(x_1, \dots, x_n)$ of Eq. (13.2.1.4) with the initial conditions

$$x_1 = \varphi_1(\xi_1, \dots, \xi_{n-1}), \quad \dots, \quad x_n = \varphi_n(\xi_1, \dots, \xi_{n-1}), \quad w = \varphi_{n+1}(\xi_1, \dots, \xi_{n-1}), \tag{13.2.2.1}$$

where the ξ_k are parameters $(k = 1, \dots, n-1)$ and the $\varphi_m(\xi_1, \dots, \xi_{n-1})$ are given functions $(m = 1, \dots, n+1)$.

2°. *Classical Cauchy problem.* Find a solution $w = w(x_1, \dots, x_n)$ of Eq. (13.2.1.4) with the initial condition

$$w = \psi(x_2, \dots, x_n) \quad \text{at} \quad x_1 = 0, \tag{13.2.2.2}$$

where $\psi(x_2, \dots, x_n)$ is a given function.

It is convenient to represent the classical Cauchy problem as a generalized Cauchy problem by rewriting the initial condition (13.2.2.2) in the parametric form

$$x_1 = 0, \quad x_2 = \xi_1, \quad \dots, \quad x_n = \xi_{n-1}, \quad w = \psi(\xi_1, \dots, \xi_{n-1}).$$

► Solution procedure for the Cauchy problem.

The solution procedure for the Cauchy problem (13.2.1.4), (13.2.2.1) involves several steps. First of all, one determines independent integrals (13.2.1.5) of the characteristic system (13.2.1.6). After that, to find the constants of integration $C_1, \dots, C_n$, the initial data (13.2.2.1) should be substituted into the integrals (13.2.1.5),

$$u_k(\varphi_1, \dots, \varphi_n, \varphi_{n+1}) = C_k, \qquad \text{where} \quad \varphi_m = \varphi_m(\xi_1, \dots, \xi_{n-1}), \quad k = 1, \dots, n. \tag{13.2.2.3}$$

On eliminating $C_1, \ldots, C_n$ from Eqs. (13.2.1.5) and (13.2.2.3), one obtains

$$u_k(x_1, \ldots, x_n, w) = u_k(\varphi_1, \ldots, \varphi_n, \varphi_{n+1}), \qquad k = 1, \ldots, n, \tag{13.2.2.4}$$

where $\varphi_m = \varphi_m(\xi_1, \ldots, \xi_{n-1})$, $m = 1, \ldots, n+1$. Relations (13.2.2.4) are a parametric form of the solution of the Cauchy problem (13.2.1.4), (13.2.2.1). In some cases, one may succeed in eliminating the parameters $\xi_1, \ldots, \xi_{n-1}$ to obtain the solution in explicit form.

► **The existence and uniqueness theorem.**

Let $f_1 = 1$, and let the other coefficients $f_2, \ldots, f_n$, g, and h in Eq. (13.2.1.4), as well as the function ψ in (13.2.2.2), be continuously differentiable functions of $x_1, \ldots, x_n$ in a domain $V = \{0 < x_1 < a,\ -\infty < x_i < \infty,\ i = 2, \ldots, n\}$. Suppose that the following inequality holds in V:

$$\sqrt{f_2^2(x_1, \ldots, x_n) + \cdots + f_n^2(x_1, \ldots, x_n)} \le k\Bigl(1 + \sqrt{x_2^2 + \cdots + x_n^2}\Bigr), \qquad k = \text{const.}$$

Then there exists a unique continuously differentiable solution of the Cauchy problem (13.2.1.4), (13.2.2.2); the existence of this solution is guaranteed in the entire domain V.

⊙ *Literature for Chapter 13*: E. Kamke (1965), I. G. Petrovsky (1991), H. Rhee, R. Aris, and N. R. Amundson (1986), R. Courant and D. Hilbert (1989), A. I. Subbotin (1991), D. Zwillinger (1998), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 14

Second-Order Linear PDEs: Classification, Problems, and Particular Solutions

14.1 Classification of Second-Order Linear Partial Differential Equations

14.1.1 Equations with Two Independent Variables

► **Examples of equations encountered in applications.**

Three basic types of partial differential equations are distinguished—*parabolic*, *hyperbolic*, and *elliptic*. The solutions of the equations pertaining to each of the types have their own characteristic qualitative differences.

The simplest example of a *parabolic equation* is the *heat equation*

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0, \qquad (14.1.1.1)$$

where the variables t and x play the role of time and the spatial coordinate, respectively. Note that Eq. (14.1.1.1) contains only one highest derivative term. Frequently encountered particular solutions of Eq. (14.1.1.1) can be found in Section 3.1.1.

The simplest example of a *hyperbolic equation* is the *wave equation*

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0, \qquad (14.1.1.2)$$

where the variables t and x play the role of time and the spatial coordinate, respectively. Note that the highest derivative terms in Eq. (14.1.1.2) differ in sign. Frequently encountered particular solutions of Eq. (14.1.1.2) can be found in Section 6.1.1.

The simplest example of an *elliptic equation* is the *Laplace equation*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \qquad (14.1.1.3)$$

where x and y play the role of the spatial coordinates. Note that the highest derivative terms in Eq. (14.1.1.3) have like signs. Frequently encountered particular solutions of Eq. (14.1.1.3) can be found in Section 9.1.1.

Any linear partial differential equation of the second-order with two independent variables can be reduced, by appropriate manipulations, to a simpler equation that has one of the three highest derivative combinations specified above in examples (14.1.1.1), (14.1.1.2), and (14.1.1.3).

► Types of equations. Characteristic equations.

Consider a second-order partial differential equation with two independent variables of the general form

$$a(x,y)\frac{\partial^2 w}{\partial x^2}+2b(x,y)\frac{\partial^2 w}{\partial x\partial y}+c(x,y)\frac{\partial^2 w}{\partial y^2}=F\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right), \qquad (14.1.1.4)$$

where a, b, c are some functions of x and y that have continuous derivatives up to the second order inclusive.*

Given a point (x,y), Eq. (14.1.1.4) is said to be

$$\begin{array}{ll} \textit{parabolic} & \text{if } b^2-ac=0,\\ \textit{hyperbolic} & \text{if } b^2-ac>0,\\ \textit{elliptic} & \text{if } b^2-ac<0 \end{array}$$

at this point.

To reduce Eq. (14.1.1.4) to canonical form, one should write out the characteristic equation

$$a\,(dy)^2-2b\,dx\,dy+c\,(dx)^2=0,$$

which splits into two equations

$$a\,dy-\left(b+\sqrt{b^2-ac}\right)dx=0 \qquad (14.1.1.5)$$

and

$$a\,dy-\left(b-\sqrt{b^2-ac}\right)dx=0, \qquad (14.1.1.6)$$

and then find their general integrals.

Remark 14.1. The characteristic equations (14.1.1.5)–(14.1.1.6) can be used if $a\not\equiv 0$. If $a\equiv 0$, the simpler equations

$$dx=0,$$
$$2b\,dy-c\,dx=0$$

should be used; the first equation has the obvious general solution $x=C$.

*The right-hand side of Eq. (14.1.1.4) may be nonlinear. The classification and the procedure of reducing such equations to a canonical form are only determined by the left-hand side of the equation.

► **Canonical form of parabolic equations (case $b^2 - ac = 0$).**

In this case, Eqs. (14.1.1.5) and (14.1.1.6) coincide and have a common general integral

$$\varphi(x, y) = C.$$

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \eta(x, y),$$

where $\eta = \eta(x, y)$ is any twice differentiable function that satisfies the condition of non-degeneracy of the Jacobian $\frac{D(\xi,\eta)}{D(x,y)}$ in the given domain, we reduce Eq. (14.1.1.4) to the canonical form

$$\frac{\partial^2 w}{\partial \eta^2} = F_1\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right). \tag{14.1.1.7}$$

For η one can take $\eta = x$ or $\eta = y$.

It is obvious that the transformed equation (14.1.1.7) has only one highest-derivative term, just as the heat equation (14.1.1.1).

Remark 14.2. In the degenerate case where the function F_1 does not depend on the derivative $\partial_\xi w$, Eq. (14.1.1.7) is an ordinary differential equation in the variable η, and ξ serves as a parameter.

► **Canonical forms of hyperbolic equations (case $b^2 - ac > 0$).**

The general integrals

$$\varphi(x, y) = C_1, \qquad \psi(x, y) = C_2$$

of Eqs. (14.1.1.5) and (14.1.1.6) are real and distinct. These integrals determine two distinct families of real characteristics.

1°. *First canonical form.* By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y),$$

we reduce Eq. (14.1.1.4) to

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = F_2\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right). \tag{14.1.1.8}$$

This is the so-called *first canonical form of a hyperbolic equation.*

2°. *Second canonical form.* The transformation

$$\xi = t + z, \qquad \eta = t - z$$

brings Eq. (14.1.1.8) to another canonical form,

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial z^2} = F_3\left(t, z, w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial z}\right), \tag{14.1.1.9}$$

where $F_3 = 4F_2$. This is the so-called *second canonical form of a hyperbolic equation*. Apart from notation, the left-hand side of the last equation coincides with that of the wave equation (14.1.1.2).

In some cases, reduction of an equation to canonical form permits one to find its general solution.

Example 14.1. The equation

$$kx\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y} = 0$$

is a special case of Eq. (14.1.1.4) with $a = kx$, $b = \frac{1}{2}$, $c = 0$, and $F = 0$. The characteristic equations

$$kx\,dy - dx = 0,$$
$$dy = 0$$

have the general integrals $ky - \ln|x| = C_1$ and $y = C_2$. Switching to the new independent variables

$$\xi = ky - \ln|x|, \qquad \eta = y$$

reduces the original equation to the canonical form

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = k\frac{\partial w}{\partial \xi}.$$

Integrating with respect to ξ yields the linear first-order equation

$$\frac{\partial w}{\partial \eta} = kw + f(\eta),$$

where $f(\eta)$ is an arbitrary function. Its general solution is expressed as

$$w = e^{k\eta} g(\xi) + e^{k\eta} \int e^{-k\eta} f(\eta)\,d\eta,$$

where $g(\xi)$ is an arbitrary function.

Remark 14.3. For higher-order hyperbolic equations, see Section 11.6.3.

► Canonical form of elliptic equations (case $b^2 - ac < 0$).

In this case, the general integrals of Eqs. (14.1.1.5) and (14.1.1.6) are complex conjugate; these determine two families of complex characteristics.

Let the general integral of Eq. (14.1.1.5) have the form

$$\varphi(x,y) + i\psi(x,y) = C, \qquad i^2 = -1,$$

where $\varphi(x,y)$ and $\psi(x,y)$ are real-valued functions.

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x,y), \qquad \eta = \psi(x,y),$$

we reduce Eq. (14.1.1.4) to the canonical form

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = F_4\biggl(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\biggr).$$

Apart from notation, the left-hand side of the last equation coincides with that of the Laplace equation (14.1.1.3).

Remark 14.4. For higher-order elliptic equations, see Section 11.6.2.

► Linear constant coefficient partial differential equations.

1°. When reduced to a canonical form, linear homogeneous constant coefficient partial differential equations

$$a\frac{\partial^2 w}{\partial x^2}+2b\frac{\partial^2 w}{\partial x\partial y}+c\frac{\partial^2 w}{\partial y^2}+p\frac{\partial w}{\partial x}+q\frac{\partial w}{\partial y}+sw=0 \qquad (14.1.1.10)$$

admit further simplifications. In general, the substitution

$$w(x,y)=\exp(\beta_1\xi+\beta_2\eta)u(\xi,\eta) \qquad (14.1.1.11)$$

can be used. Here ξ and η are new variables used to reduce Eq. (14.1.1.10) to canonical form (see above); the coefficients β_1 and β_2 in (14.1.1.11) are chosen so that there is only one first derivative remaining in a parabolic equation or both first derivatives vanish in a hyperbolic or an elliptic equation. For final results, see Table 14.1.

TABLE 14.1
Reduction of linear homogeneous constant coefficient partial differential equations (14.1.1.10) using the transformation (14.1.1.11); the constants k and k_1 are given by formulas (14.1.1.12)

Type of equation, conditions on coefficients	Variables ξ and η in transformation (14.1.1.11)	Coefficients β_1 and β_2 in transformation (14.1.1.11)	Reduced equation
Parabolic equation, $a=b=0$, $c\neq 0$, $p\neq 0$	$\xi=-\frac{c}{p}x$, $\eta=y$	$\beta_1=\frac{4cs-q^2}{4c^2}$, $\beta_2=-\frac{q}{2c}$	$u_\xi-u_{\eta\eta}=0$
Parabolic equation, $b^2-ac=0$ ($aq-bp\neq 0$, $\lvert a\rvert+\lvert b\rvert\neq 0$)	$\xi=\frac{a(ay-bx)}{bp-aq}$, $\eta=x$	$\beta_1=\frac{4as-p^2}{4a^2}$, $\beta_2=-\frac{p}{2a}$	$u_\xi-u_{\eta\eta}=0$
Hyperbolic equation, $a\neq 0$, $D=b^2-ac>0$	$\xi=ay-\bigl(b+\sqrt{D}\bigr)x$, $\eta=ay-\bigl(b-\sqrt{D}\bigr)x$	$\beta_{1,2}=\frac{aq-bp}{4aD}\pm\frac{p}{4a\sqrt{D}}$	$u_{\xi\eta}+ku=0$
Hyperbolic equation, $a=0$, $b\neq 0$	$\xi=x$, $\eta=2by-cx$	$\beta_1=\frac{cp-2bq}{4b^2}$, $\beta_2=-\frac{p}{4b^2}$	$u_{\xi\eta}+k_1u=0$
Elliptic equation, $D=b^2-ac<0$	$\xi=ay-bx$, $\eta=\sqrt{\lvert D\rvert}\,x$	$\beta_1=\frac{aq-bp}{2aD}$, $\beta_2=-\frac{p}{2a\sqrt{\lvert D\rvert}}$	$u_{\xi\xi}+u_{\eta\eta}+4ku=0$
Ordinary differential equation, $b^2-ac=0$, $aq-bp=0$	$\xi=ay-bx$, $\eta=x$	$\beta_1=\beta_2=0$	$aw_{\eta\eta}+pw_\eta+sw=0$

2°. The coefficients k and k_1 in the reduced hyperbolic and elliptic equations (see the last row in Table 14.1) are expressed as

$$k=\frac{2bpq-aq^2-cp^2}{16a(b^2-ac)^2}-\frac{s}{4a(b^2-ac)},\qquad k_1=\frac{s}{4b^2}+\frac{cp^2-2bpq}{16b^4}. \qquad (14.1.1.12)$$

If the coefficients in Eq. (14.1.1.10) satisfy the relation

$$2bpq-aq^2-cp^2-4s(b^2-ac)=0,$$

then $k = 0$; in this case with $a \neq 0$, the general solution of the corresponding hyperbolic equation has the form

$$w(x,y) = \exp(\beta_1\xi + \beta_2\eta)\bigl[f(\xi) + g(\eta)\bigr], \qquad D = b^2 - ac > 0,$$
$$\xi = ay - \bigl(b + \sqrt{D}\,\bigr)x, \qquad \eta = ay - \bigl(b - \sqrt{D}\,\bigr)x,$$
$$\beta_1 = \frac{aq - bp}{4aD} + \frac{p}{4a\sqrt{D}}, \qquad \beta_2 = \frac{aq - bp}{4aD} - \frac{p}{4a\sqrt{D}},$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions.

3°. In the degenerate case $b^2 - ac = 0$, $aq - bp = 0$ (where the original equation is reduced to an ordinary differential equation; see the last row in Table 14.1), the general solution of Eq. (14.1.1.10) is expressed as

$$w = \exp\left(-\frac{px}{2a}\right)\left[f(ay-bx)\exp\left(\frac{x\sqrt{\lambda}}{2a}\right) + g(ay-bx)\exp\left(-\frac{x\sqrt{\lambda}}{2a}\right)\right] \quad \text{if } \lambda = p^2 - 4as > 0,$$
$$w = \exp\left(-\frac{px}{2a}\right)\left[f(ay-bx)\sin\left(\frac{x\sqrt{|\lambda|}}{2a}\right) + g(ay-bx)\cos\left(\frac{x\sqrt{|\lambda|}}{2a}\right)\right] \quad \text{if } \lambda = p^2 - 4as < 0,$$
$$w = \exp\left(-\frac{px}{2a}\right)\left[f(ay-bx) + xg(ay-bx)\right] \quad \text{if } p^2 - 4as = 0,$$

where $f(z)$ and $g(z)$ are arbitrary functions.

14.1.2 Equations with Many Independent Variables

Let us look at a second-order partial differential equation with n independent variables $x_1, \ldots, x_n$ of the form

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2 w}{\partial x_i \partial x_j} = F\left(\mathbf{x}, w, \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}\right), \tag{14.1.2.1}$$

where the a_{ij} are some functions that have continuous derivatives of order ≤ 2 with respect to all variables and $\mathbf{x} = \{x_1, \ldots, x_n\}$. (The right-hand side of Eq. (14.1.2.1) may be nonlinear. The left-hand side only is required for the classification of this equation.)

At a point $\mathbf{x} = \mathbf{x}_0$, the following quadratic form is assigned to Eq. (14.1.2.1):

$$Q = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}_0)\xi_i\xi_j. \tag{14.1.2.2}$$

An appropriate linear nonsingular transformation

$$\xi_i = \sum_{k=1}^{n} \beta_{ik}\eta_k \qquad (i = 1, \ldots, n) \tag{14.1.2.3}$$

reduces the quadratic form (14.1.2.2) to the canonical form

$$Q=\sum_{i=1}^{n}c_i\eta_i^2, \tag{14.1.2.4}$$

where the coefficients c_i assume the values 1, -1, and 0. The number of negative and zero coefficients in (14.1.2.4) does not depend on the way in which the quadratic form is reduced to the canonical form.

Table 14.2 presents the basic criteria according to which equations with many independent variables are classified.

TABLE 14.2
Classification of equations with many independent variables

Type of Eq. (14.1.2.1) at a point $\mathbf{x}=x_0$	Coefficients of the canonical form (14.1.2.4)
Parabolic (in the broad sense)	At least one of the coefficients c_i is zero
Hyperbolic (in the broad sense)	All c_i are nonzero and some c_i differ in sign
Elliptic	All c_i are nonzero and have like signs

Suppose all coefficients of the highest derivatives in (14.1.2.1) are constant, $a_{ij}=\text{const}$. By introducing the new independent variables $y_1,\ldots,y_n$ in accordance with the formulas $y_i=\sum\limits_{k=1}^{n}\beta_{ik}x_k$, where the β_{ik} are the coefficients of the linear transformation (14.1.2.3), we reduce Eq. (14.1.2.1) to the canonical form

$$\sum_{i=1}^{n}c_i\frac{\partial^2 w}{\partial y_i^2}=F_1\biggl(\mathbf{y},w,\frac{\partial w}{\partial y_1},\ldots,\frac{\partial w}{\partial y_n}\biggr), \tag{14.1.2.5}$$

where $\mathbf{y}=\{y_1,\ldots,y_n\}$; the coefficients c_i are the same as in the quadratic form (14.1.2.4).

Remark 14.5. Among the parabolic equations, it is conventional to distinguish the parabolic equations in the narrow sense, i.e., the equations for which only one of the coefficients, c_k, is zero, while the other c_i is the same, and in this case the right-hand side of Eq. (14.1.2.5) must contain the first partial derivative with respect to y_k.

Remark 14.6. In turn, the hyperbolic equations are divided into *normal hyperbolic equations*—for which all c_i but one have like signs—and *ultrahyperbolic equations*—for which there are two or more positive c_i and two or more negative c_i.

Specific equations of parabolic, elliptic, and hyperbolic types will be discussed further in Section 14.2.

⊙ *Literature for Section 14.2*: S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), R. Courant and D. Hilbert (1989), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), I. G. Petrovsky (1991), W. A. Strauss (1992), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

14.2 Basic Problems of Mathematical Physics

14.2.1 Initial and Boundary Conditions. Cauchy Problem. Boundary Value Problems

Every equation of mathematical physics describes infinitely many qualitatively similar phenomena or processes. This follows from the fact that differential equations have infinitely many particular solutions. The specific solution that describes the physical phenomenon under study is separated from the set of particular solutions of the given differential equation by means of the initial and boundary conditions.

Throughout this section, we consider linear equations in the n-dimensional Euclidean space $\mathbb{R}^n$ or in an open domain $V \in \mathbb{R}^n$ (exclusive of the boundary) with a sufficiently smooth boundary $S = \partial V$.

► **Second-order parabolic equations. Initial and boundary conditions.**

In general, a linear second-order partial differential equation of the parabolic type with n independent variables can be written as

$$\frac{\partial w}{\partial t} - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t), \tag{14.2.1.1}$$

where

$$L_{\mathbf{x}}[w] \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, t) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}, t) \frac{\partial w}{\partial x_i} + c(\mathbf{x}, t) w, \tag{14.2.1.2}$$

$$\mathbf{x} = \{x_1, \dots, x_n\}, \qquad \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, t) \xi_i \xi_j \geq \sigma \sum_{i=1}^{n} \xi_i^2, \quad \sigma > 0.$$

Parabolic equations describe unsteady thermal, diffusion, and other phenomena dependent on time t.

Equation (14.2.1.1) is said to be homogeneous if $\Phi(\mathbf{x}, t) \equiv 0$.

Cauchy problem ($t \geq 0, \mathbf{x} \in \mathbb{R}^n$). Find a function w that satisfies Eq. (14.2.1.1) for $t > 0$ and the initial condition

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0. \tag{14.2.1.3}$$

*Boundary value problem** ($t \geq 0, \mathbf{x} \in V$). Find a function w that satisfies Eq. (14.2.1.1) for $t > 0$, the initial condition (14.2.1.3), and the boundary condition

$$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}, t) \quad \text{at} \quad \mathbf{x} \in S \qquad (t > 0). \tag{14.2.1.4}$$

In general, $\Gamma_{\mathbf{x}}$ is a first-order linear differential operator in the space variables $\mathbf{x}$ with coefficient depending on $\mathbf{x}$ and t. The basic types of boundary conditions are described in Section 14.2.2.

The initial condition (14.2.1.3) is said to be homogeneous if $f(\mathbf{x}) \equiv 0$. The boundary condition (14.2.1.4) is said to be homogeneous if $g(\mathbf{x}, t) \equiv 0$.

**Boundary value problems* for parabolic and hyperbolic equations are sometimes called *mixed* or *initial-boundary value problems*.

► **Second-order hyperbolic equations. Initial and boundary conditions.**

Consider a second-order linear partial differential equation of the hyperbolic type with n independent variables of the general form

$$\frac{\partial^2 w}{\partial t^2} + \varphi(\mathbf{x}, t)\frac{\partial w}{\partial t} - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t), \tag{14.2.1.5}$$

where the linear differential operator $L_{\mathbf{x}}$ is defined in (14.2.1.2). Hyperbolic equations describe unsteady wave processes, which depend on time t.

Equation (14.2.1.5) is said to be homogeneous if $\Phi(\mathbf{x}, t) \equiv 0$.

Cauchy problem ($t \geq 0$, $\mathbf{x} \in \mathbb{R}^n$). Find a function w that satisfies Eq. (14.2.1.5) for $t > 0$ and the initial conditions

$$\begin{aligned} w &= f_0(\mathbf{x}) \quad \text{at} \quad t = 0, \\ \partial_t w &= f_1(\mathbf{x}) \quad \text{at} \quad t = 0. \end{aligned} \tag{14.2.1.6}$$

Boundary value problem ($t \geq 0$, $\mathbf{x} \in V$). Find a function w that satisfies Eq. (14.2.1.5) for $t > 0$, the initial conditions (14.2.1.6), and boundary condition (14.2.1.4).

The initial conditions (14.2.1.6) are said to be homogeneous if $f_0(\mathbf{x}) \equiv 0$ and $f_1(\mathbf{x}) \equiv 0$.

Generalized Cauchy problem. In the generalized Cauchy problem for a hyperbolic equation with two independent variables, the values of the unknown function and its first derivatives are prescribed on a curve in the (x, t) plane. Alternatively, the values of the unknown function and its derivative along the normal to this curve may be prescribed. For more details, see Section 17.2.4.

Goursat problem. On the characteristics of a hyperbolic equation with two independent variables, the values of the unknown function w are prescribed; for details, see Section 17.2.5.

► **Second-order elliptic equations. Boundary conditions.**

In general, a second-order linear partial differential equation of elliptic type with n independent variables can be written as

$$L_{\mathbf{x}}[w] = \Phi(\mathbf{x}), \tag{14.2.1.7}$$

where

$$\begin{gathered} L_{\mathbf{x}}[w] \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial w}{\partial x_i} + c(\mathbf{x})w, \\ \sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\xi_i\xi_j \geq \sigma \sum_{i=1}^{n} \xi_i^2, \qquad \sigma > 0. \end{gathered} \tag{14.2.1.8}$$

Elliptic equations describe steady-state thermal, diffusion, and other phenomena independent of time t.

Equation (14.2.1.7) is said to be homogeneous if $\Phi(\mathbf{x}) \equiv 0$.

Boundary value problem. Find a function w that satisfies Eq. (14.2.1.7) and the boundary condition

$$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}) \quad \text{at} \quad \mathbf{x} \in S. \tag{14.2.1.9}$$

In general, $\Gamma_{\mathbf{x}}$ is a first-order linear differential operator in the space variables $\mathbf{x}$. The basic types of boundary conditions are described below in Section 14.2.2.

The boundary condition (14.2.1.9) is said to be homogeneous if $g(\mathbf{x}) \equiv 0$. The boundary value problem (14.2.1.7)–(14.2.1.9) is said to be homogeneous if $\Phi \equiv 0$ and $g \equiv 0$.

► Adjoint and self-adjoint differential operators. Green formula.

The linear differential operator

$$L^*_{\mathbf{x}}[w] \equiv \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\big[a_{ij}(\mathbf{x})w\big] - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big[b_i(\mathbf{x})w\big] + c(\mathbf{x})w$$

is called the *adjoint* of $L_{\mathbf{x}}$; see (14.2.1.8). One has $(L^*_{\mathbf{x}})^* = L_{\mathbf{x}}$.

If the operator $L_{\mathbf{x}}$ coincides with $L^*_{\mathbf{x}}$, then it is said to be *self-adjoint*. A self-adjoint operator can be reduced to the form

$$L_{\mathbf{x}}[w] \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(\mathbf{x})\frac{\partial w}{\partial x_j}\right] + c(\mathbf{x})w. \qquad (14.2.1.10)$$

One has

$$uL_{\mathbf{x}}[w] - wL^*_{\mathbf{x}}[u] = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i},$$

where

$$P_i = \sum_{j=1}^{n}\left[ua_{ij}\frac{\partial w}{\partial x_J} - w\frac{\partial}{\partial x_j}(a_{ij}u)\right] + b_i uw.$$

Let V be a bounded domain whose boundary is a piecewise smooth surface S. Then the *Green formula*

$$\int_V \big(uL_{\mathbf{x}}[w] - wL^*_{\mathbf{x}}[u]\big)\,dV = \int_S \sum_{i=1}^{n} P_i \cos(\nu, x_i)\,dS$$

holds, where $dV = dx_1 \ldots dx_n$ and ν is the outward normal direction on S.

Example 14.2. For the 3D Laplace operator, we have $\Delta = \Delta^*$, and the Green formula acquires the form

$$\int_V (u\Delta w - w\Delta u)\,dV = \int_S \Big(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\Big)\,dS,$$

where $\partial/\partial\nu$ is the outward normal derivative. In applications, one often uses the special case of this formula with $u = 1$.

14.2.2 First, Second, Third, and Mixed Boundary Value Problems

For any (parabolic, hyperbolic, and elliptic) second-order partial differential equations, it is conventional to distinguish four basic types of boundary value problems depending on the form of the boundary conditions (14.2.1.4) [see also the analogous condition (14.2.1.9)].

For simplicity, here we confine ourselves to the case where the coefficients a_{ij} of equations (14.2.1.1) and (14.2.1.5) have the special form

$$a_{ij}(\mathbf{x},t) = a(\mathbf{x},t)\delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases}$$

This situation is rather frequent in applications; such coefficients are used to describe various phenomena (processes) in isotropic media.

First boundary value problem. The function $w(\mathbf{x},t)$ takes prescribed values at the boundary S of the domain:

$$w(\mathbf{x},t) = g_1(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in S. \qquad (14.2.2.1)$$

Second boundary value problem. The derivative along the (outward) normal is prescribed at the boundary S of the domain:

$$\frac{\partial w}{\partial N} = g_2(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in S. \qquad (14.2.2.2)$$

In heat transfer problems, where w is temperature, the left-hand side of the boundary condition (14.2.2.2) is proportional to the heat flux per unit area of the surface S.

Third boundary value problem. A linear relationship between the unknown function and its normal derivative is prescribed at the boundary S of the domain:

$$\frac{\partial w}{\partial N} + k(\mathbf{x},t)w = g_3(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in S. \qquad (14.2.2.3)$$

Usually, it is assumed that $k(\mathbf{x},t) = \text{const}$. In mass transfer problems, where w is the concentration, the boundary condition (14.2.2.3) with $g_3 \equiv 0$ describes a surface chemical reaction of the first order.

Mixed boundary value problems. Conditions of various types listed above are set at various portions of the boundary S.

If $g_1 \equiv 0$, $g_2 \equiv 0$, or $g_3 \equiv 0$, then the respective boundary conditions (14.2.2.1), (14.2.2.2), (14.2.2.3) are said to be homogeneous.

Boundary conditions for various boundary value problems for parabolic and hyperbolic equations in two independent variables x and t are displayed in Table 14.3. The equation coefficients are assumed to be continuous, with the coefficients of the highest derivatives being nonzero in the range $x_1 \le x \le x_2$ considered.

Remark 14.7. For elliptic equations, the first boundary value problem is often called the *Dirichlet problem*, and the second boundary value problem is called the *Neumann problem*.

⊙ *Literature for Section 14.2*: S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), R. Courant and D. Hilbert (1989), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), I. G. Petrovsky (1991), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

TABLE 14.3
Boundary conditions for various boundary value problems specified by parabolic and hyperbolic equations in two independent variables ($x_1 \le x \le x_2$)

Type of problem	Boundary condition at $x = x_1$	Boundary condition at $x = x_2$
First boundary value problem	$w = g_1(t)$	$w = g_2(t)$
Second boundary value problem	$\partial_x w = g_1(t)$	$\partial_x w = g_2(t)$
Third boundary value problem	$\partial_x w + \beta_1 w = g_1(t)$ ($\beta_1 < 0$)	$\partial_x w + \beta_2 w = g_2(t)$ ($\beta_2 > 0$)
Mixed boundary value problem	$w = g_1(t)$	$\partial_x w = g_2(t)$
Mixed boundary value problem	$\partial_x w = g_1(t)$	$w = g_2(t)$

14.3 Properties and Particular Solutions of Linear Equations

14.3.1 Homogeneous Linear Equations. Basic Properties of Particular Solutions

► Preliminary remarks.

For brevity, in this section a homogeneous linear partial differential equation will be written as

$$\mathcal{L}[w] = 0. \tag{14.3.1.1}$$

For second-order linear parabolic and hyperbolic equations, the linear differential operator $\mathcal{L}[w]$ is defined by the left-hand side of Eqs. (14.2.1.1) and (14.2.1.5), respectively. It is assumed that Eq. (14.3.1.1) is an arbitrary homogeneous linear partial differential equation of any order in the variables t, x_1, …, x_n with sufficiently smooth coefficients.

The linear operator $\mathcal{L}$ possesses the properties

$$\mathcal{L}[w_1 + w_2] = \mathcal{L}[w_1] + \mathcal{L}[w_2],$$
$$\mathcal{L}[Aw] = A\mathcal{L}[w], \qquad A = \text{const}.$$

An arbitrary homogeneous linear equation (14.3.1.1) has the trivial solution $w \equiv 0$.

A function w is called a *classical solution* of Eq. (14.3.1.1) if w, when substituted into (14.3.1.1), turns the equation into an identity and if all partial derivatives of w that occur in (14.3.1.1) are continuous; the notion of a classical solution is directly linked to the range of the independent variables. In what follows, we usually write "solution" instead of "classical solution" for brevity.

► Use of particular solutions for the construction of other solutions.

Below are basic properties of particular solutions of homogeneous linear equations.

1°. Let $w_1 = w_1(\mathbf{x}, t)$, $w_2 = w_2(\mathbf{x}, t)$, …, $w_k = w_k(\mathbf{x}, t)$, where $\mathbf{x} = (x_1, \dots, x_n)$, be any particular solutions of the homogeneous equation (14.3.1.1). Then the linear combination

$$w = A_1 w_1 + A_2 w_2 + \cdots + A_k w_k \tag{14.3.1.2}$$

with arbitrary constants A_1, A_2, …, A_k is a solution of Eq. (14.3.1.1) as well; in physics, this property is known as the *linear superposition principle*.

Assume that $\{w_k\}$ is an infinite sequence of solutions of Eq. (14.3.1.1). Then the series $\sum_{k=1}^{\infty} w_k$, irrespective of its convergence, is called a *formal solution* of (14.3.1.1). If the solutions w_k are classical, the series is uniformly convergent, and the sum of the series has all the necessary partial derivatives, then the sum of the series is a classical solution of Eq. (14.3.1.1).

2°. Let the coefficients of the linear differential operator $\mathcal{L}$ be independent of time t. If Eq. (14.3.1.1) has a particular solution $\widetilde{w} = \widetilde{w}(\mathbf{x}, t)$, then the function $\widetilde{w}(\mathbf{x}, t + a)$, where a is an arbitrary constant, is a solution of the equation as well.

If the coefficients of $\mathcal{L}$ are independent of only one space coordinate, say x_k, and Eq. (14.3.1.1) has a particular solution $\widetilde{w} = \widetilde{w}(\mathbf{x}, t)$, then the function $\widetilde{w}(\mathbf{x}, t)|_{x_k \Rightarrow x_k + b}$, where b is an arbitrary constant, is a solution of the equation as well.

3°. Let the coefficients of the linear differential operator $\mathcal{L}$ be independent of time t. If Eq. (14.3.1.1) has a particular solution $\widetilde{w} = \widetilde{w}(\mathbf{x}, t)$, then the partial derivatives of $\widetilde{w}$ with respect to time,*

$$\frac{\partial \widetilde{w}}{\partial t}, \quad \frac{\partial^2 \widetilde{w}}{\partial t^2}, \quad \dots, \quad \frac{\partial^k \widetilde{w}}{\partial t^k}, \quad \dots,$$

are solutions of Eq. (14.3.1.1) as well.

4°. Let the coefficients of the linear differential operator $\mathcal{L}$ be independent of the space variables $x_1, \dots, x_n$. If Eq. (14.3.1.1) has a particular solution $\widetilde{w} = \widetilde{w}(\mathbf{x}, t)$, then the partial derivatives of $\widetilde{w}$ with respect to the space coordinates

$$\frac{\partial \widetilde{w}}{\partial x_1}, \quad \frac{\partial \widetilde{w}}{\partial x_2}, \quad \frac{\partial \widetilde{w}}{\partial x_3}, \quad \dots, \quad \frac{\partial^2 \widetilde{w}}{\partial x_1^2}, \quad \frac{\partial^2 \widetilde{w}}{\partial x_1 \partial x_2}, \quad \dots, \quad \frac{\partial^{k+m} \widetilde{w}}{\partial x_2^k \partial x_3^m}, \quad \dots$$

are solutions of Eq. (14.3.1.1) as well.

If the coefficients of $\mathcal{L}$ are independent of only one space coordinate, say x_1, and Eq. (14.3.1.1) has a particular solution $\widetilde{w} = \widetilde{w}(\mathbf{x}, t)$, then the partial derivatives

$$\frac{\partial \widetilde{w}}{\partial x_1}, \quad \frac{\partial^2 \widetilde{w}}{\partial x_1^2}, \quad \dots, \quad \frac{\partial^k \widetilde{w}}{\partial x_1^k}, \quad \dots$$

are solutions of Eq. (14.3.1.1) as well.

5°. Let the coefficients of the linear differential operator $\mathcal{L}$ be constant and let Eq. (14.3.1.1) have a particular solution $\widetilde{w} = \widetilde{w}(\mathbf{x}, t)$. Then any particular derivatives of $\widetilde{w}$ with respect to time and the space coordinates (inclusive mixed derivatives)

$$\frac{\partial \widetilde{w}}{\partial t}, \quad \frac{\partial \widetilde{w}}{\partial x_1}, \quad \dots, \quad \frac{\partial^2 \widetilde{w}}{\partial x_2^2}, \quad \frac{\partial^2 \widetilde{w}}{\partial t \partial x_1}, \quad \dots, \quad \frac{\partial^k \widetilde{w}}{\partial x_3^k}, \quad \dots$$

are solutions of Eq. (14.3.1.1).

*Here and in what follows, it is assumed that the particular solution $\widetilde{w}$ is differentiable sufficiently many times with respect to t and $x_1, \dots, x_n$ (or the parameters).

6°. Suppose that Eq. (14.3.1.1) has a particular solution dependent on a parameter μ, $\widetilde{w} = \widetilde{w}(\mathbf{x}, t; \mu)$, and the coefficients of the linear differential operator $\mathcal{L}$ are independent of μ (but can depend on time and the space coordinates). Then, by differentiating $\widetilde{w}$ with respect to μ, one obtains other solutions of Eq. (14.3.1.1),

$$\frac{\partial \widetilde{w}}{\partial \mu}, \quad \frac{\partial^2 \widetilde{w}}{\partial \mu^2}, \quad \ldots, \quad \frac{\partial^k \widetilde{w}}{\partial \mu^k}, \quad \ldots$$

Let some constants $\mu_1, \ldots, \mu_k$ belong to the range of the parameter μ. Then the sum

$$w = A_1 \widetilde{w}(\mathbf{x}, t; \mu_1) + \cdots + A_k \widetilde{w}(\mathbf{x}, t; \mu_k), \tag{14.3.1.3}$$

where $A_1, \ldots, A_k$ are arbitrary constants, is a solution of the homogeneous linear equation (14.3.1.1) as well. The number of terms in sum (14.3.1.3) can be finite as well as infinite.

7°. Another efficient way of constructing solutions involves the following. The particular solution $\widetilde{w}(\mathbf{x}, t; \mu)$, which depends on the parameter μ (as before, it is assumed that the coefficients of the linear differential operator $\mathcal{L}$ are independent of μ), is first multiplied by an arbitrary function $\varphi(\mu)$. Then the resulting expression is integrated with respect to μ over some interval $[\alpha, \beta]$. Thus, one obtains a new function,

$$\int_\alpha^\beta \widetilde{w}(\mathbf{x}, t; \mu)\varphi(\mu)\, d\mu,$$

which is also a solution of the original homogeneous linear equation.

The properties listed in Items 1°–7° enable one to use known particular solutions to construct other particular solutions of homogeneous linear equations of mathematical physics.

Example 14.3. The linear equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw$$

has the particular solution

$$\widetilde{w}(x, t) = \exp[\mu x + (a\mu^2 + b)t],$$

where μ is an arbitrary constant. Differentiating this equation with respect to μ (see Item 6°) yields another solution

$$\widetilde{w}(x, t) = (x + 2a\mu t)\exp[\mu x + (a\mu^2 + b)t].$$

Example 14.4. The wave equation

$$\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial x^2}$$

has particular solutions

$$\widetilde{w}_1(x, t) = e^{\mu(x+at)}, \qquad \widetilde{w}_2(x, t) = e^{\mu(x-at)},$$

where μ is an arbitrary constant. According to Item 7°, the equation also has more general solutions of the form

$$\widetilde{w}_1(x,t)=\int_\beta^\alpha \varphi_1(\mu)e^{\mu(x+at)}\,d\mu=F_1(x+at),\qquad \widetilde{w}_2(x,t)=\int_\beta^\alpha \varphi_2(\mu)e^{\mu(x-at)}\,d\mu=F_2(x-at).$$

Since the functions $\varphi_1(\mu)$ and $\varphi_2(\mu)$ can be chosen arbitrarily, we can assume the functions $F_1(z_1)$ and $F_2(z_2)$ to be arbitrary as well. (This can readily be verified by a straightforward substitution into the original equation.)

By the linear superposition principle (see Item 1°), one can add the resulting solutions. As a result, we arrive at the *d'Alembert formula*

$$w = F_1(x+at) + F_2(x-at).$$

14.3.2 Separable Solutions. Solutions in the Form of Infinite Series

► Multiplicative and additive separable solutions.

1°. Many homogeneous linear partial differential equations have solutions that can be represented as products of functions depending on distinct arguments. Such solutions are referred to as *multiplicative separable solutions*; very commonly these solutions are briefly, but less accurately, called just *separable solutions*.

Table 14.4 presents the most commonly encountered types of homogeneous linear differential equations with many independent variables that admit exact separable solutions. Linear combinations of particular solutions that correspond to various values of the separation parameters, λ, β_1, …, β_n, are solutions of the equations in question as well. For brevity, the word “operator” is used to denote “linear differential operator.”

For a constant coefficient equation (see the first row in Table 14.4), the separation parameters must satisfy the algebraic equation

$$D(\lambda,\ \beta_1,\ \dots,\beta_n) = 0, \tag{14.3.2.1}$$

which results from substituting the solution into Eq. (14.3.1.1). In physical applications, Eq. (14.3.2.1) is usually referred to as the *dispersion equation*. Any n out of the $n+1$ separation parameters in (14.3.2.1) can be treated as arbitrary.

Example 14.5. Consider the linear equation

$$\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw.$$

A particular solution is sought in the form

$$w = A\exp(\beta x + \lambda t).$$

This results in the dispersion equation $\lambda^2 + k\lambda = a^2\beta^2 + b\beta + c$, where one of the two parameters β or λ can be treated as arbitrary.

For more complex multiplicative separable solutions of this equation and related equations, see Section 15.1.1.

Note that constant coefficient equations also admit more sophisticated solutions; see the second and third rows, the last column.

TABLE 14.4

Homogeneous linear partial differential equations that admit multiplicative separable solutions

No.	Form of Eq. (14.3.1.1)	Form of particular solutions
1	Equation coefficients are constant	$w(\mathbf{x},t)=A\exp(\lambda t+\beta_1x_1+\dots+\beta_nx_n)$, $\lambda,\beta_1,\dots,\beta_n$ are related by an algebraic equation
2	Equation coefficients are independent of time t	$w(\mathbf{x},t)=e^{\lambda t}\psi(\mathbf{x})$, λ is an arbitrary constant, $\mathbf{x}=\{x_1,\dots,x_n\}$
3	Equation coefficients are independent of the coordinates $x_1,\dots,x_n$	$w(\mathbf{x},t)=\exp(\beta_1x_1+\dots+\beta_nx_n)\psi(t)$, $\beta_1,\dots,\beta_n$ are arbitrary constants
4	Equation coefficients are independent of the coordinates $x_1,\dots,x_k$	$w(\mathbf{x},t)=\exp(\beta_1x_1+\dots+\beta_kx_k)\psi(t,x_{k+1},\dots,x_n)$, $\beta_1,\dots,\beta_k$ are arbitrary constants
5	$L_t[w]+L_\mathbf{x}[w]=0$, operator L_t depends only on t, operator $L_\mathbf{x}$ depends only on $\mathbf{x}$	$w(\mathbf{x},t)=\varphi(t)\psi(\mathbf{x})$, $\varphi(t)$ satisfies the equation $L_t[\varphi]+\lambda\varphi=0$, $\psi(\mathbf{x})$ satisfies the equation $L_\mathbf{x}[\psi]-\lambda\psi=0$
6	$L_t[w]+L_1[w]+\dots+L_n[w]=0$, operator L_t depends only on t, operator L_k depends only on x_k	$w(\mathbf{x},t)=\varphi(t)\psi_1(x_1)\dots\psi_n(x_n)$, $\varphi(t)$ satisfies the equation $L_t[\varphi]+\lambda\varphi=0$, $\psi_k(x_k)$ satisfies the equation $L_k[\psi_k]+\beta_k\psi_k=0$, $\lambda+\beta_1+\dots+\beta_n=0$
7	$f_0(x_1)L_t[w]+\sum\limits_{k=1}^{n}f_k(x_1)L_k[w]=0$, operator L_t depends only on t, operator L_k depends only on x_k	$w(\mathbf{x},t)=\varphi(t)\psi_1(x_1)\dots\psi_n(x_n)$, $L_t[\varphi]+\lambda\varphi=0$, $L_k[\psi_k]+\beta_k\psi_k=0,\ \ k=2,\dots,n$, $f_1(x_1)L_1[\psi_1]-\Big[\lambda f_0(x_1)+\sum\limits_{k=2}^{n}\beta_kf_k(x_1)\Big]\psi_1=0$
8	$\dfrac{\partial w}{\partial t}+L_1[w]+\dots+L_n[w]=0$, where $L_k[w]=\sum\limits_{s=0}^{m_k}f_{ks}(x_k,t)\dfrac{\partial^s w}{\partial x_k^s}$	$w(\mathbf{x},t)=\psi_1(x_1,t)\psi_2(x_2,t)\dots\psi_n(x_n,t)$, $\dfrac{\partial\psi_k}{\partial t}+L_k[\psi_k]=\lambda_k(t)\psi_k,\ \ k=1,\dots,n$, $\lambda_1(t)+\lambda_2(t)+\dots+\lambda_n(t)=0$

The eighth row of Table 14.4 presents the case of *incomplete separation of variables* where the solution is separated with respect to the space variables $x_1,\ \dots,\ x_n$, but is not separated with respect to time t.

Remark 14.8. For stationary equations that do not depend on t, one should set $\lambda=0$, $L_t[w]\equiv 0$, and $\varphi(t)\equiv 1$ in rows 1, 6, and 7 of Table 14.4.

Remark 14.9. Multiplicative separable solutions play an important role in the theory of linear partial differential equations; they are used for finding solutions of stationary and nonstationary boundary value problems; see Chapter 15.

2°. Linear partial differential equations of the form

$$L_t[w]+L_\mathbf{x}[w]+kw=f(\mathbf{x})+g(t),$$

where L_t is a linear differential operator that depends on only t and $L_\mathbf{x}$ is a linear differential operator that depends on only $\mathbf{x}$, have solutions that can be represented as the sum of functions depending on distinct arguments,

$$w=u(\mathbf{x})+v(t).$$

Such solutions are referred to as *additive separable solutions*.

Example 14.6. The equation in Example 14.5 admits an exact additive separable solution $w = u(x) + v(t)$ with $u(x)$ and $v(t)$ described by the linear constant coefficient ordinary differential equations

$$a^2 u''_{xx} + bu'_x + cu = C,$$
$$v''_{tt} + kv'_t - cv = C,$$

where C is an arbitrary constant, which are easy to integrate. A more general partial differential equation with variable coefficients $a = a(x)$, $b = b(x)$, $k = k(t)$, and $c = \text{const}$ admits an additive separable solution as well.

► Solutions in the form of infinite series in t.

1°. The equation

$$\frac{\partial w}{\partial t} = M[w],$$

where M is an arbitrary linear differential operator of the second (or any) order that only depends on the space variables, has the formal series solution

$$w(\mathbf{x}, t) = f(\mathbf{x}) + \sum_{k=1}^{\infty} \frac{t^k}{k!} M^k[f(\mathbf{x})], \qquad M^k[f] = M\bigl[M^{k-1}[f]\bigr],$$

where $f(\mathbf{x})$ is an arbitrary infinitely differentiable function. This solution satisfies the initial condition $w(\mathbf{x}, 0) = f(\mathbf{x})$.

Example 14.7. Consider the heat equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2}.$$

In this case we have $M = a\frac{\partial^2}{\partial x^2}$. Therefore the formal series solution has the form

$$w(x, t) = f(x) + \sum_{k=1}^{\infty} \frac{(at)^k}{k!} f_x^{(2n)}(x), \qquad f_x^{(m)} = \frac{d^m}{dx^m} f(x).$$

If the function $f(x)$ is a polynomial of degree n, then so is the solution. For example, by setting $f(x) = Ax^2 + Bx + C$, we obtain the particular solution

$$w(x, t) = A(x^2 + 2at) + Bx + C.$$

2°. The equation

$$\frac{\partial^2 w}{\partial t^2} = M[w],$$

where M is a linear differential operator, just as in Item 1°, has a formal solution represented by the sum of two series as

$$w(\mathbf{x}, t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} M^k[f(\mathbf{x})] + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} M^k[g(\mathbf{x})],$$

where $f(\mathbf{x})$ and $g(\mathbf{x})$ are arbitrary infinitely differentiable functions. This solution satisfies the initial conditions $w(\mathbf{x}, 0) = f(\mathbf{x})$ and $\partial_t w(\mathbf{x}, 0) = g(\mathbf{x})$.

14.3.3 Nonhomogeneous Linear Equations and Their Properties

For brevity, we write a nonhomogeneous linear partial differential equation in the form

$$\mathcal{L}[w] = \Phi(\mathbf{x}, t). \tag{14.3.3.1}$$

The linear differential operator $\mathcal{L}$ is defined above (see the beginning of Section 14.3.1).

Below are the simplest properties of particular solutions of the nonhomogeneous equation (14.3.3.1).

1°. If $\widetilde{w}_\Phi(\mathbf{x}, t)$ is a particular solution of the nonhomogeneous equation (14.3.3.1) and $\widetilde{w}_0(\mathbf{x}, t)$ is a particular solution of the corresponding homogeneous equation (14.3.1.1), then the sum

$$A\widetilde{w}_0(\mathbf{x}, t) + \widetilde{w}_\Phi(\mathbf{x}, t),$$

where A is an arbitrary constant, is a solution of the nonhomogeneous equation (14.3.3.1) as well. The following, more general statement holds: The general solution of the nonhomogeneous equation (14.3.3.1) is the sum of the general solution of the corresponding homogeneous equation (14.3.1.1) and any particular solution of the nonhomogeneous equation (14.3.3.1).

2°. Suppose that w_1 and w_2 are solutions of nonhomogeneous linear equations with the same left-hand side and distinct right-hand sides; i.e.,

$$\mathcal{L}[w_1] = \Phi_1(\mathbf{x}, t), \qquad \mathcal{L}[w_2] = \Phi_2(\mathbf{x}, t).$$

Then the function $w = w_1 + w_2$ is a solution of the equation

$$\mathcal{L}[w] = \Phi_1(\mathbf{x}, t) + \Phi_2(\mathbf{x}, t).$$

14.3.4 General Solutions of Some Hyperbolic Equations

► **D'Alembert's solution of the wave equation.**

The wave equation

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{14.3.4.1}$$

has the general solution

$$w = \varphi(x + at) + \psi(x - at), \tag{14.3.4.2}$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary twice continuously differentiable functions. The solution (14.3.4.2) has the physical interpretation of two *traveling waves* of arbitrary shape that propagate to the left and to the right along the x-axis at a constant speed a ($a > 0$).

► **Laplace cascade method for hyperbolic equation in two variables.**

A general linear hyperbolic equation with two independent variables can be reduced to an equation of the form (see Section 14.1.1):

$$\frac{\partial^2 w}{\partial x \partial y} + a(x, y)\frac{\partial w}{\partial x} + b(x, y)\frac{\partial w}{\partial y} + c(x, y)w = f(x, y). \tag{14.3.4.3}$$

Sometimes it is possible to obtain formulas determining all solutions of Eq. (14.3.4.3). First consider two special cases.

1°. *First special case.* Suppose that the identity

$$g \equiv \frac{\partial a}{\partial x} + ab - c \equiv 0 \qquad (14.3.4.4)$$

is valid; for brevity, the arguments of the functions are omitted. Then Eq. (14.3.4.3) can be rewritten in the form

$$\frac{\partial u}{\partial x} + bu = f, \qquad (14.3.4.5)$$

where

$$u = \frac{\partial w}{\partial y} + aw. \qquad (14.3.4.6)$$

Equation (14.3.4.5) is a linear first-order ordinary differential equation in x for u (the variable y appears in the equation as a parameter) and is easy to integrate. Further, substituting u into (14.3.4.6) yields a linear first-order ordinary differential equation in y for w (now x appears in the equation as a parameter). On solving this equation, one obtains the general solution of the original equation (14.3.4.3) subject to condition (14.3.4.4):

$$w = \exp\Bigl(-\int a\,dy\Bigr)\Bigl\{\varphi(x) + \int \Bigl[\psi(y) + \int f \exp\Bigl(\int b\,dx\Bigr)dx\Bigr] \exp\Bigl(\int a\,dy - \int b\,dx\Bigr)dy\Bigr\},$$

where $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

2°. *Second special case.* Suppose the identity

$$h \equiv \frac{\partial b}{\partial y} + ab - c \equiv 0 \qquad (14.3.4.7)$$

holds true. Proceeding in the same ways as in the first special case, one obtains the general solution of (14.3.4.3):

$$w = \exp\Bigl(-\int b\,dx\Bigr)\Bigl\{\psi(y) + \int \Bigl[\varphi(x) + \int f \exp\Bigl(\int a\,dy\Bigr)dy\Bigr] \exp\Bigl(\int b\,dx - \int a\,dy\Bigr)dx\Bigr\}.$$

3°. *Laplace cascade method.* If $g \neq 0$, consider the new equation of the form (14.3.4.3),

$$\mathrm{L}_1[w_1] \equiv \frac{\partial^2 w_1}{\partial x \partial y} + a_1(x,y)\frac{\partial w_1}{\partial x} + b_1(x,y)\frac{\partial w_1}{\partial y} + c_1(x,y)w_1 = f_1(x,y), \qquad (14.3.4.8)$$

where

$$a_1 = a - \frac{\partial \ln g}{\partial y}, \quad b_1 = b, \quad c_1 = c - \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - b\frac{\partial \ln g}{\partial y}, \quad f_1 = \Bigl(a - \frac{\partial \ln g}{\partial y}\Bigr)f.$$

Whenever a solution w_1 is found, the corresponding solution of the original equation (14.3.4.3) can be obtained by the formula

$$w = \frac{1}{g}\Bigl(\frac{\partial w_1}{\partial x} + bw_1 - f\Bigr).$$

For Eq. (14.3.4.8), the functions similar to (14.3.4.4) and (14.3.4.7) are expressed as

$$g_1 = 2g - h - \frac{\partial^2 \ln g}{\partial x \partial y}, \quad h_1 = h.$$

If $g_1 \equiv 0$, then the function w_1 can be found using the technique described above. If $g_1 \not\equiv 0$, one proceeds to the construction, in the same way as above, of the equation $\mathrm{L}_2[w_2] = f_2$, and so on. If $h \not\equiv 0$, then a similar chain of equations may be constructed: $\mathrm{L}_1^*[w_1^*] = f_1^*$, $\mathrm{L}_2^*[w_2^*] = f_2^*$, etc.

If at some step g_k or h_k vanishes, then it is possible to obtain the general solution of Eq. (14.3.4.3).

Example 14.8. Consider the *Euler–Darboux equation*

$$\frac{\partial^2 w}{\partial x \partial y} - \frac{\alpha}{x-y}\frac{\partial w}{\partial x} + \frac{\beta}{x-y}\frac{\partial w}{\partial y} = 0.$$

We will show that its general solution can be obtained if at least one of the numbers α or β is an integer.

With the notation adopted for Eq. (14.3.4.3) and the function (14.3.4.4), we have

$$a(x,y) = -\frac{\alpha}{x-y}, \quad b(x,y) = \frac{\beta}{x-y}, \quad c(x,y) = f(x,y) = 0, \quad g = \frac{\alpha(1-\beta)}{(x-y)^2},$$

which means that $g \equiv 0$ if $\alpha = 0$ or $\beta = 1$. If $g \not\equiv 0$, then we construct Eq. (14.3.4.8), where

$$a_1 = -\frac{2+\alpha}{x-y}, \quad b_1 = \frac{\beta}{x-y}, \quad c_1 = -\frac{\alpha+\beta}{(x-y)^2}.$$

If follows that

$$g_1 = \frac{(1+\alpha)(2-\beta)}{(x-y)^2},$$

and hence $g_1 \equiv 0$ at $\alpha = -1$ or $\beta = 2$. Similarly, it can be shown that $g_k \equiv 0$ at $\alpha = -k$ or $\beta = k+1$ ($k = 0,\ 1,\ 2,\ \ldots$). If we use the other sequence of auxiliary equations, $\mathrm{L}_k^*[w_k^*] = f_k^*$, it can be shown that the above holds for $\alpha = 1,\ 2,\ \ldots$ and $\beta = 0,\ -1,\ -2,\ \ldots$

⊙ *Literature for Section 14.3*: S. G. Mikhlin (1970), W. A. Strauss (1992), G. A. Korn and T. M. Korn (2000), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 15

Separation of Variables and Integral Transform Methods

15.1 Separation of Variables (Fourier Method)

15.1.1 Description of Separation of Variables. General Stage of Solution

► **Scheme of solving boundary value problems by separation of variables.**

Many linear problems of mathematical physics can be solved by separation of variables. Figure 15.1 depicts the scheme of application of this method to solve boundary value problems for second-order homogeneous linear equations of the parabolic and hyperbolic type with homogeneous boundary conditions and nonhomogeneous initial conditions. For simplicity, problems with two independent variables x and t are considered with $x_1 \le x \le x_2$ and $t \ge 0$.

The scheme presented in Fig. 15.1 applies to boundary value problems for second-order linear homogeneous partial differential equations of the form

$$\alpha(t)\frac{\partial^2 w}{\partial t^2} + \beta(t)\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + \big[c(x) + \gamma(t)\big]w \qquad (15.1.1.1)$$

with homogeneous linear boundary conditions

$$\begin{aligned} s_1\partial_x w + k_1 w = 0 \quad &\text{at} \quad x = x_1,\\ s_2\partial_x w + k_2 w = 0 \quad &\text{at} \quad x = x_2 \end{aligned} \qquad (15.1.1.2)$$

and arbitrary initial conditions

$$w = f_0(x) \quad \text{at} \quad t = 0, \qquad (15.1.1.3)$$

$$\partial_t w = f_1(x) \quad \text{at} \quad t = 0. \qquad (15.1.1.4)$$

For parabolic equations, which correspond to $\alpha(t) \equiv 0$ in (15.1.1.1), only the initial condition (15.1.1.3) is set.

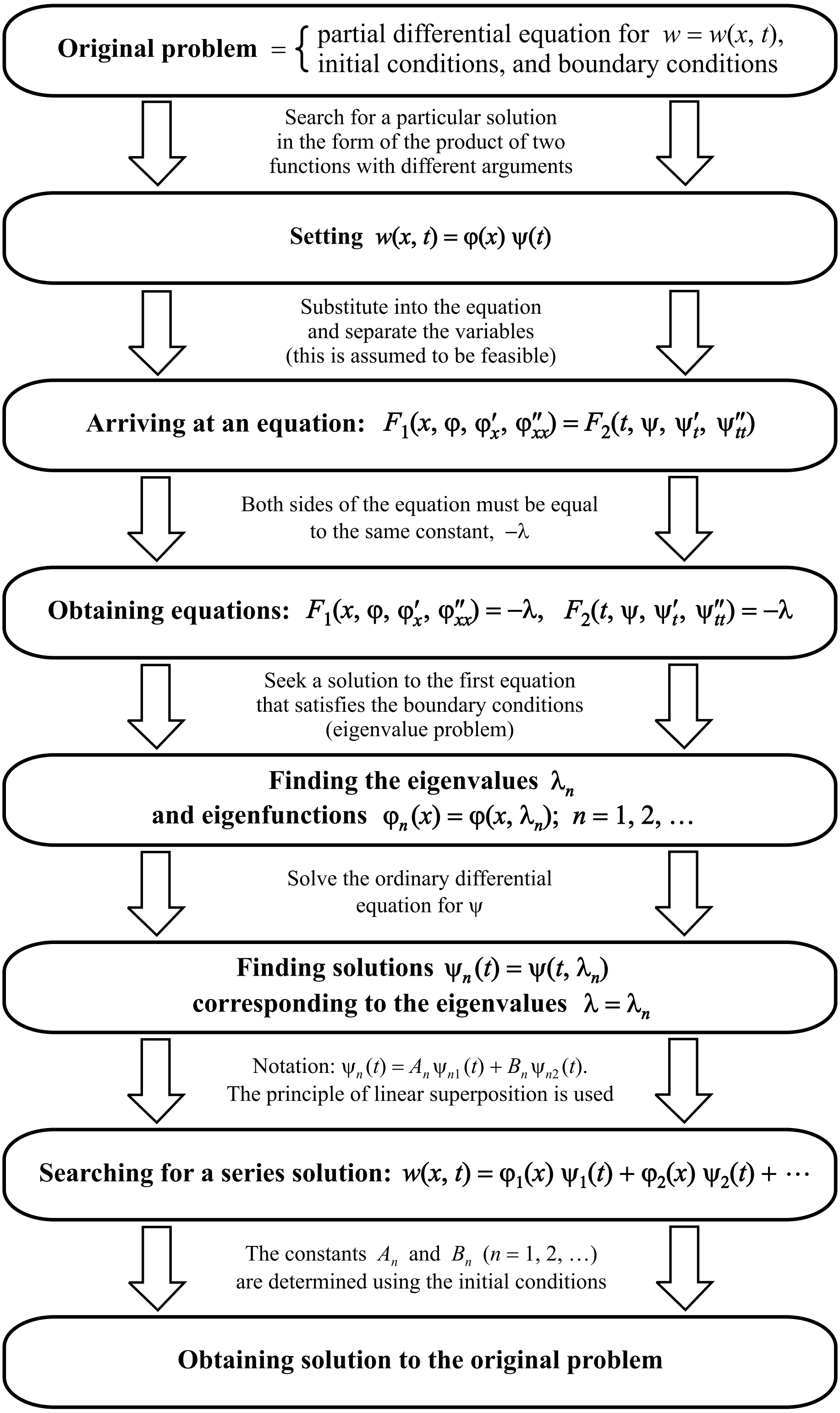


Figure 15.1: Scheme of solving linear boundary value problems by separation of variables (for parabolic equations, the function F_2 does not depend on ψ''_{tt}, and all $B_n = 0$).

Below we consider the basic stages of separation of variables in more detail. We assume that the coefficients of Eq. (15.1.1.1) and the boundary conditions (15.1.1.2) meet the following requirements:

$$\alpha(t),\ \beta(t),\ \gamma(t),\ a(x),\ b(x),\ c(x)\ \text{are continuous functions},$$
$$\alpha(t) \geq 0, \quad a(x) > 0, \quad |s_1| + |k_1| > 0, \quad |s_2| + |k_2| > 0.$$

Remark 15.1. Separation of variables is also used to solve linear boundary value problems for elliptic equations of the form (15.1.1.1) with $\alpha(t) < 0$ and $a(x) > 0$ and with the boundary conditions (15.1.1.2) in x and similar boundary conditions in t. In this case, all results obtained for the general stage of solution described below remain valid; for details, see Section 15.1.4.

Remark 15.2. In various applications, equations of the form (15.1.1.1) may arise with the coefficient $b(x)$ going to infinity at the boundary, $b(x) \to \infty$ as $x \to x_1$, with the other coefficients being continuous. In this case, the first boundary condition in (15.1.1.2) should be replaced with a condition of boundedness of the solution as $x \to x_1$. This may occur in spatial problems with central or axial symmetry where the solution depends only on the radial coordinate.

► Derivation of equations and boundary conditions for particular solutions.

The approach is based on searching for particular solutions of Eq. (15.1.1.1) in the product form

$$w(x,t) = \varphi(x)\,\psi(t). \tag{15.1.1.5}$$

After separation of the variables and elementary manipulations, one arrives at the following linear ordinary differential equations for the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$:

$$a(x)\varphi''_{xx} + b(x)\varphi'_x + [\lambda + c(x)]\varphi = 0, \tag{15.1.1.6}$$
$$\alpha(t)\psi''_{tt} + \beta(t)\psi'_t + [\lambda - \gamma(t)]\psi = 0. \tag{15.1.1.7}$$

These equations contain a free parameter λ called the *separation constant*. With the notation adopted in Fig. 15.1, Eqs. (15.1.1.6) and (15.1.1.7) can be rewritten as follows: $\varphi F_1(x,\varphi,\varphi'_x,\varphi''_{xx}) + \lambda\varphi = 0$ and $\psi F_2(t,\psi,\psi'_t,\psi''_{tt}) + \lambda\psi = 0$.

Substituting (15.1.1.5) into (15.1.1.2) yields the boundary conditions for $\varphi = \varphi(x)$:

$$\begin{aligned} s_1\varphi'_x + k_1\varphi = 0 \quad &\text{at} \quad x = x_1,\\ s_2\varphi'_x + k_2\varphi = 0 \quad &\text{at} \quad x = x_2. \end{aligned} \tag{15.1.1.8}$$

The homogeneous linear ordinary differential equation (15.1.1.6) in conjunction with the homogeneous linear boundary conditions (15.1.1.8) forms an eigenvalue problem.

► Solution of eigenvalue problems. Orthogonality of eigenfunctions.

Suppose that $\widetilde{\varphi}_1 = \widetilde{\varphi}_1(x,\lambda)$ and $\widetilde{\varphi}_2 = \widetilde{\varphi}_2(x,\lambda)$ are linearly independent particular solutions of Eq. (15.1.1.6). Then the general solution of this equation can be represented as the linear combination

$$\varphi = C_1\widetilde{\varphi}_1(x,\lambda) + C_2\widetilde{\varphi}_2(x,\lambda), \tag{15.1.1.9}$$

where C_1 and C_2 are arbitrary constants.

Substituting the solution (15.1.1.9) into the boundary conditions (15.1.1.8) yields the following homogeneous linear algebraic system of equations for C_1 and C_2:

$$\begin{aligned}\varepsilon_{11}(\lambda)C_1+\varepsilon_{12}(\lambda)C_2&=0,\\ \varepsilon_{21}(\lambda)C_1+\varepsilon_{22}(\lambda)C_2&=0,\end{aligned}\qquad(15.1.1.10)$$

where $\varepsilon_{ij}(\lambda)=\bigl[s_i(\widetilde{\varphi}_j)'_x+k_i\widetilde{\varphi}_j\bigr]_{x=x_i}$. For system (15.1.1.10) to have nontrivial solutions, its determinant must be zero; we have

$$\varepsilon_{11}(\lambda)\varepsilon_{22}(\lambda)-\varepsilon_{12}(\lambda)\varepsilon_{21}(\lambda)=0.\qquad(15.1.1.11)$$

By solving the transcendental equation (15.1.1.11) for λ, one obtains the *eigenvalues* $\lambda=\lambda_n$, where $n=1,\,2,\,\ldots$ For these values of λ, Eq. (15.1.1.6) has nontrivial solutions,

$$\varphi_n(x)=\varepsilon_{12}(\lambda_n)\widetilde{\varphi}_1(x,\lambda_n)-\varepsilon_{11}(\lambda_n)\widetilde{\varphi}_2(x,\lambda_n),\qquad(15.1.1.12)$$

which are called *eigenfunctions* (these functions are defined up to a constant factor).

To facilitate the subsequent analysis, we represent Eq. (15.1.1.6) in the form

$$[p(x)\varphi'_x]'_x+[\lambda\rho(x)-q(x)]\varphi=0,\qquad(15.1.1.13)$$

where

$$p(x)=\exp\biggl[\int\frac{b(x)}{a(x)}\,dx\biggr],\quad q(x)=-\frac{c(x)}{a(x)}p(x),\quad \rho(x)=\frac{1}{a(x)}p(x).\qquad(15.1.1.14)$$

It follows from the adopted assumptions that $p(x)$, $p'_x(x)$, $q(x)$, and $\rho(x)$ are continuous functions, with $p(x)>0$ and $\rho(x)>0$.

The eigenvalue problem (15.1.1.13), (15.1.1.8) is known to possess the following properties:

1. All eigenvalues λ_1, λ_2, … are real, and $\lambda_n\to\infty$ as $n\to\infty$; consequently, the number of negative eigenvalues is finite.

2. The system of eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$, … is orthogonal on the interval $x_1\le x\le x_2$ with weight $\rho(x)$; i.e.,

$$\int_{x_1}^{x_2}\rho(x)\varphi_n(x)\varphi_m(x)\,dx=0\quad\text{for}\quad n\neq m.\qquad(15.1.1.15)$$

3. If

$$q(x)\ge 0,\quad s_1k_1\le 0,\quad s_2k_2\ge 0,\qquad(15.1.1.16)$$

there are no negative eigenvalues. If $q\equiv 0$ and $k_1=k_2=0$, the least eigenvalue is $\lambda_1=0$ and the corresponding eigenfunction is $\varphi_1=\text{const}$. Otherwise, all eigenvalues are positive, provided that conditions (15.1.1.16) are satisfied; the first inequality in (15.1.1.16) is satisfied if $c(x)\le 0$.

Section 3.8.9 (see the text after the Sturm–Liouville problem (5)) presents some estimates for the eigenvalues λ_n and eigenfunctions $\varphi_n(x)$.

◆ *The procedure for constructing solutions of nonstationary boundary value problems is further different for parabolic and hyperbolic equations; see Sections 15.1.2 and 15.1.3 below for results (elliptic and higher-order equations are treated in Sections 15.1.4 and 15.1.5).*

15.1.2 Problems for Parabolic Equations: Final Stage of Solution

► **Series solutions of boundary value problems for parabolic equations.**

Consider the problem for the parabolic equation

$$\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + \big[c(x) + \gamma(t)\big]w \tag{15.1.2.1}$$

(this equation is obtained from (15.1.1.1) in the case $\alpha(t) \equiv 0$ and $\beta(t) = 1$) with homogeneous linear boundary conditions (15.1.1.2) and the initial condition (15.1.1.3).

First, one seeks particular solutions of Eq. (15.1.2.1) in the product form (15.1.1.5), where the function $\varphi(x)$ is obtained by solving an eigenvalue problem for the ordinary differential equation (15.1.1.6) with the boundary conditions (15.1.1.8). The solution of Eq. (15.1.1.7) with $\alpha(t) \equiv 0$ and $\beta(t) = 1$ corresponding to the eigenvalues $\lambda = \lambda_n$ and satisfying the normalizing conditions $\psi_n(0) = 1$ has the form

$$\psi_n(t) = \exp\left[-\lambda_n t + \int_0^t \gamma(\xi)\,d\xi\right]. \tag{15.1.2.2}$$

Then the solution of the nonstationary boundary value problem (15.1.2.1), (15.1.1.2), (15.1.1.3) is sought in the series form

$$w(x,t) = \sum_{n=1}^{\infty} A_n \varphi_n(x)\,\psi_n(t), \tag{15.1.2.3}$$

where the A_n are arbitrary constants and the functions $w_n(x,t) = \varphi_n(x)\,\psi_n(t)$ are particular solutions of Eq. (15.1.2.1) satisfying the boundary conditions (15.1.1.2). By the linear superposition principle, the series (15.1.2.3) is a solution of the original partial differential equation as well and satisfies the boundary conditions.

To determine the coefficients A_n, we substitute the series (15.1.2.3) into the initial condition (15.1.1.3), thus obtaining

$$\sum_{n=1}^{\infty} A_n \varphi_n(x) = f_0(x).$$

Multiplying this equation by $\rho(x)\varphi_n(x)$, where the weight function $\rho(x)$ is defined in (15.1.1.14), integrating the resulting relation with respect to x over the interval $x_1 \le x \le x_2$, and taking into account the properties (15.1.1.15), we find

$$A_n = \frac{1}{\|\varphi_n\|^2}\int_{x_1}^{x_2} \rho(x)\varphi_n(x) f_0(x)\,dx, \quad \|\varphi_n\|^2 = \int_{x_1}^{x_2} \rho(x)\varphi_n^2(x)\,dx. \tag{15.1.2.4}$$

Relations (15.1.2.3), (15.1.2.2), (15.1.2.4), and (15.1.1.12) give a formal solution of the nonstationary boundary value problem (15.1.2.1), (15.1.1.2), (15.1.1.3).

Example 15.1. Consider the first (Dirichlet) boundary value problem on the interval $0 \le x \le l$ for the heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \tag{15.1.2.5}$$

with the general initial condition (15.1.1.3) and the homogeneous boundary conditions

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l. \tag{15.1.2.6}$$

The function $\psi(t)$ in the particular solution (15.1.1.5) is found from (15.1.2.2), where $\gamma(t) = 0$:

$$\psi_n(t) = \exp(-\lambda_n t). \tag{15.1.2.7}$$

The functions $\varphi_n(x)$ are determined by solving the eigenvalue problem (15.1.1.6), (15.1.1.8) with $a(x) = 1$, $b(x) = c(x) = 0$, $s_1 = s_2 = 0$, $k_1 = k_2 = 1$, $x_1 = 0$, and $x_2 = l$:

$$\varphi''_{xx} + \lambda\varphi = 0; \qquad \varphi = 0 \quad \text{at} \quad x = 0, \qquad \varphi = 0 \quad \text{at} \quad x = l.$$

So we obtain the eigenfunctions and eigenvalues:

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1,\ 2,\ \ldots \tag{15.1.2.8}$$

The solution of problem (15.1.2.5)–(15.1.2.6), (15.1.1.3) is given by formulas (15.1.2.3) and (15.1.2.4). Taking into account the fact that $\|\varphi_n\|^2 = l/2$, we obtain

$$w(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\frac{n^2\pi^2 t}{l^2}\right), \qquad A_n = \frac{2}{l}\int_0^l f_0(\xi) \sin\left(\frac{n\pi\xi}{l}\right) d\xi. \tag{15.1.2.9}$$

If the function $f_0(x)$ is twice continuously differentiable and the compatibility conditions (see below) are satisfied, then the series (15.1.2.9) is convergent and admits termwise differentiation, once with respect to t and twice with respect to x. In this case, formula (15.1.2.9) gives the classical smooth solution of problem (15.1.2.5)–(15.1.2.6), (15.1.1.3). [If $f_0(x)$ is not as smooth as indicated or if the compatibility conditions are not met, then the series (15.1.2.9) may converge to a discontinuous function, thus giving only a generalized solution.]

Remark 15.3. For the solution of linear nonhomogeneous parabolic equations with nonhomogeneous boundary conditions, see Section 17.1.

► Conditions of compatibility of initial and boundary conditions.

Suppose that the function w has a continuous derivative with respect to t and two continuous derivatives with respect to x and is a solution of problem (15.1.2.1), (15.1.1.2), (15.1.1.3). Then the boundary conditions (15.1.1.2) and the initial condition (15.1.1.3) must be consistent; namely, the following compatibility conditions must hold:

$$[s_1 f'_0 + k_1 f_0]_{x=x_1} = 0, \quad [s_2 f'_0 + k_2 f_0]_{x=x_2} = 0. \tag{15.1.2.10}$$

If $s_1 = 0$ or $s_2 = 0$, then the additional compatibility conditions

$$\begin{aligned} &[a(x)f''_0 + b(x)f'_0]_{x=x_1} = 0 \quad \text{if} \quad s_1 = 0,\\ &[a(x)f''_0 + b(x)f'_0]_{x=x_2} = 0 \quad \text{if} \quad s_2 = 0 \end{aligned} \tag{15.1.2.11}$$

should hold as well; the primes denote derivatives with respect to x.

15.1.3 Problems for Hyperbolic Equations: Final Stage of Solution

► **Series solution of boundary value problems for hyperbolic equations.**

For hyperbolic equations, the solution of the boundary value problem (15.1.1.1)–(15.1.1.4) is sought in the series form

$$w(x,t)=\sum_{n=1}^{\infty}\varphi_n(x)\big[A_n\psi_{n1}(t)+B_n\psi_{n2}(t)\big]. \tag{15.1.3.1}$$

Here A_n and B_n are arbitrary constants. The functions $\psi_{n1}(t)$ and $\psi_{n2}(t)$ are particular solutions of the linear equation (15.1.1.7) for ψ (with $\lambda=\lambda_n$) that satisfy the conditions

$$\psi_{n1}(0)=1,\quad \psi'_{n1}(0)=0;\qquad \psi_{n2}(0)=0,\quad \psi'_{n2}(0)=1. \tag{15.1.3.2}$$

The functions $\varphi_n(x)$ and λ_n are determined by solving the eigenvalue problem (15.1.1.6), (15.1.1.8).

Substituting solution (15.1.3.1) into the initial conditions (15.1.1.3)–(15.1.1.4) yields

$$\sum_{n=1}^{\infty}A_n\varphi_n(x)=f_0(x),\qquad \sum_{n=1}^{\infty}B_n\varphi_n(x)=f_1(x).$$

Multiplying these equations by $\rho(x)\varphi_n(x)$, where the weight function $\rho(x)$ is defined in (15.1.1.14), then integrating the resulting relations with respect to x on the interval $x_1\le x\le x_2$, and taking into account properties (15.1.1.15), we obtain the coefficients of series (15.1.3.1) in the form

$$\begin{aligned}A_n&=\frac{1}{\|\varphi_n\|^2}\int_{x_1}^{x_2}\rho(x)\varphi_n(x)f_0(x)\,dx,\\ B_n&=\frac{1}{\|\varphi_n\|^2}\int_{x_1}^{x_2}\rho(x)\varphi_n(x)f_1(x)\,dx.\end{aligned} \tag{15.1.3.3}$$

The quantity $\|\varphi_n\|$ is defined in (15.1.2.4).

Relations (15.1.3.1), (15.1.1.12), and (15.1.3.3) give a formal solution of the nonstationary boundary value problem (15.1.1.1)–(15.1.1.4) for $\alpha(t)>0$.

Example 15.2. Consider a mixed boundary value problem on the interval $0\le x\le l$ for the wave equation

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial x^2} \tag{15.1.3.4}$$

with the general initial conditions (15.1.1.3)–(15.1.1.4) and the homogeneous boundary conditions

$$w=0\quad\text{at}\quad x=0,\qquad \partial_x w=0\quad\text{at}\quad x=l. \tag{15.1.3.5}$$

The functions $\psi_{n1}(t)$ and $\psi_{n2}(t)$ are determined by the linear equation [see (15.1.1.7) with $\alpha(t)=1$, $\beta(t)=\gamma(t)=0$, and $\lambda=\lambda_n$]

$$\psi''_{tt}+\lambda\psi=0$$

with the initial conditions (15.1.3.2). We find

$$\psi_{n1}(t) = \cos\bigl(\sqrt{\lambda_n}\, t\bigr), \quad \psi_{n2}(t) = \frac{1}{\sqrt{\lambda_n}} \sin\bigl(\sqrt{\lambda_n}\, t\bigr). \tag{15.1.3.6}$$

The functions $\varphi_n(x)$ are determined by solving the eigenvalue problem (15.1.1.6), (15.1.1.8) with $a(x) = 1$, $b(x) = c(x) = 0$, $s_1 = k_2 = 0$, $s_2 = k_1 = 1$, $x_1 = 0$, and $x_2 = l$:

$$\varphi''_{xx} + \lambda\varphi = 0; \qquad \varphi = 0 \quad \text{at} \quad x = 0, \qquad \varphi'_x = 0 \quad \text{at} \quad x = l.$$

So we obtain the eigenfunctions and eigenvalues:

$$\varphi_n(x) = \sin(\mu_n x), \quad \mu_n = \sqrt{\lambda_n} = \frac{\pi(2n-1)}{2l}, \quad n = 1,\, 2,\, \dots \tag{15.1.3.7}$$

The solution of problem (15.1.3.4)–(15.1.3.5), (15.1.1.3)–(15.1.1.4) is given by formulas (15.1.3.1) and (15.1.3.3). Taking into account the fact that $\|\varphi_n\|^2 = l/2$, we have

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} \Bigl[A_n \cos(\mu_n t) + B_n \frac{1}{\mu_n} \sin(\mu_n t)\Bigr] \sin(\mu_n x), \quad \mu_n = \frac{\pi(2n-1)}{2l}, \\ A_n &= \frac{2}{l} \int_0^l f_0(x) \sin(\mu_n x)\, dx, \quad B_n = \frac{2}{l} \int_0^l f_1(x) \sin(\mu_n x)\, dx. \end{aligned} \tag{15.1.3.8}$$

If $f_0(x)$ and $f_1(x)$ have three and two continuous derivatives, respectively, and the compatibility conditions are met (see below), then the series (15.1.3.8) is convergent and admits double termwise differentiation. In this case, formula (15.1.3.8) gives the classical smooth solution of problem (15.1.3.4)–(15.1.3.5), (15.1.1.3)–(15.1.1.4).

Remark 15.4. For the solution of linear nonhomogeneous hyperbolic equations with nonhomogeneous boundary conditions, see Section 17.2.

► Conditions of compatibility of initial and boundary conditions.

Suppose w is a twice continuously differentiable solution of problem (15.1.1.1)–(15.1.1.4). Then conditions (15.1.2.10) and (15.1.2.11) must hold. In addition, the following conditions of compatibility of the boundary conditions (15.1.1.2) and initial condition (15.1.1.4) must be satisfied:

$$[s_1 f_1' + k_1 f_1]_{x=x_1} = 0, \quad [s_2 f_1' + k_2 f_1]_{x=x_2} = 0.$$

15.1.4 Solution of Boundary Value Problems for Elliptic Equations

► Solution of a special problem for elliptic equations.

Now consider a boundary value problem for the elliptic equation

$$a(x)\frac{\partial^2 w}{\partial x^2} + \alpha(y)\frac{\partial^2 w}{\partial y^2} + b(x)\frac{\partial w}{\partial x} + \beta(y)\frac{\partial w}{\partial y} + \bigl[c(x) + \gamma(y)\bigr] w = 0 \tag{15.1.4.1}$$

with homogeneous boundary conditions (15.1.1.2) in x and the following mixed (homogeneous and nonhomogeneous) boundary conditions in y:

$$\begin{aligned} \sigma_1 \partial_y w + \nu_1 w &= 0 \quad &&\text{at} \quad y = y_1, \\ \sigma_2 \partial_y w + \nu_2 w &= f(x) \quad &&\text{at} \quad y = y_2. \end{aligned} \tag{15.1.4.2}$$

We assume that the coefficients of Eq. (15.1.4.1) and of the boundary conditions (15.1.1.2) and (15.1.4.2) meet the following requirements:

$$a(x),\ b(x),\ c(x),\ \alpha(y),\ \beta(y),\ \text{and}\ \gamma(t)\ \text{are continuous functions,}$$
$$a(x)>0,\ \alpha(y)>0,\ |s_1|+|k_1|>0,\ |s_2|+|k_2|>0,\ |\sigma_1|+|\nu_1|>0,\ |\sigma_2|+|\nu_2|>0.$$

The approach is based on searching for particular solutions of Eq. (15.1.4.1) in the product form

$$w(x,y)=\varphi(x)\,\psi(y). \tag{15.1.4.3}$$

As before, we first arrive at the eigenvalue problem (15.1.1.6), (15.1.1.8) for the function $\varphi=\varphi(x)$; the solution procedure is detailed in Section 15.1.1. Further on, we assume the λ_n and $\varphi_n(x)$ have been found. The functions $\psi_n=\psi_n(y)$ are determined by solving the linear ordinary differential equation

$$\alpha(y)\psi''_{yy}+\beta(y)\psi'_y+[\gamma(y)-\lambda_n]\psi=0 \tag{15.1.4.4}$$

subject to the homogeneous boundary condition

$$\sigma_1\partial_y\psi+\nu_1\psi=0\quad\text{at}\quad y=y_1, \tag{15.1.4.5}$$

which is a consequence of the first condition in (15.1.4.2). The functions ψ_n are determined up to a constant factor.

Using the linear superposition principle, we seek the solution of the boundary value problem (15.1.4.1), (15.1.4.2), (15.1.1.2) in the series form

$$w(x,y)=\sum_{n=1}^{\infty}A_n\varphi_n(x)\psi_n(y), \tag{15.1.4.6}$$

where A_n are arbitrary constants. By construction, the series (15.1.4.6) will satisfy equation (15.1.4.1) with the boundary conditions (15.1.1.2) and the first boundary condition in (15.1.4.2). To find the series coefficients A_n, we substitute (15.1.4.6) into the second boundary condition (15.1.4.2) to obtain

$$\sum_{n=1}^{\infty}A_nB_n\varphi_n(x)=f(x),\qquad B_n=\sigma_2\frac{d\psi_n}{dy}\bigg|_{y=y_2}+\nu_2\psi_n(y_2). \tag{15.1.4.7}$$

Further, we follow the same procedure as in Section 15.1.2. Specifically, multiplying equation (15.1.4.7) by $\rho(x)\varphi_n(x)$, then integrating the resulting relation with respect to x over the interval $x_1\le x\le x_2$, and taking into account properties (15.1.1.15), we obtain

$$A_n=\frac{1}{B_n\|\varphi_n\|^2}\int_{x_1}^{x_2}\rho(x)\varphi_n(x)f(x)\,dx,\quad \|\varphi_n\|^2=\int_{x_1}^{x_2}\rho(x)\varphi_n^2(x)\,dx, \tag{15.1.4.8}$$

where the weight function $\rho(x)$ is defined in (15.1.1.14).

Example 15.3. Consider the first (Dirichlet) boundary value problem for the Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \qquad (15.1.4.9)$$

with the boundary conditions

$$\begin{aligned} &w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \qquad \text{at} \quad x = l_1; \\ &w = 0 \quad \text{at} \quad y = 0, \qquad w = f(x) \quad \text{at} \quad y = l_2 \end{aligned} \qquad (15.1.4.10)$$

in a rectangular domain $0 \le x \le l_1$, $0 \le y \le l_2$.

Particular solutions of Eq. (15.1.4.9) are sought in the form (15.1.4.3). We have the following eigenvalue problem for $\varphi(x)$:

$$\varphi''_{xx} + \lambda\varphi = 0; \qquad \varphi = 0 \quad \text{at} \quad x = 0, \qquad \varphi = 0 \quad \text{at} \quad x = l_1.$$

On solving this problem, we find the eigenfunctions with respective eigenvalues,

$$\varphi_n(x) = \sin(\mu_n x), \qquad \mu_n = \sqrt{\lambda_n} = \frac{\pi n}{l_1}, \qquad n = 1, 2, \ldots \qquad (15.1.4.11)$$

The functions $\psi_n = \psi_n(y)$ are determined by solving the following problem for a linear ordinary differential equation with homogeneous boundary conditions:

$$\psi''_{yy} - \lambda_n \psi = 0; \qquad \psi = 0 \quad \text{at} \quad y = 0. \qquad (15.1.4.12)$$

It is a special case of problem (15.1.4.4)–(15.1.4.5) with $\alpha(y) = 1$, $\beta(y) = \gamma(y) = 0$, $\sigma_1 = 0$, and $\nu_1 = 1$. The nontrivial solutions of problem (15.1.4.12) are expressed as

$$\psi_n(y) = \sinh(\mu_n y), \qquad \mu_n = \sqrt{\lambda_n} = \frac{\pi n}{l_1}, \qquad n = 1, 2, \ldots \qquad (15.1.4.13)$$

Using formulas (15.1.4.6), (15.1.4.8), (15.1.4.11), and (15.1.4.13) and taking into account the relations $B_n = \psi_n(l_2) = \sinh(\mu_n l_2)$, $\rho(x) = 1$, and $\|\varphi_n\|^2 = l/2$, we find the solution of the original problem (15.1.4.9)–(15.1.4.10) in the form

$$w(x, y) = \sum_{n=1}^{\infty} A_n \sin(\mu_n x) \sinh(\mu_n y), \qquad A_n = \frac{2}{l_1 \sinh(\mu_n l_2)} \int_0^{l_1} f(x) \sin(\mu_n x)\, dx, \qquad \mu_n = \frac{\pi n}{l_1}.$$

► Generalization to the case of nonhomogeneous boundary conditions.

Now consider the linear boundary value problem for the elliptic equation (15.1.4.1) with general nonhomogeneous boundary conditions

$$\begin{aligned} &s_1 \partial_x w + k_1 w = f_1(y) \quad \text{at} \quad x = x_1, \qquad s_2 \partial_x w + k_2 w = f_2(y) \quad \text{at} \quad x = x_2, \\ &\sigma_1 \partial_y w + \nu_1 w = f_3(x) \quad \text{at} \quad y = y_1, \qquad \sigma_2 \partial_y w + \nu_2 w = f_4(x) \quad \text{at} \quad y = y_2. \end{aligned} \qquad (15.1.4.14)$$

The solution of this problem is the sum of solutions of four simpler auxiliary problems for Eq. (15.1.4.1), each corresponding to three homogeneous and one nonhomogeneous boundary conditions in (15.1.4.14); see Table 15.1. Each auxiliary problem is solved by the procedure given in Section 15.1.1, starting from the search for solutions in the form of a product of functions with distinct arguments (15.1.4.3), determined by Eqs. (15.1.1.6) and (15.1.4.4). The separation parameter λ is determined by the solution of an eigenvalue problem with homogeneous boundary conditions; see Table 15.1. The solution of each of the auxiliary problems is sought in the series form (15.1.4.6).

Remark 15.5. For the solution of linear nonhomogeneous elliptic equations with nonhomogeneous boundary conditions, see Section 17.3.

TABLE 15.1
Description of auxiliary problems for Eq. (15.1.4.1) and problems for associated functions $\varphi(x)$ and $\psi(y)$ that determine particular solutions of the form (15.1.4.3).
The abbreviation HBC stands for "homogeneous boundary condition"

Auxiliary problem	Functions vanishing in the boundary conditions (15.1.4.14)	Eigenvalue problem with homogeneous boundary conditions	Another problem with one homogeneous boundary condition (for λ_n found)
Problem 1	$f_2(y) = f_3(x) = f_4(x) = 0$, function $f_1(y)$ prescribed	functions $\psi_n(y)$ and values λ_n to be determined	functions $\varphi_n(x)$ satisfy an HBC at $x = x_2$
Problem 2	$f_1(y) = f_3(x) = f_4(x) = 0$, function $f_2(y)$ prescribed	functions $\psi_n(y)$ and values λ_n to be determined	functions $\varphi_n(x)$ satisfy an HBC at $x = x_1$
Problem 3	$f_1(y) = f_2(y) = f_4(x) = 0$, function $f_3(x)$ prescribed	functions $\varphi_n(x)$ and values λ_n to be determined	functions $\psi_n(y)$ satisfy an HBC at $y = y_2$
Problem 4	$f_1(y) = f_2(y) = f_3(x) = 0$, function $f_4(x)$ prescribed	functions $\varphi_n(x)$ and values λ_n to be determined	functions $\psi_n(y)$ satisfy an HBC at $y = y_1$

15.1.5 Solution of Boundary Value Problems for Higher-Order Equations

Separation of variables can also be used to solve boundary value problems described by linear higher-order PDEs. The main stages in the application of the method are the same as for second-order PDEs. Let us illustrate this by specific examples.

Example 15.4. Consider the third-order equation

$$\frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} - b\frac{\partial^2 w}{\partial t \partial x^2} = 0 \qquad (0 < t < \infty,\ 0 < x < l) \tag{15.1.5.1}$$

with the general initial condition (15.1.1.3) and the homogeneous boundary conditions (15.1.2.6). Equation (15.1.5.1) occurs in filtration theory and hydrodynamics and becomes the heat equation for $b = 0$.

The approach is based on searching particular solutions of Eq. (15.1.5.1) in the product form $w = \varphi(x)\psi(t)$, which yields the relation $\varphi\psi'_t - a\varphi''_{xx}\psi - b\varphi''_{xx}\psi'_t = 0$. We divide it by $\varphi''_{xx}\psi'_t$ and after elementary manipulations obtain

$$\frac{\varphi}{\varphi''_{xx}} = a\frac{\psi}{\psi'_t} + b. \tag{15.1.5.2}$$

The left-hand side of Eq. (15.1.5.2) depends only on x; the right-hand side, only on t. Hence, for (15.1.5.2) to be true, its left- and right-hand sides must be equal to one and the same constant, which we denote by $-\lambda$. As a result, we obtain the following linear ODEs for the functions φ and ψ:

$$\lambda\varphi''_{xx} + \varphi = 0, \tag{15.1.5.3}$$

$$(b + \lambda)\psi'_t + a\varphi = 0. \tag{15.1.5.4}$$

The function $\varphi = \varphi(x)$ satisfying Eq. (15.1.5.3) should also satisfy the homogeneous boundary conditions $\varphi(0) = \varphi(l) = 0$ (because we seek particular solutions $w = \varphi(x)\psi(t)$ satisfying the boundary conditions (15.1.2.6)). The solution of this eigenvalue problem has the form

$$\varphi_n(x) = \sin(\beta_n x), \quad \lambda_n = \frac{1}{\beta_n^2}, \quad \beta_n = \frac{\pi n}{l}, \quad n = 1, 2, \ldots \tag{15.1.5.5}$$

The solution of Eq. (15.1.5.4) corresponding to the eigenvalue $\lambda = \lambda_n$ and satisfying the normalizing condition $\psi_n(0) = 1$ has the form

$$\psi_n(t) = \exp\biggl(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\biggr). \qquad (15.1.5.6)$$

The solution of the original boundary value problem is sought in the series form (15.1.2.3). To determine the coefficients A_n, we substitute the series (15.1.2.3) into the initial condition (15.1.1.3). An argument similar to that given in Example 15.1 permits eventually obtaining the desired solution in the form

$$w(x,t) = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) \exp\biggl(-\frac{a\beta_n^2 t}{1 + b\beta_n^2}\biggr),$$
$$A_n = \frac{2}{l}\int_0^l f_0(x)\sin(\beta_n x)\,dx, \qquad \beta_n = \frac{\pi n}{l}.$$

Example 15.5. Consider the fourth-order equation

$$\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0 \qquad (0 < t < \infty,\ 0 < x < l) \qquad (15.1.5.7)$$

with the general initial conditions (15.1.1.3)–(15.1.1.4) and the homogeneous boundary conditions

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l. \qquad (15.1.5.8)$$

Equation (15.1.5.7) is encountered when studying forced (transverse) vibrations of elastic rods; the boundary conditions (15.1.5.8) correspond to the case in which both ends of the rod are hinged.

The search for particular solutions of Eq. (15.1.5.7) of the form $w = \varphi(x)\psi(t)$ with subsequent separation of variables leads to the ODEs

$$\varphi''''_{xxxx} - \lambda\varphi = 0, \qquad (15.1.5.9)$$
$$\psi''_{tt} + a^2\lambda\psi = 0, \qquad (15.1.5.10)$$

where λ is the separation constant.

The general solution of Eq. (15.1.5.9) is

$$\varphi = C_1\cos(\beta x) + C_2\sin(\beta x) + C_3\cosh(\beta x) + C_4\sinh(\beta x), \qquad \lambda = \beta^4, \qquad (15.1.5.11)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. The functions $\varphi = \varphi(x)$ should satisfy the homogeneous boundary conditions $\varphi(0) = \varphi''(0) = \varphi(l) = \varphi''(l) = 0$, which follow from the form of the particular solution $w = \varphi(x)\psi(t)$ and the boundary conditions (15.1.5.8). The solution of this eigenvalue problem is determined by formula (15.1.5.11) with $C_1 = C_3 = C_4 = 0$ and has the form

$$\varphi_n(x) = \sin(\beta_n x), \qquad \lambda_n = \beta_n^4, \qquad \beta_n = \frac{\pi n}{l}, \qquad n = 1,\ 2,\ \ldots \qquad (15.1.5.12)$$

The solutions of Eq. (15.1.5.10) corresponding to the eigenvalues $\lambda = \lambda_n$ and satisfying the normalizing conditions (15.1.3.2) have the form

$$\psi_{n1}(t) = \cos\bigl(a\beta_n^2 t\bigr), \quad \psi_{n2}(t) = \frac{1}{a\beta_n^2}\sin\bigl(a\beta_n^2 t\bigr). \qquad (15.1.5.13)$$

The solution of the original boundary value problem is sought in the series form (15.1.3.1). To determine the coefficients A_n and B_n, we substitute the series (15.1.3.1) into the initial conditions (15.1.1.3)–(15.1.1.4). An argument similar to that given in Section 15.1.3 permits eventually obtaining the desired solution in the form

$$w(x,t) = \sum_{n=1}^{\infty} \sin(\beta_n x)\Big[A_n \cos(a\beta_n^2 t) + B_n \frac{1}{a\beta_n^2} \sin(a\beta_n^2 t)\Big],$$

$$A_n = \frac{2}{l}\int_0^l f_0(x)\sin(\beta_n x)\,dx, \qquad B_n = \frac{2}{l}\int_0^l f_1(x)\sin(\beta_n x)\,dx, \qquad \beta_n = \frac{\pi n}{l}.$$

⊙ *Literature for Section 15.1*: S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), H. S. Carslaw and J. C. Jaeger (1984), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), E. Zauderer (1989), I. Sneddon (1972, 1995), A. N. Tikhonov and A. A. Samarskii (1990), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

15.2 Integral Transform Method

Linear problems of mathematical physics are very often solved using various integral transforms. The Laplace transform and the Fourier transform are in most common use. These and some other integral transforms are discussed below (see also Chapter 28).

15.2.1 Laplace Transform and Its Application in Mathematical Physics

► Laplace and inverse Laplace transforms. Laplace transforms for derivatives.

1°. The *Laplace transform* of an arbitrary (complex-valued) function $f(t)$ of a real variable t ($t \geq 0$) is defined by

$$\widetilde{f}(p) = \mathfrak{L}\big\{f(t)\big\}, \quad \text{where} \quad \mathfrak{L}\big\{f(t)\big\} \equiv \int_0^\infty e^{-pt} f(t)\,dt, \qquad (15.2.1.1)$$

and $p = s + i\sigma$ is a complex variable, $i^2 = -1$.

The Laplace transform exists for any continuous or piecewise continuous function satisfying the condition $|f(t)| < Me^{\sigma_0 t}$ with some $M > 0$ and $\sigma_0 \geq 0$. In the following, σ_0 often means the greatest lower bound of the possible values of σ_0 in this estimate; this value is called the *growth exponent* of the function $f(t)$.

2°. Given the transform $\widetilde{f}(p)$, the function $f(t)$ can be found by means of the *inverse Laplace transform*

$$f(t) = \mathfrak{L}^{-1}\big\{\widetilde{f}(p)\big\}, \quad \text{where} \quad \mathfrak{L}^{-1}\big\{\widetilde{f}(p)\big\} \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(p) e^{pt}\,dp, \qquad (15.2.1.2)$$

and the integration path is parallel to the imaginary axis and lies to the right of all singularities of $\widetilde{f}(p)$, which corresponds to $c > \sigma_0$.

The integral in the inversion formula (15.2.1.2) is understood in the sense of the *Cauchy principal value*:

$$\int_{c-i\infty}^{c+i\infty} \widetilde{f}(p)e^{pt}\,dp = \lim_{\omega\to\infty}\int_{c-i\omega}^{c+i\omega} \widetilde{f}(p)e^{pt}\,dp.$$

In the domain $t < 0$, formula (15.2.1.2) gives $f(t) \equiv 0$.

Formula (15.2.1.2) holds for continuous functions. If $f(t)$ has a (finite) jump discontinuity at a point $t = t_0 > 0$, then the left-hand side of (15.2.1.2) is equal to $\frac{1}{2}[f(t_0 - 0) + f(t_0 + 0)]$ at this point (for $t_0 = 0$, the first term in the square brackets must be omitted).

3°. Consider the important case in which the transform is a rational function of the form

$$\widetilde{f}(p) = \frac{R(p)}{Q(p)},$$

where $Q(p)$ and $R(p)$ are polynomials in the variable p and the degree of $Q(p)$ exceeds that of $R(p)$.

Assume that the zeros of the denominator are simple; i.e.,

$$Q(p) \equiv \text{const}\,(p - \lambda_1)(p - \lambda_2)\dots(p - \lambda_n).$$

Then the inverse transform can be determined by the formula

$$f(x) = \sum_{k=1}^{n} \frac{R(\lambda_k)}{Q'(\lambda_k)} \exp(\lambda_k x),$$

where the primes denote the derivatives.

If $Q(p)$ has multiple zeros, i.e., if

$$Q(p) \equiv \text{const}\,(p - \lambda_1)^{s_1}(p - \lambda_2)^{s_2}\dots(p - \lambda_m)^{s_m},$$

then

$$f(x) = \sum_{k=1}^{m} \frac{1}{(s_k - 1)!} \lim_{p \to s_k} \frac{d^{s_k - 1}}{dp^{s_k - 1}} \big[(p - \lambda_k)^{s_k} \widetilde{f}(p) e^{px}\big].$$

► Main properties of the Laplace transform.

The main properties of the correspondence between functions and their Laplace transforms are gathered in Table 15.2.

The Laplace transforms of some functions are listed in Table 15.3; for more detailed tables, see Section 28.1 and the list of references at the end of this chapter.

Such tables are convenient to use when solving linear problems for partial differential equations.

To solve nonstationary boundary value problems, the following Laplace transform formulas for derivatives will be required:

$$\begin{aligned} \mathfrak{L}\big\{f'(t)\big\} &= p\widetilde{f}(p) - f(0), \\ \mathfrak{L}\big\{f''(t)\big\} &= p^2\widetilde{f}(p) - pf(0) - f'(0), \end{aligned} \qquad (15.2.1.3)$$

where $f(0)$ and $f'(0)$ are the initial conditions.

TABLE 15.2
Main properties of the Laplace transform

No.	Function	Laplace transform	Operation
1	$af_1(t)+bf_2(t)$	$a\widetilde{f}_1(p)+b\widetilde{f}_2(p)$	Linearity
2	$f(t/a),\ a>0$	$a\widetilde{f}(ap)$	Scaling
3	$f(t-a)$, $f(\xi)\equiv 0$ for $\xi<0$	$e^{-ap}\widetilde{f}(p)$	Shift of the argument
4	$t^n f(t);\ n=1,2,\ldots$	$(-1)^n\widetilde{f}_p^{(n)}(p)$	Differentiation of the transform
5	$\dfrac{1}{t}f(t)$	$\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	Integration of the transform
6	$e^{at}f(t)$	$\widetilde{f}(p-a)$	Shift in the complex plane
7	$f'_t(t)$	$p\widetilde{f}(p)-f(+0)$	Differentiation
8	$f_t^{(n)}(t)$	$p^n\widetilde{f}(p)-\sum\limits_{k=1}^{n}p^{n-k}f_t^{(k-1)}(+0)$	Differentiation
9	$t^m f_t^{(n)}(t),\ m=1,2,\ldots$	$(-1)^m\dfrac{d^m}{dp^m}\Big[p^n\widetilde{f}(p)-\sum\limits_{k=1}^{n}p^{n-k}f_t^{(k-1)}(+0)\Big]$	Differentiation
10	$\dfrac{d^n}{dt^n}\big[t^m f(t)\big],\ \ m\ge n$	$(-1)^m p^n\dfrac{d^m}{dp^m}\widetilde{f}(p)$	Differentiation
11	$\displaystyle\int_0^t f(z)\,dz$	$\dfrac{\widetilde{f}(p)}{p}$	Integration
12	$\displaystyle\int_0^t f_1(z)f_2(t-z)\,dz$	$\widetilde{f}_1(p)\widetilde{f}_2(p)$	Convolution

TABLE 15.3
Laplace transforms of some functions

No.	Function, $f(t)$	Laplace transform, $\widetilde{f}(p)$	Remarks
1	1	$1/p$	
2	t^n	$\dfrac{n!}{p^{n+1}}$	$n=1,\ 2,\ \ldots$
3	t^a	$\Gamma(a+1)p^{-a-1}$	$a>-1$
4	e^{-at}	$(p+a)^{-1}$	
5	$t^a e^{-bt}$	$\Gamma(a+1)(p+b)^{-a-1}$	$a>-1$
6	$\sinh(at)$	$\dfrac{a}{p^2-a^2}$	
7	$\cosh(at)$	$\dfrac{p}{p^2-a^2}$	
8	$\ln t$	$-\dfrac{1}{p}(\ln p+\mathcal{C})$	$\mathcal{C}=0.5772\ldots$ is the Euler constant
9	$\sin(at)$	$\dfrac{a}{p^2+a^2}$	
10	$\cos(at)$	$\dfrac{p}{p^2+a^2}$	
11	$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{1}{p}\exp\big(-a\sqrt{p}\big)$	$a\ge 0$
12	$J_0(at)$	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(t)$ is the Bessel function

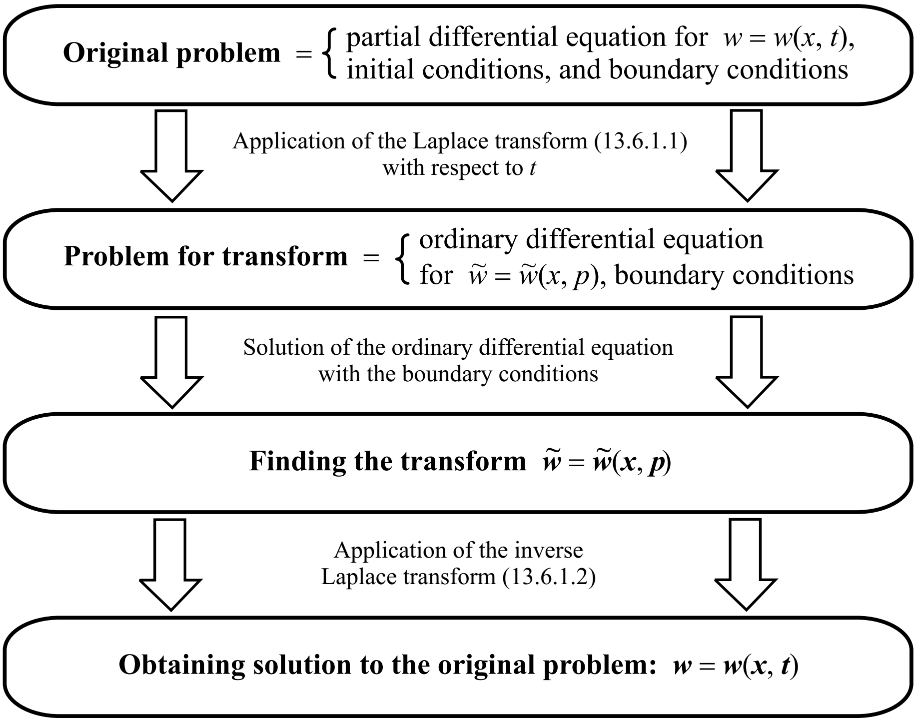


Figure 15.2: Solution procedure for linear boundary value problems using the Laplace transform.

► Solution procedure for linear problems using the Laplace transform.

Figure 15.2 shows schematically how one can utilize the Laplace transforms to solve boundary value problems for linear parabolic or hyperbolic equations with two independent variables in the case where the equation coefficients are independent of t (the same procedure can be applied for solving linear problems characterized by higher-order equations). Here and henceforth, the short notation $\widetilde{w}(x,p) = \mathfrak{L}\{w(x,t)\}$ will be used; the arguments of $\widetilde{w}$ may be omitted.

It is significant that with the Laplace transform, the original problem for a partial differential equation is reduced to a simpler problem for an ordinary differential equation with parameter p; the derivatives with respect to t are replaced by appropriate algebraic expressions taking into account the initial conditions; see formulas (15.2.1.3).

► Solving linear problems for parabolic equations with the Laplace transform.

Consider a linear nonstationary boundary value problem for the parabolic equation

$$\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + c(x)w + \Phi(x,t) \tag{15.2.1.4}$$

with the initial condition (15.1.1.3) and the general nonhomogeneous boundary conditions

$$\begin{aligned} s_1\partial_x w + k_1 w &= g_1(t) \quad \text{at} \quad x = x_1, \\ s_2\partial_x w + k_2 w &= g_2(t) \quad \text{at} \quad x = x_2. \end{aligned} \tag{15.2.1.5}$$

The application of the Laplace transform results in the problem defined by the ordinary differential equation in x (p is treated as a parameter)

$$a(x)\frac{\partial^2\widetilde{w}}{\partial x^2}+b(x)\frac{\partial\widetilde{w}}{\partial x}+[c(x)-p]\widetilde{w}+f_0(x)+\widetilde{\Phi}(x,p)=0 \qquad (15.2.1.6)$$

with the boundary conditions

$$\begin{aligned} s_1\partial_x\widetilde{w}+k_1\widetilde{w}&=\widetilde{g}_1(p) \quad \text{at} \quad x=x_1,\\ s_2\partial_x\widetilde{w}+k_2\widetilde{w}&=\widetilde{g}_2(p) \quad \text{at} \quad x=x_2. \end{aligned} \qquad (15.2.1.7)$$

Notation employed: $\widetilde{\Phi}(x,p)=\mathfrak{L}\{\Phi(x,t)\}$ and $\widetilde{g}_n(p)=\mathfrak{L}\{g_n(t)\}$ $(n=1,\,2)$. On solving problem (15.2.1.6)–(15.2.1.7), one should apply the inverse Laplace transform (15.2.1.2) to the resulting solution $\widetilde{w}=\widetilde{w}(x,p)$ to obtain the solution $w=w(x,t)$ of the original problem.

Example 15.6. Consider the first boundary value problem for the heat equation,

$$\begin{aligned} &\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2} && (x>0,\ t>0),\\ &w=0 \quad \text{at} \quad t=0 && \text{(initial condition)},\\ &w=w_0 \quad \text{at} \quad x=0 && \text{(boundary condition)},\\ &w\to 0 \quad \text{at} \quad x\to\infty && \text{(boundary condition)}. \end{aligned}$$

We apply the Laplace transform with respect to t. Let us multiply the equation, the initial condition, and the boundary conditions by e^{-pt} and then integrate with respect to t from zero to infinity. Taking into account the relations

$$\begin{aligned} &\mathfrak{L}\{\partial_t w\}=p\widetilde{w}-w|_{t=0}=p\widetilde{w} && \text{(the first property (15.2.1.3) and the initial condition are used)},\\ &\mathfrak{L}\{w_0\}=w_0\mathfrak{L}\{1\}=w_0/p && \text{(property 1 in Table 15.2 and the relation } \mathfrak{L}\{1\}=1/p\\ & && \text{are used; see property 1 in Table 15.3)}, \end{aligned}$$

we arrive at the following problem for a second-order linear ordinary differential equation with parameter p:

$$\widetilde{w}''_{xx}-p\widetilde{w}=0, \qquad (15.2.1.8)$$

$$\begin{aligned} &\widetilde{w}=w_0/p \quad \text{at} \quad x=0 && \text{(boundary condition)},\\ &\widetilde{w}\to 0 \quad \text{at} \quad x\to\infty && \text{(boundary condition)}. \end{aligned} \qquad (15.2.1.9)$$

Integrating the equation yields the general solution $\widetilde{w}=A_1(p)e^{-x\sqrt{p}}+A_2(p)e^{x\sqrt{p}}$. Using the boundary conditions, we determine the constants, $A_1(p)=w_0/p$ and $A_2(p)=0$. Thus, we have

$$\widetilde{w}=\frac{w_0}{p}e^{-x\sqrt{p}}.$$

Let us apply the inverse Laplace transform to both sides of this relation. We refer to Table 15.3, row 11 (where a must be replaced by x), to find the inverse transform of the right-hand side. Finally, we obtain the solution of the original problem in the form

$$w=w_0\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

► **Solving linear problems for hyperbolic equations by the Laplace transform.**

Consider a linear nonstationary boundary value problem defined by the hyperbolic equation

$$\frac{\partial^2 w}{\partial t^2} + \varphi(x)\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + c(x)w + \Phi(x,t) \qquad (15.2.1.10)$$

with the initial conditions (15.1.1.3), (15.1.1.4) and general boundary conditions (15.2.1.5). The application of the Laplace transform results in the problem defined by the ordinary differential equation for x (p is treated as a parameter)

$$a(x)\frac{\partial^2 \widetilde{w}}{\partial x^2} + b(x)\frac{\partial \widetilde{w}}{\partial x} + [c(x) - p^2 - p\varphi(x)]\widetilde{w} + f_0(x)[p + \varphi(x)] + f_1(x) + \widetilde{\Phi}(x,p) = 0 \qquad (15.2.1.11)$$

with the boundary conditions (15.2.1.7). On solving this problem, one should apply the inverse Laplace transform to the resulting solution $\widetilde{w} = \widetilde{w}(x,p)$.

15.2.2 Fourier Transform and Its Application in Mathematical Physics

► **Standard and alternative forms of the Fourier transform.**

1°. The *Fourier transform* is defined as follows:

$$\widetilde{f}(u) = \mathfrak{F}\{f(x)\}, \quad \text{where} \quad \mathfrak{F}\{f(x)\} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iux}\,dx, \qquad (15.2.2.1)$$

where $i^2 = -1$. This relation is meaningful for any function $f(x)$ absolutely integrable on the interval $(-\infty, \infty)$.

Given $\widetilde{f}(u)$, the function $f(x)$ can be found by the *inverse Fourier transform*

$$f(x) = \mathfrak{F}^{-1}\{\widetilde{f}(u)\}, \quad \text{where} \quad \mathfrak{F}^{-1}\{\widetilde{f}(u)\} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}(u)e^{iux}\,du, \qquad (15.2.2.2)$$

where the integral is understood in the sense of the Cauchy principal value.

Formula (15.2.2.2) holds for continuous functions. If $f(x)$ has a (finite) jump discontinuity at a point $x = x_0$, then the left-hand side of (15.2.2.2) is equal to $\frac{1}{2}\big[f(x_0 - 0) + f(x_0 + 0)\big]$ at this point.

Remark 15.6. It is natural to consider the Fourier transform in the class of complex-valued functions of a real argument, because in the general case $\widetilde{f}(u)$ takes complex values even for a real function $f(x)$.

2°. The main properties of the correspondence between functions and their Fourier transforms are gathered in Table 15.4. The function $f(x)$ and its derivatives are assumed to vanish sufficiently rapidly as $x \to \infty$. (For example, for properties 5 and 6 to hold one needs the function $f(x)$ and its derivatives to decay faster than $|x|^{-1}$).

3°. Sometimes the *alternative Fourier transform* is used (and called merely the *Fourier transform*), which corresponds to the renaming $e^{-iux} \rightleftharpoons e^{iux}$ on the right-hand sides of (15.2.2.1) and (15.2.2.2).

TABLE 15.4
Main properties of the Fourier transform

No.	Function	Fourier transform	Operation
1	$af_1(x)+bf_2(x)$	$a\widetilde{f}_1(u)+b\widetilde{f}_2(u)$	Linearity
2	$f(x/a),\ a>0$	$a\widetilde{f}(au)$	Scaling
3	$f(x-a)$	$e^{-iau}\widetilde{f}(u)$	Shift
4	$x^n f(x);\ n=1,2,\dots$	$i^n\widetilde{f}_u^{(n)}(u)$	Differentiation of the transform
5	$f''_{xx}(x)$	$-u^2\widetilde{f}(u)$	Differentiation
6	$f_x^{(n)}(x)$	$(iu)^n\widetilde{f}(u)$	Differentiation
7	$\int_{-\infty}^{\infty} f_1(\xi)f_2(x-\xi)\,d\xi$	$\widetilde{f}_1(u)\widetilde{f}_2(u)$	Convolution

► Asymmetric form of the Fourier transform.

Often it is more convenient to define the Fourier transform by

$$\check{f}(u)=\mathcal{F}\bigl\{f(x)\bigr\},\quad \text{where}\quad \mathcal{F}\bigl\{f(x)\bigr\}\equiv\int_{-\infty}^{\infty} f(x)e^{-iux}\,dx. \tag{15.2.2.3}$$

In this case, the *Fourier inversion formula* reads

$$f(x)=\mathcal{F}^{-1}\bigl\{\check{f}(u)\bigr\},\quad \text{where}\quad \mathcal{F}^{-1}\bigl\{\check{f}(u)\bigr\}\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\check{f}(u)e^{iux}\,du. \tag{15.2.2.4}$$

► n-dimensional Fourier transform.

1°. The Fourier transform admits n-dimensional generalization:

$$\widetilde{f}(\mathbf{u})=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(\mathbf{x})e^{-i(\mathbf{u}\cdot\mathbf{x})}\,dV_x,\qquad (\mathbf{u}\cdot\mathbf{x})=u_1x_1+\dots+u_nx_n, \tag{15.2.2.5}$$

where $f(\mathbf{x})=f(x_1,\dots,x_n)$, $\widetilde{f}(\mathbf{u})=\widetilde{f}(u_1,\dots,u_n)$, and $dV_x=dx_1\dots dx_n$.

The corresponding inversion formula is

$$f(\mathbf{x})=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\widetilde{f}(\mathbf{u})e^{i(\mathbf{u}\cdot\mathbf{x})}\,dV_u,\qquad dV_u=du_1\dots du_n. \tag{15.2.2.6}$$

2°. Often it is more convenient to define the Fourier transform by the asymmetric form of the Fourier transform

$$\check{f}(\mathbf{u})=\int_{\mathbb{R}^n} f(\mathbf{x})e^{-i(\mathbf{u}\cdot\mathbf{x})}\,dV_x. \tag{15.2.2.7}$$

In this case, the Fourier inversion formula has the form

$$f(\mathbf{x})=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\check{f}(\mathbf{u})e^{i(\mathbf{u}\cdot\mathbf{x})}\,dV_u. \tag{15.2.2.8}$$

The Fourier transforms (15.2.2.5) and (15.2.2.7) are frequently used in the theory of linear partial differential equations with constant coefficients ($\mathbf{x}\in\mathbb{R}^n$).

► **Solving linear problems of mathematical physics by the Fourier transform.**

The Fourier transform is usually employed to solve boundary value problems for linear partial differential equations whose coefficients are independent of the space variable x, $-\infty < x < \infty$.

The scheme for solving linear boundary value problems with the help of the Fourier transform is similar to that used in solving problems with the help of the Laplace transform. With the Fourier transform, the derivatives with respect to x in the equation are replaced by appropriate algebraic expressions; see Property 5 or 6 in Table 15.4. In the case of two independent variables, the problem for a partial differential equation is reduced to a simpler problem for an ordinary differential equation with parameter u. On solving the latter problem, one determines the transform. After that, by applying the inverse Fourier transform, one obtains the solution of the original boundary value problem.

Example 15.7. Consider the following Cauchy problem for the heat equation:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \qquad (-\infty < x < \infty),$$
$$w = f(x) \quad \text{at} \quad t = 0 \qquad \text{(initial condition)}.$$

We apply the Fourier transform with respect to the space variable x. Setting $\widetilde{w} = \mathfrak{F}\{w(x,t)\}$ and taking into account the relation $\mathfrak{F}\{\partial_{xx} w\} = -u^2\widetilde{w}$ (see Property 5 in Table 15.4), we arrive at the following problem for a linear first-order ordinary differential equation in t with parameter u:

$$\widetilde{w}'_t + u^2\widetilde{w} = 0,$$
$$\widetilde{w} = \widetilde{f}(u) \quad \text{at} \quad t = 0,$$

where $\widetilde{f}(u)$ is defined by (15.2.2.1). On solving this problem for the transform $\widetilde{w}$, we find

$$\widetilde{w} = \widetilde{f}(u)e^{-u^2 t}.$$

Let us apply the inversion formula to both sides of this equation. After some calculations, we obtain the solution of the original problem in the form

$$\begin{aligned} w &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(u)e^{-u^2 t} e^{iux}\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi)e^{-iu\xi} d\xi\right] e^{-u^2 t + iux} du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} e^{-u^2 t + iu(x-\xi)} du = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi. \end{aligned}$$

At the last stage, we have used the relation $\displaystyle\int_{-\infty}^{\infty} \exp\bigl(-a^2u^2 + bu\bigr)\, du = \frac{\sqrt{\pi}}{|a|} \exp\left(\frac{b^2}{4a^2}\right)$.

Example 15.8. Consider the Cauchy problem for the equation of transverse vibrations of elastic rods

$$\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0 \qquad (15.2.2.9)$$

with the initial conditions

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = 0 \quad \text{at} \quad t = 0. \qquad (15.2.2.10)$$

To solve problem (15.2.2.9)–(15.2.2.10), we use the asymmetric form of the Fourier transform (15.2.2.3) with respect to the space variable x. By setting $\check{w} = \mathcal{F}\{w(x,t)\}$ and by taking into account the relation $\mathcal{F}\{\partial_{xxxx}w\} = u^4\check{w}$ (see Property 6 with $n = 4$ in Table 15.4), we arrive at the following problem for a linear second-order ordinary differential equation in t with parameter u:

$$\begin{gathered}\check{w}''_{tt} + a^2u^4\check{w} = 0,\\ \check{w} = F(u) \quad \text{at} \quad t = 0, \qquad \check{w}'_t = 0 \quad \text{at} \quad t = 0,\end{gathered} \tag{15.2.2.11}$$

where $F(u) = \mathcal{F}\{f(x)\}$. The solution of problem (15.2.2.11) has the form

$$\check{w} = F(u)\cos(au^2t). \tag{15.2.2.12}$$

We apply the inverse Fourier transform (15.2.2.4) to (15.2.2.12) and obtain, after easy transformations,

$$\begin{aligned} w &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\cos(au^2t)e^{iux}\,du\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty} f(\xi)e^{-iu\xi}\,d\xi\biggr)\cos(au^2t)e^{iux}\,du\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\biggl(\int_{-\infty}^{\infty}\cos(au^2t)e^{iu(x-\xi)}\,du\biggr)\,d\xi\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\biggl(\int_0^{\infty}\cos(au^2t)\cos[u(x-\xi)]\,du\biggr)\,d\xi. \end{aligned}$$

The inner integral can be evaluated (see the tables of definite integrals in Section 27.2.5),

$$\int_0^{\infty}\cos(au^2t)\cos[u(x-\xi)]\,du = \sqrt{\frac{\pi}{8at}}\left[\cos\biggl(\frac{(x-\xi)^2}{4at}\biggr) + \sin\biggl(\frac{(x-\xi)^2}{4at}\biggr)\right].$$

As a result, we find the solution of the original problem (the *Boussinesq solution*) in the form

$$\begin{aligned} w &= \sqrt{\frac{1}{8\pi at}}\int_{-\infty}^{\infty} f(\xi)\left[\cos\biggl(\frac{(x-\xi)^2}{4at}\biggr) + \sin\biggl(\frac{(x-\xi)^2}{4at}\biggr)\right]d\xi\\ &= \sqrt{\frac{1}{8\pi at}}\int_{-\infty}^{\infty} f(x-\eta)\left[\cos\biggl(\frac{\eta^2}{4at}\biggr) + \sin\biggl(\frac{\eta^2}{4at}\biggr)\right]d\eta\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-2\sqrt{at}\,\zeta)\bigl[\cos(\zeta^2) + \sin(\zeta^2)\bigr]\,d\zeta. \end{aligned}$$

15.2.3 Fourier Sine and Cosine Transforms

► Fourier sine transform.

1°. Let a function $f(x)$ be integrable on the half-line $0 \le x < \infty$. The *Fourier sine transform* is defined by

$$\widetilde{f}_{\mathrm{s}}(u) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(xu)\,dx, \qquad 0 < u < \infty. \tag{15.2.3.1}$$

For given $\widetilde{f}_{\mathrm{s}}(u)$, the function $f(x)$ can be found by means of the *inverse Fourier sine transform*

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\widetilde{f}_{\mathrm{s}}(u)\sin(xu)\,du, \qquad 0 < x < \infty. \tag{15.2.3.2}$$

The Fourier sine transform (15.2.3.1) is briefly denoted by $\widetilde{f}_{\rm s}(u) = \mathfrak{F}_{\rm s}\{f(x)\}$.

It follows from formula (15.2.3.2) that the Fourier sine transform has the property $\mathfrak{F}_{\rm s}^2 = 1$.

Parseval's relation for the Fourier sine transform:

$$\int_0^\infty \mathfrak{F}_{\rm s}\{f(x)\}\mathfrak{F}_{\rm s}\{g(x)\}\,du = \int_0^\infty f(x)g(x)\,dx.$$

There are tables of Fourier sine transforms (see Section 28.4 and the references listed at the end of the current chapter).

2°. Often it is more convenient to apply the asymmetric form of the Fourier sine transform defined by the following two formulas:

$$\check{f}_{\rm s}(u) = \int_0^\infty f(x)\sin(xu)\,dx, \qquad f(x) = \frac{2}{\pi}\int_0^\infty \check{f}_{\rm s}(u)\sin(xu)\,du. \tag{15.2.3.3}$$

The asymmetric form of Fourier sine transform (15.2.3.3) is briefly denoted by $\check{f}_{\rm s}(u) = \check{\mathfrak{F}}_{\rm s}\{f(x)\}$.

Some properties of the Fourier sine transform:

$$\begin{aligned} \check{\mathfrak{F}}_{\rm s}\{f''(x)\} &= -u^2\check{\mathfrak{F}}_{\rm s}\{f(x)\} + uf(0), \\ \check{\mathfrak{F}}_{\rm s}\{x^{2n}f(x)\} &= (-1)^n\frac{d^{2n}}{du^{2n}}\check{\mathfrak{F}}_{\rm s}\{f(x)\}, \quad n = 1,\, 2,\, \dots \end{aligned} \tag{15.2.3.4}$$

Here $f(x)$ and their derivatives are assumed to vanish sufficiently rapidly (for example, exponentially) as $x \to \infty$.

► Fourier cosine transform.

1°. Let a function $f(x)$ be integrable on the half-line $0 \le x < \infty$. The *Fourier cosine transform* is defined by

$$\widetilde{f}_{\rm c}(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(xu)\,dx, \qquad 0 < u < \infty. \tag{15.2.3.5}$$

For given $\widetilde{f}_{\rm c}(u)$, the function can be found by means of the *Fourier cosine inversion formula*

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \widetilde{f}_{\rm c}(u)\cos(xu)\,du, \qquad 0 < x < \infty. \tag{15.2.3.6}$$

The Fourier cosine transform (15.2.3.5) is denoted for brevity by $\widetilde{f}_{\rm c}(u) = \mathfrak{F}_{\rm c}\{f(x)\}$.

It follows from formula (15.2.3.6) that the Fourier cosine transform has the property $\mathfrak{F}_{\rm c}^2 = 1$.

Parseval's relation for the Fourier cosine transform:

$$\int_0^\infty \mathfrak{F}_{\rm c}\{f(x)\}\mathfrak{F}_{\rm c}\{g(x)\}\,du = \int_0^\infty f(x)g(x)\,dx.$$

There are tables of the Fourier cosine transform (see Section 28.3 and the references listed at the end of the current chapter).

2°. Often the asymmetric form of the Fourier cosine transform is applied, which is given by the pair of formulas

$$\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(xu)\,dx, \qquad f(x) = \frac{2}{\pi}\int_0^\infty \check{f}_{\rm c}(u)\cos(xu)\,du. \tag{15.2.3.7}$$

The asymmetric form of Fourier cosine transform (15.2.3.7) is briefly denoted by $\check{f}_{\rm c}(u) = \check{\mathfrak{F}}_{\rm c}\{f(x)\}$.

Some properties of the Fourier cosine transform:

$$\begin{aligned} \check{\mathfrak{F}}_{\rm c}\{f''(x)\} &= -u^2\check{\mathfrak{F}}_{\rm c}\{f(x)\} - f'(0), \\ \check{\mathfrak{F}}_{\rm c}\{x^{2n}f(x)\} &= (-1)^n\frac{d^{2n}}{du^{2n}}\check{\mathfrak{F}}_{\rm c}\{f(x)\}, \quad n = 1,\,2,\,\dots \end{aligned} \tag{15.2.3.8}$$

Here $f(x)$ and their derivatives are assumed to vanish sufficiently rapidly (for example, exponentially) as $x \to \infty$.

► Solving mathematical physics problems by the Fourier sine and cosine transforms.

The Fourier sine and cosine transforms are usually employed to solve boundary value problems for linear partial differential equations whose coefficients are independent of the space variable x, $0 \le x < \infty$.

If a boundary condition of the first kind is given at $x = 0$, then one uses the Fourier sine transform. If a boundary condition of the second kind is given at $x = 0$, then one uses the Fourier cosine transform. Let us illustrate the solution construction procedure by specific examples.

Example 15.9. Consider the heat equation

$$\frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} = 0 \qquad (0 < t < \infty,\ 0 < x < \infty) \tag{15.2.3.9}$$

with the initial and boundary conditions

$$w = 0 \quad \text{at} \quad t = 0, \qquad w = f(t) \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty. \tag{15.2.3.10}$$

It is assumed that $a > 0$.

To solve problem (15.2.3.9)–(15.2.3.10), we use the asymmetric form of the Fourier sine transform (15.2.3.3). With regard to the first relation in (15.2.3.4) and the boundary conditions (15.2.3.10) (we also assume that $\partial_x w \to 0$ as $x \to \infty$), we find the Fourier sine transform of the second derivative of the unknown function,

$$\int_0^\infty \partial_{xx}w\sin(xu)\,dx = -u^2W(t,u) + uf(t), \tag{15.2.3.11}$$

where $W(t,u) = \check{\mathfrak{F}}_{\rm s}\{w\}$. As a result, the second-order PDE (15.2.3.9) leads to the first-order ordinary differential equation

$$W'_t + au^2W = auf(t) \tag{15.2.3.12}$$

with the initial condition $W = 0$ at $t = 0$. The solution of Eq. (15.2.3.12) with this condition has the form

$$W(t,u) = a\int_0^t uf(\tau)\exp[-au^2(t-\tau)]\,d\tau. \tag{15.2.3.13}$$

By using the inverse transform (15.2.3.3), we obtain the solution of the original problem (15.2.3.9)–(15.2.3.10),

$$w(x,t) = \frac{2}{\pi}\int_0^\infty \sin(xu)W(t,u)\,du = \int_0^t G(x,t-\tau)f(\tau)\,d\tau, \tag{15.2.3.14}$$

where

$$G(x,t) = \frac{2a}{\pi}\int_0^\infty u\sin(xu)\exp(-au^2t)\,du.$$

The integral determining the function G can be computed (see the tables of definite integrals in Section 27.2.5). As a result, we obtain

$$G(x,t) = \frac{x}{2\sqrt{\pi a}\,t^{3/2}}\exp\left(-\frac{x^2}{4at}\right). \tag{15.2.3.15}$$

Formulas (15.2.3.14)–(15.2.3.15) give the solution of problem (15.2.3.9)–(15.2.3.10).

Example 15.10. Consider the third-order equation

$$\frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} - b\frac{\partial^2 w}{\partial t \partial x^2} = 0 \qquad (0 < t < \infty,\ 0 < x < \infty) \tag{15.2.3.16}$$

with the initial and boundary conditions (15.2.3.10). Equation (15.2.3.16) is a generalization of the heat equation (15.2.3.9) (they coincide for $b = 0$) and arises in filtration theory and hydrodynamics. In the following, we assume that $a > 0$, $b \geq 0$, and the initial and boundary data satisfy the consistency condition $f(0) = 0$.

To solve this problem, just as in the preceding example, we use the asymmetric form of the Fourier sine transform (15.2.3.3). The transform of the second derivative is given by formula (15.2.3.11); further, by differentiating this formula with respect to t, we find the Fourier sine transform of the mixed derivative $\partial_{xxt}w$. As a result, the third-order PDE (15.2.3.16) leads to the first-order ordinary differential equation

$$(1 + bu^2)W_t' + au^2W = u[af(t) + bf_t'(t)] \tag{15.2.3.17}$$

with the initial condition $W = 0$ at $t = 0$. The solution of Eq. (15.2.3.17) with this condition has the form

$$W(t,u) = \int_0^t \frac{u}{1+bu^2}\exp\left[-\frac{au^2(t-\tau)}{1+bu^2}\right][af(\tau) + bf'(\tau)]\,d\tau. \tag{15.2.3.18}$$

By using the inverse transform (15.2.3.3), we obtain the solution of the original problem (15.2.3.16), (15.2.3.10),

$$\begin{gathered} w(x,t) = \frac{2}{\pi}\int_0^\infty \sin(xu)W(t,u)\,du = \int_0^t G(x,t-\tau)\Big[f(\tau) + \frac{b}{a}f'(\tau)\Big]\,d\tau, \\ G(x,t) = \frac{2a}{\pi}\int_0^\infty \frac{u\sin(xu)}{1+bu^2}\exp\left(-\frac{au^2t}{1+bu^2}\right)du. \end{gathered} \tag{15.2.3.19}$$

Example 15.11. Consider the following boundary value problem for the Helmholtz equation:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - k^2w = 0 \qquad (0 < x < \infty,\ 0 < y < a), \tag{15.2.3.20}$$

$$\partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w \to 0 \quad \text{as} \quad x \to \infty, \tag{15.2.3.21}$$

$$w = f(x) \quad \text{at} \quad y = 0, \qquad w = 0 \quad \text{at} \quad y = a. \tag{15.2.3.22}$$

To solve this problem, we use the asymmetric form of the Fourier cosine transform (15.2.3.7). In view of the first relation in (15.2.3.8) and the boundary conditions (15.2.3.21) (we also assume that $\partial_x w \to 0$ as $x \to \infty$), for the function $W(y,u) = \check{\mathfrak{F}}_{\mathrm{c}}\{w\}$ we obtain the ordinary differential equation

$$W''_{yy} - (u^2 + k^2)W = 0 \tag{15.2.3.23}$$

with the boundary conditions

$$W = \check{f}_{\mathrm{c}}(u) \quad \text{at} \quad y = 0, \qquad W = 0 \quad \text{at} \quad y = a, \tag{15.2.3.24}$$

where $\check{f}_{\mathrm{c}}(u)$ is the asymmetric form of the Fourier cosine transform for the function $f(x)$.

The general solution of Eq. (15.2.3.23) has the form

$$W = C_1(u)\exp\bigl(-y\sqrt{u^2+k^2}\,\bigr) + C_2(u)\exp\bigl(y\sqrt{u^2+k^2}\,\bigr).$$

By substituting this expression into the boundary conditions (15.2.3.23), we determine the coefficients $C_1(u)$ and $C_2(u)$. As a result, after simple transformations we find the solution of problem (15.2.3.23)–(15.2.3.24),

$$W = \check{f}_{\mathrm{c}}(u)\frac{\sinh\bigl[(a-y)\sqrt{u^2+k^2}\,\bigr]}{\sinh\bigl(a\sqrt{u^2+k^2}\,\bigr)}. \tag{15.2.3.25}$$

Now the inverse transform (15.2.3.7) provides the solution of the original problem (15.2.3.23)–(15.2.3.24):

$$\begin{aligned} w(x,y) &= \frac{2}{\pi}\int_0^\infty \check{f}_{\mathrm{c}}(u)\frac{\sinh\bigl[(a-y)\sqrt{u^2+k^2}\,\bigr]}{\sinh\bigl(a\sqrt{u^2+k^2}\,\bigr)}\cos(xu)\,du \\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(\xi)\frac{\sinh\bigl[(a-y)\sqrt{u^2+k^2}\,\bigr]}{\sinh\bigl(a\sqrt{u^2+k^2}\,\bigr)}\cos(xu)\cos(\xi u)\,d\xi\,du. \end{aligned}$$

15.2.4 Mellin, Hankel, and Other Integral Transforms

► Mellin transform.

1°. Suppose that a function $f(x)$ is defined for positive x and satisfies the conditions

$$\int_0^1 |f(x)|\,x^{\sigma_1-1}\,dx < \infty, \qquad \int_1^\infty |f(x)|\,x^{\sigma_2-1}\,dx < \infty$$

for some real numbers σ_1 and σ_2, $\sigma_1 < \sigma_2$.

The Mellin transform of $f(x)$ is defined by

$$\hat{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx, \tag{15.2.4.1}$$

where $s = \sigma + i\tau$ is a complex variable ($\sigma_1 < \sigma < \sigma_2$).

Given $\hat{f}(s)$, the function $f(x)$ can be found by means of the *inverse Mellin transform*

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\hat{f}(s)x^{-s}\,ds \qquad (\sigma_1 < \sigma < \sigma_2), \tag{15.2.4.2}$$

where the integration path is parallel to the imaginary axis of the complex plane s and the integral is understood in the sense of the Cauchy principal value.

Formula (15.2.4.2) holds for continuous functions. If $f(x)$ has a (finite) jump discontinuity at a point $x = x_0 > 0$, then the right-hand side of (15.2.4.2) evaluates to $\frac{1}{2}\big[f(x_0-0)+f(x_0+0)\big]$ at this point (for $x_0=0$, the first term in the square brackets must be omitted).

2°. The main properties of the correspondence between the functions and their Mellin transforms are gathered in Table 15.5.

TABLE 15.5
Main properties of the Mellin transform

No.	Function	Mellin transform	Operation
1	$af_1(x)+bf_2(x)$	$a\hat{f}_1(s)+b\hat{f}_2(s)$	Linearity
2	$f(ax),\ a>0$	$a^{-s}\hat{f}(s)$	Scaling
3	$x^a f(x)$	$\hat{f}(s+a)$	Shift of the argument of the transform
4	$f(x^2)$	$\frac{1}{2}\hat{f}\big(\frac{1}{2}s\big)$	Squared argument
5	$f(1/x)$	$\hat{f}(-s)$	Inversion of the argument of the transform
6	$x^\lambda f\big(ax^\beta\big),\ \ a>0, \beta\neq 0$	$\dfrac{1}{\beta}a^{-\frac{s+\lambda}{\beta}}\hat{f}\Big(\dfrac{s+\lambda}{\beta}\Big)$	Power law transform
7	$f'_x(x)$	$-(s-1)\hat{f}(s-1)$	Differentiation
8	$xf'_x(x)$	$-s\hat{f}(s)$	Differentiation
9	$f_x^{(n)}(x)$	$(-1)^n\dfrac{\Gamma(s)}{\Gamma(s-n)}\hat{f}(s-n)$	Multiple differentiation
10	$\Big(x\dfrac{d}{dx}\Big)^n f(x)$	$(-1)^n s^n\hat{f}(s)$	Multiple differentiation
11	$x^\alpha\displaystyle\int_0^\infty t^\beta f_1(xt)f_2(t)\,dt$	$\hat{f}_1(s+\alpha)\hat{f}_2(1-s-\alpha+\beta)$	Complicated integration
12	$x^\alpha\displaystyle\int_0^\infty t^\beta f_1\Big(\dfrac{x}{t}\Big)f_2(t)\,dt$	$\hat{f}_1(s+\alpha)\hat{f}_2(s+\alpha+\beta+1)$	Complicated integration

There are extensive tables of direct and inverse Mellin transforms (see the references listed at the end of the current chapter), which are useful when solving specific differential and integral equations.

3°. By $\mathfrak{M}\{f(x),s\}$ we denote the Mellin transform (15.2.4.1), and by $\mathfrak{L}\{f(t),p\}$ we denote the Laplace transform (15.2.1.1).

The Mellin transform is related to the Laplace transform by

$$\mathfrak{M}\{f(x),s\}=\mathfrak{L}\{f(e^x),-s\}+\mathfrak{L}\{f(e^{-x}),s\}. \qquad (15.2.4.3)$$

Formula (15.2.4.3) permits one to apply the much more common tables of direct and inverse Laplace transforms.

► Hankel transform.

1°. The *Hankel transform* is defined as follows:

$$\widetilde{f}_\nu(u) = \int_0^\infty x J_\nu(ux) f(x)\,dx, \qquad 0 < u < \infty, \tag{15.2.4.4}$$

where $\nu > -\frac{1}{2}$ and $J_\nu(x)$ is the Bessel function of the first kind of order ν (see Section 30.6).

For given $\widetilde{f}_\nu(u)$, the function $f(x)$ can be found by means of the *Hankel inversion formula*

$$f(x) = \int_0^\infty u J_\nu(ux) \widetilde{f}_\nu(u)\,du, \qquad 0 < x < \infty. \tag{15.2.4.5}$$

Note that if $f(x) = O(x^\alpha)$ as $x \to 0$, where $\alpha + \nu + 2 > 0$, and $f(x) = O(x^\beta)$ as $x \to \infty$, where $\beta + \frac{3}{2} < 0$, then the integral (15.2.4.4) is convergent.

The inversion formula (15.2.4.5) holds for continuous functions. If $f(x)$ has a (finite) jump discontinuity at a point $x = x_0$, then the left-hand side of (15.2.4.5) is equal to $\frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$ at this point.

For brevity, we denote the Hankel transform (15.2.4.4) by $\widetilde{f}_\nu(u) = \mathcal{H}_\nu\{f(x)\}$.

2°. It follows from formula (15.2.4.5) that the Hankel transform has the property $\mathcal{H}_\nu^2 = 1$.

Other properties of the Hankel transform:

$$\mathcal{H}_\nu\left\{\frac{1}{x} f(x)\right\} = \frac{u}{2\nu}\mathcal{H}_{\nu-1}\{f(x)\} + \frac{u}{2\nu}\mathcal{H}_{\nu+1}\{f(x)\},$$

$$\mathcal{H}_\nu\{f'(x)\} = \frac{(\nu-1)u}{2\nu}\mathcal{H}_{\nu+1}\{f(x)\} - \frac{(\nu+1)u}{2\nu}\mathcal{H}_{\nu-1}\{f(x)\},$$

$$\mathcal{H}_\nu\left\{f''(x) + \frac{1}{x} f'(x) - \frac{\nu^2}{x^2} f(x)\right\} = -u^2 \mathcal{H}_\nu\{f(x)\}.$$

The conditions

$$\lim_{x\to 0}\bigl[x^\nu f(x)\bigr] = 0, \quad \lim_{x\to 0}\bigl[x^{\nu+1} f'(x)\bigr] = 0, \quad \lim_{x\to \infty}\bigl[x^{1/2} f(x)\bigr] = 0, \quad \lim_{x\to \infty}\bigl[x^{1/2} f'(x)\bigr] = 0$$

are assumed to hold for the last formula.

Parseval's relation for the Hankel transform:

$$\int_0^\infty u \mathcal{H}_\nu\{f(x)\}\mathcal{H}_\nu\{g(x)\}\,du = \int_0^\infty x f(x) g(x)\,dx, \qquad \nu > -\frac{1}{2}.$$

► Summary table of integral transforms.

Table 15.6 summarizes the integral transforms considered above and also lists some other integral transforms; for the constraints imposed on the functions and parameters occurring in the integrand, see the references given at the end of this section.

⊙ *Literature for Section 15.2*: H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965), S. G. Mikhlin (1970), J. W. Miles (1971), V. S. Vladimirov (1971, 1988), F. Oberhettinger (1972, 1974, 1980), I. Sneddon (1972, 1995), F. Oberhettinger and L. Badii (1973), B. Davis (1978), R. Bellman and R. Roth

TABLE 15.6
Summary table of integral transforms

Integral transform	Definition	Inversion formula
Laplace transform	$\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$	$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
Laplace–Carson transform	$\widetilde{f}(p)=p\int_0^\infty e^{-px}f(x)\,dx$	$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\frac{\widetilde{f}(p)}{p}\,dp$
Two-sided Laplace transform	$\widetilde{f}_*(p)=\int_{-\infty}^\infty e^{-px}f(x)\,dx$	$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}_*(p)\,dp$
Fourier transform	$\widetilde{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-iux}f(x)\,dx$	$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{iux}\widetilde{f}(u)\,du$
Fourier sine transform	$\widetilde{f}_{\rm s}(u)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(xu)f(x)\,dx$	$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(xu)\widetilde{f}_{\rm s}(u)\,du$
Fourier cosine transform	$\widetilde{f}_{\rm c}(u)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos(xu)f(x)\,dx$	$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos(xu)\widetilde{f}_{\rm c}(u)\,du$
Hartley transform	$\widetilde{f}_{\rm h}(u)=\int_{-\infty}^\infty (\cos xu+\sin xu)f(x)\,dx$	$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty (\cos xu+\sin xu)\widetilde{f}_{\rm h}(u)\,du$
Mellin transform	$\widehat{f}(s)=\int_0^\infty x^{s-1}f(x)\,dx$	$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\widehat{f}(s)\,ds$
Hankel transform	$\widehat{f}_\nu(w)=\int_0^\infty xJ_\nu(xw)f(x)\,dx$	$f(x)=\int_0^\infty wJ_\nu(xw)\widehat{f}_\nu(w)\,dw$
Y-transform	$F_\nu(u)=\int_0^\infty \sqrt{ux}\,Y_\nu(ux)f(x)\,dx$	$f(x)=\int_0^\infty \sqrt{ux}\,H_\nu(ux)F_\nu(u)\,du$
Meijer transform (K-transform)	$\widehat{f}(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{sx}\,K_\nu(sx)f(x)\,dx$	$f(x)=\frac{1}{i\sqrt{2\pi}}\int_{c-i\infty}^{c+i\infty} \sqrt{sx}\,I_\nu(sx)\widehat{f}(s)\,ds$
Kontorovich–Lebedev transform	$F(\tau)=\int_0^\infty K_{i\tau}(x)f(x)\,dx$	$f(x)=\frac{2}{\pi^2 x}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(x)F(\tau)\,d\tau$

Notation: $i=\sqrt{-1}$; $J_\mu(x)$ and $Y_\mu(x)$ are the Bessel functions of the first and the second kind, respectively; $I_\mu(x)$ and $K_\mu(x)$ are the modified Bessel functions of the first and the second kind, respectively; and $H_\nu(x)=\sum_{j=0}^\infty \frac{(-1)^j(x/2)^{\nu+2j+1}}{\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(\nu+j+\frac{3}{2}\right)}$ is the Struve function.

(1984), Yu. A. Brychkov and A. P. Prudnikov (1989), R. Courant and D. Hilbert (1989), J. R. Hanna and J. H. Rowland (1990), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992a, 1992b), A. Pinkus and S. Zafrany (1997), D. Zwillinger (1998), R. Bracewell (1999), A. V. Manzhirov and A. D. Polyanin (1999), A. D. Polyanin (2002), R. J. Beerends and H. G. ter Morschem, and J. C. van den Berg (2003), D. G. Duffy (2004), L. Debnath and B. Bhatta (2007), A. D. Polyanin and A. V. Manzhirov (2007, 2008).

Chapter 16

Cauchy Problem. Fundamental Solutions

16.1 Dirac Delta Function. Fundamental Solutions

16.1.1 Dirac Delta Function and Its Properties

► **Properties of the one-dimensional Dirac delta function.**

The *Dirac delta function* $\delta(x)$ is the *singular generalized function* (*distribution*) acting by the rule

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x)\,dx = \varphi(0)$$

for an arbitrary function $\varphi(x)$ continuous at the point $x = 0$.

The Dirac delta function plays an important role in the theory of linear PDEs. The rigorous definition of this function as the limit of delta sequences of regular distributions, as well as its physical interpretation, can be found in Chapter 21 (see also the references therein).

Basic properties of the one-dimensional Dirac delta function:

1. $f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$,
2. $\delta(x) = \delta(-x)$,
3. $\delta(cx) = |c|^{-1}\delta(x), \quad c = \text{const} \quad (c \neq 0)$,
4. $\vartheta'(x) = \delta(x)$,
5. $\displaystyle\int_a^b f(y)\delta(x-y)\,dy = \begin{cases} f(x) & \text{if } a < x < b, \\ 0 & \text{if } x < a \text{ or } x > b, \end{cases}$

where $f(x)$ is any continuous function and $\vartheta(x)$ is the *Heaviside unit step function* ($\vartheta = 0$ for $x \le 0$ and $\vartheta = 1$ for $x > 0$).

Let a continuous function $f(x)$ have only simple zeros $x_1, x_2, \ldots, x_m$. Then

$$\delta(f(x)) = \sum_{k=1}^{m} \frac{\delta(x-x_k)}{|f'(x_k)|}.$$

The number m of zeros can be infinite.

Let the derivative $f^{(n)}(x)$ be continuous for $a < x < b$. Then

$$\int_a^b f(y)\delta^{(n)}(x-y)\,dy = f^{(n)}(x), \qquad n = 1,\,2,\,\ldots\,,$$

$$\int_a^b f(y)\delta^{(n)}(y-x)\,dy = (-1)^n f^{(n)}(x).$$

► Properties of the n-dimensional Dirac delta function.

The n-dimensional Dirac delta function possesses the following basic properties:

$$1. \quad \delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\ldots\delta(x_n),$$

$$2. \quad \int_{\mathbb{R}^n} \Phi(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\,dV_y = \Phi(\mathbf{x}),$$

where $\delta(x_k)$ is the one-dimensional Dirac delta function, $\Phi(\mathbf{x})$ is an arbitrary continuous function, and $dV_y = dy_1\ldots dy_n$.

16.1.2 Fundamental Solutions. Constructing Particular Solutions

► Fundamental solution of a differential operator.

Consider the linear nonhomogeneous equation

$$L_{\mathbf{x}}[w] = \Phi(\mathbf{x}). \tag{16.1.2.1}$$

Here $\mathbf{x} \in \mathbb{R}^n$ and $L_{\mathbf{x}}$ is a linear differential operator of the second (or any) order of general form whose coefficients may depend on $\mathbf{x}$.

A generalized function (distribution) $\mathscr{E}_\mathrm{e} = \mathscr{E}_\mathrm{e}(\mathbf{x},\mathbf{y})$ that satisfies the equation

$$L_{\mathbf{x}}[\mathscr{E}_\mathrm{e}] = \delta(\mathbf{x}-\mathbf{y}) \tag{16.1.2.2}$$

with a special right-hand side is called a *fundamental solution* corresponding to the operator $L_{\mathbf{x}}$. (Sometimes the function $\mathscr{E}_\mathrm{e}$ will also be called the fundamental solution of Eq. (16.1.2.1).) In (16.1.2.2), $\delta(\mathbf{x})$ is the n-dimensional Dirac delta function and the vector quantity $\mathbf{y} = \{y_1,\ldots,y_n\}$ appears in Eq. (16.1.2.2) as an n-dimensional free parameter. It is assumed that $\mathbf{y} \in \mathbb{R}^n$.

Remark 16.1. The fundamental solution $\mathscr{E}_\mathrm{e}$ is not unique; it is defined up to an additive term $w_0 = w_0(\mathbf{x})$ that is an arbitrary solution of the homogeneous equation $L_{\mathbf{x}}[w_0] = 0$.

For constant coefficient equations, a fundamental solution can be found by means of the n-dimensional Fourier transform (see Section 15.2.2 and Example 16.1).

► Using fundamental solutions for constructing particular solutions.

The fundamental solution $\mathscr{E}_\mathrm{e} = \mathscr{E}_\mathrm{e}(\mathbf{x},\mathbf{y})$ can be used to construct a particular solution of the linear nonhomogeneous equation (16.1.2.1); this particular solution is expressed as follows:

$$w(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{y})\mathscr{E}_\mathrm{e}(\mathbf{x},\mathbf{y})\,dV_y. \tag{16.1.2.3}$$

It is assumed that the function $\Phi(\mathbf{x})$ is continuous and sufficiently rapidly decays as $|\mathbf{x}| \to \infty$, thus ensuring the convergence of the integral in (16.1.2.3) and the existence of the derivatives determining the operator $L_{\mathbf{x}}$.

For constant coefficient linear equations, it is customary to use the fundamental solution $\mathscr{E}_{\rm e} = \mathscr{E}_{\rm e}(\mathbf{x})$ satisfying Eq. (16.1.2.2) with $\mathbf{y} = \mathbf{0}$. In this case, a particular solution of Eq. (16.1.2.1) can be represented as

$$w(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{y})\mathscr{E}_{\rm e}(\mathbf{x}-\mathbf{y})\,dV_y = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y})\mathscr{E}_{\rm e}(\mathbf{y})\,dV_y. \tag{16.1.2.4}$$

Remark 16.2. The right-hand sides of Eqs. (16.1.2.1) and (16.1.2.2) are sometimes prefixed with the minus sign. In this case, formula (16.1.2.3) remains valid. See also the footnote in Section 17.4.3.

► Examples of fundamental solutions.

Example 16.1. Consider the 3D Poisson equation

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_3^2} = \Phi(x_1, x_2, x_3). \tag{16.1.2.5}$$

In terms of Eq. (16.1.2.1), we have $L_{\mathbf{x}}[w] = \Delta w$ in this case, where Δ is the 3D Laplace operator.

Let us find the fundamental solution of the Laplace operator. It satisfies the following Poisson equation with a singular right-hand side:

$$\frac{\partial^2 \mathscr{E}_{\rm e}}{\partial x_1^2} + \frac{\partial^2 \mathscr{E}_{\rm e}}{\partial x_2^2} + \frac{\partial^2 \mathscr{E}_{\rm e}}{\partial x_3^2} = \delta(x_1)\,\delta(x_2)\,\delta(x_3). \tag{16.1.2.6}$$

We use the asymmetric form of the 3D Fourier transform (15.2.2.7):

$$\check{\mathscr{E}}_{\rm e}(\mathbf{u}) = \int_{\mathbb{R}^3} \mathscr{E}_{\rm e}(\mathbf{x})e^{-i(\mathbf{u}\cdot\mathbf{x})}\,dV_x, \quad (\mathbf{u}\cdot\mathbf{x}) = u_1x_1 + u_2x_2 + u_3x_3, \quad dV_x = dx_1\,dx_2\,dx_3.$$

As a result, the PDE (16.1.2.6) is reduced to the simple algebraic equation $-|\mathbf{u}|^2\check{\mathscr{E}}_{\rm e} = 1$, the solution of which has the form

$$\check{\mathscr{E}}_{\rm e} = -\frac{1}{|\mathbf{u}|^2}, \qquad |\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2. \tag{16.1.2.7}$$

Further, by applying the Fourier inversion formula (15.2.2.8), we obtain

$$\mathscr{E}_{\rm e}(\mathbf{x}) = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{u}|^2}e^{i(\mathbf{u}\cdot\mathbf{x})}\,dV_u, \qquad dV_u = du_1\,du_2\,du_3. \tag{16.1.2.8}$$

To compute the triple integral (16.1.2.8), we use spherical coordinates in the space of the variables (u_1, u_2, u_3), the azimuthal axis being the direction of the vector $\mathbf{x}$. We have

$$u_1 = \rho\sin\theta\cos\varphi, \quad u_2 = \rho\sin\theta\sin\varphi, \quad u_3 = \rho\cos\theta,$$
$$|\mathbf{u}| = \rho, \quad (\mathbf{u}\cdot\mathbf{x}) = \rho r\cos\theta, \quad r = |\mathbf{x}|, \quad \sqrt{g} = \rho^2\sin\theta.$$

The chain of computations given below, which starts from formula (16.1.2.8), permits one to find the fundamental solution,

$$\begin{aligned}\mathscr{E}_{\rm e}(\mathbf{x}) &= -\frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{\rho^2}e^{ir\rho\cos\theta}\rho^2\sin\theta\,d\varphi\,d\theta\,d\rho\\ &= -\frac{1}{(2\pi)^2}\int_0^\infty\int_0^\pi e^{ir\rho\cos\theta}\sin\theta\,d\theta\,d\rho = -\frac{1}{(2\pi)^2}\int_0^\infty\int_{-1}^1 e^{ir\rho\mu}\,d\mu\,d\rho\\ &= -\frac{1}{2\pi^2 r}\int_0^\infty\frac{\sin(r\rho)}{\rho}\,d\rho = -\frac{1}{2\pi^2 r}\int_0^\infty\frac{\sin\lambda}{\lambda}\,d\lambda = -\frac{1}{4\pi r}.\end{aligned} \tag{16.1.2.9}$$

A particular solution of the Poisson equation (16.1.2.5) can be found with the use of formulas (16.1.2.4) and (16.1.2.9),

$$w(x_1,x_2,x_3) = -\frac{1}{4\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{\Phi(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}}.$$

Remark 16.3. For the 2D Laplace operator

$$\Delta w = \frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}$$

the fundamental solution has the form

$$\mathscr{E}_{\rm e}(x_1,x_2) = \frac{1}{2\pi}\ln r, \qquad r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$

Example 16.2. The 2D Helmholtz operator

$$L_{\mathbf{x}}[w] = \frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}+\lambda w$$

has the following fundamental solutions:

$$\begin{aligned}\mathscr{E}_{\rm e}(x_1,x_2) &= -\frac{1}{2\pi}K_0(kr) \quad \text{if} \quad \lambda=-k^2<0,\\ \mathscr{E}_{\rm e}(x_1,x_2) &= \frac{i}{4}H_0^{(2)}(kr) \quad \text{if} \quad \lambda=k^2>0,\end{aligned}$$

where $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$, $K_0(z)$ is the modified Bessel function of the second kind, $H_0^{(2)}(z)$ is the Hankel function of the second kind of order 0, $k>0$, and $i^2=-1$.

Example 16.3. The 3D Helmholtz operator

$$L_{\mathbf{x}}[w] = \frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}+\frac{\partial^2 w}{\partial x_3^2}+\lambda w$$

has the following fundamental solutions:

$$\begin{aligned}\mathscr{E}_{\rm e}(x_1,x_2,x_3) &= -\frac{1}{4\pi r}\exp(-kr) \quad \text{if} \quad \lambda=-k^2<0,\\ \mathscr{E}_{\rm e}(x_1,x_2,x_3) &= -\frac{1}{4\pi r}\exp(-ikr) \quad \text{if} \quad \lambda=k^2>0,\end{aligned}$$

where $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$, $k>0$, and $i^2=-1$.

Remark 16.4. For fundamental solutions of higher-order elliptic and hyperbolic equations of the general type, see Sections 11.6.2 and 11.6.3.

⊙ *Literature for Section 16.1*: G. E. Shilov (1965), I. M. Gelfand, G. E. Shilov (1959), V. S. Vladimirov (1971, 1988), S. G. Krein (1972), V. S. Vladimirov, V. P. Mikhailov et al. (1974), A. G. Butkovskiy (1979, 1982), L. Hörmander (1983, 1990), R. P. Kanwal (1983), A. N. Tikhonov and A. A. Samarskii (1990), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

16.2 Representation of the Solution of the Cauchy Problem via the Fundamental Solution

16.2.1 Cauchy Problem for Ordinary Differential Equations

▶ Representation of the fundamental solution of an ordinary differential operator.

Consider the linear ordinary differential equation

$$L[w] \equiv \sum_{k=0}^{m} a_k(t) w_t^{(k)} = f(t), \qquad w_t^{(k)} \equiv \frac{d^k w}{dt^k}. \tag{16.2.1.1}$$

One can show that a fundamental solution of the operator L, i.e., a function satisfying the equation $L[\mathscr{E}_\mathrm{e}] = \delta(t)$, is given by the formula

$$\mathscr{E}_\mathrm{e}(t) = \vartheta(t) Z(t), \tag{16.2.1.2}$$

where $\vartheta(t)$ is the Heaviside unit step function and the function $Z(t)$ satisfies the homogeneous equation $L[Z] = 0$ with the special initial conditions

$$Z(0) = Z_t'(0) = \cdots = Z_t^{(m-2)}(0) = 0, \quad Z_t^{(m-1)}(0) = \frac{1}{a_m(0)}. \tag{16.2.1.3}$$

Example 16.4. Fundamental solutions of some operators:

Operator	*Fundamental solution*
$\dfrac{d}{dt} + a$	$\mathscr{E}_\mathrm{e}(t) = \vartheta(t) e^{-at}$,
$\dfrac{d^2}{dt^2} + a^2$	$\mathscr{E}_\mathrm{e}(t) = \vartheta(t) \dfrac{\sin(at)}{a}$,
$\dfrac{d^2}{dt^2} - a^2$	$\mathscr{E}_\mathrm{e}(t) = \dfrac{1}{2a} \exp(-a\|t\|), \quad a > 0.$

To obtain the last fundamental solution, one first uses formula (16.2.1.2), which gives $\mathscr{E}_\mathrm{e}(t) = \vartheta(t)\frac{\sinh(at)}{a}$. By subtracting the particular solution $\frac{1}{2a}\exp(at)$ of the homogeneous equation from this expression (recall that the fundamental solution is not unique in that one can add a solution of the homogeneous equation to it; see Remark 16.1), we obtain the expression given above.

The fundamental solution $\mathscr{E}_\mathrm{e} = \mathscr{E}_\mathrm{e}(t)$ can be used to construct a particular solution of the linear nonhomogeneous equation (16.2.1.1) for arbitrary continuous $f(t)$ (which should vanish sufficiently rapidly as $|t| \to \infty$); this particular solution is expressed as follows:

$$w(t) = \int_{-\infty}^{\infty} f(\tau) \mathscr{E}_\mathrm{e}(t - \tau)\, d\tau. \tag{16.2.1.4}$$

This formula, up to an obvious change in notation, is a special case of (16.1.2.4).

► Classical and generalized Cauchy problems for ordinary differential equations.

Consider the Cauchy problem for the constant coefficient linear equation (16.2.1.1) (with $a_k = \text{const}$) with the initial conditions

$$w_t^{(k)} = b_k \quad \text{at} \quad t = 0; \qquad k = 0,\, 1,\, \dots,\, m-1, \tag{16.2.1.5}$$

where the b_k are some constants.

Let $w(t)$ be the classical solution of the Cauchy problem for $t > 0$. We extend the functions $w(t)$ and $f(t)$ by zero into the domain $t < 0$. Let us denote the extended functions by w_+ and f_+. We have

$$w_+ = \vartheta(t)w(t), \qquad f_+ = \vartheta(t)f(t). \tag{16.2.1.6}$$

By successively differentiating the first relation in (16.2.1.6) and by taking into account the formulas $\vartheta'_t(t) = \delta(t)$ and $a(t)\delta(t) = a(0)\delta(t)$ and the initial conditions (16.2.1.5), we find the derivatives

$$(w_+)'_t = \vartheta(t)w'_t(t) + b_0\delta(t), \quad \dots, \quad (w_+)_t^{(k)} = \vartheta(t)w_t^{(k)}(t) + \sum_{j=0}^{k-1} b_j\delta_t^{(k-j-1)}(t),$$

where $k = 2,\, \dots,\, m$. By using these formulas, we compute $L[w_+]$,

$$L[w_+] = \vartheta(t)L[w] + \sum_{k=0}^{m-1} c_k\delta_t^{(k)}(t) = f_+(t) + \sum_{k=0}^{m-1} c_k\delta_t^{(k)}(t), \qquad c_k = \sum_{j=0}^{m-k-1} a_{k+j+1}b_j.$$

We see that the function w_+ satisfies the equation

$$L[w_+] = f_+(t) + \sum_{k=0}^{m-1} c_k\delta_t^{(k)}(t) \tag{16.2.1.7}$$

on $\mathbb{R}^1$.

Thus, the classical Cauchy problem in the domain $t \ge 0$ for Eq. (16.2.1.1) with the initial conditions (16.2.1.5) can be reduced to the generalized Cauchy problem defined by Eq. (16.2.1.7) in the domain $-\infty < t < \infty$. (The right-hand side of this equation contains complete information on the initial conditions in the classical problem.)

► Representation of the solution of the Cauchy problem.

To construct the solution of Eq. (16.2.1.7), we use formula (16.2.1.4), in which $f(t)$ should be replaced by $f_+(t) + \sum_{k=0}^{m-1} c_k\delta_t^{(k)}(t)$. We find the fundamental solution $\mathscr{E}_\mathrm{e}$ of the operator L by formula (16.2.1.2), where $Z(t)$ is the solution of the homogeneous equation

$L[Z] = 0$ with the special initial conditions (16.2.1.3). We have

$$\begin{aligned} w_+ &= \int_{-\infty}^{\infty} \mathscr{E}_\mathrm{e}(t-\tau)\Bigg[f_+(\tau) + \sum_{k=0}^{m-1} c_k \delta_\tau^{(k)}(\tau)\Bigg]\,d\tau \\ &= \int_{-\infty}^{\infty} \mathscr{E}_\mathrm{e}(t-\tau) f_+(\tau)\,d\tau + \sum_{k=0}^{m-1} c_k [\mathscr{E}_\mathrm{e}(t)]_t^{(k)} \\ &= \vartheta(t)\int_0^t Z(t-\tau) f(\tau)\,d\tau + \vartheta(t)\sum_{k=0}^{m-1} c_k Z_t^{(k)}(t). \end{aligned}$$

Here we have taken into account the relations

$$[\mathscr{E}_\mathrm{e}(t)]_t^{(k)} = \vartheta(t) Z_t^{(k)}(t) \qquad (k = 0,\, 1,\, \ldots,\, n-1),$$

which follow from (16.2.1.3). Thus, we have obtained a representation of the solution of the Cauchy problem via the function $Z(t)$ and the initial conditions. (Recall that the functions w_+ and w coincide in the domain $t > 0$.)

16.2.2 Cauchy Problem for Parabolic Equations

► **Formula for the solution of the Cauchy problem. General case.**

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_n\}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

Consider a nonhomogeneous linear equation of the parabolic type with an arbitrary right-hand side,

$$\frac{\partial w}{\partial t} - L_\mathbf{x}[w] = \Phi(\mathbf{x}, t), \tag{16.2.2.1}$$

where $t > 0$; the second-order linear differential operator $L_\mathbf{x}$ does not contain derivatives with respect to t and has the form (14.2.1.2).

The solution of the Cauchy problem for Eq. (16.2.2.1) with an arbitrary initial condition,

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0, \tag{16.2.2.2}$$

can be represented as the sum of two integrals,

$$w(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \tau)\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)\,dV_y\,d\tau + \int_{\mathbb{R}^n} f(\mathbf{y})\mathscr{E}(\mathbf{x}, \mathbf{y}, t, 0)\,dV_y, \tag{16.2.2.3}$$
$$dV_y = dy_1 \ldots dy_n.$$

Here $\mathscr{E} = \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)$ is the *fundamental solution of the Cauchy problem.* It satisfies the homogeneous linear equation

$$\frac{\partial \mathscr{E}}{\partial t} - L_\mathbf{x}[\mathscr{E}] = 0 \tag{16.2.2.4}$$

for $t > \tau \geq 0$ and the nonhomogeneous initial condition of the special form

$$\mathscr{E} = \delta(\mathbf{x} - \mathbf{y}) \quad \text{at} \quad t = \tau. \tag{16.2.2.5}$$

The quantities τ and $\mathbf{y}$ appear in problem (16.2.2.4)–(16.2.2.5) as free parameters, and $\delta(\mathbf{x}) = \delta(x_1) \ldots \delta(x_n)$ is the n-dimensional Dirac delta function.

Remark 16.5. If the coefficients of the differential operator $L_\mathbf{x}$ in (16.2.2.4) are independent of time t, then the fundamental solution of the Cauchy problem only depends on three arguments, $\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)=\mathscr{E}(\mathbf{x},\mathbf{y},t-\tau)$.

► Relation between the fundamental solutions $\mathscr{E}_\mathrm{e}$ and $\mathscr{E}$.

Let $\mathscr{E}=\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)$ be the fundamental solution of the Cauchy problem satisfying the homogeneous linear equation (16.2.2.4) with the initial condition (16.2.2.5). Then the function

$$\mathscr{E}_\mathrm{e}(\mathbf{x},\mathbf{y},t,\tau)=\vartheta(t-\tau)\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau), \tag{16.2.2.6}$$

where $\vartheta(t)$ is the *Heaviside unit step function* ($\vartheta=0$ for $t\le 0$ and $\vartheta=1$ for $t>0$), is a fundamental solution corresponding to the operator $\partial_t-L_\mathbf{x}$; i.e., it satisfies the linear equation

$$\frac{\partial\mathscr{E}_\mathrm{e}}{\partial t}-L_\mathbf{x}[\mathscr{E}_\mathrm{e}]=\delta(t-\tau)\delta(\mathbf{x}-\mathbf{y})$$

with a singular right-hand side. Formula (16.2.2.6) can be proved by a straightforward verification involving the computation of $[\partial_t-L_\mathbf{x}][\vartheta(t-\tau)\mathscr{E}]$ with regard to the relations $\vartheta'_t(t)=\delta(t)$ and $\mathscr{E}(\mathbf{x},t)\delta(t)=\mathscr{E}(\mathbf{x},0)\delta(t)$ and the initial condition (16.2.2.5).

In view of formula (16.2.2.6), we do not distinguish between the fundamental solutions $\mathscr{E}$ and $\mathscr{E}_\mathrm{e}$ for linear parabolic equations in what follows and omit the factor $\vartheta(t-\tau)$ in the function $\mathscr{E}_\mathrm{e}$.

► Formula for the solution of the Cauchy problem. Constant coefficient PDEs.

For constant coefficient linear parabolic equations, one customarily uses the fundamental solution of the Cauchy problem $\mathscr{E}=\mathscr{E}(\mathbf{x},t)$ depending on only two arguments and satisfying Eq. (16.2.2.4) and the simpler initial condition (16.2.2.5) with $\mathbf{y}=\mathbf{0}$ and $\tau=0$. The solution of the Cauchy problem for Eq. (16.2.2.1) with an arbitrary initial condition (16.2.2.2) can be represented as

$$w(\mathbf{x},t)=\int_0^t\int_{\mathbb{R}^n}\Phi(\mathbf{y},\tau)\mathscr{E}(\mathbf{x}-\mathbf{y},t-\tau)\,dV_y\,d\tau+\int_{\mathbb{R}^n}f(\mathbf{y})\mathscr{E}(\mathbf{x}-\mathbf{y},t)\,dV_y. \tag{16.2.2.7}$$

Remark 16.6. For equations of the form (16.2.2.1), where $L_\mathbf{x}$ is a higher-order differential operator in the space variables $x_1,\dots,x_n$ that does not contain t-derivatives, the fundamental solution of the Cauchy problem $\mathscr{E}=\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau)$ also satisfies Eq. (16.2.2.4) with the initial condition (16.2.2.5). In this case, the solution of the Cauchy problem (16.2.2.1)–(16.2.2.2) can also be determined by formula (16.2.2.3) (or, for constant coefficient linear equations, by formula (16.2.2.7)).

Example 16.5. Let us find the fundamental solution of the Cauchy problem for the linearized *Burgers–Korteweg–de Vries equation*

$$\begin{gathered}\frac{\partial\mathscr{E}}{\partial t}+a\frac{\partial^3\mathscr{E}}{\partial x^3}-b\frac{\partial^2\mathscr{E}}{\partial x^2}=0,\\ \mathscr{E}=\delta(x)\quad\text{at}\quad t=0,\end{gathered} \tag{16.2.2.8}$$

where $a\ge 0$ and $b\ge 0$ ($|a|+|b|\ne 0$).

We use the asymmetric form of the Fourier transform (15.2.2.3) with respect to the space variable x. By setting $E = \mathcal{F}\{\mathscr{E}(x,t)\}$ and by taking into account the relations $\mathcal{F}\{\partial_{xx}\mathscr{E}\} = -u^2E$ and $\mathcal{F}\{\partial_{xxx}\mathscr{E}\} = -iu^3E$ (see Property 5 and Property 6 with $n=3$ in Table 15.4), we arrive at the following problem for a linear first-order ordinary differential equation in t with parameter u:

$$E'_t + (bu^2 - iau^3)E = 0, \qquad E = 1 \quad \text{at} \quad t = 0. \tag{16.2.2.9}$$

The solution of this problem is

$$E = \exp[(-bu^2 + iau^3)t]. \tag{16.2.2.10}$$

By applying the inverse Fourier transform (15.2.2.4) to (16.2.2.10), after simple transformations we obtain

$$\begin{aligned}\mathscr{E} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[(-bu^2 + iau^3)t]e^{ixu}\,du\\ &= \frac{1}{\pi}\int_0^{\infty} \exp(-bu^2)\cos(au^3t + xu)\,du.\end{aligned} \tag{16.2.2.11}$$

For the special case of $a = 0$, we have the heat equation. The computation of the integral in (16.2.2.11) gives

$$\mathscr{E} = \frac{1}{2\sqrt{\pi bt}}\exp\biggl(-\frac{x^2}{4bt}\biggr).$$

For the special case of $b = 0$, we have the linearized *Korteweg–de Vries equation*. The fundamental solution of the Cauchy problem acquires the form

$$\mathscr{E} = \frac{1}{\pi}\int_0^{\infty}\cos(au^3t + xu)\,du = \frac{1}{(3at)^{1/3}}\operatorname{Ai}(z), \qquad z = \frac{x}{(3at)^{1/3}},$$

where $\operatorname{Ai}(z) = \frac{1}{\pi}\int_0^{\infty}\cos\bigl(\frac{1}{3}\xi^3 + z\xi\bigr)\,d\xi$ is the Airy function.

► Fundamental solution allowing incomplete separation of variables.

Consider the special case where the differential operator $L_{\mathbf{x}}$ in Eq. (16.2.2.1) can be represented as the sum

$$L_{\mathbf{x}}[w] = L_1[w] + \cdots + L_n[w], \tag{16.2.2.12}$$

where each term depends on a single space coordinate and time,

$$L_k[w] \equiv a_k(x_k,t)\frac{\partial^2 w}{\partial x_k^2} + b_k(x_k,t)\frac{\partial w}{\partial x_k} + c_k(x_k,t)w, \qquad k = 1,\,\ldots,\,n.$$

Equations of this form are often encountered in applications. The fundamental solution of the Cauchy problem for the n-dimensional equation (16.2.2.1) with the operator (16.2.2.12) can be represented in the product form

$$\mathscr{E}(\mathbf{x},\mathbf{y},t,\tau) = \prod_{k=1}^{n}\mathscr{E}_k(x_k,y_k,t,\tau), \tag{16.2.2.13}$$

where $\mathscr{E}_k = \mathscr{E}_k(x_k,y_k,t,\tau)$ are the fundamental solutions satisfying the one-dimensional equations

$$\frac{\partial \mathscr{E}_k}{\partial t} - L_k[\mathscr{E}_k] = 0 \qquad (k = 1,\,\ldots,\,n)$$

with the initial conditions

$$\mathscr{E}_k = \delta(x_k - y_k) \quad \text{at} \quad t = \tau.$$

In this case, the fundamental solution of the Cauchy problem (16.2.2.13) admits incomplete separation of variables; the fundamental solution is separated in the space variables $x_1, \ldots, x_n$ but not in time t.

Example 16.6. Consider the 2D heat equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} = 0.$$

The fundamental solutions of the Cauchy problems for the corresponding one-dimensional heat equations are expressed as

Equations		*Fundamental solutions*
$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x_1^2} = 0$	$\Longrightarrow$	$\mathscr{E}_1(x_1, y_1, t, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x_1 - y_1)^2}{4(t-\tau)}\right],$
$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x_2^2} = 0$	$\Longrightarrow$	$\mathscr{E}_2(x_2, y_2, t, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x_2 - y_2)^2}{4(t-\tau)}\right].$

The product of $\mathscr{E}_1$ and $\mathscr{E}_2$ is the fundamental solution of the 2D heat equation,

$$\mathscr{E}(x_1, x_2, y_1, y_2, t, \tau) = \frac{1}{4\pi(t-\tau)} \exp\left[-\frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{4(t-\tau)}\right]. \qquad (16.2.2.14)$$

In the above formulas for fundamental solutions, one can set $y_1 = y_2 = \tau = 0$, denote the left-hand side of (16.2.2.14) by $\mathscr{E}(x_1, x_2, t)$, and use formula (16.2.2.7) for constructing the solution of the Cauchy problem.

Example 16.7. The fundamental solution of the Cauchy problem for the equation

$$\frac{\partial w}{\partial t} - \sum_{k=1}^{n} a_k(t) \frac{\partial^2 w}{\partial x_k^2} = 0, \qquad 0 < a_k(t) < \infty,$$

is given by formula (16.2.2.13) with

$$\mathscr{E}_k(x_k, y_k, t, \tau) = \frac{1}{2\sqrt{\pi T_k}} \exp\left[-\frac{(x_k - y_k)^2}{4T_k}\right], \quad T_k = \int_\tau^t a_k(\eta)\, d\eta.$$

In the derivation of this formula, we have taken into account the fact that the corresponding one-dimensional equations could be reduced to the ordinary constant coefficient heat equation by passing from x_k, t to the new variables x_k, T_k.

16.2.3 Cauchy Problem for Hyperbolic Equations

► Formula for the solution of the Cauchy problem. General case.

Consider a nonhomogeneous linear equation of the hyperbolic type with an arbitrary right-hand side,

$$\frac{\partial^2 w}{\partial t^2} + \varphi(\mathbf{x}, t) \frac{\partial w}{\partial t} - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t), \qquad (16.2.3.1)$$

where the second-order linear differential operator $L_{\mathbf{x}}$ is defined by relation (14.2.1.2) with $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$.

The solution of the Cauchy problem for Eq. (16.2.3.1) with the general initial conditions

$$w = f_0(\mathbf{x}) \quad \text{at} \quad t = 0, \qquad \partial_t w = f_1(\mathbf{x}) \quad \text{at} \quad t = 0 \tag{16.2.3.2}$$

can be represented as the sum

$$w(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \tau) \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)\, dV_y\, d\tau - \int_{\mathbb{R}^n} f_0(\mathbf{y}) \left[\frac{\partial}{\partial \tau} \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau) \right]_{\tau=0} dV_y$$
$$+ \int_{\mathbb{R}^n} \big[f_1(\mathbf{y}) + f_0(\mathbf{y}) \varphi(\mathbf{y}, 0) \big] \mathscr{E}(\mathbf{x}, \mathbf{y}, t, 0)\, dV_y, \qquad dV_y = dy_1 \ldots dy_n.$$

Here $\mathscr{E} = \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)$ is the fundamental solution of the Cauchy problem, which satisfies, for $t > \tau \geq 0$, the homogeneous linear equation

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} + \varphi(\mathbf{x}, t) \frac{\partial \mathscr{E}}{\partial t} - L_{\mathbf{x}}[\mathscr{E}] = 0 \tag{16.2.3.3}$$

with the semihomogeneous initial conditions of special form

$$\mathscr{E} = 0 \quad \text{at} \quad t = \tau, \qquad \partial_t \mathscr{E} = \delta(\mathbf{x} - \mathbf{y}) \quad \text{at} \quad t = \tau. \tag{16.2.3.4}$$

The quantities τ and $\mathbf{y}$ appear in problem (16.2.3.3)–(16.2.3.4) as free parameters ($\mathbf{y} \in \mathbb{R}^n$).

Remark 16.7. If the coefficients of the differential operator $L_{\mathbf{x}}$ in (16.2.3.3) are independent of time t, then the fundamental solution of the Cauchy problem depends on only three arguments, $\mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau) = \mathscr{E}(\mathbf{x}, \mathbf{y}, t - \tau)$. Here $\frac{\partial}{\partial \tau} \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)\big|_{\tau=0} = -\frac{\partial}{\partial t} \mathscr{E}(\mathbf{x}, \mathbf{y}, t)$.

► Relation between the fundamental solutions $\mathscr{E}_{\rm e}$ and $\mathscr{E}$.

Let $\mathscr{E} = \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau)$ be the fundamental solution of the Cauchy problem satisfying the homogeneous linear equation (16.2.3.3) with the initial conditions (16.2.3.4). Then the function

$$\mathscr{E}_{\rm e}(\mathbf{x}, \mathbf{y}, t, \tau) = \vartheta(t - \tau) \mathscr{E}(\mathbf{x}, \mathbf{y}, t, \tau), \tag{16.2.3.5}$$

where $\vartheta(t)$ is the Heaviside unit step function, is a fundamental solution corresponding to the operator $\partial_{tt} + \varphi(\mathbf{x}, t)\partial_t - L_{\mathbf{x}}$ and satisfying the nonhomogeneous linear equation

$$\frac{\partial^2 \mathscr{E}_{\rm e}}{\partial t^2} + \varphi(\mathbf{x}, t) \frac{\partial \mathscr{E}_{\rm e}}{\partial t} - L_{\mathbf{x}}[\mathscr{E}_{\rm e}] = \delta(t - \tau) \delta(\mathbf{x} - \mathbf{y})$$

with a singular right-hand side. Formula (16.2.3.5) can be proved by a straightforward verification involving the computation of $[\partial_{tt} + \varphi(\mathbf{x}, t)\partial_t - L_{\mathbf{x}}][\vartheta(t - \tau)\mathscr{E}]$ with regard to the relations $\vartheta'_t(t - \tau) = \delta(t - \tau)$ and $a(\mathbf{x}, t)\delta(t - \tau) = a(\mathbf{x}, \tau)\delta(t - \tau)$ and the initial conditions (16.2.3.4).

In view of formula (16.2.3.5), we do not distinguish the fundamental solutions $\mathscr{E}$ and $\mathscr{E}_{\rm e}$ for linear hyperbolic equations in what follows and omit the factor $\vartheta(t - \tau)$ in the function $\mathscr{E}_{\rm e}$.

► Formula for the solution of the Cauchy problem. Constant coefficient PDEs.

For constant coefficient linear hyperbolic equations, one customarily uses the fundamental solution of the Cauchy problem $\mathscr{E} = \mathscr{E}(\mathbf{x}, t)$ depending on only two arguments and satisfying Eq. (16.2.3.3) and the simpler initial conditions (16.2.3.4) with $\mathbf{y} = \mathbf{0}$ and $\tau = 0$. In this case, the solution of the Cauchy problem for Eq. (16.2.3.1) with arbitrary initial conditions (16.2.3.2) can be represented as

$$w(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \tau)\mathscr{E}(\mathbf{x} - \mathbf{y}, t - \tau)\, dV_y\, d\tau + \frac{\partial}{\partial t}\int_{\mathbb{R}^n} f_0(\mathbf{y})\mathscr{E}(\mathbf{x} - \mathbf{y}, t)\, dV_y + \int_{\mathbb{R}^n} \bigl[f_1(\mathbf{y}) + f_0(\mathbf{y})\varphi(\mathbf{y}, 0)\bigr]\mathscr{E}(\mathbf{x} - \mathbf{y}, t)\, dV_y. \quad (16.2.3.6)$$

Example 16.8. For the one-, two-, and three-dimensional wave equations, the fundamental solutions of the Cauchy problem have the forms

Equations		*Fundamental solutions*
$\dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x^2} = 0$	$\Longrightarrow$	$\mathscr{E}(x, t) = \dfrac{1}{2}\vartheta(t - \lvert x\rvert);$
$\dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x_1^2} - \dfrac{\partial^2 w}{\partial x_2^2} = 0$	$\Longrightarrow$	$\mathscr{E}(x_1, x_2, t) = \dfrac{\vartheta(t - \rho)}{2\pi\sqrt{t^2 - \rho^2}};$
$\dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x_1^2} - \dfrac{\partial^2 w}{\partial x_2^2} - \dfrac{\partial^2 w}{\partial x_3^2} = 0$	$\Longrightarrow$	$\mathscr{E}(x_1, x_2, x_3, t) = \dfrac{1}{2\pi}\delta\bigl(t^2 - r^2\bigr),$

where $\vartheta(z)$ is the Heaviside unit step function ($\vartheta = 0$ for $z \le 0$ and $\vartheta = 1$ for $z > 0$), $\rho = \sqrt{x_1^2 + x_2^2}$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, and $\delta(z)$ is the Dirac delta function.

Example 16.9. The one-dimensional Klein–Gordon equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} - bw$$

has the following fundamental solutions of the Cauchy problem:

$$\mathscr{E}(x, t) = \tfrac{1}{2}\vartheta\bigl(t - |x|\bigr) J_0\bigl(k\sqrt{t^2 - x^2}\,\bigr) \quad \text{for} \quad b = k^2 > 0,$$
$$\mathscr{E}(x, t) = \tfrac{1}{2}\vartheta\bigl(t - |x|\bigr) I_0\bigl(k\sqrt{t^2 - x^2}\,\bigr) \quad \text{for} \quad b = -k^2 < 0,$$

where $\vartheta(z)$ is the Heaviside unit step function, $J_0(z)$ is the Bessel function, $I_0(z)$ is the modified Bessel function, and $k > 0$.

Example 16.10. The 2D Klein–Gordon equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} - bw$$

has the following fundamental solutions of the Cauchy problem:

$$\mathscr{E}(x_1, x_2, t) = \vartheta(t - \rho)\frac{\cos\bigl(k\sqrt{t^2 - \rho^2}\,\bigr)}{2\pi\sqrt{t^2 - \rho^2}} \quad \text{for} \quad b = k^2 > 0,$$
$$\mathscr{E}(x_1, x_2, t) = \vartheta(t - \rho)\frac{\cosh\bigl(k\sqrt{t^2 - \rho^2}\,\bigr)}{2\pi\sqrt{t^2 - \rho^2}} \quad \text{for} \quad b = -k^2 < 0,$$

where $\vartheta(z)$ is the Heaviside unit step function and $\rho = \sqrt{x_1^2 + x_2^2}$.

16.2.4 Higher-Order Linear PDEs. Generalized Cauchy Problem

► **Reduction of a classical Cauchy problem to a generalized Cauchy problem.**

Let us describe a procedure for reducing a classical Cauchy problem to a generalized Cauchy problem for linear partial differential equations of the form

$$L[w] \equiv a\frac{\partial^2 w}{\partial t^2} + b\frac{\partial w}{\partial t} + M_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t) \qquad (16.2.4.1)$$

with the initial conditions

$$w = f_0(\mathbf{x}) \quad \text{at} \quad t = 0, \qquad \partial_t w = f_1(\mathbf{x}) \quad \text{at} \quad t = 0. \qquad (16.2.4.2)$$

Here a and b are constants, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $M_{\mathbf{x}}$ is a constant coefficient linear differential operator of arbitrary order in the space variables $x_1, \dots, x_n$ and does not contain t-derivatives.

Let $w(\mathbf{x}, t)$ be a classical solution of the Cauchy problem (16.2.4.1)–(16.2.4.2) for $t > 0$. (We assume that this solution exists.) Let us extend the functions $w(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, t)$ by zero into the domain $t < 0$. Denoting the extended functions by w_+ and Φ_+, we have

$$w_+ = \vartheta(t) w(\mathbf{x}, t), \qquad \Phi_+ = \vartheta(t)\Phi(\mathbf{x}, t). \qquad (16.2.4.3)$$

By differentiating the first relation in (16.2.4.3) twice and by taking into account the formulas $\vartheta'_t(t) = \delta(t)$ and $\varphi(\mathbf{x}, t)\delta(t) = \varphi(\mathbf{x}, 0)\delta(t)$ and the initial conditions (16.2.4.2), we find the t-derivatives

$$\frac{\partial w_+}{\partial t} = \vartheta(t)\frac{\partial w}{\partial t} + f_0(\mathbf{x})\delta(t), \qquad \frac{\partial^2 w_+}{\partial t^2} = \vartheta(t)\frac{\partial^2 w}{\partial t^2} + f_1(\mathbf{x})\delta(t) + f_0(\mathbf{x})\delta'_t(t). \quad (16.2.4.4)$$

By using formulas (16.2.4.3) and (16.2.4.4) and the relation $M_{\mathbf{x}}[w_+] = \vartheta(t) M_{\mathbf{x}}[w]$, we compute $L[w_+]$,

$$\begin{aligned} L[w_+] &= \vartheta(t)(a\partial_{tt} + b\partial_t + M_{\mathbf{x}})[w] + \big[af_1(\mathbf{x}) + bf_0(\mathbf{x})\big]\delta(t) + af_0(\mathbf{x})\delta'_t(t) \\ &= \vartheta(t) L[w] + \big[af_1(\mathbf{x}) + bf_0(\mathbf{x})\big]\delta(t) + af_0(\mathbf{x})\delta'_t(t) \\ &= \Phi_+(\mathbf{x}, t) + \big[af_1(\mathbf{x}) + bf_0(\mathbf{x})\big]\delta(t) + af_0(\mathbf{x})\delta'_t(t). \end{aligned}$$

It follows that the function w_+ satisfies the equation

$$L[w_+] = \Phi_+(\mathbf{x}, t) + \big[af_1(\mathbf{x}) + bf_0(\mathbf{x})\big]\delta(t) + af_0(\mathbf{x})\delta'_t(t). \qquad (16.2.4.5)$$

Thus, we have reduced the classical Cauchy problem in the domain $\{t \ge 0, \mathbf{x} \in \mathbb{R}^n\}$ for Eq. (16.2.4.1) with the initial conditions (16.2.4.2) to the generalized Cauchy problem determined by Eq. (16.2.4.5) in the domain $\mathbb{R}^{n+1} = \{-\infty < t < \infty, \mathbf{x} \in \mathbb{R}^n\}$. (The right-hand side of this equation contains complete information about the initial conditions in the classical problem.)

Remark 16.8. The solution of the generalized Cauchy problem for the n-dimensional evolution equation into which (16.2.4.1) degenerates for $a = 0$ (in this case, only the first initial condition in (16.2.4.2) should be retained) satisfies (16.2.4.5) with $a = 0$.

► Representation of the solution of the Cauchy problem via the fundamental solution.

Let $a \neq 0$, and let $\mathscr{E} = \mathscr{E}(\mathbf{x}, t)$ be the fundamental solution of the Cauchy problem satisfying the homogeneous linear equation (16.2.4.1) with $\Phi(\mathbf{x}, t) = 0$ and the initial conditions (16.2.3.4) with $\mathbf{y} = \mathbf{0}$ and $\tau = 0$. Then the function

$$\mathscr{E}_{\mathrm{e}}(\mathbf{x}, t) = \frac{1}{a}\vartheta(t)\mathscr{E}(\mathbf{x}, t), \qquad (16.2.4.6)$$

where $\vartheta(t)$ is the Heaviside unit step function, is a fundamental solution corresponding to the operator $L = a\partial_{tt} + b\partial_t + M_{\mathbf{x}}$ and satisfying the nonhomogeneous linear equation $L[\mathscr{E}_{\mathrm{e}}] = \delta(t)\delta(\mathbf{x})$.

The solution of the Cauchy problem can be obtained with the use of formula (16.1.2.4) in which $\mathbb{R}^n$ should be replaced by $\mathbb{R}^{n+1}$ with $x_{n+1} = t$ and the function $\Phi(\mathbf{x})$ should be replaced by the right-hand side of Eq. (16.2.4.5). In view of (16.2.4.6), we have the chain of relations

$$\begin{aligned}
w_+ &= \int_{\mathbb{R}^n}\int_{-\infty}^{\infty} \mathscr{E}_{\mathrm{e}}(\mathbf{x}-\mathbf{y}, t-\tau)\Big\{\Phi_+(\mathbf{y},\tau) + \big[af_1(\mathbf{y}) + bf_0(\mathbf{y})\big]\delta(\tau) \\
&\quad + af_0(\mathbf{y})\delta'_\tau(\tau)\Big\}\,d\tau\,dV_y = \int_{\mathbb{R}^n}\int_{-\infty}^{\infty} \mathscr{E}_{\mathrm{e}}(\mathbf{x}-\mathbf{y}, t-\tau)\Phi_+(\mathbf{y},\tau)\,d\tau\,dV_y \\
&\quad + \int_{\mathbb{R}^n} \mathscr{E}_{\mathrm{e}}(\mathbf{x}-\mathbf{y}, t)\big[af_1(\mathbf{y}) + bf_0(\mathbf{y})\big]\,dV_y + \frac{\partial}{\partial t}\int_{\mathbb{R}^n} \mathscr{E}_{\mathrm{e}}(\mathbf{x}-\mathbf{y}, t)af_0(\mathbf{y})\,dV_y \\
&= \vartheta(t)\bigg\{\frac{1}{a}\int_{\mathbb{R}^n}\int_0^t \mathscr{E}(\mathbf{x}-\mathbf{y}, t-\tau)\Phi(\mathbf{y},\tau)\,d\tau\,dV_y \\
&\quad + \frac{1}{a}\int_{\mathbb{R}^n} \mathscr{E}(\mathbf{x}-\mathbf{y}, t)\big[af_1(\mathbf{y}) + bf_0(\mathbf{y})\big]\,dV_y + \frac{\partial}{\partial t}\int_{\mathbb{R}^n} \mathscr{E}(\mathbf{x}-\mathbf{y}, t)f_0(\mathbf{y})\,dV_y\bigg\}.
\end{aligned}$$

Thus, we have obtained a representation of the solution of the Cauchy problem (16.2.4.1)–(16.2.4.2) via the fundamental solution $\mathscr{E}(\mathbf{x}, t)$ and the initial conditions. (Recall that the functions w_+ and w coincide in the domain $t > 0$.)

Example 16.11. The Cauchy problem for the fourth-order constant coefficient partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + \Delta\Delta w = \Phi(\mathbf{x}, t)$$

with the initial conditions (16.2.4.2) is a special case of the Cauchy problem considered above for Eq. (16.2.4.1) with $a = 1$, $b = 0$, and $M_{\mathbf{x}}[w] = \Delta\Delta w$.

► Cauchy problem for a more general class of linear PDEs with mixed derivatives.

Consider the Cauchy problem for a constant coefficient partial differential equation of the form

$$L[w] \equiv \frac{\partial^2}{\partial t^2}L_{2,\mathbf{x}}[w] + \frac{\partial}{\partial t}L_{1,\mathbf{x}}[w] + L_{0,\mathbf{x}}[w] = \Phi(\mathbf{x}, t) \qquad (t > 0) \qquad (16.2.4.7)$$

with the initial conditions (16.2.4.2). Here the $L_{m,\mathbf{x}}[w]$ ($m = 0,\ 1,\ 2$) are constant coefficient linear differential operators of arbitrary order in the space variables $x_1, \dots, x_n$ and do not contain t-derivatives.

Let us introduce the functions w_+ and Φ_+ extended by zero into the domain $t<0$ by formulas (16.2.4.3). By using the expressions (16.2.4.4) for the derivatives, we obtain

$$L_{0,\mathbf{x}}[w_+]=\vartheta(t)L_{0,\mathbf{x}}[w],\quad \frac{\partial}{\partial t}L_{1,\mathbf{x}}[w_+]=\vartheta(t)\frac{\partial}{\partial t}L_{1,\mathbf{x}}[w]+\delta(t)L_{1,\mathbf{x}}[f_0(\mathbf{x})],$$
$$\frac{\partial^2}{\partial t^2}L_{2,\mathbf{x}}[w_+]=\vartheta(t)\frac{\partial^2}{\partial t^2}L_{2,\mathbf{x}}[w]+\delta(t)L_{2,\mathbf{x}}[f_1(\mathbf{x})]+\delta_t'(t)L_{2,\mathbf{x}}[f_0(\mathbf{x})].$$

We compute $L[w_+]$ by using these relations and formulas (16.2.4.3). As a result, we find that the function w_+ satisfies the equation

$$L[w_+]=\Phi_+(\mathbf{x},t)+\delta(t)\bigl\{L_{2,\mathbf{x}}[f_1(\mathbf{x})]+L_{1,\mathbf{x}}[f_0(\mathbf{x})]\bigr\}+\delta_t'(t)L_{2,\mathbf{x}}[f_0(\mathbf{x})] \qquad (16.2.4.8)$$

in $\mathbb{R}^{n+1}$.

The solution of the Cauchy problem for equation (16.2.4.7) with the initial conditions (16.2.4.2) can be obtained with the use of formula (16.1.2.4) with $\mathbb{R}^n$ replaced by $\mathbb{R}^{n+1}$, $x_{n+1}=t$, and the function $\Phi(\mathbf{x})$ replaced by the right-hand side of Eq. (16.2.4.8).

► **Relation between the fundamental solutions $\mathscr{E}_\mathrm{e}$ and $\mathscr{E}$. General case.**

Consider the Cauchy problem for the linear partial differential equation

$$L[w]\equiv\sum_{k=0}^{m}\frac{\partial^k}{\partial t^k}L_{k,\mathbf{x}}[w]=\Phi(\mathbf{x},t), \qquad (16.2.4.9)$$

more general than (16.2.4.7), with the initial conditions

$$\begin{gathered} w\big|_{t=0}=f_0(\mathbf{x}),\quad \partial_t w\big|_{t=0}=f_1(\mathbf{x}),\quad \dots,\\ \partial_t^{(m-2)}w\big|_{t=0}=f_{m-2}(\mathbf{x}),\quad \partial_t^{(m-1)}w\big|_{t=0}=f_{m-1}(\mathbf{x}). \end{gathered} \qquad (16.2.4.10)$$

Here the $L_{k,\mathbf{x}}[w]$ are constant coefficient linear differential operators of arbitrary order in the space variables $x_1,\dots,x_n$ and do not contain t-derivatives. We assume that the Cauchy problem (16.2.4.9)–(16.2.4.10) is well posed.

Let $\mathscr{E}=\mathscr{E}(\mathbf{x},t)$ be the fundamental solution of the Cauchy problem satisfying the homogeneous linear equation (16.2.4.9) with $\Phi(\mathbf{x},t)=0$ and the initial conditions of the special form

$$\mathscr{E}\big|_{t=0}=0,\quad \partial_t\mathscr{E}\big|_{t=0}=0,\quad \dots,\quad \partial_t^{(m-2)}\mathscr{E}\big|_{t=0}=0,\quad \partial_t^{(m-1)}\mathscr{E}\big|_{t=0}=\delta(\mathbf{x}), \qquad (16.2.4.11)$$

and let $\mathscr{E}_\mathrm{e}=\mathscr{E}_\mathrm{e}(\mathbf{x},t)$ be a fundamental solution corresponding to L and satisfying the nonhomogeneous linear equation $L[\mathscr{E}_\mathrm{e}]=\delta(t)\delta(\mathbf{x})$. Then

$$\vartheta(t)\mathscr{E}(\mathbf{x},t)=L_{m,\mathbf{x}}[\mathscr{E}_\mathrm{e}(\mathbf{x},t)], \qquad (16.2.4.12)$$

where $\vartheta(t)$ is the Heaviside unit step function.

The solution of the Cauchy problem for the homogeneous equation (16.2.4.9) with initial conditions of the special form

$$w\big|_{t=0}=0,\quad \partial_t w\big|_{t=0}=0,\quad \dots,\quad \partial_t^{(m-2)}w\big|_{t=0}=0,\quad \partial_t^{(m-1)}w\big|_{t=0}=f_{m-1}(\mathbf{x})$$

is given by the formula

$$w = \int_0^t \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x}-\mathbf{y}, t-\tau)\Phi(\mathbf{y},\tau)\,dV_y\,d\tau + \int_{\mathbb{R}^n} \mathscr{E}(\mathbf{x}-\mathbf{y}, t) f_{m-1}(\mathbf{y})\,dV_y. \quad (16.2.4.13)$$

By taking into account relation (16.2.4.13) between the fundamental solutions, we can successively reduce the second integral in (16.2.4.13) to the form

$$\int_{\mathbb{R}^n} \mathscr{E}(\mathbf{x}-\mathbf{y}, t) f_{m-1}(\mathbf{y})\,dV_y = \int_{\mathbb{R}^n} \mathscr{E}(\mathbf{y}, t) f_{m-1}(\mathbf{x}-\mathbf{y})\,dV_y$$
$$= \int_{\mathbb{R}^n} L_{m,\mathbf{y}}[\mathscr{E}_{\rm e}(\mathbf{y}, t)] f_{m-1}(\mathbf{x}-\mathbf{y})\,dV_y = \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x}-\mathbf{y}, t) L_{m,\mathbf{y}}[f_{m-1}(\mathbf{y})]\,dV_y.$$

Here we have assumed that the function $f_{m-1}(\mathbf{x})$ sufficiently rapidly decays as $|\mathbf{x}| \to \infty$.

► **On a third-order partial differential equation with mixed derivatives.**

1°. The constant coefficient third-order partial differential equation

$$L[w] \equiv \frac{\partial w}{\partial t} - a\Delta w - b\frac{\partial}{\partial t}\Delta w = \Phi(\mathbf{x}, t), \quad (16.2.4.14)$$

which is a special case of Eq. (16.2.4.7) with

$$L_{2,\mathbf{x}}[w] = 0, \quad L_{1,\mathbf{x}}[w] = (1 - b\Delta)[w], \quad L_{0,\mathbf{x}}[w] = -a\Delta w, \quad (16.2.4.15)$$

arises in filtration theory.

Consider the Cauchy problem for Eq. (16.2.4.14) with the initial condition

$$w = f_0(\mathbf{x}) \quad \text{at} \quad t = 0. \quad (16.2.4.16)$$

We use formulas (16.2.4.3) to introduce the functions w_+ and Φ_+ and then substitute the expressions (16.2.4.15) into (16.2.4.8). As a result, we obtain the equation

$$L[w_+] = \Phi_+(\mathbf{x}, t) + \delta(t)(1 - b\Delta)[f_0(\mathbf{x})], \quad (16.2.4.17)$$

which corresponds to the generalized Cauchy problem.

The fundamental solutions of the Cauchy problem and the operator L are related by the formula

$$\vartheta(t)\mathscr{E}(\mathbf{x}, t) = (1 - b\Delta)[\mathscr{E}_{\rm e}(\mathbf{x}, t)], \quad (16.2.4.18)$$

which follows from (16.2.4.12) (with $m = 1$) and (16.2.4.15).

The solution of the Cauchy problem for equation (16.2.4.14) with the initial conditions (16.2.4.16) can be found with the use of formula (16.1.2.4) with $\mathbb{R}^n$ replaced by $\mathbb{R}^{n+1}$, $x_{n+1} = t$, and the function $\Phi(\mathbf{x})$ replaced by the right-hand side of Eq. (16.2.4.17). As a result, we obtain

$$w = \int_0^t \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x}-\mathbf{y}, t-\tau)\Phi(\mathbf{y},\tau)\,dV_y\,d\tau + \int_{\mathbb{R}^n} \mathscr{E}_{\rm e}(\mathbf{x}-\mathbf{y}, t)\big[f_0(\mathbf{y}) - b\Delta_y f_0(\mathbf{y})\big]\,dV_y,$$

where Δ_y is the Laplace operator in the integration variables $y_1, \dots, y_n$.

2°. Let us find the fundamental solution $\mathscr{E}_\mathrm{e} = \mathscr{E}_\mathrm{e}(\mathbf{x}, t)$ of L satisfying Eq. (16.2.4.14) with a singular right-hand side $\Phi(\mathbf{x}, t) = \delta(t)\delta(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$. We use the asymmetric form (15.2.2.7) of the 3D Fourier transform in the space variables,

$$\check{\mathscr{E}}_\mathrm{e}(\mathbf{u}, t) = \int_{\mathbb{R}^3} \mathscr{E}_\mathrm{e}(\mathbf{x}, t) e^{-i(\mathbf{u}\cdot\mathbf{x})}\, dV_x, \qquad (\mathbf{u}\cdot\mathbf{x}) = u_1x_1 + u_2x_2 + u_3x_3, \qquad dV_x = dx_1dx_2dx_3.$$

As a result, we obtain the ordinary differential equation $(\check{\mathscr{E}}_\mathrm{e})'_t + a|\mathbf{u}|^2\check{\mathscr{E}}_\mathrm{e} + b|\mathbf{u}|^2(\check{\mathscr{E}}_\mathrm{e})'_t = \delta(t)$, or

$$\frac{d}{dt}\check{\mathscr{E}}_\mathrm{e} + \frac{a|\mathbf{u}|^2}{1+b|\mathbf{u}|^2}\check{\mathscr{E}}_\mathrm{e} = \frac{\delta(t)}{1+b|\mathbf{u}|^2}, \qquad |\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2.$$

The solution of this equation has the form (see the fundamental solution of the first operator in Example 16.4)

$$\check{\mathscr{E}}_\mathrm{e} = \frac{\vartheta(t)}{1+b|\mathbf{u}|^2}\exp\biggl(-\frac{a|\mathbf{u}|^2 t}{1+b|\mathbf{u}|^2}\biggr). \tag{16.2.4.19}$$

Further, by applying the Fourier inversion formula (15.2.2.8), we obtain

$$\mathscr{E}_\mathrm{e}(\mathbf{x}, t) = \frac{\vartheta(t)}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{i(\mathbf{u}\cdot\mathbf{x})}}{1+b|\mathbf{u}|^2}\exp\biggl(-\frac{a|\mathbf{u}|^2 t}{1+b|\mathbf{u}|^2}\biggr) dV_u, \qquad dV_u = du_1du_2du_3. \tag{16.2.4.20}$$

To compute the triple integral (16.2.4.20), just as in Example 16.1, we proceed to spherical coordinates in the space of the variables (u_1, u_2, u_3), the azimuthal axis being the direction of the vector $\mathbf{x}$. The chain of computations given below, which starts from formula (16.2.4.20), permits one to find the fundamental solution

$$\begin{aligned}
\mathscr{E}_\mathrm{e}(\mathbf{x}, t) &= \frac{\vartheta(t)}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{e^{ir\rho\cos\theta}}{1+b\rho^2}\exp\biggl(-\frac{a\rho^2 t}{1+b\rho^2}\biggr)\rho^2\sin\theta\, d\varphi\, d\theta\, d\rho \\
&= \frac{\vartheta(t)}{(2\pi)^2}\int_0^\infty\int_0^\pi \frac{\rho^2 e^{ir\rho\cos\theta}}{1+b\rho^2}\exp\biggl(-\frac{a\rho^2 t}{1+b\rho^2}\biggr)\sin\theta\, d\theta\, d\rho \\
&= \frac{\vartheta(t)}{(2\pi)^2}\int_0^\infty\int_{-1}^1 \frac{\rho^2 e^{ir\rho\mu}}{1+b\rho^2}\exp\biggl(-\frac{a\rho^2 t}{1+b\rho^2}\biggr) d\mu\, d\rho \\
&= \frac{\vartheta(t)}{2\pi^2 r}\int_0^\infty \frac{\rho\sin(r\rho)}{1+b\rho^2}\exp\biggl(-\frac{a\rho^2 t}{1+b\rho^2}\biggr) d\rho.
\end{aligned}$$

3°. The fundamental solution $\mathscr{E}_\mathrm{e} = \mathscr{E}_\mathrm{e}(\mathbf{x}, t)$ of L satisfying Eq. (16.2.4.14) with a singular right-hand side $\Phi(\mathbf{x}, t) = \delta(t)\delta(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2$ is determined in a similar way,

$$\begin{aligned}
\mathscr{E}_\mathrm{e}(\mathbf{x}, t) &= \frac{\vartheta(t)}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(\mathbf{u}\cdot\mathbf{x})}}{1+b|\mathbf{u}|^2}\exp\biggl(-\frac{a|\mathbf{u}|^2 t}{1+b|\mathbf{u}|^2}\biggr) du_1\, du_2 \\
&= \frac{\vartheta(t)}{(2\pi)^2}\int_0^\infty\int_0^{2\pi} \frac{e^{ir\rho\cos\varphi}}{1+b\rho^2}\exp\biggl(-\frac{a\rho^2 t}{1+b\rho^2}\biggr)\rho\, d\varphi\, d\rho \\
&= \frac{\vartheta(t)}{2\pi}\int_0^\infty \frac{\rho J_0(r\rho)}{1+b\rho^2}\exp\biggl(-\frac{a\rho^2 t}{1+b\rho^2}\biggr) d\rho,
\end{aligned}$$

where $J_0(z)$ is the Bessel function. In the derivation of this formula, we have used the relation

$$\int_0^{2\pi} e^{iz\cos\varphi}\,d\varphi = \int_0^{2\pi} \cos(z\cos\varphi)\,d\varphi = 2\pi J_0(z).$$

⊙ *Literature for Section 16.2*: G. E. Shilov (1965), V. S. Vladimirov (1971, 1988), S. G. Krein (1972), V. S. Vladimirov, V. P. Mikhailov et al. (1974), A. G. Butkovskiy (1979, 1982), L. Hörmander (1983, 1990), R. Courant and D. Hilbert (1989), A. N. Tikhonov and A. A. Samarskii (1990), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 17

Boundary Value Problems. Green's Function

17.1 Boundary Value Problems for Parabolic Equations with One Space Variable. Green's Function

17.1.1 Representation of Solutions via the Green's Function

► **Statement of the problem** $(t \geq 0, x_1 \leq x \leq x_2)$.

In general, a nonhomogeneous linear differential equation of the parabolic type with variable coefficients in one dimension can be written as

$$\frac{\partial w}{\partial t} - L_x[w] = \Phi(x,t), \tag{17.1.1.1}$$

where

$$L_x[w] \equiv a(x,t)\frac{\partial^2 w}{\partial x^2} + b(x,t)\frac{\partial w}{\partial x} + c(x,t)w, \qquad a(x,t) > 0. \tag{17.1.1.2}$$

Consider the nonstationary boundary value problem for Eq. (17.1.1.1) with an initial condition of general form

$$w = f(x) \quad \text{at} \quad t = 0, \tag{17.1.1.3}$$

and arbitrary nonhomogeneous linear boundary conditions

$$\alpha_1 \frac{\partial w}{\partial x} + \beta_1 w = g_1(t) \quad \text{at} \quad x = x_1, \tag{17.1.1.4}$$

$$\alpha_2 \frac{\partial w}{\partial x} + \beta_2 w = g_2(t) \quad \text{at} \quad x = x_2. \tag{17.1.1.5}$$

By appropriately choosing the coefficients α_1, α_2, β_1, and β_2 in (17.1.1.4) and (17.1.1.5), we obtain the first, second, third, and mixed boundary value problems for Eq. (17.1.1.1).

► **Representation of the problem solution in terms of the Green's function.**

The solution of the nonhomogeneous linear boundary value problem (17.1.1.1)–(17.1.1.5) can be represented as

$$w(x,t)=\int_0^t\int_{x_1}^{x_2}\Phi(y,\tau)G(x,y,t,\tau)\,dy\,d\tau+\int_{x_1}^{x_2}f(y)G(x,y,t,0)\,dy\\+\int_0^t g_1(\tau)a(x_1,\tau)\Lambda_1(x,t,\tau)\,d\tau+\int_0^t g_2(\tau)a(x_2,\tau)\Lambda_2(x,t,\tau)\,d\tau. \qquad (17.1.1.6)$$

Here $G(x,y,t,\tau)$ is the Green's function that satisfies, for $t>\tau\geq 0$, the homogeneous equation

$$\frac{\partial G}{\partial t}-L_x[G]=0 \qquad (17.1.1.7)$$

with the nonhomogeneous initial condition of special form

$$G=\delta(x-y)\quad\text{at}\quad t=\tau \qquad (17.1.1.8)$$

and the homogeneous boundary conditions

$$\alpha_1\frac{\partial G}{\partial x}+\beta_1 G=0\quad\text{at}\quad x=x_1, \qquad (17.1.1.9)$$

$$\alpha_2\frac{\partial G}{\partial x}+\beta_2 G=0\quad\text{at}\quad x=x_2. \qquad (17.1.1.10)$$

The quantities y and τ appear in problem (17.1.1.7)–(17.1.1.10) as free parameters with $x_1\leq y\leq x_2$, and $\delta(x)$ is the Dirac delta function.

The initial condition (17.1.1.8) implies the limit relation

$$f(x)=\lim_{t\to\tau}\int_{x_1}^{x_2}f(y)G(x,y,t,\tau)\,dy$$

for any continuous function $f=f(x)$.

The functions $\Lambda_1(x,t,\tau)$ and $\Lambda_2(x,t,\tau)$ involved in the integrands of the last two terms in solution (17.1.1.6) can be expressed via the Green's function $G(x,y,t,\tau)$. The corresponding formulas for $\Lambda_m(x,t,\tau)$ are given in Table 17.1 for the basic types of boundary value problems.

It is significant that the Green's function G and the functions Λ_1, Λ_2 are independent of the functions Φ, f, g_1, and g_2 that characterize various inhomogeneities of the boundary value problem.

If the coefficients of Eq. (17.1.1.1)–(17.1.1.2) are independent of time t, i.e., if the conditions

$$a=a(x),\quad b=b(x),\quad c=c(x) \qquad (17.1.1.11)$$

hold, then the Green's function depends on only three arguments,

$$G(x,y,t,\tau)=G(x,y,t-\tau).$$

In this case, the functions Λ_m depend on only two arguments, $\Lambda_m=\Lambda_m(x,t-\tau)$, $m=1,\ 2$.

TABLE 17.1
Expressions of the functions $\Lambda_1(x,t,\tau)$ and $\Lambda_2(x,t,\tau)$ involved in the integrands of the last two terms in solution (17.1.1.6)

Type of problem	Form of boundary conditions	Functions $\Lambda_m(x,t,\tau)$
First boundary value problem ($\alpha_1=\alpha_2=0,\ \beta_1=\beta_2=1$)	$w=g_1(t)$ at $x=x_1$ $w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t,\tau)=\partial_y G(x,y,t,\tau)\big\|_{y=x_1}$ $\Lambda_2(x,t,\tau)=-\partial_y G(x,y,t,\tau)\big\|_{y=x_2}$
Second boundary value problem ($\alpha_1=\alpha_2=1,\ \beta_1=\beta_2=0$)	$\partial_x w=g_1(t)$ at $x=x_1$ $\partial_x w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t,\tau)=-G(x,x_1,t,\tau)$ $\Lambda_2(x,t,\tau)=G(x,x_2,t,\tau)$
Third boundary value problem ($\alpha_1=\alpha_2=1,\ \beta_1<0,\ \beta_2>0$)	$\partial_x w+\beta_1 w=g_1(t)$ at $x=x_1$ $\partial_x w+\beta_2 w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t,\tau)=-G(x,x_1,t,\tau)$ $\Lambda_2(x,t,\tau)=G(x,x_2,t,\tau)$
Mixed boundary value problem ($\alpha_1=\beta_2=0,\ \alpha_2=\beta_1=1$)	$w=g_1(t)$ at $x=x_1$ $\partial_x w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t,\tau)=\partial_y G(x,y,t,\tau)\big\|_{y=x_1}$ $\Lambda_2(x,t,\tau)=G(x,x_2,t,\tau)$
Mixed boundary value problem ($\alpha_1=\beta_2=1,\ \alpha_2=\beta_1=0$)	$\partial_x w=g_1(t)$ at $x=x_1$ $w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t,\tau)=-G(x,x_1,t,\tau)$ $\Lambda_2(x,t,\tau)=-\partial_y G(x,y,t,\tau)\big\|_{y=x_2}$

Formula (17.1.1.6) remains valid for the problem with boundary conditions of the third kind if $\beta_1=\beta_1(t)$ and $\beta_2=\beta_2(t)$. Here the relation between Λ_m $(m=1,\ 2)$ and the Green's function G is the same as that in the case of constants β_1 and β_2; the Green's function itself is now different.

Remark 17.1. In the first, second, and third boundary value problems that are considered on the interval $x_1\le x<\infty$, a condition of boundedness of the solution as $x\to\infty$ is set out. In this case, the solution is calculated by formula (17.1.1.6) with $\Lambda_2=0$ and Λ_1 specified in Table 17.1.

► Formulas for calculating Green's functions.

Consider the parabolic equation of a special form

$$\frac{\partial w}{\partial t}=a(x)\frac{\partial^2 w}{\partial x^2}+b(x)\frac{\partial w}{\partial x}+\bigl[c(x)+\gamma(t)\bigr]w+\Phi(x,t), \qquad (17.1.1.12)$$

which is a special case of Eq. (17.1.1.1) with the operator (17.1.1.2), where $a(x,t)=a(x)$, $b(x,t)=b(x)$, and $c(x,t)=c(x)+\gamma(t)$. It is assumed that $a(x)>0$.

The solution of the nonhomogeneous linear boundary value problem for Eq. (17.1.1.12) subject to the initial and boundary conditions (17.1.1.3)–(17.1.1.5) is found by formula (17.1.1.6), where the Green's function is given by

$$G(x,y,t,\tau)=\rho(y)\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(y)}{\|\varphi_n\|^2}\exp\left[-\lambda_n(t-\tau)+\int_\tau^t\gamma(\xi)\,d\xi\right], \qquad (17.1.1.13)$$

where the λ_n and $\varphi_n(x)$ are the eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem for the linear ordinary differential equation (15.1.1.6) with the homogeneous linear boundary conditions (15.1.1.8) for $s_n=\alpha_n$ and $k_n=\beta_n$, and

$$\rho(y)=\frac{1}{a(y)}\exp\left[\int\frac{b(y)}{a(y)}\,dy\right],\qquad \|\varphi_n\|^2=\int_{x_1}^{x_2}\rho(x)\varphi_n^2(x)\,dx, \qquad (17.1.1.14)$$

Table 17.2 lists Green's functions for some problems for the nonhomogeneous heat equation with $a(x) = a = \text{const}$ and $b(x) = c(x) = \gamma(t) \equiv 0$ in (17.1.1.12). The function $G(x, y, t-\tau)$ in this table should be substituted for $G(x, y, t, \tau)$ in formula (17.1.1.6).

TABLE 17.2
The Green's functions for some boundary value problems for the nonhomogeneous heat equation $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \Phi(x, t)$

Type of problem	Green's function, $G(x, y, t)$
First boundary value problem $(0 \le x < \infty)$	$G(x,y,t) = \frac{1}{2\sqrt{\pi a t}}\Bigl\{\exp\Bigl[-\frac{(x-y)^2}{4at}\Bigr] - \exp\Bigl[-\frac{(x+y)^2}{4at}\Bigr]\Bigr\}$
Second boundary value problem $(0 \le x < \infty)$	$G(x,y,t) = \frac{1}{2\sqrt{\pi a t}}\Bigl\{\exp\Bigl[-\frac{(x-y)^2}{4at}\Bigr] + \exp\Bigl[-\frac{(x+y)^2}{4at}\Bigr]\Bigr\}$
First boundary value problem $(0 \le x \le l)$	$G(x,y,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\Bigl(\frac{n\pi x}{l}\Bigr)\sin\Bigl(\frac{n\pi y}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr)$ $= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\Bigl\{\exp\Bigl[-\frac{(x-y+2nl)^2}{4at}\Bigr] - \exp\Bigl[-\frac{(x+y+2nl)^2}{4at}\Bigr]\Bigr\}$ The first series converges rapidly at large t and the second series at small t
Second boundary value problem $(0 \le x \le l)$	$G(x,y,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\Bigl(\frac{n\pi x}{l}\Bigr)\cos\Bigl(\frac{n\pi y}{l}\Bigr)\exp\Bigl(-\frac{an^2\pi^2 t}{l^2}\Bigr)$ $= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\Bigl\{\exp\Bigl[-\frac{(x-y+2nl)^2}{4at}\Bigr] + \exp\Bigl[-\frac{(x+y+2nl)^2}{4at}\Bigr]\Bigr\}$ The first series converges rapidly at large t and the second series at small t
Mixed boundary value problem $(0 \le x \le l)$; w set at $x = 0$ and $\partial_x w$ set at $x = l$	$G(x,y,t) = \frac{2}{l}\sum_{n=0}^{\infty}\sin\Bigl[\frac{\pi(2n+1)x}{2l}\Bigr]\sin\Bigl[\frac{\pi(2n+1)y}{2l}\Bigr]\exp\Bigl[-\frac{a\pi^2(2n+1)^2 t}{4l^2}\Bigr]$ $= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}(-1)^n\Bigl\{\exp\Bigl[-\frac{(x-y+2nl)^2}{4at}\Bigr] - \exp\Bigl[-\frac{(x+y+2nl)^2}{4at}\Bigr]\Bigr\}$

17.1.2 Problems for Equation $s(x)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\Bigl[p(x)\frac{\partial w}{\partial x}\Bigr] - q(x)w + \Phi(x, t)$

► **General formulas for solving nonhomogeneous boundary value problems.**

Consider linear equations of the special form

$$s(x)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\Bigl[p(x)\frac{\partial w}{\partial x}\Bigr] - q(x)w + \Phi(x, t). \qquad (17.1.2.1)$$

They are often encountered in heat and mass transfer theory and chemical engineering sciences. Throughout this subsection, we assume that the functions s, p, p'_x, and q are continuous and $s > 0$, $p > 0$, and $x_1 \le x \le x_2$.

The solution of Eq. (17.1.2.1) under the initial condition (17.1.1.3) and the arbitrary linear nonhomogeneous boundary conditions (17.1.1.4)–(17.1.1.5) can be represented as the sum

$$w(x,t) = \int_0^t \int_{x_1}^{x_2} \Phi(\xi,\tau)\mathcal{G}(x,\xi,t-\tau)\,d\xi\,d\tau + \int_{x_1}^{x_2} s(\xi)f(\xi)\mathcal{G}(x,\xi,t)\,d\xi$$
$$+ p(x_1)\int_0^t g_1(\tau)\Lambda_1(x,t-\tau)\,d\tau + p(x_2)\int_0^t g_2(\tau)\Lambda_2(x,t-\tau)\,d\tau. \qquad (17.1.2.2)$$

Here the modified Green's function is given by

$$\mathcal{G}(x,\xi,t) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)}{\|y_n\|^2}\exp(-\lambda_n t), \qquad \|y_n\|^2 = \int_{x_1}^{x_2} s(x)y_n^2(x)\,dx, \quad (17.1.2.3)$$

where λ_n and $y_n(x)$ are the eigenvalues and the corresponding eigenfunctions of the following Sturm–Liouville problem for a second-order linear ordinary differential equation:

$$[p(x)y'_x]'_x + [\lambda s(x) - q(x)]y = 0,$$
$$\alpha_1 y'_x + \beta_1 y = 0 \quad \text{at} \quad x = x_1, \qquad (17.1.2.4)$$
$$\alpha_2 y'_x + \beta_2 y = 0 \quad \text{at} \quad x = x_2.$$

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ appearing in the integrands of the last two terms in solution (17.1.2.2) are expressed via the Green's function (17.1.2.3). The corresponding formulas are given in Table 17.3 for the basic types of boundary value problems.

TABLE 17.3
Expressions of the functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ involved in the integrands of the last two terms in solutions (17.1.2.2) and (17.2.1.5); the modified Green's function $\mathcal{G}(x,\xi,t)$ for parabolic equations of the form (17.1.2.1) is found by formula (17.1.2.3), and that for hyperbolic equations of the form (17.2.2.1), by formula (17.2.2.3)

Type of problem	Form of boundary conditions	Functions $\Lambda_m(x,t)$
First boundary value problem ($\alpha_1=\alpha_2=0,\ \beta_1=\beta_2=1$)	$w=g_1(t)$ at $x=x_1$ $w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t)=\partial_\xi\mathcal{G}(x,\xi,t)\big\|_{\xi=x_1}$ $\Lambda_2(x,t)=-\partial_\xi\mathcal{G}(x,\xi,t)\big\|_{\xi=x_2}$
Second boundary value problem ($\alpha_1=\alpha_2=1,\ \beta_1=\beta_2=0$)	$\partial_x w=g_1(t)$ at $x=x_1$ $\partial_x w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t)=-\mathcal{G}(x,x_1,t)$ $\Lambda_2(x,t)=\mathcal{G}(x,x_2,t)$
Third boundary value problem ($\alpha_1=\alpha_2=1,\ \beta_1<0,\ \beta_2>0$)	$\partial_x w+\beta_1 w=g_1(t)$ at $x=x_1$ $\partial_x w+\beta_2 w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t)=-\mathcal{G}(x,x_1,t)$ $\Lambda_2(x,t)=\mathcal{G}(x,x_2,t)$
Mixed boundary value problem ($\alpha_1=\beta_2=0,\ \alpha_2=\beta_1=1$)	$w=g_1(t)$ at $x=x_1$ $\partial_x w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t)=\partial_\xi\mathcal{G}(x,\xi,t)\big\|_{\xi=x_1}$ $\Lambda_2(x,t)=\mathcal{G}(x,x_2,t)$
Mixed boundary value problem ($\alpha_1=\beta_2=1,\ \alpha_2=\beta_1=0$)	$\partial_x w=g_1(t)$ at $x=x_1$ $w=g_2(t)$ at $x=x_2$	$\Lambda_1(x,t)=-\mathcal{G}(x,x_1,t)$ $\Lambda_2(x,t)=-\partial_\xi\mathcal{G}(x,\xi,t)\big\|_{\xi=x_2}$

► **Properties of the Sturm–Liouville problem (17.1.2.4). Heat equation example.**

1°. There are infinitely many eigenvalues. All eigenvalues are real and distinct and can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$ (therefore, there can exist only finitely many negative eigenvalues). Each eigenvalue is of multiplicity 1.

2°. Distinct eigenfunctions $y_n(x)$ and $y_m(x)$ are orthogonal with weight $s(x)$ on the interval $x_1 \leq x \leq x_2$:

$$\int_{x_1}^{x_2} s(x) y_n(x) y_m(x)\, dx = 0 \quad \text{for} \quad n \neq m.$$

3°. If the conditions

$$q(x) \geq 0, \quad \alpha_1\beta_1 \leq 0, \quad \alpha_2\beta_2 \geq 0 \qquad (17.1.2.5)$$

are satisfied, there are no negative eigenvalues. If $q \equiv 0$ and $\beta_1 = \beta_2 = 0$, then $\lambda_1 = 0$ is the least eigenvalue, with the corresponding eigenfunction $\varphi_1 = \text{const}$. Otherwise, all eigenvalues are positive, provided that conditions (17.1.2.5) are satisfied.

Other general and special properties of the Sturm–Liouville problem (17.1.2.4) are given in Section 3.8.9; various asymptotic and approximate formulas for the eigenvalues and eigenfunctions can be found there as well.

Example 17.1. Consider the first boundary value problem in the domain $0 \leq x \leq l$ for the heat equation with a source

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} - bw$$

under the initial condition (17.1.1.3) and boundary conditions

$$\begin{aligned} w &= g_1(t) \quad \text{at} \quad x = 0,\\ w &= g_2(t) \quad \text{at} \quad x = l. \end{aligned} \qquad (17.1.2.6)$$

The above equation is a special case of Eq. (17.1.2.1) with $s(x) = 1$, $p(x) = a$, $q(x) = b$, and $\Phi(x,t) = 0$. The corresponding Sturm–Liouville problem (17.1.2.4) has the form

$$ay''_{xx} + (\lambda - b)y = 0, \qquad y = 0 \quad \text{at} \quad x = 0, \qquad y = 0 \quad \text{at} \quad x = l.$$

The eigenfunctions and eigenvalues are found to be

$$y_n(x) = \sin\left(\frac{\pi n x}{l}\right), \quad \lambda_n = b + \frac{a\pi^2 n^2}{l^2}, \quad n = 1,\, 2,\, \ldots$$

Using formula (17.1.2.3) and taking into account the fact that $\|y_n\|^2 = l/2$, we obtain the Green's function

$$\mathcal{G}(x,\xi,t) = \frac{2}{l}e^{-bt}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n \xi}{l}\right)\exp\left(-\frac{a\pi^2 n^2}{l^2}t\right).$$

By substituting this expression into (17.1.2.2) with $p(x_1) = p(x_2) = s(\xi) = 1$, $x_1 = 0$, and $x_2 = l$ and by taking into account the formulas

$$\Lambda_1(x,t) = \partial_\xi \mathcal{G}(x,\xi,t)\big|_{\xi=x_1}, \quad \Lambda_2(x,t) = -\partial_\xi \mathcal{G}(x,\xi,t)\big|_{\xi=x_2}$$

(see the first row in Table 17.3), one obtains the solution of the problem in question.

◆ *The solutions of various boundary value problems for parabolic equations with one space variable can be found in Chapter 3.*

⊙ *Literature for Section 17.1*: P. M. Morse and H. Feshbach (1953, Vol. 2), S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), H. S. Carslaw and J. C. Jaeger (1984), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), I. G. Petrovsky (1991), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

17.2 Boundary Value Problems for Hyperbolic Equations with One Space Variable. Green's Function. Goursat Problem

17.2.1 Representation of Solutions via the Green's Function

► **Statement of the problem ($t \geq 0, x_1 \leq x \leq x_2$).**

In general, a one-dimensional nonhomogeneous linear differential equation of hyperbolic type with variable coefficients is written as

$$\frac{\partial^2 w}{\partial t^2} + \sigma(x,t)\frac{\partial w}{\partial t} - L_x[w] = \Phi(x,t), \tag{17.2.1.1}$$

where the operator $L_x[w]$ is defined by (17.1.1.2).

Consider the nonstationary boundary value problem for Eq. (17.2.1.1) with the initial conditions

$$\begin{aligned} w &= f_0(x) \quad \text{at} \quad t = 0, \\ \partial_t w &= f_1(x) \quad \text{at} \quad t = 0 \end{aligned} \tag{17.2.1.2}$$

and arbitrary nonhomogeneous linear boundary conditions

$$\alpha_1 \frac{\partial w}{\partial x} + \beta_1 w = g_1(t) \quad \text{at} \quad x = x_1, \tag{17.2.1.3}$$

$$\alpha_2 \frac{\partial w}{\partial x} + \beta_2 w = g_2(t) \quad \text{at} \quad x = x_2. \tag{17.2.1.4}$$

► **Representation of the problem solution in terms of the Green's function.**

The solution of problem (17.2.1.1)–(17.2.1.4) can be represented as the sum

$$\begin{aligned} w(x,t) &= \int_0^t \int_{x_1}^{x_2} \Phi(y,\tau) G(x,y,t,\tau)\, dy\, d\tau \\ &\quad - \int_{x_1}^{x_2} f_0(y) \left[\frac{\partial}{\partial \tau} G(x,y,t,\tau)\right]_{\tau=0} dy + \int_{x_1}^{x_2} \bigl[f_1(y) + f_0(y)\sigma(y,0)\bigr] G(x,y,t,0)\, dy \\ &\quad + \int_0^t g_1(\tau) a(x_1,\tau) \Lambda_1(x,t,\tau)\, d\tau + \int_0^t g_2(\tau) a(x_2,\tau) \Lambda_2(x,t,\tau)\, d\tau. \end{aligned} \tag{17.2.1.5}$$

Here the Green's function $G(x,y,t,\tau)$ is determined by solving the homogeneous equation

$$\frac{\partial^2 G}{\partial t^2} + \sigma(x,t)\frac{\partial G}{\partial t} - L_x[G] = 0 \tag{17.2.1.6}$$

with the semihomogeneous initial conditions

$$G = 0 \qquad \text{at} \quad t = \tau, \tag{17.2.1.7}$$

$$G_t = \delta(x-y) \quad \text{at} \quad t = \tau, \tag{17.2.1.8}$$

and the homogeneous boundary conditions

$$\alpha_1 \frac{\partial G}{\partial x} + \beta_1 G = 0 \quad \text{at} \quad x = x_1, \tag{17.2.1.9}$$

$$\alpha_2 \frac{\partial G}{\partial x} + \beta_2 G = 0 \quad \text{at} \quad x = x_2. \tag{17.2.1.10}$$

The quantities y and τ appear in problem (17.2.1.6)–(17.2.1.10) as free parameters ($x_1 \le y \le x_2$), and $\delta(x)$ is the Dirac delta function.

The functions $\Lambda_1(x,t,\tau)$ and $\Lambda_2(x,t,\tau)$ involved in the integrands of the last two terms in the solution (17.2.1.5) can be expressed via the Green's function $G(x,y,t,\tau)$. The corresponding formulas for $\Lambda_m(x,t,\tau)$ are given in Table 17.1 for the main types of boundary value problems.

It is significant that the Green's function G and Λ_1, Λ_2 are independent of the functions Φ, f_0, f_1, g_1, and g_2 that characterize various inhomogeneities of the boundary value problem.

If the coefficients of Eq. (17.2.1.1) are independent of time t, then the Green's function depends on only three arguments, $G(x,y,t,\tau) = G(x,y,t-\tau)$. In this case, one can set $\frac{\partial}{\partial \tau} G(x,y,t,\tau)\big|_{\tau=0} = -\frac{\partial}{\partial t} G(x,y,t)$ in the solution (17.2.1.5).

► **Formulas for the calculation of Green's functions.**

Consider the hyperbolic equation of the special form

$$\frac{\partial^2 w}{\partial t^2} + \beta(t)\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + \big[c(x) + \gamma(t)\big]w + \Phi(x,t), \tag{17.2.1.11}$$

which is a special case of Eq. (17.2.1.1) with the operator (17.1.1.2) where $\sigma(x,t) = \beta(t)$, $a(x,t) = a(x)$, $b(x,t) = b(x)$, and $c(x,t) = c(x) + \gamma(t)$. We assume that $a(x) > 0$.

The solution of the nonhomogeneous linear boundary value problem for Eq. (17.2.1.11) subject to the initial and boundary conditions (17.2.1.2)–(17.2.1.4) is found by formula (17.2.1.5), where the Green's function is given by

$$G(x,y,t,\tau) = \rho(y) \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(y)}{\|\varphi_n\|^2} \psi_n(t,\tau). \tag{17.2.1.12}$$

Here the λ_n and $\varphi_n(x)$ are the eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem for the linear ordinary differential equation (15.1.1.6) with the homogeneous linear boundary conditions (15.1.1.8) for $s_n = \alpha_n$ and $k_n = \beta_n$; $\rho(y)$ and $\|\varphi_n\|$ are determined by (17.1.1.14), and $\psi = \psi_n(t,\tau)$ is the solution of Eq. (15.1.1.7) with $\alpha(t) = 1$ and $\lambda = \lambda_n$ that satisfies the initial conditions

$$\psi = 0 \quad \text{at} \quad t = \tau, \qquad \psi'_t = 1 \quad \text{at} \quad t = \tau.$$

In the special case $\beta(t) = \gamma(t) = 0$, we have $\psi_n(t,\tau) = \lambda_n^{-1/2} \sin\big[\lambda_n^{1/2}(t-\tau)\big]$.

Table 17.4 lists Green's functions for some problems for the nonhomogeneous wave equation, which corresponds to $a(x) = a^2 = \text{const}$, $b(x) = c(x) = \beta(t) = \gamma(t) \equiv 0$ in (17.2.1.11). One should substitute $G(x,y,t-\tau)$ from this table for $G(x,y,t,\tau)$ into formula (17.2.1.5).

TABLE 17.4
Green's functions for some boundary value problems for the nonhomogeneous wave equation $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x,t)$ in the bounded domain $0 \le x \le l$

Type of problem	Green's function
First boundary value problem	$G(x,\xi,t) = \dfrac{2}{a\pi}\sum\limits_{n=1}^{\infty}\dfrac{1}{n}\sin\Bigl(\dfrac{n\pi x}{l}\Bigr)\sin\Bigl(\dfrac{n\pi\xi}{l}\Bigr)\sin\Bigl(\dfrac{n\pi at}{l}\Bigr)$
Second boundary value problem	$G(x,\xi,t) = \dfrac{t}{l} + \dfrac{2}{a\pi}\sum\limits_{n=1}^{\infty}\dfrac{1}{n}\cos\Bigl(\dfrac{n\pi x}{l}\Bigr)\cos\Bigl(\dfrac{n\pi\xi}{l}\Bigr)\sin\Bigl(\dfrac{n\pi at}{l}\Bigr)$
Mixed boundary value problem; w is set at $x=0$ and $\partial_x w$ is set at $x=l$	$G(x,\xi,t) = \dfrac{2}{al}\sum\limits_{n=1}^{\infty}\dfrac{1}{\lambda_n}\sin(\lambda_n x)\sin(\lambda_n\xi)\sin(\lambda_n at),\quad \lambda_n = \dfrac{\pi(2n+1)}{2l}$

17.2.2 Problems for Equation $s(x)\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\Bigl[p(x)\dfrac{\partial w}{\partial x}\Bigr] - q(x)w + \Phi(x,t)$

► **General formulas for solving nonhomogeneous boundary value problems.**

Consider linear equations of the special form

$$s(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\Bigl[p(x)\frac{\partial w}{\partial x}\Bigr] - q(x)w + \Phi(x,t). \tag{17.2.2.1}$$

It is assumed that the functions s, p, p'_x, and q are continuous and the inequalities $s>0$, $p>0$ hold for $x_1 \le x \le x_2$.

The solution of Eq. (17.2.2.1) under the general initial conditions (17.2.1.2) and the arbitrary linear nonhomogeneous boundary conditions (17.2.1.3)–(17.2.1.4) can be represented as the sum

$$\begin{aligned} w(x,t) &= \int_0^t\int_{x_1}^{x_2}\Phi(\xi,\tau)\mathcal{G}(x,\xi,t-\tau)\,d\xi\,d\tau \\ &\quad + \frac{\partial}{\partial t}\int_{x_1}^{x_2} s(\xi)f_0(\xi)\mathcal{G}(x,\xi,t)\,d\xi + \int_{x_1}^{x_2} s(\xi)f_1(\xi)\mathcal{G}(x,\xi,t)\,d\xi \\ &\quad + p(x_1)\int_0^t g_1(\tau)\Lambda_1(x,t-\tau)\,d\tau + p(x_2)\int_0^t g_2(\tau)\Lambda_2(x,t-\tau)\,d\tau. \end{aligned} \tag{17.2.2.2}$$

Here the modified Green's function is determined by

$$\mathcal{G}(x,\xi,t) = \sum_{n=1}^{\infty}\frac{y_n(x)y_n(\xi)\sin\bigl(t\sqrt{\lambda_n}\,\bigr)}{\|y_n\|^2\sqrt{\lambda_n}}, \qquad \|y_n\|^2 = \int_{x_1}^{x_2} s(x)y_n^2(x)\,dx, \tag{17.2.2.3}$$

where the λ_n and $y_n(x)$ are the eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem for the second-order linear ordinary differential equation

$$\begin{gathered} [p(x)y'_x]'_x + [\lambda s(x) - q(x)]y = 0, \\ \alpha_1 y'_x + \beta_1 y = 0 \quad \text{at} \quad x = x_1, \\ \alpha_2 y'_x + \beta_2 y = 0 \quad \text{at} \quad x = x_2. \end{gathered} \tag{17.2.2.4}$$

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ that occur in the integrands of the last two terms in solution (17.2.2.2) can be expressed via the Green's function of (17.2.2.3). The corresponding formulas for $\Lambda_m(x,t)$ are given in Table 17.3 for the basic types of boundary value problems.

► **Properties of the Sturm–Liouville problem. The Klein–Gordon equation.**

The general and special properties of the Sturm–Liouville problem (17.2.2.4) are given in Section 3.8.9; various asymptotic and approximate formulas for the eigenvalues and eigenfunctions can be found there as well.

Example 17.2. Consider the second boundary value problem in the domain $0 \le x \le l$ for the Klein–Gordon equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw$$

under the initial conditions (17.2.1.2) and the boundary conditions

$$\partial_x w = g_1(t) \quad \text{at} \quad x = 0,$$
$$\partial_x w = g_2(t) \quad \text{at} \quad x = l.$$

The Klein–Gordon equation is a special case of Eq. (17.2.2.1) with $s(x) = 1$, $p(x) = a^2$, $q(x) = b$, and $\Phi(x,t) = 0$. The corresponding Sturm–Liouville problem (17.2.2.4) has the form

$$a^2 y''_{xx} + (\lambda - b)y = 0, \qquad y'_x = 0 \quad \text{at} \quad x = 0, \qquad y'_x = 0 \quad \text{at} \quad x = l.$$

The eigenfunctions and eigenvalues are found to be

$$y_{n+1}(x) = \cos\left(\frac{\pi n x}{l}\right), \qquad \lambda_{n+1} = b + \frac{a\pi^2 n^2}{l^2}, \qquad n = 0,\ 1,\ \ldots$$

By using formula (17.2.2.4) and by taking into account the fact that $\|y_1\|^2 = l$ and $\|y_n\|^2 = l/2$ ($n = 1,\ 2,\ \ldots$), we find the Green's function

$$\mathcal{G}(x,\xi,t) = \frac{1}{l\sqrt{b}} \sin\bigl(t\sqrt{b}\,\bigr) + \frac{2}{l} \sum_{n=1}^{\infty} \cos\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi n \xi}{l}\right) \frac{\sin\bigl(t\sqrt{(a\pi n/l)^2 + b}\,\bigr)}{\sqrt{(a\pi n/l)^2 + b}}.$$

By substituting this expression into (17.2.2.2) with $p(x_1) = p(x_2) = s(\xi) = 1$, $x_1 = 0$, and $x_2 = l$ and by taking into account the formulas

$$\Lambda_1(x,t) = -\mathcal{G}(x,x_1,t), \qquad \Lambda_2(x,t) = \mathcal{G}(x,x_2,t)$$

(see the second row in Table 17.3), one obtains the solution of the problem in question.

◆ *Solutions of various boundary value problems for hyperbolic equations with one space variable can be found in Chapter 6.*

17.2.3 Problems for Equation $\dfrac{\partial^2 w}{\partial t^2} + a(t)\dfrac{\partial w}{\partial t} = b(t)\Big\{\dfrac{\partial}{\partial x}\Big[p(x)\dfrac{\partial w}{\partial x}\Big] - q(x)w\Big\} + \Phi(x,t)$

► **General formulas for solving nonhomogeneous boundary value problems.**

Consider the *generalized telegraph equation* of the form

$$\frac{\partial^2 w}{\partial t^2} + a(t)\frac{\partial w}{\partial t} = b(t)\left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right] - q(x)w\right\} + \Phi(x,t). \tag{17.2.3.1}$$

It is assumed that the functions p, p'_x, and q are continuous and $p > 0$ for $x_1 \le x \le x_2$.

The solution of Eq. (17.2.3.1) under the general initial conditions (17.2.1.2) and the arbitrary linear nonhomogeneous boundary conditions (17.2.1.3)–(17.2.1.4) can be represented as the sum

$$\begin{aligned} w(x,t) &= \int_0^t \int_{x_1}^{x_2} \Phi(\xi,\tau) G(x,\xi,t,\tau)\, d\xi\, d\tau \\ &\quad - \int_{x_1}^{x_2} f_0(\xi) \left[\frac{\partial}{\partial \tau} G(x,\xi,t,\tau) \right]_{\tau=0} d\xi + \int_{x_1}^{x_2} \big[f_1(\xi) + a(0) f_0(\xi) \big] G(x,\xi,t,0)\, d\xi \\ &\quad + p(x_1) \int_0^t g_1(\tau) b(\tau) \Lambda_1(x,t,\tau)\, d\tau + p(x_2) \int_0^t g_2(\tau) b(\tau) \Lambda_2(x,t,\tau)\, d\tau. \end{aligned} \tag{17.2.3.2}$$

Here the Green's function is determined by

$$G(x,\xi,t,\tau) = \sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{\|y_n\|^2} U_n(t,\tau), \qquad \|y_n\|^2 = \int_{x_1}^{x_2} y_n^2(x)\, dx, \tag{17.2.3.3}$$

where the λ_n and $y_n(x)$ are the eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem for the following second-order linear ordinary differential equation with homogeneous boundary conditions:

$$\begin{gathered} [p(x) y'_x]'_x + [\lambda - q(x)] y = 0, \\ \alpha_1 y'_x + \beta_1 y = 0 \quad \text{at} \quad x = x_1, \\ \alpha_2 y'_x + \beta_2 y = 0 \quad \text{at} \quad x = x_2. \end{gathered} \tag{17.2.3.4}$$

The functions $U_n = U_n(t,\tau)$ are determined by solving the Cauchy problem for the linear ordinary differential equation

$$\begin{gathered} U''_n + a(t) U'_n + \lambda_n b(t) U_n = 0, \\ U_n \big|_{t=\tau} = 0, \quad U'_n \big|_{t=\tau} = 1. \end{gathered} \tag{17.2.3.5}$$

The prime denotes the derivative with respect to t, and τ is the free parameter occurring in the initial conditions.

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ that occur in the integrands of the last two terms in solution (17.2.3.2) can be expressed via the Green's function of (17.2.3.3). The corresponding formulas will be specified below when studying specific boundary value problems.

The general and special properties of the Sturm–Liouville problem (17.2.3.4) are detailed in Section 3.8.9. Asymptotic and approximate formulas for the eigenvalues and the eigenfunctions can be found there as well.

► First, second, third, and mixed boundary value problems.

1°. *First boundary value problem.* The solution of Eq. (17.2.3.1) with the initial conditions (17.2.1.2) and boundary conditions (17.2.1.3)–(17.2.1.4) for $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$ is given by relations (17.2.3.2) and (17.2.3.3), where

$$\Lambda_1(x,t,\tau) = \frac{\partial}{\partial \xi} G(x,\xi,t,\tau) \Big|_{\xi=x_1}, \quad \Lambda_2(x,t,\tau) = -\frac{\partial}{\partial \xi} G(x,\xi,t,\tau) \Big|_{\xi=x_2}.$$

2°. *Second boundary value problem.* The solution of Eq. (17.2.3.1) with the initial conditions (17.2.1.2) and the boundary conditions (17.2.1.3)–(17.2.1.4) for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$ is given by relations (17.2.3.2) and (17.2.3.3) with

$$\Lambda_1(x,t,\tau) = -G(x,x_1,t,\tau), \quad \Lambda_2(x,t,\tau) = G(x,x_2,t,\tau).$$

3°. *Third boundary value problem.* The solution of Eq. (17.2.3.1) with the initial conditions (17.2.1.2) and the boundary conditions (17.2.1.3)–(17.2.1.4) for $\alpha_1 = \alpha_2 = 1$ and $\beta_1\beta_2 \neq 0$ is given by relations (17.2.3.2) and (17.2.3.3), in which

$$\Lambda_1(x,t,\tau) = -G(x,x_1,t,\tau), \quad \Lambda_2(x,t,\tau) = G(x,x_2,t,\tau).$$

4°. *Mixed boundary value problem.* The solution of Eq. (17.2.3.1) with the initial conditions (17.2.1.2) and the boundary conditions (17.2.1.3)–(17.2.1.4) for $\alpha_1 = \beta_2 = 0$ and $\alpha_2 = \beta_1 = 1$ is given by relations (17.2.3.2) and (17.2.3.3) with

$$\Lambda_1(x,t,\tau) = \frac{\partial}{\partial\xi}G(x,\xi,t,\tau)\Big|_{\xi=x_1}, \quad \Lambda_2(x,t,\tau) = G(x,x_2,t,\tau).$$

5°. *Mixed boundary value problem.* The solution of Eq. (17.2.3.1) with the initial conditions (17.2.1.2) and the boundary conditions (17.2.1.3)–(17.2.1.4) for $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$ is given by relations (17.2.3.2) and (17.2.3.3) with

$$\Lambda_1(x,t,\tau) = -G(x,x_1,t,\tau), \quad \Lambda_2(x,t,\tau) = -\frac{\partial}{\partial\xi}G(x,\xi,t,\tau)\Big|_{\xi=x_2}.$$

17.2.4 Generalized Cauchy Problem with Initial Conditions Set along a Curve. Riemann Function

► **Statement of the generalized Cauchy problem. Basic property of the solution.**

Consider the general linear hyperbolic equation in two independent variables reduced to the first canonical form (see Section 14.1.1)

$$\frac{\partial^2 w}{\partial x\partial y} + a(x,y)\frac{\partial w}{\partial x} + b(x,y)\frac{\partial w}{\partial y} + c(x,y)w = f(x,y), \tag{17.2.4.1}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$, and $f(x,y)$ are continuous functions.

Let a curve segment in the xy-plane be defined by

$$y = \varphi(x) \qquad (\alpha \le x \le \beta), \tag{17.2.4.2}$$

where $\varphi(x)$ is continuously differentiable, with $\varphi'(x) \neq 0$ and $\varphi'(x) \neq \infty$.

The *generalized Cauchy problem* for Eq. (17.2.4.1) with initial conditions defined along the curve (17.2.4.2) is stated as follows: find a solution of Eq. (17.2.4.1) that satisfies the conditions

$$w(x,y)|_{y=\varphi(x)} = g(x), \qquad \frac{\partial w}{\partial x}\Big|_{y=\varphi(x)} = h_1(x), \qquad \frac{\partial w}{\partial y}\Big|_{y=\varphi(x)} = h_2(x), \tag{17.2.4.3}$$

where $g(x)$, $h_1(x)$, and $h_2(x)$ are given continuous functions related by the compatibility condition

$$g'_x(x) = h_1(x) + h_2(x)\varphi'_x(x). \qquad (17.2.4.4)$$

Basic property of the generalized Cauchy problem: the value of the solution at any point $M(x_0, y_0)$ depends only on the values of the functions $g(x)$, $h_1(x)$, and $h_2(x)$ on the arc AB cut off on the given curve (17.2.4.2) by the characteristics $x = x_0$ and $y = y_0$, and on the values of $a(x,y)$, $b(x,y)$, $c(x,y)$, and $f(x,y)$ in the curvilinear triangle AMB; see Fig. 17.1. The influence domain of the solution at $M(x_0, y_0)$ is shaded for clarity.

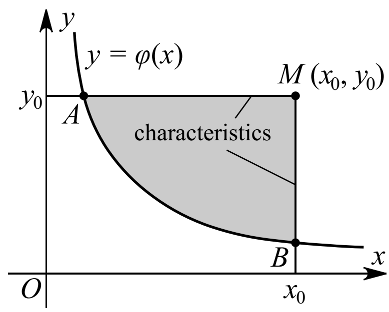


Figure 17.1: Influence domain of the solution of the generalized Cauchy problem at a point M.

Remark 17.2. Rather than setting two derivatives in the boundary conditions (17.2.4.3), it suffices to set either of them, with the other being uniquely determined from the compatibility condition (17.2.4.4).

Remark 17.3. Instead of the last two boundary conditions in (17.2.4.3), the value of the derivative along the normal to the curve (17.2.4.2) can be used,

$$\frac{\partial w}{\partial n}\bigg|_{y=\varphi(x)} \equiv \frac{1}{\sqrt{1+[\varphi'_x(x)]^2}}\left[\frac{\partial w}{\partial y} - \varphi'_x(x)\frac{\partial w}{\partial x}\right]_{y=\varphi(x)} = h_3(x). \qquad (17.2.4.5)$$

Denoting $w_x|_{y=\varphi(x)} = h_1(x)$ and $w_y|_{y=\varphi(x)} = h_2(x)$, we have

$$h_2(x) - \varphi'_x(x)h_1(x) = h_3(x)\sqrt{1+[\varphi'_x(x)]^2}. \qquad (17.2.4.6)$$

The functions $h_1(x)$ and $h_2(x)$ can be found from (17.2.4.4) and (17.2.4.6). Further, by substituting their expressions into (17.2.4.3), one arrives at the standard statement of the generalized Cauchy problem, where the compatibility condition for the initial data (17.2.4.4) will be satisfied automatically.

▶ Riemann function.

The *Riemann function* $\mathcal{R} = \mathcal{R}(x, y; x_0, y_0)$ corresponding to Eq. (17.2.4.1) is defined as the solution of the equation

$$\frac{\partial^2 \mathcal{R}}{\partial x \partial y} - \frac{\partial}{\partial x}\Big[a(x,y)\mathcal{R}\Big] - \frac{\partial}{\partial y}\Big[b(x,y)\mathcal{R}\Big] + c(x,y)\mathcal{R} = 0 \qquad (17.2.4.7)$$

with the conditions

$$\mathcal{R} = \exp\left[\int_{y_0}^{y} a(x_0,\xi)\,d\xi\right] \quad \text{at} \quad x = x_0, \qquad \mathcal{R} = \exp\left[\int_{x_0}^{x} b(\xi,y_0)\,d\xi\right] \quad \text{at} \quad y = y_0 \tag{17.2.4.8}$$

on the characteristics $x=x_0$ and $y=y_0$. Here (x_0, y_0) is an arbitrary point in the domain of Eq. (17.2.4.1). The points x_0 and y_0 appear in problem (17.2.4.7)–(17.2.4.8) as parameters in the boundary conditions only.

THEOREM. *If the functions a, b, c and the partial derivatives a_x, b_y are continuous, then the Riemann function $\mathcal{R}(x,y;x_0,y_0)$ exists. Moreover, the function $\mathcal{R}(x_0,y_0,x,y)$ obtained by swapping the parameters and the arguments is a solution of the homogeneous equation (17.2.4.1) with $f=0$.*

Remark 17.4. It is significant that the Riemann function depends neither on the shape of the curve (17.2.4.2) nor on the initial data set on it (17.2.4.3).

Example 17.3. The Riemann function for the equation $w_{xy}=0$ is just $\mathcal{R}\equiv 1$.

Example 17.4. The Riemann function for the equation

$$w_{xy} + cw = 0 \qquad (c = \text{const}) \tag{17.2.4.9}$$

can be expressed via the Bessel function $J_0(z)$ as

$$\mathcal{R} = J_0\big(\sqrt{4c(x_0-x)(y_0-y)}\,\big).$$

Remark 17.5. Any linear constant coefficient partial differential equation of the hyperbolic type in two independent variables can be reduced to an equation of the form (17.2.4.9); see the end of Section 14.1.1.

► Solution of the generalized Cauchy problem via the Riemann function.

Given the Riemann function, the solution of the generalized Cauchy problem (17.2.4.1)–(17.2.4.3) at any point (x_0, y_0) can written as

$$\begin{aligned} w(x_0,y_0) = \frac{1}{2}(w\mathcal{R})_A + \frac{1}{2}(w\mathcal{R})_B + \frac{1}{2}\int_{AB}\left(\mathcal{R}\frac{\partial w}{\partial x} - w\frac{\partial \mathcal{R}}{\partial x} + 2bw\mathcal{R}\right)dx \\ - \frac{1}{2}\int_{AB}\left(\mathcal{R}\frac{\partial w}{\partial y} - w\frac{\partial \mathcal{R}}{\partial y} + 2aw\mathcal{R}\right)dy + \iint_{\Delta AMB} f\mathcal{R}\,dx\,dy. \end{aligned}$$

The first two terms on the right-hand side are evaluated at the points A and B (see Fig. 17.1). The third and fourth terms are curvilinear integrals over the arc AB; the arc is defined by Eq. (17.2.4.2), and the integrands involve quantities defined by the initial conditions (17.2.4.3). The last integral is taken over the curvilinear triangular domain AMB.

17.2.5 Goursat Problem (a Problem with Initial Data on Characteristics)

► Statement of the Goursat problem. Basic property of the solution.

The *Goursat problem* for Eq. (17.2.4.1) is stated as follows: find a solution of Eq. (17.2.4.1) with the conditions

$$w(x,y)|_{x=x_1} = g(y), \qquad w(x,y)|_{y=y_1} = h(x) \tag{17.2.5.1}$$

on the characteristics, where $g(y)$ and $h(x)$ are given continuous functions that match each other at the point of intersection of the characteristics, so that

$$g(y_1) = h(x_1).$$

Basic properties of the Goursat problem: the value of the solution at a point $M(x_0, y_0)$ depends only on the values of $g(y)$ at the segment AN (which is part of the characteristic $x = x_1$), the values of $h(x)$ at the segment BN (which is part of the characteristic $y = y_1$), and the values of the functions $a(x,y)$, $b(x,y)$, $c(x,y)$, and $f(x,y)$ in the rectangle $NAMB$; see Fig. 17.2. The influence domain of the solution at the point $M(x_0, y_0)$ is shaded for clarity.

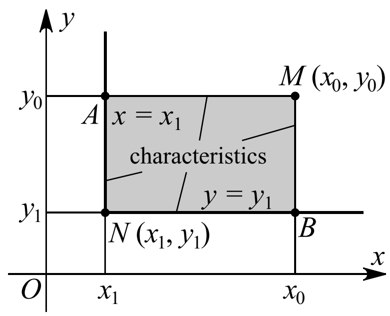


Figure 17.2: Influence domain of the solution of the Goursat problem at a point M.

► Solution representation for the Goursat problem via the Riemann function.

Given the Riemann function (see Section 17.2.4), the solution of the Goursat problem (17.2.4.1), (17.2.5.1) at any point (x_0, y_0) can be written as

$$w(x_0, y_0) = (w\mathcal{R})_N + \int_N^A \mathcal{R}\bigl(g'_y + bg\bigr)\,dy + \int_N^B \mathcal{R}\bigl(h'_x + ah\bigr)\,dx + \iint_{NAMB} f\mathcal{R}\,dx\,dy.$$

The first term on the right-hand side is evaluated at the point of intersection of the characteristics (x_1, y_1). The second and third terms are integrals along the characteristics $y = y_1$ $(x_1 \le x \le x_0)$ and $x = x_1$ $(y_1 \le y \le y_0)$; these involve the initial data of (17.2.5.1). The last integral is taken over the rectangular domain $NAMB$ defined by the inequalities $x_1 \le x \le x_0$, $y_1 \le y \le y_0$.

The Goursat problem for hyperbolic equations reduced to the second canonical form (see Section 14.1.1) can be treated in a similar way.

Example 17.5. Consider the Goursat problem for the wave equation

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0$$

with the boundary conditions prescribed on its characteristics

$$\begin{aligned} w &= f(x) \quad \text{for} \quad x - at = 0 \quad (0 \le x \le b),\\ w &= g(x) \quad \text{for} \quad x + at = 0 \quad (0 \le x \le c), \end{aligned} \tag{17.2.5.2}$$

where $f(0) = g(0)$.

By substituting the values set on the characteristics (17.2.5.2) into the general solution $w = \varphi(x-at)+\psi(x+at)$ of the wave equation, we arrive at a system of linear algebraic equations for $\varphi(x)$ and $\psi(x)$. As a result, the solution of the Goursat problem is obtained in the form

$$w(x,t) = f\left(\frac{x+at}{2}\right) + g\left(\frac{x-at}{2}\right) - f(0).$$

The solution propagation domain is the parallelogram bounded by the four lines

$$x-at=0, \quad x+at=0, \quad x-at=2c, \quad x+at=2b.$$

⊙ *Literature for Section 17.2*: P. M. Morse and H. Feshbach (1953, Vol. 2), S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), R. Courant and D. Hilbert (1989), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), I. G. Petrovsky (1991), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

17.3 Boundary Value Problems for Elliptic Equations with Two Space Variables

17.3.1 Problems and the Green's Functions for Equation $a(x)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + b(x)\frac{\partial w}{\partial x} + c(x)w = -\Phi(x,y)$

► **Statements of boundary value problems.**

Consider 2D boundary value problems for the equation

$$a(x)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + b(x)\frac{\partial w}{\partial x} + c(x)w = -\Phi(x,y) \qquad (17.3.1.1)$$

with general boundary conditions in x,

$$\begin{aligned} \alpha_1\frac{\partial w}{\partial x} - \beta_1 w &= f_1(y) \quad \text{at} \quad x=x_1,\\ \alpha_2\frac{\partial w}{\partial x} + \beta_2 w &= f_2(y) \quad \text{at} \quad x=x_2, \end{aligned} \qquad (17.3.1.2)$$

and various boundary conditions in y. It is assumed that the coefficients of Eq. (17.3.1.1) and the boundary conditions (17.3.1.2) meet the requirements

$$a(x),\ b(x),\ c(x) \text{ are continuous } (x_1 \le x \le x_2); \quad a>0, \quad |\alpha_1|+|\beta_1|>0, \quad |\alpha_2|+|\beta_2|>0.$$

► **Relations for the Green's function.**

In the general case, the Green's function can be represented as

$$G(x,y,\xi,\eta) = \rho(\xi)\sum_{n=1}^{\infty}\frac{u_n(x)u_n(\xi)}{\|u_n\|^2}\Psi_n(y,\eta;\lambda_n). \qquad (17.3.1.3)$$

Here

$$\rho(x)=\frac{1}{a(x)}\exp\left[\int\frac{b(x)}{a(x)}\,dx\right],\qquad \|u_n\|^2=\int_{x_1}^{x_2}\rho(x)u_n^2(x)\,dx, \tag{17.3.1.4}$$

and the λ_n and $u_n(x)$ are the eigenvalues and eigenfunctions of the homogeneous boundary value problem for the ordinary differential equation

$$a(x)u''_{xx}+b(x)u'_x+[\lambda+c(x)]u=0, \tag{17.3.1.5}$$

$$\alpha_1u'_x-\beta_1u=0\quad\text{at}\quad x=x_1, \tag{17.3.1.6}$$

$$\alpha_2u'_x+\beta_2u=0\quad\text{at}\quad x=x_2. \tag{17.3.1.7}$$

The functions $\Psi_n=\Psi_n(y,\eta;\lambda_n)$ for some boundary conditions in y are specified in Table 17.5 (see also the extended Table 9.5). For unbounded domains, the condition of boundedness of the solution as $y\to\pm\infty$ is set.

TABLE 17.5
The functions Ψ_n in (17.3.1.3) for various boundary conditions. Notation: $\sigma_n=\sqrt{\lambda_n}$

Domain	Boundary conditions	Function $\Psi_n(y,\eta;\lambda_n)$
$-\infty<y<\infty$	$\lvert w\rvert<\infty$ as $y\to\pm\infty$	$\dfrac{1}{2\sigma_n}e^{-\sigma_n\lvert y-\eta\rvert}$
$0\le y<\infty$	$w=0$ at $y=0$, $\lvert w\rvert<\infty$ as $y\to\infty$	$\dfrac{1}{\sigma_n}\begin{cases}e^{-\sigma_ny}\sinh(\sigma_n\eta) & \text{if } y>\eta,\\ e^{-\sigma_n\eta}\sinh(\sigma_ny) & \text{if } \eta>y\end{cases}$
$0\le y<\infty$	$\partial_yw=0$ at $y=0$, $\lvert w\rvert<\infty$ as $y\to\infty$	$\dfrac{1}{\sigma_n}\begin{cases}e^{-\sigma_ny}\cosh(\sigma_n\eta) & \text{if } y>\eta,\\ e^{-\sigma_n\eta}\cosh(\sigma_ny) & \text{if } \eta>y\end{cases}$
$0\le y\le h$	$w=0$ at $y=0$, $w=0$ at $y=h$	$\dfrac{1}{\sigma_n\sinh(\sigma_nh)}\begin{cases}\sinh(\sigma_n\eta)\sinh[\sigma_n(h-y)] & \text{if } y>\eta,\\ \sinh(\sigma_ny)\sinh[\sigma_n(h-\eta)] & \text{if } \eta>y\end{cases}$
$0\le y\le h$	$\partial_yw=0$ at $y=0$, $\partial_yw=0$ at $y=h$	$\dfrac{1}{\sigma_n\sinh(\sigma_nh)}\begin{cases}\cosh(\sigma_n\eta)\cosh[\sigma_n(h-y)] & \text{if } y>\eta,\\ \cosh(\sigma_ny)\cosh[\sigma_n(h-\eta)] & \text{if } \eta>y\end{cases}$

Equation (17.3.1.5) can be rewritten in self-adjoint form as

$$[p(x)u'_x]'_x+[\lambda\rho(x)-q(x)]u=0, \tag{17.3.1.8}$$

where the functions $p(x)$ and $q(x)$ are given by

$$p(x)=\exp\left[\int\frac{b(x)}{a(x)}\,dx\right],\quad q(x)=-\frac{c(x)}{a(x)}\exp\left[\int\frac{b(x)}{a(x)}\,dx\right]$$

and $\rho(x)$ is defined in (17.3.1.4).

The eigenvalue problem (17.3.1.8), (17.3.1.6), (17.3.1.7) possesses the following properties:

1°. All eigenvalues $\lambda_1,\ \lambda_2,\ \ldots$ are real and $\lambda_n\to\infty$ as $n\to\infty$.

2°. The system of eigenfunctions $\{u_1(x), u_2(x), \dots\}$ is orthogonal on the interval $x_1 \le x \le x_2$ with weight $\rho(x)$; that is,

$$\int_{x_1}^{x_2} \rho(x) u_n(x) u_m(x)\, dx = 0 \quad \text{for} \quad n \ne m.$$

3°. If the conditions

$$q(x) \ge 0, \quad \alpha_1\beta_1 \ge 0, \quad \alpha_2\beta_2 \ge 0 \qquad (17.3.1.9)$$

are satisfied, there are no negative eigenvalues. If $q \equiv 0$ and $\beta_1 = \beta_2 = 0$, then the least eigenvalue is $\lambda_0 = 0$ and the corresponding eigenfunction is $u_0 = \text{const}$; in this case, the summation in (17.3.1.3) should start from $n = 0$. In the other cases, if conditions (17.3.1.9) are satisfied, all eigenvalues are positive; for example, the first inequality in (17.3.1.9) holds if $c(x) \le 0$.

Section 3.8.9 presents some relations for estimating the eigenvalues λ_n and the eigenfunctions $u_n(x)$.

Example 17.6. Consider the boundary value problem for the Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

in the strip $0 \le x \le l$, $-\infty < y < \infty$ with the mixed boundary conditions

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad \frac{\partial w}{\partial x} = f_2(y) \quad \text{at} \quad x = l.$$

This equation is the special case of Eq. (17.3.1.1) with $a(x) = 1$ and $b(x) = c(x) = \Phi(x,t) = 0$. The corresponding Sturm–Liouville problem (17.3.1.5)–(17.3.1.7) is written as

$$u''_{xx} + \lambda y = 0, \qquad u = 0 \quad \text{at} \quad x = 0, \qquad u'_x = 0 \quad \text{at} \quad x = l.$$

The eigenfunctions and eigenvalues are found as

$$u_n(x) = \sin\left[\frac{\pi(2n-1)x}{l}\right], \qquad \lambda_n = \frac{\pi^2(2n-1)^2}{l^2}, \qquad n = 1,\, 2,\, \dots$$

By using formulas (17.3.1.3) and (17.3.1.4) and by taking into account the identities $\rho(\xi) = 1$ and $\|y_n\|^2 = l/2$ $(n = 1,\, 2,\, \dots)$ and the expression for Ψ_n in the first row in Table 17.5, we obtain the Green's function in the form

$$G(x,y,\xi,\eta) = \frac{1}{l}\sum_{n=1}^{\infty} \frac{1}{\sigma_n} \sin(\sigma_n x)\sin(\sigma_n \xi) e^{-\sigma_n|y-\eta|}, \qquad \sigma_n = \sqrt{\lambda_n} = \frac{\pi(2n-1)}{l}.$$

17.3.2 Representation of Solutions of Boundary Value Problems via Green's Functions

► First boundary value problem.

The solution of the first boundary value problem for Eq. (17.3.1.1) with the boundary conditions

$$w = f_1(y) \quad \text{at} \quad x = x_1, \qquad w = f_2(y) \quad \text{at} \quad x = x_2,$$
$$w = f_3(x) \quad \text{at} \quad y = 0, \qquad w = f_4(x) \quad \text{at} \quad y = h$$

can be expressed via the Green's function (17.3.1.3) as

$$
\begin{aligned}
w(x,y) &= a(x_1)\int_0^h f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=x_1} d\eta \\
&\quad - a(x_2)\int_0^h f_2(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=x_2} d\eta \\
&\quad + \int_{x_1}^{x_2} f_3(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi - \int_{x_1}^{x_2} f_4(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=h} d\xi \\
&\quad + \int_{x_1}^{x_2}\int_0^h \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi.
\end{aligned}
$$

► **Second boundary value problem.**

The solution of the second boundary value problem for Eq. (17.3.1.1) with the boundary conditions

$$
\begin{aligned}
&\partial_x w = f_1(y) \quad \text{at} \quad x = x_1, \qquad \partial_x w = f_2(y) \quad \text{at} \quad x = x_2, \\
&\partial_y w = f_3(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = f_4(x) \quad \text{at} \quad y = h
\end{aligned}
$$

can be expressed via the Green's function (17.3.1.3) as

$$
\begin{aligned}
w(x,y) &= -a(x_1)\int_0^h f_1(\eta)G(x,y,x_1,\eta)\,d\eta + a(x_2)\int_0^h f_2(\eta)G(x,y,x_2,\eta)\,d\eta \\
&\quad - \int_{x_1}^{x_2} f_3(\xi)G(x,y,\xi,0)\,d\xi + \int_{x_1}^{x_2} f_4(\xi)G(x,y,\xi,h)\,d\xi \\
&\quad + \int_{x_1}^{x_2}\int_0^h \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi.
\end{aligned}
$$

► **Third boundary value problem.**

The solution of the third boundary value problem for Eq. (17.3.1.1) can be represented via the Green's function in the same way as the solution of the second boundary value problem. (The Green's function is now different.)

◆ *The solutions of various boundary value problems for elliptic equations with two space variables can be found in Chapter 9.*

⊙ *Literature for Section 17.3*: P. M. Morse and H. Feshbach (1953, Vol. 2), S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), R. Courant and D. Hilbert (1989), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), I. G. Petrovsky (1991), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

17.4 Boundary Value Problems with Many Space Variables. Representation of Solutions via the Green's Function

17.4.1 Problems for Parabolic Equations

► **Statement of the problem.**

In general, a nonhomogeneous linear differential equation of the parabolic type in n space variables has the form

$$\frac{\partial w}{\partial t} - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t), \tag{17.4.1.1}$$

where

$$L_{\mathbf{x}}[w] \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, t)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}, t)\frac{\partial w}{\partial x_i} + c(\mathbf{x}, t)w,$$
$$\mathbf{x} = \{x_1, \dots, x_n\}, \quad \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, t)\xi_i\xi_j \ge \sigma \sum_{i=1}^{n} \xi_i^2, \quad \sigma > 0. \tag{17.4.1.2}$$

Let V be some simply connected domain in $\mathbb{R}^n$ with sufficiently smooth boundary $S = \partial V$. Consider the nonstationary boundary value problem for Eq. (17.4.1.1) in the domain V with an arbitrary initial condition

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0, \tag{17.4.1.3}$$

and a nonhomogeneous linear boundary condition

$$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in S. \tag{17.4.1.4}$$

In the general case, $\Gamma_{\mathbf{x}}$ is a first-order linear differential operator in the space coordinates with coefficients depending on $\mathbf{x}$ and t.

► **Representation of the problem solution via the Green's function.**

The solution of the nonhomogeneous linear boundary value problem defined by (17.4.1.1)–(17.4.1.4) can be represented as the sum

$$w(\mathbf{x}, t) = \int_0^t \int_V \Phi(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t, \tau)\, dV_y\, d\tau + \int_V f(\mathbf{y}) G(\mathbf{x}, \mathbf{y}, t, 0)\, dV_y$$
$$+ \int_0^t \int_S g(\mathbf{y}, \tau) H(\mathbf{x}, \mathbf{y}, t, \tau)\, dS_y\, d\tau, \tag{17.4.1.5}$$

where $G(\mathbf{x}, \mathbf{y}, t, \tau)$ is the Green's function; for $t > \tau \ge 0$, it satisfies the homogeneous equation

$$\frac{\partial G}{\partial t} - L_{\mathbf{x}}[G] = 0 \tag{17.4.1.6}$$

with the nonhomogeneous initial condition of the special form

$$G = \delta(\mathbf{x} - \mathbf{y}) \quad \text{at} \quad t = \tau \tag{17.4.1.7}$$

and the homogeneous boundary condition

$$\Gamma_{\mathbf{x}}[G] = 0 \quad \text{for} \quad \mathbf{x} \in S. \tag{17.4.1.8}$$

The vector $\mathbf{y} = \{y_1, \dots, y_n\}$ appears in problem (17.4.1.6)–(17.4.1.8) as an n-dimensional free parameter ($\mathbf{y} \in V$), and $\delta(\mathbf{x} - \mathbf{y}) = \delta(x_1 - y_1) \dots \delta(x_n - y_n)$ is the n-dimensional Dirac delta function. The Green's function G is independent of the functions Φ, f, and g that characterize various inhomogeneities of the boundary value problem. In (17.4.1.5), the integration is performed everywhere with respect to $\mathbf{y}$ with $dV_y = dy_1 \dots dy_n$.

The function $H(\mathbf{x}, \mathbf{y}, t, \tau)$ involved in the integrand of the last term in solution (17.4.1.5) can be expressed via the Green's function $G(\mathbf{x}, \mathbf{y}, t, \tau)$. The corresponding formulas for $H(\mathbf{x}, \mathbf{y}, t, \tau)$ are given in Table 17.6 for the three basic types of boundary value problems; in the third boundary value problem, the coefficient k can depend on $\mathbf{x}$ and t. The boundary conditions of the second and third kind, as well as the solution of the first boundary value problem, involve operators of differentiation along the conormal of operator (17.4.1.2); these operators act as follows:

$$\frac{\partial G}{\partial M_x} \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, t) N_j \frac{\partial G}{\partial x_i}, \qquad \frac{\partial G}{\partial M_y} \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{y}, \tau) N_j \frac{\partial G}{\partial y_i}, \tag{17.4.1.9}$$

where $\mathbf{N} = \{N_1, \dots, N_n\}$ is the unit outward normal to the surface S. In the special case where $a_{ii}(\mathbf{x}, t) = 1$ and $a_{ij}(\mathbf{x}, t) = 0$ for $i \neq j$, the operator (17.4.1.9) coincides with the ordinary operator of differentiation along the outward normal to S.

TABLE 17.6
The form of the function $H(\mathbf{x}, \mathbf{y}, t, \tau)$ for the basic types of nonstationary boundary value problems

Type of problem	Form of boundary condition (17.4.1.4)	Function $H(\mathbf{x}, \mathbf{y}, t, \tau)$
First boundary value problem	$w = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$	$H(\mathbf{x}, \mathbf{y}, t, \tau) = -\dfrac{\partial G}{\partial M_y}(\mathbf{x}, \mathbf{y}, t, \tau)$
Second boundary value problem	$\dfrac{\partial w}{\partial M_x} = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$	$H(\mathbf{x}, \mathbf{y}, t, \tau) = G(\mathbf{x}, \mathbf{y}, t, \tau)$
Third boundary value problem	$\dfrac{\partial w}{\partial M_x} + kw = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$	$H(\mathbf{x}, \mathbf{y}, t, \tau) = G(\mathbf{x}, \mathbf{y}, t, \tau)$

If the coefficient of Eq. (17.4.1.6) and the boundary condition (17.4.1.8) are independent of t, then the Green's function depends on only three arguments, $G(\mathbf{x}, \mathbf{y}, t, \tau) = G(\mathbf{x}, \mathbf{y}, t - \tau)$.

Remark 17.6. Let S_i ($i = 1, \dots, p$) be distinct portions of the surface S such that $S = \sum_{i=1}^{p} S_i$, and let boundary conditions of various types be set on the S_i,

$$\Gamma_{\mathbf{x}}^{(i)}[w] = g_i(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in S_i, \qquad i = 1, \dots, p. \tag{17.4.1.10}$$

Then formula (17.4.1.5) remains valid but the last term in (17.4.1.5) should be replaced by the sum

$$\sum_{i=1}^{p} \int_0^t \int_{S_i} g_i(\mathbf{y}, \tau) H_i(\mathbf{x}, \mathbf{y}, t, \tau)\, dS_y\, d\tau. \tag{17.4.1.11}$$

17.4.2 Problems for Hyperbolic Equations

► Statement of the problem.

The general nonhomogeneous linear differential hyperbolic equation in n space variables can be written as

$$\frac{\partial^2 w}{\partial t^2} + \sigma(\mathbf{x}, t) \frac{\partial w}{\partial t} - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t), \tag{17.4.2.1}$$

where the operator $L_{\mathbf{x}}[w]$ is explicitly defined in (17.4.1.2).

Consider the nonstationary boundary value problem for Eq. (17.4.2.1) in the domain V with arbitrary initial conditions

$$w = f_0(\mathbf{x}) \quad \text{at} \quad t = 0, \tag{17.4.2.2}$$

$$\partial_t w = f_1(\mathbf{x}) \quad \text{at} \quad t = 0 \tag{17.4.2.3}$$

and the nonhomogeneous linear boundary condition (17.4.1.4).

► Representation of the solution via the Green's function.

The solution of the nonhomogeneous linear boundary value problem defined by (17.4.2.1)–(17.4.2.3), (17.4.1.4) can be represented as the sum

$$\begin{aligned} w(\mathbf{x}, t) = &\int_0^t \int_V \Phi(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t, \tau)\, dV_y\, d\tau - \int_V f_0(\mathbf{y}) \left[\frac{\partial}{\partial \tau} G(\mathbf{x}, \mathbf{y}, t, \tau) \right]_{\tau=0} dV_y \\ &+ \int_V \big[f_1(\mathbf{y}) + f_0(\mathbf{y}) \sigma(\mathbf{y}, 0) \big] G(\mathbf{x}, \mathbf{y}, t, 0)\, dV_y \\ &+ \int_0^t \int_S g(\mathbf{y}, \tau) H(\mathbf{x}, \mathbf{y}, t, \tau)\, dS_y\, d\tau. \end{aligned} \tag{17.4.2.4}$$

Here $G(\mathbf{x}, \mathbf{y}, t, \tau)$ is the Green's function; for $t > \tau \geq 0$ it satisfies the homogeneous equation

$$\frac{\partial^2 G}{\partial t^2} + \sigma(\mathbf{x}, t) \frac{\partial G}{\partial t} - L_{\mathbf{x}}[G] = 0 \tag{17.4.2.5}$$

with the semihomogeneous initial conditions

$$G = 0 \qquad \text{at} \quad t = \tau,$$

$$\partial_t G = \delta(\mathbf{x} - \mathbf{y}) \quad \text{at} \quad t = \tau,$$

and the homogeneous boundary condition (17.4.1.8).

If the coefficients of Eq. (17.4.2.5) and the boundary condition (17.4.1.8) are independent of time t, then the Green's function depends on only three arguments, $G(\mathbf{x}, \mathbf{y}, t, \tau) =$

$G(\mathbf{x}, \mathbf{y}, t-\tau)$. In this case, one can set $\frac{\partial}{\partial\tau}G(\mathbf{x}, \mathbf{y}, t, \tau)\big|_{\tau=0} = -\frac{\partial}{\partial t}G(\mathbf{x}, \mathbf{y}, t)$ in solution (17.4.2.4).

The function $H(\mathbf{x}, \mathbf{y}, t, \tau)$ involved in the integrand of the last term in solution (17.4.2.4) can be expressed via the Green's function $G(\mathbf{x}, \mathbf{y}, t, \tau)$. The corresponding formulas for H are given in Table 17.6 for the three basic types of boundary value problems; in the third boundary value problem, the coefficient k can depend on $\mathbf{x}$ and t.

Remark 17.7. Let S_i ($i = 1, \dots, p$) be distinct portions of the surface S such that $S = \sum_{i=1}^{p} S_i$, and let boundary conditions of various types (17.4.1.10) be set on the S_i. Then formula (17.4.2.4) remains valid but the last term in (17.4.2.4) should be replaced by the sum (17.4.1.11).

17.4.3 Problems for Elliptic Equations

► Statement of the problem.

In general, a nonhomogeneous linear elliptic equation can be written as*

$$-L_{\mathbf{x}}[w] = \Phi(\mathbf{x}), \tag{17.4.3.1}$$

where

$$L_{\mathbf{x}}[w] \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial w}{\partial x_i} + c(\mathbf{x})w. \tag{17.4.3.2}$$

2D problems correspond to $n = 2$; 3D problems, to $n = 3$.

Consider Eq. (17.4.3.1)–(17.4.3.2) in a domain V and assume that the equation is supplemented with the general linear boundary condition

$$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in S. \tag{17.4.3.3}$$

The solution of the stationary problem (17.4.3.1)–(17.4.3.3) can be obtained by passing in (17.4.1.5) to the limit as $t \to \infty$. To this end, one should start from Eq. (17.4.1.1), whose coefficients are independent of t, and take the homogeneous initial condition (17.4.1.3) with $f(\mathbf{x}) = 0$ and the stationary boundary condition (17.4.1.4).

► Representation of the solution via the Green's function.

The solution of the linear boundary value problem (17.4.3.1)–(17.4.3.3) can be represented as the sum

$$w(\mathbf{x}) = \int_V \Phi(\mathbf{y})G(\mathbf{x}, \mathbf{y})\, dV_y + \int_S g(\mathbf{y})H(\mathbf{x}, \mathbf{y})\, dS_y. \tag{17.4.3.4}$$

Here the Green's function $G(\mathbf{x}, \mathbf{y})$ satisfies the nonhomogeneous equation of the special form

$$-L_{\mathbf{x}}[G] = \delta(\mathbf{x} - \mathbf{y}) \tag{17.4.3.5}$$

*In Sections 17.4.3, 17.4.4, and 17.5.1 for clarity in comparing the solutions of equations of various types, the left-hand side of Eq. (17.4.3.1) is taken with the minus sign. For this choice of the sign, the parabolic and hyperbolic equations (17.4.1.1) and (17.4.2.1) become the elliptic equation (17.4.3.1) if one sets $\partial/\partial t \equiv 0$. Note that Eqs. (14.2.1.7) and (16.1.2.1) differ in the sign of the left-hand side from Eq. (17.4.3.1).

with the homogeneous boundary condition

$$\Gamma_{\mathbf{x}}[G] = 0 \quad \text{for} \quad \mathbf{x} \in S. \tag{17.4.3.6}$$

The vector $\mathbf{y} = \{y_1, \dots, y_n\}$ appears in problem (17.4.3.5), (17.4.3.6) as an n-dimensional free parameter ($\mathbf{y} \in V$). Note that G is independent of the functions Φ and g characterizing various inhomogeneities of the original boundary value problem.

The function $H(\mathbf{x}, \mathbf{y})$ involved in the integrand of the second term in solution (17.4.3.4) can be expressed via the Green's function $G(\mathbf{x}, \mathbf{y})$. The corresponding formulas for H are given in Table 17.7 for the three basic types of boundary value problems. The boundary conditions of the second and third kind, as well as the solution of the first boundary value problem, involve operators of differentiation along the conormal of the operator (17.4.3.2); these operators are defined by (17.4.1.9); in this case, the coefficients a_{ij} depend on $\mathbf{x}$ alone.

TABLE 17.7
The form of the function $H(\mathbf{x}, \mathbf{y})$ involved in the integrand of the last term in the solution (17.4.3.4) for the basic types of stationary boundary value problems

Type of problem	Form of boundary condition (17.4.3.3)	Function $H(\mathbf{x}, \mathbf{y})$
First boundary value problem	$w = g(\mathbf{x})$ for $\mathbf{x} \in S$	$H(\mathbf{x}, \mathbf{y}) = -\dfrac{\partial G}{\partial M_y}(\mathbf{x}, \mathbf{y})$
Second boundary value problem	$\dfrac{\partial w}{\partial M_x} = g(\mathbf{x})$ for $\mathbf{x} \in S$	$H(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y})$
Third boundary value problem	$\dfrac{\partial w}{\partial M_x} + kw = g(\mathbf{x})$ for $\mathbf{x} \in S$	$H(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y})$

Remark 17.8. For the second boundary value problem with $c(\mathbf{x}) \equiv 0$, the Green's function thus defined must not necessarily exist (e.g., see Section 9.1.1).

17.4.4 Comparison of the Solution Structures for Boundary Value Problems for Equations of Various Types

Table 17.8 lists brief statements of boundary value problems for second-order equations of elliptic, parabolic, and hyperbolic types. The coefficients of the differential operators $L_{\mathbf{x}}$ and $\Gamma_{\mathbf{x}}$ in the space variables $x_1, \dots, x_n$ are assumed to be independent of time t; these operators are the same for the problems under consideration.

Below are the respective general formulas defining the solutions of these problems with zero initial conditions ($f = f_0 = f_1 = 0$):

$$w_0(\mathbf{x}) = \int_V \Phi(\mathbf{y}) G_0(\mathbf{x}, \mathbf{y})\, dV_y + \int_S g(\mathbf{y}) H\bigl[G_0(\mathbf{x}, \mathbf{y})\bigr]\, dS_y,$$

$$w_1(\mathbf{x}, t) = \int_0^t \int_V \Phi(\mathbf{y}, \tau) G_1(\mathbf{x}, \mathbf{y}, t-\tau)\, dV_y\, d\tau + \int_0^t \int_S g(\mathbf{y}, \tau) H\bigl[G_1(\mathbf{x}, \mathbf{y}, t-\tau)\bigr]\, dS_y\, d\tau,$$

$$w_2(\mathbf{x}, t) = \int_0^t \int_V \Phi(\mathbf{y}, \tau) G_2(\mathbf{x}, \mathbf{y}, t-\tau)\, dV_y\, d\tau + \int_0^t \int_S g(\mathbf{y}, \tau) H\bigl[G_2(\mathbf{x}, \mathbf{y}, t-\tau)\bigr]\, dS_y\, d\tau,$$

TABLE 17.8
Statements of boundary value problems for equations of various types

Type of equation	Form of equation	Initial conditions	Boundary conditions
Elliptic	$-L_{\mathbf{x}}[w] = \Phi(\mathbf{x})$	not set	$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x})$ for $\mathbf{x} \in S$
Parabolic	$\partial_t w - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t)$	$w = f(\mathbf{x})$ at $t = 0$	$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$
Hyperbolic	$\partial_{tt} w - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t)$	$w = f_0(\mathbf{x})$ at $t = 0$, $\partial_t w = f_1(\mathbf{x})$ at $t = 0$	$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$

where the G_n are the Green's functions, and the subscripts 0, 1, and 2 refer to the elliptic, parabolic, and hyperbolic problem, respectively. All solutions involve the same operator $H[G]$; it is explicitly defined in Sections 17.4.1–17.4.3 (see also Sections 17.1.1 and 17.2.1) for various boundary conditions.

It is apparent that the solutions of the parabolic and hyperbolic problems with zero initial conditions have the same structure. The structure of the solution of a problem for a parabolic equation differs from that for an elliptic equation by the additional integration with respect to t.

⊙ *Literature for Section 17.4*: S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), S. J. Farlow (1982), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), R. Courant and D. Hilbert (1989), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), I. G. Petrovsky (1991), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

17.5 Construction of the Green's Functions. General Formulas and Relations

17.5.1 Green's Functions of Boundary Value Problems for Equations of Various Types in Bounded Domains

► **Expressions of the Green's function in terms of infinite series.**

Table 17.9 lists the Green's functions of boundary value problems for second-order equations of various types in a bounded domain V. It is assumed that $L_{\mathbf{x}}$ is a second-order linear self-adjoint differential operator of the form (14.2.1.10) in the space variables $x_1, \dots, x_n$, and $\Gamma_{\mathbf{x}}$ is a zero- or first-order linear boundary operator that can define a boundary condition of the first, second, or third kind; the coefficients of the operators $L_{\mathbf{x}}$ and $\Gamma_{\mathbf{x}}$ can depend on the space variables but are independent of time t. The coefficients λ_k and the functions $u_k(\mathbf{x})$ are determined by solving the homogeneous eigenvalue problem

$$L_{\mathbf{x}}[u] + \lambda u = 0, \tag{17.5.1.1}$$

$$\Gamma_{\mathbf{x}}[u] = 0 \quad \text{for} \quad \mathbf{x} \in S. \tag{17.5.1.2}$$

It is apparent from Table 17.9 that, given the Green's function in the problem for a parabolic (or hyperbolic) equation, one can easily construct the Green's functions of the

TABLE 17.9
Green's functions of boundary value problems for equations of various types in bounded domains. In all problems, the operators $L_{\mathbf{x}}$ and $\Gamma_{\mathbf{x}}$ are the same; $\mathbf{x} = \{x_1, \dots, x_n\}$

Equation	Initial and boundary conditions	Green's function
Elliptic equation $-L_{\mathbf{x}}[w] = \Phi(\mathbf{x})$	$\Gamma_{\mathbf{x}}[w] = g(\mathbf{x})$ for $\mathbf{x} \in S$ (no initial condition required)	$G(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \frac{u_k(\mathbf{x}) u_k(\mathbf{y})}{\lVert u_k \rVert^2 \lambda_k}, \quad \lambda_k \neq 0$
Parabolic equation $\partial_t w - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t)$	$w = f(\mathbf{x})$ at $t = 0$ $\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$	$G(\mathbf{x}, \mathbf{y}, t) = \sum_{k=1}^{\infty} \frac{u_k(\mathbf{x}) u_k(\mathbf{y})}{\lVert u_k \rVert^2} \exp(-\lambda_k t)$
Hyperbolic equation $\partial_{tt} w - L_{\mathbf{x}}[w] = \Phi(\mathbf{x}, t)$	$w = f_0(\mathbf{x})$ at $t = 0$ $w = f_1(\mathbf{x})$ at $t = 0$ $\Gamma_{\mathbf{x}}[w] = g(\mathbf{x}, t)$ for $\mathbf{x} \in S$	$G(\mathbf{x}, \mathbf{y}, t) = \sum_{k=1}^{\infty} \frac{u_k(\mathbf{x}) u_k(\mathbf{y})}{\lVert u_k \rVert^2 \sqrt{\lambda_k}} \sin\bigl(t\sqrt{\lambda_k}\,\bigr)$

corresponding problems for elliptic and hyperbolic (or parabolic) equations. In particular, the Green's function of the problem for an elliptic equation can be expressed via the Green's function of the problem for a parabolic equation as follows:

$$G_0(\mathbf{x}, \mathbf{y}) = \int_0^{\infty} G_1(\mathbf{x}, \mathbf{y}, t)\, dt. \tag{17.5.1.3}$$

Here the fact that all λ_k are positive is taken into account; for the second boundary value problem, it is assumed that $\lambda = 0$ is not an eigenvalue of problem (17.5.1.1)–(17.5.1.2).

► **Some remarks and generalizations.**

Remark 17.9. Formula (17.5.1.3) can also be used if the domain V is infinite. In this case, one should make sure that the integral on the right-hand side is convergent.

Remark 17.10. Suppose that the equations given in the first column of Table 17.9 contain $-L_{\mathbf{x}}[w] - \beta w$ instead of $-L_{\mathbf{x}}[w]$ with β being a free parameter. Then the λ_k in the expressions of the Green's function in the third column of Table 17.9 should be replaced by $\lambda_k - \beta$; just as previously, the λ_k and $u_k(\mathbf{x})$ were determined by solving the eigenvalue problem (17.5.1.1)–(17.5.1.2).

Remark 17.11. The formulas for the Green's functions presented in Table 17.9 will also hold for boundary value problems described by equations of the fourth or higher orders in the space variables, provided that the eigenvalue problem for Eq. (17.5.1.1) supplemented with appropriate boundary conditions is self-adjoint.

17.5.2 Green's Functions Admitting Incomplete Separation of Variables

► **Boundary value problems for rectangular domains.**

1°. Consider the parabolic equation

$$\frac{\partial w}{\partial t} = L_1[w] + \cdots + L_n[w] + \Phi(\mathbf{x}, t), \tag{17.5.2.1}$$

where each term $L_m[w]$ is a second-order linear differential operator in only one space variable x_m with coefficients dependent on x_m and t,

$$L_m[w] \equiv a_m(x_m,t)\frac{\partial^2 w}{\partial x_m^2} + b_m(x_m,t)\frac{\partial w}{\partial x_m} + c_m(x_m,t)w, \qquad m = 1,\,\ldots,\,n.$$

For Eq. (17.5.2.1), we set the initial condition of general form

$$w = f(\mathbf{x}) \quad \text{at} \quad t = 0. \tag{17.5.2.2}$$

Consider the domain $V = \{\alpha_m \le x_m \le \beta_m,\ m = 1,\,\ldots,\,n\}$, which is an n-dimensional parallelepiped. We set the following boundary conditions at the faces of the parallelepiped:

$$\begin{aligned} s_m^{(1)}\frac{\partial w}{\partial x_m} + k_m^{(1)}(t)w &= g_m^{(1)}(\mathbf{x},t) \quad \text{at} \quad x_m = \alpha_m,\\ s_m^{(2)}\frac{\partial w}{\partial x_m} + k_m^{(2)}(t)w &= g_m^{(2)}(\mathbf{x},t) \quad \text{at} \quad x_m = \beta_m. \end{aligned} \tag{17.5.2.3}$$

By appropriately choosing the coefficients $s_m^{(1)}$, $s_m^{(2)}$ and the functions $k_m^{(1)} = k_m^{(1)}(t)$, $k_m^{(2)} = k_m^{(2)}(t)$, we can obtain boundary conditions of the first, second, or third kind. For infinite domains, the boundary conditions corresponding to $\alpha_m = -\infty$ or $\beta_m = \infty$ are omitted.

2°. The Green's function of the nonstationary n-dimensional boundary value problem (17.5.2.1)–(17.5.2.3) can be represented in the product form

$$G(\mathbf{x},\mathbf{y},t,\tau) = \prod_{m=1}^{n} G_m(x_m,y_m,t,\tau), \tag{17.5.2.4}$$

where the Green's functions $G_m = G_m(x_m,y_m,t,\tau)$ satisfy the one-dimensional equations

$$\frac{\partial G_m}{\partial t} - L_m[G_m] = 0 \qquad (m = 1,\,\ldots,\,n)$$

with the initial conditions

$$G_m = \delta(x_m - y_m) \quad \text{at} \quad t = \tau$$

and the homogeneous boundary conditions

$$\begin{aligned} s_m^{(1)}\frac{\partial G_m}{\partial x_m} + k_m^{(1)}(t)G_m &= 0 \quad \text{at} \quad x_m = \alpha_m,\\ s_m^{(2)}\frac{\partial G_m}{\partial x_m} + k_m^{(2)}(t)G_m &= 0 \quad \text{at} \quad x_m = \beta_m. \end{aligned}$$

Here y_m and τ are free parameters ($\alpha_m \le y_m \le \beta_m$ and $t \ge \tau \ge 0$), and $\delta(x)$ is the Dirac delta function.

It can be seen that the Green's function (17.5.2.4) admits incomplete separation of variables; it separates in the space variables $x_1,\,\ldots,\,x_n$ but not in time t.

Example 17.7. Consider the boundary value problem for the nonhomogeneous 2D heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \Phi(x_1, x_2, t)$$

with initial condition (17.5.2.2) and the nonhomogeneous mixed boundary conditions

$$\begin{aligned} &w = g_1(x_2,t) \quad \text{at} \quad x_1 = 0, \qquad && w = h_1(x_2,t) \quad \text{at} \quad x_1 = l_1;\\ &\frac{\partial w}{\partial x_2} = g_2(x_1,t) \quad \text{at} \quad x_2 = 0, \qquad && \frac{\partial w}{\partial x_2} = h_2(x_1,t) \quad \text{at} \quad x_2 = l_2. \end{aligned}$$

The Green's functions of the corresponding homogeneous one-dimensional heat equations with homogeneous boundary conditions are expressed as

Equations and boundary conditions *Green's functions*

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x_1^2}, \quad w = 0 \ \text{at} \ x_1 = 0,\, l_1 \quad \Longrightarrow \quad G_1 = \frac{2}{l_1}\sum_{m=1}^{\infty} \sin(\lambda_m x_1)\sin(\lambda_m y_1) e^{-\lambda_m^2 (t-\tau)};$$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x_2^2}, \quad \frac{\partial w}{\partial x_2} = 0 \ \text{at} \ x_2 = 0,\, l_2 \quad \Longrightarrow \quad G_2 = \frac{1}{l_2} + \frac{2}{l_2}\sum_{n=1}^{\infty} \sin(\sigma_n x_2)\sin(\sigma_n y_2) e^{-\sigma_n^2 (t-\tau)},$$

where $\lambda_m = m\pi/l_1$ and $\sigma_n = n\pi/l_2$. Multiplying G_1 and G_2 together gives the Green's function for the original 2D problem:

$$\begin{aligned} G(x_1,x_2,y_1,y_2,t,\tau) = \frac{4}{l_1 l_2}&\left[\sum_{m=1}^{\infty} \sin(\lambda_m x_1)\sin(\lambda_m y_1) e^{-\lambda_m^2 (t-\tau)}\right]\\ &\times\left[\frac{1}{2} + \sum_{n=1}^{\infty} \sin(\sigma_n x_2)\sin(\sigma_n y_2) e^{-\sigma_n^2 (t-\tau)}\right]. \end{aligned}$$

► Boundary value problems for an arbitrary cylindrical domain.

1°. Consider the parabolic equation

$$\frac{\partial w}{\partial t} = L_{\mathbf{x}}[w] + M_z[w] + \Phi(\mathbf{x}, z, t), \tag{17.5.2.5}$$

where $L_{\mathbf{x}}$ is an arbitrary second-order linear differential operator in $x_1, \dots, x_n$ with coefficients depending on $\mathbf{x}$ and t and M_z is an arbitrary second-order linear differential operator in z with coefficients depending on z and t.

For Eq. (17.5.2.5), we set the general initial condition (17.5.2.2), where $f(\mathbf{x})$ should be replaced by $f(\mathbf{x}, z)$.

We assume that the space variables belong to a cylindrical domain $V = \{\mathbf{x} \in D,\ z_1 \le z \le z_2\}$ with arbitrary cross-section D. We set the boundary conditions*

$$\begin{aligned} &\Gamma_1[w] = g_1(\mathbf{x},t) && \text{at} \quad z = z_1 && (\mathbf{x} \in D),\\ &\Gamma_2[w] = g_2(\mathbf{x},t) && \text{at} \quad z = z_2 && (\mathbf{x} \in D),\\ &\Gamma_3[w] = g_3(\mathbf{x},z,t) && \text{for} \quad \mathbf{x} \in \partial D && (z_1 \le z \le z_2), \end{aligned} \tag{17.5.2.6}$$

where the linear boundary operators Γ_k ($k = 1,\, 2,\, 3$) can define boundary conditions of the first, second, or third kind; in the last case, the coefficients of the differential operators Γ_k can depend on t.

*If $z_1 = -\infty$ or $z_2 = \infty$, the corresponding boundary condition is to be omitted.

2°. The Green's function of problem (17.5.2.5)–(17.5.2.6), (17.5.2.2) can be represented in the product form

$$G(\mathbf{x},\mathbf{y},z,\zeta,t,\tau) = G_L(\mathbf{x},\mathbf{y},t,\tau)G_M(z,\zeta,t,\tau), \tag{17.5.2.7}$$

where $G_L = G_L(\mathbf{x},\mathbf{y},t,\tau)$ and $G_M = G_M(z,\zeta,t,\tau)$ are auxiliary Green's functions; these can be determined from the following two simpler problems with fewer independent variables:

Problem on the cross-section D:

$$\begin{cases} \dfrac{\partial G_L}{\partial t} = L_\mathbf{x}[G_L] & \text{for} \quad \mathbf{x}\in D, \\ G_L = \delta(\mathbf{x}-y) & \text{at} \quad t=\tau, \\ \Gamma_3[G_L] = 0 & \text{for} \quad \mathbf{x}\in\partial D, \end{cases}$$

Problem on the interval $z_1 \le z \le z_2$:

$$\begin{cases} \dfrac{\partial G_M}{\partial t} = M_z[G_M] & \text{for} \quad z_1<z<z_2, \\ G_M = \delta(z-\zeta) & \text{at} \quad t=\tau, \\ \Gamma_k[G_M] = 0 & \text{at} \quad z=z_k \ \ (k=1,\,2). \end{cases}$$

Here $\mathbf{y}$, ζ, and τ are free parameters ($\mathbf{y}\in D$, $z_1\le\zeta\le z_2$, and $t\ge\tau\ge 0$).

It can be seen that the Green's function (17.5.2.7) admits incomplete separation of variables; it separates in the space variables $\mathbf{x}$ and z but not in time t.

17.5.3 Construction of Green's Functions via Fundamental Solutions

► **Elliptic equations. Fundamental solution.**

Consider the elliptic equation

$$L_\mathbf{x}[w] + \frac{\partial^2 w}{\partial z^2} = \Phi(\mathbf{x},z), \tag{17.5.3.1}$$

where $\mathbf{x} = \{x_1,\dots,x_n\} \in \mathbb{R}^n$, $z\in\mathbb{R}^1$, and $L_\mathbf{x}[w]$ is a linear differential operator that depends on $x_1,\dots,x_n$ but is independent of z. For the subsequent analysis, it is significant that the homogeneous equation (with $\Phi\equiv 0$) does not change under the replacement of z by $-z$ and z by $z+\text{const}$.

Let $\mathscr{E}_\text{e} = \mathscr{E}_\text{e}(\mathbf{x},\mathbf{y},z-\zeta)$ be the fundamental solution of Eq. (17.5.3.1), which means that

$$L_\mathbf{x}[\mathscr{E}_\text{e}] + \frac{\partial^2 \mathscr{E}_\text{e}}{\partial z^2} = \delta(\mathbf{x}-\mathbf{y})\delta(z-\zeta).$$

Here $\mathbf{y} = \{y_1,\dots,y_n\}\in\mathbb{R}^n$ and $\zeta\in\mathbb{R}^1$ are free parameters.

The fundamental solution of Eq. (17.5.3.1) is an even function in the last argument; i.e.,

$$\mathscr{E}_\text{e}(\mathbf{x},\mathbf{y},z) = \mathscr{E}_\text{e}(\mathbf{x},\mathbf{y},-z).$$

Below we present relations that permit one to express the Green's functions of some boundary value problems for Eq. (17.5.3.1) via its fundamental solution.

► Domain: $\mathbf{x} \in \mathbb{R}^n$, $0 \le z < \infty$. Problems for elliptic equations.

1°. *First boundary value problem.* The boundary condition:

$$w = f(\mathbf{x}) \quad \text{at} \quad z = 0.$$

Green's function:

$$G(\mathbf{x}, \mathbf{y}, z, \zeta) = \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta) - \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta). \qquad (17.5.3.2)$$

Domain of the free parameters: $\mathbf{y} \in \mathbb{R}^n$ and $0 \le \zeta < \infty$.

Example 17.8. Consider the first boundary value problem in the half-space $-\infty < x_1, x_2 < \infty$, $0 \le x_3 < \infty$ for the 3D Laplace equation

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_3^2} = 0$$

under the boundary condition

$$w = f(x_1, x_2) \quad \text{at} \quad x_3 = 0.$$

The fundamental solution of the Laplace equation has the form

$$\mathscr{E}_\mathrm{e} = \frac{1}{4\pi\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}.$$

In terms of the notation adopted for Eq. (17.5.3.1) and its fundamental solution, we have $x_3 = z$, $y_3 = \zeta$, and $\mathscr{E} = \mathscr{E}(x_1, y_1, x_2, y_2, z - \zeta)$. Using formula (17.5.3.2), we obtain the Green's function for the first boundary value problem in the half-space:

$$\begin{aligned} G(x_1, y_1, x_2, y_2, z, \zeta) &= \mathscr{E}_\mathrm{e}(x_1, y_1, x_2, y_2, z - \zeta) - \mathscr{E}_\mathrm{e}(x_1, y_1, x_2, y_2, z + \zeta) \\ &= \frac{1}{4\pi\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \\ &\quad - \frac{1}{4\pi\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2}}. \end{aligned}$$

2°. *Second boundary value problem.* The boundary condition:

$$\partial_z w = f(\mathbf{x}) \quad \text{at} \quad z = 0.$$

Green's function:

$$G(\mathbf{x}, \mathbf{y}, z, \zeta) = \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta) + \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta).$$

Example 17.9. The Green's function of the second boundary value problem for the 3D Laplace equation in the half-space $-\infty < x_1, x_2 < \infty$, $0 \le x_3 < \infty$ is expressed as

$$\begin{aligned} G(x_1, y_1, x_2, y_2, z, \zeta) &= \frac{1}{4\pi\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \\ &\quad + \frac{1}{4\pi\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2}}. \end{aligned}$$

It is obtained using the same reasoning as in Example 17.8.

$3°$. *Third boundary value problem.* The boundary condition:

$$\partial_z w - kw = f(\mathbf{x}) \quad \text{at} \quad z = 0.$$

Green's function:

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}, z, \zeta) &= \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta) + \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta) - 2k \int_0^\infty e^{-ks} \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta + s)\, ds \\ &= \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta) + \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta) - 2k \int_{z+\zeta}^\infty e^{-k(\sigma - z - \zeta)} \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, \sigma)\, d\sigma. \end{aligned}$$

► **Domain: $\mathbf{x} \in \mathbb{R}^n$, $0 \le z \le l$. Problems for elliptic equations.**

$1°$. *First boundary value problem.* Boundary conditions:

$$w = f_1(\mathbf{x}) \quad \text{at} \quad z = 0, \qquad w = f_2(\mathbf{x}) \quad \text{at} \quad z = l.$$

Green's function:

$$G(\mathbf{x}, \mathbf{y}, z, \zeta) = \sum_{n=-\infty}^{\infty} \big[\mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl) - \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl)\big]. \qquad (17.5.3.3)$$

Domain of the free parameters: $\mathbf{y} \in \mathbb{R}^n$ and $0 \le \zeta \le l$.

$2°$. *Second boundary value problem.* Boundary conditions:

$$\partial_z w = f_1(\mathbf{x}) \quad \text{at} \quad z = 0, \qquad \partial_z w = f_2(\mathbf{x}) \quad \text{at} \quad z = l.$$

Green's function:

$$G(\mathbf{x}, \mathbf{y}, z, \zeta) = \sum_{n=-\infty}^{\infty} \big[\mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl) + \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl)\big]. \qquad (17.5.3.4)$$

$3°$. *Mixed boundary value problem.* The unknown function and its derivative are prescribed at the left and right end, respectively:

$$w = f_1(\mathbf{x}) \quad \text{at} \quad z = 0, \qquad \partial_z w = f_2(\mathbf{x}) \quad \text{at} \quad z = l.$$

Green's function:

$$G(\mathbf{x}, \mathbf{y}, z, \zeta) = \sum_{n=-\infty}^{\infty} (-1)^n \big[\mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl) - \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl)\big]. \quad (17.5.3.5)$$

$4°$. *Mixed boundary value problem.* The derivative and the unknown function itself are prescribed at the left and right end, respectively:

$$\partial_z w = f_1(\mathbf{x}) \quad \text{at} \quad z = 0, \qquad w = f_2(\mathbf{x}) \quad \text{at} \quad z = l.$$

Green's function:

$$G(\mathbf{x}, \mathbf{y}, z, \zeta) = \sum_{n=-\infty}^{\infty} (-1)^n \big[\mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl) + \mathscr{E}_\mathrm{e}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl)\big]. \quad (17.5.3.6)$$

Remark 17.12. One should make sure that the series (17.5.3.3)–(17.5.3.6) are convergent; in particular, for the 3D Laplace equation, the series (17.5.3.3), (17.5.3.5), and (17.5.3.6) are convergent, and the series (17.5.3.4) is divergent.

► Boundary value problems for parabolic equations.

Let $\mathbf{x} \in \mathbb{R}^n$, $z \in \mathbb{R}^1$, and $t \geq 0$. Consider the parabolic equation

$$\frac{\partial w}{\partial t} = L_{\mathbf{x}}[w] + \frac{\partial^2 w}{\partial z^2} + \Phi(\mathbf{x}, z, t), \tag{17.5.3.7}$$

where $L_{\mathbf{x}}$ is an arbitrary linear differential operator in $x_1, \dots, x_n$ with coefficients depending on $\mathbf{x}$ and t but independent of z.

Let $\mathscr{E} = \mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta, t, \tau)$ be the fundamental solution of the Cauchy problem for Eq. (17.5.3.7); i.e.,

$$\frac{\partial \mathscr{E}}{\partial t} = L_{\mathbf{x}}[\mathscr{E}] + \frac{\partial^2 \mathscr{E}}{\partial z^2} \quad \text{for} \quad t > \tau \geq 0,$$
$$\mathscr{E} = \delta(\mathbf{x} - \mathbf{y})\,\delta(z - \zeta) \quad \text{at} \quad t = \tau.$$

Here $\mathbf{y} \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^1$, and $\tau \geq 0$ are free parameters.

The fundamental solution of the Cauchy problem possesses the property

$$\mathscr{E}(\mathbf{x}, \mathbf{y}, z, t, \tau) = \mathscr{E}(\mathbf{x}, \mathbf{y}, -z, t, \tau).$$

TABLE 17.10
Representation of the Green's functions of some nonstationary boundary value problems via the fundamental solution of the Cauchy problem

Boundary value problems	Boundary conditions	Green's functions
First problem $\mathbf{x} \in \mathbb{R}^n$, $z \in \mathbb{R}^1$	$G = 0$ at $z = 0$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta, t, \tau) - \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta, t, \tau)$
Second problem $\mathbf{x} \in \mathbb{R}^n$, $z \in \mathbb{R}^1$	$\partial_z G = 0$ at $z = 0$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta, t, \tau) + \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta, t, \tau)$
Third problem $\mathbf{x} \in \mathbb{R}^n$, $z \in \mathbb{R}^1$	$\partial_z G - kG = 0$ at $z = 0$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta, t, \tau) + \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta, t, \tau) - 2k \int_0^\infty e^{-ks} \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta + s, t, \tau)\, ds$
First problem $\mathbf{x} \in \mathbb{R}^n$, $0 \leq z \leq l$	$G = 0$ at $z = 0$, $G = 0$ at $z = l$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \sum\limits_{n=-\infty}^{\infty} \bigl[\mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl, t, \tau) - \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl, t, \tau)\bigr]$
Second problem $\mathbf{x} \in \mathbb{R}^n$, $0 \leq z \leq l$	$\partial_z G = 0$ at $z = 0$, $\partial_z G = 0$ at $z = l$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \sum\limits_{n=-\infty}^{\infty} \bigl[\mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl, t, \tau) + \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl, t, \tau)\bigr]$
Mixed problem $\mathbf{x} \in \mathbb{R}^n$, $0 \leq z \leq l$	$G = 0$ at $z = 0$, $\partial_z G = 0$ at $z = l$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \sum\limits_{n=-\infty}^{\infty} (-1)^n \bigl[\mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl, t, \tau) - \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl, t, \tau)\bigr]$
Mixed problem $\mathbf{x} \in \mathbb{R}^n$, $0 \leq z \leq l$	$\partial_z G = 0$ at $z = 0$, $G = 0$ at $z = l$	$G(\mathbf{x}, \mathbf{y}, z, \zeta, t, \tau) = \sum\limits_{n=-\infty}^{\infty} (-1)^n \bigl[\mathscr{E}(\mathbf{x}, \mathbf{y}, z - \zeta + 2nl, t, \tau) + \mathscr{E}(\mathbf{x}, \mathbf{y}, z + \zeta + 2nl, t, \tau)\bigr]$

Table 17.10 presents formulas that permit one to express the Green's functions of some nonstationary boundary value problems for Eq. (17.5.3.7) via the fundamental solution of the Cauchy problem.

⊙ *Literature for Section 17.5*: S. G. Mikhlin (1970), V. S. Vladimirov (1971, 1988), A. G. Butkovskiy (1982), S. J. Farlow (1982), R. Leis (1986), A. A. Dezin (1987), R. Haberman (1987), R. Courant and D. Hilbert (1989), E. Zauderer (1989), A. N. Tikhonov and A. A. Samarskii (1990), W. A. Strauss (1992), R. B. Guenther and J. W. Lee (1996), D. Zwillinger (1998), I. Stakgold (2000), C. Constanda (2002), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 18

Duhamel's Principles. Some Transformations

18.1 Duhamel's Principles in Nonstationary Problems

18.1.1 Problems for Homogeneous Linear Equations

► **Parabolic equations with two independent variables.**

Consider the problem for the homogeneous linear equation of parabolic type

$$\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + c(x)w \qquad (18.1.1.1)$$

with the homogeneous initial condition

$$w = 0 \quad \text{at} \quad t = 0 \qquad (18.1.1.2)$$

and the boundary conditions

$$s_1 \partial_x w + k_1 w = g(t) \qquad \text{at} \quad x = x_1, \qquad (18.1.1.3)$$

$$s_2 \partial_x w + k_2 w = 0 \qquad \text{at} \quad x = x_2. \qquad (18.1.1.4)$$

By appropriately choosing the values of the coefficients s_1, s_2, k_1, and k_2 in (18.1.1.3) and (18.1.1.4), one can obtain the first, second, third, and mixed boundary value problems for Eq. (18.1.1.1).

The solution of problem (18.1.1.1)–(18.1.1.4) with the nonstationary boundary condition (18.1.1.3) at $x = x_1$ can be expressed by the formula (*Duhamel's first principle*)

$$w(x,t) = \frac{\partial}{\partial t}\int_0^t u(x, t-\tau)\, g(\tau)\, d\tau = \int_0^t \frac{\partial u}{\partial t}(x, t-\tau)\, g(\tau)\, d\tau \qquad (18.1.1.5)$$

via the solution $u(x,t)$ of the auxiliary problem for Eq. (18.1.1.1) with the initial and boundary conditions (18.1.1.2) and (18.1.1.4) for u instead of w and with the following simpler stationary boundary condition at $x = x_1$:

$$s_1 \partial_x u + k_1 u = 1 \quad \text{at} \quad x = x_1. \qquad (18.1.1.6)$$

Remark 18.1. A similar formula holds for the homogeneous boundary condition at $x = x_1$ and a nonhomogeneous nonstationary boundary condition at $x = x_2$.

Example 18.1. Consider the first boundary value problem for the heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \tag{18.1.1.7}$$

with the homogeneous initial condition (18.1.1.2) and the boundary condition

$$w = g(t) \quad \text{at} \quad x = 0. \tag{18.1.1.8}$$

(The second boundary condition is not required in this case; $0 \le x < \infty$.)

First, consider the following auxiliary problem for the heat equation with the homogeneous initial condition and a simpler boundary condition:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u = 0 \quad \text{at} \quad t = 0, \qquad u = 1 \quad \text{at} \quad x = 0.$$

This problem has a self-similar solution of the form

$$w = w(z), \qquad z = xt^{-1/2},$$

where the function $w(z)$ is determined by the following ordinary differential equation and boundary conditions:

$$u''_{zz} + \tfrac{1}{2} z u'_z = 0, \qquad u = 1 \quad \text{at} \quad z = 0, \qquad u = 0 \quad \text{at} \quad z = \infty.$$

Its solution is expressed as

$$u(z) = \operatorname{erfc}\left(\frac{z}{2}\right) \quad \Longrightarrow \quad u(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right),$$

where $\operatorname{erfc} z = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_z^\infty \exp(-\xi^2)\, d\xi$ is the complementary error function. By substituting the expression obtained for $u(x,t)$ into (18.1.1.5), we obtain the solution of the first boundary value problem for the heat equation (18.1.1.7) with the initial condition (18.1.1.2) and an arbitrary boundary condition (18.1.1.8) in the form

$$w(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \exp\left[-\frac{x^2}{4(t-\tau)}\right] \frac{g(\tau)\, d\tau}{(t-\tau)^{3/2}}.$$

► **Hyperbolic equations with two independent variables.**

Consider the problem for the homogeneous linear hyperbolic equation

$$\frac{\partial^2 w}{\partial t^2} + \varphi(x)\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + c(x)w \tag{18.1.1.9}$$

with the homogeneous initial conditions

$$\begin{aligned} w &= 0 \quad \text{at} \quad t = 0, \\ \partial_t w &= 0 \quad \text{at} \quad t = 0, \end{aligned} \tag{18.1.1.10}$$

and the boundary conditions (18.1.1.3) and (18.1.1.4).

The solution of problem (18.1.1.9), (18.1.1.10), (18.1.1.3), (18.1.1.4) with the nonstationary boundary condition (18.1.1.3) at $x = x_1$ can be expressed by formula (18.1.1.5) in terms of the solution $u(x, t)$ of the auxiliary problem for Eq. (18.1.1.9) with the initial conditions (18.1.1.10) and the boundary condition (18.1.1.4) for u instead of w, and the simpler stationary boundary condition (18.1.1.6) at $x = x_1$.

In this case, Remark 18.1 remains valid.

► **Second-order equations with several independent variables.**

Duhamel's first principle can also be used to solve homogeneous linear equations of the parabolic or hyperbolic type with many space variables,

$$\frac{\partial^k w}{\partial t^k} = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial w}{\partial x_i} + c(\mathbf{x})w, \qquad (18.1.1.11)$$

where $k = 1$, 2 and $\mathbf{x} = \{x_1, \dots, x_n\}$.

Let V be some bounded domain in $\mathbb{R}^n$ with sufficiently smooth surface $S = \partial V$. The solution of the boundary value problem for Eq. (18.1.1.11) in V with the homogeneous initial conditions (18.1.1.2) if $k = 1$ or (18.1.1.10) if $k = 2$ and with the nonhomogeneous linear boundary condition

$$\Gamma_{\mathbf{x}}[w] = g(t) \quad \text{for} \quad \mathbf{x} \in S \qquad (18.1.1.12)$$

is given by

$$w(\mathbf{x}, t) = \frac{\partial}{\partial t} \int_0^t u(\mathbf{x}, t - \tau)\, g(\tau)\, d\tau = \int_0^t \frac{\partial u}{\partial t}(\mathbf{x}, t - \tau)\, g(\tau)\, d\tau.$$

Here $u(\mathbf{x}, t)$ is the solution of the auxiliary problem for Eq. (18.1.1.11) with the same initial conditions, (18.1.1.2) or (18.1.1.10) for u instead of w and with the simpler stationary boundary condition

$$\Gamma_{\mathbf{x}}[u] = 1 \quad \text{for} \quad \mathbf{x} \in S.$$

Note that Eq. (18.1.1.12) can represent a boundary condition of the first, second, or third kind; the coefficients of the operator $\Gamma_{\mathbf{x}}$ are assumed to be independent of t.

18.1.2 Problems for Nonhomogeneous Linear Equations

► **Parabolic equations.**

The solution of the nonhomogeneous linear equation

$$\frac{\partial w}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial w}{\partial x_i} + c(\mathbf{x})w + \Phi(\mathbf{x}, t)$$

with the homogeneous initial condition (18.1.1.2) and the homogeneous boundary condition

$$\Gamma_{\mathbf{x}}[w] = 0 \quad \text{for} \quad \mathbf{x} \in S \qquad (18.1.2.1)$$

can be represented in the form (*Duhamel's second principle*)

$$w(\mathbf{x},t)=\int_0^t U(\mathbf{x},t-\tau,\tau)\,d\tau. \tag{18.1.2.2}$$

Here $U(\mathbf{x},t,\tau)$ is the solution of the auxiliary problem for the homogeneous equation

$$\frac{\partial U}{\partial t}=\sum_{i,j=1}^{n}a_{ij}(\mathbf{x})\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(\mathbf{x})\frac{\partial U}{\partial x_i}+c(\mathbf{x})U$$

with the boundary condition (18.1.2.1), in which w must be substituted by U, and the nonhomogeneous initial condition

$$U=\Phi(\mathbf{x},\tau)\quad\text{at}\quad t=0,$$

where τ is a parameter.

Note that Eq. (18.1.2.1) can represent a boundary condition of the first, second, or third kind; the coefficients of the operator $\Gamma_{\mathbf{x}}$ are assumed to be independent of t.

► Hyperbolic equations.

The solution of the nonhomogeneous linear equation

$$\frac{\partial^2 w}{\partial t^2}+\varphi(\mathbf{x})\frac{\partial w}{\partial t}=\sum_{i,j=1}^{n}a_{ij}(\mathbf{x})\frac{\partial^2 w}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(\mathbf{x})\frac{\partial w}{\partial x_i}+c(\mathbf{x})w+\Phi(\mathbf{x},t)$$

with the homogeneous initial conditions (18.1.1.10) and homogeneous boundary condition (18.1.2.1) can be expressed by formula (18.1.2.2) in terms of the solution $U=U(\mathbf{x},t,\tau)$ of the auxiliary problem for the homogeneous equation

$$\frac{\partial^2 U}{\partial t^2}+\varphi(\mathbf{x})\frac{\partial U}{\partial t}=\sum_{i,j=1}^{n}a_{ij}(\mathbf{x})\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(\mathbf{x})\frac{\partial U}{\partial x_i}+c(\mathbf{x})U$$

with the homogeneous initial and boundary conditions (18.1.1.2) and (18.1.2.1), where w should be replaced by U, and the nonhomogeneous initial condition

$$\partial_t U=\Phi(\mathbf{x},\tau)\quad\text{at}\quad t=0,$$

where τ is a parameter.

Note that (18.1.2.1) can represent a boundary condition of the first, second, or third kind.

⊙ *Literature for Section 18.1*: R. Courant and D. Hilbert (1989), G. A. Korn and T. M. Korn (2000), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

18.2 Transformations Simplifying Initial and Boundary Conditions

18.2.1 Transformations That Lead to Homogeneous Boundary Conditions

A linear problem with arbitrary nonhomogeneous boundary conditions,

$$\Gamma_{\mathbf{x}}^{(k)}[w] = g_k(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in S_k, \tag{18.2.1.1}$$

can be reduced to a linear problem with homogeneous boundary conditions. To this end, one should make the change of variable

$$w(\mathbf{x}, t) = \psi(\mathbf{x}, t) + u(\mathbf{x}, t), \tag{18.2.1.2}$$

where u is the new unknown function, ψ is any function that satisfies the nonhomogeneous boundary conditions (18.2.1.1), and

$$\Gamma_{\mathbf{x}}^{(k)}[\psi] = g_k(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in S_k. \tag{18.2.1.3}$$

Table 18.1 gives examples of such transformations for linear boundary value problems with one space variable for parabolic and hyperbolic equations. In the third boundary value problem, it is assumed that $k_1 < 0$ and $k_2 > 0$.

TABLE 18.1
Simple transformations of the form $w(x, t) = \psi(x, t) + u(x, t)$ that lead to homogeneous boundary conditions in problems with one space variable ($0 \le x \le l$)

No.	Problems	Boundary conditions	Function $\psi = \psi(x, t)$
1	First boundary value problem	$w = g_1(t)$ at $x = 0$ $w = g_2(t)$ at $x = l$	$\psi = g_1(t) + \dfrac{x}{l}\bigl[g_2(t) - g_1(t)\bigr]$
2	Second boundary value problem	$\partial_x w = g_1(t)$ at $x = 0$ $\partial_x w = g_2(t)$ at $x = l$	$\psi = xg_1(t) + \dfrac{x^2}{2l}\bigl[g_2(t) - g_1(t)\bigr]$
3	Third boundary value problem	$\partial_x w + k_1 w = g_1(t)$ at $x = 0$ $\partial_x w + k_2 w = g_2(t)$ at $x = l$	$\psi = \dfrac{(k_2 x - 1 - k_2 l)g_1(t) + (1 - k_1 x)g_2(t)}{k_2 - k_1 - k_1 k_2 l}$
4	Mixed boundary value problem	$w = g_1(t)$ at $x = 0$ $\partial_x w = g_2(t)$ at $x = l$	$\psi = g_1(t) + xg_2(t)$
5	Mixed boundary value problem	$\partial_x w = g_1(t)$ at $x = 0$ $w = g_2(t)$ at $x = l$	$\psi = (x - l)g_1(t) + g_2(t)$

Note that the selection of the function ψ is of a purely algebraic nature and is not related to the equation in question; there are infinitely many suitable functions ψ that satisfy condition (18.2.1.3). Transformations of the form (18.2.1.2) can often be used at the first stage of solving boundary value problems.

18.2.2 Transformations That Lead to Homogeneous Initial and Boundary Conditions

A linear problem with nonhomogeneous initial and boundary conditions can be reduced to a linear problem with homogeneous initial and boundary conditions. To this end, one should introduce a new dependent variable u by formula (18.2.1.2), where the function ψ should satisfy nonhomogeneous initial and boundary conditions.

Below we specify some simple functions ψ that can be used in the transformation (18.2.1.2) to obtain boundary value problems with homogeneous initial and boundary conditions. To be specific, we consider a parabolic equation with one space variable and the general initial condition

$$w = f(x) \quad \text{at} \quad t = 0. \tag{18.2.2.1}$$

1. First boundary value problem: the initial condition is (18.2.2.1), and the boundary conditions are given in row 1 of Table 18.1. Suppose that the initial and boundary conditions are compatible; i.e., $f(0) = g_1(0)$ and $f(l) = g_2(0)$. Then in the transformation (18.2.1.2) one can take

$$\psi(x,t) = f(x) + g_1(t) - g_1(0) + \frac{x}{l}\big[g_2(t) - g_1(t) + g_1(0) - g_2(0)\big].$$

2. Second boundary value problem: the initial condition is (18.2.2.1), and the boundary conditions are given in row 2 of Table 18.1. Suppose that the initial and boundary conditions are compatible; i.e., $f'(0) = g_1(0)$ and $f'(l) = g_2(0)$. Then in the transformation (18.2.1.2) one can set

$$\psi(x,t) = f(x) + x\big[g_1(t) - g_1(0)\big] + \frac{x^2}{2l}\big[g_2(t) - g_1(t) + g_1(0) - g_2(0)\big].$$

3. Third boundary value problem: the initial condition is (18.2.2.1), and the boundary conditions are given in row 3 of Table 18.1. If the initial and boundary conditions are compatible, then in the transformation (18.2.1.2) one can take

$$\psi(x,t) = f(x) + \frac{(k_2 x - 1 - k_2 l)[g_1(t) - g_1(0)] + (1 - k_1 x)[g_2(t) - g_2(0)]}{k_2 - k_1 - k_1 k_2 l},$$

where $k_1 < 0$ and $k_2 > 0$.

4. Mixed boundary value problem: the initial condition is (18.2.2.1), and the boundary conditions are given in row 4 of Table 18.1. Suppose that the initial and boundary conditions are compatible; i.e., $f(0) = g_1(0)$ and $f'(l) = g_2(0)$. Then in the transformation (18.2.1.2) one can set

$$\psi(x,t) = f(x) + g_1(t) - g_1(0) + x\big[g_2(t) - g_2(0)\big].$$

5. Mixed boundary value problem: the initial condition is (18.2.2.1), and the boundary conditions are given in row 5 of Table 18.1. Suppose that the initial and boundary conditions are compatible; i.e., $f'(0) = g_1(0)$ and $f(l) = g_2(0)$. Then in the transformation (18.2.1.2) one can take

$$\psi(x,t) = f(x) + (x - l)\big[g_1(t) - g_1(0)\big] + g_2(t) - g_2(0).$$

⊙ *Literature for Section 18.2*: A. N. Tikhonov and A. A. Samarskii (1990), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 19

Systems of Linear Coupled PDEs. Decomposition Methods

19.1 Asymmetric and Symmetric Decompositions

19.1.1 Asymmetric Decomposition. Order of Decomposition

► **Preliminary remarks. Systems of coupled equations to be considered.**

Linear systems of coupled equations arise in various fields of continuum mechanics and physics (elasticity, thermoelasticity, porous elasticity, hydrodynamics of viscous and viscoelastic incompressible fluids as well as of viscous compressible barotropic fluids and gases, electrodynamics, etc.).

For a linear system of coupled equations, splitting the system into a few simpler subsystems (or, best of all, reducing it to several independent equations) is considered a great achievement. A representation of solutions of a system of coupled equations via solutions of independent (uncoupled) equations will be referred to as a *complete decomposition* of the original system; the representation of solutions of a system of coupled equations via solutions of a few simpler equations (of which only part is independent) will be referred to as an *incomplete decomposition*, or a *partial decomposition*.

Decomposition dramatically simplifies the qualitative study and interpretation of the most important physical properties of coupled 3D equations, thus permitting an efficient study of their wave and dissipative properties. Moreover, in a number of cases, decomposition permits one to find exact analytical solutions of the corresponding boundary value problems and initial-boundary value problems and substantially simplifies the application of numerical methods, which permits one to use the corresponding standard routines for simpler independent equations or subsystems.

In this section, we consider systems involving one vector equation and one scalar equation of the form

$$\mathrm{L}[\mathbf{u}] + \nabla \mathrm{K}[\mathbf{u}, p] = \mathbf{f}, \qquad (19.1.1.1)$$

$$\mathrm{M}[\mathbf{u}, p] = f_4, \qquad (19.1.1.2)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$ and $p = p(\mathbf{x}, t)$ are the unknown functions, $\mathbf{x} = (x, y, z)$, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1, f_2, f_3)$ and $f_4 = f_4(\mathbf{x}, t)$ are given functions, L, K and M are constant coefficient linear differential operators in (x, y, z, t), (x, y, z) are Cartesian coordinates, and $\nabla = (\partial_x, \partial_y, \partial_z)$ is the gradient operator.

In the special case of $\mathrm{K}[\mathbf{u}, p] = \mathrm{K}[\mathbf{u}]$, system (19.1.1.1)–(19.1.1.2) may consist of a single vector equation for **u**; the absence of the scalar equation (19.1.1.2) corresponds to $\mathrm{M}[\mathbf{u}, p] \equiv 0$ and $f_4 \equiv 0$.

Linear systems of coupled equations of the form (19.1.1.1)–(19.1.1.2) occur in several other areas of continuum mechanics and physics (see the summary Table 19.1).

1°. Such systems describe the slow motion of viscous incompressible and viscous compressible barotropic fluids (see rows 1–4 in Table 19.1), where **u** is the fluid velocity, p is the pressure, ρ is the fluid density, ν is the kinematic viscosity, b is the unperturbed velocity in the incoming flow, μ and λ are the dynamic coefficients of viscosity, and k is the compressibility factor.

2°. The operators L and K corresponding to the Maxwell viscoelastic fluid equations (for creeping flows) as well as to the linearized hyperbolic Navier–Stokes equations are given in row 5 in Table 19.1. Here τ is the relaxation time, and the remaining notation is the same as in row 1 (see above).

For row 6 in Table 19.1, the operator L corresponding to various models of incompressible viscoelastic fluids is given in Table 12.2.

3°. Systems of the form (19.1.1.1)–(19.1.1.2) are widely used in the linear theory of elasticity, thermoelasticity, and thermoviscoelasticity (see rows 7–13 in Table 19.1), where **u** is the displacement field, ρ is the medium density, $p = T$ is the temperature (in rows 8–10), β is the thermomechanical modulus, λ and μ (λ_0 and μ_0) are the Lamé elastic moduli, λ_1 and μ_1 are the viscosity coefficients, I_λ and I_μ are the relaxation integral operators, and a and $\widetilde{a}$ are thermal diffusivities. The last column in row 9 is an equation based on the Cattaneo hyperbolic heat conduction model, and the last column of row 10 corresponds to nondissipative thermoelasticity of the Green–Naghdi type (see Section 12.15), which describes the effect of the second sound. (This effect arises, for example, in liquid helium and some solids at low temperatures.)

4°. Systems of the form (19.1.1.1)–(19.1.1.2) also occur in electrodynamics (see rows 14 and 15 in Table 19.1).

► Asymmetric decomposition. The order of decomposition

Every solution of system (19.1.1.1)–(19.1.1.2) can be represented in the form

$$\mathbf{u} = \nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, \qquad p = p, \tag{19.1.1.3}$$

where $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors corresponding to the Cartesian coordinates y and z, two scalar functions $v_2 = v_2(\mathbf{x}, t)$ and $v_3 = v_3(\mathbf{x}, t)$ satisfy two independent linear equations of the same type,

$$\mathrm{L}\,[v_2] = f_2 - \partial_y F, \quad \mathrm{L}\,[v_3] = f_3 - \partial_z F, \tag{19.1.1.4}$$

$$F = F(\mathbf{x}, t) = F(x, y, z, t) = \int_0^x f_1(x_1, y, z, t)\, dx_1, \tag{19.1.1.5}$$

TABLE 19.1
Various linear systems of the form (19.1.1.1)–(19.1.1.2), occurring in continuum mechanics and physics. Notation: $\mathbf{u} = (u_1, u_2, u_3)$

No.	Equation names	Operator $\mathrm{L}[u]$	Operator $\mathrm{K}[\mathbf{u}, p]$	Equation (19.1.1.2)
1	Stokes (viscous incompressible fluid), see Eqs. (1)–(2) in Section 12.7	$u_t - \nu\Delta u$	p/ρ	$\operatorname{div}\mathbf{u} = 0$
2	Oseen (viscous incompressible fluid), see Eqs. (1) in Section 12.8	$u_t + bu_x - \nu\Delta u$	p/ρ	$\operatorname{div}\mathbf{u} = 0$
3	Stokes (viscous compressible fluid), see Eqs. (1)–(2) in Section 12.12	$\rho_0 u_t - \mu\Delta u$	$p - (\mu + \lambda)\operatorname{div}\mathbf{u}$	$kp_t + \rho_0 \operatorname{div}\mathbf{u} = 0$
4	Stokes (viscous compressible fluid), see Eqs. (14) in Section 12.12	$u_{tt} - \nu\Delta u_t$	$-c^2 \operatorname{div}\mathbf{u}$ $-(\nu + \varkappa)\operatorname{div}\mathbf{u}_t$	Absent
5	Maxwell (viscoelastic incompressible fluid), see Eqs. (1) in Section 12.9	$\tau u_{tt} + u_t - \nu\Delta u$	p/ρ	$\operatorname{div}\mathbf{u} = 0$
6	Viscoelastic incompressible fluid (general model)	$\mathrm{L}[u]$ (any operator)	p/ρ	$\operatorname{div}\mathbf{u} = 0$
7	Navier (linear elasticity), see Eqs. (1) in Section 12.6	$\rho u_{tt} - \mu\Delta u$	$-(\mu + \lambda)\operatorname{div}\mathbf{u}$	Absent
8	Thermoelasticity, see Eqs. (1)–(2) in Section 12.14; p is the temperature	$\rho u_{tt} - \mu\Delta u$	$\alpha p - (\mu + \lambda)\operatorname{div}\mathbf{u}$	$p_t = a\Delta p - \beta\,(\operatorname{div}\mathbf{u})_t$
9	Thermoelasticity (with hyperbolic heat conductivity); p is the temperature	$\rho u_{tt} - \mu\Delta u$	$\alpha p - (\mu + \lambda)\operatorname{div}\mathbf{u}$	$\tau p_{tt} + p_t$ $= a\Delta p - \beta\,(\operatorname{div}\mathbf{u})_t$
10	Thermoelasticity (Green–Naghdi model), see Eqs. (1)–(2) in Section 12.15; p is the temperature	$\rho u_{tt} - \mu\Delta u$	$\alpha p - (\mu + \lambda)\operatorname{div}\mathbf{u}$	$p_{tt} = \widetilde{a}\Delta p - \beta\,(\operatorname{div}\mathbf{u})_{tt}$
11	Linear viscoelasticity, see Eqs. (1) in Section 12.16	$\rho u_{tt} - \mu_0\Delta u$ $-\mu_1\Delta u_t$	$-(\mu_0 + \lambda_0)\operatorname{div}\mathbf{u}$ $-(\mu_1 + \lambda_1)\operatorname{div}\mathbf{u}_t$	Absent
12	Linear viscoelasticity (general model)	$\rho u_{tt} - \mathrm{I}_\mu[\Delta u]$	$-(\mathrm{I}_\mu + \mathrm{I}_\lambda)[\operatorname{div}\mathbf{u}]$	Absent

TABLE 19.1 (*continued*)
Various linear systems of the form (19.1.1.1)–(19.1.1.2), occurred in continuum mechanics and physics. Notation: $\mathbf{u} = (u_1, u_2, u_3)$

No.	Equation names	Operator $\mathrm{L}[u]$	Operator $\mathrm{K}[\mathbf{u}, p]$	Equation (19.1.1.2)
13	Linear thermoviscoelasticity (general model)	$\rho u_{tt} - \mathrm{I}_\mu[\Delta u]$	$\alpha p - (\mathrm{I}_\mu + \mathrm{I}_\lambda)[\operatorname{div}\mathbf{u}]$	$p_t = a\Delta p - \beta\,(\operatorname{div}\mathbf{u})_t$
14	Maxwell (electrodynamics), see Eqs. (5) with $\mathbf{H} = \mathbf{u}$ in Section 12.17	$\mu(\varepsilon u_{tt} + \lambda u_t) - \Delta u$	0	$\operatorname{div}\mathbf{u} = 0$
15	Maxwell (electrodynamics), see Eqs. (6) with $\mathbf{E} = \mathbf{u}$ in Section 12.17	$\mu(\varepsilon u_{tt} + \lambda u_t) - \Delta u$	$\operatorname{div}\mathbf{u}$	$\operatorname{div}\mathbf{u} = \rho/\varepsilon$

and the functions $\varphi = \varphi(\mathbf{x}, t)$ and $p = p(\mathbf{x}, t)$ are determined from the system of equations

$$\mathrm{L}\,[\varphi] + \mathrm{K}\,[\nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, p] = F, \qquad (19.1.1.6)$$

$$\mathrm{M}\,[\nabla\varphi + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3, p] = f_4. \qquad (19.1.1.7)$$

To avoid excessive change of notation in (19.1.1.3) and in what follows, we write $p = p$ instead of $p = \widetilde{p}$, where $\widetilde{p}$ is the new unknown function.

System (19.1.1.4)–(19.1.1.7) consists of two independent linear equations (19.1.1.4) and the subsystem of two coupled equations (19.1.1.6)–(19.1.1.7) and is substantially simpler than the original system of four coupled linear equations (19.1.1.1)–(19.1.1.2).

The representation (19.1.1.3) of components of the vector $\mathbf{u}$ includes first derivatives of the new unknown function φ, which corresponds to a first-order decomposition. In the general case, the *decomposition order* is defined as the maximum order of derivatives of the new unknown functions occurring in the representation of the components of the vector $\mathbf{u}$.

Formulas (19.1.1.3) and (19.1.1.5) and Eqs. (19.1.1.4), (19.1.1.6), and (19.1.1.7) determine a first-order partial decomposition of the linear system (19.1.1.1)–(19.1.1.2).

19.1.2 Symmetric Decomposition. Invariant Transformations

► Symmetric decomposition

Every solution of system (19.1.1.1)–(19.1.1.2) can also be represented in the symmetric form

$$\mathbf{u} = \nabla\varphi + \mathbf{v}, \qquad p = p, \qquad (19.1.2.1)$$

where the vector function $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the independent linear equation

$$\mathrm{L}[\mathbf{v}] = \mathbf{f} - \nabla G \qquad (19.1.2.2)$$

and the functions φ and p are described by the system of equations

$$\mathrm{L}\,[\varphi] + \mathrm{K}\,[\nabla\varphi + \mathbf{v}, p] = G, \qquad (19.1.2.3)$$

$$\mathrm{M}\,[\nabla\varphi + \mathbf{v}, p] = f_4. \qquad (19.1.2.4)$$

Equations (19.1.2.3)–(19.1.2.4) contain an arbitrary scalar function $G = G(\mathbf{x}, t)$.

Formulas (19.1.2.1) and Eqs. (19.1.2.2)–(19.1.2.4) determine a first-order partial decomposition of the linear system (19.1.1.1)–(19.1.1.2).

Remark 19.1. The representation (19.1.2.1) of the components of the vector **v** contains one extra (additional) unknown function compared with the representation (19.1.1.3). This permits slightly simplifying Eqs. (19.1.2.2)–(19.1.2.4) by imposing an additional condition on the components (v_1, v_2, v_3) and by choosing an appropriate function G. In particular, without loss of generality one can set $v_1 = 0$ and $G = F$ (i.e., $f_1 = G_x$) in (19.1.2.2)–(19.1.2.4), which results in the representation (19.1.1.3)–(19.1.1.7).

Linear equations of continuum mechanics and physics are special cases of system (19.1.1.1)–(19.1.1.2) of the special form

$$\mathrm{L}[\mathbf{u}] + \nabla\big(\sigma p + \mathrm{K}_1[\operatorname{div}\mathbf{u}]\big) = \mathbf{f}, \tag{19.1.2.5}$$

$$\mathrm{M}_1[p] + \mathrm{M}_2[\operatorname{div}\mathbf{u}] = f_4, \tag{19.1.2.6}$$

where σ is a constant and K_1, M_1, and M_2 are linear differential, integral, or integro-differential operators in t, x, y, and z.

Every solution of system (19.1.2.5)–(19.1.2.6) can be represented in the form (19.1.2.1) with the vector $\mathbf{v} = (v_1, v_2, v_3)$ satisfying the independent linear equation (19.1.2.2), the functions φ and p described by the system of equations

$$\mathrm{L}[\varphi] + \sigma p + \mathrm{K}_1[\Delta\varphi] + \mathrm{K}_1[\operatorname{div}\mathbf{v}] = G, \tag{19.1.2.7}$$

$$\mathrm{M}_1[p] + \mathrm{M}_2[\Delta\varphi] + \mathrm{M}_2[\operatorname{div}\mathbf{v}] = f_4, \tag{19.1.2.8}$$

and $G = G(\mathbf{x}, t)$ being an arbitrary function.

These equations contain the function $\operatorname{div}\mathbf{v}$. In what follows, using Remark 19.1, we simplify (19.1.2.7)–(19.1.2.8) by imposing the additional condition

$$\operatorname{div}\mathbf{v} = 0 \tag{19.1.2.9}$$

(a differential constraint) on the components of the vector **v**.

We point out that there are various ways to satisfy Eq. (19.1.2.9) (see Sections 19.2.1 and 19.2.2 below), which generate various types of decompositions of the system in question and lead to various numbers of the new unknown functions.

► Invariant transformations.

1°. Equations (19.1.2.5)–(19.1.2.6) are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \operatorname{curl}\boldsymbol{\Psi}^\circ, \qquad p = \widetilde{p},$$

where $\boldsymbol{\Psi}^\circ$ is an arbitrary solution of the equation $\mathrm{L}[\boldsymbol{\Psi}^\circ] = \mathbf{0}$.

2°. Equations (19.1.2.5)–(19.1.2.6) with $\sigma \neq 0$ are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi, \qquad p = \widetilde{p} - \frac{1}{\sigma}(\mathrm{L} + \mathrm{K}_1\Delta)[\Phi],$$

where the function Φ is an arbitrary solution of the equation

$$(\mathrm{M}_1\mathrm{L} + \mathrm{M}_1\mathrm{K}_1\Delta - \sigma\mathrm{M}_2\Delta)[\Phi] = 0.$$

3°. Equations (19.1.2.5)–(19.1.2.6) with $\sigma = 0$ are invariant under the transformation

$$\mathbf{u} = \widetilde{\mathbf{u}} + \nabla\Phi, \quad p = \widetilde{p} + \Theta,$$

where the functions Φ and Θ are arbitrary solutions of the equations

$$(\mathrm{L} + \mathrm{K}_1\Delta)[\Phi] = 0, \qquad \mathrm{M}_1[\Theta] + \mathrm{M}_2[\Phi] = 0.$$

19.2 First-Order Decompositions. Examples

19.2.1 Systems of Linear PDEs without Mass Forces ($\mathbf{f} = \mathbf{0}$)

► **Decomposition on the basis of two stream functions.**

In the absence of mass forces ($\mathbf{f} = \mathbf{0}$), it is convenient to set $G = 0$ in (19.1.2.2) and (19.1.2.7). In this case, the right-hand side of the vector equation (19.1.2.2) is zero, which gives three identical homogeneous equations for the components v_1, v_2, and v_3.

Let $\psi^{(1)}$ and $\psi^{(2)}$ be arbitrary solutions of two identical linear homogeneous equations

$$\mathrm{L}\,[\psi^{(1)}] = 0, \quad \mathrm{L}\,[\psi^{(2)}] = 0. \tag{19.2.1.1}$$

Then the formulas

$$v_1 = \psi^{(1)}_y, \quad v_2 = -\psi^{(1)}_x + \psi^{(2)}_z, \quad v_3 = -\psi^{(2)}_y, \tag{19.2.1.2}$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are arbitrary solutions of Eqs. (19.2.1.1), give a general representation of the solution of the homogeneous vector equation (19.1.2.2) (with $\mathbf{f} = \mathbf{0}$ and $G = 0$) and the scalar equation (19.1.2.9).

Every solution of system (19.1.2.5)–(19.1.2.6) with $\mathbf{f} = \mathbf{0}$ can be represented by formulas (19.1.2.1) and (19.2.1.2), where the functions $\psi^{(1)}$ and $\psi^{(2)}$ satisfy Eqs. (19.2.1.1) and the functions φ and p are described by the equations

$$\mathrm{L}\,[\varphi] + \sigma p + \mathrm{K}_1\,[\Delta\varphi] = 0, \tag{19.2.1.3}$$
$$\mathrm{M}_1\,[p] + \mathrm{M}_2\,[\Delta\varphi] = f_4. \tag{19.2.1.4}$$

These equations are a consequence of (19.1.2.7)–(19.1.2.9) with $G = 0$.

Remark 19.2. Formulas (19.2.1.2), where $\psi^{(1)}$ and $\psi^{(2)}$ are arbitrary functions, give the general solution of Eq. (19.1.2.9). The functions $\psi^{(1)}$ and $\psi^{(2)}$ can be interpreted as two stream functions that permit one to eliminate the continuity equation from the 3D incompressible fluid equations (where the fluid velocity is denoted by $\mathbf{v}$). In the special case of $\psi^{(2)} = 0$, in (19.2.1.2) we obtain the usual representation of the fluid velocity components for 2D planar flows ($v_3 = 0$) with a single stream function.

For $\sigma \neq 0$, one can eliminate p from (19.2.1.4) with the use of (19.2.1.3) and obtain a separate (independent) equation for φ,

$$-\mathrm{M}_1\mathrm{L}\,[\varphi] + (\sigma\mathrm{M}_2 - \mathrm{M}_1\mathrm{K}_1)\,[\Delta\varphi] = \sigma f_4. \tag{19.2.1.5}$$

Here p can be found without quadratures by the formula

$$p = -\frac{1}{\sigma}\Big(\mathrm{L}\,[\varphi] + \mathrm{K}_1\,[\Delta\varphi]\Big). \tag{19.2.1.6}$$

Thus, the general solution of system (19.1.2.5)–(19.1.2.6) with $\mathbf{f} = \mathbf{0}$ consisting of four 3D coupled equations can be expressed via the solution of three independent equations (two identical equations of the form (19.2.1.1) for the stream functions $\psi^{(1)}$ and $\psi^{(2)}$ and Eq. (19.2.1.5) for φ). The above-described representation of the solution corresponds to a first-order complete decomposition of the original system.

Example 19.1. The magnetic field intensity $\mathbf{H}$ in a medium is described by the overdetermined system of homogeneous equations (which are a corollary of the Maxwell equations)

$$\begin{gathered}\mu(\varepsilon\mathbf{H}_{tt} + \lambda\mathbf{H}_t) - \Delta\mathbf{H} = \mathbf{0},\\ \operatorname{div}\mathbf{H} = 0,\end{gathered} \tag{19.2.1.7}$$

where ε and μ are the dielectric constant and the magnetic permeability of the medium and λ is the medium conductivity. (For a perfect dielectric, i.e., for a nonconducting medium, one should set $\lambda = 0$.)

Equations (19.2.1.7) are a special case of the overdetermined system consisting of the homogeneous vector equation (19.1.2.2) and the scalar equation (19.1.2.9) with

$$\mathbf{v} = \mathbf{H},\quad \mathrm{L}[\mathbf{v}] = \mu(\varepsilon\mathbf{v}_{tt} + \lambda\mathbf{v}_t) - \Delta\mathbf{v},\quad \mathbf{f} = \mathbf{0},\quad G = 0.$$

Hence the general solution of system (19.2.1.7) is determined by formulas (19.2.1.2), where $\psi^{(1)}$ and $\psi^{(2)}$ are arbitrary solutions of Eqs. (19.2.1.1) with $\mathrm{L}[\psi] = \mu(\varepsilon\psi_{tt} + \lambda\psi_t) - \Delta\psi$.

Example 19.2. The Stokes equations describing slow motions of a viscous compressible barotropic fluid in the absence of mass forces have the form

$$\begin{gathered}\rho_0\mathbf{u}_t = -\nabla p + \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u},\\ p_t + c^2\rho_0\operatorname{div}\mathbf{u} = 0,\end{gathered} \tag{19.2.1.8}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity, ρ_0 is the unperturbed density, p is the pressure, μ and λ are the dynamic viscosity coefficients, and $c = \sqrt{p'(\rho)}\,|_{\rho=\rho_0}$ is the speed of sound.

Equations (19.2.1.8) are the special case of Eqs. (19.1.2.5)–(19.1.2.6) for

$$\begin{gathered}\mathrm{L}[u] = \rho_0 u_t - \mu\Delta u,\quad \mathrm{K}_1[q] = -(\mu+\lambda)q,\quad \mathrm{M}_1[p] = p_t,\quad \mathrm{M}_2[q] = \rho_0 c^2 q,\\ \sigma = 1,\quad \mathbf{f} = \mathbf{0},\quad f_4 = 0.\end{gathered} \tag{19.2.1.9}$$

Hence the solution of these equations can be represented by formulas (19.1.2.1) and (19.2.1.2), where the unknown functions $\psi^{(1)}$, $\psi^{(2)}$, and φ satisfy the three independent equations

$$\rho_0\psi^{(1)}_t - \mu\Delta\psi^{(1)} = 0,\qquad \rho_0\psi^{(2)}_t - \mu\Delta\psi^{(2)} = 0, \tag{19.2.1.10}$$

$$\rho_0\varphi_{tt} - (2\mu+\lambda)\Delta\varphi_t - \rho_0 c^2\Delta\varphi = 0 \tag{19.2.1.11}$$

and the pressure p is expressed via φ by the formula

$$p = (2\mu+\lambda)\Delta\varphi - \rho_0\varphi_t. \tag{19.2.1.12}$$

To derive Eqs. (19.2.1.10)–(19.2.1.11) and formula (19.2.1.12), one substitutes relations (19.2.1.9) into (19.2.1.1), (19.2.1.5), and (19.2.1.6).

► **The summary table of equations for determining the functions p and φ.**

Table 19.2 shows the final equations (19.1.2.3) and (19.1.2.4) with $G = 0$ which are stated for the functions p and φ for systems (19.1.1.1)–(19.1.1.2) listed in Table 19.1 in the absence of mass forces ($\mathbf{f} = \mathbf{0}$). The numbering of systems in Tables 19.1 and 19.2 is the same. The components v_1, v_2, v_3 of the vector function $\mathbf{v}$ are determined independently by formulas (19.2.1.2), where $\psi^{(1)}$ and $\psi^{(2)}$ are arbitrary solutions of Eq. (19.2.1.1), and the unknowns u_1, u_2, and u_3 can be computed from Eq. (19.1.2.1). In the last column, we use the notation ψ for both $\psi^{(1)}$ and $\psi^{(2)}$.

In what follows, some comments and examples pertaining to Table 19.2 are considered.

1°. For viscous and viscoelastic incompressible fluids (see rows 1, 2 and 5, 6 in Table 19.2), the pressure is determined without quadratures. In the right part of the corresponding formulas for p, one can add an arbitrary function $p_0(t)$ of time.

2°. For all systems listed in Table 19.2 and containing the pressure p, it is possible to obtain a separate equation for φ and a formula for p.

3°. Let us simplify the equations for ψ given in line 4 of Table 19.2. To this end, we integrate it with respect to time t. As a result, we obtain

$$\psi_t - \nu\Delta\psi = q(t), \tag{19.2.1.13}$$

where $q(t)$ is an arbitrary function. The general solution of Eq. (19.2.1.13) can be represented as follows:

$$\psi = \widetilde{\psi} + \int q(t)\,dt, \tag{19.2.1.14}$$

where $\widetilde{\psi}$ is the general solution of the homogeneous equation (19.2.1.13) with $q(t) \equiv 0$. Formula (19.2.1.2) determining the components of the velocity (19.1.2.1) includes the partial derivatives of ψ with respect to the spatial variables alone. The second term in (19.2.1.14), which depends on time and does not depend on spatial coordinates, does not affect the final result. Therefore, one can set $q(t) \equiv 0$ in Eq. (19.2.1.13) without loss of generality.

4°. For the solvability of the overdetermined system for the electric field (see Eqs. (6) in Section 12.17) two equations for the function φ (see row 15 in Table 19.2) have to be consistent,

$$\varepsilon\varphi_{tt} + \lambda\varphi_t = 0, \qquad \Delta\varphi = \rho/\varepsilon. \tag{19.2.1.15}$$

The consistency condition (19.2.1.15) specifies the permissible density distribution of the electric charge, which is determined by formula (4) in Section 12.17. In this case, the consistent solution of Eqs. (19.2.1.15) has the form

$$\varphi = \varphi_0(\mathbf{x}) + \varphi_1(\mathbf{x})\exp(-\lambda t/\varepsilon),$$

where the functions $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ satisfied the Laplace and Poisson equations

$$\Delta\varphi_0 = 0, \qquad \Delta\varphi_1 = f_1(\mathbf{x})/\varepsilon.$$

TABLE 19.2
Equations for functions p and φ for systems of the form (19.1.1.1)–(19.1.1.2), occurring in continuum mechanics and physics. The operators L and K as well as equation (19.1.1.2) are described in Table 19.1.

No.	Equation names	Equation (19.1.2.3)	Equation (19.1.2.4)	Equations (19.2.1.1)
1	Stokes (viscous incompressible fluid)	$p=-\rho(\varphi_t-\nu\Delta\varphi)$	$\Delta\varphi=0$	$\psi_t=\nu\Delta\psi$
2	Oseen (viscous incompressible fluid)	$p=-\rho(\varphi_t+b\varphi_x-\nu\Delta\varphi)$	$\Delta\varphi=0$	$\psi_t+b\psi_x=\nu\Delta\psi$
3	Stokes (viscous compressible fluid)	$p=-\rho_0\varphi_t+(2\mu+\lambda)\Delta\varphi$	$kp_t+\rho_0\Delta\varphi=0$	$\rho_0\psi_t=\mu\Delta\psi$
4	Stokes (viscous compressible fluid)	$\varphi_{tt}-c^2\Delta\varphi-(2\nu+\varkappa)\Delta\varphi_t=0$	Absent	$\psi_{tt}=\nu\Delta\psi_t$
5	Maxwell (viscoelastic incompressible fluid)	$p=-\rho(\tau\varphi_{tt}+\varphi_t-\nu\Delta\varphi)$	$\Delta\varphi=0$	$\tau\psi_{tt}+\psi_t=\nu\Delta\psi$
6	Viscoelastic incompressible fluid (general model)	$p=-\rho\mathrm{L}[\varphi]$	$\Delta\varphi=0$	$\mathrm{L}[\psi]=0$
7	Navier (linear elasticity)	$\rho\varphi_{tt}=(2\mu+\lambda)\Delta\varphi$	Absent	$\rho\psi_{tt}=\mu\Delta\psi$
8	Thermoelasticity; p is the temperature	$\alpha p=-\rho\varphi_{tt}+(2\mu+\lambda)\Delta\varphi$	$p_t=a\Delta p-\beta\Delta\varphi_t$	$\rho\psi_{tt}=\mu\Delta\psi$
9	Thermoelasticity (with hyperbolic heat conductivity); p is the temperature	$\alpha p=-\rho\varphi_{tt}+(2\mu+\lambda)\Delta\varphi$	$\tau p_{tt}+p_t=a\Delta p-\beta\Delta\varphi_t$	$\rho\psi_{tt}=\mu\Delta\psi$
10	Thermoelasticity (Green–Naghdi model); p is the temperature	$\alpha p=-\rho\varphi_{tt}+(2\mu+\lambda)\Delta\varphi$	$p_{tt}-\widetilde{a}\Delta p=-\beta\Delta\varphi_{tt}$	$\rho\psi_{tt}=\mu\Delta\psi$
11	Linear viscoelasticity	$\rho\varphi_{tt}=(\mu_0+2\lambda_0)\Delta\varphi+(\mu_1+2\lambda_1)\Delta\varphi_t$	Absent	$\rho\psi_{tt}=\mu_0\Delta\psi+\mu_1\Delta\psi_t$
12	Linear viscoelasticity (general model)	$\rho\varphi_{tt}=(\mathrm{I}_\mu+2\mathrm{I}_\lambda)[\Delta\varphi]$	Absent	$\rho\psi_{tt}=\mathrm{I}_\mu[\Delta\psi]$
13	Linear thermoviscoelasticity (general model)	$\alpha p=-\rho\varphi_{tt}+(\mathrm{I}_\mu+2\mathrm{I}_\lambda)[\Delta\varphi]$	$p_t=a\Delta p-\beta\Delta\varphi_t$	$\rho\psi_{tt}=\mathrm{I}_\mu[\Delta\psi]$
14	Maxwell (electrodynamics)	Satisfied identically	$\varphi=0$	$\mu(\varepsilon\psi_{tt}+\lambda\psi_t)=\Delta\psi$
15	Maxwell (electrodynamics)	$\varepsilon\varphi_{tt}+\lambda\varphi_t=0$	$\Delta\varphi=\rho/\varepsilon$	$\mu(\varepsilon\psi_{tt}+\lambda\psi_t)=\Delta\psi$

► **Using the Stokes–Helmholtz representation of the vector u.**

The solution is sought in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad p = p, \tag{19.2.1.16}$$

where φ and $\boldsymbol{\Psi}$ are the new unknown scalar and vector function, respectively. In this case, $\mathbf{v} = \operatorname{curl}\boldsymbol{\Psi}$, and hence the additional condition (19.1.2.9) is satisfied.

By substituting (19.2.1.16) into system (19.1.2.5)–(19.1.2.6) with $\mathbf{f} = \mathbf{0}$, we obtain the vector equation

$$\mathrm{L}[\boldsymbol{\Psi}] = 0 \tag{19.2.1.17}$$

and two equations (19.2.1.3)–(19.2.1.4).

The representation via the stream functions in the form (19.1.2.1), (19.2.1.2) is a special case of the Stokes–Helmholtz decomposition (19.2.1.16) with one zero component of the vector $\boldsymbol{\Psi}$,

$$\boldsymbol{\Psi} = (\Psi_1, 0, \Psi_3), \quad \Psi_1 = \psi^{(2)}, \quad \Psi_3 = \psi^{(1)}. \tag{19.2.1.18}$$

Therefore, the representation (19.1.2.1), (19.2.1.2) can be referred to as the *truncated Stokes–Helmholtz decomposition*.

We point out that the Stokes–Helmholtz representation of the vector **u** leads to the vector equation (19.2.1.17), which is equivalent to three independent scalar equations for its components Ψ_1, Ψ_2, and Ψ_3, while the truncated Stokes–Helmholtz decomposition (19.1.2.1), (19.2.1.2) gives two scalar equations for the stream functions $\psi^{(1)}$ and $\psi^{(2)}$ (i.e., *fewer equations*).

► **Special types of decompositions using representations of the vector u via three scalar functions.**

Let us describe four decompositions of system (19.1.2.5)–(19.1.2.6) with $\mathbf{f} = \mathbf{0}$ based on the representation of solutions with the use of the formulas

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}_n, \qquad p = p, \tag{19.2.1.19}$$

where the vector functions $\boldsymbol{\Psi}_n$ $(n = 1,\, 2,\, 3,\, 4)$ are expressed in a special way via two scalar functions $\psi^{(1)}$ and $\psi^{(2)}$,

1. $\boldsymbol{\Psi}_1 = \mathbf{a}\psi^{(1)} + \mathbf{b}\psi^{(2)} \qquad (\mathbf{a}\cdot\mathbf{b} \neq 0)$,
2. $\boldsymbol{\Psi}_2 = \mathbf{a}\psi^{(1)} + \operatorname{curl}(\mathbf{b}\psi^{(2)})$,
3. $\boldsymbol{\Psi}_3 = \mathbf{a}\psi^{(1)} + \mathbf{x}\psi^{(2)}$,
4. $\boldsymbol{\Psi}_4 = \mathbf{x}\psi^{(1)} + \operatorname{curl}(\mathbf{x}\psi^{(2)})$.

Here **a** and **b** are arbitrary nonzero constant vectors. The vector functions $\boldsymbol{\Psi}_3$ and $\boldsymbol{\Psi}_4$ can be used for systems (19.1.2.5)–(19.1.2.6) with an operator L of the special form

$$\mathrm{L}[u] = \mathrm{L}_1[u] + \mathrm{L}_2[\Delta u],$$

where L_1 and L_2 are linear differential operators in time t. For operators of this form, the following two identities have been used in the derivation of the definitive results stated below:

$$\Delta(\mathbf{x}g) \equiv \mathbf{x}\Delta g + 2\nabla g, \qquad \operatorname{curl}[\Delta(\mathbf{x}g)] \equiv \operatorname{curl}(\mathbf{x}\Delta g),$$

where g is an arbitrary scalar function.

For all solutions of system (19.1.2.5)–(19.1.2.6) based on formula (19.2.1.19) and for four distinct representations of the vector function $\mathbf{\Psi}_n$, the unknown functions $\psi^{(1)}$ and $\psi^{(2)}$ satisfy two independent (identical) equations (19.2.1.1), and the functions φ and p are described by Eqs. (19.2.1.3)–(19.2.1.4).

Note that the above-described four special representations of solutions determined by formula (19.2.1.19) contain one unknown function less than the Stokes–Helmholtz representation (19.2.1.16).

19.2.2 Systems of Linear PDEs with Mass Forces

► Simplest case of potential mass forces.

If the mass forces are potential, i.e., if

$$\mathbf{f} = \nabla F, \tag{19.2.2.1}$$

then one should set $G = F$ in (19.1.2.2)–(19.1.2.3). Then the right-hand sides of equations (19.1.2.2) are zero, just as in the absence of mass forces.

The solution of system (19.1.2.5)–(19.1.2.6), (19.2.2.1) can be represented by the formulas (19.1.2.1) and (19.2.1.2), where the functions $\psi^{(1)}$ and $\psi^{(2)}$ satisfy Eqs. (19.2.1.1) and the functions φ and p are described by Eq. (19.2.1.3) whose right-hand side is supplemented with the function F and by Eq. (19.2.1.4).

► Using the Stokes–Helmholtz representation of the mass force.

Let us use the Stokes–Helmholtz representation of the mass force in the form of the sum of a potential and a solenoidal vector,

$$\mathbf{f} = \nabla\gamma + \operatorname{curl}\boldsymbol{\omega}. \tag{19.2.2.2}$$

Remark 19.3. If the vector function $\mathbf{f}$ is given, then the scalar function γ and the vector function $\boldsymbol{\omega}$ can be sought, say, in the form

$$\gamma = \operatorname{div}\mathbf{U}, \qquad \boldsymbol{\omega} = -\operatorname{curl}\mathbf{U}. \tag{19.2.2.3}$$

By substituting (19.2.2.3) into (19.2.2.2) and by taking into account the identity $\operatorname{curl}\operatorname{curl}\mathbf{U} = \nabla\operatorname{div}\mathbf{U} - \Delta\mathbf{U}$, we obtain the Poisson vector equation

$$\Delta\mathbf{U} = \mathbf{f} \tag{19.2.2.4}$$

for the *Lamé vector potential* $\mathbf{U}$.

We seek a solution of system (19.1.2.5)–(19.1.2.6) in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad p = p. \tag{19.2.2.5}$$

As a result, in view of (19.2.2.2), for the unknown variables φ, $\boldsymbol{\Psi}$, and p we obtain the equations

$$\begin{aligned} \mathrm{L}[\boldsymbol{\Psi}] &= \boldsymbol{\omega}, \\ \mathrm{L}[\varphi] + \sigma p + \mathrm{K}_1[\Delta\varphi] &= \gamma, \\ \mathrm{M}_1[p] + \mathrm{M}_2\,[\Delta\varphi] &= f_4. \end{aligned} \tag{19.2.2.6}$$

For $\sigma \neq 0$, one can eliminate p from the last two equations in (19.2.2.6) and obtain an independent equation for φ. The function p can be expressed via φ without quadratures with the help of the second equation in (19.2.2.6). Thus, in this case we have a complete decomposition of system (19.1.2.5)–(19.1.2.6).

Example 19.3. The coupled equations of linear elastodynamics in the small-strain approximation for an isotropic medium read

$$\rho\mathbf{u}_{tt} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \rho\mathbf{f}, \tag{19.2.2.7}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector determining the strain of the body, t is time, ρ is the medium density, λ and μ are the Lamé elastic moduli (μ is the shear modulus), and $\mathbf{f} = (f_1, f_2, f_3)$ is the mass force.

The division by ρ reduces system (19.2.2.7) to the special case of system (19.1.2.5)–(19.1.2.6) with

$$\mathrm{L}[u] = u_{tt} - c_2^2\Delta u, \quad \sigma = 0, \quad \mathrm{K}_1[q] = (c_2^2 - c_1^2)q, \quad \mathrm{M}_1 = \mathrm{M}_2 = 0, \quad f_4 = 0, \tag{19.2.2.8}$$

where $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ is the velocity of longitudinal waves (or p-waves, or primary waves, because these waves are registered first in geophysics) and $c_2 = \sqrt{\mu/\rho}$ is the velocity of transverse waves (or s-waves, or secondary waves).

Let the mass force be written in the form of the Stokes–Helmholtz decomposition (19.2.2.2). Then the solution of the coupled equations (19.2.2.7) can be represented by the first formula in (19.2.2.5) for $\mathbf{u}$. By substituting (19.2.2.8) into (19.2.2.6), we obtain inhomogeneous wave equations for the scalar and vector functions φ and $\boldsymbol{\Psi}$,

$$\Box_1[\varphi] = \gamma, \qquad \Box_2[\boldsymbol{\Psi}] = \boldsymbol{\omega}. \tag{19.2.2.9}$$

Here and in what follows, the *d'Alembert operators* $\Box_1$ and $\Box_2$ are defined by

$$\Box_1 \equiv \partial_t^2 - c_1^2\Delta, \qquad \Box_2 \equiv \partial_t^2 - c_2^2\Delta. \tag{19.2.2.10}$$

The first formula in (19.2.2.5) and Eqs. (19.2.2.9) are called the Green–Lamé representation and provide a complete decomposition of system (19.2.2.7).

Example 19.4. The Stokes equations, which describe slow motions of a viscous incompressible fluid, have the form

$$\begin{aligned} \mathbf{u}_t &= -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{19.2.2.11}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, u_1, u_2, and u_3 are the fluid velocity components in the Cartesian coordinates, t is time, ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, and $\mathbf{f} = (f_1, f_2, f_3)$ is the mass force.

System (19.2.2.11) can be rewritten in the form (19.1.2.5)–(19.1.2.6) with

$$\mathrm{L}[u]=u_t-\nu\Delta u,\quad \sigma=1/\rho,\quad \mathrm{K}_1=\mathrm{M}_1=0,\quad \mathrm{M}_2[q]=q,\quad f_4=0. \tag{19.2.2.12}$$

Assume that the mass force is given by the Stokes–Helmholtz representation (19.2.2.2). The solution of the Stokes equations (19.2.2.11) can be represented in the form (19.2.2.5). By substituting (19.2.2.12) into the first and third equations in (19.2.2.6), we obtain independent equations for $\mathbf{\Psi}$ and φ,

$$\begin{aligned}\mathbf{\Psi}_t-\nu\Delta\mathbf{\Psi}&=\boldsymbol{\omega},\\ \Delta\varphi&=0.\end{aligned} \tag{19.2.2.13}$$

We derive the formula

$$p=\rho(\nu\Delta\varphi-\varphi_t+\gamma)=\rho(\gamma-\varphi_t) \tag{19.2.2.14}$$

expressing the pressure p via the function φ from the second equation in (19.2.2.6) with regard to (19.2.2.12).

One can add an arbitrary function $p_0(t)$ to the right-hand side of this formula.

Formulas (19.2.2.5) and (19.2.2.14) and Eqs. (19.2.2.13) determine a complete decomposition of system (19.2.2.11).

19.3 Higher-Order Decompositions

19.3.1 Decomposition of Systems Consisting of One Vector Equation

Consider the vector equation

$$\mathrm{L}[\mathbf{u}]+\nabla\mathrm{K}[\operatorname{div}\mathbf{u}]=\mathbf{f}, \tag{19.3.1.1}$$

which is a system of three coupled scalar equations. Here we assume that L and K are linear differential operators with constant coefficients.

We seek a solution of the vector equation (19.3.1.1) in the form

$$\mathbf{u}=\mathrm{Q}_1[\mathbf{w}]+\nabla\mathrm{Q}_2[\operatorname{div}\mathbf{w}], \tag{19.3.1.2}$$

where $\mathbf{w}$ is the new unknown vector function and Q_1 and Q_2 are linear differential operators with constant coefficients to be determined. We substitute (19.3.1.2) into (19.3.1.1) with regard to the relation $\operatorname{div}\mathbf{u}=\big(\mathrm{Q}_1+\Delta\mathrm{Q}_2\big)[\operatorname{div}\mathbf{w}]$ and obtain, after some transformations and a rearrangement of terms to which the gradient operator is applied,

$$\mathrm{LQ}_1[\mathbf{w}]+\nabla\big[\mathrm{KQ}_1+(\mathrm{L}+\Delta\mathrm{K})\mathrm{Q}_2\big][\operatorname{div}\mathbf{w}]=\mathbf{f}. \tag{19.3.1.3}$$

By equating the expression in brackets after the gradient symbol to zero, we obtain the operator equation $\mathrm{KQ}_1+(\mathrm{L}+\Delta\mathrm{K})\mathrm{Q}_2=0$, whose solution can be represented in the form

$$\mathrm{Q}_1=\mathrm{L}+\Delta\mathrm{K},\qquad \mathrm{Q}_2=-\mathrm{K}.$$

By substituting these operators into (19.3.1.2) and (19.3.1.3), we obtain the following representation of the solution of Eq. (19.3.1.1):

$$\mathbf{u}=\big(\mathrm{L}+\Delta\mathrm{K}\big)[\mathbf{w}]-\nabla\mathrm{K}[\operatorname{div}\mathbf{w}], \tag{19.3.1.4}$$

where the vector function $\mathbf{w}$ satisfies the equation

$$\mathrm{L}\big(\mathrm{L}+\Delta\mathrm{K}\big)[\mathbf{w}]=\mathbf{f}. \qquad (19.3.1.5)$$

The representation of the solution in the form (19.3.1.4) depends on three differential operators L, K, and Δ. The order of these operators determines the order of the decomposition. The representation (19.3.1.4) of the solution does not require the preliminary decomposition (19.2.2.2) of the mass force $\mathbf{f}$ into potential and solenoidal parts. The vector equation (19.3.1.5) consists of three independent scalar equations; i.e., we have obtained a complete decomposition of the original system (19.3.1.1) in this case. Other equivalent representations of solutions can be obtained from (19.3.1.4) and (19.3.1.5) by the substitution $\mathbf{w}=\mathrm{R}[\widetilde{\mathbf{w}}]$, where R is a linear operator.

Example 19.5. The vector equation (19.2.2.7) of elasticity theory is the special case of equation (19.3.1.1), in which the determining operators have the form

$$\mathrm{L}[u]=u_{tt}-c_2^2\Delta u, \quad \mathrm{K}[q]=(c_2^2-c_1^2)q. \qquad (19.3.1.6)$$

By substituting (19.3.1.6) into (19.3.1.4)–(19.3.1.5), we obtain the *Cauchy–Kovalevskaya solution*

$$\mathbf{u}=\Box_1[\mathbf{w}]+(c_1^2-c_2^2)\nabla\,\mathrm{div}\,\mathbf{w}, \qquad (19.3.1.7)$$

where the vector function $\mathbf{w}$ satisfies the equation

$$\Box_2\Box_1[\mathbf{w}]=\mathbf{f}. \qquad (19.3.1.8)$$

The d'Alembert operators $\Box_1$ and $\Box_2$ are defined in (19.2.2.10).

Remark 19.4. The general solution of the homogeneous equation (19.3.1.5) (for $\mathbf{f}=\mathbf{0}$) can be represented as the sum $\mathbf{w}=\mathbf{w}_1+\mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$\mathrm{L}[\mathbf{w}_1]=\mathbf{0}, \quad \big(\mathrm{L}+\Delta\mathrm{K}\big)[\mathbf{w}_2]=\mathbf{0}.$$

19.3.2 Decomposition of Systems Consisting of a Vector Equation and a Scalar Equation (the First Approach)

Consider the system consisting of the vector and scalar equations (19.1.2.5)–(19.1.2.6), where $\sigma\neq 0$ and L, K_1, M_1, and M_2 are linear differential operators with constant coefficients ($\mathrm{M}_1\not\equiv 0$ and $\mathrm{M}_2\not\equiv 0$).

We seek the solution in the form

$$\mathbf{u}=\mathrm{Q}_1[\mathbf{w}]+\nabla\big(\varphi+\mathrm{Q}_2[\mathrm{div}\,\mathbf{w}]\big), \quad p=\mathrm{Q}_3[\varphi]+\mathrm{Q}_4[\mathrm{div}\,\mathbf{w}], \qquad (19.3.2.1)$$

where $\mathbf{w}$ and φ are the new unknown vector and scalar functions and Q_1, Q_2, Q_3, and Q_4 are linear differential operators with constant coefficients to be determined. We substitute (19.3.2.1) into (19.1.2.5)–(19.1.2.6), take into account the relation

$$\mathrm{div}\,\mathbf{u}=\Delta\varphi+\big(\mathrm{Q}_1+\Delta\mathrm{Q}_2\big)[\mathrm{div}\,\mathbf{w}],$$

and obtain, after some transformations and a rearrangement of terms to which the gradient operator is applied,

$$\begin{aligned}\mathrm{LQ}_1[\mathbf{w}] + \nabla\Big\{&\underbrace{\sigma\mathrm{Q}_3[\varphi] + \mathrm{L}[\varphi] + \Delta\mathrm{K}_1[\varphi]}\\ &+ \underbrace{\sigma\mathrm{Q}_4[\operatorname{div}\mathbf{w}] + \mathrm{LQ}_2[\operatorname{div}\mathbf{w}] + \mathrm{K}_1\big(\mathrm{Q}_1 + \Delta\mathrm{Q}_2\big)[\operatorname{div}\mathbf{w}]}\Big\} = \mathbf{f},\\ \mathrm{M}_1\mathrm{Q}_3[\varphi] + \Delta\mathrm{M}_2[\varphi] + &\underbrace{\mathrm{M}_1\mathrm{Q}_4[\operatorname{div}\mathbf{w}] + \mathrm{M}_2\big(\mathrm{Q}_1 + \Delta\mathrm{Q}_2\big)[\operatorname{div}\mathbf{w}]} = g.\end{aligned}$$

By equating the sums of such terms (underbraced) containing φ and $\operatorname{div}\mathbf{w}$ with zero, we arrive at the equations

$$\begin{aligned}&\sigma\mathrm{Q}_3[\varphi] + \mathrm{L}[\varphi] + \Delta\mathrm{K}_1[\varphi] = 0,\\ &\mathrm{M}_1\mathrm{Q}_3[\varphi] + \Delta\mathrm{M}_2[\varphi] = g,\\ &\sigma\mathrm{Q}_4 + \mathrm{LQ}_2 + \mathrm{K}_1\big(\mathrm{Q}_1 + \Delta\mathrm{Q}_2\big) = 0,\\ &\mathrm{M}_1\mathrm{Q}_4 + \mathrm{M}_2\big(\mathrm{Q}_1 + \Delta\mathrm{Q}_2\big) = 0.\end{aligned}$$

Hence we find the operators

$$\begin{aligned}&\mathrm{Q}_1 = \mathrm{M}_1\mathrm{L} + \Delta\mathrm{M}_1\mathrm{K}_1 - \sigma\Delta\mathrm{M}_2, \quad \mathrm{Q}_2 = \sigma\mathrm{M}_2 - \mathrm{M}_1\mathrm{K}_1,\\ &\mathrm{Q}_3 = -\frac{1}{\sigma}\big(\mathrm{L} + \Delta\mathrm{K}_1\big), \quad \mathrm{Q}_4 = -\mathrm{M}_2\mathrm{L}\end{aligned} \tag{19.3.2.2}$$

determining the form of the solution (19.3.2.1). As a result, we obtain independent equations for the unknown scalar and vector functions $\mathbf{w}$ and φ,

$$\mathrm{LQ}_1[\mathbf{w}] = \mathbf{f}, \tag{19.3.2.3}$$

$$\mathrm{Q}_1[\varphi] = -\sigma f_4. \tag{19.3.2.4}$$

Remark 19.5. The general solution of the homogeneous equation (19.3.2.3) (for $\mathbf{f} = \mathbf{0}$) can be represented as the sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are arbitrary solutions of the two simpler equations

$$\mathrm{L}[\mathbf{w}_1] = \mathbf{0}, \quad \mathrm{Q}_1[\mathbf{w}_2] = \mathbf{0}.$$

Remark 19.6. Examples of decompositions based on formulas (19.3.2.1) can be found in Sections 12.12.3, 12.14.3, and 12.15.3, where specific systems of the form (19.1.2.5)–(19.1.2.6) are considered.

19.3.3 Decomposition of Systems Consisting of a Vector Equation and a Scalar Equation (the Second Approach)

We again consider a system of the form (19.1.2.5)–(19.1.2.6) with $f_4 = 0$, where L, K_1, M_1, and M_2 are constant coefficient linear differential operators ($\mathrm{M}_1 \not\equiv 0$ and $\mathrm{M}_2 \not\equiv 0$).

Let us reduce system (19.1.2.5)–(19.1.2.6) to a single equation of the form (19.3.1.1). To this end, set

$$\mathbf{u} = \mathrm{M}_1[\mathbf{v}], \quad p = -\mathrm{M}_2[\operatorname{div}\mathbf{v}], \tag{19.3.3.1}$$

where $\mathbf{v}$ is the new unknown vector function. By substituting (19.3.3.1) into (19.1.2.5), we obtain an equation of the form (19.3.1.1),

$$\mathrm{LM}_1[\mathbf{v}] + \nabla\big(\mathrm{K}_1\mathrm{M}_1 - \sigma\mathrm{M}_2\big)[\operatorname{div}\mathbf{v}] = \mathbf{f}. \tag{19.3.3.2}$$

Equation (19.1.2.6) with $f_4 = 0$ is satisfied identically if we substitute the expressions (19.3.3.1) into it.

The comparison of (19.3.3.2) with (19.3.1.1) results in the following representation of the solution of the vector equation (19.3.3.2):

$$\mathbf{v} = \big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\mathbf{w}] + \nabla\big\{\big(\sigma\mathrm{M}_2 - \mathrm{K}_1\mathrm{M}_1\big)[\operatorname{div}\mathbf{w}]\big\}, \tag{19.3.3.3}$$

where the vector function $\mathbf{w}$ satisfies the equation

$$\mathrm{LM}_1\big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\mathbf{w}] = \mathbf{f}, \tag{19.3.3.4}$$

which consists of three independent scalar equations for the components of the vector $\mathbf{w}$.

We point out that the above-described complete decomposition of the original system consisting of four equations (19.1.2.5)–(19.1.2.6) for $f_4 = 0$ gives a representation of the solution via three components of the vector $\mathbf{w}$.

Remark 19.7. The solution (19.3.3.3) and Eq. (19.3.3.4) can be represented in terms of the original unknown $\mathbf{u} = \mathrm{M}_1[\mathbf{v}]$ (see (19.3.3.1)). To this end, let us apply the operator M_1 to (19.3.3.3) and introduce the new unknown variable $\widetilde{\mathbf{w}} = \mathrm{M}_1[\mathbf{w}]$. As a result, we obtain

$$\mathbf{u} = \big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\widetilde{\mathbf{w}}] + \nabla\big\{\big(\sigma\mathrm{M}_2 - \mathrm{K}_1\mathrm{M}_1\big)[\operatorname{div}\widetilde{\mathbf{w}}]\big\}.$$

Equation (19.3.3.4) in terms of $\widetilde{\mathbf{w}} = \mathrm{M}_1[\mathbf{w}]$ can be reduced to the simpler form

$$\mathrm{L}\big(\mathrm{LM}_1 + \Delta\mathrm{K}_1\mathrm{M}_1 - \sigma\Delta\mathrm{M}_2\big)[\widetilde{\mathbf{w}}] = \mathbf{f}.$$

◆ *Examples of decompositions of various coupled linear PDEs can be found in Sections 12.6–12.17, where specific systems of continuum mechanics and physics are considered.*

⊙ *Literature for Chapter 19*: S. Kowalevski (1885), H. Lamb (1945), P. F. Papkovich (1932), U. Neuber (1934), M. G. Slobodianskii (1959), M. E. Gurtin and E. Sternberg (1962), P. Chadwick and E. A. Trowbridge (1967), O. A. Ladyzhenskaya (1969), A. C. Eringen and E. S. Suhubi (1975), A. D. Polyanin and A. I. Zhurov (2013), I. I. Lipatov and A. D. Polyanin (2013), A. D. Polyanin and S. A. Lychev (2014 a,b, 2015), S. A. Lychev and A. D. Polyanin (2015).

Chapter 20

Some Asymptotic Results and Formulas

This chapter serves as a very brief introduction to asymptotic methods, which can be used if the differential equation, system of equations, or problem to be solved contains a small or large parameter. Examples of such problems include:

1°. The *Schrödinger equation* in quantum mechanics. The equation contains *Planck's constant* $\hbar$. If we are in a situation where Planck's constant can be viewed as a small parameter, then one can try to solve problems for the Schrödinger equation asymptotically as $\hbar \to 0$. This is known as the *semiclassical approximation*.

2°. The wave equation $u_{tt} - c^2 \Delta u = 0$ itself does not contain a small parameter. But one often considers high-frequency solutions (for the case of constant coefficients, the simplest solution of this sort is the plane wave $u = A\cos(\omega t - kx + \phi_0)$, $\omega/k = c$, with large wave number k), and then the reciprocal wave number $1/k$ is a small parameter.

The topic is very broad, and we only hint at two directions in asymptotic theory. One is *regular perturbation theory*, where the terms containing the small parameter can be viewed as lower-order terms. We consider the simplest problem of regular perturbation theory, the problem on the eigenvalues of the perturbed operator (which often occurs in quantum mechanics). In *singular perturbation theory*, the small parameter resides in the leading part of the equations (e.g., it multiplies the derivatives). Of the variety of problems belonging there, we discuss semiclassical asymptotic solutions of the Schrödinger equation. Finally, we present the stationary phase method, which is an important tool in obtaining semiclassical asymptotic solutions.

For more detailed information and other types of problems, the reader is encouraged to consult the literature cited at the end of the chapter.

20.1 Regular Perturbation Theory Formulas for the Eigenvalues

20.1.1 Statement of the Problem

Let $A(\varepsilon) = A + \varepsilon B$, $\varepsilon \in [0,1]$, be a family of self-adjoint operators in a Hilbert space $\mathbf{H}$ with inner product $(\,\cdot\,,\,\cdot\,)$ and norm $\|u\| = (u,u)^{1/2}$. Assume that the unperturbed operator $A = A(0)$ has an simple isolated eigenvalue λ_0 with the corresponding eigenvector $\varphi_0 \in \mathbf{H}$. This means that

$$A\varphi_0 = \lambda_0\varphi_0, \qquad \|\varphi_0\| = 1, \tag{20.1.1.1}$$

and any other vector $u \in \mathbf{H}$ satisfying $Au = \lambda_0 u$ is proportional to φ_0. Under certain assumptions [e.g., see Kato (1995)] for sufficiently small ε, the perturbed operator $A(\varepsilon)$ has a simple isolated eigenvalue $\lambda(\varepsilon)$ with the corresponding eigenvector $\varphi(\varepsilon)$,

$$(A + \varepsilon B)\varphi(\varepsilon) = \lambda(\varepsilon)\varphi(\varepsilon), \qquad \|\varphi(\varepsilon)\| = 1, \tag{20.1.1.2}$$

such that $\lambda(\varepsilon)$ and $\varphi(\varepsilon)$ smoothly depend on ε, $\lambda(0) = \lambda_0$, and $\varphi(0) = \varphi_0$. Given λ_0 and φ_0, the problem is to find the coefficients in the expansions

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \qquad \varphi(\varepsilon) = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots \tag{20.1.1.3}$$

without actually solving the (presumably complicated) equation (20.1.1.2).

20.1.2 Formulas for the Coefficients of the Expansion

The first few coefficients in the expansions (20.1.1.3) can be successively step by step computed by the following formulas.

Step 1. $\lambda_1 = (\varphi_0, B\varphi_0)$, and φ_1 is found from the equations

$$(A - \lambda_0)\varphi_1 = (\lambda_1 - B)\varphi_0, \qquad \mathrm{Re}(\varphi_1, \varphi_0) = 0. \tag{20.1.2.1}$$

Step 2. $\lambda_2 = (\varphi_1, (B - \lambda_1)\varphi_0)$, $\lambda_3 = (\varphi_1, (B - \lambda_1)\varphi_1)$, and φ_2 is found from the equations

$$(A - \lambda_0)\varphi_2 = (\lambda_1 - B)\varphi_1 + \lambda_2\varphi_0, \qquad \mathrm{Re}(\varphi_2, \varphi_0) = -\frac{1}{2}(\varphi_1, \varphi_1). \tag{20.1.2.2}$$

Step 3. $\lambda_4 = (\varphi_2, (B - \lambda_1)\varphi_1) + \lambda_2(\varphi_0, \varphi_2)$.

Remark 20.1. The solvability of the first equations in (20.1.2.1) and (20.1.2.2) follows from the orthogonality of their right-hand sides to φ_0. If, for example, there exists an orthonormal basis $\{e_0, e_1, e_2, \dots\}$ of eigenvectors of A in $\mathbf{H}$ with the corresponding eigenvalues μ_j, $\mu_0 = \lambda_0$, $e_0 = \varphi_0$, then the solutions of (20.1.2.1) and (20.1.2.2) can be written in the form

$$\varphi_1 = \sum_{j=1}^{\infty} \frac{(e_j, (\lambda_1 - B)\varphi_0)}{\mu_j - \mu_0} e_j, \qquad \varphi_2 = -\frac{1}{2}(\varphi_1, \varphi_1)\varphi_0 + \sum_{j=1}^{\infty} \frac{(e_j, (\lambda_1 - B)\varphi_1)}{\mu_j - \mu_0} e_j.$$

20.2 Singular Perturbation Theory

20.2.1 Cauchy Problem for the Schrödinger Equation

► Schrödinger equation.

For the nonstationary Schrödinger equation

$$-i\hbar\frac{\partial\psi}{\partial t} + \widehat{H}\psi = 0, \qquad \widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(x), \tag{20.2.1.1}$$

describing the evolution of the ψ-function of a quantum particle of mass m in the potential field $V(x)$, consider the Cauchy problem

$$\psi|_{t=0} = \psi_0(x) \qquad \text{(the initial state of the particle).} \tag{20.2.1.2}$$

► Exact solution for the quantum oscillator.

Let the quantum Hamiltonian $\widehat{H}$ have the special form

$$\widehat{H} = \frac{1}{2}\left(-\hbar^2\Delta + x^2\right), \qquad x^2 = x_1^2 + \cdots + x_n^2. \tag{20.2.1.3}$$

This Hamiltonian is known as the *quantum oscillator*; for simplicity, we assume that the mass and the elasticity coefficient are equal to unity.

The Cauchy problem (20.2.1.1)–(20.2.1.2) for the Hamiltonian (20.2.1.3) has the fundamental solution

$$\mathscr{E}(x,y,t,\hbar) = \frac{e^{-i\pi n/2}}{(2\pi\hbar\sin t)^{n/2}}\exp\left\{\frac{i}{2\hbar\sin t}\left[(x^2+y^2)\cos t - 2xy\right]\right\}, \qquad t\in(0,\pi), \tag{20.2.1.4}$$

so that the exact solution of this Cauchy problem is given by

$$\psi(x,t,\hbar) = \int\cdots\int \mathscr{E}(x,y,t,\hbar)\psi_0(y,\hbar)\,dy, \tag{20.2.1.5}$$

where, as usual, $dy = dy_1\cdots dy_n$.

► Semiclassical approximation.

Now assume that we are interested in solutions for which the Planck constant $\hbar$ can be viewed as a small parameter. This is the subject of the so-called *semiclassical approximation*. The simplest typical initial data for the semiclassical approximation has the form

$$\psi_0(x,\hbar) = e^{\frac{i}{\hbar}S_0(x)}\varphi_0(x) \tag{20.2.1.6}$$

where S_0 and φ_0 are smooth functions, S_0 is real-valued, and $\varphi_0(x)$ is zero outside a sufficiently large ball in $\mathbb{R}^n$. Such a function is known as a “WKB element” (derived from the names of Wentzel, Kramers, and Brillouin, who were apparently the first to consider this approximation). It turns out that, at least for small t, the solution of the Schrödinger equation (20.2.1.1) with this initial data can be sought asymptotically in the form

$$\psi(x,t,\hbar) = e^{\frac{i}{\hbar}S(x,t)}a(x,t). \tag{20.2.1.7}$$

More precisely, the procedure is as follows.

1°. Consider the classical Hamiltonian

$$H(x,p) = \frac{p^2}{2m} + V(x)$$

corresponding to the quantum Hamiltonian $\widehat{H}$ in (20.2.1.1) and solve the Cauchy problem for the Hamiltonian system of ODEs

$$x'_t = \frac{\partial H}{\partial p}(x,p) = \frac{p}{m}, \qquad p'_t = -\frac{\partial H}{\partial x}(x,p) = -\frac{\partial V}{\partial x}(x) \tag{20.2.1.8}$$

with the initial data

$$x|_{t=0} = x_0, \qquad p|_{t=0} = \frac{\partial S_0}{\partial x}(x_0). \tag{20.2.1.9}$$

Denote the solution by $x = X(x_0,t)$, $p = P(x_0,t)$.

2°. Construct the function

$$\Phi(x_0,t) = S_0(x_0) + \int_0^t \left(\sum_{j=1}^n P_j(x_0,\tau)\frac{\partial H}{\partial p_j}(X(x_0,\tau),P(x_0,\tau)) - H(X(x_0,\tau),P(x_0,\tau))\right) d\tau \tag{20.2.1.10}$$

and the Jacobian

$$\mathcal{J}(x_0,t) = \frac{\partial(X_1(x_0,t),\dots,X_n(x_0,t))}{\partial(x_1,\dots,x_n)} \equiv \det\left\|\frac{\partial X_i}{\partial x_j}\right\|. \tag{20.2.1.11}$$

Note that $\mathcal{J}(x_0,0) = 1$, and so the Jacobian is nonzero for sufficiently small t whenever $x_0 \in \operatorname{supp}(\varphi_0)$, where $\operatorname{supp}(\varphi_0)$ is the *support* of φ_0, that is, the closure of the set of points x_0 where $\varphi_(x_0)$ is nonzero.

3°. Solve the equation

$$x = X(x_0,t) \tag{20.2.1.12}$$

for x_0, thus obtaining the function $x_0 = x_0(x,t)$. (The equation is solvable by the implicit function theorem provided that the Jacobian (20.2.1.11) is nonzero.) Set

$$S(x,t) = \Phi(x_0(x,t),t). \tag{20.2.1.13}$$

This function is called the *action*, or *eikonal*.

4°. Now set

$$\psi(x,t) = \frac{e^{\frac{i}{\hbar}S(x,t)}\varphi_0(x_0(x,t))}{\sqrt{\mathcal{J}(x_0(x,t),t)}}. \tag{20.2.1.14}$$

This function, known as the WKB solution, gives the asymptotics of the solution of problem (20.2.1.1)–(20.2.1.2) with accuracy $O(\hbar)$.

Remark 20.2. This solution is not valid (or even defined) everywhere; on the contrary, it only holds until the Jacobian is zero. The points (x_0, t) where the Jacobian is zero are known as *focal points*. The corresponding points $(x, t) = (X(x_0, t), t)$ are known as *caustics* (sometimes they are also called focal points). See below about how to construct the asymptotics of the solution in a neighborhood of caustics.

Example 20.1. Consider the Cauchy problem with the oscillator Hamiltonian (20.2.1.3) for $n = 1$ and with the initial data (20.2.1.6) such that

$$S_0(x) = \frac{1}{2}x^2.$$

The Cauchy problem for the Hamiltonian system has the form

$$x'_t = p, \quad p'_t = -x, \quad x|_{t=0} = p|_{t=0} = x_0,$$

and its solution is given by

$$X(x_0,t)=x_0(\cos t+\sin t)=\sqrt{2}x_0 \sin\Bigl(t+\frac{\pi}{4}\Bigr), \quad P(x_0,t)=x_0(\cos t-\sin t)=\sqrt{2}x_0 \sin\Bigl(\frac{\pi}{4}-t\Bigr).$$

Next,

$$\mathcal{J}(x_0, t) = \sqrt{2}\sin\Bigl(t + \frac{\pi}{4}\Bigr), \qquad \Phi(x_0, t) = \frac{x_0^2}{2}(1 - \sin 2t).$$

Finally, we obtain the WKB asymptotic solution in the form

$$\psi(x, t) = \frac{e^{\frac{i}{\hbar}\frac{x^2(1-\sin 2t)}{\sin t+\cos t}}\varphi_0\Bigl(\dfrac{x}{\sin t + \cos t}\Bigr)}{\sqrt{\sin t + \cos t}}.$$

This solution is valid for $0 \le t \le \frac{3\pi}{4} - \varepsilon$, where $\varepsilon > 0$ is arbitrary but fixed (independent of $\hbar$).

► Solution in a neighborhood of focal points.

As we have seen from the preceding example, the WKB solution formula fails to be true in a neighborhood of the points where the Jacobian $J(x_0, t)$ vanishes. Thus, we need a different representation of the solution near focal points. We will describe this representation only for $n = 1$ (and refer the reader to the cited literature for the general case). The procedure is as follows.

1°. Let (x_{0*}, t_*) be a focal point (that is, $\mathcal{J}(x_{0*}, t_*) = 0$). Then the Jacobian

$$J_1(x_0, t) = \frac{\partial P}{\partial x_0}(x_0, t) \tag{20.2.1.15}$$

is necessarily nonzero at (x_{0*}, t_*). Then, by the implicit function theorem, we can solve the equation $p = P(x_0, t)$ for x_0, thus obtaining a function $x_0 = x_0(p, t)$.

2°. Construct the functions

$$\widetilde{\Phi}(x_0, t) = \Phi(x_0, t) - P(x_0, t)X(x_0, t) \tag{20.2.1.16}$$

and

$$\widetilde{S}(p, t) = \widetilde{\Phi}(x_0(p, t), t). \tag{20.2.1.17}$$

3°. For a function $\varphi(x_0, t)$ supported in a neighborhood of the point (x_{0*}, t_*), define the function

$$[\widetilde{K}\phi](x,t,\hbar) = \left[\bar{\mathcal{F}}^{1/\hbar}_{p\to x}\left(e^{\frac{i}{\hbar}S(p,t)}\frac{\varphi(x_0,t)}{\sqrt{|J_1(x_0,t)|}}\right)\bigg|_{x_0=x_0(p,t)}\right](x,t,\hbar), \qquad (20.2.1.18)$$

where $\bar{\mathcal{F}}^{1/\hbar}_{p\to x}$ is the Fourier transform with parameter $\hbar$ defined below in Section 20.2.3.

That is what the solution should look like near focal points. So far, we have only one problem: for this function to satisfy the equation, the function $\phi(x_0, t)$ should be independent of t, but then the expression (20.2.1.18) is not defined, because the function $x_0(p, t)$ is not defined everywhere on the support of φ. This problem will be dealt with in the next item.

► **Pasting the local solutions together.**

In fact, now we have two kinds of “local solutions,” one of the form (20.2.1.18) and the other of the form (20.2.1.14). Neither of them, taken alone, is a solution (even asymptotic) of our Cauchy problem. So we paste them together as follows.

1°. Cover the domain of the variables (x_0, t) by open disks U_j such that, for each j, either $\mathcal{J}(x_0, t) \neq 0$ in the entire U_j (these U_j will be called *nonsingular charts*) or $\mathcal{J}_1(x_0, t) \neq 0$ in the entire U_j (these U_j will be called *singular charts*). Assume also that $\mathcal{J}(x_{0j}, t_j) \neq 0$, where (x_{0j}, t_j) is the center of the disk U_j.

2°. For each disk U_j, define an integer m_j (the *Maslov index* of U_j) as follows. Let

$$\mathcal{J}_\varepsilon(x_0,t) = \frac{\partial(X(x_0,t) + i\varepsilon P(x_0,t))}{\partial x_0}, \qquad \varepsilon > 0. \qquad (20.2.1.19)$$

If U_j is nonsingular, set

$$m_j = \frac{1}{\pi}\lim_{\varepsilon\to+0}\int_0^{t_j}\mathcal{J}_\varepsilon^{-1}(x_{0j},\tau)\frac{\partial\mathcal{J}_\varepsilon}{\partial t}(x_{0j},\tau)\,d\tau. \qquad (20.2.1.20)$$

If U_j is singular, set

$$m_j = \frac{1}{\pi}\lim_{\varepsilon\to+0}\int_0^{t_j}\mathcal{J}_\varepsilon^{-1}(x_{0j},\tau)\frac{\partial\mathcal{J}_\varepsilon}{\partial t}(x_{0j},\tau)\,d\tau + \begin{cases} 0 & \text{if } \mathcal{J}(x_{0j},t_j)\mathcal{J}(x_{0j},t_j) > 0,\\ 1 & \text{if } \mathcal{J}(x_{0j},t_j)\mathcal{J}(x_{0j},t_j) < 0.\end{cases} \qquad (20.2.1.21)$$

3°. For any function $\varphi(x_0, t)$ with support in a nonsingular disk U_j, set

$$[K_j\varphi](x,t,\hbar) = e^{-i\frac{\pi}{2}m_j}\frac{e^{\frac{i}{\hbar}\Phi(x_0(x,t),t)}\varphi(x_0(x,t),t)}{\sqrt{|\mathcal{J}(x_0(x,t),t)|}}. \qquad (20.2.1.22)$$

For any function $\varphi(x_0, t)$ with support in a singular disk U_j, set

$$[K_j\varphi](x,t,\hbar) = e^{-i\frac{\pi}{2}m_j}\left[\bar{\mathcal{F}}^{1/\hbar}_{p\to x}\left(e^{\frac{i}{\hbar}\widetilde{\Phi}(x_0,t)}\frac{\varphi(x_0,t)}{\sqrt{|J_1(x_0,t)|}}\right)\bigg|_{x_0=x_0(p,t)}\right](x,t,\hbar). \qquad (20.2.1.23)$$

Here the functions $x_0(x, t)$ and $x_0(p, t)$ give the solutions of the respective equations $x = X(x_0, t)$ and $p = P(x_0, t)$ in the corresponding disks U_j.

4°. Now the asymptotics of the solution of the Cauchy problem (20.2.1.1), (20.2.1.6) is given by the formula

$$\psi(x,t,\hbar) = \sum_j K_j[e_j(x_0,t)\varphi_0(x_0)](x,t,\hbar), \tag{20.2.1.24}$$

where $\{e_j\}$ is a partition of unity subordinate to the cover $\{U_j\}$:

$$e_j(x_0,t) \in C^\infty, \qquad \operatorname{supp} e_j \subset U_j, \qquad \sum_j e_j(x_0,t) = 1.$$

Here $\operatorname{supp} e_j$ is the support of e_j, i.e., the closure of the set of points where the function $e_j(x_0,t)$ is nonzero.

The function thus constructed is independent of the partition of unity with accuracy $O(\hbar)$. This follows from the fact that, for any function φ such that, simultaneously, $\operatorname{supp}\varphi \subset U_j$ and $\operatorname{supp}\varphi \subset U_k$, one has

$$K_j\varphi = K_k\varphi + O(\hbar).$$

This can be proved by the stationary phase method presented in the next subsection.

20.2.2 Stationary Phase Method

► **Simplest case.**

Consider the integral

$$I(\hbar) = \int e^{\frac{i}{\hbar}\Phi(\theta)} a(\theta)\,d\theta,$$

where $\Phi(\theta)$ and $a(\theta)$ are smooth functions, $\Phi(\theta)$ is real-valued, and $a(\theta)$ vanishes outside some finite interval.* Let us study the behavior of this integral as $\hbar \to 0$.

1°. If $\Phi'(\theta) \neq 0$ anywhere, then $I(\hbar) = O(\hbar^\infty)$ (i.e., $|I(\hbar)| \leq C_N h^N$ for any $N > 0$).

2°. Assume that $\Phi'(\theta_*) = 0$ at some point θ_* (such a point is called a *stationary point*), $\Phi''(\theta_*) \neq 0$ (the stationary point is *nondegenerate*), and $\Phi'(\theta) \neq 0$ for $\theta \neq \theta_*$ (there are no other stationary points). Then $I(\hbar)$ has the asymptotics

$$I(\hbar) = \sqrt{2\pi\hbar}\, e^{\frac{i\pi}{4}\operatorname{sign}\Phi''(\theta_*)} e^{\frac{i}{\hbar}\Phi(\theta_*)} \left(\frac{a(\theta_*)}{\sqrt{|\Phi''(\theta_*)|}} + O(\hbar) \right), \tag{20.2.2.1}$$

where $\operatorname{sign}\xi = \begin{cases} 1, & \xi>0, \\ -1, & \xi<0 \end{cases}$ is the sign function.

3°. If $\Phi(\theta)$ has several stationary points $\theta_*^{(1)}, \ldots, \theta_*^{(s)}$, each of which is nondegenerate, then the asymptotics of $I(\hbar)$ is given by the sum of contributions (20.2.2.1) of these stationary points:

$$I(\hbar) = \sqrt{2\pi\hbar} \sum_{j=1}^{s} e^{\frac{i\pi}{4}\operatorname{sign}\Phi''(\theta_*^{(j)})} e^{\frac{i}{\hbar}\Phi(\theta_*^{(j)})} \left(\frac{a(\theta_*^{(j)})}{\sqrt{|\Phi''(\theta_*^{(j)})|}} + O(\hbar) \right).$$

*The last requirement, which is only needed to guarantee the convergence of the integral, can be weakened.

Example 20.2. Consider the integral $\int e^{\frac{i}{2\hbar}\theta^2} a(\theta)\,d\theta$. The phase function $\Phi(\theta) = \theta^2/2$ has the unique stationary point $\theta_* = 0$, and $\Phi''(0) = 1$. By (20.2.2.1),

$$\int e^{\frac{i}{2\hbar}\theta^2} a(\theta)\,d\theta = \sqrt{2\pi\hbar}\,e^{\frac{i\pi}{4}} a(0) + O(\hbar^{3/2}).$$

► Dependence on parameters.

The functions Φ and a often depend on parameters, $\Phi = \Phi(x,\theta)$ and $a = a(x,\theta)$, where x is a one- or many-dimensional parameter varying in some domain in $\mathbb{R}^n$. Consider the parameter-dependent integral

$$I(x,\hbar) = \int e^{\frac{i}{\hbar}\Phi(x,\theta)} a(x,\theta)\,d\theta,$$

Assume that, for some parameter value $x = x_*$, the function $\Phi(x,\theta)$ has a stationary point θ_* and this point is nondegenerate; that is,

$$\Phi'_\theta(x,\theta) \equiv \frac{\partial\Phi}{\partial\theta}(x,\theta) = 0 \qquad (20.2.2.2)$$

for $x = x_*$ and $\theta = \theta_*$, and $\Phi''_{\theta\theta}(x_*,\theta_*) \neq 0$. Then, by the implicit function theorem, equation (20.2.2.2) defines a unique smooth function $\theta = \Theta(x)$ in a neighborhood of x_* such that $\Theta(x_*) = \theta_*$ and there are no stationary points near (x_*,θ_*) other than $(x,\Theta(x))$. Now if there are no other stationary points on the support of $a(x,\theta)$ (for example, if a is supported in a small neighborhood of (x_*,θ_*)), then $I(x,\hbar)$ has the asymptotics

$$I(x,\hbar) = \sqrt{2\pi\hbar}\,e^{\frac{i\pi}{4}\,\mathrm{sign}\,\Phi''_{\theta\theta}(x_*,\theta_*)} e^{\frac{i}{\hbar}\Phi(x,\Theta(x))}\left(\frac{a(x,\Theta(x))}{\sqrt{|\Phi''_{\theta\theta}(x,\Theta(x))|}} + O(\hbar)\right).$$

Example 20.3. Consider the integral $\int e^{\frac{i}{\hbar}(\theta^2/2 - x\theta)} a(\theta)\,d\theta$. The stationary point equation reads

$$\theta - x = 0 \qquad \Longrightarrow \qquad \Theta(x) = x.$$

Moreover, $\Phi''_{\theta\theta}(x,x) = 1$, and we obtain

$$\int e^{\frac{i}{\hbar}\left(\frac{\theta^2}{2} - x\theta\right)} a(\theta)\,d\theta = \sqrt{2\pi\hbar}\,e^{\frac{i\pi}{4}} e^{-\frac{ix^2}{2\hbar}} a(x) + O(\hbar^{3/2}).$$

► General case.

Now consider the general case of a multiple oscillatory integral.

Let $\Phi(x,\theta)$ be a smooth real-valued function of the variables $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $\theta = (\theta_1,\dots,\theta_m) \in \mathbb{R}^m$, let $a(x,\theta)$ be a smooth function of the same parameters such that $a(x,\theta) = 0$ for $|\theta| = \sqrt{\theta_1^2 + \cdots + \theta_m^2} > R$, and let $\hbar > 0$ be a positive parameter. The function $I(x,\hbar)$ defined as the parametric integral

$$I(x,\hbar) = \frac{1}{(2\pi\hbar)^{m/2}} \int \cdots \int e^{\frac{i}{\hbar}\Phi(x,\theta)} a(x,\theta)\,d\theta, \qquad (20.2.2.3)$$

where i is the imaginary unit and $d\theta = d\theta_1 \cdots d\theta_m$, is called the *oscillatory integral with phase function* Φ *and amplitude* a. We sometimes write $I[\Phi,a](x,\hbar)$ instead of $I(x,\hbar)$ to emphasize the dependence of the integral (20.2.2.3) on the functions Φ and a occurring in its definition. The problem is to find the asymptotics of the integral (20.2.2.3) as $\hbar \to 0$.

► **Stationary points.**

A *stationary point* of the integral (20.2.2.3) (or of the phase function $\Phi(x,\theta)$) is a point (x_*,θ_*) such that

$$\frac{\partial\Phi}{\partial\theta_j}(x_*,\theta_*)=0,\qquad j=1,\dots,m. \tag{20.2.2.4}$$

A stationary point (x_*,θ_*) is said to be *nondegenerate* if

$$\det\Phi''_{\theta\theta}(x_*,\theta_*)\equiv\det\left\|\frac{\partial^2\Phi}{\partial\theta_j\partial\theta_k}(x_*,\theta_*)\right\|\neq 0. \tag{20.2.2.5}$$

If (x_*,θ_*) is a nondegenerate stationary point, then, by the implicit function theorem applied to system (20.2.2.4), there exists a smooth function $\theta=\Theta(x)$ such that, in a small neighborhood of (x_*,θ_*), all stationary points of the phase function $\Phi(x,\theta)$ have the form $(x,\Theta(x))$. In particular, $\theta_*=\Theta(x_*)$. We will denote the matrix on the left-hand side in (20.2.2.5) by $\Phi''_{\theta\theta}(x_*,\theta_*)$.

► **Oscillatory integrals without stationary points.**

Assume that there are no stationary points on the support $\operatorname{supp}a(x,\theta)$ of the amplitude $a(x,\theta)$. Then $I(x,\hbar)=O(\hbar^\infty)$. This means that $I(x,\hbar)=O(\hbar^N)$ for every $N>0$, however large.

► **Oscillatory integrals with nondegenerate stationary points.**

Assume that, for each x, the phase function $\Phi(x,\theta)$ has a unique stationary point $(x,\Theta(x))$ on the support $\operatorname{supp}a$ and this stationary point is nondegenerate, $\det\Phi''_{\theta\theta}(x,\Theta(x))\neq 0$. Then the integral (20.2.2.3) has the asymptotics

$$I(x,\hbar)=\left[e^{\frac{i\pi m}{4}\operatorname{sign}(\Phi''_{\theta\theta}(x,\theta))}\frac{e^{\frac{i}{\hbar}\Phi(x,\theta)}a(x,\theta)}{\sqrt{|\det(\Phi''_{\theta\theta}(x,\theta))|}}\right]\Bigg|_{\theta=\Theta(x)}+O(\hbar), \tag{20.2.2.6}$$

where

$$\operatorname{sign}(\Phi''_{\theta\theta}(x,\theta))=\sigma_+(\Phi''_{\theta\theta}(x,\theta))-\sigma_-(\Phi''_{\theta\theta}(x,\theta))$$

is the *signature* of the real symmetric matrix $\Phi''_{\theta\theta}(x,\theta)$, i.e., the difference between the numbers of its positive and negative eigenvalues.

Example 20.4. Consider the integral

$$I(x,\hbar)=\frac{1}{(2\pi\hbar)^n}\iint e^{\frac{i}{\hbar}p(x-y)}\chi(p)a(y)\,dy\,dp,$$

where $x=(x_1,\dots,x_n)$, $p=(p_1,\dots,p_n)$, $y=(y_1,\dots,y_n)$, and

$$p(x-y)=\sum_{j=1}^n p_j(x_j-y_j).$$

Here $m = 2n$ and $\theta = (y, p)$. The stationary point equations and the matrix $\Phi''_{\theta\theta}$ have the form

$$p = 0, \qquad x = y, \qquad \Phi''_{\theta\theta}(x, y, p) = \begin{pmatrix} 0 & -E \\ -E & 0 \end{pmatrix},$$

where E is the $n \times n$ identity matrix. The signature of this matrix is zero (it has exactly n positive and n negative eigenvalues), and so the stationary phase formula (20.2.2.6) gives $I(x, h) = \chi(0)a(x) + O(\hbar)$. Now we take a function $\chi(p)$ such that $\chi(0) = 1$, make the change of integration variable $p = \xi/\hbar$ and rewrite the result in the form

$$\frac{1}{(2\pi)^n} \iint e^{i\xi(x-y)} \chi(\hbar\xi) a(y)\, dy\, d\xi = a(x) + O(\hbar).$$

By letting $\hbar \to 0$, we obtain the Fourier transform inversion formula.

20.2.3 Fourier Transform with a Parameter

► Fourier transform.

When working with differential equations containing a small parameter $\hbar$, it is expedient to use a version of the Fourier transform which contains that parameter as well. It is called the $1/\hbar$-Fourier transform. This transform and its inverse in the case of one variable are defined by

$$[\mathcal{F}^{1/\hbar}_{x\to p} u](p) = \frac{e^{-i\pi/4}}{(2\pi\hbar)^{1/2}} \int e^{-\frac{i}{\hbar}px} u(x)\, dx, \qquad (20.2.3.1)$$

$$[\bar{\mathcal{F}}^{1/\hbar}_{p\to x} v](x) = \frac{e^{i\pi/4}}{(2\pi\hbar)^{1/2}} \int e^{\frac{i}{\hbar}px} v(p)\, dp. \qquad (20.2.3.2)$$

The corresponding n-dimensional versions ($x, p \in \mathbb{R}^n$) are

$$[\mathcal{F}^{1/\hbar}_{x\to p} u](p) = \frac{e^{-i\pi n/4}}{(2\pi\hbar)^{n/2}} \int e^{-\frac{i}{\hbar}px} u(x)\, dx, \quad [\bar{\mathcal{F}}^{1/\hbar}_{p\to x} v](x) = \frac{e^{i\pi n/4}}{(2\pi\hbar)^{n/2}} \int e^{\frac{i}{\hbar}px} v(p)\, dp,$$

where now $px = p_1x_1 + \cdots + p_nx_n$, $dx = dx_1 \cdots dx_n$, and $dp = dp_1 \cdots dp_n$.

► Commutation with the operators x and $-i\hbar\partial/\partial x$.

The $1/\hbar$-Fourier transform enjoys the usual commutation formulas

$$\Big[\mathcal{F}^{1/\hbar}_{x\to p}\Big(-i\hbar\frac{\partial u}{\partial x}\Big)\Big](p) = p[\mathcal{F}^{1/\hbar}_{x\to p} u](p), \qquad [\mathcal{F}^{1/\hbar}_{x\to p}(xu)](p) = i\hbar\frac{\partial}{\partial p}[\mathcal{F}^{1/\hbar}_{x\to p} u](p).$$

⊙ *Literature for Chapter 20*: J. Brüning, V. V. Grushin, and S. Yu. Dobrokhotov (2012), I. M. Gelfand (1989), T. Kato (1995), V. P. Maslov (1972), V. P. Maslov (1976), V. P. Maslov (1994), A. S. Mishchenko, V. E. Shatalov, and B. Yu. Sternin (1990)

Chapter 21

Elements of Theory of Generalized Functions

21.1 Generalized Functions of One Variable

21.1.1 Important Terminological Remark

The main subject of this chapter—generalized functions—has in fact two names in the literature. One name, "generalized functions," tends to be more popular among engineers and physicists; it probably goes back to Dirac and was firmly introduced in practice by Gelfand and Shilov. The other name, "distributions," mostly preferred by mathematicians, stems from the seminal work of Laurent Schwartz. We use both terms interchangeably but prefer "distributions"—just because it is shorter, one word instead of two.

21.1.2 Test Function Space

Functions vanishing outside a finite interval are said to be compactly supported. The closure of the set of points where a function $\varphi(x)$ is nonzero is called the support of φ and is denoted by $\operatorname{supp}\varphi$. The set $\{\varphi(x)\}$ of smooth compactly supported functions (sometimes referred to as *test functions*) equipped with an appropriate convergence (see below) is called the *test function space* and is denoted by $\mathcal{K}$.

Example 21.1. The following function lies in $\mathcal{K}$:

$$\varphi_a(x) = \begin{cases} \exp\left(-\dfrac{a^2}{a^2 - x^2}\right) & \text{for } |x| < a, \\ 0 & \text{for } |x| \ge a. \end{cases}$$

It has the support $\operatorname{supp}\varphi_a(x) = \{|x| \le a\}$.

A sequence of functions $\varphi_n(x) \in \mathcal{K}$ is said to converge to a function $\varphi(x) \in \mathcal{K}$ if (i) there exists a finite interval containing the supports of all $\varphi_n(x)$; (ii) $\frac{d^m}{dx^m}\varphi_n(x) \rightrightarrows \frac{d^m}{dx^m}\varphi(x)$ as $n \to \infty$ for $m = 0,\ 1,\ \dots$. In this case, one writes $\varphi_n(x) \xrightarrow{\mathcal{K}} \varphi(x)$ as $n \to \infty$.

21.1.3 Distribution Space. Dirac Delta Function

► Regular and singular distributions. Some theorems.

The *distribution space*, or *space of generalized functions* $\mathcal{K}'$, is introduced as follows. Ordinary locally integrable functions $f(x)$ are identified in $\mathcal{K}'$ with continuous linear functionals (referred to as *regular distributions*, or *regular generalized functions*) of the form

$$(f,\varphi)=\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx, \qquad (21.1.3.1)$$

where $\varphi(x)\in\mathcal{K}$ is arbitrary. Every regular distribution corresponds to a unique (up to values of a set of measure zero) locally integrable function on $\mathbb{R}^1$.

Continuous linear functionals on $\mathcal{K}$ that cannot be represented in the form (21.1.3.1) with a locally integrable f are called *singular generalized functions*, or *singular distributions*.

The convergence in $\mathcal{K}'$ is defined as the *weak convergence of functionals*: a sequence of distributions $f_n(x)\in\mathcal{K}'$ converges to a distribution $f(x)\in\mathcal{K}'$ if $(f_n,\varphi)\to(f,\varphi)$ as $n\to\infty$ for each $\varphi(x)\in\mathcal{K}$.

THEOREM ON THE COMPLETENESS OF THE SPACE $\mathcal{K}'$. *Let $f_n\in\mathcal{K}'$ be a sequence of distributions such that the numerical sequence (f_n,φ) converges as $n\to\infty$ for each $\varphi\in\mathcal{K}$. Then the functional f defined by*

$$(f,\varphi)=\lim_{n\to\infty}(f_n,\varphi), \qquad \varphi\in\mathcal{K},$$

is linear and continuous on $\mathcal{K}$ as well; *i.e., $f\in\mathcal{K}'$.*

THEOREM ON THE REPRESENTATION OF DISTRIBUTIONS. *Each distribution $f\in\mathcal{K}'$ is a weak limit of test functions $f_n\in\mathcal{K}$; i.e., the set $\mathcal{K}$ is dense in $\mathcal{K}'$.*

► Dirac delta function and delta sequences.

The theorem on the representation of distributions permits one to construct singular distributions with the use of appropriate sequences of test functions.

Example 21.2. Consider the sequence of integrable discontinuous functions

$$f_n(x)=\begin{cases}\varepsilon^{-1} & \text{for } 0\le x\le\varepsilon,\\ 0 & \text{for } x<0 \text{ or } x>\varepsilon,\end{cases} \qquad \text{where} \quad \varepsilon=\frac{1}{n}. \qquad (21.1.3.2)$$

Let us show that

$$\lim_{n\to\infty}(f_n,\varphi)=\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\varphi(x)\,dx=\varphi(0). \qquad (21.1.3.3)$$

Indeed, it follows from the continuity of $\varphi(x)$ that for an arbitrarily small $\Delta>0$ there exists an $\varepsilon_0>0$ such that $|\varphi(x)-\varphi(0)|<\Delta$ for $|x|<\varepsilon_0$. Then

$$\left|\int_{-\infty}^{\infty} f_n(x)\varphi(x)\,dx-\varphi(0)\right|=\frac{1}{\varepsilon}\left|\int_0^{\varepsilon}\bigl[\varphi(x)-\varphi(0)\bigr]\,dx\right|\le\frac{1}{\varepsilon}\int_0^{\varepsilon}|\varphi(x)-\varphi(0)|\,dx<\frac{\Delta}{\varepsilon}\int_0^{\varepsilon}dx=\Delta$$

for all $\varepsilon=1/n\le\varepsilon_0$, as desired.

Thus, the weak limit of the sequence $f_n(x)$ as $n \to \infty$ is the functional that takes each continuous function $\varphi(x)$ to the value $\varphi(0)$ of that function at the point $x = 0$. This functional is denoted by $\delta(x)$ and is called the *Dirac delta function*.

Symbolically, the action of the delta function is denoted by

$$(\delta, \varphi) = \varphi(0). \tag{21.1.3.4}$$

A sequence of functions $f_n(x)$ is called a *delta sequence* if it converges to the delta function as $n \to \infty$. The sequence (21.1.3.2) of discontinuous functions is a delta sequence. The following sequences are examples of delta sequences of continuously differentiable functions:

$$f_n(x) = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}, \quad f_n(x) = \frac{1}{2\sqrt{\pi\varepsilon}} \exp\left(-\frac{x^2}{4\varepsilon}\right), \quad f_n(x) = \frac{1}{\pi x} \sin\frac{x}{\varepsilon}, \quad \varepsilon = \frac{1}{n}.$$

Remark 21.1. One can use the following result to obtain various delta sequences. Let $g(x) \geq 0$ be an arbitrary function satisfying the normalization condition

$$\int_{-\infty}^{\infty} g(x)\,dx = 1.$$

Then the sequence of functions

$$g_n(x) = ng(nx)$$

is a delta sequence as $n \to \infty$.

Remark 21.2. *Physical interpretation.* In physics, the Dirac delta function $\delta(x)$ is often defined as a function that vanishes for all real $x \neq 0$, is infinite at $x = 0$, and satisfies the normalization condition

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

This definition permits intuitively representing the spatial density of physical variables (mass, charge, heat source intensity, force, etc.) lumped or applied at a single point in space. For example, the delta function can be identified with the mass density distribution for which there is a unit lump mass at the point $x = 0$ and the mass is zero at all other points. If there is a lump mass m at a point $x = x_0$, then its density is $m\delta(x - x_0)$.

The main properties of the one-dimensional Dirac delta function are as follows:

1. $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$,
2. $\displaystyle\int_a^b f(y)\delta(x - y)\,dy = \begin{cases} f(x) & \text{if } a < x < b, \\ 0 & \text{if } x < a \text{ or } x > b, \end{cases}$

where $f(x)$ is any continuous function.

21.1.4 Derivatives of Distributions. Some Formulas

► Definitions of the first and higher derivatives. Schwartz theorem.

The derivative of a distribution is by definition the functional f' given by the formula

$$(f', \varphi) = -(f, \varphi'). \tag{21.1.4.1}$$

One can readily verify that f' is a continuous linear functional. For functions $f(x)$ differentiable in the ordinary sense, relation (21.1.4.1) follows from the integration by parts formula by virtue of (21.1.3.1), so that the two notions of derivative coincide in this case.

Distributions introduced in this manner are infinitely differentiable (in the sense of distributions), the kth derivative being given by the formula

$$(f^{(k)}, \varphi) = (-1)^k (f, \varphi^{(k)}). \tag{21.1.4.2}$$

Example 21.3. Let us find the generalized derivative of the *Heaviside unit step function*

$$\vartheta(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 0. \end{cases} \tag{21.1.4.3}$$

Using formula (21.1.4.1) with regard to (21.1.3.1), we obtain

$$(\vartheta', \varphi) = -(\vartheta, \varphi') = -\int_{-\infty}^{\infty} \vartheta(x)\varphi'(x)\,dx = -\int_0^{\infty} \varphi'(x)\,dx = \varphi(0). \tag{21.1.4.4}$$

The comparison of (21.1.4.4) with (21.1.3.4) gives

$$(\vartheta', \varphi) = (\delta, \varphi) \quad \Longrightarrow \quad \vartheta'(x) = \delta(x). \tag{21.1.4.5}$$

Thus, the delta function can also be defined as the generalized derivative of the Heaviside unit step function.

This example shows that the derivative of a regular distribution (= the Heaviside unit step function) can be a singular distribution (= the Dirac delta function). Note that the Heaviside unit step function itself can be defined as the derivative of the continuous function that is zero for $x \le 0$ and is equal to x for $x > 0$. This fact admits the following important generalization.

SCHWARTZ THEOREM. *Each function in $\mathcal{K}'$ is the generalized derivative (of some order) of some regular distribution specified by a continuous function.*

► Generalized derivative of a function that has discontinuities of the first kind.

Assume that a function $f(x)$ is continuous and differentiable everywhere except at a point x_0, where it has a discontinuity of the first kind. The *jump of $f(x)$ at x_0* is the number

$$[f]_{x_0} = f(x_0+0) - f(x_0-0),$$

where $f(x_0+0) = \lim\limits_{x\to x_0,\ x>x_0} f(x)$ and $f(x_0-0) = \lim\limits_{x\to x_0,\ x<x_0} f(x)$. The generalized derivative of this function is computed by the formula

$$f' = \{f'(x)\} + [f]_{x_0}\delta(x-x_0), \tag{21.1.4.6}$$

where $\{f'(x)\}$ is the classical derivative (defined everywhere except for the point x_0). The generalized derivative of a function $f(x)$ that has several isolated discontinuities of the first kind at some points x_k can be written as follows:

$$f' = \{f'(x)\} + \sum_k [f]_{x_k}\delta(x-x_k).$$

► **Integrals containing derivatives of the delta function.**

Let the derivative $f^{(n)}(x)$ be continuous for $a < x < b$. Then

$$\int_a^b f(y)\delta^{(n)}(x-y)\,dy = f^{(n)}(x), \qquad n = 1,\,2,\,\dots\,,$$
$$\int_a^b f(y)\delta^{(n)}(y-x)\,dy = (-1)^n f^{(n)}(x).$$

21.1.5 Operations on Distributions and Some Transformations

Operations on and transformations of distributions are first introduced for regular distributions on the basis of formula (21.1.3.1) and are then extended by definition to singular distributions (to construct which one can use sequences of regular distributions and the completeness theorem).

► **Addition and multiplication of distributions.**

1°. *Addition of distributions.* One can add distributions:

$$(f_1 + f_2, \varphi) = (f_1, \varphi) + (f_2, \varphi).$$

2°. *Multiplication of distributions.* Let $h = h(x)$ be an infinitely differentiable function. Then, by definition,

$$(hf, \varphi) = (f, h\varphi).$$

Example 21.4. Let us prove that

$$h(x)\delta(x) = h(0)\delta(x). \qquad (21.1.5.1)$$

Indeed, $(h\delta, \varphi) = (\delta, h\varphi) = h(0)\varphi(0) = (h(0)\delta, \varphi)$ for all $\varphi \in \mathcal{K}$.

Remark 21.3. In the general case, one cannot introduce a well-defined product of a distribution by a discontinuous function or, so much the more, by another distribution.

Note that the notion of direct product of distributions of distinct arguments is well defined in the theory of distributions of several variables (see below).

► **Linear and general changes of variables.**

1°. *Linear change of variables.* Let a and b be constants ($a \neq 0$). By definition,

$$\bigl(f(ax+b), \varphi(x)\bigr) = \biggl(f(x), \frac{1}{|a|}\varphi\Bigl(\frac{x-b}{a}\Bigr)\biggr).$$

In particular, it follows that

$$\delta(ax) = |a|^{-1}\delta(x), \qquad a \neq 0.$$

2°. *Change of variables of general form.* Let $t(x)$ be a strictly monotone infinitely differentiable function; i.e., there exists an inverse function, $y = t(x) \implies x = \tau(y)$. Then $f(t(x))$ is understood as the functional acting by the rule

$$\bigl(f(t(x)), \varphi(x)\bigr) = \biggl(f(x), \frac{\varphi(\tau(x))}{|t'(\tau(x))|}\biggr).$$

21.1.6 Tempered Distributions and Fourier Transform

► Tempered distributions.

The space $\mathcal{J}$ of *rapidly decaying test functions* contains all infinitely differentiable functions $\varphi(x)$ decaying, together with all of their derivatives, more rapidly than any power of $|x|^{-1}$ as $|x| \to \infty$. The convergence in $\mathcal{J}$ is defined as follows: a sequence of functions $\varphi_n \in \mathcal{J}$ converges to a function $\varphi \in \mathcal{J}$ if and only if $x^\beta \frac{d^m}{dx^m}\varphi_n(x) \rightrightarrows x^\beta \frac{d^m}{dx^m}\varphi(x)$ as $n \to \infty$ for any β and $m = 0,\ 1,\ \dots$.

We have the following consequences of these definitions: (i) $\mathcal{J}$ is a linear space; (ii) $\mathcal{K} \subset \mathcal{J}$; (iii) the convergence in $\mathcal{K}$ implies the convergence in $\mathcal{J}$; (iv) $\mathcal{K}$ is dense in $\mathcal{J}$, which means that for each $\varphi \in \mathcal{J}$ there exists a sequence $\varphi_n \in \mathcal{K}$ such that $\varphi_n \to \varphi$ as $n \to \infty$.

The space $\mathcal{J}'$ of *tempered distributions* consists of continuous linear functionals on the test function space $\mathcal{J}$. The convergence in $\mathcal{J}'$ is defined as the weak convergence of a sequence of functionals.

One can prove that (i) $\mathcal{J}' \subset \mathcal{K}'$; (ii) the convergence in $\mathcal{J}'$ implies the convergence in $\mathcal{K}'$.

Example 21.5. Let us describe a broad class of tempered distributions. Let $f(x)$ be a locally integrable function of polynomial (tempered) growth at infinity, so that it satisfies the condition

$$\int_{-\infty}^{\infty} |f(x)|(1+|x|)^{-m}\,dx < \infty$$

for some $m \geq 0$. Then this function defines a regular functional $f \in \mathcal{J}'$ by formula (21.1.3.1), where $\varphi(x) \in \mathcal{J}$.

► Fourier transform of tempered distributions.

Since the test functions $\mathcal{J}$ are absolutely integrable on $\mathbb{R}^1$, it follows that they have well-defined Fourier transforms

$$\widetilde{\varphi}(u) = \mathfrak{F}\bigl\{\varphi(x)\bigr\}, \quad \text{where} \quad \mathfrak{F}\bigl\{\varphi(x)\bigr\} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-iux}\,dx.$$

One can show that the Fourier transform $\mathfrak{F}$ (as well as the inverse Fourier transform) is a one-to-one transformation of $\mathcal{J}$ into $\mathcal{J}$ (in the class of complex-valued functions of real argument of the form $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$, where $\varphi_{1,2}(x)$ belong to the space of rapidly decaying test functions).

The Fourier transform of an absolutely integrable function $f(x)$ defines a generalized function $\bigl(\mathfrak{F}\{f\}, \varphi\bigr) \in \mathcal{J}$. By changing the order of integration, we obtain the chain of equalities

$$\begin{aligned}\bigl(\mathfrak{F}\{f\}, \varphi\bigr) &= \int_{-\infty}^{\infty} \mathfrak{F}\{f(x)\}\varphi(u)\,du = \int_{-\infty}^{\infty} \biggl(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iux}\,dx\biggr)\varphi(u)\,du \\ &= \int_{-\infty}^{\infty} \biggl(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(u)e^{-iux}\,dx\biggr) f(x)\,dx = \int_{-\infty}^{\infty} f(x)\mathfrak{F}\{\varphi(u)\}\,dx;\end{aligned}$$

i.e.,

$$\bigl(\mathfrak{F}\{f\},\varphi\bigr) = \bigl(f,\mathfrak{F}\{\varphi\}\bigr).$$

The last relation is taken to be the definition of the Fourier transform of any tempered distribution.

21.1.7 Generalized Solutions of Linear Ordinary Differential Equations

Consider a linear ordinary differential equation

$$L[w] \equiv \sum_{k=0}^{m} a_k(x) w_x^{(k)} = f(x), \qquad w_x^{(k)} \equiv \frac{d^k w}{dx^k}. \tag{21.1.7.1}$$

The *adjoint* of the operator L is the operator L^* defined by the formula

$$L^*[w] \equiv \sum_{k=0}^{m} (-1)^k \bigl[a_k(x)w(x)\bigr]_x^{(k)}. \tag{21.1.7.2}$$

One has the following identity (which can be obtained by integration by parts) for classical solutions having m continuous derivatives and satisfying Eq. (21.1.7.1):

$$(w, L^*[\varphi]) = \int_{-\infty}^{\infty} w(x) L^*[\varphi(x)]\,dx = \int_{-\infty}^{\infty} L\,[w(x)]\varphi(x)\,dx = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx = (f,\varphi);$$

i.e.,

$$(w, L^*[\varphi]) = (f,\varphi). \tag{21.1.7.3}$$

This identity is taken to be the definition of a generalized solution: a generalized solution of the ordinary differential equation (21.1.7.1) is a functional w such that Eq. (21.1.7.3) holds for each test function $\varphi \in \mathcal{K}$.

21.2 Generalized Functions of Several Variables

This section gives selected information, most often used in the theory of linear PDEs, about distributions of several variables.

21.2.1 Some Definitions. Partial Derivatives. Direct Product. Linear Transformations

► **Test function and distribution spaces of several variables.**

The space $\mathcal{K}(\mathbb{R}^n)$ of test functions of several variables consists of compactly supported (in every direction) functions $\varphi = \varphi(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n)$, that have all possible continuous partial derivatives of any order.

The following is an example of a test function of several variables:

$$\varphi_a(\mathbf{x}) = \begin{cases} \exp\left(-\dfrac{a^2}{a^2 - \mathbf{x}^2}\right) & \text{for } |\mathbf{x}| < a, \\ 0 & \text{for } |\mathbf{x}| \geq a, \end{cases}$$

where $\mathbf{x}^2 = x_1^2 + \cdots + x_n^2$ and $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$.

The notion of distributions as continuous linear functionals over the test function space can be defined without extra effort in the same way as this was done in the one-dimensional case. In particular, a regular distribution f of several variables can be identified with a continuous linear functional of the form

$$(f, \varphi) = \int_{\mathbb{R}^n} f(\mathbf{x})\varphi(\mathbf{x})\, dV_x, \qquad dV_x = dx_1 \ldots dx_n.$$

The space of distributions of n variables is denoted by $\mathcal{K}'(\mathbb{R}^n)$.

► Partial derivatives of distributions.

Let $\mathbf{k} = (k_1, \ldots, k_n)$ be a vector with integer nonnegative components k_j. By $\mathrm{D}^{\mathbf{k}} f$ we denote the partial derivative of a function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ of order $|\mathbf{k}| = k_1 + \cdots + k_n$,

$$\mathrm{D}^{\mathbf{k}} f = \frac{\partial^{|\mathbf{k}|} f(x_1, \ldots, x_n)}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \qquad \mathrm{D}^0 f = f(\mathbf{x}).$$

The partial derivatives of distributions are defined as follows:

$$\bigl(\mathrm{D}^{\mathbf{k}} f, \varphi\bigr) = (-1)^{|\mathbf{k}|}\bigl(f, \mathrm{D}^{\mathbf{k}}\varphi\bigr)$$

for each $\varphi(\mathbf{x}) \in \mathcal{K}(\mathbb{R}^n)$.

► Linear transformations of distributions.

Let A be a nonsingular matrix (i.e., $\det A \neq 0$), and let A^{-1} be the inverse matrix. Then the distribution $f(A\mathbf{x} + \mathbf{b})$, where $\mathbf{b}$ is an arbitrary constant vector, is defined as follows:

$$\bigl(f(A\mathbf{x} + \mathbf{b}), \varphi(x)\bigr) = \frac{1}{|\det A|}\Bigl(f(x), \varphi\bigl(A^{-1}(\mathbf{x} - \mathbf{b})\bigr)\Bigr).$$

In particular, $\delta(\lambda\mathbf{x}) = \lambda^{-n}\delta(\mathbf{x})$, where $\lambda \neq 0$ is a constant.

► Direct product of distributions.

Let $f(\mathbf{x}) \in \mathcal{K}'(\mathbb{R}^n)$ and $g(\mathbf{y}) \in \mathcal{K}'(\mathbb{R}^m)$ be distributions of distinct variables. The product of these distributions is the distribution $f(\mathbf{x})g(\mathbf{y}) \in \mathcal{K}'(\mathbb{R}^{n+m})$ acting on test functions $\varphi(\mathbf{x}, \mathbf{y}) \in \mathcal{K}(\mathbb{R}^{n+m})$ by the rule

$$\bigl(f(\mathbf{x})g(\mathbf{y}), \varphi(\mathbf{x}, \mathbf{y})\bigr) = \bigl(f(\mathbf{x}), \bigl(g(\mathbf{x}), \varphi(\mathbf{x}, \mathbf{y})\bigr)\bigr),$$

which is an analog of the *Fubini theorem* on the coincidence of repeated integrals with the multiple integral.

21.2.2 Dirac Delta Function. Generalized Solutions of Linear PDEs

► Properties of the n-dimensional Dirac delta function.

1°. The n-dimensional Dirac delta function possesses the following basic properties:

$$\begin{aligned}\delta(\mathbf{x}) &= \delta(x_1)\delta(x_2)\dots\delta(x_n),\\ \int_{\mathbb{R}^n}\Phi(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\,dV_y &= \Phi(\mathbf{x}),\end{aligned} \tag{21.2.2.1}$$

where $\delta(x_k)$ is the one-dimensional Dirac delta function, $\Phi(\mathbf{x})$ is an arbitrary continuous function, and $dV_y = dy_1\dots dy_n$.

2°. The two-dimensional Dirac delta function in polar coordinates ρ, φ:

$$\delta(\mathbf{x}-\mathbf{x}_*) = \frac{1}{\rho}\delta(\rho-\rho_*)\delta(\varphi-\varphi_*).$$

3°. The three-dimensional Dirac delta function in cylindrical coordinates ρ, φ, z:

$$\delta(\mathbf{x}-\mathbf{x}_*) = \frac{1}{\rho}\delta(\rho-\rho_*)\delta(\varphi-\varphi_*)\delta(z-z_*).$$

4°. The three-dimensional Dirac delta function in spherical coordinates r, θ, φ:

$$\delta(\mathbf{x}-\mathbf{x}_*) = \frac{1}{r^2\sin\theta}\delta(r-r_*)\delta(\theta-\theta_*)\delta(\varphi-\varphi_*).$$

► Generalized solutions of linear PDEs.

Consider the constant coefficient linear partial differential equation

$$L[w] \equiv \sum_{|\mathbf{k}|\le m} a_{\mathbf{k}}\mathrm{D}^{\mathbf{k}}w = \Phi(\mathbf{x}). \tag{21.2.2.2}$$

The adjoint of L is the operator L^* defined by the formula

$$L^*[w] \equiv \sum_{|\mathbf{k}|\le m} (-1)^{|\mathbf{k}|}a_{\mathbf{k}}\mathrm{D}^{\mathbf{k}}w.$$

A general solution of the partial differential equation (21.2.2.2) is a distribution w that satisfies the integral identity

$$(w, L^*[\varphi]) = (\Phi, \varphi)$$

for each test function $\varphi \in \mathcal{K}$.

⊙ *References for Chapter* 21: L. Schwartz (1950, 1951), G. E. Shilov (1965), I. M. Gelfand and G. E. Shilov (1959), V. S. Vladimirov (1971, 1988), S. G. Krein (1972), A. G. Butkovskiy (1979), L. Hörmander (1983, 1990), R. P. Kanwal (1983), Yu. A. Brychkov and A. P. Prudnikov (1989).

Part III

Symbolic and Numerical Solutions with Maple, Mathematica, and MATLAB®

Inna Shingareva
Carlos Lizárraga-Celaya

Mathematics and Physics Departments
University of Sonora, Mexico

Chapter 22

Linear Partial Differential Equations with Maple

22.1 Introduction

The theory of linear partial differential equations (PDEs) is one of the most important fields of mathematics due to numerous applications in many branches of science and engineering. Linear PDEs have been a research subject for more than three centuries [see Debnath (2007)], and nowadays they are an area of intensive mathematical and scientific research.

With the development of modern computers, supercomputers, computer algebra systems (such as Maple and Mathematica), and interactive software providing a programming environment for scientific computations (such as MATLAB), using modern powerful computational methods for analytical, symbolic, numerical, and graphical solutions of PDEs has become commonplace in mathematical research [see Akritas (1989), Calmet and van Hulzen (1983), Shingareva and Lizárraga-Celaya (2011), Davenport, Siret, and Tournier (1993), and Wester (1999)].

In this chapter, following the most important ideas and methods, we propose and develop new computer algebra ideas and methods to obtain analytical, symbolic, numerical, and graphical solutions for studying linear partial differential equations. We compute analytical and numerical solutions in terms of predefined functions (which are an implementation of known methods for solving linear PDEs) and develop new procedures for constructing new solutions using Maple. We show a very helpful role that computer algebra systems play in the analytical derivation of numerical methods, computing numerical solutions, and comparing numerical and analytical solutions.

22.1.1 Preliminary Remarks

In general, a partial differential equation can be written in the form

$$\mathcal{F}(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0, \qquad (22.1.1.1)$$

where $x, y, \ldots$ are independent variables, u is an unknown function (dependent variable) of these independent variables, and $u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots$ are partial derivatives of u. We use subscripts on dependent variables to denote differentiation: $u_x = \partial u/\partial x$, $u_{xy} = \partial^2 u/\partial x \partial y$, etc.

Equation (22.1.1.1) is defined in a domain D of the n-dimensional space $\mathbb{R}^n$ of the independent variables $x, y, \ldots$. We seek solutions of equation (22.1.1.1), i.e., functions $u = u(x, y, \ldots)$ that satisfy equation (22.1.1.1) identically in D. By introducing additional conditions, one can find various particular solutions.

For example, first- and second-order partial differential equations, say, in two independent variables $\mathbf{x} = (x_1, x_2) = (x, y)$, can be represented as

$$\mathcal{F}(x, y, u, u_x, u_y) = 0, \quad \mathcal{F}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{22.1.1.2}$$

respectively. These equations are defined in a domain $\mathcal{D}$, where $(x, y) \in \mathcal{D} \subset \mathbb{R}^2$, $\mathcal{F}$ is a given function, and $u = u(x, y)$ is the unknown function (*dependent variable*) of the independent variables (x, y). These equations can be written in *standard notation* as

$$\mathcal{F}(x, y, u, p, q) = 0, \quad \mathcal{F}(x, y, u, u_x, u_y, p, q, r) = 0, \tag{22.1.1.3}$$

where $p = u_x$ and $q = u_y$ (for first-order PDEs) and $p = u_{xx}$, $q = u_{xy}$, and $r = u_{yy}$ (for second-order PDEs).

A second-order linear partial differential equation in n independent variables can be written in the general form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + F u = G, \tag{22.1.1.4}$$

where A_{ij} ($A_{ij} = A_{ji}$), B_i, F, and G are functions of the n independent variables x_i.

In this chapter, we use the following notation introduced by convention in Maple:

- `_Cn` (`n=1,2,...`) for arbitrary constants
- `_Fn` for arbitrary functions
- `_c[n]` for arbitrary constants when separating the variables
- `_s` for a parameter in the characteristic system
- `&where` for the solution structure
- $\_\varepsilon$ for a Lie group parameter

We also use the following notation for Maple solutions:

- `Eqn` for equations (`n=1,2,...`)
- `PDEn/ODEn` for PDEs/ODEs
- `Soln` for solutions
- `Trn` for transformations
- `Sysn` for systems
- `IC`, `BC`, `IBC` for initial and/or boundary conditions
- `Ln` for lists of expressions
- `Gn` for graphs of solutions

22.1.2 Brief Introduction to Maple

Maple is a general-purpose computer algebra system (CAS) in which symbolic computation can readily be combined with exact and approximate (floating-point) numerical computation as well as with arbitrary-precision numerical computation. Maple provides powerful scientific graphics capabilities [for details, see Kreyszig (2000), Corless (1995), Heck (2003), Richards (2002), Abel and Braselton (2005), Meade et al. (2009), Shingareva and Lizárraga-Celaya (2009), etc.].

The first concept of Maple and its initial versions were developed by the Symbolic Computation Group at the University of Waterloo in the early 1980s. The Maplesoft company was created in 1988. Maple was mainly developed in research labs at Waterloo University and at the University of Western Ontario [see Char et al. (1990), Geddes, Czapor, and Labahn (1992)], with important contributions from worldwide research groups at other universities.

The most important features of Maple are as follows: fast symbolic and numerical computation and interactive visualization; easy usage; easy incorporation of new user-defined capabilities; understandable open-source software development path; availability on almost all operating systems; powerful programming language, intuitive syntax, and easy debugging; extensive library of mathematical functions and specialized packages; free resources and the collaborative character of development (for example, see the Maple website www.maplesoft.com (MWS), Maple Application Center (MWS/applications), Teacher Resource Center (MWS/TeacherResource), Student Help Center (MWS/studentcenter), and Maple Community (MWS/community)).

Maple consists of three parts: the interface, the kernel (basic computational engine), and the library. The interface and the kernel (written in C programming language) form a smaller part of the system (they are loaded when a Maple session is started). The interface handles the input of mathematical expressions, output display, plotting of functions, and support of other user communication with the system. The user interface is the Maple worksheet. The kernel interprets the user input, carries out the basic algebraic operations, and deals with storage management. The library consists of two parts: the main library and additional packages. The main library (written in the Maple programming language) includes many functions in which most of the common mathematical knowledge of Maple resides.

Basic concepts. The *prompt symbol* ($>$) serves for typing a Maple function; the semicolon ($;$) or colon ($:$) symbol[1] followed by pressing `Enter` at the end of a function serves for evaluating the Maple function, displaying the result, and inserting a new prompt.

Maple contains a complete *online help system* and a command line help system, which can be used, e.g., by typing `?NameOfFunction` or `help(NameOfFunction);` or `?help`; reference information can also be accessed by using the `Help` menu, by highlighting a function and then pressing `Ctrl-F1` or `F1` or `F2` (for Release ≥ 9), and by pressing `Ctrl-F2`.

Maple worksheets are files that keep track of the working process and organize it as a collection of expandable groups (see `?worksheet` and `?shortcut`). It is recommended to begin a new worksheet (or a new problem) with the `restart` statement for cleaning Maple

[1] In earlier versions of Maple and in *Classic Worksheet Maple*, we have to end a function with a colon or semicolon. In these chapters, we follow this tradition in every example and problem.

memory. All examples and problems presented in the book are assumed to begin with `restart`.

Previous results (during a session) can be referred to with the symbols `%` (the last result), `%%` (the next-to-last result), `%%...%` ($k+1$ times) (the kth-to-last result).

Comments start with the sharp symbol `#` and include all subsequent characters until the end of the line. Text can also be inserted with `Insert` → `Text`.

Incorrect response. If you get no response or an incorrect response, you may have entered or executed a function incorrectly. Correct the function or interrupt the computation (the Stop button in the Tool Bar menu).

Maple source code can be viewed for most of the functions, general and specialized (package functions); e.g., `interface(verboseproc=2); print(factor);`

Palettes can be used for building or editing mathematical expressions without needing to remember the Maple syntax.

The Maplet User Interface (for Release ≥ 8) consists of *Maplet applications*, which are collections of windows, dialogs, and actions (see `?Maplets`).

A number of specialized functions are available in various specialized *packages (subpackages)* (see `?index[package]`, `with`).

Numerical approximations: numerical approximation of `expr` to 10 significant digits, `evalf(expr)`; global change of precision, `Digits:=n` (see `?environment`); local change of the precision, `evalf(expr,n)`; numerical approximation of `expr` using a binary hardware floating-point system, `evalhf(expr)`; performing numerical approximations using the hardware or software floating-point systems, `UseHardwareFloats:=value` (for details, see `?UseHardwareFloats`, `?environment`).

22.1.3 Maple Language

Maple language is a high-level programming language, which is well structured and comprehensible. It supports a large collection of data structures, or Maple objects (functions, sequences, sets, lists, arrays, tables, matrices, vectors, etc.), and operations on these objects (type-testing, selection, composition, etc.). Maple procedures in the library are available in readable form. The library can be supplemented with locally developed user programs and packages.

Arithmetic operators: `+ - * / ^`; *logic operators*: `and`, `or`, `xor`, `implies`, `not`; *relation operators*: <, <=, >, >=, =, <>.

A variable name is a combination of letters, digits, and/or the underline symbol (_), starting from a letter; e.g., `a12_new`.

Abbreviations for the longer Maple functions or any expressions: `alias`, for example, `alias(H=Heaviside); diff(H(t),t);` to remove this abbreviation, type `alias(H=H);`

Maple is case sensitive; i.e., it distinguishes between lowercase and uppercase letters; e.g., `evalf(Pi)` and `evalf(pi)` are distinct commands.

Various reserved keywords, symbols, names, and functions; these words cannot be used as variable names, e.g., operator keywords, additional language keywords, global names that start with (_) (see `?reserved`, `?ininames`, `?inifncs`, `?names`).

The assignment/unassignment operators: a variable can be "free" (with no assigned value) or can be assigned any value (symbolic or numeric) by the assignment operators `a:=b` or `assign(a=b)`. To unassign (clear) an assigned variable (see `?:=` and `?'`), type, e.g., `x:='x'`, `evaln(x)`, or `unassign('x')`.

The difference between the *(:=) and (=) operators* is as follows: the operator `A:=B` is used to assign `B` to the variable `A`, and the operator `A=B` is used to indicate equality (not assignment) between the left- and right-hand sides (see `?rhs`), for example, `Equation:=A=B; Equation; rhs(Equation); lhs(Equation);`

The range operator `(..)`, an expression of type range `expr1..expr2`, for example, `a[i]$ i=1..9; plot(sin(x),x=-Pi..Pi);`

Statements are keyboard input instructions that are executed by Maple (e.g., `break`, `by`, `do`, `end`, `for`, `function`, `if`, `proc`, `restart`, `return`, `save`, `while`).

The statement separators semicolon (;) and colon (:). The result of a statement followed with a semicolon (;) will be displayed, and it will not be displayed if it is followed by a colon (:), e.g., `plot(sin(x),x=0..Pi); plot(sin(x),x=0..Pi):`

An expression is a valid statement and is formed as a combination of constants, variables, operators, and functions. Every expression is represented as a tree structure in which each node (and leaf) has a particular data type. For the analysis of any node and branch, the functions `type`, `whattype`, `nops`, and `op` can be used. *A boolean expression* is formed with *logical operators* and relation operators.

An equation is represented using the binary operator `(=)` and has two operands, the left-hand side, `lhs`, and the right-hand side, `rhs`.

Inequalities are represented using the relation operators and have two operands, the left-hand side, `lhs`, and the right-hand side, `rhs`.

A string is a sequence of characters having no value other than itself; it cannot be assigned to, and it will always evaluate to itself. For instance, `x:="string";` and `sqrt(x);` is an invalid function. Names and strings can be used with the `convert` and `printf` functions.

Maple is sensitive to types of brackets and quotes.

Types of brackets: parentheses for grouping expressions, `(x+9)*3`, for delimiting the arguments of functions, `sin(x)`; square brackets for constructing lists, `[a,b,c]`, vectors, matrices, arrays; curly brackets for constructing sets, `{a,b,c}`.

Types of quotes: forward-quotes to delay the evaluation of expression, `'x+9+1'`; to clear variables, `x:='x'`; back-quotes to form a symbol or a name, `` `the name:=7`; k:=5; print(`the value of k is`,k); `` double quotes to create strings; and a single double quote " to delimit strings.

Types of numbers: integer, rational, real, complex, and root, e.g., `-55`, `5/6`, `3.4`, `-2.3e4`, `Float(23,-45), 3-4*I, Complex(2/3,3); RootOf(_Z^3-2,index=1);`

Predefined constants: symbols for commonly used mathematical constants, `gamma`, `Pi`, `I`, `true`, `false`, `infinity`, `FAIL`, `exp(1)` (for details, see `?ininames`, `?constants`).

Functions or function expressions have the form `f(x)` or `expr(args)` and represent a function call, or an application of a function (or procedure) to arguments (`args`). *Active functions* (beginning with a lowercase letter) are used for computing, e.g., `diff`, `int`, `limit`. *Inert functions* (beginning with a capital letter) are used for showing steps in the problem-solving process, e.g., `Diff`, `Int`, `Limit`.

Library functions (or predefined functions) and user-defined functions.

Predefined functions: most of the well-known functions are predefined by Maple, and they are known to some Maple functions (e.g., `diff`, `evalc`, `evalf`, `expand`, `series`, `simplify`). Numerous special functions are defined (see `?FunctionAdvisor`).

User-defined functions: the functional operator (`->`) (see `?->`); e.g., the function $f(x) = \sin x$ is defined as `f:=x->sin(x);`

Alternative definitions of functions: `unapply` converts an expression to a function, and a procedure is defined with `proc`.

Evaluation of function $f(x)$ at $x = a$, $\{x = a, y = b\}$, e.g., `f(a); subs(x=a, f(x)); eval(f(x),x=a);`

In the Maple language, there are two forms of modularity: *procedures* and *modules*. *A procedure* (see `?procedure`) is a block of statements which one needs to use repeatedly. A procedure can be used to define a function (if the function is too complicated to be written with the use of the arrow operator) or to create a matrix, graph, or logical value. *A module* (see `?module`) is a generalization of the procedure concept. While a procedure groups a sequence of statements into a single statement (block of statements), a module groups related functions and data.

In the Maple language, there are essentially *two control structures*: the selection structure `if` and the repetition structure `for`.

Maple objects, sequences, lists, sets, tables, arrays, vectors, and matrices are used for representing more complicated data. *Sequences* `a1,a2,a3`, *lists* `[a1,a2,a3]`, and *sets* $\{a1, a2, a3\}$ are groups of expressions. Maple preserves the order and repetitions in sequences and lists and does not preserve them in sets. The order in sets can change during a Maple session. A *table* is a group of expressions represented in tabular form. Each entry has an index (an integer or an arbitrary expression) and a value (see `?table`). An *array* is a table with integer range of indices (see `?Array`). In Maple, arrays can be of any dimension (depending of computer memory). A *vector* is a one-dimensional array with a positive integer range of indices (see `?vector`, `?Vector`). A *matrix* is a two-dimensional array with positive integer ranges of indices (see `?matrix`, `?Matrix`).

⊙ *References for Section 22.1*: J. Calmet and J. A. van Hulzen (1983), A. G. Akritas (1989), B. W. Char, K. O. Geddes, G. H. Gonnet, M. B. Monagan, and S. M. Watt (1990), K. O. Geddes, S. R. Czapor, and G. Labahn (1992), J. H. Davenport, Y. Siret, and E. Tournier (1993), R. M. Corless (1995), M. J. Wester (1999), E. Kreyszig (2000), D. Richards (2002), A. Heck (2003), M. L. Abel and J. P. Braselton (2005), L. Debnath (2007), D. B. Meade, S. J. M. May, C-K. Cheung, and G. E. Keough (2009), I. K. Shingareva and C. Lizárraga-Celaya (2009, 2011).

22.2 Analytical Solutions and Their Visualizations

22.2.1 Constructing Analytical Solutions in Terms of Predefined Functions

The computer algebra system Maple has various predefined functions based on symbolic algorithms for constructing analytical solutions of linear PDEs [see a more detailed description in Cheb-Terrab and von Bulow (1995)]. The predefined functions implement known methods for solving PDEs, and Maple allows solving linear equations and obtaining solutions automatically (in terms of predefined functions) as well as developing new methods and procedures for constructing new solutions.

Consider the most important functions for finding all possible analytical solutions of a given problem for PDEs.

```
pdsolve(PDE);              pdsolve(PDE,build);      pdetest(Sol,PDE);
pdsolve(PDE,funcs,HINT=val,INTEGRATE,build,singsol=val);
infolevel[procname]:=val;    pdsolve(PDEsys);        with(PDEtools);
declare(funcs);            dchange(rules,PDE);     diff_table(funcs);
separability(PDE,func);      casesplit(PDEs);  charstrip(PDE, func);
Laplace(PDE,func,ops); Solve(sys, vars, ops); mapde(PDE, form, ops);
Infinitesimals(PDE);    SymmetrySolutions(Sol, Infinitesimals, ops);
TWSolutions(PDE,ops);
```

Remark. `HINT=val` gives some hints; with `build` one can construct an explicit expression for the indeterminate function `func`; with `'+'` or `'*'` one can construct a solution by separation of variables (in the form of sum or product, respectively); with `'TWS'` or `'TWS(MathFuncName)'` one can construct a traveling wave solution as a power series in $\tanh(\xi)$ or several mathematical functions (including special functions), where ξ represents a linear combination of the independent variables, etc.

`pdsolve` finding analytical solutions for a given partial differential equation `PDE` and for systems of `PDE`

`PDEtools` a collection of functions for finding analytical solutions for `PDE`, e.g.,

- `declare` declaring functions and derivatives on the screen for a simple, compact display
- `casesplit` splitting into cases and successively decoupling a system of differential equations
- `separability` determining under what conditions it is possible to obtain a complete solution by separation of variables
- `charstrip` finding the characteristic strip (for a first-order PDE)
- `Laplace` solving a second-order linear PDE (in two independent variables) using the Laplace method
- `Solve` finding exact, series, or numerical solutions for systems of algebraic or differential equations (including inequalities, initial and boundary conditions)
- `mapde` converting a PDE into a PDE of different form
- `SymmetrySolutions` finding point symmetry solutions
- `TWSolutions` constructing traveling wave solutions (for autonomous PDEs and systems of them), etc.

Let us assume that we have obtained exact solutions and wish to verify whether these solutions are exact solutions of given linear PDEs.

Example 22.1. *Linear Poisson equation. Verification of solutions.* For the two-dimensional Poisson equation 7.2.2

$$u_{xx}+u_{yy}+\Phi(x,y)=0, \qquad \Phi(x,y)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\exp(b_i x+c_j y),$$

we verify that the solution,

$$u(x,t)=-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{ij}}{b_i^2+c_j^2}\exp(b_i x+c_j y),$$

is an exact solution of the Poisson equation (with a special right-hand side) as follows:

```
with(PDEtools); declare(u(x,y)); U:=diff_table(u(x,y));
PDE1:=U[x,x]+U[y,y]+sum(sum(a[i,j]*exp(b[i]*x+c[j]*y),j=1..n),i=1..n)=0;
Sol1:=u(x,y)=-sum(sum(a[i,j]/(b[i]^2+c[j]^2)*exp(b[i]*x+c[j]*y),j=1..n),i=1..n);
Test1:=simplify(combine(pdetest(Sol1,PDE1)));
```

where a_{ij}, b_i, c_j ($i=1,\dots n$, $j=1,\dots n$) are arbitrary real constants.

Example 22.2. *Linear heat equation. Verification of solutions.* For the one-dimensional heat equation 1.1.1

$$u_t=ku_{xx},$$

we verify that

$$u(x,t)=\frac{1}{2\sqrt{\pi kt}}\exp\left(-\frac{x^2}{4kt}\right), \quad x\in\mathbb{R}, \quad t>0,$$

is the fundamental solution of this linear heat equation as follows:

```
with(PDEtools); declare(u(x,t)); U:=diff_table(u(x,t));
PDE1:=U[t]=k*U[x,x]; Sol1:=u(x,t)=1/sqrt(4*Pi*k*t)*exp(-x^2/(4*k*t));
Test1:=pdetest(Sol1,PDE1);
```

Here k is a constant.

Example 22.3. *Linear wave equation. General solution.* The general solution of the one-dimensional wave equation 4.1.1

$$u_{tt}=c^2u_{xx}$$

can be found and tested as follows:

```
with(PDEtools); declare(u(x,t)); PDE1:=diff(u(x,t),t$2)=c^2*diff(u(x,t),x$2);
Sol1:=pdsolve(PDE1); Test1:=pdetest(Sol1,PDE1);
```

where the Maple result reads:

$Sol1 := u(x,t) = \_F1(ct+x)+\_F2(ct-x)$

Example 22.4. *Linear Euler equation. Method of characteristics.* The first-order linear Euler equation

$$xu_x+yu_y=nu$$

can be solved by the method of characteristics as follows:

```
with(PDEtools):  declare(u(x,y));
PDE1:=x*diff(u(x,y),x)+y*diff(u(x,y),y)=n*u(x,y);
sysCh:=charstrip(PDE1,u(x,y)); funcs:=indets(sysCh,Function);
Sol1:=dsolve(sysCh,funcs,explicit); GenSol1:=pdsolve(PDE1);
```

where the Maple result reads:

$$sysCh := \{u_{\_s} = nu(\_s), x_{\_s} = x(\_s), y_{\_s} = y(\_s)\}, \quad funcs := \{u(\_s), x(\_s), y(\_s)\}$$

$$Sol1 := \{u(\_s) = \_C3\,e^{n\_s}, x(\_s) = \_C2\,e^{\_s}, y(\_s) = \_C1\,e^{\_s}\}, \quad GenSol1 := u(x,y) = \_F1\left(\frac{y}{x}\right)x^n$$

Remark. For the given equation (`PDE1`), we first obtain the characteristic system (depending on the parameter `_s`) via `charstrip` and then solve this system via `dsolve`. Additionally we find the general solution of the linear Euler equation.

Example 22.5. *Linear wave equation with two space variables. Analytical solutions.* For the wave equation with two space variables in the rectangular Cartesian system of coordinates,

$$u_{tt} = c^2(u_{xx} + u_{yy}),$$

which arises in many physical problems and describes wave processes in various media (e.g., shallow water waves, linearized supersonic flow in a gas, sound waves in space, electrical oscillations in a conductor, torsional vibration of a rod, longitudinal vibrations of a bar, waves in magnetohydrodynamics, etc.), we can construct various forms of analytical solutions of the wave equation, e.g.,

```
with(PDEtools): declare(u(x,y,t)); infolevel[pdsolve]:=5;
PDE1:=diff(u(x,y,t),t$2)=c^2*(diff(u(x,y,t),x$2)+diff(u(x,y,t),y$2));
casesplit(PDE1); separability(PDE1,u(x,y,t)); separability(PDE1,u(x,y,t),`*`);
Sol1:=pdsolve(PDE1,HINT=`+`,INTEGRATE); Sol2:=pdsolve(PDE1,HINT=`*`,INTEGRATE);
Sol3:=pdsolve(PDE1,HINT='TWS'); Sol4:=pdsolve(PDE1,HINT='TWS(tan)');
for i from 1 to 4 do Test||i:=pdetest(Sol||i,PDE1) od;
```

with the Maple output

$$Sol1 := (u(x,y,t) = \_F1(x) + \_F2(y) + \_F3(t))\,\&\mathtt{where}\left[\left\{\left\{\_F1(x) = 1/2\left(-\_c_2 + \frac{\_c_3}{c^2}\right)x^2 + \_C1\,x + \_C2\right\},\right.\right.$$
$$\left.\left.\{\_F2(y) = 1/2\,\_c_2 y^2 + \_C3\,y + \_C4\}, \{\_F3(t) = 1/2\,\_c_3 t^2 + \_C5\,t + \_C6\}\right\}\right]$$

$$Sol2 := (u(x,y,t) = \_F1(x)\,\_F2(y)\,\_F3(t))\,\&\mathtt{where}\left[\left\{\left\{\_F1(x) = \_C1\,e^{\sqrt{\_c_1}x} + \_C2\,e^{-\sqrt{\_c_1}x}\right\},\right.\right.$$
$$\left.\left.\left\{\_F2(y) = \_C3\,e^{\sqrt{\_c_2}y} + \_C4\,e^{-\sqrt{\_c_2}y}\right\}, \left\{\_F3(t) = \_C5\sin\left(c\sqrt{-\_c_1 - \_c_2}\,t\right) + \_C6\cos\left(c\sqrt{-\_c_1 - \_c_2}\,t\right)\right\}\right\}\right]$$

$$Sol3 := u(x,y,t) = \left(\tanh\left(-t\sqrt{\_C2^2 + \_C3^2}\,c + x\,\_C2 + \_C3\,y + \_C1\right)\right)^3 \_C8 + \tanh\left(-t\sqrt{\_C2^2 + \_C3^2}\,c + x\,\_C2 + \_C3\,y + \_C1\right)\_C6 + \_C5$$

$$Sol4 := u(x,y,t) = \left(\tan\left(-t\sqrt{\_C2^2 + \_C3^2}\,c + x\,\_C2 + \_C3\,y + \_C1\right)\right)^3 \_C8 + \tan\left(-t\sqrt{\_C2^2 + \_C3^2}\,c + x\,\_C2 + \_C3\,y + \_C1\right)\_C6 + \_C5$$

Remark. First, we can split the problem into all related regular cases (the general case and all possible singular cases). In our situation, we have the general case. Then we check the separability conditions (additive, by default, and multiplicative): if there is a separable solution, the result is 0. For our problem, we have 0 in both cases (additive and multiplicative). Finally, we construct exact solutions: additive and multiplicative separable solutions with `HINT=‘+‘` and `HINT=‘*‘`[2]; traveling wave solutions with `HINT='TWS'` and `HINT='TWS(tan)'`.

[2] Solving a PDE by separation of variables, we can indicate the option `INTEGRATE` for the integration of the set of resulting ODEs.

If we try to construct separable solutions with the functional `HINT`, e.g., `HINT=f(C1*x+C2*y+C3*t)+g(C1*x+C2*y+C4*t)`, then Maple cannot find the solution of this form (the solution $f(x\sin a+y\sin a+bt)+g(x\sin a+y\sin a-bt)$), and, according to the general strategy [see Cheb-Terrab and von Bulow (1995)], applies another method and gives the result that coincides with one of the known solutions, i.e., the traveling wave solution (`Sol3`). However, in a simpler case, e.g., with `HINT=f(x,y)`, Maple finds the solution $u(x,y,t) = \_F1(y-Ix)+\_F2(y+Ix)$.

Example 22.6. *Linear elastic beam equation. Separable and self-similar solutions* By applying predefined Maple functions to the linear elastic beam equation

$$u_{tt}+\alpha^2 u_{xxxx}=0,$$

we can find additive separable and self-similar solutions and verify that these solutions are exact solutions:

```
with(PDEtools): declare(u(x,t));
PDE1:=diff(u(x,t),t$2)+alpha^2*diff(u(x,t),x$4);
Sol1:=pdsolve(PDE1); Sol2:=collect(pdsolve(PDE1,build),[cos,sin]);
Sol3:=pdsolve(PDE1,HINT=`+`,build); Sol4:=[SimilaritySolutions(PDE1)];
for i from 1 to 4 do Test||i:=pdetest(Sol||i,PDE1); od;
```

with the Maple output

$$Sol1 := (u(x,t)=\_F1(x)+\_F2(t))\ \&\text{where}\left[\left\{\_F2_{t,t}(t)=\_c_2,\ \_F1_{x,x,x,x}(x)=-\frac{\_c_2}{\alpha^2}\right\}\right]$$

$$Sol2 := u(x,t)=\left(\_C3\,e^{i\sqrt[4]{\_c_1}x}\_C6+\_C4\,e^{\sqrt[4]{\_c_1}x}\_C6+\frac{\_C2\,\_C6}{e^{\sqrt[4]{\_c_1}x}}+\frac{\_C1\,\_C6}{e^{i\sqrt[4]{\_c_1}x}}\right)\cos(\alpha\sqrt{\_c_1}t)$$
$$+\left(\_C3\,e^{i\sqrt[4]{\_c_1}x}\_C5+\_C4\,e^{\sqrt[4]{\_c_1}x}\_C5+\frac{\_C2\,\_C5}{e^{\sqrt[4]{\_c_1}x}}+\frac{\_C1\,\_C5}{e^{i\sqrt[4]{\_c_1}x}}\right)\sin(\alpha\sqrt{\_c_1}t)$$

$$Sol3 := u(x,t)=-\frac{1}{24}\frac{\_c_2x^4}{\alpha^2}+\frac{1}{6}\_C1x^3+\frac{1}{2}\_C2x^2+\_C3x+\frac{1}{2}\_c_2t^2+\_C5t+\_C6+\_C4$$

$$Sol4 := \Bigg[u(x,t)=\_C1t+\_C2,\ u(x,t)=\frac{1}{6}\_C1x^3+\frac{1}{2}\_C2x^2+\_C3x+\_C4,$$
$$u(x,t)=\left(\_C3\,\text{hypergeom}\left(\left[\frac{1}{2},1\right],\left[\frac{3}{4},\frac{5}{4},\frac{3}{2}\right],-\frac{1}{64}\frac{x^4}{\alpha^2t^2}\right)x^2+\_C4\,\text{FresnelS}\left(\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\alpha}\sqrt{\frac{t}{x^2}}}\right)t\right.$$
$$\left.+\_C2\,\text{FresnelC}\left(\frac{1}{2}\frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\alpha}\sqrt{\frac{t}{x^2}}}\right)t+\_C1t\right)t^{-1}\Bigg]$$

Remark. We start from the function `pdsolve(PDE1)` (without options) and obtain the structure of the exact solution (multiplicative separable solution) `Sol1`, for which we build an explicit expression in `Sol2`. Then we find the additive separable solution `Sol3` via the option `HINT`. And then we construct similarity solutions in `Sol4`.

Example 22.7. *Three-dimensional wave equation. Traveling wave solutions.* By applying Maple predefined functions to the three-dimensional wave equation

$$u_{tt}=c^2(u_{xx}+u_{yy}+u_{zz}),$$

(where c is a parameter), we can find traveling wave solutions and verify that these solutions are exact solutions of the given PDE as follows:

```
with(PDEtools): declare(u(x,y,z,t)); PDE1:=diff(u(x,y,z,t),t$2)=
 c^2*(diff(u(x,y,z,t),x$2)+diff(u(x,y,z,t),y$2)+diff(u(x,y,z,t),z$2));
Sol1:=pdsolve(PDE1,HINT='TWS'); Test1:=pdetest(Sol1,PDE1);
```

with Maple results

$$Sol1 := u(x,y,z,t) = \_C9\left(\tanh\left(-\sqrt{\_C2^2+\_C3^2+\_C4^2}ct+\_C2x+\_C3y+\_C4z+\_C1\right)\right)^3 + \_C7\tanh\left(-\sqrt{\_C2^2+\_C3^2+\_C4^2}ct+\_C2x+\_C3y+\_C4z+\_C1\right)+\_C6$$

In general, hyperbolic linear PDEs can have interesting traveling wave solutions. However, for nonlinear equations the effects of nonlinear and higher-order terms interact and can lead to interesting solutions [see Shingareva and Lizárraga-Celaya (2011), Shingareva and Lizárraga-Celaya (pp. 1622–1765, 2012)]. By applying the function `TWSolutions` (with various options) of the package `PDEtools`, one can construct various types of traveling wave solutions in Maple; for example, we have

```
infolevel[TWSolutions]:=2;  Fs:=TWSolutions(functions_allowed);
Sol21:=TWSolutions(PDE1,singsol=false,functions=Fs); N:=nops([Sol21]);
for i from 1 to N do op(i,[Sol21]) od;
Sol22:=TWSolutions(PDE1,singsol=false,parameters=[c],functions=[cot],
 remove_redundant=true); Sol23:=TWSolutions(PDE1,function=identity,output=ODE);
```

with the Maple results

$$Sol22 := \left\{c=c, u(x,y,z,t) = \left(\cot\left(-\sqrt{\_C2^2+\_C3^2+\_C4^2}ct+\_C2x+\_C3y+\_C4z+\_C1\right)\right)^3\_C9 + \cot\left(-\sqrt{\_C2^2+\_C3^2+\_C4^2}ct+\_C2x+\_C3y+\_C4z+\_C1\right)\_C7+\_C6\right\}\ldots$$

$$Sol23 := \left[\left\{\left(\_C4^2-c^2\left(\_C1^2+\_C2^2+\_C3^2\right)\right)u_{\tau,\tau}\right\},\left\{\tau=\_C1x+\_C2y+\_C3z+\_C4t+\_C5, u(\tau)=u(x,y,z,t)\right\}\right]$$

Remark. By adding the function `infolevel` with values (0–5), we can obtain increasingly detailed information on each of the steps of the solution process. Then, with the aid of the predefined function `TWSolutions`, we can find various types of traveling wave solutions (of PDEs and systems) according to a list of functions `Fs` (instead of the `tanh` function, by default). The algorithm is based on the tanh-function method and its various extensions [see Shingareva and Lizárraga-Celaya (2011)]. Including the options `parameters` (splitting into cases with respect to the parameters), `singsol` (avoiding singular solutions), `functions` (expanding in series of different functions), `remove_redundant=true` (removing all redundant solutions, i.e., all particular cases of more general solutions being constructed), we can obtain numerous traveling wave solutions (`Sol21`). For example, by specifying the `cot` function (and other options), we can obtain various traveling wave solutions. (We present just one solution in `Sol22`.)

Sometimes it is useful to obtain a reduction (without computing a solution) that converts a PDE into an ODE form and the corresponding ODE.

By default, the function `TWSolutions` generates solutions of the form $f_i(\tau)=\sum_{k=0}^{n_i} A_{ik}\tau^k$, where the n_i are finite, the A_{ik} are constants with respect to x_j, and $\tau = \tanh(\sum_{i=0}^{j} C_i x_i)$. In `Sol23`, we obtain an ODE by using a simpler form, i.e., by choosing the identity as the function (the option `function=identity`), where τ becomes $\tau = \sum_{i=0}^{j} C_i x_i$. In our case, we have $u(x,y,z,t) = u(\tau)$, where $\tau = C_1x + C_2y + C_3z + C_4t + C_5$, and the ODE acquires the form $\left(C_4^2 - c^2\left(C_1^2+C_2^2+C_3^2\right)\right)u_{\tau,\tau} = 0$ (`Sol23`).

Exact solutions of various types of linear PDEs. Finally, by applying Maple predefined functions, we find exact solutions (general, traveling wave, separable, similarity, and symmetry) and characteristic systems of various types of linear PDEs (where a, b, c are arbitrary parameters). The results are presented in Table 22.1.

Table 22.1.
Maple predefined functions and exact solutions of various types of linear PDEs

No.	Linear PDEs	Exact solution	Maple function
1	$xu_x + yu_y = u$	$\{u\_s = u(\_s), x\_s = x(\_s), y\_s = y(\_s)\}$	`charstrip(Eq1,u(x,y))`
2	$u_{tt} = c^2 u_{xx}$	$u = \_F1\left(\frac{ct+x}{c}\right) + \_F2\left(\frac{ct-x}{c}\right)$	`Laplace(Eq2,u(x,t))`
3	$u_{tt} + c^2 u_{xx} = 0$	$u == \frac{\_C1\sin(ct) + \_C2\cos(ct) - e^{-x}}{e^{-x}}$	`SimilaritySolutions(Eq3,S[2])`
4	$u_x - u_y = u$	$u = \_F1(x+y)e^x$	`pdsolve(Eq4)`
5	$u_{tt} - c^2 u_{xx} + a^2 u = 0$	$u = \_C4\sin\left(\sqrt{\_C2^2c^2 + a^2}t + \_C2x + \_C1\right)$	`pdsolve(Eq5,HINT=TWS(sin))`
6	$u_t + au_x + bu_{xxx} = 0$	$u=\_c_2t+\_C3+\_C4+\frac{\left(\sin\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)\_C1-\cos\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)\_C2\right)\sqrt{b}}{\sqrt{a}} - \frac{\_c_2x}{a}$	`pdsolve(Eq6,HINT=f(x)+g(t),ops)`
7	$u_x - u_y = 1$	$u = x + \_F1(x+y)$	`pdsolve(Eq7,build)`
8	$u_t = ku_{xx}$	$\left\{u = \_C3\left(e^{2\sqrt{\_c_1}x}\_C1 + \_C2\right)e^{\_c_1k(t-\_\varepsilon)-\sqrt{\_c_1}x}\right\}$	`SymmetrySolutions(Sol80,S8[1])`
9	$u_{tt} - c^2 u_{xx} + a^2 u = 0$	$(u = f(x)g(t))$ &where $\left[\left\{f_{xx} = \_c_1f(x), g_{tt}(t) = c^2\_c_1g(t) - a^2g(t)\right\}\right]$	`pdsolve(Eq9,HINT=f(x)*g(t))`
10	$u_{tt} = c^2 u_{xx}$	$\left\{u=\_C7(\tanh(\_C2ct+\_C2x+\_C1))^3+\_C5\tanh(\_C2ct+\_C2x+\_C1)+\_C4\right\}$	`TWSolutions(Eq10,singsol=false)`
11	$x^2u_x + y^2u_y = (x+y)u$	$u = \_F1\left(\frac{-y+x}{yx}\right)xy$	`Solve(Eq11,u(x,y))`

where the linear partial differential equations and some additional solutions are defined in Maple as follows:

```
with(PDEtools): declare(u(x,t));
Eq1:=x*diff(u(x,y),x)+y*diff(u(x,y),y)=u(x,y);
Sol1:=charstrip(Eq1,u(x,y));
Eq2:=diff(u(x,t),t$2)=c^2*diff(u(x,t),x$2); Sol2:=Laplace(Eq2,u(x,t));
Eq3:=diff(u(x,t),t$2)+c^2*diff(u(x,t),x$2)=0; Infinitesimals(Eq3);
S:=Infinitesimals(Eq3,specialize_Fn); Sol3:=SimilaritySolutions(Eq3,S[2]);
Eq4:=diff(u(x,y),x)-diff(u(x,y),y)=u(x,y); Sol4:=pdsolve(Eq4);
Eq5:=diff(u(x,t),t$2)-c^2*diff(u(x,t),x$2)+a^2*u(x,t)=0;
Sol5:=pdsolve(Eq5,HINT=TWS(sin));
Eq6:=diff(u(x,t),t)+a*diff(u(x,t),x)+b*diff(u(x,t),x$3)=0;
ops:=INTEGRATE,build; Sol6:=collect(pdsolve(Eq6,HINT=f(x)+g(t),ops),[a,b]);
Eq7:=diff(u(x,y),x)-diff(u(x,y),y)=1; Sol7:=pdsolve(Eq7,build);
Eq8:=diff(u(x,t),t)=k*diff(u(x,t),x$2); Sol80:=pdsolve(Eq8,build);
S8:=[Infinitesimals(Eq8)]; Sol8:=SymmetrySolutions(Sol80,S8[1]);
Eq9:=diff(u(x,t),t$2)-c^2*diff(u(x,t),x$2)+a^2*u(x,t)=0;
Sol9:=pdsolve(Eq9,HINT=f(x)*g(t));
Eq10:=diff(u(x,t),t$2)=c^2*diff(u(x,t),x$2);
Sol10:=[TWSolutions(Eq10,singsol=false)]; Sol10[3];
Eq11:=x^2*diff(u(x,y),x)+y^2*diff(u(x,y),y)=(x+y)*u(x,y);
Sol11:=Solve(Eq11,u(x,y));
```

Remark. In the above table, for the linearized KdV equation (`Eq6`), an additive separable solution is presented. The automatic integration of the ODE found by separation of variables is performed with the option `INTEGRATE`, and an explicit expression for the unknown function is constructed with the option `build`. The infinitesimals introduced for solving the third and the eighth equations refer to the list containing the components of the infinitesimal generator of a symmetry group. Note that the infinitesimals for equation 3 involve arbitrary functions; therefore, to work with these infinitesimals, we add the option `specialize_Fn`. The solution of equation 8 contains an arbitrary constant $\_\varepsilon$ (Lie group parameter). Equations 6 and 9 are solved by separation of variables; in this case, according to Maple notation, the arbitrary constants are denoted by $\_c_1$ and $\_c_2$ (so as to distinguish them from the usual arbitrary constants $\_C_1$, $\_C_2$, etc.).

22.2.2 Constructing General Solutions via the Method of Characteristics

Consider the method for finding general solutions of first-order linear equations, the *Lagrange method of characteristics*. This method allows reducing a PDE to a system of ODEs along which the given PDE with some initial data (the Cauchy data) is integrable. Once the system of ODEs is found, it can be solved along the *characteristic curves* and transformed into a general solution of the original PDE. This method was first proposed by Lagrange in 1772 and 1779 for solving first-order linear and nonlinear PDEs.

In general, a first-order PDE in two independent variables (x, y) can be written in the form

$$\mathcal{F}(x,y,u,u_x,u_y)=0 \quad \text{or} \quad \mathcal{F}(x,y,u,p,q)=0, \qquad (22.2.2.1)$$

where $\mathcal{F}$ is a given function, $u=u(x,y)$ is the unknown function of the independent variables x and y ($(x,y)\in D\subset\mathbb{R}^2$), $u_x=p$, and $u_y=q$. Equation (22.2.2.1) is said to be linear if the function $\mathcal{F}$ is linear in the variables u, u_x, and u_y and the coefficients of these variables are functions of the independent variables x and y alone. The most general first-order linear PDE has the form

$$A(x,y)u_x+B(x,y)u_y+C(x,y)u=G(x,y). \qquad (22.2.2.2)$$

The solution $u(x,y)$ of Eq. (22.2.2.1) can be visualized geometrically as a surface, called an *integral surface* in the (x,y,u)-space.

A *direction vector field* or the *characteristic direction*, (A,B,C), is a *tangent vector* to the integral surface $f(x,y,u)$=0 at a point (x,y,u).

A *characteristic curve* is a curve in (x,y,u)-space such that the tangent at each point coincides with the characteristic direction field (A,B,C).

The parametric equations of the characteristic curve can be written in the form $x=x(t)$, $y=y(t)$, $u=u(t)$, and the tangent vector to this curve $(dx/dt,dy/dt,du/dt)$ is equal to (A,B,C).

The *characteristic equations* of Eq. (22.2.2.2) in parametric form is the system of ODEs $dx/dt=A(x,y)$, $dy/dt=B(x,y)$, $du/dt=C(x,y)$.

The *characteristic equations* of Eq. (22.2.2.2) in nonparametric form are $dx/A(x,y)=dy/B(x,y)=du/C(x,y)$. The slopes of the characteristics are determined by the equation $dy/dx=B(x,y)/A(x,y)$.

Example 22.8. *First-order linear equations. Direction vector fields.* By applying predefined Maple functions to a first-order linear PDE of the form

$$u_x+u_t+u=0, \quad xu_x+tu_t+(x^2+t^2)=0,$$

where $\{x \in \mathbb{R}, t \geq 0\}$, we can construct the direction vector fields for the given first-order linear PDE as follows:

```
with(plots): R1:=-Pi..Pi: N1:=10: V1:=[-24,72]; V2:=[-59,-29];
A1:=`THICK`;setoptions3d(fieldplot3d,grid=[N1,N1,N1],axes=boxed);
r1:=(x,t,u)->[1,1,-u]; r2:=(x,t,u)->[x,t,-(x^2+t^2)];
fieldplot3d(r1(x,t,u),x=R1,t=R1,u=R1,arrows=SLIM,orientation=V1);
fieldplot3d(r2(x,t,u),x=R1,t=R1,u=R1,arrows=A1,orientation=V2);
```

The *general solution* (or general integral) of the given first-order PDE is an equation of the form

$$f(\phi,\psi)=0, \tag{22.2.2.3}$$

where f is an arbitrary function of the known functions, $\phi = \phi(x,y,u)$ and $\psi=\psi(x,y,u)$, and provides a solution of this partial differential equation. The functions $\phi(x,y,u) = C_1$, $\psi(x,y,u) = C_2$ (where C_1 and C_2 are constants) are solution curves of the characteristic equations or the families of characteristic curves of Eq. (22.2.2.2).

Example 22.9. *First-order equation. General solution.* By applying the method of characteristics to the first-order linear PDE

$$xu_x + yu_y = u,$$

where $\{x \in \mathbb{R}, y \in \mathbb{R}\}$, we can prove that the general solution of the given first-order PDE takes the form $f(y/x, u/x) = 0$ or $u(x,y) = xg\,(y/x)$:

```
with(PDEtools): declare(u(x,y)); U:=diff_table(u(x,y));
F:=x; G:=y; H:=U[]; PDE:=F*U[x]+G*U[y]=H; CharEqs:=[dx/F,dy/G,du/H];
Eq10:=map(exp,combine(int(1/G,y)-int(1/F,x)=C10) assuming x>0);
Eq20:=map(exp,combine(int(1/u,u)-int(1/F,x)=C20) assuming x>0);
Eq11:=subs(rhs(Eq10)=C1,Eq10); Eq21:=subs(rhs(Eq20)=C2,Eq20);
GenSol:=f(lhs(Eq11),lhs(Eq21))=0; Test1:=pdetest(subs(u=U[],GenSol),PDE);
pdsolve(PDE,u(x,y)); GenSol1:=op(2,lhs(GenSol))=g(op(1,lhs(GenSol)));
GenSol2:=isolate(GenSol1,u); Test2:=pdetest(subs(u=U[],GenSol2),PDE);
```

Remark. The system of characteristic equations $dx/x = dy/y = du/u$ gives the integral surfaces $\phi = y/x = C_1$, $\psi = u/x = C_2$ (`Eq11`, `Eq21`, and C_1, C_2 are arbitrary constants); therefore, according to Eq. (22.2.2.3), the general solution of the linear PDE is $f(y/x, u/x) = 0$ (`GenSol`), where f is an arbitrary function. This solution can be verified (`Test1`). By applying the predefined function `pdsolve`, we can find the result in a different form: $u\,(x,y) = xg\left(\frac{y}{x}\right)$. This form of general solution can be obtained (`GenSol2`) and verified (`Test2`).

Example 22.10. *Linear Euler equation. General solution.* By applying the method of characteristics to the linear Euler equation

$$xu_x + yu_y = nu,$$

where $\{x \in \mathbb{R}, y \in \mathbb{R}\}$, we can prove that the general solution of the given first-order PDE takes the form $f(y/x, u/x^n) = 0$ or $u(x,y) = x^n g\,(y/x)$ as follows:

```
interface(showassumed=0): assume(x>0,y>0,u>0,n>0); with(PDEtools):
declare(u(x,y)); U:=diff_table(u(x,y)); F:=x; G:=y; H:=n*U[];
PDE:=F*U[x]+G*U[y]=H; CharEqs:=[dx/F,dy/G,du/H];
Eq10:=map(exp,combine(int(1/G,y)-int(1/F,x)=C10) assuming x>0);
Eq20:=map(exp,int(1/(n*u),u)-int(1/F,x)=C20);
```

```
Eq200:=simplify(map(combine,Eq20)); Eq11:=subs(rhs(Eq10)=C1,Eq10);
Eq21:=subs(rhs(Eq200)=C21,Eq200); Eq22:=simplify(lhs(Eq21)^n)=C2;
GenSol:=f(lhs(Eq11),lhs(Eq22))=0; Test1:=pdetest(subs(u=U[],GenSol),PDE);
pdsolve(PDE,u(x,y)); GenSol1:=op(2,lhs(GenSol))=g(op(1,lhs(GenSol)));
GenSol2:=simplify(isolate(GenSol1,u)); Test2:=pdetest(subs(u=U[],GenSol2),PDE);
```

Remark. The system of characteristic equations $dx/x = dy/y = du/(nu)$ gives the integral surfaces: $\phi = y/x = C_1$, $\psi = x^{-n}u = C_2$ (`Eq11`, `Eq22`, and C_1, C_2 are arbitrary constants); therefore, according to Eq. (22.2.2.3), the general solution of the linear PDE is $f(y/x, x^{-n}u) = 0$ (`GenSol`), where f is an arbitrary function. This solution can be verified (`Test1`). By applying the predefined function `pdsolve`, we can find the result in a different form: $u(x,y) = x^n g\left(\frac{y}{x}\right)$. This form of general solution can be obtained (`GenSol2`) and verified (`Test2`).

22.2.3 Constructing General Solutions via Transformations to Canonical Forms

First, consider the general first-order linear partial differential equation

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = G(x,y).$$

By introducing a new transformation by the equations $\xi = \xi(x,y)$, $\eta = \eta(x,y)$,[3] we can transform the original equation into a *canonical* (or standard) form

$$u_\xi + \alpha(\xi,\eta)u = \beta(\xi,\eta),$$

where $\alpha(\xi,\eta) = \tilde{C}/\tilde{A}$ and $\beta(\xi,\eta) = \tilde{G}/\tilde{A}$, $\tilde{A} = u\xi_x + B\xi_y$, $\tilde{B} = A\eta_x + B\eta_y$, and the equation is $\tilde{A}u_\xi + \tilde{B}u_\eta + \tilde{C}u = \tilde{G}$. This equation can be integrated, and the general solution of the original equation can be found.

Example 22.11. *Linear first-order equation. Canonical form. General solution.* Considering the linear first-order PDE

$$u_x + u_y = u,$$

where $\{x \in \mathbb{R}, y \in \mathbb{R}\}$, we can obtain the transformation $\xi = -x + y$, $\eta = x$ (`Eq_xi`, `Eq_eta`) and the canonical form $u_\eta = u$ (`CanForm`). By integrating this equation, we find the general solution $u(x,y) = e^x F(-x+y)$ (where $F(-x+y)$ is an arbitrary function). Finally, we can find and test the general solution using predefined functions as follows:

```
with(VectorCalculus): with(PDEtools): declare(u(x,y));
U:=diff_table(u(x,y)); V:=diff_table(u(xi,eta)); A:=1; B:=1; C:=-1; G:=0;
PDE:=A*U[x]+B*U[y]+C*U[]=0; CharEqs:=[dx/A,dy/B,du/U[]];
Eq_xi:=xi=int(1/B,y)-int(1/A,x); Eq_eta:=eta=x;
TestJ:=diff(rhs(Eq_xi),x)*diff(rhs(Eq_eta),y)-diff(rhs(Eq_xi),y)*diff(rhs(Eq_eta),x);
Vx:= V[xi]*diff(rhs(Eq_xi),x)+V[eta]*diff(rhs(Eq_eta),x);
Vy:= V[xi]*diff(rhs(Eq_xi),y)+V[eta]*diff(rhs(Eq_eta),y);
CanForm:=A*Vx+B*Vy=-C*V[]; CanForm1:=CanForm/(rhs(CanForm));
Eq1:=int(lhs(CanForm1),eta)=int(rhs(CanForm1),eta)+ln(F(xi)); Eq2:=isolate(Eq1,V[]);
GenSol:=U[]=subs({Eq_xi,Eq_eta},rhs(Eq2)); GenSolPred:=pdsolve(PDE,U[]);
Test1:=pdetest(GenSolPred,PDE); Test2:=pdetest(GenSol,PDE);
```

[3] Here the functions ξ and η are continuously differentiable and the Jacobian, $J(x,y) = \xi_x\eta_y - \xi_y\eta_x$, is nonzero in a domain $\mathcal{D}$.

For linear second-order PDEs, we consider the classification of equations (that does not depend on their solutions and is determined by the coefficients of the highest derivatives) and the reduction of a given equation to appropriate canonical forms.

Let us introduce the new variables $a{=}\mathcal{F}_p$, $b{=}\frac{1}{2}\mathcal{F}_q$, $c{=}\mathcal{F}_r$, and calculate the discriminant $\delta{=}b^2{-}ac$ at some point. Depending on the sign of the discriminant δ, the type of the equation at a specific point can be *parabolic* (if $\delta{=}0$), *hyperbolic* (if $\delta > 0$), and *elliptic* (if $\delta < 0$). Let us call the equations

$$u_{y_1y_2}{=}f_1(y_1,y_2,u,u_{y_1},u_{y_2}), \quad u_{z_1z_1}-u_{z_2z_2}{=}f_2(z_1,z_2,u,u_{z_1},u_{z_2}),$$

respectively, the *first canonical form* and the *second canonical form* for hyperbolic PDEs.

Example 22.12. *Linear second-order equation. Classification. Canonical forms.* Considering the linear second-order PDE

$$-2y^2u_{xx}+\tfrac{1}{2}x^2u_{yy}=0,$$

where $\{x\in\mathbb{R}, y\in\mathbb{R}\}$, we can verify that this equation is *hyperbolic* everywhere (except at the point $x{=}0$, $y{=}0$) and prove that a change of variables, $\xi=-\frac{1}{2}x^2+y^2$ and $\eta=\frac{1}{2}x^2+y^2$, transforms the PDE, respectively, to the *first* and *second canonical forms*:

$$v_{\eta\xi}+\frac{v_\xi\eta-v_\eta\xi}{2(\eta^2-\xi^2)}{=}0, \qquad v_{\lambda\lambda}-v_{\mu\mu}+\frac{1}{2}\Big(\frac{v_\lambda}{\lambda}-\frac{v_\mu}{\mu}\Big)=0.$$

Classification. In the standard notation (22.1.1.3), this linear equation takes the form $F_1=-2y^2p+\frac{1}{2}x^2r=0$. The new variables are $a{=}-2y^2$, $b{=}0$, $c{=}\frac{1}{2}x^2$ (`tr2(F1)`) and the discriminant $\delta{=}b^2{-}ac{=}x^2y^2$ (`delta1`) is positive except at the point $x{=}0$, $y{=}0$.

```
with(PDEtools): declare(u(x,y),F1(p,r,q)); U:=diff_table(u(x,y));
PDE1:=-2*y^2*U[x,x]+x^2*U[y,y]/2=0;
tr1:=(x,y,U)->{U[x,x]=p,U[y,y]=r,U[x,y]=q};
tr2:=F->{a=diff(lhs(F(p,q,r)),p),b=1/2*diff(lhs(F(p,q,r)),q),c=diff(lhs(F(p,q,r)),r)};
delta:=b^2-a*c; F1:=(p,r,q)->subs(tr1(x,y,U),PDE1); F1(p,r,q); tr2(F1);
delta1:=subs(tr2(F1),delta)-rhs(F1(p,r,q));
is(delta1,'positive'); coulditbe(delta1,'positive');
```

The same result can be obtained with the principal part coefficient matrix as follows:

```
interface(showassumed=0): assume(x<0 or x>0, y<0 or y>0);
with(LinearAlgebra): A1:=Matrix([[-2*y^2,0],[0,x^2/2]]);
D1:=Determinant(A1); is(D1,'negative'); coulditbe(D1,'negative');
```

Remark. Here we calculate the determinant `D1` of the matrix `A1`. The PDEs can be classified according to the eigenvalues of the matrix `A1`, i.e., depending on the sign of `D1`: if `D1=0`, parabolic, if `D1<0`, hyperbolic, and `D1>0`, elliptic equations.

Canonical forms. We find a change of variables that transforms the PDE to the first and second canonical forms as follows:

```
with(LinearAlgebra): with(VectorCalculus): with(PDEtools): declare(v(xi,eta));
interface(showassumed=0): vars:=x,y; varsN:=xi,eta;
Op1:=Expr->subs(y=y(x),Expr); Op2:=Expr->subs(y(x)=y,Expr);
m1:=simplify((-A1[1,2]+sqrt(-D1))/A1[1,1],radical,symbolic);
m2:=simplify((-A1[1,2]-sqrt(-D1))/A1[1,1],radical,symbolic);
```

```
Eq1:=dsolve(diff(y(x),x)=-Op1(m1),y(x));
Eq11:=lhs(Eq1[1])^2=rhs(Eq1[1])^2; Eq12:=solve(Eq11,_C1); g1:=Op2(Eq12);
Eq2:=dsolve(diff(y(x),x)=-Op1(m2),y(x)); Eq21:=lhs(Eq2[1])^2=rhs(Eq2[1])^2;
Eq22:=solve(Eq21,_C1); g2:=Op2(Eq22); Jg:=Jacobian(Vector(2,[g1,g2]),[vars]);
dv:=Gradient(v(varsN),[varsN]); ddv:=Hessian(v(varsN),[varsN]);
ddu:=Jg^%T.ddv.Jg+add(dv[i]*Hessian(g||i,[vars]),i=1..2);
Eq3:=simplify(Trace(A1.ddu))=0;
tr1:={isolate(subs(isolate(g1=xi,x^2),g2=eta),y^2),
      isolate(subs(isolate(g1=xi,y^2),g2=eta),x^2)};
CanForm1:=collect(expand(subs(tr1,Eq3)),diff(v(varsN),varsN));
c1:=coeff(lhs(CanForm1),diff(v(varsN),varsN));
CanFormF:=collect(CanForm1/c1,diff(v(varsN),varsN));
CanForm2:=expand(expand(dchange({xi=lambda+mu,eta=mu-lambda},CanFormF))*(-4));
```

Example 22.13. *Linear second-order equation. Classification. Canonical forms.* Considering the linear second-order PDE

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0,$$

where $\{x\in\mathbb{R},y\in\mathbb{R}\}$; we can verify that this equation is *parabolic* everywhere and prove that the canonical forms of the PDE are $x^2v_{\xi\xi}=0$ and $v_{\xi\xi}=0$, respectively.

```
with(LinearAlgebra): with(VectorCalculus): with(PDEtools): declare(v(xi,eta));
Op1:=Expr->subs(y=y(x),Expr); Op2:=Expr->subs(y(x)=y,Expr); vars:=x,y; varsN:=xi,eta;
A1:=Matrix([[x^2,x*y],[x*y,y^2]]); D1:=Determinant(A1);
m1:=simplify((-A1[1,2]+sqrt(-D1))/A1[1,1],radical,symbolic);
Eq1:=dsolve(diff(y(x),x)=-Op1(m1),y(x)); Eq11:=solve(Eq1,_C1);
g1:=Op2(Eq11); g2:=x; Jg:=Jacobian(Vector(2,[g1,g2]),[vars]);
dv:=Gradient(v(varsN),[varsN]); ddv:=Hessian(v(varsN),[varsN]);
ddu:=Jg^%T.ddv.Jg+add(dv[i]*Hessian(g||i,[vars]),i=1..2);
CanForm1:=simplify(Trace(A1.ddu))=0; CanForm2:=expand(CanForm1/x^2);
```

22.2.4 Constructing Analytical Solutions of Cauchy Problems

If we consider a mathematical problem in an unbounded domain, then the solution can be determined uniquely by prescribing initial conditions. The corresponding problem is called the *initial value problem*, or the *Cauchy problem*.

Mathematical problems can be *well-posed* or *ill-posed*. A mathematical problem is well-posed if it possesses the following features: *existence* (there exists at least one solution), *uniqueness* (there exists at most one solution), and *continuity* (the solution depends continuously on the data).

In Maple, it is possible to construct exact solutions for linear (and nonlinear) partial differential equations subject to initial conditions with the aid of the predefined function `pdsolve`. Let us solve some Cauchy problems.

Example 22.14. *Linear first-order PDEs. Initial value problem (IVP). Exact solution.* Considering the Cauchy problem for the linear first-order partial differential equation,

$$yu_x+xu_y=u, \quad u(x,0)=F(x),$$

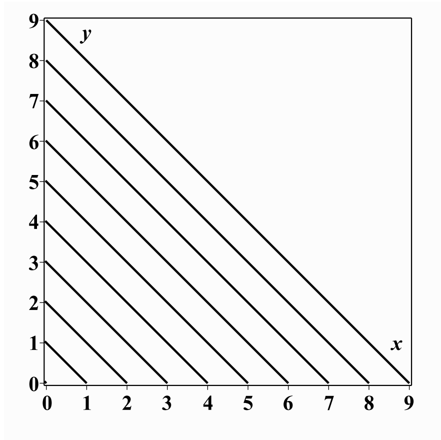


Figure 22.1 Characteristics for the linear equation $u_x - u_y = 1$.

and applying Maple predefined functions, we can solve the Cauchy problem for this equation and verify that the solution obtained, $u(x,y) = \dfrac{F(\sqrt{x^2-y^2})(x+y)}{\sqrt{x^2-y^2}}$, is an exact solution of this initial value problem as follows:

```
with(PDEtools): declare(F(x)); PDE1:=y*diff(u(x,y),x)+x*diff(u(x,y),y)=u(x,y);
IC1:=u(x,0)=F(x); sys1:=[PDE1,IC1]; Sol:=pdsolve(sys1);
Test1:= simplify(pdetest(Sol,sys1),symbolic);
```

Using different Maple notation, we can obtain the same result:

```
with(PDEtools): declare(F(x));  U:=diff_table(u(x,y)): PDE1:=y*U[x]+x*U[y]=U[];
IC1:=eval(U[],t=0)=f(x); IC1:=eval(U[],y=0)=F(x); sys1:=[PDE1,IC1];
Sol:=pdsolve(sys1); Test1:= simplify(pdetest(Sol,sys1),symbolic);
```

Example 22.15. *Linear first-order equation. Method of characteristics. Classical Cauchy problem.* Considering the first-order linear partial differential equation with the Cauchy data,

$$u_x - u_y = 1, \quad u(x,0) = x^n \qquad (n \in \mathbb{N}),$$

and applying the method of characteristics, we can obtain the general and particular solutions of this equation,

$$u(x,y) = f(x+y) + x, \quad u(x,y) = (y+x)^n - y,$$

respectively, and plot the characteristic curves (see Fig. 22.1) as follows:

```
with(plots): xR:=0..9; yR:=0..9; ODE:=diff(U(x),x)=1; tr1:=U(x)=u; tr2:=y(x)=y;
Sol_Ch:=dsolve({ODE,U(0)=C[2]}); Eq_Ch:=diff(y(x),x)=-1;
Cur_Ch:=dsolve({Eq_Ch, y(0)=C[1]});
display([seq(plot([subs(C[1]=y,eval(y(x),Cur_Ch)),x,x=xR],color=blue,thickness=2),
         y=yR)],view=[yR,xR],scaling=constrained);
Eq1:=subs(tr1,isolate(Sol_Ch,C[2])); Eq2:=subs(tr2,isolate(Cur_Ch,C[1]));
Eq3:=rhs(Eq1)=f(rhs(Eq2)); F:=unapply(lhs(subs({y=0,u=x^n},Eq3)),x);
u:=solve(Eq3,u); Sol1:=simplify(subs( f(rhs(Eq2))=F(rhs(Eq2)),u));
```

The same result can be obtained applying predefined functions:

```
with(PDEtools): declare(v(x,y)); PDE:=diff(v(x,y),x)-diff(v(x,y),y)=1;
IC:=v(x,0)=x^n; sys1:=[PDE,IC]; Sol2:=pdsolve(sys1);
Test:= simplify(pdetest(Sol2,sys1),symbolic);
```

Example 22.16. *First-order linear equation. Method of characteristics. Classical Cauchy problem.* Let us solve the first-order linear equation with Cauchy data,

$$u_t - xu_x = u, \quad u(x,0) = f(x),$$

by applying the method of characteristics. This equation can be obtained from the Fokker–Planck equation [see Bluman et al. (2010)],[4]

$$u_t = u_{xx} + (xu)_x,$$

by neglecting the term u_{xx}.

We can obtain the solution of this Cauchy problem

$$u_1(x,t) = f(xe^t)\, e^t$$

and plot the characteristic curves as follows:

```
with(plots); interface(showassumed=0); assume(t>0); assume(X[0],constant);
tR:=0..6; xR:=-10..10; Ops1:=color=blue,thickness=2; Ops2:=view=[xR,tR];
tr1:=x=x(t); tr2:=x(t)=x; tr3:=f(X[0])=f1(X[0]); f1:=x->x;
ODE1:=diff(x(t),t)=-x(t); ODE2:=diff(U(t),t)=U(t);
Sol2:=dsolve(ODE2,U(t)); Sol21:=subs(_C1=C,rhs(Sol2));
IniCond:=u(X[0],0)=f(X[0]); Const2:=evala(subs(t=0,Sol21))=rhs(IniCond);
Sol22:=U(t)=subs(Const2,Sol21); ODE11:=combine(subs(Sol22,ODE1));
Sol1:=dsolve(ODE11,x(t)); X0:=expand(subs(t=0,Sol1));
Const1:=isolate(X0,_C1); Chars:=subs(Const1,x(0)=X[0],Sol1);
GenSol:=rhs(Sol22); Char1:=subs(tr2,tr3,Chars);
trX0:=simplify(isolate(Char1,X[0]));
u1:=unapply(subs(trX0,GenSol),x,t); u1(x,t);
u2:=unapply(subs(f(X[0])=f1(rhs(trX0)),GenSol),x,t); u2(x,t);
display([seq(plot([subs(X[0]=x,eval(x(t),Char1)),t,t=tR],Ops1),x=xR)],Ops2);
display([seq(plot([u2(x,t),t,t=tR],Ops1),x=xR)],Ops2);
```

We show that the *implicit form of the solution* (or parametric representation of the solution) of this Cauchy problem has the Maple form

$Sol22 := U(t) = f(X_0)\,\mathrm{e}^t \quad Chars := x(t) = X_0\,\mathrm{e}^{-t}$

The characteristic curves for $f(x) = x$ are presented in Fig. 22.2.

Note that there are some difficulties in stating Cauchy problems for parabolic and elliptic equations. By way of example, consider the *Hadamard example* [see Hadamard (1952)].

[4] The Fokker–Planck equation arises in various applications of statistical mechanics; it describes the evolution of the probability distribution function.

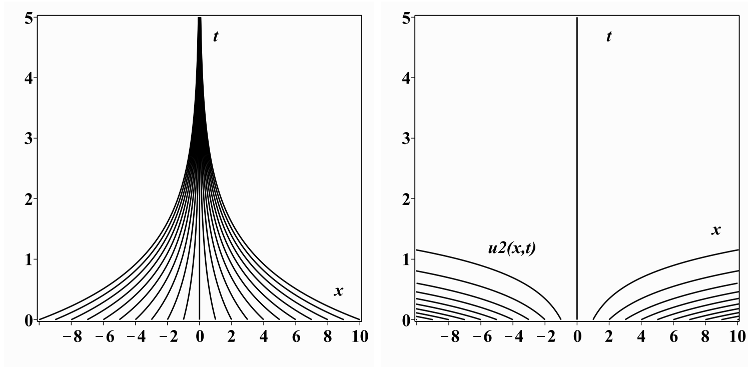


Figure 22.2 Characteristic curves for $u_t - xu_x = u$ and the solution $u_2(x,t)$ (for $f(x) = x$).

Example 22.17. *Laplace equation. Hadamard example. Cauchy problem.* Considering an elliptic equation, e.g., the Laplace equation with the initial conditions proposed by Hadamard [see Hadamard (1952)],

$$u_{xx} + u_{yy} = 0, \qquad u(x,0) = 0, \quad u_y(x,0) = A_n \sin(nx), \qquad A_n = \frac{1}{n^p} \quad (p > 0),$$

we obtain the solution of this Cauchy problem (`SolF`)

$$u(x,y) = \sinh(ny)\sin(nx)n^{-1-p}$$

and verify that this solution is an exact solution of the Laplace equation as follows:

```
PDE1:=diff(u(x,y),x$2)+diff(u(x,y),y$2)=0; A[n]:= 1/n^p;
SGen:=pdsolve(PDE1,u(x,y),HINT=sin(n*x)*h(y),build); SGen1:=eval(subs(_c[1]=1,SGen));
Eq1:=eval(rhs(SGen1),y=0)=0; Eq2:=eval(diff(rhs(SGen1),y),y=0)=A[n]*sin(n*x);
trConst:=op(solve({Eq1,Eq2},[_C1,_C2]));
Sol1:=subs(trConst,SGen1); SolF:=simplify(convert(Sol1,trig));
Test1:=pdetest(SolF,PDE1);
```

However, the solution of this problem does not depend continuously on the initial data (i.e., a small change in the initial data produces a small change in the solution). These difficulties arise in connection with the Cauchy–Kowalevski theorem [for details, see Petrovsky (1991)]. This can be shown as follows:

```
limit(A[n]*sin(n*x),n=infinity) assuming p>0;
for i from 1 to 30 by 10 do plot(subs({n=i,y=1,p=1},rhs(SolF)),x=0..10); od;
for i from 1 to 10 by 3 do plot(subs({n=i,x=1,p=1},rhs(SolF)),y=0..1); od;
plot(subs({n=9,x=1,p=1},rhs(SolF)),y=0..10);
```

The function (`Eq2`), which describes the initial data, tends to zero as n tends to infinity, i.e., $\lim_{n\to\infty} A_n \sin(nx) = 0$. If $n \to \infty$, the solution (`SolF`) represents oscillations for any $y \neq 0$ (the first sequence of plots). Also, if $n \to \infty$, the amplitude of the solution tends to infinity as y tends to infinity for any $x \neq 0$ (the second sequence of plots). If n is a fixed number, the solution tends to infinity as y tends to infinity for any $x \neq 0$ (for which $\sin(nx) \neq 0$). Therefore, this problem is an example of an ill-posed problem.

22.2.5 Constructing Analytical Solutions of Boundary Value Problems

A *boundary value problem* is a problem of finding an unknown function that satisfies a given partial differential equation and particular boundary conditions. Boundary value problems are associated with partial differential equations of elliptic type and are considerably more difficult to solve.

Boundary value problems can be divided into two classes: *interior* and *exterior boundary value problems*. For exterior boundary value problems, part of the boundary is at infinity and their solutions should satisfy an additional requirement (boundedness at infinity). Interior boundary value problems can have various types of boundary conditions. The most important types of boundary conditions are the *Dirichlet condition* (the solution u is given on the boundary), the *Neumann condition* ($\partial u/\partial\mathbf{n}$ is given on the boundary), the *mixed condition* or the *Robin condition* ($\partial u/\partial\mathbf{n}+hu$ is given on the boundary, or boundary conditions of distinct types are given on distinct parts of the boundary),[5] and the *periodic boundary conditions*.

In Maple, exact solutions of linear partial differential equations with boundary conditions can be constructed with the aid of the predefined function `pdsolve` (for details, see `?pdsolve[boundaryconditions]`). Consider various types of boundary value problems.

Example 22.18. *Laplace equation. Dirichlet boundary value problem (BVP).* Considering the Laplace equation and the Dirichlet boundary conditions on the rectangular region

$$u_{xx}+u_{yy}=0,\quad 0<x<a,\quad 0<y<b;$$
$$u(x,0)=x^2,\quad u(x,b)=x^2-b^2,\quad u(0,y)=-y^2,\quad u(a,y)=a^2-y^2,$$

we obtain the general solution (`SGen`) of this boundary value problem (by applying the option `HINT=g(x)+h(y)` for additive separable form of solution)

$$u(x,y)=-\frac{1}{2}\_c_2x^2+\_C1x+\_C2+\frac{1}{2}\_c_2y^2+\_C3y+\_C4.$$

Then, considering the given boundary conditions, we get the constants (`trConst`), obtain the final form of the analytical solution

$$u(x,t)=x^2-y^2,$$

and verify that this solution is an exact solution of the Laplace equation as follows:

```
PDE1:=diff(u(x,y),x$2)+diff(u(x,y),y$2)=0;
SGen:=pdsolve(PDE1,u(x,y),HINT=g(x)+h(y),build);  SGen1:=eval(subs(_c[2]=-2,SGen));
Eq1:=eval(rhs(SGen1),y=0)=x^2; Eq2:=eval(rhs(SGen1),y=b)=x^2-b^2;
Eq3:=eval(rhs(SGen1),x=0)=-y^2; Eq4:=eval(rhs(SGen1),x=a)=a^2-y^2;
S1:=op(solve({Eq1,Eq2,Eq3,Eq4},[_C1,_C2,_C3,_C4]));
trConst:=[op(1..3,subs(_C4=0,S1)),_C4=0];
SolF:=subs(trConst,SGen1); Test1:=pdetest(SolF,PDE1);
```

Example 22.19. *Linear Poisson equation. Dirichlet boundary value problem (BVP).* Consider the Poisson equation and the Dirichlet boundary conditions on the rectangular region

$$u_{xx}+u_{yy}=f(x,y),\quad a<x<b,\quad c<y<d;$$
$$u(x,c)=f_1(x),\quad u(x,d)=f_2(x),\quad u(a,y)=f_3(y),\quad u(b,y)=f_4(y).$$

[5]Here $\partial u/\partial\mathbf{n}$ is the *directional derivative* of the solution u along the *outward normal* to the boundary, and h is a given continuous function on the boundary.

Such boundary value problems describe a steady-state process u(x, y) in a bounded rectangular object. Let us choose $f(x,y) = \cos x \sin y$, $f_1(x) = f_2(x) = 0$, $f_3(y) = f_4(y) = -\frac{1}{2}\sin y$, $a = 0$, $b = \pi$, $c = 0$, $d = 2\pi$. By applying Maple predefined functions, we obtain the analytical solution $u(x,y) = -\frac{1}{2}\cos x \sin y$ (Sol12) of the boundary value problem, visualize it in the given region, and verify that this solution is an exact solution of the linear Poisson equation as follows:

```
with(linalg): with(PDEtools): f:=(x,y)->cos(x)*sin(y);
PDE1:=laplacian(u(x,y),[x,y])-f(x,y)=0;
Sol1:=pdsolve(PDE1,build); Test1:=pdetest(Sol1,PDE1);
Sol11:=unapply(subsop(1=0,2=0,rhs(Sol1)),x,y); Sol12:=expand(Sol11(x,y));
Sol11(x,0); Sol11(x,Pi); Sol11(0,y); Sol11(2*Pi,y);
plot3d(Sol12,x=0..Pi,y=0..2*Pi,shading=zhue);
```

22.2.6 Constructing Analytical Solutions of Initial-Boundary Value Problems

If a mathematical problem is to find an unknown function satisfying a PDE (defined at an appropriate domain) and appropriate supplementary conditions (initial and boundary conditions), this is known as an *initial-boundary value problem*. The boundary conditions describe the unknown function at prescribed boundary points. The initial condition prescribes the unknown function at a certain initial time t (e.g., $t = t_0$, $t = 0$).

In Maple, it is possible to construct exact solutions for an increasing number of linear (and nonlinear) partial differential equations subject to initial and boundary conditions with the aid of the predefined function `pdsolve` (see `?pdsolve[boundaryconditions]`). Let us solve some initial-boundary value problems.

Example 22.20. *Linear telegraph equation. Initial-boundary value problem (IBVP).* Considering the initial-boundary value problem for the linear telegraph equation,

$$u_{xx} = u_{tt} + u_t - u, \qquad x > 0, \quad t > 0;$$
$$u(x,0) = e^x, \quad u_t(x,0) = -2e^x, \quad u(0,t) = e^{-2t}, \quad u_x(0,t) = e^{-2t}, \quad x \geq 0, \quad t \geq 0,$$

and applying Maple predefined functions, we obtain the solution of this initial-boundary value problem (Sol2)

$$u(x,t) = e^{-2t+x}$$

and verify that this solution is an exact solution of the linear telegraph equation as follows:

```
with(PDEtools): declare(u(x,t)); U:=diff_table(u(x,t)):
PDE1:=U[x,x]=U[t,t]+U[t]-U[]; Sol1:=pdsolve({PDE1},{u(x,t)},build);
IC1:=u(x,0)=exp(x),D[2](u)(x,0)=-2*exp(x);
BC1:=u(0,t)=exp(-2*t),D[1](u)(0,t)=exp(-2*t); Sol2:=pdsolve([PDE1,IC1,BC1]);
T1:=pdetest(Sol2,PDE1); T2:=simplify(subs(t=0,Sol2));
T3:=D[2](u)(x,0)=simplify(subs(t=0,diff(rhs(Sol2),t)));
T4:=simplify(subs(x=0,Sol2)); T5:=D[1](u)(0,t)=simplify(subs(x=0,diff(rhs(Sol2),x)));
```

Example 22.21. *Linear wave equation. Initial-boundary value problem (IBVP).* Considering a semiinfinite vibrating string with a fixed end, we solve the following initial-boundary value problem for the linear wave equation:

$$u_{tt} = c^2 u_{xx}, \qquad x > 0, \quad t > 0;$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad u(0,t) = 0 \quad x \geq 0, \quad t \geq 0.$$

The boundary condition at $x = 0$ produces a wave moving to the right with velocity c. By applying Maple predefined functions with appropriate assumptions ($c > 0$, $x > 0$, $t > 0$), we obtain the solution of this initial-boundary value problem (`Sol2`)

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(ct+x) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\zeta), & x < ct, \\ \frac{1}{2}\left[f(-ct+x) + f(ct+x)\right] + \frac{1}{2c}\int_{-ct+x}^{ct+x} g(\zeta), & x > ct, \end{cases}$$

and verify that this solution is an exact solution of the linear wave equation as follows:

```
with(PDEtools): declare(u(x,t)); U:=diff_table(u(x,t)): PDE1:=U[t,t]=c^2*U[x,x];
Sol1:=pdsolve({PDE1},{u(x,t)},build); IC1:=u(x,0)=f(x),D[2](u)(x,0)=g(x);
BC1:=u(0,t)=0; Sol2:=pdsolve([PDE1,IC1, BC1]) assuming c>0, x>0, t>0;
T1:=pdetest(u(x,t)=op(2,op(2,Sol2)),PDE1);
T2:=pdetest(u(x,t)=op(4,op(2,Sol2)),PDE1);
T3:=simplify(subs(t=0,Sol2)) assuming c>0, x>0;
T4:=D[2](u)(x,0)=simplify(subs(t=0,diff(rhs(Sol2),t))) assuming c>0, x>0;
T5:=simplify(subs(x=0,Sol2)) assuming c>0, t>0;
```

22.2.7 Constructing Analytical Solutions of Systems of Linear PDEs

Computer algebra system Maple has various predefined functions based on symbolic algorithms for constructing analytical solutions of systems of linear PDEs. These functions allow solving linear systems and obtaining solutions automatically as well as developing new methods and procedures for constructing new solutions. Just as before, we consider the most important functions for finding analytical solutions of a given nonlinear system.

```
pdsolve(PDESys,[DepVars],HINT=val,singsol=val,mindim=N,parameters=P);
pdetest(Sol,PDESys);   pdsolve([PDESys,ICs],[DepVars],series,order=N);
with(PDEtools);    separability(PDESys,DepVars);    casesplit(PDESys);
splitsys(PDESys,DepVars);                   ConsistencyTest(PDESys,ops);
TWSolutions(PDESys,ops);               SimilaritySolutions(PDESys,ops);
ReducedForm(PDESys1,PDESys2);
```

Remark. `ICs` are initial conditions; `DepVars` is a set or list of dependent variables, indicating the solving ordering, where slight changes in the solving ordering can lead to different solutions or make a system unsolvable (by `pdsolve`) a solvable system.

`pdsolve` finding analytical solutions for a given PDE system; relevant options are: `HINT`, some hints; `singsol`, computing or not computing singular solutions; `mindim`, a minimum dimension of the solution space; `parameters`, indicating the solving variables with less priority; `series`, computing formal power series solutions,

`PDEtools` a collection of functions for finding analytical solutions for `PDESys`, e.g.,

`splitsys` splitting a PDE system into subsets (each one with equations coupled among themselves but not coupled to the equations of the other subsets)

`ConsistencyTest` testing whether a given system of equations is or is not consistent

`ReducedForm` reducing one given PDE system with respect to another given PDE system, etc.

Example 22.22. *Linear hyperbolic system. General solution.* Consider the linear first-order hyperbolic system

$$u_t + k_1 u_x + k_2 v_x = 0, \quad v_t + k_2 u_x + k_1 v_x = 0,$$

where k_1 and k_2 are parameters ($k_1, k_2 \in \mathbb{R}$). By applying Maple predefined functions, we solve this linear hyperbolic system as follows:

```
with(PDEtools): declare((u,v)(x,t));
U,V:=diff_table(u(x,t)),diff_table(v(x,t)):
Sys1:=[U[t]+k[1]*U[x]+k[2]*V[x]=0,V[t]+k[2]*U[x]+k[1]*V[x]=0];
Sol1:=collect(pdsolve(Sys1,[U[],V[]]),[t]); pdetest(Sol1,Sys1);
```

We obtain the general solution (`Sol1`) of this linear system, and in the Maple notation it reads:

$$Sol1 := \{u(x,t) = -\_F2((k_1-k_2)t-x) + \_F1((k_1+k_2)t-x) + \_C1,\ v(x,t) = \_F1((k_1+k_2)t-x) + \_F2((k_1-k_2)t-x)\}$$

It can be shown that the solution obtained consists of two independent parts, one propagating with speed $k_1 + k_2$ and another with speed $k_1 - k_2$.

By applying Maple predefined functions, we will show how to find analytical solutions of the Cauchy problem and boundary value problem for linear second-order systems.

Example 22.23. *Linear second-order system. Cauchy problem.* Considering the linear second-order system and the Cauchy data,

$$v_t - u_{xx} = 0, \quad u_t + v_{xx} = 0, \qquad u(x,0) = e^x, \quad v(x,0) = e^{-x},$$

and applying Maple predefined functions, we solve the Cauchy problem for this system and verify that the solution obtained, $u(x,t) = (\mathrm{e}^x - x - 1)\cos(t) - \mathrm{e}^{-x}\sin(t) + x + 1$, $v(x,t) = \mathrm{e}^x\cos(t) + (\mathrm{e}^x - x - 1)\sin(t)$ (`S2`), is an exact solution of this initial value problem as follows:

```
with(PDEtools): declare((u,v)(x,t));
U,V:=diff_table(u(x,t)),diff_table(v(x,t)): Sys1:=V[t]-U[x,x]=0,
U[t]+V[x,x]=0; Sol1:=pdsolve([Sys1],[U[],V[]],build);
Sol2:=subs(_c[1]=1,Sol1); IC1:=eval(rhs(Sol2[1]),t=0)=exp(x);
IC2:=eval(rhs(Sol2[2]),t=0)=exp(-x);
trConst1:=_C6=1,_C4=1,_C3=0,_C1=0,_C7=1,_C8=1;
IC11:=subs(trConst1,IC1); IC12:=subs(trConst1,IC2);
trConst2:=op(solve({IC11,IC12}));
S20:=subs(trConst2,trConst1,Sol2);
S2:=map(collect,expand(S20),[cos,sin,exp]);
Test1:=pdetest(Sol2,[Sys1]);
Test2:=simplify(subs(trConst2,trConst1,{IC1,IC2}));
```

Example 22.24. *Linear second-order system. Boundary value problem.* Considering the linear second-order system and the boundary conditions,

$$\begin{aligned} &v_t - u_{xx} = 0, \quad u_t + v_{xx} = 0, \\ &u(0,t) = 2[\cos(t) - \sin(t)], \quad v(0,t) = 2[\cos(t) + \sin(t)], \\ &u(1,t) = \frac{1+\mathrm{e}^2}{\mathrm{e}}[\cos(t) - \sin(t)], \quad v(1,t) = \frac{1+\mathrm{e}^2}{\mathrm{e}}[\cos(t) + \sin(t)], \end{aligned}$$

and applying Maple predefined functions, we solve the boundary value problem for this system and verify that the solution obtained (`SolF`),

$$u(x,t) = \frac{1+e^{2x}}{e^x}[\cos(t) - \sin(t)], \quad v(x,t) = \frac{1+e^{2x}}{e^x}[\cos(t) + \sin(t)],$$

is an exact solution of this initial value problem as follows:

```
with(PDEtools): declare((u,v)(x,t)); U,V:=diff_table(u(x,t)),diff_table(v(x,t)):
Sys1:=V[t]-U[x,x]=0, U[t]+V[x,x]=0; BC1:=u(0,t)=2*(cos(t)-sin(t));
BC2:=v(0,t)=2*(cos(t)+sin(t)); BC3:=u(1,t)=((1+exp(1)^2)/exp(1))*(cos(t)-sin(t));
BC4:=v(1,t)=((1+exp(1)^2)/exp(1))*(cos(t)+sin(t));
Sol1:=pdsolve([Sys1],[U[],V[]],build); Sol2:=subs(_c[1]=1,Sol1);
S1:=simplify(subs(BC1,BC2,subs(x=0,Sol2)));
S2:=simplify(subs(BC3,BC4,subs(x=1,Sol2))); Sys2:= S1 union S2;
trConst1:=_C3=0,_C1=0,_C4=1,_C6=1; Sys3:=subs(trConst1, Sys2);
vars3:=remove(has,indets(Sys3),t); trConst2:=solve(Sys3,vars3);
SolF0:=subs(trConst1,trConst2,Sol2);
SolF:=factor(map(collect,expand(SolF0),[cos,sin]));
Test1:=pdetest(SolF,[Sys1]); Test2:=seq(subs(x=0,rhs(SolF[i]))=rhs(BC||i),i=1..2);
Test3:=seq(subs(x=1,rhs(SolF[i]))=rhs(BC||(i+2)),i=1..2);
simplify([Test2]); factor(combine([Test3]));
```

⊙ *References for Section 22.2*: J. Hadamard (1952), I. Petrovsky (1991), E. S. Cheb-Terrab and K. von Bulow (1995), G. W. Bluman, A. F. Cheviakov, and S. C. Anco (2010), I. K. Shingareva and C. Lizárraga-Celaya (2011, 2012).

22.3 Analytical Solutions of Mathematical Problems

22.3.1 Constructing Separable Solutions

Separation of variables is one of the most important methods for solving linear PDEs, in which the structure of a PDE allows us to seek *multiplicative separable* or *additive separable* exact solutions, e.g., $u(x,t) = \phi(x) \circ \psi(t)$ (where the multiplication or addition is denoted by $\circ$). Numerous problems in linear partial differential equations can be solved by separation of variables. This method has been generalized [see, e.g., recent papers by Galaktionov (1990, 1995), Polyanin and Zhurov (1998), Polyanin and Manzhirov (2007)], and nowadays it is one of the classical methods in mathematics and physics.

Let us start from first-order linear equations, which can be solved by separation of variables without considering Fourier series.

Example 22.25. *First-order linear equation. Separable solution. Cauchy problem.* Consider the first-order linear PDE and the Cauchy data

$$au_x + bu_y = 0, \quad u(0,y) = \alpha e^{-\beta y},$$

where $a, b \in \mathbb{R}$ are parameters. By applying separation of variables and by seeking exact solutions in the form $u(x,t) = \phi(x)\psi(y)$, we arrive at the following equations (`Eq61`, `Eq62`):

$$-\frac{a\phi'_x}{b\phi(x)} = C_1, \quad \frac{\psi'_y}{\psi(y)} = C_1.$$

```
with(PDEtools): declare(u(x,y),W(x,y),phi(x),psi(y));
interface(showassumed=0): assume(n,'integer',n>0): tr1:=phi(x)*psi(y);
PDE1:=u->a*diff(u(x,y),x) +b*diff(u(x,y),y); IC1:= u(0,y)=alpha*exp(-beta*y);
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,y)=tr1,Eq2));
Eq4:=expand(Eq3/phi(x)/psi(y)); Eq5:=isolate(Eq4,psi(y));
Eq61:=rhs(Eq5)=_C1; Eq62:=lhs(Eq5)=_C1;
```

Then we seek exact solutions of these equations as follows:

```
Sol1:=dsolve(Eq61,phi(x)); Sol2:=dsolve(Eq62,psi(y));
GenSol:=u(x,y)=simplify(subs(Sol1,Sol2,tr1)); trC3:=_C2^2=_C3;
GenSol1:=subs(trC3,GenSol); Eq8:=subs(x=0,rhs(GenSol1))=rhs(IC1);
trC13:=_C1=-beta,_C3=alpha; SolF:=subs(trC13,GenSol1);
Test1:=pdetest(SolF,PDE1(u)); Test2:=subs(x=0,rhs(SolF))=rhs(IC1);
```

We find that $\phi(x) = C_2 e^{-C_1 bx/a}$ (`Sol1`), $\psi(y) = C_2 e^{C_1 y}$ (`Sol2`), and the exact solution acquires the form (`SolF`)

$$u(x,y) = \alpha e^{-\beta(ay-bx)/a}.$$

Separation of variables combined with the *linear superposition principle* can be applied for solving a large class of initial-boundary value problems for linear partial differential equations. According to the method, the partial differential equation is reduced to two ordinary differential equations (for $\phi(x)$ and $\psi(y)$). A similar idea can be applied to equations in several independent variables. This method is also known as the *Fourier method* or the *eigenfunction expansion method*.

First, let us obtain a solution of the initial-boundary value problem with the aid of the predefined Maple function `pdsolve`, which applies separation of variables and the Fourier method.

Example 22.26. *Linear heat equation. Separable solution. Fourier series. Initial-boundary value problem.*

Consider the linear heat equation with the initial and boundary conditions

$$u_t = cu_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where c is a real parameter, $L = 1$, and $f(x) = x^3$.

If we apply the predefined Maple function `pdsolve` for the linear heat equation (which invokes separation of variables in this case), we can obtain the multiplicative separable solution (`Sol1`)

$$u(x,t) = \_C_1\_C_3\,\mathrm{e}^{\sqrt{\_c_1}x}\mathrm{e}^{c\_c_1 t} + \_C_2\_C_3\frac{\mathrm{e}^{c\_c_1 t}}{\mathrm{e}^{\sqrt{\_c_1}x}}$$

and solve the given initial-boundary value problem step by step.

If we apply the predefined Maple function `pdsolve` for the given initial-boundary value problem (which invokes separation of variables and the Fourier method in this case), we can solve this problem (automatically) and obtain the particular solution (`SolF`),

$$u(x,t) = \sum_{k=1}^{\infty} \frac{2(-1)^{1+k}\left((\pi k)^2 - 6\right)}{(\pi k)^3}\sin(\pi kx)\mathrm{e}^{-c(\pi k)^2 t}.$$

Finally, we verify (symbolically, numerically, and graphically) that this solution satisfies the linear heat equation and the initial and boundary conditions as follows:

```
with(PDEtools): with(plots): declare(u(x,t)); U:=diff_table(u(x,t)):
PDE1:=U[t]=c*U[x,x];  Sol1:=pdsolve({PDE1},{u(x,t)},build); f:=x->x^3;
IC1:=u(x,0)=f(x); BC1:=u(0,t)=0, u(1,t)=0;
Sol2:=pdsolve([PDE1,IC1,BC1]); SolF:=simplify(subs(_Z1=k,Sol2));
T1:=pdetest(SolF,PDE1); T2:=simplify(subs(t=0,SolF));
T3:=simplify(subs(x=0,SolF)); T4:=simplify(subs(x=1,SolF));
G1:=plot(f(x),x=0..1,color=magenta):
G2:=plot(evalf(sum(op(1,rhs(T2)),k=1..100)),x=0..1,color=blue): display({G1,G2});
```

Example 22.27. *Linear wave equation. Separable solution. Fourier series. Initial-boundary value problem.*

Consider the second-order linear hyperbolic PDE with the initial and boundary conditions

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \ \ (0 \le x \le L), \quad u(0,t) = 0, \quad u(L,t) = 0 \ \ (t \ge 0),$$

where $c^2 = \frac{1}{4}$, $L = 1$, $f(x) = 0$, $g(x) = \sin(x) - \sin(3\pi x)$. This problem describes a vibrating string (with constant tension T and density ρ, $c^2 = T/\rho$) stretched along the x-axis from 0 to L and fixed at the endpoints. The initial displacement $f(x)$ is zero, and the initial velocity is $g(x)$.

By separation of variables, i.e., by seeking the exact solution in the form $u(x,t) = \phi(x)\psi(t)$, we obtain the two ODEs

$$\phi''_{xx} - \lambda\phi = 0, \quad \psi''_{tt} - \lambda c^2 \psi = 0.$$

Then, separating the boundary conditions, we solve the eigenvalue problem and obtain the solution

$$u(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right),$$

which satisfies the original wave equation and the boundary conditions. Since the wave equation is linear and homogeneous, by the superposition principle, the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

is a solution as well. We assume the following properties of the solution: it converges and is twice continuously differentiable with respect to x and t. Since each term of the series satisfies the boundary conditions, it follows that the series satisfies these conditions. From the two initial conditions, we can determine the constants A_n and B_n. By differentiating the solution with respect to t, we obtain

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right).$$

These equations are satisfied if $f(x)$ and $g(x)$ can be represented by *Fourier sine series*. According to the Fourier theory, the formulas for the coefficients A_n and B_n read

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Finally, we obtain the analytical and graphical solutions of the initial-boundary value problem as follows:

```
with(plots): L:=1: c:=1/4: N:=10: IC1:=sin(x)-sin(3*Pi*x);
u:=(x,t)->Sum((An*cos(n*Pi*c*t/L)+Bn*sin(n*Pi*c*t/L))*sin(n*Pi*x/L),n=1..N);
u(x,t); An1:=solve(u(x,0)=0,An); EqIC:=expand(eval(diff(u(x,t),t),t=0));
Eq1:=expand(Bn*Pi*add(n*sin(n*Pi*x),n=1..N),sin);
for i from 1 to N do
 C[i]:=select(has,Eq1,sin(i*Pi*x));  IC[i]:=select(has,IC1,sin(i*Pi*x));
 B[i]:=solve(C[i]=IC[i],Bn);
end do:
U:=unapply(add(subs(Bn=B[n],An=An1,op(1,u(x,t))),n=1..N),x,t);
animate(U(x,t),x=0..L,t=0.1..5,frames=100,color=blue,thickness=3);
G:=animate(subs(t=j,U(x,t)),x=0..L,j=0.1..5,frames=12,tickmarks=[2,2],color=blue):
display(G);
```

Example 22.28. *Linear heat equation. Separable solution. Fourier series. Initial-boundary value problem.*

Consider the second-order linear parabolic PDE with the initial and boundary conditions

$$u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $k = 1/30$, $L = 1$, $f(x) = \sin^4(\pi x)$. This problem describes a homogeneous rod (of length L). We assume that the rod is sufficiently thin (i.e., the heat is distributed equally over the cross section at time t), the surface of the rod is insulated (i.e., there is no heat loss through the boundary), and the temperature distribution of the rod is given by the solution of the initial-boundary value problem.

By separation of variables, i.e., by seeking the exact solution in the form $u(x,t) = \phi(x)\psi(t)$, we obtain the two ODEs

$$\phi''_{xx} + \lambda^2\phi = 0, \quad \psi'_t + \lambda^2 k\psi = 0.$$

Then, by separating the boundary conditions, we solve the eigenvalue problem and obtain the solution

$$u(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt},$$

which satisfies the original heat equation and the boundary conditions. Since the heat equation is linear and homogeneous, by the superposition principle, the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt}$$

is a solution as well. This solution satisfies the initial condition if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients A_n are the Fourier coefficients:

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Finally, by recursively computing the Fourier coefficients and the desired solution, we obtain a formal series solution of the initial-boundary value problem and visualize it as follows:

```
with(plots): f:=x->sin(Pi*x)^4; L:= 1; k:=1/30: N:=100: NF1:=30: NF2:=12: NT:=7:
A:=proc(n) option remember;  evalf(2/L*Int(f(x)*sin(n*Pi*x/L),x=0..L)) end:
u:=proc(n) option remember; u(n-1)+A(n)*exp(-(n*Pi/L)^2*k*t)*sin(n*Pi*x/L) end:
A(1); u(1):=A(1)*exp(-(Pi/L)^2*k*t)*sin(Pi*x/L); u(N);
A:=animate(subs(t=j,u(N)),x=0..L,j=0..NT,frames=NF2,color=blue): display(A);
animate(subs(t=j,u(N)),x=0..L,j=0..NT,frames=NF1,color=blue);
```

22.3.2 Constructing Analytical Solutions via Integral Transform Methods

Integral transform methods are the most important methods for constructing analytical solutions of mathematical problems (initial and/or boundary value problems) described by linear partial differential equations.

The main idea of the methods is to transform the original mathematical problem into a simpler form whose solutions can be obtained and then inverted (by applying the corresponding inverse integral transform) for representing the solutions in terms of the original variables. Formal definitions of the most important integral transforms, together with their general forms and properties, can be found in Debnath (2007).

In Maple, integral transforms (e.g., Fourier, Hankel, Hilbert, Laplace, and Mellin integral transforms) can be studied with the aid of the `inttrans` package. Let us solve some initial-boundary value problems.

Example 22.29. *Linear wave equation. Laplace transform. Initial-boundary value problem.* Consider the second-order linear hyperbolic PDE with the initial and boundary conditions

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x < \infty, \quad t > 0,$$
$$u(x,0) = h(x), \quad u_t(x,0) = g(x) \ \ (0 < x < \infty), \quad u(0,t) = Af(t), \quad u(x,t) \to 0 \ \text{as} \ x \to \infty,$$

where $h(x) = 0$, $g(x) = 0$, and A is a constant ($A \in \mathbb{R}$). This problem describes transverse vibrations of a semiinfinite string. The initial displacement $h(x)$ and the initial velocity $g(x)$ are zero; i.e., the string is at rest in its equilibrium position.

Let $U(x,s)$ be the Laplace transform of $u(x,t)$. By transforming the equation of motion and by substituting the initial conditions, we obtain (`Eq4`)

$$s^2 U - c^2 U_{xx} = 0.$$

The solution of this ordinary differential equation is (`Sol`)

$$U = C_1 e^{-sx/c} + C_2 e^{sx/c}.$$

By transforming the boundary conditions, we obtain (`BC3`)

$$U(0,s) = AF(s), \quad \lim_{x\to 0} U(x,s) = 0,$$

According to the second condition, we have $C_2 = 0$, and by applying the first condition, we obtain (`Sol3`)

$$U(x,s) = AF(s)e^{-sx/c}.$$

By applying the inverse Laplace transform, we obtain the solution (`SolF`)

$$u(x,t) = AH\left(\frac{ct-x}{c}\right) f\left(\frac{ct-x}{c}\right) \quad \text{or} \quad u(x,t) = \begin{cases} 0, & t < x/c, \\ Af\big((ct-x)/c\big), & t \ge x/c, \end{cases}$$

where H is the Heaviside unit step function. This solution represents a wave propagating at a velocity c with the characteristic $x = ct$.

Finally, we obtain the the displacement of a semiinfinite string as follows:

```
with(inttrans): with(PDEtools): interface(showassumed=0): assume(c>0,s>0);
declare(u(x,t),U(x));
Eq1:=diff(u(x,t),t$2)-c^2*diff(u(x,t),x$2); Eq2:=laplace(Eq1,t,s);
```

```
Eq3:=subs(laplace(u(x,t),t,s)=U(x),Eq2);
IC1:={u(x,0)=0,D[2](u)(x,0)=0}; IC2:=laplace(IC1,t,s);
BC1:=u(0,t)=A*f(t); BC2:=laplace(BC1,t,s);
BC3:=subs({laplace(u(0,t),t,s)=U(0),laplace(f(t),t,s)=F(s)},BC2);
Eq4:=subs(IC1,Eq3); Sol:=dsolve(Eq4,U(x)); Sol1:=rhs(Sol);
l1:=simplify(limit(op(1,Sol1),x=infinity));
l2:=simplify(limit(op(2,Sol1),x=infinity));
Sol2:=subs(_C1=0,Sol1); Sol3:=subs(_C2=op(2,BC3),Sol2);
U1:=subs(F(s)=laplace(f(t),t,s),Sol3);
SolF:=simplify(invlaplace(U1,s,t)) assuming t>0 and x>0;
NumericEventHandler(invalid_operation = `Heaviside/EventHandler`(value_at_zero = 1));
SolF1:=convert(SolF,piecewise,t);
```

22.3.3 Constructing Analytical Solutions in Terms of Green's Functions

It is well known that the *linear superposition principle* is one of the most important methods for representing solutions of linear PDEs with initial and/or boundary conditions in terms of *eigenfunctions* or *Green's functions*.[6]

As was seen earlier (Section 22.3.1), the *infinite series representation* of solutions of mathematical problems (involving linear PDEs) can be obtained by applying the *eigenfunction expansion method*. The *integral representation* of solutions can be obtained by applying the *Green's function method*. Integral representations have some advantages over infinite series representations (e.g., the description of the general analytical structure of a solution and the evaluation of a solution).

Consider the linear Poisson equation in a volume V with surface S on which Dirichlet boundary conditions are posed. The Green's function $\mathcal{G}(X,X_0)$ ($X=(x,y,z)$ and $X_0=(x_0,y_0,z_0)$) associated with the boundary value problem is a function of two variables X (the position vector) and X_0 (a fixed location) defined as the solution to

$$\nabla^2\mathcal{G}(X,X_0)=\delta(X-X_0) \text{ in } V; \qquad \mathcal{G}(X,X_0)=0 \text{ on } S.$$

If the volume V is the whole space, the Green's function is called the *fundamental solution*. Since $\mathcal{G}(X,X_0)$ is symmetric, i.e., $\mathcal{G}(X,X_0)=\mathcal{G}(X_0,X)$, this fact can serve to verify that $\mathcal{G}(X,X_0)$ is computed correctly.

Example 22.30. *Linear Laplace equation. Green's function. Boundary value problem.*

Consider the Dirichlet boundary value problem for the Laplace equation on the semiinfinite plane $V=\{y>0\}$,

$$\nabla^2u(x,y)=u_{xx}+u_{yy}=0 \text{ in } V, \qquad u(x,y)=f(x) \text{ on } S,$$

where $V=\{y>0\}$ and $S=\{y=0\}$. To construct the Green's function and find the solution of the boundary value problem, we apply the *method of images*; i.e., we seek a Green's function $\mathcal{G}(X,X_0)$ such that, for $X,X_0\in V$,

$$\mathcal{G}(X,X_0)=v(X,X_0)+w(X,X_0), \quad \text{where} \quad \nabla^2v(X,X_0)=-\delta(X-X_0) \text{ and } \nabla^2w(X,X_0)=0.$$

[6]The auxiliary function known today as *Green's function* was first introduced by George Green in 1828 [see Green (1828)].

Here $X = (x,y)$, $X_0 = (x_0,y_0)$, the function $v(X,X_0)$ is the *free space Green's function* (does not depend on the boundary conditions), and the function $w(X,X_0)$ satisfies the Laplace equation and the boundary conditions (and is regular at $X = X_0$); i.e., $\nabla^2 w(X,X_0) = 0$ in V and $w(X,X_0) = -v(X,X_0)$ (i.e., $\mathcal{G}(X,X_0) = 0$) on S for the Dirichlet boundary conditions.

It is well known that the 2D free space function $v(X,X_0)$ is

$$v(X,X_0) = -\frac{1}{4\pi}\ln\left((x-x_0)^2+(y-y_0)^2\right).$$

If to $v(X,X_0)$ we add the function

$$w(X,X_0) = \frac{1}{4\pi}\ln\left((x-x_0)^2+(y+y_0)^2\right),$$

which satisfies the Laplace equation $\nabla^2 w(X,X_0) = 0$ in V and is regular at $x = x_0$ and $y = y_0$, then we obtain the Green's function (`G11`)

$$\mathcal{G}(X,X_0) = v(X,X_0) + w(X,X_0) = -\frac{1}{4\pi}\ln\left(\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right).$$

Then, setting $y = 0$, we obtain $\mathcal{G}(X,X_0) = 0$ (`Test1`). The solution of the boundary value problem is

$$u(x_0,y_0) = -\int_S f(x)\frac{\partial\mathcal{G}}{\partial n}\,dS.$$

By computing $\dfrac{\partial\mathcal{G}}{\partial n}$ for the boundary $y = 0$, we obtain the derivative (`DGn`)

$$\left.\frac{\partial\mathcal{G}}{\partial n}\right|_S = -\left.\frac{\partial\mathcal{G}}{\partial y}\right|_{y=0} = -\frac{y_0}{\pi[(x-x_0)^2+y_0^2]}$$

and the solution (`SolF`) of the boundary value problem

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}.$$

Finally, we verify that the Green's function obtained is symmetric and visualize it (see Fig. 22.3) as follows:

```
with(plots): with(Student[Precalculus]): interface(showassumed=0):
assume(x>0,y>0,x0>0,y0>0); Rx:=-10..10; Ry:=0..10; Ops1:=grid=[100,100];
v:=-1/(4*Pi)*ln((x-x0)^2+(y-y0)^2); w:=1/(4*Pi)*ln((x-x0)^2+(y+y0)^2);
G1:=CompleteSquare(combine(v+w));
G11:=op(1,G1)*op(3,G1)*(-1)*combine(op(2,G1)*(-1));
Test1:=eval(G11,y=0); DGn:=eval(-diff(G11,y),y=0);
Sol1:=u(x0,y0)=-int(DGn*f(x),x=-infinity..infinity);
SolF:=CompleteSquare(expand(subs(x=s,x0=x,y0=y,Sol1)));
G11; G2:=unapply(G11,x,y,x0,y0); G2(x,y,x0,y0); G2(x0,y0,x,y);
assume(x>x0,y>y0); is(G2(x,y,x0,y0)=G2(x0,y0,x,y));
assume(x0>x,y0>y); is(G2(x,y,x0,y0)=G2(x0,y0,x,y));
plot3d(G2(x,y,1,1),x=Rx,y=Ry,axes=boxed,orientation=[-40,55],color=grey,Ops1);
contourplot(G2(x,y,1,1),x=Rx,y=Ry,grid=[150,150],axes=boxed,contours=20);
```

Green's functions can also be constructed by applying Laplace transforms [see Cole et al. (2011)], the eigenfunction expansion method [see Debnath (2007), Polyanin and Manzhirov (2007), Polyanin (2002)], and conformal mappings of the complex plane (for solving 2D problems).

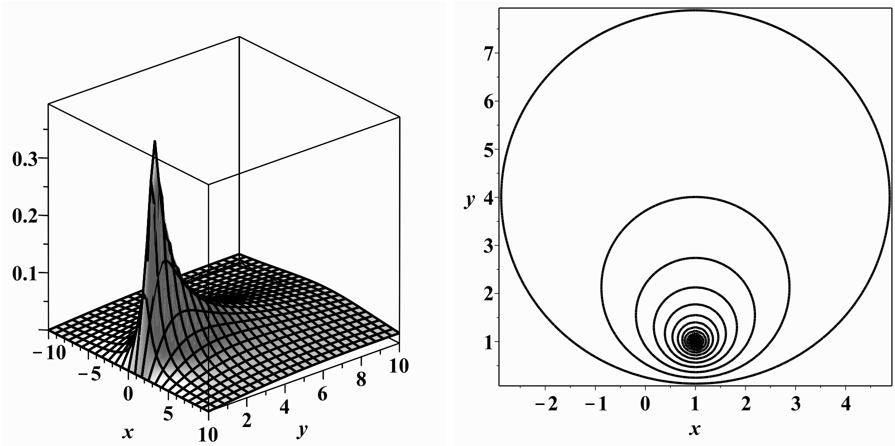


Figure 22.3 Green's function $\mathcal{G}(X,X_0)$ for the semiinfinite plane $y>0$ and $x_0=1$, $y_0=1$.

Now consider an extension of the theory of Green's functions, namely, the construction of modified Green's functions and solutions of initial-boundary value problems.

Example 22.31. *Linear Klein–Gordon equation. Modified Green's function. Initial-boundary value problem.*

Consider the initial-boundary value problem for the Klein–Gordon equation with the following initial conditions and Neumann boundary conditions:

$$u_{tt}=a^2u_{xx}-bu, \qquad x_1\le x\le x_2,$$
$$u(x,0)=f_1(x), \quad u_t(x,0)=f_2(x), \quad u_x(0,t)=g_1(t), \quad u_x(L,t)=g_2(t),$$

where a and b are real parameters ($a>0$, $b>0$), $x_1=0$, $x_2=L$, $f_1(x)$=1, $f_2(x)$=0, $g_1(t)$=1, $g_2(t)$=0. The linear Klein–Gordon equation is a special case of the equation

$$s(x)u_{tt}=(p(x)u_x)_x-q(x)u+\phi(x,t),$$

where $s(x)=1$, $p(x)=a^2$, $q(x)=b$, and $\phi(x,t)=0$. By applying the *eigenfunction expansion method*, we will construct the modified Green's function. The corresponding Sturm–Liouville problem has the form

$$a^2\phi''_{xx}+(\lambda-b)\phi=0, \qquad \phi'_x=0 \text{ at } x=0, \quad \phi'_x=0 \text{ at } x=L.$$

By solving this eigenvalue problem, we find the eigenvalues and the corresponding eigenfunctions (`EVal`, `EFun`),

$$\phi_{n+1}(x)=\cos\left(\frac{\pi nx}{L}\right), \quad \lambda_{n+1}=b+\left(\frac{\pi na}{L}\right)^2, \quad n=0,1,\dots.$$

According to Polyanin (2002),

$$\mathcal{G}(x,\xi,t)=\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)\sin(t\sqrt{\lambda_n})}{\|\phi_n\|^2\sqrt{\lambda_n}},$$

where $\|\phi_n\|^2=\int_{x_1}^{x_2}s(x)\phi_n^2(x)\,dx$. Then the modified Green's function (`G1`) becomes

$$\mathcal{G}(x,\xi,t)=\frac{\sin(t\sqrt{b})}{L\sqrt{b}}+\sum_{n=1}^{\infty}\frac{2}{L}\frac{\cos(\pi nx/L)\cos(\pi n\xi/L)\sin(t\sqrt{b+(\pi na/L)^2})}{\sqrt{b+(\pi na/L)^2}},$$

which is a spectral representation of the Green's function for this problem. To visualize the Green's function, we use finitely many terms of the series (GAppr).

Finally, by constructing the solution of the IBVP according to Polyanin (2002),

$$u(x,t) = \int_0^t \int_{x_1}^{x_2} \Phi(\xi,\tau)\mathcal{G}(x,\xi,t-\tau)\,d\xi\,d\tau + \frac{\partial}{\partial t}\int_{x_1}^{x_2} s(\xi) f_1(\xi)\mathcal{G}(x,\xi,t)\,d\xi$$
$$+\int_{x_1}^{x_2} s(\xi) f_2(\xi)\mathcal{G}(x,\xi,t)\,d\xi + p(x_1)\int_0^t g_1(\tau)(-\mathcal{G}(x,x_1,t-\tau))\,d\tau + p(x_2)\int_0^t g_2(\tau)\mathcal{G}(x,x_2,t-\tau)\,d\tau,$$

we obtain the solution to the given problem and visualize it using finitely many terms of the series (SolF) for various values of t (see Fig. 22.4) as follows:

```
with(plots): with(LinearAlgebra): nInf:=n=1..infinity; s:=x->1; p:=x->a^2; q:=x->b;
Phi:=(x,t)->0; L1:=1; a1:=1; b1:=1; Rx:=0..L1; Rxi:=0..L1; t1:=10; x1:=0; x2:=L1;
TrInf:=infinity=29; f1:=x->1; f2:=x->0; g1:=t->1; g2:=t->0; Ops1:=grid=[150,150];
tauT:=tau=0..t; assume(L>0,a>0); assume(n::integer,n>0); interface(showassumed=0):
xiX:=xi=x1..x2; Sol1:=dsolve(a^2*diff(phi(x),x$2)+(lambda-b)*phi(x)=0, phi(x));
phi:=unapply(rhs(Sol1),x); phi(x);
BC1:=[eval(diff(phi(x),x),x=0)=0, eval(diff(phi(x),x),x=L)=0];
MCoef:=GenerateMatrix(BC1,[_C1,_C2])[1]; CharEq:=Determinant(MCoef)=0;
Sols:=[solve(CharEq,AllSolutions = true,lambda)];
Trlambda:=lambda=subs(_Z1=n,expand(Sols[2]));
MN:=map(xi->simplify(subs(Trlambda,xi)),MCoef); NS:=NullSpace(MN);
Funs:=simplify(subs(_C1=NS[1][1],_C2=NS[1][2],Trlambda,phi(x)));
EVal:=unapply(rhs(Trlambda),n); EFun:=unapply(Funs,n,x); EVal(n); EFun(n,x);
NEFun:=int(s(x)*EFun(n,x)^2,x=0..L); NEFun0:=int(s(x)*EFun(0,x)^2,x=0..L);
GTerm0:=EFun(0,x)*EFun(0,xi)*sin(t*sqrt(EVal(0)))/sqrt(EVal(0))/NEFun0;
G1:=GTerm0+sum(EFun(n,x)*EFun(n,xi)*sin(t*sqrt(EVal(n)))/sqrt(EVal(n))/NEFun,nInf);
GAppr:=unapply(subs(TrInf,G1),x,xi,t,L,a,b);  GAppr(x,xi,t,L1,a1,b1):
plot3d(GAppr(x,xi,t1,L1,a1,b1),x=Rx,xi=Rxi,axes=frame, orientation=[20,45],
 color=grey,style=patchcontour,grid=[100,100],resolution=600,numpoints=1000);
contourplot(GAppr(x,xi,t1,L1,a1,b1),x=Rx,xi=Rxi,axes=boxed,contours=20,Ops1);
G2:=unapply(G1,x,xi,t); G2(x,xi,t);
u:=Int(Int(Phi(xi,tau)*G2(x,xi,t-tau),xiX),tauT)
 +diff(Int(s(xi)*f1(xi)*G2(x,xi,t),xiX),t)+Int(s(xi)*f2(xi)*G2(x,xi,t),xi=x1..x2)
 +p(x1)*Int(g1(tau)*(-G2(x,x1,t-tau)),tauT)+p(x2)*Int(g2(tau)*G2(x,x2,t-tau),tauT);
u1:=subs(L=L1,a=a1,b=b1,infinity=3,u):
SolF:=unapply(collect(map(value,u1),[cos]),x,t);
plot3d(SolF(x,t),x=Rx,t=0..t1,axes=boxed, color=grey,style=patchcontour);
contourplot(SolF(x,t),x=Rx,t=0..t1,grid=[150,150],axes=boxed,contours=20);
for i from 0 to t1 do G||i:=plot(SolF(x,i),x=Rx,color=black); od:
display(seq(G||i,i=0..t1));
plot([SolF(x,0),SolF(x,1),SolF(x,5),SolF(x,t1)],x=Rx,color=black,
 linestyle=[solid,dot,dash,dashdot],legend=[typeset("u(x,0)"),typeset("u(x,1)"),
 typeset("u(x,5)"),typeset("u(x,10)")]);
```

⊙ *References for Section 22.3*: G. Green (1828), V. A. Galaktionov (1990, 1995), A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007), L. Debnath (2007), K. D. Cole, J. V. Beck, A. Haji-Sheikh, and B. Litkouhi (2011).

22.4 Numerical Solutions and Their Visualizations

It is well known that there are numerous linear partial differential equations (arising when modeling real-world problems) and associated mathematical problems (even with simple

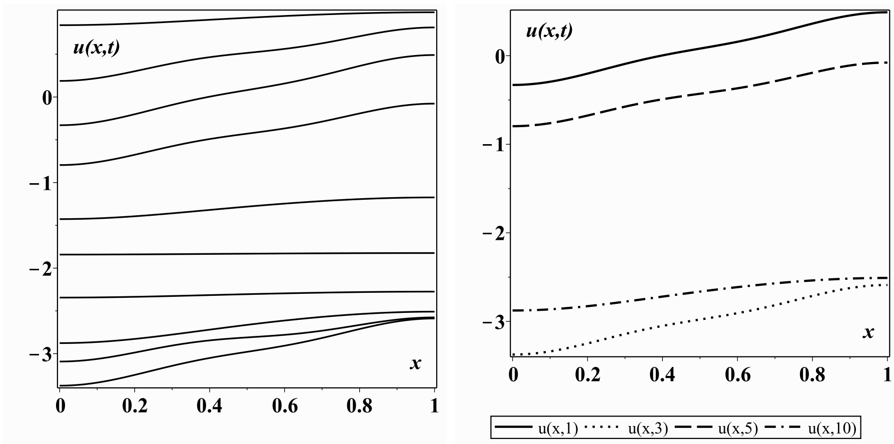


Figure 22.4 The solution $u(x,t)$ of the IBVP for the linear Klein–Gordon equation for various values of t, $t \in [0,10]$.

boundary and/or initial conditions) that cannot be solved analytically. Therefore, one has to develop and apply approximation methods. Nowadays, approximation methods are becoming very important and useful in applications due to the increasing development of modern computers, supercomputers, and computer algebra systems. In this section, some of the most important approximation approaches to the solution of linear partial differential equations and associated mathematical problems are discussed with the aid of the computer algebra system Maple.

22.4.1 Constructing Numerical Solutions in Terms of Predefined Functions

First, consider the predefined Maple functions with the aid of which we can obtain approximate numerical solutions when solving various linear time-based PDE problems. With the aid of the predefined function `pdsolve` (the option `numeric`), we can numerically solve initial-boundary value problems.

The predefined function `pdsolve` allows us to solve a single (higher-order) PDE and PDE systems by the *default methods* or specify a *particular method* for solving a single PDE. It is possible to pose Dirichlet, Neumann, Robin, or periodic boundary conditions.

```
infolevel[all]:=5;        Sol:=pdsolve(PDEs,IBCs,numeric,funcs,ops);
Num_vals:=Sol:-value();              Sol:-plot3d(func,t=t0..t1,ops);
Num_vals(num1,num2);       Sol:-animate(func,t=t0..t1,x=x0..x1,ops);
pdsolve(PDE,IBCs,numeric,numericalbcs=val,method=M1,startup=M2,ops);
pdsolve(PDE,ICs,series,order=num,ops);
with(PDEtools): PDEplot(PDE,ICs,ranges,ops);
```

`pdsolve,numeric` finding numerical solutions of a partial differential equation `PDE` or a system of `PDEs`

`plot3d,animate` visualizing the numerical solution `Sol` obtained by `pdsolve,numeric` `value` displaying numerical values of a numerical solution `Sol`

`pdsolve,series` finding formal power series solutions for systems of differential equations

`PDEplot` plotting the solution of a first-order linear (or nonlinear) partial differential equation

Remark. The solution obtained is represented as a module (similar to a procedure, with the operator `:-`), which can be used for obtaining visualizations (`plot`, `plot3d`, `animate`, `animate3d`) and numerical values (`value`); for more detail, see `?pdsolve[numeric]`.

Numerical and graphical solutions by default methods. Numerical solutions can be obtained in Maple automatically (without specifying a numerical method) by the default θ-methods [see Larsson and Thomée (2008), Morton and Mayers (1995)]. The θ-method is a generalization of the known finite difference approximations (explicit and implicit) by introducing a parameter θ ($0 \le \theta \le 1$) and by taking a weighted average of the two formulas, where the special case $\theta = \frac{1}{2}$ corresponds to the Crank–Nicolson method [see Crank and Nicolson (1947)] and $\theta = 0$ and $\theta = 1$ are just the explicit and implicit methods, respectively.

It should be noted that in Maple (Release 18 or earlier) one can numerically solve only evolution equations via predefined functions.

Example 22.32. *Linear wave equation. Initial-boundary value problem. Numerical, graphical, and exact solutions.* We find numerical, graphical, and exact solutions of the following initial-boundary value problem for the linear wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x), \quad u(0,t) = g_1(t), \quad u(L,t) = g_2(t),$$

where $c = 1/10$, $L = 1$, $f_1(x) = A\sin(B\pi x)$, $f_2(x) = 0$, $g_1(t) = 0$, $g_2(t) = 0$, $A = 3$, and $B = 2$. First, we solve the this initial-boundary value problem numerically and visualize it as follows:

```
with(plots): with(PDEtools): declare(u(x,t)); A:=3: B:=2: c:=1/10: L:=1:
tR:=0..1; xR:=0..L; NF:=30; NP:=50; N:=3;
L1:=[red,blue,green]; L2:=[1/8,3/8,5/8]; Ops:=spacestep=1/256,timestep=1/60;
f1:=x->A*sin(B*Pi*x); f2:=x->0; g1:=t->0; g2:=t->0;
PDE1:=diff(u(x,t),t$2)-c^2*diff(u(x,t),x$2)=0;
IC:={u(x,0)=f1(x),D[2](u)(x,0)=f2(x)}; BC:={u(0,t)=g1(t),u(L,t)=g2(t)};
Sol1:=pdsolve(PDE1,IC union BC,numeric,u(x,t),time=t,range=0..L,Ops);
for i from 1 to N do G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP): od:
display({seq(G||i,i=1..N)}); Num_vals1:=Sol1:-value(); Num_vals1(1/2,1/8);
Sol1:-plot3d(u(x,t),t=tR,shading=zhue,axes=boxed);
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

Then we construct the exact solution $u(x,t) = A\sin(B\pi x)\cos(c\pi t)$ (`SolF`) of this initial-boundary value problem and verify that this solution is an exact solution of the problem (`T1`–`T5`) as follows:

```
SolEx:=pdsolve(PDE1,HINT=`*`,build);
Eq1:=expand(subs(t=0,rhs(SolEx)))=f1(x);
Eq2:=expand(subs(t=0,diff(rhs(SolEx),t)))=f2(x);
Eq3:=expand(subs(x=0,rhs(SolEx)))=g1(t);
```

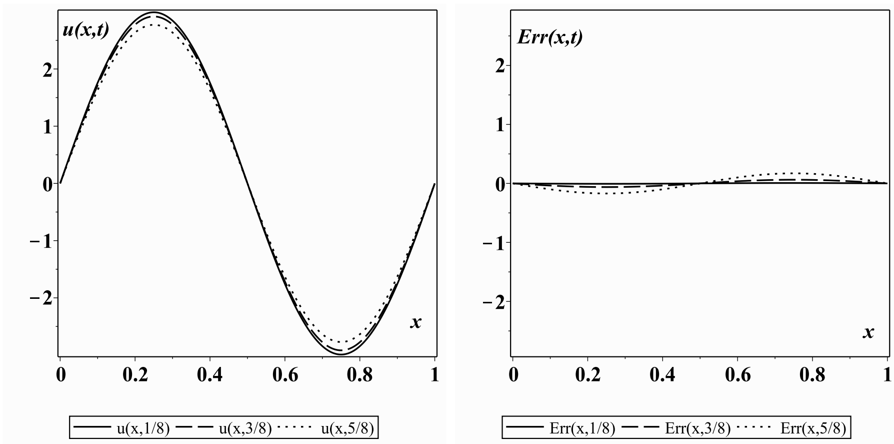


Figure 22.5 The numerical solution $u(x,t)$ of the IBVP for the linear wave equation and the corresponding errors (between the exact and numerical solutions) at times $t = 1/8, 3/8, 5/8$.

```
Eq4:=expand(subs(x=L,rhs(SolEx)))=g2(t);
Sols:=[solve({Eq1,Eq2,Eq3,Eq4},{_C1,_C2,_C3,_C4,_c[1]})] assuming _c[1]<0;
Sol3:=subs(Sols[2],SolEx);   SolF:=simplify(convert(Sol3,trig));
T1:= pdetest(u(x,t)=rhs(SolEx),PDE1); T2:=simplify(subs(t=0,Sols[2],SolEx));
T3:=D[2](u)(x,0)=simplify(subs(t=0,Sols[2],diff(rhs(SolEx),t)));
T4:=simplify(subs(x=0,Sols[2],rhs(SolEx)));
T5:=simplify(subs(x=L,Sols[2],rhs(SolEx)));
```

Finally, we compare the numerical and exact solutions obtained as follows:

```
Trt:=t=3/8; Trx:=x=1/2; p1:=Sol1:-plot(Trt,color=blue,thickness=2):
p2:=plot(subs(Trt,rhs(SolF)),x=xR,color=green,thickness=2):
display({p1,p2});
Num_vals1(1/2,3/8); evalf(subs(Trx,Trt,rhs(SolF)),16);
p3:=Sol1:-plot(u-rhs(SolF),t=1/8,numpoints=NP,color=L1[1]):
p4:=Sol1:-plot(u-rhs(SolF),Trt,  numpoints=NP,color=L1[2]):
p5:=Sol1:-plot(u-rhs(SolF),t=5/8,numpoints=NP,color=L1[3]):
display({p3,p4,p5});
```

The numerical solution of this initial-boundary value problem and the corresponding errors (between the exact and numerical solutions) at times $t = 1/8, 3/8, 5/8$ are shown in Fig. 22.5.

22.4.2 Numerical Methods Embedded in Maple

In Maple, one can obtain numerical solutions specifying one of the eleven *classical methods*, specifying numerical boundary conditions and finite difference schemes for two-stage methods.

The classical numerical methods embedded in Maple are the forward-time forward-space (backward-space) method, centered-time forward-space (backward-space) method, backward-time forward-space (backward-space) method, forward-time centered-space (or Euler) method, centered-time centered-space (or Crank–Nicolson) method, backward-time centered-space (or backward Euler) method, box method, Lax–Friedrichs method, Lax–Wendroff method, leapfrog method, and DuFort–Frankel method.

However, there is some restriction with respect to these classical methods: a single PDE must be parabolic or hyperbolic and of first order in time. PDEs that are greater than first order in time can be solved by converting to an equivalent first-order system.

More detailed information about numerical methods embedded in Maple is presented in Table 22.2.

Table 22.2.
Numerical methods embedded in Maple with a brief description and some references

Numerical method	Brief description	References
`FTime1Space[b]` `FTime1Space[f]`	Explicit 1-stage method for first-order time/space PDEs. Accuracy: $O(h,k)$. Stability: $k<ah$ (a depends upon the problem). `[b/f]`: to describe right/left TW, (to specify BC at the left/right boundary). Numerical BCs are not required.	Lapidus and Pinder (1999) LeVeque (2007) Morton and Mayers (1995)
`CTime1Space[b]` `CTime1Space[f]`	Implicit 1-stage method for PDEs: $F(u,u_x,u_t,u_{xt})=0$. Accuracy: $O(h,k^2)$. Stability: unconditionally stable for many problems. `[b/f]`: to describe right/left TW, (to specify BC at the left/right boundary). Numerical BCs are not required.	Lapidus and Pinder (1999) LeVeque (2007) Morton and Mayers (1995)
`BTime1Space[b]` `BTime1Space[f]`	Implicit 1-stage method for PDEs: $F(u,u_x,u_t,u_{xt})=0$. Accuracy: $O(h,k)$. Stability: unconditionally stable for many problems. `[b/f]`: to describe right/left TW, (to specify BC at the left/right boundary). Numerical BCs are not required.	Lapidus and Pinder (1999) LeVeque (2007) Morton and Mayers (1995)
`Euler` or `FTimeCSpace`	Explicit 1-stage method for PDEs: 1-order in t, n-order in space (no mixed derivative). Accuracy: $O(h^2,k)$. Stability: some restriction on h and k. Numerical BCs are required depending upon the order of the PDE in space.	Lapidus and Pinder (1999) Strikwerda (2004) Morton and Mayers (1995)
`CrankNicholson` or `CTimeCSpace`	Implicit 1-stage method for PDEs: 1-order in t, n-order in space (no mixed derivative). Accuracy: $O(h^2,k^2)$. Stability: unconditionally stable for many problems. Numerical BCs are required depending upon the order of the PDE in space.	Lapidus and Pinder (1999) Thomas (1995) Morton and Mayers (1995)
`BackwardEuler` or `BTimeCSpace`	Implicit 1-stage method for PDEs: 1-order in t, n-order in space (no mixed derivative). Accuracy: $O(h^2,k)$. Stability: unconditionally stable for many problems. Numerical BCs are required depending upon the order of the PDE in space.	Lapidus and Pinder (1999) Strikwerda (2004) Morton and Mayers (1995)
`Box[b]` `Box[f]`	Implicit 1-stage method for PDEs: 1-order in t, n-order in space (no mixed derivative). Accuracy: $O(h^2,k^2)$. Stability: unconditionally stable for many problems. Numerical BCs are required depending upon the order of the PDE in space.	Strikwerda (2004) LeVeque (2007) Larsson and Thomée (2008)
`LaxFriedrichs`	Explicit 1-stage method for PDEs: 1-order in time, odd-order in space (no mixed derivative). Accuracy: $O(h^2,k,M)$. Stability: restriction of the form $k<ah^P$. Numerical BCs are required depending upon the order of the PDE in space.	Larsson and Thomée (2008) LeVeque (2007) Strikwerda (2004)
`LaxWendroff`	Explicit 1-stage method for linear PDEs that are first-order in time and space. Accuracy: $O(h^2,k,h,k^2)$. Stability: restriction of the form $k<ah$. Numerical BC is required so that one BC is specified for each boundary.	Larsson and Thomée (2008) LeVeque (2007) Strikwerda (2004)
`Leapfrog`	Explicit 2-stage method for PDEs: 1-order in time, n-order in space (no mixed derivative). Accuracy: $O(h^2,k^2)$. Startup method. Stability: restriction. Numerical BCs are required depending upon the order of the PDE in space.	Morton and Mayers (1995) Strikwerda (2004) Thomas (1995)
`DuFortFrankel`	Explicit 2-stage method for linear/nonlinear PDEs: 1-order in t, even-order in space (no mixed derivative). Accuracy: $O(h^2,k^2,Q)$. Startup method is required. No numerical BCs are required. Stability: restriction of the form $k<ah^P$.	Larsson and Thomée (2008) LeVeque (2007) Strikwerda (2004)

The abbreviated Maple names of the embedded numerical methods are presented in the first column of the table. The following abbreviations in the table are adopted: `F`, forward; `B`, backward; `C`, centered; `b`, backward; `f`, forward; `TW`, traveling wave; `BC(s)`, boundary condition(s); h, space step; k, time step; P, the differential order of the PDE in the spatial variable; *ceil*, smallest integer greater than or equal to a number; $M = h^{2N}/k$, $Q = k^2/h^{2N}$, and $N = ceil(\frac{1}{2}P)$. For example, for the *forward-time backward-space method*, the abbreviated Maple name is `FTime1Space[b]` and the corresponding complete Maple name is `ForwardTime1Space[backward]`.[7]

The *forward-time 1-space [forward/backward]* method is an *explicit 1-stage* method for finding solutions to first-order time/space PDEs.

The finite difference scheme corresponding to the *forward-time 1-space [backward]* method and used to compute the value, e.g., at the mesh point $(i,1)$, can be obtained by applying differencing to the PDE about the point (x_i,t_j) and by substituting the following discretizations:

$$u_t = \frac{U_{i,1}-U_{i,0}}{k}, \quad u_x = \frac{U_{i,0}-U_{i-1,0}}{h}, \quad u = U_{i,0},$$

where we introduce the notation $u_{i,j} = u(x_i,t_j)$ for the exact solution and $U_{i,j}$ for the discrete approximation (with uniform spacing h and k), so that $U_{i,j} \approx u(x_i,t_j)$. In particular, $U_{i,0} = u(x_i,t_0)$ and $U_{i,1} = u(x_i,t_1)$.

For the *forward-time 1-space [forward]* method the discretizations read

$$u_t = \frac{U_{i,1}-U_{i,0}}{k}, \quad u_x = \frac{U_{i+1,0}-U_{i,0}}{h}, \quad u = U_{i,0}.$$

The *centered-time 1-space [forward/backward]* method is an *implicit 1-stage* method for finding solutions to PDEs containing the derivatives u_x, u_t, u_{xt}.

The finite difference scheme corresponding to the *centered-time 1-space [backward]* method and used to compute the value at the mesh point $(i,1)$ can be obtained by applying differencing to the PDE about the point $(x_i,t_j+k/2)$ and by substituting the following discretizations:

$$u_{xt} = \frac{U_{i,1}-U_{i-1,1}-U_{i,0}+U_{i-1,0}}{kh}, \; u_t = \frac{U_{i,1}-U_{i,0}}{k}, \; u_x = \frac{U_{i,1}-U_{i-1,1}+U_{i,0}-U_{i-1,0}}{2h},$$
$$u = \frac{U_{i,1}+U_{i,0}}{2}, \quad x = x_i, \quad t = t_j+k/2. \qquad (22.4.2.1)$$

For the *centered-time 1-space [forward]* method, the discretizations read

$$u_{xt} = \frac{U_{i+1,1}-U_{i,1}-U_{i+1,0}+U_{i,0}}{kh}, \; u_t = \frac{U_{i,1}-U_{i,0}}{k}, \; u_x = \frac{U_{i+1,1}-U_{i,1}+U_{i+1,0}-U_{i,0}}{2h},$$
$$u = \frac{U_{i,1}+U_{i,0}}{2}, \quad x = x_i, \quad t = t_j+k/2. \qquad (22.4.2.2)$$

The *backward-time 1-space [forward/backward]* method is an *implicit 1-stage* method for finding solutions of PDEs containing the derivatives u_x, u_t, u_{xt}.

The finite difference scheme corresponding to the *backward-time 1-space [backward]* method and used to compute the value at the mesh point $(i,1)$ can be obtained by applying differencing to the PDE about the point (x_i,t_j+k) and by substituting the following discretizations:

$$u_{xt} = \frac{U_{i,1}-U_{i-1,1}-U_{i,0}+U_{i-1,0}}{kh}, \quad u_t = \frac{U_{i,1}-U_{i,0}}{k},$$
$$u_x = \frac{U_{i,1}-U_{i-1,1}}{h}, \quad u = U_{i,1}, \quad x = x_i, \; t = t_j+k. \qquad (22.4.2.3)$$

[7]The Maple names `CrankNicholson`, `LaxFriedrichs`, `LaxWendroff`, and `DuFortFrankel` correspond to numerical methods known in scientific literature: Crank–Nicolson, Lax–Friedrichs, Lax–Wendroff, and DuFort–Frankel.

For the *backward-time 1-space [forward]* method, the discretizations read

$$u_{xt} = \frac{U_{i+1,1} - U_{i,1} - U_{i+1,0} + U_{i,0}}{kh}, \quad u_t = \frac{U_{i,1} - U_{i,0}}{k},$$
$$u_x = \frac{U_{i+1,1} - U_{i,1}}{h}, \quad u = \frac{U_{i,1} + U_{i,0}}{2}, \quad x = x_i,\ t = t_j + k. \tag{22.4.2.4}$$

The *forward-time centered-space* (or Euler) method is an *explicit 1-stage* method for finding solutions of PDEs that are first order in time and arbitrary order in space (with no mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ can be obtained by applying central differencing to the PDE about the point (x_i, t_j).

The *centered-time centered-space* (or Crank–Nicolson) method is an *implicit 1-stage* method for finding solutions of PDEs that are first order in time and arbitrary order in space (with no mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ can be obtained by applying central differencing to the PDE about the point $(x_i, t_j + k/2)$.

The *backward-time centered-space* method is an *implicit 1-stage method* for finding solutions of PDEs that are first order in time and arbitrary order in space (with mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ can be obtained by applying central differencing to the PDE about the point $(x_i, t_j + k)$.

The *box* method is an *implicit 1-stage method* for finding solutions of PDEs that are first order in time and arbitrary order in space (with mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ depends on the extra boundary condition (or numerical boundary condition for even-order problems): if the extra boundary condition is given on the left boundary, then the finite difference scheme can be obtained by applying central differencing of the PDE about the point $(x_i - h/2, t_j + k/2)$; if the extra boundary condition is given on the right boundary, then the finite difference scheme can be obtained by applying central differencing of the PDE about the point $(x_i + h/2, t_j + k/2)$.

The *Lax–Friedrichs* method is an *explicit 1-stage method* for finding solutions of PDEs that are first order in time and odd order in space (with no mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ is similar to that of the *Euler scheme*, where the value $U_{i,0}$ is replaced by the interpolation of the value at $U_{i,0}$ using all other points $U_{j,0}$ in the *Euler stencil* $(j \neq i)$.

The *Lax–Wendroff* method is an *explicit 1-stage method* for finding solutions of *linear PDEs* that are first order in time and space. The finite difference scheme of the method can be derived in a variety of ways, for example, by using the idea of multistep methods: at the first step, by considering expressions for half the time step and by applying central differencing to approximate the derivative u_x, and at the second step, by using the Lax method [for more details, e.g., see Richtmyer (1963), Strikwerda (2004), and Thomas (1995)].

The *Leapfrog* method is an *explicit 2-stage method* for finding solutions of PDEs that are first order in time and arbitrary order in space (with no mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ can be obtained by applying central differencing to compute all spatial derivatives and the discretization

$$u_t = \frac{U_{i,1} - U_{i,-1}}{2k}$$

for the time derivative.

The *DuFort-Frankel* method is an *explicit 2-stage method* for finding solutions of *linear PDEs* that are first order in time and even order in space (with no mixed partial derivatives). The finite difference scheme for computing the value at the mesh point $(i,1)$ is similar to that of the Leapfrog scheme, where the value $U_{i,0}$ is replaced by the average of the values $U_{i,1}$ and $U_{i,-1}$ for all occurrences in the scheme.

Example 22.33. *Linear advection equation. Initial-boundary value problem. Numerical, graphical, and exact solutions. Single-stage numerical method.* Consider the following initial-boundary value problem for the linear *advection equation* (or first-order wave equation, or first-order hyperbolic equation):

$$u_t + cu_x = 0; \qquad u(x,0) = f(x), \quad u(0,t) = g(t).$$

where $f(x) = \sin(x)$, $g(t) = 0$, and c is a real parameter ($c > 0$).

First, we obtain the numerical and graphical solutions of this initial-boundary value problem by applying the single-stage explicit `ForwardTime1SpaceBackward` method as follows:

```
with(PDEtools): with(plots): declare(u(x,t)); NF:=30; NP:=100; L:=1; T:=1;
xR:=0..L; tR:=0..T; c:=0.7; St:=1/100; Sx:=1/100; N:=3;
Ops:=timestep=St,spacestep=Sx; L1:=[red,blue,magenta]; L2:=[0.1,0.15,0.2];
f:=x->sin(x); g:=t->0; M1:=ForwardTime1Space[backward];
PDE1:=diff(u(x,t),t)+c*diff(u(x,t),x)=0; IBC:={u(x,0)=f(x),u(0,t)=g(t)};
Sol1:=pdsolve(PDE1,IBC,numeric,time=t,range=xR,method=M1,Ops);
Num_vals1:=Sol1:-value(); Num_vals1(0.5,0.5);
for i from 1 to N do G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)});
Sol1:-animate(u(x,t),x=xR,t=tR,view=[xR,tR],frames=NF,numpoints=NP,thickness=3);
```

Remark. We choose the backward method (`ForwardTime1Space[backward]`) for first-order linear PDEs (where the boundary condition is given at the left boundary) that describe right-traveling waves. We can find numerical and graphical solutions of this initial-boundary value problem, e.g., at times $t = 0.1, 0.15, 0.2$ (`L2`).

Then we construct the exact solution (`SolF`)

$$u(x,t) = \frac{1428571429\,x \sin x}{1000000000\,t + 1428571429\,x}$$

of this initial-boundary value problem and verify that this solution is an exact solution of the given initial-boundary value problem (`T1`–`T3`) as follows:

```
SolEx:=pdsolve(PDE1,HINT=`+`,build);
Eq1:=expand(subs(t=0,rhs(SolEx)))=f(x);  Eq2:=expand(subs(x=0,rhs(SolEx)))=g(t);
Sols:=solve({Eq1,Eq2},{_c[2],_C1,_C2});  SolF:=simplify(subs(Sols,SolEx));
T1:= pdetest(u(x,t)=rhs(SolEx),PDE1);    T2:=simplify(subs(t=0,Sols,SolEx));
T3:=simplify(subs(x=0,Sols,SolEx));
```

Finally, by comparing the numerical and exact solutions obtained, we visualize the solutions and the corresponding errors between them at various times (e.g., $t = 0.1, 0.3, 0.5$) as follows:

```
Trt:=t=0.1; Trx:=x=0.5; p1:=Sol1:-plot(Trt,color=blue,thickness=2):
p2:=plot(subs(Trt,rhs(SolF)),x=xR,color=green,thickness=2): display({p1,p2});
Num_vals1(0.5,0.1); evalf(subs(Trx,Trt,rhs(SolF)),16);
p3:=Sol1:-plot(u-rhs(SolF),t=0.1,numpoints=NP,color=L1[1]):
p4:=Sol1:-plot(u-rhs(SolF),t=0.3,numpoints=NP,color=L1[2]):
p5:=Sol1:-plot(u-rhs(SolF),t=0.5,numpoints=NP,color=L1[3]): display({p3,p4,p5});
```

Example 22.34. *Linear heat equation. Initial-boundary value problem. Numerical, graphical, and exact solutions. Crank–Nicolson method.* Consider the following initial-boundary value problem for the linear one-dimensional heat equation:

$$u_t = k u_{xx}, \qquad 0 < x < L, \quad t > 0, \qquad u(x,0) = f(x), \quad u(0,t) = g_1(t), \quad u(L,t) = g_2(t),$$

where $L = 1$, $k = 0.1$, $f(x) = x(1-x)$, $g_1(t) = 0$, and $g_2(t) = 0$. By constructing the numerical and graphical solutions of this problem, we apply one of the successful *implicit finite difference schemes* based on six grid points, the `CrankNicholson` method [see Crank and Nicolson (1947)], as follows:

```
with(PDEtools): with(plots): declare(u(x,t)); NF:=30; NP:=100;
L:=1; k:=0.1; xR:=0..L; tR:=0..5; S:=1/300;
Ops:=timestep=S,spacestep=S; N:=2; L1:=[red,blue]; L2:=[0,0.5];
f:=x->x*(1-x); g1:=t->0; g2:=t->0;
PDE1:=diff(u(x,t),t)=k*diff(u(x,t),x$2); IBC:={u(x,0)=f(x),
u(0,t)=g1(t), u(L,t)=g2(t)}; M1:=CrankNicholson;
Sol1:=pdsolve(PDE1,IBC,numeric,method=M1,Ops);
Num_vals1:=Sol1:-value(); for i from 1 to N do
G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)}); Num_vals1:=Sol1:-value();
Num_vals1(1/2,Pi);
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

Then we construct the exact solution (`SolF`)

$$u(x,t) = x(1-x)\frac{e^{2x+0.1t} - e^{0.1t}}{e^{2x} - 1}$$

of this initial-boundary value problem and verify that this solution is an exact solution of the given initial-boundary value problem (`T1`–`T4`) as follows:

```
SolEx:=pdsolve(PDE1,HINT=`*`,build);  Eq1:=expand(subs(t=0,rhs(SolEx)))=f(x);
Eq2:=expand(subs(x=0,rhs(SolEx))=g1(t)); Eq3:=expand(subs(x=L,rhs(SolEx))=g2(t));
Sols:=solve({Eq1,Eq2},{_C1,_C2,_C3,_c[1]});
SolF:=simplify(subs(Sols,_c[1]=1,SolEx));
T1:= pdetest(u(x,t)=rhs(SolEx),PDE1);
T2:=expand(simplify(subs(Sols,t=0,_c[1]=1,SolEx)));
T3:=simplify(subs(Sols,x=0,_c[1]=1,rhs(SolEx)));
T4:=simplify(subs(Sols,x=L,_c[1]=1,rhs(SolEx)));
```

Finally, by comparing the numerical and exact solutions obtained, we visualize the solutions and the corresponding errors between them at distinct times (e.g., $t = 0.2, 0.4$) and examine the errors by constructing the error function as follows:

```
Trt:=t=0.1; Trx:=x=0.5; p1:=Sol1:-plot(Trt,color=blue,thickness=2):
p2:=plot(subs(Trt,rhs(SolF)),x=xR,color=green,thickness=2): display({p1,p2});
Num_vals1(0.5,0.1); evalf(subs(Trx,Trt,rhs(SolF)),16);
p3:=Sol1:-plot(u-rhs(SolF),t=0.2,numpoints=NP,color=L1[1]):
p4:=Sol1:-plot(u-rhs(SolF),t=0.4,numpoints=NP,color=L1[2]): display({p3,p4});
errfunc:=rhs(Sol1:-value(u(x, t)-rhs(SolF),t=0.1,output=listprocedure)[3]);
for i from 0.01 to 1 by 0.1 do abs(errfunc(i)); od;
```

Example 22.35. *Linear advection equation. Initial-boundary value problem. Numerical and graphical solutions. Du Fort–Frankel method. Startup method. Numerical boundary condition.* Consider the initial-boundary value problem for the linear advection equation

$$u_t + c u_x = 0, \quad -L < x < L, \quad t > 0, \qquad u(x,0) = f(x), \quad u_x(-L,t) = 0,$$

where

$$L = 10, \quad c = 0.5, \quad f(x) = \begin{cases} 0, & x < -1, \\ x+1, & -1 \le x \le 0, \\ 1, & x > 0. \end{cases}$$

For constructing the numerical and graphical solutions of this problem, we apply the two-stage explicit Du Fort–Frankel method. Therefore, we have to indicate how to compute the additional stage required for two-stage methods, i.e., the `startup` option. Since the advection equation is of odd order in space, we have to indicate a numerical boundary condition (NBC), which we can choose to be an NBC that forces the value of the solution on the right boundary to be the same as the value of the solution at the first interior point; i.e., `u[1,n]-u[1,n-1]`. Then we find the numerical and graphical solutions of this initial-boundary value problem at times $t = 0, 0.5, 0.9$ as follows:

```
with(PDEtools): with(plots): declare(u(x,t)); NF:=30; NP:=100; c:=0.5;
xR:=-10..10; tR:=0..9; S:=1/300; Ops:=timestep=S,spacestep=S; N:=3;
L1:=[red,blue,magenta]; L2:=[0,0.5,0.9];
f:=x->piecewise(x<-1,0,x>=-1 and x<=0,x+1,1); M1:=DuFortFrankel;
PDE1:=diff(u(x,t),t)+c*diff(u(x,t),x)=0; IBC:={u(x,0)=f(x),(D[1](u))(-L,t)=0};
Sol1:=pdsolve(PDE1,IBC,type=numeric,time=t,range=xR,numericalbcs=u[1,n]-u[1,n-1],
      method=M1,startup=Euler,Ops);
for i from 1 to N do
 G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP,thickness=2):
od:
display({seq(G||i,i=1..N)});
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=4);
```

22.4.3 Numerical Solutions of Initial-Boundary Value Problems

Now we show a helpful role of computer algebra systems for generating and applying various finite difference approximations for constructing numerical solutions of linear PDEs.

To approximate linear PDEs by finite differences, we have to generate a *mesh* (or grid) in a domain $\mathcal{D}$; e.g., $\mathcal{D} = \{a < x < b, c < t < d\}$. The mesh can be of various types, e.g., rectangular, along the characteristics, polar, etc. We assume (for simplicity) that the sets of lines of the mesh are equally spaced and the dependent variable in a given PDE is $u(x,t)$.

We write h and k for the line spacings and define the *mesh points* as follows: $X_i = a + ih$, $T_j = c + jk$ $(i = 0,\dots,NX,\ j = 0,\dots,NT)$, and $h = (b-a)/NX$, $k = (d-c)/NT$. We calculate approximations to the solution at these mesh points; these approximate values will be denoted by $U_{i,j} \approx u(X_i, T_j)$. We approximate the derivatives in a given equation by finite differences (of various types) and then solve the resulting difference equations.

Example 22.36. *Linear heat equation. Initial-boundary value problem. Forward/backward finite difference (FD) methods. Crank–Nicolson method.* Consider the following initial-boundary value problem for the linear heat equation:

$$u_t = \nu u_{xx}, \quad 0 < x < L, \quad t > 0, \qquad u(x,0) = f(x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $f(x) = \sin(\pi x/L)$, $L = 1$, and $\nu = 1$. By applying the forward/backward difference methods and the Crank–Nicolson method, we construct the approximate numerical solution of the given initial-boundary value problem.

We generate the rectangular mesh $X = ih$, $T = jk$ $(i = 0,\dots,NX,\ j = 0,\dots,NT,\ h = L/NX,$ $k = T/NT)$. We denote the approximate solution of $u(x,t)$ at the mesh point (i,j) by $U_{i,j}$.

In the *forward difference method*, the second derivative u_{xx} is replaced by a central difference approximation (CDA) and the first derivative u_t by a forward difference approximation (FWDA) as follows:

$$u_{xx}(x_i,t_j) \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}, \quad u_t(x_i,t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{k}.$$

The final FD scheme for the linear heat equation is

$$U_{i,j+1} = (1-2r)U_{i,j} + r(U_{i+1,j} + U_{i-1,j}),$$

where $r{=}\nu k/h^2$. In this explicit FD scheme, the unknown value $U_{i,j+1}$ (at the $(j{+}1)$st step) is determined from the three known values $U_{i-1,j}$, $U_{i,j}$, and $U_{i+1,j}$ (at the jth step). This FD scheme is unstable for $r{>}0.5$. We find the numerical solution as follows:

```
with(plots): nu:=1: NX:=15: NT:=100: L:=1.: T:=0.2; h:=L/NX; k:=T/NT; r:=nu*k/h^2;
f:=x->evalf(sin(Pi*x)); for i from 0 to NX do X[i]:=i*h od:
IC:={seq(U(i,0)=f(X[i]),i=0..NX)};
BC:={seq(U(0,j)=0,j=0..NT), seq(U(NX,j)=0,j=0..NT)}: IBC:=IC union BC:
FD:=(i,j)->(1-2*r)*U(i,j)+r*(U(i+1,j)+U(i-1,j));
for j from 0 to NT do for i from 1 to NX-1 do U(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([X[i],subs(IBC,U(i,j))],i=0..NX)],color=blue):
Ops1:=thickness=3,labels=["X","U"];
display([seq(G(j),j=0..NT)],insequence=true,Ops1);
```

This FD scheme can be represented in the matrix form

$$U_i{=}MU_{i-1},$$

where $U_0{=}(f(X_1),\ldots,f(X_{NX-1}))$, and M is the $NX{\times}NX$ tridiagonal band matrix (with $1{-}2r$ along the main diagonal, r along the first subdiagonals, and zeros elsewhere). We obtain the numerical solution using this matrix representation of the FD scheme as follows:

```
with(plots): with(LinearAlgebra): nu:=1: NX:=40: NT:=800: L:=1.: T:=0.2;
h:=L/NX; k:=T/NT; r:=nu*k/h^2; NG:=90; interface(rtablesize=NX):
M:=BandMatrix([r,1-2*r,r],1,NX-1); f:=x->evalf(sin(Pi*x));
U0:=Vector([[seq(f(i*h),i=1..NX-1)]]);
for k from 1 to NG do
 U||k:=M.U||(k-1); G||k:=plot([[0,0],seq([i/NX,U||k[i]],i=1..NX-1),[L,0]]);
od:
display([seq(G||i,i=1..NG)],insequence=true,thickness=3);
```

In the *backward difference method*, the second derivative u_{xx} is replaced by a central difference approximation (CDA) and the first derivative u_t by a backward difference approximation (BWDA) as follows:

$$u_{xx}(x_i,t_j) \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}, \quad u_t(x_i,t_j) \approx \frac{U_{i,j} - U_{i,j-1}}{k}.$$

The final FD scheme for the linear diffusion equation is

$$(1+2r)U_{i,j} - r(U_{i+1,j} + U_{i-1,j}) - U_{i,j-1} = 0,$$

where $r{=}\nu k/h^2$. In this *implicit FD scheme*, we have to solve these difference equations numerically at each of the interior mesh points at each jth step (where $j{=}1,\ldots,NT$) with the initial and boundary conditions. This FD scheme is unconditionally stable. We calculate the approximate numerical solution of the initial-boundary value problem by applying the backward difference method as follows:

```
with(plots): nu:=1: NX:=50: NT:=50: L:=1.: T:=0.2; h:=L/NX; k:=T/NT; r:=nu*k/h^2;
for i from 0 to NX do X[i]:=i*h; od:  f:=i->evalf(sin(Pi*X[i])):
IBC:={seq(U[i,0]=f(i),i=0..NX), seq(U[0,j]=0,j=0..NT),seq(U[NX,j]=0,j=0..NT)}:
Sol0:=IBC; FD:=(i,j)->(1+2*r)*U[i,j]-r*(U[i+1,j]+U[i-1,j])-U[i,j-1];
for j from 1 to NT do
 Eqs||j:={seq(FD(i,j)=0,i=1..NX-1)}; Eqs1||j:=subs(Sol||(j-1),IBC,Eqs||j);
 vars||j:={seq(U[i,j],i=1..NX-1)};   Sol||j:=fsolve(Eqs1||j,vars||j);
od:
G:=j->plot([seq([X[i],subs(Sol||j,IBC,U[i,j])],i=0..NX)],color=blue,thickness=3):
display([seq(G(j),j=0..NT)],insequence=true,labels=["X","U"]);
```

The *Crank–Nicolson method* is obtained by centered differencing in time about the point $(x_i, t_{j+1/2})$. To do so, we average the central difference approximations in space at time t_j and t_{j+1} as follows:

$$u_{xx}(x_i, t_{j+1/2}) \approx \frac{1}{2}\left(\frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{h^2} + \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}\right),$$

$$u_t(x_i, t_{j+1/2}) \approx \frac{U_{i,j+1} - U_{i,j}}{k}.$$

The final FD scheme for the linear heat equation is

$$-rU_{i-1,j+1} + 2(1+r)U_{i,j+1} - rU_{i+1,j+1} = rU_{i-1,j} + 2(1-r)U_{i,j} + rU_{i+1,j},$$

where $r = \nu k/h^2$. In this FD scheme we have three unknown values of U at the $(j+1)$st time step and three known values at the jth time step. This FD scheme is unconditionally stable. We calculate the approximate numerical solution of the initial-boundary value problem by applying the Crank–Nicolson method [Crank and Nicolson (1947)] as follows:

```
with(PDEtools): with(plots): declare(v(x1,t1)); L:=1; T:=0.2: nu:=1:
NX:=20: NT:=20: NX1:=NX-1: NX2:=NX-2: h:=L/NX; k:=T/NT; r:=nu*k/(h^2);
SX:=h; ST:=k; tR:=0..T; xR:=0..L; NF:=30: NP:=100: tk:=0.2:
Ops1:=spacestep=SX,timestep=ST: F:=i->sin(Pi*i);
IC:={v(x1,0)=F(x1)}; BC:={v(0,t1)=0,v(L,t1)=0}; U[NX-1]:=0:
PDE1:=diff(v(x1,t1),t1)-nu*diff(v(x1,t1),x1$2)=0;
for i from 1 to NX1 do U[i-1]:=evalf(F(i*h)); od:
LM[0]:=1+r: UM[0]:=-r/(2*LM[0]):
for i from 2 to NX2 do LM[i-1]:=1+r+r*UM[i-2]/2; UM[i-1]:=-r/(2*LM[i-1]); od:
LM[NX1-1]:=1+r+0.5*r*UM[NX2-1]:
for j from 1 to NT do
 t:=j*k; Z[0]:=((1-r)*U[0]+r*U[1]/2)/LM[0];
 for i from 2 to NX1 do
  Z[i-1]:=((1-r)*U[i-1]+0.5*r*(U[i]+U[i-2]+Z[i-2]))/LM[i-1];
 od:
 U[NX1-1]:=Z[NX1-1]:
 for i1 to NX2 do i:=NX2-i1+1; U[i-1]:=Z[i-1]-UM[i-1]*U[i]; od:
od:
```

Finally, we compare the approximate numerical solution with the exact solution of this problem,

$$u(x,t) = \exp(-\pi^2 t)\sin(\pi x) \quad \text{at} \quad (x_k, t_k), \quad 0 < x < L, \quad t > 0.$$

Additionally, we obtain the numerical solution using the Maple predefined function `pdsolve` (with the option `method=CrankNicholson`) and compare the resulting two numerical solutions with the exact solution as follows:

```
M1:=CrankNicholson; ExSol:=(x,t)->exp(-Pi^2*t)*sin(Pi*x):
printf(`Crank-Nicolson Method\n`); for i from 1 to NX1 do
 X:=i*h;
 printf(`%3d %11.8f %13.8f %13.8f,%13.8f\n`,i,X,U[i-1],evalf(ExSol(X,tk)),
  U[i-1]-evalf(ExSol(X,tk)));
od:
NSol1:=pdsolve(PDE1,IC union BC,numeric,v(x1,t1),Ops1,time=t1,range=0..L,method=M1):
printf(`Crank-Nicolson Method\n`); vtk:=NSol1:-value(t1=tk,output=listprocedure):
vVal:=rhs(op(3,vtk)):
for i from 1 to NX1 do
 X1:=i*h:
 printf(`%3d %11.8f %13.8f %13.8f,%13.8f\n`,i,evalf(X1),vVal(X1),
  evalf(ExSol(X1,tk)),abs(vVal(X1)-evalf(ExSol(X1,tk))));
od:
```

Note that to ensure the coincidence of the numerical solution (using `pdsolve`, `numeric`, `method`) with our solution (with FD scheme), one has to establish the coincidence between the parameters of the two solutions: `SX:=h`, `ST:=k`, `spacestep=SX`, `timestep=ST`.

Example 22.37. *Linear wave equation. Explicit difference methods.* Consider the initial-boundary value problem for the linear wave equation describing the motion of a fixed string:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \quad u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $f_1(x) = 0$, $f_2(x) = \sin(4\pi x)$, $L = 0.5$, $c = 1/(4\pi)$. By applying the explicit central finite difference method, we construct the approximate numerical solution of the initial-boundary value problem.

In the *explicit central difference method*, each second derivative is replaced by a central difference approximation as follows:

$$u_{xx}(x_i,t_j) \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}, \quad u_{tt}(x_i,t_j) \approx \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{k^2}.$$

The FD scheme for the linear wave equation is

$$U_{i,j+1} = 2(1-r)U_{i,j} + r(U_{i+1,j} + U_{i-1,j}) - U_{i,j-1},$$

where $r{=}(ck/h)^2$. In this FD scheme, we have one unknown value $U_{i,j+1}$ that depends explicitly on the four known values $U_{i,j}$, $U_{i+1,j}$, $U_{i-1,j}$, $U_{i,j-1}$ at the previous time steps (j and $j{-}1$). To start the process, we have to know the values of U at the time steps $j{=}0$ and $j{=}1$. Thus, we can define the initial conditions at these time steps: $U_{i,0}{=}f1(X_i)$ and $U(X_i,0)_t{\approx}(U_{i,1} - U_{i,0})/k{=}f2(X_i)$, $U_{i,1}{=}f1(X_i){+}kf2(X_i)$. This FD scheme is stable for $r \le 1$. We find the approximate numerical solution of the initial-boundary value problem by applying the explicit central finite difference method as follows:

```
c:=1/(4*Pi); L:=0.5; T:=1.5; NX:=40; NT:=40; NX1:=NX+1; NX2:=NX-1; NT1:=NT+1;
NT2:=NT-1; h:=L/NX; k:=T/NT; r:=evalf(c*k/h); F1:=i->0; F2:=i->sin(4*Pi*i);
for j from 2 to NT1 do U[0,j-1]:=0; U[NX1-1,j-1]:=0; od:
U[0,0]:=evalf(F1(0)); U[NX1-1,0]:=evalf(F1(L));
for i from 2 to NX do
 U[i-1,0]:=F1(h*(i-1));
 U[i-1,1]:=(1-r^2)*F1(h*(i-1))+r^2*(F1(i*h)+F1(h*(i-2)))/2+k*F2(h*(i-1));
od:
for j from 2 to NT do for i from 2 to NX do
 U[i-1,j]:=evalf(2*(1-r^2)*U[i-1,j-1]+r^2*(U[i,j-1]+U[i-2,j-1])-U[i-1,j-2]);
```

```
od; od;
printf(` i X(i) U(X(i),NT)\n`);
for i from 1 to NX1 do
 X[i-1]:=(i-1)*h: printf(`%3d %11.8f %13.8f\n`,i,X[i-1],U[i-1,NT1-1]);
od:
Points:=[seq([X[i-1],U[i-1,NT1-1]],i=1..NX1)];
plot(Points,style=point,color=blue,symbol=circle);
```

We construct and visualize the same approximate numerical solution of the initial-boundary value problem by applying the explicit central finite difference method and by following a different style of programming as follows:

```
with(plots): c:=evalf(1/(4*Pi)); L:=0.5: T:=1.5: NX:=40: NT:=40:
h:=L/NX; k:=T/NT; r:=(c*k/h)^2; f1:=x->0: f2:=x->evalf(sin(4*Pi*x)):
IC:={seq(U1(i,0)=f1(i*h),i=1..NX-1),
     seq(U1(i,1)=f1(i*h)+k*f2(i*h),i=1..NX-1)}:
BC:={seq(U1(0,j)=0,j=0..NT),seq(U1(NX,j)=0,j=0..NT)}: IBC:=IC union BC:
FD:=(i,j)->2*(1-r)*U1(i,j)+r*(U1(i+1,j)+U1(i-1,j))-U1(i,j-1);
for j from 1 to NT-1 do
 for i from 1 to NX-1 do
  U1(i,j+1):=subs(IBC,FD(i,j));
od:
od:
G:=j->plot([seq([i*h,subs(IBC,U1(i,j))],i=0..NX)],color=blue):
display([seq(G(j),j=0..NT)],insequence=true,thickness=3,labels=["X","U"]);
```

22.4.4 Numerical Solutions of Boundary Value Problems

Since Maple (Release 18 or earlier) is only capable (via predefined functions) of numerically solving evolution equations, we apply finite difference methods for constructing numerical and graphical solutions of boundary value problems for elliptic equations, e.g., the linear Poisson equation. It should be noted that MATLAB can numerically solve various boundary value problems for linear and nonlinear elliptic equations by applying predefined functions and embedded methods, e.g., solve scalar linear and nonlinear elliptic PDEs and systems of PDEs in two space dimensions as well as linear and nonlinear problems defined on a more complicated geometry.

Example 22.38. *Linear Poisson equation. Boundary value problem. Central difference scheme.* Consider the two-dimensional linear Poisson equation

$$u_{xx}+u_{yy}=f(x,y), \quad \mathcal{D}=\{a\le x\le b,\ c\le y\le d\},$$

with the boundary conditions

$$u(x,c)=f_1(x), \quad u(x,d)=f_2(x), \quad u(a,y)=f_3(y), \quad u(b,y)=f_4(y).$$

Such boundary value problems describe a steady-state process $u(x,y)$ in a bounded rectangular object. Let us choose $f(x,y)=\sin x\cos y$, $f_1(x)=f_2(x)=-\frac{1}{2}\sin x$, $f_3(y)=f_4(y)=0$, $a=0$, $b=\pi$, $c=0$, $d=2\pi$. For the linear Poisson equation, by applying the explicit finite difference scheme, we obtain the approximate numerical solution of the boundary value problem, visualize it in $\mathcal{D}$, and compare with the exact solution (which can be obtained via predefined functions) as follows:

By applying Maple's predefined functions, we find the exact solution

$$u(x,y) = -\tfrac{1}{2}\sin x \cos y$$

of the boundary value problem for the linear Poisson equation (Sol32) and visualize it in $\mathcal{D}$ as follows:

```
with(linalg): with(PDEtools): f:=(x,y)->sin(x)*cos(y);
PDE1:=laplacian(u(x,y),[x,y])-f(x,y)=0;
Sol1:=pdsolve(PDE1,build); Test1:=pdetest(Sol1,PDE1);
Sol11:=unapply(subsop(1=0,2=0,rhs(Sol1)),x,y);
Sol12:=expand(Sol11(x,y));
plot3d(Sol12,x=0..Pi,y=0..2*Pi,shading=zhue);
Sol11(x,0);  Sol11(x,2*Pi); Sol11(0,y); Sol11(Pi,y);
```

Let us generate a rectangular mesh: $x{=}a{+}ih$, $y{=}c{+}jk$ (where $i = 0,\dots,NX$, $j = 0,\dots,NY$, $h = (b-a)/NX$, $k = (d-c)/NY$). We denote the approximate solution of $u(x,y)$ at the mesh point (i,j) by $U_{i,j}$. The second derivatives in the Poisson equation are replaced by a central difference approximation (CDA) as follows:

$$u_{xx}(x_i,y_j) \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}, \quad u_{yy}(x_i,y_j) \approx \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{k^2}.$$

The FD scheme acquires the form

$$2(1+r)U_{i,j} - U_{i+1,j} - U_{i-1,j} - rU_{i,j+1} - rU_{i,j-1} = \sin(ih)\cos(jk),$$

where $r{=}(h/k)^2$.

Finally, we construct the approximate numerical solution of the boundary value problem by applying the above explicit finite-difference scheme and plot the numerical solution inside the domain as follows:

```
with(plots): a:=0; b:=Pi; c:=0; d:=2*Pi; NX:=20; NY:=20;
h:=(b-a)/NX; k:=(d-c)/NY; r:=(h/k)^2; Ops1:=orientation=[50,70];
XY:=seq(x[i]=a+i*h,i=0..NX),seq(y[j]=c+j*k,j=0..NY);
FD:=(i,j)->2*(1+r)*U[i,j]-U[i+1,j]-U[i-1,j]-r*U[i,j+1]-r*U[i,j-1]
    -cos(j*k)*sin(i*h)=0;
F1:=i->-1/2*sin(i*h); F2:=i->-1/2*sin(i*h); F3:=j->0; F4:=j->0;
BC:=seq(U[i,0]=F1(i),i=0..NX),seq(U[i,NY]=F2(i),i=0..NX),
    seq(U[0,j]=F3(j),j=0..NY),seq(U[NX,j]=F4(j),j=0..NY);
Eqs:={seq(seq(FD(i,j),i=1..NX-1),j=1..NY-1)}: Eqs1:=subs(BC,Eqs):
vars:={seq(seq(U[i,j],i=1..NX-1),j=1..NY-1)}:
Sol:=evalf(fsolve(Eqs1,vars));
Points:=[seq(seq([x[i],y[j],U[i,j]],i=0..NX),j=0..NY)]:
Points1:=subs({XY,BC,op(Sol)},Points):
pointplot3d(Points1,symbol=solidsphere,shading=z,axes=frame,Ops1);
```

22.4.5 Numerical Solutions of Cauchy Problems

There are various methods to obtain approximate solutions of Cauchy problems involving linear partial differential equations. The most convenient and accurate method for solving *Cauchy problems* for hyperbolic equations is the method of characteristics (see Sections 22.2.2, 22.2.4). Finite difference methods for hyperbolic equations are not very convenient (e.g., in the case of discontinuous initial data) [see Richtmyer and Morton (1994), Collatz (1966)]. Let us solve some Cauchy problems by applying various methods.

Example 22.39. *Linear inhomogeneous advection equation. Cauchy problem. Exact and approximate (power series) solutions.* Consider the initial value problem for the linear inhomogeneous advection equation

$$u_t + c\,u_x = F(x,t), \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x),$$

where $f(x)$ and $F(x,t)$ are the functions of general form or the functions specified as $f(x) = \sin x$, $F(x,t) = xt$ and $f(x) = \cos\left(e^{(-x)^3}\right)$, $F(x,t) = \sin\left(e^{(-xt)^5}\right)$.

We obtain the exact solution of the Cauchy problem; in the Maple notation it reads (`Sol1`)

$$u(x,t) = \frac{1}{c}\left(f(x-ct)c + \int^{x} G\left(\_a, \frac{1}{c}(ct + \_a - x)\right) d\_a - \int^{x-ct} G\left(\_a, \frac{1}{c}(ct + \_a - x)\right), d\_a\right),$$

the exact solution (`Sol2`) of the Cauchy problem with the simple specified functions,

$$u(x,t) = -\frac{1}{6}ct^3 + \frac{1}{2}t^2 x - \sin(ct - x),$$

and the approximate (power series) solution (e.g., up to the third order) (`Sol1Ser`) of the Cauchy problem with the complicated specified functions,

$$u(x,t) = \frac{1}{6c^3}\left(-15c^2\cos(1)x^2 t + 5c\cos(1)x^3 + 6c^3\cos(1) + 6c^2\sin(1)x\right).$$

```
with(PDEtools); declare(u(x,t)); PDE1:=diff(u(x,t),t)+c*diff(u(x,t),x)=F(x,t);
IC1:=u(x,0)=f(x); Sol1:=pdsolve([PDE1,IC1],u(x,t)); Test1:=pdetest(Sol1,PDE1);
PDE2:=diff(u(x,t),t)+c*diff(u(x,t),x)=x*t; IC2:=u(x,0)=sin(x);
Sol2:=pdsolve([PDE2,IC2],u(x,t)); Test2:=pdetest(Sol2,PDE2);
Sol1Ser:=pdsolve([PDE1,IC1],series,u(x,t),order=3);
F:=(x,t)->sin(exp(-x*t)^5); f:=x->cos(exp(-x)^3); Sol1Ser; evalf(Sol1Ser);
Sol2Ser:=pdsolve([PDE2,IC2],series,u(x,t));
```

Example 22.40. *Linear inhomogeneous advection equation. Cauchy problem. Method of characteristics. Numerical and graphical solutions.* Consider the initial value problem for the linear inhomogeneous advection equation (as in the previous example)

$$u_t + c\,u_x = F(x,t), \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x),$$

where $f(x) = \sin x$ and $F(x,t) = xt$.

By applying the method of characteristics, we obtain the initial value ODE problem and its solution (`ODEs`, `Sol1`),

$$\frac{dx(r)}{r} = \frac{c}{r}, \quad \frac{dt(r)}{r} = \frac{1}{r}, \quad \frac{dz(r)}{r} = \frac{x(r)t(r)}{r}, \quad x(1) = s, \quad t(1) = 0, \quad z(1) = h(s),$$

$$x(r) = c\ln r + s, \quad t(r) = \ln r, \quad z(r) = \frac{1}{3}c\ln r^3 + \frac{1}{2}\ln r^2 s + h(s).$$

Finding z (in terms of x and t), we have (`Sol2`)

$$s = x - ct, \quad r = e^t, \quad z = -\frac{1}{6}ct^3 + \frac{1}{2}t^2 x + h(x-ct),$$

and the solution $u(x,t) = -\frac{1}{6}ct^3 + \frac{1}{2}t^2 x + h(x-ct)$ (see the previous example), which satisfies the original PDE and the initial data (`Test1`, `Test2`). Finally, we visualize the solution of the given Cauchy problem (Fig. 22.6) by using numerical ODE integration methods (with the Maple predefined function `PDEplot`). For example, we obtain the solution of the Cauchy problem for $c = 2$ and $h(s) = \sin s$ as follows:

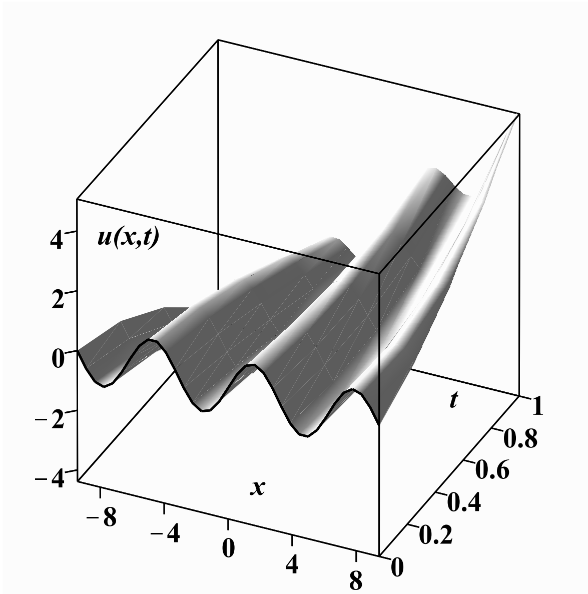


Figure 22.6 Numerical solution of the Cauchy problem $u_t + c\,u_x = xt$, $u(x,0) = \sin x$.

```
with(PDEtools): TrVars:=x(r)=x,t(r)=t,z(r)=z;
PDE1:=diff(u(x,t),t)+c*diff(u(x,t),x)=x*t;
ODEs:=diff(x(r),r)=c/r,diff(t(r),r)=1/r,diff(z(r),r)=x(r)*t(r)/r;
Sol1:=dsolve({ODEs,x(1)=s,t(1)=0,z(1)=h(s)},{x(r),t(r),z(r)});
Sol2:=op(solve(subs(TrVars, Sol1),[s,r,z])); Sol2[3];
W:=unapply(rhs(Sol2[3]),x,t);
Test1:=is(diff(W(x,t),t)+c*diff(W(x,t),x)=x*t); Test2:=is(W(s,0)=h(s));
c:=2; h:=s->sin(s);
PDEplot(PDE1,[s,0,h(s)],s=-3*Pi..3*Pi,x=-3*Pi..3*Pi,t=0..1,color=h(x),
 orientation=[-65,58],numchar=15,axes=boxed,style=patchnogrid);
```

22.4.6 Numerical Solutions of Systems of Linear PDEs

In this section, we show how to obtain numerical and graphical solutions of systems of linear partial differential equations in Maple with the aid of the predefined function `pdsolve`.

Example 22.41. *Linear first-order system (equivalent to the linear wave equations). Numerical and graphical solutions.* Introducing the dependent variables $u(x,t)$ and $v(x,t)$ according to the formulas $u = w_t$ and $v = c^2 w_x$, we can transform the linear wave equation, $w_{tt} = c^2 w_{xx}$, to the equivalent system of linear first-order equations $u_t = v_x$, $v_t = c^2 u_x$. Consider the initial-boundary value problem for this linear first-order system (describing standing waves):

$$u_t = v_x, \quad v_t = c^2 u_x, \quad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad v(x,0) = 0, \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $L = 2\pi$, $f(x) = \sin x$.

We obtain numerical solutions (`Sol1`, `P1`) of the given initial-boundary value problem, visualize it at various times (`G1`–`G3`, `GU`, `GV`) and in parametric form (`PU`, `PV`), construct the animation of a standing wave profile at some time (e.g., $t = 20$), and visualize the solutions in 3D (`GU3D`, `GV3D`) as follows:

```
Digits:=30: with(PDEtools):  with(plots):
Ops1:=numpoints=100: Ops2:=color=magenta: Ops3:=color=blue:
Ops4:=color="BlueViolet": Ops5:=axes=boxed,shading=zhue,orientation=[40,50];
c:=1/2; L:=evalf(2*Pi); a:=0: b:=L: Tf:=20;
U,V:=diff_table(u(x,t)),diff_table(v(x,t)): S:=1/30;
Ops:=spacestep=S,timestep=S; f:=x->sin(x); L1:=[0.1,0.2,0.5]; NL1:=nops(L1);
sys1:={U[t]=V[x],V[t]=c^2*U[x]}; IBC1:={u(x,0)=f(x),v(x,0)=0, u(0,t)=0,u(L,t)=0};
Sol1:=pdsolve(sys1,IBC1,[u,v],numeric,time=t,range=a..b,Ops);
P1:=Sol1:-value(u,t=0); P1(0.1,0);
for i from 1 to NL1 do G||i:=Sol1:-plot(t=L1[i],Ops1,Ops||(i+1)); od:
display({G1,G2,G3});
GU:=Sol1:-plot(u(x,t),t=Tf,Ops1,Ops2): GV:=Sol1:-plot(v(x,t),t=Tf,Ops1,Ops3):
display({GU,GV});
PU:=r->eval(u,(Sol1:-value(u,t=Tf))(r)): PV:=r->eval(v,(Sol1:-value(v,t=Tf))(r)):
plot([PU,PV,a..b],Ops2,labels=[u,v],Ops1);
Sol1:-animate(t=Tf,Ops1,color=blue,thickness=3);
GU3D:=Sol1:-plot3d(u(x,t),t=0..Tf): GV3D:=Sol1:-plot3d(v(x,t),t=0..Tf):
display(GU3D,Ops5); display(GV3D,Ops5);
```

⊙ *References for Section 22.4*: J. Crank and P. Nicolson (1947), L. O. Collatz (1966), R. D. Richtmyer and K. W. Morton (1967), K. W. Morton and D. F. Mayers (1995), J. W. Thomas (1995), L. Lapidus and G. F. Pinder (1999), L. Strikwerda (2004), R. J. LeVeque (2007), S. Larsson and V. Thomée (2008), I. K. Shingareva and C. Lizárraga-Celaya (2011).

Chapter 23

Linear Partial Differential Equations with Mathematica

23.1 Introduction

23.1.1 Some Notational Conventions

In this chapter, we use the following conventions introduced in Mathematica:

`C[n]` (`n=1,2,...`) stands for arbitrary constants or arbitrary functions.

In general, arbitrary parameters (e.g., F_1, F_2, ...) can be specified by applying the option `GeneratedParameters->(Subscript[F,#]&)` of the predefined function `DSolve`.

We also introduce the following notation (where `n=1,2,...`) for Mathematica solutions:

`eqn` for equations
`pden/oden` for PDEs/ODEs
`trn` for transformations
`sysn` for systems
`ic`, `bc`, `ibc` for initial and/or boundary conditions
`listn` for lists of expressions
`gn` for graphs of solutions

23.1.2 Brief Introduction to Mathematica

Mathematica is a general-purpose computer algebra system in which symbolic computation can readily be combined with exact, approximate (floating-point), and arbitrary-precision numerical computation. Mathematica provides powerful scientific graphics capabilities [for details, see Bahder (1995), Getz and Helmstedt (2004), Gray (1994), Gray and Glynn (1991), Green et al. (1994), Ross (1995), Shingareva and Lizárraga-Celaya (2009), Vvedensky (1993), Zimmerman and Olness (1995), etc.].

The first concept of Mathematica and its first versions were developed by Stephen Wolfram in 1979–88. The Wolfram Research company, which continues to develop Mathematica, was founded in 1987.

The most important features of Mathematica are fast symbolic, numerical, acoustic, and parallel computation; static and dynamic computation, and interactive visualization;

incorporation of new user-defined capabilities; availability on almost all operating systems; powerful and logical programming language; extensive library of mathematical functions and specialized packages; interactive mathematical typesetting system; free resources (e.g., see the Mathematica Learning Center `www.wolfram.com/support/learn`, Wolfram Demonstrations Project `demonstrations.wolfram.com`, and Wolfram Information Center `library.wolfram.com`).

Mathematica consists of two basic parts: the *kernel* (computational engine) and the *interface* (*front end*). These two parts are separate but communicate with each other via the *MathLink* protocol. The kernel interprets user input and performs all computations. The kernel assigns the labels `In[number]` to the input expression and `Out[number]` to the output. These labels can be used for keeping the computation order. In this chapter, we do not include these labels in the examples. The result of kernel operation can be viewed with the function `InputForm`. The interface between the user and the kernel is called the *front end* and is used to display the input and the output generated by the kernel. The medium of the front end is the Mathematica *notebook*.

There are significant changes to numerous Mathematica functions incorporated in the new versions. The most important differences between Release < 6 and Release ≥ 6 are described in the literature [e.g., see Shingareva and Lizárraga-Celaya (2009)].[1]

Mathematica 10 (launched in 2014) is the first version based on the *complete Wolfram Language* and has over 700 new functions (e.g., finite element analysis, enhanced PDEs, symbolic delay differential equations, hybrid differential equations, highly automated machine learning, integrated geometric computation, advanced geographic computation, expanded random process framework, integration with the *Wolfram Cloud*, introducing *Mathematica Online* version, and access to the expanded *Wolfram Knowledgebase*).

Basic concepts. If we type a Mathematica command and press the `RightEnter` key or `Shift+ Enter` (or `Enter` to continue the command on the next line), Mathematica evaluates the command, displays the result, and inserts a horizontal line (for the next input).

Mathematica contains many sources of online help, e.g., Wolfram Documentation Center, Wolfram Demonstrations Project (for Release ≥ 6), Mathematica Virtual Book (for Release ≥ 7), and the `Help` menu; one can mark a function and press F1; type `?func`, `??func`, `Options[func]`; use the symbols (`?`) and (`*`); e.g., `?Inv*`, `?*Plot`, or `?*our*`.

Mathematica notebooks are electronic documents that may contain Mathematica output, text, and graphics (see `?Notebook`). It is possible to work with several notebooks simultaneously. A Mathematica notebook consists of a list of cells. Cells are indicated by brackets along the right edge of the notebook. Cells can contain subcells, and so on. The kernel evaluates a notebook cell by cell. There are *various types of cells:* input cells (for evaluation) and text cells (for comments); Title, Subtitle, Section, Subsection, etc., can be found in the menu `Format` → `Style`.

Previous results (during a session) can be referred to with symbols `%` (the last result), `%%` (the next-to-last result), and so on.

Comments can be included within the characters `(*comments*)`.

Incorrect response: if some functions take an "infinite" computation time, you may

[1]A complete list of all changes can be found in the Documentation Center and on the Wolfram Web Site `www.wolfram.com`.

have entered or executed the command incorrectly. To terminate a computation, you can use `Evaluation` → `Quit Kernel` → `Local`.

Palettes can be used for building or editing mathematical expressions, texts, and graphics and allow one to access the most common mathematical symbols by mouse clicks.

In Mathematica, there exist many specialized functions and modules that are not loaded initially. They must be loaded separately from files in the Mathematica directory. These files are of the form `filename.m`. The full name of a package consists of a `context` and a `short` name and is written as `` context`short ``. To load a package corresponding to a context, type `` <<context` ``. To get a list of all functions in a package, type `` Names["context`*"] ``.

Numerical approximations: `N[expr]`, `expr//N` (numerical approximation of `expr` to 6 significant digits); `N[expr,n]`, `NumberForm[expr,n]` (numerical approximation of the expression to n significant digits); `ScientificForm[expr,n]` (scientific notation of numerical approximation of `expr` to n significant digits).

23.1.3 Mathematica Language

Mathematica language is a very powerful programming language based on systems of transformation rules and on functional, procedural, and object-oriented programming techniques [see Maeder (1996)]. This distinguishes it from traditional programming languages. It supports a large collection of data structures, or Mathematica objects (functions, sequences, sets, lists, arrays, tables, matrices, vectors, etc.), and operations on these objects (type-testing, selection, composition, etc.). The library can be extended with custom programs and packages.

In Mathematica 10, a new concept has been introduced, namely, the *complete Wolfram Language*. It has a vast depth of built-in algorithms and knowledge, all accessible automatically through a unified symbolic language. The main idea of the Wolfram Language is to build as much knowledge (about algorithms and the real world) as possible into the language.

Symbols refers to tokens with specified names, e.g., expressions, functions, objects, optional values, results, and argument names. The *name of a symbol* is a combination of letters, digits, or certain special characters, not starting from a digit, e.g., `a12new`. Once defined, a symbol retains its value until it is changed or removed.

Expression is a symbol that represents an ordinary Mathematica expression `expr` in readable form. The head of `expr` can be obtained with `Head[expr]`. The structure and various forms of `expr` can be analyzed with `TreeForm`, `FullForm[expr]`, `InputForm[expr]`. A *boolean expression* is formed with *logical operators* and relation operators.

Basic arithmetic operators and the corresponding functions:
`+ - * / ^`, `Plus`, `Subtract`, `Minus`, `Times`, `Divide`, and `Power`.

Logic and relation operators and their equivalent functions: `&&`, `||`, `!`, `=>`, `==`, `!=` `<`, `>`, `<=` `>=`, `And`, `Or`, `Xor`, `Not`, `Implies`, `Equal`, `Unequal`, `Less`, `Greater`, `LessEqual`, and `GreaterEqual`.

Mathematica is case sensitive in that it distinguishes lowercase and uppercase letters; e.g., `Sin[Pi]` and `sin[Pi]` are not the same. All predefined Mathematica functions begin with a capital letter. Some functions (e.g., PlotPoints) use more than one capital. To avoid conflicts, it is best to begin with a lowercase letter for all user-defined symbols.

The result of each calculation is displayed unless the output is suppressed by using a semicolon (;), e.g., Plot[Sin[x],x,0,2*Pi]; a=9; b=3; c=a*b.

Patterns: Mathematica language is based on pattern matching. A pattern is an expression that contains an underscore character (_). The pattern can stand for any expression. Patterns can be constructed from templates; e.g., x_, x_/;cond, pattern?test, x_:IniValue, x^n_, x_^n_, f[x_], f_[x_].

Basic transformation rules: ->, :>, =, :=, ^:=, ^=.

The rule lhs->rhs transforms lhs to rhs. Mathematica regards the left-hand side as a pattern. The rule lhs:>rhs transforms lhs to rhs, evaluating rhs only after the rule is actually used. The assignment lhs=rhs (or Set) specifies that the rule lhs->rhs should be used whenever it applies. The assignment lhs:=rhs (or SetDelayed) specifies that lhs:>rhs should be used whenever it applies, i.e., lhs:=rhs does not evaluate rhs immediately but leaves it unevaluated until the rule is actually called. The rule lhs^:=rhs assigns rhs to be the delayed value of lhs, and associates the assignment with symbols that occur at level one in lhs. The rule lhs^=rhs assigns rhs to be the value of lhs, and associates the assignment with symbols that occur at level one in lhs. Transformation rules are useful for making substitutions without making the definitions permanent and are applied to an expression using the operator /. (ReplaceAll) or //. (ReplaceRepeated).

The difference between the operators (=) and (==) is as follows: the operator lhs=rhs is used to assign rhs to lhs, and the equality operator lhs==rhs indicates equality (not assignment) between lhs and rhs.

Unassignment of definitions:
Clear[symb], ClearAll[symb], Remove[symb], symb=.;
Clear["Global`*"]; ClearAll["Global`*"]; Remove["`*"];
(to clear all global symbols defined in a Mathematica session),
?symb, ?`* (to recall a symbol's definition)

ClearAll["Global'*"]; Remove["Global'*"]; is a useful initialization to start working on a problem.

An *equation* is represented using the binary operator == and has two operands, the left-hand side lhs and the right-hand side rhs.

Inequalities are represented using relational operators and have two operands, the left-hand side lhs and the right-hand side rhs.

A *string* is a sequence of characters having no value other than itself and can be used as a label for graphs, tables, and other displays. The strings are enclosed within double-quotes, e.g., "abc".

Data types: every expression is represented as a tree structure in which each node (and leaf) has a particular data type. A variety of functions can be used for the analysis of any node and branch, e.g., Length, Part, and a group of functions ending in the letter Q (DigitQ, IntegerQ, etc.).

Types of brackets: parentheses for grouping, (x+9)*3; square brackets for function arguments, Sin[x]; curly brackets for lists, {a,b,c}.

Types of quotes: back-quotes for context mark, format string character, number mark, precision mark, and accuracy mark; double-quotes for strings.

Types of numbers: integer, rational, real, complex, and root; e.g., `-5, 5/6, -2.3^-4, ScientificForm[-2.3^-4], 3-4*I, Root[#^2+#+1&,2]`.

Mathematical constants: symbols for definitions of selected mathematical constants, e.g., `Catalan`, `Degree`, `E`, `EulerGamma`, `I`, `Pi`, `Infinity`, and `GoldenRatio`; for example, `{60Degree//N, N[E,30]}`.

Two classes of functions: *pure functions* and functions defined in terms of a variable (*predefined* and *user-defined* functions).

Pure functions are defined without a reference to any specific variable. The arguments are labeled `#1,#2,...`, and an ampersand `&` is used at the end of definition. Most of the mathematical functions are predefined. Mathematica includes all common special functions of mathematical physics.

The *names* of mathematical functions are complete English words or the traditional abbreviations (for a few very common functions), e.g., `Conjugate` and `Mod`. Mathematical functions named after persons have names of the form `PersonSymbol`, for example, the Legendre polynomials $P_n(x)$, `LegendreP[n,x]`.

User-defined functions are defined using the pattern `x_`; e.g., the function $f(x) =$`expr` of one variable is defined as `f[x_]:=expr;`

Evaluation of a function or an expression without assigning a value can be performed using the replacement operator `/.`, e.g., `f[a]`, `expr/.x->a`.

Function application: `expr//func` is equivalent to `fun[expr]`.

A module is a local object that consists of several functions which one needs to use repeatedly (see `?Module`). A module can be used to define a function (if the function is too complicated to write by using the notation `f[x_]:=expr`), to create a matrix, a graph, a logical value, etc. *Block* is similar to `Module`; the main difference between them is that `Block` treats the values assigned to symbols as local but the names as global, whereas `Module` treats the names of local variables as local. *With* is similar to `Module`, the important difference between them is that `With` uses local constants that are evaluated only once, while `Module` uses local variables whose values may change many times.

In Mathematica language, there are *two types of control structures*: selection structures `If`, `Which`, `Switch` and repetition structures `Do`, `While`, `For`.

Mathematica objects: *lists* are fundamental objects in Mathematica. All other objects (e.g., sets, matrices, tables, vectors, arrays, tensors, and objects containing data of mixed type) are represented as lists. A list is an ordered set of objects separated by commas and enclosed in curly braces, `{elements}`, or defined with the function `List[elements]`. *Nested lists* are lists that contain other lists. There are many functions that manipulate lists, and here we review some of the most basic ones. *Sets* are represented as lists. *Vectors* are represented as lists; vectors are simple lists. Vectors can be expressed as single columns with `ColumnForm[list,horiz,vert]`. *Tables, matrices, and tensors* are represented as nested lists. There is no difference between the way they are stored: they can be generated by using the functions `MatrixForm[list]` or `TableForm[list]` or by using the nested list functions. Matrices and tables can also be conveniently generated by using the *Palettes* or `Insert` menu. *A matrix* is a list of vectors. *A tensor* is a list of matrices of the same dimension.

23.1.4 Dynamic Computation and Visualization in Mathematica Notebook

In Mathematica (for Release ≥ 6), a new kind of manipulation of Mathematica expressions (e.g., computation and visualization), *dynamic computation and visualization*, has been introduced, allowing the creation of dynamic and control interfaces of various types. Numerous new functions for producing interactive elements (or various dynamic and control interfaces) have been developed within a Mathematica notebook (for more detail, see the Documentation Center, "Introduction to Manipulate," "Introduction to Dynamic," "Dynamic and Control," "Interactive Manipulation," "How to: Build an Interactive Application," etc.).

Let us mention the most important of them:

```
Dynamic[expr]       Slider[Dynamic[x]]
Slider[x,{x1,x2,xStep}] Manipulate[expr,{x,x1,x2,xStep}]
TabView[{expr1,expr2,...}] SlideView[{expr1,expr2,...}]
DynamicModule[{x=x0,...},expr] Manipulator[expr,{x,x1,x2}]
Animator[x,{x1,x2,dx}]    Pane[expr]
```

`Dynamic`, `DynamicModule` representing an object that displays as a dynamically updated current value of `expr`; the object can be interactively changed or edited.

`Slider`, `Slider,Dynamic` representing sliders of various configurations.

`Manipulate`, `Manipulator` generating a version of `expr` with controls added to allow interactive manipulations of the value of x, etc.

Example 23.1. *Linear first-order equation. Cauchy problem. Dynamic and control objects.* Let us create various dynamic and control objects, for example, for the exact solution $u(x,y) = (y+x)^n - y$ of the Cauchy problem

$$u_x - u_y = 1, \quad u(x,0) = x^n, \qquad n \in \mathbb{N},$$

as follows:

```
F[x_,y_,n_]:=(x+y)^n-y; {r=2, k=9} {Slider[Dynamic[y]],
Dynamic[Plot[F[x,y,k],{x,-r,r},PlotRange->{{-r,r},{-r,r}},
 ImageSize->500]]}
DynamicModule[{y=0.5},{Slider[Dynamic[y]],Dynamic[Plot[F[x,y,k],{x,-r,r},
 PlotRange->{{-r,r},{-r,r}},ImageSize->500]]}]
TabView[Table[Plot[F[x,y,k],{x,-r,r},PlotRange->{{-r,r},{-r,r}}],{y,-r,r,0.1}]]
SlideView[Table[Plot[F[x,y,k],{x,-r,r},PlotRange->{{-r,r},{-r,r}}],{y,-r,r,0.1}]]
Manipulate[Expand[F[x,y,k]],{k,3,10,1}]
```

Example 23.2. *Linear first-order equation. Cauchy problem. Dynamic object without controls and animation frame.* Let us create a dynamic object without controls and an animation frame, for example, for the exact solution $u(x,y) = -\frac{1}{2}\cos x \sin y$ of the boundary value problem

$$u_{xx} + u_{yy} = \cos x \sin y \quad (0 < x < \pi, \quad 0 < y < 2\pi), \tag{23.1.4.1}$$

$$u(x,0) = 0, \quad u(x,2\pi) = 0, \quad u(0,y) = -\frac{1}{2}\sin y, \quad u(\pi,y) = -\frac{1}{2}\sin y, \tag{23.1.4.2}$$

as follows:

```
F[x_,y_]:=-1/2*Cos[x]*Sin[y];
t=0; r=2*Pi;
Row[{Pane[Animator[Dynamic[y],{-r,r}],{0,0}],
  Dynamic[Plot[F[x,y],{x,-r,r},
  ImageSize->300,PlotRange->{{-r,r},{-r,r}}]]}]
```

⊙ *References for Section 23.1*: J. Calmet and J. A. van Hulzen (1983), A. G. Akritas (1989), T. Gray and J. Glynn (1991), J. H. Davenport, Y. Siret, and E. Tournier (1993), D. D. Vvedensky (1993), J. W. Gray (1994), E. Green, B. Evans, and J. Johnson (1994), T. B. Bahder (1995), C. C. Ross (1995), R. L. Zimmerman and F. Olness (1995), M. J. Wester (1999), S. Wolfram (2002, 2003), C. Getz and J. Helmstedt (2004), L. Debnath (2007), I. K. Shingareva and C. Lizárraga-Celaya (2009, 2011).

23.2 Analytical Solutions and Their Visualizations

It is well known that analytical solutions are often impossible in practice or investigations, since numerical solutions only can be obtained. However, it is important to understand the general theory in order to perform a sophisticated research (e.g., certain types of equations need appropriate boundary conditions and solution methods, problems may be ill posed, etc.). In this section, we consider various methods for constructing analytical solutions of linear partial differential equations and their systems.

23.2.1 Constructing Analytical Solutions in Terms of Predefined Functions

In the computer algebra system Mathematica, analytical and symbolic solutions of a given linear partial differential equation can be found with the aid of the predefined function `DSolve`:

```
DSolve[PDE,u,{x1,..,xn}]  DSolve[PDE,u[x1,...,xn],{x1,...,xn}]
DSolve[PDE, u[x1,...,xn], {x1,...,xn}, GeneratedParameters->C]
```

`DSolve` finding analytical solutions of a PDE for the function `u` with independent variables `x1,...xn` (“pure function” solution).

`DSolve` finding analytical solutions of a PDE for the function `u[x1,...xn]` with independent variables `x1,...xn`.

`DSolve`, `GeneratedParameters` finding analytical solutions of a PDE for the function `u[x1,...xn]` with independent variables `x1,...xn` and specifying the arbitrary constants.

The function `DSolve` can solve the following classes of linear partial differential equations (with two or more independent variables and one dependent variable): most first-order PDEs and a limited number of second-order PDEs.

First, let us assume that we have obtained exact solutions and we wish to verify whether these solutions are exact solutions of given linear PDEs.

Example 23.3. *Linear Poisson equation. Verification of solutions.* For the two-dimensional Poisson equation with a special right-hand side

$$u_{xx}+u_{yy}=\Phi(x,y), \qquad \Phi(x,y)=-\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\exp(b_i x+c_j y),$$

we verify that

$$u(x,t)=-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{ij}}{b_i^2+c_j^2}\exp(b_i x+c_j y)$$

is its exact solution as follows:

```
pde1=D[u[x,y],{x,2}]+D[u[x,y],{y,2}]+Sum[a[i,j]*Exp[b[i]*x+c[j]*y],{j,1,n},{i,1,n}]==0
sol1=u->Function[{x,y},-Sum[a[i,j]/(b[i]^2+c[j]^2)*Exp[b[i]*x+c[j]*y],{j,n},{i,n}]]
test1=FullSimplify[pde1/.sol1/.n->9]
FullSimplify[sol2[[1,1]]-sol2[[2,2,1]]-sol2[[3,2,1]]]
```

where a_{ij}, b_i, and c_j ($i=1,\dots n$, $j=1,\dots n$) are arbitrary real constants.

Example 23.4. *Linear heat equation. Verification of solutions.* For the one-dimensional heat equation (1.1.1)

$$u_t=ku_{xx},$$

we verify that

$$u(x,t)=\frac{1}{2\sqrt{\pi kt}}\exp\left(-\frac{x^2}{4kt}\right), \quad x\in\mathbb{R}, \quad t>0,$$

is the fundamental solution of this linear heat equation as follows:

```
pde1:=D[u[x,t],t]==k*D[u[x,t],{x,2}]
sol1=u->Function[{x,t},1/Sqrt[4*Pi*k*t]*Exp[-x^2/(4*k*t)]]
test1=FullSimplify[pde1/.sol1]
```

where k is a constant.

Example 23.5. *Linear wave equation. General solution.* The general solution of the one-dimensional wave equation (4.1.1)

$$u_{tt}=c^2u_{xx}$$

can be found and tested as follows:

```
pde1=D[u[x,t],{t,2}]-c^2*D[u[x,t],{x,2}]==0
sol1=Assuming[{c\[Element] Reals && c>0},
  DSolve[pde1,u,{x,t}]//Simplify]//First
sol2=u->Rationalize[N[PowerExpand[sol1[[1,2]]]]]
test1=Cancel[PowerExpand[pde1/.u->sol1[[1,2]]]] Print["sol2=",sol2]
HoldForm[sol2]==sol2 sol3=u[x,t]/.sol2
sol4=(u[x,t]/.sol2)/.{Table[C[i][var_]->Subscript[F,i][var],{i,1,2}]}//FullSimplify
  {sol1,sol2,sol3,sol4}
```

where the Mathematica result reads:

$$\text{sol1} = \left\{u \to \text{Function}\left[\{x,t\}, C[1]\left[t - \frac{x}{\sqrt{c^2}}\right] + C[2]\left[t + \frac{x}{\sqrt{c^2}}\right]\right]\right\},$$

$$\text{sol2} = u \to \text{Function}\left[\{x,t\}, C[1]\left[t - \frac{x}{c}\right] + C[2]\left[t + \frac{x}{c}\right]\right],$$

$$\text{sol3} = C[1]\left[t - \frac{x}{c}\right] + C[2]\left[t + \frac{x}{c}\right],$$

$$\text{sol4} = \left\{F_1\left[t - \frac{x}{c}\right] + F_2\left[t + \frac{x}{c}\right]\right\}.$$

According to the Mathematica notation, `sol1` is a "pure function" solution for $u(x,t)$ (where `C[1]` and `C[2]` are arbitrary functions), `sol2` is a "pure function" simplified solution, `sol3` represents the solution $u(x,t)$, and `sol4` represents the solution $u(x,t)$ in a more convenient form, with arbitrary functions F_1 and F_2.

23.2.2 Constructing General Solutions via the Method of Characteristics

Consider a method for finding general solutions of first-order linear equations, the *Lagrange method of characteristics*. This method allows reducing a PDE to a system of ODEs along which the given PDE with some initial data (Cauchy data) is integrable. Once the system of ODEs is found, it can be solved along the *characteristic curves* and transformed into a general solution of the original PDE. This method was originally proposed by Lagrange in 1772 and 1779 for solving first-order linear and nonlinear PDEs.

In general, a first-order PDE in two independent variables (x, y) can be written in the form

$$\mathcal{F}(x,y,u,u_x,u_y) = 0 \quad \text{or} \quad \mathcal{F}(x,y,u,p,q) = 0, \tag{23.2.2.1}$$

where $\mathcal{F}$ is a given function, $u = u(x,y)$ is an unknown function of the independent variables x and y ($(x,y) \in D \subset \mathbb{R}^2$), and $u_x = p$, $u_y = q$. Equation (23.2.2.1) is said to be linear if the function $\mathcal{F}$ is linear in the variables u, u_x, and u_y and the coefficients of these variables are functions of the independent variables x and y alone. The most general first-order linear PDE has the form

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = G(x,y). \tag{23.2.2.2}$$

The solution $u(x,y)$ of Eq. (23.2.2.1) can be visualized geometrically as a surface, called an *integral surface* in the (x,y,u)-space.

A *direction vector field* or the *characteristic direction*, (A,B,C), is a *tangent vector* to the integral surface $f(x,y,u)=0$ at the point (x,y,u).

A *characteristic curve* is a curve in (x,y,u)-space such that the tangent at each point coincides with the characteristic direction field (A,B,C).

The parametric equations of the characteristic curve can be written in the form $x=x(t)$, $y=y(t)$, $u=u(t)$, and the tangent vector to this curve $(dx/dt, dy/dt, du/dt)$ is equal to (A,B,C).

The *characteristic equations* of Eq. (23.2.2.2) in parametric form are the system of ODEs $dx/dt = A(x,y)$, $dy/dt = B(x,y)$, $du/dt = C(x,y)$.

The *characteristic equations* of Eq. (23.2.2.2) in nonparametric form are $dx/A(x,y) = dy/B(x,y) = du/C(x,y)$. The slope of the characteristics is determined by the equation $dy/dx = B(x,y)/A(x,y)$.

Example 23.6. *First-order linear equations. Direction vector fields.* By applying Mathematica predefined functions to a first-order linear PDE of the form

$$u_x + u_t + u = 0, \quad xu_x + tu_t + (x^2 + t^2) = 0,$$

where $\{x \in \mathbb{R}, t \geq 0\}$, we can construct the direction vector fields for the given first-order linear PDE as follows:

```
r1[x_,t_,u_]:={1,1,-u}; r2[x_,t_,u_]:={x,t,-(x^2+t^2)};
{n1=10, p=Pi, v1={1,-3,1}, v2={1,2,3}}
SetOptions[VectorPlot3D,VectorColorFunction->Hue,VectorPoints->
  {n1,n1,n1},VectorStyle->Arrowheads[0.02],PlotRange->All];
VectorPlot3D[r1[x,t,u],{x,-p,p},{t,-p,p},{u,-p,p},ViewPoint->v1]
VectorPlot3D[r2[x,t,u],{x,-p,p},{t,-p,p},{u,-p,p},ViewPoint->v2]
```

The *general solution* (or general integral) of a given first-order PDE is an equation of the form

$$f(\phi, \psi)=0, \tag{23.2.2.3}$$

where f is an arbitrary function of the known functions $\phi = \phi(x,y,u)$ and $\psi=\psi(x,y,u)$, and provides a solution of this partial differential equation. The functions $\phi(x,y,u) = C_1$, $\psi(x,y,u) = C_2$ (where C_1 and C_2 are constants) are the solution curves of the characteristic equations, or the families of characteristic curves of Eq. (23.2.2.2).

Example 23.7. *First-order equation. General solution.* By applying the method of characteristics to the first-order linear PDE

$$xu_x + yu_y = u,$$

where $\{x \in \mathbb{R}, y \in \mathbb{R}\}$, we can prove that the general solution of the given first-order PDE has the form

$$f\left(\frac{y}{x}, \frac{u}{x}\right) = 0 \quad \text{or} \quad u(x,y) = xg\left(\frac{y}{x}\right):$$

```
{fU=u[x,y], fF=x, fG=y, fH=u[x,y]} {pde=fF*D[fU,x]+fG*D[fU,y]==fH,
charEqs={dx/fF,dy/fG,du/fH}}
eq10=Map[Exp,Integrate[1/fG,y]-Integrate[1/fF,x]==C10]
eq20=Map[Exp,Integrate[1/u,u]-Integrate[1/fF,x]==C20]
{eq11=eq10/.eq10[[2]]->C1, eq21=eq20/.eq20[[2]]->C2}
genSol=f[eq11[[1]],eq21[[1]]]==0 {genSol1=(genSol/.u->u[x,y]),
gs1=(genSol/.u->u[x,y])[[1]]}
test1=Simplify[pde/.{u[x,y]->gs1}/.{D[u[x,y],x]->D[gs1,x]}/.
 {D[u[x,y],y]->D[gs1,y]}/.-u[x,y]->-pde[[1]]]
Equal[genSol1, test1] {DSolve[pde,u,{x,y}],
genSol1=genSol[[1,2]]==g[genSol[[1,1]]]} {genSol2=Solve[genSol1,u],
gs2=genSol2[[1,1,2]]}
test2=Simplify[pde/.{u[x,y]->gs2}/.{D[u[x,y],x]->D[gs2,x]}/.{D[u[x,y],y]->D[gs2,y]}]
```

Remark. The system of characteristic equations $dx/x = dy/y = du/u$ gives the integral surfaces (eq11, eq21):

$$\phi = \frac{y}{x} = C_1, \quad \psi = \frac{u}{x} = C_2,$$

where C_1 and C_2 are arbitrary constants. Hence, according to Eq. (23.2.2.3), the general solution of the linear PDE is $f(y/x, u/x) = 0$ (genSol), where f is an arbitrary function. This solution can be verified (test1). By applying the predefined function DSolve, we find the result in a different form:

$$u(x,y) = xg\left(\frac{y}{x}\right).$$

This form of the general solution can be obtained (genSol2) and verified (test2).

Example 23.8. *Linear Euler equation. General solution.* By applying the method of characteristics to the linear Euler equation

$$xu_x + yu_y = nu,$$

where $\{x \in \mathbb{R}, y \in \mathbb{R}\}$, we can prove that the general solution of the given first-order PDE takes the form

$$f\left(\frac{y}{x}, \frac{u}{x^n}\right) = 0 \quad \text{or} \quad u(x,y) = x^n g\left(\frac{y}{x}\right)$$

as follows:

```
{fU=u[x,y], fF=x, fG=y, fH=n*u[x,y]}
{pde=fF*D[fU,x]+fG*D[fU,y]==fH, charEqs={dx/fF,dy/fG,du/fH}} eq10=
Map[Exp,Integrate[1/fG,y]-Integrate[1/fF,x]==C10] eq20=
Map[Exp,Integrate[1/(n*u),u]-Integrate[1/fF,x]==C20]
{eq11=eq10/.eq10[[2]]->C1, eq21=eq20/.eq20[[2]]->C2,
 eq22=PowerExpand[(eq21[[1]])^n==C2]}
genSol=f[eq11[[1]],eq22[[1]]]==0 {genSol1=(genSol/.u->u[x,y]),
gs1=(genSol/.u->u[x,y])[[1]]} {tr1=u[x,y]->gs1,
test11=FullSimplify[pde/.{tr1}/.{D[tr1,x]}/.{D[tr1,y]}],
 test12=test11[[1]]->0, test13=Thread[(test11/.test12)/n,Equal], genSol1==test13}
{DSolve[pde,u,{x,y}], genSol1=genSol[[1,2]]==g[genSol[[1,1]]]}
{genSol2 = Solve[genSol1, u], gs2=genSol2[[1,1,2]],
tr2=u[x,y]->gs2} test2=Simplify[pde/.{tr2}/.{D[tr2,x]}/.{D[tr2,y]}]
```

The system of characteristic equations $dx/x = dy/y = du/(nu)$ gives the integral surfaces (eq11, eq22):

$$\phi = \frac{y}{x} = C_1, \quad \psi = \frac{u}{x^n} = C_2,$$

where C_1 and C_2 are arbitrary constants. Hence, according to Eq. (23.2.2.3), the general solution of the linear PDE is $f(y/x, x^{-n}u) = 0$ (genSol), where f is an arbitrary function. This solution can be verified (test11, test12, test13). By applying the predefined function DSolve, we find the result in a different form:

$$u(x,y) = x^n g\left(\frac{y}{x}\right).$$

This form of the general solution can be obtained (genSol2) and verified (test2).

23.2.3 Constructing General Solutions via Conversion to Canonical Forms

Consider the general first-order linear partial differential equation

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = G(x,y).$$

By introducing a new transformation by the equations $\xi = \xi(x,y)$, $\eta = \eta(x,y)$[2] we can reduce the original equation to the *canonical* (or standard) form

$$u_\xi + \alpha(\xi,\eta)u = \beta(\xi,\eta),$$

where $\alpha(\xi,\eta) = \tilde{C}/\tilde{A}$ and $\beta(\xi,\eta) = \tilde{G}/\tilde{A}$ with $\tilde{A} = u\xi_x + B\xi_y$ and $\tilde{B} = A\eta_x + B\eta_y$, so that the equation is $\tilde{A}u_\xi + \tilde{B}u_\eta + \tilde{C}u = \tilde{G}$. This equation can be integrated, and the general solution of the original equation can be found.

Example 23.9. *Linear first-order equation. Canonical form. General solution.* Considering the linear first-order PDE

$$u_x + u_y = u,$$

where $\{x \in \mathbb{R}, y \in \mathbb{R}\}$, we can obtain the transformation $\xi = -x + y$, $\eta = x$ (`eqxi`, `eqeta`) and the canonical form $u_\eta = u$ (`canForm`). By integrating this equation, we find the general solution

$$u(x,y) = e^x F(-x+y),$$

where $F(-x+y)$ is an arbitrary function. Finally, we can find and test the general solution using predefined functions as follows:

```
{aA=1,bB=1,cC=-1,gG=0}
pde=aA*D[u[x,y],x]+bB*D[u[x,y],y]+cC*u[x,y]==0
charEqs={dx/aA,dy/bB,du/u[x,y]}
{eqxi=xi==Integrate[1/bB,y]-Integrate[1/aA,x], eqeta=eta==x}
testJ=D[eqxi[[2]],x]*D[eqeta[[2]],y]-D[eqxi[[2]],y]*D[eqeta[[2]],x]
ux=D[u[xi,eta],xi]*D[eqxi[[2]],x]+D[u[xi,eta],eta]*D[eqeta[[2]],x]
uy=D[u[xi,eta],xi]*D[eqxi[[2]],y]+D[u[xi,eta],eta]*D[eqeta[[2]],y]
{canForm=aA*ux+bB*uy==-cC*u[xi,eta],
canForm1=Thread[canForm/canForm[[2]],Equal]}
eq1=Integrate[canForm1[[1]],eta]==Integrate[canForm1[[2]],eta]+Log[F[xi]]
eq2=u[xi,eta]->(Solve[eq1,u[xi,eta]])[[1,1,2,1]]
genSol=u[x,y]==eq2[[2]]/.{eqxi//ToRules}/.{eqeta//ToRules}
{genSolPred=DSolve[pde,u,{x,y}], gSol=ToRules[genSol[[1,1]]]}
{test1=pde/.genSolPred, test2=pde/.gSol/.D[gSol,x]/.D[gSol,y]}
```

For linear second-order PDEs, we consider the classification of equations (which does not depend on their solutions and is determined by the coefficients of the highest derivatives) and the reduction of a given equation to appropriate canonical forms.

Let us introduce the new variables $a = \mathcal{F}_p$, $b = \frac{1}{2}\mathcal{F}_q$, $c = \mathcal{F}_r$, and calculate the discriminant $\delta = b^2 - ac$ at some point. Depending on the sign of the discriminant δ, the type of equation at a specific point can be *parabolic* (if $\delta = 0$), *hyperbolic* (if $\delta > 0$), or *elliptic* (if $\delta < 0$). Let us call the following equations

$$u_{y_1y_2} = f_1(y_1,y_2,u,u_{y_1},u_{y_2}), \quad u_{z_1z_1} - u_{z_2z_2} = f_2(z_1,z_2,u,u_{z_1},u_{z_2})$$

[2]Here the functions ξ and η are continuously differentiable and the Jacobian $J(x,y) = \xi_x\eta_y - \xi_y\eta_{,x}$ is nonzero in a domain $\mathcal{D}$.

the *first canonical form* and the *second canonical form* for hyperbolic PDEs, respectively.

Example 23.10. *Linear second-order equation. Classification. Canonical forms.* Considering the linear second-order PDEs

$$-2y^2u_{xx}+\tfrac{1}{2}x^2u_{yy}=0,$$

where $\{x\in\mathbb{R}, y\in\mathbb{R}\}$, we can verify that this equation is *hyperbolic* everywhere (except at the point x=0, y=0) and prove that the change of variables $\xi=-\frac{1}{2}x^2+y^2$, $\eta=\frac{1}{2}x^2+y^2$ transforms the PDE to the *first* and *second canonical forms*, respectively,

$$v_{\eta\xi}+\frac{v_\xi\eta-v_\eta\xi}{2(\eta^2-\xi^2)}=0,\qquad v_{\lambda\lambda}-v_{\mu\mu}+\frac{1}{2}\left(\frac{v_\lambda}{\lambda}-\frac{v_\mu}{\mu}\right)=0.$$

1. Classification. In standard notation, this linear equation takes the form

$$F_1=-2y^2p+\frac{1}{2}x^2r=0.$$

The new variables are $a=-2y^2$, $b=0$, $c=\frac{1}{2}x^2$ (`tr2(f1)`) and the discriminant $\delta=b^2-ac=x^2y^2$ (`delta1`) is positive except at the point $x=0$, $y=0$.

```
pde1=-2*y^2*D[u[x,y],{x,2}]+x^2*D[u[x,y],{y,2}]/2==0
tr1[x_,y_,u_]:={D[u[x,y],{x,2}]->p,D[u[x,y],{y,2}]->r,D[u[x,y],{x,y}]->q};
tr2[f_]:={a->D[f[p,q,r][[1]],p],b->1/2*D[f[p,q,r][[1]],q],c->D[f[p,q,r][[1]],r]};
f1[p_,r_,q_]:=pde1/.tr1[x,y,u]; delta=b^2-a*c {f1[p,r,q], tr2[f1],
delta1=delta/.tr2[f1]-f1[p,r,q][[2]]} {Reduce[delta1>0],
FindInstance[delta1>0,{x,y}]}
```

The same result can be obtained with the principal part coefficient matrix as follows:

```
{a1={{-2*y^2,0},{0,x^2/2}},d1=Det[a1],Reduce[d1<0],
FindInstance[d1<0,{x,y}]}
```

Here we calculate the determinant `d1` of the matrix `a1`. PDEs can be classified according to the eigenvalues of the matrix `a1`, i.e., depending on the sign of `d1`: if `d1=0`, then it is parabolic; if `d1<0`, then it is hyperbolic; and if `d1>0`, then it is elliptic.

2. Canonical forms. We find a change of variables that transforms the PDE to the first and second canonical forms as follows:

```
jacobianM[f_List?VectorQ,x_List]:=Outer[D,f,x]/;Equal@@(Dimensions/@{f,x});
hessianH[f_,x_List?VectorQ]:=D[f,{x,2}];
gradF[f_,x_List?VectorQ]:=D[f,{x}]; op1[expr_]:=expr/.y->y[x];
op2[expr_]:=expr/.y[x]->y; {vars=Sequence[x,y],
varsN=Sequence[xi,eta]}
m1=Assuming[{x>0,y>0},Simplify[(-a1[[1,2]]+Sqrt[-d1])/a1[[1,1]]]]
m2=Assuming[{x>0,y>0},Simplify[(-a1[[1,2]]-Sqrt[-d1])/a1[[1,1]]]]
{eq1=DSolve[D[y[x],x]==-op1[m1],y[x],x],
eq11=eq1[[1,1,1]]^2==eq1[[1,1,2]]^2}
{eq12=Solve[eq11,C[1]][[1,1,2]], g[1]=Expand[op2[eq12]*2]}
{eq2=DSolve[D[y[x],x]==-op1[m2],y[x],x],
eq21=eq2[[1,1,1]]^2==eq2[[1,1,2]]^2}
{eq22=Solve[eq21,C[1]][[1,1,2]], g[2]=Expand[op2[eq22]*2]}
{jg=jacobianM[{g[1],g[2]},{vars}], dv=gradF[v[varsN],{varsN}]}
ddv=hessianH[v[varsN],{varsN}]
```

```
ddu=Transpose[jg].ddv.jg+Sum[dv[[i]]*hessianH[g[i],{vars}],{i,1,2}]
{eq3=Simplify[Tr[a1.ddu]]==0, tr0={y^2->Y,x^2->X},
tr01={Y->y^2,X->x^2}}
tr1=Flatten[{Expand[Solve[First[g[2]==eta/.{Solve[g[1]==xi/.tr0,X]/.tr01}/.
 tr0],Y]/.tr01],Expand[Solve[First[g[1]==xi/.
 {Solve[g[2]==eta/.tr0,Y]/.tr01}/.tr0],X]/.tr01]}]
nForm=Collect[Expand[eq3/.tr1],D[v[varsN],varsN]]
c1=Coefficient[nForm[[1]],D[v[varsN],varsN]]
normalFormF=Collect[Thread[nForm/c1,Equal],D[v[varsN],varsN]]
nF[x_,t_]:=D[D[u[x,t],x],t]+(2*t*D[u[x,t],x]-2*x*D[u[x,t],t])/(4*t^2-4*x^2)==0;
tr2={xi->lambda+mu,eta->mu-lambda}; nF[xi,eta]
nFT[v_]:=((Simplify[nF[xi,eta]/.u->Function[{xi,eta},
 u[(xi-eta)/2,(xi+eta)/2]]])/.tr2//ExpandAll)/.{u->v}; canonicalForm=nFT[v]
```

Example 23.11. *Linear second-order equation. Classification. Canonical forms.* Considering the linear second-order PDE

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0,$$

where $\{x\in\mathbb{R},y\in\mathbb{R}\}$, we can verify that this equation is *parabolic* everywhere and prove that the canonical forms of the PDE are $x^2v_{\xi\xi}=0$ and $v_{\xi\xi}=0$, respectively.

```
jacobianM[f_List?VectorQ,
x_List]:=Outer[D,f,x]/;Equal@@(Dimensions/@{f,x});
hessianH[f_,x_List?VectorQ]:=D[f,{x,2}];
gradF[f_,x_List?VectorQ]:=D[f,{x}]; op1[expr_]:=expr/.y->y[x];
op2[expr_]:=expr/.y[x]->y; {vars=Sequence[x,y],
varsN=Sequence[xi,eta]} {a1={{x^2,x*y},{x*y,y^2}}, d1=Det[a1]}
m1=Assuming[{x>0,y>0},Simplify[(-a1[[1,2]]+Sqrt[-d1])/a1[[1,1]]]]
eq1=DSolve[D[y[x],x]==-op1[m1],y[x],x]/.Rule->Equal//First
{eq11=Solve[eq1,C[1]][[1,1,2]], g[1]=op2[Eq11], g[2]=x}
{jg=jacobianM[{g[1],g[2]},{vars}], dv=gradF[v[varsN],{varsN}]}
ddv=hessianH[v[varsN],{varsN}]
ddu=Transpose[jg].ddv.jg+Sum[dv[[i]]*hessianH[g[i],{vars}],{i,1,2}]
{norF=Simplify[Tr[a1.ddu]]==0, canF=Thread[norF/x^2,Equal]//Expand}
```

23.2.4 Constructing Analytical Solutions of Cauchy Problems

If we consider a mathematical problem in an unbounded domain, the solution can be determined uniquely by prescribing initial conditions. The corresponding problem is called the *initial value problem* or the *Cauchy problem*.

Mathematical problems can be *well-posed* or *ill-posed* problems. A mathematical problem is well posed if it possesses the following features: *existence* (there exists at least one solution), *uniqueness* (there exists at most one solution), and *continuity* (the solution depends continuously on the data).

In Mathematica, it is possible to construct exact solutions for some linear partial differential equations subject to initial conditions with the aid of the predefined function `DSolve`. Let us solve some initial value problems.

Example 23.12. *Linear first-order PDEs. Initial value problem (IVP). Exact solution.* Considering the Cauchy problem for the linear first-order partial differential equation,

$$yu_x+xu_y=u(x,y),\quad u(x,0)=F(x),$$

and applying Mathematica predefined functions, we can solve the Cauchy problem for this equation and verify that of the two solutions obtained,

$$u(x,y) = \frac{\sqrt{x^2-y^2}F\left(\sqrt{x^2-y^2}\right)}{x+y}, \quad u(x,y) = \frac{(x+y)F\left(\sqrt{x^2-y^2}\right)}{\sqrt{x^2-y^2}},$$

the second one is an exact solution of this initial value problem as follows:

```
{pde1=y*D[u[x,y],x]+x*D[u[x,y],y]==u[x,y], ic1=u[x,0]==F[x],
sys1={pde1,ic1}} {sol1=DSolve[sys1,u[x,y],{x,y}],
sol2=DSolve[sys1,u[x,y],{x,y}]//Simplify,
 sol3=ComplexExpand[DSolve[sys1,u[x,y],{x,y}]]//Simplify}
{s1=sol3[[1,1,2]], s2=sol3[[2,1,2]]}
t1PDEl=pde1[[1]]/.u[x,y]->s1/.D[u[x,y],x]->D[s1,x]/.D[u[x,y],y]->D[s1,y]//Simplify
t1PDEr=pde1[[2]]/.u[x,y]->s1//Simplify {t1PDEl===t1PDEr,
testIC=ic1/.u[x,0]->s1/.y->0//Simplify}
t2PDEl=pde1[[1]]/.u[x,y]->s2/.D[u[x,y],x]->D[s2,x]/.D[u[x,y],y]->D[s2,y]//Simplify
t2PDEr=pde1[[2]]/.u[x,y]->s2//Simplify {t2PDEl===t2PDEr,
testIC=ic1/.u[x,0]->s2/.y->0//Simplify}
```

Note that the predefined function `DSolve` always treats all variables as complex, so the general form of the solution is complex (`sol1`). However, we have to obtain real solutions, so we can add real initial conditions, some assumptions, and simplify the resulting solution (e.g., `Simplify`, `ComplexExpand`, `sol2`, `sol3`).

Example 23.13. *Linear first-order equation. Method of characteristics. Classical Cauchy problem.* Considering the first-order linear partial differential equation with the Cauchy data,

$$u_x - u_y = 1, \quad u(x,0) = x^n \qquad (n \in \mathbb{N}),$$

and applying the method of characteristics, we can obtain the general and particular solutions of this equation,

$$u(x,y) = f(y+x)+x, \quad u(x,y) = (y+x)^n - y,$$

respectively, and plot the characteristic curves as follows:

```
SetOptions[ParametricPlot,ImageSize->200,PlotStyle->{Hue[0.7],Thickness[0.01]}];
{ode=D[uN[x],x]==1, solCh=DSolve[ode,uN[x],x], eqCh=D[y[x],x]==-1}
{curCh=DSolve[eqCh,y[x],x]//First, tr1=uN[x]->u, tr2=y[x]->y}
g=Table[ParametricPlot[{(curCh[[1,2]]/.C[1]->y),x},{x,0,9}],{y,0,9}];
Show[g,PlotRange->{{0,9},{0,9}},AspectRatio->1]
eq1=Solve[(solCh/.Rule->Equal)[[1]]/.tr1,C[1]]
eq2=Solve[(curCh/.Rule->Equal)[[1]]/.tr2,C[1]]
eq3=eq1[[1,1,2]]==f[eq2[[1,1,2]]]
fN[xN_]:=eq3[[1]]/.{y->0,u->x^n}/.x->xN; fN[x]
sol1=Solve[eq3,u]/.f[eq2[[1,1,2]]]->fN[eq2[[1,1,2]]]
```

The same result can be obtained by applying the predefined functions:

```
{pde=D[v[x,y],x]-D[v[x,y],y]==1, ic=v[x,0]==x^n, sys1={pde,ic}}
{sol2=DSolve[sys1,v,{x,y}], test=sys1/.sol2}
```

Example 23.14. *First-order linear equation. Method of characteristics. Classical Cauchy problem.* Let us solve the first-order linear equation with the Cauchy data

$$u_t - xu_x = u, \quad u(x,0) = f(x)$$

by the method of characteristics. This equation can be obtained from the Fokker–Planck equation [see Bluman et al. (2010)][3]

$$u_t = u_{xx} + (xu)_x,$$

by neglecting the term u_{xx}.

We can obtain the solution of this Cauchy problem

$$u_1(x,t) = f(xe^t)\,e^t$$

and plot the characteristic curves as follows:

```
SetOptions[ParametricPlot,ImageSize->300,PlotStyle->{Hue[0.7],Thickness[0.001]}];
{tF=Pi/2, xF=2*Pi, tr1=x->x[t], tr2=x[t]->x,
tr3=f[xN[0]]->f1[xN[0]]} f1[x_]:=Cos[x]; f1[x]
{ode1=D[x[t],t]==-x[t], ode2=D[uN[t],t]==uN[t],
sol2=DSolve[ode2,uN[t],t]} {sol21=sol2[[1,1,2]],
iniCond=u[xN[0],0]==f[xN[0]]} {const2=(sol21/.t->0)->iniCond[[2]],
sol22=uN[t]->(sol21/.const2)} {ode11=ode1/.sol22,
sol1=DSolve[ode11,x[t],t], xN0=sol1/.t->0//First}
{const1=Solve[xN0/.Rule->Equal,C[1]],
chars=sol1/.const1/.x[0]->xN[0]} {genSol=sol22[[2]],
char1=chars/.tr2/.tr3//First//First}
trxN0=Solve[char1/.Rule->Equal,xN[0]]
u1[xN_,tN_]:=genSol/.trxN0/.{x->xN,t->tN}; u1[x,t]
u2[x1_,t1_]:=genSol/.f[xN[0]]->f1[trxN0[[1,1,2]]]/.{x->x1,t->t1};
u2[x,t]
g1=Table[ParametricPlot[{(char1[[1,2]]/.xN[0]->x),t},{t,0,tF}],{x,-xF,xF}];
Show[g1,PlotRange->{{-xF,xF},{0,tF}},AspectRatio->1]
g2=Table[ParametricPlot[{u2[x,t],t},{t,0,tF}],{x,-xF,xF}];
Show[g2,PlotRange->{{-xF,xF},{0,tF}},AspectRatio->1]
```

We show that the *implicit form of the solution* (or parametric representation of the solution) of this Cauchy problem in the Mathematica notation reads

$sol22 := uN[t] = f[xN[0]]\,\mathrm{e}^t \quad chars := x[t] = xN[0]\,\mathrm{e}^{-t}$

The characteristic curves for $f(x) = \cos x$ are presented in Fig. 23.1.

23.2.5 Constructing Analytical Solutions of Boundary Value Problems

In Mathematica, by applying the predefined function `DSolve`, it is possible to find analytical solutions for *restricted classes of linear PDEs* (e.g., linear first-order PDEs and linear homogeneous second-order PDEs of the form $au_{xx} + bu_{xy} + cu_{yy} = 0$). However, in this section, we solve a specific boundary value problem associated with linear PDEs that do not belong to such restricted classes.

[3]The Fokker–Planck equation arises in various applications of statistical mechanics; it describes the evolution of the probability distribution function.

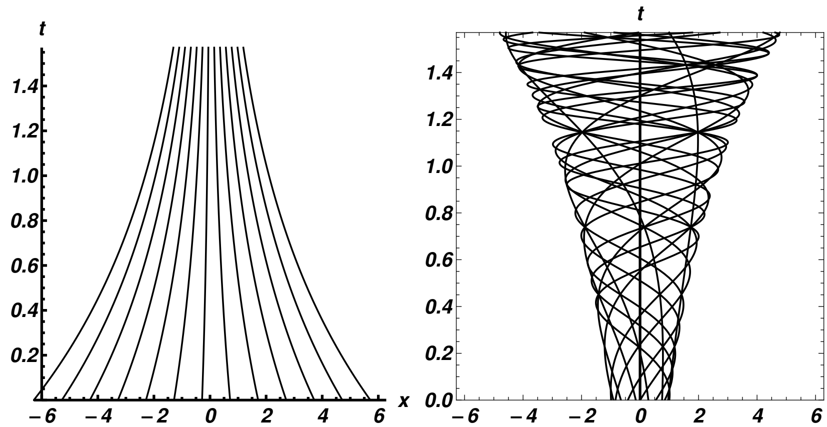


Figure 23.1 Characteristic curves for $u_t - xu_x = u$ and the solution $u_2(x,t)$ (for $f(x) = \cos x$).

Example 23.15. *Linear Poisson equation with polynomial right-hand side. Polynomial boundary conditions. Exact solution.* Consider the Dirichlet boundary value problem of a special form, i.e., the linear Poisson equation with polynomial right-hand side and polynomial boundary conditions:

$$u_{xx} + u_{yy} = q(x,y), \quad 0 < x < m, \;\; 0 < y < m,$$
$$u(0,y) = 0, \quad u(m,y) = f(y), \quad u(x,0) = 0, \quad u(x,m) = 0,$$

where m is a parameter ($m \in \mathbb{Q}$, $m > 0$); e.g., $m = 5/2$. The right-hand side $q(x,y)$ of the Poisson equation is a polynomial of degree n of the form

$$q(x,y) = \sum_{k=0}^{n} a_k x^{n-k} y^k,$$

where the coefficients a_k are determined in the solution process. Polynomial boundary conditions are given, e.g., $f(y) = y^3 - y$.

Since it is not possible to solve such problems with the aid of the predefined function `DSolve`, we propose a new method for solving Dirichlet boundary value problems of special form on the basis of some ideas [see Muleshkov et al. (2002)] for constructing exact solutions of the Poisson equation. This method can also be applied for another configurations of the problem, e.g., when the coefficients a_k are given and the boundary conditions are determined in the solution process. Looking for a solution of the given Poisson problem in the form

$$u(x,y) = \sum_{k=0}^{n} p_k x^{n-k+2} y^k,$$

we find the coefficients a_k and p_k (`trAP`) that satisfy the boundary conditions and the Poisson equation

$$a_0 = 0, \quad a_1 = \frac{2(50(48+125y-48y^2) - 125x(-25+8y^2) + 4x^2(-48-125y+48y^2))}{25(-625+16x^2y^2)},$$
$$a_2 = 1, \quad a_3 = \frac{x(-15625x^2 + 400x^4y^2 + 6250y^3 + 8xy^3(-48-125y+48y^2))}{75y^3(-625+16x^2y^2)},$$
$$p_0 = 0, \quad p_1 = \frac{48+125y-4(12+5x)y^2}{-1875+48x^2y^2}$$

and the exact solution

$$\frac{x^3y(-5+2y)(3125y+2x(-240+y(-721+2y(-5+50x+48y))))}{150(-625+16x^2y^2)},$$

and verify that this solution satisfies the given Poisson problem (testPDE, testBCs) as follows:

```
{n=3, m=5/2, pde=D[u[x,y],{x,2}]+D[u[x,y],{y,2}]==q[x,y]}
q[x_,y_]:=Sum[a[k]*x^(n-k)*y^k,{k,0,n}]; q[x,y]
u[x_,y_]:=Sum[p[k]*x^(n-k+2)*y^k,{k,0,n}]; f[y_]:=y^3-y; {u[x,y],
f[y]} lRange={Range[0,n],Range[0,n]}
eq1=Tuples[lRange]/.Append[CoefficientRules[q[x,y],{x,y}],{_,_}->0]
eq2=Tuples[lRange]/.Append[CoefficientRules[pde[[1]],{x,y}],{_,_}->0]
{eq3=Thread[eq2==eq1], trPk=Solve[eq3]//First, u[x_,y_]=u[x,
y]/.trPk} Map[Simplify, {u[0,y],u[m,y],u[x,0],u[x,m]}]
{bc1=u[0,y]==0, bc2=u[m,y]==f[y], bc3=u[x,0]==0, bc4=u[x,m]==0}
{p0=Solve[bc3,p[0]]//First, a2=a[2]->1,
p10=Solve[bc2/.p0/.a2,p[1]]//First,
 a1=Solve[bc4/.p0/.p10/.a2,a[1]]//Simplify//First, p1=p10/.a1//Simplify}
{bcs={u[0,y],u[m,y],u[x,0],u[x,m]},
trCoeff=Flatten[{a1,p0,p1,a2}]//Simplify} testBCs=Map[#1/.trCoeff&,
bcs]//Simplify {pde1=pde/.trCoeff//Simplify, a0=a[0]->0,
pde2=pde1/.a0//FullSimplify,
 a3=Solve[pde2,a[3]]//First, pde/.a3/.a0/.trCoeff//FullSimplify}
trAP=Flatten[{a0,a1,a2,a3,p0,p1}]//Simplify
{testPDE=pde/.trAP//Simplify, solF=u[x, y]/.trAP//FullSimplify}
```

23.2.6 Constructing Analytical Solutions of Initial-Boundary Value Problems

In Mathematica, by applying the predefined function DSolve, it is possible to find analytical solutions for *restricted classes of linear PDEs* (e.g., linear first-order PDEs and linear homogeneous second-order PDEs of the form $au_{xx}+bu_{xy}+cu_{yy}=0$). However, in this section we solve some specific boundary value problems associated with linear PDEs that do not belong to such restricted classes.

If a mathematical problem consists in finding an unknown solution of a PDE (defined at an appropriate domain) satisfying appropriate supplementary conditions (initial and boundary conditions), this is known as an *initial-boundary value problem*. The boundary conditions describe the unknown function at prescribed boundary points. The initial condition prescribes the unknown function at a certain initial time t (e.g., $t=t_0$, $t=0$).

In Mathematica, it is possible to construct exact solutions for a restricted number of linear partial differential equations with the aid of the predefined function DSolve (e.g., single first-order equations and some linear homogeneous second-order equations). However, we cannot solve initial-boundary value problems with the aid of the function DSolve even for simple linear first-order or second-order equations, e.g.,

```
pde1={D[u[x,t],t]==D[u[x,t],x], u[x,0]==f[x], u[0,t]==g[t]}
pde2={D[u[x,t],t]==D[u[x,t],{x,2}],
u[x,0]==f[x],(D[u[x,t],t]/.t->0)==g[x],u[0,t]==0}
{DSolve[pde1,u,{x,t}], DSolve[pde2,u,{x,t}]}
```

In this section, we solve a specific initial-boundary value problem associated with linear PDEs that do not belong to such restricted classes.

Example 23.16. *Linear telegraph equation. Initial-boundary value problem (IBVP).* Consider the initial-boundary value problem for the linear telegraph equation

$$u_{xx} = u_{tt} + u_t - u, \qquad x > 0, \quad t > 0,$$
$$u(x,0) = e^x, \quad u_t(x,0) = -2e^x, \quad u(0,t) = e^{-2t}, \quad u_x(0,t) = e^{-2t}, \quad x \geq 0, \quad t \geq 0.$$

It is not possible to solve such problems automatically via the predefined function `DSolve`, but by using separation of variables [for more details, see Shingareva and Lizárraga-Celaya (2011)], we can obtain the solution of this initial-boundary value problem (`SolF`)

$$u(x,t) = e^{-2t+x}$$

and verify that this solution is an exact solution of the linear telegraph equation (`testPDE`, `testIC1`, `testIC2`, `testBC1`, `testBC1`) as follows:

```
tr1=w[x,t]->phi[x]*psi[t];  trC=c->1;
tr1D[v_]:=Table[D[tr1,{v,i}],{i,1,2}];
pde1[u_]:=D[u,{x,2}]==D[u,{t,2}]+D[u,t]-u;
{eq2=Expand[pde1[w[x,t]]],
eq3=PowerExpand[eq2/.tr1/.tr1D[t]/.tr1D[x]],
 eq4=Factor[Thread[eq3/(phi[x]*psi[t]),Equal]], eq5=Map[Simplify,eq4]}
{solPhi=DSolve[eq5[[1]]==c^2,phi,x],
solPsi=DSolve[eq5[[2]]==c^2,psi,t]}
solU=(phi[x]/.solPhi)*(psi[t]/.solPsi)//Simplify//First
Map[Simplify, {ic1=(solU/.t->0)==Exp[x],
ic2=(D[solU,t]/.t->0)==-2*Exp[x],
 bc1=(solU/.x->0)==Exp[-2*t], bc2=(D[solU,x]/.x->0)==Exp[-2*t]}]
{sys1=Map[#/.trC&,{ic1,ic2,bc1,bc2}]//Simplify,
trCoeff=Solve[sys1,{C[1],C[2]}]} solF=solU/.trCoeff/.trC
{testPDE=pde1[solF]//FullSimplify, testIC1=(solF/.t->0)==ic1[[2]],
 testIC2=(D[solF,t]/.t->0)==ic2[[2]], testBC1=(solF/.x->0)==bc1[[2]],
 testBC2=(D[solF,x]/.x->0)==bc2[[2]]}
```

23.2.7 Constructing Analytical Solutions of Systems of Linear PDEs

In the computer algebra system Mathematica, analytical (symbolic) solutions of restricted classes of systems of linear partial differential equations can be found with the aid of the predefined function `DSolve`:

```
DSolve[PDESys,u, {x1,x2}]     DSolve[PDEsys,u[x1,x2],{x1,x2}]
DSolve[PDEsys,{u1[x1,...,xn],...,un[x1,...,xn]},{x1,...,xn}]
```

`DSolve`, finding analytical solutions of a system of PDEs for the function `u` with two independent variables `x1,x2` (“pure function” solution).

`DSolve`, finding analytical solutions of a system of PDEs for the function `u[x1,x2]` with independent variables `x1,x2`.

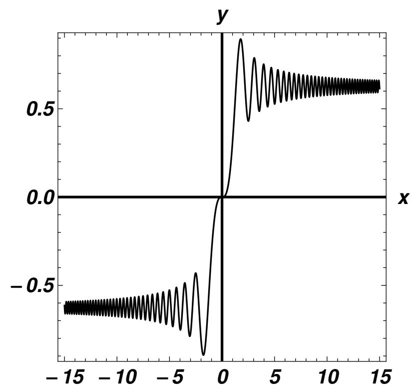


Figure 23.2 Exact solution of the linear system $u_x = \sin(x^2) + y$, $u_y = \cos(y^2) + x$ ($y = 0$, $C_1 = 0$).

`DSolve`, finding analytical solutions of an uncoupled system of PDEs for the functions `u1[x1,...xn],...,un[x1,...xn]` with independent variables `x1,...xn`.

The function `DSolve` can solve some simple classes of systems of linear partial differential equations, e.g., first-order systems with two independent and one dependent variable and first-order uncoupled systems with two or more independent variables and two or more dependent variables. By applying Mathematica predefined functions, we will show how to find analytical solutions of such linear first-order systems.

Example 23.17. *Linear first-order system. Exact solution.* Considering the linear first-order system with two independent variables and one dependent variable,

$$u_x = \sin(x^2) + y, \quad u_y = \cos(y^2) + x,$$

and applying Mathematica predefined functions, we solve this system and verify that the solution obtained (`sol1`),

$$u(x,y) = xy + C_1 + \sqrt{\pi/2}\,\text{FresnelC}\left[\sqrt{2/\pi}y\right] + \sqrt{\pi/2}\,\text{FresnelS}\left[\sqrt{2/\pi}x\right]$$

is an exact solution of this system as follows:

```
sys1={D[u[x,y],x]==Sin[x^2]+y, D[u[x,y],y]==Cos[y^2]+x}
sol1=DSolve[sys1,u[x,y],{x,y}]//First
Plot[sol1[[1,2]]/.y->0/.C[1]->0,{x,-15,15}]
```

The exact solution of this system for $y = 0$ and $C_1 = 0$ is presented in Fig. 23.2.

By applying the elimination procedure, we present an alternative (equivalent) approach to solving this linear system as follows:

```
Off[Eliminate::ifun]; sys={D[u[x,y],x]==Sin[x^2]+y,
D[u[x,y],y]==Cos[y^2]+x}; {v=Eliminate[sys,y],
sol1=DSolve[v[[2]],u[x,y],{x,y}]//First} {u[x_,y_]=sol1[[1, 2]],
sys1=Simplify[sys]} {sol2=DSolve[sys1[[2]],C[1][y],y]//First,
u[x,y]/.sol2}
```

Example 23.18. *Linear first-order system. Exact solution.* Considering the linear first-order uncoupled system with two independent variables and two dependent variables,

$$u_t + u_x = \sin t \cos x, \quad v_t + 2v_x = \sin x \sin t,$$

and applying Mathematica predefined functions, we solve this system of PDEs and verify that the solution obtained (`sol3`),

$$u(x,t) = -\frac{1}{4}\cos(x+t) + F_1(t-x) + \frac{1}{2}x\sin(t-x), \quad v(x,t) = F_2(t-x/2) - \frac{1}{2}\sin(t-x) - \frac{1}{6}\sin(t+x),$$

is an exact solution of this linear system as follows:

```
sys1={D[u[x,t],t]+D[u[x,t],x]==Sin[t]*Cos[x],D[v[x,t],t]+2*D[v[x,t],x]==Sin[x]*Sin[t]}
sol1=DSolve[sys1,{u,v},{x,t}]//First test1=sys1/.sol1//FullSimplify
sol2={sol1[[1]]/.C[1]->F1,sol1[[2]]/.C[1]->F2}
Map[ExpandAll,{sol2[[1,2,2]],sol2[[2,2,2]]}]
```

Since the predefined function `DSolve` represents the two arbitrary functions as `C[1]`, it is necessary to specify new names of the arbitrary functions for each solution function (in our case, `F1` and `F2`).

⊙ *References for Section 23.2*: A. S. Muleshkov, M. A. Golberg, A. H.-D. Cheng, and C. S. Chen (2002), G. W. Bluman, A. F. Cheviakov, and S. C. Anco (2010), I. K. Shingareva and C. Lizárraga-Celaya (2011).

23.3 Analytical Solutions of Mathematical Problems

23.3.1 Constructing Separable Solutions

Separation of variables is one of the most important methods for solving linear PDEs, in which the structure of a PDE allows us to seek *multiplicative separable* or *additive separable* exact solutions; e.g., $u(x,t) = \phi(x) \circ \psi(t)$ (where the multiplication or addition is denoted by $\circ$). Numerous problems in linear partial differential equations can be solved by separation of variables. This method was recently generalized [e.g., see Galaktionov (1990, 1995), Polyanin and Zhurov (1998), Polyanin and Manzhirov (2007)] and nowadays it is a classical method in mathematics and physics.

Let us start from first-order linear equations that can be solved by separation of variables without considering Fourier series.

Example 23.20. *First-order linear equation. Separable solution. Cauchy problem.* Consider the first-order linear PDE and the Cauchy data

$$au_x + bu_y = 0, \quad u(0,y) = \alpha e^{-\beta y},$$

where $a, b \in \mathbb{R}$ are parameters. By applying separation of variables and by seeking exact solutions in the form $u(x,t) = \phi(x)\psi(y)$, we arrive at the following equations (`Eq51`, `Eq52`):

$$\frac{a\phi'_x}{b\phi(x)} = -C_1, \quad \frac{\psi'_y}{\psi(y)} = C_1.$$

```
tr1=w[x,y]->phi[x]*psi[y]; pde1[u_]=a*D[u[x,y],x]+b*D[u[x,y],y]==0;
pde1[u] {ic1=u[0,y]==alpha*Exp[-beta*y], eq2=pde1[w]//Expand}
eq3=eq2/.w[x,y]->tr1/.D[tr1,x]/.D[tr1,y]//Expand
eq4=Thread[eq3/tr1[[2]]/b,Equal]//Expand {eq51=eq4[[1,1]]==-C[1],
eq52=eq4[[1,2]]==C[1]}
```

Then we seek exact solutions of these equations as follows:

```
{sol1=DSolve[eq51,phi[x],x]//First,
sol2=DSolve[eq52,psi[y],y]//First} genSol=Simplify[tr1/.sol1/.sol2]
{trC3=C[2]^2->C[3], genSol1=genSol/.trC3}
{eq8=(genSol1[[2]]/.x->0)==ic1[[2]],
trC13={C[1]->-beta,C[3]->alpha}} solF=genSol1/.trC13//Factor
test1=pde1[w]/.solF/.D[solF,x]/.D[solF,y]//Expand
test2=(solF[[2]]/.x->0)==ic1[[2]]
```

We find that $\phi(x) = C_2 e^{-C_1 bx/a}$ (`sol1`), $\psi(y) = C_2 e^{C_1 y}$ (`sol2`), and the exact solution acquires the form (`solF`)

$$u(x,y) = \alpha e^{-\beta(ay-bx)/a}.$$

Separation of variables combined with the *linear superposition principle* can be applied for solving a large class of initial-boundary value problems for linear partial differential equations. According to the method, the partial differential equation is reduced to two ordinary differential equations (for $\phi(x)$ and $\psi(y)$). Similar ideas can be applied to equations in several independent variables. This method is also known as the *Fourier method* or the *eigenfunction expansion method*.

Example 23.21. *Linear wave equation. Separable solution. Fourier series. Initial-boundary value problem.*

Consider a second-order linear hyperbolic PDE with the initial and boundary conditions

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \ \ (0 \le x \le L), \quad u(0,t) = 0, \quad u(L,t) = 0 (t \ge 0),$$

where $c^2 = \frac{1}{4}$, $L = 1$, $f(x) = 0$, $g(x) = \sin(x) - \sin(3\pi x)$. This problem describes a vibrating string (with constant tension T and density ρ, $c^2 = T/\rho$) stretched along the x-axis from 0 to L and fixed at the endpoints. The initial displacement $f(x)$ is zero, and the initial velocity is $g(x)$.

By applying separation of variables, i.e., by seeking the exact solution in the form $u(x,t) = \phi(x)\psi(t)$, we obtain the two ODEs

$$\phi''_{xx} - \lambda\phi = 0, \quad \psi''_{tt} - \lambda c^2 \psi = 0.$$

Then, separating the boundary conditions, we solve the eigenvalue problem and obtain the solution

$$u(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right),$$

which satisfies the original wave equation and the boundary conditions. Since the wave equation is linear and homogeneous, by the superposition principle, the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

is a solution as well. We assume the following properties of the solution: it converges and is twice continuously differentiable with respect to x and t. Since each term of the series satisfies the boundary conditions, it follows that the series satisfies these conditions. From the two initial conditions, we can determine the constants A_n and B_n. By differentiating the solution with respect to t, we obtain

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right).$$

These equations are satisfied if $f(x)$ and $g(x)$ can be represented by *Fourier sine series*. According to Fourier theory, the formulas for the coefficients A_n and B_n read

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Finally, we obtain the analytical and graphical solutions of the initial-boundary value problem as follows:

```
{lL=1, c=1/4, nN=10, ic1=Sin[x]-Sin[3*Pi*x]}
u[x_,t_]=Sum[(an*Cos[n*Pi*c*t/lL]+bn*Sin[n*Pi*c*t/lL])*Sin[n*Pi*x/lL],{n,1,nN}];
{u[x,t], an1=Solve[u[x,0]==0,an]//First,
eqIC=D[u[x,t],t]/.t->0//Expand}
eq1=bn*Pi*Sum[n*Sin[n*Pi*x],{n,1,nN}]//Expand
cC=Table[Select[eq1,MemberQ[#1,Sin[n*Pi*x]]&],{n,1,nN}]
iIC=Table[Select[ic1,MemberQ[#1,Sin[n*Pi*x]]&],{n,1,nN}]
bB=Table[Solve[cC[[n]]==iIC[[n]],bn]//First,{n,1,nN}]
{sol1=u[x,t]/.an1,
sol2=DeleteCases[Table[sol1[[i]]/.bB[[i]],{i,1,nN}],0]}
U[x_,t_]=sol2//First; U[x,t]
Animate[Plot[U[x,t],{x,0,lL},PlotRange->{-0.2,0.2},PlotStyle->Hue[0.7]],{t,0.1,5}]
gG=Partition[Table[Plot[Evaluate[U[x,t]],{x,0,lL},Frame->True,ImageSize->300,
 PlotRange->{-0.2,0.2},DisplayFunction->Identity],{t,0.1,5,0.4}],4];
GraphicsGrid[gG]
```

Example 23.22. *Linear heat equation. Separable solution. Fourier series. Initial-boundary value problem.*

Consider the second-order linear parabolic PDE with the initial and boundary conditions

$$u_t = k u_{xx}, \qquad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $k = 1/30$, $L = 1$, $f(x) = \sin^4(\pi x)$. This problem describes a homogeneous rod (of length L). We assume that the rod is sufficiently thin (i.e., the heat is distributed equally over the cross-section at time t), the surface of the rod is insulated (i.e., there is no heat loss through the boundary), and the temperature distribution of the rod is given by the solution of the given initial-boundary value problem.

By applying separation of variables, i.e., by seeking the exact solution in the form $u(x,t) = \phi(x)\psi(t)$, we obtain the two ODEs

$$\phi''_{xx} + \lambda^2\phi = 0, \quad \psi'_t + \lambda^2 k\psi = 0.$$

Then, by separating the boundary conditions, we solve the eigenvalue problem and obtain the solution

$$u(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt},$$

which satisfies the original heat equation and the boundary conditions. Since the heat equation is linear and homogeneous, by the superposition principle, the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt}$$

is also a solution. This solution satisfies the initial condition if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients A_n are the Fourier coefficients,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Finally, by recursively computing the Fourier coefficients and the desired solution, we obtain a formal series solution of the initial-boundary value problem and visualize it as follows:

```
SetOptions[Plot,ImageSize->300,PlotStyle->{Hue[0.7],Thickness[0.01]},
 PlotRange->{0,1}]; f[x_]:=Sin[Pi*x]^4; {lL=1,k=1/30,nN=20,nNT=5}
{aA=Table[0,{i,1,nN}], u=Table[0,{i,1,nN}]}
Do[aA[[i]]=(2./lL)*Integrate[f[x]*Sin[i*Pi*x/lL],{x,0,lL}]//N,{i,1,nN}];
aA[[1]] u[[1]]=aA[[1]]*Exp[-(Pi/lL)^2*k*t]*Sin[Pi*x/lL]
Do[u[[i]]=u[[i-1]]+aA[[i]]*Exp[-(i*Pi/lL)^2*k*t]*Sin[i*Pi*x/lL]//N,{i,2,nN}]
U[x_,t_]=u[[nN]]; U[x,t]
Animate[Plot[Evaluate[U[x,t]],{x,0,lL}],{t,0,nNT}]
gG=Partition[Table[Plot[Evaluate[U[x,t]],{x,0,lL},Frame->True,
 DisplayFunction->Identity],{t,0,nNT,0.4}],4]; GraphicsGrid[gG]
```

23.3.2 Constructing Analytical Solutions via Integral Transform Methods

Integral transform methods are the most important methods for constructing analytical solutions of mathematical problems (initial and/or boundary value problems) described by linear partial differential equations.

The main idea of the methods consists in transforming the original mathematical problem to a simpler form whose solutions can be obtained and inverted (by applying inverse integral transforms) for representing the solutions in terms of the original variables. The formal definitions of the most important integral transforms and their general forms and properties can be found in Debnath (2007).

In Mathematica, integral transforms (Fourier, Laplace, etc.) can be studied with the aid of the predefined functions `FourierTransform`, `InverseFourierTransform`, `LaplaceTransform`, `InverseLaplaceTransform`, etc. Let us solve an initial-boundary value problem for the linear wave equation.

Example 23.23. *Linear wave equation. Laplace transform. Initial-boundary value problem.* Consider the second-order linear hyperbolic PDE with the initial and boundary conditions

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x < \infty, \quad t > 0,$$
$$u(x,0) = h(x), \quad u_t(x,0) = g(x) \ \ (0 < x < \infty), \quad u(0,t) = a\sin t, \quad u(x,t) \to 0 \ \text{ as } \ x \to \infty,$$

where $h(x) = 0$, $g(x) = 0$, and a is constant ($a \in \mathbb{R}$). This problem describes transverse vibrations of a semiinfinite string. The initial displacement $h(x)$ and the initial velocity $g(x)$ are zero (i.e., the string is at rest in its equilibrium position).

Let $U(x,s)$ be the Laplace transform of $u(x,t)$. By transforming the equation of motion and by substituting the initial conditions, we obtain (`eq3`)

$$s^2U = c^2U_{xx}.$$

The solution of this ordinary differential equation is (`sol`)

$$U = C_1e^{sx/c} + C_2e^{-sx/c}.$$

By transforming the boundary conditions, we obtain (`bC2`)

$$U(0,s) = \frac{a}{1+s^2}, \quad \lim_{x\to 0} U(x,s) = 0,$$

According to the second condition, we have $C_2 = 0$, and by applying the first condition, we obtain (`sol3`)

$$U(x,s) = \frac{a}{1+s^2}\, e^{-sx/c}.$$

By applying the inverse Laplace transform, we obtain the solution (`solF`)

$$u(x,t) = aH(t-x/c)\sin(t-x/c),$$

where H is the Heaviside unit step function. This solution represents a wave propagating at a velocity c with the characteristic $x = ct$.

Finally, we obtain the displacement of the semiinfinite string as follows:

```
eq1=D[u[x,t],{t,2}]==c^2*D[u[x,t],{x,2}]
eq2=LaplaceTransform[eq1,t,s]/.{LaplaceTransform[u[x,t],t,s]->uU[x],
 LaplaceTransform[D[u[x,t],{x,2}],t,s]->uU''[x]}
ic1={u[x,0]->0,(D[u[x,t],t]/.{t->0})->0}
ic2=Map[LaplaceTransform[#,t,s]&,ic1,{2}] {eq3=(eq2/.ic1),
bc1=u[0,t]==a*Sin[t]}
bc2=LaplaceTransform[bc1,t,s]/.{LaplaceTransform[u[0,t],t,s]->uU[0]}
{sol=DSolve[eq3,uU[x],x], sol1=uU[x]/.sol}
l1=Limit[sol1[[1,1]],x->Infinity,Assumptions->{c>0,s>0}]//Simplify
l2=Limit[sol1[[1,2]],x->Infinity,Assumptions->{c>0,s>0}]//Simplify
{sol2=sol1/.{C[1]->0}, sol3=sol2/.{C[2]->bc2[[2]]},
uU1=sol3//ExpandAll}
solF=Assuming[{s>0,c>0,x>0,t>0},InverseLaplaceTransform[uU1,s,t]]
```

Remark. It should be noted that if we consider the more general boundary condition $u(0,t) = af(t)$ (instead of $u(0,t) = a\sin t$), then we can find the exact solution (`SolF`)

$$u(x,t) = aH\left(t - \frac{x}{c}\right) f\left(t - \frac{x}{c}\right)$$

in Mathematica 6, 7, 8 and we cannot find this form of solution in Mathematica 9, 10. The corresponding Mathematica script is

```
eq1=D[u[x,t],{t,2}]==c^2*D[u[x,t],{x,2}]
eq2=LaplaceTransform[eq1,t,s]/.{LaplaceTransform[u[x,t],t,s]->u1[x],
 LaplaceTransform[D[u[x,t],{x,2}],t,s]->u1''[x]}
ic1={u[x,0]->0,(D[u[x,t],t]/.{t->0})->0}
```

```
ic2=Map[LaplaceTransform[#,t,s]&,ic1,{2}] {eq3=(eq2/.ic1),
bc1=u[0,t]==a*f[t]}
bc2=LaplaceTransform[bc1,t,s]/.{LaplaceTransform[u[0,t],t,s]->u1[0],
 LaplaceTransform[f[t],t,s]->f1[s]}
{sol=DSolve[eq3,u1[x],x], sol1=u1[x]/.sol}
l1=Limit[sol1[[1,1]],x->Infinity,Assumptions->{c>0,s>0}]//Simplify
l2=Limit[sol1[[1,2]],x->Infinity,Assumptions->{c>0,s>0}]//Simplify
{sol2=sol1/.{C[1]->0}, sol3=sol2/.{C[2]->bc2[[2]]}}
u2=sol3/.f1[s]->LaplaceTransform[f[t],t,s]
solF=InverseLaplaceTransform[u2,s,t]
```

23.3.3 Constructing Analytical Solutions in Terms of Green's Functions

It is well known that the *linear superposition principle* is one of the most important methods for representing solutions of linear PDEs with initial and/or boundary conditions in terms of *eigenfunctions* or *Green's functions*.[4]

As was seen earlier (Section 23.3.1), the *infinite series representation* of solutions of mathematical problems (involving linear PDEs) can be obtained by applying the *eigenfunction expansion method*. The *integral representation* of solutions can be obtained by applying the *Green's function method*. Integral representations have some advantages over infinite series representations (e.g., the description of the general analytical structure of the solution and the evaluation of a solution).

Consider the linear Poisson equation in a volume V with surface S on which the Dirichlet boundary conditions are imposed. The Green's function $\mathcal{G}(X,X_0)$ ($X=(x,y,z)$ and $X_0=(x_0,y_0,z_0)$) associated with the boundary value problem is a function of two variables X (the position vector) and X_0 (a fixed location) defined as the solution to

$$\nabla^2\mathcal{G}(X,X_0)=\delta(X-X_0)\ \text{ in }\ V; \qquad \mathcal{G}(X,X_0)=0\ \text{ on }\ S.$$

If the volume V is the entire space, then the Green's function is called the *fundamental solution*. Since $\mathcal{G}(X,X_0)$ is a symmetric function, $\mathcal{G}(X,X_0)=\mathcal{G}(X_0,X)$ (under interchange of the arguments), this fact can serve to verify that $\mathcal{G}(X,X_0)$ is correctly calculated.

Example 23.24. *Linear Laplace equation. Green's function. Boundary value problem.*

Consider the Dirichlet boundary value problem for the Laplace equation on the semiinfinite plane $V=\{y>0\}$:

$$\nabla^2u(x,y)=u_{xx}+u_{yy}=0\ \text{ in }\ V, \qquad u(x,y)=f(x)\ \text{ on }\ S,$$

where $V=\{y>0\}$ and $S=\{y=0\}$. By applying the *method of images*, i.e., by seeking a Green's function $\mathcal{G}(X,X_0)$ such that, in V,

$$\mathcal{G}(X,X_0)=v(X,X_0)+w(X,X_0), \quad \text{where} \quad \nabla^2v(X,X_0)=-\delta(X-X_0)\ \text{ and }\ \nabla^2w(X,X_0)=0,$$

we will construct the Green's function and find the solution of the boundary value problem. Here $X=(x,y)$, $X_0=(x_0,y_0)$, the function $v(X,X_0)$ is the *free space Green's function* (does not depend on the boundary conditions), the function $w(X,X_0)$ satisfies the Laplace equation and the boundary

[4] Auxiliary functions now known as *Green's functions* were first introduced by George Green in 1828 [see Green (1828)].

conditions (and is regular at $X = X_0$), i.e., $\nabla^2 w(X,X_0) = 0$ in V and $w(X,X_0) = -v(X,X_0)$ (i.e., $\mathcal{G}(X,X_0) = 0$) on S for the Dirichlet boundary conditions.

It is well known that the 2D free space function $v(X,X_0)$ is

$$v(X,X_0) = -\frac{1}{4\pi}\ln\left((x-x_0)^2+(y-y_0)^2\right).$$

If to $v(X,X_0)$ we add the function

$$w(X,X_0) = \frac{1}{4\pi}\ln\left((x-x_0)^2+(y+y_0)^2\right),$$

which satisfies the Laplace equation $\nabla^2 w(X,X_0) = 0$ in V and is regular at $x = x_0$ and $y = y_0$, then we obtain the Green's function (`g11`)

$$\mathcal{G}(X,X_0) = v(X,X_0)+w(X,X_0) = -\frac{1}{4\pi}\ln\left(\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right).$$

Then, by setting $y = 0$, we have $\mathcal{G}(X,X_0) = 0$ (`test1`). The solution of the boundary value problem is

$$u(x_0,y_0) = -\int_S f(x)\frac{\partial\mathcal{G}}{\partial n}\,dS.$$

By computing $\frac{\partial\mathcal{G}}{\partial n}$ for the boundary $y = 0$, we obtain the derivative (`dGn`)

$$\left.\frac{\partial\mathcal{G}}{\partial n}\right|_S = -\left.\frac{\partial\mathcal{G}}{\partial y}\right|_{y=0} = -\frac{y_0}{\pi[(x-x_0)^2+y_0^2]}$$

and the solution (`solF`) of the boundary value problem

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}.$$

Finally, we verify that the Green's function obtained is symmetric and visualize it as follows:

```
{v=-1/(4*Pi)*Log[(x-x0)^2+(y-y0)^2],
w=1/(4*Pi)*Log[(x-x0)^2+(y+y0)^2]} {z1=ExpandAll[v+w],
gG0=Simplify[z1,{z1[[1,3,1]]>0}]}
z2=Simplify[ExpandAll[gG0[[3]]*(-1)//PowerExpand],{z1[[1,3,1]]>0}]
z3=gG0[[1]]*gG0[[2]]*(-1)*z2
z33=(z3[[3,1,1]]//FullSimplify)*(z3[[3,1,2]]//FullSimplify)
{g11=z3[[1]]*z3[[2]]*Log[z33], test1=g11/.y->0,
dGn=-D[g11,y]/.y->0}
sol1=u[x0,y0]=-Integrate[dGn*f[x],{x,-Infinity,Infinity}]
solF=sol1/.x->s/.x0->x/.y0->y g2[x_,y_,x0_,y0_]=g11;
{g2[x,y,x0,y0], g2[x0,y0,x,y]}
Simplify[g2[x,y,x0,y0]-g2[x0,y0,x,y]//FullSimplify,{x>x0,y>y0}]
Simplify[g2[x,y,x0,y0]-g2[x0,y0,x,y]//FullSimplify,{x0>x,y0>y}]
Plot3D[g2[x,y,1,1],{x,-10,10},{y,0,10},PlotPoints->{100,100},
 PlotRange->All,BoxRatios->{3,3,2},ViewPoint->{2,1,2}]
ContourPlot[g2[x,y,1,1],{x,-5,5},{y,0,8},PlotPoints->150,Contours->20]
```

Green's functions can also be constructed by applying Laplace transforms [see Cole et al. (2011)], the eigenfunction expansion method [see Debnath (2007), Polyanin and

Manzhirov (2007), Polyanin (2002)], and conformal mappings of the complex plane (for solving 2D problems).

Now let us consider an extension of the theory of Green's functions, namely, the construction of modified Green's functions and solutions of initial-boundary value problems.

Example 23.25. *Linear Klein–Gordon equation. Modified Green's function. Initial-boundary value problem.*

Consider the initial-boundary value problem for the Klein–Gordon equation with the following initial conditions and Neumann boundary conditions:

$$u_{tt} = a^2 u_{xx} - bu \quad x_1 \le x \le x_2,$$
$$u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x), \quad u_x(0,t) = g_1(t), \quad u_x(L,t) = g_2(t),$$

where a and b are real parameters ($a > 0$, $b > 0$), $x_1 = 0$, $x_2 = L$, $f_1(x)$=1, $f_2(x)$=0, $g_1(t)$=1, and $g_2(t)$=0. The linear Klein–Gordon equation is a special case of the equation

$$s(x)u_{tt} = (p(x)u_x)_x - q(x)u + \phi(x,t),$$

where $s(x) = 1$, $p(x) = a^2$, $q(x) = b$, and $\Phi(x,t) = 0$. By applying the *eigenfunction expansion method*, we will construct the modified Green's function. The corresponding Sturm–Liouville problem has the form

$$a^2\phi''_{xx} + (\lambda - b)\phi = 0, \qquad \phi'_x = 0 \text{ at } x = 0, \quad \phi'_x = 0 \text{ at } x = L.$$

By solving this eigenvalue problem, we find the eigenvalues and the corresponding eigenfunctions (`eVal`, `eFun`)

$$\phi_{n+1}(x) = \cos\left(\frac{\pi n x}{L}\right), \quad \lambda_{n+1} = b + \left(\frac{\pi n a}{L}\right)^2, \quad n = 0, 1, \ldots.$$

According to Polyanin (2002),

$$\mathcal{G}(x,\xi,t) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)\sin(t\sqrt{\lambda_n})}{\|\phi_n\|^2\sqrt{\lambda_n}},$$

where $\|\phi_n\|^2 = \int_{x_1}^{x_2} s(x)\phi_n^2(x)\,dx$, we construct the modified Green's function (`g1`)

$$\mathcal{G}(x,\xi,t) = \frac{\sin(t\sqrt{b})}{L\sqrt{b}} + \sum_{n=1}^{\infty} \frac{2}{L} \frac{\cos(\pi n x/L)\cos(\pi n \xi/L)\sin(t\sqrt{b + (\pi n a/L)^2})}{\sqrt{b + (\pi n a/L)^2}},$$

i.e., a spectral representation of the Green's function for this problem. To visualize the Green's function (see Fig. 23.3), we use finitely many terms of the series (`gAppr`).

```
SetOptions[Plot3D,Boxed->True,PlotPoints->{100,100},PlotRange->All,BoxRatios->{3,3,2},
 ViewPoint->{2,1,2}]; SetOptions[ContourPlot,PlotPoints->100,Contours->20];
s[x_]= 1; p[x_]=a^2; q[x_]=b; pPhi[x_,t_]=0; f1[x_]=1; f2[x_]=0;
g1[t_]=1; g2[t_]=0;
{lL1=1,a1=1,b1=1,t1=10,x1=0,x2=lL1,Inf=29,Inf1=3}
assumps={a>0,lL>0,n\[Element] Integers,n>0}
sol1=DSolve[a^2*D[phi[x],{x,2}]+(lambda-b)*phi[x]==0,phi[x],x]//ExpToTrig//First
sol11=FullSimplify[sol1[[1,2,1]]+sol1[[1,2,3]],{lambda>b}]//ExpToTrig
sol12=FullSimplify[sol1[[1,2,2]]+sol1[[1,2,4]],{lambda>b}]//ExpToTrig
sol2=sol11[[1]]+sol12[[2]]/(-I) phi[x_]=sol2; {phi[x],
bc1={(D[phi[x],x]/.x->0)==0,(D[phi[x],x]/.x->lL)==0}}
```

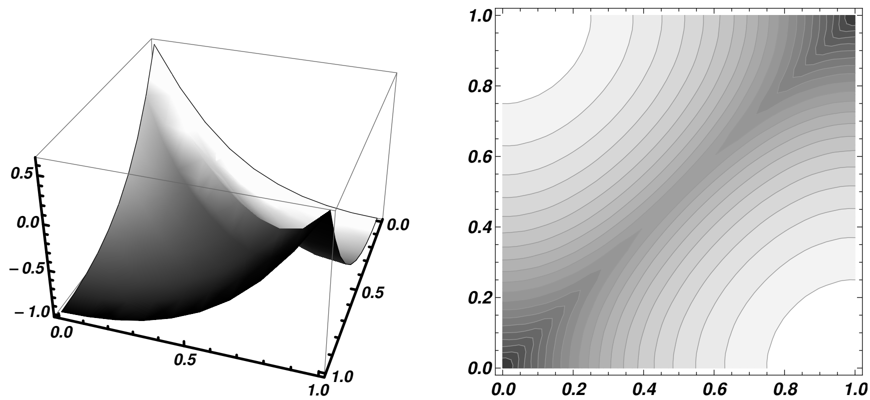


Figure 23.3 The modified Green's function $\mathcal{G}(x,\xi,t)$ for the initial-boundary value problem ($n=29$).

```
{mCoef=Normal[CoefficientArrays[bc1,{C[2],C[1]}]],
mCoef[[2]]//MatrixForm} charEq=Det[mCoef[[2]]]==0//Factor
sols=Reduce[charEq && lL>0 && a>0 && b>0 && lambda>0 &&
lambda>b,lambda]
sols1=(sols[[2,2,5]]/.C[1]->0/.Pi->Pi*n)//FullSimplify
{trlambda=sols1/.Equal->Rule, mn=mCoef/.trlambda}
{mn1=Simplify[mn[[2]],assumps], ns=NullSpace[mn1]}
funs=Simplify[phi[x]/.C[1]->ns[[1,2]]/.C[2]->ns[[1,1]]/.trlambda,assumps]
{eVal[n_]=trlambda[[2]], eFun[n_,x_]=funs, eVal[n], eFun[n,x]}
nEFun=Simplify[Integrate[s[x]*eFun[n,x]^2,{x,0,lL}],n\[Element]
Integers] nEFun0=Integrate[s[x]*eFun[0,x]^2,{x,0,lL}]
gTerm0=eFun[0,x]*eFun[0,xi]*Sin[t*Sqrt[eVal[0]]]/Sqrt[eVal[0]]/nEFun0
gIntnd=eFun[n,x]*eFun[n,xi]*Sin[t*Sqrt[eVal[n]]]/Sqrt[eVal[n]]/nEFun
gAppr[x_,xi_,t_,lL_,a_,b_]=gTerm0+Sum[gIntnd,{n,1,Inf}];
gAppr[x,xi,t,lL1,a1,b1]
Plot3D[Evaluate[gAppr[x,xi,t1,lL1,a1,b1]],{x,0,lL1},{xi,0,lL1}]
ContourPlot[Evaluate[gAppr[x,xi,t1,lL1,a1,b1]],{x,0,lL1},{xi,0,lL1}]
```

Finally, by constructing the solution of the IBVP according to Polyanin (2002),

$$\begin{aligned} u(x,t) &= \int_0^t \int_{x_1}^{x_2} \Phi(\xi,\tau)\mathcal{G}(x,\xi,t-\tau)\,d\xi\,d\tau + \frac{\partial}{\partial t}\int_{x_1}^{x_2} s(\xi) f_1(\xi)\mathcal{G}(x,\xi,t)\,d\xi \\ &+ \int_{x_1}^{x_2} s(\xi) f_2(\xi)\mathcal{G}(x,\xi,t)\,d\xi + p(x_1)\int_0^t g_1(\tau)(-\mathcal{G}(x,x_1,t-\tau))\,d\tau \\ &+ p(x_2)\int_0^t g_2(\tau)\mathcal{G}(x,x_2,t-\tau)\,d\tau, \end{aligned}$$

we obtain the solution of the given problem and visualize it by using finitely many terms of the series (`solF`) for various values of t as follows:

```
gG1=(gTerm0+Sum[eFun[n,x]*eFun[n,xi]*Sin[t*Sqrt[eVal[n]]]/Sqrt[eVal[n]]/nEFun,
 {n,1,Inf1}])/.lL->lL1/.a->a1/.b->b1
gG2[x_,xi_,t_]=gG1;
u=Integrate[Integrate[pPhi[xi,tau]*gG2[x,xi,t-tau],{xi,x1,x2}],{tau,0,t}]+
 D[Integrate[s[xi]*f1[xi]*gG2[x,xi,t],{xi,x1,x2}],t]+Integrate[
 s[xi]*f2[xi]*gG2[x,xi,t],{xi,x1,x2}]+p[x1]*Integrate[g1[tau]*(-gG2[x,x1,t-tau]),
 {tau,0,t}]+p[x2]*Integrate[g2[tau]*gG2[x,x2,t-tau],{tau,0,t}]
```

```
solF[x_,t_]=Collect[u/.a->a1,{Cos[_]}]
Plot3D[Evaluate[solF[x,t]],{x,0,lL1},{t,0,t1}]
ContourPlot[Evaluate[solF[x,t]],{x,0,lL1},{t,0,t1}]
gs=Table[g[i]=Plot[Evaluate[solF[x,i]],{x,0,lL1},Frame->False,AspectRatio->1,
 PlotStyle->{Blue,Thickness[0.005]},PlotPoints->100],{i,0,t1}];
Show[gs,PlotRange->All]
Plot[{solF[x,1],solF[x,3],solF[x,5],solF[x,t1]},{x,0,lL1},
 PlotStyle->{{Blue,Thickness[0.008]},{Blue,Dotted,Thickness[0.005]},
 {Blue,Dashed,Thickness[0.009]},{Blue,DotDashed,Thickness[0.008]}},
 PlotPoints->100,PlotLegends->Placed[{"u[x,1]","u[x,3]","u[x,5]","u[x,10]"},Below]]
```

⊙ *References for Section 23.3*: G. Green (1828), V. A. Galaktionov (1990, 1995), A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2007), L. Debnath (2007), K. D. Cole, J. V. Beck, A. Haji-Sheikh, and B. Litkouhi (2011).

23.4 Numerical Solutions and Their Visualizations

It is well known that there are numerous linear partial differential equations (arising from modeling of real-world problems) and associated mathematical problems (even with simple boundary and/or initial conditions) that cannot be solved analytically. Presently, it is necessary to develop and apply approximation methods. Nowadays, approximation methods are becoming very important and useful in applications due to the increasing development of modern computers, supercomputers, and computer algebra systems. In this section, some of the most important approximation approaches to the solution of linear partial differential equations and associated mathematical problems are discussed with the aid of the computer algebra system Mathematica.

23.4.1 Constructing Numerical Solutions in Terms of Predefined Functions

First, consider the predefined Mathematica functions with the aid of which we can obtain approximate numerical solutions when solving various linear time-based PDE problems. With the aid of the predefined function `NDSolve`, it is possible to obtain approximate numerical solutions of various linear PDE problems (initial-boundary value problems). Note that in Mathematica (Release $\leq$ 10) it is only possible to solve evolution equations numerically by using `NDSolve`. It is possible to impose Dirichlet, Neumann, Robin, or periodic boundary conditions.

```
NDSolve[{PDEs,IC,BC}, DepVars, IndVars, Ops] NDSolve[{PDE,IC,BC},
u, {x,x1,x2},{t,t1,t2},...] NDSolve[{PDEs,IC,BC}, {u1,...,un},
{x,x1,x2},{t,t1,t2},...]
```

`NDSolve`, finding numerical solutions to PDE problems (initial-boundary value problems), where `DepVars` and `IndVars` are the dependent and independent variables, respectively.

Now we schematically write out the numerical solution `sol` with the aid of `NDSolve` and then use it to obtain various visualizations (e.g., `Plot`, `Plot3D`, `Animate`) and numerical values (`numVals`).

```
sol=NDSolve[{PDE,IC,BC},u,{x,x1,x2},{t,t1,t2},ops]
Plot[Evaluate[u[x,tk]/.sol],{x,x1,x2},ops]
numVals=Evaluate[u[xk,tk]/.sol]
Plot3D[Evaluate[u[x,t]/.sol],{x,x1,x2},{t,t1,t2},ops]
Animate[Plot[Evaluate[u[x,t]/.sol],{x,x1,x2},ops],{t,t1,t2},ops]
```

Example 23.26. *Linear wave equation. Initial-boundary value problem. Numerical, graphical, and exact solutions.* We find numerical, graphical, and exact solutions of the following initial-boundary value problem for the linear wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x), \quad u(0,t) = g_1(t), \quad u(L,t) = g_2(t),$$

where $c = 1/10$, $L = 1$, $f_1(x) = A\sin(B\pi x)$, $f_2(x) = 0$, $g_1(t) = 0$, $g_2(t) = 0$, $A = 3$, $B = 2$. First, we construct the numerical solution of this initial-boundary value problem and visualize it as follows:

```
<<DifferentialEquations`InterpolatingFunctionAnatomy`
 {aA=3,bB=2,lL=1,tF=1,s=1/100,nP=100,nN=3,lL1={Red,Blue,Green},lL2={1/8,3/8,5/8}}
SetOptions[Plot,ImageSize->500,PlotRange->{All,{-5,5}},PlotPoints->nP*2,
 PlotStyle->{Blue,Thickness[0.01]}]; SetOptions[Plot3D,ImageSize->500,
 PlotRange->All]; c=N[1/10]; f1[x_]=aA*Sin[bB*Pi*x]; f2[x_]=0; g1[t_]=0; g2[t]=0;
pde1=D[u[x,t],{t,2}]-c^2*D[u[x,t],{x,2}]==0
ic1={u[x,0]==f1[x],(D[u[x,t],t]/.t->0)==f2[x]}
bc1={u[0,t]==g1[t],u[lL,t]==g2[t]}
sol1=NDSolve[Flatten[{pde1,ic1,bc1}],u,{x,0,lL},{t,0,tF},MaxStepSize->s,
 PrecisionGoal->2]; f1=u/.First[sol1];
 Map[Length,InterpolatingFunctionCoordinates[f1]]
Do[g[i]=Plot[Evaluate[u[x,lL2[[i]]]/.sol1],{x,0,lL},PlotStyle->{lL1[[i]],
 Thickness[0.01]}],{i,1,nN}]; Show[Table[g[i],{i,1,nN}]]
numVals=Evaluate[u[0.1,1/2]/.sol1]; Print[numVals];
Plot3D[Evaluate[u[x,t]/.sol1],{x,0,lL},{t,0,tF},ColorFunction->Function[
 {x,y},Hue[x]],ViewPoint->{-1,2,1},ImageSize->500]
ContourPlot[Evaluate[u[x,t]/.sol1],{x,0,lL},{t,0,tF},ColorFunction->Hue,
 ImageSize->300]
Animate[Plot[Evaluate[u[x,t]/.sol1,{x,0,lL}],PlotRange->{-3,3}],{t,0,tF,0.001},
 AnimationRate->0.5]
```

Knowing the exact solution, $u(x,t) = A\sin(B\pi x)\cos(c\pi t)$ (for details, see the previous chapter) of this initial-boundary value problem, we can compare the numerical and exact solutions as follows:

```
uEx[x_,t_]=aA*Sin[bB*Pi*x]*Cos[c*Pi*t]; {trt=t->3/8,trx=x->1/2}
p1=Plot[Evaluate[u[x,t]/.sol1]/.trt,{x,0,lL},PlotStyle->{lL1[[1]],Thickness[0.001]},
 ImageSize->500]; p2=Plot[Evaluate[uEx[x,lL2[[2]]]],{x,0,lL},PlotStyle->{lL1[[2]],
 Thickness[0.001]},ImageSize->500]; Show[{p1,p2}]
numVals=Evaluate[u[1/2,3/8]/.sol1]; Print[numVals];
N[uEx[1/2.,3/8.],16]
Do[p[i+2]=Plot[Evaluate[(u[x,t]/.sol1/.t->lL2[[i]])-uEx[x,lL2[[i]]]],{x,0,lL},
 PlotStyle->{lL1[[i]],Thickness[0.001]},PlotRange->All,ImageSize->500],{i,1,nN}];
Show[Table[p[i+2],{i,1,nN}]]
```

23.4.2 Numerical Methods Embedded in Mathematica

In Mathematica, approximate numerical solutions of various linear PDE problems (initial-boundary value problems) can be obtained with the aid of the predefined function `NDSolve`, which implements the *method of lines*. In addition, it is possible to explicitly specify the method of lines and appropriate suboptions for this method for solving mathematical problems, i.e., the option `Method`, which can be written in general form as follows: `Method->{MethodName, MethodOptions}`. For instance, we can specify `MethodOfLines` and other suboptions (e.g., `MinPoints` for a smaller grid spacing, etc.).

```
NDSolve[{PDE,IC,BC},u,{x,x1,x2},{t,t1,t2},Method->{m,subOps}]
Options[NDSolve`MethodOfLines]
```

`NDSolve,Method` finding numerical solutions to PDE problems by the method of lines with some specific suboptions `subOps`.

`Method` The option `Method` and the most important suboptions:

```
Method->{"MethodOfLines","SpatialDiscretization"->{"TensorProductGrid",
 "MinPoints"->val,"MaxPoints"->val,"MaxStepSize"->val,"PrecisionGoal"->val,
 "DifferenceOrder"->val}}
```

By applying the method of lines, it is possible to solve *restricted classes* of mathematical problems: problems which require a temporal variable with an initial condition (in at least one independent variable). For example, the method of lines cannot solve elliptic equations. For numerous initial-boundary value problems, the method of lines is quite effective, fast, and general (for more details, see the Documentation Center, "Numerical Solution of Partial Differential Equations").

Example 23.27. *Linear heat equation. Initial-boundary value problem. Numerical, graphical, and exact solutions. Method of Lines.* Consider the following initial-boundary value problem for the linear one-dimensional heat equation:

$$u_t = k\,u_{xx}, \quad 0 < x < L, \quad t > 0, \qquad u(x,0) = f(x), \quad u(0,t) = g_1(t), \quad u(L,t) = g_2(t),$$

where $L = 1$, $k = 0.1$, $f(x) = x(1-x)$, $g_1(t) = 0$, and $g_2(t) = 0$. Constructing the numerical and graphical solutions of this problem, we explicitly specify the method of lines with several suboptions as follows:

```
f[x_]=x*(1-x); g1[t_]=0; g2[t_]=0;
{k=0.1,sS=1/300.,nP=100,nNF=4,lL=1,lL1={Red,Blue},lL2={0,0.5},nN=2}
SetOptions[Plot,ImageSize->500,PlotRange->All,PlotPoints->nP*2,
 PlotStyle->{Blue,Thickness[0.01]}];
{pde1=D[u[x,t],t]-k*D[u[x,t],{x,2}]==0,
 ic1={u[x,0]==f[x]}, bc1={u[0,t]==g1[t],u[lL,t]==g2[t]}}
sol1=NDSolve[{pde1,ic1,bc1},u,{x,0,lL},{t,0,nNF},AccuracyGoal->1,
 PrecisionGoal->1,MaxStepSize->sS,MaxSteps->{300,Infinity},
 Method->{"MethodOfLines","SpatialDiscretization"->{"TensorProductGrid"}}];
Do[g[i]=Plot[Evaluate[u[x,lL2[[i]]]/.sol1],{x,0,lL},PlotStyle->{lL1[[i]],
 Thickness[0.01]}],{i,1,nN}]; Show[Table[g[i],{i,1,nN}]]
```

```
numVals1=Evaluate[u[1/2,Pi]/.sol1]; Print[numVals1];
Plot3D[Evaluate[u[x,t]/.sol1],{x,0,lL},{t,0,nNF},ColorFunction->Function[
 {x,t},Hue[x^2+t^2]],BoxRatios->1,ViewPoint->{-1,2,1},ImageSize->300]
ContourPlot[Evaluate[u[x,t]/.sol1],{x,0,lL},{t,0,nNF},
 ColorFunction->Hue,ImageSize->300]
Animate[Plot[Evaluate[u[x,t]/.sol1,{x,0,lL}],PlotRange->{0,0.3}],
 {t,0,nNF},AnimationRate->0.5]
```

Finally, knowing the exact solution, $u(x,t) = x(1-x)\dfrac{e^{2x+0.1t} - e^{0.1t}}{e^{2x}-1}$ (for details, see the previous chapter) of the initial-boundary value problem, we can compare the numerical and exact solutions and analyze the corresponding errors between them as follows:

```
uEx[x_,t_]=x*(1-x)*(Exp[2*x+0.1*t]-Exp[0.1*t])/(Exp[2*x]-1);
{trt=t->0.1, trx=x->0.5, lL3={0.2,0.4}}
p1=Plot[Evaluate[u[x,t]/.sol1]/.trt,{x,0,lL},PlotStyle->{lL1[[1]],
 Thickness[0.001]},ImageSize->500]; p2=Plot[Evaluate[uEx[x,lL2[[2]]]],{x,0,lL},
 PlotStyle->{lL1[[2]],Thickness[0.001]},ImageSize->500]; Show[{p1,p2}]
numVals=Evaluate[u[0.5,0.1]/.sol1]; Print[numVals];
N[uEx[0.5,0.1],16]
Do[p[i+2]=Plot[Evaluate[(u[x,t]/.sol1/.t->lL3[[i]])-uEx[x,lL3[[i]]]],{x,0,lL},
 PlotStyle->{lL1[[i]],Thickness[0.001]},PlotRange->All,ImageSize->500],{i,1,nN}];
Show[Table[p[i+2],{i,1,nN}]]
PaddedForm[Table[Abs[N[(u[x,t]/.sol1/.trt/.x->i)-uEx[i,0.1],16]],
 {i,0.01,lL,0.1}]//TableForm,{16,10}]
```

23.4.3 Numerical Solutions of Initial-Boundary Value Problems

Now let us show the helpful role of computer algebra systems for generating and applying various finite difference approximations for constructing numerical solutions of linear PDEs.

To approximate linear PDEs by finite differences, we have to generate a *mesh* (or grid) in a domain $\mathcal{D}$, e.g., $\mathcal{D} = \{a < x < b, c < t < d\}$. The mesh can be of various types, e.g., rectangular, along the characteristics, polar, etc. We assume (for simplicity) that the sets of lines of the mesh are equally spaced and the dependent variable in a given PDE is $u(x,t)$.

We write h and k for the line spacings and define the *mesh points* as follows: $X_i = a+ih$, $T_j = c + jk$ $(i = 0,\dots,NX,\ j = 0,\dots,NT)$ and $h = (b-a)/NX,\ k = (d-c)/NT$. We calculate approximations to the solution at these mesh points; these approximate points will be denoted by $U_{i,j} \approx u(X_i,T_j)$. We approximate the derivatives in a given equation by finite differences (of various types) and then solve the resulting difference equations.

Example 23.28. *Linear heat equation. Initial-boundary value problem. Forward/backward FD methods. Crank–Nicolson method.* Consider the following initial-boundary value problem for the linear heat equation:

$$u_t = \nu u_{xx}, \quad 0 < x < L, \quad t > 0, \qquad u(x,0) = f(x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $f(x) = \sin(\pi x/L)$, $L = 1$, $\nu = 1$. By applying the forward/backward difference methods and the Crank–Nicolson method, we construct an approximate numerical solution of the given initial-boundary value problem.

We generate the rectangular mesh $X = ih$, $T = jk$ $(i = 0,\dots,NX,\ j = 0,\dots,NT,\ h = L/NX,$ $k = T/NT)$. We denote the approximate solution of $u(x,t)$ at the mesh point (i,j) by $U_{i,j}$.

In the *forward difference method*, the second derivative u_{xx} is replaced by a central difference approximation (CDA) and the first derivative u_t by a forward difference approximation (FWDA). The final FD scheme for the linear heat equation is

$$U_{i,j+1} = (1-2r)U_{i,j} + r(U_{i+1,j} + U_{i-1,j}),$$

where $r{=}\nu k/h^2$. In this explicit FD scheme, the unknown value $U_{i,j+1}$ (at the $(j{+}1)$th step) is determined from the three known values $U_{i-1,j}$, $U_{i,j}$, and $U_{i+1,j}$ (at the jth step). This FD scheme is unstable for $r{>}0.5$. We find the numerical solution as follows:

```
SetOptions[ListPlot,ImageSize->300,PlotRange->{{0,1},{0,1}},Joined->True];
f[x_]:=N[Sin[Pi*x]];
{nu=1,nNX=15,nNT=100,lL=1,tT=0.2,h=lL/nNX,k=tT/nNT,r=nu*k/h^2}
Table[xX[i]=i*h,{i,0,nNX}]; ic=Table[uU[i,0]->f[xX[i]],{i,0,nNX}];
bc={Table[uU[0,j]->0,{j,0,nNT}],Table[uU[nNX,j]->0,{j,0,nNT}]};
ibc=Flatten[{ic,bc}]
fd[i_,j_]:=(1-2*r)*uU[i,j]+r*(uU[i+1,j]+uU[i-1,j]);
Do[uU[i,j+1]=fd[i,j]/.ibc,{j,0,nNT},{i,1,nNX-1}];
g[j_]:=ListPlot[Table[{xX[i],uU[i,j]/.ibc},{i,0,nNX}],
 PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[j],{j,0,nNT}]]; ListAnimate[grs]
```

This FD scheme can be represented in the matrix form

$$U_i{=}MU_{i-1},$$

where $U_0{=}(f(X_1),\dots,f(X_{NX-1}))$, and M is an $NX{\times}NX$ tridiagonal band matrix (with $1{-}2r$ along the main diagonal, r along the subdiagonal and superdiagonal, and zero everywhere else). We obtain the numerical solution by using this matrix representation of the FD scheme as follows:

```
SetOptions[ListPlot,ImageSize->500,PlotStyle->{Blue,Thickness[0.01]},
 PlotRange->{{0,1},{0,1}},Joined->True]; f[x_]:=N[Sin[Pi*x]];
{nu=1,nNX=40,nNT=800,lL=1,tT=0.2,h=lL/nNX,k=tT/nNT,r=nu*k/h^2,nNG=90}
mat=SparseArray[{Band[{2,1}]->r,Band[{1,1}]->1-2*r,Band[{1,2}]->r},{nNX-1,nNX-1}];
Print[MatrixForm[mat]]; uU[0]=Table[f[i*h],{i,1,nNX-1}]
Do[uU[k]=mat.uU[k-1]; gr[k]={{0,0}};
Do[gr[k]=Append[gr[k],{i/nNX,uU[k][[i]]}],{i,1,nNX-1}];
gr[k]=Append[gr[k],{lL,0}]; g[k]=ListPlot[gr[k]],{k,1,nNG}];
grs=Evaluate[Table[g[j],{j,1,nNG}]]; ListAnimate[grs]
```

In the *backward difference method*, the second derivative u_{xx} is replaced by a central difference approximation (CDA) and the first derivative u_t by a backward difference approximation (BWDA). The final FD scheme for the linear diffusion equation is

$$(1+2r)U_{i,j} - r(U_{i+1,j} + U_{i-1,j}) - U_{i,j-1} = 0,$$

where $r = \nu k/h^2$. In this *implicit FD scheme*, we have to solve these difference equations numerically at each of the internal mesh points at each jth step (where $j = 1,\dots,NT$) with the initial and boundary conditions. This FD scheme is unconditionally stable. We calculate the approximate numerical solution of the initial-boundary value problem by applying the backward difference method as follows:

```
SetOptions[ListPlot,PlotRange->{{0,lL},{0,lL}},Joined->True];
{nu=1,nNX=50,nNT=50,lL=1,tT=0.2,h=lL/nNX,k=tT/nNT,r=nu*k/h^2}
```

```
Table[xX[i]=i*h,{i,0,nNX}]; f[i_]:=N[Sin[Pi*xX[i]]];
ibc={Table[uU1[i,0]->f[i],{i,0,nNX}],Table[uU1[0,j]->0,{j,0,nNT}],
 Table[uU1[nNX,j]->0,{j,0,nNT}]}//Flatten
sol[0]=ibc;
fd[i_,j_]:=(1+2*r)*uU1[i,j]-r*(uU1[i+1,j]+uU1[i-1,j])-uU1[i,j-1];
Do[eqs[j]=Table[Expand[fd[i,j]]==0,{i,1,nNX-1}];
eqs1[j]=eqs[j]/.sol[j-1]/.ibc; vars[j]=Table[uU1[i,j],{i,1,nNX-1}];
sol[j]=NSolve[eqs1[j],vars[j]],{j,1,nNT}];
g[j_]:=ListPlot[Table[{xX[i],uU1[i,j]/.Flatten[sol[j]]/.ibc},{i,0,nNX}],
 PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=Evaluate[Table[g[j],{j,1,nNT}]]; ListAnimate[grs]
```

The *Crank–Nicolson method* is obtained by centered differencing in time about the point $(x_i,t_{j+1/2})$. To do so, we average the central difference approximations in space at time t_j and t_{j+1}. The final FD scheme for the linear heat equation is

$$-rU_{i-1,j+1}+2(1+r)U_{i,j+1}-rU_{i+1,j+1}=rU_{i-1,j}+2(1-r)U_{i,j}+rU_{i+1,j},$$

where $r{=}\nu k/h^2$. In this FD scheme, we have three unknown values of U at the $(j{+}1)$th time step and three known values at the jth time step. This FD scheme is unconditionally stable. We calculate the approximate numerical solution of the initial-boundary value problem by applying the Crank–Nicolson method [see Crank and Nicolson (1947)] as follows:

```
fF[i_]:=Sin[Pi*i]; {lL=1,tT=0.2,nu=1,nNX=20,nNT=20,nNX1=nNX-1,
nNX2=nNX-2,h=lL/nNX,k=tT/nNT,r=nu*k/(h^2),tk=0.2}
{ic={v[x1,0]==fF[x1]}, bc={v[0,t1]==0,v[lL,t1]==0}}
{lLM=Table[0,{i,0,nNX}],uU=Table[0,{i,0,nNX}],uUM=Table[0,{i,0,nNX}],
zZ=Table[0,{i,0,nNX}], uU[[nNX-1]]=0}
pde1=D[v[x1,t1],t1]-nu*D[v[x1,t1],{x1,2}]==0
Do[uU[[i-1]]=N[fF[i*h]],{i,1,nNX1}];
{lLM[[0]]=1+r,uUM[[0]]=-r/(2*lLM[[0]])}
Do[lLM[[i-1]]=1+r+r*uUM[[i-2]]/2;
uUM[[i-1]]=-r/(2*lLM[[i-1]]),{i,2,nNX2}];
lLM[[nNX1-1]]=1+r+0.5*r*uUM[[nNX2-1]] Do[t=j*k;
zZ[[0]]=((1-r)*uU[[0]]+r*uU[[1]]/2)/lLM[[0]];
Do[zZ[[i-1]]=((1-r)*uU[[i-1]]+0.5*r*(uU[[i]]+uU[[i-2]]+zZ[[i-2]]))/lLM[[i-1]],
 {i,2,nNX1}]; uU[[nNX1-1]]=zZ[[nNX1-1]];
Do[i=nNX2-i1+1;
uU[[i-1]]=zZ[[i-1]]-uUM[[i-1]]*uU[[i]],{i1,1,nNX2}],{j,1,nNT}];
```

Finally, we compare the approximate numerical solution with the exact solution of this problem

$$u(x,t)=\exp(-\pi^2 t)\sin(\pi x) \quad \text{at} \quad (x_k,t_k), \quad 0<x<L, \quad t>0,$$

as follows:

```
nD=10; extSol[x1_,t1_]:=Exp[-Pi^2*t]*Sin[Pi*x]/.{x->x1,t->t1};
Print["Crank-Nicolson Method"]; Do[xX=i*h; Print[i,"
",PaddedForm[N[xX,nD],{12,10}]," ",
 PaddedForm[uU[[i-1]],{12,10}]," ", PaddedForm[N[extSol[xX,tk],nD],{12,10}]," ",
 PaddedForm[uU[[i-1]]-N[extSol[xX,tk],nD],{12,10}]],{i,1,nNX1}];
```

Example 23.29. *Linear wave equation. Explicit difference methods.* Consider the initial-boundary value problem for the linear wave equation describing the motion of a fixed string:

$$u_{tt}=c^2u_{xx}, \quad 0<x<L, \quad t>0, \quad u(x,0)=f_1(x), \quad u_t(x,0)=f_2(x), \quad u(0,t)=0, \quad u(L,t)=0,$$

where $f_1(x) = 0$, $f_2(x) = \sin(4\pi x)$, $L = 0.5$, and $c = 1/(4\pi)$. By applying the explicit central finite difference method, we construct the approximate numerical solution of the initial-boundary value problem.

In the *explicit central difference method*, each second derivative is replaced by a central difference approximation. The FD scheme for the linear wave equation is

$$U_{i,j+1} = 2(1-r)U_{i,j} + r(U_{i+1,j} + U_{i-1,j}) - U_{i,j-1},$$

where $r = (ck/h)^2$. In this FD scheme, we have one unknown value $U_{i,j+1}$ that depends explicitly on the four known values $U_{i,j}$, $U_{i+1,j}$, $U_{i-1,j}$, $U_{i,j-1}$ at the previous time steps (j and $j-1$). To start the process, we have to know the values of U at the time steps $j = 0$ and $j = 1$. So we can define the initial conditions at these time steps: $U_{i,0} = f1(X_i)$ and $U(X_i, 0)_t \approx (U_{i,1} - U_{i,0})/k = f2(X_i)$, $U_{i,1} = f1(X_i) + kf2(X_i)$. This FD scheme is stable for $r \leq 1$. We find the approximate numerical solution of the initial-boundary value problem by applying the explicit central finite difference method as follows:

```
SetOptions[ListPlot,PlotRange->All,Joined->False]; fF1[x_]:=0;
fF2[x_]:=Sin[4*Pi*x]; {c=N[1/(4*Pi)],lL=0.5,tT=1.5,nNX=40,
 nNX1=nNX+1,nNT=40,nNT1=nNT+1,h=lL/nNX,k=tT/nNT,r=N[c*k/h]}
fF1i[i_]:=fF1[h*(i-1)]; fF2i[i_]:=fF2[h*(i-1)];
uU=Table[0,{nNT1},{nNX1}]; For[i=1,i<=nNT1,i++,uU[[i,1]]=fF1i[i]];
For[i=2,i<=nNT,i++,uU[[i,2]]=(1-r^2)*fF1i[i]+r^2*(fF1i[i+1]
 +fF1i[i-1])/2+k*fF2i[i]];
For[j=3,j<=nNX1,j++, For[i=2,i<=nNT,i++,
 uU[[i,j]]=2*(1-r^2)*uU[[i,j-1]]+r^2*(uU[[i+1,j-1]]+uU[[i-1,j-1]])
 -uU[[i,j-2]]//N];];
Print[" i xX[i] uU[xX[i],nNT1]", "\n"];
For[i=1,i<=nNT1,i++,Print[PaddedForm[i,2],PaddedForm[h*(i-1),7]," ",
 PaddedForm[uU[[i,nNT1]],10]]];
points=Table[{h*(i-1),uU[[i,nNT1]]},{i,1,nNT1}]
ListPlot[points,PlotStyle->{Blue,PointSize[0.02]}]
Print[NumberForm[TableForm[Transpose[Chop[uU]]],3]];
ListPlot3D[uU,ViewPoint->{3,1,3},ColorFunction->Hue]
```

We construct and visualize the same approximate numerical solution of the initial-boundary value problem by applying the explicit central finite difference method and by following a different style of programming as follows:

```
SetOptions[ListPlot,ImageSize->500,PlotRange->{{0,0.5},{-1,1}},Joined->True];
fF1[x_]:=0; fF2[x_]:=N[Sin[4*Pi*x]];
{c=N[1/(4*Pi)],lL=0.5,tT=1.5,nNX=40,nNT=40,h=lL/nNX,k=tT/nNT,r=(c*k/h)^2}
ic={Table[uU1[i,0]->fF1[i*h],{i,1,nNX-1}],
 Table[uU1[i,1]->fF1[i*h]+k*fF2[i*h],{i,1,nNX-1}]};
bc={Table[uU1[0,j]->0,{j,0,nNT}],Table[uU1[nNX,j]->0,{j,0,nNT}]};
ibc=Flatten[{ic,bc}]
fd[i_,j_]:=2*(1-r)*uU1[i,j]+r*(uU1[i+1,j]+uU1[i-1,j])-uU1[i,j-1];
Do[uU1[i,j+1]=fd[i,j]/.ibc,{j,1,nNT-1},{i,1,nNX-1}];
g[j_]:=ListPlot[Table[{i*h,uU1[i,j]/.ibc},{i,0,nNX}],
 PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
grs=N[Table[g[j],{j,0,nNT}]]; ListAnimate[grs]
```

23.4.4 Numerical Solutions of Boundary Value Problems

Since in Mathematica (Release ≤ 10) it is only possible to solve evolution equations numerically (via predefined functions), we apply finite difference methods and the method of lines for constructing numerical and graphical solutions of boundary value problems for elliptic equations, e.g., the linear Poisson equation.

Example 23.30. *Linear Poisson equation. Boundary value problem. Central difference scheme.* Consider the two-dimensional linear Poisson equation

$$u_{xx} + u_{yy} = f(x,y), \quad \mathcal{D} = \{a \leq x \leq b,\ c \leq y \leq d\}$$

with the boundary conditions

$$u(x,c) = f_1(x), \quad u(x,d) = f_2(x), \quad u(a,y) = f_3(y), \quad u(b,y) = f_4(y).$$

Such boundary value problems describe a steady-state process $u(x,y)$ in a bounded rectangular object. Let us choose $f(x,y) = \cos x \sin y$, $f_1(x) = f_2(x) = 0$, $f_3(y) = f_4(y) = -\frac{1}{2}\sin y$, $a = 0$, $b = \pi$, $c = 0$, and $d = 2\pi$. For the linear Poisson equation, by applying the explicit finite difference scheme, we obtain the approximate numerical solution of the boundary value problem, visualize it in $\mathcal{D}$, and compare with the exact solution $u(x,y) = -\frac{1}{2}\cos x \sin y$ (see the previous chapter).

Let us generate the rectangular mesh $x = a + ih$, $y = c + jk$ (where $i = 0,\ldots,NX$, $j = 0,\ldots,NY$, $h = (b-a)/NX$, and $k = (d-c)/NY$). We denote the approximate solution of $u(x,y)$ at the mesh point (i,j) by $U_{i,j}$. The second derivatives in the Poisson equation are replaced by the central difference approximation (CDA). The FD scheme takes the form

$$2(1+r)U_{i,j} - U_{i+1,j} - U_{i-1,j} - rU_{i,j+1} - rU_{i,j-1} = \cos(ih)\sin(jk),$$

where $r = (h/k)^2$.

We construct an approximate numerical solution of the boundary value problem by applying the above explicit finite-difference scheme and plot the numerical solution inside the domain as follows:

```
SetAttributes[{x,y},NHoldAll]; {nD=10,a=0.,b=N[Pi],c=0.,d=N[2.*Pi],
 nNX=20,nNY=260,h=N[(b-a)/nNX],k=N[(d-c)/nNY],r=N[(h/k)^2]}
xX=Table[x[i]->a+i*h,{i,0,nNX}]//N;
yY=Table[y[j]->c+j*k,{j,0,nNY}]//N;
fd[i_,j_]=2.*(1+r)*uU[i,j]-uU[i+1,j]-uU[i-1,j]-r*uU[i,j+1]-r*uU[i,j-1]
 -Cos[i*h]*Sin[j*k]; f1[i_]=0.; f2[i_]=0.; f3[j_]=N[-0.5*Sin[j*k]];
f4[j_] = N[-0.5*Sin[j*k]];
bc=Flatten[{Table[uU[i,0]->f1[i],{i,0,nNX}],Table[uU[i,nNY]->f2[i],
 {i,0,nNX}],Table[uU[0,j]->f3[j],{j,0,nNY}],Table[uU[nNX,j]->f4[j],
 {j,0,nNY}]}]; eqs=Flatten[Table[fd[i,j]==0.,{i,1,nNX-1},{j,1,nNY-1}]];
 eqs1=Flatten[eqs/.bc]; vars=Flatten[Table[uU[i,j],{i,1,nNX-1},{j,1,nNY-1}]];
sol=NSolve[eqs1,vars,nD];
points=Table[{x[i],y[j],uU[i,j]},{i,0,nNX},{j,0,nNY}];
points1=Flatten[N[points/.xX/.yY/.bc/.sol[[1]],nD],1];
g3D=ListPointPlot3D[points1,PlotStyle->PointSize[0.01],BoxRatios->{1,1,1},
 PlotStyle->{{PointSize[0.05]},{PointSize[0.05]}},PlotRange->All];
gCP=ListContourPlot[points1]; GraphicsRow[{g3D,gCP},ImageSize->500]
uEx[x_,y_]=-0.5*Cos[x]*Sin[y]
g1=Plot3D[uEx[x,y],{x,0.,Pi},{y,0.,2.*Pi},PlotRange->All,BoxRatios->{3,3,2}];
g2=ContourPlot[uEx[x,y],{x,0.,Pi},{y,0.,2.*Pi},PlotPoints->100,Contours->20];
GraphicsRow[{g1,g2},ImageSize->500]
```

We restate our problem by following the approach proposed by Silebi and Schiesser (1992), i.e., by adding the time derivative to the steady-state elliptic partial differential equation and by applying the method of lines and the predefined function `NDSolve`. Finally, we obtain the steady-state solution by increasing the time interval (e.g., $tT = 50$) as follows:

```
f1[x_]:=0.; f2[x_]:=0.; f3[y_]:=-N[0.5*Sin[y]];
f4[y_]:=-N[0.5*Sin[y]]; {a=0.,b=N[Pi],c=0.,d=N[2.*Pi],tT=50}
{pde1=D[u[x,y,t],t]==D[u[x,y,t],{x,2}]+D[u[x,y,t],{y,2}]-Cos[x]*Sin[y],
 ic={u[x,y,0]==-N[0.5*Sin[y]]}, bc={u[x,c,t]==f1[x],u[x,d,t]==f2[x],
 u[a,y,t]==f3[y],u[b,y,t]==f4[y]}}
sol1=NDSolve[{pde1,ic,bc},u,{x,a,b},{y,c,d},{t,0,tT},Method->
 {"MethodOfLines","SpatialDiscretization"->{"TensorProductGrid",
 "DifferenceOrder"->"Pseudospectral"}}]
g1=Plot3D[Evaluate[u[x,y,tT]/.sol1],{x,a,b},{y,c,d},PlotRange->All];
g2=ContourPlot[Evaluate[u[x,y,tT]/.sol1],{x,a,b},{y,c,d}];
GraphicsRow[{g1,g2},ImageSize->500]
```

23.4.5 Numerical Solutions of Cauchy Problems

There are various methods to obtain approximate solutions of Cauchy problems involving linear partial differential equations. The most convenient and accurate method for solving *Cauchy problems* for hyperbolic equations is the method of characteristics (see Sections 23.2.2, 23.2.4). The finite difference methods for hyperbolic equations are not very convenient (e.g., in the case of discontinuous initial data) [see Richtmyer and Morton (1994), Collatz (1966)].

Example 23.31. *Linear inhomogeneous advection equation. Cauchy problem. Method of characteristics. Graphical solutions.* Consider the initial value problem for the linear inhomogeneous advection equation

$$u_t + c\,u_x = F(x,t), \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x),$$

where $f(x) = \sin x$ and $F(x,t) = xt$.

By applying the method of characteristics, we obtain the initial value ODE problem and its solution (`odes`, `sol1`):

$$\frac{dx(r)}{r} = \frac{c}{r}, \quad \frac{dt(r)}{r} = \frac{1}{r}, \quad \frac{dz(r)}{r} = \frac{x(r)t(r)}{r}, \quad x(1) = s, \quad t(1) = 0, \quad z(1) = h(s),$$
$$x(r) = c\ln r + s, \quad t(r) = \ln r, \quad z(r) = \frac{1}{3}c\ln r^3 + \frac{1}{2}\ln r^2 s + h(s).$$

Finding z (in terms of x and t), we have (`sol2`)

$$s = x - ct, \quad r = e^t, \quad z = -\frac{1}{6}ct^3 + \frac{1}{2}t^2 x + h(x-ct),$$

and the solution $u(x,t) = -\frac{1}{6}ct^3 + \frac{1}{2}t^2 x + h(x-ct)$ satisfies the original PDE and the initial data (`test1`, `test2`). Finally, we visualize the solution of the given Cauchy problem (e.g., for $c = 2$ and $h(s) = \sin s$) as follows:

```
{trVars={x[r]->x,t[r]->t,z[r]->z},
pde1=D[u[x,t],t]+c*D[u[x,t],x]==x*t} odes={D[x[r],r]==c/r,
```

```
D[t[r],r]==1/r, D[z[r],r]==x[r]*t[r]/r} sol1=DSolve[{odes,x[1]==s,
t[1]==0, z[1]==h[s]},{x[r],t[r],z[r]},r]//First
sol2=Reduce[(sol1/.trVars)/.Rule->Equal,{s,r}]
{sol2[[2]],sol2[[3]],sol2[[4]],trPar=c->2} w[x_,t_]=sol2[[2,2]];
w[x, t] {test1=Equal[D[w[x,t],t]+c*D[w[x,t],x]==x*t],
test2=Equal[w[s,0]==h[s]]} h[s_]=Sin[s]; h[s]
Plot3D[Evaluate[w[x,t]/.trPar],{x,-3*Pi,3*Pi},{t,0,1},Mesh->False,ColorFunction->h[x],
 PlotRange->All,BoxRatios->{1,1,1},PlotStyle->{{PointSize[0.05]},{PointSize[0.05]}}]
ContourPlot[Evaluate[w[x,t]/.trPar],{x,-3*Pi,3*Pi},{t,0,1},PlotPoints->100,
 Contours->20,PlotRange->All]
```

23.4.6 Numerical Solutions of Systems of Linear PDEs

In this section, we show how to obtain numerical and graphical solutions of systems of linear partial differential equations in Mathematica with the aid of the predefined function `NDSolve`.

Example 23.32. *Linear first-order system (equivalent to the linear wave equations). Numerical and graphical solutions.* Introducing the dependent variables $u(x,t)$ and $v(x,t)$ according to the formulas $u = w_t$ and $v = c^2 w_x$, we can transform the linear wave equation, $w_{tt} = c^2 w_{xx}$, to the equivalent system of linear first-order equations $u_t = v_x$, $v_t = c^2 u_x$. Consider the initial-boundary value problem for this linear first-order system (describing standing waves):

$$u_t = v_x, \quad v_t = c^2 u_x, \quad 0 < x < L, \quad t > 0,$$
$$u(x,0) = f(x), \quad v(x,0) = 0, \quad u(0,t) = 0, \quad u(L,t) = 0,$$

where $L = 2\pi$, $f(x) = \sin x$.

We obtain a numerical solution (`sol1`, `w`, `q`) of the given initial-boundary value problem, visualize it at various times (`g1`–`g3`, `gu`, `gv`) in parametric form, construct the animation of a standing wave profile, and visualize the solutions in 3D as follows:

```
{c=0.5,lL=N[2*Pi],a=0,b=lL,tT=20,lL1={0.1,0.2,0.5},lL2={Magenta,Blue,Red},
 nNL1=Length[lL1]}
f[x_]=Sin[x];
sys1={D[u[x,t],t]==D[v[x,t],x],D[v[x,t],t]==c^2*D[u[x,t],x]}
ibc1={u[x,0]==f[x],v[x,0]==0,u[0,t]==0,u[lL,t]==0}
{sol1=NDSolve[{sys1,ibc1},{u,v},{x,a,b},{t,0,tT}], w=u/.sol1[[1]],
q=v/.sol1[[1]]}
Plot3D[Evaluate[w[x,t]],{x,a,b},{t,0,tT},BoxRatios->1,PlotRange->All]
Plot3D[Evaluate[q[x,t]],{x,a,b},{t,0,tT},BoxRatios->1,PlotRange->All]
{w[0.1,0], q[0.1,0]}
Do[g[i]=Plot[Evaluate[w[x,lL1[[i]]]],{x,0,lL},PlotStyle->{lL2[[i]],
 Thickness[0.001]}],{i,1,nNL1}]; Show[Table[g[i],{i,1,nNL1}],ImageSize->500]
gu=Plot[Evaluate[w[x,tT]],{x,0,lL},PlotStyle->{lL2[[1]],Thickness[0.007]}];
gv=Plot[Evaluate[q[x,tT]],{x,0,lL},PlotStyle->{lL2[[2]],Thickness[0.007]}];
Show[{gu,gv}]
ParametricPlot[{Evaluate[w[r,tT],q[r,tT]]},{r,a,b},AspectRatio->1]
Animate[Plot[w[x,t],{x,a,b},PlotRange->{-1,1},PlotStyle->Thickness[0.001],
 PlotRange->All],{t,0,tT},AnimationRate->0.5]
```

⊙ *References for Section 23.4*: J. Crank and P. Nicolson (1947), L. O. Collatz (1966), R. D. Richtmyer and K. W. Morton (1967), K. W. Morton and D. F. Mayers (1995), J. W. Thomas (1995), L. Lapidus and G. F. Pinder (1999), L. Strikwerda (2004), R. J. LeVeque (2007), S. Larsson and V. Thomée (2008), I. K. Shingareva and C. Lizárraga-Celaya (2011).

Chapter 24

Linear Partial Differential Equations with MATLAB®

24.1 Introduction

In the previous two chapters, we paid special attention to analytical solutions of linear partial differential equations and systems in the context of computer algebra systems Maple and Mathematica. However, in real-world problems, the functions and data in PDE problems are often defined at discrete points and the equations are too complicated, so that it is not possible to construct analytical solutions. Therefore, one has to study and develop numerical approximation methods for linear PDEs [e.g., see Crank and Nicolson (1947), Larsson and Thomée (2008), Lax (1968), LeVeque (2007), Li and Chen (2009)].

Nowadays, for this purpose one can use computers and supercomputers equipped with convenient and powerful computational software such as MATLAB, an interactive programming environment for scientific computing, which provides integrated numeric computation and graphical visualization in a high-level programming language. MATLAB's excellent graphics capabilities can help one better understand the results and the solution properties.

In this chapter, we turn our attention to numerical methods for solving linear partial differential equations. Following the most important ideas and methods, we apply and develop numerical methods to obtain numerical and graphical solutions of linear partial differential equations. We compute numerical solutions via MATLAB predefined functions (which implement well-known methods for solving partial differential equations) and develop new procedures for constructing numerical solutions (e.g., by applying the method of characteristics and finite difference approximations) with the aid of MATLAB.

24.1.1 Preliminary Remarks

In this chapter, we consider second-order linear partial differential equations in one and two space dimensions. These PDEs are classified into three groups, *elliptic*, *parabolic*, and *hyperbolic* equations. The most significant difference is the order of the derivative of the unknown function u with respect to time t. If the derivative is absent, the PDE is elliptic. The first and second derivatives correspond to parabolic and hyperbolic equations, respectively. These three classes of equations are associated with equilibrium states, diffusion states, and oscillating systems, respectively. Since analytical solutions of such PDEs with initial and boundary conditions are usually difficult to find, we consider numerical methods for solving these equations.

In general, an elliptic PDE can be written in the form

$$-\operatorname{div}(c\nabla u) + au = f(\mathbf{x},t), \tag{24.1.1.1}$$

where c, a, and f are scalar (complex-valued) functions defined on a bounded domain D, $x_1, x_2, \ldots, x_n, t$ are independent variables, u is the unknown scalar (complex-valued) function (dependent variable) of these independent variables, $u = u(x_1, x_2, \ldots, x_n, t) = u(\mathbf{x}, t)$, ∇u is the gradient of the unknown function u, and $\operatorname{div}(c\nabla u)$ is the divergence of $c\nabla u$. If c is a constant, we can simplify this to $\operatorname{div}(c\nabla u) = c\Delta u$, where Δ is the Laplacian operator. In this case, the elliptic PDE has the form

$$-c\left(u_{x_1x_1} + u_{x_2x_2} + \cdots u_{x_nx_n}\right)u + au = f(\mathbf{x},t).$$

We will use the subscripts on dependent variables to denote differentiations; e.g., $u_{x_1} = \partial u/\partial x_1$, $u_{x_1x_1} = \partial^2 u/\partial x_1 \partial x_1$.

Equation (24.1.1.1) is defined in a bounded domain D of the n-dimensional space $\mathbb{R}^n$ in the independent variables $x_1, x_2, \ldots, x_n$. We will find numerical solutions of Eq. (24.1.1.1), i.e., functions $u = u(x_1, x_2, \ldots, x_n, t)$ that satisfy Eq. (24.1.1.1) in D and additional initial and boundary conditions.

The general form of a parabolic PDE can be written as

$$du_t - \operatorname{div}(c\nabla u) + au = f(\mathbf{x},t), \tag{24.1.1.2}$$

where d, c, and a are scalar functions. If c is a constant, then the equation can be simplified as

$$du_t - c\left(u_{x_1x_1} + u_{x_2x_2} + \cdots u_{x_nx_n}\right)u + au = f(\mathbf{x},t).$$

The general form of a hyperbolic PDE can be written as

$$du_{tt} - \operatorname{div}(c\nabla u) + au = f(\mathbf{x},t). \tag{24.1.1.3}$$

If c is a constant, then the equation can be simplified as

$$du_{tt} - c\left(u_{x_1x_1} + u_{x_2x_2} + \cdots u_{x_nx_n}\right)u + au = f(\mathbf{x},t).$$

For linear parabolic and hyperbolic PDEs, the coefficients c, a, f, and d can depend on time t.

24.1.2 Brief Introduction to MATLAB

MATLAB (short for "matrix laboratory") is not a general purpose programming language like Maple and Mathematica.

MATLAB is an interactive programming environment that provides powerful high-performance numerical computing, excellent graphics visualization, symbolic computing capabilities, and capabilities for writing new software programs using a high-level programming language.

Symbolic Math Toolbox (Release ≥ 4.9) is based on the muPAD symbolic kernel and provides symbolic computations and variable-precision arithmetic. Earlier versions of *Symbolic Math Toolbox* are based on the Maple symbolic kernel.

Simulink (short for "simulation and link"), also included in MATLAB, offers modeling, simulation, and analysis of dynamical systems (e.g., signal processing, control, communications, etc.) under a *graphical user interface* (GUI) environment.

The first concept of MATLAB and the original version (written in Fortran) were developed by Prof. Cleve Moler at the University of New Mexico in the late 1970s to provide his students with simple interactive access to LINPACK and EISPACK software (without having to learn Fortran).[1] Over the next several years, this original version of MATLAB spread within the applied mathematics community. In early 1983, Jack Little, together with Cleve Moler and Steve Bangert, developed a professional version of MATLAB (written in C and integrated with graphics). The company MathWorks was created in 1984 and headquartered in Natick, Massachusetts, to continue its development.

The most important features of MATLAB are as follows: interactive user interface; a combination of comprehensive mathematical and graphics functions with a powerful high-level language in an easy-to-use environment; fast numerical computation and visualization, especially for performing matrix operations [e.g., see Higham (2008)]; great flexibility in data manipulation; symbolic computing capabilities via the Symbolic Math Toolbox (Release < 4.9 or Release ≥ 4.9), based on the Maple or muPAD symbolic kernel, respectively; easy usability; the basic data element is an array that does not require dimensioning; a large library of functions for a broad range of applications; easy incorporation of new user-defined capabilities (toolboxes consisting of M-files and written for specific applications); understandability and availability on almost all operating systems; a powerful programming language with intuitive and concise syntax and easy debugging; Simulink, as an integral part of MATLAB, provides modeling, simulation, and analysis of dynamical systems; free resources (e.g., MathWorks Web Site `www.mathworks.com`, MathWorks Education Web Site `www.mathworks.com/education`, and MATLAB group `comp.soft-sys.matlab`, etc.).

MATLAB consists of five parts: *Development Environment*, a set of tools that facilitate using MATLAB functions and files (e.g., graphical user interfaces and the workspace); *Mathematical Function Library*, a vast collection of computational algorithms; the MATLAB *language*, a high-level matrix/array language (with flow control statements, functions, data structures, input/output, and object-oriented programming features); the MATLAB *graphics system*, which includes high-level functions (for 2D/3D data visualization, image processing, animation, etc.) and low-level functions (for fully customizing the appearance of graphics and constructing complete graphical user interfaces); *Application Program Interface* (API), a library for writing C and Fortran programs that interact with MATLAB.

Basic concepts. The *prompt symbol* >> serves to type a MATLAB function; typing a statement and pressing `Return` or `Enter` at the end of the function serves to evaluate the

[1]LINPACK and EISPACK are collections of Fortran subroutines developed by Cleve Moler and his colleagues for solving linear equations and eigenvalue problems, respectively.

MATLAB function, display the result, and insert a new prompt; the semicolon (;) symbol at the end of a MATLAB function serves only to evaluate the function without displaying any result.

In contrast to Maple and Mathematica, it is not possible in MATLAB to move the cursor to the desired line, but corrections for sufficiently simple problems can be made by pressing the `up`- and `down-arrow` keys for scrolling through the list of functions (recently used) and then `left`- and `right-arrow` keys for changing the text. Also, corrections can be made by using `copy/paste` of the previous lines located in the `Command Window` or `Command History`.

The *previous result* (during a session) can be referred to with the variable `ans` (the last result). MATLAB prints the *answer* and assigns the value to `ans`, which can be used for further calculations.

MATLAB has many forms of help: a complete *online help system* with tutorials and reference information for all the functions; the command line help system, which can be accessed by using the `Help` menu, by pressing `F1`, by selecting `Help->Demos`, and by entering `Help` and selecting `Functions->Alphabetical List` or `Index`, `Search`, `MATLAB->Mathematics`; or by typing `helpbrowser`, `lookfor` (e.g., `lookfor plot`); by typing `help FunctionName`, `doc FunctionName`, etc.

In MATLAB 7, a new feature for typing function names correctly has been added. It is possible to type only the first few letters of the function and then to press the `TAB` key (to observe all available functions and complete typing the function).

MATLAB *desktop* contains tools (graphical user interfaces) for managing files, variables, and applications. The default configuration of desktop includes various tools, e.g., `Command Window`, `Command History`, `Current Directory`, `Workspace`, `Find Files` (for more details, see `demo MATLAB desktop`), etc. It is possible to modify the arrangement of tools and documents.

For a new problem, it is best to begin with the statement `clear all` for cleaning all variables from MATLAB's memory. All examples and problems in the book are assumed to begin with `clear all`.

A MATLAB program can be typed at the prompt `>>` or, alternatively (e.g., for more complicated problems), by creating an *M-file* (with `.m` extension) using MATLAB *editor* (or using another text editor). MATLAB editor is invoked by typing `edit` at the prompt.

M-files are files that contain code in the MATLAB language. There are two kinds of M-files: *script M-files* (which do not accept input arguments or return output data) and *function M-files* (which can accept input arguments and return output arguments).

In the process of working with various M-files, it is necessary to define the path, which can be done by selecting `File->Set Path->Add Folder` or via the `cd` function.

The *structure* of a MATLAB program or source code is as follows: the *main program* or *script* and the necessary *user-defined functions*. The execution starts by typing the file name of the main program.

Incorrect response. If you get no response or an incorrect response, you may have entered or executed a function incorrectly. To correct the function or interrupt the computation by entering debug mode and setting breakpoints, select `Debug->Open M-files when Debugging` and `Debug->Stop if Errors/Warnings` on the Desktop menu.

It is also possible to detect erroneous or unexpected behavior in a program with the aid of MATLAB functions, e.g., `break`, `warning`, and `error`.

Palettes can be used, e.g., for building or editing graphs (`Figure Palette`), displaying the names of GUI components (`Component Palette`), etc.

MATLAB *graphical user interface development environment* (GUIDE) provides a set of tools for creating graphical user interfaces (GUI). These tools greatly simplify the process of constructing GUI, e.g., customizing the layout of the GUI components (panels, buttons, menus, etc.), and programming the GUI.

MATLAB consists of a family of add-on *toolboxes*, which are collections of MATLAB functions (M-files) that extend the MATLAB environment to solve particular classes of problems.

Toolboxes can be *standard* and *specialized* (see `Contents` in `Help`). Nowadays, a vast number of specialized toolboxes are available. MATLAB can be augmented by a number of toolboxes consisting of M-files and written for specific applications.

24.1.3 MATLAB Language

MATLAB language is a high-level procedural dynamic and imperative programming language (similar to Fortran 77, C, and C++), with powerful matrix/array operations, control statements, functions, data structures, input/output, and object-oriented programming features. On the other hand, MATLAB language is an interpreted language, similar to Maple and Mathematica [e.g., see Shingareva and Lizárraga-Celaya (2009)]; i.e., the instructions are translated into machine language and executed in real time (one at a time). MATLAB language allows programming-in-the-small (coding or creating programs for performing small-scale tasks) and programming-in-the-large (creating complete large and complex application programs). It supports a large collection of data structures or MATLAB classes and operations among these classes.

In linear algebra, there exist two distinct types of operations with vectors/matrices: operations based on the mathematical structure of vector spaces and element-by-element operations on vectors/matrices as data arrays. This difference can be made in the name of the operation or the name of the data structure. In MATLAB, separate operations are defined (for matrix and array manipulation), but the data structures `array` and `vector/matrix` are the same. Note that, for example, in Maple the situation is opposite: the operations are the same, but the data structures are different.

Arithmetic operators: scalar operators (`+ - * / ^`), matrix multiplication/power (`* ^`), array multiplication/power (`.* .^`), left/right matrix division (`\ /`), and array division (`./`).

Logical operators: and (`&`), or (`|`), exclusive or (`xor`), not (`~`).

Relational operators: less/greater than (`< >`), less/greater than or equal to (`<= >=`), equal/not equal (`== ~=`).

A variable name is a character string of letters, digits, and underscores that begins with a letter and whose length is bounded by `N=namelengthmax` (e.g., $N = 63$). Punctuation marks are not allowed (see `genvarname` function). Variable declaration is not necessary in MATLAB, but all variables must be given initial values, e.g., `a12_new=9`. A variable can change in the calculation process, e.g., from integer to real (and vice versa).

MATLAB is case sensitive; i.e., it distinguishes between lowercase and uppercase letters, as in `pi` and `Pi`.

Various reserved keywords, symbols, names, and functions, for example, reserved keywords and function names, cannot be used as variable names (see `isvarname`, `which -all`, `isreserved`, `iskeyword`).

A string variable is enclosed in single quotes and belongs to the `char` class (e.g., `x='string'`), so that the function `sin(x)` is invalid. Strings can be used with converting, formatting, and parsing functions (e.g., see `cellstr`, `char`, `sprintf`, `fprintf`, `strfind`, `findstr`).

MATLAB provides three basic types of variables: *local variables*, *global variables*, and *persistent variables*.

The operator "set equal to" (=). A variable in MATLAB (in contrast to Maple and Mathematica) cannot be "free" (with no assigned value) and must be assigned any initial value by the operator "set equal to" (=).

The difference between the operators "set equal to" (=) and "equal" (==). The operator `var=val` is used to assign `val` to the variable `var`, and the operator `val1==val2` compares two values; e.g., `A=3; B=3; A==B`.

Statements are keyboard input instructions that are executed by MATLAB (e.g., `for i=1:N s=s+i*2; end`). A MATLAB statement may begin at any position in a line and may continue indefinitely in the same line, or may continue in the next line, by typing three dots (...) at the end of the current line. White spaces between words in a statement are ignored; a number cannot be split into two pieces separated by a space.

The statement separator semicolon (;) The result of a statement followed with a semicolon (;) will not be displayed. If the semicolon is omitted, the results will be printed on the screen, e.g., `x=-pi:pi/3:pi;` and `x=-pi:pi/3:pi`.

Multiple statements in a line: two or more statements may be written in the same line if they are separated with semicolons.

Comments can be included with the percent sign `%` and all characters following it up to the end of a line. Comments at the start of a code have a special significance: they are used by MATLAB to provide the entry for the help manual for a particular script. The block comment operators `%{ %}` can be used for writing comments that require more than one line.

An expression is a valid statement and is formed as a combination of numbers, variables, operators and functions. The arithmetic operators have different precedences (increasing precedence `+ - * / ^`). Precedence is altered by parentheses (the expressions within parentheses are evaluated before the expressions outside the parentheses).

A boolean or *logical expression* is formed with *logical* and *relational operators*, e.g., `x>0`. Logical expressions are used in `if`, `switch`, and `while` statements. The logical values *true* and *false* are represented by the numerical values *1* and *0*, respectively.

A regular expression is a string of characters that defines a *pattern*; e.g., `'Math?e\w*'`. Regular and dynamic expressions can be used to search text for a group of words that matches the pattern (e.g., for parsing or replacing a subset of characters within a text).

MATLAB is sensitive to types of brackets and quotes (see `help paren`, `help punct`).

Types of brackets:

Square brackets, `[ ]`, for constructing vectors and matrices, such as `A1=[1 2 3]`, `A2=[1, 2, 3]`, `A3=[1, 2; 4, 5]`, and for multiple assignment statements, for example, `A4=[1,5;2,6]`, `[L,U]=lu(A4)`.

Parentheses, `( )`, for grouping expressions, `(5+9)*3`, for delimiting the arguments of functions, `sin(5)`, for vector and matrix elements, `A1(2)`, `A3(1,1)`, `A2([1 2])`, and in logical expressions, `A1(A1>2)`.

Braces, `{ }`, for working with cell arrays, `C1={int8(3) 2.59 'A'}`, `C1{1}`, `X(2,1)={[1 3; 4 6]}`.

Dot-parentheses, `.( )`, for working with a structure via a dynamic field name, `S.F1=1; S.F2=2; F='F1'; val1=S.(F)`.

Quotes:

Forward-quotes, (`' '`), for creating strings, e.g., `T='the name=7;' k=5; disp('the value of k is'); disp(k)`.

Single forward-quote and dot single forward-quote, (`' .'`), for matrix transposition (the complex conjugate/nonconjugate transpose of a matrix), `A1=[1+i,i;-i,1-i]; A1'; A.'`.

Types of numbers. Numbers are stored (by default) as double-precision floating point (class `double`). To operate with integers, it is necessary to convert from double to the integer type (e.g., classes `int8`, `int16`, `int32`), `x=int16(12.3)`, `str='MATLAB'`, `int8(str)`. Mathematical operations that involve integers and floating-point numbers result in an integer data type. Real numbers can be stored as *double-precision floating point* (by default) or *single-precision floating point*, e.g., `x1=3.25`, `x2=single(x1)`, `x3=double(x2)` (for details, see `whos`, `isfloat`, `class`). Complex numbers can be created as `z1=1+2*i`, `z2=complex(1,2)`. Rational numbers can be formed by setting the `format` to `rational`, e.g., `x=3.25; format rational x format`. To check the current format setting, type `get(0,'format')`.

Predefined constants: symbols for commonly used mathematical constants, e.g., `true`, `false`, `pi`, `i`, `j`, `Inf`, `inf`, `NaN` (not a number), `exp(1)`, the Euler constant γ, `-psi(1)`, `eps`.

In MATLAB, there are *predefined functions* and *user-defined functions*.

Predefined functions are divided into *built-in functions* and *library functions*.

Built-in functions are precompiled executable programs and run much more efficiently (see `help elfun`, `help elmat`).

Library functions are stored as M-files (in libraries or toolboxes), which are available in readable form (see `which`, `type`, `exist`). MATLAB can be supplemented with locally user-developed M-files and toolboxes.

Many functions are overloaded (i.e., have an additional implementation of an existing function) so that they handle distinct classes (e.g., `which -all plot`).

Numerous *special functions* are defined, e.g., `help bessel`, `help specfun`.

User-defined functions can be created as M-files (see `help 'function'`) or as anonymous functions.

User-defined function written in an M-file (with the extension `.m`) must contain only one function. It is best to have the same name for the *function name* and the *file name*. The process of creating functions is as follows: create and save an M-file using a text editor, then call the function in the main program (or in `Command Window`).

Functions written in M-files have the form

```
function OArg=FunName(IArg); FunBody; ,or
function [OArg1,OArg2,...]=FunName(IArg1, IArg2,...); FunBody;
```

where `OArg` and `IArg` are output arguments and input arguments, respectively.
For example, the function $y = \sin x$ is defined as `function f=SinFun(x); f=sin(x);`

Evaluation of functions: `FunName(Args)`.
For example, for the sine function we have `cd('c:/mypath'); SinFun(pi/2); type SinFun`.

Anonymous functions serve to create simple functions without storing functions to files. Anonymous functions can be constructed either in the `Command Window` or in any function or script; e.g., the function $f(x) = \sin x$ is defined as `f=@(x) sin(x); f(pi/2)`.

A *function handle*, `@`, is one of the standard MATLAB data types that provides calling functions indirectly, e.g., calling a subfunction when outside the file that defines that subfunction (see class `function_handle`) .

Nested functions are allowed in MATLAB; i.e., one or more functions or *subfunctions* within another function can be defined in MATLAB. In this case, the `end` statements are necessary.

The MATLAB language has the following *control structures*: the selection structures `if`, `switch`, `try` and the repetition structures `for`, `while`.

MATLAB does not have a *module system* in the traditional form: it has a system based on storing *scripts* and *functions* in M-files and placing them into *directories* (see `cd` function for changing the current directory, `help ..`).

MATLAB *data structures* or *classes*, vectors, matrices, and arrays, are used for representing more complicated data. There are 15 fundamental classes, which are the form of a matrix or array: `double`, `single`, `int8`, `uint8`, `int16`, `uint16`, `int32`, `uint32`, `int64`, `uint64`, `char`, `logical`, `function_handle`, `struct`, and `cell`. The numerical values are represented (by default) as floating-point double precision (`float double`). It is possible to construct various *composite data types* (e.g., sequences, lists, sets, tables, etc.) using the classes `struct` and `cell`.

Vectors are ordered lists of numbers separated by commas or spaces inside `[ ]`; no dimensioning is required. But *vector and array indices* can only be positive and nonzero. The notation `X=[1:0.1:9]` stands for a vector of numbers from 1 to 9 in increments of 0.1 (see `help colon`).

Matrices are rectangular arrays of numbers (row/column vectors are special cases of matrices).

⊙ *References for Section 24.1*: J. Crank and P. Nicolson (1947), P. D. Lax (1968), R. J. LeVeque (2007), S. Larsson and V. Thomée (2008), N. J. Higham (2008), J. Li and Y. T. Chen (2009), I. K. Shingareva and C. Lizárraga-Celaya (2009).

24.2 Numerical Solutions of Linear PDEs

In this section, we consider the construction of numerical and graphical solutions of various mathematical problems (initial-boundary value problems, boundary value problems, and initial value problems) using predefined MATLAB functions and default methods. In particular, we construct numerical and graphical solutions of scalar linear PDEs in one space dimension (e.g., the linear heat equation, the convection–diffusion equations, and the linear Euler equation) and scalar linear PDEs in two space dimensions (e.g., the linear Poisson equation, the linear heat equation with radiation, and the linear nonhomogeneous wave equation). Additionally, we construct numerical and graphical solutions of linear problems defined on a more complicated geometry, which can consist of various solid objects, e.g., for the linearized Poisson–Boltzmann equation defined on an irregular domain [see Harries (1998)].

24.2.1 Constructing Numerical Solutions via Predefined Functions

1D Partial Differential Equations

First, consider the predefined MATLAB functions with the aid of which we can obtain approximate numerical solutions solving various linear PDE problems. Applying the predefined function `pdepe` (provided in MATLAB `PDEToolbox`),[2] we can numerically solve initial-boundary value problems for a class of linear parabolic PDEs in one space variable x and time t.[3]

Consider a single linear PDE in one space dimension. In this case, the predefined function `pdepe` allows us to solve initial-boundary value problems for a single parabolic PDE in 1D (see `help pdepe`). These PDEs involve an unknown function u that depends on a scalar space variable x and a scalar time variable t.

In MATLAB notation, a general class of linear parabolic PDEs defined in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$ is represented in the form

$$c(x,t)u_t = x^{-m}\partial_x\big(x^m f(x,t,u,u_x)\big) + s(x,t,u,u_x), \qquad (24.2.1.1)$$

where x_L, x_R, t_0, t_f are given constants, $f(x,t,u,u_x)$ is the flux, $s(x,t,u,u_x)$ is the source term, $m \in \{0,1,2\}$ (which corresponds to slab, cylindrical, or spherical symmetry, respectively), and $c(x,t) > 0$ for parabolic equations.

The initial condition at the initial time $t = t_0$ has the form

$$u(x,t_0) = u_0(x) \qquad (24.2.1.2)$$

and the boundary conditions at $x = x_L$ and $x = x_R$ (for $t_0 \le t \le t_f$) have the form

$$p(x_L,t,u) + q(x_L,t)f(x_L,t,u,u_x) = 0, \quad p(x_R,t,u) + q(x_R,t)f(x_R,t,u,u_x) = 0. \quad (24.2.1.3)$$

One can impose the Dirichlet, Neumann, Robin, or periodic boundary conditions.

```
sol=pdepe(m,@PDEfun,@ICfun,@BCfun,xMesh,tSpan,ops)
uk=sol(j,:,k);  [uOut,DuOutDx]=pdeval(m,xMesh,uk,xOut)
```

`PDEfun`, `ICfun`, `BCfun` are function handles.

`[c,f,s]=PDEfun(x,t,u,DuDx)` evaluates the quantities defining the partial differential equation, where `PDEfun` is the function name and the functions `c, f, s` can be calculated from the given arguments.

[2]The name `pdepe` stands for parabolic-elliptic partial differential equations.

[3]A single elliptic PDE in one space dimension is treated as an ODE.

`u0=ICfun(x)` evaluates the initial conditions.

`[pL,qL,pR,qR]=BCfun(xL,uL,xR,uR,t)` evaluates the boundary conditions at time t.

`uk` approximates component k of the solution at time `tSpan(j)` and mesh points `xMesh`.

`pdeval` evaluates the function u and the derivative u_x `DuDx` at the array of points `xOut` at points that are not in `xMesh`; it stores them in `uOut` and `DuOutDx`, respectively.

The numerical solution obtained is represented as a three-dimensional array `sol`, where `sol(:,:,k)` approximates component k of the solution, `sol(i,:,k)` approximates component k of the solution at time `tSpan(i)` and mesh points `xMesh(:)`, and `sol(i,j,k)` approximates component k of the solution at time `tSpan(i)` and the mesh point `xMesh(j)`.

Numerical solutions can be used for obtaining visualizations (solution surface, solution profile, or animation), e.g., by using `plot` and `surf` (for more detail, see `help graphics`).

For more advanced applications, it is best to specify *options* that change the default integration parameters: error control and step size (for details, see `odeset`, `RelTol`, `AbsTol`, `NormControl`, `InitialStep`, `MaxStep`).

Example 24.1. *Linear heat equation.* Let us find numerical and graphical solutions of the following initial-boundary value problem for the linear heat equation (as in Problems 28 and 22 of the Maple and Mathematica chapters, respectively):

$$u_t = \nu u_{xx}, \quad u(x,0) = A\sin^4(\pi x), \quad u(0,t) = 0, \quad u(L,t) = 0,$$

in the domain $\mathcal{D} = \{0 \le x \le L,\ 0 \le t \le T\}$ $(L = 1)$ with initial amplitude $A = 1$ and kinematic viscosity $\nu = 1/30$.

First, let us rewrite the heat equation with initial and boundary conditions in the form (24.2.1.1)–(24.2.1.3). We create three separate function M-files for each set of functions, i.e., `PDE1.m` for the functions c, f, and s, `IC1.m` for the function $u(x,0) = u_0(x)$, and `BC1.m` for the functions p and q. The `pdepe` function will combine these M-files and construct the solution of the problem.

For the heat equation, we have $m = 0$, $c(x,t) = 1$, $f(x,t,u,u_x) = \nu u_x$, and $s(x,t) = 0$; for the initial condition, we have $u_0(x) = A\sin^4(\pi x)$; and for the boundary conditions, we have $p(0,t,u) = u$, $q(0,t) = 0$, $p(L,t,u) = u$, and $q(L,t) = 0$. These functions are specified in the function M-files as follows:

```
function [c,f,s]=PDE1(x,t,u,DuDx)
   nu=1/30; c=1; f=nu*DuDx; s=0;
function u0=IC1(x)
   A=1; u0=A*sin(pi*x)^4;
function [pL,qL,pR,qR]=BC1(xL,uL,xR,uR,t)
   pL=uL; qL=0; pR=uR; qR=0;
```

Then, for constructing numerical and graphical solutions of this initial-boundary value problem for the heat equation, we compose the script M-file `SolPDE1` as follows:

```
clear all; close all; echo on; format long; m=0; a=0; b=1; Nx=101;
t0=0; tf=40; Nt=161; x=linspace(a,b,Nx); t=linspace(t0,tf,Nt);
sol=pdepe(m,@PDE1,@IC1,@BC1,x,t); u=sol(:,:,1); n=2:10; m=1:10;
u(m,n)' figure(1);
 surf(x,t,u);  rotate3d on; title('Heat equation. Surface plot of solution');
 xlabel('Distance x'); ylabel('Time t');
```

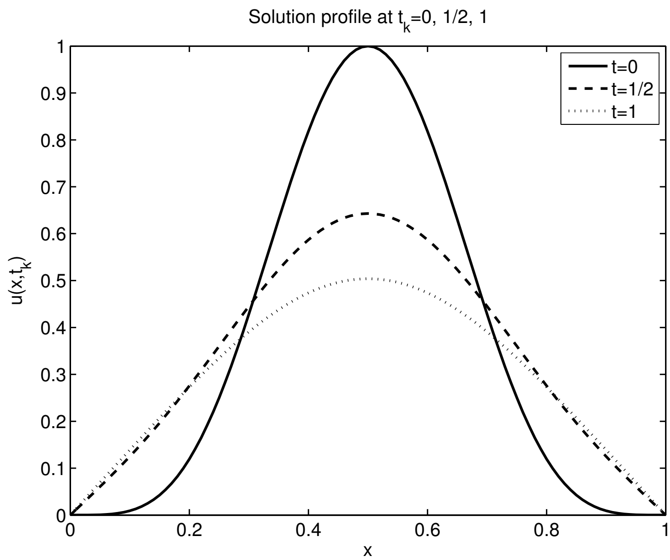


Figure 24.1 Surface profiles of the solution of the linear heat equation at $t_k = 0, \frac{1}{2}, 1$.

```
figure(2);
 plot(x,u(1,:),'k-',x,u(3,:),'k--',x,u(5,:),'k:','LineWidth',2);
 set(gca,'FontSize',14); set(gca,'FontName','Arial'); set(gca,'LineWidth',1);
 title('Solution profile at t_k=0, 1/2, 1');
 xlabel('x'); ylabel('u(x,t_k)'); legend('t=0','t=1/2','t=1');
figure(3);
 G=plot(x,u(1,:),'erase','xor');
 for k=2:length(t) set(G,'xdata',x,'ydata',u(k,:)); pause(0.7); end
 title('Animation of the solution'); xlabel('x'); ylabel('u(x,t_k)');
echo off
```

In this script M-file, we choose a grid of x and t values (`linsolve` functions), solve the linear equation (`pdepe` function), and extract the solution u from `sol` as `u=sol(:,:,1)`. Then we print some numerical values of the solution, plot it as a surface with interactive rotation (the `rotate3d` function), plot the surface profile at $t_k = 0, 1/2, 1$, and produce animations of the surface profile for $t \in [0, 40]$.

Before working with these new files that are located in a new working directory in the `Command Window`, we can make this directory the current working directory in MATLAB by using `cd` function, for example `cd('c:/lpde')`, or by selecting this directory in the `Desktop->Current Directory`, or by adding it to the path of directories that MATLAB searches to find files `File->SetPath->AddFolder`, or by using `addpath('c:/lpde')`. Finally, we run the file `SolPDE1` in the `Command Window`.

The text and numerical values display in the same `Command Window`, but graphics appear in separate graphics windows (`Figure1`, `Figure2`, and `Figure3`). Three surface profiles of the solution obtained at times $t_k = 0, \frac{1}{2}, 1$ are shown in Fig. 24.1.

Example 24.2. *Linear convection–diffusion equation.* Consider the linear convection–diffusion equation of the form

$$u_t = u_{xx} - k u_x,$$

where $k \geq 0$ is a constant. We find numerical and graphical solutions of the equation under the following boundary and initial conditions:

$$u_t = u_{xx} - ku_x \quad \text{in } \mathcal{D}, \quad u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = \begin{cases} 1, & x < a, \\ w(x), & a < x < b, \\ 0, & x > b, \end{cases}$$

where $k = 1$, $L = 10$, a and b are real parameters, and $w(x)$ is a real-valued function; e.g., we take $a = -1$, $b = 1$, $w(x) = \cos(x)$, and $\mathcal{D} = \{0 \le x \le 10,\ 0 \le t \le 10\}$.

As before, we rewrite the linear convection–diffusion equation with initial and boundary conditions in the form (24.2.1.1)–(24.2.1.3) and create three separate function M-files, `PDE2.m` for the functions c, f, and s, `IC2.m` for the function $u(x,0) = u_0(x)$, and `BC2.m` for the functions p and q.

For the convection–diffusion equation, we have $m = 0$, $c(x,t) = 1$, $f(x,t,u,u_x) = u_x$, and $s(x,t,u,u_x) = -ku_x$.

For the boundary conditions, we have $p(0,t,u) = u$, $q(0,t) = 0$, $p(L,t,u) = u$, $q(L,t) = 0$, and the discontinuous initial data $u_0(x)$ defined above.

These functions are specified in the function M-files as follows:

```
function [c,f,s]=PDE2(x,t,u,DuDx)
   c=1; f=DuDx; k=1; s=-k*DuDx;
function [pL,qL,pR,qR]=BC2(xL,uL,xR,uR,t)
   pL=uL; qL=0; pR=uR; qR=0;
function u0=IC2(x)
   a=-1; b=1;
   if x<a u0=1; elseif x>a && x<b u0=cos(x); else u0=0; end
```

Then, for constructing numerical and graphical solutions of this initial-boundary value problem for the linear convection–diffusion equation, we compose the script M-file `SolPDE2` as follows:

```
clear all; close all; echo on; format long; m=0; a=0; b=10; Nx=101;
t0=0; tf=10; Nt=101; x=linspace(a,b,Nx); t=linspace(t0,tf,Nt);
sol=pdepe(m,@PDE2,@IC2,@BC2,x,t); u=sol(:,:,1); figure(1);
 surf(x,t,u);  rotate3d on;
 title('Convection--diffusion equation. Surface plot of solution');
 xlabel('Distance x'); ylabel('Time t');
figure(2);
 plot(x,u(10,:),'k-',x,u(30,:),'k--',x,u(101,:),'k:','LineWidth',2);
 set(gca,'FontSize',14); set(gca,'FontName','Arial');
 set(gca,'LineWidth',1);
 title('Solution profile at t_k=0.9, 2.9, 10')
 xlabel('x'); ylabel('u(x,t_k)'); legend('t=0.9','t=2.9','t=10');
figure(3);
 G=plot(x,u(10,:),'erase','xor');
 for k=2:length(t) set(G,'xdata',x,'ydata',u(k,:)); pause(0.7); end
 title('Animation of the solution'); xlabel('x'); ylabel('u(x,t_k)');
echo off
```

We choose a grid of x and t values (`linsolve` functions), solve the linear convection–diffusion equation (`pdepe` function), and extract the solution u from `sol` as `u=sol(:,:,1)`. Then we plot the solution as a surface (with interactive rotation, `rotate3d` function), plot the surface profile at $t_k = 0.9, 2.9, 10$, and produce animations of the surface profile for $t \in [0,10]$. Three surface profiles of the solution obtained at times $t_k = 0.9, 2.9, 10$ are shown in Fig. 24.2.

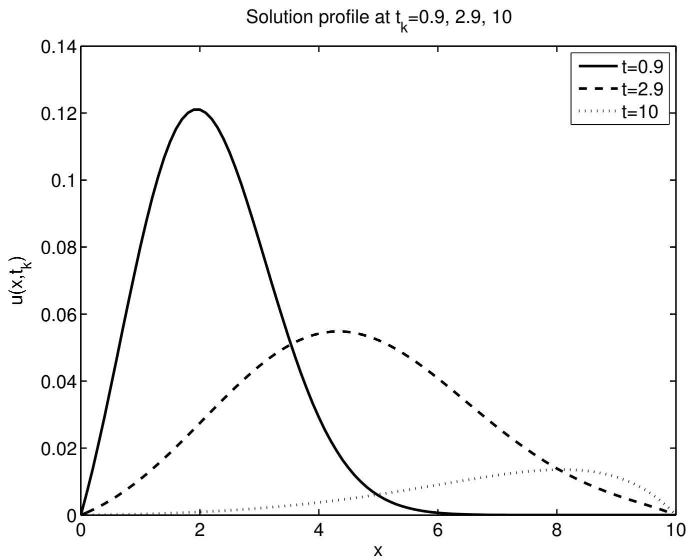


Figure 24.2 Surface profile evolution of the convection–diffusion equation for $t_k = 0.9, 2.9, 10$.

2D Partial Differential Equations

The `PDE Toolbox` allows one to solve 2D linear partial differential equations via a graphical user interface (GUI). The solution domains can be complicated and drawn by *GUI tools* using basic elements (circles, ellipses, rectangles, and polygons), their combinations, and basic set operations (union, difference, and intersection).

A solution procedure contains four stages: *PDE* for describing a partial differential equation, *ICs* for setting the initial conditions (ICs), *BCs* for setting the boundary conditions (BCs), *Domain* for drawing the solution domain.

The PDE GUI has the following functions:

Menu system for calling functions directly from the menu items and toolbar buttons.

Toolbar for defining the solution domains, setting parameters in the PDE, solving the PDE, visualizing the results.

Set formula for defining set operations (union, intersect, difference).

Solution regions for drawing solution domains, solving 2D PDE within the solution domain, visualizing the results (in 3D).

pdetool	Draw	Boundary	PDE	Mesh	Solve	Plot
pdeinit	pdecirc	pdeellip	pdepoly	pderect	pdebound	
assemb	initmesh	refinemesh	assempde	pdenonlin	parabolic	
hyperbolic	pdeeig	pdesurf	pdeplot	pdemesh	pdecont	

`pdetool` starts the `PDE Toolbox` that invokes the *graphical user interface* (GUI).

`Draw`, `Boundary`, `PDE`, `Mesh`, `Solve`, `Plot` are various icons (or modes) in the GUI menu corresponding to different stages of the solution process.

`pdeinit`, `pderect`, `pdebound`, `initmesh`, `assempde`, `pdenonlin`, `pdeplot` are examples of functions (which correspond to different modes of the GUI interface) for writing programs.

For starting the `PDE Toolbox`, one should type `pdetool` in the `Command Window` (in the MATLAB prompt). This invokes the graphical user interface (in a separate window), which is appropriate for simple or standard PDE problems. For more complicated problems and full control over the numerical process, it is best to write programs and call them from the prompt or M-files.

The `PDE Toolbox` allows solving three classes of PDE (see Section 24.1.1) and the eigenvalue problems defined on a bounded domain $\mathcal{D} \in \mathbb{R}^2$:

$$\begin{aligned} -\nabla\cdot(c\nabla u)+au &= f, && \text{elliptic equations.}\\ du_t-\nabla\cdot(c\nabla u)+au &= f, && \text{parabolic equations.}\\ du_{tt}-\nabla\cdot(c\nabla u)+au &= f, && \text{hyperbolic equations.}\\ -\nabla\cdot(c\nabla u)+au &= \lambda du, && \text{eigenvalue problems.} \end{aligned}$$

Here c is the scalar coefficient, λ is an unknown eigenvalue, and a, d, f, and u are complex-valued functions defined on $\mathcal{D}$. For linear PDEs, the coefficients a, d, and c do not depend on the unknown solution u. For hyperbolic and parabolic PDEs, the coefficients can depend on time.

The `PDE Toolbox` allows one to define the following boundary conditions:

$$h(\mathbf{x},t)u\Big|_{\partial\mathcal{D}} = r(\mathbf{x},t), \qquad \left[\frac{\partial}{\partial\mathbf{n}}(c\nabla u)+qu\right]\Bigg|_{\partial\mathcal{D}} = g,$$

which, respectively, are referred to as the Dirichlet (or essential) boundary conditions, generalized Neumann (or natural) boundary conditions,[4] and the mixed boundary conditions (a combination of Dirichlet and generalized Neumann conditions). Here $\partial\mathcal{D}$ denotes the boundary of the solution domain, $\frac{\partial}{\partial\mathbf{n}}$ is the partial derivative of vector $\mathbf{x}$ in the normal direction, and g, q, h, and r are complex-valued functions defined on the boundary $\partial\mathcal{D}$. For linear PDEs, the coefficients g, q, h, and r do not depend on the unknown solution u. For eigenvalue problems, $g=0$, $r=0$, and for hyperbolic and parabolic PDEs, the coefficients can depend on time.

Let us briefly describe the solution process for simple problems via the graphical user interface (GUI). There are various icons (or modes) in the GUI menu that correspond to various stages of the solution process:

`Draw`, for defining the domain $\mathcal{D}$ and the geometry (using the constructive solid geometry (CSG) model paradigm), e.g., by combining various solid objects (rectangle, circle, etc.) using set formulas.

[4]In some contexts, the generalized Neumann boundary conditions are also referred to as the Robin boundary conditions.

`Boundary`, for specifying the boundary conditions (e.g., distinct types of boundary conditions on distinct boundary segments).

`PDE`, for interactively specifying the type of the PDE and the coefficients d, c, a, f (e.g., specifying the coefficients for each subdomain independently).

`Mesh`, for generating and plotting meshes and controlling the parameters of the automated mesh generator.

`Solve`, for invoking and controlling the solvers (e.g., adaptive mode); for specifying the initial conditions and the times for which the solution should be constructed (for parabolic or hyperbolic PDEs); for specifying the interval in which to search for eigenvalues (for eigenvalue problems).

`Plot`, for visualizing numerical solutions (inside the GUI and in separate figures); for simultaneously plotting three distinct solution properties (using color, height, and vector field plots); for plotting surfaces, meshes, contours, and arrows (quivers); for producing animations of solutions (for parabolic or hyperbolic PDEs).

Remark. After solving a problem, it is possible to refine the mesh (by returning to `Mesh`) and then solve the problem again and again. This can also be done by using an adaptive mesh refiner and solver.

Example 24.3. *Linear Poisson equation.* Consider the boundary value problem for the two-dimensional linear Poisson equation

$$u_{xx}+u_{yy}=x+y, \quad u(x,0)=0, \quad u(x,L_2)=0, \quad u(0,y)=0, \quad u(L_1,y)=w_1(y),$$

describing a potential field $u(x,y)$ in a bounded domain $\mathcal{D}=\{0\le x\le L_1,\ 0\le y\le L_2\}$. Here $L_1=3$, $L_2=6$, and $w_1(y)=\cos(y)$.

First, we solve this problem with the aid of the graphical user interface (GUI).

1°. We draw the domain $\mathcal{D}=\{0\le x\le 3,\ 0\le y\le 6\}$. By selecting `Options->Grid`, we generate the grid. Then, by selecting `Options->Axes Limits`, we change the size of the default domain; i.e., we put the new size $[-1,4]$ for the x-axis and $[-1,8]$ for the y-axis, `Apply`, and `Close`. By selecting `rectangle`-icon, we draw a rectangle with the aid of the mouse (click-drag); its name `R1` is introduced by MATLAB. The exact coordinates of the cursor appear in the top right of the window. To check and correct our coordinates, we double-click on `R1` and enter the exact coordinates.

2°. We specify the boundary conditions. If we select `Boundary->Boundary Mode`, the boundary will appear as a red arrow, indicating the type of condition (red, blue, and green for Dirichlet, Neumann, and mixed conditions) and the direction. Then, by double-clicking on the bottom horizontal line, we set the first boundary condition $u(x,0)=0$ by introducing $h=1$ and $r=0$ in the corresponding menu (since the Dirichlet boundary conditions have the form $hu=r$). Next, we specify the remaining three boundary conditions. Then, by selecting `File->Save As`, we save this domain as M-file `LinearPoisson1.m`.

3°. By selecting `PDE->PDE Specification`, we define the type of the PDE: `Elliptic` (the type of the linear Poisson equation), `c=-1`, `a=0`, and `f=x+y`.

4°. By selecting `Mesh->Initialize Mesh`, we generate a triangular finite element mesh. Then, by selecting a `triangle-inside-a-triangle` icon, we can refine the initial mesh.

5°. By selecting `Solve->Parameters`, we specify some parameters for solving the linear partial differential equation: `adaptive mode`, `maximum number of triangles`, `maximum number of refinements`,

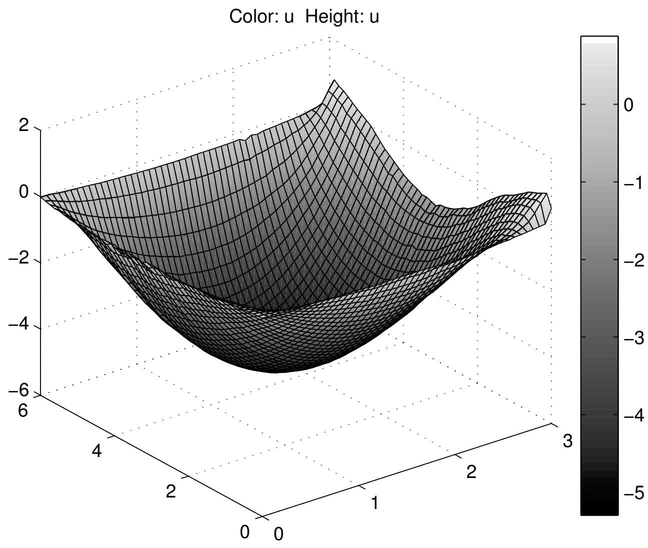


Figure 24.3 Surface plot of the solution of the linear Poisson equation.

etc. Then, selecting `Solve->Solve PDE`, we obtain the numerical solution. By default, the solution appears as a color-scale contour plot.

6°. By selecting `Plot->Parameters`, we can set various options for generating a figure, e.g., `Height (3-D plot)`, `Plot in x-y grid`, `Color`, `colormap` (`gray`), and `Show Mesh`. Then, by selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained (in the MATLAB graphics editor) and create the final plot shown in Fig. 24.3).

Remark. The MATLAB graphics editor has various options for creating an appropriate figure (e.g., interactive rotation, `Tools->Rotate 3D`).

In addition, we show how to solve this problem in another way, i.e., by writing an M-file and by applying the *export facilities* of the GUI. This can be very useful when solving complicated problems.

1°. We type `pdetool` and open the M-file (created recently) `LinearPoisson1.m`. We have created the domain $\mathcal{D}$ and defined boundary conditions using graphical user interface (GUI). Then we export the *decomposed geometry* (i.e., the domain $\mathcal{D}$) and the *boundary conditions* to the workspace by selecting `Boundary->Export Decomposed Geometry, Boundary Conditions`. By default, the decomposed geometry and boundary conditions are described by g and d, respectively. In our case, g is `Decomposed Geometry Matrix` and d is `Boundary Condition Matrix`.

2°. We create the triangular mesh using the graphical user interface (GUI). Then, selecting `Mesh -> Export Mesh`, we export the mesh to the workspace. By default, the mesh is described by p, e, and t. In our case, p is `Point Matrix`, e is `Edge Matrix`, and t is `Triangle Matrix`.

3°. Now we can solve the linear Poisson equation with the following M-file `LinearPoisson2.m`:

```
echo on; format long; g, b, p, e, t c=-1; a=0; f='x+y';
[p, e, t]=refinemesh(g, p, e, t); u=assempde(b, p, e, t, c, a, f);
pdeplot(p,e,t,'zdata',u,'mesh','on') echo off
```

In this script M-file, we use the export variables, g,b,p,e,t, the PDE coefficients, `c=-1`, `a=0`, `f=x+y`, for solving PDEs with the aid of the `assempde` function. This is the basic *Partial Differential*

Equation Toolbox function, which assembles a PDE problem by using the finite element formulation. Also, we refine the mesh, and the new mesh is returned using the same matrix variables. Finally, we draw the solution and the mesh.

This problem can also be solved with the `pdenonlin` function and an additional parameter, the tolerance, $tol{=}1e{-}2$, as follows:

```
echo on; format long; g, b, p, e, t c=-1; a=0; f='x+y';  tol=1e-2;
[p, e, t]=refinemesh(g, p, e, t); u=pdenonlin(b, p, e, t, c, a, f,
'Tol',tol) pdeplot(p, e, t, 'zdata', u, 'mesh', 'on') echo off
```

Remark. The M-file `LinearPoisson1.m` can be observed by `type LinearPoisson1` and modified using MATLAB editor.

24.2.2 Numerical Methods Embedded in MATLAB

In MATLAB, partial differential equations or systems in one space dimension (1D) and time, which arise in many applications, can be solved with the aid of the predefined function `pdepe` (provided in MATLAB `PDEToolbox`).

Implementing the *method of lines* (in a general setting), the function `pdepe` converts the PDE into an ODE using a second-order accurate *spatial discretization* [for details, see Skeel and Berzins (1990), Schiesser and Griffiths (2009), Lee and Schiesser (2004)] (i.e., by replacing only the spatial derivatives with finite differences) based on a fixed set of nodes or a mesh `xMesh` (represented as an array), where `xMesh(1)=a`, `xMesh(end)=b`, and `xMesh(i)<xMesh(i+1)` (for i=2, ..., `end-1`). Then the resulting ODEs are integrated (with a stiff ODE solver `ode15s`) to obtain approximate solutions at times `tSpan`, where `tSpan(1)=t_0`, `tSpan(end)=t_f`, and `tSpan(i)<tSpan(i+1)` (for i=2, ..., `end-1`).

Partial differential equations or systems in two space dimensions (2D) and time, which arise in a wide variety of phenomena in all branches of science and engineering [e.g., see Schiesser (1994), Cook, Malkus, and Plesha (1989)], can be solved with the aid of the `PDEToolbox`, which allows one to define a PDE problem (geometry, boundary conditions, and a PDE or a system of PDEs), numerically solve the problem (generate meshes, discretize the equations, and find approximation to the solution), and visualize the results. PDEs are solved using the *Finite Element Method* (FEM).

24.2.3 Numerical Solutions of Cauchy Problems

In this section, we consider the method of characteristics, which allows us to reduce a PDE to a system of ODEs along which the given PDE with some initial data (the Cauchy data) is integrable. Once the system of ODEs is found, it can be solved along the characteristic curves and transformed into a general solution for the original PDE [e.g., see Schiesser and Griffiths (2009)].

Example 24.4. *Linear first-order equation. Method of characteristics. Classical Cauchy problem.* Consider the initial value problem

$$u_x - u_y = 1, \quad u(x,0) = x^n.$$

By applying the method of characteristics, we can obtain

$$\frac{dU(x)}{dx} = 1, \quad U(x) = x + X_0, \quad \frac{dY(x)}{dx} = -1,$$

and the characteristic curves are defined as $Y(x) = -x + X_0$, where $x \in [x_0, x_f]$. Also we can find the solution of this Cauchy problem, $u(x,y) = (x+y)^n - y$, and plot the characteristic curves as follows:

```
clear all; close all;
x=0:0.1:9; hold on; figure(1) for x0=0:1:9
ChCur=-x+x0; plot(ChCur,x,'k-','LineWidth',2); end hold off;
set(gca,'XLim',[0 9],'YLim',[0 9]);
```

Example 24.5. *Linear Euler equation. Method of characteristics. Cauchy problem.* Consider the initial value problem for the linear Euler equation

$$xu_x + yu_y = nu, \quad u(x,1) = 1 + e^{-|x|},$$

where $n = 1$. By applying the method of characteristics, we obtain the characteristic ODEs with the initial conditions

$$\frac{dX(t)}{dt} = X(t), \quad \frac{dY(t)}{dt} = Y(t), \quad \frac{dU(t)}{dt} = U(t),$$
$$X(0) = s, \quad Y(0) = 1, \quad U(0) = 1 + e^{-|s|}.$$

We can also find the solution of this Cauchy problem,

$$\left(se^t, \quad e^t, \quad (1 + e^{-|s|})e^t\right),$$

and plot the characteristic curves and the solution (whose graph is an integral surface). See Fig. 24.4.

```
s=linspace(-5,5,21); t=linspace(0,1,20); axesM=[-5 5 0 5 0 5];
f=[s; ones(size(s)); 1+exp(-abs(s))];
for i=1:length(s)
 [t,x]=ode45(@CharODEs,t,f(:,i)');
 X(i,:)=x(:,1); Y(i,:)=x(:,2); U(i,:)=x(:,3);
end
plot3(f(1,:),f(2,:),f(3,:),'k','LineWidth',2);
axis(axesM) hold on
X([find(X<axesM(1)|X>axesM(2)|Y<axesM(3)|Y>axesM(4))])=NaN;
surf(X,Y,U), shading flat plot3(X',Y',zeros(size(U')),'k')
xlabel x, ylabel y
```

We solve this Cauchy problem numerically with the aid of the MATLAB function `ode45`. This function is the most widely used initial value problem solver for ODEs, where the variable-step 4/5 Runge–Kutta–Felhberg algorithm is implemented. We describe the ODEs using the following *function M-file*:

```
function dxdt=CharODEs(t,x) dxdt=x;
```

Remark. It should be noted that the Maple function `PDEplot` (see Chapter 15) plots solutions of Cauchy problems for general scalar first-order PDEs.

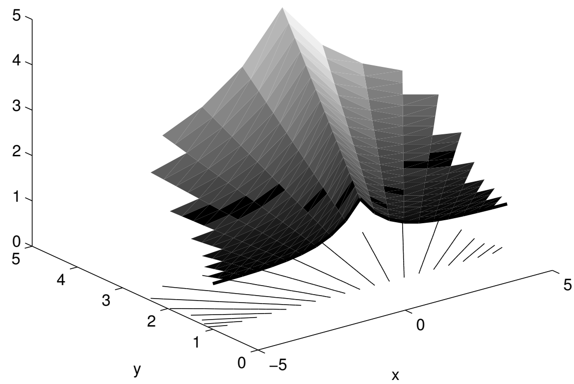


Figure 24.4 Solution surface and characteristic curves for the linear Euler equation.

24.2.4 Numerical Solutions of Initial-Boundary Value Problems

In this section, we apply the MATLAB `PDEToolbox` to solve some initial-boundary value problems for parabolic and hyperbolic equations with the aid of the graphical user interface (GUI). The procedure for solving linear parabolic and hyperbolic PDEs is similar to the case of elliptic PDEs. Some modifications should be introduced with respect to the PDE description (its coefficients, solve parameters, and visualizations).

For example, for parabolic PDEs, the *solve parameters* are *time* (a MATLAB vector of times at which a solution of the parabolic PDE should be generated), *the initial value* $u(t_0)$ (the initial value can be a constant or a column vector of values on the nodes of the current mesh), *relative* and *absolute tolerance* (relative and absolute tolerance parameters for the ODE solver applied to solve the time-dependent part of the parabolic PDE problem). For hyperbolic PDEs, the *solve parameters* are the same except for the initial values: these are $u(t_0)$ and $u'(t_0)$.

Example 24.6. *Linear heat equation with radiation.* Consider the initial-boundary value problem for the two-dimensional linear heat equation

$$u_t = \nu(u_{xx} + u_{yy}) - \mu(u - u_0), \quad 0 < x < L_1, \quad 0 < y < L_2, \quad t > 0,$$
$$u(x,y,0) = w(x,y), \quad u(0,y,t) = 0, \quad u(L_1,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,L_2,t) = 0,$$

describing the temperature distribution $u(x,y,t)$ in a rectangular plate $\mathcal{D} = \{0 \le x \le L_1,\ 0 \le y \le L_2\}$ with radiation (from the surface). Here ν, μ, u_0 are constants. Let $L_1 = L_2 = 4$, $\nu = 0.01$, $\mu = 100$, $u_0 = 0.2$, and $w(x,y) = \mathrm{e}^{-x}\cos y$.

1°. We draw the domain $\mathcal{D} = \{0 \le x \le 4,\ 0 \le y \le 4\}$:

`Options->Grid`: we generate the grid.

`Options->Axes Limits`: we change the size of the default domain, i.e., we put the new size $[0,5]$ for the x-axis and $[0,5]$ for the y-axis, `Apply`, and `Close`.

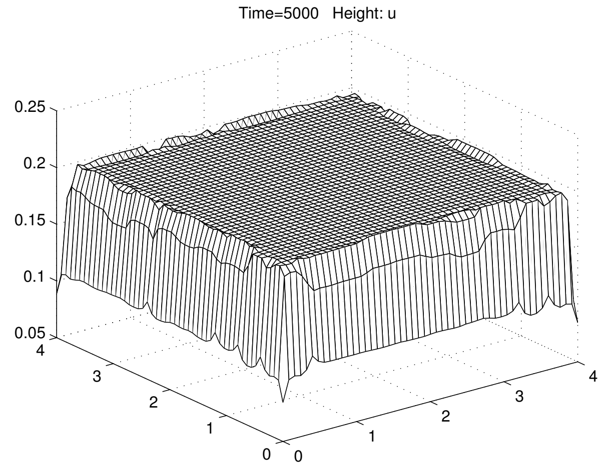


Figure 24.5 The mesh plot of the solution of the linear heat equation ($t = 5000$).

`rectangle`-icon: we draw a rectangle with the aid of the mouse (click-drag); its name `R1` is introduced by MATLAB. The exact coordinates of the cursor appear in the top right of the window. To check and correct our coordinates, we double-click on `R1` and enter the exact coordinates.

2°. We specify the boundary conditions. If we select `Boundary->Boundary Mode`, the boundary will appear as a red arrow, indicating the type of condition (Dirichlet conditions) and the direction. By double-clicking on the bottom horizontal line, we set the first boundary condition $u(x,0,t) = 0$ by introducing $h = 1$ and $r = 0$ in the corresponding menu.[5] Next, we specify the remaining three boundary conditions.

`File->Save As`: we save this domain as M-file `LinearHeatRad2D.m`.

3°. We specify the PDE: selecting `PDE->PDE Specification`, we define the type of the PDE: `Parabolic` (the type of the linear heat equation) and the coefficients: `c=0.01`, `a=100`, `f=20` ($f = \mu u_0 = 100 \times 0.2 = 20$), and `d=1`.

4°. We generate a triangular finite element mesh: selecting `Mesh->Initialize Mesh`.

`triangle-inside-a-triangle` icon: we can refine the initial mesh.

5°. We specify parameters for solving the linear parabolic equation: `Solve->Parameters`. `Time` (the time range): `0:100:5000`; `u(t0)` (the initial condition): `exp(-x).*cos(y)`; `Relative tolerance`: `0.01`, `Absolute tolerance`: `0.001`.

`Solve->Solve PDE`: we obtain the numerical solution. By default, the solution appears as a color-scale contour plot.

6°. We can set various options for generating a figure: `Plot->Parameters`, e.g., `Height (3-D plot)`, `Plot in x-y grid`, `Color`, `colormap (gray)`, and `Show Mesh`.

[5]The Dirichlet boundary conditions have the form $hu = r$.

By selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained (in the MATLAB graphics editor) and create the final plot shown in Fig. 24.5).

Example 24.7. *Linear nonhomogeneous wave equation.* Consider the initial-boundary value problem for the two-dimensional linear wave equation

$$\begin{aligned}
&u_{tt} = \nu^2(u_{xx}+u_{yy})+F(x,y,t), \quad 0<x<L_1, \quad 0<y<L_2, \quad t>0,\\
&u(x,y,0)=w_1(x,y), \quad u_t(x,y,0)=w_2(x,y),\\
&u(0,y,t)=0, \quad u(L_1,y,t)=0, \quad u(x,0,t)=0, \quad u(x,L_2,t)=0,
\end{aligned}$$

describing the vibrations of a membrane of length L_1 and width L_2. Let us find the displacement function $u(x,y,t)$. Let $L_1=L_2=\pi$, $\nu=1$, $F(x,y,t)=xy\sin t$, $w_1(x,y)=\cos x\sin y$, and $w_2(x,y)=\cos y\sin x$.

1°. We draw the domain $\mathcal{D}=\{0\le x\le\pi,\ 0\le y\le\pi\}$:

`Options->Grid`: we generate the grid.

`Options->Axes Limits`: we change the size of the default domain, i.e., we put the new size $[0,4]$ for the x-axis and $[0,4]$ for the y-axis, `Apply`, and `Close`.

`rectangle`-icon: we draw a rectangle with the aid of the mouse (click-drag); its name `R1` is introduced by MATLAB. The exact coordinates of the cursor appear in the top right of the window. To check and correct our coordinates, we double-click on `R1` and enter the exact coordinates (`0`, `0`, `pi`, `pi`).

2°. We specify the boundary conditions: `Boundary->Boundary Mode`, the Dirichlet conditions. By double-clicking on the bottom horizontal line, we set the first boundary condition $u(x,0,t)=0$ by introducing $h=1$ and $r=0$ in the corresponding menu. Next, we specify the remaining three boundary conditions.

`File->Save As`: we save this domain as M-file `LinearWave2D.m`.

3°. We specify the PDE: by selecting `PDE->PDE Specification`, we define the type of the PDE: `Hyperbolic` (since deal with a linear wave equation) and set the coefficients: `c=1.0`, `a=0.0`, `f=x.*y.*sin(t)`, and `d=1`.

4°. We generate a triangular finite element mesh by selecting `Mesh->Initialize Mesh`.

`triangle-inside-a-triangle` icon: we can refine the initial mesh.

5°. We specify parameters for solving the linear hyperbolic equation: `Solve->Parameters. Time` (the time range): `0:7`; `u(t0)` (the initial condition): `cos(x).*sin(y)`; `u'(t0)` (the second initial condition): `cos(y).*sin(x)`; `Relative tolerance`: `0.01`, `Absolute tolerance`: `0.001`.

`Solve->Solve PDE`: we obtain the numerical solution. By default, the solution appears as a color-scale contour plot.

6°. We can set various options for generating the figure: `Plot->Parameters`, e.g., `Height (3-D plot)`, `Plot in x-y grid`, `Color`, `colormap (gray)`, and `Show Mesh`.

By selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained (in the MATLAB graphics editor) and create the final plot shown in Fig. 24.6).

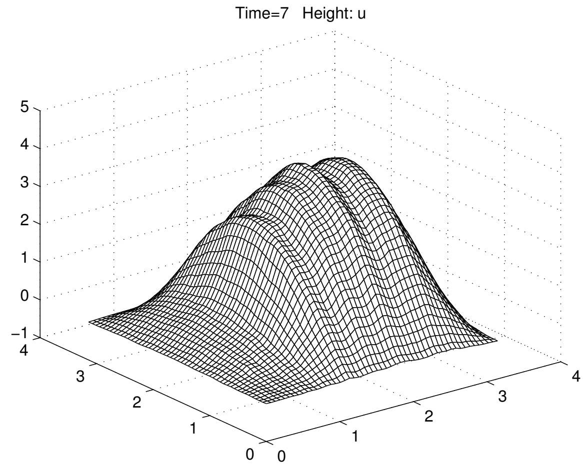


Figure 24.6 The mesh plot of the solution of the linear nonhomogeneous wave equation ($t = 7$).

24.2.5 Numerical Solutions of Boundary Value Problems

The `PDE Toolbox` provided by MATLAB allows one to describe and work with more complicated geometries, which can consist of various solid objects. Various geometric models can be constructed interactively (using `pdetool` GUI) or via M-files (using `pdegeom` and `pdebound`).

We can devise a *Constructive Solid Geometry* (CSG) model of the geometry by drawing solid objects that can overlap. By default, each object is assigned a unique name (e.g., `C1`, `R1`), which can be changed. There are four types of solid objects: *circle*, *polygon*, *rectangle*, and *ellipse*, which represent the set of points inside an object. It is possible to move, rotate, cut, and paste selected objects.

The CSG models are described by the *Geometry Description Matrix* `gd`. The current `gd` matrix can be viewed in the main workspace by selecting the `Draw->Export Geometry Description`, `Set Formula`, `Labels`. Each column in the `gd` matrix corresponds to an object in the CSG model.

By default, the resulting geometric model is the union of all objects. However, the solid objects can be combined by typing a *set formula* (which is displayed in the GUI and can be changed, e.g., `R1-C1-C2`). The resulting geometrical model is the set of points for which the set formula evaluates to true.

A set formula `sf` is expressed with the set of variables (that correspond to the name of each object) and the operators `+ * -` (that correspond to the set operations: union, intersection, and set difference, respectively).

Example 24.8. *Linearized Poisson–Boltzmann equation. Constructive solid geometry.* Consider the following boundary value problem for the two-dimensional linearized Poisson–Boltzmann equation defined on an irregular domain $\mathcal{D} = R_1 - \Omega$:

$$u_{xx} + u_{yy} = \nu u - F(x,y), \tag{24.2.5.1}$$

$$u(x,-1) = x^2 + 1 + e^{-x}\sin(-1), \quad u(x,1) = x^2 + 1 + e^{-x}\sin(1), \tag{24.2.5.2}$$

$$u_x(-1,y) = 0, \quad u_y(1,y) = 0, \tag{24.2.5.3}$$

$$u\big|_{\partial E_1} = 0.1, \quad u\big|_{\partial E_2} = 2.0, \tag{24.2.5.4}$$

where ν is a positive constant (e.g., $\nu = 1$) and $F(x,y) = x^2 + y^2 + e^{-x}\sin(y)$.

R_1 is a rectangle, $R_1 = [-1,1]\times[-1,1]$ with boundary ∂R_1. Ω is an arbitrary multiply connected domain within R_1 with boundary $\partial\Omega$. In our case, it is the interior region of the two ellipses E_1 and E_2 defined by the equations $(x+0.35)^2/0.25^2 + (y+0.35)^2/0.3^2 - 1 = 0$ and $(x-0.35)^2/0.2^2 + (y-0.35)^2/0.2^2 - 1 = 0$.

The linearized Poisson–Boltzmann equation is defined outside Ω but within the rectangle R_1.

Along the boundary $\partial\Omega$, the Dirichlet boundary condition (24.2.5.4) is defined. According to different applications, different boundary conditions can be imposed along the boundary of the rectangle R_1; e.g., we set the Dirichlet boundary conditions (24.2.5.2) and the Neumann boundary conditions (24.2.5.3).

The linearized Poisson–Boltzmann equation (24.2.5.1) has been used in many applications, e.g., models describing bio-molecular processes and electrostatic interactions between colloidal particles [e.g., see Fogolari, Zuccato, Esposito, and Viglino (1999)].

We solve this problem with the aid of the graphical user interface (GUI) in the `Generic Scalar` mode (by selecting `Options->Application->Generic Scalar`).

1°. We draw the domain $\mathcal{D} = R_1 - \Omega = R_1 - E_1 - E_2$:

- `Options->Grid` and `Options->Axes Limits`: we generate the grid.
- We draw the rectangle `R1`, $[-1,-1]\times[1,1]$.
- We draw the the ellipse `E1` with the parameters `X-center`, -0.35, `Y-center`, -0.35, `A-semiaxes`, 0.25, `B-semiaxes`, 0.3, `Rotation`, 0.
- We draw the the ellipse `E2` with the parameters `X-center`, 0.35, `Y-center`, 0.35, `A-semiaxes`, 0.2, `B-semiaxes`, 0.2, and `Rotation`.
- We draw the domain $\mathcal{D} = R_1 - \Omega = R_1 - E_1 - E_2$ by typing the set formula `R1-E1-E2`.

2°. We specify the boundary conditions:

- If we select `Boundary->Boundary Mode`, the boundaries will appear as red arrows, indicating the type of conditions (i.e., Dirichlet conditions) and the direction.
- By double-clicking on the bottom horizontal line, we set the first boundary condition in (24.2.5.2), $u(x,-1) = x^2 + 1 + e^{-x}\sin(-1)$, by introducing `h=1` as well as `r=x.^2+1+exp(-x).*sin(-1)` in the corresponding menu.
- We specify the remaining Dirichlet boundary conditions on the upper horizontal line by typing `h=1` and `r=x.^2+1+exp(-x).*sin(1)`, and on the two ellipses `E1` and `E2` by typing, respectively, `h=1`, `r=0.1` and `h=1`, `r=2.0`.
- We specify the two remaining Neumann boundary conditions by typing `q=0` and `g=0`.
- By selecting `File->Save As`, we save this domain as M-file `LineqrizedPBEqGeom2D.m`.

3°. By selecting `PDE->PDE Specification`, we define the type of the PDE:

- `Elliptic`, `c=-1.0`, `a=-1.0`, `f=-(x.^2+y.^2+exp(-x).*sin(y))`.

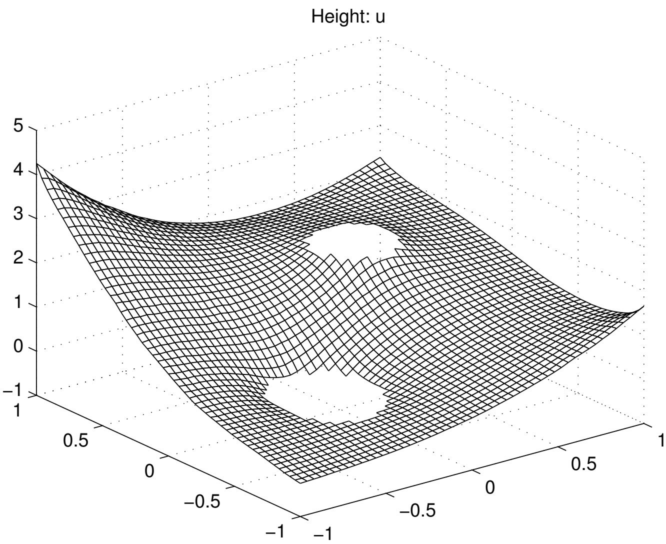


Figure 24.7 The mesh plot of the solution of the linearized Poisson–Boltzmann equation.

4°. By selecting `Mesh->Initialize Mesh`, we generate a triangular finite element mesh.

- By selecting `Mesh->Refine Mesh`, we can refine the initial mesh (e.g., two times).
- By selecting `Mesh->Jiggle Mesh`, we can jiggle the mesh (for improving the triangle quality).

5°. By selecting `Solve->Parameters`, we specify some parameters for solving a linear PDE:

- `Adaptive mode`, `Maximum number of triangles`, `Maximum number of refinements`.
- `Triangle selection method`, `Relative tolerance`, `Refinement method`.
- By selecting `Solve->Solve PDE`, we obtain the numerical solution.

6°. By selecting `Plot->Parameters`, we can set various options for generating a figure, e.g., `Height (3-D plot)`, `Color`, `colormap` (gray), `Show Mesh`.

- By selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained and create the final plot shown in Fig. 24.7.

Example 24.9. *Linear Poisson equation. Comparison of analytical and numerical solutions.* Consider the following boundary value problem for the two-dimensional linear Poisson equation defined on the domain $\mathcal{D} = \{0 \le x \le 1,\ 0 \le y \le 1\}$:

$$-(u_{xx} + u_{yy}) = F(x,y), \quad 0 < x < 1, \quad 0 < y < 1, \tag{24.2.5.5}$$

$$u(x,0) = 0, \quad u(x,1) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0, \tag{24.2.5.6}$$

where $F(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$ and $u(x,y) = \sin(\pi x)\sin(\pi y)$ is the analytical solution of this boundary value problem.

As before (see the previous examples), we solve this problem with the aid of the graphical user interface (GUI) and then compare the numerical and analytical solutions.

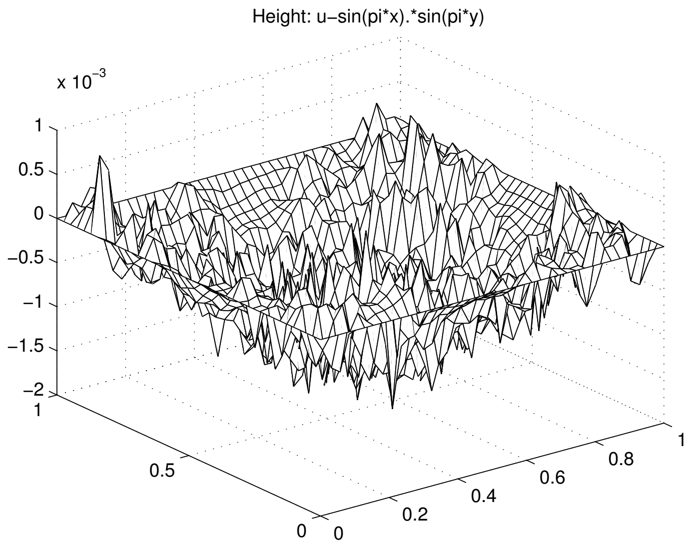


Figure 24.8 The difference between the numerical and analytical solutions of the linear Poisson equation.

1°. We draw the domain $\mathcal{D} = \{0 \le x \le 1,\ 0 \le y \le 1\}$.

2°. We specify the homogeneous Dirichlet boundary conditions.

3°. We specify the PDE: `PDE->PDE Specification`:

- `Elliptic`, `c=1.0`, `a=0.0`, `f=2*pi^2.*sin(pi*x).*sin(pi*y)`.

4°. We generate a triangular finite element mesh.

5°. We specify some parameters for solving a linear PDE and obtain the numerical solution.

6°. We plot the difference between the numerical and analytical solutions:

- `Plot->Parameters`, change the entry `u` into `user entry` in the `Height (3-D plot)` row and write `u-sin(pi*x).*sin(pi*y)` into the corresponding field in the `User entry` column.
- We set various options for generating a figure (`Height (3-D plot)`, `Color`, `colormap` (gray), `Show Mesh`).
- By selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained and create the final plot shown in Fig. 24.8.

⊙ *References for Section 24.2*: R. D. Cook, D. S. Malkus, and M. E. Plesha (1989), R. D. Skeel and M. Berzins (1990), W. E. Schiesser (1994), D. Harries (1998), F. Fogolari, P. Zuccato, G. Esposito, and P. Viglino (1999), H. J. Lee and W. E. Schiesser (2004), W. E. Schiesser and G. W. Griffiths (2009).

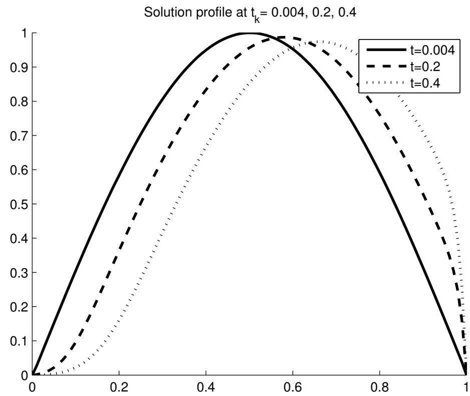


Figure 24.9 Linear diffusion–convection equation: solution profiles at $t_k = 0.004,\ 0.2,\ 0.4$.

24.3 Constructing Finite Difference Approximations

Now we show the helpful role of MATLAB when applying and developing various finite difference approximations to construct numerical solutions of initial value problems and initial-boundary value problems for linear PDEs (e.g., parabolic and hyperbolic equations) [for details, see Li and Chen (2009), Mathews and Fink (1999), Yang, Cao, Chung, and Morris (2005)].

To approximate a linear PDE by finite differences, we have to generate a mesh in a domain $\mathcal{D}$; e.g., $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$.

We assume (for simplicity) that the sets of lines of the mesh are equally spaced and the dependent variable in the given PDE is $u(x,t)$. We write h and k for the line spacings and define the mesh points as follows:

$$x_j = x_L + (j-1)h, \quad t_n = t_0 + (n-1)k,$$

where $j = 1,\ldots,NX+1$, $n = 1,\ldots,NT+1$, $h = (x_R - x_L)/NX$, and $k = (t_f - t_0)/NT$.

We calculate approximations to the solution at these mesh points; these approximations will be denoted by $u_{j,n} \approx u(x_j,t_n)$. We approximate the derivatives in the given equation by finite differences (of various types) and then solve the resulting difference equations.

24.3.1 Explicit Finite Difference Solutions

Let us study some numerical methods, namely, explicit and implicit finite difference methods, for solving linear partial differential equations.

Example 24.10. *Linear diffusion–convection equation. Forward difference method.* Consider the initial-boundary value problem for the linear diffusion–convection equation

$$u_t = \nu u_{xx} - \mu u_x, \quad u(x,0) = \sin(\pi x), \quad u(x_L,t) = 0, \quad u(x_R,t) = 0$$

in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$, where $x_L = 0$, $x_R = 1$, $t_0 = 0$, $t_f = 0.4$, $\nu = 0.009$, $\mu = 0.4$.

Let us generate the rectangular mesh

$$x_j = x_L + (j-1)h, \quad t_n = nk,$$

where $j = 1,\dots,NX+1$, $n = 1,\dots,NT$, $h = (x_R - x_L)/NX$, and $k = t_f/NT$. We denote the approximate solution of $u(x,t)$ at the mesh point (j,n) by $u_{j,n}$.

In the *forward difference method*, the second derivative u_{xx} is replaced by the central difference approximation (CDA), and the first derivatives u_t and u_x are replaced by the forward difference approximation (FWDA). The finite difference scheme for the linear heat equation has the form

$$u_{j,n+1} = u_{j,n} + r(u_{j+1,n} - 2u_{j,n} + u_{j-1,n}) - (k/h)\mu(u_{j+1,n} + u_{j,n}),$$

where $r = \nu k/h^2$.

In this *explicit finite difference scheme*, the unknown value $u_{j,n+1}$ (at the $(n+1)$st step) is determined from three known values $u_{j-1,n}$, $u_{j,n}$, and $u_{j+1,n}$ (at the nth step).

We construct an approximate numerical solution of the initial-boundary value problem by applying the forward difference method, and plot (see Fig. 24.9) the solution profiles at various times (for example, $t_k = 0.004,\ 0.2,\ 0.4$) as follows:

```
clear all; close all; nu=0.009; L=1; xL=0; xR=1; NX=100; t0=0;
tf=0.4; NT=100; h=(xR-xL)/NX; k=(tf-t0)/NT; r=nu*k/h^2; mu=0.4;
x=xL:h:xR; u=zeros(NX+1,NT); f=sin(pi*x/L); for n=1:NT  t=n*k;
  if n==1
   for j=2:NX u(j,n)=f(j)+r*(f(j+1)-2*f(j)+f(j-1))-k/h*mu*(f(j+1)-f(j)); end;
   u(1,n)=0; u(NX+1,n)=0;
  else
   for j=2:NX
    u(j,n)=u(j,n-1)+r*(u(j+1,n-1)-2*u(j,n-1)+u(j-1,n-1))-...
           k/h*mu*(u(j+1,n-1)-u(j,n-1));
   end;
   u(1,n)=0; u(NX+1,n)=0;
  end
end; T=1*k:k:NT*k; figure(1);
 colormap(gray); surf(x,T,u'); xlabel('x'); ylabel('y'); zlabel('u');
 title('Numerical solution of the linear diffusion--convection equation');
figure(2);
 hold on; plot(x,u(:,1),'k-',x,u(:,50),'k--',x,u(:,NT),'k:','LineWidth',2);
 title('Solution profile at t_k= 0.004, 0.2, 0.4');
 legend('t=0.004','t=0.2','t=0.4');
figure(3);
 G=plot(x,u(:,1),'LineWidth',3,'erase','xor');
 for j=2:NT set(G,'xdata',x,'ydata',u(:,j)); pause(0.1); end
 title('Animation of solution'); xlabel('x'); ylabel('u(x,t_k)');
```

Example 24.11. *Linear nonhomogeneous wave equation. Central difference method.* Consider the initial-boundary value problem for the linear nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + F(x,t), \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(x_L,t) = 0, \quad u(x_R,t) = 0$$

describing the motion of a fixed string in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$. We take $x_L = 0$, $x_R = 0.5$, $t_0 = 0$, $t_f = 1.7$, $c = 1/4$, $F(x,t) = x\sin t$, $f(x) = 0$, and $g(x) = \sin(4\pi x)$.

In the *explicit central difference method*, each second derivative is replaced by the central difference approximation (CDA). The finite difference scheme for the linear nonhomogeneous wave equation has the form

$$u_{j,n+1} = 2(1-r)u_{j,n} + r(u_{j+1,n} + u_{j-1,n}) - u_{j,n-1} + j\sin(n)(ck)^2,$$

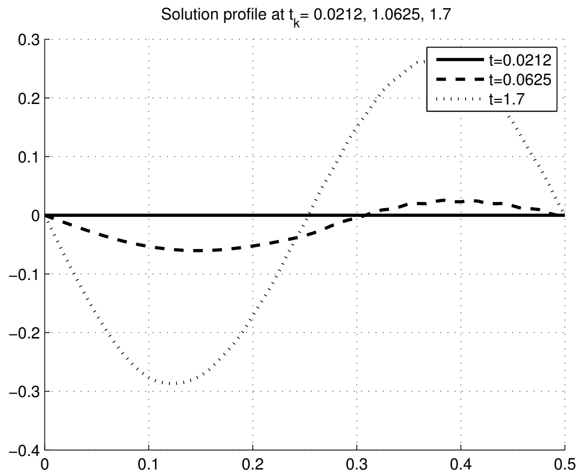


Figure 24.10 Linear nonhomogeneous wave equation: solution profiles at $t_k = 0.0212,\ 1.0625,\ 1.7$.

where $r = (ck/h)^2$. In this finite difference scheme, we have one unknown value $u_{j,n+1}$ that depends explicitly on the four known values $u_{j,n}$, $u_{j+1,n}$, $u_{j-1,n}$, $u_{j,n-1}$ at the previous time steps n and $n-1$. To start the process, we have to know the values of u at the time steps $n = 0$ and $n = 1$. Thus, we can define the initial conditions at these time steps: $u_{j,0} = f(x_j)$ and $u(x_j, 0)_t \approx (u_{j,1} - u_{j,0})/k = g(x_j)$, $u_{j,1} = f(x_j) + kg(x_j)$.

We construct an approximate numerical solution of the initial-boundary value problem by applying the explicit finite difference method and plot the numerical solution in $\mathcal{D}$ as follows:

```
clear all; close all; c=0.25; xL=0; xR=0.5; NX=60; t0=0; tf=1.7;
NT=80; h=(xR-xL)/NX; k=(tf-t0)/NT; r=(c*k/h)^2; x=xL:h:xR;
u=zeros(NX+1,NT); f=0; g=sin(4*pi*x); for n=1:NT
 t=n*k;
  if n==1
   for j=2:NX u(j,n)=f; end; u(1,n)=0; u(NX+1,n)=0;
  elseif n==2
   for j=2:NX  u(j,n)=f+k*g(j); end; u(1,n)=0; u(NX+1,n)=0;
  else
   for j=2:NX
    u(j,n)=2*(1-r)*u(j,n-1)+r*(u(j+1,n-1)+u(j-1,n-1))-u(j,n-2)+j*sin(n)*(c*k)^2;
   end; u(1,n)=0; u(NX+1,n)=0;
  end
end; T=1*k:k:NT*k; figure(1);
 colormap(gray); surf(x,T,u'); xlabel('x'); ylabel('y'); zlabel('u');
 title('Numerical solution of linear nonhomogeneous wave equation');
figure(2);
 hold on; plot(x,u(:,1),'k-',x,u(:,50),'k--',x,u(:,NT),'k:','LineWidth',2);
 title('Solution profile at t_k= 0.0212, 1.0625, 1.7');
 legend('t=0.0212','t=0.0625','t=1.7');
figure(3);
 G=plot(x,u(:,1),'LineWidth',3,'erase','xor');
 for n=1:NT uMax=max(u(:,j)); uMin=min(u(:,j)); end
 Nu=2; delta=(uMax-uMin)/Nu; axis([xL,xR,uMin-delta,uMax+delta]);
 for j=2:NT set(G,'xdata',x,'ydata',u(:,j)); pause(0.1); end
 title('Animation of solution'); xlabel('x'); ylabel('u(x,t_k)');
```

24.3.2 Implicit Finite Difference Solutions

In this section, we consider another class of finite difference methods for solving linear equations, *implicit methods*. In particular, we will construct a finite difference scheme of the *backward Euler method* (or the backward time centered space method) for solving linear heat equations.

The backward Euler method is an implicit one-stage method for finding solutions of PDEs that are first order in time and arbitrary order in space (with mixed partial derivatives).

Example 24.12. *Linear heat equation. Backward Euler method.* Consider the initial-boundary value problem for the linear heat equation

$$u_t = \nu u_{xx}, \qquad u(x,0) = \sin(\pi x), \quad u(x_L,t) = 0, \quad u(x_R,t) = 0,$$

in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$, where $x_L = 0$, $x_R = 1$, $t_0 = 0$, $t_f = 0.1$, and $\nu = 1$.

Let us generate the rectangular mesh

$$x_j = x_L + (j-1)h, \quad t_n = nk,$$

where $j = 1,\dots,NX+1$, $n = 1,\dots,NT$, $h = (x_R - x_L)/NX$, and $k = t_f/NT$. We denote the approximate solution of $u(x,t)$ at the mesh point (j,n) by $u_{j,n}$.

In the *backward Euler method*, the second derivative u_{xx} is replaced by the central difference approximation (CDA), and the first derivative u_t is replaced by the backward difference approximation (BWDA). The finite difference scheme for the linear heat equation has the form

$$u_{j,n-1} = -ru_{j-1,n} + (1+2r)u_{j,n} - ru_{j+1,n},$$

where $r = \nu k/h^2$.

If the values of $u_{0,n}$ and $u_{NX,n}$ at both endpoints are given from the Dirichlet boundary conditions, then the above finite difference equation can be transformed into the following system of equations:

$$\begin{bmatrix} u_{1,n-1} + ru_{0,n} \\ u_{2,n-1} \\ u_{3,n-1} \\ \cdot \\ u_{NX-2,n-1} \\ u_{NX-1,n-1} + ru_{NX,n} \end{bmatrix} = \begin{bmatrix} 1+2r & -r & 0 & \cdot & 0 & 0 \\ -r & 1+2r & -r & \cdot & 0 & 0 \\ 0 & -r & 1+2r & \cdot & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1+2r & -r \\ 0 & 0 & 0 & \cdot & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ \cdot \\ u_{NX-2,n} \\ u_{NX-1,n} \end{bmatrix}.$$

We construct an approximate numerical solution of the initial-boundary value problem by applying the forward difference method and plot (see Fig. 24.11) the solution profiles at various times (for example, $t_k = 0, 0.049, 0.1$) as follows:

```
clear all; close all; nu=1; f=inline('sin(pi*x)','x');
g1=inline('0'); g2=inline('0'); xL=0; xR=1; t0=0; tf=0.1; NX=25;
NT=100; h=(xR-xL)/NX; k=(tf-t0)/NT; x=[0:NX]'*h;  t=[0:NT]*k; for
j=1:NX+1, u(j,1)=f(x(j)); end for n=1:NT+1, u([1
NX+1],n)=[g1(t(n)); g2(t(n))]; end; r=nu*k/h/h; for j=1:NX-1
A(j,j)=1+2*r;  if j>1, A(j-1,j)=-r; A(j,j-1)=-r; end; end for
i=2:NT+1
 b=[r*u(1,i); zeros(NX-3,1); r*u(NX+1,i)]+u(2:NX,i-1); u(2:NX,i)=A\b;
end figure(1);
 colormap(gray); surf(x,t,u'); xlabel('x'); ylabel('y'); zlabel('u');
 title('Numerical solution of linear heat equation');
```

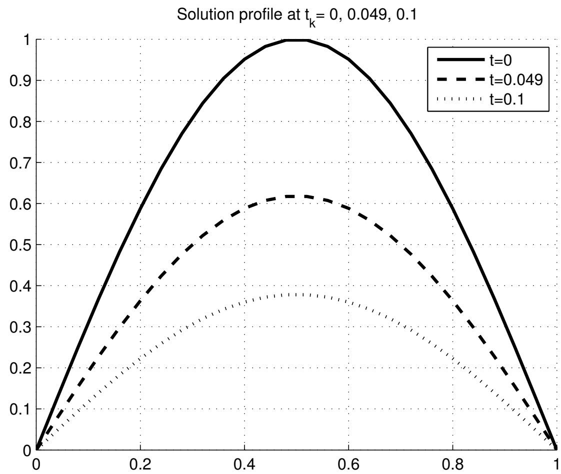


Figure 24.11 Linear heat equation: solution profiles at $t_k = 0, 0.049, 0.1$.

```
figure(2);
 hold on; plot(x,u(:,1),'k-',x,u(:,50),'k--',x,u(:,NT),'k:','LineWidth',2);
 title('Solution profile at t_k= 0, 0.049, 0.1');
 legend('t=0','t=0.049','t=0.1');
figure(3);
 G=plot(x,u(:,1),'LineWidth',3,'erase','xor');
 for n=1:NT uMax=max(u(:,j)); uMin=min(u(:,j)); end
 Nu=2; delta=(uMax-uMin)/Nu; axis([xL,xR,uMin-delta,uMax+delta]);
 for j=2:NT set(G,'xdata',x,'ydata',u(:,j)); pause(0.1); end
 title('Animation of solution');
 xlabel('x'); ylabel('u(x,t_k)');
```

⊙ *References for Section 24.3*: J. H. Mathews and K. D. Fink (1999), Yang, Cao, Chung, and Morris (2005), J. Li and Y. T. Chen (2009).

24.4 Numerical Solutions of Systems of Linear PDEs

The MATLAB `PDE Toolbox` can also deal with systems of partial differential equations. Let us consider systems of linear partial differential equations in one space dimension and two space dimensions [for details, see Young (1971), Lapidus and Pinder (1999), Knabner and Angerman (2003), Larsson and Thomée (2008), Li and Chen (2009)].

24.4.1 Linear Systems of 1D PDEs

The PDE solver `pdepe` (provided in MATLAB PDE Toolbox) can be applied for numerically solving general one-dimensional partial differential equations. Now we consider systems of linear partial differential equations in one space dimension. In this case, the predefined function `pdepe` allows us to solve initial-boundary value problems for systems of 1D parabolic-elliptic PDEs. There must be at least one parabolic equation in the system (see `help pdepe`). The class of 1D parabolic-elliptic PDEs defined in $\mathcal{D} = \{a \le x \le b, t_0 \le t \le t_f\}$ to which the function `pdepe` can be applied has the form

$$C(x,t)\mathbf{u}_t = x^{-m}\partial_x\big(x^m\,\mathbf{f}(x,t,\mathbf{u},\mathbf{u}_x)\big) + \mathbf{s}(x,t,\mathbf{u},\mathbf{u}_x), \qquad (24.4.1.1)$$

where $\mathbf{u}$ is the unknown vector function that depends on the scalar space variable x and the scalar time variable t; the flux function $\mathbf{f}$ and the source function $\mathbf{s}$ are vector functions; the integer $m \in \{0,1,2\}$ corresponds to slab, cylindrical, and spherical symmetry, respectively; the function C is a diagonal matrix whose diagonal entries are zero or positive (which corresponds to elliptic or parabolic equations, respectively).[6]

The initial condition at $t = t_0$ and for $a \le x \le b$ and a given function $\mathbf{u_0}$ is defined as follows:

$$\mathbf{u}(x,t_0) = \mathbf{u}_0(x). \qquad (24.4.1.2)$$

The boundary conditions at $x = a$ and $x = b$ and for $t_0 \le t \le t_f$ have the form

$$\mathbf{p}(a,t,\mathbf{u}) + \mathbf{q}(a,t)\mathbf{f}(a,t,\mathbf{u},\mathbf{u}_x) = 0, \;\; \mathbf{p}(b,t,\mathbf{u}) + \mathbf{q}(b,t)\mathbf{f}(b,t,\mathbf{u},\mathbf{u}_x) = 0, \qquad (24.4.1.3)$$

where $\mathbf{p}$ and $\mathbf{q}$ are given vector functions.

Since the predefined function `pdepe` implements a second-order spatial discretization method based on the `xMesh` values, it follows that the choice of `xMesh` is important and can affect the accuracy and cost of the numerical solution (e.g., it is best to define closely spaced `xMesh` points for domains where the solution can vary rapidly with respect to x). The time points in $[t_0, t_f]$ at which the solution is obtained are given in the vector `tSpan`, where `tSpan(1)=t_0`, `tSpan(end)=t_f`, and `tSpan(i)<tSpan(i+1)` (for `i=2,...,end-1`). Since the time integration in `pdepe` is performed by the stiff ODE solver `ode15s`, the actual time step values are chosen dynamically and do not affect the accuracy and cost.

Example 24.13. *Linear system of 1D parabolic equations.* Consider the following system of linear one-dimensional partial differential equations:

$$u_t = \alpha_1 u_{xx} - g(u-v),$$
$$v_t = \alpha_2 v_{xx} + g(u-v),$$

where $u(x,t)$ and $v(x,t)$ are the unknown functions and α_1 and α_2 are real constants. The function $g(u-v)$ is defined as $g(u-v) = (u-v)(\delta_1 - \delta_2)$, where δ_1 and δ_2 are real constants.

We find numerical and graphical solutions of this linear system under the following boundary and initial conditions:

$$u_x(0,t) = 0, \quad v(0,t) = 0, \quad u(L,t) = 1, \quad v_x(L,t) = 0,$$
$$u(x,0) = 1, \quad v(x,0) = 0.$$

To apply the `pdepe` function, we denote (u,v) by (u_1,u_2) and rewrite the linear system with initial and boundary conditions in the form (24.4.1.1)–(24.4.1.3) as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{\partial}{\partial x} \begin{pmatrix} \alpha_1 u_{1x} \\ \alpha_2 u_{2x} \end{pmatrix} + \begin{pmatrix} -g(u_1-u_2) \\ g(u_1-u_2) \end{pmatrix}$$

[6] According to this notation, there must be at least one parabolic equation.

and

$$m = 0, \quad C(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{f}(x,t,\mathbf{u},\mathbf{u}_x) = \begin{pmatrix} \alpha_1 u_{1x} \\ \alpha_2 u_{2x} \end{pmatrix}, \quad \mathbf{s}(x,t,\mathbf{u},\mathbf{u}_x) = \begin{pmatrix} -g(u_1 - u_2) \\ g(u_1 - u_2) \end{pmatrix}.$$

Then we choose $\mathcal{D} = \{0 \le x \le L,\ 0 \le t \le T\}$ $(L = 1,\ T = 2)$, $\alpha_1 = 0.01$, $\alpha_2 = 0.5$, $\delta_1 = 5$, $\alpha_2 = -5$.

Creating three separate function M-files, `SYS1.m`, `ICSYS1.m`, and `BCSYS1.m`, we specify all parameters and functions as follows:

```
function [c,f,s]=SYS1(x,t,u,DuDx)
  alpha1=0.01; alpha2=0.5; h=u(1)-u(2); delta1=5.;  delta2=-5.;
  g=delta1*h-delta2*h;
  c=[1;1]; f=[alpha1*DuDx(1); alpha2*DuDx(2)]; s=g*[-1;1];
function u0=ICSYS1(x)
   u0(1)=1; u0(2)=0; u0=[u0(1),u0(2)];
function [pL,qL,pR,qR]=BCSYS1(xL,uL,xR,uR,t)
  pL=[0;uL(2)]; qL=[1;0]; pR=[uR(1)-1;0]; qR=[0;1];
```

Then, for constructing numerical and graphical solutions of this initial-boundary value problem, we compose a script M-file `SolSYS1` as follows:

```
clear all; close all; echo on; format long; m=0; a=0; b=1; Nx=31;
t0=0; tf=2; Nt=101; x=linspace(a,b,Nx); t=linspace(t0,tf,Nt);
sol=pdepe(m,@SYS1,@ICSYS1,@BCSYS1,x,t); u1=sol(:,:,1);
u2=sol(:,:,2); figure; subplot(2,1,1); colormap(gray);
  surf(x,t,u1);  rotate3d on; view(-39,27);
  set(gca,'FontSize',14); set(gca,'FontName','Arial');
  set(gca,'LineWidth',1); title('u_1(x,t)'); xlabel('x');ylabel('t');
  subplot(2,1,2); surf(x,t,u2);  rotate3d on; view(-45,25);
  set(gca,'FontSize',14); set(gca,'FontName','Arial');
  set(gca,'LineWidth',1); title('u_2(x,t)'); xlabel('x');ylabel('t');
figure; subplot(2,1,1);
  plot(x,u1(2,:),'k-',x,u1(21,:),'k--',x,u1(Nt,:),'k:','LineWidth',2);
  title('Solution profile at t_k=0.02,0.4,2');
  xlabel('x'); ylabel('u1(x,t_k)'); legend('t=0.02','t=0.4','t=2');
  subplot(2,1,2);
  plot(x,u2(2,:),'k-',x,u2(21,:),'k--',x,u2(Nt,:),'k:','LineWidth',2);
  title('Solution profile at t_k=0.02,0.4,2');
  xlabel('x'); ylabel('u2(x,t_k)'); legend('t=0.02','t=0.4','t=2');
figure;  subplot(2,1,1); G=plot(x,u1(2,:),'erase','xor');
  for k=2:length(t) set(G,'xdata',x,'ydata',u1(k,:)); pause(0.2); end
  title('Animation of u_1(x,t_k)'); xlabel('x'); ylabel('u1(x,t_k)');
subplot(2,1,2); G=plot(x,u2(2,:),'erase','xor');
  for k=2:length(t) set(G,'xdata',x,'ydata',u2(k,:)); pause(0.2); end
  title('Animation of u_2(x,t_k)'); xlabel('x'); ylabel('u2(x,t_k)');
  echo off
```

In this case, we extract the solution u from `sol` as `u1=sol(:,:,1)`, `u2=sol(:,:,2)`. Then we plot the solution as a surface (with interactive rotation), plot the surface profile at $t_k = 0.02,\ 0.4,\ 2$, and produce animations of the surface profile for $t \in [0,2]$. Three surface profiles of the solution obtained at times $t_k = 0.02,\ 0.4,\ 2$ are shown in Fig. 24.12.

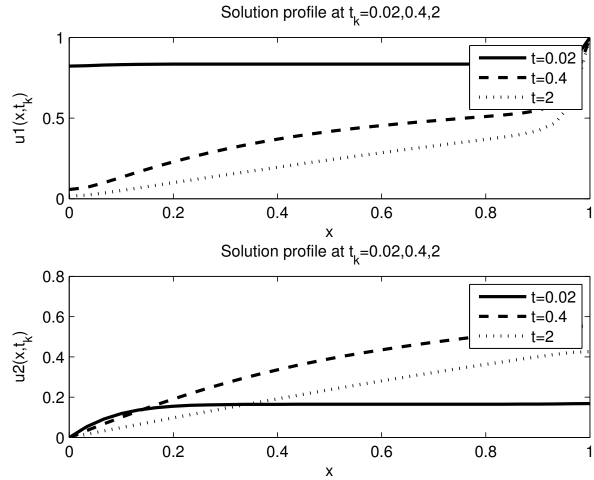


Figure 24.12 Surface profiles of the solution at $t_k = 0.02, 0.4, 2$.

24.4.2 Linear Systems of 2D PDEs

Now consider systems of N linear elliptic PDEs in two space dimensions (2D) and time over the domain $\mathcal{D}$, i.e., systems of the form

$$-\nabla \cdot (c \otimes \nabla \mathbf{u}) + a\mathbf{u} = \mathbf{f},$$

where c is an $N \times N \times 2 \times 2$ tensor, a is an $N \times N$ matrix, and $\mathbf{f}$ and $\mathbf{u}$ are column vectors of length N, respectively, describing the flux and unknown functions. In the MATLAB notation, $\nabla \cdot (c \otimes \nabla \mathbf{u})$ means the $N \times 1$ matrix with $(i,1)$st entry

$$\sum_{j=1}^{N} \left(\frac{\partial c_{ij11}}{\partial x} \frac{\partial u_j}{\partial x} + \frac{\partial c_{ij12}}{\partial x} \frac{\partial u_j}{\partial y} + \frac{\partial c_{ij21}}{\partial y} \frac{\partial u_j}{\partial x} + \frac{\partial c_{ij22}}{\partial y} \frac{\partial u_j}{\partial y} \right).$$

In general, the boundary conditions are mixed (i.e., for each point on the boundary there is a combination of Dirichlet and generalized Neumann conditions) and have the form

$$h\mathbf{u} = \mathbf{r}, \quad \mathbf{n} \cdot (c \otimes \nabla \mathbf{u}) + q\mathbf{u} = \mathbf{g} + h'\mu,$$

where h is an $M \times N$ matrix ($M \geq 0$) for M Dirichlet conditions, $h'\mu$ is a source in the generalized Neumann condition, and μ are the Lagrange multipliers. The data h, $\mathbf{r}$, q, and $\mathbf{g}$ are to be specified. In the MATLAB notation, $\mathbf{n} \cdot (c \otimes \nabla \mathbf{u})$ means the $N \times 1$ matrix with $(i,1)$st entry

$$\sum_{j=1}^{N} \left(\cos\alpha\, c_{ij11} \frac{\partial u_j}{\partial x} + \cos\alpha\, c_{ij12} \frac{\partial u_j}{\partial y} + \sin\alpha\, c_{ij21} \frac{\partial u_j}{\partial x} + \sin\alpha\, c_{ij22} \frac{\partial u_j}{\partial y} \right),$$

where the outward normal vector on the boundary is $\mathbf{n} = (\cos\alpha, \sin\alpha)$.

The generalized Neumann boundary condition is defined for $M = 0$, the Dirichlet boundary condition is defined for $M = N$, and the mixed boundary condition is defined for $0 < M < N$. These systems can be solved with the aid of `PDE Toolbox`.

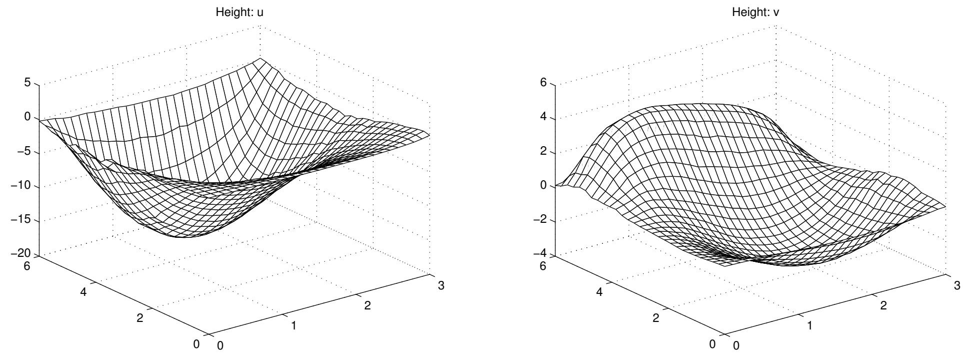


Figure 24.13 Surface plot of the solutions $u_1(x,y)$ and $u_2(x,y)$.

Consider the special case of the two-dimensional system of the form

$$\begin{aligned} &-\nabla\cdot(c_{11}\nabla u_1)-\nabla\cdot(c_{12}\nabla u_2)+a_{11}u_1+a_{12}u_2=f_1,\\ &-\nabla\cdot(c_{21}\nabla u_1)-\nabla\cdot(c_{22}\nabla u_2)+a_{21}u_1+a_{22}u_2=f_2. \end{aligned} \tag{24.4.2.1}$$

The Dirichlet boundary conditions are

$$h_{11}u_1+h_{12}u_2=r_1,\quad h_{21}u_1+h_{22}u_2=r_2. \tag{24.4.2.2}$$

The generalized Neumann boundary conditions are

$$\begin{aligned} &\mathbf{n}\cdot(c_{11}\nabla u_1)+\mathbf{n}\cdot(c_{12}\nabla u_2)+q_{11}u_1+q_{12}u_2=g_1,\\ &\mathbf{n}\cdot(c_{21}\nabla u_1)+\mathbf{n}\cdot(c_{22}\nabla u_2)+q_{21}u_1+q_{22}u_2=g_2. \end{aligned}$$

The mixed boundary conditions are

$$\begin{aligned} &h_{11}u_1+h_{12}u_2=r_1,\\ &\mathbf{n}\cdot(c_{11}\nabla u_1)+\mathbf{n}\cdot(c_{12}\nabla u_2)+q_{11}u_1+q_{12}u_2=g_1+h_{11}\mu,\\ &\mathbf{n}\cdot(c_{21}\nabla u_1)+\mathbf{n}\cdot(c_{22}\nabla u_2)+q_{21}u_1+q_{22}u_2=g_2+h_{12}\mu. \end{aligned}$$

Example 24.14. *System of linear elliptic equations.* Consider the boundary value problem for two linear Poisson equations in two space variables:

$$\begin{aligned} &(u_1)_{xx}+(u_1)_{yy}=u_2+g_1(x,y),\\ &(u_2)_{xx}+(u_2)_{yy}=-u_1+g_2(x,y),\\ &u_1(x,0)=0,\quad u_1(x,6)=0,\quad u_1(0,y)=0,\quad u_1(3,y)=\cos(y),\\ &u_2(x,0)=0,\quad u_2(x,6)=0,\quad u_2(0,y)=0,\quad u_2(3,y)=\sin(y), \end{aligned}$$

describing a potential field $\mathbf{u}(x,y)$ in a bounded domain $\mathcal{D}=\{0\le x\le 3,\ 0\le y\le 6\}$. Let $g_1(x,y)=x^2+y^2$ and $g_2(x,y)=x^2-y^2$.

We solve this problem with the aid of the graphical user interface (GUI).

1°. We draw the domain $\mathcal{D}=\{0\le x\le 3,\ 0\le y\le 6\}$.

- By selecting `Options->Application->Generic System`, we indicate that we work with a *generic system*, i.e., a system of two equations (it is possible to work with systems of arbitrary dimension by writing programs).

2°. We specify the boundary conditions for a system of two equations.

- When selecting `Boundary->Boundary Mode`, the boundaries and their directions will appear as *red arrows*. According to equations (24.4.2.2), we set the Dirichlet boundary conditions (e.g., $u_1(x,0)=0$ by introducing $h_{11}=1$, $h_{12}=0$, $h_{21}=0$, $h_{22}=1$ and $r_1=r_2=0$).
- By selecting `File->Save As`, we save this domain as M-file `LinSys2D.m`.

3°. By selecting `PDE->PDE Specification`, we define the type of the PDE system:

- `Elliptic`, the coefficients of the PDE system: $c_{11}=-1$, $c_{12}=c_{21}=0$, $c_{22}=-1$, $a_{11}=0$, $a_{12}=-1$, $a_{21}=1$, $a_{22}=0$, $f_1=x^2+y^2$, and $f_2=x^2-y^2$, or, according to the MATLAB syntax, `x.^2+y.^2` (with the array operation `.^`).

4°. By selecting `Mesh->Initialize Mesh`, we generate a triangular finite element mesh.

5°. By selecting `Solve->Parameters`, we specify some parameters for solving a linear system.

- By selecting `Solve->Solve PDE`, we obtain the numerical solution. By default, the first solution $u_1(x,y)$ appears (or `u`, in MATLAB notation). By selecting `Plot -> Parameters -> Property` (`v`), we plot the second solution $u_2(x,y)$.

6°. By selecting `Plot->Parameters` and by setting various options (`Height (3-D plot)`, `Plot in x-y grid`, `Color`, `colormap` (`gray`), `Show Mesh`), we create the final figures shown in Fig. 24.13.

⊙ *References for Section 24.4*: D. M. Young (1971), L. Lapidus and G. F. Pinder (1999), P. Knabner and L. Angerman (2003), S. Larsson and V. Thomée (2008), J. Li and Y. T. Chen (2009).

Part IV

Tables and Supplements

Chapter 25

Elementary Functions and Their Properties

★ *Throughout Chapter 25 it is assumed that n is a positive integer unless otherwise specified.*

25.1 Power, Exponential, and Logarithmic Functions

25.1.1 Properties of the Power Function

Basic properties of the power function:

$$x^\alpha x^\beta = x^{\alpha+\beta}, \quad (x_1 x_2)^\alpha = x_1^\alpha x_2^\alpha, \quad (x^\alpha)^\beta = x^{\alpha\beta},$$

for any α and β, where $x > 0$, $x_1 > 0$, $x_2 > 0$.

Differentiation and integration formulas:

$$(x^\alpha)' = \alpha x^{\alpha-1}, \qquad \int x^\alpha\,dx = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1} + C & \text{if } \alpha \neq -1, \\ \ln|x| + C & \text{if } \alpha = -1. \end{cases}$$

The Taylor series expansion in a neighborhood of an arbitrary point:

$$x^\alpha = \sum_{n=0}^{\infty} C_\alpha^n x_0^{\alpha-n} (x - x_0)^n \quad \text{for} \quad |x - x_0| < |x_0|,$$

where $C_\alpha^n = \dfrac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$ are binomial coefficients.

25.1.2 Properties of the Exponential Function

Basic properties of the exponential function:

$$a^{x_1} a^{x_2} = a^{x_1+x_2}, \quad a^x b^x = (ab)^x, \quad (a^{x_1})^{x_2} = a^{x_1 x_2},$$

where $a > 0$ and $b > 0$.

Number e, *base of natural (Napierian) logarithms*, and the function e^x:

$$e = \lim_{n\to\infty}\Bigl(1+\frac{1}{n}\Bigr)^n = 2.718281\ldots, \qquad e^x = \lim_{n\to\infty}\Bigl(1+\frac{x}{n}\Bigr)^n.$$

The formula for passing from an arbitrary base a to the base e of natural logarithms:

$$a^x = e^{x\ln a}.$$

The inequality

$$a^{x_1} > a^{x_2} \qquad\Longleftrightarrow\qquad \begin{cases} x_1 > x_2 & \text{if } a > 1,\\ x_1 < x_2 & \text{if } 0 < a < 1.\end{cases}$$

The limit relations for any $a > 1$ and $b > 0$:

$$\lim_{x\to+\infty}\frac{a^x}{|x|^b} = \infty, \qquad \lim_{x\to-\infty} a^x|x|^b = 0.$$

Differentiation and integration formulas:

$$(e^x)' = e^x, \qquad \int e^x\,dx = e^x + C;$$
$$(a^x)' = a^x\ln a, \quad \int a^x\,dx = \frac{a^x}{\ln a} + C.$$

The expansion in power series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

25.1.3 Properties of the Logarithmic Function

By definition, the logarithmic function is the inverse of the exponential function. The following equivalence relation holds:

$$y = \log_a x \qquad\Longleftrightarrow\qquad x = a^y,$$

where $a > 0$, $a \neq 1$.

Basic properties of the logarithmic function:

$$a^{\log_a x} = x, \qquad \log_a(x_1x_2) = \log_a x_1 + \log_a x_2,$$
$$\log_a(x^k) = k\log_a x, \qquad \log_a x = \frac{\log_b x}{\log_b a},$$

where $x > 0$, $x_1 > 0$, $x_2 > 0$, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$.

The simplest inequality:

$$\log_a x_1 > \log_a x_2 \qquad\Longleftrightarrow\qquad \begin{cases} x_1 > x_2 & \text{if } a > 1,\\ x_1 < x_2 & \text{if } 0 < a < 1.\end{cases}$$

For any $b > 0$, the following limit relations hold:

$$\lim_{x\to+\infty} \frac{\log_a x}{x^b} = 0, \qquad \lim_{x\to+0} x^b \log_a x = 0.$$

The logarithmic function with the base e (*base of natural logarithms* or *Napierian base*) is denoted by

$$\log_e x = \ln x,$$

where $e = \lim\limits_{n\to\infty}\Bigl(1+\dfrac{1}{n}\Bigr)^n = 2.718281\ldots$

Formula for passing from an arbitrary base a to the Napierian base e:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Differentiation and integration formulas:

$$(\ln x)' = \frac{1}{x}, \qquad \int \ln x\,dx = x\ln x - x + C.$$

Expansion in power series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}, \qquad |x| < 1;$$

$$\ln\Bigl(\frac{x+1}{x-1}\Bigr) = \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \cdots = 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)x^{2k-1}}, \qquad |x| > 1;$$

$$\ln x = 2\Bigl(\frac{x-1}{x+1}\Bigr) + \frac{2}{3}\Bigl(\frac{x-1}{x+1}\Bigr)^3 + \frac{2}{5}\Bigl(\frac{x-1}{x+1}\Bigr)^5 + \frac{2}{7}\Bigl(\frac{x-1}{x+1}\Bigr)^7 + \cdots$$
$$= 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\Bigl(\frac{x-1}{x+1}\Bigr)^{2k-1}, \qquad x > 0.$$

25.2 Trigonometric Functions

25.2.1 Simplest Relations

$$\begin{aligned}
&\sin^2 x + \cos^2 x = 1, && \tan x \cot x = 1,\\
&\sin(-x) = -\sin x, && \cos(-x) = \cos x,\\
&\tan x = \frac{\sin x}{\cos x}, && \cot x = \frac{\cos x}{\sin x},\\
&\tan(-x) = -\tan x, && \cot(-x) = -\cot x,\\
&1 + \tan^2 x = \frac{1}{\cos^2 x}, && 1 + \cot^2 x = \frac{1}{\sin^2 x}.
\end{aligned}$$

25.2.2 Reduction Formulas

$$\sin(x \pm 2n\pi) = \sin x, \qquad \cos(x \pm 2n\pi) = \cos x,$$
$$\sin(x \pm n\pi) = (-1)^n \sin x, \qquad \cos(x \pm n\pi) = (-1)^n \cos x,$$
$$\sin\Big(x \pm \frac{2n+1}{2}\pi\Big) = \pm(-1)^n \cos x, \qquad \cos\Big(x \pm \frac{2n+1}{2}\pi\Big) = \mp(-1)^n \sin x,$$
$$\sin\Big(x \pm \frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}(\sin x \pm \cos x), \qquad \cos\Big(x \pm \frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}(\cos x \mp \sin x),$$
$$\tan(x \pm n\pi) = \tan x, \qquad \cot(x \pm n\pi) = \cot x,$$
$$\tan\Big(x \pm \frac{2n+1}{2}\pi\Big) = -\cot x, \qquad \cot\Big(x \pm \frac{2n+1}{2}\pi\Big) = -\tan x,$$
$$\tan\Big(x \pm \frac{\pi}{4}\Big) = \frac{\tan x \pm 1}{1 \mp \tan x}, \qquad \cot\Big(x \pm \frac{\pi}{4}\Big) = \frac{\cot x \mp 1}{1 \pm \cot x},$$

where $n = 1,\ 2,\ \ldots$

25.2.3 Relations between Trigonometric Functions of Single Argument

$$\sin x = \pm\sqrt{1-\cos^2 x} = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} = \pm\frac{1}{\sqrt{1+\cot^2 x}},$$
$$\cos x = \pm\sqrt{1-\sin^2 x} = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \pm\frac{\cot x}{\sqrt{1+\cot^2 x}},$$
$$\tan x = \pm\frac{\sin x}{\sqrt{1-\sin^2 x}} = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{1}{\cot x},$$
$$\cot x = \pm\frac{\sqrt{1-\sin^2 x}}{\sin x} = \pm\frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\tan x}.$$

The sign before the radical is determined by the quarter in which the argument takes its values.

25.2.4 Addition and Subtraction of Trigonometric Functions

$$\sin x + \sin y = 2\sin\Big(\frac{x+y}{2}\Big)\cos\Big(\frac{x-y}{2}\Big),$$
$$\sin x - \sin y = 2\sin\Big(\frac{x-y}{2}\Big)\cos\Big(\frac{x+y}{2}\Big),$$
$$\cos x + \cos y = 2\cos\Big(\frac{x+y}{2}\Big)\cos\Big(\frac{x-y}{2}\Big),$$
$$\cos x - \cos y = -2\sin\Big(\frac{x+y}{2}\Big)\sin\Big(\frac{x-y}{2}\Big),$$
$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y),$$
$$\sin^2 x - \cos^2 y = -\cos(x+y)\cos(x-y),$$
$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}, \quad \cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y},$$
$$a\cos x + b\sin x = r\sin(x+\varphi) = r\cos(x-\psi).$$

Here $r = \sqrt{a^2 + b^2}$, $\sin\varphi = a/r$, $\cos\varphi = b/r$, $\sin\psi = b/r$, and $\cos\psi = a/r$.

25.2.5 Products of Trigonometric Functions

$$\begin{aligned}\sin x \sin y &= \tfrac{1}{2}[\cos(x-y) - \cos(x+y)],\\ \cos x \cos y &= \tfrac{1}{2}[\cos(x-y) + \cos(x+y)],\\ \sin x \cos y &= \tfrac{1}{2}[\sin(x-y) + \sin(x+y)].\end{aligned}$$

25.2.6 Powers of Trigonometric Functions

$$\begin{aligned}\cos^2 x &= \tfrac{1}{2}\cos 2x + \tfrac{1}{2}, & \sin^2 x &= -\tfrac{1}{2}\cos 2x + \tfrac{1}{2},\\ \cos^3 x &= \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x, & \sin^3 x &= -\tfrac{1}{4}\sin 3x + \tfrac{3}{4}\sin x,\\ \cos^4 x &= \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}, & \sin^4 x &= \tfrac{1}{8}\cos 4x - \tfrac{1}{2}\cos 2x + \tfrac{3}{8},\\ \cos^5 x &= \tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x, & \sin^5 x &= \tfrac{1}{16}\sin 5x - \tfrac{5}{16}\sin 3x + \tfrac{5}{8}\sin x,\end{aligned}$$

$$\begin{aligned}\cos^{2n} x &= \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cos[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^n,\\ \cos^{2n+1} x &= \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \cos[(2n-2k+1)x],\\ \sin^{2n} x &= \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^{n-k} C_{2n}^k \cos[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^n,\\ \sin^{2n+1} x &= \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^{n-k} C_{2n+1}^k \sin[(2n-2k+1)x].\end{aligned}$$

Here $n = 1, 2, \ldots$ and $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients $(0! = 1)$.

25.2.7 Addition Formulas

$$\begin{aligned}\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y, & \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y,\\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, & \cot(x \pm y) &= \frac{1 \mp \tan x \tan y}{\tan x \pm \tan y}.\end{aligned}$$

25.2.8 Trigonometric Functions of Multiple Arguments

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \qquad \sin 2x = 2\sin x \cos x,$$
$$\cos 3x = -3\cos x + 4\cos^3 x, \qquad \sin 3x = 3\sin x - 4\sin^3 x,$$
$$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x, \qquad \sin 4x = 4\cos x\,(\sin x - 2\sin^3 x),$$
$$\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x, \qquad \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$$
$$\cos(2nx) = 1 + \sum_{k=1}^{n} (-1)^k 4^k \frac{n^2(n^2-1)\dots[n^2-(k-1)^2]}{(2k)!} \sin^{2k} x,$$
$$\cos[(2n+1)x] = \cos x \left\{1 + \sum_{k=1}^{n} (-1)^k \times \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!} \sin^{2k} x\right\},$$
$$\sin(2nx) = 2n\cos x \left[\sin x + \sum_{k=1}^{n} (-1)^k 4^k \frac{(n^2-1)(n^2-2^2)\dots(n^2-k^2)}{(2k-1)!} \sin^{2k-1} x\right],$$
$$\sin[(2n+1)x] = (2n+1)\left\{\sin x + \sum_{k=1}^{n} (-1)^k \times \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \sin^{2k+1} x\right\},$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}, \quad \tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}, \quad \tan 4x = \frac{4\tan x - 4\tan^3 x}{1-6\tan^2 x + \tan^4 x},$$

where $n = 1,\ 2,\ \dots$

25.2.9 Trigonometric Functions of Half Argument

$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}, \qquad \cos^2 \frac{x}{2} = \frac{1+\cos x}{2},$$
$$\tan \frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}, \qquad \cot \frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x},$$
$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}.$$

25.2.10 Differentiation Formulas

$$\frac{d\sin x}{dx} = \cos x, \qquad \frac{d\cos x}{dx} = -\sin x, \qquad \frac{d\tan x}{dx} = \frac{1}{\cos^2 x}, \qquad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x}.$$

25.2.11 Integration Formulas

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C,$$
$$\int \tan x\,dx = -\ln|\cos x| + C, \qquad \int \cot x\,dx = \ln|\sin x| + C,$$

where C is an arbitrary constant.

25.2.12 Expansion in Power Series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad (|x| < \infty),$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad (|x| < \infty),$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1} + \cdots \qquad (|x| < \pi/2),$$
$$\cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \cdots + \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1} + \cdots \right) \qquad (0 < |x| < \pi),$$
$$\frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} + \cdots \qquad (|x| < \pi/2),$$
$$\frac{1}{\sin x} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \cdots + \frac{(-1)^{n-1} 2(2^{2n-1}-1) B_{2n}}{(2n)!} x^{2n-1} + \cdots \qquad (0 < |x| < \pi),$$

where B_n and E_n are Bernoulli and Euler numbers (see Sections 30.1.3 and 30.1.4).

25.2.13 Representation in the Form of Infinite Products

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \ldots \left(1 - \frac{x^2}{n^2\pi^2}\right) \ldots$$
$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{9\pi^2}\right)\left(1 - \frac{4x^2}{25\pi^2}\right) \ldots \left(1 - \frac{4x^2}{(2n+1)^2\pi^2}\right) \ldots$$

25.2.14 Euler and de Moivre Formulas. Relationship with Hyperbolic Functions

$$e^{y+ix} = e^y(\cos x + i \sin x), \quad (\cos x + i \sin x)^n = \cos(nx) + i \sin(nx), \quad i^2 = -1,$$
$$\sin(ix) = i \sinh x, \quad \cos(ix) = \cosh x, \quad \tan(ix) = i \tanh x, \quad \cot(ix) = -i \coth x.$$

25.3 Inverse Trigonometric Functions

25.3.1 Definitions of Inverse Trigonometric Functions

Inverse trigonometric functions (*arc functions*) are the functions that are inverse to the trigonometric functions. Since the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$ are periodic, the corresponding inverse functions, denoted by $\text{Arcsin}\, x$, $\text{Arccos}\, x$, $\text{Arctan}\, x$, $\text{Arccot}\, x$, are multivalued. The following relations define the multivalued inverse trigonometric functions:

$$\sin\big(\text{Arcsin}\, x\big) = x, \quad \cos\big(\text{Arccos}\, x\big) = x,$$
$$\tan\big(\text{Arctan}\, x\big) = x, \quad \cot\big(\text{Arccot}\, x\big) = x.$$

These functions admit the following verbal definitions: $\text{Arcsin}\, x$ is the angle whose sine is equal to x; $\text{Arccos}\, x$ is the angle whose cosine is equal to x; $\text{Arctan}\, x$ is the angle whose tangent is equal to x; $\text{Arccot}\, x$ is the angle whose cotangent is equal to x.

The principal (single-valued) branches of the inverse trigonometric functions are denoted by

$$\begin{aligned}
&\arcsin x \equiv \sin^{-1} x && \text{(arcsine is the inverse of sine)},\\
&\arccos x \equiv \cos^{-1} x && \text{(arccosine is the inverse of cosine)},\\
&\arctan x \equiv \tan^{-1} x && \text{(arctangent is the inverse of tangent)},\\
&\operatorname{arccot} x \equiv \cot^{-1} x && \text{(arccotangent is the inverse of cotangent)}
\end{aligned}$$

and are determined by the inequalities

$$\begin{aligned}
&-\tfrac{\pi}{2} \le \arcsin x \le \tfrac{\pi}{2}, \quad 0 \le \arccos x \le \pi \qquad (-1 \le x \le 1);\\
&-\tfrac{\pi}{2} < \arctan x < \tfrac{\pi}{2}, \quad 0 < \operatorname{arccot} x < \pi \qquad (-\infty < x < \infty).
\end{aligned}$$

The following equivalent relations can be taken as definitions of single-valued inverse trigonometric functions:

$$\begin{aligned}
&y = \arcsin x, \quad -1 \le x \le 1 && \Longleftrightarrow \quad x = \sin y, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2};\\
&y = \arccos x, \quad -1 \le x \le 1 && \Longleftrightarrow \quad x = \cos y, \quad 0 \le y \le \pi;\\
&y = \arctan x, \quad -\infty < x < +\infty && \Longleftrightarrow \quad x = \tan y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2};\\
&y = \operatorname{arccot} x, \quad -\infty < x < +\infty && \Longleftrightarrow \quad x = \cot y, \quad 0 < y < \pi.
\end{aligned}$$

The multivalued and the single-valued inverse trigonometric functions are related by the formulas

$$\begin{aligned}
\operatorname{Arcsin} x &= (-1)^n \arcsin x + \pi n,\\
\operatorname{Arccos} x &= \pm \arccos x + 2\pi n,\\
\operatorname{Arctan} x &= \arctan x + \pi n,\\
\operatorname{Arccot} x &= \operatorname{arccot} x + \pi n,
\end{aligned}$$

where $n = 0, \pm 1, \pm 2, \ldots$

25.3.2 Simplest Formulas

$$\begin{aligned}
&\sin(\arcsin x) = x, \quad \cos(\arccos x) = x,\\
&\tan(\arctan x) = x, \quad \cot(\operatorname{arccot} x) = x.
\end{aligned}$$

25.3.3 Some Properties

$$\begin{aligned}
&\arcsin(-x) = -\arcsin x, && \arccos(-x) = \pi - \arccos x,\\
&\arctan(-x) = -\arctan x, && \operatorname{arccot}(-x) = \pi - \operatorname{arccot} x,
\end{aligned}$$

$$\arcsin(\sin x) = \begin{cases} x - 2n\pi & \text{if } 2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2},\\ -x + 2(n+1)\pi & \text{if } (2n+1)\pi - \frac{\pi}{2} \le x \le 2(n+1)\pi + \frac{\pi}{2}, \end{cases}$$

$$\arccos(\cos x) = \begin{cases} x - 2n\pi & \text{if } 2n\pi \le x \le (2n+1)\pi,\\ -x + 2(n+1)\pi & \text{if } (2n+1)\pi \le x \le 2(n+1)\pi, \end{cases}$$

$$\arctan(\tan x) = x - n\pi \quad \text{if} \quad n\pi - \tfrac{\pi}{2} < x < n\pi + \tfrac{\pi}{2},$$
$$\operatorname{arccot}(\cot x) = x - n\pi \quad \text{if} \quad n\pi < x < (n+1)\pi.$$

25.3.4 Relations between Inverse Trigonometric Functions

$$\arcsin x + \arccos x = \tfrac{\pi}{2}, \quad \arctan x + \operatorname{arccot} x = \tfrac{\pi}{2};$$

$$\arcsin x = \begin{cases} \arccos\sqrt{1-x^2} & \text{if } 0 \le x \le 1, \\ -\arccos\sqrt{1-x^2} & \text{if } -1 \le x \le 0, \\ \arctan\dfrac{x}{\sqrt{1-x^2}} & \text{if } -1 < x < 1, \\ \operatorname{arccot}\dfrac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0; \end{cases}$$

$$\arccos x = \begin{cases} \arcsin\sqrt{1-x^2} & \text{if } 0 \le x \le 1, \\ \pi - \arcsin\sqrt{1-x^2} & \text{if } -1 \le x \le 0, \\ \arctan\dfrac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1, \\ \operatorname{arccot}\dfrac{x}{\sqrt{1-x^2}} & \text{if } -1 < x < 1; \end{cases}$$

$$\arctan x = \begin{cases} \arcsin\dfrac{x}{\sqrt{1+x^2}} & \text{for any } x, \\ \arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if } x \ge 0, \\ -\arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if } x \le 0, \\ \operatorname{arccot}\dfrac{1}{x} & \text{if } x > 0; \end{cases}$$

$$\operatorname{arccot} x = \begin{cases} \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if } x > 0, \\ \pi - \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if } x < 0, \\ \arctan\dfrac{1}{x} & \text{if } x > 0, \\ \pi + \arctan\dfrac{1}{x} & \text{if } x < 0. \end{cases}$$

25.3.5 Addition and Subtraction of Inverse Trigonometric Functions

$$\arcsin x + \arcsin y = \arcsin\bigl(x\sqrt{1-y^2} + y\sqrt{1-x^2}\,\bigr) \quad \text{for} \quad x^2 + y^2 \le 1,$$
$$\arccos x \pm \arccos y = \pm\arccos\bigl[xy \mp \sqrt{(1-x^2)(1-y^2)}\,\bigr] \quad \text{for} \quad x \pm y \ge 0,$$
$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy} \quad \text{for} \quad xy < 1,$$
$$\arctan x - \arctan y = \arctan\frac{x-y}{1+xy} \quad \text{for} \quad xy > -1.$$

25.3.6 Differentiation Formulas

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}},$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}.$$

25.3.7 Integration Formulas

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C,$$
$$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + C,$$
$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C,$$
$$\int \operatorname{arccot} x\,dx = x\operatorname{arccot} x + \frac{1}{2}\ln(1+x^2) + C,$$

where C is an arbitrary constant.

25.3.8 Expansion in Power Series

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\frac{x^5}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}\frac{x^7}{7} + \cdots$$
$$+ \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{x^{2n+1}}{2n+1} + \cdots \qquad (|x|<1),$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots \qquad (|x|\le 1),$$
$$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots + (-1)^n\frac{1}{(2n-1)x^{2n-1}} + \cdots \qquad (|x|>1).$$

The expansions for $\arccos x$ and $\operatorname{arccot} x$ can be obtained from the relations $\arccos x = \frac{\pi}{2} - \arcsin x$ and $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$.

25.4 Hyperbolic Functions

25.4.1 Definitions of Hyperbolic Functions

Hyperbolic functions are defined in terms of the exponential functions as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

25.4.2 Simplest Relations

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh x \coth x = 1,$$
$$\sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x,$$
$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$
$$\tanh(-x) = -\tanh x, \qquad \coth(-x) = -\coth x,$$
$$1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \coth^2 x - 1 = \frac{1}{\sinh^2 x}.$$

25.4.3 Relations between Hyperbolic Functions of Single Argument ($x \geq 0$)

$$\sinh x = \sqrt{\cosh^2 x - 1} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} = \frac{1}{\sqrt{\coth^2 x - 1}},$$
$$\cosh x = \sqrt{\sinh^2 x + 1} = \frac{1}{\sqrt{1 - \tanh^2 x}} = \frac{\coth x}{\sqrt{\coth^2 x - 1}},$$
$$\tanh x = \frac{\sinh x}{\sqrt{\sinh^2 x + 1}} = \frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \frac{1}{\coth x},$$
$$\coth x = \frac{\sqrt{\sinh^2 x + 1}}{\sinh x} = \frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x}.$$

25.4.4 Addition and Subtraction of Hyperbolic Functions

$$\sinh x + \sinh y = 2 \sinh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right),$$
$$\sinh x - \sinh y = 2 \sinh\left(\frac{x-y}{2}\right) \cosh\left(\frac{x+y}{2}\right),$$
$$\cosh x + \cosh y = 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right),$$
$$\cosh x - \cosh y = 2 \sinh\left(\frac{x+y}{2}\right) \sinh\left(\frac{x-y}{2}\right),$$
$$\sinh^2 x - \sinh^2 y = \cosh^2 x - \cosh^2 y = \sinh(x+y) \sinh(x-y),$$
$$\sinh^2 x + \cosh^2 y = \cosh(x+y) \cosh(x-y),$$
$$(\cosh x \pm \sinh x)^n = \cosh(nx) \pm \sinh(nx),$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}, \quad \coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y},$$

where $n = 0, \pm 1, \pm 2, \ldots$

25.4.5 Products of Hyperbolic Functions

$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)],$$
$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)],$$
$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)].$$

25.4.6 Powers of Hyperbolic Functions

$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$, $\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$,

$\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$, $\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x$,

$\cosh^4 x = \frac{1}{8}\cosh 4x + \frac{1}{2}\cosh 2x + \frac{3}{8}$, $\sinh^4 x = \frac{1}{8}\cosh 4x - \frac{1}{2}\cosh 2x + \frac{3}{8}$,

$\cosh^5 x = \frac{1}{16}\cosh 5x + \frac{5}{16}\cosh 3x + \frac{5}{8}\cosh x$, $\sinh^5 x = \frac{1}{16}\sinh 5x - \frac{5}{16}\sinh 3x + \frac{5}{8}\sinh x$,

$$\cosh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cosh[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^n,$$

$$\cosh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \cosh[(2n-2k+1)x],$$

$$\sinh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^k C_{2n}^k \cosh[2(n-k)x] + \frac{(-1)^n}{2^{2n}} C_{2n}^n,$$

$$\sinh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^k C_{2n+1}^k \sinh[(2n-2k+1)x].$$

Here $n = 1, 2, \ldots$ and C_m^k are binomial coefficients.

25.4.7 Addition Formulas

$$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}, \quad \coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}.$$

25.4.8 Hyperbolic Functions of Multiple Argument

$\cosh 2x = 2\cosh^2 x - 1$, $\sinh 2x = 2\sinh x \cosh x$,

$\cosh 3x = -3\cosh x + 4\cosh^3 x$, $\sinh 3x = 3\sinh x + 4\sinh^3 x$,

$\cosh 4x = 1 - 8\cosh^2 x + 8\cosh^4 x$, $\sinh 4x = 4\cosh x(\sinh x + 2\sinh^3 x)$,

$\cosh 5x = 5\cosh x - 20\cosh^3 x + 16\cosh^5 x$, $\sinh 5x = 5\sinh x + 20\sinh^3 x + 16\sinh^5 x$.

$$\cosh(nx) = 2^{n-1}\cosh^n x + \frac{n}{2}\sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}}{k+1} C_{n-k-2}^{k-2} 2^{n-2k-2} (\cosh x)^{n-2k-2},$$

$$\sinh(nx) = \sinh x \sum_{k=0}^{[(n-1)/2]} 2^{n-k-1} C_{n-k-1}^k (\cosh x)^{n-2k-1}.$$

Here C_m^k are binomial coefficients and $[A]$ stands for the integer part of the number A.

25.4.9 Hyperbolic Functions of Half Argument

$$\sinh\frac{x}{2} = \operatorname{sign} x\sqrt{\frac{\cosh x - 1}{2}}, \qquad \cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}},$$

$$\tanh\frac{x}{2} = \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}, \quad \coth\frac{x}{2} = \frac{\sinh x}{\cosh x - 1} = \frac{\cosh x + 1}{\sinh x}.$$

25.4.10 Differentiation Formulas

$$\frac{d\sinh x}{dx} = \cosh x, \qquad \frac{d\cosh x}{dx} = \sinh x,$$

$$\frac{d\tanh x}{dx} = \frac{1}{\cosh^2 x}, \qquad \frac{d\coth x}{dx} = -\frac{1}{\sinh^2 x}.$$

25.4.11 Integration Formulas

$$\int \sinh x\,dx = \cosh x + C, \qquad \int \cosh x\,dx = \sinh x + C,$$

$$\int \tanh x\,dx = \ln\cosh x + C, \quad \int \coth x\,dx = \ln|\sinh x| + C,$$

where C is an arbitrary constant.

25.4.12 Expansion in Power Series

$$\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots \qquad (|x|<\infty),$$

$$\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}+\cdots \qquad (|x|<\infty),$$

$$\tanh x = x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\cdots+(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)|B_{2n}|x^{2n-1}}{(2n)!}+\cdots \qquad (|x|<\pi/2),$$

$$\coth x = \frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}-\cdots+(-1)^{n-1}\frac{2^{2n}|B_{2n}|x^{2n-1}}{(2n)!}+\cdots \qquad (|x|<\pi),$$

$$\frac{1}{\cosh x} = 1-\frac{x^2}{2}+\frac{5x^4}{24}-\frac{61x^6}{720}+\cdots+\frac{E_{2n}}{(2n)!}x^{2n}+\cdots \qquad (|x|<\pi/2),$$

$$\frac{1}{\sinh x} = \frac{1}{x}-\frac{x}{6}+\frac{7x^3}{360}-\frac{31x^5}{15120}+\cdots+\frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}x^{2n-1}+\cdots \qquad (0<|x|<\pi),$$

where B_n and E_n are Bernoulli and Euler numbers (see Sections 30.1.3 and 30.1.4).

25.4.13 Representation in the Form of Infinite Products

$$\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\ldots\left(1+\frac{x^2}{n^2\pi^2}\right)\ldots$$

$$\cosh x = \left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{9\pi^2}\right)\left(1+\frac{4x^2}{25\pi^2}\right)\ldots\left(1+\frac{4x^2}{(2n+1)^2\pi^2}\right)\ldots$$

25.4.14 Relationship with Trigonometric Functions

$$\sinh(ix) = i\sin x, \qquad \cosh(ix) = \cos x, \qquad i^2 = -1,$$
$$\tanh(ix) = i\tan x, \qquad \coth(ix) = -i\cot x.$$

25.5 Inverse Hyperbolic Functions

25.5.1 Definitions of Inverse Hyperbolic Functions

Inverse hyperbolic functions are the functions that are inverse to hyperbolic functions. The following notation is used for inverse hyperbolic functions:

$$\begin{aligned} &\operatorname{arcsinh} x \equiv \sinh^{-1} x &&(\text{inverse of hyperbolic sine}),\\ &\operatorname{arccosh} x \equiv \cosh^{-1} x &&(\text{inverse of hyperbolic cosine}),\\ &\operatorname{arctanh} x \equiv \tanh^{-1} x &&(\text{inverse of hyperbolic tangent}),\\ &\operatorname{arccoth} x \equiv \coth^{-1} x &&(\text{inverse of hyperbolic cotangent}). \end{aligned}$$

Inverse hyperbolic functions can be expressed in terms of logarithmic functions:

$$\operatorname{arcsinh} x = \ln\bigl(x+\sqrt{x^2+1}\bigr) \quad (x \text{ is any}); \qquad \operatorname{arccosh} x = \ln\bigl(x+\sqrt{x^2-1}\bigr) \quad (x \ge 1);$$
$$\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x} \quad (|x|<1); \qquad \operatorname{arccoth} x = \frac{1}{2}\ln\frac{x+1}{x-1} \quad (|x|>1).$$

Here only one (principal) branch of the function $\operatorname{arccosh} x$ is listed, the function itself being double-valued. In order to write out both branches of $\operatorname{arccosh} x$, the symbol $\pm$ should be placed before the logarithm on the right-hand side of the formula.

25.5.2 Simplest Relations

$$\operatorname{arcsinh}(-x) = -\operatorname{arcsinh} x, \quad \operatorname{arctanh}(-x) = -\operatorname{arctanh} x, \quad \operatorname{arccoth}(-x) = -\operatorname{arccoth} x.$$

25.5.3 Relations between Inverse Hyperbolic Functions

$$\operatorname{arcsinh} x = \operatorname{arccosh}\sqrt{x^2+1} = \operatorname{arctanh}\frac{x}{\sqrt{x^2+1}},$$
$$\operatorname{arccosh} x = \operatorname{arcsinh}\sqrt{x^2-1} = \operatorname{arctanh}\frac{\sqrt{x^2-1}}{x},$$
$$\operatorname{arctanh} x = \operatorname{arcsinh}\frac{x}{\sqrt{1-x^2}} = \operatorname{arccosh}\frac{1}{\sqrt{1-x^2}} = \operatorname{arccoth}\frac{1}{x}.$$

25.5.4 Addition and Subtraction of Inverse Hyperbolic Functions

$$\operatorname{arcsinh} x \pm \operatorname{arcsinh} y = \operatorname{arcsinh}\Bigl(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\Bigr),$$
$$\operatorname{arccosh} x \pm \operatorname{arccosh} y = \operatorname{arccosh}\Bigl[xy \pm \sqrt{(x^2-1)(y^2-1)}\Bigr],$$
$$\operatorname{arcsinh} x \pm \operatorname{arccosh} y = \operatorname{arcsinh}\Bigl[xy \pm \sqrt{(x^2+1)(y^2-1)}\Bigr],$$
$$\operatorname{arctanh} x \pm \operatorname{arctanh} y = \operatorname{arctanh}\frac{x \pm y}{1 \pm xy}, \quad \operatorname{arctanh} x \pm \operatorname{arccoth} y = \operatorname{arctanh}\frac{xy \pm 1}{y \pm x}.$$

25.5.5 Differentiation Formulas

$$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{x^2+1}}, \qquad \frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}},$$
$$\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{1-x^2} \quad (x^2<1), \qquad \frac{d}{dx}\operatorname{arccoth} x = \frac{1}{1-x^2} \quad (x^2>1).$$

25.5.6 Integration Formulas

$$\int \operatorname{arcsinh} x\,dx = x\operatorname{arcsinh} x - \sqrt{1+x^2} + C,$$
$$\int \operatorname{arccosh} x\,dx = x\operatorname{arccosh} x - \sqrt{x^2-1} + C,$$
$$\int \operatorname{arctanh} x\,dx = x\operatorname{arctanh} x + \frac{1}{2}\ln(1-x^2) + C,$$
$$\int \operatorname{arccoth} x\,dx = x\operatorname{arccoth} x + \frac{1}{2}\ln(x^2-1) + C,$$

where C is an arbitrary constant.

25.5.7 Expansion in Power Series

$$\operatorname{arcsinh} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\frac{x^5}{5} - \cdots + (-1)^n \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{x^{2n+1}}{2n+1} + \cdots \qquad (|x|<1),$$
$$\operatorname{arcsinh} x = \ln(2x) + \frac{1}{2}\frac{1}{2x^2} + \frac{1\times 3}{2\times 4}\frac{1}{4x^4} + \cdots + \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{1}{2nx^{2n}} + \cdots \qquad (|x|>1),$$
$$\operatorname{arccosh} x = \ln(2x) - \frac{1}{2}\frac{1}{2x^2} - \frac{1\times 3}{2\times 4}\frac{1}{4x^4} - \cdots - \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{1}{2nx^{2n}} - \cdots \qquad (|x|>1),$$
$$\operatorname{arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots \qquad (|x|<1),$$
$$\operatorname{arccoth} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots + \frac{1}{(2n+1)x^{2n+1}} + \cdots \qquad (|x|>1).$$

⊙ *References for Chapter* 25: M. Abramowitz and I. A. Stegun (1964), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986), D. G. Zill and J. M. Dewar (1990), M. Kline (1998), R. Courant and F. John (1999), I. S. Gradshteyn and I. M. Ryzhik (2000), G. A. Korn and T. M. Korn (2000), C. H. Edwards and D. Penney (2002), D. Zwillinger (2002), E. W. Weisstein (2003), I. N. Bronshtein and K. A. Semendyayev (2004), M. Sullivan (2004), H. Anton, I. Bivens, and S. Davis (2005), R. Adams (2006).

Chapter 26

Finite Sums and Infinite Series

26.1 Finite Numerical Sums

26.1.1 Progressions

Arithmetic progression:

1. $\displaystyle\sum_{k=0}^{n-1}(a+bk) = an + \frac{bn(n-1)}{2}.$

Geometric progression:

2. $\displaystyle\sum_{k=1}^{n} aq^{k-1} = a\frac{q^n-1}{q-1}.$

Arithmetic-geometric progression:

3. $\displaystyle\sum_{k=0}^{n-1}(a+bk)q^k = \frac{a(1-q^n)-b(n-1)q^n}{1-q} + \frac{bq(1-q^{n-1})}{(1-q)^2}.$

26.1.2 Sums of Powers of Natural Numbers Having the Form $\sum k^m$

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2.$

4. $\displaystyle\sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).$

5. $\displaystyle\sum_{k=1}^{n} k^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1).$

6. $\displaystyle\sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + \frac{n^m}{2} + \frac{1}{2}C_m^1 B_2 n^{m-1} + \frac{1}{4}C_m^3 B_4 n^{m-3} + \frac{1}{6}C_m^5 B_6 n^{m-5} + \cdots.$

Here C_m^k are binomial coefficients and B_{2k} are Bernoulli numbers (see Section 30.1.3); the last term in the sum contains n or n^2.

26.1.3 Alternating Sums of Powers of Natural Numbers, $\sum(-1)^k k^m$

1. $\displaystyle\sum_{k=1}^{n}(-1)^k k = (-1)^n\left[\frac{n-1}{2}\right]$; $[m]$ stands for the integer part of m.

2. $\displaystyle\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n\frac{n(n+1)}{2}.$

3. $\displaystyle\sum_{k=1}^{n}(-1)^k k^3 = \frac{1}{8}\bigl[1+(-1)^n(4n^3+6n^2-1)\bigr].$

4. $\displaystyle\sum_{k=1}^{n}(-1)^k k^4 = (-1)^n\frac{1}{2}(n^4+2n^3-n).$

5. $\displaystyle\sum_{k=1}^{n}(-1)^k k^5 = \frac{1}{4}\bigl[-1+(-1)^n(2n^5+5n^4-5n^2+1)\bigr].$

26.1.4 Other Sums Containing Integers

1. $\displaystyle\sum_{k=0}^{n}(2k+1) = (n+1)^2.$

2. $\displaystyle\sum_{k=0}^{n}(2k+1)^2 = \frac{1}{3}(n+1)(2n+1)(2n+3).$

3. $\displaystyle\sum_{k=1}^{n}k(k+1) = \frac{1}{3}n(n+1)(n+2).$

4. $\displaystyle\sum_{k=1}^{n}(k+a)(k+b) = \frac{1}{6}n(n+1)(2n+1+3a+3b)+nab.$

5. $\displaystyle\sum_{k=1}^{n}k\,k! = (n+1)!-1.$

6. $\displaystyle\sum_{k=0}^{n}(-1)^k(2k+1) = (-1)^n(n+1).$

7. $\displaystyle\sum_{k=0}^{n}(-1)^k(2k+1)^2 = 2(-1)^n(n+1)^2-\frac{1}{2}\bigl[1+(-1)^n\bigr].$

26.1.5 Sums Containing Binomial Coefficients

◆ *Throughout Section 26.1.5, it is assumed that* $m = 1, 2, 3, \ldots$

1. $\displaystyle\sum_{k=0}^{n} C_n^k = 2^n.$

2. $\displaystyle\sum_{k=0}^{n} C_{m+k}^m = C_{n+m+1}^{m+1}.$

3. $\displaystyle\sum_{k=0}^{n} (-1)^k C_m^k = (-1)^n C_{m-1}^n.$

4. $\displaystyle\sum_{k=0}^{n} (k+1) C_n^k = 2^{n-1}(n+2).$

5. $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} k C_n^k = 0.$

6. $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} C_n^k = \sum_{m=1}^{n} \frac{1}{m}.$

7. $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1} C_n^k = \frac{n}{n+1}.$

8. $\displaystyle\sum_{k=0}^{n} \frac{1}{k+1} C_n^k = \frac{2^{n+1}-1}{n+1}.$

9. $\displaystyle\sum_{k=0}^{n} \frac{a^{k+1}}{k+1} C_n^k = \frac{(a+1)^{n+1}-1}{n+1}.$

10. $\displaystyle\sum_{k=0}^{p} C_n^k C_m^{p-k} = C_{n+m}^p;$ m and p are natural numbers.

11. $\displaystyle\sum_{k=0}^{n-p} C_n^k C_n^{p+k} = \frac{(2n)!}{(n-p)!\,(n+p)!}.$

12. $\displaystyle\sum_{k=0}^{n} (C_n^k)^2 = C_{2n}^n.$

13. $\displaystyle\sum_{k=0}^{2n} (-1)^k (C_{2n}^k)^2 = (-1)^n C_{2n}^n.$

14. $\displaystyle\sum_{k=0}^{2n+1} (-1)^k (C_{2n+1}^k)^2 = 0.$

15. $\displaystyle\sum_{k=1}^{n} k (C_n^k)^2 = \frac{(2n-1)!}{[(n-1)!]^2}.$

26.1.6 Other Numerical Sums

1. $\displaystyle\sum_{k=1}^{n-1} \sin\frac{\pi k}{n} = \cot\frac{\pi}{2n}.$

2. $\displaystyle\sum_{k=1}^{n} \sin^{2m}\frac{\pi k}{2n} = \frac{n}{2^{2m}}C_{2m}^{m} + \frac{1}{2}, \quad m < 2n.$

3. $\displaystyle\sum_{k=0}^{n-1} (-1)^k \cos^m \frac{\pi k}{n} = \frac{1}{2}\big[1 - (-1)^{m+n}\big], \quad m = 0,\, 1,\, \dots,\, n-1.$

4. $\displaystyle\sum_{k=0}^{n-1} (-1)^k \cos^n \frac{\pi k}{n} = \frac{n}{2^{n-1}}.$

26.2 Finite Functional Sums

26.2.1 Sums Involving Hyperbolic Functions

1. $\displaystyle\sum_{k=0}^{n-1} \sinh(kx+a) = \sinh\left(\frac{n-1}{2}x + a\right)\frac{\sinh(nx/2)}{\sinh(x/2)}.$

2. $\displaystyle\sum_{k=0}^{n-1} \cosh(kx+a) = \cosh\left(\frac{n-1}{2}x + a\right)\frac{\sinh(nx/2)}{\sinh(x/2)}.$

3. $\displaystyle\sum_{k=0}^{n-1} (-1)^k \sinh(kx+a) = \frac{1}{2\cosh(x/2)}\left[\sinh\left(a - \frac{x}{2}\right) + (-1)^n \sinh\left(\frac{2n-1}{2}x + a\right)\right].$

4. $\displaystyle\sum_{k=0}^{n-1} (-1)^k \cosh(kx+a) = \frac{1}{2\cosh(x/2)}\left[\cosh\left(a - \frac{x}{2}\right) + (-1)^n \cosh\left(\frac{2n-1}{2}x + a\right)\right].$

5. $\displaystyle\sum_{k=1}^{n-1} k\sinh(kx+a) = -\frac{1}{\sinh^2(x/2)}\Big\{n\sinh[(n-1)x+a] - (n-1)\sinh(nx+a) - \sinh a\Big\}.$

6. $\displaystyle\sum_{k=1}^{n-1} k\cosh(kx+a) = -\frac{1}{\sinh^2(x/2)}\Big\{n\cosh[(n-1)x+a] - (n-1)\cosh(nx+a) - \cosh a\Big\}.$

7. $\displaystyle\sum_{k=1}^{n-1} (-1)^k k\sinh(kx+a) = \frac{1}{\cosh^2(x/2)}\Big\{(-1)^{n-1} n\sinh[(n-1)x+a] + (-1)^{n-1}(n-1)\sinh(nx+a) - \sinh a\Big\}.$

8. $\displaystyle\sum_{k=1}^{n-1} (-1)^k k\cosh(kx+a) = \frac{1}{\cosh^2(x/2)}\Big\{(-1)^{n-1} n\cosh[(n-1)x+a] + (-1)^{n-1}(n-1)\cosh(nx+a) - \cosh a\Big\}.$

9. $\displaystyle\sum_{k=0}^{n} C_n^k \sinh(kx+a) = 2^n \cosh^n\frac{x}{2}\,\sinh\left(\frac{nx}{2} + a\right).$

10. $\displaystyle\sum_{k=0}^{n} C_n^k \cosh(kx+a) = 2^n \cosh^n\frac{x}{2}\,\cosh\left(\frac{nx}{2} + a\right).$

11. $\displaystyle\sum_{k=1}^{n-1} a^k \sinh(kx) = \frac{a\sinh x - a^n \sinh(nx) + a^{n+1}\sinh[(n-1)x]}{1 - 2a\cosh x + a^2}.$

12. $\displaystyle\sum_{k=0}^{n-1} a^k \cosh(kx) = \frac{1 - a\cosh x - a^n \cosh(nx) + a^{n+1}\cosh[(n-1)x]}{1 - 2a\cosh x + a^2}.$

13. $\displaystyle\sum_{k=1}^{n} \frac{1}{2^k}\tanh\frac{x}{2^k} = \coth x - \frac{1}{2^n}\coth\frac{x}{2^n}.$

14. $\displaystyle\sum_{k=0}^{n-1} 2^k \tanh(2^k x) = 2^n \coth(2^n x) - \coth x.$

26.2.2 Sums Involving Trigonometric Functions

1. $\displaystyle\sum_{k=1}^{n} \sin(2kx) = \sin[(n+1)x]\sin(nx)\operatorname{cosec} x.$

2. $\displaystyle\sum_{k=0}^{n} \cos(2kx) = \sin[(n+1)x]\cos(nx)\operatorname{cosec} x.$

3. $\displaystyle\sum_{k=1}^{n} \sin[(2k-1)x] = \sin^2(nx)\operatorname{cosec} x.$

4. $\displaystyle\sum_{k=1}^{n} \cos[(2k-1)x] = \sin(nx)\cos(nx)\operatorname{cosec} x.$

5. $\displaystyle\sum_{k=0}^{n-1} \sin(kx+a) = \sin\left(\frac{n-1}{2}x + a\right)\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}.$

6. $\displaystyle\sum_{k=0}^{n-1} \cos(kx+a) = \cos\left(\frac{n-1}{2}x + a\right)\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}.$

7. $\displaystyle\sum_{k=0}^{2n-1} (-1)^k \cos(kx+a) = \sin\left(\frac{2n-1}{2}x + a\right)\sin(nx)\sec\frac{x}{2}.$

8. $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} \sin[(2k-1)x] = (-1)^{n+1}\frac{\sin(2nx)}{2\cos x}.$

9. $\displaystyle\sum_{k=1}^{n} (-1)^k \cos(2kx) = -\frac{1}{2} + (-1)^n\frac{\cos[(2n+1)x]}{2\cos x}.$

10. $\displaystyle\sum_{k=1}^{n} \sin^2(kx) = \frac{n}{2} - \frac{\cos[(n+1)x]\sin(nx)}{2\sin x}.$

11. $\displaystyle\sum_{k=1}^{n} \cos^2(kx) = \frac{n}{2} + \frac{\cos[(n+1)x]\sin(nx)}{2\sin x}.$

12. $\displaystyle\sum_{k=1}^{n-1} k\sin(2kx) = \frac{\sin(2nx)}{4\sin^2 x} - \frac{n\cos[(2n-1)x]}{2\sin x}.$

13. $\displaystyle\sum_{k=1}^{n-1} k\cos(2kx) = \frac{n\sin[(2n-1)x]}{2\sin x} - \frac{1-\cos(2nx)}{4\sin^2 x}.$

14. $\displaystyle\sum_{k=1}^{n-1} a^k \sin(kx) = \frac{a\sin x - a^n\sin(nx) + a^{n+1}\sin[(n-1)x]}{1-2a\cos x + a^2}.$

15. $\displaystyle\sum_{k=0}^{n-1} a^k \cos(kx) = \frac{1 - a\cos x - a^n\cos(nx) + a^{n+1}\cos[(n-1)x]}{1-2a\cos x + a^2}.$

16. $\displaystyle\sum_{k=0}^{n} C_n^k \sin(kx+a) = 2^n\cos^n\frac{x}{2}\sin\left(\frac{nx}{2}+a\right).$

17. $\displaystyle\sum_{k=0}^{n} C_n^k \cos(kx+a) = 2^n\cos^n\frac{x}{2}\cos\left(\frac{nx}{2}+a\right).$

18. $\displaystyle\sum_{k=0}^{n} (-1)^k C_n^k \sin(kx+a) = (-2)^n\sin^n\frac{x}{2}\sin\left(\frac{nx}{2}+\frac{\pi n}{2}+a\right).$

19. $\displaystyle\sum_{k=0}^{n} (-1)^k C_n^k \cos(kx+a) = (-2)^n\sin^n\frac{x}{2}\cos\left(\frac{nx}{2}+\frac{\pi n}{2}+a\right).$

20. $\displaystyle\sum_{k=1}^{n} \left(2^k\sin^2\frac{x}{2^k}\right)^2 = \left(2^n\sin^2\frac{x}{2^n}\right)^2 - \sin^2 x.$

21. $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k}\tan\frac{x}{2^k} = \frac{1}{2^n}\cot\frac{x}{2^n} - 2\cot(2x).$

26.3 Infinite Numerical Series

26.3.1 Progressions

1. $\displaystyle\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q}, \quad |q|<1.$

2. $\displaystyle\sum_{k=0}^{\infty} (a+bk)q^k = \frac{a}{1-q} + \frac{bq}{(1-q)^2}, \quad |q|<1.$

26.3.2 Other Numerical Series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2.$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}=1-2\ln 2.$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\frac{3}{4}.$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)}=-\frac{1}{4}.$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}.$

8. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}.$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}.$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{2a}\coth(\pi a)-\frac{1}{2a^2}.$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}=-\frac{\pi}{2a}\cot(\pi a)+\frac{1}{2a^2}.$

13. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=\frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|;$ the B_{2n} are Bernoulli numbers (see Section 30.1.3).

14. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n}}=\frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}|B_{2n}|;$ the B_{2n} are Bernoulli numbers.

15. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2n}}=\frac{(2^{2n-1}-1)\pi^{2n}}{2(2n)!}|B_{2n}|;$ the B_{2n} are Bernoulli numbers.

16. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k2^k}=\ln 2.$

17. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{n^{2k}}=\frac{n^2}{n^2+1}.$

18. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}=e=2.71828\ldots$

19. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=\frac{1}{e}=0.36787\ldots$

20. $\displaystyle\sum_{k=1}^{\infty}\frac{k}{(k+1)!}=1.$

26.4 Infinite Functional Series

26.4.1 Power Series

1. $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1.$

2. $\displaystyle\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}, \quad |x| < 1.$

3. $\displaystyle\sum_{k=1}^{\infty} k^2x^k = \frac{x(x+1)}{(1-x)^3}, \quad |x| < 1.$

4. $\displaystyle\sum_{k=1}^{\infty} k^3x^k = \frac{x(1+4x+x^2)}{(1-x)^4}, \quad |x| < 1.$

5. $\displaystyle\sum_{k=0}^{\infty} (\pm 1)^k k^n x^k = \left(x\frac{d}{dx}\right)^n \frac{1}{1 \mp x}, \quad |x| < 1.$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x), \quad -1 \le x < 1.$

7. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k} = \ln(1+x), \quad |x| < 1.$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1} = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1.$

9. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^{2k-1}}{2k-1} = \arctan x, \quad |x| \le 1.$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2} = -\int_0^x \frac{\ln(1-t)}{t}\,dt, \quad |x| \le 1.$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} = x + (1-x)\ln(1-x), \quad |x| \le 1.$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{x^{k+2}}{k(k+2)} = \frac{x}{2} + \frac{x^2}{4} + \frac{1}{2}(1-x^2)\ln(1-x), \quad |x| \le 1.$

13. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x,$ x is any number.

14. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh x,$ x is any number.

15. $\displaystyle\sum_{k=0}^{\infty} (-1)^k\frac{x^{2k}}{(2k)!} = \cos x,$ x is any number.

16. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh x,$ x is any number.

17. $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin x$, x is any number.

18. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{k+1}}{k!\,(k+1)} = e^x - 1$, x is any number.

19. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{k+2}}{k!\,(k+2)} = (x-1)e^x + 1$, x is any number.

20. $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{k!\,(2k+1)} = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}\,x$, x is any number.

21. $\displaystyle\sum_{k=0}^{\infty} \frac{(k+a)^n}{k!} x^k = \left[\frac{d^n}{dt^n}\exp(at + xe^t)\right]_{t=0}$, x is any number.

22. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1} = \tan x$; the B_{2k} are Bernoulli numbers, $|x| < \pi/2$.

23. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1} = \tanh x$; the B_{2k} are Bernoulli numbers, $|x| < \pi/2$.

24. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} = \frac{1}{x} - \cot x$; the B_{2k} are Bernoulli numbers, $0 < |x| < \pi$.

25. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} = \coth x - \frac{1}{x}$; the B_{2k} are Bernoulli numbers, $|x| < \pi$.

26.4.2 Trigonometric Series in One Variable Involving Sine

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\sin(kx) = \frac{1}{2}(\pi - x)$, $0 < x < 2\pi$.

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\sin(kx) = \frac{1}{2}x$, $-\pi < x < \pi$.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{a^k}{k}\sin(kx) = \arctan\frac{a\sin x}{1 - a\cos x}$, $0 < x < 2\pi$, $|a| \le 1$.

4. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin(kx) = \frac{\pi}{4}\cos\frac{x}{2} - \sin\frac{x}{2}\ln\left(\cot^2\frac{x}{4}\right)$, $0 < x < 2\pi$.

5. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\sin(kx) = -\frac{1}{4}\cos\frac{x}{2}\ln\left(\cot^2\frac{x+\pi}{4}\right) - \frac{\pi}{4}\sin\frac{x}{2}$, $-\pi < x < \pi$.

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\sin(kx) = -\int_0^x \ln\left(2\sin\frac{t}{2}\right)dt$, $0 \le x < \pi$.

7. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\sin(kx) = -\int_0^x \ln\left(2\cos\frac{t}{2}\right)dt$, $-\pi < x < \pi$.

8. $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sin(kx) = (\pi - x)\sin^2\frac{x}{2} + \sin x \ln\left(2\sin\frac{x}{2}\right), \quad 0 \le x \le 2\pi.$$

9. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \sin(kx) = -x\cos^2\frac{x}{2} + \sin x \ln\left(2\cos\frac{x}{2}\right), \quad -\pi \le x \le \pi.$$

10. $$\sum_{k=1}^{\infty} \frac{k}{k^2+a^2} \sin(kx) = \frac{\pi}{2\sinh(\pi a)} \sinh[a(\pi - x)], \quad 0 < x < 2\pi.$$

11. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+a^2} \sin(kx) = \frac{\pi}{2\sinh(\pi a)} \sinh(ax), \quad -\pi < x < \pi.$$

12. $$\sum_{k=1}^{\infty} \frac{k}{k^2-a^2} \sin(kx) = \frac{\pi}{2\sin(\pi a)} \sin[a(\pi - x)], \quad 0 < x < 2\pi.$$

13. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2-a^2} \sin(kx) = \frac{\pi}{2\sin(\pi a)} \sin(ax), \quad -\pi < x < \pi.$$

14. $$\sum_{k=2}^{\infty} (-1)^{k} \frac{k}{k^2-1} \sin(kx) = \frac{1}{4}\sin x + \frac{1}{2}x\cos x, \quad -\pi < x < \pi.$$

15. $$\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sin(kx) = \frac{(-1)^{n-1}(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1}\left(\frac{x}{2\pi}\right),$$
where $0 \le x \le 2\pi$ for $n = 1, 2, \dots$; $0 < x < 2\pi$ for $n = 0$; and the $B_n(x)$ are Bernoulli polynomials (see Section 30.18.1).

16. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n+1}} \sin(kx) = \frac{(-1)^{n-1}(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1}\left(\frac{x+\pi}{2\pi}\right),$$
where $-\pi < x \le \pi$ for $n = 0, 1, \dots$; the $B_n(x)$ are Bernoulli polynomials.

17. $$\sum_{k=1}^{\infty} \frac{1}{k!} \sin(kx) = \exp(\cos x)\sin(\sin x), \quad x \text{ is any number.}$$

18. $$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sin(kx) = -\exp(-\cos x)\sin(\sin x), \quad x \text{ is any number.}$$

19. $$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \sin(kx) = \sin\left(\sin\frac{x}{2}\right)\sinh\left(\cos\frac{x}{2}\right), \quad x \text{ is any number.}$$

20. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \sin(kx) = -\sin\left(\cos\frac{x}{2}\right)\sinh\left(\sin\frac{x}{2}\right), \quad x \text{ is any number.}$$

21. $$\sum_{k=0}^{\infty} \frac{a^k}{k!} \sin(kx) = \exp(k\cos x)\sin(k\sin x), \quad |a| \le 1,\ x \text{ is any number.}$$

22. $$\sum_{k=0}^{\infty} a^k \sin(kx) = \frac{a\sin x}{1 - 2a\cos x + a^2}, \quad |a| < 1,\ x \text{ is any number.}$$

23. $$\sum_{k=1}^{\infty} ka^k \sin(kx) = \frac{a(1-a^2)\sin x}{(1 - 2a\cos x + a^2)^2}, \quad |a| < 1,\ x \text{ is any number.}$$

24. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\sin(kx+a)=\frac{1}{2}(\pi-x)\cos a-\ln\left(2\sin\frac{x}{2}\right)\sin a,\quad 0<x<2\pi.$

25. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sin(kx+a)=\frac{1}{2}x\cos a+\ln\left(2\cos\frac{x}{2}\right)\sin a,\quad -\pi<x<\pi.$

26. $\displaystyle\sum_{k=1}^{\infty}\frac{\sin[(2k-1)x]}{2k-1}=\frac{\pi}{4},\quad 0<x<\pi.$

27. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin[(2k-1)x]}{2k-1}=\frac{1}{2}\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right),\quad -\frac{\pi}{2}<x<\frac{\pi}{2}.$

28. $\displaystyle\sum_{k=1}^{\infty}a^{2k-1}\frac{\sin[(2k-1)x]}{2k-1}=\frac{1}{2}\arctan\frac{2a\sin x}{1-a^2},\quad 0<x<2\pi,\ |a|\le 1.$

29. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}a^{2k-1}\frac{\sin[(2k-1)x]}{2k-1}=\frac{1}{4}\ln\frac{1+2a\sin x+a^2}{1-2a\sin x+a^2},\quad 0<x<\pi,\ |a|\le 1.$

30. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k}\frac{\sin[(k+1)x]}{k(k+1)}=\sin x-\frac{1}{2}x(1+\cos x)-\sin x\ln\left|2\cos\frac{x}{2}\right|.$

31. $\displaystyle\sum_{k=0}^{\infty}a^{2k+1}\sin[(2k+1)x]=\frac{a(1+a^2)\sin x}{(1+a^2)^2-4a^2\cos^2 x},\quad |a|<1,\ x \text{ is any number.}$

32. $\displaystyle\sum_{k=0}^{\infty}(-1)^{k}a^{2k+1}\sin[(2k+1)x]=\frac{a(1-a^2)\sin x}{(1+a^2)^2-4a^2\sin^2 x},\quad |a|<1,\ x \text{ is any number.}$

33. $\displaystyle\sum_{k=1}^{\infty}\frac{\sin[2(k+1)x]}{k(k+1)}=\sin(2x)-(\pi-2x)\sin^2 x-\sin x\cos x\ln(4\sin^2 x),\quad 0\le x\le\pi.$

34. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k}\frac{\sin[(2k+1)x]}{(2k+1)^2}=\begin{cases}\frac{1}{4}\pi x & \text{if } -\frac{1}{2}\pi\le x\le\frac{1}{2}\pi,\\ \frac{1}{4}\pi(\pi-x) & \text{if } \frac{1}{2}\pi\le x\le\frac{3}{2}\pi.\end{cases}$

26.4.3 Trigonometric Series in One Variable Involving Cosine

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\cos(kx)=-\ln\left(2\sin\frac{x}{2}\right),\quad 0<x<2\pi.$

2. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cos(kx)=\ln\left(2\cos\frac{x}{2}\right),\quad -\pi<x<\pi.$

3. $\displaystyle\sum_{k=1}^{\infty}\frac{a^k}{k}\cos(kx)=\ln\frac{1}{\sqrt{1-2a\cos x+a^2}},\quad 0<x<2\pi,\ |a|\le 1.$

4. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2k+1}\cos(kx)=\frac{\pi}{4}\sin\frac{x}{2}+\cos\frac{x}{2}\ln\left(\cot^2\frac{x}{4}\right),\quad 0<x<2\pi.$

5. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos(kx)=-\frac{1}{4}\sin\frac{x}{2}\ln\left(\cot^2\frac{x+\pi}{4}\right)+\frac{\pi}{4}\cos\frac{x}{2},\quad -\pi<x<\pi.$

6. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}\cos(kx)=\frac{1}{12}(3x^2-6\pi x+2\pi^2),\quad 0\le x\le 2\pi.$

7. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx)=\frac{1}{12}(3x^2-\pi^2),\quad -\pi\le x\le \pi.$

8. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\cos(kx)=\frac{1}{2}(x-\pi)\sin x-2\sin^2\frac{x}{2}\ln\left(2\sin\frac{x}{2}\right)+1,\quad 0\le x\le 2\pi.$

9. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\cos(kx)=-\frac{1}{2}x\sin x-2\cos^2\frac{x}{2}\ln\left(2\cos\frac{x}{2}\right)+1,\quad -\pi\le x\le \pi.$

10. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+a^2}\cos(kx)=\frac{\pi}{2a\sinh(\pi a)}\cosh[a(\pi-x)]-\frac{1}{2a^2},\quad 0\le x\le 2\pi.$

11. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2-a^2}\cos(kx)=-\frac{\pi}{2a\sin(\pi a)}\cos[a(\pi-x)]+\frac{1}{2a^2},\quad 0\le x\le 2\pi.$

12. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k^2-1}\cos(kx)=\frac{1}{2}-\frac{1}{4}\cos x-\frac{1}{2}x\sin x,\quad -\pi\le x\le \pi.$

13. $\displaystyle\sum_{k=2}^{\infty}\frac{k}{k^2-1}\cos(kx)=-\frac{1}{2}-\frac{1}{4}\cos x-\cos x\ln\left(2\sin\frac{x}{2}\right),\quad 0< x< 2\pi.$

14. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\cos(kx)=\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}\left(\frac{x}{2\pi}\right),$
where $0\le x\le 2\pi$ for $n=1,\,2,\,\dots$; the $B_n(x)$ are Bernoulli polynomials (see Section 30.18.1).

15. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2n}}\cos(kx)=\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}\left(\frac{x+\pi}{2\pi}\right),$
where $-\pi\le x\le \pi$ for $n=1,\,2,\,\dots$; the $B_n(x)$ are Bernoulli polynomials.

16. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}\cos(kx)=\exp(\cos x)\cos(\sin x),\quad x$ is any number.

17. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\cos(kx)=\exp(-\cos x)\cos(\sin x),\quad x$ is any number.

18. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k)!}\cos(kx)=\cos\left(\sin\frac{x}{2}\right)\cosh\left(\cos\frac{x}{2}\right),\quad x$ is any number.

19. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\cos(kx)=\cos\left(\cos\frac{x}{2}\right)\cosh\left(\sin\frac{x}{2}\right),\quad x$ is any number.

20. $\displaystyle\sum_{k=0}^{\infty}\frac{a^k}{k!}\cos(kx)=\exp(a\cos x)\cos(a\sin x),\quad |a|\le 1,\ x$ is any number.

21. $\displaystyle\sum_{k=0}^{\infty}a^k\cos(kx)=\frac{1-a\cos x}{1-2a\cos x+a^2},\quad |a|<1,\ x$ is any number.

22. $\displaystyle\sum_{k=1}^{\infty} ka^k \cos(kx) = \frac{a(1+a^2)\cos x - 2a^2}{(1-2a\cos x + a^2)^2}, \quad |a|<1, \ x \text{ is any number.}$

23. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\cos(kx+a) = \frac{1}{2}(x-\pi)\sin a - \ln\left(2\sin\frac{x}{2}\right)\cos a, \quad 0<x<2\pi.$

24. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\cos(kx+a) = -\frac{1}{2}x\sin a + \ln\left(2\cos\frac{x}{2}\right)\cos a, \quad -\pi<x<\pi.$

25. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos[(2k-1)x]}{2k-1} = \frac{1}{2}\ln\cot\frac{x}{2}, \quad 0<x<\pi.$

26. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}\frac{\cos[(2k-1)x]}{2k-1} = \frac{\pi}{4}, \quad 0<x<\pi.$

27. $\displaystyle\sum_{k=1}^{\infty} a^{2k-1}\frac{\cos[(2k-1)x]}{2k-1} = \frac{1}{4}\ln\frac{1+2a\cos x+a^2}{1-2a\cos x+a^2}, \quad 0<x<2\pi, \ |a|\le 1.$

28. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}a^{2k-1}\frac{\cos[(2k-1)x]}{2k-1} = \frac{1}{2}\arctan\frac{2a\cos x}{1-a^2}, \quad 0<x<\pi, \ |a|\le 1.$

29. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos[(2k-1)x]}{(2k-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2}-|x|\right), \quad -\pi\le x\le\pi.$

30. $\displaystyle\sum_{k=1}^{\infty} (-1)^k\frac{\cos[(k+1)x]}{k(k+1)} = \cos x - \frac{1}{2}x\sin x - (1+\cos x)\ln\left|2\cos\frac{x}{2}\right|.$

31. $\displaystyle\sum_{k=0}^{\infty} a^{2k+1}\cos[(2k+1)x] = \frac{a(1-a^2)\cos x}{(1+a^2)^2-4a^2\cos^2 x}, \quad |a|<1, \ x \text{ is any number.}$

32. $\displaystyle\sum_{k=0}^{\infty} (-1)^k a^{2k+1}\cos[(2k+1)x] = \frac{a(1+a^2)\cos x}{(1+a^2)^2-4a^2\sin^2 x}, \quad |a|<1, \ x \text{ is any number.}$

33. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos[2(k+1)x]}{k(k+1)} = \cos(2x) - \left(\frac{\pi}{2}-x\right)\sin(2x) + \sin^2 x\ln(4\sin^2 x), \quad 0\le x\le\pi.$

26.4.4 Trigonometric Series in Two Variables

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\sin(kx)\sin(ky) = \frac{1}{2}\ln\left|\sin\frac{x+y}{2}\,\mathrm{cosec}\,\frac{x-y}{2}\right|, \quad x\pm y\ne 0, 2\pi, 4\pi, \dots$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\sin(kx)\sin(ky) = \frac{1}{2}\ln\left|\cos\frac{x+y}{2}\sec\frac{x-y}{2}\right|, \quad x\pm y\ne \pi, 3\pi, 5\pi, \dots$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)\sin(ky) = \begin{cases} \frac{1}{2}x(\pi-y) & \text{if } -y\le x\le y, \\ \frac{1}{2}y(\pi-x) & \text{if } y\le x\le 2\pi-y. \end{cases}$ Here $0<y<\pi$.

4. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}\sin(kx)\sin(ky) = \frac{1}{2}xy, \quad |x\pm y|\le\pi.$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{a^k}{k} \sin(kx)\sin(ky) = \frac{1}{4} \ln \frac{4a\sin^2[(x+y)/2] + (a-1)^2}{4a\sin^2[(x-y)/2] + (a-1)^2}, \quad 0 < a < 1.$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \sin^2(kx)\sin^2(ky) = \frac{1}{2}\pi x, \quad 0 \le x \le y \le \frac{\pi}{2}.$

7. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \cos(kx)\cos(ky) = -\frac{1}{2} \ln\bigl|2(\cos x - \cos y)\bigr|, \quad x \pm y \neq 0,\ 2\pi,\ 4\pi,\ \ldots$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cos(kx)\cos(ky) = -\frac{1}{2} \ln\bigl|2(\cos x + \cos y)\bigr|, \quad x \pm y \neq \pi,\ 3\pi,\ 5\pi,\ \ldots$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \sin(kx)\cos(ky) = \begin{cases} -\frac{1}{2} & \text{if } 0 < x < y, \\ \frac{1}{4}(\pi - 2y) & \text{if } x = y, \\ \frac{1}{2}(\pi - x) & \text{if } y < x < \pi. \end{cases}$ Here $0 < y < \pi$.

10. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \cos(kx)\cos(ky) = \begin{cases} \frac{1}{12}\bigl[3x^2 + 3(y-\pi)^2 - \pi^2\bigr] & \text{if } 0 \le x \le y, \\ \frac{1}{12}\bigl[3y^2 + 3(x-\pi)^2 - \pi^2\bigr] & \text{if } y \le x \le \pi. \end{cases}$
Here $0 < y < \pi$.

11. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx)\cos(ky)$
$$= \begin{cases} \frac{1}{12}\bigl[3(x^2 + y^2) - \pi^2\bigr] & \text{if } -(\pi - y) \le x \le \pi - y, \\ \frac{1}{12}\bigl[3(x-\pi)^2 + 3(y-\pi)^2 - \pi^2\bigr] & \text{if } \pi - y \le x \le \pi + y. \end{cases}$$
Here $0 < y < \pi$.

⊙ *References for Chapter* 26: H. B. Dwight (1961), V. Mangulis (1965), E. R. Hansen (1975), I. S. Gradshteyn and I. M. Ryzhik (2000), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986), D. Zwillinger (2002).

Chapter 27

Indefinite and Definite Integrals

27.1 Indefinite Integrals

★ *Throughout Section 27.1, the integration constant C is omitted for brevity.*

27.1.1 Integrals Involving Rational Functions

► **Integrals involving $a + bx$.**

1. $\displaystyle\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx|.$

2. $\displaystyle\int (a+bx)^n dx = \frac{(a+bx)^{n+1}}{b(n+1)}, \qquad n \neq -1.$

3. $\displaystyle\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\bigl(a+bx-a\ln|a+bx|\bigr).$

4. $\displaystyle\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\Bigl[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\Bigr].$

5. $\displaystyle\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\ln\Bigl|\frac{a+bx}{x}\Bigr|.$

6. $\displaystyle\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\Bigl|\frac{a+bx}{x}\Bigr|.$

7. $\displaystyle\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\Bigl(\ln|a+bx| + \frac{a}{a+bx}\Bigr).$

8. $\displaystyle\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\Bigl(a+bx-2a\ln|a+bx| - \frac{a^2}{a+bx}\Bigr).$

9. $\displaystyle\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\Bigl|\frac{a+bx}{x}\Bigr|.$

10. $\displaystyle\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\Bigl[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\Bigr].$

► **Integrals involving $a + x$ and $b + x$.**

1. $\displaystyle\int \frac{a+x}{b+x}\,dx = x + (a-b)\ln|b+x|.$

2. $\displaystyle\int \frac{dx}{(a+x)(b+x)} = \frac{1}{a-b}\ln\left|\frac{b+x}{a+x}\right|$, $a \neq b$. For $a=b$, see Integral 2 with $n=-2$ in the paragraph 'Integrals involving $a+bx$' above.

3. $\displaystyle\int \frac{x\,dx}{(a+x)(b+x)} = \frac{1}{a-b}\bigl(a\ln|a+x| - b\ln|b+x|\bigr).$

4. $\displaystyle\int \frac{dx}{(a+x)(b+x)^2} = \frac{1}{(b-a)(b+x)} + \frac{1}{(a-b)^2}\ln\left|\frac{a+x}{b+x}\right|.$

5. $\displaystyle\int \frac{x\,dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2}\ln\left|\frac{a+x}{b+x}\right|.$

6. $\displaystyle\int \frac{x^2\,dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(a-b)^2}\ln|a+x| + \frac{b^2-2ab}{(b-a)^2}\ln|b+x|.$

7. $\displaystyle\int \frac{dx}{(a+x)^2(b+x)^2} = -\frac{1}{(a-b)^2}\left(\frac{1}{a+x} + \frac{1}{b+x}\right) + \frac{2}{(a-b)^3}\ln\left|\frac{a+x}{b+x}\right|.$

8. $\displaystyle\int \frac{x\,dx}{(a+x)^2(b+x)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{a+x} + \frac{b}{b+x}\right) + \frac{a+b}{(a-b)^3}\ln\left|\frac{a+x}{b+x}\right|.$

9. $\displaystyle\int \frac{x^2\,dx}{(a+x)^2(b+x)^2} = -\frac{1}{(a-b)^2}\left(\frac{a^2}{a+x} + \frac{b^2}{b+x}\right) + \frac{2ab}{(a-b)^3}\ln\left|\frac{a+x}{b+x}\right|.$

► **Integrals involving a^2+x^2.**

1. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}.$

2. $\displaystyle\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\arctan\frac{x}{a}.$

3. $\displaystyle\int \frac{dx}{(a^2+x^2)^3} = \frac{x}{4a^2(a^2+x^2)^2} + \frac{3x}{8a^4(a^2+x^2)} + \frac{3}{8a^5}\arctan\frac{x}{a}.$

4. $\displaystyle\int \frac{dx}{(a^2+x^2)^{n+1}} = \frac{x}{2na^2(a^2+x^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(a^2+x^2)^n}; \quad n=1,\,2,\,\ldots$

5. $\displaystyle\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2).$

6. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^2} = -\frac{1}{2(a^2+x^2)}.$

7. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^3} = -\frac{1}{4(a^2+x^2)^2}.$

8. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2n(a^2+x^2)^n}; \quad n=1,\,2,\,\ldots$

9. $\displaystyle\int \frac{x^2\,dx}{a^2+x^2} = x - a\arctan\frac{x}{a}.$

10. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^2} = -\frac{x}{2(a^2+x^2)} + \frac{1}{2a}\arctan\frac{x}{a}.$

11. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^3} = -\frac{x}{4(a^2+x^2)^2} + \frac{x}{8a^2(a^2+x^2)} + \frac{1}{8a^3}\arctan\frac{x}{a}.$

12. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^{n+1}} = -\frac{x}{2n(a^2+x^2)^n} + \frac{1}{2n}\int \frac{dx}{(a^2+x^2)^n}; \quad n = 1,\,2,\,\ldots$

13. $\displaystyle\int \frac{x^3\,dx}{a^2+x^2} = \frac{x^2}{2} - \frac{a^2}{2}\ln(a^2+x^2).$

14. $\displaystyle\int \frac{x^3\,dx}{(a^2+x^2)^2} = \frac{a^2}{2(a^2+x^2)} + \frac{1}{2}\ln(a^2+x^2).$

15. $\displaystyle\int \frac{x^3\,dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2(n-1)(a^2+x^2)^{n-1}} + \frac{a^2}{2n(a^2+x^2)^n}; \quad n = 2,\,3,\,\ldots$

16. $\displaystyle\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\frac{x^2}{a^2+x^2}.$

17. $\displaystyle\int \frac{dx}{x(a^2+x^2)^2} = \frac{1}{2a^2(a^2+x^2)} + \frac{1}{2a^4}\ln\frac{x^2}{a^2+x^2}.$

18. $\displaystyle\int \frac{dx}{x(a^2+x^2)^3} = \frac{1}{4a^2(a^2+x^2)^2} + \frac{1}{2a^4(a^2+x^2)} + \frac{1}{2a^6}\ln\frac{x^2}{a^2+x^2}.$

19. $\displaystyle\int \frac{dx}{x^2(a^2+x^2)} = -\frac{1}{a^2x} - \frac{1}{a^3}\arctan\frac{x}{a}.$

20. $\displaystyle\int \frac{dx}{x^2(a^2+x^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(a^2+x^2)} - \frac{3}{2a^5}\arctan\frac{x}{a}.$

21. $\displaystyle\int \frac{dx}{x^3(a^2+x^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(a^2+x^2)} - \frac{1}{a^6}\ln\frac{x^2}{a^2+x^2}.$

22. $\displaystyle\int \frac{dx}{x^2(a^2+x^2)^3} = -\frac{1}{a^6x} - \frac{x}{4a^4(a^2+x^2)^2} - \frac{7x}{8a^6(a^2+x^2)} - \frac{15}{8a^7}\arctan\frac{x}{a}.$

23. $\displaystyle\int \frac{dx}{x^3(a^2+x^2)^3} = -\frac{1}{2a^6x^2} - \frac{1}{a^6(a^2+x^2)} - \frac{1}{4a^4(a^2+x^2)^2} - \frac{3}{2a^8}\ln\frac{x^2}{a^2+x^2}.$

► **Integrals involving $a^2 - x^2$.**

1. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|.$

2. $\displaystyle\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right|.$

3. $\displaystyle\int \frac{dx}{(a^2-x^2)^3} = \frac{x}{4a^2(a^2-x^2)^2} + \frac{3x}{8a^4(a^2-x^2)} + \frac{3}{16a^5}\ln\left|\frac{a+x}{a-x}\right|.$

4. $\displaystyle\int \frac{dx}{(a^2-x^2)^{n+1}} = \frac{x}{2na^2(a^2-x^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(a^2-x^2)^n}; \quad n = 1,\,2,\,\ldots$

5. $\displaystyle\int \frac{x\,dx}{a^2-x^2} = -\frac{1}{2}\ln|a^2-x^2|.$

6. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^2} = \frac{1}{2(a^2-x^2)}.$

7. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^3} = \frac{1}{4(a^2-x^2)^2}.$

8. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^{n+1}} = \frac{1}{2n(a^2-x^2)^n}; \quad n = 1,\,2,\,\ldots$

9. $\displaystyle\int \frac{x^2\,dx}{a^2 - x^2} = -x + \frac{a}{2}\ln\left|\frac{a+x}{a-x}\right|.$

10. $\displaystyle\int \frac{x^2\,dx}{(a^2 - x^2)^2} = \frac{x}{2(a^2 - x^2)} - \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right|.$

11. $\displaystyle\int \frac{x^2\,dx}{(a^2 - x^2)^3} = \frac{x}{4(a^2 - x^2)^2} - \frac{x}{8a^2(a^2 - x^2)} - \frac{1}{16a^3}\ln\left|\frac{a+x}{a-x}\right|.$

12. $\displaystyle\int \frac{x^2\,dx}{(a^2 - x^2)^{n+1}} = \frac{x}{2n(a^2 - x^2)^n} - \frac{1}{2n}\int \frac{dx}{(a^2 - x^2)^n}; \quad n = 1,\, 2,\, \ldots$

13. $\displaystyle\int \frac{x^3\,dx}{a^2 - x^2} = -\frac{x^2}{2} - \frac{a^2}{2}\ln|a^2 - x^2|.$

14. $\displaystyle\int \frac{x^3\,dx}{(a^2 - x^2)^2} = \frac{a^2}{2(a^2 - x^2)} + \frac{1}{2}\ln|a^2 - x^2|.$

15. $\displaystyle\int \frac{x^3\,dx}{(a^2 - x^2)^{n+1}} = -\frac{1}{2(n-1)(a^2 - x^2)^{n-1}} + \frac{a^2}{2n(a^2 - x^2)^n}; \quad n = 2,\, 3, \ldots$

16. $\displaystyle\int \frac{dx}{x(a^2 - x^2)} = \frac{1}{2a^2}\ln\left|\frac{x^2}{a^2 - x^2}\right|.$

17. $\displaystyle\int \frac{dx}{x(a^2 - x^2)^2} = \frac{1}{2a^2(a^2 - x^2)} + \frac{1}{2a^4}\ln\left|\frac{x^2}{a^2 - x^2}\right|.$

18. $\displaystyle\int \frac{dx}{x(a^2 - x^2)^3} = \frac{1}{4a^2(a^2 - x^2)^2} + \frac{1}{2a^4(a^2 - x^2)} + \frac{1}{2a^6}\ln\left|\frac{x^2}{a^2 - x^2}\right|.$

► **Integrals involving $a^3 + x^3$.**

1. $\displaystyle\int \frac{dx}{a^3 + x^3} = \frac{1}{6a^2}\ln\frac{(a+x)^2}{a^2 - ax + x^2} + \frac{1}{a^2\sqrt{3}}\arctan\frac{2x - a}{a\sqrt{3}}.$

2. $\displaystyle\int \frac{dx}{(a^3 + x^3)^2} = \frac{x}{3a^3(a^3 + x^3)} + \frac{2}{3a^3}\int \frac{dx}{a^3 + x^3}.$

3. $\displaystyle\int \frac{x\,dx}{a^3 + x^3} = \frac{1}{6a}\ln\frac{a^2 - ax + x^2}{(a+x)^2} + \frac{1}{a\sqrt{3}}\arctan\frac{2x - a}{a\sqrt{3}}.$

4. $\displaystyle\int \frac{x\,dx}{(a^3 + x^3)^2} = \frac{x^2}{3a^3(a^3 + x^3)} + \frac{1}{3a^3}\int \frac{x\,dx}{a^3 + x^3}.$

5. $\displaystyle\int \frac{x^2\,dx}{a^3 + x^3} = \frac{1}{3}\ln|a^3 + x^3|.$

6. $\displaystyle\int \frac{dx}{x(a^3 + x^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{a^3 + x^3}\right|.$

7. $\displaystyle\int \frac{dx}{x(a^3 + x^3)^2} = \frac{1}{3a^3(a^3 + x^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{a^3 + x^3}\right|.$

8. $\displaystyle\int \frac{dx}{x^2(a^3 + x^3)} = -\frac{1}{a^3x} - \frac{1}{a^3}\int \frac{x\,dx}{a^3 + x^3}.$

9. $\displaystyle\int \frac{dx}{x^2(a^3 + x^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(a^3 + x^3)} - \frac{4}{3a^6}\int \frac{x\,dx}{a^3 + x^3}.$

► **Integrals involving $a^3 - x^3$.**

1. $\displaystyle\int \frac{dx}{a^3 - x^3} = \frac{1}{6a^2}\ln\frac{a^2 + ax + x^2}{(a-x)^2} + \frac{1}{a^2\sqrt{3}}\arctan\frac{2x+a}{a\sqrt{3}}.$
2. $\displaystyle\int \frac{dx}{(a^3 - x^3)^2} = \frac{x}{3a^3(a^3 - x^3)} + \frac{2}{3a^3}\int\frac{dx}{a^3 - x^3}.$
3. $\displaystyle\int \frac{x\,dx}{a^3 - x^3} = \frac{1}{6a}\ln\frac{a^2 + ax + x^2}{(a-x)^2} - \frac{1}{a\sqrt{3}}\arctan\frac{2x+a}{a\sqrt{3}}.$
4. $\displaystyle\int \frac{x\,dx}{(a^3 - x^3)^2} = \frac{x^2}{3a^3(a^3 - x^3)} + \frac{1}{3a^3}\int\frac{x\,dx}{a^3 - x^3}.$
5. $\displaystyle\int \frac{x^2\,dx}{a^3 - x^3} = -\frac{1}{3}\ln|a^3 - x^3|.$
6. $\displaystyle\int \frac{dx}{x(a^3 - x^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{a^3 - x^3}\right|.$
7. $\displaystyle\int \frac{dx}{x(a^3 - x^3)^2} = \frac{1}{3a^3(a^3 - x^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{a^3 - x^3}\right|.$
8. $\displaystyle\int \frac{dx}{x^2(a^3 - x^3)} = -\frac{1}{a^3 x} + \frac{1}{a^3}\int\frac{x\,dx}{a^3 - x^3}.$
9. $\displaystyle\int \frac{dx}{x^2(a^3 - x^3)^2} = -\frac{1}{a^6 x} - \frac{x^2}{3a^6(a^3 - x^3)} + \frac{4}{3a^6}\int\frac{x\,dx}{a^3 - x^3}.$

► **Integrals involving $a^4 \pm x^4$.**

1. $\displaystyle\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}}\ln\frac{a^2 + ax\sqrt{2} + x^2}{a^2 - ax\sqrt{2} + x^2} + \frac{1}{2a^3\sqrt{2}}\arctan\frac{ax\sqrt{2}}{a^2 - x^2}.$
2. $\displaystyle\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2}\arctan\frac{x^2}{a^2}.$
3. $\displaystyle\int \frac{x^2\,dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}}\ln\frac{a^2 + ax\sqrt{2} + x^2}{a^2 - ax\sqrt{2} + x^2} + \frac{1}{2a\sqrt{2}}\arctan\frac{ax\sqrt{2}}{a^2 - x^2}.$
4. $\displaystyle\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right| + \frac{1}{2a^3}\arctan\frac{x}{a}.$
5. $\displaystyle\int \frac{x\,dx}{a^4 - x^4} = \frac{1}{4a^2}\ln\left|\frac{a^2 + x^2}{a^2 - x^2}\right|.$
6. $\displaystyle\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\arctan\frac{x}{a}.$

27.1.2 Integrals Involving Irrational Functions

► **Integrals involving $x^{1/2}$.**

1. $\displaystyle\int \frac{x^{1/2}\,dx}{a^2 + b^2 x} = \frac{2}{b^2}x^{1/2} - \frac{2a}{b^3}\arctan\frac{bx^{1/2}}{a}.$
2. $\displaystyle\int \frac{x^{3/2}\,dx}{a^2 + b^2 x} = \frac{2x^{3/2}}{3b^2} - \frac{2a^2 x^{1/2}}{b^4} + \frac{2a^3}{b^5}\arctan\frac{bx^{1/2}}{a}.$

3. $\displaystyle\int \frac{x^{1/2}\,dx}{(a^2+b^2x)^2} = -\frac{x^{1/2}}{b^2(a^2+b^2x)} + \frac{1}{ab^3}\arctan\frac{bx^{1/2}}{a}.$

4. $\displaystyle\int \frac{x^{3/2}\,dx}{(a^2+b^2x)^2} = \frac{2x^{3/2}}{b^2(a^2+b^2x)} + \frac{3a^2x^{1/2}}{b^4(a^2+b^2x)} - \frac{3a}{b^5}\arctan\frac{bx^{1/2}}{a}.$

5. $\displaystyle\int \frac{dx}{(a^2+b^2x)x^{1/2}} = \frac{2}{ab}\arctan\frac{bx^{1/2}}{a}.$

6. $\displaystyle\int \frac{dx}{(a^2+b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} - \frac{2b}{a^3}\arctan\frac{bx^{1/2}}{a}.$

7. $\displaystyle\int \frac{dx}{(a^2+b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2+b^2x)} + \frac{1}{a^3b}\arctan\frac{bx^{1/2}}{a}.$

8. $\displaystyle\int \frac{x^{1/2}\,dx}{a^2-b^2x} = -\frac{2}{b^2}x^{1/2} + \frac{2a}{b^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

9. $\displaystyle\int \frac{x^{3/2}\,dx}{a^2-b^2x} = -\frac{2x^{3/2}}{3b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{a^3}{b^5}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

10. $\displaystyle\int \frac{x^{1/2}\,dx}{(a^2-b^2x)^2} = \frac{x^{1/2}}{b^2(a^2-b^2x)} - \frac{1}{2ab^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

11. $\displaystyle\int \frac{x^{3/2}\,dx}{(a^2-b^2x)^2} = \frac{3a^2x^{1/2}-2b^2x^{3/2}}{b^4(a^2-b^2x)} - \frac{3a}{2b^5}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

12. $\displaystyle\int \frac{dx}{(a^2-b^2x)x^{1/2}} = \frac{1}{ab}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

13. $\displaystyle\int \frac{dx}{(a^2-b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} + \frac{b}{a^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

14. $\displaystyle\int \frac{dx}{(a^2-b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2-b^2x)} + \frac{1}{2a^3b}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

► Integrals involving $(a+bx)^{p/2}$.

1. $\displaystyle\int (a+bx)^{p/2}\,dx = \frac{2}{b(p+2)}(a+bx)^{(p+2)/2}.$

2. $\displaystyle\int x(a+bx)^{p/2}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{(p+4)/2}}{p+4} - \frac{a(a+bx)^{(p+2)/2}}{p+2}\right].$

3. $\displaystyle\int x^2(a+bx)^{p/2}\,dx = \frac{2}{b^3}\left[\frac{(a+bx)^{(p+6)/2}}{p+6} - \frac{2a(a+bx)^{(p+4)/2}}{p+4} + \frac{a^2(a+bx)^{(p+2)/2}}{p+2}\right].$

► Integrals involving $(x^2+a^2)^{1/2}$.

1. $\displaystyle\int (x^2+a^2)^{1/2}\,dx = \frac{1}{2}x(a^2+x^2)^{1/2} + \frac{a^2}{2}\ln\bigl[x+(x^2+a^2)^{1/2}\bigr].$

2. $\displaystyle\int x(x^2+a^2)^{1/2}\,dx = \frac{1}{3}(a^2+x^2)^{3/2}.$

3. $\displaystyle\int (x^2+a^2)^{3/2}\,dx = \frac{1}{4}x(a^2+x^2)^{3/2} + \frac{3}{8}a^2x(a^2+x^2)^{1/2} + \frac{3}{8}a^4\ln\bigl|x+(x^2+a^2)^{1/2}\bigr|.$

4. $\displaystyle\int \frac{1}{x}(x^2+a^2)^{1/2}\,dx = (a^2+x^2)^{1/2} - a\ln\left|\frac{a+(x^2+a^2)^{1/2}}{x}\right|.$

5. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\bigl[x+(x^2+a^2)^{1/2}\bigr].$

6. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+a^2}} = (x^2+a^2)^{1/2}.$

7. $\displaystyle\int (x^2+a^2)^{-3/2}\,dx = a^{-2}x(x^2+a^2)^{-1/2}.$

► **Integrals involving $(x^2-a^2)^{1/2}$.**

1. $\displaystyle\int (x^2-a^2)^{1/2}\,dx = \frac{1}{2}x(x^2-a^2)^{1/2} - \frac{a^2}{2}\ln\bigl|x+(x^2-a^2)^{1/2}\bigr|.$

2. $\displaystyle\int x(x^2-a^2)^{1/2}\,dx = \frac{1}{3}(x^2-a^2)^{3/2}.$

3. $\displaystyle\int (x^2-a^2)^{3/2}\,dx = \frac{1}{4}x(x^2-a^2)^{3/2} - \frac{3}{8}a^2x(x^2-a^2)^{1/2} + \frac{3}{8}a^4\ln\bigl|x+(x^2-a^2)^{1/2}\bigr|.$

4. $\displaystyle\int \frac{1}{x}(x^2-a^2)^{1/2}\,dx = (x^2-a^2)^{1/2} - a\arccos\left|\frac{a}{x}\right|.$

5. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\bigl|x+(x^2-a^2)^{1/2}\bigr|.$

6. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-a^2}} = (x^2-a^2)^{1/2}.$

7. $\displaystyle\int (x^2-a^2)^{-3/2}\,dx = -a^{-2}x(x^2-a^2)^{-1/2}.$

► **Integrals involving $(a^2-x^2)^{1/2}$.**

1. $\displaystyle\int (a^2-x^2)^{1/2}\,dx = \frac{1}{2}x(a^2-x^2)^{1/2} + \frac{a^2}{2}\arcsin\frac{x}{a}.$

2. $\displaystyle\int x(a^2-x^2)^{1/2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2}.$

3. $\displaystyle\int (a^2-x^2)^{3/2}\,dx = \frac{1}{4}x(a^2-x^2)^{3/2} + \frac{3}{8}a^2x(a^2-x^2)^{1/2} + \frac{3}{8}a^4\arcsin\frac{x}{a}.$

4. $\displaystyle\int \frac{1}{x}(a^2-x^2)^{1/2}\,dx = (a^2-x^2)^{1/2} - a\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$

5. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}.$

6. $\displaystyle\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -(a^2-x^2)^{1/2}.$

7. $\displaystyle\int (a^2-x^2)^{-3/2}\,dx = a^{-2}x(a^2-x^2)^{-1/2}.$

► **Integrals involving arbitrary powers. Reduction formulas.**

1. $\displaystyle\int \frac{dx}{x(ax^n+b)} = \frac{1}{bn}\ln\left|\frac{x^n}{ax^n+b}\right|.$
2. $\displaystyle\int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{2}{an}\ln\left|\frac{x^{n/2}}{\sqrt{x^n+a^2}+a}\right|.$
3. $\displaystyle\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{an}\arccos\left|\frac{a}{x^{n/2}}\right|.$
4. $\displaystyle\int \frac{dx}{x\sqrt{ax^{2n}+bx^n}} = -\frac{2\sqrt{ax^{2n}+bx^n}}{bnx^n}.$

★ *The parameters* a, b, p, m, *and* n *below in Integrals 5–8 can assume arbitrary values, except those for which denominators vanish in successive applications of a formula. Notation:* $w = ax^n + b$.

5. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{m+np+1}\left(x^{m+1}w^p + npb\int x^m w^{p-1}\,dx\right).$
6. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{bn(p+1)}\left[-x^{m+1}w^{p+1} + (m+n+np+1)\int x^m w^{p+1}\,dx\right].$
7. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{b(m+1)}\left[x^{m+1}w^{p+1} - a(m+n+np+1)\int x^{m+n} w^{p}\,dx\right].$
8. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{a(m+np+1)}\left[x^{m-n+1}w^{p+1} - b(m-n+1)\int x^{m-n} w^{p}\,dx\right].$

27.1.3 Integrals Involving Exponential Functions

1. $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax}.$
2. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}.$
3. $\displaystyle\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right).$
4. $\displaystyle\int x^2e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right).$
5. $\displaystyle\int x^ne^{ax}\,dx = e^{ax}\left[\frac{1}{a}x^n - \frac{n}{a^2}x^{n-1} + \frac{n(n-1)}{a^3}x^{n-2} - \cdots + (-1)^{n-1}\frac{n!}{a^n}x + (-1)^n\frac{n!}{a^{n+1}}\right],$
   $n = 1,\ 2,\ \ldots$
6. $\displaystyle\int P_n(x)e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k}{a^{k+1}}\frac{d^k}{dx^k}P_n(x),$ where $P_n(x)$ is an arbitrary polynomial of degree n.
7. $\displaystyle\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\ln|a+be^{px}|.$
8. $$\int \frac{dx}{ae^{px}+be^{-px}} = \begin{cases} \dfrac{1}{p\sqrt{ab}}\arctan\left(e^{px}\sqrt{\dfrac{a}{b}}\right) & \text{if } ab>0, \\ \dfrac{1}{2p\sqrt{-ab}}\ln\left(\dfrac{b+e^{px}\sqrt{-ab}}{b-e^{px}\sqrt{-ab}}\right) & \text{if } ab<0. \end{cases}$$

9. $$\int \frac{dx}{\sqrt{a+be^{px}}} = \begin{cases} \dfrac{1}{p\sqrt{a}} \ln \dfrac{\sqrt{a+be^{px}}-\sqrt{a}}{\sqrt{a+be^{px}}+\sqrt{a}} & \text{if } a>0, \\ \dfrac{2}{p\sqrt{-a}} \arctan \dfrac{\sqrt{a+be^{px}}}{\sqrt{-a}} & \text{if } a<0. \end{cases}$$

27.1.4 Integrals Involving Hyperbolic Functions

► **Integrals involving $\cosh x$.**

1. $$\int \cosh(a+bx)\,dx = \frac{1}{b}\sinh(a+bx).$$

2. $$\int x\cosh x\,dx = x\sinh x - \cosh x.$$

3. $$\int x^2\cosh x\,dx = (x^2+2)\sinh x - 2x\cosh x.$$

4. $$\int x^{2n}\cosh x\,dx = (2n)!\sum_{k=1}^{n}\left[\frac{x^{2k}}{(2k)!}\sinh x - \frac{x^{2k-1}}{(2k-1)!}\cosh x\right].$$

5. $$\int x^{2n+1}\cosh x\,dx = (2n+1)!\sum_{k=0}^{n}\left[\frac{x^{2k+1}}{(2k+1)!}\sinh x - \frac{x^{2k}}{(2k)!}\cosh x\right].$$

6. $$\int x^p\cosh x\,dx = x^p\sinh x - px^{p-1}\cosh x + p(p-1)\int x^{p-2}\cosh x\,dx.$$

7. $$\int \cosh^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sinh 2x.$$

8. $$\int \cosh^3 x\,dx = \sinh x + \tfrac{1}{3}\sinh^3 x.$$

9. $$\int \cosh^{2n} x\,dx = C_{2n}^{n}\frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^{k}\frac{\sinh[2(n-k)x]}{2(n-k)}, \qquad n=1,\,2,\,\dots$$

10. $$\int \cosh^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^{k}\frac{\sinh[(2n-2k+1)x]}{2n-2k+1} = \sum_{k=0}^{n} C_{n}^{k}\frac{\sinh^{2k+1} x}{2k+1},$$
$n=1,\,2,\dots$

11. $$\int \cosh^p x\,dx = \frac{1}{p}\sinh x\cosh^{p-1} x + \frac{p-1}{p}\int \cosh^{p-2} x\,dx.$$

12. $$\int \cosh ax\cosh bx\,dx = \frac{1}{a^2-b^2}(a\cosh bx\sinh ax - b\cosh ax\sinh bx).$$

13. $$\int \frac{dx}{\cosh ax} = \frac{2}{a}\arctan\bigl(e^{ax}\bigr).$$

14. $$\int \frac{dx}{\cosh^{2n} x} = \frac{\sinh x}{2n-1}\left[\frac{1}{\cosh^{2n-1} x} + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\,\frac{1}{\cosh^{2n-2k-1} x}\right], \quad n=1,\,2,\,\dots$$

15. $$\int \frac{dx}{\cosh^{2n+1} x} = \frac{\sinh x}{2n}\left[\frac{1}{\cosh^{2n} x} + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \frac{1}{\cosh^{2n-2k} x}\right] + \frac{(2n-1)!!}{(2n)!!} \arctan\sinh x,$$
$n = 1,\ 2,\ \dots$

16. $$\int \frac{dx}{a + b\cosh x} = \begin{cases} -\dfrac{\operatorname{sign} x}{\sqrt{b^2 - a^2}} \arcsin \dfrac{b + a\cosh x}{a + b\cosh x} & \text{if } a^2 < b^2, \\ \dfrac{1}{\sqrt{a^2 - b^2}} \ln \dfrac{a + b + \sqrt{a^2 - b^2}\tanh(x/2)}{a + b - \sqrt{a^2 - b^2}\tanh(x/2)} & \text{if } a^2 > b^2. \end{cases}$$

► **Integrals involving $\sinh x$.**

1. $$\int \sinh(a + bx)\,dx = \frac{1}{b}\cosh(a + bx).$$

2. $$\int x\sinh x\,dx = x\cosh x - \sinh x.$$

3. $$\int x^2 \sinh x\,dx = (x^2 + 2)\cosh x - 2x\sinh x.$$

4. $$\int x^{2n} \sinh x\,dx = (2n)!\left[\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}\cosh x - \sum_{k=1}^{n} \frac{x^{2k-1}}{(2k-1)!}\sinh x\right].$$

5. $$\int x^{2n+1} \sinh x\,dx = (2n+1)!\sum_{k=0}^{n}\left[\frac{x^{2k+1}}{(2k+1)!}\cosh x - \frac{x^{2k}}{(2k)!}\sinh x\right].$$

6. $$\int x^p \sinh x\,dx = x^p\cosh x - px^{p-1}\sinh x + p(p-1)\int x^{p-2}\sinh x\,dx.$$

7. $$\int \sinh^2 x\,dx = -\tfrac{1}{2}x + \tfrac{1}{4}\sinh 2x.$$

8. $$\int \sinh^3 x\,dx = -\cosh x + \tfrac{1}{3}\cosh^3 x.$$

9. $$\int \sinh^{2n} x\,dx = (-1)^n C_{2n}^n \frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k C_{2n}^k \frac{\sinh[2(n-k)x]}{2(n-k)}, \quad n = 1,\ 2,\ \dots$$

10. $$\int \sinh^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k C_{2n+1}^k \frac{\cosh[(2n-2k+1)x]}{2n-2k+1} = \sum_{k=0}^{n}(-1)^{n+k} C_n^k \frac{\cosh^{2k+1} x}{2k+1}, \quad n = 1,\ 2,\ \dots$$

11. $$\int \sinh^p x\,dx = \frac{1}{p}\sinh^{p-1} x\cosh x - \frac{p-1}{p}\int \sinh^{p-2} x\,dx.$$

12. $$\int \sinh ax \sinh bx\,dx = \frac{1}{a^2 - b^2}\bigl(a\cosh ax\sinh bx - b\cosh bx\sinh ax\bigr).$$

13. $$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right|.$$

14. $$\int \frac{dx}{\sinh^{2n} x} = \frac{\cosh x}{2n-1}\left[-\frac{1}{\sinh^{2n-1} x} + \sum_{k=1}^{n-1} (-1)^{k-1} \frac{A_{nk}}{\sinh^{2n-2k-1} x}\right],$$
$$A_{nk} = \frac{2^k (n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \qquad n = 1,\, 2,\, \dots$$

15. $$\int \frac{dx}{\sinh^{2n+1} x} = \frac{\cosh x}{2n}\left[-\frac{1}{\sinh^{2n} x} + \sum_{k=1}^{n-1} (-1)^{k-1} \frac{A_{nk}}{\sinh^{2n-2k} x}\right]$$
$$+ (-1)^n \frac{(2n-1)!!}{(2n)!!} \ln \tanh \frac{x}{2}, \quad A_{nk} = \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k (n-1)(n-2)\dots(n-k)},$$
$n = 1,\, 2,\, \dots$

16. $$\int \frac{dx}{a + b \sinh x} = \frac{1}{\sqrt{a^2+b^2}} \ln \frac{a \tanh(x/2) - b + \sqrt{a^2+b^2}}{a \tanh(x/2) - b - \sqrt{a^2+b^2}}.$$

17. $$\int \frac{Ax + B \sinh x}{a + b \sinh x}\, dx = \frac{B}{b} x + \frac{Ab - Ba}{b\sqrt{a^2+b^2}} \ln \frac{a \tanh(x/2) - b + \sqrt{a^2+b^2}}{a \tanh(x/2) - b - \sqrt{a^2+b^2}}.$$

► **Integrals involving $\tanh x$ or $\coth x$.**

1. $$\int \tanh x\, dx = \ln \cosh x.$$

2. $$\int \tanh^2 x\, dx = x - \tanh x.$$

3. $$\int \tanh^3 x\, dx = -\tfrac{1}{2} \tanh^2 x + \ln \cosh x.$$

4. $$\int \tanh^{2n} x\, dx = x - \sum_{k=1}^{n} \frac{\tanh^{2n-2k+1} x}{2n-2k+1}, \qquad n = 1,\, 2,\, \dots$$

5. $$\int \tanh^{2n+1} x\, dx = \ln \cosh x - \sum_{k=1}^{n} \frac{(-1)^k C_n^k}{2k \cosh^{2k} x} = \ln \cosh x - \sum_{k=1}^{n} \frac{\tanh^{2n-2k+2} x}{2n-2k+2},$$
$n = 1,\, 2,\, \dots$

6. $$\int \tanh^p x\, dx = -\frac{1}{p-1} \tanh^{p-1} x + \int \tanh^{p-2} x\, dx.$$

7. $$\int \coth x\, dx = \ln |\sinh x|.$$

8. $$\int \coth^2 x\, dx = x - \coth x.$$

9. $$\int \coth^3 x\, dx = -\tfrac{1}{2} \coth^2 x + \ln |\sinh x|.$$

10. $$\int \coth^{2n} x\, dx = x - \sum_{k=1}^{n} \frac{\coth^{2n-2k+1} x}{2n-2k+1}, \qquad n = 1,\, 2,\, \dots$$

11. $$\int \coth^{2n+1} x\, dx = \ln |\sinh x| - \sum_{k=1}^{n} \frac{C_n^k}{2k \sinh^{2k} x} = \ln |\sinh x| - \sum_{k=1}^{n} \frac{\coth^{2n-2k+2} x}{2n-2k+2},$$
$n = 1,\, 2,\, \dots$

12. $$\int \coth^p x\, dx = -\frac{1}{p-1} \coth^{p-1} x + \int \coth^{p-2} x\, dx.$$

27.1.5 Integrals Involving Logarithmic Functions

1. $\displaystyle\int \ln ax\,dx = x\ln ax - x.$

2. $\displaystyle\int x\ln x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2.$

3. $$\int x^p\ln ax\,dx = \begin{cases} \dfrac{1}{p+1}x^{p+1}\ln ax - \dfrac{1}{(p+1)^2}x^{p+1} & \text{if } p\neq -1, \\ \tfrac{1}{2}\ln^2 ax & \text{if } p = -1. \end{cases}$$

4. $\displaystyle\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x.$

5. $\displaystyle\int x(\ln x)^2\,dx = \tfrac{1}{2}x^2(\ln x)^2 - \tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2.$

6. $$\int x^p(\ln x)^2\,dx = \begin{cases} \dfrac{x^{p+1}}{p+1}(\ln x)^2 - \dfrac{2x^{p+1}}{(p+1)^2}\ln x + \dfrac{2x^{p+1}}{(p+1)^3} & \text{if } p\neq -1, \\ \tfrac{1}{3}\ln^3 x & \text{if } p = -1. \end{cases}$$

7. $\displaystyle\int (\ln x)^n\,dx = \frac{x}{n+1}\sum_{k=0}^{n}(-1)^k(n+1)n\ldots(n-k+1)(\ln x)^{n-k}, \qquad n = 1,\,2,\,\ldots$

8. $\displaystyle\int (\ln x)^q\,dx = x(\ln x)^q - q\int (\ln x)^{q-1}\,dx,\ q\neq -1.$

9. $\displaystyle\int x^n(\ln x)^m\,dx = \frac{x^{n+1}}{m+1}\sum_{k=0}^{m}\frac{(-1)^k}{(n+1)^{k+1}}(m+1)m\ldots(m-k+1)(\ln x)^{m-k},$
$n, m = 1,\,2,\,\ldots$

10. $\displaystyle\int x^p(\ln x)^q\,dx = \frac{1}{p+1}x^{p+1}(\ln x)^q - \frac{q}{p+1}\int x^p(\ln x)^{q-1}\,dx, \qquad p,\,q\neq -1.$

11. $\displaystyle\int \ln(a+bx)\,dx = \frac{1}{b}(ax+b)\ln(ax+b) - x.$

12. $\displaystyle\int x\ln(a+bx)\,dx = \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right)\ln(a+bx) - \frac{1}{2}\left(\frac{x^2}{2} - \frac{a}{b}x\right).$

13. $\displaystyle\int x^2\ln(a+bx)\,dx = \frac{1}{3}\left(x^3 - \frac{a^3}{b^3}\right)\ln(a+bx) - \frac{1}{3}\left(\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2}\right).$

14. $\displaystyle\int \frac{\ln x\,dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab}\ln\frac{x}{a+bx}.$

15. $\displaystyle\int \frac{\ln x\,dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b}\ln\frac{x}{a+bx}.$

16. $$\int \frac{\ln x\,dx}{\sqrt{a+bx}} = \begin{cases} \dfrac{2}{b}\left[(\ln x - 2)\sqrt{a+bx} + \sqrt{a}\,\ln\dfrac{\sqrt{a+bx}+\sqrt{a}}{\sqrt{a+bx}-\sqrt{a}}\right] & \text{if } a > 0, \\ \dfrac{2}{b}\left[(\ln x - 2)\sqrt{a+bx} + 2\sqrt{-a}\,\arctan\dfrac{\sqrt{a+bx}}{\sqrt{-a}}\right] & \text{if } a < 0. \end{cases}$$

17. $\displaystyle\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\arctan(x/a).$

18. $\displaystyle\int x\ln(x^2+a^2)\,dx = \tfrac{1}{2}\left[(x^2+a^2)\ln(x^2+a^2) - x^2\right].$

19. $\int x^2 \ln(x^2+a^2)\,dx = \frac{1}{3}\big[x^3 \ln(x^2+a^2) - \frac{2}{3}x^3 + 2a^2 x - 2a^3 \arctan(x/a)\big].$

27.1.6 Integrals Involving Trigonometric Functions

► **Integrals involving $\cos x$ ($n = 1,\ 2,\ \ldots$).**

1. $\int \cos(a+bx)\,dx = \dfrac{1}{b}\sin(a+bx).$

2. $\int x\cos x\,dx = \cos x + x\sin x.$

3. $\int x^2 \cos x\,dx = 2x\cos x + (x^2-2)\sin x.$

4. $\int x^{2n}\cos x\,dx = (2n)!\left[\sum_{k=0}^{n}(-1)^k \dfrac{x^{2n-2k}}{(2n-2k)!}\sin x + \sum_{k=0}^{n-1}(-1)^k \dfrac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\right].$

5. $\int x^{2n+1}\cos x\,dx = (2n+1)!\sum_{k=0}^{n}\left[(-1)^k \dfrac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x + \dfrac{x^{2n-2k}}{(2n-2k)!}\cos x\right].$

6. $\int x^p \cos x\,dx = x^p \sin x + px^{p-1}\cos x - p(p-1)\int x^{p-2}\cos x\,dx.$

7. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x.$

8. $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x.$

9. $\int \cos^{2n} x\,dx = \dfrac{1}{2^{2n}}C_{2n}^{n}x + \dfrac{1}{2^{2n-1}}\sum_{k=0}^{n-1}C_{2n}^{k}\dfrac{\sin[(2n-2k)x]}{2n-2k}.$

10. $\int \cos^{2n+1} x\,dx = \dfrac{1}{2^{2n}}\sum_{k=0}^{n}C_{2n+1}^{k}\dfrac{\sin[(2n-2k+1)x]}{2n-2k+1}.$

11. $\int \dfrac{dx}{\cos x} = \ln\left|\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right|.$

12. $\int \dfrac{dx}{\cos^2 x} = \tan x.$

13. $\int \dfrac{dx}{\cos^3 x} = \dfrac{\sin x}{2\cos^2 x} + \dfrac{1}{2}\ln\left|\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right|.$

14. $\int \dfrac{dx}{\cos^n x} = \dfrac{\sin x}{(n-1)\cos^{n-1} x} + \dfrac{n-2}{n-1}\int \dfrac{dx}{\cos^{n-2} x}, \quad n > 1.$

15. $\int \dfrac{x\,dx}{\cos^{2n} x} = \sum_{k=0}^{n-1}A_{nk}\dfrac{(2n-2k)x\sin x - \cos x}{\cos^{2n-2k+1} x} + \dfrac{2^{n-1}(n-1)!}{(2n-1)!!}\big(x\tan x + \ln|\cos x|\big),$

$$A_{nk} = \frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)}\,\frac{1}{(2n-2k+1)(2n-2k)}.$$

16. $\int \cos ax\,\cos bx\,dx = \dfrac{\sin\big[(b-a)x\big]}{2(b-a)} + \dfrac{\sin\big[(b+a)x\big]}{2(b+a)}, \quad a \neq \pm b.$

17. $$\int \frac{dx}{a+b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}} \arctan \dfrac{(a-b)\tan(x/2)}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\ \dfrac{1}{\sqrt{b^2-a^2}} \ln \left| \dfrac{\sqrt{b^2-a^2}+(b-a)\tan(x/2)}{\sqrt{b^2-a^2}-(b-a)\tan(x/2)} \right| & \text{if } b^2 > a^2. \end{cases}$$

18. $$\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2} \int \frac{dx}{a+b\cos x}.$$

19. $$\int \frac{dx}{a^2+b^2\cos^2 x} = \frac{1}{a\sqrt{a^2+b^2}} \arctan \frac{a\tan x}{\sqrt{a^2+b^2}}.$$

20. $$\int \frac{dx}{a^2-b^2\cos^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \arctan \dfrac{a\tan x}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\ \dfrac{1}{2a\sqrt{b^2-a^2}} \ln \left| \dfrac{\sqrt{b^2-a^2}-a\tan x}{\sqrt{b^2-a^2}+a\tan x} \right| & \text{if } b^2 > a^2. \end{cases}$$

21. $$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b}{a^2+b^2}\sin bx + \frac{a}{a^2+b^2}\cos bx\right).$$

22. $$\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\cos^2 x + 2\sin x\cos x + \frac{2}{a}\right).$$

23. $$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1} x}{a^2+n^2}(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2} \int e^{ax}\cos^{n-2} x\,dx.$$

► **Integrals involving $\sin x$ ($n = 1,\ 2,\ \ldots$).**

1. $$\int \sin(a+bx)\,dx = -\frac{1}{b}\cos(a+bx).$$

2. $$\int x\sin x\,dx = \sin x - x\cos x.$$

3. $$\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x.$$

4. $$\int x^3\sin x\,dx = (3x^2-6)\sin x - (x^3-6x)\cos x.$$

5. $$\int x^{2n}\sin x\,dx = (2n)!\left[\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k}}{(2n-2k)!}\cos x + \sum_{k=0}^{n-1}(-1)^k\frac{x^{2n-2k-1}}{(2n-2k-1)!}\sin x\right].$$

6. $$\int x^{2n+1}\sin x\,dx = (2n+1)!\sum_{k=0}^{n}\left[(-1)^{k+1}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\cos x + (-1)^k\frac{x^{2n-2k}}{(2n-2k)!}\sin x\right].$$

7. $$\int x^p\sin x\,dx = -x^p\cos x + px^{p-1}\sin x - p(p-1)\int x^{p-2}\sin x\,dx.$$

8. $$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$$

9. $$\int x\sin^2 x\,dx = \tfrac{1}{4}x^2 - \tfrac{1}{4}x\sin 2x - \tfrac{1}{8}\cos 2x.$$

10. $$\int \sin^3 x\,dx = -\cos x + \tfrac{1}{3}\cos^3 x.$$

11. $$\int \sin^{2n} x\,dx = \frac{1}{2^{2n}} C_{2n}^{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k C_{2n}^{k} \frac{\sin[(2n-2k)x]}{2n-2k},$$
where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients $(0! = 1)$.

12. $$\int \sin^{2n+1} x\,dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n+k+1} C_{2n+1}^{k} \frac{\cos[(2n-2k+1)x]}{2n-2k+1}.$$

13. $$\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right|.$$

14. $$\int \frac{dx}{\sin^2 x} = -\cot x.$$

15. $$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|.$$

16. $$\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}, \qquad n > 1.$$

17. $$\int \frac{x\,dx}{\sin^{2n} x} = -\sum_{k=0}^{n-1} A_{nk} \frac{\sin x + (2n-2k)x\cos x}{\sin^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!}\big(\ln|\sin x| - x\cot x\big),$$
$$A_{nk} = \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)}\,\frac{1}{(2n-2k+1)(2n-2k)}.$$

18. $$\int \sin ax\,\sin bx\,dx = \frac{\sin[(b-a)x]}{2(b-a)} - \frac{\sin[(b+a)x]}{2(b+a)}, \qquad a \neq \pm b.$$

19. $$\int \frac{dx}{a+b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}} \arctan\dfrac{b + a\tan x/2}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\[2ex] \dfrac{1}{\sqrt{b^2-a^2}} \ln\left|\dfrac{b - \sqrt{b^2-a^2} + a\tan x/2}{b + \sqrt{b^2-a^2} + a\tan x/2}\right| & \text{if } b^2 > a^2. \end{cases}$$

20. $$\int \frac{dx}{(a+b\sin x)^2} = \frac{b\cos x}{(a^2-b^2)(a+b\sin x)} + \frac{a}{a^2-b^2}\int \frac{dx}{a+b\sin x}.$$

21. $$\int \frac{dx}{a^2+b^2\sin^2 x} = \frac{1}{a\sqrt{a^2+b^2}} \arctan\frac{\sqrt{a^2+b^2}\,\tan x}{a}.$$

22. $$\int \frac{dx}{a^2-b^2\sin^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \arctan\dfrac{\sqrt{a^2-b^2}\,\tan x}{a} & \text{if } a^2 > b^2, \\[2ex] \dfrac{1}{2a\sqrt{b^2-a^2}} \ln\left|\dfrac{\sqrt{b^2-a^2}\,\tan x + a}{\sqrt{b^2-a^2}\,\tan x - a}\right| & \text{if } b^2 > a^2. \end{cases}$$

23. $$\int \frac{\sin x\,dx}{\sqrt{1+k^2\sin^2 x}} = -\frac{1}{k}\arcsin\frac{k\cos x}{\sqrt{1+k^2}}.$$

24. $$\int \frac{\sin x\,dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{k}\ln\left|k\cos x + \sqrt{1-k^2\sin^2 x}\,\right|.$$

25. $$\int \sin x\sqrt{1+k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1+k^2\sin^2 x} - \frac{1+k^2}{2k}\arcsin\frac{k\cos x}{\sqrt{1+k^2}}.$$

26. $$\int \sin x\sqrt{1-k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1-k^2\sin^2 x}$$
$$-\frac{1-k^2}{2k}\ln\left|k\cos x + \sqrt{1-k^2\sin^2 x}\,\right|.$$

27. $\displaystyle\int e^{ax}\sin bx\,dx = e^{ax}\Big(\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\Big).$

28. $\displaystyle\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^2+4}\Big(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\Big).$

29. $\displaystyle\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx.$

► Integrals involving $\sin x$ and $\cos x$.

1. $\displaystyle\int \sin ax\cos bx\,dx = -\frac{\cos[(a+b)x]}{2(a+b)} - \frac{\cos\big[(a-b)x\big]}{2(a-b)}, \qquad a\neq\pm b.$

2. $\displaystyle\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc}\arctan\Big(\frac{c}{b}\tan ax\Big).$

3. $\displaystyle\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\ln\Big|\frac{c\tan ax + b}{c\tan ax - b}\Big|.$

4. $\displaystyle\int \frac{dx}{\cos^{2n}x\sin^{2m}x} = \sum_{k=0}^{n+m-1} C_{n+m-1}^{k}\frac{\tan^{2k-2m+1}x}{2k-2m+1}, \qquad n,m = 1,\ 2,\ \ldots$

5. $\displaystyle\int \frac{dx}{\cos^{2n+1}x\sin^{2m+1}x} = C_{n+m}^{m}\ln|\tan x| + \sum_{k=0}^{n+m} C_{n+m}^{k}\frac{\tan^{2k-2m}x}{2k-2m}, \quad n,m=1,\ 2,\ \ldots$

► Reduction formulas.

★ *The parameters p and q below can assume any values, except those for which the denominators on the right-hand side vanish.*

1. $\displaystyle\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}x\cos^q x\,dx.$

2. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q-1}x}{p+q} + \frac{q-1}{p+q}\int \sin^p x\cos^{q-2}x\,dx.$

3. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p-1}x\cos^{q-1}x}{p+q}\Big(\sin^2 x - \frac{q-1}{p+q-2}\Big) + \frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int \sin^{p-2}x\cos^{q-2}x\,dx.$

4. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q+1}x}{p+1} + \frac{p+q+2}{p+1}\int \sin^{p+2}x\cos^q x\,dx.$

5. $\displaystyle\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p+1}x\cos^{q+1}x}{q+1} + \frac{p+q+2}{q+1}\int \sin^p x\cos^{q+2}x\,dx.$

6. $\displaystyle\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q+2}x\,dx.$

7. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q-1}x}{p+1} + \frac{q-1}{p+1}\int \sin^{p+2}x\cos^{q-2}x\,dx.$

► **Integrals involving $\tan x$ and $\cot x$.**

1. $\displaystyle\int \tan x\,dx = -\ln|\cos x|.$

2. $\displaystyle\int \tan^2 x\,dx = \tan x - x.$

3. $\displaystyle\int \tan^3 x\,dx = \tfrac{1}{2}\tan^2 x + \ln|\cos x|.$

4. $\displaystyle\int \tan^{2n} x\,dx = (-1)^n x - \sum_{k=1}^{n} \frac{(-1)^k(\tan x)^{2n-2k+1}}{2n-2k+1}, \quad n = 1,\,2,\,\ldots$

5. $\displaystyle\int \tan^{2n+1} x\,dx = (-1)^{n+1}\ln|\cos x| - \sum_{k=1}^{n} \frac{(-1)^k(\tan x)^{2n-2k+2}}{2n-2k+2}, \quad n = 1,\,2,\,\ldots$

6. $\displaystyle\int \frac{dx}{a+b\tan x} = \frac{1}{a^2+b^2}\bigl(ax + b\ln|a\cos x + b\sin x|\bigr).$

7. $\displaystyle\int \frac{\tan x\,dx}{\sqrt{a+b\tan^2 x}} = \frac{1}{\sqrt{b-a}}\arccos\biggl(\sqrt{1-\frac{a}{b}}\,\cos x\biggr), \quad b>a,\ b>0.$

8. $\displaystyle\int \cot x\,dx = \ln|\sin x|.$

9. $\displaystyle\int \cot^2 x\,dx = -\cot x - x.$

10. $\displaystyle\int \cot^3 x\,dx = -\tfrac{1}{2}\cot^2 x - \ln|\sin x|.$

11. $\displaystyle\int \cot^{2n} x\,dx = (-1)^n x + \sum_{k=1}^{n} \frac{(-1)^k(\cot x)^{2n-2k+1}}{2n-2k+1}, \quad n = 1,\,2,\,\ldots$

12. $\displaystyle\int \cot^{2n+1} x\,dx = (-1)^n\ln|\sin x| + \sum_{k=1}^{n} \frac{(-1)^k(\cot x)^{2n-2k+2}}{2n-2k+2}, \quad n = 1,\,2,\,\ldots$

13. $\displaystyle\int \frac{dx}{a+b\cot x} = \frac{1}{a^2+b^2}\bigl(ax - b\ln|a\sin x + b\cos x|\bigr).$

27.1.7 Integrals Involving Inverse Trigonometric Functions

1. $\displaystyle\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2}.$

2. $\displaystyle\int \Bigl(\arcsin\frac{x}{a}\Bigr)^2 dx = x\Bigl(\arcsin\frac{x}{a}\Bigr)^2 - 2x + 2\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}.$

3. $\displaystyle\int x\arcsin\frac{x}{a}\,dx = \frac{1}{4}(2x^2-a^2)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2-x^2}.$

4. $\displaystyle\int x^2\arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}.$

5. $\displaystyle\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}.$

6. $\displaystyle\int \Bigl(\arccos\frac{x}{a}\Bigr)^2 dx = x\Bigl(\arccos\frac{x}{a}\Bigr)^2 - 2x - 2\sqrt{a^2-x^2}\,\arccos\frac{x}{a}.$

7. $\displaystyle\int x \arccos\frac{x}{a}\,dx = \frac{1}{4}(2x^2 - a^2)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2 - x^2}.$

8. $\displaystyle\int x^2 \arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2}.$

9. $\displaystyle\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2 + x^2).$

10. $\displaystyle\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2 + a^2)\arctan\frac{x}{a} - \frac{ax}{2}.$

11. $\displaystyle\int x^2\arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2 + x^2).$

12. $\displaystyle\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} + \frac{a}{2}\ln(a^2 + x^2).$

13. $\displaystyle\int x\operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}(x^2 + a^2)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}.$

14. $\displaystyle\int x^2\operatorname{arccot}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arccot}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2 + x^2).$

27.2 Definite Integrals

★ *Throughout Section 27.2 it is assumed that n is a positive integer, unless otherwise specified.*

27.2.1 Integrals Involving Power-Law Functions

► **Integrals over a finite interval.**

1. $\displaystyle\int_0^1 \frac{x^n\,dx}{x+1} = (-1)^n\left[\ln 2 + \sum_{k=1}^{n}\frac{(-1)^k}{k}\right].$

2. $\displaystyle\int_0^1 \frac{dx}{x^2 + 2x\cos\beta + 1} = \frac{\beta}{2\sin\beta}.$

3. $\displaystyle\int_0^1 \frac{(x^a + x^{-a})\,dx}{x^2 + 2x\cos\beta + 1} = \frac{\pi\sin(a\beta)}{\sin(\pi a)\sin\beta}, \quad |a| < 1,\ \beta \neq (2n+1)\pi.$

4. $\displaystyle\int_0^1 x^a(1-x)^{1-a}\,dx = \frac{\pi a(1-a)}{2\sin(\pi a)}, \quad -1 < a < 1.$

5. $\displaystyle\int_0^1 \frac{dx}{x^a(1-x)^{1-a}} = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1.$

6. $\displaystyle\int_0^1 \frac{x^a\,dx}{(1-x)^a} = \frac{\pi a}{\sin(\pi a)}, \quad -1 < a < 1.$

7. $\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \equiv B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q > 0.$

8. $\displaystyle\int_0^1 x^{p-1}(1-x^q)^{-p/q}\,dx = \frac{\pi}{q\sin(\pi p/q)}, \quad q > p > 0.$

9. $\displaystyle\int_0^1 x^{p+q-1}(1-x^q)^{-p/q}\,dx = \frac{\pi p}{q^2\sin(\pi p/q)}, \quad q > p.$

10. $\displaystyle\int_0^1 x^{q/p-1}(1-x^q)^{-1/p}\,dx = \frac{\pi}{q\sin(\pi/p)}, \quad p > 1,\ q > 0.$

11. $\displaystyle\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \pi\cot(\pi p), \quad |p| < 1.$

12. $\displaystyle\int_0^1 \frac{x^{p-1}-x^{-p}}{1+x}\,dx = \frac{\pi}{\sin(\pi p)}, \quad |p| < 1.$

13. $\displaystyle\int_0^1 \frac{x^{p}-x^{-p}}{x-1}\,dx = \frac{1}{p} - \pi\cot(\pi p), \quad |p| < 1.$

14. $\displaystyle\int_0^1 \frac{x^{p}-x^{-p}}{1+x}\,dx = \frac{1}{p} - \frac{\pi}{\sin(\pi p)}, \quad |p| < 1.$

15. $\displaystyle\int_0^1 \frac{x^{1+p}-x^{1-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\Bigl(\frac{\pi p}{2}\Bigr) - \frac{1}{p}, \quad |p| < 1.$

16. $\displaystyle\int_0^1 \frac{x^{1+p}-x^{1-p}}{1+x^2}\,dx = \frac{1}{p} - \frac{\pi}{2\sin(\pi p/2)}, \quad |p| < 1.$

17. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1+a^2x)(1-x)}} = \frac{2}{a}\arctan a.$

18. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-a^2x)(1-x)}} = \frac{1}{a}\ln\frac{1+a}{1-a}.$

19. $\displaystyle\int_{-1}^1 \frac{dx}{(a-x)\sqrt{1-x^2}} = \frac{\pi}{\sqrt{a^2-1}}, \quad 1 < a.$

20. $\displaystyle\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}} = \frac{2\,(2n)!!}{(2n+1)!!}, \quad n = 1,\ 2,\ \dots$

21. $\displaystyle\int_0^1 \frac{x^{n-1/2}\,dx}{\sqrt{1-x}} = \frac{\pi\,(2n-1)!!}{(2n)!!}, \quad n = 1,\ 2,\ \dots$

22. $\displaystyle\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\,\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}, \quad n = 1,\ 2,\ \dots$

23. $\displaystyle\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)}, \quad n = 1,\ 2,\ \dots$

24. $\displaystyle\int_0^1 \frac{x^{\lambda-1}\,dx}{(1+ax)(1-x)^\lambda} = \frac{\pi}{(1+a)^\lambda\sin(\pi\lambda)}, \quad 0 < \lambda < 1, \quad a > -1.$

25. $\displaystyle\int_0^1 \frac{x^{\lambda-1/2}\,dx}{(1+ax)^\lambda(1-x)^\lambda} = 2\pi^{-1/2}\Gamma\bigl(\lambda+\tfrac{1}{2}\bigr)\Gamma\bigl(1-\lambda\bigr)\cos^{2\lambda}k\,\frac{\sin[(2\lambda-1)k]}{(2\lambda-1)\sin k},$
$k = \arctan\sqrt{a},\ -\tfrac{1}{2} < \lambda < 1, \quad a > 0.$

► **Integrals over an infinite interval.**

1. $\displaystyle\int_0^\infty \frac{dx}{ax^2+b} = \frac{\pi}{2\sqrt{ab}}.$

2. $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}.$

3. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x+1} = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1.$

4. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(1+ax)^2} = \frac{\pi(1-\lambda)}{a^\lambda \sin(\pi\lambda)}, \quad 0 < \lambda < 2.$

5. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(x+a)(x+b)} = \frac{\pi(a^{\lambda-1} - b^{\lambda-1})}{(b-a)\sin(\pi\lambda)}, \quad 0 < \lambda < 2.$

6. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}(x+c)\,dx}{(x+a)(x+b)} = \frac{\pi}{\sin(\pi\lambda)}\left(\frac{a-c}{a-b}a^{\lambda-1} + \frac{b-c}{b-a}b^{\lambda-1}\right), \quad 0 < \lambda < 1.$

7. $\displaystyle\int_0^\infty \frac{x^\lambda\,dx}{(x+1)^3} = \frac{\pi\lambda(1-\lambda)}{2\sin(\pi\lambda)}, \quad -1 < \lambda < 2.$

8. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi\big(b^{\lambda-2} - a^{\lambda-2}\big)}{2\big(a^2-b^2\big)\sin(\pi\lambda/2)}, \quad 0 < \lambda < 4.$

9. $\displaystyle\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x}\,dx = \pi[\cot(\pi p) - \cot(\pi q)], \quad p, q > 0.$

10. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(1+ax)^{n+1}} = (-1)^n \frac{\pi C^n_{\lambda-1}}{a^\lambda \sin(\pi\lambda)}, \quad C^n_{\lambda-1} = \frac{(\lambda-1)(\lambda-2)\ldots(\lambda-n)}{n!},$
$0 < \lambda < n+1.$

11. $\displaystyle\int_0^\infty \frac{x^m\,dx}{(a+bx)^{n+1/2}} = 2^{m+1} m!\,\frac{(2n-2m-3)!!}{(2n-1)!!}\frac{a^{m-n+1/2}}{b^{m+1}}, \quad a, b > 0, \quad m < b - \tfrac{1}{2},$
$n, m = 1,\ 2,\ \ldots$

12. $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{\pi}{2}\frac{(2n-3)!!}{(2n-2)!!}\frac{1}{a^{2n-1}}, \quad n = 1,\ 2,\ \ldots$

13. $\displaystyle\int_0^\infty \frac{(x+1)^{\lambda-1}}{(x+a)^{\lambda+1}}\,dx = \frac{1-a^{-\lambda}}{\lambda(a-1)}, \quad a > 0.$

14. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x^b+1} = \frac{\pi}{b\sin(\pi a/b)}, \quad 0 < a \le b.$

15. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{(x^b+1)^2} = \frac{\pi(a-b)}{b^2\sin[\pi(a-b)/b]}, \quad a < 2b.$

16. $\displaystyle\int_0^\infty \frac{x^{\lambda-1/2}\,dx}{(x+a)^\lambda(x+b)^\lambda} = \sqrt{\pi}\big(\sqrt{a}+\sqrt{b}\,\big)^{1-2\lambda}\frac{\Gamma(\lambda-1/2)}{\Gamma(\lambda)}, \quad \lambda > 0.$

17. $\displaystyle\int_0^\infty \frac{1-x^a}{1-x^b}\,x^{c-1}\,dx = \frac{\pi\sin A}{b\sin C\sin(A+C)}, \quad A = \frac{\pi a}{b}, \quad C = \frac{\pi c}{b}; \quad a+c < b,$
$c > 0.$

18. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{(1+x^2)^{1-b}} = \tfrac{1}{2}B\big(\tfrac{1}{2}a, 1-b-\tfrac{1}{2}a\big), \quad \tfrac{1}{2}a + b < 1, \quad a > 0.$

19. $\displaystyle\int_0^\infty \frac{x^{2m}\,dx}{(ax^2+b)^n} = \frac{\pi(2m-1)!!\,(2n-2m-3)!!}{2\,(2n-2)!!\,a^m b^{n-m-1}\sqrt{ab}}, \quad a, b > 0, \quad n > m+1.$

20. $\displaystyle\int_0^\infty \frac{x^{2m+1}\,dx}{(ax^2+b)^n} = \frac{m!\,(n-m-2)!}{2(n-1)!a^{m+1}b^{n-m-1}}, \quad ab > 0, \quad n > m+1 \ge 1.$

21. $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+ax^p)^\nu} = \frac{1}{pa^{\mu/p}} B\Big(\frac{\mu}{p}, \nu - \frac{\mu}{p}\Big), \quad p>0, \quad 0<\mu<p\nu.$

22. $\displaystyle\int_0^\infty \big(\sqrt{x^2+a^2}-x\big)^n\,dx = \frac{na^{n+1}}{n^2-1}, \quad n=2,\,3,\,\ldots$

23. $\displaystyle\int_0^\infty \frac{dx}{\big(x+\sqrt{x^2+a^2}\,\big)^n} = \frac{n}{a^{n-1}(n^2-1)}, \quad n=2,\,3,\,\ldots$

24. $\displaystyle\int_0^\infty x^m\big(\sqrt{x^2+a^2}-x\big)^n\,dx = \frac{m!\,na^{n+m+1}}{(n-m-1)(n-m+1)\ldots(n+m+1)},$
$n,m=1,\,2,\,\ldots,\quad 0\le m\le n-2.$

25. $\displaystyle\int_0^\infty \frac{x^m\,dx}{\big(x+\sqrt{x^2+a^2}\,\big)^n} = \frac{m!\,n}{(n-m-1)(n-m+1)\ldots(n+m+1)a^{n-m-1}},$
$n=2,\,3,\,\ldots$

27.2.2 Integrals Involving Exponential Functions

1. $\displaystyle\int_0^\infty e^{-ax}\,dx = \frac{1}{a}, \quad a>0.$

2. $\displaystyle\int_0^1 x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} - e^{-a}\sum_{k=0}^{n}\frac{n!}{k!}\frac{1}{a^{n-k+1}}, \quad a>0,\ \ n=1,\,2,\,\ldots$

3. $\displaystyle\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}, \quad a>0,\ \ n=1,\,2,\,\ldots$

4. $\displaystyle\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}, \quad a>0.$

5. $\displaystyle\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \frac{\Gamma(\nu)}{\mu^\nu}, \quad \mu,\nu>0.$

6. $\displaystyle\int_0^\infty \frac{dx}{1+e^{ax}} = \frac{\ln 2}{a}.$

7. $\displaystyle\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\Big(\frac{2\pi}{p}\Big)^{2n}\frac{B_{2n}}{4n}, \quad n=1,\,2,\,\ldots;$ the B_m are Bernoulli numbers (see Section 30.1.3).

8. $\displaystyle\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = (1-2^{1-2n})\Big(\frac{2\pi}{p}\Big)^{2n}\frac{|B_{2n}|}{4n}, \quad n=1,\,2,\,\ldots;$ the B_m are Bernoulli numbers.

9. $\displaystyle\int_{-\infty}^\infty \frac{e^{-px}\,dx}{1+e^{-qx}} = \frac{\pi}{q\sin(\pi p/q)}, \quad q>p>0 \text{ or } 0>p>q.$

10. $\displaystyle\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{bx}+e^{-bx}}\,dx = \frac{\pi}{2b\cos\Big(\dfrac{\pi a}{2b}\Big)}, \quad b>a.$

11. $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\frac{\pi p}{p+q}, \quad p,q>0.$

12. $\displaystyle\int_0^\infty \big(1-e^{-\beta x}\big)^\nu e^{-\mu x}\,dx = \frac{1}{\beta}B\Big(\frac{\mu}{\beta},\nu+1\Big).$

13. $\displaystyle\int_0^\infty \exp(-ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0.$

14. $\displaystyle\int_0^\infty x^{2n+1}\exp(-ax^2)\,dx = \frac{n!}{2a^{n+1}}, \quad a > 0, \quad n = 1,\,2,\,\ldots$

15. $\displaystyle\int_0^\infty x^{2n}\exp(-ax^2)\,dx = \frac{1\times 3\times\cdots\times(2n-1)\sqrt{\pi}}{2^{n+1}a^{n+1/2}}, \quad a > 0, \quad n = 1,\,2,\,\ldots$

16. $\displaystyle\int_{-\infty}^\infty \exp(-a^2x^2 \pm bx)\,dx = \frac{\sqrt{\pi}}{|a|}\exp\Bigl(\frac{b^2}{4a^2}\Bigr).$

17. $\displaystyle\int_0^\infty \exp\Bigl(-ax^2 - \frac{b}{x^2}\Bigr)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\bigl(-2\sqrt{ab}\,\bigr), \quad a, b > 0.$

18. $\displaystyle\int_0^\infty \exp(-x^a)\,dx = \frac{1}{a}\Gamma\Bigl(\frac{1}{a}\Bigr), \quad a > 0.$

27.2.3 Integrals Involving Hyperbolic Functions

1. $\displaystyle\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2|a|}.$

2. $$\int_0^\infty \frac{dx}{a + b\cosh x} = \begin{cases} \dfrac{2}{\sqrt{b^2-a^2}}\arctan\dfrac{\sqrt{b^2-a^2}}{a+b} & \text{if } |b| > |a|, \\ \dfrac{1}{\sqrt{a^2-b^2}}\ln\dfrac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2+b^2}} & \text{if } |b| < |a|. \end{cases}$$

3. $\displaystyle\int_0^\infty \frac{x^{2n}\,dx}{\cosh ax} = \Bigl(\frac{\pi}{2a}\Bigr)^{2n+1}|E_{2n}|, \quad a > 0,$ the E_m are Euler numbers (see Section 30.1.4).

4. $\displaystyle\int_0^\infty \frac{x^{2n}\,dx}{\cosh^2 ax} = \frac{\pi^{2n}(2^{2n}-2)}{|a|(2a)^{2n}}|B_{2n}|,$ the B_m are Bernoulli numbers (see Section 30.1.3).

5. $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b\cos\Bigl(\dfrac{\pi a}{2b}\Bigr)}, \quad b > |a|.$

6. $\displaystyle\int_0^\infty x^{2n}\frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2n}}{da^{2n}}\frac{1}{\cos\bigl(\frac{1}{2}\pi a/b\bigr)}, \quad b > |a|, \quad n = 1,\,2,\,\ldots$

7. $\displaystyle\int_0^\infty \frac{\cosh ax\cosh bx}{\cosh(cx)}\,dx = \frac{\pi}{c}\frac{\cos\Bigl(\dfrac{\pi a}{2c}\Bigr)\cos\Bigl(\dfrac{\pi b}{2c}\Bigr)}{\cos\Bigl(\dfrac{\pi a}{c}\Bigr) + \cos\Bigl(\dfrac{\pi b}{c}\Bigr)}, \quad c > |a| + |b|.$

8. $\displaystyle\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{2a^2}, \quad a > 0.$

9. $\displaystyle\int_0^\infty \frac{dx}{a + b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}, \quad ab \neq 0.$

10. $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\Bigl(\frac{\pi a}{2b}\Bigr), \quad b > |a|.$

11. $\displaystyle\int_0^\infty x^{2n}\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2n}}{dx^{2n}}\tan\Bigl(\frac{\pi a}{2b}\Bigr), \quad b > |a|, \quad n = 1,\, 2,\, \ldots$

12. $\displaystyle\int_0^\infty \frac{x^{2n}}{\sinh^2 ax}\,dx = \frac{\pi^{2n}}{a^{2n+1}}|B_{2n}|, \quad a > 0;$ the B_m are Bernoulli numbers.

27.2.4 Integrals Involving Logarithmic Functions

1. $\displaystyle\int_0^1 x^{a-1}\ln^n x\,dx = (-1)^n n!\,a^{-n-1}, \quad a > 0, \quad n = 1,\, 2,\, \ldots$

2. $\displaystyle\int_0^1 \frac{\ln x}{x+1}\,dx = -\frac{\pi^2}{12}.$

3. $\displaystyle\int_0^1 \frac{x^n\ln x}{x+1}\,dx = (-1)^{n+1}\left[\frac{\pi^2}{12} + \sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right], \quad n = 1,\, 2,\, \ldots$

4. $\displaystyle\int_0^1 \frac{x^{\mu-1}\ln x}{x+a}\,dx = \frac{\pi a^{\mu-1}}{\sin(\pi\mu)}\bigl[\ln a - \pi\cot(\pi\mu)\bigr], \quad 0 < \mu < 1.$

5. $\displaystyle\int_0^1 |\ln x|^\mu\,dx = \Gamma(\mu+1), \quad \mu > -1.$

6. $\displaystyle\int_0^\infty x^{\mu-1}\ln(1+ax)\,dx = \frac{\pi}{\mu a^\mu\sin(\pi\mu)}, \quad -1 < \mu < 0.$

7. $\displaystyle\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}, \quad n = 1,\, 2,\, \ldots$

8. $\displaystyle\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{1}{2n+1}\left[\ln 4 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right], \quad n = 0,\, 1,\, \ldots$

9. $\displaystyle\int_0^1 x^{n-1/2}\ln(1+x)\,dx = \frac{2\ln 2}{2n+1} + \frac{4(-1)^n}{2n+1}\left[\pi - \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right], \quad n = 1,\, 2,\, \ldots$

10. $\displaystyle\int_0^\infty \ln\frac{a^2+x^2}{b^2+x^2}\,dx = \pi(a-b), \quad a, b > 0.$

11. $\displaystyle\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\,dx = -\frac{\pi^2\cos(\pi p/q)}{q^2\sin^2(\pi p/q)}, \quad 0 < p < q.$

12. $\displaystyle\int_0^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}(\mathcal{C} + \ln\mu), \quad \mu > 0, \quad \mathcal{C} = 0.5772\ldots$

27.2.5 Integrals Involving Trigonometric Functions

► **Integrals over a finite interval.**

1. $\displaystyle\int_0^{\pi/2} \cos^{2n} x\,dx = \frac{\pi}{2}\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}, \quad n = 1,\, 2,\, \ldots$

2. $\displaystyle\int_0^{\pi/2} \cos^{2n+1} x\,dx = \frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)}, \quad n = 1,\, 2,\, \ldots$

3. $$\int_0^{\pi/2} x\cos^n x\,dx = -\sum_{k=0}^{m-1}\frac{(n-2k+1)(n-2k+3)\dots(n-1)}{(n-2k)(n-2k+2)\dots n}\frac{1}{n-2k} + \begin{cases}\dfrac{\pi}{2}\dfrac{(2m-2)!!}{(2m-1)!!} & \text{if } n=2m-1,\\ \dfrac{\pi^2}{8}\dfrac{(2m-1)!!}{(2m)!!} & \text{if } n=2m,\end{cases}\qquad m=1,\,2,\,\dots$$

4. $$\int_0^{\pi}\frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n}\frac{(2n-2k-1)!!\,(2k-1)!!}{(n-k)!\,k!}\left(\frac{a+b}{a-b}\right)^k,$$ $a>|b|$.

5. $$\int_0^{\pi/2}\sin^{2n}x\,dx = \frac{\pi}{2}\frac{1\times 3\times\dots\times(2n-1)}{2\times 4\times\dots\times(2n)},\qquad n=1,\,2,\,\dots$$

6. $$\int_0^{\pi/2}\sin^{2n+1}x\,dx = \frac{2\times 4\times\dots\times(2n)}{1\times 3\times\dots\times(2n+1)},\qquad n=1,\,2,\,\dots$$

7. $$\int_0^{\pi} x\sin^{\mu}x\,dx = \frac{\pi^2}{2^{\mu+1}}\frac{\Gamma(\mu+1)}{\left[\Gamma\left(\mu+\frac{1}{2}\right)\right]^2},\qquad \mu>-1.$$

8. $$\int_0^{\pi/2}\frac{\sin x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2k}\ln\frac{1+k}{1-k}.$$

9. $$\int_0^{\pi/2}\sin^{2n+1}x\cos^{2m+1}x\,dx = \frac{n!\,m!}{2(n+m+1)!},\qquad n,m=1,\,2,\,\dots$$

10. $$\int_0^{\pi/2}\sin^{p-1}x\cos^{q-1}x\,dx = \tfrac{1}{2}B\left(\tfrac{1}{2}p,\,\tfrac{1}{2}q\right).$$

11. $$\int_0^{2\pi}(a\sin x+b\cos x)^{2n}\,dx = 2\pi\frac{(2n-1)!!}{(2n)!!}\left(a^2+b^2\right)^n,\qquad n=1,\,2,\,\dots$$

12. $$\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{a^2+1-2a\cos x}} = \begin{cases}2 & \text{if } 0\le a\le 1,\\ 2/a & \text{if } 1<a.\end{cases}$$

13. $$\int_0^{\pi/2}(\tan x)^{\pm\lambda}\,dx = \frac{\pi}{2\cos\left(\frac{1}{2}\pi\lambda\right)},\qquad |\lambda|<1.$$

14. $$\int_0^{a}\frac{\cos(xt)\,dt}{\sqrt{a^2-t^2}} = \frac{\pi}{2}J_0(ax),$$ $J_0(z)$ is the Bessel function (see Section 30.6).

15. $$\int_0^{a}\frac{t\sin(xt)\,dt}{\sqrt{a^2-t^2}} = \frac{\pi}{2}aJ_1(ax),$$ $J_1(z)$ is the Bessel function.

16. $$\int_0^{2\pi}\cos(a\cos x)\,dx = 2\pi J_0(a),$$ $J_0(z)$ is the Bessel function.

17. $$\int_0^{2\pi}\sin(a\cos x)\,dx = 0.$$

► **Integrals over an infinite interval.**

1. $$\int_0^{\infty}\frac{\cos ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}},\qquad a>0.$$

2. $$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\left|\frac{b}{a}\right|, \quad ab \neq 0.$$

3. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \tfrac{1}{2}\pi(b-a), \quad a, b \geq 0.$$

4. $$\int_0^\infty x^{\mu-1}\cos ax\,dx = a^{-\mu}\Gamma(\mu)\cos\left(\tfrac{1}{2}\pi\mu\right), \quad a > 0, \quad 0 < \mu < 1.$$

5. $$\int_0^\infty \frac{\cos ax}{b^2 + x^2}\,dx = \frac{\pi}{2b}e^{-ab}, \quad a, b > 0.$$

6. $$\int_0^\infty \frac{\cos ax}{b^4 + x^4}\,dx = \frac{\pi\sqrt{2}}{4b^3}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left[\cos\left(\frac{ab}{\sqrt{2}}\right) + \sin\left(\frac{ab}{\sqrt{2}}\right)\right], \quad a, b > 0.$$

7. $$\int_0^\infty \frac{\cos ax}{(b^2 + x^2)^2}\,dx = \frac{\pi}{4b^3}(1 + ab)e^{-ab}, \quad a, b > 0.$$

8. $$\int_0^\infty \frac{\cos ax\,dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi\left(be^{-ac} - ce^{-ab}\right)}{2bc\left(b^2 - c^2\right)}, \quad a, b, c > 0.$$

9. $$\int_0^\infty \cos\left(ax^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$$

10. $$\int_0^\infty \cos\left(ax^p\right)dx = \frac{\Gamma(1/p)}{pa^{1/p}}\cos\frac{\pi}{2p}, \quad a > 0, \quad p > 1.$$

11. $$\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a.$$

12. $$\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi}{2}|a|.$$

13. $$\int_0^\infty \frac{\sin ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}, \quad a > 0.$$

14. $$\int_0^\infty x^{\mu-1}\sin ax\,dx = a^{-\mu}\Gamma(\mu)\sin\left(\tfrac{1}{2}\pi\mu\right), \quad a > 0, \quad 0 < \mu < 1.$$

15. $$\int_0^\infty \sin\left(ax^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$$

16. $$\int_0^\infty \sin\left(ax^p\right)dx = \frac{\Gamma(1/p)}{pa^{1/p}}\sin\frac{\pi}{2p}, \quad a > 0, \quad p > 1.$$

17. $$\int_0^\infty \frac{\sin x\cos ax}{x}\,dx = \begin{cases} \frac{\pi}{2} & \text{if } |a| < 1, \\ \frac{\pi}{4} & \text{if } |a| = 1, \\ 0 & \text{if } 1 < |a|. \end{cases}$$

18. $$\int_0^\infty \frac{\tan ax}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a.$$

19. $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2 + b^2}, \quad a > 0.$$

20. $$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2 + b^2}, \quad a > 0.$$

21. $$\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2 + b^2)^2}, \quad a > 0.$$

22. $\displaystyle\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}, \quad a>0.$

23. $\displaystyle\int_0^\infty x^n e^{-ax}\sin bx\,dx = (-1)^n\frac{\partial^n}{\partial a^n}\left(\frac{b}{a^2+b^2}\right), \quad a>0,\ n=1,\,2,\,\ldots.$

24. $\displaystyle\int_0^\infty x^n e^{-ax}\cos bx\,dx = (-1)^n\frac{\partial^n}{\partial a^n}\left(\frac{a}{a^2+b^2}\right), \quad a>0,\ n=1,\,2,\,\ldots.$

25. $\displaystyle\int_0^\infty \exp\bigl(-ax^2\bigr)\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\exp\Bigl(-\frac{b^2}{4a}\Bigr), \quad a>0.$

26. $\displaystyle\int_0^\infty x\exp\bigl(-ax^2\bigr)\sin bx\,dx = \frac{\sqrt{\pi}\,b}{4a^{3/2}}\,\exp\Bigl(-\frac{b^2}{4a}\Bigr), \quad a>0.$

27. $\displaystyle\int_0^\infty \cos(ax^2)\cos bx\,dx = \sqrt{\frac{\pi}{8a}}\left[\cos\left(\frac{b^2}{4a}\right)+\sin\left(\frac{b^2}{4a}\right)\right], \quad a,b>0.$

28. $\displaystyle\int_0^\infty \cos(ax^2)\sin bx\,dx = \sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{b^2}{4a}\right)C\left(\frac{b^2}{4a}\right)-\sin\left(\frac{b^2}{4a}\right)S\left(\frac{b^2}{4a}\right)\right],$
$a,b>0$ and $C(z)$ and $S(z)$ are Fresnel integrals.

29. $\displaystyle\int_0^\infty \sin(ax^2)\cos bx\,dx = \sqrt{\frac{\pi}{8a}}\left[\cos\left(\frac{b^2}{4a}\right)-\sin\left(\frac{b^2}{4a}\right)\right], \quad a,b>0.$

30. $\displaystyle\int_0^\infty \sin(ax^2)\sin bx\,dx = \sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{b^2}{4a}\right)C\left(\frac{b^2}{4a}\right)+\sin\left(\frac{b^2}{4a}\right)S\left(\frac{b^2}{4a}\right)\right],$
$a,b>0$ and $C(z)$ and $S(z)$ are Fresnel integrals.

31. $\displaystyle\int_0^\infty \frac{1}{x^2}\sin(ax^2)\cos bx\,dx = \frac{b\pi}{2}\left[S\left(\frac{b^2}{4a}\right)-C\left(\frac{b^2}{4a}\right)+\sqrt{\pi a}\,\sin\left(\frac{b^2}{4a}+\frac{\pi}{4}\right)\right],$
$a,b>0$ and $C(z)$ and $S(z)$ are Fresnel integrals.

32. $\displaystyle\int_0^\infty (\cos ax+\sin ax)\cos(b^2x^2)\,dx = \frac{1}{b}\sqrt{\frac{\pi}{8}}\exp\left(-\frac{a^2}{2b}\right), \quad a,b>0.$

33. $\displaystyle\int_0^\infty \bigl(\cos ax+\sin ax\bigr)\sin(b^2x^2)\,dx = \frac{1}{b}\sqrt{\frac{\pi}{8}}\exp\Bigl(-\frac{a^2}{2b}\Bigr), \quad a,b>0.$

27.2.6 Integrals Involving Bessel Functions

► Integrals over an infinite interval.

1. $\displaystyle\int_0^\infty J_\nu(ax)\,dx = \frac{1}{a}, \quad a>0,\ \operatorname{Re}\nu>-1.$

2. $\displaystyle\int_0^\infty \cos(xu)J_0(tu)\,du = \begin{cases}\dfrac{1}{\sqrt{t^2-x^2}} & \text{if } x<t,\\ 0 & \text{if } x>t.\end{cases}$

3. $\displaystyle\int_0^\infty \sin(xu)J_0(tu)\,du = \begin{cases}0 & \text{if } x<t,\\ \dfrac{1}{\sqrt{x^2-t^2}} & \text{if } x>t.\end{cases}$

4. $\displaystyle\int_0^\infty \cos(xu)J_1(tu)\,du = \begin{cases}\dfrac{1}{t} & \text{if } x<t,\\ -\dfrac{t}{\sqrt{x^2-t^2}\,(x+\sqrt{x^2-t^2}\,)} & \text{if } x>t.\end{cases}$

5. $\displaystyle\int_0^\infty \frac{\sin(tu)J_0(au)}{u^2+b^2}\,du = \frac{\sinh(bt)}{b}K_0(ab), \quad b>0,\ 0<t<a,\ K_0(z)$ is the modified Bessel function (see Section 30.7).

6. $\displaystyle\int_0^\infty \frac{u\sin(tu)J_0(au)}{u^2+b^2}\,du = \frac{\pi}{2}e^{-bt}I_0(ab), \quad b>0,\ a<t<\infty,\ I_0(z)$ is the modified Bessel function.

7. $\displaystyle\int_0^\infty \frac{\sin(tu)J_1(au)}{u^2+b^2}\,du = \frac{\pi}{2b}e^{-bt}I_1(ab), \quad b>0,\ a<t<\infty,\ I_1(z)$ is the modified Bessel function.

8. $\displaystyle\int_0^\infty \frac{u\sin(tu)J_1(au)}{u^2+b^2}\,du = \sinh(bt)K_1(ab), \quad b>0,\ 0<t<a,\ K_1(z)$ is the modified Bessel function.

9. $\displaystyle\int_0^\infty \frac{J_1(au)}{\sqrt{u^2+b^2}}\,du = \frac{1-e^{-ab}}{ab}, \quad a>0,\ \operatorname{Re} b>0.$

► **Other integrals.**

1. $\displaystyle\int_0^1 uJ_0(xu)\,du = \frac{J_1(x)}{x}.$

2. $\displaystyle\int_0^a \frac{J_1(bx)\,dx}{\sqrt{a^2-x^2}} = \frac{1-\cos(ab)}{ab}, \quad a>0.$

3. $\displaystyle\int_0^t \frac{uJ_0(xu)\,du}{\sqrt{t^2-u^2}} = \frac{\sin(xt)}{x}.$

4. $\displaystyle\int_t^\infty \frac{J_1(xu)\,du}{\sqrt{u^2-t^2}} = \frac{\sin(xt)}{x}, \quad x>0,\ t>0.$

⊙ *References for Chapter* 27: H. B. Dwight (1961), I. S. Gradshteyn and I. M. Ryzhik (2000), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986, 1988), D. Zwillinger (2002), I. N. Bronshtein and K. A. Semendyayev (2004).

Chapter 28

Integral Transforms

28.1 Tables of Laplace Transforms

28.1.1 General Formulas

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$af_1(x)+bf_2(x)$	$a\widetilde{f}_1(p)+b\widetilde{f}_2(p)$
2	$f(x/a),\ \ a>0$	$a\widetilde{f}(ap)$
3	$\begin{cases} 0 & \text{if } 0<x<a, \\ f(x-a) & \text{if } x>a \end{cases}$	$e^{-ap}\widetilde{f}(p)$
4	$x^n f(x);\ \ n=1,\,2,\,\ldots$	$(-1)^n \dfrac{d^n}{dp^n}\widetilde{f}(p)$
5	$\dfrac{1}{x}f(x)$	$\int_p^\infty \widetilde{f}(q)\,dq$
6	$e^{ax}f(x)$	$\widetilde{f}(p-a)$
7	$\sinh(ax)f(x)$	$\frac{1}{2}\big[\widetilde{f}(p-a)-\widetilde{f}(p+a)\big]$
8	$\cosh(ax)f(x)$	$\frac{1}{2}\big[\widetilde{f}(p-a)+\widetilde{f}(p+a)\big]$
9	$\sin(\omega x)f(x)$	$-\frac{i}{2}\big[\widetilde{f}(p-i\omega)-\widetilde{f}(p+i\omega)\big],\ \ i^2=-1$
10	$\cos(\omega x)f(x)$	$\frac{1}{2}\big[\widetilde{f}(p-i\omega)+\widetilde{f}(p+i\omega)\big],\ \ i^2=-1$
11	$f(x^2)$	$\dfrac{1}{\sqrt{\pi}}\int_0^\infty \exp\Big(-\dfrac{p^2}{4t^2}\Big)\widetilde{f}(t^2)\,dt$
12	$x^{a-1}f\Big(\dfrac{1}{x}\Big),\ \ a>-1$	$\int_0^\infty (t/p)^{a/2}J_a\big(2\sqrt{pt}\,\big)\widetilde{f}(t)\,dt$
13	$f(a\sinh x),\ \ a>0$	$\int_0^\infty J_p(at)\widetilde{f}(t)\,dt$
14	$f(x+a)=f(x)$ (periodic function)	$\dfrac{1}{1-e^{ap}}\int_0^a f(x)e^{-px}\,dx$
15	$f(x+a)=-f(x)$ (antiperiodic function)	$\dfrac{1}{1+e^{-ap}}\int_0^a f(x)e^{-px}\,dx$

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
16	$f'_x(x)$	$p\widetilde{f}(p)-f(+0)$
17	$f_x^{(n)}(x)$	$p^n\widetilde{f}(p)-\sum_{k=1}^{n}p^{n-k}f_x^{(k-1)}(+0)$
18	$x^m f_x^{(n)}(x),\ \ m\ge n$	$\left(-\frac{d}{dp}\right)^m\left[p^n\widetilde{f}(p)\right]$
19	$\frac{d^n}{dx^n}\left[x^m f(x)\right],\ \ m\ge n$	$(-1)^m p^n\frac{d^m}{dp^m}\widetilde{f}(p)$
20	$\int_0^x f(t)\,dt$	$\frac{\widetilde{f}(p)}{p}$
21	$\int_0^x (x-t)f(t)\,dt$	$\frac{1}{p^2}\widetilde{f}(p)$
22	$\int_0^x (x-t)^\nu f(t)\,dt,\quad \nu>-1$	$\Gamma(\nu+1)p^{-\nu-1}\widetilde{f}(p)$
23	$\int_0^x e^{-a(x-t)}f(t)\,dt$	$\frac{1}{p+a}\widetilde{f}(p)$
24	$\int_0^x \sinh\big[a(x-t)\big]f(t)\,dt$	$\frac{a\widetilde{f}(p)}{p^2-a^2}$
25	$\int_0^x \sin\big[a(x-t)\big]f(t)\,dt$	$\frac{a\widetilde{f}(p)}{p^2+a^2}$
26	$\int_0^x f_1(t)f_2(x-t)\,dt$	$\widetilde{f}_1(p)\widetilde{f}_2(p)$
27	$\int_0^x \frac{1}{t}f(t)\,dt$	$\frac{1}{p}\int_p^\infty \widetilde{f}(q)\,dq$
28	$\int_x^\infty \frac{1}{t}f(t)\,dt$	$\frac{1}{p}\int_0^p \widetilde{f}(q)\,dq$
29	$\int_0^\infty \frac{1}{\sqrt{t}}\sin\big(2\sqrt{xt}\,\big)f(t)\,dt$	$\frac{\sqrt{\pi}}{p\sqrt{p}}\widetilde{f}\Big(\frac{1}{p}\Big)$
30	$\frac{1}{\sqrt{x}}\int_0^\infty \cos\big(2\sqrt{xt}\,\big)f(t)\,dt$	$\frac{\sqrt{\pi}}{\sqrt{p}}\widetilde{f}\Big(\frac{1}{p}\Big)$
31	$\int_0^\infty \frac{1}{\sqrt{\pi x}}\exp\Big(-\frac{t^2}{4x}\Big)f(t)\,dt$	$\frac{1}{\sqrt{p}}\widetilde{f}\big(\sqrt{p}\big)$
32	$\int_0^\infty \frac{t}{2\sqrt{\pi x^3}}\exp\Big(-\frac{t^2}{4x}\Big)f(t)\,dt$	$\widetilde{f}\big(\sqrt{p}\big)$
33	$f(x)-a\int_0^x f\big(\sqrt{x^2-t^2}\,\big)J_1(at)\,dt$	$\widetilde{f}\big(\sqrt{p^2+a^2}\,\big)$
34	$f(x)+a\int_0^x f\big(\sqrt{x^2-t^2}\,\big)I_1(at)\,dt$	$\widetilde{f}\big(\sqrt{p^2-a^2}\,\big)$

28.1.2 Expressions with Power-Law Functions

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	1	$\dfrac{1}{p}$
2	$\begin{cases}0 & \text{if } 0<x<a,\\ 1 & \text{if } a<x<b,\\ 0 & \text{if } b<x\end{cases}$	$\dfrac{1}{p}\bigl(e^{-ap}-e^{-bp}\bigr)$
3	x	$\dfrac{1}{p^2}$
4	$\dfrac{1}{x+a}$	$-e^{ap}\,\mathrm{Ei}(-ap)$
5	$x^n,\quad n=1,\,2,\,\dots$	$\dfrac{n!}{p^{n+1}}$
6	$x^{n-1/2},\quad n=1,\,2,\,\dots$	$\dfrac{1\cdot 3\dots(2n-1)\sqrt{\pi}}{2^n p^{n+1/2}}$
7	$\dfrac{1}{\sqrt{x+a}}$	$\sqrt{\dfrac{\pi}{p}}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
8	$\dfrac{\sqrt{x}}{x+a}$	$\sqrt{\dfrac{\pi}{p}}-\pi\sqrt{a}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
9	$(x+a)^{-3/2}$	$2a^{-1/2}-2(\pi p)^{1/2}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
10	$x^{1/2}(x+a)^{-1}$	$(\pi/p)^{1/2}-\pi a^{1/2}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
11	$x^{-1/2}(x+a)^{-1}$	$\pi a^{-1/2}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
12	$x^{\nu},\quad \nu>-1$	$\Gamma(\nu+1)p^{-\nu-1}$
13	$(x+a)^{\nu},\quad \nu>-1$	$p^{-\nu-1}e^{-ap}\Gamma(\nu+1,ap)$
14	$x^{\nu}(x+a)^{-1},\quad \nu>-1$	$ke^{ap}\Gamma(-\nu,ap),\quad k=a^{\nu}\Gamma(\nu+1)$
15	$(x^2+2ax)^{-1/2}(x+a)$	$ae^{ap}K_1(ap)$

28.1.3 Expressions with Exponential Functions

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	e^{-ax}	$(p+a)^{-1}$
2	xe^{-ax}	$(p+a)^{-2}$
3	$x^{\nu-1}e^{-ax},\quad \nu>0$	$\Gamma(\nu)(p+a)^{-\nu}$
4	$\dfrac{1}{x}\bigl(e^{-ax}-e^{-bx}\bigr)$	$\ln(p+b)-\ln(p+a)$
5	$\dfrac{1}{x^2}\bigl(1-e^{-ax}\bigr)^2$	$(p+2a)\ln(p+2a)+p\ln p-2(p+a)\ln(p+a)$
6	$\exp\bigl(-ax^2\bigr),\quad a>0$	$(\pi b)^{1/2}\exp\bigl(bp^2\bigr)\,\mathrm{erfc}\bigl(p\sqrt{b}\,\bigr),\quad a=\dfrac{1}{4b}$
7	$x\exp\bigl(-ax^2\bigr)$	$2b-2\pi^{1/2}b^{3/2}p\,\mathrm{erfc}\bigl(p\sqrt{b}\,\bigr),\quad a=\dfrac{1}{4b}$
8	$\exp(-a/x),\quad a\ge 0$	$2\sqrt{a/p}\,K_1\bigl(2\sqrt{ap}\,\bigr)$
9	$\sqrt{x}\exp(-a/x),\quad a\ge 0$	$\tfrac{1}{2}\sqrt{\pi/p^3}\bigl(1+2\sqrt{ap}\,\bigr)\exp\bigl(-2\sqrt{ap}\,\bigr)$
10	$\dfrac{1}{\sqrt{x}}\exp(-a/x),\quad a\ge 0$	$\sqrt{\pi/p}\exp\bigl(-2\sqrt{ap}\,\bigr)$

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
11	$\dfrac{1}{x\sqrt{x}} \exp(-a/x), \quad a > 0$	$\sqrt{\pi/a}\exp\bigl(-2\sqrt{ap}\bigr)$
12	$x^{\nu-1} \exp(-a/x), \quad a > 0$	$2(a/p)^{\nu/2} K_\nu\bigl(2\sqrt{ap}\bigr)$
13	$\exp\bigl(-2\sqrt{ax}\bigr)$	$p^{-1} - (\pi a)^{1/2} p^{-3/2} e^{a/p} \operatorname{erfc}\bigl(\sqrt{a/p}\bigr)$
14	$\dfrac{1}{\sqrt{x}} \exp\bigl(-2\sqrt{ax}\bigr)$	$(\pi/p)^{1/2} e^{a/p} \operatorname{erfc}\bigl(\sqrt{a/p}\bigr)$

28.1.4 Expressions with Hyperbolic Functions

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$\sinh(ax)$	$\dfrac{a}{p^2 - a^2}$
2	$\sinh^2(ax)$	$\dfrac{2a^2}{p^3 - 4a^2 p}$
3	$\dfrac{1}{x}\sinh(ax)$	$\dfrac{1}{2}\ln\dfrac{p+a}{p-a}$
4	$x^{\nu-1}\sinh(ax), \quad \nu > -1$	$\frac{1}{2}\Gamma(\nu)\bigl[(p-a)^{-\nu} - (p+a)^{-\nu}\bigr]$
5	$\sinh\bigl(2\sqrt{ax}\bigr)$	$\dfrac{\sqrt{\pi a}}{p\sqrt{p}} e^{a/p}$
6	$\sqrt{x}\sinh\bigl(2\sqrt{ax}\bigr)$	$\pi^{1/2} p^{-5/2}\bigl(\frac{1}{2}p + a\bigr) e^{a/p} \operatorname{erf}\bigl(\sqrt{a/p}\bigr) - a^{1/2} p^{-2}$
7	$\dfrac{1}{\sqrt{x}}\sinh\bigl(2\sqrt{ax}\bigr)$	$\pi^{1/2} p^{-1/2} e^{a/p} \operatorname{erf}\bigl(\sqrt{a/p}\bigr)$
8	$\dfrac{1}{\sqrt{x}}\sinh^2\bigl(\sqrt{ax}\bigr)$	$\frac{1}{2}\pi^{1/2} p^{-1/2}\bigl(e^{a/p} - 1\bigr)$
9	$\cosh(ax)$	$\dfrac{p}{p^2 - a^2}$
10	$\cosh^2(ax)$	$\dfrac{p^2 - 2a^2}{p^3 - 4a^2 p}$
11	$x^{\nu-1}\cosh(ax), \quad \nu > 0$	$\frac{1}{2}\Gamma(\nu)\bigl[(p-a)^{-\nu} + (p+a)^{-\nu}\bigr]$
12	$\cosh\bigl(2\sqrt{ax}\bigr)$	$\dfrac{1}{p} + \dfrac{\sqrt{\pi a}}{p\sqrt{p}} e^{a/p} \operatorname{erf}\bigl(\sqrt{a/p}\bigr)$
13	$\sqrt{x}\cosh\bigl(2\sqrt{ax}\bigr)$	$\pi^{1/2} p^{-5/2}\bigl(\frac{1}{2}p + a\bigr) e^{a/p}$
14	$\dfrac{1}{\sqrt{x}}\cosh\bigl(2\sqrt{ax}\bigr)$	$\pi^{1/2} p^{-1/2} e^{a/p}$
15	$\dfrac{1}{\sqrt{x}}\cosh^2\bigl(\sqrt{ax}\bigr)$	$\frac{1}{2}\pi^{1/2} p^{-1/2}\bigl(e^{a/p} + 1\bigr)$

28.1.5 Expressions with Logarithmic Functions

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$\ln x$	$-\frac{1}{p}(\ln p+\mathcal{C})$, $\mathcal{C}=0.5772\ldots$ is the Euler constant
2	$\ln(1+ax)$	$-\frac{1}{p}e^{p/a}\,\mathrm{Ei}(-p/a)$
3	$\ln(x+a)$	$\frac{1}{p}\bigl[\ln a-e^{ap}\,\mathrm{Ei}(-ap)\bigr]$
4	$x^n\ln x,\quad n=1,\,2,\,\ldots$	$\frac{n!}{p^{n+1}}\bigl(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln p-\mathcal{C}\bigr)$, $\mathcal{C}=0.5772\ldots$ is the Euler constant
5	$\frac{1}{\sqrt{x}}\ln x$	$-\sqrt{\pi/p}\,\bigl[\ln(4p)+\mathcal{C}\bigr]$
6	$x^{n-1/2}\ln x,\quad n=1,\,2,\,\ldots$	$\frac{k_n}{p^{n+1/2}}\bigl[2+\frac{2}{3}+\frac{2}{5}+\cdots+\frac{2}{2n-1}-\ln(4p)-\mathcal{C}\bigr]$, $k_n=1\cdot3\cdot5\ldots(2n-1)\frac{\sqrt{\pi}}{2^n},\quad \mathcal{C}=0.5772\ldots$
7	$x^{\nu-1}\ln x,\quad \nu>0$	$\Gamma(\nu)p^{-\nu}\bigl[\psi(\nu)-\ln p\bigr]$, $\psi(\nu)$ is the logarithmic derivative of the gamma function
8	$(\ln x)^2$	$\frac{1}{p}\bigl[(\ln x+\mathcal{C})^2+\frac{1}{6}\pi^2\bigr],\quad \mathcal{C}=0.5772\ldots$
9	$e^{-ax}\ln x$	$-\frac{\ln(p+a)+\mathcal{C}}{p+a},\quad \mathcal{C}=0.5772\ldots$

28.1.6 Expressions with Trigonometric Functions

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$\sin(ax)$	$\frac{a}{p^2+a^2}$
2	$\lvert\sin(ax)\rvert,\quad a>0$	$\frac{a}{p^2+a^2}\coth\Bigl(\frac{\pi p}{2a}\Bigr)$
3	$\sin^{2n}(ax),\quad n=1,\,2,\,\ldots$	$\frac{a^{2n}(2n)!}{p\bigl[p^2+(2a)^2\bigr]\bigl[p^2+(4a)^2\bigr]\ldots\bigl[p^2+(2na)^2\bigr]}$
4	$\sin^{2n+1}(ax),\quad n=1,\,2,\,\ldots$	$\frac{a^{2n+1}(2n+1)!}{\bigl[p^2+a^2\bigr]\bigl[p^2+3^2a^2\bigr]\ldots\bigl[p^2+(2n+1)^2a^2\bigr]}$
5	$x^n\sin(ax),\quad n=1,\,2,\,\ldots$	$\frac{n!\,p^{n+1}}{\bigl(p^2+a^2\bigr)^{n+1}}\sum_{0\le 2k\le n}(-1)^k C_{n+1}^{2k+1}\Bigl(\frac{a}{p}\Bigr)^{2k+1}$
6	$\frac{1}{x}\sin(ax)$	$\arctan\Bigl(\frac{a}{p}\Bigr)$
7	$\frac{1}{x}\sin^2(ax)$	$\frac{1}{4}\ln\bigl(1+4a^2p^{-2}\bigr)$

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
8	$\dfrac{1}{x^2}\sin^2(ax)$	$a\arctan(2a/p)-\frac{1}{4}p\ln\bigl(1+4a^2p^{-2}\bigr)$
9	$\sin\bigl(2\sqrt{ax}\bigr)$	$\dfrac{\sqrt{\pi a}}{p\sqrt{p}}e^{-a/p}$
10	$\dfrac{1}{x}\sin\bigl(2\sqrt{ax}\bigr)$	$\pi\,\mathrm{erf}\bigl(\sqrt{a/p}\bigr)$
11	$\cos(ax)$	$\dfrac{p}{p^2+a^2}$
12	$\cos^2(ax)$	$\dfrac{p^2+2a^2}{p(p^2+4a^2)}$
13	$x^n\cos(ax),\quad n=1,\,2,\,\ldots$	$\dfrac{n!\,p^{n+1}}{(p^2+a^2)^{n+1}}\sum\limits_{0\le 2k\le n+1}(-1)^kC_{n+1}^{2k}\Bigl(\dfrac{a}{p}\Bigr)^{2k}$
14	$\dfrac{1}{x}\bigl[1-\cos(ax)\bigr]$	$\frac{1}{2}\ln\bigl(1+a^2p^{-2}\bigr)$
15	$\dfrac{1}{x}\bigl[\cos(ax)-\cos(bx)\bigr]$	$\dfrac{1}{2}\ln\dfrac{p^2+b^2}{p^2+a^2}$
16	$\sqrt{x}\cos\bigl(2\sqrt{ax}\bigr)$	$\frac{1}{2}\pi^{1/2}p^{-5/2}(p-2a)e^{-a/p}$
17	$\dfrac{1}{\sqrt{x}}\cos\bigl(2\sqrt{ax}\bigr)$	$\sqrt{\pi/p}\,e^{-a/p}$
18	$\sin(ax)\sin(bx)$	$\dfrac{2abp}{\bigl[p^2+(a+b)^2\bigr]\bigl[p^2+(a-b)^2\bigr]}$
19	$\cos(ax)\sin(bx)$	$\dfrac{b(p^2-a^2+b^2)}{\bigl[p^2+(a+b)^2\bigr]\bigl[p^2+(a-b)^2\bigr]}$
20	$\cos(ax)\cos(bx)$	$\dfrac{p(p^2+a^2+b^2)}{\bigl[p^2+(a+b)^2\bigr]\bigl[p^2+(a-b)^2\bigr]}$
21	$\dfrac{ax\cos(ax)-\sin(ax)}{x^2}$	$p\arctan\dfrac{a}{x}-a$
22	$e^{bx}\sin(ax)$	$\dfrac{a}{(p-b)^2+a^2}$
23	$e^{bx}\cos(ax)$	$\dfrac{p-b}{(p-b)^2+a^2}$
24	$\sin(ax)\sinh(ax)$	$\dfrac{2a^2p}{p^4+4a^4}$
25	$\sin(ax)\cosh(ax)$	$\dfrac{a(p^2+2a^2)}{p^4+4a^4}$
26	$\cos(ax)\sinh(ax)$	$\dfrac{a(p^2-2a^2)}{p^4+4a^4}$
27	$\cos(ax)\cosh(ax)$	$\dfrac{p^3}{p^4+4a^4}$

28.1.7 Expressions with Special Functions

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$\operatorname{erf}(ax)$	$\frac{1}{p}\exp\bigl(b^2p^2\bigr)\operatorname{erfc}(bp), \quad b=\frac{1}{2a}$
2	$\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)$	$\frac{\sqrt{a}}{p\sqrt{p+a}}$
3	$e^{ax}\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)$	$\frac{\sqrt{a}}{\sqrt{p}\,(p-a)}$
4	$\operatorname{erf}\bigl(\frac{1}{2}\sqrt{a/x}\,\bigr)$	$\frac{1}{p}\bigl[1-\exp\bigl(-\sqrt{ap}\,\bigr)\bigr]$
5	$\operatorname{erfc}\bigl(\sqrt{ax}\,\bigr)$	$\frac{\sqrt{p+a}-\sqrt{a}}{p\sqrt{p+a}}$
6	$e^{ax}\operatorname{erfc}\bigl(\sqrt{ax}\,\bigr)$	$\frac{1}{p+\sqrt{ap}}$
7	$\operatorname{erfc}\bigl(\frac{1}{2}\sqrt{a/x}\,\bigr)$	$\frac{1}{p}\exp\bigl(-\sqrt{ap}\,\bigr)$
8	$\operatorname{Ci}(x)$	$\frac{1}{2p}\ln(p^2+1)$
9	$\operatorname{Si}(x)$	$\frac{1}{p}\operatorname{arccot}p$
10	$\operatorname{Ei}(-x)$	$-\frac{1}{p}\ln(p+1)$
11	$J_0(ax)$	$\frac{1}{\sqrt{p^2+a^2}}$
12	$J_\nu(ax), \quad \nu>-1$	$\frac{a^\nu}{\sqrt{p^2+a^2}\,\bigl(p+\sqrt{p^2+a^2}\,\bigr)^\nu}$
13	$x^nJ_n(ax), \quad n=1,\,2,\,\ldots$	$1\cdot3\cdot5\ldots(2n-1)a^n\bigl(p^2+a^2\bigr)^{-n-1/2}$
14	$x^\nu J_\nu(ax), \quad \nu>-\frac{1}{2}$	$2^\nu\pi^{-1/2}\Gamma\bigl(\nu+\frac{1}{2}\bigr)a^\nu\bigl(p^2+a^2\bigr)^{-\nu-1/2}$
15	$x^{\nu+1}J_\nu(ax), \quad \nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma\bigl(\nu+\frac{3}{2}\bigr)a^\nu p\bigl(p^2+a^2\bigr)^{-\nu-3/2}$
16	$J_0\bigl(2\sqrt{ax}\,\bigr)$	$\frac{1}{p}e^{-a/p}$
17	$\sqrt{x}J_1\bigl(2\sqrt{ax}\,\bigr)$	$\frac{\sqrt{a}}{p^2}e^{-a/p}$
18	$x^{\nu/2}J_\nu\bigl(2\sqrt{ax}\,\bigr), \quad \nu>-1$	$a^{\nu/2}p^{-\nu-1}e^{-a/p}$
19	$I_0(ax)$	$\frac{1}{\sqrt{p^2-a^2}}$
20	$I_\nu(ax), \quad \nu>-1$	$\frac{a^\nu}{\sqrt{p^2-a^2}\,\bigl(p+\sqrt{p^2-a^2}\,\bigr)^\nu}$
21	$x^\nu I_\nu(ax), \quad \nu>-\frac{1}{2}$	$2^\nu\pi^{-1/2}\Gamma\bigl(\nu+\frac{1}{2}\bigr)a^\nu\bigl(p^2-a^2\bigr)^{-\nu-1/2}$
22	$x^{\nu+1}I_\nu(ax), \quad \nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma\bigl(\nu+\frac{3}{2}\bigr)a^\nu p\bigl(p^2-a^2\bigr)^{-\nu-3/2}$

No.	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
23	$I_0\big(2\sqrt{ax}\,\big)$	$\dfrac{1}{p}e^{a/p}$
24	$\dfrac{1}{\sqrt{x}}I_1\big(2\sqrt{ax}\,\big)$	$\dfrac{1}{\sqrt{a}}\big(e^{a/p}-1\big)$
25	$x^{\nu/2}I_\nu\big(2\sqrt{ax}\,\big),\quad \nu>-1$	$a^{\nu/2}p^{-\nu-1}e^{a/p}$
26	$Y_0(ax)$	$-\dfrac{2}{\pi}\dfrac{\operatorname{arcsinh}(p/a)}{\sqrt{p^2+a^2}}$
27	$K_0(ax)$	$\dfrac{\ln\big(p+\sqrt{p^2-a^2}\,\big)-\ln a}{\sqrt{p^2-a^2}}$

⊙ Literature for Section 28.1: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965), F. Oberhettinger and L. Badii (1973), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992a, Vol. 4).

28.2 Tables of Inverse Laplace Transforms

28.2.1 General Formulas

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}\widetilde{f}(p)\,dp$
1	$\widetilde{f}(p+a)$	$e^{-ax}f(x)$
2	$\widetilde{f}(ap),\quad a>0$	$\dfrac{1}{a}f\Big(\dfrac{x}{a}\Big)$
3	$\widetilde{f}(ap+b),\quad a>0$	$\dfrac{1}{a}\exp\Big(-\dfrac{b}{a}x\Big)f\Big(\dfrac{x}{a}\Big)$
4	$\widetilde{f}(p-a)+\widetilde{f}(p+a)$	$2f(x)\cosh(ax)$
5	$\widetilde{f}(p-a)-\widetilde{f}(p+a)$	$2f(x)\sinh(ax)$
6	$e^{-ap}\widetilde{f}(p),\quad a\ge 0$	$\begin{cases}0 & \text{if } 0\le x<a,\\ f(x-a) & \text{if } a<x.\end{cases}$
7	$p\widetilde{f}(p)$	$\dfrac{df(x)}{dx}\quad$ if $f(+0)=0$
8	$\dfrac{1}{p}\widetilde{f}(p)$	$\int_0^x f(t)\,dt$
9	$\dfrac{1}{p+a}\widetilde{f}(p)$	$e^{-ax}\int_0^x e^{at}f(t)\,dt$
10	$\dfrac{1}{p^2}\widetilde{f}(p)$	$\int_0^x (x-t)f(t)\,dt$
11	$\dfrac{\widetilde{f}(p)}{p(p+a)}$	$\dfrac{1}{a}\int_0^x\big[1-e^{a(x-t)}\big]f(t)\,dt$
12	$\dfrac{\widetilde{f}(p)}{(p+a)^2}$	$\int_0^x (x-t)e^{-a(x-t)}f(t)\,dt$
13	$\dfrac{\widetilde{f}(p)}{(p+a)(p+b)}$	$\dfrac{1}{b-a}\int_0^x\big[e^{-a(x-t)}-e^{-b(x-t)}\big]f(t)\,dt$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
14	$\dfrac{\widetilde{f}(p)}{(p+a)^2+b^2}$	$\dfrac{1}{b}\displaystyle\int_0^x e^{-a(x-t)}\sin\bigl[b(x-t)\bigr]f(t)\,dt$
15	$\dfrac{1}{p^n}\widetilde{f}(p),\quad n=1,\,2,\,\ldots$	$\dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1}f(t)\,dt$
16	$\widetilde{f}_1(p)\widetilde{f}_2(p)$	$\displaystyle\int_0^x f_1(t)f_2(x-t)\,dt$
17	$\dfrac{1}{\sqrt{p}}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty \frac{\cos\bigl(2\sqrt{xt}\,\bigr)}{\sqrt{\pi x}}f(t)\,dt$
18	$\dfrac{1}{p\sqrt{p}}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty \frac{\sin\bigl(2\sqrt{xt}\,\bigr)}{\sqrt{\pi t}}f(t)\,dt$
19	$\dfrac{1}{p^{2\nu+1}}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty (x/t)^\nu J_{2\nu}\bigl(2\sqrt{xt}\,\bigr)f(t)\,dt$
20	$\dfrac{1}{p}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty J_0\bigl(2\sqrt{xt}\,\bigr)f(t)\,dt$
21	$\dfrac{1}{p}\widetilde{f}\Bigl(p+\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^x J_0\bigl(2\sqrt{xt-t^2}\,\bigr)f(t)\,dt$
22	$\dfrac{1}{p^{2\nu+1}}\widetilde{f}\Bigl(p+\dfrac{a}{p}\Bigr),\quad -\dfrac{1}{2}<\nu\le 0$	$\displaystyle\int_0^x \Bigl(\frac{x-t}{at}\Bigr)^\nu J_{2\nu}\bigl(2\sqrt{axt-at^2}\,\bigr)f(t)\,dt$
23	$\widetilde{f}\bigl(\sqrt{p}\bigr)$	$\displaystyle\int_0^\infty \frac{t}{2\sqrt{\pi x^3}}\exp\Bigl(-\frac{t^2}{4x}\Bigr)f(t)\,dt$
24	$\dfrac{1}{\sqrt{p}}\widetilde{f}\bigl(\sqrt{p}\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\displaystyle\int_0^\infty \exp\Bigl(-\frac{t^2}{4x}\Bigr)f(t)\,dt$
25	$\widetilde{f}\bigl(p+\sqrt{p}\bigr)$	$\dfrac{1}{2\sqrt{\pi}}\displaystyle\int_0^x \frac{t}{(x-t)^{3/2}}\exp\Bigl[-\frac{t^2}{4(x-t)}\Bigr]f(t)\,dt$
26	$\widetilde{f}\bigl(\sqrt{p^2+a^2}\,\bigr)$	$f(x)-a\displaystyle\int_0^x f\bigl(\sqrt{x^2-t^2}\,\bigr)J_1(at)\,dt$
27	$\widetilde{f}\bigl(\sqrt{p^2-a^2}\,\bigr)$	$f(x)+a\displaystyle\int_0^x f\bigl(\sqrt{x^2-t^2}\,\bigr)I_1(at)\,dt$
28	$\dfrac{\widetilde{f}\bigl(\sqrt{p^2+a^2}\,\bigr)}{\sqrt{p^2+a^2}}$	$\displaystyle\int_0^x J_0\bigl(a\sqrt{x^2-t^2}\,\bigr)f(t)\,dt$
29	$\dfrac{\widetilde{f}\bigl(\sqrt{p^2-a^2}\,\bigr)}{\sqrt{p^2-a^2}}$	$\displaystyle\int_0^x I_0\bigl(a\sqrt{x^2-t^2}\,\bigr)f(t)\,dt$
30	$\widetilde{f}\bigl(\sqrt{(p+a)^2-b^2}\,\bigr)$	$e^{-ax}f(x)+be^{-ax}\displaystyle\int_0^x f\bigl(\sqrt{x^2-t^2}\,\bigr)I_1(bt)\,dt$
31	$\widetilde{f}(\ln p)$	$\displaystyle\int_0^\infty \frac{x^{t-1}}{\Gamma(t)}f(t)\,dt$
32	$\dfrac{1}{p}\widetilde{f}(\ln p)$	$\displaystyle\int_0^\infty \frac{x^t}{\Gamma(t+1)}f(t)\,dt$
33	$\widetilde{f}(p-ia)+\widetilde{f}(p+ia),\;\; i^2=-1$	$2f(x)\cos(ax)$
34	$i\bigl[\widetilde{f}(p-ia)-\widetilde{f}(p+ia)\bigr],\;\; i^2=-1$	$2f(x)\sin(ax)$
35	$\dfrac{d\widetilde{f}(p)}{dp}$	$-xf(x)$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
36	$\dfrac{d^n\widetilde{f}(p)}{dp^n}$	$(-x)^n f(x)$
37	$p^n\dfrac{d^m\widetilde{f}(p)}{dp^m}, \quad m \geq n$	$(-1)^m\dfrac{d^n}{dx^n}\bigl[x^m f(x)\bigr]$
38	$\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	$\dfrac{1}{x}f(x)$
39	$\dfrac{1}{p}\displaystyle\int_0^p \widetilde{f}(q)\,dq$	$\displaystyle\int_x^\infty \frac{f(t)}{t}\,dt$
40	$\dfrac{1}{p}\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	$\displaystyle\int_0^x \frac{f(t)}{t}\,dt$

28.2.2 Expressions with Rational Functions

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{p}$	1
2	$\dfrac{1}{p+a}$	e^{-ax}
3	$\dfrac{1}{p^2}$	x
4	$\dfrac{1}{p(p+a)}$	$\dfrac{1}{a}\bigl(1-e^{-ax}\bigr)$
5	$\dfrac{1}{(p+a)^2}$	xe^{-ax}
6	$\dfrac{p}{(p+a)^2}$	$(1-ax)e^{-ax}$
7	$\dfrac{1}{p^2-a^2}$	$\dfrac{1}{a}\sinh(ax)$
8	$\dfrac{p}{p^2-a^2}$	$\cosh(ax)$
9	$\dfrac{1}{(p+a)(p+b)}$	$\dfrac{1}{a-b}\bigl(e^{-bx}-e^{-ax}\bigr)$
10	$\dfrac{p}{(p+a)(p+b)}$	$\dfrac{1}{a-b}\bigl(ae^{-ax}-be^{-bx}\bigr)$
11	$\dfrac{1}{p^2+a^2}$	$\dfrac{1}{a}\sin(ax)$
12	$\dfrac{p}{p^2+a^2}$	$\cos(ax)$
13	$\dfrac{1}{(p+b)^2+a^2}$	$\dfrac{1}{a}e^{-bx}\sin(ax)$
14	$\dfrac{p}{(p+b)^2+a^2}$	$e^{-bx}\Bigl[\cos(ax)-\dfrac{b}{a}\sin(ax)\Bigr]$
15	$\dfrac{1}{p^3}$	$\frac{1}{2}x^2$
16	$\dfrac{1}{p^2(p+a)}$	$\dfrac{1}{a^2}\bigl(e^{-ax}+ax-1\bigr)$
17	$\dfrac{1}{p(p+a)(p+b)}$	$\dfrac{1}{ab(a-b)}\bigl(a-b+be^{-ax}-ae^{-bx}\bigr)$
18	$\dfrac{1}{p(p+a)^2}$	$\dfrac{1}{a^2}\bigl(1-e^{-ax}-axe^{-ax}\bigr)$
19	$\dfrac{1}{(p+a)(p+b)(p+c)}$	$\dfrac{(c-b)e^{-ax}+(a-c)e^{-bx}+(b-a)e^{-cx}}{(a-b)(b-c)(c-a)}$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
20	$\dfrac{p}{(p+a)(p+b)(p+c)}$	$\dfrac{a(b-c)e^{-ax}+b(c-a)e^{-bx}+c(a-b)e^{-cx}}{(a-b)(b-c)(c-a)}$
21	$\dfrac{p^2}{(p+a)(p+b)(p+c)}$	$\dfrac{a^2(c-b)e^{-ax}+b^2(a-c)e^{-bx}+c^2(b-a)e^{-cx}}{(a-b)(b-c)(c-a)}$
22	$\dfrac{1}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\bigl[e^{-ax}-e^{-bx}+(a-b)xe^{-bx}\bigr]$
23	$\dfrac{p}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\bigl\{-ae^{-ax}+\bigl[a+b(b-a)x\bigr]e^{-bx}\bigr\}$
24	$\dfrac{p^2}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\bigl[a^2e^{-ax}+b(b-2a-b^2x+abx)e^{-bx}\bigr]$
25	$\dfrac{1}{(p+a)^3}$	$\frac{1}{2}x^2e^{-ax}$
26	$\dfrac{p}{(p+a)^3}$	$x\bigl(1-\frac{1}{2}ax\bigr)e^{-ax}$
27	$\dfrac{p^2}{(p+a)^3}$	$\bigl(1-2ax+\frac{1}{2}a^2x^2\bigr)e^{-ax}$
28	$\dfrac{1}{p(p^2+a^2)}$	$\dfrac{1}{a^2}\bigl[1-\cos(ax)\bigr]$
29	$\dfrac{1}{p\bigl[(p+b)^2+a^2\bigr]}$	$\dfrac{1}{a^2+b^2}\left\{1-e^{-bx}\left[\cos(ax)+\dfrac{b}{a}\sin(ax)\right]\right\}$
30	$\dfrac{1}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\left[e^{-ax}+\dfrac{a}{b}\sin(bx)-\cos(bx)\right]$
31	$\dfrac{p}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\bigl[-ae^{-ax}+a\cos(bx)+b\sin(bx)\bigr]$
32	$\dfrac{p^2}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\bigl[a^2e^{-ax}-ab\sin(bx)+b^2\cos(bx)\bigr]$
33	$\dfrac{1}{p^3+a^3}$	$\dfrac{e^{-ax}-e^{ax/2}}{3a^2}\bigl[\cos(kx)-\sqrt{3}\sin(kx)\bigr]$, $k=\frac{1}{2}a\sqrt{3}$
34	$\dfrac{p}{p^3+a^3}$	$-\dfrac{e^{-ax}-e^{ax/2}}{3a}\bigl[\cos(kx)+\sqrt{3}\sin(kx)\bigr]$, $k=\frac{1}{2}a\sqrt{3}$
35	$\dfrac{p^2}{p^3+a^3}$	$\frac{1}{3}e^{-ax}+\frac{2}{3}e^{ax/2}\cos(kx),\quad k=\frac{1}{2}a\sqrt{3}$
36	$\dfrac{1}{(p+a)\bigl[(p+b)^2+c^2\bigr]}$	$\dfrac{e^{-ax}-e^{-bx}\cos(cx)+ke^{-bx}\sin(cx)}{(a-b)^2+c^2}$, $k=\dfrac{a-b}{c}$
37	$\dfrac{p}{(p+a)\bigl[(p+b)^2+c^2\bigr]}$	$\dfrac{-ae^{-ax}+ae^{-bx}\cos(cx)+ke^{-bx}\sin(cx)}{(a-b)^2+c^2}$, $k=\dfrac{b^2+c^2-ab}{c}$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
38	$\dfrac{p^2}{(p+a)\big[(p+b)^2+c^2\big]}$	$\dfrac{a^2e^{-ax}+(b^2+c^2-2ab)e^{-bx}\cos(cx)+ke^{-bx}\sin(cx)}{(a-b)^2+c^2}$, $k=-ac-bc+\dfrac{ab^2-b^3}{c}$
39	$\dfrac{1}{p^4}$	$\frac{1}{6}x^3$
40	$\dfrac{1}{p^3(p+a)}$	$\dfrac{1}{a^3}-\dfrac{1}{a^2}x+\dfrac{1}{2a}x^2-\dfrac{1}{a^3}e^{-ax}$
41	$\dfrac{1}{p^2(p+a)^2}$	$\dfrac{1}{a^2}x\big(1+e^{-ax}\big)+\dfrac{2}{a^3}\big(e^{-ax}-1\big)$
42	$\dfrac{1}{p^2(p+a)(p+b)}$	$-\dfrac{a+b}{a^2b^2}+\dfrac{1}{ab}x+\dfrac{1}{a^2(b-a)}e^{-ax}+\dfrac{1}{b^2(a-b)}e^{-bx}$
43	$\dfrac{1}{(p+a)^2(p+b)^2}$	$\dfrac{1}{(a-b)^2}\Big[e^{-ax}\Big(x+\dfrac{2}{a-b}\Big)+e^{-bx}\Big(x-\dfrac{2}{a-b}\Big)\Big]$
44	$\dfrac{1}{(p+a)^4}$	$\frac{1}{6}x^3e^{-ax}$
45	$\dfrac{p}{(p+a)^4}$	$\frac{1}{2}x^2e^{-ax}-\frac{1}{6}ax^3e^{-ax}$
46	$\dfrac{1}{p^2(p^2+a^2)}$	$\dfrac{1}{a^3}\big[ax-\sin(ax)\big]$
47	$\dfrac{1}{p^4-a^4}$	$\dfrac{1}{2a^3}\big[\sinh(ax)-\sin(ax)\big]$
48	$\dfrac{p}{p^4-a^4}$	$\dfrac{1}{2a^2}\big[\cosh(ax)-\cos(ax)\big]$
49	$\dfrac{p^2}{p^4-a^4}$	$\dfrac{1}{2a}\big[\sinh(ax)+\sin(ax)\big]$
50	$\dfrac{p^3}{p^4-a^4}$	$\dfrac{1}{2}\big[\cosh(ax)+\cos(ax)\big]$
51	$\dfrac{1}{p^4+a^4}$	$\dfrac{1}{a^3\sqrt{2}}\big(\cosh\xi\sin\xi-\sinh\xi\cos\xi\big)$, $\xi=\dfrac{ax}{\sqrt{2}}$
52	$\dfrac{p}{p^4+a^4}$	$\dfrac{1}{a^2}\sin\Big(\dfrac{ax}{\sqrt{2}}\Big)\sinh\Big(\dfrac{ax}{\sqrt{2}}\Big)$
53	$\dfrac{p^2}{p^4+a^4}$	$\dfrac{1}{a\sqrt{2}}\big(\cos\xi\sinh\xi+\sin\xi\cosh\xi\big)$, $\xi=\dfrac{ax}{\sqrt{2}}$
54	$\dfrac{1}{(p^2+a^2)^2}$	$\dfrac{1}{2a^3}\big[\sin(ax)-ax\cos(ax)\big]$
55	$\dfrac{p}{(p^2+a^2)^2}$	$\dfrac{1}{2a}x\sin(ax)$
56	$\dfrac{p^2}{(p^2+a^2)^2}$	$\dfrac{1}{2a}\big[\sin(ax)+ax\cos(ax)\big]$
57	$\dfrac{p^3}{(p^2+a^2)^2}$	$\cos(ax)-\frac{1}{2}ax\sin(ax)$
58	$\dfrac{1}{[(p+b)^2+a^2]^2}$	$\dfrac{1}{2a^3}e^{-bx}\big[\sin(ax)-ax\cos(ax)\big]$
59	$\dfrac{1}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{1}{a^2-b^2}\Big[\dfrac{1}{a}\sinh(ax)-\dfrac{1}{b}\sinh(bx)\Big]$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
60	$\dfrac{p}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{\cosh(ax)-\cosh(bx)}{a^2-b^2}$
61	$\dfrac{p^2}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{a\sinh(ax)-b\sinh(bx)}{a^2-b^2}$
62	$\dfrac{p^3}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{a^2\cosh(ax)-b^2\cosh(bx)}{a^2-b^2}$
63	$\dfrac{1}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{1}{b^2-a^2}\Big[\dfrac{1}{a}\sin(ax)-\dfrac{1}{b}\sin(bx)\Big]$
64	$\dfrac{p}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{\cos(ax)-\cos(bx)}{b^2-a^2}$
65	$\dfrac{p^2}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{-a\sin(ax)+b\sin(bx)}{b^2-a^2}$
66	$\dfrac{p^3}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{-a^2\cos(ax)+b^2\cos(bx)}{b^2-a^2}$
67	$\dfrac{1}{p^n},\quad n=1,\,2,\,\dots$	$\dfrac{1}{(n-1)!}x^{n-1}$
68	$\dfrac{1}{(p+a)^n},\quad n=1,\,2,\,\dots$	$\dfrac{1}{(n-1)!}x^{n-1}e^{-ax}$
69	$\dfrac{1}{p(p+a)^n},\quad n=1,\,2,\,\dots$	$a^{-n}\big[1-e^{-ax}e_n(ax)\big]$, $e_n(z)=1+\dfrac{z}{1!}+\dots+\dfrac{z^n}{n!}$
70	$\dfrac{1}{p^{2n}+a^{2n}},\quad n=1,\,2,\,\dots$	$-\dfrac{1}{na^{2n}}\displaystyle\sum_{k=1}^{n}\exp(a_kx)\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\;\; b_k=a\sin\varphi_k,\;\; \varphi_k=\dfrac{\pi(2k-1)}{2n}$
71	$\dfrac{1}{p^{2n}-a^{2n}},\quad n=1,\,2,\,\dots$	$\dfrac{1}{na^{2n-1}}\sinh(ax)+\dfrac{1}{na^{2n}}\displaystyle\sum_{k=2}^{n}\exp(a_kx)$ $\times\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\;\; b_k=a\sin\varphi_k,\;\; \varphi_k=\dfrac{\pi(k-1)}{n}$
72	$\dfrac{1}{p^{2n+1}+a^{2n+1}},\quad n=0,\,1,\,\dots$	$\dfrac{e^{-ax}}{(2n+1)a^{2n}}-\dfrac{2}{(2n+1)a^{2n+1}}\displaystyle\sum_{k=1}^{n}\exp(a_kx)$ $\times\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\;\; b_k=a\sin\varphi_k,\;\; \varphi_k=\dfrac{\pi(2k-1)}{2n+1}$
73	$\dfrac{1}{p^{2n+1}-a^{2n+1}},\quad n=0,\,1,\,\dots$	$\dfrac{e^{ax}}{(2n+1)a^{2n}}+\dfrac{2}{(2n+1)a^{2n+1}}\displaystyle\sum_{k=1}^{n}\exp(a_kx)$ $\times\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\;\; b_k=a\sin\varphi_k,\;\; \varphi_k=\dfrac{2\pi k}{2n+1}$
74	$\dfrac{Q(p)}{P(p)}$, $P(p)=(p-a_1)\dots(p-a_n)$; $Q(p)$ is a polynomial of degree $\le n-1$; $a_i\ne a_j$ if $i\ne j$	$\displaystyle\sum_{k=1}^{n}\dfrac{Q(a_k)}{P'(a_k)}\exp(a_kx)$ (prime stands for differentiation)

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
75	$\dfrac{Q(p)}{P(p)}$, $P(p) = (p-a_1)^{m_1}\dots(p-a_n)^{m_n}$; $Q(p)$ is a polynomial of degree $< m_1+m_2+\dots+m_n-1$; $a_i \neq a_j$ if $i \neq j$	$\displaystyle\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\Phi_{kl}(a_k)}{(m_k-l)!\,(l-1)!}x^{m_k-l}\exp(a_k x)$, $\Phi_{kl}(p) = \dfrac{d^{l-1}}{dp^{l-1}}\left[\dfrac{Q(p)}{P_k(p)}\right]$, $P_k(p) = \dfrac{P(p)}{(p-a_k)^{m_k}}$
76	$\dfrac{Q(p)+pR(p)}{P(p)}$, $P(p) = (p^2+a_1^2)\dots(p^2+a_n^2)$; $Q(p)$ and $R(p)$ are polynomials of degree $\le 2n-2$; $a_l \neq a_j$, $l \neq j$	$\displaystyle\sum_{k=1}^{n}\frac{Q(ia_k)\sin(a_k x)+a_k R(ia_k)\cos(a_k x)}{a_k P_k(ia_k)}$, $P_m(p) = \dfrac{P(p)}{p^2+a_m^2}$, $i^2=-1$

28.2.3 Expressions with Square Roots

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}$
2	$\sqrt{p-a}-\sqrt{p-b}$	$\dfrac{e^{bx}-e^{ax}}{2\sqrt{\pi x^3}}$
3	$\dfrac{1}{\sqrt{p+a}}$	$\dfrac{1}{\sqrt{\pi x}}e^{-ax}$
4	$\sqrt{\dfrac{p+a}{p}}-1$	$\frac{1}{2}ae^{-ax/2}\left[I_1\left(\frac{1}{2}ax\right)+I_0\left(\frac{1}{2}ax\right)\right]$
5	$\dfrac{\sqrt{p+a}}{p+b}$	$\dfrac{e^{-ax}}{\sqrt{\pi x}}+(a-b)^{1/2}e^{-bx}\operatorname{erf}\left[(a-b)^{1/2}x^{1/2}\right]$
6	$\dfrac{1}{p\sqrt{p}}$	$2\sqrt{\dfrac{x}{\pi}}$
7	$\dfrac{1}{(p+a)\sqrt{p+b}}$	$(b-a)^{-1/2}e^{-ax}\operatorname{erf}\left[(b-a)^{1/2}x^{1/2}\right]$
8	$\dfrac{1}{\sqrt{p}\,(p-a)}$	$\dfrac{1}{\sqrt{a}}e^{ax}\operatorname{erf}\left(\sqrt{ax}\right)$
9	$\dfrac{1}{p^{3/2}(p-a)}$	$a^{-3/2}e^{ax}\operatorname{erf}\left(\sqrt{ax}\right)-2a^{-1}\pi^{-1/2}x^{1/2}$
10	$\dfrac{1}{\sqrt{p}+a}$	$\pi^{-1/2}x^{-1/2}-ae^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
11	$\dfrac{a}{p\left(\sqrt{p}+a\right)}$	$1-e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
12	$\dfrac{1}{p+a\sqrt{p}}$	$e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\right)$
13	$\dfrac{1}{\left(\sqrt{p}+\sqrt{a}\right)^2}$	$1-\dfrac{2}{\sqrt{\pi}}(ax)^{1/2}+(1-2ax)e^{ax}\left[\operatorname{erf}\left(\sqrt{ax}\right)-1\right]$
14	$\dfrac{1}{p\left(\sqrt{p}+\sqrt{a}\right)^2}$	$\dfrac{1}{a}+\left(2x-\dfrac{1}{a}\right)e^{ax}\operatorname{erfc}\left(\sqrt{ax}\right)-\dfrac{2}{\sqrt{\pi a}}\sqrt{x}$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
15	$\dfrac{1}{\sqrt{p}\,(\sqrt{p}+a)^2}$	$2\pi^{-1/2}x^{1/2}-2axe^{a^2x}\operatorname{erfc}(a\sqrt{x}\,)$
16	$\dfrac{1}{(\sqrt{p}+a)^3}$	$\dfrac{2}{\sqrt{\pi}}(a^2x+1)\sqrt{x}-ax(2a^2x+3)e^{a^2x}\operatorname{erfc}(a\sqrt{x}\,)$
17	$p^{-n-1/2},\quad n=1,\,2,\,\dots$	$\dfrac{2^n}{1\cdot 3\dots(2n-1)\sqrt{\pi}}\,x^{n-1/2}$
18	$(p+a)^{-n-1/2}$	$\dfrac{2^n}{1\cdot 3\dots(2n-1)\sqrt{\pi}}\,x^{n-1/2}e^{-ax}$
19	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(ax)$
20	$\dfrac{1}{\sqrt{p^2-a^2}}$	$I_0(ax)$
21	$\dfrac{1}{\sqrt{p^2+ap+b}}$	$\exp\bigl(-\tfrac{1}{2}ax\bigr)J_0\bigl[(b-\tfrac{1}{4}a^2)^{1/2}\,x\bigr]$
22	$\bigl(\sqrt{p^2+a^2}-p\bigr)^{1/2}$	$\dfrac{1}{\sqrt{2\pi x^3}}\,\sin(ax)$
23	$\dfrac{1}{\sqrt{p^2+a^2}}\bigl(\sqrt{p^2+a^2}+p\bigr)^{1/2}$	$\dfrac{\sqrt{2}}{\sqrt{\pi x}}\,\cos(ax)$
24	$\dfrac{1}{\sqrt{p^2-a^2}}\bigl(\sqrt{p^2-a^2}+p\bigr)^{1/2}$	$\dfrac{\sqrt{2}}{\sqrt{\pi x}}\,\cosh(ax)$
25	$\bigl(\sqrt{p^2+a^2}+p\bigr)^{-n}$	$na^{-n}x^{-1}J_n(ax)$
26	$\bigl(\sqrt{p^2-a^2}+p\bigr)^{-n}$	$na^{-n}x^{-1}I_n(ax)$
27	$\bigl(p^2+a^2\bigr)^{-n-1/2}$	$\dfrac{(x/a)^nJ_n(ax)}{1\cdot 3\cdot 5\dots(2n-1)}$
28	$\bigl(p^2-a^2\bigr)^{-n-1/2}$	$\dfrac{(x/a)^nI_n(ax)}{1\cdot 3\cdot 5\dots(2n-1)}$

28.2.4 Expressions with Arbitrary Powers

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$(p+a)^{-\nu},\ \ \nu>0$	$\dfrac{1}{\Gamma(\nu)}x^{\nu-1}e^{-ax}$
2	$\bigl[(p+a)^{1/2}+(p+b)^{1/2}\bigr]^{-2\nu},\ \ \nu>0$	$\dfrac{\nu}{(a-b)^\nu}x^{-1}\exp\bigl[-\tfrac{1}{2}(a+b)x\bigr]I_\nu\bigl[\tfrac{1}{2}(a-b)x\bigr]$
3	$\bigl[(p+a)(p+b)\bigr]^{-\nu},\ \ \nu>0$	$\dfrac{\sqrt{\pi}}{\Gamma(\nu)}\Bigl(\dfrac{x}{a-b}\Bigr)^{\nu-1/2}\exp\Bigl(-\dfrac{a+b}{2}x\Bigr)I_{\nu-1/2}\Bigl(\dfrac{a-b}{2}x\Bigr)$
4	$\bigl(p^2+a^2\bigr)^{-\nu-1/2},\ \ \nu>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+\frac{1}{2})}x^\nu J_\nu(ax)$
5	$\bigl(p^2-a^2\bigr)^{-\nu-1/2},\ \ \nu>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+\frac{1}{2})}x^\nu I_\nu(ax)$
6	$p\bigl(p^2+a^2\bigr)^{-\nu-1/2},\ \ \nu>0$	$\dfrac{a\sqrt{\pi}}{(2a)^\nu\Gamma\bigl(\nu+\frac{1}{2}\bigr)}x^\nu J_{\nu-1}(ax)$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
7	$p\bigl(p^2-a^2\bigr)^{-\nu-1/2},\ \ \nu>0$	$\dfrac{a\sqrt{\pi}}{(2a)^\nu\Gamma\bigl(\nu+\frac{1}{2}\bigr)}x^\nu I_{\nu-1}(ax)$
8	$\bigl[(p^2+a^2)^{1/2}+p\bigr]^{-\nu} = a^{-2\nu}\bigl[(p^2+a^2)^{1/2}-p\bigr]^\nu,\ \ \nu>0$	$\nu a^{-\nu}x^{-1}J_\nu(ax)$
9	$\bigl[(p^2-a^2)^{1/2}+p\bigr]^{-\nu} = a^{-2\nu}\bigl[p-(p^2-a^2)^{1/2}\bigr]^\nu,\ \ \nu>0$	$\nu a^{-\nu}x^{-1}I_\nu(ax)$
10	$p\bigl[(p^2+a^2)^{1/2}+p\bigr]^{-\nu},\ \ \nu>1$	$\nu a^{1-\nu}x^{-1}J_{\nu-1}(ax)-\nu(\nu+1)a^{-\nu}x^{-2}J_\nu(ax)$
11	$p\bigl[(p^2-a^2)^{1/2}+p\bigr]^{-\nu},\ \ \nu>1$	$\nu a^{1-\nu}x^{-1}I_{\nu-1}(ax)-\nu(\nu+1)a^{-\nu}x^{-2}I_\nu(ax)$
12	$\dfrac{\bigl(\sqrt{p^2+a^2}+p\bigr)^{-\nu}}{\sqrt{p^2+a^2}},\ \ \nu>-1$	$a^{-\nu}J_\nu(ax)$
13	$\dfrac{\bigl(\sqrt{p^2-a^2}+p\bigr)^{-\nu}}{\sqrt{p^2-a^2}},\ \ \nu>-1$	$a^{-\nu}I_\nu(ax)$

28.2.5 Expressions with Exponential Functions

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$p^{-1}e^{-ap},\quad a>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ 1 & \text{if } a<x\end{cases}$
2	$p^{-1}\bigl(1-e^{-ap}\bigr),\quad a>0$	$\begin{cases}1 & \text{if } 0<x<a,\\ 0 & \text{if } a<x\end{cases}$
3	$p^{-1}\bigl(e^{-ap}-e^{-bp}\bigr),\quad 0\le a<b$	$\begin{cases}0 & \text{if } 0<x<a,\\ 1 & \text{if } a<x<b,\\ 0 & \text{if } b<x\end{cases}$
4	$p^{-2}\bigl(e^{-ap}-e^{-bp}\bigr),\quad 0\le a<b$	$\begin{cases}0 & \text{if } 0<x<a,\\ x-a & \text{if } a<x<b,\\ b-a & \text{if } b<x\end{cases}$
5	$(p+b)^{-1}e^{-ap},\quad a>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ e^{-b(x-a)} & \text{if } a<x\end{cases}$
6	$p^{-\nu}e^{-ap},\quad \nu>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ \dfrac{(x-a)^{\nu-1}}{\Gamma(\nu)} & \text{if } a<x\end{cases}$
7	$p^{-1}\bigl(e^{ap}-1\bigr)^{-1},\quad a>0$	$f(x)=n$ if $na<x<(n+1)a;\ \ n=0,1,2,\dots$
8	$e^{a/p}-1$	$\sqrt{\dfrac{a}{x}}\,I_1\bigl(2\sqrt{ax}\,\bigr)$
9	$p^{-1/2}e^{a/p}$	$\dfrac{1}{\sqrt{\pi x}}\cosh\bigl(2\sqrt{ax}\,\bigr)$
10	$p^{-3/2}e^{a/p}$	$\dfrac{1}{\sqrt{\pi a}}\sinh\bigl(2\sqrt{ax}\,\bigr)$
11	$p^{-5/2}e^{a/p}$	$\sqrt{\dfrac{x}{\pi a}}\cosh\bigl(2\sqrt{ax}\,\bigr)-\dfrac{1}{2\sqrt{\pi a^3}}\sinh\bigl(2\sqrt{ax}\,\bigr)$
12	$p^{-\nu-1}e^{a/p},\quad \nu>-1$	$(x/a)^{\nu/2}I_\nu\bigl(2\sqrt{ax}\,\bigr)$
13	$1-e^{-a/p}$	$\sqrt{\dfrac{a}{x}}\,J_1\bigl(2\sqrt{ax}\,\bigr)$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
14	$p^{-1/2}e^{-a/p}$	$\dfrac{1}{\sqrt{\pi x}}\cos\bigl(2\sqrt{ax}\,\bigr)$
15	$p^{-3/2}e^{-a/p}$	$\dfrac{1}{\sqrt{\pi a}}\sin\bigl(2\sqrt{ax}\,\bigr)$
16	$p^{-5/2}e^{-a/p}$	$\dfrac{1}{2\sqrt{\pi a^3}}\sin\bigl(2\sqrt{ax}\,\bigr) - \sqrt{\dfrac{x}{\pi a}}\cos\bigl(2\sqrt{ax}\,\bigr)$
17	$p^{-\nu-1}e^{-a/p}, \quad \nu > -1$	$(x/a)^{\nu/2}J_\nu\bigl(2\sqrt{ax}\,\bigr)$
18	$\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{\sqrt{a}}{2\sqrt{\pi}}\,x^{-3/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
19	$p\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{\sqrt{a}}{8\sqrt{\pi}}\,(a-6x)x^{-7/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
20	$\dfrac{1}{p}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \ge 0$	$\operatorname{erfc}\Bigl(\dfrac{\sqrt{a}}{2\sqrt{x}}\Bigr)$
21	$\sqrt{p}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{1}{4\sqrt{\pi}}\,(a-2x)x^{-5/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
22	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \ge 0$	$\dfrac{1}{\sqrt{\pi x}}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
23	$\dfrac{1}{p\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \ge 0$	$\dfrac{2\sqrt{x}}{\sqrt{\pi}}\exp\Bigl(-\dfrac{a}{4x}\Bigr) - \sqrt{a}\operatorname{erfc}\Bigl(\dfrac{\sqrt{a}}{2\sqrt{x}}\Bigr)$
24	$\dfrac{\exp\bigl(-k\sqrt{p^2+a^2}\,\bigr)}{\sqrt{p^2+a^2}}, \quad k > 0$	$\begin{cases} 0 & \text{if } 0 < x < k, \\ J_0\bigl(a\sqrt{x^2-k^2}\,\bigr) & \text{if } k < x \end{cases}$
25	$\dfrac{\exp\bigl(-k\sqrt{p^2-a^2}\,\bigr)}{\sqrt{p^2-a^2}}, \quad k > 0$	$\begin{cases} 0 & \text{if } 0 < x < k, \\ I_0\bigl(a\sqrt{x^2-k^2}\,\bigr) & \text{if } k < x \end{cases}$

28.2.6 Expressions with Hyperbolic Functions

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{p\sinh(ap)}, \quad a > 0$	$f(x) = 2n$ if $a(2n-1) < x < a(2n+1)$; $n = 0, 1, 2, \ldots$ $(x > 0)$
2	$\dfrac{1}{p^2\sinh(ap)}, \quad a > 0$	$f(x) = 2n(x-an)$ if $a(2n-1) < x < a(2n+1)$; $n = 0, 1, 2, \ldots$ $(x > 0)$
3	$\dfrac{\sinh(a/p)}{\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi x}}\bigl[\cosh\bigl(2\sqrt{ax}\,\bigr) - \cos\bigl(2\sqrt{ax}\,\bigr)\bigr]$
4	$\dfrac{\sinh(a/p)}{p\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi a}}\bigl[\sinh\bigl(2\sqrt{ax}\,\bigr) - \sin\bigl(2\sqrt{ax}\,\bigr)\bigr]$
5	$p^{-\nu-1}\sinh(a/p), \quad \nu > -2$	$\tfrac{1}{2}(x/a)^{\nu/2}\bigl[I_\nu\bigl(2\sqrt{ax}\,\bigr) - J_\nu\bigl(2\sqrt{ax}\,\bigr)\bigr]$
6	$\dfrac{1}{p\cosh(ap)}, \quad a > 0$	$f(x) = \begin{cases} 0 & \text{if } a(4n-1) < x < a(4n+1), \\ 2 & \text{if } a(4n+1) < x < a(4n+3), \end{cases}$ $n = 0, 1, 2, \ldots$ $(x > 0)$
7	$\dfrac{1}{p^2\cosh(ap)}, \quad a > 0$	$x - (-1)^n(x-2an)$ if $2n-1 < x/a < 2n+1$; $n = 0, 1, 2, \ldots$ $(x > 0)$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
8	$\dfrac{\cosh(a/p)}{\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi x}}\bigl[\cosh\bigl(2\sqrt{ax}\,\bigr)+\cos\bigl(2\sqrt{ax}\,\bigr)\bigr]$
9	$\dfrac{\cosh(a/p)}{p\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi a}}\bigl[\sinh\bigl(2\sqrt{ax}\,\bigr)+\sin\bigl(2\sqrt{ax}\,\bigr)\bigr]$
10	$p^{-\nu-1}\cosh(a/p),\quad \nu>-1$	$\frac{1}{2}(x/a)^{\nu/2}\bigl[I_\nu\bigl(2\sqrt{ax}\,\bigr)+J_\nu\bigl(2\sqrt{ax}\,\bigr)\bigr]$
11	$\dfrac{1}{p}\tanh(ap),\quad a>0$	$f(x)=(-1)^{n-1}$ if $2a(n-1)<x<2an$, $n=1,\,2,\,\ldots$
12	$\dfrac{1}{p}\coth(ap),\quad a>0$	$f(x)=(2n-1)$ if $2a(n-1)<x<2an$, $n=1,\,2,\,\ldots$
13	$\operatorname{arccoth}(p/a)$	$\dfrac{1}{x}\sinh(ax)$

28.2.7 Expressions with Logarithmic Functions

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{p}\ln p$	$-\ln x-\mathcal{C},\quad \mathcal{C}=0.5772\ldots$ is the Euler constant
2	$p^{-n-1}\ln p$	$\bigl(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln x-\mathcal{C}\bigr)\dfrac{x^n}{n!}$, $\mathcal{C}=0.5772\ldots$ is the Euler constant
3	$p^{-n-1/2}\ln p$	$k_n\bigl[2+\frac{2}{3}+\frac{2}{5}+\cdots+\frac{2}{2n-1}-\ln(4x)-\mathcal{C}\bigr]x^{n-1/2}$, $k_n=\dfrac{2^n}{1\cdot3\cdot5\ldots(2n-1)\sqrt{\pi}},\quad \mathcal{C}=0.5772\ldots$
4	$p^{-\nu}\ln p,\quad \nu>0$	$\dfrac{1}{\Gamma(\nu)}x^{\nu-1}\bigl[\psi(\nu)-\ln x\bigr]$, $\psi(\nu)$ is the logarithmic derivative of the gamma function
5	$\dfrac{1}{p}(\ln p)^2$	$(\ln x+\mathcal{C})^2-\frac{1}{6}\pi^2,\quad \mathcal{C}=0.5772\ldots$
6	$\dfrac{1}{p^2}(\ln p)^2$	$x\bigl[(\ln x+\mathcal{C}-1)^2+1-\frac{1}{6}\pi^2\bigr]$
7	$\dfrac{\ln(p+b)}{p+a}$	$e^{-ax}\bigl\{\ln(b-a)-\operatorname{Ei}\bigl[(a-b)x\bigr]\bigr\}$
8	$\dfrac{\ln p}{p^2+a^2}$	$\dfrac{1}{a}\cos(ax)\operatorname{Si}(ax)+\dfrac{1}{a}\sin(ax)\bigl[\ln a-\operatorname{Ci}(ax)\bigr]$
9	$\dfrac{p\ln p}{p^2+a^2}$	$\cos(ax)\bigl[\ln a-\operatorname{Ci}(ax)\bigr]-\sin(ax)\operatorname{Si}(ax)$
10	$\ln\dfrac{p+b}{p+a}$	$\dfrac{1}{x}\bigl(e^{-ax}-e^{-bx}\bigr)$
11	$\ln\dfrac{p^2+b^2}{p^2+a^2}$	$\dfrac{2}{x}\bigl[\cos(ax)-\cos(bx)\bigr]$
12	$p\ln\dfrac{p^2+b^2}{p^2+a^2}$	$\dfrac{2}{x}\bigl[\cos(bx)+bx\sin(bx)-\cos(ax)-ax\sin(ax)\bigr]$
13	$\ln\dfrac{(p+a)^2+k^2}{(p+b)^2+k^2}$	$\dfrac{2}{x}\cos(kx)(e^{-bx}-e^{-ax})$
14	$p\ln\Bigl(\dfrac{1}{p}\sqrt{p^2+a^2}\Bigr)$	$\dfrac{1}{x^2}\bigl[\cos(ax)-1\bigr]+\dfrac{a}{x}\sin(ax)$
15	$p\ln\Bigl(\dfrac{1}{p}\sqrt{p^2-a^2}\Bigr)$	$\dfrac{1}{x^2}\bigl[\cosh(ax)-1\bigr]-\dfrac{a}{x}\sinh(ax)$

28.2.8 Expressions with Trigonometric Functions

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{\sin(a/p)}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}\sinh\bigl(\sqrt{2ax}\bigr)\sin\bigl(\sqrt{2ax}\bigr)$
2	$\dfrac{\sin(a/p)}{p\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi a}}\cosh\bigl(\sqrt{2ax}\bigr)\sin\bigl(\sqrt{2ax}\bigr)$
3	$\dfrac{\cos(a/p)}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}\cosh\bigl(\sqrt{2ax}\bigr)\cos\bigl(\sqrt{2ax}\bigr)$
4	$\dfrac{\cos(a/p)}{p\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi a}}\sinh\bigl(\sqrt{2ax}\bigr)\cos\bigl(\sqrt{2ax}\bigr)$
5	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\bigr)\sin\bigl(\sqrt{ap}\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\sin\Bigl(\dfrac{a}{2x}\Bigr)$
6	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\bigr)\cos\bigl(\sqrt{ap}\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\cos\Bigl(\dfrac{a}{2x}\Bigr)$
7	$\arctan\dfrac{a}{p}$	$\dfrac{1}{x}\sin(ax)$
8	$\dfrac{1}{p}\arctan\dfrac{a}{p}$	$\operatorname{Si}(ax)$
9	$p\arctan\dfrac{a}{p}-a$	$\dfrac{1}{x^2}\bigl[ax\cos(ax)-\sin(ax)\bigr]$
10	$\arctan\dfrac{2ap}{p^2+b^2}$	$\dfrac{2}{x}\sin(ax)\cos\bigl(x\sqrt{a^2+b^2}\bigr)$

28.2.9 Expressions with Special Functions

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\exp\bigl(ap^2\bigr)\operatorname{erfc}\bigl(p\sqrt{a}\bigr)$	$\dfrac{1}{\sqrt{\pi a}}\exp\Bigl(-\dfrac{x^2}{4a}\Bigr)$
2	$\dfrac{1}{p}\exp\bigl(ap^2\bigr)\operatorname{erfc}\bigl(p\sqrt{a}\bigr)$	$\operatorname{erf}\Bigl(\dfrac{x}{2\sqrt{a}}\Bigr)$
3	$\operatorname{erfc}\bigl(\sqrt{ap}\bigr),\quad a>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ \dfrac{\sqrt{a}}{\pi x\sqrt{x-a}} & \text{if } a<x\end{cases}$
4	$e^{ap}\operatorname{erfc}\bigl(\sqrt{ap}\bigr)$	$\dfrac{\sqrt{a}}{\pi\sqrt{x}\,(x+a)}$
5	$\dfrac{1}{\sqrt{p}}e^{ap}\operatorname{erfc}\bigl(\sqrt{ap}\bigr)$	$\dfrac{1}{\sqrt{\pi(x+a)}}$
6	$\operatorname{erf}\bigl(\sqrt{a/p}\bigr)$	$\dfrac{1}{\pi x}\sin\bigl(2\sqrt{ax}\bigr)$
7	$\dfrac{1}{\sqrt{p}}\exp(a/p)\operatorname{erf}\bigl(\sqrt{a/p}\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\sinh\bigl(2\sqrt{ax}\bigr)$
8	$\dfrac{1}{\sqrt{p}}\exp(a/p)\operatorname{erfc}\bigl(\sqrt{a/p}\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\exp\bigl(-2\sqrt{ax}\bigr)$
9	$p^{-a}\gamma(a,bp),\quad a,b>0$	$\begin{cases}x^{a-1} & \text{if } 0<x<b,\\ 0 & \text{if } b<x\end{cases}$

No.	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
10	$\gamma(a,b/p), \quad a>0$	$b^{a/2}x^{a/2-1}J_a\bigl(2\sqrt{bx}\,\bigr)$
11	$a^{-p}\gamma(p,a)$	$\exp\bigl(-ae^{-x}\bigr)$
12	$K_0(ap), \quad a>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ (x^2-a^2)^{-1/2} & \text{if } a<x \end{cases}$
13	$K_\nu(ap), \quad a>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ \dfrac{\cosh\bigl[\nu\operatorname{arccosh}(x/a)\bigr]}{\sqrt{x^2-a^2}} & \text{if } a<x \end{cases}$
14	$K_0\bigl(a\sqrt{p}\bigr)$	$\dfrac{1}{2x}\exp\Bigl(-\dfrac{a^2}{4x}\Bigr)$
15	$\dfrac{1}{\sqrt{p}}K_1\bigl(a\sqrt{p}\,\bigr)$	$\dfrac{1}{a}\exp\Bigl(-\dfrac{a^2}{4x}\Bigr)$

⊙ Literature for Section 28.2: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), I. I. Hirschman and D. V. Widder (1955), V. A. Ditkin and A. P. Prudnikov (1965), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992b, Vol. 5).

28.3 Tables of Fourier Cosine Transforms

28.3.1 General Formulas

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u)=\displaystyle\int_0^\infty f(x)\cos(ux)\,dx$
1	$af_1(x)+bf_2(x)$	$a\check{f}_{1\rm c}(u)+b\check{f}_{2\rm c}(u)$
2	$f(ax), \quad a>0$	$\dfrac{1}{a}\check{f}_{\rm c}\Bigl(\dfrac{u}{a}\Bigr)$
3	$x^{2n}f(x), \quad n=1,\,2,\,\ldots$	$(-1)^n\dfrac{d^{2n}}{du^{2n}}\check{f}_{\rm c}(u)$
4	$x^{2n+1}f(ax), \quad n=0,\,1,\,\ldots$	$(-1)^n\dfrac{d^{2n+1}}{du^{2n+1}}\check{f}_{\rm s}(u),\ \check{f}_{\rm s}(u)=\displaystyle\int_0^\infty f(x)\sin(xu)\,dx$
5	$f(ax)\cos(bx), \quad a,b>0$	$\dfrac{1}{2a}\Bigl[\check{f}_{\rm c}\Bigl(\dfrac{u+b}{a}\Bigr)+\check{f}_{\rm c}\Bigl(\dfrac{u-b}{a}\Bigr)\Bigr]$

28.3.2 Expressions with Power-Law Functions

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u)=\displaystyle\int_0^\infty f(x)\cos(ux)\,dx$
1	$\begin{cases} 1 & \text{if } 0<x<a, \\ 0 & \text{if } a<x \end{cases}$	$\dfrac{1}{u}\sin(au)$
2	$\begin{cases} x & \text{if } 0<x<1, \\ 2-x & \text{if } 1<x<2, \\ 0 & \text{if } 2<x \end{cases}$	$\dfrac{4}{u^2}\cos u\,\sin^2\dfrac{u}{2}$
3	$\dfrac{1}{a+x}, \quad a>0$	$-\sin(au)\operatorname{si}(au)-\cos(au)\operatorname{Ci}(au)$
4	$\dfrac{1}{a^2+x^2}, \quad a>0$	$\dfrac{\pi}{2a}e^{-au}$ (the integral is understood in the sense of Cauchy principal value)

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
5	$\dfrac{1}{a^2 - x^2}, \quad a > 0$	$\dfrac{\pi \sin(au)}{2u}$
6	$\dfrac{a}{a^2 + (b+x)^2} + \dfrac{a}{a^2 + (b-x)^2}$	$\pi e^{-au}\cos(bu)$
7	$\dfrac{b+x}{a^2 + (b+x)^2} + \dfrac{b-x}{a^2 + (b-x)^2}$	$\pi e^{-au}\sin(bu)$
8	$\dfrac{1}{a^4 + x^4}, \quad a > 0$	$\frac{1}{2}\pi a^{-3}\exp\Bigl(-\dfrac{au}{\sqrt{2}}\Bigr)\sin\Bigl(\dfrac{\pi}{4} + \dfrac{au}{\sqrt{2}}\Bigr)$
9	$\dfrac{1}{(a^2 + x^2)(b^2 + x^2)}, \quad a, b > 0$	$\dfrac{\pi}{2}\,\dfrac{ae^{-bu} - be^{-au}}{ab(a^2 - b^2)}$
10	$\dfrac{x^{2m}}{(x^2 + a)^{n+1}}$, $n, m = 1, 2, \ldots;\ n + 1 > m \geq 0$	$(-1)^{n+m}\dfrac{\pi}{2n!}\,\dfrac{\partial^n}{\partial a^n}\bigl(a^{1/\sqrt{m}} e^{-u\sqrt{a}}\bigr)$
11	$\dfrac{1}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2u}}$
12	$\begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } 0 < x < a, \\ 0 & \text{if } a < x \end{cases}$	$2\sqrt{\dfrac{\pi}{2u}}\,C(au)$, $C(u)$ is the Fresnel integral
13	$\begin{cases} 0 & \text{if } 0 < x < a, \\ \dfrac{1}{\sqrt{x}} & \text{if } a < x \end{cases}$	$\sqrt{\dfrac{\pi}{2u}}\bigl[1 - 2C(au)\bigr]$, $C(u)$ is the Fresnel integral
14	$\begin{cases} 0 & \text{if } 0 < x < a, \\ \dfrac{1}{\sqrt{x - a}} & \text{if } a < x \end{cases}$	$\sqrt{\dfrac{\pi}{2u}}\bigl[\cos(au) - \sin(au)\bigr]$
15	$\dfrac{1}{\sqrt{a^2 + x^2}}$	$K_0(au)$
16	$\begin{cases} \dfrac{1}{\sqrt{a^2 - x^2}} & \text{if } 0 < x < a, \\ 0 & \text{if } a < x \end{cases}$	$\dfrac{\pi}{2}\,J_0(au)$
17	$x^{-\nu}, \quad 0 < \nu < 1$	$\sin\bigl(\frac{1}{2}\pi\nu\bigr)\Gamma(1 - \nu)u^{\nu-1}$

28.3.3 Expressions with Exponential Functions

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	e^{-ax}	$\dfrac{a}{a^2 + u^2}$
2	$\dfrac{1}{x}\bigl(e^{-ax} - e^{-bx}\bigr)$	$\dfrac{1}{2}\ln\dfrac{b^2 + u^2}{a^2 + u^2}$
3	$\sqrt{x}e^{-ax}$	$\frac{1}{2}\sqrt{\pi}\,(a^2 + u^2)^{-3/4}\cos\Bigl(\frac{3}{2}\arctan\dfrac{u}{a}\Bigr)$

No.	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
4	$\dfrac{1}{\sqrt{x}}e^{-ax}$	$\sqrt{\dfrac{\pi}{2}}\left[\dfrac{a+(a^2+u^2)^{1/2}}{a^2+u^2}\right]^{1/2}$
5	$x^n e^{-ax}, \quad n=1,\,2,\,\ldots$	$\dfrac{a^{n+1}n!}{(a^2+u^2)^{n+1}}\sum\limits_{0\le 2k\le n+1}(-1)^k C_{n+1}^{2k}\left(\dfrac{u}{a}\right)^{2k}$
6	$x^{n-1/2}e^{-ax}, \quad n=1,\,2,\,\ldots$	$k_n u\dfrac{\partial^n}{\partial a^n}\dfrac{1}{r\sqrt{r-a}}$, where $r=\sqrt{a^2+u^2}$, $k_n=(-1)^n\sqrt{\pi/2}$
7	$x^{\nu-1}e^{-ax}$	$\Gamma(\nu)(a^2+u^2)^{-\nu/2}\cos\left(\nu\arctan\dfrac{u}{a}\right)$
8	$\dfrac{x}{e^{ax}-1}$	$\dfrac{1}{2u^2}-\dfrac{\pi^2}{2a^2\sinh^2(\pi a^{-1}u)}$
9	$\dfrac{1}{x}\left(\dfrac{1}{2}-\dfrac{1}{x}+\dfrac{1}{e^x-1}\right)$	$-\dfrac{1}{2}\ln\left(1-e^{-2\pi u}\right)$
10	$\exp(-ax^2)$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\exp\left(-\dfrac{u^2}{4a}\right)$
11	$\dfrac{1}{\sqrt{x}}\exp\left(-\dfrac{a}{x}\right)$	$\sqrt{\dfrac{\pi}{2u}}\,e^{-\sqrt{2au}}\left[\cos\left(\sqrt{2au}\right)-\sin\left(\sqrt{2au}\right)\right]$
12	$\dfrac{1}{x\sqrt{x}}\exp\left(-\dfrac{a}{x}\right)$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\sqrt{2au}}\cos\left(\sqrt{2au}\right)$

28.3.4 Expressions with Hyperbolic Functions

No.	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\dfrac{1}{\cosh(ax)}, \quad a>0$	$\dfrac{\pi}{2a\cosh\left(\frac{1}{2}\pi a^{-1}u\right)}$
2	$\dfrac{1}{\cosh^2(ax)}, \quad a>0$	$\dfrac{\pi u}{2a^2\sinh\left(\frac{1}{2}\pi a^{-1}u\right)}$
3	$\dfrac{\cosh(ax)}{\cosh(bx)}, \quad \lvert a\rvert<b$	$\dfrac{\pi}{b}\left[\dfrac{\cos\left(\frac{1}{2}\pi ab^{-1}\right)\cosh\left(\frac{1}{2}\pi b^{-1}u\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}\right]$
4	$\dfrac{1}{\cosh(ax)+\cos b}$	$\dfrac{\pi\sinh\left(a^{-1}bu\right)}{a\sin b\sinh\left(\pi a^{-1}u\right)}$
5	$\exp(-ax^2)\cosh(bx), \quad a>0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\exp\left(\dfrac{b^2-u^2}{4a}\right)\cos\left(\dfrac{abu}{2}\right)$
6	$\dfrac{x}{\sinh(ax)}$	$\dfrac{\pi^2}{4a^2\cosh^2\left(\frac{1}{2}\pi a^{-1}u\right)}$
7	$\dfrac{\sinh(ax)}{\sinh(bx)}, \quad \lvert a\rvert<b$	$\dfrac{\pi}{2b}\dfrac{\sin\left(\pi ab^{-1}\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}$
8	$\dfrac{1}{x}\tanh(ax), \quad a>0$	$\ln\left[\coth\left(\frac{1}{4}\pi a^{-1}u\right)\right]$

28.3.5 Expressions with Logarithmic Functions

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u)=\int_0^\infty f(x)\cos(ux)\,dx$
1	$\begin{cases}\ln x & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$-\dfrac{1}{u}\,\mathrm{Si}(u)$
2	$\dfrac{\ln x}{\sqrt{x}}$	$-\sqrt{\dfrac{\pi}{2u}}\Bigl[\ln(4u)+\mathcal{C}+\dfrac{\pi}{2}\Bigr]$, $\mathcal{C}=0.5772\ldots$ is the Euler constant
3	$x^{\nu-1}\ln x,\quad 0<\nu<1$	$\Gamma(\nu)\cos\Bigl(\dfrac{\pi\nu}{2}\Bigr)u^{-\nu}\Bigl[\psi(\nu)-\dfrac{\pi}{2}\tan\Bigl(\dfrac{\pi\nu}{2}\Bigr)-\ln u\Bigr]$
4	$\ln\Bigl\lvert\dfrac{a+x}{a-x}\Bigr\rvert,\quad a>0$	$\dfrac{2}{u}\bigl[\cos(au)\,\mathrm{Si}(au)-\sin(au)\,\mathrm{Ci}(au)\bigr]$
5	$\ln\bigl(1+a^2/x^2\bigr),\quad a>0$	$\dfrac{\pi}{u}\bigl(1-e^{-au}\bigr)$
6	$\ln\dfrac{a^2+x^2}{b^2+x^2},\quad a,b>0$	$\dfrac{\pi}{u}\bigl(e^{-bu}-e^{-au}\bigr)$
7	$e^{-ax}\ln x,\quad a>0$	$-\dfrac{aC+\frac{1}{2}a\ln(u^2+a^2)+u\arctan(u/a)}{u^2+a^2}$
8	$\ln\bigl(1+e^{-ax}\bigr),\quad a>0$	$\dfrac{a}{2u^2}-\dfrac{\pi}{2u\sinh\bigl(\pi a^{-1}u\bigr)}$
9	$\ln\bigl(1-e^{-ax}\bigr),\quad a>0$	$\dfrac{a}{2u^2}-\dfrac{\pi}{2u}\coth\bigl(\pi a^{-1}u\bigr)$

28.3.6 Expressions with Trigonometric Functions

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u)=\int_0^\infty f(x)\cos(ux)\,dx$
1	$\dfrac{\sin(ax)}{x},\quad a>0$	$\begin{cases}\frac{1}{2}\pi & \text{if } u<a,\\ \frac{1}{4}\pi & \text{if } u=a,\\ 0 & \text{if } u>a\end{cases}$
2	$x^{\nu-1}\sin(ax),\quad a>0,\ \lvert\nu\rvert<1$	$\pi\,\dfrac{(u+a)^{-\nu}-\lvert u+a\rvert^{-\nu}\,\mathrm{sign}(u-a)}{4\Gamma(1-\nu)\cos\bigl(\frac{1}{2}\pi\nu\bigr)}$
3	$\dfrac{x\sin(ax)}{x^2+b^2},\quad a,b>0$	$\begin{cases}\frac{1}{2}\pi e^{-ab}\cosh(bu) & \text{if } u<a,\\ -\frac{1}{2}\pi e^{-bu}\sinh(ab) & \text{if } u>a\end{cases}$
4	$\dfrac{\sin(ax)}{x(x^2+b^2)},\quad a,b>0$	$\begin{cases}\frac{1}{2}\pi b^{-2}\bigl[1-e^{-ab}\cosh(bu)\bigr] & \text{if } u<a,\\ \frac{1}{2}\pi b^{-2}e^{-bu}\sinh(ab) & \text{if } u>a\end{cases}$
5	$e^{-bx}\sin(ax),\quad a,b>0$	$\dfrac{1}{2}\Bigl[\dfrac{a+u}{(a+u)^2+b^2}+\dfrac{a-u}{(a-u)^2+b^2}\Bigr]$
6	$\dfrac{1}{x}\sin^2(ax),\quad a>0$	$\dfrac{1}{4}\ln\Bigl\lvert 1-4\dfrac{a^2}{u^2}\Bigr\rvert$
7	$\dfrac{1}{x^2}\sin^2(ax),\quad a>0$	$\begin{cases}\frac{1}{4}\pi(2a-u) & \text{if } u<2a,\\ 0 & \text{if } u>2a\end{cases}$
8	$\dfrac{1}{x}\sin\Bigl(\dfrac{a}{x}\Bigr),\quad a>0$	$\dfrac{\pi}{2}J_0\bigl(2\sqrt{au}\bigr)$

No.	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
9	$\dfrac{1}{\sqrt{x}}\sin\bigl(a\sqrt{x}\bigr)\sin\bigl(b\sqrt{x}\bigr), \quad a,b>0$	$\sqrt{\dfrac{\pi}{u}}\,\sin\Bigl(\dfrac{ab}{2u}\Bigr)\sin\Bigl(\dfrac{a^2+b^2}{4u}-\dfrac{\pi}{4}\Bigr)$
10	$\sin\bigl(ax^2\bigr), \quad a>0$	$\sqrt{\dfrac{\pi}{8a}}\left[\cos\Bigl(\dfrac{u^2}{4a}\Bigr)-\sin\Bigl(\dfrac{u^2}{4a}\Bigr)\right]$
11	$\exp\bigl(-ax^2\bigr)\sin\bigl(bx^2\bigr), \quad a>0$	$\dfrac{\sqrt{\pi}}{(A^2+B^2)^{1/4}}\exp\Bigl(-\dfrac{Au^2}{A^2+B^2}\Bigr)\sin\Bigl(\varphi-\dfrac{Bu^2}{A^2+B^2}\Bigr),$ $A=4a,\ B=4b,\ \varphi=\frac{1}{2}\arctan(b/a)$
12	$\dfrac{1-\cos(ax)}{x}, \quad a>0$	$\dfrac{1}{2}\ln\Bigl\lvert 1-\dfrac{a^2}{u^2}\Bigr\rvert$
13	$\dfrac{1-\cos(ax)}{x^2}, \quad a>0$	$\begin{cases}\frac{1}{2}\pi(a-u) & \text{if } u<a,\\ 0 & \text{if } u>a\end{cases}$
14	$x^{\nu-1}\cos(ax), \quad a>0,\ 0<\nu<1$	$\frac{1}{2}\Gamma(\nu)\cos\bigl(\frac{1}{2}\pi\nu\bigr)\bigl[\lvert u-a\rvert^{-\nu}+(u+a)^{-\nu}\bigr]$
15	$\dfrac{\cos(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases}\frac{1}{2}\pi b^{-1}e^{-ab}\cosh(bu) & \text{if } u<a,\\ \frac{1}{2}\pi b^{-1}e^{-bu}\cosh(ab) & \text{if } u>a\end{cases}$
16	$e^{-bx}\cos(ax), \quad a,b>0$	$\dfrac{b}{2}\left[\dfrac{1}{(a+u)^2+b^2}+\dfrac{1}{(a-u)^2+b^2}\right]$
17	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)$	$\sqrt{\dfrac{\pi}{u}}\,\sin\Bigl(\dfrac{a^2}{4u}+\dfrac{\pi}{4}\Bigr)$
18	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)\cos\bigl(b\sqrt{x}\bigr)$	$\sqrt{\dfrac{\pi}{u}}\,\cos\Bigl(\dfrac{ab}{2u}\Bigr)\sin\Bigl(\dfrac{a^2+b^2}{4u}+\dfrac{\pi}{4}\Bigr)$
19	$\exp\bigl(-bx^2\bigr)\cos(ax), \quad b>0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{b}}\exp\Bigl(-\dfrac{a^2+u^2}{4b}\Bigr)\cosh\Bigl(\dfrac{au}{2b}\Bigr)$
20	$\cos\bigl(ax^2\bigr), \quad a>0$	$\sqrt{\dfrac{\pi}{8a}}\,\bigl[\cos\bigl(\frac{1}{4}a^{-1}u^2\bigr)+\sin\bigl(\frac{1}{4}a^{-1}u^2\bigr)\bigr]$
21	$\exp\bigl(-ax^2\bigr)\cos\bigl(bx^2\bigr), \quad a>0$	$\dfrac{\sqrt{\pi}}{(A^2+B^2)^{1/4}}\exp\Bigl(-\dfrac{Au^2}{A^2+B^2}\Bigr)\cos\Bigl(\varphi-\dfrac{Bu^2}{A^2+B^2}\Bigr),$ $A=4a,\ B=4b,\ \varphi=\frac{1}{2}\arctan(b/a)$

28.3.7 Expressions with Special Functions

No.	Original function, $f(x)$	Cosine transform, $\check{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$
1	$\operatorname{Ei}(-ax)$	$-\dfrac{1}{u}\arctan\Bigl(\dfrac{u}{a}\Bigr)$
2	$\operatorname{Ci}(ax)$	$\begin{cases}0 & \text{if } 0<u<a,\\ -\frac{\pi}{2u} & \text{if } a<u\end{cases}$
3	$\operatorname{si}(ax)$	$-\dfrac{1}{2u}\ln\Bigl\lvert\dfrac{u+a}{u-a}\Bigr\rvert, \quad u\neq a$

No.	Original function, $f(x)$	Cosine transform, $\check{f}_{\rm c}(u)=\int_0^\infty f(x)\cos(ux)\,dx$
4	$J_0(ax), \quad a>0$	$\begin{cases} \dfrac{1}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
5	$J_\nu(ax), \quad a>0,\ \nu>-1$	$\begin{cases} \dfrac{\cos[\nu\arcsin(u/a)]}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ -\dfrac{a^\nu\sin(\pi\nu/2)}{\xi(u+\xi)^\nu} & \text{if } a<u, \end{cases}$ where $\xi=\sqrt{u^2-a^2}$
6	$\dfrac{1}{x}J_\nu(ax), \quad a>0,\ \nu>0$	$\begin{cases} \nu^{-1}\cos[\nu\arcsin(u/a)] & \text{if } 0<u<a, \\ \dfrac{a^\nu\cos(\pi\nu/2)}{\nu\bigl(u+\sqrt{u^2-a^2}\bigr)^\nu} & \text{if } a<u \end{cases}$
7	$x^{-\nu}J_\nu(ax), \quad a>0,\ \nu>-\frac{1}{2}$	$\begin{cases} \dfrac{\sqrt{\pi}\,\bigl(a^2-u^2\bigr)^{\nu-1/2}}{(2a)^\nu\Gamma\bigl(\nu+\frac{1}{2}\bigr)} & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
8	$x^{\nu+1}J_\nu(ax)$, $a>0,\ -1<\nu<-\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \dfrac{2^{\nu+1}\sqrt{\pi}\,a^\nu u}{\Gamma\bigl(-\nu-\frac{1}{2}\bigr)(u^2-a^2)^{\nu+3/2}} & \text{if } a<u \end{cases}$
9	$J_0\bigl(a\sqrt{x}\,\bigr), \quad a>0$	$\dfrac{1}{u}\sin\left(\dfrac{a^2}{4u}\right)$
10	$\dfrac{1}{\sqrt{x}}J_1\bigl(a\sqrt{x}\,\bigr), \quad a>0$	$\dfrac{4}{a}\sin^2\left(\dfrac{a^2}{8u}\right)$
11	$x^{\nu/2}J_\nu\bigl(a\sqrt{x}\,\bigr)$, $a>0,\ -1<\nu<\frac{1}{2}$	$\left(\dfrac{a}{2}\right)^\nu u^{-\nu-1}\sin\left(\dfrac{a^2}{4u}-\dfrac{\pi\nu}{2}\right)$
12	$J_0\bigl(a\sqrt{x^2+b^2}\,\bigr)$	$\begin{cases} \dfrac{\cos\bigl(b\sqrt{a^2-u^2}\,\bigr)}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
13	$Y_0(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\dfrac{1}{\sqrt{u^2-a^2}} & \text{if } a<u \end{cases}$
14	$x^\nu Y_\nu(ax), \quad a>0,\ \|\nu\|<\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\dfrac{(2a)^\nu\sqrt{\pi}}{\Gamma\bigl(\frac{1}{2}-\nu\bigr)(u^2-a^2)^{\nu+1/2}} & \text{if } a<u \end{cases}$
15	$K_0\bigl(a\sqrt{x^2+b^2}\,\bigr), \quad a,b>0$	$\dfrac{\pi}{2\sqrt{u^2+a^2}}\exp\bigl(-b\sqrt{u^2+a^2}\,\bigr)$

⊙ Literature for Section 28.3: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965), F. Oberhettinger (1980).

28.4 Tables of Fourier Sine Transforms

28.4.1 General Formulas

No.	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$af_1(x) + bf_2(x)$	$a\check{f}_{1s}(u) + b\check{f}_{2s}(u)$
2	$f(ax), \quad a > 0$	$\frac{1}{a}\check{f}_s\Bigl(\frac{u}{a}\Bigr)$
3	$x^{2n}f(x), \quad n = 1, 2, \ldots$	$(-1)^n \frac{d^{2n}}{du^{2n}}\check{f}_s(u)$
4	$x^{2n+1}f(ax), \quad n = 0, 1, \ldots$	$(-1)^{n+1}\frac{d^{2n+1}}{du^{2n+1}}\check{f}_c(u), \quad \check{f}_c(u) = \int_0^\infty f(x)\cos(xu)\,dx$
5	$f(ax)\cos(bx), \quad a, b > 0$	$\frac{1}{2a}\Bigl[\check{f}_s\Bigl(\frac{u+b}{a}\Bigr) + F_s\Bigl(\frac{u-b}{a}\Bigr)\Bigr]$

28.4.2 Expressions with Power-Law Functions

No.	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\begin{cases} 1 & \text{if } 0 < x < a, \\ 0 & \text{if } a < x \end{cases}$	$\frac{1}{u}\bigl[1 - \cos(au)\bigr]$
2	$\begin{cases} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \\ 0 & \text{if } 2 < x \end{cases}$	$\frac{4}{u^2}\sin u\,\sin^2\frac{u}{2}$
3	$\frac{1}{x}$	$\frac{\pi}{2}$
4	$\frac{1}{a + x}, \quad a > 0$	$\sin(au)\,\mathrm{Ci}(au) - \cos(au)\,\mathrm{si}(au)$
5	$\frac{x}{a^2 + x^2}, \quad a > 0$	$\frac{\pi}{2}e^{-au}$
6	$\frac{1}{x(a^2 + x^2)}, \quad a > 0$	$\frac{\pi}{2a^2}\bigl(1 - e^{-au}\bigr)$
7	$\frac{a}{a^2 + (x - b)^2} - \frac{a}{a^2 + (x + b)^2}$	$\pi e^{-au}\sin(bu)$
8	$\frac{x + b}{a^2 + (x + b)^2} - \frac{x - b}{a^2 + (x - b)^2}$	$\pi e^{-au}\cos(bu)$
9	$\frac{x}{(x^2 + a^2)^n}, \quad a > 0, \ n = 1, 2, \ldots$	$\frac{\pi u e^{-au}}{2^{2n-2}(n-1)!\,a^{2n-3}}\sum_{k=0}^{n-2}\frac{(2n-k-4)!}{k!\,(n-k-2)!}(2au)^k$
10	$\frac{x^{2m+1}}{(x^2 + a)^{n+1}}$, $n, m = 0, 1, \ldots; \ 0 \le m \le n$	$(-1)^{n+m}\frac{\pi}{2n!}\frac{\partial^n}{\partial a^n}\bigl(a^m e^{-u\sqrt{a}}\,\bigr)$
11	$\frac{1}{\sqrt{x}}$	$\sqrt{\frac{\pi}{2u}}$
12	$\frac{1}{x\sqrt{x}}$	$\sqrt{2\pi u}$
13	$x(a^2 + x^2)^{-3/2}$	$uK_0(au)$
14	$\frac{\bigl(\sqrt{a^2 + x^2} - a\bigr)^{1/2}}{\sqrt{a^2 + x^2}}$	$\sqrt{\frac{\pi}{2u}}\,e^{-au}$
15	$x^{-\nu}, \quad 0 < \nu < 2$	$\cos\bigl(\frac{1}{2}\pi\nu\bigr)\Gamma(1 - \nu)u^{\nu-1}$

28.4.3 Expressions with Exponential Functions

No.	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$e^{-ax}, \quad a>0$	$\dfrac{u}{a^2+u^2}$
2	$x^n e^{-ax}, \quad a>0,\ n=1,\,2,\,\ldots$	$n!\Big(\dfrac{a}{a^2+u^2}\Big)^{n+1}\sum_{k=0}^{[n/2]}(-1)^k C_{n+1}^{2k+1}\Big(\dfrac{u}{a}\Big)^{2k+1}$
3	$\dfrac{1}{x}e^{-ax}, \quad a>0$	$\arctan\dfrac{u}{a}$
4	$\sqrt{x}e^{-ax}, \quad a>0$	$\dfrac{\sqrt{\pi}}{2}(a^2+u^2)^{-3/4}\sin\Big(\dfrac{3}{2}\arctan\dfrac{u}{a}\Big)$
5	$\dfrac{1}{\sqrt{x}}e^{-ax}, \quad a>0$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{\big(\sqrt{a^2+u^2}-a\big)^{1/2}}{\sqrt{a^2+u^2}}$
6	$\dfrac{1}{x\sqrt{x}}e^{-ax}, \quad a>0$	$\sqrt{2\pi}\,\big(\sqrt{a^2+u^2}-a\big)^{1/2}$
7	$x^{n-1/2}e^{-ax}, \quad a>0,\ n=1,\,2,\,\ldots$	$(-1)^n\sqrt{\dfrac{\pi}{2}}\,\dfrac{\partial^n}{\partial a^n}\left[\dfrac{\big(\sqrt{a^2+u^2}-a\big)^{1/2}}{\sqrt{a^2+u^2}}\right]$
8	$x^{\nu-1}e^{-ax}, \quad a>0,\ \nu>-1$	$\Gamma(\nu)(a^2+u^2)^{-\nu/2}\sin\Big(\nu\arctan\dfrac{u}{a}\Big)$
9	$x^{-2}\big(e^{-ax}-e^{-bx}\big), \quad a,b>0$	$\dfrac{u}{2}\ln\Big(\dfrac{u^2+b^2}{u^2+a^2}\Big)+b\arctan\Big(\dfrac{u}{b}\Big)-a\arctan\Big(\dfrac{u}{a}\Big)$
10	$\dfrac{1}{e^{ax}+1}, \quad a>0$	$\dfrac{1}{2u}-\dfrac{\pi}{2a\sinh(\pi u/a)}$
11	$\dfrac{1}{e^{ax}-1}, \quad a>0$	$\dfrac{\pi}{2a}\coth\Big(\dfrac{\pi u}{a}\Big)-\dfrac{1}{2u}$
12	$\dfrac{e^{x/2}}{e^x-1}$	$-\tfrac{1}{2}\tanh(\pi u)$
13	$x\exp\big(-ax^2\big)$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}\,u\exp\Big(-\dfrac{u^2}{4a}\Big)$
14	$\dfrac{1}{x}\exp\big(-ax^2\big)$	$\dfrac{\pi}{2}\operatorname{erf}\Big(\dfrac{u}{2\sqrt{a}}\Big)$
15	$\dfrac{1}{\sqrt{x}}\exp\Big(-\dfrac{a}{x}\Big)$	$\sqrt{\dfrac{\pi}{2u}}\,e^{-\sqrt{2au}}\big[\cos\big(\sqrt{2au}\,\big)+\sin\big(\sqrt{2au}\,\big)\big]$
16	$\dfrac{1}{x\sqrt{x}}\exp\Big(-\dfrac{a}{x}\Big)$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\sqrt{2au}}\sin\big(\sqrt{2au}\,\big)$

28.4.4 Expressions with Hyperbolic Functions

No.	Original function, $f(x)$	Sine transform, $\check{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\dfrac{1}{\sinh(ax)}, \quad a>0$	$\dfrac{\pi}{2a}\tanh\big(\tfrac{1}{2}\pi a^{-1}u\big)$
2	$\dfrac{x}{\sinh(ax)}, \quad a>0$	$\dfrac{\pi^2\sinh\big(\tfrac{1}{2}\pi a^{-1}u\big)}{4a^2\cosh^2\big(\tfrac{1}{2}\pi a^{-1}u\big)}$
3	$\dfrac{1}{x}e^{-bx}\sinh(ax), \quad b>\lvert a\rvert$	$\tfrac{1}{2}\arctan\Big(\dfrac{2au}{u^2+b^2-a^2}\Big)$
4	$\dfrac{1}{x\cosh(ax)}, \quad a>0$	$\arctan\big[\sinh\big(\tfrac{1}{2}\pi a^{-1}u\big)\big]$

No.	Original function, $f(x)$	Sine transform, $\check{f}_{\mathrm{s}}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
5	$1-\tanh\left(\frac{1}{2}ax\right), \quad a>0$	$\dfrac{1}{u} - \dfrac{\pi}{a\sinh\left(\pi a^{-1}u\right)}$
6	$\coth\left(\frac{1}{2}ax\right)-1, \quad a>0$	$\dfrac{\pi}{a}\coth\left(\pi a^{-1}u\right) - \dfrac{1}{u}$
7	$\dfrac{\cosh(ax)}{\sinh(bx)}, \quad \lvert a\rvert<b$	$\dfrac{\pi}{2b}\,\dfrac{\sinh\left(\pi b^{-1}u\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}$
8	$\dfrac{\sinh(ax)}{\cosh(bx)}, \quad \lvert a\rvert<b$	$\dfrac{\pi}{b}\,\dfrac{\sin\left(\frac{1}{2}\pi ab^{-1}\right)\sinh\left(\frac{1}{2}\pi b^{-1}u\right)}{\cos\left(\pi ab^{-1}\right)+\cosh\left(\pi b^{-1}u\right)}$

28.4.5 Expressions with Logarithmic Functions

No.	Original function, $f(x)$	Sine transform, $\check{f}_{\mathrm{s}}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\begin{cases}\ln x & \text{if } 0<x<1,\\ 0 & \text{if } 1<x\end{cases}$	$\dfrac{1}{u}\left[\mathrm{Ci}(u)-\ln u-\mathcal{C}\right]$, $\mathcal{C}=0.5772\ldots$ is the Euler constant
2	$\dfrac{\ln x}{x}$	$-\frac{1}{2}\pi(\ln u+\mathcal{C})$
3	$\dfrac{\ln x}{\sqrt{x}}$	$-\sqrt{\dfrac{\pi}{2u}}\left[\ln(4u)+\mathcal{C}-\dfrac{\pi}{2}\right]$
4	$x^{\nu-1}\ln x, \quad \lvert\nu\rvert<1$	$\dfrac{\pi u^{-\nu}\left[\psi(\nu)+\frac{\pi}{2}\cot\left(\frac{\pi\nu}{2}\right)-\ln u\right]}{2\Gamma(1-\nu)\cos\left(\frac{\pi\nu}{2}\right)}$
5	$\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert, \quad a>0$	$\dfrac{\pi}{u}\sin(au)$
6	$\ln\dfrac{(x+b)^2+a^2}{(x-b)^2+a^2}, \quad a,b>0$	$\dfrac{2\pi}{u}e^{-au}\sin(bu)$
7	$e^{-ax}\ln x, \quad a>0$	$\dfrac{a\arctan(u/a)-\frac{1}{2}u\ln(u^2+a^2)-e^{C}u}{u^2+a^2}$
8	$\dfrac{1}{x}\ln\left(1+a^2x^2\right), \quad a>0$	$-\pi\,\mathrm{Ei}\left(-\dfrac{u}{a}\right)$

28.4.6 Expressions with Trigonometric Functions

No.	Original function, $f(x)$	Sine transform, $\check{f}_{\mathrm{s}}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
1	$\dfrac{\sin(ax)}{x}, \quad a>0$	$\dfrac{1}{2}\ln\left\lvert\dfrac{u+a}{u-a}\right\rvert$
2	$\dfrac{\sin(ax)}{x^2}, \quad a>0$	$\begin{cases}\frac{1}{2}\pi u & \text{if } 0<u<a,\\ \frac{1}{2}\pi a & \text{if } u>a\end{cases}$
3	$x^{\nu-1}\sin(ax), \quad a>0,\ -2<\nu<1$	$\pi\,\dfrac{\lvert u-a\rvert^{-\nu}-\lvert u+a\rvert^{-\nu}}{4\Gamma(1-\nu)\sin\left(\frac{1}{2}\pi\nu\right)}, \quad \nu\neq 0$

No.	Original function, $f(x)$	Sine transform, $\check{f}_{\rm s}(u)=\int_0^\infty f(x)\sin(ux)\,dx$
4	$\dfrac{\sin(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} \frac{1}{2}\pi b^{-1}e^{-ab}\sinh(bu) & \text{if } 0<u<a, \\ \frac{1}{2}\pi b^{-1}e^{-bu}\sinh(ab) & \text{if } u>a \end{cases}$
5	$\dfrac{\sin(\pi x)}{1-x^2}$	$\begin{cases} \sin u & \text{if } 0<u<\pi, \\ 0 & \text{if } u>\pi \end{cases}$
6	$e^{-ax}\sin(bx), \quad a>0$	$\dfrac{a}{2}\left[\dfrac{1}{a^2+(b-u)^2}-\dfrac{1}{a^2+(b+u)^2}\right]$
7	$x^{-1}e^{-ax}\sin(bx), \quad a>0$	$\dfrac{1}{4}\ln\dfrac{(u+b)^2+a^2}{(u-b)^2+a^2}$
8	$\dfrac{1}{x}\sin^2(ax), \quad a>0$	$\begin{cases} \frac{1}{4}\pi & \text{if } 0<u<2a, \\ \frac{1}{8}\pi & \text{if } u=2a, \\ 0 & \text{if } u>2a \end{cases}$
9	$\dfrac{1}{x^2}\sin^2(ax), \quad a>0$	$\frac{1}{4}(u+2a)\ln\lvert u+2a\rvert+\frac{1}{4}(u-2a)\ln\lvert u-2a\rvert$ $-\frac{1}{2}u\ln u$
10	$\exp\bigl(-ax^2\bigr)\sin(bx), \quad a>0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\exp\left(-\dfrac{u^2+b^2}{4a}\right)\sinh\left(\dfrac{bu}{2a}\right)$
11	$\dfrac{1}{x}\sin(ax)\sin(bx), \ a\ge b>0$	$\begin{cases} 0 & \text{if } 0<u<a-b, \\ \frac{\pi}{4} & \text{if } a-b<u<a+b, \\ 0 & \text{if } a+b<u \end{cases}$
12	$\sin\left(\dfrac{a}{x}\right), \quad a>0$	$\dfrac{\pi\sqrt{a}}{2\sqrt{u}}J_1\bigl(2\sqrt{au}\bigr)$
13	$\dfrac{1}{\sqrt{x}}\sin\left(\dfrac{a}{x}\right), \quad a>0$	$\sqrt{\dfrac{\pi}{8u}}\left[\sin\bigl(2\sqrt{au}\bigr)-\cos\bigl(2\sqrt{au}\bigr)+\exp\bigl(-2\sqrt{au}\bigr)\right]$
14	$\exp\bigl(-a\sqrt{x}\bigr)\sin\bigl(a\sqrt{x}\bigr), \ a>0$	$a\sqrt{\dfrac{\pi}{8}}\,u^{-3/2}\exp\left(-\dfrac{a^2}{2u}\right)$
15	$\dfrac{\cos(ax)}{x}, \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \frac{1}{4}\pi & \text{if } u=a, \\ \frac{1}{2}\pi & \text{if } a<u \end{cases}$
16	$x^{\nu-1}\cos(ax), \quad a>0, \ \lvert\nu\rvert<1$	$\dfrac{\pi(u+a)^{-\nu}-\operatorname{sign}(u-a)\lvert u-a\rvert^{-\nu}}{4\Gamma(1-\nu)\cos\bigl(\frac{1}{2}\pi\nu\bigr)}$
17	$\dfrac{x\cos(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} -\frac{1}{2}\pi e^{-ab}\sinh(bu) & \text{if } u<a, \\ \frac{1}{2}\pi e^{-bu}\cosh(ab) & \text{if } u>a \end{cases}$
18	$\dfrac{1-\cos(ax)}{x^2}, \quad a>0$	$\dfrac{u}{2}\ln\left\lvert\dfrac{u^2-a^2}{u^2}\right\rvert+\dfrac{a}{2}\ln\left\lvert\dfrac{u+a}{u-a}\right\rvert$
19	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)$	$\sqrt{\dfrac{\pi}{u}}\cos\left(\dfrac{a^2}{4u}+\dfrac{\pi}{4}\right)$
20	$\dfrac{1}{\sqrt{x}}\cos\bigl(a\sqrt{x}\bigr)\cos\bigl(b\sqrt{x}\bigr), \ a,b>0$	$\sqrt{\dfrac{\pi}{u}}\cos\left(\dfrac{ab}{2u}\right)\cos\left(\dfrac{a^2+b^2}{4u}+\dfrac{\pi}{4}\right)$

28.4.7 Expressions with Special Functions

No.	Original function, $f(x)$	Sine transform, $\check{f}_s(u)=\int_0^\infty f(x)\sin(ux)\,dx$
1	$\operatorname{erfc}(ax), \quad a>0$	$\dfrac{1}{u}\left[1-\exp\left(-\dfrac{u^2}{4a^2}\right)\right]$
2	$\operatorname{ci}(ax), \quad a>0$	$-\dfrac{1}{2u}\ln\left\|1-\dfrac{u^2}{a^2}\right\|$
3	$\operatorname{si}(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\frac{1}{2}\pi u^{-1} & \text{if } a<u \end{cases}$
4	$J_0(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \dfrac{1}{\sqrt{u^2-a^2}} & \text{if } a<u \end{cases}$
5	$J_\nu(ax), \quad a>0,\ \nu>-2$	$\begin{cases} \dfrac{\sin\left[\nu\arcsin(u/a)\right]}{\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ \dfrac{a^\nu\cos(\pi\nu/2)}{\xi(u+\xi)^\nu} & \text{if } a<u, \end{cases}$ where $\xi=\sqrt{u^2-a^2}$
6	$\dfrac{1}{x}J_0(ax), \quad a>0,\ \nu>0$	$\begin{cases} \arcsin(u/a) & \text{if } 0<u<a, \\ \pi/2 & \text{if } a<u \end{cases}$
7	$\dfrac{1}{x}J_\nu(ax), \quad a>0,\ \nu>-1$	$\begin{cases} \nu^{-1}\sin\left[\nu\arcsin(u/a)\right] & \text{if } 0<u<a, \\ \dfrac{a^\nu\sin(\pi\nu/2)}{\nu\left(u+\sqrt{u^2-a^2}\right)^\nu} & \text{if } a<u \end{cases}$
8	$x^\nu J_\nu(ax), \quad a>0,\ -1<\nu<\frac{1}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \dfrac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)\left(u^2-a^2\right)^{\nu+1/2}} & \text{if } a<u \end{cases}$
9	$x^{-1}e^{-ax}J_0(bx), \quad a>0$	$\arcsin\left(\dfrac{2u}{\sqrt{(u+b)^2+a^2}+\sqrt{(u-b)^2+a^2}}\right)$
10	$\dfrac{J_0(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} b^{-1}\sinh(bu)K_0(ab) & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
11	$\dfrac{xJ_0(ax)}{x^2+b^2}, \quad a,b>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \frac{1}{2}\pi e^{-bu}I_0(ab) & \text{if } a<u \end{cases}$
12	$\dfrac{\sqrt{x}J_{2n+1/2}(ax)}{x^2+b^2}$, $a,b>0, \quad n=0,\,1,\,2,\,\ldots$	$\begin{cases} (-1)^n\sinh(bu)K_{2n+1/2}(ab) & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
13	$\dfrac{x^\nu J_\nu(ax)}{x^2+b^2}$, $a,b>0, \quad -1<\nu<\frac{5}{2}$	$\begin{cases} b^{\nu-1}\sinh(bu)K_\nu(ab) & \text{if } 0<u<a, \\ 0 & \text{if } a<u \end{cases}$
14	$\dfrac{x^{1-\nu}J_\nu(ax)}{x^2+b^2}$, $a,b>0, \quad \nu>-\frac{3}{2}$	$\begin{cases} 0 & \text{if } 0<u<a, \\ \frac{1}{2}\pi b^{-\nu}e^{-bu}I_\nu(ab) & \text{if } a<u \end{cases}$
15	$J_0\left(a\sqrt{x}\right), \quad a>0$	$\dfrac{1}{u}\cos\left(\dfrac{a^2}{4u}\right)$

No.	Original function, $f(x)$	Sine transform, $\check{f}_{\rm s}(u) = \int_0^\infty f(x)\sin(ux)\,dx$
16	$\dfrac{1}{\sqrt{x}}J_1\bigl(a\sqrt{x}\,\bigr), \quad a>0$	$\dfrac{2}{a}\sin\left(\dfrac{a^2}{4u}\right)$
17	$x^{\nu/2}J_\nu\bigl(a\sqrt{x}\,\bigr), \quad a>0,\ -2<\nu<\frac{1}{2}$	$\dfrac{a^\nu}{2^\nu u^{\nu+1}}\cos\left(\dfrac{a^2}{4u}-\dfrac{\pi\nu}{2}\right)$
18	$Y_0(ax), \quad a>0$	$\begin{cases} \dfrac{2\arcsin(u/a)}{\pi\sqrt{a^2-u^2}} & \text{if } 0<u<a, \\ \dfrac{2\bigl[\ln\bigl(u-\sqrt{u^2-a^2}\bigr)-\ln a\bigr]}{\pi\sqrt{u^2-a^2}} & \text{if } a<u \end{cases}$
19	$Y_1(ax), \quad a>0$	$\begin{cases} 0 & \text{if } 0<u<a, \\ -\dfrac{u}{a\sqrt{u^2-a^2}} & \text{if } a<u \end{cases}$
20	$K_0(ax), \quad a>0$	$\dfrac{\ln\bigl(u+\sqrt{u^2+a^2}\,\bigr)-\ln a}{\sqrt{u^2+a^2}}$
21	$xK_0(ax), \quad a>0$	$\dfrac{\pi u}{2(u^2+a^2)^{3/2}}$
22	$x^{\nu+1}K_\nu(ax), \quad a>0,\ \nu>-\frac{3}{2}$	$\sqrt{\pi}\,(2a)^\nu\Gamma\bigl(\nu+\frac{3}{2}\bigr)u(u^2+a^2)^{-\nu-3/2}$

⊙ Literature for Section 28.4: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), I. I. Hirschman and D. V. Widder (1955), V. A. Ditkin and A. P. Prudnikov (1965), F. Oberhettinger (1980).

Chapter 29

Curvilinear Coordinates, Vectors, Operators, and Differential Relations

29.1 Arbitrary Curvilinear Coordinate Systems

29.1.1 General Nonorthogonal Curvilinear Coordinates

► **Metric tensor. Arc length and volume elements in curvilinear coordinates.**

The curvilinear coordinates x^1, x^2, x^3 are defined as functions of the rectangular Cartesian coordinates x, y, z:

$$x^1 = x^1(x, y, z), \quad x^2 = x^2(x, y, z), \quad x^3 = x^3(x, y, z).$$

Using these formulas, one can express x, y, z in terms of the curvilinear coordinates x^1, x^2, x^3 as follows:

$$x = x(x^1, x^2, x^3), \quad y = y(x^1, x^2, x^3), \quad z = z(x^1, x^2, x^3).$$

The *metric tensor components* g_{ij} are determined by the formulas

$$g_{ij}(x^1, x^2, x^3) = \frac{\partial x}{\partial x^i}\frac{\partial x}{\partial x^j} + \frac{\partial y}{\partial x^i}\frac{\partial y}{\partial x^j} + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j};$$
$$g_{ij}(x^1, x^2, x^3) = g_{ji}(x^1, x^2, x^3); \qquad i, j = 1,\ 2,\ 3.$$

The arc length *dl* between close points $(x, y, z) \equiv (x^1, x^2, x^3)$ and $(x+dx,\ y+dy,\ z+dz) \equiv (x^1 + dx^1,\ x^2 + dx^2,\ x^3 + dx^3)$ is expressed as

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}(x^1, x^2, x^3)\, dx^i\, dx^j.$$

The volume of the elementary parallelepiped with vertices at the eight points (x^1, x^2, x^3), $(x^1 + dx^1,\ x^2,\ x^3)$, $(x^1,\ x^2 + dx^2,\ x^3)$, $(x^1,\ x^2,\ x^3 + dx^3)$, $(x^1 + dx^1,\ x^2 + dx^2,\ x^3)$,

$(x^1 + dx^1,\, x^2,\, x^3 + dx^3)$, $(x^1,\, x^2 + dx^2,\, x^3 + dx^3)$, $(x^1 + dx^1,\, x^2 + dx^2,\, x^3 + dx^3)$ is given by

$$dV = \frac{\partial(x,y,z)}{\partial(x^1,x^2,x^3)}\,dx^1\,dx^2\,dx^3 = \pm\sqrt{\det|g_{ij}|}\,dx^1\,dx^2\,dx^3.$$

Here the plus sign corresponds to the standard situation where the tangent vectors to the coordinate lines x^1, x^2, x^3, pointing in the direction of growth of the respective coordinate, form a right-handed triple, just as unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ of a right-handed rectangular Cartesian coordinate system.

► **Vector components in Cartesian and curvilinear coordinate systems.**

The unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ of a rectangular Cartesian coordinate system* x, y, z and the unit vectors $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ of a curvilinear coordinate system x^1, x^2, x^3 are connected by the linear relations

$$\begin{aligned}
\vec{i}_n &= \frac{1}{\sqrt{g_{nn}}}\left(\frac{\partial x}{\partial x^n}\vec{i} + \frac{\partial y}{\partial x^n}\vec{j} + \frac{\partial z}{\partial x^n}\vec{k}\right), \qquad n = 1,2,3;\\
\vec{i} &= \sqrt{g_{11}}\,\frac{\partial x^1}{\partial x}\vec{i}_1 + \sqrt{g_{22}}\,\frac{\partial x^2}{\partial x}\vec{i}_2 + \sqrt{g_{33}}\,\frac{\partial x^3}{\partial x}\vec{i}_3;\\
\vec{j} &= \sqrt{g_{11}}\,\frac{\partial x^1}{\partial y}\vec{i}_1 + \sqrt{g_{22}}\,\frac{\partial x^2}{\partial y}\vec{i}_2 + \sqrt{g_{33}}\,\frac{\partial x^3}{\partial y}\vec{i}_3;\\
\vec{k} &= \sqrt{g_{11}}\,\frac{\partial x^1}{\partial z}\vec{i}_1 + \sqrt{g_{22}}\,\frac{\partial x^2}{\partial z}\vec{i}_2 + \sqrt{g_{33}}\,\frac{\partial x^3}{\partial z}\vec{i}_3.
\end{aligned}$$

In the general case, the vectors $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ are not orthogonal and change their direction from point to point.

The components v_x, v_y, v_z of a vector $\vec{v}$ in a rectangular Cartesian coordinate system x, y, z and the components v_1, v_2, v_3 of the same vector in a curvilinear coordinate system x^1, x^2, x^3 are related by

$$\begin{aligned}
\vec{v} &= v_x\vec{i} + v_y\vec{j} + v_z\vec{k} = v_1\vec{i}_1 + v_2\vec{i}_2 + v_3\vec{i}_3,\\
v_n &= \sqrt{g_{nn}}\left(\frac{\partial x^n}{\partial x}v_x + \frac{\partial x^n}{\partial y}v_y + \frac{\partial x^n}{\partial z}v_z\right), \qquad n = 1,2,3;\\
v_x &= \frac{\partial x}{\partial x^1}\frac{v_1}{\sqrt{g_{11}}} + \frac{\partial x}{\partial x^2}\frac{v_2}{\sqrt{g_{22}}} + \frac{\partial x}{\partial x^3}\frac{v_3}{\sqrt{g_{33}}};\\
v_y &= \frac{\partial y}{\partial x^1}\frac{v_1}{\sqrt{g_{11}}} + \frac{\partial y}{\partial x^2}\frac{v_2}{\sqrt{g_{22}}} + \frac{\partial y}{\partial x^3}\frac{v_3}{\sqrt{g_{33}}};\\
v_z &= \frac{\partial z}{\partial x^1}\frac{v_1}{\sqrt{g_{11}}} + \frac{\partial z}{\partial x^2}\frac{v_2}{\sqrt{g_{22}}} + \frac{\partial z}{\partial x^3}\frac{v_3}{\sqrt{g_{33}}}.
\end{aligned}$$

*Here and henceforth the coordinate axes and the respective coordinates of points in space are denoted by the same letters.

29.1.2 General Orthogonal Curvilinear Coordinates

► **Orthogonal coordinates. Length, area, and volume elements.**

A system of coordinates is orthogonal if

$$g_{ij}(x^1, x^2, x^3) = 0 \quad \text{for} \quad i \neq j.$$

In this case the third invariant of the metric tensor is given by

$$g = \det|g_{ij}| = g_{11} g_{22} g_{33}.$$

The *Lamé coefficients* L_k of orthogonal curvilinear coordinates are expressed in terms of the components of the metric tensor as

$$L_i = \sqrt{g_{ii}} = \sqrt{\left(\frac{\partial x}{\partial x^i}\right)^2 + \left(\frac{\partial y}{\partial x^i}\right)^2 + \left(\frac{\partial z}{\partial x^i}\right)^2}, \qquad i = 1, 2, 3.$$

Arc length element:

$$dl = \sqrt{(L_1\,dx^1)^2 + (L_2\,dx^2)^2 + (L_3\,dx^3)^2} = \sqrt{g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2}.$$

The area elements ds_i of the respective coordinate surfaces $x^i = \text{const}$ are given by

$$\begin{aligned} ds_1 &= dl_2\,dl_3 = L_2 L_3\,dx^2\,dx^3 = \sqrt{g_{22}g_{33}}\,dx^2\,dx^3, \\ ds_2 &= dl_1\,dl_3 = L_1 L_3\,dx^1\,dx^3 = \sqrt{g_{11}g_{33}}\,dx^1\,dx^3, \\ ds_3 &= dl_1\,dl_2 = L_1 L_2\,dx^1\,dx^2 = \sqrt{g_{11}g_{22}}\,dx^1\,dx^2. \end{aligned}$$

Volume element:

$$dV = L_1 L_2 L_3\,dx^1\,dx^2\,dx^3 = \sqrt{g_{11}g_{22}g_{33}}\;dx^1\,dx^2\,dx^3.$$

► **Basic differential relations in orthogonal curvilinear coordinates.**

In what follows, we present the basic differential operators in the orthogonal curvilinear coordinates x^1, x^2, x^3. The corresponding unit vectors are denoted by $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$.

The gradient of a scalar f is expressed as

$$\operatorname{grad} f \equiv \nabla f = \frac{1}{\sqrt{g_{11}}}\frac{\partial f}{\partial x^1}\vec{i}_1 + \frac{1}{\sqrt{g_{22}}}\frac{\partial f}{\partial x^2}\vec{i}_2 + \frac{1}{\sqrt{g_{33}}}\frac{\partial f}{\partial x^3}\vec{i}_3.$$

Divergence of a vector $\vec{v} = v_1\vec{i}_1 + v_2\vec{i}_2 + v_3\vec{i}_3$:

$$\operatorname{div}\vec{v} \equiv \nabla\cdot\vec{v} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(v_1\sqrt{\frac{g}{g_{11}}}\right) + \frac{\partial}{\partial x^2}\left(v_2\sqrt{\frac{g}{g_{22}}}\right) + \frac{\partial}{\partial x^3}\left(v_3\sqrt{\frac{g}{g_{33}}}\right)\right].$$

Gradient of a scalar f along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)f = \frac{v_1}{\sqrt{g_{11}}}\frac{\partial f}{\partial x^1} + \frac{v_2}{\sqrt{g_{22}}}\frac{\partial f}{\partial x^2} + \frac{v_3}{\sqrt{g_{33}}}\frac{\partial f}{\partial x^3}.$$

Gradient of a vector $\vec{w}$ along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)\vec{w} = \vec{i}_1(\vec{v}\cdot\nabla)w_1 + \vec{i}_2(\vec{v}\cdot\nabla)w_2 + \vec{i}_3(\vec{v}\cdot\nabla)w_3.$$

Curl of a vector $\vec{v}$:

$$\begin{aligned}\operatorname{curl}\vec{v} \equiv \nabla\times\vec{v} = \vec{i}_1\frac{\sqrt{g_{11}}}{\sqrt{g}}\left[\frac{\partial}{\partial x^2}\left(v_3\sqrt{g_{33}}\right) - \frac{\partial}{\partial x^3}\left(v_2\sqrt{g_{22}}\right)\right]\\ + \vec{i}_2\frac{\sqrt{g_{22}}}{\sqrt{g}}\left[\frac{\partial}{\partial x^3}\left(v_1\sqrt{g_{11}}\right) - \frac{\partial}{\partial x^1}\left(v_3\sqrt{g_{33}}\right)\right]\\ + \vec{i}_3\frac{\sqrt{g_{33}}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(v_2\sqrt{g_{22}}\right) - \frac{\partial}{\partial x^2}\left(v_1\sqrt{g_{11}}\right)\right].\end{aligned}$$

Remark 29.1. Sometimes $\operatorname{curl}\vec{v}$ is denoted by $\operatorname{rot}\vec{v}$.

Laplace operator of a scalar f:

$$\Delta f \equiv \nabla^2 f = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{\sqrt{g}}{g_{11}}\frac{\partial f}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{\sqrt{g}}{g_{22}}\frac{\partial f}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{\sqrt{g}}{g_{33}}\frac{\partial f}{\partial x^3}\right)\right].$$

► **Some other differential relations.**

Let f, g and $\vec{v}$, $\vec{u}$ be arbitrary sufficiently smooth functions and vector functions. Then, the following differential relations hold:

$$\begin{aligned}\operatorname{curl}\nabla f &= 0,\\ \operatorname{div}\operatorname{curl}\vec{v} &= 0,\\ \operatorname{div}\nabla f &= \Delta f,\\ \operatorname{curl}\operatorname{curl}\vec{v} &= \nabla\operatorname{div}\vec{v} - \Delta\vec{v},\\ \operatorname{div}(f\vec{v}) &= f\operatorname{div}\vec{v} + \nabla f\cdot\vec{v},\\ \operatorname{curl}(\vec{v}\times\vec{u}) &= \vec{u}\cdot\operatorname{curl}\vec{v} - \vec{v}\cdot\operatorname{curl}\vec{u},\\ \operatorname{curl}(f\vec{v}) &= \nabla f\times\vec{v} + f\operatorname{curl}\vec{v},\\ \Delta(fg) &= f\Delta g + g\Delta f + 2\nabla f\cdot\nabla g,\\ \Delta(f\vec{v}) &= f\Delta\vec{v} + \vec{v}\Delta f + 2(\nabla f\cdot\nabla)\vec{v}.\end{aligned}$$

29.2 Cartesian, Cylindrical, and Spherical Coordinate Systems

29.2.1 Cartesian Coordinates

► **A vector in Cartesian coordinates and the metric tensor components.**

A vector $\vec{v}$ in rectangular Cartesian coordinates x, y, z:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}.$$

Metric tensor components:

$$g_{xx} = g_{yy} = g_{zz} = 1, \quad \sqrt{g} = 1.$$

► **Basic differential relations.**

Gradient of a scalar f:

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

Divergence of a vector $\vec{v}$:

$$\operatorname{div}\vec{v} \equiv \nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

Gradient of a scalar f along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)f = v_x\frac{\partial f}{\partial x} + v_y\frac{\partial f}{\partial y} + v_z\frac{\partial f}{\partial z}.$$

Gradient of a vector $\vec{w}$ along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)\vec{w} = \vec{i}(\vec{v}\cdot\nabla)w_x + \vec{j}(\vec{v}\cdot\nabla)w_y + \vec{k}(\vec{v}\cdot\nabla)w_z.$$

Curl of a vector $\vec{v}$:

$$\operatorname{curl}\vec{v} \equiv \nabla\times\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{k}.$$

Laplacian of a scalar f:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

29.2.2 Cylindrical Coordinates

► **Transformations of coordinates and vectors. The metric tensor components.**

The Cartesian coordinates are expressed in terms of the cylindrical ones as

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z$$
$$(0 \le \rho < \infty,\ \ 0 \le \varphi < 2\pi,\ \ -\infty < z < \infty).$$

The cylindrical coordinates are expressed in terms of the Cartesian ones as

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\varphi = y/x, \quad z = z \qquad (\sin\varphi = y/\rho).$$

Remark 29.2. Sometimes (including this handbook) the cylindrical coordinate ρ is denoted by r.

Coordinate surfaces:

$x^2 + y^2 = \rho^2$ (right circular cylinders with their axis coincident with the z-axis),
$y = x\tan\varphi$ (half-planes through the z-axis),
$z = z$ (planes perpendicular to the z-axis).

Direct and inverse transformations of the components of a vector $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} = v_\rho\vec{i}_\rho + v_\varphi\vec{i}_\varphi + v_z\vec{i}_z$:

$$v_\rho = v_x\cos\varphi + v_y\sin\varphi, \qquad v_x = v_\rho\cos\varphi - v_\varphi\sin\varphi,$$
$$v_\varphi = -v_x\sin\varphi + v_y\cos\varphi, \qquad v_y = v_\rho\sin\varphi + v_\varphi\cos\varphi,$$
$$v_z = v_z; \qquad v_z = v_z.$$

Metric tensor components:

$$g_{\rho\rho} = 1, \quad g_{\varphi\varphi} = \rho^2, \quad g_{zz} = 1, \quad \sqrt{g} = \rho.$$

► **Basic differential relations.**

Gradient of a scalar f:

$$\nabla f = \frac{\partial f}{\partial\rho}\vec{i}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\varphi}\vec{i}_\varphi + \frac{\partial f}{\partial z}\vec{i}_z.$$

Divergence of a vector $\vec{v}$:

$$\operatorname{div}\vec{v} \equiv \nabla\cdot\vec{v} = \frac{1}{\rho}\frac{\partial(\rho v_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z}.$$

Gradient of a scalar f along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)f = v_\rho\frac{\partial f}{\partial\rho} + \frac{v_\varphi}{\rho}\frac{\partial f}{\partial\varphi} + v_z\frac{\partial f}{\partial z}.$$

Gradient of a vector $\vec{w}$ along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)\vec{w} = (\vec{v}\cdot\nabla)w_\rho\vec{i}_\rho + (\vec{v}\cdot\nabla)w_\varphi\vec{i}_\varphi + (\vec{v}\cdot\nabla)w_z\vec{i}_z.$$

Curl of a vector $\vec{v}$:

$$\operatorname{curl}\vec{v} \equiv \nabla\times\vec{v} = \left(\frac{1}{\rho}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\vec{i}_\rho + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial\rho}\right)\vec{i}_\varphi + \frac{1}{\rho}\left[\frac{\partial(\rho v_\varphi)}{\partial\rho} - \frac{\partial v_\rho}{\partial\varphi}\right]\vec{i}_z.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Remark 29.3. The cylindrical coordinates ρ, φ are also used as *polar coordinates* on the xy plane.

29.2.3 Spherical Coordinates

► **Transformations of coordinates and vectors. The metric tensor components.**

The Cartesian coordinates are expressed in terms of the spherical ones as

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$
$$(0 \le r < \infty, \;\; 0 \le \theta \le \pi, \;\; 0 \le \varphi < 2\pi).$$

The spherical coordinates are expressed in terms of the Cartesian ones as

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos\frac{z}{r}, \quad \tan\varphi = \frac{y}{x} \qquad \left(\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}\right).$$

Coordinate surfaces:

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2 && \text{(spheres)},\\ x^2 + y^2 - z^2\tan^2\theta &= 0 && \text{(circular cones)},\\ y &= x\tan\varphi && \text{(half-planes through the } z\text{-axis)}. \end{aligned}$$

Transformation of the components of a vector $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} = v_r\vec{i}_r + v_\theta\vec{i}_\theta + v_\varphi\vec{i}_\varphi$:

$$\begin{aligned} v_r &= v_x\sin\theta\cos\varphi + v_y\sin\theta\sin\varphi + v_z\cos\theta,\\ v_\theta &= v_x\cos\theta\cos\varphi + v_y\cos\theta\sin\varphi - v_z\sin\theta,\\ v_\varphi &= -v_x\sin\varphi + v_y\cos\varphi. \end{aligned}$$

The inverse transformation is

$$\begin{aligned} v_x &= v_r\sin\theta\cos\varphi + v_\theta\cos\theta\cos\varphi - v_\varphi\sin\varphi,\\ v_y &= v_r\sin\theta\sin\varphi + v_\theta\cos\theta\sin\varphi + v_\varphi\cos\varphi,\\ v_z &= v_r\cos\theta - v_\theta\sin\theta. \end{aligned}$$

The metric tensor components are

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2\sin^2\theta, \quad \sqrt{g} = r^2\sin\theta.$$

► **Basic differential relations.**

Gradient of a scalar f:

$$\nabla f = \frac{\partial f}{\partial r}\vec{i}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\vec{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\vec{i}_\varphi.$$

Divergence of a vector $\vec{v}$:

$$\operatorname{div}\vec{v} \equiv \nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\bigl(r^2 v_r\bigr) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\bigl(\sin\theta\, v_\theta\bigr) + \frac{1}{r\sin\varphi}\frac{\partial v_\varphi}{\partial\varphi}.$$

Gradient of a scalar f along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)f = v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi}.$$

Gradient of a vector $\vec{w}$ along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)\vec{w} = (\vec{v}\cdot\nabla)w_r\vec{i}_r + (\vec{v}\cdot\nabla)w_\theta\vec{i}_\theta + (\vec{v}\cdot\nabla)w_\varphi\vec{i}_\varphi.$$

Curl of a vector $\vec{v}$:

$$\operatorname{curl}\vec{v} \equiv \nabla\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, v_\varphi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right]\vec{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial(rv_\varphi)}{\partial r}\right]\vec{i}_\theta + \frac{1}{r}\left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\vec{i}_\varphi.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

29.3 Other Special Orthogonal Coordinates

29.3.1 Coordinates of a Prolate Ellipsoid of Revolution

► **Transformations of coordinates. The metric tensor components.**

1°. Transformations of coordinates:

$$x^2 = a^2(\sigma^2-1)(1-\tau^2)\cos^2\varphi, \quad y^2 = a^2(\sigma^2-1)(1-\tau^2)\sin^2\varphi, \quad z = a\sigma\tau$$
$$(\sigma\ge 1\ge\tau\ge -1,\ 0\le\varphi<2\pi).$$

Coordinate surfaces (the z-axis is the axis of revolution):

$$\frac{x^2+y^2}{a^2(\sigma^2-1)} + \frac{z^2}{a^2\sigma^2} = 1 \qquad \text{(prolate ellipsoids of revolution)},$$
$$\frac{x^2+y^2}{a^2(\tau^2-1)} + \frac{z^2}{a^2\tau^2} = 1 \qquad \text{(hyperboloid of revolution of two sheets)},$$
$$y = x\tan\varphi \qquad \text{(half-planes through the } z\text{-axis)}.$$

Metric tensor components:

$$g_{\sigma\sigma} = a^2\frac{\sigma^2-\tau^2}{\sigma^2-1}, \quad g_{\tau\tau} = a^2\frac{\sigma^2-\tau^2}{1-\tau^2}, \quad g_{\varphi\varphi} = a^2(\sigma^2-1)(1-\tau^2), \quad \sqrt{g} = a^3(\sigma^2-\tau^2).$$

2°. Special coordinate system u, v, φ:

$$\sigma = \cosh u, \quad \tau = \cos v, \quad \varphi = \varphi \quad (0\le u<\infty,\ 0\le v\le\pi,\ 0\le\varphi<2\pi);$$
$$x = a\sinh u\sin v\cos\varphi, \quad y = a\sinh u\sin v\sin\varphi, \quad z = a\cosh u\cos v.$$

Metric tensor components:

$$g_{uu} = g_{vv} = a^2(\sinh^2 u + \sin^2 v), \quad g_{\varphi\varphi} = a^2\sinh^2 u\sin^2 v.$$

► **Basic differential relations.**

Gradient of a scalar f:

$$\nabla f = \frac{1}{a}\sqrt{\frac{\sigma^2-1}{\sigma^2-\tau^2}}\frac{\partial f}{\partial\sigma}\vec{i}_\sigma + \frac{1}{a}\sqrt{\frac{1-\tau^2}{\sigma^2-\tau^2}}\frac{\partial f}{\partial\tau}\vec{i}_\tau + \frac{1}{a\sqrt{(1-\tau^2)(\sigma^2-1)}}\frac{\partial f}{\partial\varphi}\vec{i}_\varphi.$$

Divergence of a vector $\vec{v}$:

$$\nabla\cdot\vec{v} = \frac{1}{a(\sigma^2-\tau^2)}\biggl\{\frac{\partial}{\partial\sigma}\Bigl[v_\sigma\sqrt{(\sigma^2-\tau^2)(\sigma^2-1)}\Bigr] + \frac{\partial}{\partial\tau}\Bigl[v_\tau\sqrt{(\sigma^2-\tau^2)(1-\tau^2)}\Bigr] + \frac{\partial}{\partial\varphi}\biggl[v_\varphi\frac{\sigma^2-\tau^2}{\sqrt{(\sigma^2-1)(1-\tau^2)}}\biggr]\biggr\}.$$

Gradient of a scalar f along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)f = \frac{v_\sigma}{a}\sqrt{\frac{\sigma^2-1}{\sigma^2-\tau^2}}\frac{\partial f}{\partial\sigma} + \frac{v_\tau}{a}\sqrt{\frac{1-\tau^2}{\sigma^2-\tau^2}}\frac{\partial f}{\partial\tau} + \frac{v_\varphi}{a\sqrt{(\sigma^2-1)(1-\tau^2)}}\frac{\partial f}{\partial\varphi}.$$

Gradient of a vector $\vec{w}$ along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)\vec{w} = (\vec{v}\cdot\nabla)w_\sigma\vec{i}_\sigma + (\vec{v}\cdot\nabla)w_\tau\vec{i}_\tau + (\vec{v}\cdot\nabla)w_\varphi\vec{i}_\varphi.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{a^2(\sigma^2-\tau^2)}\biggl\{\frac{\partial}{\partial\sigma}\biggl[(\sigma^2-1)\frac{\partial f}{\partial\sigma}\biggr] + \frac{\partial}{\partial\tau}\biggl[(1-\tau^2)\frac{\partial f}{\partial\tau}\biggr] + \frac{\sigma^2-\tau^2}{(\sigma^2-1)(1-\tau^2)}\frac{\partial^2 f}{\partial\varphi^2}\biggr\}.$$

29.3.2 Coordinates of an Oblate Ellipsoid of Revolution

► **Transformations of coordinates. The metric tensor components.**

1°. Transformations of coordinates:

$$x^2 = a^2(1+\sigma^2)(1-\tau^2)\cos^2\varphi, \quad y^2 = a^2(1+\sigma^2)(1-\tau^2)\sin^2\varphi, \quad z = a\sigma\tau$$
$$(\sigma\ge 0,\ \ -1\le\tau\le 1,\ \ 0\le\varphi<2\pi).$$

Coordinate surfaces (the z-axis is the axis of revolution):

$$\frac{x^2+y^2}{a^2(1-\sigma^2)} + \frac{z^2}{a^2\sigma^2} = 1 \qquad \text{(oblate ellipsoids of revolution)},$$
$$\frac{x^2+y^2}{a^2(1-\tau^2)} - \frac{z^2}{a^2\tau^2} = 1 \qquad \text{(hyperboloid of revolution of one sheet)},$$
$$y = x\tan\varphi \qquad \text{(half-planes through the } z\text{-axis)}.$$

Components of the metric tensor:

$$g_{\sigma\sigma} = a^2\frac{\sigma^2+\tau^2}{1+\sigma^2}, \quad g_{\tau\tau} = a^2\frac{\sigma^2+\tau^2}{1-\tau^2}, \quad g_{\varphi\varphi} = a^2(1+\sigma^2)(1-\tau^2), \quad \sqrt{g} = a^3(\sigma^2+\tau^2).$$

2°. Special coordinate system u, v, φ:

$$\sigma = \sinh u, \quad \tau = \cos v, \quad \varphi = \varphi \quad (0 \le u < \infty,\ 0 \le v \le \pi,\ 0 \le \varphi < 2\pi),$$
$$x = a \cosh u \sin v \cos\varphi, \quad y = a \cosh u \sin v \sin\varphi, \quad z = a \sinh u \cos v.$$

Components of the metric tensor:

$$g_{uu} = g_{vv} = a^2(\sinh^2 u + \cos^2 v), \qquad g_{\varphi\varphi} = a^2 \cosh^2 u \, \sin^2 v.$$

► **Basic differential relations.**

Gradient of a scalar f:

$$\nabla f = \frac{1}{a}\sqrt{\frac{\sigma^2+1}{\sigma^2+\tau^2}}\frac{\partial f}{\partial \sigma}\vec{i}_\sigma + \frac{1}{a}\sqrt{\frac{1-\tau^2}{\sigma^2+\tau^2}}\frac{\partial f}{\partial \tau}\vec{i}_\tau + \frac{1}{a\sqrt{(1-\tau^2)(\sigma^2+1)}}\frac{\partial f}{\partial \varphi}\vec{i}_\varphi.$$

Divergence of a vector $\vec{v}$:

$$\nabla\cdot\vec{v} = \frac{1}{a(\sigma^2+\tau^2)}\left\{\frac{\partial}{\partial\sigma}\left[v_\sigma\sqrt{(\sigma^2+\tau^2)(\sigma^2+1)}\right] + \frac{\partial}{\partial\tau}\left[v_\tau\sqrt{(\sigma^2+\tau^2)(1-\tau^2)}\right] + \frac{\partial}{\partial\varphi}\left[v_\varphi\frac{\sigma^2+\tau^2}{\sqrt{(\sigma^2+1)(1-\tau^2)}}\right]\right\}.$$

Gradient of a scalar f along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)f = \frac{v_\sigma}{a}\sqrt{\frac{\sigma^2+1}{\sigma^2+\tau^2}}\frac{\partial f}{\partial\sigma} + \frac{v_\tau}{a}\sqrt{\frac{1-\tau^2}{\sigma^2+\tau^2}}\frac{\partial f}{\partial\tau} + \frac{v_\varphi}{a\sqrt{(\sigma^2+1)(1-\tau^2)}}\frac{\partial f}{\partial\varphi}.$$

Gradient of a vector $\vec{w}$ along a vector $\vec{v}$:

$$(\vec{v}\cdot\nabla)\vec{w} = (\vec{v}\cdot\nabla)w_\sigma\vec{i}_\sigma + (\vec{v}\cdot\nabla)w_\tau\vec{i}_\tau + (\vec{v}\cdot\nabla)w_\varphi\vec{i}_\varphi.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{a^2(\sigma^2+\tau^2)}\left\{\frac{\partial}{\partial\sigma}\left[(1+\sigma^2)\frac{\partial f}{\partial\sigma}\right] + \frac{\partial}{\partial\tau}\left[(1-\tau^2)\frac{\partial f}{\partial\tau}\right] + \frac{\sigma^2+\tau^2}{(1+\sigma^2)(1-\tau^2)}\frac{\partial^2 f}{\partial\varphi^2}\right\}.$$

29.3.3 Coordinates of an Elliptic Cylinder

1°. Transformations of coordinates:

$$x = a\sigma\tau, \quad y^2 = a^2(\sigma^2-1)(1-\tau^2), \quad z = z$$
$$(\sigma \ge 1,\ -1 \le \tau \le 1,\ -\infty < z < \infty).$$

Coordinate surfaces:

$$\frac{x^2}{a^2\sigma^2} + \frac{y^2}{a^2(\sigma^2-1)} = 1 \qquad \text{(elliptic cylinders)},$$
$$\frac{x^2}{a^2\tau^2} + \frac{y^2}{a^2(\tau^2-1)} = 1 \qquad \text{(hyperbolic cylinders)},$$
$$z = z \qquad \text{(planes parallel to the } xy\text{-plane)}.$$

Components of the metric tensor:

$$g_{\sigma\sigma} = a^2 \frac{\sigma^2 - \tau^2}{\sigma^2 - 1}, \quad g_{\tau\tau} = a^2 \frac{\sigma^2 - \tau^2}{1 - \tau^2}, \quad g_{zz} = 1.$$

2°. Special coordinate system u, v, z:

$$\begin{aligned} &\sigma = \cosh u, \quad \tau = \cos v, \quad z = z; \\ &x = a \cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z \\ &(0 \le u < \infty, \ 0 \le v \le \pi, \ -\infty < z < \infty). \end{aligned}$$

Components of the metric tensor:

$$g_{uu} = g_{vv} = a^2(\sinh^2 u + \sin^2 v), \quad g_{zz} = 1.$$

3°. Laplacian:

$$\begin{aligned} \Delta f &= \frac{1}{a^2(\sinh^2 u + \sin^2 v)} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right) + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{\sqrt{\sigma^2 - 1}}{a^2(\sigma^2 - \tau^2)} \frac{\partial}{\partial \sigma} \left(\sqrt{\sigma^2 - 1}\, \frac{\partial f}{\partial \sigma} \right) + \frac{\sqrt{1 - \tau^2}}{a^2(\sigma^2 - \tau^2)} \frac{\partial}{\partial \tau} \left(\sqrt{1 - \tau^2}\, \frac{\partial f}{\partial \tau} \right) + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

Remark 29.4. The elliptic cylinder coordinates σ, τ are also used as *elliptic coordinates* on the plane xy.

29.3.4 Conical Coordinates

Transformations of coordinates:

$$x = \pm \frac{uvw}{ab}, \quad y^2 = \frac{u^2(v^2 - a^2)(w^2 - a^2)}{a^2(a^2 - b^2)}, \quad z^2 = \frac{u^2(v^2 - b^2)(w^2 - b^2)}{b^2(b^2 - a^2)}$$
$$(b^2 > v^2 > a^2 > w^2).$$

Coordinate surfaces:

$$\begin{aligned} x^2 + y^2 + z^2 &= u^2 && \text{(spheres)}, \\ \frac{x^2}{v^2} + \frac{y^2}{v^2 - a^2} - \frac{z^2}{b^2 - v^2} &= 0 && \text{(cones with their axes coincident with the } z\text{-axis)}, \\ \frac{x^2}{w^2} - \frac{y^2}{a^2 - w^2} - \frac{z^2}{b^2 - w^2} &= 0 && \text{(cones with their axes coincident with the } x\text{-axis)}. \end{aligned}$$

Components of the metric tensor:

$$g_{uu} = 1, \quad g_{vv} = \frac{u^2(v^2 - w^2)}{(v^2 - a^2)(b^2 - v^2)}, \quad g_{ww} = \frac{u^2(v^2 - w^2)}{(a^2 - w^2)(b^2 - w^2)}.$$

29.3.5 Parabolic Cylinder Coordinates

Transformations of coordinates:

$$x = \sigma\tau, \qquad y = \frac{1}{2}(\tau^2 - \sigma^2), \qquad z = z.$$

Coordinate surfaces:

$$\frac{x^2}{\sigma^2} = 2y + \sigma^2 \qquad \text{(right parabolic cylinders with element parallel to the } z\text{-axis)},$$
$$\frac{x^2}{\tau^2} = -2y + \tau^2 \qquad \text{(right parabolic cylinders with element parallel to the } z\text{-axis)},$$
$$z = z \qquad \text{(planes parallel to the } xy\text{-plane)}.$$

Components of the metric tensor:

$$g_{\sigma\sigma} = g_{\tau\tau} = \sigma^2 + \tau^2, \qquad g_{zz} = 1.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{(\sigma^2 + \tau^2)}\left(\frac{\partial^2 f}{\partial \sigma^2} + \frac{\partial^2 f}{\partial \tau^2}\right) + \frac{\partial^2 f}{\partial z^2}.$$

Remark 29.5. The parabolic cylinder coordinates σ, τ are also used as *parabolic coordinates* on the plane xy.

29.3.6 Parabolic Coordinates

Transformations of coordinates:

$$x = \sigma\tau\cos\varphi, \qquad y = \sigma\tau\sin\varphi, \qquad z = \frac{1}{2}(\tau^2 - \sigma^2).$$

Coordinate surfaces (the z-axis is the axis of revolution):

$$\frac{x^2 + y^2}{\sigma^2} = 2z + \sigma^2 \qquad \text{(paraboloids of revolution)},$$
$$\frac{x^2 + y^2}{\tau^2} = -2z + \tau^2 \qquad \text{(paraboloids of revolution)},$$
$$y = x\tan\varphi \qquad \text{(half-planes through the } z\text{-axis)}.$$

Components of the metric tensor:

$$g_{\sigma\sigma} = g_{\tau\tau} = \sigma^2 + \tau^2, \qquad g_{\varphi\varphi} = \sigma^2\tau^2.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{(\sigma^2 + \tau^2)}\left[\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial f}{\partial \sigma}\right) + \frac{1}{\tau}\frac{\partial}{\partial \tau}\left(\tau\frac{\partial f}{\partial \tau}\right) + \left(\frac{1}{\sigma^2} + \frac{1}{\tau^2}\right)\frac{\partial^2 f}{\partial \varphi^2}\right].$$

29.3.7 Bicylindrical Coordinates

Transformations of coordinates:

$$x = \frac{a \sinh \tau}{\cosh \tau - \cos \sigma}, \quad y = \frac{a \sin \sigma}{\cosh \tau - \cos \sigma}, \quad z = z.$$

Coordinate surfaces (abbreviation RCC is "right circular cylinders"):

$$\begin{aligned} x^2 + (y - a \cot \sigma)^2 &= a^2(\cot^2 \sigma + 1) && \text{(RCC with element parallel to the } z\text{-axis)},\\ (x - a \coth \tau)^2 + y^2 &= a^2(\coth^2 \tau - 1) && \text{(RCC with element parallel to the } z\text{-axis)},\\ z &= z && \text{(planes parallel to the } xy\text{-plane)}. \end{aligned}$$

Components of the metric tensor:

$$g_{\sigma\sigma} = g_{\tau\tau} = \frac{a^2}{(\cosh \tau - \cos \sigma)^2}, \quad g_{zz} = 1.$$

Laplacian of a scalar f:

$$\Delta f = \frac{1}{a^2}(\cosh \tau - \cos \sigma)^2 \left(\frac{\partial^2 f}{\partial \sigma^2} + \frac{\partial^2 f}{\partial \tau^2} \right) + \frac{\partial^2 f}{\partial z^2}.$$

Remark 29.6. The bicylindrical coordinates σ, τ are also used as *bipolar coordinates* on the plane xy.

29.3.8 Bipolar Coordinates (in Space)

Transformations of coordinates:

$$x = \frac{a \sin \sigma \cos \varphi}{\cosh \tau - \cos \sigma}, \quad y = \frac{a \sin \sigma \sin \varphi}{\cosh \tau - \cos \sigma}, \quad z = \frac{a \sinh \tau}{\cosh \tau - \cos \sigma}$$

$$(-\infty < \tau < \infty, \ \ 0 \le \sigma < \pi, \ \ 0 \le \varphi < 2\pi).$$

Coordinate surfaces (the z-axis is the axis of revolution):

$$\begin{aligned} x^2 + y^2 + (z - a \coth \tau)^2 &= \frac{a^2}{\sinh^2 \tau} && \text{(spheres with centers on the } z\text{-axis)},\\ \left(\sqrt{x^2 + y^2} - a \cot \sigma\right)^2 + z^2 &= \frac{a^2}{\sin^2 \sigma} && \text{(surfaces obtained by revolution of circular arches}\\ & && (y - a \cot \sigma)^2 + z^2 = a^2/\sin^2 \sigma \text{ about the } z\text{-axis)},\\ y &= x \tan \varphi && \text{(half-planes through the } z\text{-axis)}. \end{aligned}$$

Components of the metric tensor:

$$g_{\sigma\sigma} = g_{\tau\tau} = \frac{a^2}{(\cosh \tau - \cos \sigma)^2}, \quad g_{\varphi\varphi} = \frac{a^2 \sin^2 \sigma}{(\cosh \tau - \cos \sigma)^2}.$$

Laplacian of a scalar f:

$$\Delta f = \frac{(\cosh \tau - \cos \sigma)^2}{a^2 \sin \sigma} \left[\frac{\partial}{\partial \tau} \left(\frac{\sin \sigma}{\cosh \tau - \cos \sigma} \frac{\partial f}{\partial \tau} \right) + \frac{\partial}{\partial \sigma} \left(\frac{\sin \sigma}{\cosh \tau - \cos \sigma} \frac{\partial f}{\partial \sigma} \right) + \frac{1}{\sin \sigma (\cosh \tau - \cos \sigma)} \frac{\partial^2 f}{\partial \varphi^2} \right].$$

29.3.9 Toroidal Coordinates

Transformations of coordinates:

$$x = \frac{a \sinh\tau \cos\varphi}{\cosh\tau - \cos\sigma}, \qquad y = \frac{a \sinh\tau \sin\varphi}{\cosh\tau - \cos\sigma}, \qquad z = \frac{a \sin\sigma}{\cosh\tau - \cos\sigma}$$
$$(-\pi \le \sigma \le \pi,\ \ 0 \le \tau < \infty,\ \ 0 \le \varphi < 2\pi).$$

Coordinate surfaces (the z-axis is the axis of revolution):

$$x^2 + y^2 + (z - a\cot\sigma)^2 = \frac{a^2}{\sin^2\sigma} \qquad \text{(spheres with centers on the } z\text{-axis)},$$
$$\bigl(\sqrt{x^2+y^2} - a\coth\tau\bigr)^2 + z^2 = \frac{a^2}{\sinh^2\tau} \qquad \text{(tori with centers on the } z\text{-axis)},$$
$$y = x\tan\varphi \qquad \text{(half-planes through the } z\text{-axis)}.$$

Components of the metric tensor:

$$g_{\sigma\sigma} = g_{\tau\tau} = \frac{a^2}{(\cosh\tau - \cos\sigma)^2}, \qquad g_{\varphi\varphi} = \frac{a^2\sinh^2\tau}{(\cosh\tau - \cos\sigma)^2}.$$

Laplacian of a scalar f:

$$\Delta f = \frac{(\cosh\tau - \cos\sigma)^2}{a^2\sinh\tau}\biggl[\frac{\partial}{\partial\sigma}\biggl(\frac{\sinh\tau}{\cosh\tau - \cos\sigma}\frac{\partial f}{\partial\sigma}\biggr) + \frac{\partial}{\partial\tau}\biggl(\frac{\sinh\tau}{\cosh\tau - \cos\sigma}\frac{\partial f}{\partial\tau}\biggr) + \frac{1}{\sinh\tau(\cosh\tau - \cos\sigma)}\frac{\partial^2 f}{\partial\varphi^2}\biggr].$$

⊙ *References for Chapter* 29: P. M. Morse and H. Feshbach (1953), D. H. Menzel (1961), P. Moon and D. E. Spencer (1988), G. A. Korn and T. M. Korn (2000), D. Zwillinger (2002), E. W. Weisstein (2003), G. B. Arfken and H. J. Weber (2005), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 30

Special Functions and Their Properties

★ *Throughout Chapter 30 it is assumed that n is a positive integer unless otherwise specified.*

30.1 Some Coefficients, Symbols, and Numbers

30.1.1 Binomial Coefficients

► **Definitions.**

$$C_n^k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad \text{where} \quad k = 1, \dots, n;$$

$$C_a^0 = 1, \quad C_a^k = \binom{a}{k} = (-1)^k \frac{(-a)_k}{k!} = \frac{a(a-1)\dots(a-k+1)}{k!}, \quad \text{where} \quad k = 1, 2, \dots$$

Here a is an arbitrary real number.

► **Generalization. Some properties.**

General case:

$$C_a^b = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, \quad \text{where} \quad \Gamma(x) \quad \text{is the gamma function.}$$

Properties:

$$C_a^0 = 1, \quad C_n^k = 0 \quad \text{for} \quad k = -1, -2, \dots \quad \text{or} \quad k > n,$$

$$C_a^{b+1} = \frac{a}{b+1} C_{a-1}^b = \frac{a-b}{b+1} C_a^b, \quad C_a^b + C_a^{b+1} = C_{a+1}^{b+1},$$

$$C^n_{-1/2} = \frac{(-1)^n}{2^{2n}} C^n_{2n} = (-1)^n \frac{(2n-1)!!}{(2n)!!},$$
$$C^n_{1/2} = \frac{(-1)^{n-1}}{n 2^{2n-1}} C^{n-1}_{2n-2} = \frac{(-1)^{n-1}}{n} \frac{(2n-3)!!}{(2n-2)!!},$$
$$C^{2n+1}_{n+1/2} = (-1)^n 2^{-4n-1} C^n_{2n}, \quad C^n_{2n+1/2} = 2^{-2n} C^{2n}_{4n+1},$$
$$C^{1/2}_n = \frac{2^{2n+1}}{\pi C^n_{2n}}, \quad C^{n/2}_n = \frac{2^{2n}}{\pi} C^{(n-1)/2}_n.$$

30.1.2 Pochhammer Symbol

► **Definition.**

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}.$$

► **Some properties ($k = 1, 2, \dots$).**

$$(a)_0 = 1, \quad (a)_{n+k} = (a)_n (a+n)_k, \quad (n)_k = \frac{(n+k-1)!}{(n-1)!},$$
$$(a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}, \quad \text{where } a \neq 1, \dots, n;$$
$$(1)_n = n!, \quad (1/2)_n = 2^{-2n} \frac{(2n)!}{n!}, \quad (3/2)_n = 2^{-2n} \frac{(2n+1)!}{n!},$$
$$(a+mk)_{nk} = \frac{(a)_{mk+nk}}{(a)_{mk}}, \quad (a+n)_n = \frac{(a)_{2n}}{(a)_n}, \quad (a+n)_k = \frac{(a)_k (a+k)_n}{(a)_n}.$$

30.1.3 Bernoulli Numbers

► **Definition.**

The *Bernoulli numbers* are defined by the recurrence relation

$$B_0 = 1, \quad \sum_{k=0}^{n-1} C^k_n B_k = 0, \quad n = 2,\, 3,\, \dots$$

Numerical values:

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30},$$
$$B_{10} = \tfrac{5}{66}, \quad \dots; \quad B_{2m+1} = 0 \quad \text{for} \quad m = 1, 2, \dots$$

All odd-numbered Bernoulli numbers but B_1 are zero; all even-numbered Bernoulli numbers have alternating signs.

The Bernoulli numbers are the values of Bernoulli polynomials at $x = 0$: $B_n = B_n(0)$.

► **Generating function.**

Generating function:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x| < 2\pi.$$

This relation may be regarded as a definition of the Bernoulli numbers.

The following expansions may be used to calculate the Bernoulli numbers:

$$\tan x = \sum_{n=1}^{\infty} |B_{2n}| \frac{2^{2n}(2^{2n}-1)}{(2n)!} x^{2n}, \qquad |x| < \frac{\pi}{2};$$

$$\cot x = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{2^{2n}}{(2n)!} x^{2n-1}, \qquad |x| < \pi.$$

30.1.4 Euler Numbers

► **Definition.**

The *Euler numbers* E_n are defined by the recurrence relation

$$\sum_{k=0}^{n} C_{2n}^{2k} E_{2k} = 0 \qquad \text{(even numbered)},$$

$$E_{2n+1} = 0 \qquad \text{(odd numbered)},$$

where $n = 0,\ 1,\ \ldots$

Numerical values:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \quad E_{10} = -50251, \quad \ldots,$$
$$E_{2n+1} = 0 \quad \text{for} \quad n = 0,\ 1,\ \ldots$$

All Euler numbers are integers, the odd-numbered Euler numbers are zero, and the even-numbered Euler numbers have alternating signs.

The Euler numbers are expressed via the values of Euler polynomials at $x = 1/2$: $E_n = 2^n E_n(1/2)$, where $n = 0,\ 1,\ \ldots$

► **Generating function. Integral representation.**

Generating function:

$$\frac{e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}, \qquad |x| < 2\pi.$$

This relation may be regarded as a definition of the Euler numbers.

Representation via a definite integral:

$$E_{2n} = (-1)^n 2^{2n+1} \int_0^{\infty} \frac{t^{2n}\,dt}{\cosh(\pi t)}.$$

30.2 Error Functions. Exponential and Logarithmic Integrals

30.2.1 Error Function and Complementary Error Function

► **Integral representations.**

Definitions:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt \quad \text{(error function, also called probability integral)},$$

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)\,dt \quad \text{(complementary error function)}.$$

Properties:

$$\operatorname{erf}(-x) = -\operatorname{erf} x; \quad \operatorname{erf}(0) = 0, \quad \operatorname{erf}(\infty) = 1; \quad \operatorname{erfc}(0) = 1, \quad \operatorname{erfc}(\infty) = 0.$$

► **Expansions as $x \to 0$ and $x \to \infty$. Definite integral.**

Expansion of $\operatorname{erf} x$ into series in powers of x as $x \to 0$:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{k!\,(2k+1)} = \frac{2}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!}.$$

Asymptotic expansion of $\operatorname{erfc} x$ as $x \to \infty$:

$$\operatorname{erfc} x = \frac{1}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \left[\sum_{m=0}^{M-1} (-1)^m \frac{\bigl(\frac{1}{2}\bigr)_m}{x^{2m+1}} + O\bigl(|x|^{-2M-1}\bigr) \right], \qquad M = 1, 2, \ldots$$

Integral:

$$\int_0^x \operatorname{erf} t\,dt = x \operatorname{erf} x - \frac{1}{2} + \frac{1}{2} \exp(-x^2).$$

30.2.2 Exponential Integral

► **Integral representations.**

Definition:

$$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt = -\int_{-x}^\infty \frac{e^{-t}}{t}\,dt \qquad \text{for} \quad x < 0,$$

$$\operatorname{Ei}(x) = \lim_{\varepsilon \to +0} \left(\int_{-\infty}^{-\varepsilon} \frac{e^t}{t}\,dt + \int_{\varepsilon}^x \frac{e^t}{t}\,dt \right) \quad \text{for} \quad x > 0.$$

Other integral representations:

$$\operatorname{Ei}(-x) = -e^{-x} \int_0^\infty \frac{x \sin t + t \cos t}{x^2 + t^2}\, dt \qquad \text{for} \quad x > 0,$$

$$\operatorname{Ei}(-x) = e^{-x} \int_0^\infty \frac{x \sin t - t \cos t}{x^2 + t^2}\, dt \qquad \text{for} \quad x < 0,$$

$$\operatorname{Ei}(-x) = -x \int_1^\infty e^{-xt} \ln t\, dt \qquad \text{for} \quad x > 0,$$

$$\operatorname{Ei}(x) = \mathcal{C} + \ln x + \int_0^x \frac{e^t - 1}{t}\, dt \qquad \text{for} \quad x > 0,$$

where $\mathcal{C} = 0.5772\ldots$ is the Euler constant.

► **Expansions as $x \to 0$ and $x \to \infty$.**

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Ei}(x) = \begin{cases} \mathcal{C} + \ln(-x) + \displaystyle\sum_{k=1}^\infty \frac{x^k}{k!\,k} & \text{if } x < 0, \\ \mathcal{C} + \ln x + \displaystyle\sum_{k=1}^\infty \frac{x^k}{k!\,k} & \text{if } x > 0. \end{cases}$$

Asymptotic expansion as $x \to \infty$:

$$\operatorname{Ei}(-x) = e^{-x} \sum_{k=1}^n (-1)^k \frac{(k-1)!}{x^k} + R_n, \qquad R_n < \frac{n!}{x^n}.$$

30.2.3 Logarithmic Integral

► **Integral representations.**

Definition:

$$\operatorname{li}(x) == \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln t} & \text{if } 0 < x < 1, \\ \displaystyle\lim_{\varepsilon \to +0} \left(\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right) & \text{if } x > 1. \end{cases}$$

► **Limiting properties. Relation to the exponential integral.**

For small x,

$$\operatorname{li}(x) \approx \frac{x}{\ln(1/x)}.$$

For large x,

$$\operatorname{li}(x) \approx \frac{x}{\ln x}.$$

Asymptotic expansion as $x \to 1$:

$$\operatorname{li}(x) = \mathcal{C} + \ln|\ln x| + \sum_{k=1}^{\infty} \frac{\ln^k x}{k!\,k}.$$

Relation to the exponential integral:

$$\operatorname{li} x = \operatorname{Ei}(\ln x), \qquad x < 1;$$
$$\operatorname{li}(e^x) = \operatorname{Ei}(x), \qquad x < 0.$$

30.3 Sine Integral and Cosine Integral. Fresnel Integrals

30.3.1 Sine Integral

► Integral representations. Properties.

Definition:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \operatorname{si}(x) = -\int_x^{\infty} \frac{\sin t}{t}\,dt = \operatorname{Si}(x) - \frac{\pi}{2}.$$

Specific values:

$$\operatorname{Si}(0) = 0, \quad \operatorname{Si}(\infty) = \frac{\pi}{2}, \quad \operatorname{si}(\infty) = 0.$$

Properties:

$$\operatorname{Si}(-x) = -\operatorname{Si}(x), \quad \operatorname{si}(x) + \operatorname{si}(-x) = -\pi, \quad \lim_{x\to-\infty} \operatorname{si}(x) = -\pi.$$

► Expansions as $x \to 0$ and $x \to \infty$.

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Si}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)\,(2k-1)!}.$$

Asymptotic expansion as $x \to \infty$:

$$\operatorname{si}(x) = -\cos x \left[\sum_{m=0}^{M-1} \frac{(-1)^m (2m)!}{x^{2m+1}} + O\big(|x|^{-2M-1}\big)\right] + \sin x \left[\sum_{m=1}^{N-1} \frac{(-1)^m (2m-1)!}{x^{2m}} + O\big(|x|^{-2N}\big)\right],$$

where $M, N = 1, 2, \ldots$

30.3.2 Cosine Integral

► **Integral representation.**

Definition:

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt = \mathcal{C} + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt,$$

where $\mathcal{C} = 0.5772\ldots$ is the Euler constant.

► **Expansions as $x \to 0$ and $x \to \infty$.**

Expansion into series in powers of x as $x \to 0$:

$$\mathrm{Ci}(x) = \mathcal{C} + \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2k\,(2k)!}.$$

Asymptotic expansion as $x \to \infty$:

$$\begin{aligned}\mathrm{Ci}(x) = \cos x &\left[\sum_{m=1}^{M-1} \frac{(-1)^m (2m-1)!}{x^{2m}} + O\big(|x|^{-2M}\big)\right]\\ &+ \sin x \left[\sum_{m=0}^{N-1} \frac{(-1)^m (2m)!}{x^{2m+1}} + O\big(|x|^{-2N-1}\big)\right],\end{aligned}$$

where $M, N = 1, 2, \ldots$

30.3.3 Fresnel Integrals

► **Integral representation.**

Definitions:

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\,dt = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \sin t^2\,dt,$$

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\,dt = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \cos t^2\,dt.$$

► **Expansions as $x \to 0$ and $x \to \infty$.**

Expansion into series in powers of x as $x \to 0$:

$$S(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(4k+3)\,(2k+1)!},$$

$$C(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(4k+1)\,(2k)!}.$$

Asymptotic expansion as $x \to \infty$:

$$S(x) = \frac{1}{2} - \frac{\cos x}{\sqrt{2\pi x}} P(x) - \frac{\sin x}{\sqrt{2\pi x}} Q(x),$$
$$C(x) = \frac{1}{2} + \frac{\sin x}{\sqrt{2\pi x}} P(x) - \frac{\cos x}{\sqrt{2\pi x}} Q(x),$$
$$P(x) = 1 - \frac{1 \times 3}{(2x)^2} + \frac{1 \times 3 \times 5 \times 7}{(2x)^4} - \cdots, \quad Q(x) = \frac{1}{2x} - \frac{1 \times 3 \times 5}{(2x)^3} + \cdots.$$

30.4 Gamma Function, Psi Function, and Beta Function

30.4.1 Gamma Function

► **Integral representations. Simplest properties.**

The gamma function, $\Gamma(z)$, is an analytic function of the complex argument z everywhere except for the points $z = 0, -1, -2, \dots$

For $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For $-(n+1) < \operatorname{Re} z < -n$, where $n = 0, 1, 2, \dots,$

$$\Gamma(z) = \int_0^\infty \left[e^{-t} - \sum_{m=0}^{n} \frac{(-1)^m}{m!} \right] t^{z-1}\, dt.$$

Simplest properties:

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!, \quad \Gamma(1) = \Gamma(2) = 1.$$

Fractional values of the argument:

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}, \qquad \Gamma\Big(n + \frac{1}{2}\Big) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,$$
$$\Gamma\Big(-\frac{1}{2}\Big) = -2\sqrt{\pi}, \qquad \Gamma\Big(\frac{1}{2} - n\Big) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}.$$

► **Euler, Stirling, and other formulas.**

Euler formula:

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)} \qquad (z \neq 0, -1, -2, \dots).$$

Symmetry formulas:

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$
$$\Gamma\Big(\frac{1}{2} + z\Big)\Gamma\Big(\frac{1}{2} - z\Big) = \frac{\pi}{\cos(\pi z)}.$$

Multiple argument formulas:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\Bigl(z+\frac{1}{2}\Bigr),$$
$$\Gamma(3z) = \frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma\Bigl(z+\frac{1}{3}\Bigr)\Gamma\Bigl(z+\frac{2}{3}\Bigr),$$
$$\Gamma(nz) = (2\pi)^{(1-n)/2}n^{nz-1/2}\prod_{k=0}^{n-1}\Gamma\Bigl(z+\frac{k}{n}\Bigr).$$

Asymptotic expansion (*Stirling formula*):

$$\Gamma(z) = \sqrt{2\pi}\,e^{-z}z^{z-1/2}\bigl[1+\tfrac{1}{12}z^{-1}+\tfrac{1}{288}z^{-2}+O(z^{-3})\bigr] \qquad (|\arg z|<\pi).$$

30.4.2 Psi Function (Digamma Function)

► **Definition. Integral representations.**

Definition:

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma'_z(z)}{\Gamma(z)}.$$

The psi function is the logarithmic derivative of the gamma function and is also called the *digamma function*.

Integral representations ($\operatorname{Re} z > 0$):

$$\psi(z) = \int_0^\infty \bigl[e^{-t}-(1+t)^{-z}\bigr]t^{-1}\,dt,$$
$$\psi(z) = \ln z + \int_0^\infty \bigl[t^{-1}-(1-e^{-t})^{-1}\bigr]e^{-tz}\,dt,$$
$$\psi(z) = -\mathcal{C} + \int_0^1 \frac{1-t^{z-1}}{1-t}\,dt,$$

where $\mathcal{C} = -\psi(1) = 0.5772\ldots$ is the Euler constant.

Values for integer argument:

$$\psi(1) = -\mathcal{C}, \qquad \psi(n) = -\mathcal{C} + \sum_{k=1}^{n-1}k^{-1} \quad (n=2,3,\ldots).$$

► **Properties. Asymptotic expansion as $z \to \infty$.**

Functional relations:

$$\psi(z) - \psi(1+z) = -\frac{1}{z},$$
$$\psi(z) - \psi(1-z) = -\pi\cot(\pi z),$$
$$\psi(z) - \psi(-z) = -\pi\cot(\pi z) - \frac{1}{z},$$
$$\psi\bigl(\tfrac{1}{2}+z\bigr) - \psi\bigl(\tfrac{1}{2}-z\bigr) = \pi\tan(\pi z),$$
$$\psi(mz) = \ln m + \frac{1}{m}\sum_{k=0}^{m-1}\psi\Bigl(z+\frac{k}{m}\Bigr).$$

Asymptotic expansion as $z \to \infty$ ($|\arg z| < \pi$):

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots = \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}},$$

where the B_{2n} are Bernoulli numbers.

30.4.3 Beta Function

► **Integral representation. Relationship with the gamma function.**

Definition:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

where $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$.

Relationship with the gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

► **Some properties.**

$$B(x,y) = B(y,x);$$
$$B(x,y+1) = \frac{y}{x}B(x+1,y) = \frac{y}{x+y}B(x,y);$$
$$B(x,1-x) = \frac{\pi}{\sin(\pi x)}, \quad 0 < x < 1;$$
$$\frac{1}{B(n,m)} = mC_{n+m-1}^{n-1} = nC_{n+m-1}^{m-1},$$

where n and m are positive integers.

30.5 Incomplete Gamma and Beta Functions

30.5.1 Incomplete Gamma Function

► Integral representations. Recurrence formulas.

Definitions:

$$\gamma(\alpha,x)=\int_0^x e^{-t}t^{\alpha-1}\,dt,\qquad \operatorname{Re}\alpha>0,$$
$$\Gamma(\alpha,x)=\int_x^\infty e^{-t}t^{\alpha-1}\,dt=\Gamma(\alpha)-\gamma(\alpha,x).$$

Recurrence formulas:

$$\gamma(\alpha+1,x)=\alpha\gamma(\alpha,x)-x^\alpha e^{-x},$$
$$\gamma(\alpha+1,x)=(x+\alpha)\gamma(\alpha,x)+(1-\alpha)x\gamma(\alpha-1,x),$$
$$\Gamma(\alpha+1,x)=\alpha\Gamma(\alpha,x)+x^\alpha e^{-x}.$$

Special cases:

$$\gamma(n+1,x)=n!\left[1-e^{-x}\left(\sum_{k=0}^{n}\frac{x^k}{k!}\right)\right],\qquad n=0,\,1,\,\ldots;$$
$$\Gamma(n+1,x)=n!\,e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!},\qquad n=0,\,1,\,\ldots;$$
$$\Gamma(-n,x)=\frac{(-1)^n}{n!}\left[\Gamma(0,x)-e^{-x}\sum_{k=0}^{n-1}(-1)^k\frac{k!}{x^{k+1}}\right],\quad n=1,\,2,\,\ldots$$

► Expansions as $x\to 0$ and $x\to\infty$. Relation to other functions.

Asymptotic expansions as $x\to 0$:

$$\gamma(\alpha,x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)},$$
$$\Gamma(\alpha,x)=\Gamma(\alpha)-\sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)}.$$

Asymptotic expansions as $x\to\infty$:

$$\gamma(\alpha,x)=\Gamma(\alpha)-x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(1-\alpha)_m}{(-x)^m}+O\bigl(|x|^{-M}\bigr)\right],$$
$$\Gamma(\alpha,x)=x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(1-\alpha)_m}{(-x)^m}+O\bigl(|x|^{-M}\bigr)\right]\quad\bigl(-\tfrac{3}{2}\pi<\arg x<\tfrac{3}{2}\pi\bigr).$$

Asymptotic formulas as $\alpha \to \infty$:

$$\gamma(x,\alpha) = \Gamma(\alpha)\Big[\Phi\big(2\sqrt{x}-\sqrt{\alpha-1}\,\big)+O\Big(\frac{1}{\sqrt{\alpha}}\Big)\Big], \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\Big(-\frac{1}{2}t^2\Big)\,dt;$$

$$\gamma(x,\alpha) = \Gamma(\alpha)\Big[\Phi\big(3\sqrt{\alpha}\,z\big)+O\Big(\frac{1}{\alpha}\Big)\Big], \quad z = \Big(\frac{x}{\alpha}\Big)^{1/3}-1+\frac{1}{9\alpha}.$$

Representation of the error function, complementary error function, and exponential integral in terms of the gamma functions:

$$\operatorname{erf} x = \frac{1}{\sqrt{\pi}}\gamma\Big(\frac{1}{2},\, x^2\Big), \quad \operatorname{erfc} x = \frac{1}{\sqrt{\pi}}\Gamma\Big(\frac{1}{2},\, x^2\Big), \quad \operatorname{Ei}(-x) = -\Gamma(0,x).$$

30.5.2 Incomplete Beta Function

► Integral representation.

Definitions:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \qquad I_x(a,b) = \frac{B_x(a,b)}{B(a,b)},$$

where $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$, and $B(a,b) = B_1(a,b)$ is the beta function.

► Some properties.

Symmetry:

$$I_x(a,b) + I_{1-x}(b,a) = 1.$$

Recurrence formulas:

$$\begin{aligned}
&I_x(a,b) = xI_x(a-1,b) + (1-x)I_x(a,b-1),\\
&(a+b)I_x(a,b) = aI_x(a+1,b) + bI_x(a,b+1),\\
&(a+b-ax)I_x(a,b) = a(1-x)I_x(a+1,b-1) + bI_x(a,b+1).
\end{aligned}$$

30.6 Bessel Functions (Cylindrical Functions)

30.6.1 Definitions and Basic Formulas

► Bessel functions of the first and the second kind.

The *Bessel function of the first kind*, $J_\nu(x)$, and the *Bessel function of the second kind*, $Y_\nu(x)$ (also called the *Neumann function*), are solutions of the Bessel equation

$$x^2y''_{xx} + xy'_x + (x^2-\nu^2)y = 0$$

and are defined by the formulas

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \quad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}. \tag{1}$$

The formula for $Y_\nu(x)$ is valid for $\nu \neq 0, \pm1, \pm2, \ldots$ (the cases $\nu \neq 0, \pm1, \pm2, \ldots$ are discussed in what follows).

The general solution of the Bessel equation has the form $Z_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$ and is called the *cylinder function*.

► **Some formulas.**

$$
\begin{gathered}
2\nu Z_\nu(x) = x[Z_{\nu-1}(x) + Z_{\nu+1}(x)],\\
\frac{d}{dx} Z_\nu(x) = \frac{1}{2}[Z_{\nu-1}(x) - Z_{\nu+1}(x)] = \pm\Big[\frac{\nu}{x} Z_\nu(x) - Z_{\nu\pm1}(x)\Big],\\
\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \quad \frac{d}{dx}[x^{-\nu} Z_\nu(x)] = -x^{-\nu} Z_{\nu+1}(x),\\
\left(\frac{1}{x}\frac{d}{dx}\right)^n [x^\nu J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x), \quad \left(\frac{1}{x}\frac{d}{dx}\right)^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x),\\
J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, \ldots
\end{gathered}
$$

► **Bessel functions for $\nu = \pm n \pm \frac{1}{2}$, where $n = 0, 1, 2, \ldots$**

$$
J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x,
$$

$$
J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{1}{x}\sin x - \cos x\Big), \qquad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(-\frac{1}{x}\cos x - \sin x\Big),
$$

$$
\begin{aligned}
J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}} \Bigg[&\sin\Big(x - \frac{n\pi}{2}\Big) \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}}\\
&+ \cos\Big(x - \frac{n\pi}{2}\Big) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\Bigg],\\
J_{-n-1/2}(x) = \sqrt{\frac{2}{\pi x}} \Bigg[&\cos\Big(x + \frac{n\pi}{2}\Big) \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}}\\
&- \sin\Big(x + \frac{n\pi}{2}\Big) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\Bigg],
\end{aligned}
$$

$$
Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x, \qquad Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,
$$

$$
Y_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x), \qquad Y_{-n-1/2}(x) = (-1)^n J_{n+1/2}(x),
$$

where $[A]$ is the integer part of the number A.

► **Bessel functions for $\nu = \pm n$, where $n = 0, 1, 2, \ldots$**

Let $\nu = n$ be an arbitrary integer. The relations

$$J_{-n}(x) = (-1)^n J_n(x), \qquad Y_{-n}(x) = (-1)^n Y_n(x)$$

are valid. The function $J_n(x)$ is given by the first formula in (1) with $\nu = n$, and $Y_n(x)$ can be obtained from the second formula in (1) by proceeding to the limit $\nu \to n$. For nonnegative n, $Y_n(x)$ can be represented in the form

$$\begin{aligned} Y_n(x) = \frac{2}{\pi} J_n(x) \ln\frac{x}{2} - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(\frac{2}{x}\Big)^{n-2k} \\ - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\Big(\frac{x}{2}\Big)^{n+2k}\frac{\psi(k+1)+\psi(n+k+1)}{k!\,(n+k)!}, \end{aligned}$$

where $\psi(1) = -\mathcal{C}$, $\psi(n) = -\mathcal{C} + \sum\limits_{k=1}^{n-1} k^{-1}$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant, and $\psi(x) = [\ln\Gamma(x)]'_x$ is the logarithmic derivative of the gamma function, also known as the digamma function.

► **Wronskians and similar formulas.**

$$\begin{aligned} W(J_\nu, J_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \quad W(J_\nu, Y_\nu) = \frac{2}{\pi x}, \\ J_\nu(x)J_{-\nu+1}(x) + J_{-\nu}(x)J_{\nu-1}(x) = \frac{2\sin(\pi\nu)}{\pi x}, \\ J_\nu(x)Y_{\nu+1}(x) - J_{\nu+1}(x)Y_\nu(x) = -\frac{2}{\pi x}. \end{aligned}$$

Here, the notation $W(f,g) = fg'_x - f'_x g$ is used.

30.6.2 Integral Representations and Asymptotic Expansions

► **Integral representations.**

The functions $J_\nu(x)$ and $Y_\nu(x)$ can be represented in the form of definite integrals (for $x > 0$):

$$\begin{aligned} \pi J_\nu(x) = \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu\int_0^\infty \exp(-x\sinh t - \nu t)\,dt, \\ \pi Y_\nu(x) = \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu)\,e^{-x\sinh t}\,dt. \end{aligned}$$

For $|\nu| < \frac{1}{2}$, $x > 0$,

$$\begin{aligned} J_\nu(x) = \frac{2^{1+\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\sin(xt)\,dt}{(t^2-1)^{\nu+1/2}}, \\ Y_\nu(x) = -\frac{2^{1+\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\cos(xt)\,dt}{(t^2-1)^{\nu+1/2}}. \end{aligned}$$

For $\nu > -\frac{1}{2}$,

$$J_\nu(x) = \frac{2(x/2)^\nu}{\pi^{1/2}\Gamma(\frac{1}{2}+\nu)} \int_0^{\pi/2} \cos(x\cos t)\sin^{2\nu} t\, dt \quad (\textit{Poisson's formula}).$$

For $\nu = 0$, $x > 0$,

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh t)\, dt, \qquad Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\, dt.$$

For integer $\nu = n = 0, 1, 2, \dots,$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\, dt \quad (\textit{Bessel's formula}),$$

$$J_{2n}(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin t)\cos(2nt)\, dt,$$

$$J_{2n+1}(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin t)\sin[(2n+1)t]\, dt.$$

► **Asymptotic expansions as $|x| \to \infty$.**

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\biggl\{\cos\Bigl(\frac{4x-2\nu\pi-\pi}{4}\Bigr)\biggl[\sum_{m=0}^{M-1}(-1)^m(\nu,2m)(2x)^{-2m} + O(|x|^{-2M})\biggr]$$
$$- \sin\Bigl(\frac{4x-2\nu\pi-\pi}{4}\Bigr)\biggl[\sum_{m=0}^{M-1}(-1)^m(\nu,2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\biggr]\biggr\},$$

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\biggl\{\sin\Bigl(\frac{4x-2\nu\pi-\pi}{4}\Bigr)\biggl[\sum_{m=0}^{M-1}(-1)^m(\nu,2m)(2x)^{-2m} + O(|x|^{-2M})\biggr]$$
$$+ \cos\Bigl(\frac{4x-2\nu\pi-\pi}{4}\Bigr)\biggl[\sum_{m=0}^{M-1}(-1)^m(\nu,2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\biggr]\biggr\},$$

where $(\nu,m) = \dfrac{1}{2^{2m}m!}(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2] = \dfrac{\Gamma(\frac{1}{2}+\nu+m)}{m!\,\Gamma(\frac{1}{2}+\nu-m)}$.

For nonnegative integer n and large x,

$$\sqrt{\pi x}\, J_{2n}(x) = (-1)^n(\cos x + \sin x) + O(x^{-2}),$$
$$\sqrt{\pi x}\, J_{2n+1}(x) = (-1)^{n+1}(\cos x - \sin x) + O(x^{-2}).$$

► **Asymptotic for large ν ($\nu \to \infty$).**

$$J_\nu(x) \simeq \frac{1}{\sqrt{2\pi\nu}}\Bigl(\frac{ex}{2\nu}\Bigr)^\nu, \quad Y_\nu(x) \simeq -\sqrt{\frac{2}{\pi\nu}}\Bigl(\frac{ex}{2\nu}\Bigr)^{-\nu},$$

where x is fixed,

$$J_\nu(\nu) \simeq \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)}\frac{1}{\nu^{1/3}}, \quad Y_\nu(\nu) \simeq -\frac{2^{1/3}}{3^{1/6}\Gamma(2/3)}\frac{1}{\nu^{1/3}}.$$

► **Integrals with Bessel functions.**

Let $F(a, b, c; x)$ denote the hypergeometric series (see Section 30.10.1). Then

$$\int_0^x x^\lambda J_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\,\Gamma(\nu+1)}\,F\biggl(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ -\frac{x^2}{4}\biggr),$$

where $\operatorname{Re}(\lambda+\nu) > -1$, and

$$\begin{aligned}\int_0^x x^\lambda Y_\nu(x)\,dx = &-\frac{\cos(\nu\pi)\Gamma(-\nu)}{2^\nu\pi(\lambda+\nu+1)}x^{\lambda+\nu+1}\,F\biggl(\frac{\lambda+\nu+1}{2},\ \nu+1,\frac{\lambda+\nu+3}{2};\ -\frac{x^2}{4}\biggr)\\ &-\frac{2^\nu\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}\,F\biggl(\frac{\lambda-\nu+1}{2},\ 1-\nu,\frac{\lambda-\nu+3}{2};\ -\frac{x^2}{4}\biggr),\end{aligned}$$

where $\operatorname{Re}\lambda > |\operatorname{Re}\nu| - 1$.

30.6.3 Zeros and Orthogonality Properties of Bessel Functions

► **Zeros of Bessel functions.**

Each of the functions $J_\nu(x)$ and $Y_\nu(x)$ has infinitely many real zeros (for real ν). All zeros are simple, except possibly for the point $x = 0$.

The zeros γ_m of $J_0(x)$, i.e., the roots of the equation $J_0(\gamma_m) = 0$, are approximately given by

$$\gamma_m = 2.4 + 3.13\,(m-1) \qquad (m = 1, 2, \dots),$$

with a maximum error of 0.2%.

► **Orthogonality properties of Bessel functions.**

1°. Let $\mu = \mu_m$ be positive roots of the Bessel function $J_\nu(\mu)$, where $\nu > -1$ and $m = 1,\ 2,\ 3,\ \dots$ Then the set of functions $J_\nu(\mu_m r/a)$ is orthogonal on the interval $0 \le r \le a$ with weight r:

$$\int_0^a J_\nu\Bigl(\frac{\mu_m r}{a}\Bigr)J_\nu\Bigl(\frac{\mu_k r}{a}\Bigr)r\,dr = \begin{cases}0 & \text{if } m \ne k,\\ \frac{1}{2}a^2\bigl[J_\nu'(\mu_m)\bigr]^2 = \frac{1}{2}a^2 J_{\nu+1}^2(\mu_m) & \text{if } m = k.\end{cases}$$

2°. Let $\mu = \mu_m$ be positive zeros of the Bessel function derivative $J_\nu'(\mu)$, where $\nu > -1$ and $m = 1,\ 2,\ 3,\ \dots$ Then the set of functions $J_\nu(\mu_m r/a)$ is orthogonal on the interval $0 \le r \le a$ with weight r:

$$\int_0^a J_\nu\Bigl(\frac{\mu_m r}{a}\Bigr)J_\nu\Bigl(\frac{\mu_k r}{a}\Bigr)r\,dr = \begin{cases}0 & \text{if } m \ne k,\\ \dfrac{1}{2}a^2\biggl(1-\dfrac{\nu^2}{\mu_m^2}\biggr)J_\nu^2(\mu_m) & \text{if } m = k.\end{cases}$$

3°. Let $\mu = \mu_m$ be positive roots of the transcendental equation $\mu J_\nu'(\mu) + sJ_\nu(\mu) = 0$, where $\nu > -1$ and $m = 1,\ 2,\ 3,\ \dots$ Then the set of functions $J_\nu(\mu_m r/a)$ is orthogonal on the interval $0 \le r \le a$ with weight r:

$$\int_0^a J_\nu\Big(\frac{\mu_m r}{a}\Big) J_\nu\Big(\frac{\mu_k r}{a}\Big) r\,dr = \begin{cases} 0 & \text{if } m \neq k, \\ \dfrac{1}{2}a^2\Big(1+\dfrac{s^2-\nu^2}{\mu_m^2}\Big) J_\nu^2(\mu_m) & \text{if } m = k. \end{cases}$$

4°. Let $\mu = \mu_m$ be positive roots of the transcendental equation

$$J_\nu(\lambda_m b) Y_\nu(\lambda_m a) - J_\nu(\lambda_m a) Y_\nu(\lambda_m b) = 0 \qquad (\nu > -1,\ m = 1,\ 2,\ 3,\ \ldots).$$

Then the set of functions

$$Z_\nu(\lambda_m r) = J_\nu(\lambda_m r) Y_\nu(\lambda_m a) - J_\nu(\lambda_m a) Y_\nu(\lambda_m r), \qquad m = 1,\ 2,\ 3,\ \ldots\ ;$$

satisfying the conditions $Z_\nu(\lambda_m a) = Z_\nu(\lambda_m b) = 0$ is orthogonal on the interval $a \le r \le b$ with weight r:

$$\int_a^b Z_\nu(\lambda_m r) Z_\nu(\lambda_k r) r\,dr = \begin{cases} 0 & \text{if } m \neq k, \\ \dfrac{2}{\pi^2 \lambda_m^2} \dfrac{J_\nu^2(\lambda_m a) - J_\nu^2(\lambda_m b)}{J_\nu^2(\lambda_m b)} & \text{if } m = k. \end{cases}$$

5°. Let $\mu = \mu_m$ be positive roots of the transcendental equation

$$J'_\nu(\lambda_m b) Y'_\nu(\lambda_m a) - J'_\nu(\lambda_m a) Y'_\nu(\lambda_m b) = 0 \qquad (\nu > -1,\ m = 1,\ 2,\ 3,\ \ldots).$$

Then the set of functions

$$Z_\nu(\lambda_m r) = J_\nu(\lambda_m r) Y'_\nu(\lambda_m a) - J'_\nu(\lambda_m a) Y_\nu(\lambda_m r), \qquad m = 1,\ 2,\ 3,\ \ldots\ ;$$

satisfying the conditions $Z'_\nu(\lambda_m a) = Z'_\nu(\lambda_m b) = 0$ is orthogonal on the interval $a \le r \le b$ with weight r:

$$\int_a^b Z_\nu(\lambda_m r) Z_\nu(\lambda_k r) r\,dr = \begin{cases} 0 & \text{if } m \neq k, \\ \dfrac{2}{\pi^2 \lambda_m^2}\bigg[\Big(1-\dfrac{\nu^2}{b^2\lambda_m^2}\Big)\dfrac{\big[J'_\nu(\lambda_m a)\big]^2}{\big[J'_\nu(\lambda_m b)\big]^2} - \Big(1-\dfrac{\nu^2}{a^2\lambda_m^2}\Big)\bigg] & \text{if } m = k. \end{cases}$$

30.6.4 Hankel Functions (Bessel Functions of the Third Kind)

► **Definition.**

The *Hankel functions of the first kind and the second kind* are related to Bessel functions by

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z),$$

where $i^2 = -1$.

► **Expansions as $z \to 0$ and $z \to \infty$.**

Asymptotics for $z \to 0$:

$$H_0^{(1)}(z) \simeq \frac{2i}{\pi} \ln z, \qquad H_\nu^{(1)}(z) \simeq -\frac{i}{\pi} \frac{\Gamma(\nu)}{(z/2)^\nu} \qquad (\operatorname{Re}\nu > 0),$$
$$H_0^{(2)}(z) \simeq -\frac{2i}{\pi} \ln z, \quad H_\nu^{(2)}(z) \simeq \frac{i}{\pi} \frac{\Gamma(\nu)}{(z/2)^\nu} \qquad (\operatorname{Re}\nu > 0).$$

Asymptotics for $|z| \to \infty$:

$$H_\nu^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp\bigl[i\bigl(z - \tfrac{1}{2}\pi\nu - \tfrac{1}{4}\pi\bigr)\bigr] \qquad (-\pi < \arg z < 2\pi),$$
$$H_\nu^{(2)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp\bigl[-i\bigl(z - \tfrac{1}{2}\pi\nu - \tfrac{1}{4}\pi\bigr)\bigr] \quad (-2\pi < \arg z < \pi).$$

30.7 Modified Bessel Functions

30.7.1 Definitions. Basic Formulas

► **Modified Bessel functions of the first and the second kind.**

The *modified Bessel functions of the first kind*, $I_\nu(x)$, and the *modified Bessel functions of the second kind*, $K_\nu(x)$ (also called the *Macdonald function*), of order ν are solutions of the modified Bessel equation

$$x^2 y''_{xx} + x y'_x - (x^2 + \nu^2) y = 0$$

and are defined by the formulas

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma(\nu + k + 1)}, \qquad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)},$$

(see below for $K_\nu(x)$ with $\nu = 0, 1, 2, \ldots$).

► **Some formulas.**

The modified Bessel functions possess the properties

$$K_{-\nu}(x) = K_\nu(x), \quad I_{-n}(x) = (-1)^n I_n(x) \quad (n = 0, 1, 2, \ldots),$$
$$2\nu I_\nu(x) = x[I_{\nu-1}(x) - I_{\nu+1}(x)], \quad 2\nu K_\nu(x) = -x[K_{\nu-1}(x) - K_{\nu+1}(x)],$$
$$\frac{d}{dx} I_\nu(x) = \frac{1}{2}[I_{\nu-1}(x) + I_{\nu+1}(x)], \quad \frac{d}{dx} K_\nu(x) = -\frac{1}{2}[K_{\nu-1}(x) + K_{\nu+1}(x)].$$

► **Modified Bessel functions for $\nu = \pm n \pm \frac{1}{2}$, where $n = 0, 1, 2, \dots$**

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x, \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x,$$

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(-\frac{1}{x}\sinh x + \cosh x\Big), \quad I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(-\frac{1}{x}\cosh x + \sinh x\Big),$$

$$I_{n+1/2}(x) = \frac{1}{\sqrt{2\pi x}}\Bigg[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} - (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}\Bigg],$$

$$I_{-n-1/2}(x) = \frac{1}{\sqrt{2\pi x}}\Bigg[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} + (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}\Bigg],$$

$$K_{\pm 1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}, \qquad K_{\pm 3/2}(x) = \sqrt{\frac{\pi}{2x}}\Big(1 + \frac{1}{x}\Big)e^{-x},$$

$$K_{n+1/2}(x) = K_{-n-1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}.$$

► **Modified Bessel functions for $\nu = n$, where $n = 0, 1, 2, \dots$**

If $\nu = n$ is a nonnegative integer, then

$$K_n(x) = (-1)^{n+1} I_n(x) \ln\frac{x}{2} + \frac{1}{2}\sum_{m=0}^{n-1} (-1)^m \Big(\frac{x}{2}\Big)^{2m-n} \frac{(n-m-1)!}{m!}$$
$$+ \frac{1}{2}(-1)^n \sum_{m=0}^{\infty} \Big(\frac{x}{2}\Big)^{n+2m} \frac{\psi(n+m+1) + \psi(m+1)}{m!\,(n+m)!},$$

where $\psi(z)$ is the logarithmic derivative of the gamma function; for $n = 0$, the first sum is dropped.

► **Wronskians and similar formulas.**

$$W(I_\nu, I_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \quad W(I_\nu, K_\nu) = -\frac{1}{x},$$

$$I_\nu(x) I_{-\nu+1}(x) - I_{-\nu}(x) I_{\nu-1}(x) = -\frac{2\sin(\pi\nu)}{\pi x},$$

$$I_\nu(x) K_{\nu+1}(x) + I_{\nu+1}(x) K_\nu(x) = \frac{1}{x},$$

where $W(f, g) = f g'_x - f'_x g$.

30.7.2 Integral Representations and Asymptotic Expansions

► **Integral representations.**

The functions $I_\nu(x)$ and $K_\nu(x)$ can be represented in terms of definite integrals:

$$I_\nu(x) = \frac{x^\nu}{\pi^{1/2}2^\nu\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}\exp(-xt)(1-t^2)^{\nu-1/2}\,dt \qquad (x>0,\ \nu>-\tfrac{1}{2}),$$

$$K_\nu(x) = \int_0^\infty \exp(-x\cosh t)\cosh(\nu t)\,dt \qquad (x>0),$$

$$K_\nu(x) = \frac{1}{\cos\bigl(\frac{1}{2}\pi\nu\bigr)}\int_0^\infty \cos(x\sinh t)\cosh(\nu t)\,dt \qquad (x>0,\ -1<\nu<1),$$

$$K_\nu(x) = \frac{1}{\sin\bigl(\frac{1}{2}\pi\nu\bigr)}\int_0^\infty \sin(x\sinh t)\sinh(\nu t)\,dt \qquad (x>0,\ -1<\nu<1).$$

For integer $\nu=n$,

$$I_n(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos t)\cos(nt)\,dt \qquad (n=0,1,2,\dots),$$

$$K_0(x) = \int_0^\infty \cos(x\sinh t)\,dt = \int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}}\,dt \qquad (x>0).$$

► **Asymptotic expansions as $x\to\infty$.**

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\biggl\{1+\sum_{m=1}^{M}(-1)^m\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m}\biggr\},$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\biggl\{1+\sum_{m=1}^{M}\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m}\biggr\}.$$

The terms of the order of $O(x^{-M-1})$ are omitted in the braces.

► **Integrals with modified Bessel functions.**

Let $F(a,b,c;x)$ denote the hypergeometric series (see Section 30.10.1). Then

$$\int_0^x x^\lambda I_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\Gamma(\nu+1)}\,F\biggl(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ \frac{x^2}{4}\biggr),$$

where $\operatorname{Re}(\lambda+\nu)>-1$, and

$$\begin{aligned}\int_0^x x^\lambda K_\nu(x)\,dx &= \frac{2^{\nu-1}\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}F\biggl(\frac{\lambda-\nu+1}{2},\ 1-\nu,\ \frac{\lambda-\nu+3}{2};\ \frac{x^2}{4}\biggr)\\ &\quad+\frac{2^{-\nu-1}\Gamma(-\nu)}{\lambda+\nu+1}x^{\lambda+\nu+1}F\biggl(\frac{\lambda+\nu+1}{2},\ 1+\nu,\ \frac{\lambda+\nu+3}{2};\ \frac{x^2}{4}\biggr),\end{aligned}$$

where $\operatorname{Re}\lambda>|\operatorname{Re}\nu|-1$.

30.8 Airy Functions

30.8.1 Definition and Basic Formulas

► **Airy functions of the first and the second kinds.**

The *Airy function of the first kind*, $\mathrm{Ai}(x)$, and the *Airy function of the second kind*, $\mathrm{Bi}(x)$, are solutions of the Airy equation

$$y''_{xx} - xy = 0$$

and are defined by the formulas

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\bigl(\tfrac{1}{3}t^3 + xt\bigr)\,dt,$$
$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \bigl[\exp\bigl(-\tfrac{1}{3}t^3 + xt\bigr) + \sin\bigl(\tfrac{1}{3}t^3 + xt\bigr)\bigr]\,dt.$$

Wronskian: $W\{\mathrm{Ai}(x), \mathrm{Bi}(x)\} = 1/\pi$.

► **Relation to the Bessel functions and the modified Bessel functions.**

$$\mathrm{Ai}(x) = \tfrac{1}{3}\sqrt{x}\,\bigl[I_{-1/3}(z) - I_{1/3}(z)\bigr] = \pi^{-1}\sqrt{\tfrac{1}{3}x}\,K_{1/3}(z), \qquad z = \tfrac{2}{3}x^{3/2},$$
$$\mathrm{Ai}(-x) = \tfrac{1}{3}\sqrt{x}\,\bigl[J_{-1/3}(z) + J_{1/3}(z)\bigr],$$
$$\mathrm{Bi}(x) = \sqrt{\tfrac{1}{3}x}\,\bigl[I_{-1/3}(z) + I_{1/3}(z)\bigr],$$
$$\mathrm{Bi}(-x) = \sqrt{\tfrac{1}{3}x}\,\bigl[J_{-1/3}(z) - J_{1/3}(z)\bigr].$$

30.8.2 Power Series and Asymptotic Expansions

► **Power series expansions as $x \to 0$.**

$$\mathrm{Ai}(x) = c_1 f(x) - c_2 g(x),$$
$$\mathrm{Bi}(x) = \sqrt{3}\,[c_1 f(x) + c_2 g(x)],$$
$$f(x) = 1 + \frac{1}{3!}x^3 + \frac{1\times 4}{6!}x^6 + \frac{1\times 4\times 7}{9!}x^9 + \cdots = \sum_{k=0}^{\infty} 3^k \bigl(\tfrac{1}{3}\bigr)_k \frac{x^{3k}}{(3k)!},$$
$$g(x) = x + \frac{2}{4!}x^4 + \frac{2\times 5}{7!}x^7 + \frac{2\times 5\times 8}{10!}x^{10} + \cdots = \sum_{k=0}^{\infty} 3^k \bigl(\tfrac{2}{3}\bigr)_k \frac{x^{3k+1}}{(3k+1)!},$$

where $c_1 = 3^{-2/3}/\Gamma(2/3) \approx 0.3550$ and $c_2 = 3^{-1/3}/\Gamma(1/3) \approx 0.2588$.

► **Asymptotic expansions as $x \to \infty$.**

For large values of x, the leading terms of asymptotic expansions of the Airy functions are

$$\begin{aligned} \operatorname{Ai}(x) &\simeq \tfrac{1}{2}\pi^{-1/2}x^{-1/4}\exp(-z), \qquad z = \tfrac{2}{3}x^{3/2}, \\ \operatorname{Ai}(-x) &\simeq \pi^{-1/2}x^{-1/4}\sin\left(z + \tfrac{\pi}{4}\right), \\ \operatorname{Bi}(x) &\simeq \pi^{-1/2}x^{-1/4}\exp(z), \\ \operatorname{Bi}(-x) &\simeq \pi^{-1/2}x^{-1/4}\cos\left(z + \tfrac{\pi}{4}\right). \end{aligned}$$

30.9 Degenerate Hypergeometric Functions (Kummer Functions)

30.9.1 Definitions and Basic Formulas

► **Degenerate hypergeometric functions $\Phi(a, b; x)$ and $\Psi(a, b; x)$.**

The *degenerate hypergeometric functions* (*Kummer functions*) $\Phi(a, b; x)$ and $\Psi(a, b; x)$ are solutions of the degenerate hypergeometric equation

$$xy''_{xx} + (b - x)y'_x - ay = 0.$$

In the case $b \neq 0, -1, -2, -3, \ldots$, the function $\Phi(a, b; x)$ can be represented as Kummer's series:

$$\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!},$$

where $(a)_k = a(a + 1)\ldots(a + k - 1)$, $(a)_0 = 1$.

Table 30.1 presents some special cases where Φ can be expressed in terms of simpler functions.

The function $\Psi(a, b; x)$ is defined as follows:

$$\Psi(a, b; x) = \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)}\Phi(a, b; x) + \frac{\Gamma(b - 1)}{\Gamma(a)}x^{1-b}\Phi(a - b + 1,\ 2 - b;\ x).$$

Table 30.2 presents some special cases where Ψ can be expressed in terms of simpler functions.

► **Kummer transformation and linear relations.**

Kummer transformation:

$$\Phi(a, b; x) = e^x\Phi(b - a, b; -x), \qquad \Psi(a, b; x) = x^{1-b}\Psi(1 + a - b, 2 - b; x).$$

TABLE 30.1
Special cases of the Kummer function $\Phi(a,b;z)$

a	b	z	Φ	Conventional notation
a	a	x	e^x	
1	2	$2x$	$\frac{1}{x}e^x \sinh x$	
a	$a+1$	$-x$	$ax^{-a}\gamma(a,x)$	Incomplete gamma function $\gamma(a,x)=\int_0^x e^{-t}t^{a-1}\,dt$
$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{\sqrt{\pi}}{2}\operatorname{erf} x$	Error function $\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt$
$-n$	$\frac{1}{2}$	$\frac{x^2}{2}$	$\frac{n!}{(2n)!}\left(-\frac{1}{2}\right)^{-n}H_{2n}(x)$	Hermite polynomials $H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$, $n=0,1,2,\dots$
$-n$	$\frac{3}{2}$	$\frac{x^2}{2}$	$\frac{n!}{(2n+1)!}\left(-\frac{1}{2}\right)^{-n}H_{2n+1}(x)$	
$-n$	b	x	$\frac{n!}{(b)_n}L_n^{(b-1)}(x)$	Laguerre polynomials $L_n^{(\alpha)}(x)=\frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$, $\alpha=b-1$, $(b)_n=b(b+1)\dots(b+n-1)$
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\Gamma(1+\nu)e^x\left(\frac{x}{2}\right)^{-\nu}I_\nu(x)$	Modified Bessel functions $I_\nu(x)$
$n+1$	$2n+2$	$2x$	$\Gamma\left(n+\frac{3}{2}\right)e^x\left(\frac{x}{2}\right)^{-n-\frac{1}{2}}I_{n+\frac{1}{2}}(x)$	

TABLE 30.2
Special cases of the Kummer function $\Psi(a,b;z)$

a	b	z	Ψ	Conventional notation
$1-a$	$1-a$	x	$e^x\Gamma(a,x)$	Incomplete gamma function $\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt$
$\frac{1}{2}$	$\frac{1}{2}$	x^2	$\sqrt{\pi}\exp(x^2)\operatorname{erfc} x$	Complementary error function $\operatorname{erfc} x=\frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt$
1	1	$-x$	$-e^{-x}\operatorname{Ei}(x)$	Exponential integral $\operatorname{Ei}(x)=\int_{-\infty}^x \frac{e^t}{t}\,dt$
1	1	$-\ln x$	$-x^{-1}\operatorname{li} x$	Logarithmic integral $\operatorname{li} x=\int_0^x \frac{dt}{t}$
$\frac{1}{2}-\frac{n}{2}$	$\frac{3}{2}$	x^2	$2^{-n}x^{-1}H_n(x)$	Hermite polynomials $H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$, $n=0,1,2,\dots$
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\pi^{-1/2}(2x)^{-\nu}e^x K_\nu(x)$	Modified Bessel functions $K_\nu(x)$

Linear relations for Φ:

$$(b-a)\Phi(a-1,b;x)+(2a-b+x)\Phi(a,b;x)-a\Phi(a+1,b;x)=0,$$
$$b(b-1)\Phi(a,b-1;x)-b(b-1+x)\Phi(a,b;x)+(b-a)x\Phi(a,b+1;x)=0,$$
$$(a-b+1)\Phi(a,b;x)-a\Phi(a+1,b;x)+(b-1)\Phi(a,b-1;x)=0,$$
$$b\Phi(a,b;x)-b\Phi(a-1,b;x)-x\Phi(a,b+1;x)=0,$$
$$b(a+x)\Phi(a,b;x)-(b-a)x\Phi(a,b+1;x)-ab\Phi(a+1,b;x)=0,$$
$$(a-1+x)\Phi(a,b;x)+(b-a)\Phi(a-1,b;x)-(b-1)\Phi(a,b-1;x)=0.$$

Linear relations for Ψ:

$$\Psi(a-1,b;x)-(2a-b+x)\Psi(a,b;x)+a(a-b+1)\Psi(a+1,b;x)=0,$$
$$(b-a-1)\Psi(a,b-1;x)-(b-1+x)\Psi(a,b;x)+x\Psi(a,b+1;x)=0,$$
$$\Psi(a,b;x)-a\Psi(a+1,b;x)-\Psi(a,b-1;x)=0,$$
$$(b-a)\Psi(a,b;x)-x\Psi(a,b+1;x)+\Psi(a-1,b;x)=0,$$
$$(a+x)\Psi(a,b;x)+a(b-a-1)\Psi(a+1,b;x)-x\Psi(a,b+1;x)=0,$$
$$(a-1+x)\Psi(a,b;x)-\Psi(a-1,b;x)+(a-c+1)\Psi(a,b-1;x)=0.$$

► **Differentiation formulas and Wronskian.**

Differentiation formulas:

$$\frac{d}{dx}\Phi(a,b;x)=\frac{a}{b}\Phi(a+1,b+1;x),\qquad \frac{d^n}{dx^n}\Phi(a,b;x)=\frac{(a)_n}{(b)_n}\Phi(a+n,b+n;x),$$
$$\frac{d}{dx}\Psi(a,b;x)=-a\Psi(a+1,b+1;x),\qquad \frac{d^n}{dx^n}\Psi(a,b;x)=(-1)^n(a)_n\Psi(a+n,b+n;x).$$

Wronskian:

$$W(\Phi,\Psi)=\Phi\Psi'_x-\Phi'_x\Psi=-\frac{\Gamma(b)}{\Gamma(a)}x^{-b}e^x.$$

► **Degenerate hypergeometric functions for $n=0,1,2,\ldots$**

$$\Psi(a,n+1;x)=\frac{(-1)^{n-1}}{n!\,\Gamma(a-n)}\biggl\{\Phi(a,n{+}1;x)\ln x+\frac{(n-1)!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r}{(1-n)_r}\frac{x^{r-n}}{r!}$$
$$+\sum_{r=0}^{\infty}\frac{(a)_r}{(n+1)_r}\bigl[\psi(a+r)-\psi(1+r)-\psi(1+n+r)\bigr]\frac{x^r}{r!}\biggr\},$$

where $n=0,\ 1,\ 2,\ \ldots$ (the last sum is dropped for $n=0$), $\psi(z)=[\ln\Gamma(z)]'_z$ is the logarithmic derivative of the gamma function,

$$\psi(1)=-\mathcal{C},\qquad \psi(n)=-\mathcal{C}+\sum_{k=1}^{n-1}k^{-1},$$

where $\mathcal{C}=0.5772\ldots$ is the Euler constant.

If $b < 0$, then the formula

$$\Psi(a,b;x) = x^{1-b}\Psi(a-b+1,\ 2-b;\ x)$$

is valid for any x.

For $b \neq 0,\ -1,\ -2,\ -3,\ \ldots$, the general solution of the degenerate hypergeometric equation can be represented in the form

$$y = C_1\Phi(a,b;x) + C_2\Psi(a,b;x),$$

and for $b = 0, -1, -2, -3, \ldots$, in the form

$$y = x^{1-b}\bigl[C_1\Phi(a-b+1,\ 2-b;\ x) + C_2\Psi(a-b+1,\ 2-b;\ x)\bigr].$$

30.9.2 Integral Representations and Asymptotic Expansions

► **Integral representations.**

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}\,dt \qquad (\text{for } \ b > a > 0),$$

$$\Psi(a,b;x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{b-a-1}\,dt \qquad (\text{for } \ a > 0,\ x > 0),$$

where $\Gamma(a)$ is the gamma function.

► **Asymptotic expansion as $|x| \to \infty$.**

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)}e^x x^{a-b}\left[\sum_{n=0}^{N}\frac{(b-a)_n(1-a)_n}{n!}x^{-n} + \varepsilon\right], \quad x > 0,$$

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(b-a)}(-x)^{-a}\left[\sum_{n=0}^{N}\frac{(a)_n(a-b+1)_n}{n!}(-x)^{-n} + \varepsilon\right], \quad x < 0,$$

$$\Psi(a,b;x) = x^{-a}\left[\sum_{n=0}^{N}(-1)^n\frac{(a)_n(a-b+1)_n}{n!}x^{-n} + \varepsilon\right], \quad -\infty < x < \infty,$$

where $\varepsilon = O(x^{-N-1})$.

► **Integrals with degenerate hypergeometric functions.**

$$\int \Phi(a,b;x)\,dx = \frac{b-1}{a-1}\Psi(a-1,b-1;x) + C,$$

$$\int \Psi(a,b;x)\,dx = \frac{1}{1-a}\Psi(a-1,b-1;x) + C,$$

$$\int x^n\Phi(a,b;x)\,dx = n!\sum_{k=1}^{n+1}\frac{(-1)^{k+1}(1-b)_k x^{n-k+1}}{(1-a)_k(n-k+1)!}\Phi(a-k,b-k;x) + C,$$

$$\int x^n\Psi(a,b;x)\,dx = n!\sum_{k=1}^{n+1}\frac{(-1)^{k+1}x^{n-k+1}}{(1-a)_k(n-k+1)!}\Psi(a-k,b-k;x) + C.$$

30.9.3 Whittaker Functions

The *Whittaker functions* $M_{k,\mu}(x)$ and $W_{k,\mu}(x)$ are linearly independent solutions of the Whittaker equation:

$$y''_{xx} + \bigl[-\tfrac{1}{4} + \tfrac{1}{2}k + \bigl(\tfrac{1}{4} - \mu^2\bigr)x^{-2}\bigr]y = 0.$$

The Whittaker functions are expressed in terms of degenerate hypergeometric functions as

$$M_{k,\mu}(x) = x^{\mu+1/2}e^{-x/2}\Phi\bigl(\tfrac{1}{2} + \mu - k,\ 1 + 2\mu;\ x\bigr),$$
$$W_{k,\mu}(x) = x^{\mu+1/2}e^{-x/2}\Psi\bigl(\tfrac{1}{2} + \mu - k,\ 1 + 2\mu;\ x\bigr).$$

30.10 Hypergeometric Functions

30.10.1 Various Representations of the Hypergeometric Function

► **Representations of the hypergeometric function via hypergeometric series.**

The *hypergeometric function* $F(\alpha, \beta, \gamma; x)$ is a solution of the Gaussian hypergeometric equation

$$x(x-1)y''_{xx} + [(\alpha+\beta+1)x - \gamma]y'_x + \alpha\beta y = 0.$$

For $\gamma \neq 0, -1, -2, -3, \dots$, the function $F(\alpha, \beta, \gamma; x)$ can be expressed in terms of the hypergeometric series:

$$F(\alpha, \beta, \gamma; x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}, \quad (\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1),$$

which certainly converges for $|x| < 1$.

If γ is not an integer, then the general solution of the hypergeometric equation can be written in the form

$$y = C_1 F(\alpha, \beta, \gamma; x) + C_2 x^{1-\gamma} F(\alpha - \gamma + 1,\ \beta - \gamma + 1,\ 2 - \gamma;\ x).$$

Table 30.3 shows some special cases where F can be expressed in terms of elementary functions.

► **Integral representation.**

For $\gamma > \beta > 0$, the hypergeometric function can be expressed in terms of a definite integral:

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt,$$

where $\Gamma(\beta)$ is the gamma function.

TABLE 30.3
Some special cases where the hypergeometric function
$F(\alpha,\beta,\gamma;z)$ can be expressed in terms of elementary functions

α	β	γ	z	F
$-n$	β	γ	x	$\displaystyle\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}$, where $n=1,2,\ldots$
$-n$	β	$-n-m$	x	$\displaystyle\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(-n-m)_k}\frac{x^k}{k!}$, where $n=1,2,\ldots$
α	β	β	x	$(1-x)^{-\alpha}$
α	$\alpha+1$	$\frac{1}{2}\alpha$	x	$(1+x)(1-x)^{-\alpha-1}$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	x	$\left(\dfrac{1+\sqrt{1-x}}{2}\right)^{-2\alpha}$
α	$\alpha+\frac{1}{2}$	2α	x	$\dfrac{1}{\sqrt{1-x}}\left(\dfrac{1+\sqrt{1-x}}{2}\right)^{1-2\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	x^2	$\dfrac{(1+x)^{1-2\alpha}-(1-x)^{1-2\alpha}}{2x(1-2\alpha)}$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	$-\tan^2 x$	$\cos^{2\alpha}x\cos(2\alpha x)$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	x^2	$\frac{1}{2}\left[(1+x)^{-2\alpha}+(1-x)^{-2\alpha}\right]$
α	$\alpha-\frac{1}{2}$	$2\alpha-1$	x	$2^{2\alpha-2}\left(1+\sqrt{1-x}\right)^{2-2\alpha}$
α	$2-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\dfrac{\sin[(2\alpha-2)x]}{(\alpha-1)\sin(2x)}$
α	$1-\alpha$	$\frac{1}{2}$	$-x^2$	$\dfrac{\left(\sqrt{1+x^2}+x\right)^{2\alpha-1}+\left(\sqrt{1+x^2}-x\right)^{2\alpha-1}}{2\sqrt{1+x^2}}$
α	$1-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\dfrac{\sin[(2\alpha-1)x]}{(\alpha-1)\sin(2x)}$
α	$1-\alpha$	$\frac{1}{2}$	$\sin^2 x$	$\dfrac{\cos[(2\alpha-1)x]}{\cos x}$
α	$-\alpha$	$\frac{1}{2}$	$-x^2$	$\frac{1}{2}\left[\left(\sqrt{1+x^2}+x\right)^{2\alpha}+\left(\sqrt{1+x^2}-x\right)^{2\alpha}\right]$
α	$-\alpha$	$\frac{1}{2}$	$\sin^2 x$	$\cos(2\alpha x)$
1	1	2	$-x$	$\dfrac{1}{x}\ln(x+1)$
$\frac{1}{2}$	1	$\frac{3}{2}$	x^2	$\dfrac{1}{2x}\ln\dfrac{1+x}{1-x}$
$\frac{1}{2}$	1	$\frac{3}{2}$	$-x^2$	$\dfrac{1}{x}\arctan x$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	x^2	$\dfrac{1}{x}\arcsin x$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\dfrac{1}{x}\operatorname{arcsinh} x$
$n+1$	$n+m+1$	$n+m+l+2$	x	$\dfrac{(-1)^m(n+m+l+1)!}{n!\,l!\,(n+m)!\,(m+l)!}\dfrac{d^{n+m}}{dx^{n+m}}\left\{(1-x)^{m+l}\dfrac{d^lF}{dx^l}\right\}$, $F=-\dfrac{\ln(1-x)}{x}$, $n,m,l=0,1,2,\ldots$

30.10.2 Basic Properties

► **Linear transformation formulas.**

$$
\begin{aligned}
&F(\alpha,\beta,\gamma;x)=F(\beta,\alpha,\gamma;x),\\
&F(\alpha,\beta,\gamma;x)=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),\\
&F(\alpha,\beta,\gamma;x)=(1-x)^{-\alpha}F\Big(\alpha,\ \gamma-\beta,\ \gamma;\ \frac{x}{x-1}\Big),\\
&F(\alpha,\beta,\gamma;x)=(1-x)^{-\beta}F\Big(\beta,\ \gamma-\alpha,\ \gamma;\ \frac{x}{x-1}\Big).
\end{aligned}
$$

► **Gauss's linear relations for contiguous functions.**

$$
\begin{aligned}
&(\beta-\alpha)F(\alpha,\beta,\gamma;x)+\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)=0,\\
&(\gamma-\alpha-1)F(\alpha,\beta,\gamma;x)+\alpha F(\alpha+1,\beta,\gamma;x)-(\gamma-1)F(\alpha,\beta,\gamma-1;x)=0,\\
&(\gamma-\beta-1)F(\alpha,\beta,\gamma;x)+\beta F(\alpha,\beta+1,\gamma;x)-(\gamma-1)F(\alpha,\beta,\gamma-1;x)=0,\\
&(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma;x)+\alpha(1-x)F(\alpha+1,\beta,\gamma;x)-(\gamma-\beta)F(\alpha,\beta-1,\gamma;x)=0,\\
&(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma;x)-(\gamma-\alpha)F(\alpha-1,\beta,\gamma;x)+\beta(1-x)F(\alpha,\beta+1,\gamma;x)=0.
\end{aligned}
$$

► **Differentiation formulas.**

$$
\begin{aligned}
&\frac{d}{dx}F(\alpha,\beta,\gamma;x)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\ \beta+1,\ \gamma+1;\ x),\\
&\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x)=\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\ \beta+n,\ \gamma+n;\ x),\\
&\frac{d^n}{dx^n}\big[x^{\gamma-1}F(\alpha,\beta,\gamma;x)\big]=(\gamma-n)_n x^{\gamma-n-1}F(\alpha,\ \beta,\ \gamma-n;\ x),\\
&\frac{d^n}{dx^n}\big[x^{\alpha+n-1}F(\alpha,\beta,\gamma;x)\big]=(\alpha)_n x^{\alpha-1}F(\alpha+n,\ \beta,\ \gamma;\ x),
\end{aligned}
$$

where $(\alpha)_n=\alpha(\alpha+1)\dots(\alpha+n-1)$.

See Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 1) for more detailed information about hypergeometric functions.

30.11 Legendre Polynomials, Legendre Functions, and Associated Legendre Functions

30.11.1 Legendre Polynomials and Legendre Functions

► **Implicit and recurrence formulas for Legendre polynomials and functions.**

The *Legendre polynomials* $P_n(x)$ and the *Legendre functions* $Q_n(x)$ are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx}-2xy'_x+n(n+1)y=0.$$

The *Legendre polynomials* $P_n(x)$ and the Legendre functions $Q_n(x)$ are defined by the formulas

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n,$$
$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{m=1}^{n}\frac{1}{m}P_{m-1}(x)P_{n-m}(x).$$

The polynomials $P_n = P_n(x)$ can be calculated using the formulas

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1),$$
$$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4-30x^2+3),$$
$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x).$$

The first five functions $Q_n = Q_n(x)$ have the form

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1,$$
$$Q_2(x) = \frac{1}{4}(3x^2-1)\ln\frac{1+x}{1-x} - \frac{3}{2}x, \quad Q_3(x) = \frac{1}{4}(5x^3-3x)\ln\frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3},$$
$$Q_4(x) = \frac{1}{16}(35x^4-30x^2+3)\ln\frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x.$$

The polynomials $P_n(x)$ have the explicit representation

$$P_n(x) = 2^{-n}\sum_{m=0}^{[n/2]}(-1)^m C_n^m C_{2n-2m}^n x^{n-2m},$$

where $[A]$ stands for the integer part of a number A.

► Integral representation. Useful formulas.

Integral representation of the Legendre polynomials (*Laplace integral*):

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \bigl(x \pm \sqrt{x^2-1}\,\cos t\bigr)^n\,dt, \qquad x > 1.$$

Integral representation of the Legendre polynomials (*Dirichlet–Mehler integral*):

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos\bigl[(n+\frac{1}{2})\psi\bigr]\,d\psi}{\sqrt{\cos\psi - \cos\theta}}, \qquad 0 < \theta < \pi, \quad n = 0,\,1,\,\ldots$$

Integral representation of the Legendre functions:

$$Q_n(x) = 2^n\int_x^\infty \frac{(t-x)^n}{(t^2-1)^{n+1}}\,dt, \qquad x > 1.$$

Properties:

$$P_n(-x) = (-1)^n P_n(x), \qquad Q_n(-x) = (-1)^{n+1} Q_n(x).$$

Recurrence relations:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$
$$(x^2-1)\frac{d}{dx}P_n(x) = n\bigl[xP_n(x) - P_{n-1}(x)\bigr] = \frac{n(n+1)}{2n+1}\bigl[P_{n+1}(x) - P_{n-1}(x)\bigr].$$

Values of the Legendre polynomials and their derivatives at $x = 0$:

$$P_{2m}(0) = (-1)^m \frac{(2m-1)!!}{2^m m!}, \quad P_{2m+1}(0) = 0,$$
$$P'_{2m}(0) = 0, \quad P'_{2m+1}(0) = (-1)^m \frac{(2m+1)!!}{2^m m!}.$$

Asymptotic formula as $n \to \infty$:

$$P_n(\cos\theta) \approx \left(\frac{2}{\pi n \sin\theta}\right)^{1/2} \sin\left[\left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right], \qquad 0 < \theta < \pi.$$

► **Zeros and orthogonality of the Legendre polynomials.**

The polynomials $P_n(x)$ (with natural n) have exactly n real distinct zeros; all zeros lie on the interval $-1 < x < 1$. The zeros of $P_n(x)$ and $P_{n+1}(x)$ alternate with each other. The function $Q_n(x)$ has exactly $n+1$ zeros, which lie on the interval $-1 < x < 1$.

The functions $P_n(x)$ form an orthogonal system on the interval $-1 \le x \le 1$, with

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{2}{2n+1} & \text{if } n = m. \end{cases}$$

► **Generating functions.**

The generating function for Legendre polynomials is

$$\frac{1}{\sqrt{1 - 2sx + s^2}} = \sum_{n=0}^{\infty} P_n(x) s^n \qquad (|s| < 1).$$

The generating function for Legendre functions is

$$\frac{1}{\sqrt{1 - 2sx + s^2}} \ln\left[\frac{x - s + \sqrt{1 - 2sx + s^2}}{\sqrt{1 - x^2}}\right] = \sum_{n=0}^{\infty} Q_n(x) s^n \qquad (|s| < 1,\ x > 1).$$

30.11.2 Associated Legendre Functions with Integer Indices and Real Argument

► Formulas for associated Legendre functions. Differential equation.

The *associated Legendre functions* $P_n^m(x)$ of order m are defined by the formulas

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x), \qquad n = 1,\ 2,\ 3,\ \ldots, \quad m = 0,\ 1,\ 2,\ \ldots$$

It is assumed by definition that $P_n^0(x) = P_n(x)$.

Properties:

$$P_n^m(x) = 0 \quad \text{if} \quad m > n, \qquad P_n^m(-x) = (-1)^{n-m}P_n^m(x).$$

The associated Legendre functions $P_n^m(x)$ have exactly $n-m$ real zeros, which lie on the interval $-1 < x < 1$.

The associated Legendre functions $P_n^m(x)$ with low indices:

$$P_1^1(x) = (1-x^2)^{1/2}, \quad P_2^1(x) = 3x(1-x^2)^{1/2}, \quad P_2^2(x) = 3(1-x^2),$$
$$P_3^1(x) = \tfrac{3}{2}(5x^2-1)(1-x^2)^{1/2}, \quad P_3^2(x) = 15x(1-x^2), \quad P_3^3(x) = 15(1-x^2)^{3/2}.$$

The associated Legendre functions $P_n^m(x)$ with $n > m$ are solutions of the linear ordinary differential equation

$$(1-x^2)y''_{xx} - 2xy'_x + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0.$$

► Orthogonality of the associated Legendre functions.

The functions $P_n^m(x)$ form an orthogonal system on the interval $-1 \le x \le 1$, with

$$\int_{-1}^{1} P_n^m(x)P_k^m(x)\,dx = \begin{cases} 0 & \text{if } n \ne k, \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{if } n = k. \end{cases}$$

The functions $P_n^m(x)$ (with $m \ne 0$) are orthogonal on the interval $-1 \le x \le 1$ with weight $(1-x^2)^{-1}$, that is,

$$\int_{-1}^{1} \frac{P_n^m(x)P_n^k(x)}{1-x^2}\,dx = \begin{cases} 0 & \text{if } m \ne k, \\ \dfrac{(n+m)!}{m(n-m)!} & \text{if } m = k. \end{cases}$$

30.11.3 Associated Legendre Functions. General Case

► Definitions. Basic formulas.

In the general case, the associated Legendre functions of the first and the second kind, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, are linearly independent solutions of the Legendre equation

$$(1-z^2)y''_{zz} - 2zy'_z + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]y = 0,$$

where the parameters ν and μ and the variable z can assume arbitrary real or complex values.

For $|1-z|<2$, the formulas

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\Big(\frac{z+1}{z-1}\Big)^{\mu/2} F\Big(-\nu,\ 1+\nu,\ 1-\mu;\ \frac{1-z}{2}\Big),$$

$$Q_\nu^\mu(z) = A\Big(\frac{z-1}{z+1}\Big)^{\frac{\mu}{2}} F\Big(-\nu,\ 1+\nu,\ 1+\mu;\ \frac{1-z}{2}\Big) + B\Big(\frac{z+1}{z-1}\Big)^{\frac{\mu}{2}} F\Big(-\nu,\ 1+\nu,\ 1-\mu;\ \frac{1-z}{2}\Big),$$

$$A = e^{i\mu\pi}\frac{\Gamma(-\mu)\,\Gamma(1+\nu+\mu)}{2\,\Gamma(1+\nu-\mu)},\quad B = e^{i\mu\pi}\frac{\Gamma(\mu)}{2},\quad i^2=-1,$$

are valid, where $F(a,b,c;z)$ is the hypergeometric series (see Section 30.10).

For $|z|>1$,

$$P_\nu^\mu(z) = \frac{2^{-\nu-1}\Gamma(-\frac{1}{2}-\nu)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)} z^{-\nu+\mu-1}(z^2-1)^{-\mu/2} F\Big(\frac{1+\nu-\mu}{2},\ \frac{2+\nu-\mu}{2},\ \frac{2\nu+3}{2};\ \frac{1}{z^2}\Big) + \frac{2^\nu\Gamma(\frac{1}{2}+\nu)}{\Gamma(1+\nu-\mu)} z^{\nu+\mu}(z^2-1)^{-\mu/2} F\Big(-\frac{\nu+\mu}{2},\ \frac{1-\nu-\mu}{2},\ \frac{1-2\nu}{2};\ \frac{1}{z^2}\Big),$$

$$Q_\nu^\mu(z) = e^{i\pi\mu}\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1}(z^2-1)^{\mu/2} F\Big(\frac{2+\nu+\mu}{2},\frac{1+\nu+\mu}{2},\frac{2\nu+3}{2};\frac{1}{z^2}\Big).$$

The functions $P_\nu(z) \equiv P_\nu^0(z)$ and $Q_\nu(z) \equiv Q_\nu^0(z)$ are called the *Legendre functions*.

For $n = 1, 2, \ldots,$

$$P_\nu^n(z) = (z^2-1)^{n/2}\frac{d^n}{dz^n}P_\nu(z), \qquad Q_\nu^n(z) = (z^2-1)^{n/2}\frac{d^n}{dz^n}Q_\nu(z).$$

► **Relations between associated Legendre functions.**

$$P_\nu^\mu(z) = P_{-\nu-1}^\mu(z), \qquad P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\Gamma(\nu-n+1)}P_\nu^{-n}(z), \quad n=0,1,2,\ldots,$$

$$P_{\nu+1}^\mu(z) = \frac{2\nu+1}{\nu-\mu+1}\, zP_\nu^\mu(z) - \frac{\nu+\mu}{\nu-\mu+1}P_{\nu-1}^\mu(z),$$

$$P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)(z^2-1)^{1/2}P_\nu^{\mu-1}(z),$$

$$(z^2-1)\frac{d}{dz}P_\nu^\mu(z) = \nu zP_\nu^\mu(z) - (\nu+m)P_{\nu-1}^\mu(z),$$

$$Q_\nu^\mu(z) = \frac{\pi}{2\sin(\mu\pi)}e^{i\pi\mu}\Big[P_\nu^\mu(z) - \frac{\Gamma(1+\nu+\mu}{\Gamma(1+\nu-\mu)}P_\nu^{-\mu}(z)\Big],$$

$$Q_\nu^\mu(z) = e^{i\pi\mu}\Big(\frac{\pi}{2}\Big)^{1/2}\Gamma(\nu+\mu+1)(z^2-1)^{-1/4}P_{-\mu-1/2}^{-\nu-1/2}\Big(\frac{z}{\sqrt{z^2-1}}\Big), \qquad \operatorname{Re} z>0.$$

► **Integral representations.**

For $\operatorname{Re}(-\mu) > \operatorname{Re}\nu > -1$,

$$P_\nu^\mu(z) = \frac{2^{-\nu}(z^2-1)^{-\mu/2}}{\Gamma(\nu+1)\Gamma(-\mu-\nu)}\int_0^\infty (z+\cosh t)^{\mu-\nu-1}(\sinh t)^{2\nu+1}\,dt,$$

where z does not lie on the real axis between -1 and ∞.

For $\mu < 1/2$,

$$P_\nu^\mu(z) = \frac{2^\mu (z^2-1)^{-\mu/2}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\mu)} \int_0^\pi \bigl(z+\sqrt{z^2-1}\,\cos t\bigr)^{\nu+\mu} (\sin t)^{-2\mu}\,dt,$$

where z does not lie on the real axis between -1 and 1.

For $\operatorname{Re}\nu > -1$ and $\operatorname{Re}(\nu+\mu+1) > 0$,

$$Q_\nu^\mu(z) = e^{\pi\mu i} \frac{\Gamma(\nu+\mu+1)(z^2-1)^{-\mu/2}}{2^{\nu+1}\Gamma(\nu+1)} \int_0^\pi \bigl(z+\cos t\bigr)^{\mu-\nu-1} (\sin t)^{2\nu+1}\,dt,$$

where z does not lie on the real axis between -1 and 1.

For $n = 0, 1, 2, \ldots,$

$$P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\pi\Gamma(\nu+1)} \int_0^\pi \bigl(z+\sqrt{z^2-1}\,\cos t\bigr)^{\nu} \cos(nt)\,dt, \qquad \operatorname{Re} z > 0;$$

$$Q_\nu^n(z) = (-1)^n \frac{\Gamma(\nu+n+1)}{2^{\nu+1}\Gamma(\nu+1)} (z^2-1)^{-n/2} \int_0^\pi (z+\cos t)^{n-\nu-1} (\sin t)^{2\nu+1}\,dt.$$

Note that $z \neq x$, $-1 < x < 1$, and $\operatorname{Re}\nu > -1$ in the latter formula for $Q_\nu^n(z)$.

► Modified associated Legendre functions.

The *modified associated Legendre functions*, on the cut $z = x$, $-1 < x < 1$, of the real axis, are defined by the formulas

$$\mathsf{P}_\nu^\mu(x) = \tfrac{1}{2}\bigl[e^{\frac{1}{2}i\mu\pi} P_\nu^\mu(x+i0) + e^{-\frac{1}{2}i\mu\pi} P_\nu^\mu(x-i0)\bigr],$$

$$\mathsf{Q}_\nu^\mu(x) = \tfrac{1}{2}e^{-i\mu\pi}\bigl[e^{-\frac{1}{2}i\mu\pi} Q_\nu^\mu(x+i0) + e^{\frac{1}{2}i\mu\pi} Q_\nu^\mu(x-i0)\bigr].$$

Notation:

$$\mathsf{P}_\nu(x) = \mathsf{P}_\nu^0(x), \qquad \mathsf{Q}_\nu(x) = \mathsf{Q}_\nu^0(x).$$

► Trigonometric expansions.

For $-1 < x < 1$, the modified associated Legendre functions can be represented in the form of the trigonometric series:

$$\mathsf{P}_\nu^\mu(\cos\theta) = \frac{2^{\mu+1}}{\sqrt{\pi}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\sin\theta)^\mu \sum_{k=0}^\infty \frac{(\frac{1}{2}+\mu)_k (1+\nu+\mu)_k}{k!\,(\nu+\frac{3}{2})_k} \sin[(2k+\nu+\mu+1)\theta],$$

$$\mathsf{Q}_\nu^\mu(\cos\theta) = \sqrt{\pi}\,2^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\sin\theta)^\mu \sum_{k=0}^\infty \frac{(\frac{1}{2}+\mu)_k (1+\nu+\mu)_k}{k!\,(\nu+\frac{3}{2})_k} \cos[(2k+\nu+\mu+1)\theta],$$

where $0 < \theta < \pi$.

► **Some relations for the modified associated Legendre functions.**

For $0 < x < 1$,

$$\mathsf{P}_\nu^\mu(-x) = \mathsf{P}_\nu^\mu(x)\cos[\pi(\nu+\mu)] - 2\pi^{-1}\mathsf{Q}_\nu^\mu(x)\sin[\pi(\nu+\mu)],$$
$$\mathsf{Q}_\nu^\mu(-x) = -\mathsf{Q}_\nu^\mu(x)\cos[\pi(\nu+\mu)] - \tfrac{1}{2}\pi\mathsf{P}_\nu^\mu(x)\sin[\pi(\nu+\mu)].$$

For $-1 < x < 1$,

$$\mathsf{P}_{\nu+1}^\mu(x) = \frac{2\nu+1}{\nu-\mu+1}x\mathsf{P}_\nu^\mu(x) - \frac{\nu+\mu}{\nu-\mu+1}\mathsf{P}_{\nu-1}^\mu(x),$$
$$\mathsf{P}_{\nu+1}^\mu(x) = \mathsf{P}_{\nu-1}^\mu(x) - (2\nu+1)(1-x^2)^{1/2}\mathsf{P}_\nu^{\mu-1}(x),$$
$$\mathsf{P}_{\nu+1}^\mu(x) = x\mathsf{P}_\nu^\mu(x) - (\nu+\mu)(1-x^2)^{1/2}\mathsf{P}_\nu^{\mu-1}(x),$$
$$\frac{d}{dx}\mathsf{P}_\nu^\mu(x) = \frac{\nu x}{x^2-1}\mathsf{P}_\nu^\mu(x) - \frac{\nu+\mu}{x^2-1}\mathsf{P}_{\nu-1}^\mu(x).$$

Wronskian:

$$\mathsf{P}_\nu^\mu(x)\frac{d}{dx}\mathsf{Q}_\nu^\mu(x) - \mathsf{Q}_\nu^\mu(x)\frac{d}{dx}\mathsf{P}_\nu^\mu(x) = \frac{k}{1-x^2}, \qquad k = 2^{2\mu}\frac{\Gamma\bigl(\frac{\nu+\mu+1}{2}\bigr)\Gamma\bigl(\frac{\nu+\mu+2}{2}\bigr)}{\Gamma\bigl(\frac{\nu-\mu+1}{2}\bigr)\Gamma\bigl(\frac{\nu-\mu+2}{2}\bigr)}.$$

For $n = 1, 2, \dots,$

$$\mathsf{P}_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}\mathsf{P}_\nu(x), \quad \mathsf{Q}_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}\mathsf{Q}_\nu(x).$$

30.12 Parabolic Cylinder Functions

30.12.1 Definitions. Basic Formulas

► **Differential equation.**

Formulas for the parabolic cylinder functions.

The *Weber parabolic cylinder function* $D_\nu(z)$ is a solution of the linear ordinary differential equation

$$y''_{zz} + \bigl(-\tfrac{1}{4}z^2 + \nu + \tfrac{1}{2}\bigr)y = 0,$$

where the parameter ν and the variable z can assume arbitrary real or complex values. Another linearly independent solution of this equation is the function $D_{-\nu-1}(iz)$; if ν is a noninteger, then $D_\nu(-z)$ can also be taken as a linearly independent solution.

The parabolic cylinder functions can be expressed in terms of degenerate hypergeometric functions as

$$D_\nu(z) = \exp\bigl(-\tfrac{1}{4}z^2\bigr)\biggl[2^{1/2}\frac{\Gamma\bigl(\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}-\frac{\nu}{2}\bigr)}\Phi\bigl(-\tfrac{\nu}{2},\tfrac{1}{2};\tfrac{1}{2}z^2\bigr) + \frac{\Gamma\bigl(-\frac{1}{2}\bigr)}{\Gamma\bigl(-\frac{\nu}{2}\bigr)}z\Phi\bigl(\tfrac{1}{2}-\tfrac{\nu}{2},\tfrac{3}{2};\tfrac{1}{2}z^2\bigr)\biggr].$$

► **Special cases.**

For nonnegative integer $\nu = n$, we have

$$D_n(z) = \frac{1}{2^{n/2}} \exp\Big(-\frac{z^2}{4}\Big) H_n\Big(\frac{z}{\sqrt{2}}\Big), \qquad n = 0,\ 1,\ 2,\ \dots;$$
$$H_n(z) = (-1)^n \exp\big(z^2\big) \frac{d^n}{dz^n} \exp\big(-z^2\big),$$

where $H_n(z)$ is the Hermitian polynomial of order n.

Connection with the error function:

$$D_{-1}(z) = \sqrt{\frac{\pi}{2}} \exp\Big(\frac{z^2}{4}\Big) \operatorname{erfc}\Big(\frac{z}{\sqrt{2}}\Big),$$
$$D_{-2}(z) = \sqrt{\frac{\pi}{2}}\, z \exp\Big(\frac{z^2}{4}\Big) \operatorname{erfc}\Big(\frac{z}{\sqrt{2}}\Big) - \exp\Big(-\frac{z^2}{4}\Big).$$

30.12.2 Integral Representations, Asymptotic Expansions, and Linear Relations

► **Integral representations and the asymptotic expansion.**

Integral representations:

$$D_\nu(z) = \sqrt{2/\pi}\, \exp\big(\tfrac{1}{4}z^2\big) \int_0^\infty t^\nu \exp\big(-\tfrac{1}{2}t^2\big) \cos\big(zt - \tfrac{1}{2}\pi\nu\big)\, dt \qquad \text{for} \quad \operatorname{Re}\nu > -1,$$
$$D_\nu(z) = \frac{1}{\Gamma(-\nu)} \exp\big(-\tfrac{1}{4}z^2\big) \int_0^\infty t^{-\nu-1} \exp\big(-zt - \tfrac{1}{2}t^2\big)\, dt \qquad \text{for} \quad \operatorname{Re}\nu < 0.$$

Asymptotic expansion as $|z| \to \infty$:

$$D_\nu(z) = z^\nu \exp\big(-\tfrac{1}{4}z^2\big) \Bigg[\sum_{n=0}^{N} \frac{(-2)^n \big(-\frac{\nu}{2}\big)_n \big(\frac{1}{2} - \frac{\nu}{2}\big)_n}{n!} \frac{1}{z^{2n}} + O\big(|z|^{-2N-2}\big)\Bigg],$$

where $|\arg z| < \frac{3}{4}\pi$ and $(a)_0 = 1$, $(a)_n = a(a+1)\dots(a+n-1)$ for $n = 1,\ 2,\ 3,\ \dots$

► **Recurrence relations.**

$$D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0,$$
$$\frac{d}{dz}D_\nu(z) + \frac{1}{2}zD_\nu(z) - \nu D_{\nu-1}(z) = 0,$$
$$\frac{d}{dz}D_\nu(z) - \frac{1}{2}zD_\nu(z) + D_{\nu+1}(z) = 0.$$

30.13 Elliptic Integrals

30.13.1 Complete Elliptic Integrals

► Definitions. Properties. Conversion formulas.

Complete elliptic integral of the first kind:

$$\mathsf{K}(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

Complete elliptic integral of the second kind:

$$\mathsf{E}(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha = \int_0^1 \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}}\, dx.$$

The argument k is called the *elliptic modulus* ($k^2 < 1$).

Notation:

$$k' = \sqrt{1 - k^2}, \qquad \mathsf{K}'(k) = \mathsf{K}(k'), \qquad \mathsf{E}'(k) = \mathsf{E}(k'),$$

where k' is the *complementary modulus*.

Properties:

$$\begin{aligned} &\mathsf{K}(-k) = \mathsf{K}(k), \qquad \mathsf{E}(-k) = \mathsf{E}(k);\\ &\mathsf{K}(k) = \mathsf{K}'(k'), \qquad \mathsf{E}(k) = \mathsf{E}'(k');\\ &\mathsf{E}(k)\mathsf{K}'(k) + \mathsf{E}'(k)\mathsf{K}(k) - \mathsf{K}(k)\mathsf{K}'(k) = \frac{\pi}{2}. \end{aligned}$$

Conversion formulas for complete elliptic integrals:

$$\begin{aligned} \mathsf{K}\left(\frac{1-k'}{1+k'}\right) &= \frac{1+k'}{2}\mathsf{K}(k),\\ \mathsf{E}\left(\frac{1-k'}{1+k'}\right) &= \frac{1}{1+k'}\big[\mathsf{E}(k) + k'\mathsf{K}(k)\big],\\ \mathsf{K}\left(\frac{2\sqrt{k}}{1+k}\right) &= (1+k)\mathsf{K}(k),\\ \mathsf{E}\left(\frac{2\sqrt{k}}{1+k}\right) &= \frac{1}{1+k}\big[2\mathsf{E}(k) - (k')^2\mathsf{K}(k)\big]. \end{aligned}$$

► Representation of complete elliptic integrals in series form.

Representation of complete elliptic integrals in the form of series in powers of the modulus k:

$$\begin{aligned} \mathsf{K}(k) &= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 k^4 + \cdots + \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n} + \cdots\right\},\\ \mathsf{E}(k) &= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\times 3}{2\times 4}\right)^2 \frac{k^4}{3} - \cdots - \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{k^{2n}}{2n-1} - \cdots\right\}. \end{aligned}$$

Representation of complete elliptic integrals in the form of series in powers of the complementary modulus $k' = \sqrt{1-k^2}$:

$$\mathsf{K}(k) = \frac{\pi}{1+k'}\biggl\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1\times 3}{2\times 4}\right)^2\left(\frac{1-k'}{1+k'}\right)^4 + \cdots + \left[\frac{(2n-1)!!}{(2n)!!}\right]^2\left(\frac{1-k'}{1+k'}\right)^{2n} + \cdots\biggr\},$$

$$\mathsf{K}(k) = \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\times 2}\right)(k')^2 + \left(\frac{1\times 3}{2\times 4}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\times 2} - \frac{2}{3\times 4}\right)(k')^4 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\times 2} - \frac{2}{3\times 4} - \frac{2}{5\times 6}\right)(k')^6 + \cdots;$$

$$\mathsf{E}(k) = \frac{\pi(1+k')}{4}\biggl\{1 + \frac{1}{2^2} - \left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{(2\times 4)^2}\left(\frac{1-k'}{1+k'}\right)^4 + \cdots + \left[\frac{(2n-3)!!}{(2n)!!}\right]^2\left(\frac{1-k'}{1+k'}\right)^{2n} + \cdots\biggr\},$$

$$\mathsf{E}(k) = 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - \frac{1}{1\times 2}\right)(k')^2 + \frac{1^2\times 3}{2^2\times 4}\left(\ln\frac{4}{k'} - \frac{2}{1\times 2} - \frac{1}{3\times 4}\right)(k')^4 + \frac{1^2\times 3^2\times 5}{2^2\times 4^2\times 6}\left(\ln\frac{4}{k'} - \frac{2}{1\times 2} - \frac{2}{3\times 4} - \frac{1}{5\times 6}\right)(k')^6 + \cdots.$$

► **Differentiation formulas. Differential equations.**

Differentiation formulas:

$$\frac{d\mathsf{K}(k)}{dk} = \frac{\mathsf{E}(k)}{k(k')^2} - \frac{\mathsf{K}(k)}{k}, \qquad \frac{d\mathsf{E}(k)}{dk} = \frac{\mathsf{E}(k) - \mathsf{K}(k)}{k}.$$

The functions $\mathsf{K}(k)$ and $\mathsf{K}'(k)$ satisfy the second-order linear ordinary differential equation

$$\frac{d}{dk}\left[k(1-k^2)\frac{d\mathsf{K}}{dk}\right] - k\mathsf{K} = 0.$$

The functions $\mathsf{E}(k)$ and $\mathsf{E}'(k) - \mathsf{K}'(k)$ satisfy the second-order linear ordinary differential equation

$$(1-k^2)\frac{d}{dk}\left(k\frac{d\mathsf{E}}{dk}\right) + k\mathsf{E} = 0.$$

30.13.2 Incomplete Elliptic Integrals (Elliptic Integrals)

► **Definitions. Properties.**

Elliptic integral of the first kind:

$$F(\varphi, k) = \int_0^{\varphi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Elliptic integral of the second kind:

$$E(\varphi, k) = \int_0^{\varphi}\sqrt{1-k^2\sin^2\alpha}\,d\alpha = \int_0^{\sin\varphi}\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx.$$

Elliptic integral of the third kind:

$$\Pi(\varphi, n, k) = \int_0^{\varphi} \frac{d\alpha}{(1 - n\sin^2\alpha)\sqrt{1 - k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{(1 - nx^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

The quantity k is called the *elliptic modulus* ($k^2 < 1$), $k' = \sqrt{1 - k^2}$ is the *complementary modulus*, and n is the *characteristic parameter*.

Complete elliptic integrals:

$$\mathsf{K}(k) = F\Big(\frac{\pi}{2}, k\Big), \qquad \mathsf{E}(k) = E\Big(\frac{\pi}{2}, k\Big),$$
$$\mathsf{K}'(k) = F\Big(\frac{\pi}{2}, k'\Big), \qquad \mathsf{E}'(k) = E\Big(\frac{\pi}{2}, k'\Big).$$

Properties of elliptic integrals:

$$F(-\varphi, k) = -F(\varphi, k), \qquad F(n\pi \pm \varphi, k) = 2n\mathsf{K}(k) \pm F(\varphi, k);$$
$$E(-\varphi, k) = -E(\varphi, k), \qquad E(n\pi \pm \varphi, k) = 2n\mathsf{E}(k) \pm E(\varphi, k).$$

► Conversion formulas.

Conversion formulas for elliptic integrals (first set):

$$F\bigg(\psi, \frac{1}{k}\bigg) = kF(\varphi, k),$$
$$E\bigg(\psi, \frac{1}{k}\bigg) = \frac{1}{k}\big[E(\varphi, k) - (k')^2F(\varphi, k)\big],$$

where the angles φ and ψ are related by $\sin\psi = k\sin\varphi$, $\cos\psi = \sqrt{1 - k^2\sin^2\varphi}$.

Conversion formulas for elliptic integrals (second set):

$$F\bigg(\psi, \frac{1-k'}{1+k'}\bigg) = (1+k')F(\varphi, k),$$
$$E\bigg(\psi, \frac{1-k'}{1+k'}\bigg) = \frac{2}{1+k'}\big[E(\varphi, k) + k'F(\varphi, k)\big] - \frac{1-k'}{1+k'}\sin\psi,$$

where the angles φ and ψ are related by $\tan(\psi - \varphi) = k'\tan\varphi$.

Transformation formulas for elliptic integrals (third set):

$$F\bigg(\psi, \frac{2\sqrt{k}}{1+k}\bigg) = (1+k)F(\varphi, k),$$
$$E\bigg(\psi, \frac{2\sqrt{k}}{1+k}\bigg) = \frac{1}{1+k}\bigg[2E(\varphi, k) - (k')^2F(\varphi, k) + 2k\frac{\sin\varphi\,\cos\varphi}{1+k\sin^2\varphi}\sqrt{1-k^2\sin^2\varphi}\bigg],$$

where the angles φ and ψ are related by $\sin\psi = \dfrac{(1+k)\sin\varphi}{1+k\sin^2\varphi}$.

► **Trigonometric expansions.**

Trigonometric expansions for small k and φ:

$$F(\varphi,k)=\frac{2}{\pi}\mathsf{K}(k)\varphi-\sin\varphi\cos\varphi\biggl(a_0+\frac{2}{3}a_1\sin^2\varphi+\frac{2\times4}{3\times5}a_2\sin^4\varphi+\cdots\biggr),$$

$$a_0=\frac{2}{\pi}\mathsf{K}(k)-1,\qquad a_n=a_{n-1}-\left[\frac{(2n-1)!!}{(2n)!!}\right]^2k^{2n};$$

$$E(\varphi,k)=\frac{2}{\pi}\mathsf{E}(k)\varphi-\sin\varphi\cos\varphi\biggl(b_0+\frac{2}{3}b_1\sin^2\varphi+\frac{2\times4}{3\times5}b_2\sin^4\varphi+\cdots\biggr),$$

$$b_0=1-\frac{2}{\pi}\mathsf{E}(k),\qquad b_n=b_{n-1}-\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{k^{2n}}{2n-1}.$$

Trigonometric expansions for $k\to1$:

$$F(\varphi,k)=\frac{2}{\pi}\mathsf{K}'(k)\ln\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right)-\frac{\tan\varphi}{\cos\varphi}\biggl(a'_0-\frac{2}{3}a'_1\tan^2\varphi+\frac{2\times4}{3\times5}a'_2\tan^4\varphi-\cdots\biggr),$$

$$a'_0=\frac{2}{\pi}\mathsf{K}'(k)-1,\qquad a'_n=a'_{n-1}-\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(k')^{2n};$$

$$E(\varphi,k)=\frac{2}{\pi}\mathsf{E}'(k)\ln\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right)+\frac{\tan\varphi}{\cos\varphi}\biggl(b'_0-\frac{2}{3}b'_1\tan^2\varphi+\frac{2\times4}{3\times5}b'_2\tan^4\varphi-\cdots\biggr),$$

$$b'_0=\frac{2}{\pi}\mathsf{E}'(k)-1,\qquad b'_n=b'_{n-1}-\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{(k')^{2n}}{2n-1}.$$

30.14 Elliptic Functions

An *elliptic function* is a function that is the inverse of an elliptic integral. An elliptic function is a doubly periodic meromorphic function of a complex variable. All its periods can be written in the form $2m\omega_1+2n\omega_2$ with integer m and n, where ω_1 and ω_2 are a pair of (primitive) half-periods. The ratio $\tau=\omega_2/\omega_1$ is a complex quantity that may be considered to have a positive imaginary part, $\operatorname{Im}\tau>0$.

Throughout the rest of this section, the following brief notation will be used: $\mathsf{K}=\mathsf{K}(k)$ and $\mathsf{K}'=\mathsf{K}(k')$ are complete elliptic integrals with $k'=\sqrt{1-k^2}$.

30.14.1 Jacobi Elliptic Functions

► **Definitions. Simple properties. Special cases.**

When the upper limit φ of the incomplete elliptic integral of the first kind

$$u=\int_0^\varphi\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}=F(\varphi,k)$$

is treated as a function of u, the following notation is used:

$$u=\operatorname{am}\varphi.$$

Naming: φ is the *amplitude* and u is the *argument*.

Jacobi elliptic functions:

$$\begin{aligned}
&\operatorname{sn} u = \sin\varphi = \sin\operatorname{am} u && (\textit{sine amplitude}),\\
&\operatorname{cn} u = \cos\varphi = \cos\operatorname{am} u && (\textit{cosine amplitude}),\\
&\operatorname{dn} u = \sqrt{1-k^2\sin^2\varphi} = \frac{d\varphi}{du} && (\textit{delta amlplitude}).
\end{aligned}$$

Along with the brief notations $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, the respective full notations are also used: $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, $\operatorname{dn}(u,k)$.

Simple properties:

$$\begin{aligned}
&\operatorname{sn}(-u) = -\operatorname{sn} u, && \operatorname{cn}(-u) = \operatorname{cn} u, && \operatorname{dn}(-u) = \operatorname{dn} u;\\
&\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, && k^2\operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, && \operatorname{dn}^2 u - k^2\operatorname{cn}^2 u = 1-k^2,
\end{aligned}$$

where $i^2 = -1$.

Jacobi functions for special values of the modulus ($k=0$ and $k=1$):

$$\begin{aligned}
&\operatorname{sn}(u,0) = \sin u, && \operatorname{cn}(u,0) = \cos u, && \operatorname{dn}(u,0) = 1;\\
&\operatorname{sn}(u,1) = \tanh u, && \operatorname{cn}(u,1) = \frac{1}{\cosh u}, && \operatorname{dn}(u,1) = \frac{1}{\cosh u}.
\end{aligned}$$

Jacobi functions for special values of the argument:

$$\begin{aligned}
&\operatorname{sn}(\tfrac12\mathsf{K},k) = \frac{1}{\sqrt{1+k'}}, && \operatorname{cn}(\tfrac12\mathsf{K},k) = \sqrt{\frac{k'}{1+k'}}, && \operatorname{dn}(\tfrac12\mathsf{K},k) = \sqrt{k'};\\
&\operatorname{sn}(\mathsf{K},k) = 1, && \operatorname{cn}(\mathsf{K},k) = 0, && \operatorname{dn}(\mathsf{K},k) = k'.
\end{aligned}$$

► **Reduction formulas.**

$$\begin{aligned}
&\operatorname{sn}(u\pm\mathsf{K}) = \pm\frac{\operatorname{cn} u}{\operatorname{dn} u}, && \operatorname{cn}(u\pm\mathsf{K}) = \mp k'\frac{\operatorname{sn} u}{\operatorname{dn} u}, && \operatorname{dn}(u\pm\mathsf{K}) = \frac{k'}{\operatorname{dn} u};\\
&\operatorname{sn}(u\pm 2\mathsf{K}) = -\operatorname{sn} u, && \operatorname{cn}(u\pm 2\mathsf{K}) = -\operatorname{cn} u, && \operatorname{dn}(u\pm 2\mathsf{K}) = \operatorname{dn} u;\\
&\operatorname{sn}(u+i\mathsf{K}') = \frac{1}{k\operatorname{sn} u}, && \operatorname{cn}(u+i\mathsf{K}') = -\frac{i}{k}\frac{\operatorname{dn} u}{\operatorname{sn} u}, && \operatorname{dn}(u+i\mathsf{K}') = -i\frac{\operatorname{cn} u}{\operatorname{sn} u};\\
&\operatorname{sn}(u+2i\mathsf{K}') = \operatorname{sn} u, && \operatorname{cn}(u+2i\mathsf{K}') = -\operatorname{cn} u, && \operatorname{dn}(u+2i\mathsf{K}') = -\operatorname{dn} u;\\
&\operatorname{sn}(u+\mathsf{K}+i\mathsf{K}') = \frac{\operatorname{dn} u}{k\operatorname{cn} u}, && \operatorname{cn}(u+\mathsf{K}+i\mathsf{K}') = -\frac{ik'}{k\operatorname{cn} u}, && \operatorname{dn}(u+\mathsf{K}+i\mathsf{K}') = ik'\frac{\operatorname{sn} u}{\operatorname{cn} u};\\
&\operatorname{sn}(u+2\mathsf{K}+2i\mathsf{K}') = -\operatorname{sn} u, && \operatorname{cn}(u+2\mathsf{K}+2i\mathsf{K}') = \operatorname{cn} u, && \operatorname{dn}(u+2\mathsf{K}+2i\mathsf{K}') = -\operatorname{dn} u.
\end{aligned}$$

► **Periods, zeros, poles, and residues.**

TABLE 30.4
Periods, zeros, poles, and residues of the Jacobian elliptic functions (m, $n = 0, \pm1, \pm2, \ldots$; $i^2 = -1$)

Functions	Periods	Zeros	Poles	Residues
$\operatorname{sn} u$	$4m\mathsf{K}+2n\mathsf{K}'i$	$2m\mathsf{K}+2n\mathsf{K}'i$	$2m\mathsf{K}+(2n+1)\mathsf{K}'i$	$(-1)^m\frac{1}{k}$
$\operatorname{cn} u$	$(4m+2n)\mathsf{K}+2n\mathsf{K}'i$	$(2m+1)\mathsf{K}+2n\mathsf{K}'i$	$2m\mathsf{K}+(2n+1)\mathsf{K}'i$	$(-1)^{m-1}\frac{i}{k}$
$\operatorname{dn} u$	$2m\mathsf{K}+4n\mathsf{K}'i$	$(2m+1)\mathsf{K}+(2n+1)\mathsf{K}'i$	$2m\mathsf{K}+(2n+1)\mathsf{K}'i$	$(-1)^{n-1}i$

▶ **Double-argument formulas.**

$$\operatorname{sn}(2u) = \frac{2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^4u} = \frac{2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u}{\operatorname{cn}^2u+\operatorname{sn}^2u\operatorname{dn}^2u},$$
$$\operatorname{cn}(2u) = \frac{\operatorname{cn}^2u-\operatorname{sn}^2u\operatorname{dn}^2u}{1-k^2\operatorname{sn}^4u} = \frac{\operatorname{cn}^2u-\operatorname{sn}^2u\operatorname{dn}^2u}{\operatorname{cn}^2u+\operatorname{sn}^2u\operatorname{dn}^2u},$$
$$\operatorname{dn}(2u) = \frac{\operatorname{dn}^2u-k^2\operatorname{sn}^2u\operatorname{cn}^2u}{1-k^2\operatorname{sn}^4u} = \frac{\operatorname{dn}^2u+\operatorname{cn}^2u\,(\operatorname{dn}^2u-1)}{\operatorname{dn}^2u-\operatorname{cn}^2u\,(\operatorname{dn}^2u-1)}.$$

▶ **Half-argument formulas.**

$$\operatorname{sn}^2\frac{u}{2} = \frac{1}{k^2}\frac{1-\operatorname{dn}u}{1+\operatorname{cn}u} = \frac{1-\operatorname{cn}u}{1+\operatorname{dn}u},$$
$$\operatorname{cn}^2\frac{u}{2} = \frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{dn}u} = \frac{1-k^2}{k^2}\frac{1-\operatorname{dn}u}{\operatorname{dn}u-\operatorname{cn}u},$$
$$\operatorname{dn}^2\frac{u}{2} = \frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{cn}u} = (1-k^2)\frac{1-\operatorname{cn}u}{\operatorname{dn}u-\operatorname{cn}u}.$$

▶ **Argument addition formulas.**

$$\operatorname{sn}(u\pm v) = \frac{\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v\pm\operatorname{sn}v\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$
$$\operatorname{cn}(u\pm v) = \frac{\operatorname{cn}u\operatorname{cn}v\mp\operatorname{sn}u\operatorname{sn}v\operatorname{dn}u\operatorname{dn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$
$$\operatorname{dn}(u\pm v) = \frac{\operatorname{dn}u\operatorname{dn}v\mp k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}u\operatorname{cn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v}.$$

▶ **Conversion formulas.**

Table 30.5 presents conversion formulas for Jacobi elliptic functions. If $k > 1$, then $k_1 = 1/k < 1$. Elliptic functions with real modulus can be reduced, using the first set of conversion formulas, to elliptic functions with a modulus lying between 0 and 1.

▶ **Descending Landen transformation (Gauss's transformation).**

Notation:

$$\mu = \left|\frac{1-k'}{1+k'}\right|, \qquad v = \frac{u}{1+\mu}.$$

Descending transformations:

$$\operatorname{sn}(u,k) = \frac{(1+\mu)\operatorname{sn}(v,\mu^2)}{1+\mu\operatorname{sn}^2(v,\mu^2)}, \qquad \operatorname{cn}(u,k) = \frac{\operatorname{cn}(v,\mu^2)\operatorname{dn}(v,\mu^2)}{1+\mu\operatorname{sn}^2(v,\mu^2)},$$
$$\operatorname{dn}(u,k) = \frac{\operatorname{dn}^2(v,\mu^2)+\mu-1}{1+\mu-\operatorname{dn}^2(v,\mu^2)}.$$

TABLE 30.5
Conversion formulas for Jacobi elliptic functions. Full notation is used: $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$

u_1	k_1	$\mathrm{sn}(u_1,k_1)$	$\mathrm{cn}(u_1,k_1)$	$\mathrm{dn}(u_1,k_1)$
ku	$\dfrac{1}{k}$	$k\,\mathrm{sn}(u,k)$	$\mathrm{dn}(u,k)$	$\mathrm{cn}(u,k)$
iu	k'	$i\dfrac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$	$\dfrac{1}{\mathrm{cn}(u,k)}$	$\dfrac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}$
$k'u$	$i\dfrac{k}{k'}$	$k'\dfrac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1}{\mathrm{dn}(u,k)}$
iku	$i\dfrac{k'}{k}$	$ik\dfrac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1}{\mathrm{dn}(u,k)}$	$\dfrac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$
$ik'u$	$\dfrac{1}{k'}$	$ik'\dfrac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$	$\dfrac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}$	$\dfrac{1}{\mathrm{cn}(u,k)}$
$(1+k)u$	$\dfrac{2\sqrt{k}}{1+k}$	$\dfrac{(1+k)\,\mathrm{sn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$	$\dfrac{\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$	$\dfrac{1-k\,\mathrm{sn}^2(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$
$(1+k')u$	$\dfrac{1-k'}{1+k'}$	$\dfrac{(1+k')\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1-(1+k')\,\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1-(1-k')\,\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}$

► **Ascending Landen transformation.**

Notation:

$$\mu=\frac{4k}{(1+k)^2},\qquad \sigma=\left|\frac{1-k}{1+k}\right|,\qquad v=\frac{u}{1+\sigma}.$$

Ascending transformations:

$$\mathrm{sn}(u,k)=(1+\sigma)\frac{\mathrm{sn}(v,\mu)\,\mathrm{cn}(v,\mu)}{\mathrm{dn}(v,\mu)},\qquad \mathrm{cn}(u,k)=\frac{1+\sigma}{\mu}\frac{\mathrm{dn}^2(v,\mu)-\sigma}{\mathrm{dn}(v,\mu)},$$
$$\mathrm{dn}(u,k)=\frac{1-\sigma}{\mu}\frac{\mathrm{dn}^2(v,\mu)+\sigma}{\mathrm{dn}(v,\mu)}.$$

► **Series representation.**

Representation of Jacobi functions in the form of power series in u:

$$\begin{aligned}
\mathrm{sn}\,u&=u-\frac{1}{3!}(1+k^2)u^3+\frac{1}{5!}(1+14k^2+k^4)u^5\\
&\quad-\frac{1}{7!}(1+135k^2+135k^4+k^6)u^7+\cdots,\\
\mathrm{cn}\,u&=1-\frac{1}{2!}u^2+\frac{1}{4!}(1+4k^2)u^4-\frac{1}{6!}(1+44k^2+16k^4)u^6+\cdots,\\
\mathrm{dn}\,u&=1-\frac{1}{2!}k^2u^2+\frac{1}{4!}k^2(4+k^2)u^4-\frac{1}{6!}k^2(16+44k^2+k^4)u^6+\cdots,\\
\mathrm{am}\,u&=u-\frac{1}{3!}k^2u^3+\frac{1}{5!}k^2(4+k^2)u^5-\frac{1}{7!}k^2(16+44k^2+k^4)u^7+\cdots.
\end{aligned}$$

These functions converge for $|u|<|\mathsf{K}(k')|$.

Representation of Jacobi functions in the form of trigonometric series:

$$\operatorname{sn} u = \frac{2\pi}{k\mathsf{K}\sqrt{q}} \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n-1}} \sin\left[(2n-1)\frac{\pi u}{2\mathsf{K}}\right],$$
$$\operatorname{cn} u = \frac{2\pi}{k\mathsf{K}\sqrt{q}} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n-1}} \cos\left[(2n-1)\frac{\pi u}{2\mathsf{K}}\right],$$
$$\operatorname{dn} u = \frac{\pi}{2\mathsf{K}} + \frac{2\pi}{\mathsf{K}} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos\left(\frac{n\pi u}{\mathsf{K}}\right),$$
$$\operatorname{am} u = \frac{\pi u}{2\mathsf{K}} + 2\sum_{n=1}^{\infty} \frac{1}{n}\frac{q^n}{1+q^{2n}} \sin\left(\frac{n\pi u}{\mathsf{K}}\right),$$

where $q = \exp(-\pi\mathsf{K}'/\mathsf{K})$, $\mathsf{K} = \mathsf{K}(k)$, $\mathsf{K}' = \mathsf{K}(k')$, and $k' = \sqrt{1-k^2}$.

► **Derivatives and integrals.**

Derivatives:

$$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \qquad \frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \qquad \frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

Integrals:

$$\int \operatorname{sn} u\, du = \frac{1}{k}\ln(\operatorname{dn} u - k \operatorname{cn} u) = -\frac{1}{k}\ln(\operatorname{dn} u + k\operatorname{cn} u),$$
$$\int \operatorname{cn} u\, du = \frac{1}{k}\arccos(\operatorname{dn} u) = \frac{1}{k}\arcsin(k\operatorname{sn} u),$$
$$\int \operatorname{dn} u\, du = \arcsin(\operatorname{sn} u) = \operatorname{am} u.$$

The arbitrary additive constant C in the integrals is omitted.

30.14.2 Weierstrass Elliptic Function

► **Infinite series representation. Some properties.**

The Weierstrass elliptic function (or Weierstrass $\wp$-function) is defined as

$$\wp(z) = \wp(z|\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{m,n}\left[\frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2}\right],$$

where the summation is assumed over all integer m and n, except for $m = n = 0$. This function is a complex, double periodic function of a complex variable z with periods $2\omega_1$ and $2\omega_1$:

$$\wp(-z) = \wp(z),$$
$$\wp(z + 2m\omega_1 + 2n\omega_2) = \wp(z),$$

where $m,\, n = 0,\, \pm 1,\, \pm 2,\, \dots$ and $\mathrm{Im}(\omega_2/\omega_1) \neq 0$. The series defining the Weierstrass $\wp$-function converges everywhere except for second-order poles located at $z_{mn} = 2m\omega_1 + 2n\omega_2$.

Argument addition formula:

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left[\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right]^2.$$

► Representation in the form of a definite integral.

The Weierstrass function $\wp = \wp(z, g_2, g_3) = \wp(z|\omega_1, \omega_2)$ is defined implicitly by the elliptic integral

$$z = \int_{\wp}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \int_{\wp}^{\infty} \frac{dt}{2\sqrt{(t-e_1)(t-e_2)(t-e_3)}}.$$

The parameters g_2 and g_3 are known as the *invariants*.

The parameters e_1, e_2, e_3, which are the roots of the cubic equation $4z^3 - g_2 z - g_3 = 0$, are related to the half-periods ω_1, ω_2 and invariants g_2, g_3 by

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_1+\omega_2), \quad e_1 = \wp(\omega_2),$$
$$e_1+e_2+e_3 = 0, \quad e_1e_2 + e_1e_3 + e_2e_3 = -\tfrac{1}{4}g_2, \quad e_1e_2e_3 = \tfrac{1}{4}g_3.$$

Homogeneity property:

$$\wp(z, g_2, g_3) = \lambda^2 \wp(\lambda z, \lambda^{-4} g_2, \lambda^{-6} g_3).$$

► Representation as a Laurent series. Differential equations.

The Weierstrass $\wp$-function can be expanded into a Laurent series:

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \frac{3g_2g_3}{6160}z^8 + \cdots = \frac{1}{z^2} + \sum_{k=2}^{\infty} a_k z^{2k-2},$$
$$a_k = \frac{3}{(k-3)(2k+1)} \sum_{m=2}^{k-2} a_m a_{k-m} \quad \text{for} \quad k \geq 4, \qquad 0 < |z| < \min(|\omega_1|, |\omega_2|).$$

The Weierstrass $\wp$-function satisfies the first-order and second-order nonlinear differential equations

$$(\wp'_z)^2 = 4\wp^3 - g_2\wp - g_3,$$
$$\wp''_{zz} = 6\wp^2 - \tfrac{1}{2}g_2.$$

► **Connection with Jacobi elliptic functions.**

Direct and inverse representations of the Weierstrass elliptic function via Jacobi elliptic functions:

$$\wp(z) = e_1 + (e_1 - e_3)\frac{\operatorname{cn}^2 w}{\operatorname{sn}^2 w} = e_2 + (e_1 - e_3)\frac{\operatorname{dn}^2 w}{\operatorname{sn}^2 w} = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2 w};$$

$$\operatorname{sn} w = \sqrt{\frac{e_1 - e_3}{\wp(z) - e_3}}, \quad \operatorname{cn} w = \sqrt{\frac{\wp(z) - e_1}{\wp(z) - e_3}}, \quad \operatorname{dn} w = \sqrt{\frac{\wp(z) - e_2}{\wp(z) - e_3}};$$

$$w = z\sqrt{e_1 - e_3} = \mathsf{K}z/\omega_1.$$

The parameters are related by

$$k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}, \quad \mathsf{K} = \omega_1\sqrt{e_1 - e_3}, \quad i\mathsf{K}' = \omega_2\sqrt{e_1 - e_3}.$$

30.15 Jacobi Theta Functions

30.15.1 Series Representation of the Jacobi Theta Functions. Simplest Properties

► **Definition of the Jacobi theta functions.**

The *Jacobi theta functions* are defined by the following series:

$$\vartheta_1(v) = \vartheta_1(v,q) = \vartheta_1(v|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin[(2n+1)\pi v]$$
$$= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2} e^{i\pi(2n-1)v},$$

$$\vartheta_2(v) = \vartheta_2(v,q) = \vartheta_2(v|\tau) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos[(2n+1)\pi v] = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} e^{i\pi(2n-1)v},$$

$$\vartheta_3(v) = \vartheta_3(v,q) = \vartheta_3(v|\tau) = 1 + 2\sum_{n=0}^{\infty} q^{n^2}\cos(2n\pi v) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2i\pi nv},$$

$$\vartheta_4(v) = \vartheta_4(v,q) = \vartheta_4(v|\tau) = 1 + 2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos(2n\pi v) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2i\pi nv},$$

where v is a complex variable and $q = e^{i\pi\tau}$ is a complex parameter (τ has a positive imaginary part).

► **Simplest properties.**

The Jacobi theta functions are periodic entire functions that possess the following properties:

$\vartheta_1(v)$	odd,	has period 2,	vanishes at $v = m + n\tau$;
$\vartheta_2(v)$	even,	has period 2,	vanishes at $v = m + n\tau + \frac{1}{2}$;
$\vartheta_3(v)$	even,	has period 1,	vanishes at $v = m + (n + \frac{1}{2})\tau + \frac{1}{2}$;
$\vartheta_4(v)$	even,	has period 1,	vanishes at $v = m + (n + \frac{1}{2})\tau$.

Here, $m,\ n = 0,\ \pm 1,\ \pm 2,\ \dots$

Remark 30.1. The theta functions are not elliptic functions. The very good convergence of their series allows the computation of various elliptic integrals and elliptic functions using the relations given above in Section 30.15.1.

30.15.2 Various Relations and Formulas. Connection with Jacobi Elliptic Functions

► **Linear and quadratic relations.**

Linear relations (first set):

$$\begin{aligned}
\vartheta_1\Big(v+\frac{1}{2}\Big) &= \vartheta_2(v), & \vartheta_2\Big(v+\frac{1}{2}\Big) &= -\vartheta_1(v),\\
\vartheta_3\Big(v+\frac{1}{2}\Big) &= \vartheta_4(v), & \vartheta_4\Big(v+\frac{1}{2}\Big) &= \vartheta_3(v),\\
\vartheta_1\Big(v+\frac{\tau}{2}\Big) &= ie^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_4(v), & \vartheta_2\Big(v+\frac{\tau}{2}\Big) &= e^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_3(v),\\
\vartheta_3\Big(v+\frac{\tau}{2}\Big) &= e^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_2(v), & \vartheta_4\Big(v+\frac{\tau}{2}\Big) &= ie^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_1(v).
\end{aligned}$$

Linear relations (second set):

$$\begin{aligned}
\vartheta_1(v|\tau+1) &= e^{i\pi/4}\vartheta_1(v|\tau), & \vartheta_2(v|\tau+1) &= e^{i\pi/4}\vartheta_2(v|\tau),\\
\vartheta_3(v|\tau+1) &= \vartheta_4(v|\tau), & \vartheta_4(v|\tau+1) &= \vartheta_3(v|\tau),\\
\vartheta_1\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big) &= \frac{1}{i}\sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_1(v|\tau), & \vartheta_2\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_4(v|\tau),\\
\vartheta_3\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_3(v|\tau), & \vartheta_4\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_2(v|\tau).
\end{aligned}$$

Quadratic relations:

$$\begin{aligned}
\vartheta_1^2(v)\vartheta_2^2(0) &= \vartheta_4^2(v)\vartheta_3^2(0) - \vartheta_3^2(v)\vartheta_4^2(0),\\
\vartheta_1^2(v)\vartheta_3^2(0) &= \vartheta_4^2(v)\vartheta_2^2(0) - \vartheta_2^2(v)\vartheta_4^2(0),\\
\vartheta_1^2(v)\vartheta_4^2(0) &= \vartheta_3^2(v)\vartheta_2^2(0) - \vartheta_2^2(v)\vartheta_3^2(0),\\
\vartheta_4^2(v)\vartheta_4^2(0) &= \vartheta_3^2(v)\vartheta_3^2(0) - \vartheta_2^2(v)\vartheta_2^2(0).
\end{aligned}$$

► **Representation of the theta functions in the form of infinite products.**

$$\vartheta_1(v) = 2q_0 q^{1/4} \sin(\pi v) \prod_{n=1}^{\infty} \left[1 - 2q^{2n} \cos(2\pi v) + q^{4n}\right],$$

$$\vartheta_2(v) = 2q_0 q^{1/4} \cos(\pi v) \prod_{n=1}^{\infty} \left[1 + 2q^{2n} \cos(2\pi v) + q^{4n}\right],$$

$$\vartheta_3(v) = q_0 \prod_{n=1}^{\infty} \left[1 + 2q^{2n-1} \cos(2\pi v) + q^{4n-2}\right],$$

$$\vartheta_4(v) = q_0 \prod_{n=1}^{\infty} \left[1 - 2q^{2n-1} \cos(2\pi v) + q^{4n-2}\right],$$

where $q_0 = \prod\limits_{n=1}^{\infty} (1 - q^{2n})$.

► **Connection with Jacobi elliptic functions.**

Representations of Jacobi elliptic functions in terms of the theta functions:

$$\operatorname{sn} w = \frac{\vartheta_3(0)}{\vartheta_2(0)} \frac{\vartheta_1(v)}{\vartheta_4(v)}, \qquad \operatorname{cn} w = \frac{\vartheta_4(0)}{\vartheta_2(0)} \frac{\vartheta_2(v)}{\vartheta_4(v)}, \qquad \operatorname{dn} w = \frac{\vartheta_4(0)}{\vartheta_3(0)} \frac{\vartheta_3(v)}{\vartheta_4(v)}, \qquad w = 2\mathsf{K}v.$$

The parameters are related by

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}, \qquad k' = \frac{\vartheta_4^2(0)}{\vartheta_3^2(0)}, \qquad \mathsf{K} = \frac{\pi}{2}\vartheta_3^2(0), \qquad \mathsf{K}' = -i\tau\mathsf{K}.$$

30.16 Mathieu Functions and Modified Mathieu Functions

30.16.1 Mathieu Functions

► **Mathieu equation and Mathieu functions.**

The Mathieu functions $\operatorname{ce}_n(x, q)$ and $\operatorname{se}_n(x, q)$ are periodical solutions of the Mathieu equation

$$y''_{xx} + (a - 2q \cos 2x)y = 0.$$

Such solutions exist for definite values of parameters a and q (those values of a are referred to as eigenvalues). The Mathieu functions are listed in Table 30.6.

► **Properties of the Mathieu functions.**

The Mathieu functions possess the following properties:

$$\operatorname{ce}_{2n}(x, -q) = (-1)^n \operatorname{ce}_{2n}\left(\tfrac{\pi}{2} - x,\ q\right), \qquad \operatorname{ce}_{2n+1}(x, -q) = (-1)^n \operatorname{se}_{2n+1}\left(\tfrac{\pi}{2} - x,\ q\right),$$
$$\operatorname{se}_{2n}(x, -q) = (-1)^{n-1} \operatorname{se}_{2n}\left(\tfrac{\pi}{2} - x,\ q\right), \quad \operatorname{se}_{2n+1}(x, -q) = (-1)^n \operatorname{ce}_{2n+1}\left(\tfrac{\pi}{2} - x,\ q\right).$$

TABLE 30.6
Mathieu functions $\text{ce}_n = \text{ce}_n(x,q)$ and $\text{se}_n = \text{se}_n(x,q)$ (for odd n, functions ce_n and se_n are 2π-periodic, and for even n, they are π-periodic); definite eigenvalues $a = a_n(q)$ and $a = b_n(q)$ correspond to each value of parameter q

Mathieu functions	Recurrence relations for coefficients	Normalization conditions
$\text{ce}_{2n} = \sum\limits_{m=0}^{\infty} A_{2m}^{2n} \cos 2mx$	$qA_2^{2n} = a_{2n} A_0^{2n}$; $qA_4^{2n} = (a_{2n}-4)A_2^{2n} - 2qA_0^{2n}$; $qA_{2m+2}^{2n} = (a_{2n}-4m^2)A_{2m}^{2n} - qA_{2m-2}^{2n}$, $m \ge 2$	$(A_0^{2n})^2 + \sum\limits_{m=0}^{\infty} (A_{2m}^{2n})^2 = \begin{cases} 2 & \text{if } n=0, \\ 1 & \text{if } n \ge 1 \end{cases}$
$\text{ce}_{2n+1} = \sum\limits_{m=0}^{\infty} A_{2m+1}^{2n+1} \cos(2m+1)x$	$qA_3^{2n+1} = (a_{2n+1}-1-q)A_1^{2n+1}$; $qA_{2m+3}^{2n+1} = [a_{2n+1}-(2m+1)^2]A_{2m+1}^{2n+1} - qA_{2m-1}^{2n+1}$, $m \ge 1$	$\sum\limits_{m=0}^{\infty} (A_{2m+1}^{2n+1})^2 = 1$
$\text{se}_{2n} = \sum\limits_{m=0}^{\infty} B_{2m}^{2n} \sin 2mx$, $\text{se}_0 = 0$	$qB_4^{2n} = (b_{2n}-4)B_2^{2n}$; $qB_{2m+2}^{2n} = (b_{2n}-4m^2)B_{2m}^{2n} - qB_{2m-2}^{2n}$, $m \ge 2$	$\sum\limits_{m=0}^{\infty} (B_{2m}^{2n})^2 = 1$
$\text{se}_{2n+1} = \sum\limits_{m=0}^{\infty} B_{2m+1}^{2n+1} \sin(2m+1)x$	$qB_3^{2n+1} = (b_{2n+1}-1-q)B_1^{2n+1}$; $qB_{2m+3}^{2n+1} = [b_{2n+1}-(2m+1)^2]B_{2m+1}^{2n+1} - qB_{2m-1}^{2n+1}$, $m \ge 1$	$\sum\limits_{m=0}^{\infty} (B_{2m+1}^{2n+1})^2 = 1$

Selecting sufficiently large number m and omitting the term with the maximum number in the recurrence relations (indicated in Table 30.6), we can obtain approximate relations for eigenvalues a_n (or b_n) with respect to parameter q. Then, equating the determinant of the corresponding homogeneous linear system of equations for coefficients A_m^n (or B_m^n) to zero, we obtain an algebraic equation for finding $a_n(q)$ (or $b_n(q)$).

For fixed real $q \neq 0$, eigenvalues a_n and b_n are all real and different, while

$$\begin{aligned} &\text{if} \quad q > 0 \quad \text{then} \quad a_0 < b_1 < a_1 < b_2 < a_2 < \cdots; \\ &\text{if} \quad q < 0 \quad \text{then} \quad a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < b_4 < \cdots. \end{aligned}$$

The eigenvalues possess the properties

$$a_{2n}(-q) = a_{2n}(q), \quad b_{2n}(-q) = b_{2n}(q), \quad a_{2n+1}(-q) = b_{2n+1}(q).$$

Tables of the eigenvalues $a_n = a_n(q)$ and $b_n = b_n(q)$ can be found in Abramowitz and Stegun (1964, Chapter 20).

The solution of the Mathieu equation corresponding to eigenvalue a_n (or b_n) has n zeros on the interval $0 \le x < \pi$ (q is a real number).

► Asymptotic expansions as $q \to 0$ and $q \to \infty$.

Listed below are two leading terms of the asymptotic expansions of the Mathieu functions $\text{ce}_n(x,q)$ and $\text{se}_n(x,q)$, as well as of the corresponding eigenvalues $a_n(q)$ and $b_n(q)$, as $q \to 0$:

$$\mathrm{ce}_0(x,q) = \frac{1}{\sqrt{2}}\Big(1 - \frac{q}{2}\cos 2x\Big), \quad a_0(q) = -\frac{q^2}{2} + \frac{7q^4}{128};$$

$$\mathrm{ce}_1(x,q) = \cos x - \frac{q}{8}\cos 3x, \quad a_1(q) = 1 + q;$$

$$\mathrm{ce}_2(x,q) = \cos 2x + \frac{q}{4}\Big(1 - \frac{\cos 4x}{3}\Big), \quad a_2(q) = 4 + \frac{5q^2}{12};$$

$$\mathrm{ce}_n(x,q) = \cos nx + \frac{q}{4}\left[\frac{\cos(n+2)x}{n+1} - \frac{\cos(n-2)x}{n-1}\right], \quad a_n(q) = n^2 + \frac{q^2}{2(n^2-1)} \quad (n \ge 3);$$

$$\mathrm{se}_1(x,q) = \sin x - \frac{q}{8}\sin 3x, \quad b_1(q) = 1 - q;$$

$$\mathrm{se}_2(x,q) = \sin 2x - q\frac{\sin 4x}{12}, \quad b_2(q) = 4 - \frac{q^2}{12};$$

$$\mathrm{se}_n(x,q) = \sin nx - \frac{q}{4}\left[\frac{\sin(n+2)x}{n+1} - \frac{\sin(n-2)x}{n-1}\right], \quad b_n(q) = n^2 + \frac{q^2}{2(n^2-1)} \quad (n \ge 3).$$

Asymptotic results as $q \to \infty$ $(-\pi/2 < x < \pi/2)$:

$$a_n(q) \approx -2q + 2(2n+1)\sqrt{q} + \tfrac{1}{4}(2n^2 + 2n + 1),$$

$$b_{n+1}(q) \approx -2q + 2(2n+1)\sqrt{q} + \tfrac{1}{4}(2n^2 + 2n + 1),$$

$$\mathrm{ce}_n(x,q) \approx \lambda_n q^{-1/4}\cos^{-n-1} x\big[\cos^{2n+1}\xi\,\exp(2\sqrt{q}\sin x) + \sin^{2n+1}\xi\,\exp(-2\sqrt{q}\sin x)\big],$$

$$\mathrm{se}_{n+1}(x,q) \approx \mu_{n+1} q^{-1/4}\cos^{-n-1} x\big[\cos^{2n+1}\xi\,\exp(2\sqrt{q}\sin x) - \sin^{2n+1}\xi\,\exp(-2\sqrt{q}\sin x)\big],$$

where λ_n and μ_n are some constants independent of the parameter q, and $\xi = \frac{1}{2}x + \frac{\pi}{4}$.

30.16.2 Modified Mathieu Functions

The modified Mathieu functions $\mathrm{Ce}_n(x,q)$ and $\mathrm{Se}_n(x,q)$ are solutions of the modified Mathieu equation

$$y''_{xx} - (a - 2q\cosh 2x)y = 0,$$

with $a = a_n(q)$ and $a = b_n(q)$ being the eigenvalues of the Mathieu equation (see Section 30.16.1).

The modified Mathieu functions are defined as

$$\mathrm{Ce}_{2n+p}(x,q) = \mathrm{ce}_{2n+p}(ix,q) = \sum_{k=0}^{\infty} A_{2k+p}^{2n+p}\cosh[(2k+p)x],$$

$$\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q) = \sum_{k=0}^{\infty} B_{2k+p}^{2n+p}\sinh[(2k+p)x],$$

where p may be equal to 0 and 1, and coefficients A_{2k+p}^{2n+p} and B_{2k+p}^{2n+p} are indicated in Section 30.16.1.

30.17 Orthogonal Polynomials

All zeros of each of the orthogonal polynomials $\mathcal{P}_n(x)$ considered in this section are real and simple. The zeros of the polynomials $\mathcal{P}_n(x)$ and $\mathcal{P}_{n+1}(x)$ are alternating.

For Legendre polynomials, see Section 30.11.1.

30.17.1 Laguerre Polynomials and Generalized Laguerre Polynomials

► **Laguerre polynomials.**

The Laguerre polynomials $L_n = L_n(x)$ satisfy the second-order linear ordinary differential equation

$$xy''_{xx} + (1-x)y'_x + ny = 0$$

and are defined by the formulas

$$L_n(x) = \frac{1}{n!}e^x \frac{d^n}{dx^n}\bigl(x^n e^{-x}\bigr) = \frac{(-1)^n}{n!}\biggl[x^n - n^2 x^{n-1} + \frac{n^2(n-1)^2}{2!}x^{n-2} + \cdots\biggr].$$

The first four polynomials have the form

$$L_0(x) = 1, \quad L_1(x) = -x+1, \quad L_2(x) = \tfrac{1}{2}(x^2 - 4x + 2),$$
$$L_3(x) = \tfrac{1}{6}(-x^3 + 9x^2 - 18x + 6).$$

To calculate $L_n(x)$ for $n \ge 2$, one can use the recurrence formulas

$$L_{n+1}(x) = \frac{1}{n+1}\bigl[(2n+1-x)L_n(x) - nL_{n-1}(x)\bigr].$$

The functions $L_n(x)$ form an orthonormal system on the interval $0 < x < \infty$ with weight e^{-x}:

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\,dx = \begin{cases} 0 & \text{if } n \ne m, \\ 1 & \text{if } n = m. \end{cases}$$

The generating function is

$$\frac{1}{1-s}\exp\Bigl(-\frac{sx}{1-s}\Bigr) = \sum_{n=0}^{\infty} L_n(x) s^n, \qquad |s| < 1.$$

► **Generalized Laguerre polynomials.**

The generalized Laguerre polynomials $L_n^\alpha = L_n^\alpha(x)$ $(\alpha > -1)$ satisfy the equation

$$xy''_{xx} + (\alpha + 1 - x)y'_x + ny = 0$$

and are defined by the formulas

$$L_n^\alpha(x) = \frac{1}{n!}x^{-\alpha}e^x \frac{d^n}{dx^n}\bigl(x^{n+\alpha}e^{-x}\bigr)$$
$$= \sum_{m=0}^{n} C_{n+\alpha}^{n-m}\frac{(-x)^m}{m!} = \sum_{m=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(m+\alpha+1)}\frac{(-x)^m}{m!\,(n-m)!}.$$

Notation: $L_n^0(x) = L_n(x)$.

Special cases:

$$L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = \alpha + 1 - x, \quad L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}.$$

To calculate $L_n^\alpha(x)$ for $n \geq 2$, one can use the recurrence formulas

$$L_{n+1}^\alpha(x) = \frac{1}{n+1}\bigl[(2n + \alpha + 1 - x)L_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x)\bigr].$$

Other recurrence formulas:

$$L_n^\alpha(x) = L_{n-1}^\alpha(x) + L_n^{\alpha-1}(x), \qquad \frac{d}{dx}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x),$$
$$x\frac{d}{dx}L_n^\alpha(x) = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x).$$

The functions $L_n^\alpha(x)$ form an orthogonal system on the interval $0 < x < \infty$ with weight $x^\alpha e^{-x}$:

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{\Gamma(\alpha+n+1)}{n!} & \text{if } n = m. \end{cases}$$

The generating function is

$$(1-s)^{-\alpha-1} \exp\Bigl(-\frac{sx}{1-s}\Bigr) = \sum_{n=0}^\infty L_n^\alpha(x) s^n, \qquad |s| < 1.$$

30.17.2 Chebyshev Polynomials and Functions

► Chebyshev polynomials of the first kind.

The *Chebyshev polynomials of the first kind* $T_n = T_n(x)$ satisfy the second-order linear ordinary differential equation

$$(1 - x^2)y''_{xx} - xy'_x + n^2 y = 0 \qquad (30.17.2.1)$$

and are defined by the formulas

$$T_n(x) = \cos(n \arccos x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}\bigl[(1-x^2)^{n-\frac{1}{2}}\bigr]$$
$$= \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m} \quad (n = 0, 1, 2, \dots),$$

where $[A]$ stands for the integer part of a number A.

An alternative representation of the Chebyshev polynomials:

$$T_n(x) = \frac{(-1)^n}{(2n-1)!!}(1-x^2)^{1/2}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}.$$

The first five Chebyshev polynomials of the first kind are

$$T_0(x)=1,\quad T_1(x)=x,\quad T_2(x)=2x^2-1,\quad T_3(x)=4x^3-3x,\quad T_4(x)=8x^4-8x^2+1.$$

The recurrence formulas:

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\qquad n\ge 2.$$

The functions $T_n(x)$ form an orthogonal system on the interval $-1<x<1$, with

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx=\begin{cases}0 & \text{if } n\ne m,\\ \frac{1}{2}\pi & \text{if } n=m\ne 0,\\ \pi & \text{if } n=m=0.\end{cases}$$

The generating function is

$$\frac{1-sx}{1-2sx+s^2}=\sum_{n=0}^{\infty}T_n(x)s^n\qquad (|s|<1).$$

The functions $T_n(x)$ have only real simple zeros, all lying on the interval $-1<x<1$.

The normalized Chebyshev polynomials of the first kind, $2^{1-n}T_n(x)$, deviate from zero least of all. This means that among all polynomials of degree n with the leading coefficient 1, it is the maximum of the modulus $\max\limits_{-1\le x\le 1}|2^{1-n}T_n(x)|$ that has the least value, the maximum being equal to 2^{1-n}.

► Chebyshev polynomials of the second kind.

The *Chebyshev polynomials of the second kind* $U_n=U_n(x)$ satisfy the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx}-3xy'_x+n(n+2)y=0$$

and are defined by the formulas

$$\begin{aligned}U_n(x)&=\frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}=\frac{2^n(n+1)!}{(2n+1)!}\frac{1}{\sqrt{1-x^2}}\frac{d^n}{dx^n}(1-x^2)^{n+1/2}\\ &=\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)!}{m!\,(n-2m)!}(2x)^{n-2m}\qquad (n=0,\,1,\,2,\,\dots).\end{aligned}$$

The first five Chebyshev polynomials of the second kind are

$$U_0(x)=1,\quad U_1(x)=2x,\quad U_2(x)=4x^2-1,$$
$$U_3(x)=8x^3-4x,\quad U_4(x)=16x^4-12x^2+1.$$

The recurrence formulas:

$$U_{n+1}(x)=2xU_n(x)-U_{n-1}(x),\qquad n\ge 2.$$

The generating function is

$$\frac{1}{1-2sx+s^2}=\sum_{n=0}^{\infty}U_n(x)s^n \qquad (|s|<1).$$

The Chebyshev polynomials of the first and second kind are related by

$$U_n(x)=\frac{1}{n+1}\frac{d}{dx}T_{n+1}(x).$$

► **Chebyshev functions of the second kind.**

The *Chebyshev functions of the second kind*,

$$\mathsf{U}_0(x)=\arcsin x,$$
$$\mathsf{U}_n(x)=\sin(n\arccos x)=\frac{\sqrt{1-x^2}}{n}\frac{dT_n(x)}{dx} \quad (n=1,2,\dots),$$

just as the Chebyshev polynomials, also satisfy the differential equation (30.17.2.1).

The first five the Chebyshev functions are

$$\mathsf{U}_0(x)=0,\quad \mathsf{U}_1(x)=\sqrt{1-x^2},\quad \mathsf{U}_2(x)=2x\sqrt{1-x^2},$$
$$\mathsf{U}_3(x)=(4x^2-1)\sqrt{1-x^2},\quad \mathsf{U}_5(x)=(8x^3-4x)\sqrt{1-x^2}.$$

The recurrence formulas:

$$\mathsf{U}_{n+1}(x)=2x\mathsf{U}_n(x)-\mathsf{U}_{n-1}(x), \qquad n\ge 2.$$

The functions $\mathsf{U}_n(x)$ form an orthogonal system on the interval $-1<x<1$, with

$$\int_{-1}^{1}\frac{\mathsf{U}_n(x)\mathsf{U}_m(x)}{\sqrt{1-x^2}}\,dx=\begin{cases}0 & \text{if } n\neq m \text{ or } n=m=0,\\ \frac{1}{2}\pi & \text{if } n=m\neq 0.\end{cases}$$

The generating function is

$$\frac{\sqrt{1-x^2}}{1-2sx+s^2}=\sum_{n=0}^{\infty}\mathsf{U}_{n+1}(x)s^n \qquad (|s|<1).$$

30.17.3 Hermite Polynomials

► **Various representations of the Hermite polynomials.**

The *Hermite polynomials* $H_n=H_n(x)$ satisfy the second-order linear ordinary differential equation

$$y''_{xx}-2xy'_x+2ny=0$$

and are defined by the formulas

$$H_n(x)=(-1)^n\exp\bigl(x^2\bigr)\frac{d^n}{dx^n}\exp\bigl(-x^2\bigr)=\sum_{m=0}^{[n/2]}(-1)^m\frac{n!}{m!\,(n-2m)!}(2x)^{n-2m}.$$

The first five polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2,$$
$$H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12.$$

Recurrence formulas:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad n \geq 2;$$
$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x).$$

Integral representation:

$$H_{2n}(x) = \frac{(-1)^n 2^{2n+1}}{\sqrt{\pi}} \exp\bigl(x^2\bigr) \int_0^\infty \exp\bigl(-t^2\bigr) t^{2n} \cos(2xt)\, dt,$$
$$H_{2n+1}(x) = \frac{(-1)^n 2^{2n+2}}{\sqrt{\pi}} \exp\bigl(x^2\bigr) \int_0^\infty \exp\bigl(-t^2\bigr) t^{2n+1} \sin(2xt)\, dt,$$

where $n = 0,\ 1,\ 2,\ \ldots$

► Orthogonality. The generating function. An asymptotic formula.

The functions $H_n(x)$ form an orthogonal system on the interval $-\infty < x < \infty$ with weight e^{-x^2}:

$$\int_{-\infty}^\infty \exp\bigl(-x^2\bigr) H_n(x) H_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \sqrt{\pi}\, 2^n n! & \text{if } n = m. \end{cases}$$

Generating function:

$$\exp\bigl(-s^2 + 2sx\bigr) = \sum_{n=0}^\infty H_n(x) \frac{s^n}{n!}.$$

Asymptotic formula as $n \to \infty$:

$$H_n(x) \approx 2^{\frac{n+1}{2}} n^{\frac{n}{2}} e^{-\frac{n}{2}} \exp\bigl(x^2\bigr) \cos\Bigl(\sqrt{2n+1}\, x - \tfrac{1}{2}\pi n\Bigr).$$

► Hermite functions.

The *Hermite functions* $h_n(x)$ are introduced by the formula

$$h_n(x) = \exp\Bigl(-\frac{1}{2}x^2\Bigr) H_n(x) = (-1)^n \exp\Bigl(\frac{1}{2}x^2\Bigr) \frac{d^n}{dx^n} \exp\bigl(-x^2\bigr), \qquad n = 0, 1, 2, \ldots$$

The Hermite functions satisfy the second-order linear ordinary differential equation

$$h''_{xx} + (2n + 1 - x^2)h = 0.$$

The functions $h_n(x)$ form an orthogonal system on the interval $-\infty < x < \infty$ with weight 1:

$$\int_{-\infty}^\infty h_n(x) h_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \sqrt{\pi}\, 2^n n! & \text{if } n = m. \end{cases}$$

30.17.4 Jacobi Polynomials and Gegenbauer Polynomials

► **Jacobi polynomials.**

The *Jacobi polynomials*, $P_n^{\alpha,\beta}(x)$, are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx}+\bigl[\beta-\alpha-(\alpha+\beta+2)x\bigr]y'_x+n(n+\alpha+\beta+1)y=0$$

and are defined by the formulas

$$\begin{aligned}P_n^{\alpha,\beta}(x)&=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Bigl[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Bigr]\\&=2^{-n}\sum_{m=0}^{n}C_{n+\alpha}^{m}C_{n+\beta}^{n-m}(x-1)^{n-m}(x+1)^m,\end{aligned}$$

where the C_b^a are binomial coefficients.

The generating function:

$$2^{\alpha+\beta}R^{-1}(1-s+R)^{-\alpha}(1+s+R)^{-\beta}=\sum_{n=0}^{\infty}P_n^{\alpha,\beta}(x)s^n,\qquad R=\sqrt{1-2xs+s^2},\qquad |s|<1.$$

The Jacobi polynomials are orthogonal on the interval $-1\le x\le 1$ with weight $(1-x)^\alpha(1+x)^\beta$:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)\,dx
=\begin{cases}0 & \text{if } n\neq m,\\ \dfrac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\dfrac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} & \text{if } n=m.\end{cases}$$

For $\alpha>-1$ and $\beta>-1$, all zeros of the polynomial $P_n^{\alpha,\beta}(x)$ are simple and lie on the interval $-1<x<1$.

► **Gegenbauer polynomials.**

The *Gegenbauer polynomials* (also called *ultraspherical polynomials*), $C_n^{(\lambda)}(x)$, are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx}-(2\lambda+1)xy'_x+n(n+2\lambda)y=0$$

and are defined by the formulas

$$\begin{aligned}C_n^{(\lambda)}(x)&=\frac{(-2)^n}{n!}\frac{\Gamma(n+\lambda)\,\Gamma(n+2\lambda)}{\Gamma(\lambda)\,\Gamma(2n+2\lambda)}(1-x^2)^{-\lambda+1/2}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2}\\&=\sum_{m=0}^{[n/2]}(-1)^m\frac{\Gamma(n-m+\lambda)}{\Gamma(\lambda)\,m!\,(n-2m)!}(2x)^{n-2m}.\end{aligned}$$

Recurrence formulas:

$$C_{n+1}^{(\lambda)}(x) = \frac{2(n+\lambda)}{n+1}\, xC_n^{(\lambda)}(x) - \frac{n+2\lambda-1}{n+1} C_{n-1}^{(\lambda)}(x);$$
$$C_n^{(\lambda)}(-x) = (-1)^n C_n^{(\lambda)}(x), \qquad \frac{d}{dx} C_n^{(\lambda)}(x) = 2\lambda C_{n-1}^{(\lambda+1)}(x).$$

The generating function:

$$\frac{1}{(1-2xs+s^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) s^n.$$

The Gegenbauer polynomials are orthogonal on the interval $-1 \le x \le 1$ with weight $(1-x^2)^{\lambda-1/2}$:

$$\int_{-1}^{1} (1-x^2)^{\lambda-1/2} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{\pi\Gamma(2\lambda+n)}{2^{2\lambda-1}(\lambda+n)n!\,\Gamma^2(\lambda)} & \text{if } n = m. \end{cases}$$

30.18 Nonorthogonal Polynomials

30.18.1 Bernoulli Polynomials

► Definition. Basic properties.

The *Bernoulli polynomials* $B_n(x)$ are introduced by the formula

$$B_n(x) = \sum_{k=0}^{n} C_n^k B_k x^{n-k} \qquad (n = 0,\ 1,\ 2,\ \dots),$$

where C_n^k are the binomial coefficients and B_n are Bernoulli numbers (see Section 30.1.3).

The Bernoulli polynomials can be defined using the recurrence relation

$$B_0(x) = 1, \qquad \sum_{k=0}^{n-1} C_n^k B_k(x) = nx^{n-1}, \quad n = 2,\ 3,\ \dots$$

The first six Bernoulli polynomials are given by

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}, \quad B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x.$$

Basic properties:

$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad B'_{n+1}(x) = (n+1)B_n(x),$$
$$B_n(1-x) = (-1)^n B_n(x), \qquad (-1)^n E_n(-x) = E_n(x) + nx^{n-1},$$

where the prime denotes a derivative with respect to x, and $n = 0,\ 1,\ \dots$

Multiplication and addition formulas:

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\Bigl(x + \frac{k}{m}\Bigr),$$

$$B_n(x+y) = \sum_{k=0}^{n} C_n^k B_k(x) y^{n-k},$$

where $n = 0,\ 1,\ \dots$ and $m = 1,\ 2,\ \dots$

► **Generating function. Fourier series expansions. Integrals.**

The generating function is expressed as

$$\frac{te^{xt}}{e^t - 1} \equiv \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi).$$

This relation may be used as a definition of the Bernoulli polynomials.

Fourier series expansions:

$$B_n(x) = -2\frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - \frac{1}{2}\pi n)}{k^n} \qquad (n = 1,\ 0 < x < 1;\ \ n > 1,\ 0 \le x \le 1);$$

$$B_{2n-1}(x) = 2(-1)^n \frac{(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n-1}} \qquad (n = 1,\ 0 < x < 1;\ \ n > 1,\ 0 \le x \le 1);$$

$$B_{2n}(x) = 2(-1)^n \frac{(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}} \qquad (n = 1,\ 2,\ \dots,\ \ 0 \le x \le 1).$$

Integrals:

$$\int_a^x B_n(t)\, dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1},$$

$$\int_0^1 B_m(t) B_n(t)\, dt = (-1)^{n-1} \frac{m!\, n!}{(m+n)!} B_{m+n},$$

where m and n are positive integers and B_n are Bernoulli numbers.

30.18.2 Euler Polynomials

► **Definition. Basic properties.**

Definition:

$$E_n(x) = \sum_{k=0}^{n} C_n^k \frac{E_k}{2^n} \Bigl(x - \frac{1}{2}\Bigr)^{n-k} \qquad (n = 0,\ 1,\ 2,\ \dots),$$

where C_n^k are the binomial coefficients and E_n are Euler numbers.

The first six Euler polynomials are given by

$$E_0(x) = 1, \quad E_1(x) = x - \tfrac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{4},$$
$$E_4(x) = x^4 - 2x^3 + x, \quad E_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{2}x^2 - \tfrac{1}{2}.$$

Basic properties:

$$E_n(x+1)+E_n(x)=2x^n, \qquad E'_{n+1}=(n+1)E_n(x),$$
$$E_n(1-x)=(-1)^nE_n(x), \qquad (-1)^{n+1}E_n(-x)=E_n(x)-2x^n,$$

where the prime denotes a derivative with respect to x, and $n=0,\,1,\,\ldots$

Multiplication and addition formulas:

$$E_n(mx)=m^n\sum_{k=0}^{m-1}(-1)^kE_n\Bigl(x+\frac{k}{m}\Bigr), \qquad n=0,\,1,\,\ldots,\ \ m=1,\,3,\,\ldots;$$

$$E_n(mx)=-\frac{2}{n+1}m^n\sum_{k=0}^{m-1}(-1)^kE_{n+1}\Bigl(x+\frac{k}{m}\Bigr), \qquad n=0,\,1,\,\ldots,\ \ m=2,\,4,\,\ldots;$$

$$E_n(x+y)=\sum_{k=0}^{n}C_n^kE_k(x)y^{n-k}, \qquad n=0,\,1,\,\ldots$$

► Generating function. Fourier series expansions. Integrals.

The generating function is expressed as

$$\frac{2e^{xt}}{e^t+1}\equiv\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \qquad (|t|<\pi).$$

This relation may be used as a definition of the Euler polynomials.

Fourier series expansions:

$$E_n(x)=4\frac{n!}{\pi^{n+1}}\sum_{k=0}^{\infty}\frac{\sin\bigl((2k+1)\pi x-\frac{1}{2}\pi n\bigr)}{(2k+1)^{n+1}} \qquad (n=0,\ 0<x<1;\ \ n>0,\ 0\le x\le 1);$$

$$E_{2n}(x)=4(-1)^n\frac{(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\frac{\sin\bigl((2k+1)\pi x\bigr)}{(2k+1)^{2n+1}} \qquad (n=0,\ 0<x<1;\ \ n>0,\ 0\le x\le 1);$$

$$E_{2n-1}(x)=4(-1)^n\frac{(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\frac{\cos\bigl((2k+1)\pi x\bigr)}{(2k+1)^{2n}} \qquad (n=1,\,2,\,\ldots,\ \ 0\le x\le 1).$$

Integrals:

$$\int_a^x E_n(t)\,dt=\frac{E_{n+1}(x)-E_{n+1}(a)}{n+1},$$
$$\int_0^1 E_m(t)E_n(t)\,dt=4(-1)^n(2^{m+n+2}-1)\frac{m!\,n!}{(m+n+2)!}B_{m+n+2},$$

where $m,\,n=0,\,1,\,\ldots$ and B_n are Bernoulli numbers. The Euler polynomials are orthogonal for even $n+m$.

Connection with the Bernoulli polynomials:

$$E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] = \frac{2}{n}\left[B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right],$$

where $n = 1,\, 2,\, \dots$

⊙ *References for Chapter* 30: H. Bateman and A. Erdélyi (1953, 1955), N. W. McLachlan (1955), M. Abramowitz and I. A. Stegun (1964), W. Magnus, F. Oberhettinger, and R. P. Soni (1966), I. S. Gradshteyn and I. M. Ryzhik (2000), G. A. Korn and T. M. Korn (2000), S. Yu. Slavyanov and W. Lay (2000), D. Zwillinger (2002), A. D. Polyanin and V. F. Zaitsev (2003), E. W. Weisstein (2003).

REFERENCES

Abel, M. L. and Braselton, J. P., *Maple by Example, 3rd Edition*, AP Professional, Boston, 2005.

Abramowitz, M. and Stegun, I. A. (Editors), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics, Washington, DC, 1964.

Acrivos, A., A note of the rate of heat or mass transfer from a small sphere freely suspended in linear shear field, *J. Fluid Mech.,* Vol. 98, No. 2, pp. 299–304, 1980.

Adams, R., *Calculus: A Complete Course, 6th Edition*, Pearson Education, Toronto, 2006.

Akritas, A. G., *Elements of Computer Algebra with Applications*, Wiley, New York, 1989.

Aksenov, A. V., Linear differential relations between solutions of the equations of Euler–Poisson–Darboux class, *Mechanics of Solids*, Vol. 36, No. 1, pp. 11–15, 2001.

Akulenko, L. D. and Nesterov, S. V., Determination of the frequencies and forms of oscillations of non-uniform distributed systems with boundary conditions of the third kind, *J. Appl. Math. & Mech. (PMM)*, Vol. 61, No. 4, p. 531–538, 1997.

Akulenko, L. D. and Nesterov, S. V., Vibration of a nonhomogeneous membrane, *Mechanics of Solids*, Vol. 34, No. 6, pp. 112–121, 1999.

Akulenko, L. D. and Nesterov, S. V., Free vibrations of a homogeneous elliptic membrane, *Mechanics of Solids*, Vol. 35, No. 1, pp. 153–162, 2000.

Akulenko, L. D., Nesterov, S. V., and Popov, A. L., Natural frequencies of an elliptic plate with clamped edge, *Mechanics of Solids*, Vol. 36, No. 1, pp. 143–148, 2001.

Andreev, V. K., Kaptsov, O. V., Pukhnachov, V. V., and Rodionov, A. A., *Applications of Group-Theoretical Methods in Hydrodynamics*, Nauka, Moscow, 1994. (English translation: Kluwer, Dordrecht, 1999.)

Anton, H., Bivens, I., and Davis, S., *Calculus: Early Transcendental Single Variable, 8th Edition*, John Wiley & Sons, New York, 2005.

Appell, P., *Traité de Mécanique Rationnelle, T. 1: Statique. Dinamyque du Point (Ed. 6)*, Gauthier-Villars, Paris, 1953.

Arfken G. B. and Weber H. J., *Mathematical Methods for Physicists, 6th Edition*, Academic Press, New York, 2005.

Arscott, F., *Periodic Differential Equations*, Macmillan (Pergamon), New York, 1964.

Arscott, F., The Whittaker–Hill equation and the wave equation in paraboloidal coordinates, *Proc. Roy. Soc. Edinburg*, Vol. A67, pp. 265–276, 1967.

Babich, V. M., Kapilevich, M. B., Mikhlin, S. G., et al., *Linear Equations of Mathematical Physics* [in Russian], Nauka, Moscow, 1964.

Bahder, T. B., *Mathematica for Scientists and Engineers*, Addison-Wesley, Redwood City, CA, 1995.

Bandelli, R., Rajagopal, K. R., and Galdi, G. P., On some unsteady motions of fluids of second grade, *Arch. Mech.*, Vol. 47, pp. 661–667, 1995.

Barenblatt, G. I., On certain boundary-value problems for the equations of seepage of a liquid in fissured rocks, *J. Appl. Math. & Mech. (PMM)*, Vol. 27, No. 2, pp. 348–350, 1963.

Barenblatt, G. I., Zheltov, Yu. P., and Kochina, I. N., Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. & Mech. (PMM)*, Vol. 24, No. 5, pp. 1286–1303, 1960.

Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, Cambridge, 1970.

Batchelor, G. K., Mass transfer from a particle suspended in fluid with a steady linear ambient velocity distribution, *J. Fluid Mech.*, Vol. 95, No. 2, pp. 369–400, 1979.

Bateman, H. and Erdélyi, A., *Higher Transcendental Functions, Vol. 1 and Vol. 2*, McGraw-Hill, New York, 1953.

Bateman, H. and Erdélyi, A., *Tables of Integral Transforms, Vol. 1 and Vol. 2*, McGraw-Hill, New York, 1954.

Bateman, H. and Erdélyi, A., *Higher Transcendental Functions, Vol. 3*, McGraw-Hill, New York, 1955.

Beerends, R. J., ter Morschem, H. G., and van den Berg, J. C., *Fourier and Laplace Transforms*, Cambridge Univ. Press, Cambridge, 2003.

Bellman, R. and Roth, R., *The Laplace Transform*, World Scientific Publ. Co., Singapore, 1984.

Belotserkovskii, O. M., and Oparin, A. A., *Numerical Experiment in Turbulence* [in Russian], Nauka, Moscow, 2000.

Benenson, W., Harris, J. W., Stocker, H., and Lutz, H., (Eds.), *Handbook of Physics*, Springer-Verlag, New York, 2002.

Berker, R., Intégration des équations du mouvement d'un fluide visqueux incompressible, In: *Encyclopedia of Physics: Fluid Dynamics II*, Vol. VIII/2, pp. 1–384 (Editors S. Flügge and C. Truesdell), Springer, Berlin, 1963.

Beyer, W. H., *CRC Standard Mathematical Tables and Formulae*, CRC Press, Boca Raton, FL, 1991.

Bitsadze, A. V. and Kalinichenko, D. F., *Collection of Problems on Mathematical Physics Equations* [in Russian], Nauka, Moscow, 1985.

Bluman, G. W., Cheviakov, A. F., and Anco, S. C., *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2010.

Bôcher, M., *Die Reihenentwickelungen der Potentialtheory*, Leipzig, 1894.

Bolotin V. V. (Editor), *Vibration in Engineering: A Handbook. Vol. 1. Vibration of Linear Systems* [in Russian], Mashinostroenie, Moscow, 1978.

Borzykh, A. A. and Cherepanov, G. P., A plane problem of the theory of convective heat transfer and mass exchange, *J. Appl. Math. & Mech. (PMM)*, Vol. 42, No. 5, pp. 848–855, 1978.

Boyer, C., The maximal kinematical invariance group for an arbitrary potential, *Helv. Phys. Acta*, Vol. 47, pp. 589–605, 1974.

Boyer, C., Lie theory and separation of variables for equation $iU_t + \Delta_2 U - (\alpha/x_1^2 + \beta/x_2^2)U = 0$, *SIAM J. Math. Anal.*, Vol. 7, pp. 230–263, 1976.

Bracewell, R., *The Fourier Transform and Its Applications, 3rd Edition*, McGraw-Hill, New York, 1999.

Brenner, H., Forced convection-heat and mass transfer at small Peclet numbers from particle of arbitrary shape, *Chem. Eng. Sci.*, Vol. 18, No. 2, pp. 109–122, 1963.

Bronshtein, I. N. and Semendyayev, K. A., *Handbook of Mathematics, 4th Edition*, Springer-Verlag, Berlin, 2004.

Brüning, J., Grushin, V. V., and Dobrokhotov, S. Yu., Approximate formulas for eigenvalues of the Laplace operator on a torus arising in linear problems with oscillating coefficients, *Russ. J. Math. Phys.*, Vol. 19, No. 3, pp. 261–272, 2012.

Brychkov, Yu. A. and Prudnikov, A. P., *Integral Transforms of Generalized Functions*, Gordon & Breach Sci. Publ., New York, 1989.

Budak, B. M., Samarskii, A. A., and Tikhonov, A. N., *Collection of Problems on Mathematical Physics* [in Russian], Nauka, Moscow, 1980.

Butkov, E., *Mathematical Physics*, Addison-Wesley, Reading, MA, 1968.

Butkovskiy, A. G., *Characteristics of Systems with Distributed Parameters* [in Russian], Nauka, Moscow, 1979.

Butkovskiy, A. G., *Green's Functions and Transfer Functions Handbook*, Halstead Press–John Wiley & Sons, New York, 1982.

Calmet, J. and van Hulzen, J. A., *Computer Algebra Systems. Computer Algebra: Symbolic and Algebraic Computations, 2nd Edition*, Springer, New York, 1983.

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1984.

Cauchy A.-L., Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique, *Ex. de Math.*, Vol. 3, pp. 160–187, 1828.

Chadwick, P. and Trowbridge, E. A., Elastic wave fields generated by scalar wave functions, *Proc. Cambridge Phil. Soc.*, No. 63, pp. 1177–1187, 1967.

Chandrasechariah, D. S., Naghdi–Hsu type solution in elastodynamics, *Acta Mechanica*, No. 76, pp. 235–241, 1988.

Char, B. W., Geddes, K. O., Gonnet, G. H., Monagan, M. B., and Watt, S. M., *Maple Reference Manual*, Waterloo Maple Publishing, Waterloo, Ontario, Canada, 1990.

Cheb-Terrab, E. S. and von Bulow, K., A computational approach for the analytical solving of partial differential equations. *Computer Physics Communications*, Vol. 90, pp. 102–116, 1995.

Cherpakov, P. V., *Theory of Regular Heat Exchange* [in Russian], Energiya, Moscow, 1975.

Christov, I. C., Stokes' first problem for some non-Newtonian fluids: Results and mistakes, *Mech. Research Comm.*, Vol. 37, No. 8, pp. 717–723, 2010.

Christov, I. C. and Christov, C. I., Comment on "On a class of exact solutions of the equations of motion of a second grade fluid" by C. Fetecâu and J. Zierep (*Acta Mech.* 150, 135–138, 2001), *Acta Mech.*, Vol. 215, pp. 25–28, 2010.

Christov, I. C. and Jordan, P. M., Comments on: "Starting solutions for some unsteady unidirectional flows of a second grade fluid," (*Int. J. Eng. Sci.* 43 (2005) 781), *Int. J. Eng. Sci.*, Vol. 51, pp. 326–332, 2012.

Cole K. D., Beck J. V., Haji-Sheikh A., and Litkouhi, B., *Methods for Obtaining Green's Functions*, In: *Heat Conduction Using Green's Functions*, Chapter 4, pp. 101–148, Taylor and Francis, Boca Raton, FL, 2011.

Collatz, L. O., *The Numerical Treatment of Differential Equations*, Springer, Berlin, 1966.

Colton, D., *Partial Differential Equations. An Introduction,* Random House, New York, 1988.

Constanda, C., *Solution Techniques for Elementary Partial Differential Equations*, Chapman & Hall/CRC Press, Boca Raton, FL, 2002.

Cook, R. D., Malkus, D. S., and Plesha, M. E., *Concepts and Applications of Finite Element Analysis*, John Wiley & Sons, New York, 1989.

Corless, R. M., *Essential Maple*, Springer, Berlin, 1995.

Courant, R. and Hilbert, D., *Methods of Mathematical Physics, Vol. 2*, Wiley–Interscience Publ., New York, 1989.

Courant, R. and John, F., *Introduction to Calculus and Analysis, Vol. 1*, Springer-Verlag, New York, 1999.

Cowin, S. C. and Nunziato, J. W., Linear elastic materials with voids, *J. Elasticity*, Vol. 12, No. 2, pp. 125–147, 1983.

Crank, J., *The Mathematics of Diffusion*, Clarendon Press, Oxford, 1975.

Crank, J. and Nicolson, P., A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, *Proc. Camb. Philos. Soc.*, Vol. 43, pp. 50–67, 1947.

Davenport, J. H., Siret, Y., and Tournier. E., *Computer Algebra Systems and Algorithms for Algebraic Computation*, Academic Press, London, 1993.

Davis, B., *Integral Transforms and Their Applications*, Springer-Verlag, New York, 1978.

Davis, E. J., Exact solutions for a class of heat and mass transfer problems, *Can. J. Chem. Eng.*, Vol. 51, No. 5, pp. 562–572, 1973.

Deavours, C. A., An exact solution for the temperature distribution in parallel plate Poiseuille flow, *Trans. ASME, J. Heat Transfer*, Vol. 96, No. 4, 1974.

Debnath, L., *Linear Partial Differential Equations for Scientists and Engineers, 4th Edition*, Birkhäuser, Boston, 2007.

Debnath, L. and Bhatta, B., *Integral Transforms and Their Applications, 2nd Edition*, Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

Dezin, A. A., *Partial Differential Equations. An Introduction to a General Theory of Linear Boundary Value Problems*, Springer-Verlag, Berlin-New York, 1987.

Ditkin, V. A. and Prudnikov, A. P., *Integral Transforms and Operational Calculus*, Pergamon Press, New York, 1965.

Doetsch, G., *Handbuch der Laplace-Transformation. Vol. I. Theorie der Laplace-Transformation*, Birkhäuser, Basel, 1950.

Doetsch, G., *Handbuch der Laplace-Transformation. Vol. II. Anwendungen der Laplace-Transformation. Part 1*, Birkhäuser, Basel, 1956.

Doetsch, G., *Handbuch der Laplace-Transformation. Vol. III. Anwendungen der Laplace-Transformation. Part 2*, Birkhäuser, Basel, 1958.

Duffy, D. G., *Transform Methods for Solving Partial Differential Equations, 2nd Edition*, Chapman & Hall/CRC Press, Boca Raton, FL, 2004.

Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, 1961.

Edwards, C. H. and Penney, D., *Calculus, 6th Edition*, Pearson Education, Toronto, 2002.

Elrick, D. E., Source functions for diffusion in uniform shear flows, *Australian J. Phys.*, Vol. 15, No. 3, p. 283–288, 1962.

Eringen, A. C. and Şuhubi, E. S., *Elastodynamics*, Academic Press, New York, 1975.

Faddeev, L. D. (Editor), *Mathematical Physics. Encyclopedia* [in Russian], Bol'shaya Rossiiskaya Entsyklopediya, Moscow, 1998.

Faminskii, A. V., On mixed problems for the Corteveg–de Vries equation with irregular boundary data, *Doklady Mathematics*, Vol. 59, No. 3, pp. 366–367, 1999.

Farlow, S. J., *Partial Differential Equations for Scientists and Engineers*, John Wiley & Sons, New York, 1982.

Fedotov, I., Gai, Y., Polyanin, A., and Shatalov, M., Analysis for an N-stepped Rayleigh bar with sections of complex geometry, *Applied Math. Modelling*, Vol. 32, pp. 1–11, 2008.

Fedotov, I. A., Polyanin, A. D., and Shatalov, M. Yu., Theory of free and forced vibrations of a rigid rod based on the Rayleigh model, *Doklady Physics*, Vol. 52, No. 11, pp. 607–612, 2007.

Fedotov, I. A., Polyanin, A. D., Shatalov, M. Yu., and Tenkam, H. M., Longitudinal vibrations of a Rayleigh–Bishop rod, *Doklady Physics*, Vol. 55, No. 12, pp. 609–614, 2010.

Fetecâu, C. and Zierep, J., On a class of exact solutions of the equations of motion of a second grade fluid, *Acta Mechanica*, Vol. 150, pp. 135–138, 2001.

Fitzpatrick, R., *Maxwell's Equations and the Principles of Electromagnetism*, Infinity Science Press, Hingham, MA, 2008.

Fogolari, F., Zuccato, P., Esposito, G., and Viglino, P., Bio-molecular electrostatics with the linearized Poisson–Boltzmann equation, *Biophysical Journal*, Vol. 76, pp. 1–16, 1999.

Gabov, S. A. and Orazov, B. B., The equation $\frac{\partial^2}{\partial t^2}[u_{xx}-u]+u_{xx}=0$ and several problems associated with it, *U.S.S.R. Comput. Math. Math. Phys.*, Vol. 26, No. 1, pp. 58–64, 1986.

Gabov, S. A. and Sveshnikov, A. G., *Linear Problems of the Theory of Nonstationary Internal Waves* [in Russian], Nauka, Moscow, 1990.

Galaktionov, V. A., On new exact blow-up solutions for nonlinear heat conduction equations, *Differential and Integral Equations*, Vol. 3, pp. 863–874, 1990.

Galaktionov, V. A., Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities, *Proc. Royal. Soc. Edinburgh Sect. A*, Vol. 125, No. 2, pp. 225–246, 1995.

Galerkin, B., Contribution a la solution ge ne rale du probleme de la the orie de l'e lasticite dans le cas de trios dimensions, *Comptes Rendus*, Vol. 190, pp. 1047–1048, 1930.

Geddes, K. O., Czapor, S. R., and Labahn G., *Algorithms for Computer Algebra*, Kluwer Academic Publishers, Boston, MA, 1992.

Gelfand, I. M., *Lectures on Linear Algebra*, Dover, New York, 1989.

Gelfand, I. M. and Shilov, G. E., *Distributions and Operations on Them* [in Russian], Fizmatlit, Moscow, 1959.

Getz, C. and Helmstedt, J., *Graphics with Mathematica: Fractals, Julia Sets, Patterns and Natural Forms*, Elsevier Science & Technology Books, Amsterdam, Boston, 2004.

Gradshteyn, I. S. and Ryzhik, I. M., *Tables of Integrals, Series, and Products*, Academic Press, Orlando, FL, 2000.

Graetz, L., Über die Warmeleitungsfähigkeit von Flüssigkeiten, *Annln. Phys.*, Bd. 18, S. 79–84, 1883.

Gray, J. W., *Mastering Mathematica: Programming Methods and Applications*, Academic Press, San Diego, 1994.

Gray, T. and Glynn, J., *Exploring Mathematics with Mathematica: Dialogs Concerning Computers and Mathematics*, Addison-Wesley, Reading, MA, 1991.

Green, A. E. and Naghdi, P. M., Thermoelasticity without energy dissipation, *J. Elasticity*, Vol. 31, No. 3, pp. 189–208, 1993.

Green, E., Evans, B., and Johnson, J., *Exploring Calculus with Mathematica*, Wiley, New York, 1994.

Green, G., *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, T. Wheelhouse, Nottingham, England, 1828.

Griffiths, D. J., *Introduction to Electrodynamics, 3rd Edition*, Benjamin Cummings, New York, 1999.

Guenther, R. B. and Lee, J. W., *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publ., Mineola, NY, 1996.

Gupalo, Yu. P., Polyanin, A. D., and Ryazantsev, Yu. S., *Mass Exchange of Reacting Particles with Flow* [in Russian], Nauka, Moscow, 1985.

Gupalo, Yu. P. and Ryazantsev, Yu. S., Mass and heat transfer from a sphere in a laminar flow, *Chem. Eng. Sci.*, Vol. 27, pp. 61–68, 1972.

Gurtin, M. E., On Helmboltz's theorem and the completeness of the Papkovich–Neuber stress functions for infinite domains, *Archive for Rational Mechanics and Analysis*, Vol. 9, No. 1, pp. 225–233, 1962.

Gurtin, M. E., The linear theory of elasticity, In: *Encyclopedia of Physics: Mechanics of Solids II*, Vol. VIa/2 (Editors S. Flügge and C. Truesdell), Springer, Berlin, 1972.

Gurtin, M. E. and Sternberg, E., On the linear theory of viscoelasticity, *Archive for Rational Mechanics and Analysis*, Vol. 11, No. 1, pp. 291–356, 1962.

Gusev, N. A., Weak and strong convergence of solutions to linearized equations of low compressible fluid [in Russian], *Vestnik Samar. Gos. Tekhn. Univ. Ser. Fiz.-Mat. Nauki*, No. 1 (22), pp. 47–52, 2011.

Haberman, R., *Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Prentice-Hall, Englewood Cliffs, NJ, 1987.

Hadamard, J., *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover Publications, New York, 1952.

Hanna, J. R. and Rowland, J. H., *Fourier Series, Transforms, and Boundary Value Problems*, Wiley-Interscience Publ., New York, 1990.

Hansen, E. R., *A Table of Series and Products*, Prentice Hall, Englewood Cliffs, NJ, 1975.

Happel, J. and Brenner, H., *Low Reynolds Number Hydrodynamics*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

Harries, D., Solving the Poisson–Boltzmann equation for two parallel cylinders, *Langmuir*, Vol. 14, pp. 3149–3152, 1998.

Heck, A., *Introduction to Maple, 3rd Edition*, Springer, New York, 2003.

Helmholtz, H., Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, *J. Reine und Angewandte Mathematik*, No. 9, pp. 1–62, 1858.

Higham, N. J., *Functions of Matrices. Theory and Computation*, SIAM, Philadelphia, 2008.

Hirschman, I. I. and Widder, D. V., *The Convolution Transform*, Princeton University Press, Princeton, 1955.

Hörmander, L., *The Analysis of Linear Partial Differential Operators. II. Differential Operators with Constant Coefficients*, Springer-Verlag, Berlin-New York, 1983.

Hörmander, L., *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis*, Springer-Verlag, Berlin, 1990.

Ibragimov N. H. (Editor), *CRC Handbook of Lie Group to Differential Equations, Vol. 1*, CRC Press, Boca Raton, FL, 1994.

Ivanov, V. I. and Trubetskov, M. K., *Handbook of Conformal Mapping with Computer-Aided Visualization*, CRC Press, Boca Raton, FL, 1994.

John, F., *Partial Differential Equations*, Springer-Verlag, New York, 1982.

Joseph, D. D., *Fluid Dynamics of Viscoelastic Liquids* (Appl. Math. Sciences, Vol. 84), Springer, New York, 1990.

Kalnins, E., On the separation of variables for the Laplace equation in two- and three-dimensional Minkowski space, *SIAM J. Math. Anal., Hung.*, Vol. 6, pp. 340–373, 1975.

Kalnins, E. and Miller, W. (Jr.), Lie theory and separation of variables, 5: The equations $iU_t + U_{xx} = 0$ and $iU_t + U_{xx} - \frac{c}{x^2}U = 0$, *J. Math. Phys.*, Vol. 15, pp. 1728–1737, 1974.

Kalnins, E. and Miller, W. (Jr.), Lie theory and separation of variables, 8: Semisubgroup coordinates for $\Psi_{tt} - \Delta_2\Psi = 0$, *J. Math. Phys.*, Vol. 16, pp. 2507–2516, 1975.

Kalnins, E. and Miller, W. (Jr.), Lie theory and separation of variables, 9: Orthogonal R-separable coordinate systems for the wave equation $\Psi_{tt} - \Delta_2\Psi = 0$, *J. Math. Phys.*, Vol. 17, pp. 331–335, 1976.

Kalnins, E. and Miller, W. (Jr.), Lie theory and separation of variables, 10: Nonorthogonal R-separable solutions of the wave equation $\Psi_{tt} - \Delta_2\Psi = 0$, *J. Math. Phys.*, Vol. 17, pp. 356–368, 1976.

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, II, Partielle Differentialgleichungen Erster Ordnung für eine gesuchte Funktion*, Akad. Verlagsgesellschaft Geest & Portig, Leipzig, 1965.

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.

Kanwal, R. P., *Generalized Functions. Theory and Technique*, Academic Press, Orlando, FL, 1983.

Karlsson, A. and Ritke, S., Time-domain theory of forerunners, *J. Opt. Soc. Am.*, Vol. A 15, pp. 487–502, 1998.

Kato, T., *Perturbation Theory for Linear Operators*, Springer, Berlin, 1995.

Kline, M., *Calculus: An Intuitive and Physical Approach, 2nd Edition*, Dover Publications, New York, 1998.

Knabner, P. and Angerman, L., *Numerical Methods for Elliptic and Parabolic Partial Differential Equations*, Springer, New York, 2003.

Korn, G. A. and Korn, T. M., *Mathematical Handbook for Scientists and Engineers, 2nd Edition*, Dover Publications, New York, 2000.

Koshlyakov, N. S., Gliner, E. B., and Smirnov, M. M., *Partial Differential Equations of Mathematical Physics* [in Russian], Vysshaya Shkola, Moscow, 1970.

Kowalevski, S., Über die Brechung des Lichtes in cristallinischen Mitteln, *Acta Math.*, No. 6, 249–304, 1885.

Krein, S. G. (Editor), *Functional Analysis* [in Russian], Nauka, Moscow, 1972.

Kreyszig, E., *Maple Computer Guide for Advanced Engineering Mathematics, 8th Edition*, Wiley, New York, 2000.

Kristensson, G., Karlsson, A., and Rikte, S., Electromagnetic wave propagation in dispersive and complex material with time-domain techniques, In: *Scattering and Inverse Scattering in Pure and Applied Science* (eds R. Pike and P. Sabatier), pp. 277–294, Academic Press, London, 2002.

Krylov, A. N., *Collected Works. III. Mathematics, Pt. 2* [in Russian], Izd-vo AN SSSR, Moscow, 1949.

Kudryashov, N. A. and Sinelshchikov, D. I., The Cauchy problem for the equation of the Burgers hierarchy, *Nonlinear Dynamics*, Vol. 76, No. 1, pp. 561–569, 2014.

Ladyzhenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flow, 2nd Edition*, Gordon & Breach, New York, 1969.

Lamb, H., *Hydrodynamics*, Dover Publ., New York, 1945.

Landau, L. D. and Lifshitz, E. M., *Quantum Mechanics. Nonrelativistic Theory* [in Russian], Nauka, Moscow, 1974.

Landau, L. D. and Lifshitz, E. M., *The Classical Theory of Fields, Fourth Edition: Volume 2* (Course of Theoretical Physics Series), Butterworth-Heinemann, Oxford, 1980.

Lapidus, L. and Pinder, G. F., *Numerical Solution of Partial Differential Equations in Science and Engineering*, Wiley-Interscience, New York, 1999.

Larsson S. and Thomée, V., *Partial Differential Equations with Numerical Methods*, Springer, New York, 2008.

Lavrent'ev, M. A. and Shabat B. V., *Methods of Complex Variable Theory* [in Russian], Nauka, Moscow, 1973.

Lavrik, V. I. and Savenkov, V. N., *Handbook of Conformal Mappings* [in Russian], Naukova Dumka, Kiev, 1970.

Lax, P. D., Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.*, Vol. 21, pp. 467–490, 1968.

Lebedev, N. N., Skal'skaya, I. P., and Uflyand, Ya. S., *Collection of Problems on Mathematical Physics* [in Russian], Gostekhizdat, Moscow, 1955.

Lee, H. J. and Schiesser, W. E., *Ordinary and Partial Differential Equation Routines in C, C++, Fortran, Java, Maple, and MATLAB*, Chapman & Hall/CRC Press, Boca Raton, FL, 2004.

Lee, K.-C. and Finlayson, D. A., Stability of plane Poiseuille and Couette flow of a Maxwell fluid, *J. Non-Newtonian Mech.*, Vol. 21, pp. 65–78, 1986.

Leis, R., *Initial-Boundary Value Problems in Mathematical Physics,* John Wiley & Sons, Chichester, 1986.

LeVeque, R. J., *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*, SIAM, Philadelphia, 2007.

Levich, V. G., *Physicochemical Hydrodynamics*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

Levitan, B. M. and Sargsyan, I. S., *Sturm–Liouville and Dirac Operators* [in Russian], Nauka, Moscow, 1988.

Li, J. and Chen, Y. T., *Computational Partial Differential Equations Using MATLAB*, Chapman & Hall/CRC Press, Boca Raton, FL, 2009.

Lipatov, I. I. and Polyanin, A. D., Decomposition and exact solutions of equations of a weakly compressible barotropic fluid, *Doklady Physics*, Vol. 58, No. 3, pp. 116–120, 2013.

Loitsyanskiy, L. G., *Mechanics of Liquids and Gases*, Begell House, New York, 1996.

Lychev, S. A. and Polyanin, A. D., Methods for constructing solutions of systems of linear equations arising in elasticity theory and hydrodynamics [in Russian], *Bulletin of the National Research Nuclear University MEPhI*, Vol. 4, No. 4, 2015.

Lykov, A. V., *Theory of Heat Conduction* [in Russian], Vysshaya Shkola, Moscow, 1967.

Mackie, A. G., *Boundary Value Problems*, Scottish Academic Press, Edinburgh, 1989.

Maeder, R. E., *Programming in Mathematica*, Addison-Wesley, Reading, MA, third edition, 1996.

Magnus, W., Oberhettinger, F., and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd Edition*, Springer-Verlag, Berlin, 1966.

Makarov, A., Smorodinsky, J., Valiev, K., and Winternitz, P., A systematic search for nonrelativistic systems with dynamical symmetries. Part I: The integrals of motion, *Nuovo Cimento*, Vol. 52A, pp. 1061–1084, 1967.

Mangulis, V., *Handbook of Series for Scientists and Engineers*, Academic Press, New York, 1965.

Manzhirov, A. V. and Polyanin, A. D., *Solution Methods for Integral Equations: Handbook* [in Russian], Faktorial, Moscow, 1999.

Marchenko, V. A., *Sturm–Liouville Operators and Applications,* Birkhauser Verlag, Basel–Boston, 1986.

Markeev, A. P., *Theoretical Mechanics* [in Russian], Nauka, Moscow, 1990.

Maslov, V. P., *Théorie des Perturbations et Méthodes Asymptotiques*, Dunod, Paris, 1972.

Maslov, V. P., *Operational Methods*, Mir, Moscow, 1976.

Maslov, V. P., *The Complex WKB Method for Nonlinear Equations I. Linear Theory*, Birkhäuser, Basel–Boston–Berlin, 1994.

Mathematical Encyclopedia, Vol. 1 (Editor-in-Chief I. M. Vinogradov) [in Russian], Sovetskaya Entsiklopediya, Moscow, 1977.

Mathematical Encyclopedia, Vol. 2 (Editor-in-Chief I. M. Vinogradov) [in Russian], Sovetskaya Entsiklopediya, Moscow, 1979.

Mathews, J. H. and Fink, K. D., *Numerical Methods Using MATLAB, 3rd Edition*, Prentice Hall, Upper Saddle River, NJ, 1999.

McLachlan, N. W., *Theory and Application of Mathieu Functions*, Clarendon Press, Oxford, 1947.

McLachlan, N. W., *Bessel Functions for Engineers*, Clarendon Press, Oxford, 1955.

Meade, D. B., May, S. J. M., Cheung, C-K., and Keough, G. E., *Getting Started with Maple, 3rd Edition*, Wiley, Hoboken, NJ, 2009.

Meixner, J. and Schäfke, F., *Mathieusche Funktionen und Sphäroidfunktionnen*, Springer, Berlin, 1965.

Melia F., *Electrodynamics*, University Of Chicago Press, Chicago, 2001.

Menzel, D. H., *Mathematical Physics*, Dover Publications, New York, 1961.

Michell, J. H., The flexure of a circular plate, *Proc. Lond. Math. Soc.*, Vol. 34, pp. 223–228, 1902.

Mikhlin, S. G., *Variational Methods in Mathematical Physics* [in Russian], Nauka, Moscow, 1970.

Miles, J. W., *Integral Transforms in Applied Mathematics*, Cambridge Univ. Press, Cambridge, 1971.

Miller, J. (Jr.) and Rubel, L. A., Functional separation of variables for Laplace equations in two dimensions, *J. Phys. A*, Vol. 26, No. 8, pp. 1901–1913, 1993.

Miller, W. (Jr.), *Symmetry and Separation of Variables*, Addison-Wesley, London, 1977.

Miranda, K., *Partial Differential Equations of Elliptic Type*, Springer, Berlin–Heidelberg–New York, 1970.

Mishchenko, A., Shatalov, V., and Sternin, B.,, *Lagrangian Manifolds and the Maslov Operator*, Springer, Berlin–Heidelberg, 1990.

Moon, P. and Spencer, D. E., *Field Theory Handbook, Including Coordinate Systems, Differential Equations and Their Solutions, 2nd Edition*, Springer, Berlin, 1988.

Morino, L., Helmholtz decomposition revised: Vorticity generation and trailing edge condition, *Computational Mech.*, No. 1, pp. 65–90, 1986.

Morse, P. M. and Feshbach, H., *Methods of Theoretical Physics, Vols. 1–2*, McGraw-Hill, New York, 1953.

Morton, K. W. and Mayers, D. F., *Numerical Solution of Partial Differential Equations: An Introduction*, Cambridge University Press, Cambridge, 1995.

Mucha, P. B. and Zajaczkowski, W. M., On a L_p-estimate for the linearized compressible Navier–Stokes equations with the Dirichlet boundary conditions, *J. Dif. Equations*, Vol. 186, No. 2, pp. 377–393, 2002.

Muleshkov, A. S., Golberg, M. A., Cheng, A. H.-D., and Chen C. S., Polynomial particular solutions for Poisson problems, In: *Boundary Elements XXIV* (Eds., Brebbia, C. S., Tadeo, A., and Popov, V.), WIT Press, Southampton, Boston, pp. 115–124, 2002.

Murphy, G. M., *Ordinary Differential Equations and Their Solutions*, D. Van Nostrand, New York, 1960.

Myint-U, T. and Debnath, L., *Partial Differential Equations for Scientists and Engineers*, North-Holland Publ., New York, 1987.

Naimark, M. A., *Linear Differential Operators* [in Russian], Nauka, Moscow, 1969.

Navier, C.-L.-M.-H., Sur les lois de l'equilibre et du mouvement des corps solides elastiques, *Bull. Soc. Philomath.*, pp. 177–181, 1821.

Neuber, U., Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätsheorie, *J. Appl. Math. & Mech.*, Vol. 14, pp. 203–212, 1934.

Niederer, U., The maximal kinematical invariance group of the harmonic oscillator, *Helv. Phys. Acta*, Vol. 46, pp. 191–200, 1973.

Nikiforov, A. F. and Uvarov, V. B., *Special Functions of Mathematical Physics. A Unified Introduction with Applications*, Birkhauser Verlag, Basel-Boston, 1988.

Nowatski, W., *The Theory of Elasticity*, Mir, Moscow, 1975.

Novikov, E. A., Concerning turbulent diffusion in a stream with a transverse gradient of velocity, *J. Appl. Math. & Mech. (PMM)*, Vol. 22, No. 3, p. 412–414, 1958.

Nusselt, W., Abhängigkeit der Wärmeübergangzahl con der Rohränge, *VDI Zeitschrift*, Bd. 54, No. 28, S. 1154–1158, 1910.

Oberhettinger, F., *Tables of Bessel Transforms*, Springer-Verlag, New York, 1972.

Oberhettinger, F., *Tables of Mellin Transforms*, Springer-Verlag, New York, 1974.

Oberhettinger, F., *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer-Verlag, Berlin, 1980.

Oberhettinger, F. and Badii, L., *Tables of Laplace Transforms*, Springer-Verlag, New York, 1973.

Oldroyd, J. G., Non-Newtonian flow of liquids and solids, Rheology, In: *Theory and Applications* (ed. F. R. Eirich), Vol. 1, pp. 653–682, Academic Press, New York, 1956.

Olver, P. J., *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

Oseen, C. W., *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademische Verlag, Leipzig, 1927.

Ovsyannikov, L. V., *Lectures on the Fundamentals of Gas Dynamics*, Nauka, Moscow, 1981.

Papkovich, P. F., Solution Générale des équations differentielles foudamentales d'élasticité exprimée par trois fonctions harmoniques, *Compt. Rend. Acad. Sci. Paris*, Vol. 195, pp. 513–515, 1932.

Paris, R. B. and Wood, A. D., Results old and new on the hyper-Bessel equation, *Proc. R. Soc. Edinburg*, Vol. A 106, pp. 259–265, 1987.

Pecknold, D. A. W., On the role of the Stokes–Helmholtz decomposition in the derivation of displacement potential in classical elasticity, *J. Elasticity*, Vol. 1, No 2, pp. 171–174, 1971.

Petrovsky, I. G., *Lectures on Partial Differential Equations*, Dover Publ., New York, 1991.

Pinkus, A. and Zafrany, S., *Fourier Series and Integral Transforms*, Cambridge University Press, Cambridge, 1997.

Pinsky, M. A., *Introduction to Partial Differential Equations with Applications*, McGraw-Hill, New York, 1984.

Polozhii, G. N., *Mathematical Physics Equations* [in Russian], Vysshaya Shkola, Moscow, 1964.

Polyanin, A. D., The structure of solutions of linear nonstationary boundary-value problems of mechanics and mathematical physics, *Doklady Physics*, Vol. 45, No. 8, pp. 415–418, 2000a.

Polyanin, A. D., Partial separation of variables in unsteady problems of mechanics and mathematical physics, *Doklady Physics*, Vol. 45, No. 12, pp. 680–684, 2000b.

Polyanin, A. D., Linear problems of heat and mass transfer: General relations and results, *Theor. Found. Chem. Eng.*, Vol. 34, No. 6, pp. 509–520, 2000c.

Polyanin, A. D., *Handbook of Linear Mathematical Physics Equations* [in Russian], Fizmatlit, Moscow, 2001.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC Press, Boca Raton, FL, 2002.

Polyanin, A. D. and Chernoutsan, A. I., (Eds.), *A Concise Handbook of Mathematics, Physics, and Engineering Sciences*, Chapman & Hall/CRC Press, Boca Raton, FL, 2011.

Polyanin, A. D. and Dilman, V. V., *Methods of Modeling Equations and Analogies in Chemical Engineering*, CRC Press, Boca Raton, FL, 1994.

Polyanin, A. D., Kutepov, A. M., Vyazmin, A. V., and Kazenin, D. A., *Hydrodynamics, Mass and Heat Transfer in Chemical Engineering*, Gordon & Breach Sci. Publ., London, 2002.

Polyanin, A. D. and Lychev, S. A., Various representations of the solutions of systems of equations of continuum mechanics, *Doklady Physics*, Vol. 59, No. 3, pp. 148–152, 2014a.

Polyanin, A. D. and Lychev, S. A., Various decomposition techniques for linear equations of continuum mechanics, *Doklady Physics*, Vol. 59, No. 10, pp. 487–490, 2014b.

Polyanin, A. D. and Lychev, S. A., Decomposition methods for coupled 3D equations of applied mathematics and continuum mechanics: Partial survey, classification, new results, and generalizations, *Applied Mathematical Modelling*, 2015 (accepted for publication).

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations*, CRC Press, Boca Raton, FL, 1998 and 2008 (2nd Edition).

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Mathematics for Engineers and Scientists*, Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

Polyanin, A. D. and Vyazmin, A. V., Integration of hydrodynamic-type linear systems, *Doklady Physics*, Vol. 57, No. 12, pp. 479–482, 2012.

Polyanin, A. D. and Vyazmin, A. V., Decomposition and exact solutions of three-dimensional nonstationary linearized equations for a viscous fluid, *Theor. Found. Chem. Eng.*, Vol. 47, No. 2, pp. 114–123, 2013a.

Polyanin, A. D. and Vyazmin, A. V., Decomposition of three-dimensional linearized equations for Maxwell and Oldroyd viscoelastic fluids and their generalizations, *Theor. Found. Chem. Eng.*, Vol. 47, No. 4, pp. 321–329, 2013b.

Polyanin, A. D., Vyazmin, A. V., Zhurov, A. I., and Kazenin, D. A., *Handbook of Exact Solutions of Heat and Mass Transfer Equations* [in Russian], Faktorial, Moscow, 1998.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, Boca Raton, FL, 1995 and 2003 (2nd Edition).

Polyanin, A. D. and Zaitsev, V. F., *Handbuch der linearen Differentialgleichungen*, Spectrum Akad. Verlag, Heidelberg, 1996.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations, 2nd Edition*, CRC Press, Boca Raton, FL, 2012.

Polyanin, A. D., Zaitsev, V. F., and Moussiaux, A., *Handbook of First Order Partial Differential Equations*, Gordon & Breach, London, 2002.

Polyanin, A. D. and Zhurov, A. I., Exact solutions to nonlinear equations of mechanics and mathematical physics, *Doklady Physics*, Vol. 43, No. 6, pp. 381–385, 1998.

Polyanin, A. D. and Zhurov, A. I., Integration of linear and some model non-linear equations of motion of incompressible fluids, *Int. J. Non-Linear Mech.*, Vol. 49, pp. 77–83, 2013.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 1, Elementary Functions*, Gordon & Breach Sci. Publ., New York, 1986.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 2, Special Functions*, Gordon & Breach Sci. Publ., New York, 1986.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 3, More Special Functions*, Gordon & Breach Sci. Publ., New York, 1988.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 4, Direct Laplace Transform*, Gordon & Breach, New York, 1992a.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 5, Inverse Laplace Transform*, Gordon & Breach, New York, 1992b.

Puri, P., Impulsive motion of a flat plate in a Rivlin–Ericksen fluid, *Rheol. Acta*, Vol. 23, pp. 451–453, 1984.

Racke, R. and Saal, J., Hyperbolic Navier–Stokes equations I: Local well-posedness, *Evolution Equations & Control Theory*, Vol. 1, No. 1, pp. 195–215, 2012.

Rajagopal, K. R., Mechanics of non-Newtonian fluids, In: *Recent Developments in Theoretical Fluid Mechanics, Pitman Research Notes in Mathematics Series*, Vol. 291, pp. 129–162, Longman, Harlow, 1993.

Rhee, H., Aris, R., and Amundson, N. R., *First Order Partial Differential Equations, Vol. 1*, Prentice Hall, Englewood Cliffs, New Jersey, 1986.

Richards, D., *Advanced Mathematical Methods with Maple*, Cambridge University Press, Cambridge, 2002.

Richtmyer, R. D., *A Survey of Difference Methods for Non-steady Fluid Dynamics*, N.C.A.R. Tech. Note, 1963.

Richtmyer, R. D. and Morton, K. W., *Difference Methods for Initial Value Problems, 2nd Edition*, Wiley-Interscience, New York, 1994.

Rimmer, P. L., Heat transfer from a sphere in a stream of small Reynolds number, *J. Fluid Mech.*, Vol. 32, No. 1, pp. 1–7, 1968.

Ross, C. C., *Differential Equations: An Introduction with Mathematica*, Springer, New York, 1995.

Rotem, Z., and Neilson, J. E., Exact solution for diffusion to flow down an incline, *Can. J. Chem. Eng.*, Vol. 47, pp. 341–346, 1966.

Schiesser, W. E., *Computational Mathematics in Engineering and Applied Science: ODEs, DAEs, and PDEs*, CRC Press, Boca Raton, FL, 1994.

Schiesser, W. E. and Griffiths, G. W., *A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab*, Cambridge University Press, Cambridge, 2009.

Schlichting, H., *Boundary Layer Theory*, McGraw-Hill, New York, 1981.

Schwartz, L., *Theorie des distributions, Tome I*, Hermann & Cie, Paris, 1950.

Schwartz, L., *Theorie des distributions. Tome II*, Hermann & Cie, Paris, 1951.

Schwartz, M., *Principles of Electrodynamics*, Dover Publications, New York, 1987.

Sedov, L. I., *Plane Problems of Hydrodynamics and Airdynamics* [in Russian], Nauka, Moscow, 1980.

Sekerzh-Zen'kovich, S. Ya., A fundamental solution of the internal-wave operator, *Sov. Phys., Dokl.*, Vol. 24, pp. 347–349, 1979.

Shilov, G. E., *Mathematical Analysis: A Second Special Course* [in Russian], Nauka, Moscow, 1965.

Shingareva, I. K. and Lizárraga-Celaya, C., *Maple and Mathematica. A Problem Solving Approach for Mathematics, 2nd Edition*, Springer, New York, 2009.

Shingareva, I. K. and Lizárraga-Celaya, C., *Solving Nonlinear Partial Differential Equations with Maple and Mathematica*, Springer, New York, 2011.

Shingareva, I. K. and Lizárraga-Celaya, C., Symbolic and Numerical Solutions of Nonlinear Partial Differential Equations with Maple, Mathematica, and MATLAB, In: *Handbook of Nonlinear Partial Differential Equations, 2nd Edition* (A. D. Polyanin and V. F. Zaitsev), CRC Press, Boca Raton, FL, pp. 1622–1765, 2012.

Silebi, C. A. and Schiesser, W. E., *Dynamic Modeling of Transport Process Systems*, Academic Press, San Diego, 1992.

Skeel, R. D. and Berzins, M., A method for the spatial discretization of parabolic equations in one space variable, *SIAM J. Scientific and Statistical Computing*, Vol. 11, pp. 1–32, 1990.

Slavyanov, S. Yu. and Lay, W., *Special Functions: A Unified Theory Based on Singularities*, Oxford University Press, Oxford, 2000.

Slobodianskii, M. G., General and complete solutions of the equations of elasticity, *J. Appl. Math. & Mech.*, Vol. 23, No. 3, pp. 666–685, 1959.

Smirnov, M. M., *Second Order Partial Differential Equations* [in Russian], Nauka, Moscow, 1964.

Smirnov, M. M., *Problems on Mathematical Physics Equations* [in Russian], Nauka, Moscow, 1975.

Smirnov, V. I., *A Course of Higher Mathematics. Vols. 2–3* [in Russian], Nauka, Moscow, 1974.

Sneddon, I., *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.

Sneddon, I., *Fourier Transforms*, McGraw-Hill, New York, 1951 and Dover Publ., New York, 1995.

Somigliana, C., Sulle equazioni della elasticita, *Ann. Mat.*, Vol. 2, No. 17, 37–64, 1889.

Stakgold, I., *Boundary Value Problems of Mathematical Physics. Vols. I and II*, SIAM, Philadelphia, 2000.

Stokes, G. G., On the dynamical theory of diffraction, *Trans. Cambridge Phil. Soc.*, No. 9, pp. 1–62, 1849.

Strauss, W. A., *Partial Differential Equations. An Introduction*, John Wiley & Sons, New York, 1992.

Strikwerda, L., *Finite Difference Schemes and Partial Differential Equations, 2nd Edition*, SIAM, Philadelphia, 2004.

Subbotin, A. I., *Minimax and Viscosity Solutions of Hamilton–Jacobi Equations* [in Russian], Nauka, Moscow, 1991.

Sullivan, M., *Trigonometry, 7th Edition*, Prentice Hall, Englewood Cliffs, NJ, 2004.

Sutton, W. G. L., On the equation of diffusion in a turbulent medium, *Proc. Poy. Soc., Ser. A*, Vol. 138, No. 988, pp. 48–75, 1943.

Sveshnikov, A. G. and Tikhonov, A. N., *Theory of Functions of One Complex Variable, 3rd Edition* [in Russian], Nauka, Moscow, 1974.

Tanner, R. I., Note on the Rayleigh problem for a visco-elastic fluid, *Z. Angew. Math. Phys. (ZAMP)*, Vol. 13, pp. 573–580, 1962.

Taylor, G. I., Viscosity of a fluid containing small drops of another fluid, *Proc. Poy. Soc., Ser. A*, Vol. 138, No. 834, pp. 41–48, 1932.

Taylor, M., *Partial Differential Equations, Vol. 3*, Springer-Verlag, New York, 1996.

Temme, N. M., *Special Functions. An Introduction to the Classical Functions of Mathematical Physics*, Wiley–Interscience Publ., New York, 1996.

Thomas, H. C., Heterogeneous ion exchange in a flowing system, *J. Amer. Chem. Soc.*, Vol. 66, pp. 1664–1666, 1944.

Thomas, J. W., *Numerical Partial Differential Equations: Finite Difference Methods*, Springer, New York, 1995.

Tikhonov, A. N. and Samarskii, A. A., *Equations of Mathematical Physics*, Dover Publ., New York, 1990.

Tomotika, S. and Tamada, K., Studies on two-dimensional transonic flows of compressible fluid, Part 1, *Quart. Appl. Math.*, Vol. 7, p. 381, 1950.

Tong, D., Starting solutions for oscillating motions of a generalized Burgers' fluid in cylindrical domains, *Acta Mechanica*, Vol. 214, No. 3–4, pp. 395–407, 2010.

Urvin, K. and Arscott, F., Theory of the Whittaker–Hill equation, *Proc. Roy. Soc.*, Vol. A69, pp. 28–44, 1970.

Vladimirov, V. S., *Equations of Mathematical Physics*, Dekker, New York, 1971.

Vladimirov, V. S., *Mathematical Physics Equations* [in Russian], Nauka, Moscow, 1988.

Vladimirov, V. S., Mikhailov, V. P., Vasharin A. A., et al., *Collection of Problems on Mathematical Physics Equations* [in Russian], Nauka, Moscow, 1974.

Vvedensky, D. D., *Partial Differential Equations with Mathematica*, Addison-Wesley, Wokingham, 1993.

Weisstein, E. W., *CRC Concise Encyclopedia of Mathematics, 2nd Edition*, CRC Press, Boca Raton, FL, 2003.

Wester, M. J., *Computer Algebra Systems: A Practical Guide*, Wiley, Chichester, UK, 1999.

Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis, Vols. 1–2, 4th Edition*, Cambridge Univ. Press, Cambridge, 1963.

Wilkinson, W. L., *Non-Newtonian Fluids*, Pergamon, Oxford, 1960.

Winter, A., On a generalisation of Airy's function, *Arch. Ration. Mech. Anal.*, Vol. 1, 242–245, 1957.

Wolfram, S., *A New Kind of Science*, Wolfram Media, Champaign, IL, 2002.

Wolfram, S., *The Mathematica Book, 5th Edition*, Wolfram Media, Champaign, IL, 2003.

Yang, W. Y., Cao, W., Chung, T. S., and Morris, J., *Applied Numerical Methods Using MATLAB*, Wiley, Hoboken, New Jersey, 2005.

Young, D. M., *Iterative Solution of Large Linear Systems*, Academic Press, New York, 1971.

Zachmanoglou, E. C. and Thoe, D. W., *Introduction to Partial Differential Equations with Applications*, Dover Publ., New York, 1986.

Zaitsev, V. F. and Polyanin, A. D., *Handbook of Partial Differential Equations: Exact Solutions* [in Russian], MP Obrazovaniya, Moscow, 1996.

Zauderer, E., *Partial Differential Equations of Applied Mathematics*, Wiley–Interscience Publ., New York, 1989.

Zill, D. G. and Dewar, J. M., *Trigonometry, 2nd Edition*, McGraw-Hill, New York, 1990.

Zimmerman, R. L. and Olness, F., *Mathematica for Physicists*, Addison-Wesley, Reading, MA, 1995.

Zwillinger, D., *Handbook of Differential Equations*, Academic Press, San Diego, 1998.

Zwillinger, D., *CRC Standard Mathematical Tables and Formulae, 31st Edition*, CRC Press, Boca Raton, FL, 2002.

Mathematics, Engineering, Physics

HANDBOOK OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS FOR ENGINEERS AND SCIENTISTS

SECOND EDITION

UPDATED, REVISED AND EXTENDED

Features

- Includes nearly 4,000 linear partial differential equations (PDEs) with solutions
- Presents solutions of numerous problems relevant to heat and mass transfer, wave theory, hydrodynamics, aerodynamics, elasticity, acoustics, electrodynamics, diffraction theory, quantum mechanics, chemical engineering sciences, electrical engineering, and other fields
- Outlines basic methods for solving various problems in science and engineering
- Contains many more linear equations, problems, and solutions than any other book currently available
- Provides a database of test problems for numerical and approximate analytical methods for solving linear PDEs and systems of coupled PDEs

New to the Second Edition

- More than 700 pages with 1,500+ new first-, second-, third-, fourth-, and higher-order linear equations with solutions
- Systems of coupled PDEs with solutions
- Some analytical methods, including decomposition methods and their applications
- Symbolic and numerical methods for solving linear PDEs with Maple, Mathematica, and MATLAB®
- Many new problems, illustrative examples, tables, and figures

To accommodate different mathematical backgrounds, the authors avoid wherever possible the use of special terminology, outline some of the methods in a schematic, simplified manner, and arrange the material in increasing order of complexity.

CRC Press
Taylor & Francis Group
an **informa** business
www.crcpress.com

6000 Broken Sound Parkway, NW
Suite 300, Boca Raton, FL 33487
711 Third Avenue
New York, NY 10017
2 Park Square, Milton Park
Abingdon, Oxon OX14 4RN, UK

K18874
ISBN: 978-1-4665-8145-6
90000
9 781466 581456
www.crcpress.com