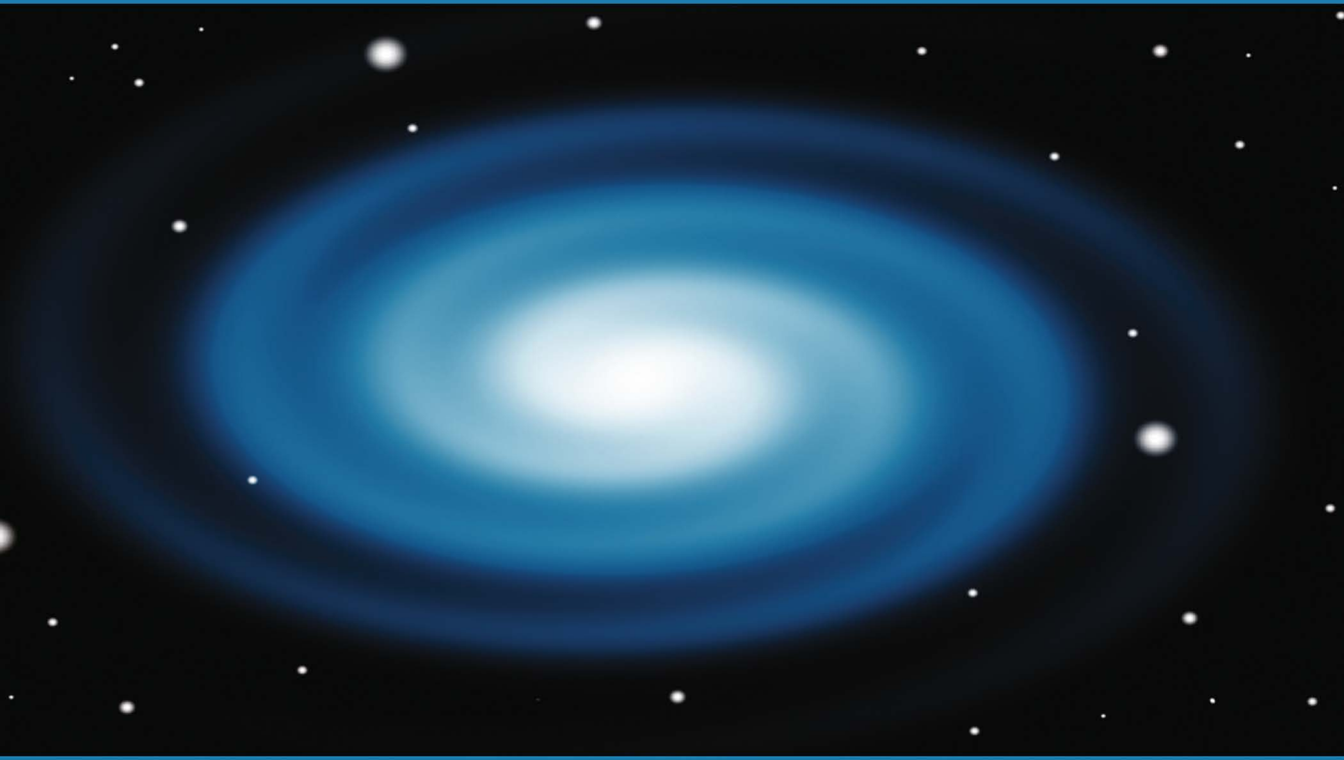
HANDBOOK OF ORDINARY DIFFERENTIAL EQUATIONS

EXACT SOLUTIONS, METHODS, AND PROBLEMS



Andrei D. Polyanin
Valentin F. Zaitsev

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Exact Solutions, Methods, and Problems

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PREFACE

The *Handbook of Ordinary Differential Equations for Scientists and Engineers*, is a unique reference for scientists and engineers, which contains over 7,000 ordinary differential equations with solutions, as well as exact, asymptotic, approximate analytical, numerical, symbolic, and qualitative methods for solving and analyzing linear and nonlinear equations. First-, second-, third-, fourth- and higher-order ordinary differential equations and systems of equations are considered. A number of new nonlinear equations, exact solutions, transformations, and methods are described. Equations arising in various applications (in the theory of heat and mass transfer, nonlinear mechanics, elasticity, hydrodynamics, theory of nonlinear oscillations, combustion theory, chemical engineering science, etc.) are considered. Analytical formulas for the effective construction of solutions are given. Special attention is paid to equations of general form that depend on arbitrary functions. Almost all other equations contain one or more arbitrary parameters (i.e., in fact, this book deals with whole families of ordinary differential equations), which can be fixed by the reader at will. A number of specific examples where the methods described in the book are used are considered. Statements of existence and uniqueness theorems as well as theorems of stability and instability of solutions are given as well. Boundary-value problems and eigenvalue problems are described. Significant attention is given to Cauchy problems with blow-up solutions as well as the important questions of nonexistence and nonuniqueness of solutions to nonlinear boundary-value problems. Elements of bifurcation theory, Lie group and discrete-group methods for ODEs, and the factorization principle are discussed. Symbolic and numerical methods for solving ODEs problems with Maple, *Mathematica*, and MATLAB® are considered.

All in all, the handbook contains much more ordinary differential equations, problems, methods, solutions, and transformations than any other book currently available. It essential that symbolic computation systems, even the most powerful ones such as Maple or Mathematica, can provide no more than 40–50% of the exact analytical solutions to ODEs given in this book (Chapters 13 through 18).

The main material is followed by a number of supplements, which present tables of integrals, finite and infinite series, and integral transforms as well as a brief description of the basic properties of elementary and special functions (Bessel, modified Bessel, hypergeometric, Legendre, etc.).

New material compared to *Handbook of Exact Solutions for Ordinary Differential Equations, 2003:*

- The total volume of the new handbook has almost doubled (increased by nearly 700 pages).
- Some first-, second-, and third-order nonlinear ODEs with solutions.
- Some analytical methods (including new methods) and standard numerical methods.
- Special numerical methods (including new methods) for solving problems with qualitative features or singularities.
- Symbolic and numerical methods with Maple, Mathematica, and MATLAB.

- Many new problems, illustrative examples, and figures.
- Elementary theory of using invariants for solving equations.
- Methods for the construction of particular solutions (including the method of differential constraints).
- Systems of coupled ordinary differential equations with solutions.
- Equations defined parametrically or implicitly (exact and numerical methods and exact solutions) as well as overdetermined systems of ODEs and underdetermined ODEs.

For the convenience of a wide audience with varying mathematical backgrounds, the authors tried to do their best to avoid special terminology whenever possible. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner, with necessary references made to books where these methods are considered in more detail. Many sections were written so that they could be read independently (moreover, many topics do not require special mathematical background for their understanding and successful practical application). This allows the reader to get to the heart of the matter quickly.

The handbook consists of parts, chapters, sections, subsections, and paragraphs. The material within sections is arranged in increasing order of complexity. An extensive table of contents and detailed index provides rapid access to the desired equations.

Isolated sections of the book can be used by university and college lecturers in practical courses and lectures on ordinary differential equations for graduate and postgraduate students. Furthermore, the second part of the book (Chapters 13–18) can be used as a database of test problems for numerical, approximate analytical, and symbolic methods for solving ordinary differential equations.

We would like to express our keen gratitude to Alexei Zhurov for fruitful discussions and valuable remarks. We are very thankful to Inna Shingareva and Carlos Lizárraga-Celaya, who wrote three chapters (19–21) of the book at our request. Also, we would like to express our deep gratitude to Vladimir Nazaikinskii for translating several chapters of this handbook.

The authors hope that the handbook will prove helpful for a wide audience of researchers, university and college teachers, engineers, and students in various fields of mathematics, physics, mechanics, control, chemistry, economics, and engineering sciences.

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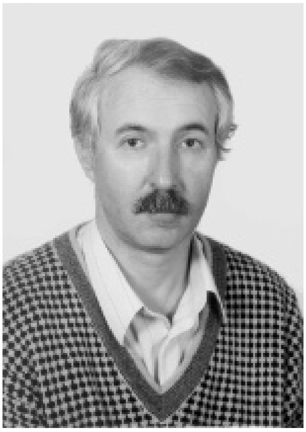

Andrei D. Polyanin, Ph.D., D.Sc., is an internationally renowned scientist of broad interests and is active in various areas of mathematics, mechanics, and chemical engineering sciences. He is one of the most prominent authors in the field of reference literature on mathematics.

Professor Polyanin graduated with honors from the Faculty of Mechanics and Mathematics at Lomonosov Moscow State University in 1974. He received his Ph.D. in 1981 and D.Sc. in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, Professor Polyanin has been working at the Institute for Problems in Mechanics of the Russian Academy of Sciences; he is also Professor of applied mathematics at Bauman Moscow State Technical University and at National Research Nuclear University MEPhI. He is a member of the Russian National Committee on Theoretical and Applied Mechanics and the Mathematics and Mechanics Expert Council of the Higher Certification Committee of the Russian Federation.

Professor Polyanin has made important contributions to the theory of differential and integral equations, mathematical physics, engineering mathematics, the theory of heat and mass transfer, and hydrodynamics. He has obtained exact solutions for several thousand ordinary differential, partial differential, delay partial differential, and integral equations.

Professor Polyanin has authored more than 30 books in English, Russian, German, and Bulgarian, as well as more than 200 research papers, and holds three patents. He has written a number of fundamental handbooks, including: A. D. Polyanin and V. F. Zaitsev's *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, 1995 and 2003); A. D. Polyanin and A. V. Manzhirov's *Handbook of Integral Equations* (CRC Press, 1998 and 2008); A. D. Polyanin's *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC Press, 2002); A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux's *Handbook of First Order Partial Differential Equations* (Taylor & Francis, 2002); A. D. Polyanin and V. F. Zaitsev's *Handbook of Nonlinear Partial Differential Equations* (Chapman & Hall/CRC Press, 2004 and 2012); A. D. Polyanin and A. V. Manzhirov's *Handbook of Mathematics for Engineers and Scientists* (Chapman & Hall/CRC Press, 2007); A. D. Polyanin and A. I. Chernoutsan's (Eds.) *A Concise Handbook of Mathematics, Physics, and Engineering Sciences* (Chapman & Hall/CRC Press, 2010), and A. D. Polyanin and A. V. Nazaikinskii's *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC Press, 2016).

Professor Polyanin is editor-in-chief of the international scientific educational website *EqWorld—The World of Mathematical Equations* and he is editor of the book series *Differential and Integral Equations and Their Applications* (Chapman & Hall/CRC Press, London/Boca Raton). Professor Polyanin is a member of the editorial board of the journals *Theoretical Foundations of Chemical Engineering*, *Mathematical Modeling and Compu-*

tational Methods (in Russian), and *Bulletin of the National Research Nuclear University MEPhI* (in Russian).

In 1991, Professor Polyanin was awarded the Chaplygin Prize of the Russian Academy of Sciences for his research in mechanics. In 2001, he received an award from the Ministry of Education of the Russian Federation.

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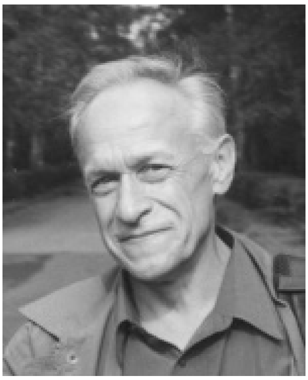

Valentin F. Zaitsev, Ph.D., D.Sc., is a noted scientist in the fields of ordinary differential equations, mathematical physics, and nonlinear mechanics.

Dr. Zaitsev graduated from the Radio Electronics Faculty of the Leningrad Polytechnical Institute (now St. Petersburg Technical University) in 1969 and received his Ph.D. degree in 1983 at the Leningrad State University. His Ph.D. thesis was devoted to the group approach of the study of some classes of ordinary differential equations. In 1992, Professor Zaitsev received his D.Sc. and his thesis was dedicated to the discrete group analysis of ordinary differential equations.

From 1971–1996, Dr. Zaitsev worked in the Research Institute for Computational Mathematics and Control Processes of the St. Petersburg State University. Since 1996, Professor Zaitsev has been a member of the staff of the Russian State Pedagogical University (St. Petersburg); he is also a professor at St. Petersburg State University (Faculty of Applied Mathematics and Control Processes).

Professor Zaitsev has made important contributions to new methods in the theory of ordinary and partial differential equations. He is the author of more than 200 scientific publications, including 27 books in English, Russian, and German, and one patent.

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BASIC NOTATION AND REMARKS

Brief Notation for Derivatives and Operators

1. Throughout this book, in the original equations, the independent variable is denoted by x, and the dependent one is denoted by y. In the given solutions, the symbols C, C_0, C_1, C_2, … stand for arbitrary integration constants. Solutions are often represented in parametric form (e.g., see Sections 13.3.1 and 14.3.1).

2. Notation for derivatives:

$$y'_x = \frac{dy}{dx}, \quad y''_{xx} = \frac{d^2y}{dx^2}, \quad y'''_{xxx} = \frac{d^3y}{dx^3}, \quad y''''_{xxxx} = \frac{d^4y}{dx^4}; \quad y_x^{(n)} = \frac{d^ny}{dx^n} \quad \text{with} \quad n \ge 5.$$

3. Brief notation for partial derivatives:

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \qquad \text{where} \quad f = f(x,y).$$

4. In some cases, we use the operator notation $\left(f\dfrac{d}{dx}\right)^n g$, which is defined by the recurrence relation

$$\left(f(x)\frac{d}{dx}\right)^n g(x) = f(x)\frac{d}{dx}\left[\left(f(x)\frac{d}{dx}\right)^{n-1} g(x)\right].$$

5. Brief operator notation corresponding to partial derivatives: $\partial_x = \dfrac{\partial}{\partial x}$, $\partial_y = \dfrac{\partial}{\partial y}$.

Special Functions

$\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)dt$	Airy function; $\operatorname{Ai}(x) = \frac{1}{\pi}\left(\frac{1}{3}x\right)^{1/2} K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$
$\operatorname{Ce}_{2n+p}(x,q) = \sum_{k=0}^{\infty} A_{2k+p}^{2n+p}\cosh[(2k+p)x]$	Even modified Mathieu functions, where $p = 0, 1$; $\operatorname{Ce}_{2n+p}(x,q) = \operatorname{ce}_{2n+p}(ix,q)$
$\operatorname{ce}_{2n}(x,q) = \sum_{k=0}^{\infty} A_{2k}^{2n}\cos 2kx$	Even π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n}(q)$ are eigenvalues
$\operatorname{ce}_{2n+1}(x,q) = \sum_{k=0}^{\infty} A_{2k+1}^{2n+1}\cos[(2k+1)x]$	Even 2π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n+1}(q)$ are eigenvalues
$D_\nu = D_\nu(x)$	Parabolic cylinder function; it satisfies the equation $y'' + \left(\nu + \frac{1}{2} - \frac{1}{4}x^2\right)y = 0$
$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-\xi^2\right)d\xi$	Error function
$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp\left(-\xi^2\right)d\xi$	Complementary error function
$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$	Hermite polynomial

$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$	Hankel function of first kind, $i^2 = -1$
$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$	Hankel function of second kind, $i^2 = -1$
$F(a,b,c;x) = 1 + \sum\limits_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$	Hypergeometric function, $(a)_n = a(a+1)\dots(a+n-1)$
$I_\nu(x) = \sum\limits_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$	Modified Bessel function of first kind
$J_\nu(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$	Bessel function of first kind
$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}$	Modified Bessel function of second kind
$L_n^s(x) = \frac{1}{n!} x^{-s} e^x \frac{d^n}{dx^n}\bigl(x^{n+s}e^{-x}\bigr)$	Generalized Laguerre polynomial
$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n$	Legendre polynomial
$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$	Associated Legendre functions
$\mathrm{Se}_{2n+p}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k+p}^{2n+p} \sinh[(2k+p)x]$	Odd modified Mathieu functions, where $p = 0, 1$; $\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q)$
$\mathrm{se}_{2n}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k}^{2n} \sin 2kx$	Odd π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = b_{2n}(q)$ are eigenvalues
$\mathrm{se}_{2n+1}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k+1}^{2n+1} \sin[(2k+1)x]$	Odd 2π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = b_{2n+1}(q)$ are eigenvalues
$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}$	Bessel function of second kind
$\gamma(\alpha,x) = \int_0^x e^{-\xi}\xi^{\alpha-1}\,d\xi$	Incomplete gamma function
$\Gamma(\alpha) = \int_0^\infty e^{-\xi}\xi^{\alpha-1}\,d\xi$	Gamma function
$\Phi(a,b;x) = 1 + \sum\limits_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$	Degenerate hypergeometric function, $(a)_n = a(a+1)\dots(a+n-1)$

Miscellaneous Remarks

1. Throughout the book, unless explicitly specified otherwise, all parameters, variables, and functions occurring in the equations considered are assumed to be real numbers.

2. If a formula or a solution contains derivatives of some functions, then the functions are assumed to be differentiable.

3. If a formula or a solution contains finite or definite integrals, then the integrals are supposed to exist and to be convergent.

4. If a relation contains an expression like $\dfrac{f(x)}{a-2}$, it is often not stated that the assumption $a \neq 2$ is adopted.

5. In solutions, expressions like $\varphi_n(x) = \dfrac{1}{n+1}x^{n+1}$ can usually be defined so as to cover the case $n = -1$ in accordance with the rule $\varphi_{-1}(x) = \ln|x|$. This is accounted for by the fact that such expressions arise from the integration of the power-law function $\varphi_n(x) = \int x^n\,dx$.

6. The order symbol O is used to compare two functions, $f = f(\varepsilon)$ and $g = g(\varepsilon)$, where ε is a small parameter. So $f = O(g)$ means that $|f/g|$ is bounded as $\varepsilon \to 0$, or f and g are of the same order of magnitude as $\varepsilon \to 0$.

7. In Chapters 13–18, when referring to a particular equation, we use notation like 14.1.2.35, which denotes Eq. 35 in Section 14.1.2.

8. The handbooks by Kamke (1977), Murphy (1960), Zaitsev and Polyanin (1993, 2001), Polyanin and Zaitsev (1995, 2003) were extensively used in compiling this book; references to these sources are frequently omitted.

9. In some sections (e.g., see 13.3, 14.3–14.6, 15.2–15.3), for the sake of brevity, solutions are represented as several formulas containing terms with the signs "±" and "∓." Two formulas are meant—one corresponds to the upper sign and the other to the lower sign. For example, the solution of equation 13.3.1.16 is written in the parametric form

$$x = af^{-1}\exp(\mp\tau^2), \quad y = af^{-1}\bigl[\exp(\mp\tau^2) \pm 2\tau f\bigr],$$

where

$$f = \int \exp(\mp\tau^2)\,d\tau - C, \quad A = \mp 2a^2.$$

This is equivalent so that the solutions of equation 13.3.1.16 are given by the two formulas:

$$x = af^{-1}\exp(-\tau^2), \quad y = af^{-1}\bigl[\exp(-\tau^2) + 2\tau f\bigr],$$

where

$$f = \int \exp(-\tau^2)\,d\tau - C, \quad A = -2a^2$$

(for the upper signs) and

$$x = af^{-1}\exp(\tau^2), \quad y = af^{-1}\bigl[\exp(\tau^2) - 2\tau f\bigr],$$

where

$$f = \int \exp(\tau^2)\,d\tau - C, \quad A = 2a^2$$

(for the lower signs).

10. To highlight portions of the text, the following symbols are used throughout the book:

- ► marks the beginning of a small section of the fourth level; such sections are referred to as paragraphs;
- ◆ indicates important information pertaining to a group of equations (Chapters 13–18);
- ⊙ indicates the literature used in the preparation of the text in subsections, paragraphs, and specific equations.

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Part I

Methods for Ordinary Differential Equations

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Chapter 1

Methods for First-Order Differential Equations

1.1 General Concepts. Cauchy Problem. Uniqueness and Existence Theorems

1.1.1 Equations Solved for the Derivative

► **Form of equations. General and particular solutions.**

A *first-order ordinary differential equation* solved for the derivative has the form[†]

$$y'_x = f(x, y). \tag{1.1.1.1}$$

In what follows, we often call an ordinary differential equation a "differential equation" or, even shorter, an "equation."

Sometimes equation (1.1.1.1) is represented in terms of differentials as $dy = f(x, y)\, dx$.

A *solution of a differential equation* is a function $y(x)$ that, when substituted into the equation, turns it into an identity. The *general solution of a differential equation* is the set of all its solutions. In some cases, the general solution can be represented as a function $y = \varphi(x, C)$ that depends on one *arbitrary constant* C; specific values of C define specific solutions of the equation (*particular solutions*). In practice, the general solution more frequently appears in *implicit form*, $\Phi(x, y, C) = 0$, or *parametric form*, $x = x(t, C)$, $y = y(t, C)$.

Geometrically, the general solution (also called the *general integral*) of an equation is a family of curves in the xy-plane depending on a single parameter C; these curves are called *integral curves* of the equation. To each particular solution (particular integral) there corresponds a single curve that passes through a given point in the plane.

For each point (x, y), the equation $y'_x = f(x, y)$ defines a value of y'_x, i.e., the slope of the integral curve that passes through this point. In other words, the equation generates a field of directions in the xy-plane. From the geometrical point of view, the problem of

[†]Unless otherwise specified, we assume here and henceforth that $y = y(x)$ and $f = f(x, y)$ are real-valued functions of real arguments.

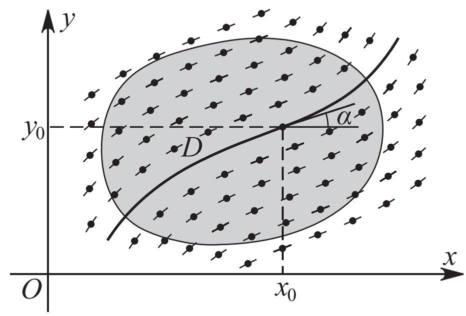

Figure 1.1: The direction field of a differential equation and the integral curve passing through a point (x_0, y_0).

solving a first-order differential equation involves finding the curves, the slopes of which at each point coincide with the direction of the field at this point.

Figure 1.1 depicts the tangent to an integral curve at a point (x_0, y_0); the slope of the integral curve at this point is determined by the right-hand side of equation (1.1.1.1): $\tan\alpha = f(x_0, y_0)$. The little lines show the field of tangents to the integral curves of the differential equation (1.1.1.1) at other points.

► Equations integrable by quadrature.

To integrate a differential equation in closed form is to represent its solution in the form of formulas written using a predefined bounded set of allowed functions and mathematical operations. A solution is expressed as a quadrature if the set of allowed functions consists of the elementary functions and the functions appearing in the equation and the allowed mathematical operations are the arithmetic operations, a finite number of function compositions, and the indefinite integral. An equation is said to be integrable by quadrature if its general solution can be expressed in terms of quadratures.

► Cauchy problem. Uniqueness and existence theorems.

The *Cauchy problem* (or the *initial value problem*): find a solution of equation (1.1.1.1) that satisfies the *initial condition*

$$y = y_0 \quad \text{at} \quad x = x_0, \tag{1.1.1.2}$$

where y_0 and x_0 are some numbers.

The geometrical meaning of the Cauchy problem is as follows: find an integral curve of equation (1.1.1.1) that passes through the point (x_0, y_0); see Fig. 1.1.

Condition (1.1.1.2) is alternatively written $y(x_0) = y_0$ or $y|_{x=x_0} = y_0$.

EXISTENCE THEOREM (PEANO). *Let the function $f(x, y)$ be continuous in an open domain D of the xy-plane. Then there is at least one integral curve of equation (1.1.1.1) that passes through each point $(x_0, y_0) \in D$; each of these curves can be extended at both ends up to the boundary of any closed domain $D_0 \subset D$ such that (x_0, y_0) belongs to the interior of D_0.*

UNIQUENESS THEOREM. *Let the function $f(x,y)$ be continuous in an open domain D and have in D a bounded partial derivative with respect to y (or the Lipschitz condition holds: $|f(x,y) - f(x,z)| \leq K|y - z|$, where K is some positive number, called the Lipschitz constant). Then there is a unique solution of equation (1.1.1.1) satisfying condition (1.1.1.2).*

► Comments on the uniqueness and existence theorems.

The violence of a condition stated in the existence and uniqueness theorems may result in the existence of one, several, or infinitely many solutions or even no solutions at all. Below we give a few simple illustrative examples for the case where the right-hand side of equation (1.1.1.1) has a singularity at the boundary of the domain.

Example 1.1. Consider the Cauchy problem

$$y'_x = y^{1/3} \quad (x > 0), \qquad y(0) = 0. \tag{1.1.1.3}$$

Since the right-hand side of the equation is a continuous function, the existence theorem states that there is a solution, at least for x close to 0. It is easy to verify that problem (1.1.1.3) two solutions: $y_1 = 0$ and $y_2 = \left(\frac{2}{3}x\right)^{3/2}$. Furthermore, the problem has infinitely many solutions (a one-parameter family) of the form

$$y(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a, \\ \left[\frac{2}{3}(x-a)\right]^{3/2} & \text{if } x \geq a, \end{cases} \tag{1.1.1.4}$$

where $a > 0$ is an arbitrary constant. Function (1.1.1.4) is differentiable everywhere, even at $x = a$, and it satisfies both the differential equation and the initial condition (1.1.1.3).

The uniqueness is violated here due to the fact that the derivative of the right-hand side of the equation with respect to y becomes infinite at $x = 0$, by virtue of the initial condition.

Example 1.2. Consider the Cauchy problem

$$y'_x = \frac{1}{2y} \quad (x > 0), \qquad y(0) = 0. \tag{1.1.1.5}$$

Here the right-hand side of the equation becomes infinite at $x = 0$ by virtue of the initial conditions.

Problem (1.1.1.5) has two solutions: $y_1 = -\sqrt{x}$ and $y_2 = \sqrt{x}$.

Example 1.3. Consider the Cauchy problem

$$y'_x = \frac{1}{\sqrt{y}} \quad (x > 0), \qquad y(0) = 0. \tag{1.1.1.6}$$

The right-hand side of the equation becomes infinite at $x = 0$ by virtue of the initial condition.

Problem (1.1.1.6) has one solution: $y = \left(\frac{3}{2}x\right)^{2/3}$.

Example 1.4. In the Cauchy problem

$$y'_x = \frac{2y}{x} \quad (x > 0), \qquad y(0) = 0, \tag{1.1.1.7}$$

the right-hand side of the equation has a fixed singularity at the boundary (becomes infinite at $x = 0$).

Problem (1.1.1.7) has infinitely many (a one-parameter family) smooth solutions: $y = ax^2$, where a is an arbitrary constant.

Example 1.5. In the Cauchy problem

$$y'_x = -\frac{1}{2y} \quad (x > 0), \quad y(0) = 0, \tag{1.1.1.8}$$

the right-hand side of the equation becomes infinite at $x = 0$ by virtue of the initial condition.

Equation (1.1.1.8) has the general integral

$$y^2 = -x + C, \tag{1.1.1.9}$$

where C is an arbitrary constant. Using the initial condition $y(0) = 0$, we get $C = 0$. With $C = 0$, the left-hand side of (1.1.1.9) is positive (nonnegative), while the right-hand side is negative for $x > 0$. Hence, the Cauchy problem (1.1.1.8) does not have a real solution.

Remark 1.1. Cauchy problems in which the right-hand side of equation (1.1.1.1) has a singularity in the interior of the domain are treated in Sections 1.7.3 and 1.7.4.

► Theorems on smoothness and parametric continuity of solutions.

THEOREM ON SMOOTHNESS OF SOLUTIONS. *Let the function $f(x, y)$ have n continuous derivatives in either argument. Then any solution $y = y(x)$ to equation (1.1.1.1) has continuous derivatives up to the $(n+1)$st order inclusive. If $f(x, y)$ is analytic, then all solutions $y = y(x)$ are also analytic.*

THEOREM ON PARAMETRIC CONTINUITY OF SOLUTIONS TO THE CAUCHY PROBLEM. *Let in the initial value problem*

$$y'_x = f(x, y, \boldsymbol{\lambda}), \quad y(x_0) = y_0(\boldsymbol{\lambda}), \tag{1.1.1.10}$$

the differential equation and/or the initial condition depend continuously on one or more parameters $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$. Then the solution $y = y(x, \boldsymbol{\lambda})$ (which is assumed to exist and be unique) depends continuously upon the parameters.

► Point transformations.

In the general case, a point transformation is defined by

$$x = F(X, Y), \quad y = G(X, Y), \tag{1.1.1.11}$$

where X is the new independent variable, $Y = Y(X)$ is the new dependent variable, and F and G are some (prescribed or unknown) functions.

The derivative y'_x under the point transformation (1.1.1.11) is calculated by

$$y'_x = \frac{G_X + G_Y Y'_X}{F_X + F_Y Y'_X},$$

where the subscripts X and Y denote the corresponding partial derivatives.

Transformation (1.1.1.11) is invertible if $F_X G_Y - F_Y G_X \neq 0$.

Point transformations are used to simplify equations and reduce them to known equations. Sometimes a point transformation enables the reduction of a nonlinear equation to a linear one.

Example 1.6. The simplest point transformations are

$$x = X + A, \quad y = Y + B \quad \textit{translation transformation};$$
$$x = AX, \quad y = BY \quad \textit{scaling transformation},$$

where A and B are arbitrary constants.

Example 1.7. The *hodograph transformation* is an important example of a point transformation. It is defined by $x = Y$, $y = X$, which means that y is taken to be the independent variable and x the dependent one. In this case, the derivative is expressed as

$$y'_x = \frac{1}{X'_Y}.$$

Other examples of point transformations can be found in Sections 1.2 and 1.4–1.6.

1.1.2 Equations Not Solved for the Derivative

► Form of equations not solved for the derivative. Existence theorem.

A first-order differential equation not solved for the derivative can generally be written as

$$F(x, y, y'_x) = 0. \tag{1.1.2.1}$$

EXISTENCE AND UNIQUENESS THEOREM. *There exists a unique solution $y = y(x)$ of equation (1.1.2.1) satisfying the conditions $y|_{x=x_0} = y_0$ and $y'_x|_{x=x_0} = t_0$, where t_0 is one of the real roots of the equation $F(x_0, y_0, t_0) = 0$ if the following conditions hold in a neighborhood of the point (x_0, y_0, t_0):*

1. The function $F(x, y, t)$ is continuous in each of the three arguments.
2. The partial derivative F_t exists and is nonzero.
3. There is a bounded partial derivative with respect to y, $|F_y| \le K$.

The solution exists for $|x - x_0| \le a$, where a is a (sufficiently small) positive number.

► Singular solutions.

1°. A point (x, y) at which the uniqueness of the solution to equation (1.1.2.1) is violated is called a *singular* point. If conditions 1 and 3 of the existence and uniqueness theorem hold, then

$$F(x, y, t) = 0, \qquad F_t(x, y, t) = 0 \tag{1.1.2.2}$$

simultaneously at each singular point. Relations (1.1.2.2) define a *t-discriminant curve* in parametric form. In some cases, the parameter t can be eliminated from (1.1.2.2) to give an equation of this curve in implicit form, $\Psi(x, y) = 0$. If a branch $y = \psi(x)$ of the curve $\Psi(x, y) = 0$ consists of singular points and, at the same time, is an integral curve, then this branch is called a *singular integral curve* and the function $y = \psi(x)$ is a *singular solution* of equation (1.1.2.1).

2°. The singular solutions can be found by identifying the *envelope of the family of integral curves*, $\Phi(x, y, C) = 0$, of equation (1.1.2.1). The envelope is part of the *C-discriminant curve*, which is defined by the equations

$$\Phi(x, y, C) = 0, \qquad \Phi_C(x, y, C) = 0.$$

The branch of the C-discriminant curve at which

(a) there exist bounded partial derivatives, $|\Phi_x| < M_1$ and $|\Phi_y| < M_2$, and
(b) $|\Phi_x| + |\Phi_y| \neq 0$

is the envelope.

⊙ *Literature for Section 1.1*: E. L. Ince (1956), G. M. Murphy (1960), L. E. El'sgol'ts (1961), P. Hartman (1964), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and Rota (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

1.2 Equations Solved for the Derivative. Simplest Techniques of Integration*

1.2.1 Equations with Separable Variables and Related Equations

► **Equations with separated variables.**

An *equation with separated variables* (a *separated equation*) has the form

$$f(y)y'_x = g(x). \tag{1.2.1.1}$$

Equivalently, the equation can be rewritten as $f(y)\,dy = g(x)\,dx$ (the right-hand side depends on x alone and the left-hand side on y alone). The general solution can be obtained by termwise integration:

$$\int f(y)\,dy = \int g(x)\,dx + C,$$

where C is an arbitrary constant.

► **Equations with separable variables.**

An *equation with separable variables* (a *separable equation*) is generally represented by

$$f_1(y)g_1(x)y'_x = f_2(y)g_2(x). \tag{1.2.1.2}$$

Dividing the equation by $f_2(y)g_1(x)$, one obtains a separated equation. Integrating yields

$$\int \frac{f_1(y)}{f_2(y)}\,dy = \int \frac{g_2(x)}{g_1(x)}\,dx + C.$$

Remark 1.2. In termwise division of the equation by $f_2(y)g_1(x)$, solutions corresponding to $f_2(y) = 0$ can be lost.

► **Related equation.**

Consider an equation of the form

$$y'_x = f(ax + by). \tag{1.2.1.3}$$

The substitution $z = ax + by$ brings it to a separable equation, $z'_x = bf(z) + a$.

*This section deals with equations of fairly general form involving arbitrary functions.

1.2.2 Homogeneous and Generalized Homogeneous Equations

▶ Homogeneous equations and equations reducible to them.

1°. A *homogeneous equation* remains the same under simultaneous scaling (dilatation) of the independent and dependent variables in accordance with the rule $x \to \alpha x$, $y \to \alpha y$, where α is an arbitrary constant ($\alpha \neq 0$). Such equations can be represented in the form

$$y'_x = f\left(\frac{y}{x}\right). \tag{1.2.2.1}$$

The substitution $u = y/x$ brings a homogeneous equation to a separable one, $xu'_x = f(u) - u$; see Section 1.2.1.

2°. The equations of the form

$$y'_x = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{1.2.2.2}$$

can be reduced to a homogeneous equation. To this end, for $a_1x + b_1y \neq k(a_2x + b_2y)$, one should use the change of variables $\xi = x - x_0$, $\eta = y - y_0$, where the constants x_0 and y_0 are determined by solving the linear algebraic system

$$a_1x_0 + b_1y_0 + c_1 = 0,$$
$$a_2x_0 + b_2y_0 + c_2 = 0.$$

As a result, one arrives at the following equation for $\eta = \eta(\xi)$:

$$\eta'_\xi = f\left(\frac{a_1\xi + b_1\eta}{a_2\xi + b_2\eta}\right).$$

On dividing the numerator and denominator of the argument of f by ξ, one obtains a homogeneous equation whose right-hand side is dependent on the ratio η/ξ only:

$$\eta'_\xi = f\left(\frac{a_1 + b_1\eta/\xi}{a_2 + b_2\eta/\xi}\right).$$

For $a_1x + b_1y = k(a_2x + b_2y)$, we have an equation of the type 1.2.1.3.

▶ Generalized homogeneous equations and equations reducible to them.

1°. A *generalized homogeneous equation* (a homogeneous equation in the generalized sense) remains the same under simultaneous scaling of the independent and dependent variables in accordance with the rule $x \to \alpha x$, $y \to \alpha^k y$, where $\alpha \neq 0$ is an arbitrary constant and k is some number. Such equations can be represented in the form

$$y'_x = x^{k-1}f(yx^{-k}). \tag{1.2.2.3}$$

The substitution $u = yx^{-k}$ brings a generalized homogeneous equation to a separable equation, $xu'_x = f(u) - ku$; see Section 1.2.1.

Example 1.8. Consider the equation

$$y'_x = ax^2y^4 + by^2. \tag{1.2.2.4}$$

Let us perform the transformation $x = \alpha\bar{x}$, $y = \alpha^k\bar{y}$ and then multiply the resulting equation by α^{1-k} to obtain

$$\bar{y}'_{\bar{x}} = a\alpha^{3(k+1)}\bar{x}^2\bar{y}^4 + b\alpha^{k+1}\bar{y}^2. \tag{1.2.2.5}$$

It is apparent that if $k = -1$, the transformed equation (1.2.2.5) is the same as the original one, up to notation. This means that equation (1.2.2.4) is generalized homogeneous of degree $k = -1$. Therefore the substitution $u = xy$ brings it to a separable equation: $xu'_x = au^4 + bu^2 + u$.

2°. The equations of the form

$$y'_x = yf(e^{\lambda x}y)$$

can be reduced to a generalized homogeneous equation. To this end, one should use the change of variable $z = e^x$ and set $\lambda = -k$.

1.2.3 Linear Equation and Bernoulli Equation

► **Linear equation.**

A first-order *linear equation* is written as

$$y'_x + f(x)y = g(x). \tag{1.2.3.1}$$

The solution is sought in the product form $y = uv$, where $v = v(x)$ is any function that satisfies the "truncated" equation $v'_x + f(x)v = 0$ [as $v(x)$ one takes the particular solution $v = e^{-F}$, where $F = \int f(x)\,dx$]. As a result, one obtains the following separable equation for $u = u(x)$: $v(x)u'_x = g(x)$. Integrating it yields the general solution:

$$y(x) = e^{-F}\Bigl(\int e^F g(x)\,dx + C\Bigr), \qquad F = \int f(x)\,dx, \tag{1.2.3.2}$$

where C is an arbitrary constant.

► **Bernoulli equation.**

A *Bernoulli equation* has the form

$$y'_x + f(x)y = g(x)y^\beta, \qquad \beta \neq 0,\ 1. \tag{1.2.3.3}$$

(For $\beta = 0$ and $\beta = 1$, it is a linear equation.) The substitution $z = y^{1-\beta}$ brings it to a linear equation, $z'_x + (1-\beta)f(x)z = (1-\beta)g(x)$. With this in view, one can obtain the general integral:

$$y^{1-a} = Ce^F + (1-\beta)e^F \int e^{-F}g(x)\,dx, \quad \text{where} \quad F = (\beta-1)\int f(x)\,dx, \tag{1.2.3.4}$$

and C is an arbitrary constant.

Example 1.9. Let us look at the Cauchy problem for the Bernoulli equation

$$y'_x = -f(x)y + g(x)y^2 \quad (x > 0), \quad y(0) = a > 0. \tag{1.2.3.5}$$

Using formula (1.2.3.4) with $\beta = 2$ and considering the initial condition, we can write the solution to problem (1.2.3.5) as

$$y = \frac{ae^{-F(x)}}{1 - aG(x)}, \quad F(x) = \int_0^x f(x)\,dx, \quad G(x) = \int_0^x e^{-F(x)}g(x)\,dx. \tag{1.2.3.6}$$

This solution does not have singularities if $g(x) \le 0$. If $g(x) > 0$, two scenarios are possible.

1°. Let $\lim_{x\to\infty} G(x) \le 1/a$. Then there is a solution for all $x > 0$.

2°. Let $\lim_{x\to\infty} G(x) > 1/a$. Then there exists a critical point, x_*, satisfying the condition $G(x_*) = 1/a$, which makes the denominator in (1.2.3.6) vanish. In this case, there is a solution, in the limited range $0 < x < x_*$, that increases indefinitely as $x \to x_*$. Such solutions are known as *blow-up solutions*.

If the equation coefficients are constant, $f(x) = b$ and $g(x) = c$, we have

$$y = \frac{ab}{ac + (b - ac)e^{bx}}. \tag{1.2.3.7}$$

The condition $ac/b > 1$ corresponds to a blow-up solution existing on the interval $0 \le x < x_*$, where $x_* = -\frac{1}{b}\ln\bigl(1 - \frac{b}{ac}\bigr)$.

By letting $b \to 0$ in (1.2.3.7), we get

$$y = \frac{a}{1 - acx}.$$

If $c > 0$, this is a blow-up solution with $x_* = 1/(ac)$.

► **A related equation.**

Consider an equation of the form

$$y'_x = f(x) + g(x)e^{\lambda y} \qquad (\lambda \ne 0).$$

The substitution $u = e^{-\lambda y}$ brings it to the linear equation $u'_x + \lambda f(x)u + \lambda g(x) = 0$.

1.2.4 Darboux Equation and Other Equations

► **Darboux equation.**

A *Darboux equation* can be represented as

$$\Bigl[f\Bigl(\frac{y}{x}\Bigr) + x^a h\Bigl(\frac{y}{x}\Bigr)\Bigr]y'_x = g\Bigl(\frac{y}{x}\Bigr) + yx^{a-1}h\Bigl(\frac{y}{x}\Bigr). \tag{1.2.4.1}$$

Using the substitution $y = xz(x)$ and taking z to be the independent variable, one obtains a Bernoulli equation

$$\bigl[g(z) - zf(z)\bigr]x'_z = xf(z) + x^{a+1}h(z),$$

which is considered in Section 1.2.3.

► **Other equations.**

1°. Consider an equation of the form

$$xy'_x = y + f(x)g(y/x).$$

The substitution $u = y/x$ brings it to a separable equation, $x^2u'_x = f(x)g(u)$; see Section 1.2.1.

2°. Consider a more complex equation

$$y'_x = -\frac{\varphi'_x}{\varphi} y + f(x)g(\varphi y), \qquad \varphi = \varphi(x). \tag{1.2.4.2}$$

The substitution $w = \varphi(x)y$ brings it to a separable equation, $w'_x = \varphi(x)f(x)g(w)$.

Example 1.10. The equation

$$y'_x = -y + f(x)g(e^x y).$$

is a special case of Eq. (1.2.4.2) with $\varphi = e^x$. Therefore, the substitution $w = e^x y$ brings it to a separable equation, $w'_x = e^x f(x)g(v)$.

◆ *Some other first-order equations integrable by quadrature are treated in Chapter 13.*

⊙ *Literature for Section 1.2*: D. M. Sintsov (1913),E. L. Ince (1956), V. V. Stepanov (1958), G. M. Murphy (1960), L. E. El'sgol'ts (1961), P. Hartman (1964), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and Rota (1978), M. Tenenbaum and H. Pollard (1985), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), R. Grimshaw (1991), M. Braun (1993), D. Zwillinger (1997), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007), V. F. Zaitsev and L. V. Linchuk (2015).

1.3 Exact Differential Equations. Integrating Factor

1.3.1 Exact Differential Equations

An *exact differential equation* has the form

$$f(x,y)\,dx + g(x,y)\,dy = 0, \quad \text{where} \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}. \tag{1.3.1.1}$$

The left-hand side of the equation is the total differential of a function of two variables $U(x,y)$.

The general integral, $U(x,y) = C$, where C is an arbitrary constant and the function U is determined from the system:

$$\frac{\partial U}{\partial x} = f, \quad \frac{\partial U}{\partial y} = g.$$

Integrating the first equation yields $U = \int f(x,y)\,dx + \Psi(y)$ (while integrating, the variable y is treated as a parameter). On substituting this expression into the second equation, one identifies the function Ψ (and hence, U). As a result, the general integral of an exact differential equation can be represented in the form

$$\int_{x_0}^{x} f(\xi,y)\,d\xi + \int_{y_0}^{y} g(x_0,\eta)\,d\eta = C, \tag{1.3.1.2}$$

where x_0 and y_0 are any numbers.

Example 1.11. Consider the equation

$$(ay^n + bx)y'_x + by + cx^m = 0, \quad \text{or} \quad (by + cx^m)\,dx + (ay^n + bx)\,dy = 0,$$

defined by the functions $f(x,y) = by + cx^m$ and $g(x,y) = ay^n + bx$. Computing the derivatives, we have

$$\frac{\partial f}{\partial y} = b, \quad \frac{\partial g}{\partial x} = b \quad \Longrightarrow \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Hence the given equation is an exact differential equation. Its solution can be found using formula (1.3.1.2) with $x_0 = y_0 = 0$:

$$\frac{a}{n+1} y^{n+1} + bxy + \frac{c}{m+1} x^{m+1} = C.$$

1.3.2 Integrating Factor

An *integrating factor* for the equation

$$f(x,y)\,dx + g(x,y)\,dy = 0$$

is a function $\mu(x,y) \not\equiv 0$ such that the left-hand side of the equation, when multiplied by $\mu(x,y)$, becomes a total differential, and the equation itself becomes an exact differential equation.

An integrating factor satisfies the first-order partial differential equation,

$$g\frac{\partial \mu}{\partial x} - f\frac{\partial \mu}{\partial y} = \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\mu,$$

which is not generally easier to solve than the original equation.

Table 1.1 lists some special cases where an integrating factor can be found in explicit form.

⊙ *Literature for Section 1.3*: G. M. Murphy (1960), N. M. Matveev (1967), E. Kamke (1977), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

1.4 Riccati Equation

1.4.1 General Riccati Equation. Simplest Integrable Cases. Polynomial Solutions

► **General Riccati equation.**

A *Riccati equation* has the general form

$$y'_x = f_2(x)y^2 + f_1(x)y + f_0(x). \tag{1.4.1.1}$$

If $f_2 \equiv 0$, we have a linear equation (1.2.3.1), and if $f_0 \equiv 0$, we have a Bernoulli equation (1.2.3.3) with $a = 2$, whose solutions were given previously. For arbitrary f_2, f_1, and f_0, the Riccati equation is not integrable by quadrature.

► **Simplest integrable cases.**

Listed below are some special cases where the Riccati equation (1.4.1.1) is integrable by quadrature.

TABLE 1.1

An integrating factor $\mu = \mu(x, y)$ for some types of ordinary differential equations $f\,dx + g\,dy = 0$, where $f = f(x, y)$ and $g = g(x, y)$. The subscripts x and y indicate the corresponding partial derivatives

No.	Conditions for f and g	Integrating factor	Remarks
1	$f = y\varphi(xy),\ g = x\psi(xy)$	$\mu = \frac{1}{xf-yg}$	$xf - yg \not\equiv 0$; $\varphi(z)$ and $\psi(z)$ are any functions
2	$f_x = g_y,\ f_y = -g_x$	$\mu = \frac{1}{f^2+g^2}$	$f + ig$ is an analytic function of the complex variable $x + iy$
3	$\frac{f_y-g_x}{g} = \varphi(x)$	$\mu = \exp\bigl[\int \varphi(x)\,dx\bigr]$	$\varphi(x)$ is any function
4	$\frac{f_y-g_x}{f} = \varphi(y)$	$\mu = \exp\bigl[-\int \varphi(y)\,dy\bigr]$	$\varphi(y)$ is any function
5	$\frac{f_y-g_x}{g-f} = \varphi(x+y)$	$\mu = \exp\bigl[\int \varphi(z)\,dz\bigr],\ \ z = x + y$	$\varphi(z)$ is any function
6	$\frac{f_y-g_x}{yg-xf} = \varphi(xy)$	$\mu = \exp\bigl[\int \varphi(z)\,dz\bigr],\ \ z = xy$	$\varphi(z)$ is any function
7	$\frac{x^2(f_y-g_x)}{yg+xf} = \varphi\bigl(\frac{y}{x}\bigr)$	$\mu = \exp\bigl[-\int \varphi(z)\,dz\bigr],\ \ z = \frac{y}{x}$	$\varphi(z)$ is any function
8	$\frac{f_y-g_x}{xg-yf} = \varphi(x^2+y^2)$	$\mu = \exp\bigl[\frac{1}{2}\int \varphi(z)\,dz\bigr],\ z = x^2+y^2$	$\varphi(z)$ is any function
9	$f_y - g_x = \varphi(x)g - \psi(y)f$	$\mu = \exp\bigl[\int \varphi(x)\,dx + \int \psi(y)\,dy\bigr]$	$\varphi(x)$ and $\psi(y)$ are any functions
10	$\frac{f_y-g_x}{g\omega_x-f\omega_y} = \varphi(\omega)$	$\mu = \exp\bigl[\int \varphi(\omega)\,d\omega\bigr]$	$\omega = \omega(x, y)$ is any function of two variables

1°. The functions f_2, f_1, and f_0 are proportional, i.e.,

$$y'_x = \varphi(x)(ay^2 + by + c),$$

where a, b, and c are constants. This equation is a separable equation; see Section 1.2.1.

2°. The Riccati equation is homogeneous:

$$y'_x = a\frac{y^2}{x^2} + b\frac{y}{x} + c.$$

See Section 1.2.2, Eq. (1.2.2.1) with $f(z) = az^2 + bz + c$.

3°. The Riccati equation is generalized homogeneous:

$$y'_x = ax^n y^2 + \frac{b}{x}y + cx^{-n-2}.$$

See Eq. (1.2.2.3) with $k = -n - 1$. The substitution $z = x^{n+1}y$ brings it to a separable equation: $xz'_x = az^2 + (b + n + 1)z + c$.

4°. The Riccati equation has the form

$$y'_x = ax^{2n}y^2 + \frac{m-n}{x}y + cx^{2m}.$$

By the substitution $y = x^{m-n}z$, the equation is reduced to a separable equation: $x^{-n-m}z'_x = az^2 + c$.

◆ *Some other Riccati equations integrable by quadrature are treated in Chapter 13 (see equations 13.2.2.1 to 13.2.9.14).*

► **Polynomial solutions of the Riccati equation.**

Let $f_2 = 1$, $f_1(x)$, and $f_0(x)$ be polynomials. If the degree of the polynomial

$$\Delta = f_1^2 - 2(f_1)'_x - 4f_0$$

is odd, the Riccati equation cannot possess a polynomial solution. If the degree of Δ is even, the equation involved may possess only the following polynomial solutions:

$$y = -\tfrac{1}{2}\bigl(f_1 \pm \bigl[\sqrt{\Delta}\,\bigr]\bigr),$$

where $\bigl[\sqrt{\Delta}\,\bigr]$ denotes an integer rational part of the expansion of $\sqrt{\Delta}$ in decreasing powers of x (for example, $\bigl[\sqrt{x^2 - 2x + 3}\,\bigr] = x - 1$).

1.4.2 Use of Particular Solutions to Construct the General Solution

► **One particular solution is known.**

Let $y_0 = y_0(x)$ be a particular solution of equation (1.4.1.1). Then the substitution $y = y_0 + 1/w$ leads to a linear equation for $w = w(x)$:

$$w'_x + \bigl[2f_2(x)y_0(x) + f_1(x)\bigr]w + f_2(x) = 0.$$

The general solution of the Riccati equation (1.4.1.1) can be written as

$$y = y_0(x) + \Phi(x)\Bigl[C - \int \Phi(x) f_2(x)\,dx\Bigr]^{-1}, \qquad (1.4.2.1)$$

where C is an arbitrary constant and

$$\Phi(x) = \exp\Bigl\{\int \bigl[2f_2(x)y_0(x) + f_1(x)\bigr]\,dx\Bigr\}. \qquad (1.4.2.2)$$

To the particular solution $y_0(x)$ there corresponds $C = \infty$.

► **Two particular solutions are known.**

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two different particular solutions of equation (1.4.1.1). Then the general solution can be calculated by

$$y = \frac{Cy_1 + U(x)y_2}{C + U(x)}, \quad \text{where} \quad U(x) = \exp\Bigl[\int f_2(y_1 - y_2)\,dx\Bigr].$$

To the particular solution $y_1(x)$, there corresponds $C = \infty$; and to $y_2(x)$, there corresponds $C = 0$.

► **Three particular solutions are known.**

Let $y_1 = y_1(x)$, $y_2 = y_2(x)$, and $y_3 = y_3(x)$ be three distinct particular solutions of equation (1.4.1.1). Then the general solution can be found without quadrature:

$$\frac{y - y_2}{y - y_1}\frac{y_3 - y_1}{y_3 - y_2} = C.$$

This means that the Riccati equation has a fundamental system of solutions.

1.4.3 Some Transformations

► **Nonlinear transformation reduces the Riccati equation to a Riccati equation.**

The transformation (φ, ψ_1, ψ_2, ψ_3, and ψ_4 are arbitrary functions)

$$x = \varphi(\xi), \quad y = \frac{\psi_4(\xi)u + \psi_3(\xi)}{\psi_2(\xi)u + \psi_1(\xi)}$$

reduces the Riccati equation (1.4.1.1) to a Riccati equation for $u = u(\xi)$.

► **Reduction of the Riccati equation to a second-order linear equation.**

The substitution

$$u(x) = \exp\Bigl(-\int f_2 y\,dx\Bigr)$$

reduces the general Riccati equation (1.4.1.1) to a second-order linear equation:

$$f_2 u''_{xx} - \bigl[(f_2)'_x + f_1 f_2\bigr]u'_x + f_0 f_2^2 u = 0,$$

which often may be easier to solve than the original Riccati equation.

► **Reduction of the Riccati equation to the canonical form.**

The general Riccati equation (1.4.1.1) can be reduced with the aid of the transformation

$$x = \varphi(\xi), \quad y = \frac{1}{F_2}w - \frac{1}{2}\frac{F_1}{F_2} + \frac{1}{2}\left(\frac{1}{F_2}\right)'_\xi, \quad \text{where} \quad F_i(\xi) = f_i(\varphi)\varphi'_\xi, \qquad (1.4.3.1)$$

to the canonical form

$$w'_\xi = w^2 + \Psi(\xi). \qquad (1.4.3.2)$$

Here the function Ψ is defined by the formula

$$\Psi(\xi) = F_0F_2 - \frac{1}{4}F_1^2 + \frac{1}{2}F_1' - \frac{1}{2}F_1\frac{F_2'}{F_2} - \frac{3}{4}\left(\frac{F_2'}{F_2}\right)^2 + \frac{1}{2}\frac{F_2''}{F_2};$$

the prime denotes differentiation with respect to ξ.

Transformation (1.4.3.1) depends on a function $\varphi = \varphi(\xi)$ that can be arbitrary. For a specific original Riccati equation, different functions φ in (1.4.3.1) will generate different functions Ψ in equation (1.4.3.2). In practice, transformation (1.4.3.1) is most frequently used with $\varphi(\xi) = \xi$.

1.4.4 Special Riccati Equation

1°. A special Riccati equation has the form

$$y'_x + ay^2 = bx^m. \tag{1.4.4.1}$$

For $m \neq -3$, the transformation

$$y = \frac{1}{x^2\eta} + \frac{1}{ax}, \qquad x = \xi^{\frac{1}{m+3}} \tag{1.4.4.2}$$

brings equation (1.4.4.1) to a similar equation

$$\eta'_\xi + a_1\eta^2 = b_1\xi^{-\frac{m+4}{m+3}}, \qquad a_1 = \frac{b}{m+3}, \quad b_1 = \frac{a}{m+3}.$$

The essential parameter m changes by the rule

$$m \longrightarrow -\frac{m+4}{m+3}. \tag{1.4.4.3}$$

(The parameters a and b are inessential, as they can be made equal to one by changing the scale of x and y.) Repeating the above transformation k times, we arrive at a special Riccati equation with the exponent

$$m_k = -\frac{(2k-1)m_0 + 4k}{km_0 + 2k + 1}, \tag{1.4.4.4}$$

where $m_0 = m$.

2°. Let us now discuss the integrability of the special Riccati equation.

For $m = 0$, it becomes separable. This equation is linked to other quadrature-integrable special Riccati equations through transformations of the form (1.4.4.2) whose exponents are obtained by substituting $m_0 = 0$ in (1.4.4.4):

$$m_k = -\frac{4k}{2k+1}, \tag{1.4.4.5}$$

where k is an arbitrary integer.

For $m = -2$, the equation becomes generalized homogeneous; with the substitution $y = 1/z$, it is reduced to the homogeneous equation $z'_x = a - b(z/x)^2$.

THEOREM (LIOUVILLE). *The values*

$$m = -\frac{4k}{2k+1} \quad (k \textit{ is any integer}) \quad \textit{and} \quad m = -2 \tag{1.4.4.6}$$

exhaust all quadrature-integrable cases of the special Riccati equation.

3°. If the exponent m in (1.4.4.1) is different from the values of (1.4.4.6), the solutions can be expressed in terms of special functions. The substitution $y(x) = z'_x/(az)$ reduces the special Riccati equation to the second-order equation

$$z''_{xx} = abx^m z, \tag{1.4.4.7}$$

whose solutions can be written as

$$z = \sqrt{x}\left[C_1 J_{\frac{1}{m+2}}\left(\frac{2\sqrt{-ab}}{m+2}x^{\frac{m+2}{2}}\right) + C_2 Y_{\frac{1}{m+2}}\left(\frac{2\sqrt{-ab}}{m+2}x^{\frac{m+2}{2}}\right)\right] \quad \text{if} \quad b<0,$$

$$z = \sqrt{x}\left[C_1 I_{\frac{1}{m+2}}\left(\frac{2\sqrt{ab}}{m+2}x^{\frac{m+2}{2}}\right) + C_2 K_{\frac{1}{m+2}}\left(\frac{2\sqrt{ab}}{m+2}x^{\frac{m+2}{2}}\right)\right] \quad \text{if} \quad b>0,$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of the first and second kind, respectively, while $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions. If $m = -2$, equation (1.4.4.7) is the Euler equation. The values (1.4.4.5) give us the set of orders ν of the Bessel functions and modified Bessel functions at which they are expressible in terms of elementary functions. This occurs at half-integer orders:

$$\nu = \frac{2k+1}{2}.$$

⊙ *Literature for Section 1.4*: G. M. Murphy (1960), N. M. Matveev (1967), W. T. Reid (1972), E. Kamke (1977), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

1.5 Abel Equations of the First Kind

1.5.1 General Form of Abel Equations of the First Kind. Simplest Integrable Cases

► General form of Abel equations of the first kind.

An *Abel equation of the first kind* has the general form

$$y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x), \qquad f_3(x) \not\equiv 0. \tag{1.5.1.1}$$

In the degenerate case $f_2(x) = f_0(x) = 0$, we have a Bernoulli equation (1.2.3.3) with $a = 3$. The Abel equation (1.5.1.1) is not integrable in closed form for arbitrary $f_n(x)$.

► Simplest integrable cases.

Listed below are some special cases where the Abel equation of the first kind is integrable by quadrature.

1°. If the functions $f_n(x)$ ($n = 0, 1, 2, 3$) are proportional, i.e., $f_n(x) = a_n g(x)$, then (1.5.1.1) is a separable equation (see Section 1.2.1).

2°. The Abel equation is homogeneous:

$$y'_x = a\frac{y^3}{x^3} + b\frac{y^2}{x^2} + c\frac{y}{x} + d.$$

See Section 1.2.2, Eq. (1.2.2.1) with $f(z) = az^3 + bz^2 + cz + d$.

3°. The Abel equation is generalized homogeneous:

$$y'_x = ax^{2n+1}y^3 + bx^n y^2 + \frac{c}{x}y + dx^{-n-2}.$$

See Eq. (1.2.2.3) with $k = -n-1$. The substitution $w = x^{n+1}y$ leads to a separable equation: $xw'_x = aw^3 + bw^2 + (c+n+1)w + d$.

4°. The Abel equation

$$y'_x = ax^{3n-m}y^3 + bx^{2n}y^2 + \frac{m-n}{x}y + dx^{2m}$$

can be reduced with the substitution $y = x^{m-n}z$ to a separable equation: $x^{-n-m}z'_x = az^3 + bz^2 + c$.

5°. Let $f_0 \equiv 0$, $f_1 \equiv 0$, and $(f_3/f_2)'_x = af_2$ for some constant a. Then the substitution $y = f_2 f_3^{-1} u$ leads to a separable equation: $u'_x = f_2^2 f_3^{-1}(u^3 + u^2 + au)$.

6°. If

$$f_0 = \frac{f_1 f_2}{3f_3} - \frac{2f_2^3}{27f_3^2} - \frac{1}{3}\frac{d}{dx}\frac{f_2}{f_3}, \qquad f_n = f_n(x),$$

then the solution of equation (1.5.1.1) is given by

$$y(x) = E\Big(C - 2\int f_3 E^2\,dx\Big)^{-1/2} - \frac{f_2}{3f_3}, \qquad \text{where} \quad E = \exp\bigg[\int\Big(f_1 - \frac{f_2^2}{3f_3}\Big)\,dx\bigg].$$

◆ *For other solvable Abel equations of the first kind, see Section 13.4.1.*

1.5.2 Some Transformations

► **Reduction of the Abel equation of the first kind to the canonical form.**

The transformation

$$y = U(x)\eta(\xi) - \frac{f_2}{3f_3}, \qquad \xi = \int f_3 U^2\,dx, \qquad \text{where} \quad U(x) = \exp\bigg[\int\Big(f_1 - \frac{f_2^2}{3f_3}\Big)\,dx\bigg],$$

brings equation (1.5.1.1) to the canonical (normal) form

$$\eta'_\xi = \eta^3 + \Phi(\xi).$$

Here the function $\Phi(\xi)$ is defined parametrically (x is the parameter) by the relations

$$\Phi = \frac{1}{f_3 U^3}\Big(f_0 - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2} + \frac{1}{3}\frac{d}{dx}\frac{f_2}{f_3}\Big), \qquad \xi = \int f_3 U^2\,dx.$$

► **Reduction to an Abel equation of the second kind.**

Let $y_0 = y_0(x)$ be a particular solution of equation (1.5.1.1). Then the substitution

$$y = y_0 + \frac{E(x)}{z(x)}, \qquad \text{where} \quad E(x) = \exp\Big[\int (3f_3 y_0^2 + 2f_2 y_0 + f_1)\,dx\Big],$$

leads to an Abel equation of the second kind:

$$zz'_x = -(3f_3 y_0 + f_2)Ez - f_3 E^2.$$

For equations of this type, see Section 1.6.

⊙ *Literature for Section 1.5*: G. M. Murphy (1960), E. Kamke (1977), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

1.6 Abel Equations of the Second Kind

1.6.1 General Form of Abel Equations of the Second Kind. Simplest Integrable Cases

► General form of Abel equations of the second kind.

An *Abel equation of the second kind* has the general form

$$[y+g(x)]y'_x = f_2(x)y^2 + f_1(x)y + f_0(x), \qquad g(x) \not\equiv 0. \tag{1.6.1.1}$$

The Abel equation (1.6.1.1) is not integrable for arbitrary $f_n(x)$ and $g(x)$. Given below are some special cases where the Abel equation of the second kind is integrable by quadrature.

► Simplest integrable cases.

1°. If $g(x) = \text{const}$ and the functions $f_n(x)$ $(n = 0,\ 1,\ 2)$ are proportional, i.e., $f_n(x) = a_n g(x)$, then (1.6.1.1) is a separable equation (see Section 1.2.1).

2°. The Abel equation is homogeneous:

$$(y+sx)y'_x = \frac{a}{x}y^2 + by + cx.$$

See Section 1.2.2. The substitution $w = y/x$ leads to a separable equation.

3°. The Abel equation is generalized homogeneous:

$$(y+sx^n)y'_x = \frac{a}{x}y^2 + bx^{n-1}y + cx^{2n-1}.$$

See Eq. (1.2.2.3) with for $k=n$. The substitution $w = yx^{-n}$ leads to a separable equation: $x(w+s)w'_x = (a-n)w^2 + (b-ns)w + c$.

4°. The Abel equation

$$(y + a_2x + c_2)y'_x = b_1y + a_1x + c_1$$

is a special case of equation (1.2.2.2) with $f(w) = w$ and $b_2 = 1$.

5°. The unnormalized Abel equation

$$[(a_1x + a_2x^n)y + b_1x + b_2x^n]y'_x = c_2y^2 + c_1y + c_0$$

can be reduced to the form (1.6.1.1) by dividing it by $(a_1x + a_2x^n)$. Taking y to be the independent variable and $x = x(y)$ to be the dependent one, we obtain the Bernoulli equation

$$(c_2y^2 + c_1y + c_0)x'_y = (a_1y + b_1)x + (a_2y + b_2)x^n.$$

See Eq. (1.2.3.3).

6°. The general solution of the Abel equation

$$(y+g)y'_x = f_2y^2 + f_1y + f_1g - f_2g^2, \qquad f_n = f_n(x), \;\; g = g(x),$$

is given by

$$y = -g + CE + E\int (f_1 + g'_x - 2f_2g)E^{-1}\,dx, \qquad \text{where} \quad E = \exp\Bigl(\int f_2\,dx\Bigr).$$

7°. If $f_1 = 2f_2g - g'_x$, the general solution of the Abel equation (1.6.1.1) has the form

$$y = -g \pm E\Bigl[2\int (f_0 + gg'_x - f_2g^2)E^{-2}\,dx + C\Bigr]^{1/2}, \qquad \text{where} \quad E = \exp\Bigl(\int f_2\,dx\Bigr).$$

◆ *For other solvable Abel equations of the second kind, see Section 13.3.*

1.6.2 Some Transformations

► **Reduction of the Abel equation of the second kind to the canonical form.**

1°. The substitution

$$w = (y+g)E, \qquad \text{where} \quad E = \exp\Bigl(-\int f_2\,dx\Bigr), \tag{1.6.2.1}$$

brings equation (1.6.1.1) to the simpler form

$$ww'_x = F_1(x)w + F_0(x), \tag{1.6.2.2}$$

where

$$F_1 = (f_1 - 2f_2g + g'_x)E, \quad F_0 = (f_0 - f_1g + f_2g^2)E^2.$$

2°. In turn, equation (1.6.2.2) can be reduced, by the introduction of the new independent variable

$$z = \int F_1(x)\,dx, \tag{1.6.2.3}$$

to the *canonical form*

$$ww'_z - w = R(z). \tag{1.6.2.4}$$

Here the function $R(z)$ is defined parametrically (x is the parameter) by the relations

$$R = \frac{F_0(x)}{F_1(x)}, \qquad z = \int F_1(x)\,dx.$$

Substitutions (1.6.2.1) and (1.6.2.3), which take the Abel equation to the canonical form, are called *canonical*.

Remark 1.3. The transformation $w = a\hat{w}$, $z = a\hat{z} + b$ brings (1.6.2.4) to a similar equation, $\hat{w}\hat{w}'_{\hat{z}} - \hat{w} = a^{-1}R(a\hat{z} + b)$. Therefore the function $R(z)$ in the right-hand side of the Abel equation (1.6.2.4) can be identified with the two-parameter family of functions $a^{-1}R(az + b)$.

Remark 1.4. Any Abel equations of the second kind related by linear (in y) transformations of the form

$$\widetilde{x} = \varphi_1(x), \qquad \widetilde{y} = \varphi_2(x)y + \varphi_3(x)$$

have identical canonical forms, up to the two-parameter family of functions specified in Remark 1.3.

► **Reduction to an Abel equation of the first kind.**

The substitution $y+g=1/u$ leads to an Abel equation of the first kind:

$$u'_x+(f_0-f_1g+f_2g^2)u^3+(f_1-2f_2g+g'_x)u^2+f_2u=0.$$

For equations of this type, see Section 1.5.

1.6.3 Use of Particular Solutions to Construct Self-Transformations and the General Solution

► **Use of particular solutions to construct self-transformations.**

1°. Let a particular solution $y_0=y_0(x)$ of an Abel equation of the second kind (1.6.1.1) be known. Then the substitution $U=1/(y-y_0)$ leads to a similar Abel equation:

$$\left(U+\frac{1}{y_0+g}\right)U'_x=\frac{y'_0-f_1-2f_2y_0}{y_0+g}U^2-\frac{f_2}{y_0+g}U. \tag{1.6.3.1}$$

If $f_0\equiv 0$, equation (1.6.1.1) has the trivial particular solution $y_0=0$. In this case, the change of variable $U=1/y$ leads to an Abel equation of the form (1.6.3.1) with $y_0=0$.

2°. Given a particular solution $y_0=y_0(x)$ of the Abel equation of the second kind

$$yy'_x=f_1(x)y+f_0(x), \tag{1.6.3.2}$$

the substitution

$$w=\frac{H(x)y}{y_0^2(y_0-y)},\qquad \text{where}\quad H(x)=\exp\left(\int\frac{f_1}{y_0}\,dx\right), \tag{1.6.3.3}$$

brings (1.6.3.2) to another, similar Abel equation:

$$ww'_x=\mathcal{F}_1(x)w+\mathcal{F}_0(x). \tag{1.6.3.4}$$

Here, the functions $\mathcal{F}_1=\mathcal{F}_1(x)$ and $\mathcal{F}_0=\mathcal{F}_0(x)$ are defined by

$$\mathcal{F}_1=\frac{(f_1y_0+3f_0)H}{y_0^4},\qquad \mathcal{F}_0=\frac{f_0H^2}{y_0^6}.$$

It is not difficult to verify by direct substitution that equation (1.6.3.4) has a particular solution:

$$w_0(x)=-\frac{H(x)}{y_0^2(x)}. \tag{1.6.3.5}$$

The transformation based on the particular solution (1.6.3.5) brings the Abel equation (1.6.3.4) to the original equation (1.6.3.2) with f_1 having the opposite sign.

Remark 1.5. In general, the canonical forms of equations (1.6.1.1) and (1.6.3.1) and also those of equations (1.6.3.2) and (1.6.3.4) are different. See Section 1.6.2.

Remark 1.6. Given k distinct particular solutions y_k of equation (1.6.3.2), k distinct Abel equations of the second kind related to (1.6.3.2) by known substitutions of the form (1.6.3.3) can be constructed.

► **Use of particular solutions to construct the general solution.**

For some Abel equations of the second kind, the general solution can be found if n of its distinct particular solutions $y_k = y_k(x)$, $k = 1, \dots, n$, are known.

Below we consider Abel equations of the canonical form

$$yy'_x - y = R(x), \tag{1.6.3.6}$$

whose general solutions can be represented in the special form:

$$\prod_{k=1}^{n} |y - y_k(x)|^{m_k} = C. \tag{1.6.3.7}$$

Here, the particular solutions $y_k = y_k(x)$ correspond to $C = 0$ (if $m_k > 0$) and $C = \infty$ (if $m_k < 0$).

The logarithmization of (1.6.3.7), followed by the differentiation of the resulting expression and rearrangement, leads to the equation

$$\sum_{j=1}^{n}\Bigg[m_j(y'_x - y'_j)\prod_{\substack{k=1\\k\neq j}}^{n}(y - y_k)\Bigg] \equiv y'_x\sum_{s=1}^{n-1}\Phi_s y^s + \sum_{s=1}^{n-1}\Psi_s y^s = 0, \tag{1.6.3.8}$$

where $y'_j = (y_j)'_x$. We require that equation (1.6.3.8) be equivalent to the Abel equation (1.6.3.6). To this end, we set:

$$\Psi_\nu = -\Phi_\nu, \quad \Psi_{\nu-1} = -R(x)\Phi_\nu \quad \text{and equate the other } \Phi_i \text{ and } \Psi_i \text{ with zero.}$$

Selecting different values $\nu = 1,\ 2,\ \dots,\ n-1$, we obtain $n-1$ systems of differential-algebraic equations; only one of the systems, corresponding to $m_k \neq 0$ for all $k = 1, \dots, n$ and $y_i \neq y_j$ for $i \neq j$, leads to a nondegenerate solution of the form (1.6.3.7). Consider the Abel equations (1.6.3.6) corresponding to the simplest solutions of the form (1.6.3.7) in more detail.

1°. *Case* $n = 2$. The system of differential-algebraic equations has the form:

$$\begin{aligned}
&m_1 + m_2 = M,\\
&m_1 y_2 + m_2 y_1 = 0,\\
&m_1 y'_1 + m_2 y'_2 = M,\\
&m_1 y'_1 y_2 + m_2 y_1 y'_2 = -MR(x),
\end{aligned} \tag{1.6.3.9}$$

where M is an arbitrary constant. It follows from the second and third equations that

$$y_1 = \frac{m_1}{m_1^2 - m_2^2}(Mx + N), \quad y_2 = -\frac{m_2}{m_1^2 - m_2^2}(Mx + N),$$

where N is an arbitrary constant. Introducing the new constants

$$A = \frac{m_1 m_2(m_1 + m_2)}{(m_1 - m_2)^2}M, \quad B = \frac{m_1 m_2(m_1 + m_2)}{(m_1 - m_2)^2}N,$$

we find from the last relation in (14) that

$$R(x) = Ax + B, \tag{1.6.3.10}$$

which means that for $n = 2$ the right-hand side of the Abel equation is a linear function of x (see equation 13.3.1.2).

The particular solutions y_1, y_2, and the corresponding exponents m_1, m_2 in the general integral (1.6.3.7), are expressed in terms of the coefficients A, B on the right-hand side (1.6.3.10) of the Abel equation (1.6.3.6) as follows:

$$y_1 = \frac{1+\sqrt{4A+1}}{2A}(Ax+B), \qquad m_1 = 2A+1+\sqrt{4A+1},$$
$$y_2 = -\frac{1+\sqrt{4A+1}}{2A+1+\sqrt{4A+1}}(Ax+B), \quad m_2 = 2A.$$

2°. *Case* $n = 3$. Equation (1.6.3.8) with $n = 3$ leads to the Abel equation (1.6.3.6) with the right-hand side

$$R(x) = -\tfrac{2}{9}x + A + Bx^{-1/2} \tag{1.6.3.11}$$

(see equation 13.3.1.3).

The particular solutions and the exponents in the general integral (1.6.3.7) are expressed as:

$$y_s = \frac{2}{3}x + \frac{2}{3}\lambda_s x^{1/2} + \frac{3B}{\lambda_s}, \quad m_s = \frac{2A}{3(2\lambda_s^2 - 3A)},$$

where the λ_s are roots of the cubic equation

$$\lambda^3 - \tfrac{9}{2}A\lambda - \tfrac{9}{2}B = 0, \quad s = 1,\ 2,\ 3.$$

3°. *Case* $n = 4$. Equation (1.6.3.8) with $n = 4$ leads to the Abel equation (1.6.3.6) with the right-hand side

$$R(x) = -\tfrac{3}{16}x + Ax^{-1/3} + Bx^{-5/3}$$

(see equation 13.3.3.61).

The particular solutions and the exponents in (1.6.3.7) are expressed as:

$$y_{1,2} = \tfrac{3}{4}x \pm \sqrt{3A + \tfrac{3}{2}\sqrt{-3B}}\,x^{1/3} + \sqrt{-3B}\,x^{-1/3}, \quad m_{1,2} = \mp\bigl(2A - \sqrt{-3B}\,\bigr),$$
$$y_{3,4} = \tfrac{3}{4}x \pm \sqrt{3A - \tfrac{3}{2}\sqrt{-3B}}\,x^{1/3} - \sqrt{-3B}\,x^{-1/3}, \quad m_{3,4} = \pm\sqrt{4A^2 + 3B}.$$

4°. *Case* $n > 4$. The equations for y_s are algebraic equations of degree n and, in the general case, are not soluble in radicals. The right-hand side of equation (1.6.3.6) is expressed as

$$R(x) = -\frac{n-1}{n^2}x + Q(x),$$

with the function $Q(x)$ bounded as $x \to \infty$ (Q can be specified in parametric form).

⊙ *Literature for Section 1.6*: B. M. Koyalovich (1894), G. M. Murphy (1960), E. Kamke (1977), V. F. Zaitsev and A. D. Polyanin (1993, 1994, 2001) A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

1.7 Classification and Specific Features of Some Classes of Solutions

◆ *The uniqueness and existence theorems stated in Section 1.1 do not say anything about qualitative features of solutions arising in specific problems. This section deals with certain classes of problems and solutions that have important distinguishing features or show pronounced unusual properties (as a rule, such problems cannot be solved with standard numerical methods).*

1.7.1 Stable and Unstable Solutions. Equilibrium Points

► **Stable, asymptotically stable, and unstable solutions.**

In many applications, the independent variable x plays the role of time.

Let $\bar{y}(x)$ be a solution of equation (1.1.1.1) with initial condition (1.1.1.2) and let $\widetilde{y}(x)$ be a solution of the equation with initial condition $y(x_0) = \widetilde{y}_0$.

1°. A solution $\bar{y}(x)$ is called (Lyapunov) *stable* if for any sufficiently small $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\widetilde{y}(x)$ that is close to $\bar{y}(x)$ initially, $|y_0 - \widetilde{y}_0| < \delta$, remains close to it at all times: $|\bar{y}(x) - \widetilde{y}(x)| < \delta$ for all $x \geq x_0$.

Any solution that is not stable is called *unstable*.

2°. A solution $\bar{y}(x)$ is called *asymptotically stable* if it is stable and, in addition, there exists a $\delta_0 > 0$ such that whenever $|y_0 - \widetilde{y}_0| < \delta_0$, we have $|\bar{y}(x) - \widetilde{y}(x)| \to 0$ as $x \to \infty$.

Remark 1.7. In stability analysis, it is normally assumed, without loss of generality, that $x_0 = 0$. (This can be achieved with the substitution $X = x - x_0$.) Further, with the change of variable

$$Y = y - \bar{y}(x),$$

the stability analysis of any solution is reduced to that of the zero solution, $Y = 0$.

► **Equilibrium points. An example.**

Simplest solutions of the form $y = \mathring{y}$, where $\mathring{y} = \text{const}$, are called *equilibrium points* (or *stationary points*). Equilibrium points (if they exist) make the right-hand side of equation (1.1.1.1) zero for any x. For simplicity and clarity, we will discuss equilibrium points of autonomous equations, whose right-hand side is independent explicitly of x,

$$y'_x = f(y). \tag{1.7.1.1}$$

Equilibrium points are roots of the algebraic (or transcendental) equation $f(\mathring{y}) = 0$.

Example 1.12. Consider the *logistic differential equation*

$$y'_x = ky(1 - y), \qquad k > 0. \tag{1.7.1.2}$$

It is one of the simplest nonlinear population mathematical models in which y denotes the dimensionless number of individuals. Solving the quadratic equation $f(\mathring{y}) = \mathring{y} - \mathring{y}^2 = 0$ gives two equilibrium points: $\mathring{y}_1 = 0$ and $\mathring{y}_2 = 1$.

The general solution of the separable equation (1.7.1.2) is

$$y = \frac{Ce^{kx}}{1 + Ce^{kx}}, \tag{1.7.1.3}$$

where C is an arbitrary constant. At $C = 0$ and $C = \infty$, we get the equilibrium points $\mathring{y}_1$ and $\mathring{y}_2$.

The solution to equation (1.7.1.2) satisfying the initial condition $y(0) = a$ corresponds to $C = \frac{y_0}{1-y_0}$ in (1.7.1.3) and is expressed as

$$y = \frac{ae^{kx}}{1 - a + ae^{kx}}. \tag{1.7.1.4}$$

The initial value in the logistic equation (1.7.1.2), describing model population dynamics, is assumed to be positive, $a > 0$. For small initial values, $a \ll 1$, the solution initially rises exponentially as ae^{kx}, which corresponds to the *Malthusian population model* with unlimited resources. However, as x increases, the rate of rise decreases gradually and the solution levels off tending to an equilibrium.

The first equilibrium solution $\mathring{y}_1 = 0$ is unstable, since all nearby solutions go away from it rapidly with an exponential rate. The second equilibrium solution $\mathring{y}_2 = 1$ is asymptotically stable, since any solution (1.7.1.4) with $a > 0$ tends exponentially to the equilibrium value, $y(x) \to \mathring{y}_2$ as $x \to \infty$.

► Theorems on stability or instability of equilibrium points. An example.

For the autonomous equation (1.7.1.1), there is a simple criterion for determining stability or instability of an equilibrium, which is stated below.

THEOREM 1 (ON STABILITY/INSTABILITY OF EQUILIBRIA). *Let $\mathring{y}$ be an equilibrium point of the autonomous differential equation* (1.7.1.1). *If $f'_y(\mathring{y}) < 0$, then $\mathring{y}$ is asymptotically stable. If $f'_y(\mathring{y}) > 0$, then $\mathring{y}$ is unstable.*

Example 1.13. Consider the differential equation

$$y'_x = y - y^3.$$

Solving the cubic equation $f(\mathring{y}) = \mathring{y} - \mathring{y}^3 = 0$ gives three equilibrium points: $\mathring{y}_1 = -1$, $\mathring{y}_2 = 0$, and $\mathring{y}_3 = 1$.

Using Theorem 1, we calculate the derivative, $f'_y(y) = 1 - 3y^2$, and its values at the equilibria:

$$f'_y(-1) = -2 < 0, \quad f'_y(0) = 1 > 0, \quad f'_y(1) = -2 < 0.$$

This suggests that equilibrium points $\mathring{y}_1 = -1$ and $\mathring{y}_3 = 1$ are both stable, while $\mathring{y}_2 = 0$ is unstable.

Theorem 1 does not answer the question whether the equilibria with $f'_y(\mathring{y}) = 0$ are stable or unstable. In this situation, the following additional criterion can be used.

THEOREM 2 (ON ASYMPTOTIC STABILITY OF EQUILIBRIA). *An equilibrium point $\mathring{y}$ of the autonomous differential equation* (1.7.1.1) *is asymptotically stable if and only if $f(y) > 0$ for $\mathring{y} - \delta < y < \mathring{y}$ and $f(y) < 0$ for $\mathring{y} < y < \mathring{y} + \delta$, where δ is a sufficiently small positive number.*

► Supplementary remarks, examples, and theorems.

1°. A solution to an ODE, $y = y(x)$, is said to be *indefinitely extensible to the right* if it exists for any $x \in [x_0, \infty)$, where x_0 is the value appearing in the statement of the Cauchy problem. A solution that is not indefinitely extensible to the right will reach the bound of the existence range at a final $x = x_*$.

Example 1.14. The domain of definition of the equation

$$y'_x = \big(1 - \sqrt{1 - ax^2y^2}\,\big) f(x, y) \tag{1.7.1.5}$$

with $a > 0$ and $0 < f(x, y) < \infty$ is given by $x^2y^2 \le 1/a$. Equation (1.7.1.5) has an equilibrium at $y = 0$. The Cauchy problem solutions for this equation with the initial condition $y(0) = y_0$ behave differently depending on the sign of y_0: if $y_0 < 0$, the solutions are indefinitely extensible to the right, while if $y_0 > 0$, these reach the bound of the existence range at a finite x and so are not indefinitely extensible to the right. The equilibrium $y = 0$ is unstable, since there is an inextensible solution in any of its neighborhoods.

Remark 1.8. Numerical solutions of the Cauchy problem for equation (1.7.1.5) with $a = 1$ and $f(x, y) \equiv 1$ and various initial conditions are presented in Section 19.4.3 (see Fig. 19.7).

2°. The definition of stability involves the initial point x_0, which is further treated as the initial time. The question arises whether the property of stability is dependent on the choice of the initial time.

THEOREM. *If an equilibrium of an equation (or system of equations) is stable for an initial time $x = x_0$, it is also stable for any subsequent time $x = x_1 > x_0$ taken as the initial.*

3°. Let us now discuss whether the property of stability of a solution is preserved under transformations of the class of equations (or system of equations) in question. In general, this property in not preserved.

Example 1.15. Consider the Cauchy problem

$$y'_x = 1, \qquad y(0) = y_0, \tag{1.7.1.6}$$

whose solution is given by

$$y = y_0 + x. \tag{1.7.1.7}$$

Let us investigate the stability of this solution.

If the initial condition is perturbed, $y(0) = y_0 + \delta$, we get the perturbed solution $\widetilde{y} = y_0 + \delta + x$. The difference between the perturbed and original solutions, $|\widetilde{y} - y| = \delta$, is indefinitely small for small δ and any y_0.

Changing in (1.7.1.6) to the new dependent variable $z = y^2$, we obtain the problem

$$z'_x = 2\sqrt{z}, \qquad z(0) = y_0^2. \tag{1.7.1.8}$$

Then, solution (1.7.1.7) becomes

$$z = (y_0 + x)^2.$$

With the perturbed initial condition, $z(0) = y_0^2 + \delta$, the solution of the transformed problem is given by

$$\widetilde{z} = \big(\sqrt{y_0^2 + \delta} + x\big)^2.$$

The difference

$$|\widetilde{z} - z| = 2\big(\sqrt{y_0^2 + \delta} - y_0\big)x + \delta$$

is unbounded as $x \to \infty$ no matter how small the initial perturbation δ was.

One can see that, in this problem, the solution stable with respect to the original variable y became unstable with respect to the new variable z.

4°. For the stability analysis to be correct, one has to understand clearly the variables by which the stability is assessed. Furthermore, if the analysis of the problem requires changing to new variables, one must guarantee the equivalence of the stability properties in terms of the original and new variables.

Transformations of variables that preserve the properties of stability between solutions are called *allowable*.

THEOREM ON ALLOWABLE TRANSFORMATIONS. *Let $y \equiv 0$ be a stable solution of the equation $y'_x = f(x,y)$ and let the transformation $y = \varphi(x,z)$ with $\varphi(x,0) \equiv 0$ satisfy the conditions:*

(i) *the partial derivative φ_z is nondegenerate in a neighborhood of $z = 0$;*

(ii) *the functions φ and φ^{-1} are uniformly continuous in x at $z = 0$.*

Then the solution $z \equiv 0$ of the equation $z'_x = \varphi_z^{-1}\big[f(x,\varphi) - \varphi_x\big]$ is stable. Otherwise, it is unstable.

1.7.2 Blow-Up Solutions

► Blow-up solutions with a power-law singularity. An example.

There are Cauchy problems whose solution tends to infinity at a certain finite value, $x = x_*$, which does not appear in the equation explicitly and is unknown in advance. Such solutions exist on a limited interval, denoted $x_0 \le x < x_*$ throughout this section, and are called *blow-up solutions*.

In general, a blow-up solution with a power-law singularity at a singular point x_* can be represented as

$$y \approx A(x_* - x)^{-\mu}, \qquad \mu > 0, \tag{1.7.2.1}$$

where A is some constant. So we have $|y(x_*)| = \infty$.

Example 1.16. Consider a model Cauchy problem for a separable ODE:

$$y'_x = \tfrac{1}{2}by^3 \quad (x > 0), \qquad y(0) = 1. \tag{1.7.2.2}$$

The exact solution to this problem is

$$y = \frac{1}{\sqrt{1 - bx}}. \tag{1.7.2.3}$$

If $b \le 0$, the solution exists and is bounded for all $x > 0$. If $b > 0$, the solution is only defined on a limited interval, $0 \le x < x_*$, where $x_* = 1/b$ is a singular point, at which the solution is infinite. This is a blow-up solution, which does not exist for $x > x_*$. One cannot see in advance from the statement of the problem (1.7.2.2) that the solution has a singularity.

Remark 1.9. In problems where the independent variable x plays the role of time, the critical value x_* is often called the *blow-up time*.

► Blow-up solutions with a logarithmic singularity. An example.

There are blow-up problems whose solution has a singularity other than (1.7.2.1). In particular, blow-up solutions with a logarithmic singularity at a point x_* can be represented as

$$y \approx A \ln\big[B(x_* - x)\big],$$

where A and $B > 0$ are some constants. We have $|y(x_*)| = \infty$.

Example 1.17. Consider a model Cauchy problem for a separable ODE:

$$y'_x = be^y \quad (x > 0), \qquad y(0) = a \tag{1.7.2.4}$$

with $a \ge 0$ and $b > 0$. The exact solution is

$$y = -\ln(e^{-a} - bx). \tag{1.7.2.5}$$

It has a logarithmic singularity at $x_* = e^{-a}/b$ and does not exist for $x > x_*$.

► **Autonomous equation.**

Let us look at the Cauchy problem for the autonomous equation (1.7.1.1) subject to the initial condition $y(0) = a > 0$. We assume that $f(y) > 0$ is a continuous function defined for all $y \geq a$. The solution to the Cauchy problem for $x > 0$ can be written in implicit form as

$$x = \int_a^y \frac{d\xi}{f(\xi)}. \tag{1.7.2.6}$$

This is a blow-up solution if and only if the definite integral (1.7.2.6) is finite for $y = \infty$. The critical value x_* is evaluated as:

$$x_* = \int_a^\infty \frac{d\xi}{f(\xi)}. \tag{1.7.2.7}$$

Sufficient criterion for the existence of a blow-up solution. Suppose that the above conditions hold as well as the limiting relation

$$\lim_{y\to\infty} \frac{f(y)}{y^{1+\sigma}} = s, \qquad 0 < s \leq \infty, \tag{1.7.2.8}$$

for some $\sigma > 0$. Then the solution to the Cauchy problem is a blow-up solution. If $f(y)$ is differentiable, then (1.7.2.8) can be replaced with the equivalent criterion

$$\lim_{y\to\infty} \bigl[y^{-\sigma} f'_y(y)\bigr] = s_1, \qquad 0 < s_1 \leq \infty \qquad (\sigma > 0).$$

Example 1.18. Consider the Cauchy problem for the power-law autonomous ODE

$$y'_x = by^k, \qquad y(0) = a, \tag{1.7.2.9}$$

where $a > 0$ and $b > 0$. The solution is given by formula (1.7.2.6), which can be solved for the unknown function and rewritten explicitly as

$$y = \bigl[a^{1-k} - b(k-1)x\bigr]^{-\frac{1}{k-1}}.$$

It is apparent that the problem has a blow-up solution if $k > 1$. The critical value x_* is given by

$$x_* = \frac{1}{a^{k-1}b(k-1)}.$$

We use criterion (1.7.2.8) in order to determine, without solving problem (1.7.2.9), the value of the parameter k corresponding to the blow-up solution. To do so, σ in (1.7.2.8) can be chosen in the form $\sigma = k - 1$ to give $s = b$. Since the condition $\sigma > 0$ must hold, we get $k > 1$.

► **Nonautonomous equations. Some estimates.**

Consider the Cauchy problem for the general equation (1.1.1.1) subject to the initial condition (1.1.1.2) with $x_0 = 0$. Suppose the conditions

$$f(x, y) \geq g(y) > 0 \quad \text{for all} \quad y \geq y_0 > 0 \;\text{ and }\; x \geq 0, \tag{1.7.2.10}$$

hold and the finite integral

$$I_g = \int_{y_0}^\infty \frac{d\xi}{g(\xi)} < \infty \tag{1.7.2.11}$$

exists. Then the solution $y = y(x)$ of the Cauchy problem (1.1.1.1)–(1.1.1.2) is a blow-up solution, with the critical value x_* satisfying the inequality

$$x_* \leq I_g. \tag{1.7.2.12}$$

This estimate follows from the inequality (see Theorem 1 in Section 1.12.1)

$$y(x) \geq y_g(x), \tag{1.7.2.13}$$

where $y(x)$ is the solution of the Cauchy problem (1.1.1.1)–(1.1.1.2), while $y_g(x)$ is the solution of the auxiliary Cauchy problem

$$y'_x = g(y) \quad (x > 0), \qquad y(0) = y_0. \tag{1.7.2.14}$$

Example 1.19. Consider the Cauchy problem for the Riccati equation

$$y'_x = y^2 + h(x) \quad (x > 0); \qquad y(0) = a > 0. \tag{1.7.2.15}$$

If $h(x) \geq 0$ for $x \geq 0$, then the following inequality holds:

$$f(x, y) \equiv y^2 + h(x) \geq g(y) \equiv y^2 > 0 \quad \text{for all} \quad y = y_0 > a.$$

Evaluating the integral (1.7.2.11) with $g(y) = y^2$ and $y_0 = a$, we get

$$I_g = \int_a^\infty \frac{d\xi}{\xi^2} = \frac{1}{a} < \infty. \tag{1.7.2.16}$$

Hence, the solution to the Cauchy problem (1.7.2.15) with $h(x) \geq 0$ is a blow-up solution, with $x_* \leq 1/a$.

Remark 1.10. In the Cauchy problem for the more complex Riccati equation

$$y'_x = y^2 + f_1(x)y + f_0(x) \quad (x > 0); \qquad y(0) = a > 0,$$

one can obtain, apart from the obvious conditions $f_1(x) \geq 0$ and $f_0(x) \geq 0$, more complex conditions for the existence of a blow-up solution:

$$a + \tfrac{1}{2} f_1(0) > 0, \quad f_0(x) \geq \tfrac{1}{4} f_1^2(x) - \tfrac{1}{2} f_1'(x) \quad \text{for} \quad x \geq 0.$$

This can be proved by substituting $u = y + \frac{1}{2} f_1(x)$ into the equation and taking into account the results obtained in Example 1.19.

Consider two cases in which the estimate (1.7.2.12) can be improved. We use the notation:

$$I_1 = \int_{y_0}^\infty \frac{d\xi}{f(0, \xi)}, \qquad I_2 = \int_{y_0}^\infty \frac{d\xi}{f(I_1, \xi)}. \tag{1.7.2.17}$$

1°. Suppose the integral I_1 in (1.7.2.17) exists and is finite. Suppose also that the conditions

$$f(x, y) > 0 \text{ and } f_x(x, y) \geq 0 \text{ for all } 0 \leq x \leq I_1 \text{ and } y \geq y_0 > 0 \tag{1.7.2.18}$$

hold. Then, the integral I_2 exists and we get

$$f(0, y) \leq f(x, y) \leq f(I_1, y) \quad \text{for} \quad 0 \leq x \leq I_1 \tag{1.7.2.19}$$

and

$$y_1(x) \le y(x) \le y_2(x) \quad \text{for} \quad 0 \le x \le I_2 \le I_1. \tag{1.7.2.20}$$

Here, $y(x)$ is the solution to the Cauchy problem (1.1.1.1)–(1.1.1.2), while $y_1(x)$ and $y_2(x)$ are the respective solutions of the auxiliary Cauchy problems

$$y'_x = f(0, y) \qquad (x > 0), \qquad y(0) = y_0; \tag{1.7.2.21}$$

$$y'_x = f(I_1, y) \qquad (x > 0), \qquad y(0) = y_0. \tag{1.7.2.22}$$

The solutions $y_1(x)$ and $y_2(x)$ can be represented in implicit form as

$$x = \int_{y_0}^{y} \frac{d\xi}{f(0,\xi)} \quad \text{and} \quad x = \int_{y_0}^{y} \frac{d\xi}{f(I_1,\xi)}. \tag{1.7.2.23}$$

The critical value x_* satisfies the bilateral estimate

$$I_2 \le x_* \le I_1. \tag{1.7.2.24}$$

2°. Suppose the integrals I_1 and I_2 defined in (1.7.2.17) exist and are finite. Suppose also that the conditions

$$f(x, y) > 0 \;\text{ and }\; f_x(x, y) \le 0 \;\text{ for all }\; 0 \le x \le I_2 \;\text{ and }\; y \ge y_0 > 0 \tag{1.7.2.25}$$

hold. Then, we get

$$f(I_1, y) \le f(x, y) \le f(0, y) \quad \text{for} \quad 0 \le x \le I_2 \tag{1.7.2.26}$$

and

$$y_2(x) \le y(x) \le y_1(x) \quad \text{for} \quad 0 \le x \le I_1 \le I_2, \tag{1.7.2.27}$$

where $y(x)$ is the solution to the Cauchy problem (1.1.1.1)–(1.1.1.2), while $y_1(x)$ and $y_2(x)$ are the respective solutions of two auxiliary Cauchy problems (1.7.2.21) and (1.7.2.22). The last two solutions can be represented in the implicit form (1.7.2.23). The critical value x_* satisfies the bilateral estimate

$$I_1 \le x_* \le I_2. \tag{1.7.2.28}$$

Example 1.20. Let us look at the Cauchy problem for the Riccati equation (1.7.2.15) once again.
1°. Suppose $h(x) \ge 0$ and $h'_x(x) \ge 0$. In this case, the first auxiliary Cauchy problem (1.7.2.21) becomes

$$y'_x = y^2 + h(0) \qquad (x > 0), \qquad y(0) = a. \tag{1.7.2.29}$$

Its solution is determined explicitly by the first relation in (1.7.2.23) with $y_0 = a$ and $f(0, y) = y^2 + h(0)$. By a simple rearrangement, this solution can be rewritten in explicit form as

$$y = \sqrt{b}\,\frac{a\cos(\sqrt{b}\,x) + \sqrt{b}\sin(\sqrt{b}\,x)}{\sqrt{b}\cos(\sqrt{b}\,x) - a\sin(\sqrt{b}\,x)}, \qquad b = h(0). \tag{1.7.2.30}$$

The singular point of this solution, I_1, is equal to the first integral in (1.7.2.17) with $y_0 = a$ and $f(0, y) = y^2 + h(0)$, is given by

$$I_1 = \frac{1}{\sqrt{b}} \arctan\frac{\sqrt{b}}{a}, \qquad b = h(0).$$

The solution to the second auxiliary Cauchy problem (1.7.2.22) can be obtain from formula (1.7.2.30) by formally replacing $h(0)$ with $h(I_1)$. This results in the following bilateral estimate for the critical value x_*:

$$I_2 \le x_* \le I_1, \qquad I_1 = \frac{1}{\sqrt{h(0)}} \arctan \frac{\sqrt{h(0)}}{a}, \quad I_2 = \frac{1}{\sqrt{h(I_1)}} \arctan \frac{\sqrt{h(I_1)}}{a}. \qquad (1.7.2.31)$$

In the limit case $h(0) \to 0$, we get

$$I_1 = 1/a, \qquad I_2 = \frac{1}{\sqrt{h(1/a)}} \arctan \frac{\sqrt{h(1/a)}}{a}.$$

If $h(x) = \text{const} > 0$, then inequalities (1.7.2.31) give the exact result $x_* = I_1 = I_2$.

In particular, if $a = 1$ and $h(x) = x^m$ with $m > 0$ in (1.7.2.15), we have $I_1 = 1$ and $I_2 = \arctan 1$. Hence, $0.785 \le x_* \le 1$.

2°. Let $h(x) \ge 0$ and $h'_x(x) \le 0$. In this case, the solution to the first auxiliary Cauchy problem (1.7.2.29) is also given by formula (1.7.2.30), while that to the second auxiliary Cauchy problem is obtained from (1.7.2.30) by formally replacing $h(0)$ with $h(I_1)$. This gives the following bilateral estimate for the critical value x_*:

$$I_1 \le x_* \le I_2,$$

where the integrals I_1 and I_2 are defined by (1.7.2.31).

Remark 1.11. It is noteworthy that, in case 2°, it does not matter how the function $f(x, y)$ and its derivative $f_x(x, y)$ behave for $x > I_2$; in particular, the right-hand side of equation (1.1.1.1) can be negative for $x > I_2$.

► An approximate method for determining the critical value x_*.

The material below describes an approximate (engineering) method for evaluating the critical value x_* in blow-up problems. Consider the recursive sequence of integrals

$$I_{n+1} = \int_{y_0}^{\infty} \frac{d\xi}{f(I_n, \xi)}, \qquad n = 0,\ 1,\ 2,\ \dots \qquad (1.7.2.32)$$

The first two terms coincide with the integrals (1.7.2.17) in which $I_0 = 0$. Going to the limit in (1.7.2.32) as $n \to \infty$, we arrive at an approximate transcendental equation for $x \approx \hat{x}_*$,

$$\hat{x}_* = \int_{y_0}^{\infty} \frac{d\xi}{f(\hat{x}_*, \xi)}, \qquad \hat{x}_* = \lim_{n\to\infty} I_n. \qquad (1.7.2.33)$$

For autonomous equations, those with $f(x, y) = f(y)$, formula (1.7.2.33) gives the exact result.

Equation (1.7.2.33) can be used for rough (engineering) estimates of the critical value $x_* \approx \hat{x}_*$ in blow-up problems in which $f(x, y) > 0$ and $\max_{x\ge 0} f(x, y)/\min_{x\ge 0} f(x, y) = O(1)$ for all $y \ge y_0$.

Example 1.21. In the Cauchy problem from example 1.20, the recursive sequence of integrals is

$$I_{n+1} = \frac{1}{\sqrt{h(I_1)}} \arctan \frac{\sqrt{h(I_n)}}{a}.$$

Going to the limit as $n \to \infty$, we get an approximate transcendental equation for $\hat{x}_*$,

$$\hat{x}_* = \frac{1}{\sqrt{h(\hat{x}_*)}} \arctan \frac{\sqrt{h(\hat{x}_*)}}{a}, \qquad \hat{x}_* = \lim_{n\to\infty} I_n. \qquad (1.7.2.34)$$

If $h(x) = \text{const}$, equation (1.7.2.34) provides the exact result.

In particular, if $a = 1$ and $h(x) = x^2$ in (1.7.2.15), we get $\hat{x}_* \approx 0.8336$.

Remark 1.12. For the numerical integration of problems with blow-up solutions, see Section 1.14.4.

1.7.3 Space Localization of Solutions

► Preliminary remarks. An example of a spatially localized solution.

There are Cauchy problems whose solution is bounded and changes only on a finite interval, $x_0 < x \le x_*$, while being constant, $y = \mathring{y} = \text{const}$, for $x > x_*$, where $\mathring{y}$ is an equilibrium point (if $y = \mathring{y}$ the right-hand side of equation (1.1.1.1) is zero). The critical value x_* does not appear in the equation and is unknown in advance.

Example 1.22. Consider a model Cauchy problem for a separable ODE:

$$y'_x = -\tfrac{3}{2}y^{1/3} \quad (x > 0), \qquad y(0) = a,$$

where $a > 0$. Its exact solution is

$$y(x) = \begin{cases} (a^{2/3} - x)^{3/2} & \text{if } 0 \le x \le a^{2/3}, \\ 0 & \text{if } x > a^{2/3}. \end{cases} \tag{1.7.3.1}$$

This solution is nonnegative and decreases monotonically from the initial value a, at $x = 0$, to zero, at $x_* = a^{2/3}$, and then remain constant for $x > x_*$. Solution (1.7.3.1) is smooth (it has a continuous first derivative for all $x \ge 0$, including the point $x = x_*$).

► Autonomous equations.

1°. Let us look at a more general, than in example 1.22, Cauchy problem for an equation with a power-law nonlinearity:

$$y'_x = -by^\mu \quad (x > 0), \qquad y(0) = a. \tag{1.7.3.2}$$

If the conditions

$$a > 0, \qquad b > 0, \qquad 0 < \mu < 1$$

hold, problem (1.7.3.2) has a spatially localized solution:

$$y(x) = \begin{cases} \bigl[a^{1-\mu} - b(1-\mu)x\bigr]^{\frac{1}{1-\mu}} & \text{if } 0 \le x \le x_*, \\ 0 & \text{if } x > x_*, \end{cases} \qquad x_* = \frac{a^{1-\mu}}{b(1-\mu)}. \tag{1.7.3.3}$$

Remark 1.13. If $a > 0$, $b > 0$, and $\mu > 1$, the solution to problem (1.7.3.2) is strictly monotonically decreasing for all $x > 0$ and the limit relation $\lim_{x\to\infty} y = 0$ holds (so the solution is not spatially localized).

2°. Consider the Cauchy problem for a general autonomous equation

$$y'_x = -f(y) \quad (x > 0), \qquad y(0) = a. \tag{1.7.3.4}$$

Let the conditions

$$a > 0, \qquad f(0) = 0, \qquad f(y) > 0 \quad \text{for} \quad 0 < y \le a$$

hold and let there exist a μ from the interval $0 < \mu < 1$ such that

$$\lim_{y\to 0}[y^{-\mu} f(y)] = s, \quad 0 < s < \infty.$$

Then, problem (1.7.3.4) has a spatially localized solution, which can be written in a mixed explicit/implicit form:

$$\begin{cases} x = \Phi(y, a) & \text{if } 0 \le x \le x_*, \\ y = 0 & \text{if } x > x_*, \end{cases} \qquad \Phi(y, a) = \int_y^a \frac{d\xi}{f(\xi)}, \quad x_* = \Phi(0, a). \tag{1.7.3.5}$$

► **Nonautonomous equations. Some estimates.**

1°. Consider the Cauchy problem for a general nonautonomous equation*

$$y'_x = -f(x, y) \quad (x > 0), \qquad y(0) = a. \tag{1.7.3.6}$$

Suppose the conditions

$$a > 0, \ \ f(x, 0) = 0, \ \ f(x, y) > 0 \ \text{ and } \ f_x(x, y) \ge 0 \ \text{ for } \ x > 0 \ \text{ and } \ 0 < y \le a \tag{1.7.3.7}$$

hold and there exists a μ from the interval $0 < \mu < 1$ such that

$$\lim_{y\to +0}[y^{-\mu} f(x, y)] = s(x), \qquad 0 < s(x) < \infty. \tag{1.7.3.8}$$

Using the notation

$$I_1 = \int_0^a \frac{d\xi}{f(0, \xi)}, \qquad I_2 = \int_0^a \frac{d\xi}{f(I_1, \xi)}, \tag{1.7.3.9}$$

we get

$$f(0, y) \le f(x, y) \le f(I_1, y) \tag{1.7.3.10}$$

and

$$y_2(x) \le y(x) \le y_1(x). \tag{1.7.3.11}$$

Here, $y(x)$ is the solution of the Cauchy problem (1.1.1.1)–(1.1.1.2), while $y_1(x)$ and $y_2(x)$ are the respective solutions of the two auxiliary Cauchy problems

$$y'_x = -f(0, y) \qquad (x > 0), \qquad y(0) = a; \tag{1.7.3.12}$$

$$y'_x = -f(I_1, y) \qquad (x > 0), \qquad y(0) = a. \tag{1.7.3.13}$$

The solutions $y_1(x)$ and $y_2(x)$ can be represented, respectively, in implicit form as

$$x = \int_y^a \frac{d\xi}{f(0, \xi)} \quad \text{and} \quad x = \int_y^a \frac{d\xi}{f(I_1, \xi)}. \tag{1.7.3.14}$$

The critical value x_* is estimated as

$$I_2 \le x_* \le I_1. \tag{1.7.3.15}$$

*The right-hand sides of equations (1.1.1.1) and (1.7.3.6) differ in sign.

2°. Consider the Cauchy problem (1.7.3.6). Suppose the conditions

$$a>0,\ \ f(x,0)=0,\ \ f(x,y)>0 \ \text{ and } \ f_x(x,y)\le 0 \ \text{ for } \ x>0,\ \ 0<y\le a \quad (1.7.3.16)$$

hold and there exists a μ from the interval $0<\mu<1$ such that the limit relation (1.7.3.8) holds. Then, we have

$$f(0,y)\ge f(x,y)\ge f(I_1,y) \qquad (1.7.3.17)$$

and

$$y_1(x)\le y(x)\le y_2(x), \qquad (1.7.3.18)$$

where I_1 is defined in (1.7.3.9), $y(x)$ is the solution of the Cauchy problem (1.1.1.1)–(1.1.1.2), while $y_1(x)$ and $y_2(x)$ are the respective solutions of the auxiliary Cauchy problems (1.7.3.12)–(1.7.3.13). The last two solutions can be represented in implicit form with formulas (1.7.3.14). The critical value x_* is estimated as

$$I_1\le x_*\le I_2. \qquad (1.7.3.19)$$

Example 1.23. Consider the Cauchy problem

$$y'_x=-b\sqrt{y}-h(x)y \quad (x>0), \qquad y(0)=a, \qquad (1.7.3.20)$$

where $a>0$, $b>0$, and $h(x)\ge 0$.

This is a special case of problem (1.7.3.6) with $f(x,y)=b\sqrt{y}+h(x)y$. Conditions (1.7.3.16) and (1.7.3.8) hold if $h'_x(x)\le 0$ and $\mu=\frac{1}{2}$. Using formulas (1.7.3.9), we evaluate the integrals

$$I_1=\frac{2}{I_0}\ln\biggl(1+\frac{\sqrt{a}}{b}I_0\biggr), \qquad I_2=\frac{2}{I_1}\ln\biggl(1+\frac{\sqrt{a}}{b}I_1\biggr), \qquad I_0=h(0). \qquad (1.7.3.21)$$

In view of (1.7.3.19), the critical value x_* can be estimated as

$$\frac{2}{I_0}\ln\biggl(1+\frac{\sqrt{a}}{b}I_0\biggr)\le x_*\le\frac{2}{I_1}\ln\biggl(1+\frac{\sqrt{a}}{b}I_1\biggr). \qquad (1.7.3.22)$$

Since the equation in question is a Bernoulli equation, problem (1.7.3.20) admits the exact solution

$$y=E^2(x)\biggl[\sqrt{a}-\frac{1}{2}b\int_0^x\frac{d\xi}{E(\xi)}\biggr]^2, \qquad E(x)=\exp\biggl[-\frac{1}{2}\int_0^x h(\xi)\,d\xi\biggr], \qquad (1.7.3.23)$$

which is valid in the range $0\le x\le x_*$, where x_* is the root of the equation

$$\int_0^{x_*}\frac{d\xi}{E(\xi)}=\frac{2\sqrt{a}}{b}. \qquad (1.7.3.24)$$

For $x>x_*$, we get $y=0$.

In the special case $h(x)=h_0=\text{const}$, the root of equation (1.7.3.24) is given by

$$x_*=\frac{2}{h_0}\ln\biggl(1+\frac{\sqrt{a}}{b}h_0\biggr). \qquad (1.7.3.25)$$

Let us dwell on the special case of problem (1.7.3.20) with $a=b=1$ and $h(x)=2/(1+x)$. Using (1.7.3.21), we find that $I_1=\ln 3\approx 1.099$ and $I_2=\frac{2}{\ln 3}\ln(1+\ln 3)\approx 1.349$. Hence,

$$1.099<x_*<1.349.$$

The exact value of x_* is determined by relation (1.7.3.24), in which $E(x)=1/(1+x)$. It follows that $x_*=\sqrt{5}-1\approx 1.136$, which is the positive root of the quadratic equation $x_*^2+2x_*-4=0$.

► **An approximate method for calculating the critical value x_*.**

Below is an approximate (engineering) method for calculating the critical value x_* in spatial localization problems. Consider the recursive sequence of integrals

$$I_{n+1} = \int_0^a \frac{d\xi}{f(I_n, \xi)}, \qquad n = 0,\ 1,\ 2,\ \dots \tag{1.7.3.26}$$

The first two terms coincide with the integrals (1.7.3.9) with $I_0 = 0$. Going to the limit in (1.7.3.26) as $n \to \infty$, we arrive at the following transcendental equation for $x \approx \hat{x}_*$:

$$\hat{x}_* = \int_0^a \frac{d\xi}{f(\hat{x}_*, \xi)}, \qquad \hat{x}_* = \lim_{n\to\infty} I_n. \tag{1.7.3.27}$$

For autonomous equations, with $f(x, y) = f(y)$, equation (1.7.3.27) provides an exact result.

1.7.4 Cauchy Problems Admitting Non-Unique Solutions

This section discusses a few Cauchy problems that have a unique solution at sufficiently small deviations form the initial value of the independent variable, $x_0 \le x \le x_0 + \delta$, while for $x > x_* \ge \delta$, where x_* is a critical value, the desired function $y = y(x)$ can be non-unique.

► **An example of a problem with a non-unique solution.**

Example 1.24. Let us look at the model Cauchy problem

$$y'_x = -|y|^{1/2} \quad (x > 0), \qquad y(0) = 1. \tag{1.7.4.1}$$

If $0 \le y < 1$, the equation is equivalent to $y'_x = -y^{1/2}$. Then the solution coincides with solution (1.7.3.3) of problem (1.7.3.2) with $a = b = 1$ and $\mu = \frac{1}{2}$:

$$y(x) = \tfrac{1}{4}(2 - x)^2, \qquad 0 \le x \le x_* = 2. \tag{1.7.4.2}$$

At $x = 2$, the function $y(x)$ and its derivative $y'_x(x)$ both become zero. Therefore, arguing as in Section 1.7.3, one can construct a spatially localized solution by extending solution (1.7.4.2) with zero to obtain

$$y(x) = \begin{cases} \frac{1}{4}(2 - x)^2 & \text{if } 0 \le x \le 2, \\ 0 & \text{if } x \ge 2. \end{cases} \tag{1.7.4.3}$$

This extension is not unique. Specifically, the one-parameter family of smooth functions

$$y(x) = \begin{cases} \frac{1}{4}(2 - x)^2 & \text{if } 0 \le x \le 2, \\ 0 & \text{if } 2 \le x \le c, \\ -\frac{1}{4}(x - c)^2 & \text{if } x \ge c, \end{cases} \tag{1.7.4.4}$$

where c is an arbitrary number such that $c \ge x_* = 2$, is also a solution. This means that the Cauchy problem (1.7.4.1) admits infinitely many smooth solutions.

The spatially localized solution (1.7.4.3) is unique if the additional condition $y \ge 0$ is imposed. Alternatively, problem (1.7.4.1) can be treated as a boundary-value problem with the additional condition

$$y(b) = -k$$

with $b \ge 2$ and $k > 0$. This problem has a unique solution given by (1.7.4.4) with $c = b - 2\sqrt{k}$.

► **Autonomous and nonautonomous equations.**

1°. Consider a more general Cauchy problem than in Example 1.24:

$$y'_x = -b|y|^{\mu} \quad (x > 0), \qquad y(0) = a. \tag{1.7.4.5}$$

Under the conditions

$$a > 0, \quad b > 0, \quad 0 < \mu < 1$$

problem (1.7.4.5) has a one-parameter family of smooth solutions

$$y(x) = \begin{cases} \bigl[a^{1-\mu} - b(1-\mu)x\bigr]^{\frac{1}{1-\mu}} & \text{if } 0 \le x \le x_*, \\ 0 & \text{if } x_* \le x \le c, \\ -\bigl[b(1-\mu)(x-c)\bigr]^{\frac{1}{1-\mu}} & \text{if } x \ge c, \end{cases} \qquad x_* = \frac{a^{1-\mu}}{b(1-\mu)},$$

where c is an arbitrary number such that $c \ge x_*$.

2°. The autonomous Cauchy problem

$$y'_x = -|y|^{\mu} g(y) \quad (x > 0), \qquad y(0) = a$$

has a non-unique solution under the conditions

$$a > 0, \quad 0 < \mu < 1, \quad 0 < g(y) < \infty \quad \text{for} \quad 0 \le y \le a.$$

3°. The nonautonomous Cauchy problem

$$y'_x = -|y|^{\mu} g(x, y) \quad (x > 0), \qquad y(0) = a$$

has a non-unique solution under the conditions

$$a > 0, \quad 0 < \mu < 1, \quad 0 < g(x, y) < \infty \ \text{for} \ \ x > 0 \ \text{and} \ 0 \le y \le a. \tag{1.7.4.6}$$

⊙ *Literature for Section 1.7*: R. Meyer-Spasche (1998), A. Goriely and C. Hyde (2000), V. F. Zhuravlev (2001), P. J. Olver (2012).

1.8 Equations Not Solved for the Derivative and Equations Defined Parametrically

1.8.1 Method of "Integration by Differentiation" for Equations Not Solved for the Derivative

In the general case, a first-order equation not solved for the derivative,

$$F(x, y, y'_x) = 0, \tag{1.8.1.1}$$

can be rewritten in the equivalent form

$$F(x, y, t) = 0, \qquad t = y'_x. \tag{1.8.1.2}$$

We look for a solution in parametric form: $x = x(t)$, $y = y(t)$. In accordance with the first relation in (1.8.1.2), the differential of F is given by

$$F_x\,dx + F_y\,dy + F_t\,dt = 0. \tag{1.8.1.3}$$

Using the relation $dy = t\,dx$, we eliminate successively dy and dx from (1.8.1.3). As a result, we obtain the system of two first-order ordinary differential equations:

$$\frac{dx}{dt} = -\frac{F_t}{F_x + tF_y}, \qquad \frac{dy}{dt} = -\frac{tF_t}{F_x + tF_y}. \tag{1.8.1.4}$$

By finding a solution of this system, one thereby obtains a solution of the original equation (1.8.1.1) in parametric form, $x = x(t)$, $y = y(t)$.

Remark 1.14. The application of the above method may lead to loss of individual solutions (satisfying the condition $F_x + tF_y = 0$); this issue should be additionally investigated.

Remark 1.15. One of the differential equations of system (1.8.1.4) can be replaced by the algebraic equation $F(x, y, t) = 0$; see equation (1.8.1.2). This technique is used subsequently in Section 1.8.2.

1.8.2 Equations Not Solved for the Derivative. Specific Equations

► **Equations of the form $y = f(y'_x)$.**

This equation is a special case of equation (1.8.1.1), with $F(x, y, t) = y - f(t)$. The procedure described in Section 1.8.1 yields

$$\frac{dx}{dt} = \frac{f'(t)}{t}, \qquad y = f(t). \tag{1.8.2.1}$$

Here the original equation is used instead of the second equation in system (1.8.1.4); this is valid because the first equation in (1.8.1.4) does not depend on y explicitly.

Integrating the first equation in (1.8.2.1) yields the solution in parametric form,

$$x = \int \frac{f'(t)}{t}\,dt + C, \qquad y = f(t).$$

► **Equations of the form $x = f(y'_x)$.**

This equation is a special case of equation (1.8.1.1), with $F(x, y, t) = x - f(t)$. The procedure described in Section 1.8.1 yields

$$x = f(t), \qquad \frac{dy}{dt} = tf'(t). \tag{1.8.2.2}$$

Here the original equation is used instead of the first equation in system (1.8.1.4); this is valid because the second equation in (1.8.1.4) does not depend on x explicitly.

Integrating the second equation in (1.8.2.1) yields the solution in parametric form,

$$x = f(t), \qquad y = \int tf'(t)\,dt + C.$$

► Clairaut's equation $y = xy'_x + f(y'_x)$.

Clairaut's equation is a special case of equation (1.8.1.1), with $F(x, y, t) = y - xt - f(t)$. It can be rewritten as

$$y = xt + f(t), \qquad t = y'_x. \tag{1.8.2.3}$$

This equation corresponds to the degenerate case $F_x + tF_y \equiv 0$, where system (1.8.1.4) cannot be obtained. One should proceed in the following way: the first relation in (1.8.2.3) gives $dy = x\,dt + t\,dx + f'(t)\,dt$; performing the substitution $dy = t\,dx$, which follows from the second relation in (1.8.2.3), one obtains

$$[x + f'(t)]\,dt = 0.$$

This equation splits into $dt = 0$ and $x + f'(t) = 0$. The solution of the first equation is obvious: $t = C$; it gives the general solution of Clairaut's equation,

$$y = Cx + f(C), \tag{1.8.2.4}$$

which is a family of straight lines. The second equation generates a solution in parametric form,

$$x = -f'(t), \qquad y = -tf'(t) + f(t), \tag{1.8.2.5}$$

which is a singular solution and is the envelope of the family of lines (1.8.2.4).

Remark 1.16. There are also "compound" solutions of Clairaut's equation; they consist of part of curve (1.8.2.5) joined with the tangents at finite points; these tangents are defined by formula (1.8.2.4).

► Lagrange's equation $y = xf(y'_x) + g(y'_x)$.

Lagrange's equation is a special case of equation (1.8.1.1), with $F(x, y, t) = y - xf(t) - g(t)$. In the special case $f(t) \equiv t$, it coincides with Clairaut's equation.

The procedure described in Section 1.8.1 yields

$$\frac{dx}{dt} + \frac{f'(t)}{f(t) - t}x = \frac{g'(t)}{t - f(t)}, \qquad y = xf(t) + g(t). \tag{1.8.2.6}$$

Here the original equation is used instead of the second equation in system (1.8.1.4); this is valid because the first equation in (1.8.1.4) does not depend on y explicitly.

The first equation of system (1.8.2.6) is linear. Its general solution has the form $x = \varphi(t)C + \psi(t)$; the functions φ and ψ are defined in Section 1.2.3 (see formula (1.2.3.2)). Substituting this solution into the second equation in (1.8.2.6), we obtain the general solution of Lagrange's equation in parametric form,

$$x = \varphi(t)C + \psi(t), \qquad y = \bigl[\varphi(t)C + \psi(t)\bigr]f(t) + g(t).$$

Remark 1.17. With the above method, solutions of the form $y = t_k x + g(t_k)$, where the t_k are roots of the equation $f(t) - t = 0$, may be lost. These solutions can be particular or singular solutions of Lagrange's equation.

1.8.3 Equations Defined Parametrically and Differential-Algebraic Equations

In general, first-order ordinary differential equations defined parametrically is written using two coupled equations of the form

$$F_1(x, y, y'_x, t) = 0, \quad F_2(x, y, y'_x, t) = 0, \tag{1.8.3.1}$$

where $y = y(x)$ is an unknown function, $t = t(x)$ is a functional parameter, $F_1(\dots)$ and $F_2(\dots)$ are given functions of their arguments.

In what follows, it will be assumed that the derivative y'_x can be isolated from one of the equations (1.8.3.1) to get $y'_x = G(x, y, t)$. By eliminating the derivative from the other equation, one can rewrite the original parametric equation in the *canonical form*

$$F(x, y, t) = 0, \quad y'_x = G(x, y, t). \tag{1.8.3.2}$$

Below we deal with the general case where the parameter t cannot be eliminated from the equations (1.8.3.2).

In the theory of *differential-algebraic equations*, equations of the form (1.8.3.2) are called systems of *semi-explicit DAEs* or systems of *ODEs with constraints*. The standard way of reducing such equations to an ordinary system of ODEs for $y = y(x)$ and $t = t(x)$ is to differentiate the first equation in (1.8.3.2) with respect to x. However, there is an alternative system of ODEs for $y = y(t)$ and $x = x(t)$ which is more convenient for seeking exact solutions to semi-explicit DAEs. This system is derived below.

By taking the full differential of the first equation, we can rewrite system (1.8.3.2) as

$$F_x\, dx + F_y\, dy + F_t\, dt = 0, \quad dy = G(x, y, t)\, dx, \tag{1.8.3.3}$$

where F_x, F_y, and F_t are the respective partial derivatives of the function $F = F(x, y, t)$. The parametric equation (1.8.3.2) can be integrated in three ways, which are outlined below.

1°. Eliminating dx from (1.8.3.3) yields a first-order ODE for $y = y(t)$:

$$(F_x + GF_y)y'_t + GF_t = 0. \tag{1.8.3.4}$$

In conjunction with the first relation in (1.8.3.2), equation (1.8.3.4) may be simpler than the original parametric equation, in which case these can be used to seek solutions in parametric form.

2°. Eliminating dy from (1.8.3.3) yields a first-order ODE for $x = x(t)$:

$$(F_x + GF_y)x'_t + F_t = 0. \tag{1.8.3.5}$$

Equation (1.8.3.5) in conjunction with the first relation in (1.8.3.2) can be simpler than the original equation, in which case these can be used to seek solutions in parametric form.

3°. With the second relation in (1.8.3.3), we eliminate dy and then dx from the first relation in (1.8.3.3) to arrive at the system of two ODEs

$$x'_t = -\frac{F_t}{F_x + GF_y}, \quad y'_t = -\frac{GF_t}{F_x + GF_y}. \tag{1.8.3.6}$$

If found, a solution to this system will be a solution to the original equation (1.8.3.2) in parametric form.

Remark 1.18. With the above techniques, isolated solutions, satisfying $F_x + GF_y = 0$, may be lost; this issue calls for further analysis (see also Example 1.27).

Example 1.25. Consider the following first-order ODE defined in parametric form:

$$x = f(t), \quad y'_x = g(t), \tag{1.8.3.7}$$

where t is the parameter, while $f(t)$ and $g(t)$ are given, sufficiently arbitrary functions.

It is a special case of equation (1.8.3.3) with $F = x - f(t)$ and $G = g(t)$. We have

$$F_x = 1, \quad F_y = 0, \quad F_t = -f'_t(t). \tag{1.8.3.8}$$

Substituting (1.8.3.8) into (1.8.3.4) gives the separated equation $y'_t - g(t)f'_t(t) = 0$. Integrating this equation and taking into account the first relation in (1.8.3.7), we arrive at the general solution to equation (1.8.3.7) in parametric form

$$x = f(t), \quad y = \int f'_t(t)g(t)\,dt + C, \tag{1.8.3.9}$$

where C is an arbitrary constant.

For solution (1.8.3.9) to exit, it suffices that the function $f(t)$ is continuously differentiable and the integral exists.

Remark 1.19. In the special case $g(t) = t$, ODE (1.8.3.7) is equivalent to $x = f(y'_x)$; its general solution can be found in Section 1.8.2.

Example 1.26. Consider the following first-order ODE defined in parametric form:

$$y = f(t), \quad y'_x = g(t). \tag{1.8.3.10}$$

It is a special case of equation (1.8.3.3) with $F = y - f(t)$ and $G = g(t)$. We have

$$F_x = 0, \quad F_y = 1, \quad F_t = -f'_t(t). \tag{1.8.3.11}$$

Substituting (1.8.3.11) into (1.8.3.5) yields the separated equation $g(t)y'_t - f'_t(t) = 0$. In view of (1.8.3.10), its solution gives the general solution to equation (1.8.3.10) in parametric form

$$x = \int \frac{f'_t(t)}{g(t)}\,dt + C, \quad y = f(t), \tag{1.8.3.12}$$

where C is an arbitrary constant.

Remark 1.20. In the special case $g(t) = t$, ODE (1.8.3.10) is equivalent to $y = f(y'_x)$; its general solution can be found in Section 1.8.2.

Example 1.27. Consider the equation

$$F(x, y, t) = 0, \quad y'_x = -\frac{F_x}{F_y}, \tag{1.8.3.13}$$

which corresponds to the degenerate case $G = -F_x/F_y$ and $F_x + GF_y \equiv 0$ (see Remark 1.18). Eliminating dy from (1.8.3.3) gives

$$F_t\,dt = 0.$$

This equation splits into $dt = 0$ and $F_t = 0$. The solution of the first equation is obvious: $t = C$; it gives the general solution to the original equation in implicit form:

$$F(x, y, C) = 0.$$

The second equation generates a singular solution, which is defined by two algebraic (or transcendental) coupled equations:

$$F(x, y, t) = 0, \quad F_t(x, y, t) = 0.$$

▶ **Transformation of standard differential equations to parametric equations.**

The standard first-order ODE

$$y'_x = f(x, y) \tag{1.8.3.14}$$

can be rewritten as a parametric ODE defined by two relations

$$f(x, y) - t = 0, \qquad y'_x = t. \tag{1.8.3.15}$$

This equation is a special case of equation (1.8.3.2) with $F(x, y, t) = f(x, y) - t$ and $G(x, y, t) = t$. It can be reduced to a standard system of first-order ODEs:

$$(f_x + tf_y)x'_t = 1, \qquad (f_x + tf_y)y'_t = t. \tag{1.8.3.16}$$

The system is obtained by substituting $F = f - t$ and $G = t$ in the equations of (1.8.3.6).

System (1.8.3.16) is convenient for the numerical integration of Cauchy problems with blow-up or square-root singularity, in which the solution, $y = y(x)$, or its first derivative become infinite at a finite value $x = x_*$ (the number x_* is unknown in advance and is to be determined in solving the problem). In such problems, the critical value $x = x_*$ for equation (1.8.3.14) corresponds to $t \to \pm\infty$ for system (1.8.3.16). See Sections 1.14.4 and 1.14.5 for the usage of system (1.8.3.16) in the numerical integration of equations of the form (1.8.3.14).

⊙ *Literature for Section 1.7*: G. M. Murphy (1960), N. M. Matveev (1967), E. Kamke (1977), K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007), A. D. Polyanin and A. I. Zhurov (2016a, 2016b, 2017a, 2017b).

1.9 Contact Transformations

1.9.1 General Form of Contact Transformations. Method for the Construction of Contact Transformations

▶ **General form of contact transformations.**

A contact transformation has the form

$$\begin{aligned} x &= F(X, Y, Y'_X), \\ y &= G(X, Y, Y'_X), \end{aligned} \tag{1.9.1.1}$$

where the functions $F(X, Y, U)$ and $G(X, Y, U)$ are chosen so that the derivative y'_x does not depend on Y''_{XX}:

$$y'_x = \frac{y'_X}{x'_X} = \frac{G_X + G_Y Y'_X + G_U Y''_{XX}}{F_X + F_Y Y'_X + F_U Y''_{XX}} = H(X, Y, Y'_X). \tag{1.9.1.2}$$

The subscripts X, Y, and U after F and G denote the respective partial derivatives (it is assumed that $F_U \not\equiv 0$ and $G_U \not\equiv 0$).

It follows from (1.9.1.2) that the relation

$$\frac{\partial G}{\partial U}\left(\frac{\partial F}{\partial X} + U\frac{\partial F}{\partial Y}\right) - \frac{\partial F}{\partial U}\left(\frac{\partial G}{\partial X} + U\frac{\partial G}{\partial Y}\right) = 0 \tag{1.9.1.3}$$

holds; the derivative is calculated by

$$y'_x = \frac{G_U}{F_U}, \tag{1.9.1.4}$$

where $G_U/F_U \not\equiv$ const.

The application of contact transformations preserves the order of differential equations. The inverse of a contact transformation can be obtained by solving system (1.9.1.1) and (1.9.1.4) for X, Y, Y'_X.

► **Method for the construction of contact transformations.**

Suppose the function $F = F(X,Y,U)$ in the contact transformation (1.9.1.1) is specified. Then relation (1.9.1.3) can be viewed as a linear partial differential equation for the second function G. The corresponding characteristic system of ordinary differential equations (see Polyanin, Zaitsev, and Moussiaux, 2002),

$$\frac{dX}{1} = \frac{dY}{U} = -\frac{F_U\,dU}{F_X + UF_Y},$$

admits the obvious first integral:

$$F(X,Y,U) = C_1, \tag{1.9.1.5}$$

where C_1 is an arbitrary constant. It follows that, to obtain the general representation of the function $G = G(X,Y,U)$, one has to deal with the ordinary differential equation

$$Y'_X = U, \tag{1.9.1.6}$$

whose right-hand side is defined in implicit form by (1.9.1.5). Let the first integral of equation (1.9.1.6) have the form

$$\Phi(X,Y,C_1) = C_2.$$

Then the general representation of $G = G(X,Y,U)$ in transformation (1.9.1.1) is given by

$$G = \Psi(F, \widetilde{\Phi}),$$

with $\Psi(F,\widetilde{\Phi})$ representing an arbitrary function of two variables, $F = F(X,Y,U)$ and $\widetilde{\Phi} = \Phi(X,Y,F)$.

1.9.2 Examples of Contact Transformations

► **Contact transformations linear in the derivative.**

Example 1.28. *Legendre transformation*:

$$x = Y'_X, \quad y = XY'_X - Y, \quad y'_x = X \quad \text{(direct transformation)};$$
$$X = y'_x, \quad Y = xy'_x - y, \quad Y'_X = x \quad \text{(inverse transformation)}.$$

This transformation is used for solving some equations. In particular, the nonlinear equation

$$(xy'_x - y)^a f(y'_x) + yg(y'_x) + xh(y'_x) = 0$$

can be reduced by the Legendre transformation to a Bernoulli equation: $[Xg(X) + h(X)]Y'_X = g(X)Y - f(X)Y^a$ (see Section 1.2.3).

Example 1.29. Contact transformation ($a \neq 0$):

$$x = Y'_X + aY, \quad y = be^{aX}Y'_X, \qquad y'_x = be^{aX} \quad \text{(direct transformation);}$$
$$X = \frac{1}{a}\ln\frac{y'_x}{b}, \quad Y = \frac{1}{a}\Big(x - \frac{y}{y'_x}\Big), \quad Y'_X = \frac{y}{y'_x} \quad \text{(inverse transformation).}$$

Remark 1.21. It is apparent from this example that a contact transformation that is linear in the derivative can have a nonlinear inverse, which is also a contact transformation.

Table 1.2 presents some other contact transformations linear in the derivative.

TABLE 1.2
Some contact transformations linear in the derivative

No.	Contact transformations	Notations and remarks
1	$x = Y'_X + \frac{a}{X}Y,\ y = X^{a+1}Y'_X - X^aY,\ y'_x = X^{a+1};$ $X = (y'_x)^{\frac{1}{a+1}},\ Y = \frac{1}{a+1}(xy'_x - y)(y'_x)^{-\frac{a}{a+1}},\ Y'_X = \frac{xy'_x + ay}{(a+1)y'_x}$	$a \neq -1$
2	$x = fY'_X + gY,\ y = (fY'_X + gY)\int\frac{\varphi}{f}\,dX - \varphi Y,\ y'_x = \int\frac{\varphi}{f}\,dX$	$f = f(X)$ and $g = g(X)$ are arbitrary functions, $\varphi = \exp\big[\int(g/f)\,dX\big]$
3	$x = Y'_X + fY + g,\ y = IY'_X + (fI - e^F)Y + gI - \int ge^F\,dX,\ y'_x = I$	$f = f(X)$ and $g = g(X)$ are arbitrary functions, $F = \int f\,dX, I = \int e^F\,dX$

► Contact transformations nonlinear in the derivative.

Example 1.30. Contact transformation ($a \neq 0$):

$$x = Y'_X + aX, \qquad y = \tfrac{1}{2}(Y'_X)^2 + aY, \qquad y'_x = Y'_X \quad \text{(direct transformation);}$$
$$X = \frac{1}{a}(x - y'_x), \quad Y = \frac{1}{2a}\big[2y - (y'_x)^2\big], \quad Y'_X = y'_x \quad \text{(inverse transformation).}$$

Example 1.31. Contact transformation ($ab \neq 0$):

$$x = a(Y'_X)^2 - bX, \qquad y = 2a(Y'_X)^3 - 3bY, \qquad y'_x = 3Y'_X \quad \text{(direct transformation);}$$
$$X = \frac{a}{9b}(y'_x)^2 - \frac{1}{b}x, \quad Y = \frac{2a}{81b}(y'_x)^3 - \frac{1}{3b}y, \quad Y'_X = \frac{1}{3}y'_x \quad \text{(inverse transformation).}$$

Table 1.3 presents some other contact transformations nonlinear in the derivative.

⊙ *Literature for Section 1.8*: D. Zwillinger (1997), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

1.10 Pfaffian Equations

1.10.1 Preliminary Remarks

A *Pfaffian equation* is an equation of the form

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0. \qquad (1.10.1.1)$$

TABLE 1.3
Some contact transformations nonlinear in the derivative

No.	Contact transformations
1	$x = Y'_X + \frac{1}{X}Y,\; y = X^2(Y'_X)^2 - Y^2,\; y'_x = 2X^2Y'_X$ (direct transformation) $X = \pm\frac{1}{x}\sqrt{xy'_x - y},\; Y = \pm\frac{xy'_x - 2y}{2\sqrt{xy'_x - y}},\; Y'_X = \frac{x^2y'_x}{2(xy'_x - y)}$ (inverse transformation)
2	$x = e^X(Y'_X - Y),\; y = (Y'_X)^2 - Y^2,\; y'_x = 2e^{-X}Y'_X$ (direct transformation) $X = \ln\Bigl(\frac{\pm x}{\sqrt{xy'_x - y}}\Bigr),\; Y = \pm\frac{2y - xy'_x}{2\sqrt{xy'_x - y}},\; Y'_X = \pm\frac{xy'_x}{2\sqrt{xy'_x - y}}$ (inverse transformation)
3	$x = (Y'_X)^2 - Y^2,\; y = Y'_X\cosh X - Y\sinh X,\; y'_x = \frac{\cosh X}{2Y'_X}$ (direct transformation)
4	$x = (Y'_X)^2 + Y^2,\; y = Y'_X\cos X + Y\sin X,\; y'_x = \frac{\cos X}{2Y'_X}$ (direct transformation)
5	$x = X - YY'_X,\; y = -Y\sqrt{(Y'_X)^2 - 1},\; y'_x = \frac{Y'_X}{\sqrt{(Y'_X)^2 - 1}}$ (direct transformation) $X = x - yy'_x,\; Y = y\sqrt{(y'_x)^2 - 1},\; Y'_X = -\frac{y'_x}{\sqrt{(y'_x)^2 - 1}}$ (inverse transformation)
6	$x = X - \frac{aY'_X}{\sqrt{(Y'_X)^2 + 1}},\; y = Y + \frac{a}{\sqrt{(Y'_X)^2 + 1}},\; y'_x = Y'_X$ (direct transformation) $X = x + \frac{ay'_x}{\sqrt{(y'_x)^2 + 1}},\; Y = y - \frac{a}{\sqrt{(y'_x)^2 + 1}},\; Y'_X = y'_x$ (inverse transformation)
7	$x = a(Y'_X)^k - bX,\; y = ak(Y'_X)^{k+1} - b(k+1)Y,\; y'_x = (k+1)Y'_X$ (direct transformation) $X = \frac{a(y'_x)^k}{b(k+1)^k} - \frac{x}{b},\; Y = \frac{ak(y'_x)^{k+1}}{b(k+1)^{k+2}} - \frac{y}{b(k+1)},\; Y'_X = \frac{y'_x}{k+1}$ (inverse transformation) Here $ab \neq 0$ and $k \neq -1$

Equation (1.10.1.1) always has a solution $x = x_0$, $y = y_0$, $z = z_0$, where x_0, y_0, and z_0 are arbitrary constants. Such simple solutions are not considered below.

We will distinguish between the following two cases:

1. Find a two-dimensional solution to the Pfaffian equation, when the three variables x, y, z are connected by a single relation (a certain condition must hold for such a solution to exist).

2. Find a one-dimensional solution to the Pfaffian equation, when the three variables x, y, z are connected by two relations.

1.10.2 Completely Integrable Pfaffian Equations

► **Condition for integrability of the Pfaffian equation by a single relation.**

Let a solution of the Pfaffian equation be representable in the form $z = z(x, y)$, where z is the unknown function and x, y are independent variables. From equation (1.10.1.1) we find the expression for the differential:

$$dz = -\frac{P}{R}\,dx - \frac{Q}{R}\,dy. \tag{1.10.2.1}$$

On the other hand, since $z = z(x, y)$, we have

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{1.10.2.2}$$

Equating the right-hand sides of (1.10.2.1) and (1.10.2.2) to each other and taking into account the independence of the differentials dx and dy, we obtain an overdetermined system of equations of the form

$$z_x = -P/R, \quad z_y = -Q/R. \tag{1.10.2.3}$$

In the general case, system (1.10.2.3) is unsolvable. To derive a necessary consistency condition for the system, let us differentiate the first equation with respect to y and the second with respect to x. Equating the expressions of the second derivatives z_{xy} and z_{yx} to each other and then eliminate the first derivatives from the resulting relations using (1.10.2.3), we obtain a necessary condition for consistency of system (1.10.2.3):

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0. \tag{1.10.2.4}$$

If condition (1.10.2.4) is satisfied identically, the Pfaffian equation (1.10.1.1) is integrable by one relation of the form

$$U(x, y, z) = C, \tag{1.10.2.5}$$

where C is an arbitrary constant. In this case, the Pfaffian equation is said to be *completely integrable*. The left-hand side of a completely integrable Pfaffian equation (1.10.1.1) can be represented in the form

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz \equiv \mu(x, y, z)\,dU(x, y, z),$$

where $\mu(x, y, z)$ is an *integrating factor*.

► Solution methods for completely integrable Pfaffian equations. Examples.

There are two main methods for the solution of completely integrable Pfaffian equations. These methods are outlined below.

First method. An integrable Pfaffian equation can be solved by the direct solution of the overdetermined system (1.10.2.3).

Example 1.32. Consider the Pfaffian equation

$$ae^{y-z}\,dx + (be^{y-z} + 1)\,dy - dz = 0. \tag{1.10.2.6}$$

In this case, we have $P = ae^{y-z}$, $Q = be^{y-z} + 1$, and $R = -1$; the consistency condition (1.10.2.4) is satisfied identically. System (1.10.2.3) has the form

$$z_x = ae^{y-z}, \quad z_y = be^{y-z} + 1. \tag{1.10.2.7}$$

To solve the first equation, let us make the change of variable $w = e^z$ to obtain the linear equation $w_x = ae^y$. Its general solution has the form $w = ae^y x + \varphi(y)$, where $\varphi(y)$ is an arbitrary function. Going back to the original variable, we find the general solution of the first equation of system (1.10.2.7):

$$z = \ln[ae^y x + \varphi(y)]. \tag{1.10.2.8}$$

Substituting this solution into the second equation of system (1.10.2.7), we obtain a linear first-order equation for $\varphi = \varphi(y)$:

$$\varphi'_y = \varphi + be^y.$$

Its general solution is $\varphi = (by + C)e^y$, where C is an arbitrary constant. Substituting this solution into (1.10.2.8), we obtain the following solution of system (1.10.2.7):

$$z = \ln[ae^y x + (by + C)e^y] = y + \ln(ax + by + C).$$

It is the solution of the Pfaffian equation (1.10.2.6).

Second method. For solving a completely integrable Pfaffian equation, the following technique can also be used: it is first assumed that $x = \text{const}$ in equation (1.10.1.1), which corresponds to $dx = 0$. Then the resulting ordinary differential equation is solved for $z = z(y)$, where x is treated as a parameter, and the constant of integration is regarded as an arbitrary function of x: $C = \varphi(x)$. Finally, by substituting the resulting solution into the original equation (1.10.1.1), one finds the function $\varphi(x)$.

Example 1.33. Consider the Pfaffian equation

$$y(xz + a)\,dx + x(y + b)\,dy + x^2 y\,dz = 0. \tag{1.10.2.9}$$

The integrability condition (1.10.2.4) is satisfied identically. Let us set $dx = 0$ in equation (1.10.2.9) to obtain the ordinary differential equation

$$(y + b)\,dy + xy\,dz = 0. \tag{1.10.2.10}$$

Treating x as a parameter, we find the general solution of the separable equation (1.10.2.10):

$$z = -\frac{1}{x}(y + b\ln|y|) + \varphi(x), \tag{1.10.2.11}$$

where $\varphi(x)$ is the constant of integration, dependent on x. On substituting (1.10.2.11) into the original equation (1.10.2.9), we arrive at a linear ordinary differential equation for $\varphi(x)$:

$$x^2\varphi'_x + x\varphi + a = 0.$$

Its general solution is expressed as

$$\varphi(x) = -\frac{a}{x}\ln|x| + \frac{C}{x},$$

where C is an arbitrary constant. Substituting this $\varphi(x)$ into (1.10.2.11) yields the solution of the Pfaffian equation (1.10.2.9):

$$z = -\frac{1}{x}(y + a\ln|x| + b\ln|y| - C).$$

This solution can be equivalently represented in the form of an integral (1.10.2.5) as

$$xz + y + a\ln|x| + b\ln|y| = C.$$

► Geometric interpretation.

Let us introduce the vector $\mathbf{F} = \{P, Q, R\}$. Then the condition of complete integrability (1.10.2.4) of the Pfaffian equation (1.10.1.1) can be written in the dot product form: $\mathbf{F} \cdot \operatorname{curl}\mathbf{F} = 0$. Solution (1.10.2.5) represents a one-parameter family of surfaces orthogonal to $\mathbf{F}$.

1.10.3 Pfaffian Equations Not Satisfying the Integrability Condition

Consider now Pfaffian equations that do not satisfy the condition of integrability by one relation. In this case, relation (1.10.2.4) is not satisfied identically and there are two different methods for the investigation of such equations.

First method. Relation (1.10.2.4) is treated as an algebraic (transcendental) equation for one of the variables. For example, solving it for z, we get a relation $z = z(x, y)$.* The direct substitution of this solution into (1.10.1.1), with x and y regarded as independent variables, answers the question whether it is a solution of the Pfaffian equation. The thus obtained solutions (if any) do not contain a free parameter.

Second method. One-dimensional solutions are sought in the form of two relations. One relation is prescribed, for example, in the form

$$z = f(x), \tag{1.10.3.1}$$

where $f(x)$ is an arbitrary function. Using it, one eliminates z from the Pfaffian equation (1.10.1.1) to obtain an ordinary differential equation for $y = y(x)$:

$$Q(x, y, f(x))y'_x + P(x, y, f(x)) + R(x, y, f(x))f'_x(x) = 0.$$

Suppose the general solution of this equation has the form

$$\Phi(x, y, C) = 0, \tag{1.10.3.2}$$

where C is an arbitrary constant. Then formulas (1.10.3.1) and (1.10.3.2) define a one-dimensional solution of the Pfaffian equation in the form of two relations involving one arbitrary function and one free parameter.

Example 1.34. Consider the Pfaffian equation

$$y\,dx + dy + x\,dz = 0. \tag{1.10.3.3}$$

Substituting $P = y$, $Q = 1$, and $R = x$ into the left-hand side of condition (1.10.2.4), we find that $x + 1 \not\equiv 0$. Therefore, equation (1.10.3.3) is not integrable by one relation.

Let us look for one-dimensional solutions by choosing one relation in the form (1.10.3.1). Consequently, we arrive at the ordinary differential equation

$$y'_x + y + xf'_x(x) = 0.$$

Its general solution has the form

$$y = Ce^{-x} - e^{-x}\int e^x x f'_x(x)\,dx, \tag{1.10.3.4}$$

where C is an arbitrary constant. Formulas (1.10.3.1) and (1.10.3.4) represent a one-dimensional solution of the Pfaffian equation (1.10.3.3) involving an arbitrary function $f(x)$ and an arbitrary constant C.

Remark 1.22. For equations that are not completely integrable, the first relation can be chosen in a more general form than (1.10.3.1):

$$z = f(x, y),$$

where $f(x, y)$ is an arbitrary function of two arguments. Using it to eliminate z from the Pfaffian equation, we obtain an ordinary differential equation for $y = y(x)$. However, it is impossible to find the general solution of this equation in closed form even for very simple equations, including equation (1.10.3.3).

⊙ *Literature for Section 1.9*: G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2012).

*There may be several such relations, and even infinitely many.

1.11 Approximate Analytic Methods for Solution of ODEs

1.11.1 Method of Successive Approximations (Picard Method)

► **Description of the method.**

The method of successive approximations consists of two stages. At the first stage, the Cauchy problem

$$y'_x = f(x, y) \qquad \text{(equation)}, \qquad (1.11.1.1)$$

$$y(x_0) = y_0 \qquad \text{(initial condition)} \qquad (1.11.1.2)$$

is reduced to the equivalent integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \qquad (1.11.1.3)$$

Then a solution of equation (1.11.1.3) is sought using the formula of successive approximations:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt; \qquad n = 0,\ 1,\ 2,\ \ldots \qquad (1.11.1.4)$$

The initial approximation $y_0(x)$ can be chosen arbitrarily; the simplest way is to take the number y_0 that appears in the initial condition (1.11.1.2). The iterative process converges as $n \to \infty$, provided the conditions of the theorems in Section 1.1.1 are satisfied (see also below).

► **Estimates of the convergence range and error of approximation.**

Suppose that the function $f(x, y)$ is continuous in the rectangle

$$|x - x_0| \le a, \qquad |y - y_0| \le b, \qquad (1.11.1.5)$$

and satisfies the *Lipschitz condition* y:

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|, \qquad K = \text{const}, \qquad (1.11.1.6)$$

with respect to y for $|x - x_0| \le a$.

Remark 1.23. The Lipschitz condition (1.11.1.6) holds, in particular, if $f = f(x, y)$ has a bounded derivative f_y such that $|f_y| \le K$.

Since the function $f(x, y)$ is continuous in the closed region (1.11.1.5), it is bounded in this region: $|f(x, y)| \le M$. Under the above conditions, a solution to problem (1.11.1.1)–(1.11.1.2) exists and is unique on the interval

$$|x - x_0| \le c, \qquad c = \min(a, b/M);$$

the solution can be found as the limit of successive approximations defined by formula (1.11.1.4).

The estimate

$$|y(x) - y_n(x)| \le \frac{M}{K} \frac{(Kc)^n}{n!} \qquad (1.11.1.7)$$

holds true; however, the bound is greatly overestimated. In practice, the successive approximations should be stopped at an n at which y_n and y_{n+1} coincide to within a selected error threshold.

Example 1.35. Consider the Cauchy problem

$$y'_x = \alpha y^p + \beta x^q, \qquad y(0) = 0, \tag{1.11.1.8}$$

where p, q, α, and β are free parameters, with $p > 0$ and $q > -1$.

Let us choose $y_0(x) = 0$ to be the initial approximation. The recurrence formula (1.11.1.4) gives the first and second approximations

$$y_1(x) = \int_0^x \beta t^q\,dt = \frac{\beta}{q+1}x^{q+1},$$
$$y_2(x) = \int_0^x \left[\frac{\alpha\beta^p}{(q+1)^p}x^{p(q+1)} + \beta t^q\right]dt = \frac{\alpha\beta^p}{(q+1)^p(pq+p+1)}x^{pq+p+1} + \frac{\beta}{q+1}x^{q+1}.$$

Let us focus on the special case $p = q = 2$ and $\alpha = \beta = 1$. The corresponding first three approximations are

$$\begin{aligned} y_1(x) &= \tfrac{1}{3}x^3,\\ y_2(x) &= \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7,\\ y_3(x) &= \int_0^x \left[\left(\tfrac{1}{3}t^3 + \tfrac{1}{63}t^7\right)^2 + t^2\right]dt = \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 + \tfrac{2}{2079}x^{11} + \tfrac{1}{59535}x^{15}. \end{aligned} \tag{1.11.1.9}$$

In the rectangle $|x| \le 1$, $|y| < 1$, which corresponds to $x_0 = y_0 = 0$ and $a = b = 1$ in (1.11.1.5), the approximations $y_2(x)$ and $y_3(x)$ are quite close to each other:

$$\left|y_3(x) - y_2(x)\right| = \left|\tfrac{2}{2079}x^{11} + \tfrac{1}{59535}x^{15}\right| \le \tfrac{2}{2079} + \tfrac{1}{59535} < 10^{-3}.$$

Hence, we can set

$$y(x) \simeq \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7. \tag{1.11.1.10}$$

To assess the accuracy of the estimate (1.11.1.7), we evaluate the constant K appearing in the Lipschitz condition (1.11.1.6) with $f = y^2 + x^2$ in the rectangle $|x| \le 1$, $|y| < 1$ (which corresponds to $x_0 = y_0 = 0$ and $a = b = 1$ in (1.11.1.5)):

$$K = \max_{|x|\le 1,\,|y|<1} |f_y| = \max_{|x|\le 1,\,|y|<1} |2y| = 2.$$

We also evaluate the constants

$$M = \max_{|x|\le 1,\,|y|<1} |f| = 2, \qquad c = \min_{a=b=1}(a, b/M) = \tfrac{1}{2}.$$

Substituting these values into (1.11.1.7) with $n = 3$ yields

$$|y(x) - y_3(x)| \le \tfrac{1}{3!} < 0.17. \tag{1.11.1.11}$$

This estimate will be shown later on to be quite rough, two orders of magnitude worse than the true error (see Example 1.36).

1.11.2 Newton–Kantorovich Method

► Description of the method.

Suppose there is a not-too-rough initial approximation of the solution to a problem. In a nutshell, the Newton–Kantorovich method serves to construct more and more accurate successive approximations each obtained by solving a linear ODE.

Suppose there is an approximate solution $\bar{y}$ to problem (1.11.1.1)–(1.11.1.2) that satisfies the condition

$$y - \bar{y} = \varepsilon, \tag{1.11.2.1}$$

where $|\varepsilon| \ll 1$ (here and henceforth, we omit the argument x of the function ε). Substituting $y = \bar{y} + \varepsilon$ into the right-hand side of equation (1.11.1.1), expanding it in the Taylor series in powers of ε, and retaining only the first two terms, we obtain

$$y'_x = f(x, \bar{y} + \varepsilon) \quad \Longrightarrow \quad y'_x = f(x, \bar{y}) + f_y(x, \bar{y})\varepsilon + o(\varepsilon). \tag{1.11.2.2}$$

Omitting terms of the order of $o(\varepsilon)$, we replace ε with the left-hand side of (1.11.2.1) and rearrange to obtain

$$y'_x - f_y(x, \bar{y})y = f(x, \bar{y}) - f_y(x, \bar{y})\bar{y}. \tag{1.11.2.3}$$

The approximate relation (1.11.2.3) forms the basis of the Newton–Kantorovich method. Specifically, given an approximation $y_n = y_n(x)$, one finds the next approximation $y_{n+1} = y_{n+1}(x)$ as the solution of the linear equation

$$y'_{n+1} - f_y(x, y_n)y_{n+1} = f(x, y_n) - f_y(x, y_n)y_n \tag{1.11.2.4}$$

satisfying the initial condition (1.11.1.2). The recursive sequence of equations (1.11.2.4) is obtained from (1.11.2.3) by the formal substitutions $y = y_{n+1}$ and $\bar{y} = y_n$.

The first-order ODE (1.11.2.4) subject to the initial condition (1.11.1.2) has the solution

$$y_{n+1}(x) = E_n(x)\left[y_0 + \int_{x_0}^{x} \frac{g_n(z)}{E_n(z)}\,dz\right], \tag{1.11.2.5}$$

where

$$E_n(x) = \exp\left[\int_{x_0}^{x} f_y(t, y_n(t))\,dt\right], \quad g_n(x) = f(x, y_n(x)) - f_y(x, y_n(x))y_n(x).$$

► Convergence condition and error estimate of the approximate method.

Suppose the function $f = f(x, y)$ and its partial derivatives f_y and f_{yy} are all continuous in the rectangle (1.11.1.5); hence, these are bounded:

$$|f_y(x, y)| \le M_1, \qquad |f_{yy}(x, y)| \le M_2. \tag{1.11.2.6}$$

We assume that the initial approximation $y_0(x)$ is defined for $|x - x_0| \le a$ and satisfies the inequality $|y_0(x) - y_0| \le b$. Denote

$$\rho = \max_{|x-x_0|\le a}\left|y_0(x) - y_0 - \int_{x_0}^{x} f(t, y_0(t))\,dt\right|. \tag{1.11.2.7}$$

If the initial approximation is close to the exact solution, then ρ must be small. We set

$$\gamma = aM_2\rho e^{2aM_1} \tag{1.11.2.8}$$

and assume that the inequalities

$$\gamma \le \frac{1}{2} \quad \text{and} \quad \frac{2\rho}{1 + \sqrt{1 - 2\gamma}} \le b \tag{1.11.2.9}$$

hold. Then the subsequent approximations (1.11.2.5) obtained by the Newton–Kantorovich method satisfy the inequality $|y_n(x) - y_0| \le b$ and, hence, are uniformly convergent to the exact solution of problem (1.11.1.1)–(1.11.1.2) on the interval $|x - x_0| \le a$. The rate of convergence is estimated by

$$|y(x) - y_n(x)| \le \frac{1}{2^{n-1}}(2\gamma)^{2^n-1}\rho. \tag{1.11.2.10}$$

The inequality

$$|y(x) - y_0(x)| \le \frac{2\rho}{1+\sqrt{1-2\gamma}} \tag{1.11.2.11}$$

enables one to assess the proximity of the initial approximation to the exact solution.

► Error estimates of approximate solutions obtained by other methods.

It is important that the inequality (1.11.2.11) can also be used to assess the accuracy of solutions obtained by other approximate analytical or asymptotic methods (or obtained empirically from physical or other considerations). To this end, the approximate solution we wish to test should be taken to be the initial approximation $y_0(x)$ in the Newton–Kantorovich method. Then one evaluates the constants M_1, M_2, ρ, and γ appearing in (1.11.2.6)–(1.11.2.8) and checks whether the inequalities (1.11.2.8) hold (these inequalities can be satisfied by varying the parameters a and b defining the size of the domain (1.11.1.5)).

Example 1.36. Let us use the Newton–Kantorovich methods to assess the accuracy of the approximate solution to problem (1.11.1.8) with $p = q = 2$ and $\alpha = \beta = 1$. The solution will be constructed on the interval $|x| \le 1$. We take

$$y_0(x) = \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 \tag{1.11.2.12}$$

as the initial approximation; it satisfies the initial condition and corresponds to the second successive approximation (see second formula in (1.11.1.9)).

Taking into account that $f(x,y) = y^2 + x^2$, we find M_1, M_2, ρ, and γ appearing in (1.11.2.6)–(1.11.2.8):

$$\begin{aligned}
&M_1 = M_2 = 2,\\
&\rho = \max_{|x|\le 1}\left|\tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 - \int_0^x \left(t^2 + \tfrac{1}{9}t^6 + \tfrac{2}{189}t^{10} + \tfrac{1}{3969}t^{14}\right) dt\right| = \tfrac{2}{2079} + \tfrac{1}{59535} < 0.001,\\
&\gamma = 2\rho e^4 < 0.110.
\end{aligned}$$

The right-hand side of (1.11.2.11) is evaluated as

$$\frac{2\rho}{1+\sqrt{1-2\gamma}} < \frac{2\times 0.001}{1+\sqrt{1-0.220}} < 0.00107.$$

It follows that the solution to the problem, $y = y(x)$, exists and is confined, for $|x| \le 1$, within the bounds

$$\left|y - \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7\right| < 0.00107. \tag{1.11.2.13}$$

It is noteworthy that the estimate (1.11.2.13) is two orders of magnitude more precise than the estimate (1.11.1.11) obtained in Example 1.35 by the method of successive approximations.

Remark 1.24. In this example, by evaluating the maximum discrepancy using the formula

$$\delta = \max_{|x|<1}\left|y_0'(x) - y_0^2(x) - x^2\right|,$$

where $y_0(x)$ is the initial approximation (1.11.2.12), we obtain $\delta = \frac{2}{3\times 63} + \frac{1}{(63)^2} < 0.011$, which is an order of magnitude more precise than the estimate of the maximum deviation of the above approximation from the exact solution (1.11.2.13).

One possible way of employing inequality (1.11.2.11) is to look for the approximate solution $y_0(x)$ as a finite polynomial (satisfying the initial condition (1.11.1.2)) with free coefficients, which are to be determined by minimizing the right-hand side of (1.11.2.7). Apart from polynomials, one can use sums of exponential, trigonometric, or other functions with undetermined coefficients. For convenience, the square of the right-hand side of (1.11.2.7) can be minimized.

1.11.3 Method of Series Expansion in the Independent Variable

► **Method of Taylor series expansion in the independent variable.**

Let the right-hand side of equation (1.11.1.1) be an infinitely differentiable function in both arguments. Then the solution of the Cauchy problem (1.11.1.1)–(1.11.1.2) can be sought in the form of the Taylor series in powers of $(x - x_0)$:

$$y(x) = y(x_0) + y'_x(x_0)(x - x_0) + \frac{y''_{xx}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{y^{(n)}_x(x_0)}{n!}(x - x_0)^n + \cdots . \tag{1.11.3.1}$$

The first coefficient $y(x_0)$ in solution (1.11.3.1) is defined by the initial condition (1.11.1.2). The values of the derivatives of $y(x)$ at $x = x_0$ are determined from equation (1.11.1.1) and its derivative equations (obtained by successive differentiation), taking into account the initial condition (1.11.1.2). In particular, setting $x = x_0$ in (1.11.1.1) and substituting (1.11.1.2), one obtains the value of the first derivative:

$$y'_x(x_0) = f(x_0, y_0). \tag{1.11.3.2}$$

Further, differentiating equation (1.11.1.1) yields

$$y''_{xx} = f_x(x, y) + f_y(x, y)y'_x. \tag{1.11.3.3}$$

On substituting $x = x_0$, as well as the initial condition (1.11.1.2) and the first derivative (1.11.3.2), into the right-hand side of this equation, one calculates the value of the second derivative:

$$y''_{xx}(x_0) = f_x(x_0, y_0) + f(x_0, y_0)f_y(x_0, y_0).$$

Likewise, one can determine the subsequent derivatives of y at $x = x_0$.

Solution (1.11.3.1) obtained by this method can normally be used in only some sufficiently small neighborhood of the point $x = x_0$.

Example 1.37. Consider the Cauchy problem for the equation

$$y'_x = e^y + \cos x \tag{1.11.3.4}$$

with the initial condition $y(0) = 0$.

Since $x_0 = 0$, we will be constructing a series in powers of x. Differentiating equation (1.11.3.4) sequentially, we find the following three derivatives:

$$\begin{aligned} y''_{xx} &= e^y y'_x - \sin x, \\ y'''_{xxx} &= e^y y''_{xx} + e^y (y'_x)^2 - \cos x, \\ y''''_{xxxx} &= e^y y'''_{xxx} + 3e^y y'_x y''_{xx} + e^y (y'_x)^3 + \sin x. \end{aligned} \tag{1.11.3.5}$$

Using equation (1.11.3.4), the initial condition $y(0) = 0$, and formulas (1.11.3.5), we evaluate the derivatives at zero:

$$\begin{aligned}
y'_x(0) &= \left[e^y + \cos x\right]_{x=y=0} = 2,\\
y''_{xx}(0) &= \left[e^y y'_x - \sin x\right]_{x=y=0,\, y'_x=2} = 2,\\
y'''_{xxx}(0) &= \left[e^y y''_{xx} + e^y (y'_x)^2 - \cos x\right]_{x=y=0,\, y'_x=y''_{xx}=2} = 5,\\
y''''_{xxxx}(0) &= \left[e^y y'''_{xxx} + 3e^y y'_x y''_{xx} + e^y (y'_x)^3 + \sin x\right]_{x=y=0,\, y'_x=y''_{xx}=2,\, y'''_{xxx}=5} = 25.
\end{aligned}$$

Substituting the values of the derivatives into series (1.11.3.1) with $x = 0$, we obtain the desired series representation of the solution:

$$y = 2x + x^2 + \tfrac{5}{6}x^3 + \tfrac{25}{24}x^4 + \cdots .$$

► **Equations with integrable singularities admitting the application of Taylor's method.**

There are differential equations (including those with integrable singularities) that cannot be directly treated with the method of Taylor series expansion but can be transformed to enable the application of this method.

1°. Consider the equation with fractional powers of the argument

$$y'_x = x^{-1/2} f_1(x, y) + f_2(x, y) + x^{1/2} f_3(x, y), \qquad (1.11.3.6)$$

where $f_j(x, y)$ ($j = 1,\ 2,\ 3$) infinitely differentiable functions in both arguments for $x \geq 0$. If $f_1(0, y) \neq 0$, the right-hand side of equation (1.11.3.6) has an integrable singularity at $x = 0$. The equation cannot be solved in terms of the Taylor series (1.11.3.1) with $x_0 = 0$. However, if one makes the change of variable $x = t^2$, equation (1.11.3.6) becomes

$$y'_t = 2f_1(t^2, y) + 2tf_2(t^2, y) + 2t^2 f_3(t^2, y).$$

This equation already enables the application of the method of Taylor series expansion. It is important to note that if $f_1(0, y) \not\equiv 0$, equation (1.11.3.6) has a singularity at $x = 0$.

2°. Equation (1.11.3.6) is a special case of the equation

$$y'_x = x^{-p} f_1(x^q, y) + f_2(x^q, y) + x^p f_3(x^q, y), \qquad (1.11.3.7)$$

where $p = m/n$, $q = k/n$; m, n, and k are positive integers with $m < n$; $f_j(\xi, y)$ ($j = 1,\ 2,\ 3$) are infinitely differentiable functions in both arguments for $x \geq 0$. The change of variable $x = z^n$ reduces this equation to

$$y'_z = nz^{n-m-1} f_1(x^k, y) + nz^{n-1} f_2(x^k, y) + nz^{n+m-1} f_3(x^k, y).$$

This equation does not have singularities and so is treatable with the method of Taylor series expansion.

3°. Consider the equation

$$y'_x = x^{\nu-1} f_1(x^\nu, y) + x^{2\nu-1} f_2(x^\nu, y) + x^{3\nu-1} f_3(x^\nu, y) \qquad (1.11.3.8)$$

where $0 < \nu < 1$, with rational or irrational values of ν allowed. If $f_1(0, y) \not\equiv 0$, equation (1.11.3.8) has an integrable singularity at $x = 0$. With the change of variable $t = x^\nu$, it can be reduced to an equation without a singularity

$$\nu y'_t = f_1(t, y) + tf_2(t, y) + t^2 f_3(t, y),$$

which is treatable with the method of Taylor series expansion; as previously, the functions $f_j(t, y)$ are assumed to be infinitely differentiable.

► Equations involving fractional powers and untreatable with Taylor's method.

In general, equations involving fractional powers of the independent variable cannot be reduced to equations treatable using the method of Taylor series expansion. A local solution to such equations in the neighborhood of $x = 0$ can be sought in the form

$$y = \sum_{j=0}^{m} A_j x^{\sigma_j} + o(x^{\sigma_j}),$$

with the coefficients σ_j and A_j to be determined successively in the analysis after substituting the series into the equation; it is assumed that $\sigma_j < \sigma_{j+1}$.

Example 1.38. Consider the equation with an integrable singularity

$$y'_x = x^{-\nu}(a_0 + a_1 x + a_2 y) + f(x, y), \quad 0 < \nu < 1, \tag{1.11.3.9}$$

where $f(x, y)$ is an analytic function expandable in a series at $x = y = 0$:

$$f(x, y) = b_0 + b_1 x + b_2 y + \cdots . \tag{1.11.3.10}$$

We look for an approximate solution to equation (1.11.3.9) satisfying the initial condition $y(0) = 0$ in the form

$$y = c_1 x^\alpha + c_2 x^\beta + c_3 x^\gamma + o(x^\gamma), \quad 0 < \alpha < \beta < \gamma, \tag{1.11.3.11}$$

for small positive x. Substituting (1.11.3.11) into (1.11.3.9) and taking into account (1.11.3.10), we obtain

$$\begin{aligned} &\underline{c_1 \alpha x^{\alpha-1}} + c_2 \beta x^{\beta-1} + c_3 \gamma x^{\gamma-1} \\ &\qquad = \underline{a_0 x^{-\nu}} + b_0 + a_2 c_1 x^{\alpha-\nu} + b_2 c_1 x^\alpha + a_1 x^{1-\nu} + a_2 c_2 x^{\beta-\nu} + \cdots . \end{aligned} \tag{1.11.3.12}$$

The leading terms of the expansion are underlined on both sides (it is assumed that $a_0 \neq 0$). We see that

$$\alpha = 1 - \nu, \quad c_1 = a_0/\alpha. \tag{1.11.3.13}$$

Substituting (1.11.3.13) into (1.11.3.12) yields

$$\underline{c_2 \beta x^{\beta-1}} + c_3 \gamma x^{\gamma-1} = b_0 + a_2 c_1 x^{1-2\nu} + (a_1 + b_2 c_1) x^{1-\nu} + a_2 c_2 x^{\beta-\nu} + \cdots . \tag{1.11.3.14}$$

Now, depending on the value of ν, a few different cases are possible, which are considered in order below.

1°. Let $0 < \nu < \frac{1}{2}$ and $b_0 \neq 0$. The first term on the right is leading for small x. By matching up the terms on the left- and right-hand sides of (1.11.3.14), we find that

$$\beta = 1, \quad \gamma = 2(1 - \nu), \quad c_2 = b_0, \quad c_3 = a_2 c_1/\gamma \quad (a_2 \neq 0). \tag{1.11.3.15}$$

The coefficients c_1 and α are defined by formulas (1.11.3.13).

2°. Let $\frac{1}{2} < \nu < 1$ and $a_2 \neq 0$. The second term on the right in (1.11.3.14) is leading for small x. By matching up the respective terms on the left- and right-hand sides, we obtain

$$\beta = 2(1 - \nu), \quad \gamma = 1, \quad c_2 = a_2 c_1/\beta, \quad c_3 = b_0 \quad (b_0 \neq 0). \tag{1.11.3.16}$$

3°. Let $\nu = \frac{1}{2}$. Then $\beta = 1$ and $\gamma = \frac{3}{2}$, and the original equation (1.11.3.9) becomes a special case of equation (1.11.3.6), to which the method of Taylor series expansion will be applied after substituting $x = t^2$.

1.11.4 Method of Regular Expansion in the Small Parameter

Consider a general first-order ordinary differential equation with a small parameter ε:

$$y'_x = f(x, y, \varepsilon). \tag{1.11.4.1}$$

Suppose the function f is representable as a series in powers of ε:

$$f(x, y, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_n(x, y). \tag{1.11.4.2}$$

One looks for a solution of the Cauchy problem for equation (1.11.4.1) with the initial condition (1.11.1.2) as $\varepsilon \to 0$ in the form of a regular expansion in powers of the small parameter:

$$y = \sum_{n=0}^{\infty} \varepsilon^n Y_n(x). \tag{1.11.4.3}$$

Relation (1.11.4.3) is substituted in equation (1.11.4.1) taking into account (1.11.4.2). Then one expands the functions f_n into a power series in ε and matches the coefficients of like powers of ε to obtain a system of equations for $Y_n(x)$:

$$Y'_0 = f_0(x, Y_0), \tag{1.11.4.4}$$

$$Y'_1 = g(x, Y_0)Y_1 + f_1(x, Y_0), \qquad g(x, y) = \frac{\partial f_0}{\partial y}. \tag{1.11.4.5}$$

Only the first two equations are written out here. The prime denotes differentiation with respect to x. The initial conditions for Y_n can be obtained from (1.11.1.2) taking into account (1.11.4.3):

$$Y_0(x_0) = y_0, \qquad Y_1(x_0) = 0.$$

Success in the application of this method is primarily determined by the possibility of constructing a solution of equation (1.11.4.4) for the leading term in the expansion of Y_0. It is significant that the remaining terms of the expansion, Y_n with $n \ge 1$, are governed by linear equations with homogeneous initial conditions.

Example 1.39. Consider the following Cauchy problem for a nonlinear equation with a small parameter:

$$y'_x + y = \varepsilon y^2, \qquad y(0) = 1. \tag{1.11.4.6}$$

The solution is sought in the form (1.11.4.3) while retaining the first three terms in the expansion:

$$y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + o(\varepsilon^2), \qquad Y_n = Y_n(x). \tag{1.11.4.7}$$

Substituting (1.11.4.7) into (1.11.4.6) and collecting the terms with like powers of ε yields

$$Y'_0 + Y_0 + \varepsilon(Y'_1 + Y_1 - Y_0^2) + \varepsilon^2(Y'_2 + Y_2 - 2Y_0Y_1) + o(\varepsilon^2) = 0. \tag{1.11.4.8}$$

Similarly, substituting (1.11.4.7) into the original condition (1.11.4.6) gives

$$Y_0(0) - 1 + \varepsilon Y_1(0) + \varepsilon^2 Y_2(0) + o(\varepsilon^2) = 0. \tag{1.11.4.9}$$

Now equating the terms with like powers of ε in (1.11.4.8) and (1.11.4.9) to zero, we arrive at the following sequence of simple linear problems:

$$\begin{aligned} &Y'_0 + Y_0 = 0, &&Y_0(0) = 1;\\ &Y'_1 + Y_1 = Y_0^2, &&Y_1(0) = 0;\\ &Y'_2 + Y_2 = 2Y_0Y_1, &&Y_2(0) = 0. \end{aligned}$$

Integrating these equations sequentially yields

$$Y_0 = e^{-x},$$
$$Y_1 = e^{-x} - e^{-2x} = e^{-x}(1 - e^{-x}),$$
$$Y_2 = e^{-x} - 2e^{-2x} + e^{-3x} = e^{-x}(1 - e^{-x})^2.$$

Substituting these expressions into (1.11.4.7), we obtain the desired solution in the form

$$y = e^{-x}\big[1 + \varepsilon(1 - e^{-x}) + \varepsilon^2(1 - e^{-x})^2\big] + o(\varepsilon^2). \qquad (1.11.4.10)$$

Let us compare the asymptotic solution to problem (1.11.4.6) with the exact solution

$$y = \frac{e^{-x}}{1 - \varepsilon(1 - e^{-x})}. \qquad (1.11.4.11)$$

By expanding expression (1.11.4.11) in a series in powers of the small parameter ε and retaining the terms up to the second order inclusive, we arrive at the expression of the asymptotic solution (1.11.4.10).

Remark 1.25. Section 3.6.2 gives an example of solving a Cauchy problem by the method of regular expansion for a second-order equation and also discusses characteristic features of the method.

Remark 1.26. The methods of scaled coordinates, two-scale expansions, and matched asymptotic expansions are also used to solve problems defined by first-order differential equations with a small parameter. The basic ideas of these methods and illustrative examples are given in Sections 3.6.3, 3.6.5, and 3.6.6.

⊙ *Literature for Section 1.10*: L. V. Kantorovich and G. P. Akilov (1959), G. M. Murphy (1960), S. G. Mikhlin and Kh. L. Smolitskii (1965), A. H. Nayfeh (1973, 1981), G. A. Korn and T. M. Korn (2000), E. Kamke (1977), D. Zwillinger (1997), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

1.12 Differential Inequalities and Solution Estimates

1.12.1 Two Theorems on Solution Estimates

THEOREM 1. *Let $y = y(x)$ and $u = u(x)$ be solutions to the differential equations*

$$y'_x = f(x, y), \qquad u'_x = g(x, u) \qquad (1.12.1.1)$$

satisfying the condition

$$y(x_0) \ge u(x_0) \qquad (1.12.1.2)$$

for some x_0. If the functions $f(x, y)$ and $g(x, y)$ are continuous in some domain G and

$$f(x, y) > g(x, y), \qquad (1.12.1.3)$$

then the inequalities

$$y(x) > u(x) \quad \text{if} \quad x > x_0, \qquad y(x) < u(x) \quad \text{if} \quad x < x_0 \qquad (1.12.1.4)$$

hold in this domain.

Remark 1.27. If at least one of the functions f or g satisfies the conditions of the uniqueness theorem from Section 1.1, then the sign $>$ in (1.12.1.3) can be replaced with $\ge$; moreover, the strict inequalities in (1.12.1.4) must be replaced with non-strict ones.

Example 1.40. Consider the special case where the right-hand side of the second equation in (1.12.1.1) has the form

$$g(x,u) = bu^\gamma \quad \text{with } b > 0 \text{ and } \gamma > 1$$

and the initial conditions for both equations are the same: $y(0) = u(0) = a > 0$. Then the inequality $y(x) > u(x)$ holds for $x > 0$. In this case, $u(x)$ is unbounded and exists on a finite interval $0 \le x < x_*$ (blow up; see also Section 1.14.4); it is given by formula (1.14.4.21) in which u must be replaced with y. It follows that whenever the conditions

$$f(x,y) \ge by^\gamma \quad \text{with } b > 0 \text{ and } \gamma > 1$$

hold, the Cauchy problem for the equation $y'_x = f(x,y)$ with the initial condition $y(0) = a > 0$ also has a blow-up solution.

THEOREM 2. Let the functions $f(x,y)$ and $g(x,y)$ be continuous in a domain G and let the inequalities

$$\begin{gathered} |f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|, \\ |f(x,y) - g(x,y)| \le \varepsilon, \end{gathered} \tag{1.12.1.5}$$

the first being the Lipschitz condition, hold in G. Then the estimate

$$|y(x) - u(x)| \le \frac{\varepsilon}{K}\bigl(e^{K|x-x_0|} - 1\bigr) \tag{1.12.1.6}$$

holds for the solutions $y = y(x)$ and $u = u(x)$ of the differential equations (1.12.1.1) that share the same initial condition:

$$y(x_0) = u(x_0). \tag{1.12.1.7}$$

This theorem enables one to obtain approximate solutions to nonlinear differential equations through the analysis of suitable simpler equations.

1.12.2 Chaplygin's Theorem and Its Applications (Bilateral Estimates of the Cauchy Problem Solution)

► Chaplygin's theorem. An illustrative example.

Consider the Cauchy problem for equation (1.11.1.1) with the initial condition (1.11.1.2). Suppose the conditions of the uniqueness theorem in Section 1.1.1 are satisfied and $y = y(x)$ is a solution to the problem.

Theorem (Chaplygin). *Suppose there are two functions, $u = u(x)$ and $v = v(x)$, that satisfy the same initial conditions as in (1.11.1.2),*

$$u(x_0) = v(x_0) = y_0, \tag{1.12.2.1}$$

and also satisfy the differential inequalities

$$\begin{aligned} &u'_x - f(x, u(x)) < 0, \\ &v'_x - f(x, v(x)) > 0 \end{aligned} \tag{1.12.2.2}$$

for $x_0 < x \le x_1$. Then the inequalities

$$u(x) < y(x) < v(x) \tag{1.12.2.3}$$

hold for $x_0 < x \le x_1$.

Chaplygin's theorem enables one to make estimates for solutions to Cauchy problems.

In order to choose the auxiliary functions $u(x)$ and $v(x)$ appearing in the statement of the theorem, one should take a test function $w = w(x, \mathbf{k})$ involving one or more parameters, $\mathbf{k} = \{k_1, \dots, k_m\}$, and satisfying the initial condition $w(x_0, \mathbf{k}) = y_0$. Then, one introduces the discrepancy

$$\delta = [y'_x - f(x, y)]_{y=w}$$

and selects two values of the parameter vector, $\mathbf{k} = \mathbf{k}_1$ and $\mathbf{k} = \mathbf{k}_2$, such that $\delta < 0$ and $\delta > 0$, respectively, in the range of the argument. One chooses $w(x, \mathbf{k}_1)$ to be $u(x)$, which gives a lower estimate, and $w(x, \mathbf{k}_2)$ to be $v(x)$, which gives an upper estimate.

Example 1.41. Consider the Cauchy problem

$$y'_x = y^2 + x^4, \qquad y(0) = 0, \tag{1.12.2.4}$$

and estimate its solution on the interval $0 < x \le 1$.

Let us take $w = w(x, k) = kx^5$ as the test function with parameter k to be varied in the analysis. Introducing the discrepancy

$$\delta = [y'_x - (y^2 + x^2)]_{y=w} = x^4(5k - 1 - k^2x^6),$$

we get

$$x^4(5k - 1 - k^2) < \delta < x^4(5k - 1). \tag{1.12.2.5}$$

For the lower estimate, we take $k = k_1 = \frac{1}{5}$, which makes the right-hand side of inequality (1.12.2.5) zero, to obtain $u(x) = w(x, k_1) = \frac{1}{5}x^5$. For the upper estimate, we take the least root of the quadratic equation $5k - 1 - k^2 = 0$, which makes the left-hand side of inequality (1.12.2.5) zero, $k_2 = \frac{5}{2} - \frac{\sqrt{21}}{2} \approx 0.2087$, to obtain $v(x) = w(x, k_2) = 0.2087\, x^5$. This suggests that, by virtue of Chaplygin's theorem, the solution of the Cauchy problem (1.12.2.4) on the interval $0 < x < 1$ satisfies the inequalities

$$\tfrac{1}{5}x^5 < y(x) < 0.2087\, x^5.$$

Remark 1.28. In the statement of Chaplygin's theorem, the initial data in (1.12.2.1) can be replaced with the inequalities

$$u(x_0) \le y_0 \le v(x_0). \tag{1.12.2.6}$$

The functions $u(x)$ and $v(x)$ are called, respectively, *a lower and an upper solution of the Cauchy problem* (1.11.1.1)–(1.11.1.2).

► First method for successive refinement of estimates.

Suppose we have a suitable pair of functions, $u(x)$ and $v(x)$, that satisfy inequalities (1.12.2.2). Is it possible to improve the bilateral estimate?

Let the second derivative f_{yy} retain its sign in a rectangular domain (1.11.1.5). Let us look at the surface $z = f(x, y)$ in the three-dimensional space (x, y, z) and focus on the curves along which the surface intersects with planes $x = \text{const}$. These curves are either concave upward, if $f_{yy} > 0$, or concave downward, if $f_{yy} < 0$. On the curve $z = f(x, y)$ with $x = \text{const}$, let us choose an arc segment defined by the values of y such that $u(x) \le y \le v(x)$ and write the equations of the tangent line at $y = u(x)$ and the chord connecting the points $y = u(x)$ and $y = v(x)$:

$$\begin{aligned} z &= f(x, u(x)) + f_y(x, u(x))(y - u(x)) \equiv M(x)y + N(x), \\ z &= f(x, u(x)) + \frac{f(x, v(x)) - f(x, u(x))}{v(x) - u(x)}(y - u(x)) \equiv Q(x)y + R(x). \end{aligned} \tag{1.12.2.7}$$

The curve $z = f(x, y)$ lies between the tangent and the chord, and hence the function $f(x, y)$ is lies between the functions $f_1 = M(x)y + N(x)$ and $f_2 = Q(x)y + R(x)$, which are linear in y. To be specific, we assume that $f_{yy} > 0$. Then the curve is above the tangent and below the chord:

$$M(x)y + N(x) < f(x, y) < Q(x)y + R(x).$$

Consider two auxiliary Cauchy problems for the first-order linear differential equations

$$(u_1)'_x = M(x)u_1 + N(x), \qquad (v_1)'_x = Q(x)v_1 + R(x) \tag{1.12.2.8}$$

subject to the initial conditions (1.12.2.1). The solutions of these problems, $u_1 = u_1(x)$ and $v_1 = v_1(x)$, are straightforward to obtain. One can easily prove that the inequalities

$$u(x) < u_1(x) < y(x) < v_1(x) < v(x) \tag{1.12.2.9}$$

hold, which suggests that the new functions $u_1(x)$ and $v_1(x)$ provide more accurate approximations of the desired function $y(x)$. In a similar fashion, starting from $u_1(x)$ and $v_1(x)$, one can obtain further, even more accurate approximations, $u_2(x)$ and $v_2(x)$, and so on. The process is rapidly converging, so that

$$v_n(x) - u_n(x) \le \frac{A}{2^{2^n}}, \tag{1.12.2.10}$$

where A is a positive constant independent of x and n.

Example 1.42. Consider the Cauchy problem

$$y'_x = y^2 + x^2, \qquad y(0) = 0. \tag{1.12.2.11}$$

Its solution will be sought in the range $0 < x \le \frac{\sqrt{2}}{2} \approx 0.7071$.

We see that $f''_{yy} = 2 > 0$. Let us show that the initial functions $u(x)$ and $v(x)$ can be taken in the form

$$u(x) = \tfrac{1}{3}x^3, \qquad v(x) = \tfrac{11}{30}x^3.$$

Indeed,

$$\begin{aligned} &u'_x - u^2(x) - x^2 = -\tfrac{1}{9}x^6 < 0,\\ &v'_x - v^2(x) - x^2 = \tfrac{1}{10}x^2 - \tfrac{121}{900}x^6 = x^2\big(\tfrac{1}{10} - \tfrac{121}{900}x^4\big) \ge x^2\big(\tfrac{1}{10} - \tfrac{121}{900} \times \tfrac{1}{4}\big) > 0.06\,x^2 > 0. \end{aligned}$$

Calculating $M(x)$, $N(x)$, $Q(x)$, and $R(x)$ by formulas (1.12.2.7), we get

$$M(x) = \tfrac{2}{3}x^3, \quad N(x) = x^2 - \tfrac{1}{9}x^6; \quad Q(x) = \tfrac{7}{10}x^3, \quad R(x) = x^2 - \tfrac{11}{90}x^6.$$

Equations (1.12.2.8) for the next approximation are written as

$$(u_1)'_x = \tfrac{2}{3}x^3u_1 + x^2 - \tfrac{1}{9}x^6, \qquad (v_1)'_x = \tfrac{7}{10}x^3v_1 + x^2 - \tfrac{11}{90}x^6.$$

Their solutions satisfying the initial conditions $u(0) = v(0) = 0$ are

$$\begin{aligned} u_1(x) &= \exp\big(\tfrac{1}{6}x^4\big)\int_0^x \big(t^2 - \tfrac{1}{9}t^6\big)\exp\big(-\tfrac{1}{6}t^4\big)\,dt,\\ v_1(x) &= \exp\big(\tfrac{7}{40}x^4\big)\int_0^x \big(t^2 - \tfrac{11}{90}t^6\big)\exp\big(-\tfrac{7}{40}t^4\big)\,dt. \end{aligned}$$

Now by expanding the exponentials into power series and retaining only the terms up to x^{11} and t^{11}, we obtain

$$u_1(x) \simeq \bar{u}_1(x) = \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 + \tfrac{2}{2079}x^{11},$$
$$v_1(x) \simeq \bar{v}_1(x) = \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 + \tfrac{1}{990}x^{11}.$$

Let us show that the functions $\bar{u}_1$ and $\bar{v}_1$ are also upper and lower bounds of the desired solution:

$$(\bar{u}_1)'_x - \bar{u}_1^2(x) - x^2 = -\bigl[\bigl(\tfrac{1}{63^2} + \tfrac{4}{3\times 2079}\bigr)x^{14} + \tfrac{4}{63\times 2079}x^{18} + \tfrac{4}{2079^2}x^{22}\bigr] < 0,$$
$$(\bar{v}_1)'_x - \bar{v}^2(x) - x^2 = \tfrac{1}{9}x^{10}\bigl\{\tfrac{1}{210} - \bigl[\bigl(\tfrac{1}{21^2} + \tfrac{1}{165}\bigr)x^4 + \tfrac{2}{7\times 990}x^8 + \tfrac{1}{330^2}x^{12}\bigr]\bigr\}.$$

The derivative of the expression in braces is negative for $0 \le x \le \frac{\sqrt{2}}{2}$; hence, the expression attains its minimum at the right endpoint of the range, $x = \frac{\sqrt{2}}{2}$. Consequently,

$$(\bar{v}_1)'_x - \bar{v}^2(x) - x^2 \ge \tfrac{1}{9}x^{10}\bigl\{\tfrac{1}{210} - \bigl[\bigl(\tfrac{1}{441} + \tfrac{1}{165}\bigr)\tfrac{1}{4} + \tfrac{2}{7\times 990} \times \tfrac{1}{16} + \tfrac{1}{330^2} \times \tfrac{1}{64}\bigr]\bigr\} > 0.$$

It follows that

$$\tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 + \tfrac{2}{2079}x^{11} < y(x) < \tfrac{1}{3}x^3 + \tfrac{1}{63}x^7 + \tfrac{1}{990}x^{11}, \qquad 0 < x \le \tfrac{\sqrt{2}}{2}.$$

If either function $\bar{u}_1(x)$ or $\bar{v}_1(x)$ is taken to be an approximate solution to problem (1.12.2.11), then the approximation error does not exceed

$$\bigl(\tfrac{1}{990} - \tfrac{2}{2079}\bigr)x^{11} \le \bigl(\tfrac{1}{990} - \tfrac{2}{2079}\bigr)2^{-11/2} \approx 10^{-6}.$$

► **Second method for successive refinement of estimates.**

Below we outline another method for successive refinement of the approximations $u(x)$ and $v(x)$, which does not require the assumption that the second derivative f_{yy} is sign invariant.

Let K be the Lipschitz constant appearing in the first inequality in (1.12.1.5). It can be shown that the functions

$$\begin{aligned} u_1(x) &= u(x) + \int_{x_0}^{x} e^{K(x-t)}\bigl[f(t,u(t)) - u'_t(t)\bigr]\,dt,\\ v_1(x) &= v(x) - \int_{x_0}^{x} e^{K(x-t)}\bigl[v'_t(t) - f(t,v(t))\bigr]\,dt \end{aligned} \qquad (1.12.2.12)$$

satisfy the initial conditions 1.12.2.1 and inequalities (1.12.2.9) hold for $x > x_0$. By repeated application of formulas 1.12.2.12, one can obtain a sequence of more and more refined approximations $u_n(x)$ and $v_n(x)$. It can be proven that $u_n(x)$ and $v_n(x)$ converge uniformly to the solution $y(x)$ are $n \to \infty$; however, the rate of convergence is lower than that provided by formula (1.12.2.10).

⊙ *Literature for Section 1.12*: I. S. Berezin and N. P. Zhidkov (1960), S. G. Mikhlin and Kh. L. Smolitskii (1965), E. Kamke (1977).

1.13 Standard Numerical Methods for Solving Ordinary Differential Equations

1.13.1 Single-Step Methods. Runge–Kutta Methods

► **Preliminary remarks.**

1°. The classes of differential equations that allow for exact general solutions (in closed form), using exact methods developed so far, are quite narrow and cover only a small portion of practical problems. For this reason, numerical methods are commonly used nowadays, which apply to wide classes of equations and enable one to obtain particular solutions to specific problems.

2°. Consider the Cauchy problem for the first-order differential equation

$$y'_x = f(x, y) \qquad (x > x_0) \tag{1.13.1.1}$$

with the initial condition

$$y(x_0) = y_0. \tag{1.13.1.2}$$

Our aim is to construct an approximate solution $y = y(x)$ of this equation on an interval $[x_0, x_*]$.

Difference solution methods for problem (1.13.1.1)–(1.13.1.2) will be understood as numerical methods based on the replacement of the differential equation (1.13.1.1) for the continuously differentiable function $y = y(x)$ with an approximate equation (or equations) for functions of discrete argument, which are defined at a discrete set of points from the interval $[x_0, x_*]$. This makes us look at difference equations of integer argument, which are a special case of functional equations.

3°. Let us split the interval $[x_0, x_*]$ into n equal segments of length $h = (x_* - x_0)/n$. Let $y_1, \ldots, y_n$ denote the approximate values of the function $y(x)$ at the partitioning points $x_1, \ldots, x_n = x_*$. The discrete set of points $X_h = \{x_0, x_1, \ldots, x_n\}$ is called a mesh, the individual points x_k are *mesh nodes*, and h is a *mesh increment* or *step size*. The collection of values of the desired quantity is called a mesh function and denoted $y^{(h)} = \{y_k,\ k = 0, 1, \ldots, n\}$.

Statement of a numerical problem: given an initial value $y_0 = y(x_0)$ and a sufficiently small h, find approximate values of the unknown function, $y_k = y(x_k)$, at the points $x_k = x_0 + kh$, $k = 1, \ldots, n$.

► Order of approximation of a numerical method.

It is convenient to represent equation (1.13.1.1) in the short operator form

$$L[y] = f. \tag{1.13.1.3}$$

Likewise, the corresponding numerical (difference) scheme approximating (1.13.1.3) can be represented as

$$L_h[y^{(h)}] = f^{(h)}, \tag{1.13.1.4}$$

with the same initial value y_0 as in (1.13.1.2). It is assumed that for any smooth function $u = u(x)$, the limiting relations

$$\lim_{h\to 0} L_h[u] = L[u] \equiv u'_x, \qquad \lim_{h\to 0} f^{(h)}|_{y=u} = f|_{y=u}$$

hold. Specific numerical schemes (1.13.1.4) will be outlined below.

Let Y_h denote the space of mesh functions defined on the mesh X_h with norm*

$$\|y^{(h)}\|_Y = \max_{0\le k\le n} |y_k|. \tag{1.13.1.5}$$

*This norm is a difference analogue of the norm in the space of continuous functions.

Let F_h denote the space of mesh functions of two arguments with norm

$$\|f^{(h)}\|_F = \max_{0\le k\le n} |f_k|, \qquad f_k = f(x_k, y_k).$$

Suppose the difference problem (1.13.1.4) has a unique solution. If the mesh function $y_{\rm e}^{(h)}$ coinciding with the exact solution at the mesh nodes is substituted in (1.13.1.4) for $y^{(h)}$, the resulting discrepancy,

$$\delta f^{(h)} = L_h[y_{\rm e}^{(h)}] - f_e^{(h)},$$

will generally be nonzero.

Definition. The difference scheme (1.13.1.4) will be said to approximate the differential problem (1.13.1.3) on the closed interval $[x_0, x_*]$ if $\|\delta f^{(h)}\|_F \to 0$ as $h \to 0$. If, in addition, the inequality

$$\|\delta f^{(h)}\|_F \le C_1 h^m \tag{1.13.1.6}$$

holds, where C_1 and m are some positive constants, then the difference scheme will be said to have *approximation of order* m with respect to the step size h; the quantity m is also called the *global order of approximation*.

Remark 1.29. Suppose that the exact value of $y = y(x)$ at $x = x_{k+1}$, which corresponds to $h \to 0$, differs from the approximate value y_{k+1} by a quantity of the order of $O\big(h^{m_l}\big)$. Then the number m_l is called the *local order of approximation* of the numerical method in question. The local and global orders of approximations are related to each other simply as $m_l = m + 1$.

Remark 1.30. Suppose one looks at a boundary value problem of the second or third kind for a second or higher order equation, where a solution derivative is involved in the boundary condition. When the problem is replaced with a suitable difference scheme, the error in approximating the boundary conditions will affect the order of approximation of the difference problem.

► Single-step methods.

In general, a single-step method is a numerical method that provides successively an approximation of the exact solution, y_{k+1}, at the point x_{k+1} based on the known approximation y_k at the point x_k.

Explicit single-step methods for solving the Cauchy problem for equation (1.13.1.1) are defined by formulas of the form

$$y_{k+1} = \Phi(f, x_k, y_k, x_{k+1}). \tag{1.13.1.7}$$

For simplicity, the dependence of Φ on h is not specified.

More complex, implicit single-step methods are defined by

$$y_{k+1} = \Phi(f, x_k, y_k, x_{k+1}, y_{k+1}), \tag{1.13.1.8}$$

with the unknown y_{k+1} appearing on both the left- and right-hand side of (1.13.1.8). The values y_{k+1} can be found using, for example, iterative methods that are suitable for solving algebraic (generally transcendental) equations. Implicit methods are usually more preferable than explicit ones because of greater stability.

► Euler Method of polygonal lines. Implicit Euler method.

1°. Suppose that the Cauchy problem for equation (1.13.1.1) is solved at the points x_0, x_1, …, x_k to obtain the set of values y_0, y_1, …, y_k. By integrating equation (1.13.1.1) over the interval $[x_k, x_{k+1}]$, we get

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(x, y(x))\, dx. \tag{1.13.1.9}$$

Many numerical methods are based on various approximations of the integrand in (1.13.1.9).

2°. For a given initial value $y_0 = y(x_0)$ the values $y_k = y(x_k)$ at the other points $x_k = x_0 + kh$ are calculated successively by the formula

$$y_{k+1} = y_k + hf(x_k, y_k) \qquad (\textit{Euler polygonal line}), \tag{1.13.1.10}$$

where $k = 0,\ 1,\ \dots,\ n-1$.

The Euler method is the simplest explicit single-step method providing a first-order approximation (with respect to the step size h). It is obtained from formula (1.13.1.9) by the replacement of the integrand with a constant, $f(x, y) \approx f(x_k, y_k)$.

The Euler method is of little or no practical use, since its accuracy is usually very low.

3°. If the integrand in (1.13.1.9) is replaced with a constant, $f(x, y) \approx f(x_{k+1}, y_{k+1})$, this will result in the following algebraic (transcendental) equation for y_{k+1}:

$$y_{k+1} = y_k + hf(x_{k+1}, y_{k+1}). \tag{1.13.1.11}$$

This equation defines the implicit Euler method with the first order of accuracy. In general, the nonlinear equation (1.13.1.11) is solved numerically using a suitable iterative method (e.g., Newton's method).

► Single-step methods with a second-order approximation.

1°. There is a modification of the Euler method of polygonal lines known as the *modified Euler method* that provides a second-order approximation and so is more accurate. With the modified Euler method, one first calculates the intermediate values

$$x_{k+\frac{1}{2}} = x_k + \tfrac{1}{2}h, \qquad y_{k+\frac{1}{2}} = y_k + \tfrac{1}{2}hf(x_k, y_k)$$

and then calculates y_{k+1} by the formula

$$y_{k+1} = y_k + hf\bigl(x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}\bigr) \equiv y_k + hf\bigl(x_k + \tfrac{1}{2}h,\ y_k + \tfrac{1}{2}hf_k\bigr), \tag{1.13.1.12}$$

where $f_k = f(x_k, y_k)$ and $k = 0,\ 1,\ \dots,\ n-1$.

2°. Another second-order single-step method for solving the Cauchy problem is defined by the recurrence formula

$$y_{k+1} = y_k + \tfrac{1}{2}h\bigl[f(x_k, y_k) + f(x_{k+1},\ y_k + f_k h)\bigr]. \tag{1.13.1.13}$$

This method is sometimes called *Heun's method*.

Example 1.43. In the simple special case where the right-hand side of the equation is independent of y, or $f(x,y) = f(x)$, the maximum error ε_{m} of the approximate solution (1.13.1.13) is estimated as

$$\varepsilon_{\mathrm{m}} \le \frac{L}{12}\max|f''_{xx}(x)|h^2,$$

where $L = x_* - x_0$ is the length of the interval where the solution is looked for ($x_0 \le x \le x_*$).

3°. Apart from formulas (1.13.1.12) and (1.13.1.13), the equation

$$y_{k+1} = y_k + \tfrac{1}{2}h\bigl[f(x_k,\ y_k) + f(x_{k+1},\ y_{k+1})\bigr] \qquad (1.13.1.14)$$

is sometimes used. It determines an approximate solution in implicit form, since y_{k+1} appears in both the left- and right-hand side of Eq. (1.13.1.14).

▶ Runge–Kutta fourth-order methods.

1°. The unknown values y_k are successively found by the formulas

$$y_{k+1} = y_k + \tfrac{1}{6}h(\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4), \qquad (1.13.1.15)$$

where

$$\varphi_1 = f(x_k,\ y_k), \qquad \varphi_2 = f(x_k + \tfrac{1}{2}h,\ y_k + \tfrac{1}{2}h\varphi_1),$$
$$\varphi_3 = f(x_k + \tfrac{1}{2}h,\ y_k + \tfrac{1}{2}h\varphi_2), \qquad \varphi_4 = f(x_k + h,\ y_k + h\varphi_3).$$

This is one of the most common methods providing fourth-order approximation.

Remark 1.31. When saying the "Runge–Kutta method" without further specifications, one usually means the classical fourth-order Runge–Kutta method defined above.

For a rough estimate of the local accuracy of formula (1.13.1.15), it is quite useful to evaluate the quantity

$$\varepsilon = h(3f_{k+1} + \varphi_1 - 2\varphi_3 - 2\varphi_4).$$

Example 1.44. In the simple special case $f(x,y) = f(x)$, meaning that the right-hand side of equation is independent of y), the maximum error ε_{m} of the approximate solution (1.13.1.15) can be estimated as

$$\varepsilon_{\mathrm{m}} \le \frac{L}{2880}\max|f''''(x)|h^4, \qquad (1.13.1.16)$$

where $L = x_* - x_0$ is the length of the interval on which the solution is looked for ($x_0 \le x \le x_*$). It is apparent that, apart from rapid decrease of the error as $h \to 0$, the right-hand side of (1.13.1.16) contains a very small numerical coefficient, $L/2880$. This is one of the causes of the high accuracy of the method.

The above scheme has the convenience that the step size h can be changed starting from any k (e.g., it can be decreased where the desired functions are rapidly changing or increased where these change slowly). In practice, the step size can be controlled using the following simple technique. For each k, one evaluates the parameter

$$\theta = \left|\frac{\varphi_2 - \varphi_3}{\varphi_1 - \varphi_2}\right|.$$

While θ is of the order of a few hundredths, the computation should be carried out with the same step size. If θ is greater than one tenth, the step size can be decreased. If θ is less than one hundredth, the step size can be increased.

2°. Although much less common, the following formula provides a fourth-order approximation and can also be efficiently used in computations:

$$y_{k+1} = y_k + \tfrac{1}{8}h(\varphi_1 + 3\varphi_2 + 3\varphi_3 + \varphi_4), \tag{1.13.1.17}$$

where

$$\begin{aligned}
&\varphi_1 = f(x_k,\, y_k), \qquad \varphi_2 = f(x_k + \tfrac{1}{3}h,\; y_k + \tfrac{1}{3}h\varphi_1),\\
&\varphi_3 = f(x_k + \tfrac{2}{3}h,\; y_k - \tfrac{1}{3}h\varphi_1 + h\varphi_2),\\
&\varphi_4 = f(x_k + h,\; y_k + h\varphi_1 - h\varphi_2 + h\varphi_3).
\end{aligned}$$

► **Runge–Kutta methods. General scheme.**

1°. All explicit numerical methods outlined above are special cases of the general approach, known the family of explicit Runge–Kutta methods, whose main ideas are outlined below. For a detailed description of this approach, see the monographs listed at the end of the current section.

The value $y_{k+1} = y(x_k + h)$ is approximated using the formula

$$y(x_k + h) \approx \xi(x_k, h) = y(x_k) + \sum_{n=1}^{p} A_n s_n(h), \tag{1.13.1.18}$$

where $s_n = s_n(h)$ is the set of functions defined as

$$\begin{aligned}
&s_1 = hf(x_k, y_k),\\
&s_2 = hf(x_k + \alpha_2 h, y_k + \beta_{2,1}s_1),\\
&s_3 = hf(x_k + \alpha_3 h, y_k + \beta_{3,1}s_1 + \beta_{3,2}s_2),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&s_p = hf(x_k + \alpha_p h, y_k + \beta_{p,1}s_1 + \cdots + \beta_{p,p-1}s_{p-1}).
\end{aligned}$$

The unknown parameters A_n, α_n, and $\beta_{n,m}$ $(0 < m < n \le p)$ are chosen from the conditions

$$\delta(0) = \delta'(0) = \delta''(0) = \cdots = \delta^{(r)}(0) = 0. \tag{1.13.1.19}$$

The function $\delta(h) = y(x_k + h) - \xi(x_k, h)$ shows the discrepancy between the approximate solution $\xi(x_k, h)$ and exact solution $y(x_k + h)$.

In the family of Runge–Kutta schemes defined by formula (1.13.1.18), one can make the approximation error indefinitely small by increasing the parameter p. The value $p = 1$ corresponds the Euler method (1.13.1.10), while $p = 4$ results in the more accurate formula (1.13.1.15).

2°. The computation error of formula (1.13.1.18) can be estimated using Runge's rule

$$y_{\rm e} - y_{k+1,h/2} \approx \frac{y_{k+1,h/2} - y_{k+1,h}}{2^{p+1} - 1},$$

where $y_{\rm e}$ is the exact solution, $y_{k+1,h}$ is the approximate solution corresponding to the step size h, and $y_{k+1,h/2}$ is the approximate solution corresponding to the step size $h/2$.

► Qualitative features of the Runge–Kutta schemes.

The Runge–Kutta schemes outlined above offer a number of important advantages:

(i) All of them are stable and provide high accuracy (except for Euler's method of polygonal lines).

(ii) In explicit schemes, the value y_{k+1} is calculated from the values obtained at previous steps using fixed formulas and a number of operations.

(iii) All of the schemes allow for straightforward generalizations to utilize variable step sizes, $h_k = x_k - x_{k-1}$; starting from any point (any k), the step size can be decreased, where the function is rapidly changing, or increased, where the function changes slowly.

(iv) No use of further methods is required (unlike, for example, the multistep methods discussed below).

The Runge–Kutta schemes are easy to extend from a single equation to a system of first-order differential equations with the formal change of the variables y, $f(x, y)$ to $\mathbf{y}$, $\mathbf{f}(x, \mathbf{y})$ (see Section 7.4 below).

The errors of the Runge–Kutta schemes are determined by the maxima of the absolute values of derivatives. If the function $f(x)$ on the right-hand side of equation (1.13.1.1) is continuous and bounded together with its fourth derivatives (as long as they are not too large in magnitude), then the fourth-order scheme (1.13.1.15) provides good results due to the very small coefficient in the remainder and rapid increase in the accuracy as the step size decreases. If $f(x)$ does not have the required derivatives (or they are unbounded), it is more reasonable to use lower-order schemes, whose order is determined by the highest continuous and bounded derivative available; for example, the simple scheme (1.13.1.12) or (1.13.1.13) can be used for the twice continuously differentiable right-hand side in (1.13.1.1).

► Remarks on the choice of the step size.

Remark 1.32. In practice, calculations are performed on the basis of any of the above recurrence formulas with two different steps h, $\frac{1}{2}h$ and an arbitrarily chosen small h. Then one compares the results obtained at common points. If these results coincide within the given order of accuracy, one assumes that the chosen step h ensures the desired accuracy of calculations. Otherwise, the step is halved and the calculations are performed with the steps $\frac{1}{2}h$ and $\frac{1}{4}h$, after which the results are compared again, etc. (Quite often, one compares the results of calculations with steps varying by ten or more times.)

Remark 1.33. For schemes with a variable step size, $h_k = x_k - x_{k-1}$, intended for the solution of equation (1.13.1.1), the formula

$$h_k = \frac{h}{1 + \alpha |f(x_{k-1}, y_{k-1})|} \tag{1.13.1.20}$$

can be used to select the step size automatically, with α being a positive numeric parameter whose value can be chosen depending on the problem (schemes with a constant step size correspond to $\alpha = 0$). The step size should be decreased when the function is rapidly changing and increased when it changes slowly.

Instead of $|f(x_{k-1}, y_{k-1})|$ in the denominator in (1.13.1.20), one can also use $|f(x_{k-\frac{1}{2}}, y_{k-\frac{1}{2}})|$, where $x_{k-\frac{1}{2}} = x_{k-1} + \frac{1}{2}h$ and $y_{k-\frac{1}{2}} = y_{k-1} + \frac{1}{2}hf(x_{k-1}, y_{k-1})$.

1.13.2 Multistep Methods

► Preliminary remarks. General scheme.

1°. Multistep numeric methods are those in which the unknown y_{k+1} at x_{k+1} is calculated based on a number of known values, y_{k-m+1}, y_{k-m+2}, …, y_k, at several previous points, x_{k-m+1}, x_{k-m+2}, …, x_k. In practice, three to five points usually suffice, which suggests that $m = 3$, 4, or 5, with $m = 1$ corresponding to single-step methods.

2°. In the implementation of multistep methods, the problem arises of how to start the method, since the values of the unknown quantity at x_1, …, x_{m-1} must already be known. This can be done with any single-step method by using the initial condition $y = y_0$ at $x = x_0$. Sometimes, on the initial segment, one uses a truncated Taylor series (see Section (1.11.3)) with sufficiently many terms retained.

Remark 1.34. The values of the unknown at the initial points x_1, …, x_{m-1} must be evaluated with an accuracy at least several times higher than it is required for the entire solution; in the Runge–Kutta method used to calculate the initial values, the step size h should be taken much smaller than in the multistep method used for subsequent values.

► Adams methods.

1°. *Extrapolation methods*. In the explicit Adams method, to calculate the integral appearing in (1.13.1.9), one replaces the integrand with an interpolation polynomial of degree $m-1$, $P_{m-1}(x)$, whose values coincide with the those of $f(x, y(x))$ at the points x_{k-m+1}, x_{k-m+2}, …, x_k. In particular, with a polynomial of degree 0 (e.g., with the integrand replaced with its value at the left endpoint of the interval at x_k), one obtains the explicit Euler method.

With a cubic interpolation polynomial determined by the last four points of the integrand, one obtains a formula representing the *Adams–Bashforth method of the fourth order*:

$$y_{k+1} = y_k + \frac{1}{24}\bigl(55f_k - 59f_{k-1} + 37f_{k-2} - 9f_{k-3}\bigr)h, \qquad (1.13.2.1)$$

where $f_s = f(x_s, y_s)$.

2°. *Interpolation methods*. In the implicit Adams method, to calculate the integral appearing in (1.13.1.9), one replaces the integrand with an interpolation polynomial of degree $m-1$, $Q_{m-1}(x)$, whose values coincide with the those of $f(x, y(x))$ at the points x_{k-m+2}, x_{k-m+3}, …, x_{k+1}.

In particular, the four-step *Adams–Moulton implicit formula* is

$$y_{k+1} = y_k + \frac{1}{24}\bigl(9f_{k+1} + 19f_k - 5f_{k-1} + f_{k-2}\bigr)h. \qquad (1.13.2.2)$$

It is apparent that (1.13.2.2) is an equation for y_{k+1}, which appears on both sides of the equation. However, one usually avoids solving this equation by replacing f_{k+1} on the right-hand side with a value calculated by an explicit formula (e.g., the Adams–Bashforth formula). This approach underlies predictor-corrector methods, which are discussed below in Section 1.13.3.

► Milne method.

The multistep Milne method of the fourth order can be implemented in two different ways:

$$y_{k+1} = y_{k-3} + \frac{4}{3}\big(2f_k - f_{k-1} + 2f_{k-2}\big) \qquad \text{(first way)},$$
$$y_{k+1} = y_{k-2} + \frac{3}{8}\big(7f_k - 3f_{k-1} + 5f_{k-2}\big) \quad \text{(second way)}.$$

To start the computation, four initial points are requited.

► Nyström method.

The multistep Nyström method of the fourth order results in the formula

$$y_{k+1} = y_{k-1} + \frac{1}{3}\big(8f_k - 5f_{k-1} + 4f_{k-2} - f_{k-3}\big)h.$$

The method requires four initial points.

► General scheme. Concluding remarks.

1°. In general, multistep methods are based on the following approximation of the derivative at the point x_k:

$$y'_x \approx \frac{1}{h}\sum_{i=0}^{s} a_i y(x_{k+1-i}). \qquad (1.13.2.3)$$

The right-hand side of the differential equation (1.13.1.1) at x_k is approximated as

$$f\big(x, y(x)\big) \approx \sum_{i=0}^{s} b_i f\big(x_{k+1-i}, y(x_{k+1-i})\big). \qquad (1.13.2.4)$$

The numeric coefficients a_i ($a_0 \neq 0$) and b_i are independent of h.

This results in finite-difference schemes of the form

$$\sum_{i=0}^{s} a_i y_{k+1-i} - h\sum_{i=0}^{s} b_i f_{k+1-i} = 0. \qquad (1.13.2.5)$$

Explicit schemes correspond to $b_0 = 0$, while implicit schemes correspond to $b_0 \neq 0$.

If scheme (1.13.2.5) is obtained with the approximations (1.13.2.3) and (1.13.2.4), then the coefficients a_i and b_i must satisfy three relations

$$\sum_{i=0}^{s} a_i = 0, \qquad \sum_{i=0}^{s}(k+1-i)a_i = 1, \qquad \sum_{i=0}^{s} b_i = 1.$$

Remark 1.35. When writing specific schemes, it is customary to set the coefficient of y_{k+1} equal to one, which is equivalent to dividing all coefficients a_i and b_i by a_0.

The error of a scheme is determined by the leading term in the expansion of the left-hand side of (1.13.2.5) as $h \to 0$. In the expansion, one must take into account the dependence of $y_k = y(x_k)$ and $f_k = f(x_k, y_k)$ on h, since $x_k = x_0 + kh$.

2°. The difference scheme (1.13.2.5) is associated with a *characteristic polynomial*

$$P(\mu) = \sum_{i=0}^{s} a_i \mu^{s-i}, \qquad (1.13.2.6)$$

which is obtained by substituting $y_k = \mu^k$ into the truncated equation (1.13.2.5) with $h = 0$. The roots of the characteristic polynomial, which are determined from the algebraic equation $P(\mu) = 0$, will be denoted μ_j, where $j = 1, \dots, s$.

Difference schemes of the form (1.13.2.5) must satisfy the α-*condition*: all roots of the characteristic polynomial (1.13.2.6) must lie in the unit circle ($|\mu_j| \le 1$), with no multiple roots on the circumference.

For any difference scheme that does not meet the α-condition, there exists an equation of the form (1.13.1.1) with a smooth right-hand side whose difference solution does not converge to the differential solution (obtained without rounding) as the step size of the mesh decreases.

Let m be the order of approximation of the difference scheme (1.13.2.5). Then the following theorem holds.

THEOREM. *In the cases that*

the scheme is explicit and $m > k$,
the scheme is implicit, s *is odd, and* $m > k + 1$,
the scheme is implicit, s *is even, and* $m > k + 2$,

there is a at least one root, among all roots of the characteristic polynomial (1.13.2.6), *whose absolute value exceeds unity.*

This theorem states that schemes of the form (1.13.2.5) with a sufficiently high order of approximation do not satisfy the α-condition.

3°. Apart from the problem of solution initiation in multistep methods, one faces the problem of step size change in the course of the solution. This problem requires nonstandard actions; it is easily solved in single-step methods. With the same accuracy, the Runge–Kutta method allows one to use 4–6 times larger step sizes than the Adams method. For this reason, the Adams and Milne methods are much less common in practice than the Runge–Kutta methods.

1.13.3 Predictor–Corrector Methods

► **Adams type predictor–corrector method.**

This method combines the explicit and implicit four-step Adams method and makes it possible to increase the accuracy of the Adams method by computing the value of $f(x, y)$ twice when determining y_{k+1} in each new step with respect to x. The following actions are performed:

(i) *Predictor step.* Starting from the values at x_{k-3}, x_{k-2}, x_{k-1}, x_k, one calculates an initial guess value $\widetilde{y}_{k+1}$ at x_{k+1} using Adams's formula (1.13.2.1).

(ii) *Intermediate step.* One calculates the intermediate value of f at the new point:

$$\widetilde{f}_{k+1} = f(x_{k+1}, \widetilde{y}_{k+1}).$$

(iii) *Corrector step.* Using the fourth-order Adams method and the values of y at x_{k-2}, x_{k-1}, x_k, x_{k+1}, one calculates a refined value at x_{k+1}:

$$y_{k+1} = y_k + \frac{1}{24}\bigl(9\widetilde{f}_{k+1} + 19f_k - 5f_{k-1} + f_{k-2}\bigr)h. \qquad (1.13.3.1)$$

The truncation error of this method is of the order of $O(h^4)$. If the refined value does not deviate from the guess value by the allowed computational error, $|\widetilde{y}_{k+1} - y_{k+1}| \le \varepsilon$, the step size h is considered acceptable.

► Milne predictor–corrector method.

The following actions are performed:

(i) *Predictor step.* Starting from the values of y at x_{k-3}, x_{k-2}, x_{k-1}, x_k, one calculates an initial guess value at x_{k+1} by the formula

$$\widetilde{y}_{k+1} = y_{k-3} + \frac{4}{3}\bigl(2f_k - f_{k-1} + 2f_{k-2}\bigr)h. \qquad (1.13.3.2)$$

(ii) *Intermediate step.* One calculates the intermediate value of f at the new point:

$$\widetilde{f}_{k+1} = f(x_{k+1}, \widetilde{y}_{k+1}).$$

(iii) *Corrector step.* The refined value is calculated by the corrector formula

$$y_{k+1} = y_{k-1} + \frac{1}{3}\bigl(\widetilde{f}_{k+1} + 4f_k + f_{k-1}\bigr)h. \qquad (1.13.3.3)$$

The truncation error of the Milne method is $O(h^5)$.

The computational error can be estimated as

$$\varepsilon = \tfrac{1}{29}|y_{k+1} - \widetilde{y}_{k+1}|. \qquad (1.13.3.4)$$

If the maximum allowed error equals δ, then condition $\delta \le \varepsilon$ must be checked at each step. If this condition does not hold, the step size should be decreased h.

► Hamming predictor–corrector method.

In fact, the Hamming method has four steps: calculation of a prediction $\widetilde{y}_{i+1}$ followed by an improvement δy_{i+1} and then calculation of a correction y^*_{i+1} followed by its refinement y_{i+1}. The prediction and its improvement are performed using not only nodal values of the derivative of $f(x, y(x))$ but also nodal values of the desired function $y(x)$ as well as auxiliary quantities.

The four steps are expressed by the formulas

$$\begin{aligned}
\widetilde{y}_{k+1} &= \frac{1}{3}(2y_{k-1} + y_{k-2}) + \frac{1}{72}(191f_k - 107f_{k-1} + 109f_{k-2} - 25f_{k-3})h,\\
\delta y_{k+1} &= \widetilde{y}_{k+1} - \frac{707}{750}(\widetilde{y}_k - y^*_k),\\
y^*_{k+1} &= \frac{1}{3}(2y_{k-1} + y_{k-2}) + \frac{1}{72}\bigl[25f(x_{k+1}, \delta y_{k+1}) + 91f_k + 43f_{k-1} + 9f_{k-2}\bigr]h,\\
y_{k+1} &= y^*_{k+1} + \frac{43}{750}(\widetilde{y}_{k+1} - y^*_{k+1}).
\end{aligned}$$

The truncation error of this method is $O(h^6)$.

1.13.4 Modified Multistep Methods (Butcher's Methods)

Just like multistep methods, the modified multistep methods use several preceding values of the unknown, y_{k-i}, to compute the current value y_k; moreover, just as in the Runge–Kutta methods, they also calculate the right-hand side several times at each step. Two examples are given below.

1°. Formulas providing an accuracy of $O(h^6)$:

$$\begin{aligned} y_{k-\frac{1}{2}} &= y_{k-2} + \tfrac{1}{8}(9f_{k-1} + 3f_{k-2})h, \\ y_k^\circ &= \tfrac{1}{5}(28y_{k-1} - 23y_{k-2}) + \tfrac{1}{15}(32f_{k-\frac{1}{2}} - 60f_{k-1} - 26f_{k-2})h, \\ y_k &= \tfrac{1}{31}(32y_{k-1} - y_{k-2}) + \tfrac{1}{93}(64f_{k-\frac{1}{2}} + 15f_k^\circ + 12f_{k-1} - f_{k-2})h, \end{aligned}$$

where $f_k^\circ = f(x_k, y_k^\circ)$.

2°. Formulas providing an accuracy of $O(h^8)$:

$$\begin{aligned} y_{k-\frac{1}{2}} &= \tfrac{1}{128}(-225y_{k-1} + 200y_{k-2} + 153y_{k-3}) + \tfrac{1}{128}(225f_{k-1} + 300f_{k-2} + 45f_{k-3})h, \\ y_k^\circ &= \tfrac{1}{31}(540y_{k-1} - 297y_{k-2} - 212y_{k-3}) \\ &\quad + \tfrac{1}{155}(384f_{k-\frac{1}{2}} - 1395f_{k-1} - 2130f_{k-2} - 309f_{k-3})h, \\ y_k &= \tfrac{1}{617}(783y_{k-1} - 135y_{k-2} - 31y_{k-3}) \\ &\quad + \tfrac{1}{3085}(2304f_{k-\frac{1}{2}} + 465f_k^\circ - 135f_{k-1} - 495f_{k-2} - 39f_{k-3})h. \end{aligned}$$

1.13.5 Stability and Convergence of Numerical Methods

► **Stability.**

Definition 1. The difference scheme (1.13.1.4) is called *stable*, if there are positive numbers h_m and δ such that for any $h < h_m$ and any mesh function of arguments, $\varepsilon^{(h)}$, that satisfies the inequality $\|\varepsilon^{(h)}\|_F < \delta$, the difference problem

$$L_h[z^{(h)}] = f^{(h)} + \varepsilon^{(h)}, \qquad (1.13.5.1)$$

obtained from (1.13.1.4) by adding the perturbation $\varepsilon^{(h)}$ to the right-hand side, has a unique solution $z^{(h)}$ and the relation holds

$$\|z^{(h)} - y^{(h)}\|_Y \le C_2\|\varepsilon^{(h)}\|_F, \qquad (1.13.5.2)$$

where C_2 is a constant independent of h.

This definition is quite general; it is valid even if L_h is nonlinear. Inequality (1.13.5.2) suggests that a small perturbation on the right-hand side of the difference scheme (1.13.1.4) causes a uniformly small (with respect to h) disturbance of the solution.

In the above case, the operator L_h is linear and the definition of stability is equivalent to the following.

Definition 2. The difference scheme (1.13.1.4) with the linear operator L_h is said to be stable if for any $f^{(h)} \in F_h$ equation (1.13.1.4) has a unique solution $y^{(h)} \in Y_h$ and the inequality

$$\|y^{(h)}\|_Y \le C_2\|f^{(h)}\|_F$$

holds; C_2 is a constant independent of h.

Remark 1.36. Stability is an intrinsic property of the difference scheme; it is not related to the original differential problem.

▶ Convergence. Lax theorem.

Definition. A solution $y^{(h)}$ of the difference equation (1.13.1.4) is said to be convergent, as $h \to 0$, to a solution $y = y(x)$ of the differential equation (1.13.1.1), both subject to the same initial condition (1.13.1.2)), if

$$\|y_e^{(h)} - y^{(h)}\|_Y \to 0 \quad \text{at} \quad h \to 0, \tag{1.13.5.3}$$

where $y_e^{(h)}$ is a mesh function coinciding with the solution y at the mesh nodes.

The relationship between approximation, stability, and convergence is set by the following theorem.

LAX THEOREM. *Let the difference scheme* (1.13.1.4) *provide an h^m approximation of the differential equation* (1.13.1.1) *on the solution y and be stable. Then the solution $y^{(h)}$ of the difference equation* (1.13.1.4) *converges to $y_e^{(h)}$ and the estimate*

$$\|y_e^{(h)} - y^{(h)}\|_Y \le C_1 C_2 h^m \tag{1.13.5.4}$$

holds true; C_1 and C_2 are the constant appearing in the estimates (1.13.1.6) *and* (1.13.5.2).

To prove this theorem, let us set $\varepsilon^{(h)} \equiv \delta f^{(h)}$ and $y_e^{(h)} \equiv z^{(h)}$ in (1.13.5.2). The we get the estimate $\|y_e^{(h)} - y^{(h)}\|_Y \le C_2 \|\delta f^{(h)}\|_F$. In view of (1.13.1.6), we immediately arrive at the required inequality (1.13.5.4).

1.13.6 Well- and Ill-Conditioned Problems

Numerical methods can only be applied to *well-conditioned problems*, in which small changes in the initial data (or the right-hand side of the equation) lead to small changes in the solution. Otherwise, for *ill-conditioned problems*, small perturbations in the initial conditions (or the right-hand side of the equation) or equivalent errors inherent in the numerical method can significantly change the solution.

Example 1.45. Consider the ordinary differential equation

$$y'_x = ay - a^2 x \qquad (0 < x \le 100) \tag{1.13.6.1}$$

with a free parameter a and subject to the initial condition

$$y(0) = 1. \tag{1.13.6.2}$$

The general solution to (1.13.6.1) is

$$y = 1 + ax + Ce^{ax}, \tag{1.13.6.3}$$

where C is an arbitrary constant.

In view of the initial condition (1.13.6.2), we get $C = 1$ and then

$$y = 1 + ax, \tag{1.13.6.4}$$

Now consider equation (1.13.6.1) with a slightly changed initial condition:

$$y(0) = 1 + \varepsilon, \tag{1.13.6.5}$$

where ε is a small positive number.

Substituting (1.13.6.3) into (1.13.6.5) yields the solution of problem (1.13.6.1), (1.13.6.5):

$$y_\varepsilon = 1 + ax + \varepsilon e^{ax}. \tag{1.13.6.6}$$

Solution (1.13.6.6) shows a qualitatively different behavior depending on the value of the parameter a. Let us look at possible situations.

1°. If $a < 0$, the exponential εe^{ax} in (1.13.6.6) decays as $x \to \infty$. The difference between solutions (1.13.6.6) and (1.13.6.3), equal to εe^{ax}, tends to zero as $\varepsilon \to 0$ for all $x \ge 0$. If $a = 0$, the difference between the solutions is also a small quantity equal to ε. Hence, the values $a \le 0$ correspond to a well-conditioned problem (1.13.6.1)–(1.13.6.2), with respect to changes in the initial data, so that $|y_\varepsilon - y| \le \varepsilon$.

2°. If $a > 0$, the difference between solutions (1.13.6.6) and (1.13.6.3) increases exponentially without bound as $x \to \infty$. In this case, for any $\varepsilon > 0$ solutions (1.13.6.3) and (1.13.6.6) diverge indefinitely far as $x \to \infty$. Hence, problem (1.13.6.1)–(1.13.6.2) is ill-conditioned for $a > 0$.

In particular, if $a = 1$, $\varepsilon = 10^{-6} \ll 1$, and $x = 10^2$, we get $y = 101$ and $y_\varepsilon \approx 2.7 \times 10^{37}$, which show that the solution has changed dramatically ($y_\varepsilon - y \gg 1$).

Remark 1.37. If $a > 0$, the solution to the equation

$$y'_x = ay - a^2 x + \varepsilon \qquad (\varepsilon \ll 1) \tag{1.13.6.7}$$

subject to the initial condition (1.13.6.2) is

$$y_\varepsilon = 1 - \frac{\varepsilon}{a} + ax + \frac{\varepsilon}{a} e^{ax}. \tag{1.13.6.8}$$

In this case, the difference between solutions (1.13.6.8) and (1.13.6.3) increases exponentially without bound as $x \to \infty$. This means that problem (1.13.6.1)–(1.13.6.2) is ill-conditioned for $a > 0$ with respect to small perturbations of the right-hand side.

⊙ *Literature for Section 1.13*: M. Abramowitz and I. A. Stegun (1964), J. C. Butcher (1965), C. W. Gear (1971), N. S. Bakhvalov (1973), J. D. Lambert (1973), E. Kamke (1977), N. N. Kalitkin (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), M. Stuart and M. S. Floater (1990), E. Hairer, S. P. Norsett, and G. Wanner (1993), W. E. Schiesser (1994), R. E. Mickens (1994), L. F. Shampine (1994), K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), J. R. Dormand (1996), E. Hairer and G. Wanner (1996), D. Zwillinger (1997), U. M. Ascher and L. R. Petzold (1998), R. Meyer-Spasche (1998), G. A. Korn and T. M. Korn (2000), S. S. Gaisaryan (2002), P. J. Rabier and W. C. Rheinboldt (2002), A. D. Polyanin and A. V. Manzhirov (2007), S C. Chapra and R. P. Canale (2010), P. G. Dlamini and M. Khumalo (2012).

1.14 Special Numerical Methods for Solving Problems with Qualitative Features or Singularities

1.14.1 Special Methods Based on Auxiliary Equations

► Preliminary remarks.

Certain problems require the use of special methods devised for fairly narrow classes of equations; these methods usually become unsuitable for other classes of equations.

The majority of special methods for equations of the form $y'_x = f(x, y)$ are based on seeking an *auxiliary equation* $u'_x = g(x, u)$ such that its solution is expressed in terms of elementary functions in a simple way, with the approximate relation $y(x) \approx u(x)$ being valid on a sufficiently large interval for the argument. In other words, one seeks an approximate solution having a fairly simple form.

Outlined below are three techniques for the construction of numerical schemes under the assumption that a suitable auxiliary solution, $u = u(x)$, has been obtained.

► First technique.

This approach is especially suitable for solutions changing their sign. One looks at the difference $w(x) = y(x) - u(x)$. Subtracting the auxiliary equation for the original one leads to the equation for the difference

$$w'_x = f(x, u(x) + w) - g(x, u(x)), \tag{1.14.1.1}$$

where, by assumption, $u(x)$ is a known function. If $u(x)$ approximates the solution sufficiently well, then $w = w(x)$ is fairly small in magnitude or subject to weak changes; hence, equation (1.14.1.1) should be easily integrable with customary numerical methods (e.g., Runge–Kutta methods).

Example 1.46. Let us look at a problem defined on the semi-infinite interval $[0, \infty)$. Suppose the asymptotic behavior of its solution, $y \simeq \varphi(x)$ as $x \to \infty$, is known; $\varphi(x)$ is a (rapidly) oscillating function. Then

$$u(x) = \frac{\delta x}{1 + \delta x}\varphi(x), \tag{1.14.1.2}$$

with $\delta \ll 1$ being a sufficiently small positive number, can be used as the auxiliary function. For small and moderate values of x, we have $|u(x)| \ll 1$, while for large x, the function $u(x)$ is asymptotically equivalent to $\varphi(x)$. As g in equation (1.14.1.1), one should take the derivative $u'_x(x)$.

On the right-hand side of (1.14.1.2), δx can be replaced with $\delta s(x)$, where $s(x) \geq 0$ is a monotonically increasing function that tends to infinity as $x \to \infty$.

► Second technique.

This approach is especially suitable for solutions that keep a constant sign and increase exponentially of by a power law. One looks at the ratio $w(x) = y(x)/u(x)$. It is not difficult to verify that the ratio satisfies the equation

$$w'_x = \frac{1}{u(x)}\big[f(x, u(x)w) - wg(x, u(x))\big], \tag{1.14.1.3}$$

where $u(x)$ is a known approximate solution. Just like in the previous case, this equation should be easy to integrate using standard numerical methods.

Remark 1.38. Both of the above techniques allow the application of high-order Runge–Kutta schemes; however, the remainder may not necessarily be small, since the solutions $y(x)$ and $u(x)$ can significantly differ and the right-hand sides of equations (1.14.1.1) and (1.14.1.3) can be large in magnitude. Nevertheless, both techniques can be applied locally to a short interval of the mesh (see the third technique below); in this way, one can construct special high-order schemes with a small remainder.

► Third technique.

Instead of a large interval, the auxiliary equation is constructed for a single mesh step, $x_k \leq x \leq x_{k+1}$, which is a small interval of length $h = x_{k+1} - x_k$. One takes an approximate solution, $u_n(x)$, satisfying the initial condition $u_n(x_n) = y_n \approx y(x_n)$. Since the step size is small, the approximate solution will be close to the exact one, so that we can set $y_n(x_{n+1}) \approx y_{n+1} = u(x_{n+1})$. Difference schemes based on this technique are intended to ensure that the auxiliary equation is solved exactly, while the original equation is solved approximately with a small error.

In some cases, to construct an approximate solution for a single-step interval, one can first solve the autonomous equation

$$v'_x = f(\sigma, v) \tag{1.14.1.4}$$

with parameters $\sigma = \text{const}$ and then use $u(x) = v(x, \sigma)$, where v is the solution of this equation, as the auxiliary function. Equation (1.14.1.4) can be obtained from (1.13.1.1) by formally replacing $f(x, y)$ with $f(\sigma, y)$ and y with v.

Example 1.47. Let us look at the nonlinear equation

$$y'_x = -\big[y^2 + \rho(x)\big], \qquad \rho(x) > 0, \tag{1.14.1.5}$$

which arises in differential sweep problems. If one sets $\rho(x) \approx \text{const} = \rho_{k+\frac{1}{2}}$ on the interval $x_k \le x \le x_{k+1}$, then the auxiliary equation becomes

$$u'_x = -(u^2 + \rho), \qquad \rho = \text{const}.$$

This is a separable equation; it is easy to integrate in terms of elementary functions. As a result, we obtain

$$\arctan \frac{u_{k+1}}{\sqrt{\rho}} - \arctan \frac{u_k}{\sqrt{\rho}} = -h\sqrt{\rho}.$$

This relation can be explicitly solved for $u_{k+1} \approx y_{k+1}$ to give the following special numerical scheme:

$$y_{k+1} = \sqrt{\rho_{k+\frac{1}{2}}}\, \frac{y_k - \sqrt{\rho_{k+\frac{1}{2}}}\, \tan\big(h\sqrt{\rho_{k+\frac{1}{2}}}\,\big)}{\sqrt{\rho_{k+\frac{1}{2}}} + y_k \tan\big(h\sqrt{\rho_{k+\frac{1}{2}}}\,\big)}, \qquad \rho_{k+\frac{1}{2}} = \rho(x_k + \tfrac{1}{2}h). \tag{1.14.1.6}$$

This scheme can be simplified under the assumption that the step size is sufficiently small, $h\sqrt{\rho} \ll 1$, to give

$$y_{k+1} = \frac{y_k - h\rho_{k+\frac{1}{2}}}{1 + hy_k}. \tag{1.14.1.7}$$

Schemes (1.14.1.6) and (1.14.1.7) ensure fairly good results even in the cases where the sweep stability condition is not satisfied and the exact solution of the problem for equation (1.14.1.5) has poles.

Remark 1.39. With the third technique, one often succeeds in constructing first- or second-order schemes with a sufficiently small remainder.

1.14.2 Numerical Integration of Equations That Contain Fixed Singular Points

► **Preliminary remarks.**

A solution can be singular at isolated points of the domain in question; this can happen if the right-hand side of equation (1.13.1.1) or its derivative becomes infinite. Let us assume that the initial point, $x = 0$, is singular. There are three main techniques for the numerical integration of such equations. These will be outlined below by looking at the example problem

$$y'_x = f(x, y) + bx^{-1/2}, \qquad y(0) = 0, \tag{1.14.2.1}$$

where $f(x, y)$ is a smooth function without singularities and $b \neq 0$ is a free parameter. The right-hand side of equation (1.14.2.1) has an integrable singularity of the order of $x^{-1/2}$ as $x \to 0$.

► First technique.

One looks for a change of variables that converts the original equation into one without singularities. In this case, it suffices to use the substitution $x=t^2$; then the singular problem (1.14.2.1) reduces to the problem without singularities

$$y'_t = 2tf(t^2, y) + 2b, \qquad y(0) = 0. \tag{1.14.2.2}$$

Remark 1.40. The substitution $x = t^2$ can be used as a transformation preceding numerical integration of equations with a fixed singularity of the form

$$y'_x = f_1(x^{1/2}, y) + x^{-1/2} f_2(x^{1/2}, y),$$

where $f_{1,2}(z, y)$ are smooth functions, with $f_2(0, y) \not\equiv 0$. For the numerical solution of the more general class of equations with a singularity

$$y'_x = x^{2\nu-1} f_1(x^\nu, y) + x^{\nu-1} f_2(x^\nu, y), \qquad 0 < \nu < 1,$$

one should use the change of variable $t = x^\nu$, which will lead to a singularity-free equation.

► Second technique.

In a small neighborhood of the singular point, one makes an asymptotic expansion of the solution (while retaining only a few terms) or constructs an equivalent approximate solution as $x \to 0$. Suppose the function $f(x, y)$ is expandable in a double Taylor series about $x = y = 0$. Then, near the singular point $x = 0$, the solution of equation (1.14.2.1) can be represented as a series in integer powers of $z = x^{1/2}$:

$$\begin{gathered} y = a_1 x^{1/2} + a_2 x + a_3 x^{3/2} + a_3 x^2 + \cdots, \\ a_1 = 2b, \quad a_2 = f(0,0), \quad a_3 = \tfrac{4}{3} b f_y(0,0). \end{gathered} \tag{1.14.2.3}$$

Let us look at a point $\bar{x}$ close to the singular point and compute the solution at $\bar{x}$ using the first few terms of the series (1.14.2.3). The point $\bar{x}$ is not singular; it can be taken as the first node of the difference mesh and used as the starting point of the computation with standard numerical methods.

Remark 1.41. It is noteworthy that if $\bar{x}$ is close to the singular point $x = 0$, the right-hand side of the equation and its partial derivatives can be quite large at $\bar{x}$ and, hence, the standard numerical methods may give a significant error near this point. It is therefore desirable to choose $\bar{x}$ as close as possible to $x = 0$. But then, to ensure a high accuracy of $y(\bar{x})$, one has to use a sufficiently good approximate solution involving more terms of the asymptotic expansion.

For more details on the methods of series expansion of solutions in the independent variable in a neighborhood of the singular point, see Section 1.11.3.

► Third technique.

This approach is based on developing a problem-specific scheme that allows the numerical integration to start directly from the singular point.

1°. Equation (1.14.2.1) can be integrated over a single step interval of the mesh to obtain

$$y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} \bigl[f(\xi, y(\xi)) + b\xi^{-1/2}\bigr]\,d\xi$$
$$\approx y_k + hf(x_k, y_k) + 2b\bigl(\sqrt{x_k + h} - \sqrt{x_k}\,\bigr), \qquad x_{k+1} = x_k + h. \qquad (1.14.2.4)$$

The first term in the integrand has been integrated approximately using the rectangle rule based on the left endpoint of the interval, while the second term has been integrated exactly.

The explicit scheme (1.14.2.4) is constructed in a similar way to first-order schemes; it becomes the Euler scheme of polygonal lines at $b = 0$.

2°. For equation (1.14.2.1), it is possible to construct more accurate numerical schemes similar to Runge–Kutta schemes by using more accurate approximations of the integral of the first term in (1.14.2.4). In particular, one can use the scheme

$$y_{k+1} = y_k + hf\bigl(x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}\bigr) + 2b\bigl(\sqrt{x_k + h} - \sqrt{x_k}\,\bigr), \qquad (1.14.2.5)$$
$$x_{k+\frac{1}{2}} = x_k + \tfrac{1}{2}h, \qquad y_{k+\frac{1}{2}} = y_k + \tfrac{1}{2}hf(x_k, y_k) + 2b\Bigl(\sqrt{x_k + \tfrac{1}{2}h} - \sqrt{x_k}\,\Bigr),$$

which becomes the modified second-order Euler scheme (1.13.1.12) at $b = 0$.

► Some generalizations.

1°. Let us look at a more general problem than (1.14.2.1),

$$y'_x = f(x, y) + \sum_{j=1}^{m} \varphi_j(x) g_j(y), \qquad y(0) = 0, \qquad (1.14.2.6)$$

assuming that $f(x, y)$ and $g_j(y)$ are smooth functions without singularities and the $\varphi_j(x)$ are functions with integrable singularities at $x = 0$ (so that $\varphi_j(0) = \infty$).

Integrating equation (1.14.2.5) over a small interval $[x_k, x_{k+1}]$ yields

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(x, y(x))\,dx + \sum_{j=1}^{m} \int_{x_k}^{x_{k+1}} \varphi_j(x) g_j(y(x))\,dx. \qquad (1.14.2.7)$$

Replacing the integrand of the first integral with $f(x_k, y(x_k))$, we get

$$\int_{x_k}^{x_{k+1}} f(x, y(x))\,dx \approx hf(x_k, y(x_k)),$$

where $h = x_{k+1} - x_k$. The remaining integrals, which contain singularities, can approximately be calculated as follows:

$$\int_{x_k}^{x_{k+1}} \varphi_j(x) g_j(y(x))\,dx \approx \int_{x_k}^{x_{k+1}} \varphi_j(x) g_j(y(x_k))\,dx$$
$$= g_j(y_k) \int_{x_k}^{x_{k+1}} \varphi_j(x)\,dx = [\Phi_j(x_{k+1}) - \Phi_j(x_k)]g_j(y_k),$$

where $\Phi_j(x) = \int_0^x \varphi_j(x)\,dx$ and $y_k = y(x_k)$. This results in the difference scheme

$$y_{k+1} = y_k + hf(x_k, y_k) + \sum_{j=1}^{m} [\Phi_j(x_{k+1}) - \Phi_j(x_k)]g_j(y_k). \qquad (1.14.2.8)$$

If $\varphi_j(x) \equiv 0$ $(j = 1, \ldots, m)$, it becomes Euler's scheme of polygonal lines of the first order of accuracy.

One can construct more accurate numerical schemes analogous to Runge–Kutta methods by using more accurate approximations of the integrals (e.g., see Item 2° below for an example).

2°. As a further generalization, let us look at the problem

$$y'_x = f(x, y) + \sum_{j=1}^{m} \varphi_j(x) g_j(x, y), \qquad y(0) = 0, \tag{1.14.2.9}$$

where $f(x, y)$ and $g_j(x, y)$ are singularity-free smooth functions satisfying the conditions $g_j(0, y) \not\equiv 0$, while the $\varphi_j(x)$ are functions with integrable singularities at $x = 0$ (so that $\varphi_j(0) = \infty$).

A simple Euler-type difference scheme for equation (1.14.2.9) is

$$y_{k+1} = y_k + h f(x_k, y_k) + \sum_{j=1}^{m} [\Phi_j(x_{k+1}) - \Phi_j(x_k)] g_j(x_k, y_k). \tag{1.14.2.10}$$

Remark 1.42. Note that the order of accuracy of schemes (1.14.2.4), (1.14.2.5), (1.14.2.8), and (1.14.2.10) is unknown in advance, since the derivatives of the right-hand sides of (1.14.2.1), (1.14.2.6), and (1.14.2.9) are unbounded; this question calls for further investigation and is not discussed here.

1.14.3 Numerical Integration of Equations Defined Parametrically or Implicitly

► **Numerical integration of equations defined parametrically.**

Consider the Cauchy problem for an equation defined parametrically using two relations (see Section 1.8.3):

$$F(x, y, t) = 0, \qquad y'_x = G(x, y, t) \qquad \text{(equation)}; \tag{1.14.3.1}$$

$$y = y_0 \quad \text{at} \quad x = x_0 \qquad \text{(initial condition)}. \tag{1.14.3.2}$$

Let us look at the general case where the parameter t cannot be eliminated from equations (1.14.3.1). Below we describe the main ideas of two methods for the solution of such problems.

First method. We start from equations (1.14.3.1). Let $y_F = y_F(x, t)$ denote a solution of the first equation (which is algebraic or transcendental) and let $y_G = y_G(x, t)$ denote a solution of the second (differential) equation subject to the initial condition (1.14.3.2). We also use the notation

$$\Delta(x, t) = y_G(x, t) - y_F(x, t). \tag{1.14.3.3}$$

By fixing a value of the parameter, $t = t_k$, and finding the corresponding solutions $y_F(x, t_k)$ and $y_G(x, t_k)$ (for example, y_F can be constructed by the iterative Newton method and y_G by the Runge–Kutta method). Further, by varying x, we find an x_k such that the right-hand side of (1.14.3.3) becomes zero, $\Delta(x_k, t_k) = 0$. To this x_k there corresponds the value of the desired function $y_k = y_F(x_k, t_k) = y_G(x_k, t_k)$. Thus, each value t_k in the (x, y) plane is associated with the point (x_k, y_k) at which the curves $y_F = y_F(x, t_k)$ and $y_G = y_G(x, t_k)$ intersect. By taking another value of the parameter, t_{k+1}, we find a different point (x_{k+1}, y_{k+1}). The combination of discrete points (x_k, y_k) with $k = 0, 1, 2, \ldots$ determines an approximate solution $y = y(x)$ to problem (1.14.3.1)–(1.14.3.2). The initial value of the parameter, $t = t_0$, is found from the algebraic (or transcendental) equation

$$F(x_0, y_0, t_0) = 0, \tag{1.14.3.4}$$

where x_0 and y_0 are the quantities appearing in the initial condition (1.14.3.2).

This method is especially easy to use if the first equation is explicitly solvable for y or x.

Second method. Using the method outlined in Section 1.8.3, we reduce equation (1.14.3.1) to a standard system of first-order differential equations for $x = x(t)$ and $y = y(t)$ (see system (1.8.3.6)):

$$x'_t = -\frac{F_t}{F_x + GF_y}, \qquad y'_t = -\frac{GF_t}{F_x + GF_y}. \tag{1.14.3.5}$$

Suppose that $F_x + GF_y \neq 0$. Then system (1.14.3.5) with the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0, \tag{1.14.3.6}$$

where t_0 is found from the algebraic (or transcendental) equation (1.14.3.4), can be solved numerically with, for example, the Runge–Kutta method (see Section 7.4.1 for relevant formulas). The solution will simultaneously be a solution to the original problem (1.14.3.1)–(1.14.3.2) in parametric form.

This method is much more effective than the first one.

Remark 1.43. The algebraic (or transcendental) equation (1.14.3.4) can generally have more than one different root. Then the original system (1.14.3.1)–(1.14.3.2) will have the same number of different solutions.

Remark 1.44. Sometimes, it is more convenient to replace the second equation in (1.14.3.1) with the equivalent equation $y'_x = G_1(x, y, t)$, where $G_1(x, y, t) = G(x, y, t) + F(x, y, t)H(x, y, t)$ and $H(x, y, t)$ is some function.

► Numerical integration of equations defined implicitly.

Let us look at the Cauchy problem for the equation defined implicitly as

$$F(x, y, y'_x) = 0 \tag{1.14.3.7}$$

subject to the initial conditions (1.14.3.2).

The substitution $y'_x = t$ reduces equation (1.14.3.7) to the parametric equation

$$F(x, y, t) = 0, \qquad y'_x = t \tag{1.14.3.8}$$

and initial conditions (1.14.3.2).

Problem (1.14.3.8), (1.14.3.2) is a special case of problem (1.14.3.1)–(1.14.3.2) with $G(x, y, t) = t$; it can be solved using the numerical methods described above.

Remark 1.45. For special forms of equation (1.14.3.7), it may be more convenient to change the definition of the parameter t. For example, in the implicit equation

$$F(y, y'_x + f(x,y)) = G(x,y),$$

the parameter can be introduced as $t = y'_x + f(x,y)$. This will result in the parametric equation

$$F(y,t) - G(x,y) = 0, \qquad t = y'_x + f(x,y).$$

► **Differential-algebraic equations.**

Parametrically defined nonlinear differential equations of the form (1.14.3.1) are a special class of coupled (DAEs for short). For numerical methods for DAEs other than those described above, see the books by Hairer, Lubich, and Roche (1989), Schiesser (1994), Hairer and Wanner (1996), Brenan, Campbell, and Petzold (1996), Ascher and Petzold (1998), an Rabier and Rheinboldt (2002).

1.14.4 Numerical Solution of Blow-Up Problems

► **Preliminary remarks. Blow-up solutions with a power-law singularity.**

There are problems whose solution tends to infinity at some finite value of the independent variable, $x = x_*$, which is unknown in advance. Such solutions exist only on the bounded interval $x_0 \le x < x_*$ and are called *blow-up solutions*. A practically important question arises in treating such problems: How can one determine the singular point x_* with numerical methods?

Example 1.48. Let us look at the model Cauchy problem for a separable ODE

$$y'_x = y^2 \quad (x > 0), \qquad y(0) = a, \tag{1.14.4.1}$$

where $a > 0$. The exact solution of this problem is

$$y = \frac{a}{1 - ax}. \tag{1.14.4.2}$$

It has a power-law singularity (a pole) at $x_* = 1/a$ and does not exist for $x > x_*$.

If we solve problem (1.14.4.1) using, for example, the first-order Euler method of polygonal lines with a constant step size h, we obtain a numerical solution which is positive, monotonically increases, and exists for arbitrarily large x_k. By the form of the numerical solution, it is impossible to conclude that the exact solution has a pole—it looks like the exact solution rapidly increases and exists for any $x > 0$. The same qualitative behavior is given by explicit high-order Runge–Kutta schemes. Furthermore, the standard implicit schemes also fail to determine the right qualitative behavior of solutions in such case.

In general, blow-up solutions with a power-law singularity can be represented in the vicinity of the singular point x_* as

$$y \approx A(x_* - x)^{-\mu}, \qquad \mu > 0,$$

where A is some constant. For blow-up solutions, we get $y(x_*) = \infty$.

Below we outline a few numerical methods for solving problems of the form (1.13.1.1)–(1.13.1.2) having blow-up solutions. We assume that $f(x,y) > 0$ for $x \ge x_0$ and $y \ge y_0 > 0$.

▶ Method based on the hodograph transformation.

For monotonic blow-up solutions, having performed the hodograph transformation, we will be solving the following Cauchy problem for $x = x(y)$:

$$x'_y = \frac{1}{f(x,y)} \quad (y > y_0), \qquad x(y_0) = x_0. \tag{1.14.4.3}$$

The computations will be carried out using the explicit fourth-order Runge–Kutta scheme. The existence of an asymptote $x = x_*$ can be numerically established at large y.

Example 1.49. In the model problem (1.14.4.1), the hodograph transformation results in the exact solution

$$x = \frac{1}{a} - \frac{1}{y}.$$

It satisfies the initial condition $x(a) = 0$, does not have singularities, monotonically increases for $y > a$, and tends to the limit value $x_* = \lim_{y\to\infty} x(y) = 1/a$.

Remark 1.46. This method is also suitable for the numerical integration of higher-order equations when dealing with Cauchy problems having blow-up solutions.

▶ Method based on the use of the differential variable $t = y'_x$.

Suppose that $f(x,y) \geq 0$. Let us introduce the auxiliary *differential variable* $t = y'_x$ and rewrite problem (1.13.1.1)–(1.13.1.2) in parametric form:

$$f(x,y) - t = 0, \qquad y'_x = t \quad (t > t_0); \tag{1.14.4.4}$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = f(x_0, y_0). \tag{1.14.4.5}$$

Using the results of Section 1.14.3, we treat these equations as system (1.14.3.5) with $F = f(x,y) - t$ and $G = t$ to obtain the Cauchy problem for the system of two first-order equations

$$x'_t = \frac{1}{f_x + t f_y}, \qquad y'_t = \frac{t}{f_x + t f_y} \quad (t > t_0) \tag{1.14.4.6}$$

subject to the initial conditions (1.14.4.5).

Assuming that the conditions $f_x + t f_y > 0$ at $t_0 < t < \infty$ hold, we solve problem (1.14.4.6), (1.14.4.5) numerically with, for example, the Runge–Kutta method (for the relevant formulas, see Section 7.4.1) or other standard numerical methods. In this case, no blow-up related difficulties will occur, since x'_t rapidly tends to zero as $t \to \infty$. The resulting solution will also be a solution to the original parametric problem (1.13.1.1)–(1.13.1.2). The upper bound of the existence domain of the solution, $x = x_*$, is determined numerically for sufficiently large t.

Example 1.50. In the model problem (1.14.4.1), the introduction of the new differential variable $t = y'_x$ leads to the following Cauchy problem for a system of equations:

$$x'_t = \frac{1}{2ty}, \qquad y'_t = \frac{1}{2y};$$
$$x(t_0) = 0, \quad y(t_0) = a, \quad t_0 = a^2.$$

The exact solution of this problem is

$$x = \frac{1}{a} - \frac{1}{\sqrt{t}}, \qquad y = \sqrt{t} \qquad (t \geq a). \tag{1.14.4.7}$$

It does not have singularities; the function $x = x(t)$ monotonically increases for $t > a$ and tends to the desired limit value $x_* = \lim_{t\to\infty} x(t) = 1/a$, while $y = y(t)$ monotonically increases without limit.

Example 1.51. In the more general Cauchy problem

$$y'_x = y^k, \qquad y(0) = a,$$

where $a > 0$ and $k > 1$, the introduction of the new variable $t = y'_x$ leads to a system whose solution is expressed as

$$x = \frac{1}{k-1}\left(a^{1-k} - t^{\frac{1-k}{k}}\right), \qquad y = t^{\frac{1}{k}} \qquad (t \geq a^k). \tag{1.14.4.8}$$

Its behavior is qualitatively similar to that of solution (1.14.4.7).

Remark 1.47. Solutions (1.14.4.7) and (1.14.4.8) approach their asymptotic value $x \to x_*$ as $t \to \infty$ quite slowly. To speed up the process, one can substitute $\exp(\lambda\tau)$ for t, with $\lambda > 0$, which is equivalent to introducing the new variable $\tau = \frac{1}{\lambda}\ln y'_x$.

► Method based on the arc length transformation and its modification.

In problems with a blow-up solution, $y = y(x)$, the right-hand side of equation (1.13.1.1), equal to $f(x, y)$ and determining the derivative y'_x, tends to infinity as $x \to x_*$. The fact that $f(x, y)$ becomes infinite at a finite value of the independent variable, x_*, which is unknown in advance, is the main reason for the failure of standard numerical methods.

1°. This issue can be avoided by replacing the original problem for a single equation (1.13.1.1)–(1.13.1.2) with an equivalent problem for a system of two equations:

$$x'_s = \frac{1}{\sqrt{1 + f^2(x, y)}}, \qquad y'_s = \frac{f(x, y)}{\sqrt{1 + f^2(x, y)}}; \qquad x(0) = x_0, \qquad y(0) = y_0. \tag{1.14.4.9}$$

Problem (1.14.4.9), unlike (1.13.1.1)–(1.13.1.2), does not have singularities. As $x \to x_*$, we have $f(x, y(x)) \to \infty$, and hence, $x'_s \to 0$ and $y'_s \to 1$. When obtained, the solution $x = x(s)$, $y = y(s)$ determines the solution of the original problem in parametric form.

Problem (1.14.4.9) can be solved numerically using, for example, the Runge–Kutta method (for the relevant formula, see Section 7.4.1).

Remark 1.48. The auxiliary variable s appearing in the autonomous system (1.14.4.9) is expressed in terms of the solution to the original problem as follows:

$$s = \int_{x_0}^{x} \sqrt{1 + f^2(x, y(x))}\, dx = \int_{x_0}^{x} \sqrt{1 + [y'_x(x)]^2}\, dx. \tag{1.14.4.10}$$

It has a clear geometrical meaning; specifically, s is the arc length of the desired curve $y = y(x)$ in the (x, y) plane, counted off from the initial point (x_0, y_0). The following limit property holds true: $s \to \infty$ as $x \to x_*$.

Relation (1.14.4.10) is called the *arc length transformation*.

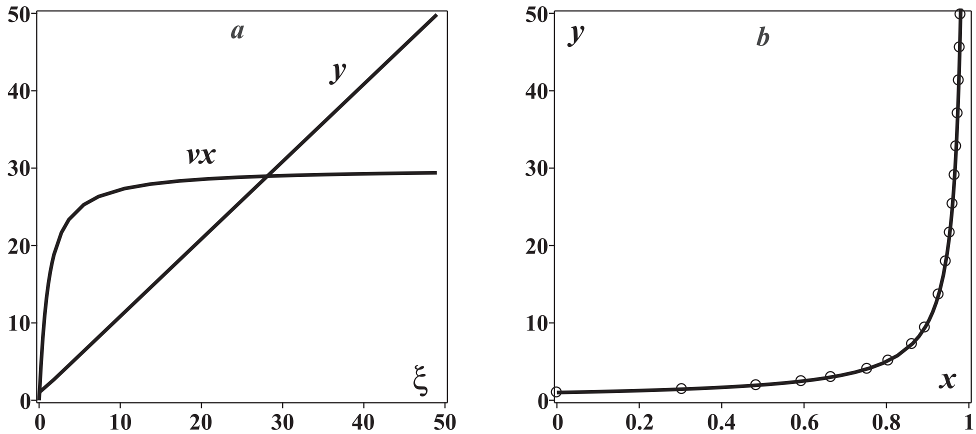

Figure 1.2: *a*, numerical solution $x = x(\xi)$, $y = y(\xi)$ of the Cauchy problem (1.14.4.11) with the scale factor $\nu = 30$; *b*, exact solution (1.14.4.2) with $a = 1$, solid line; numerical solution of problem (1.14.4.11), open circles.

Example 1.52. In the model problem (1.14.4.1) with $a = 1$, the equivalent problem for the system of equations (1.14.4.9) takes the form

$$x'_\xi = \frac{1}{\sqrt{1+y^4}}, \quad y'_\xi = \frac{y^2}{\sqrt{1+y^4}}; \qquad x(0) = 0, \quad y(0) = 1. \tag{1.14.4.11}$$

The second equation of this system is a separable equation whose solution is not expressed in terms of elementary functions.

Figure 1.2 shows a numerical solution of the Cauchy problem (1.14.4.11) in parametric form and compares the numerical solution with the exact solution (1.14.4.2).

2°. The above method allows various modifications. For the numerical solution, one can use, for example, the following simpler problem instead of (1.14.4.9):

$$\begin{gathered} x'_\tau = \frac{1}{1+|f(x,y)|}, \quad y'_\tau = \frac{f(x,y)}{1+|f(x,y)|}; \\ x(0) = x_0, \quad y(0) = y_0. \end{gathered} \tag{1.14.4.12}$$

It is equivalent to the original problem (1.13.1.1)–(1.13.1.2). (The modulus sign in the denominators is used for generality, since problem (1.14.4.12) can also be used in the case of $f < 0$ for the numerical investigation of problems having solutions with roots singularities; see Section 1.14.5.)

Example 1.53. In the model problem (1.14.4.1) with $a = 1$, the equivalent problem for the system of equations (1.14.4.12) results in the parametric solution

$$x = 1 + \tfrac{1}{2}\tau - \tfrac{1}{2}\sqrt{\tau^2+4}, \quad y = \tfrac{1}{2}\tau + \tfrac{1}{2}\sqrt{\tau^2+4} \qquad (\tau \ge 0).$$

This solution satisfies the initial conditions $x(0) = 0$ and $y(0) = 1$ and does not have singularities. The function $x(\tau)$ monotonically increases and tends to the desired limit value $x_* = \lim_{\tau\to\infty} x(\tau) = 1$. The function $y(\tau)$ monotonically increases and tends to infinity as $\tau \to \infty$; moreover, $\lim_{\tau\to\infty} y(\tau)/\tau = 1$.

▶ Method based on nonlocal transformations.

Introducing a new *nonlocal variable* by the formula

$$\xi = \int_{x_0}^{x} g(x,y)\,dx, \qquad y = y(x), \tag{1.14.4.13}$$

leads the Cauchy problem for one equation (1.13.1.1)–(1.13.1.2) to the equivalent problem for the autonomous system of equations

$$\begin{gathered} x'_\xi = \frac{1}{g(x,y)}, \qquad y'_\xi = \frac{f(x,y)}{g(x,y)} \qquad (\xi > 0); \\ x(0) = x_0, \qquad y(0) = y_0. \end{gathered} \tag{1.14.4.14}$$

The *regularizing function* $g = g(x,y)$ must satisfy the following conditions:

$$g > 0 \text{ if } x \ge x_0,\ y \ge y_0; \qquad g \to \infty \text{ as } y \to \infty; \qquad f/g = k \text{ as } y \to \infty, \tag{1.14.4.15}$$

where $k = \text{const} > 0$ (and the limiting case $k = \infty$ is also allowed); otherwise the function g can be chosen rather arbitrarily. It follows from (1.14.4.13) and the second condition (1.14.4.15) that $x'_\xi \to 0$ as $\xi \to \infty$.

A blow-up problem of the form (1.13.1.1)–(1.13.1.2) can be solved using the equivalent system (1.14.4.14). With this equivalent system, the unknown singular point, $x = x_*$, of the solution to the original problem (1.13.1.1)–(1.13.1.2) becomes the known point at infinity $\xi = \infty$ of system (1.14.4.14). The Cauchy problem (1.14.4.14) can be integrated numerically by applying the Runge–Kutta method or another standard numerical method.

Here are a few possible ways of how the regularizing function g in system (1.14.4.14) can be chosen.

1°. The special case $g = f$ is equivalent to the hodograph transformation with an additional translation of the dependent variable, which gives $\xi = y - y_0$.

2°. We can take $g = \bigl(c + |f|^s\bigr)^{1/s}$ with $c > 0$ and $s > 0$. In this case, $k = 1$ in (1.14.4.15). For $c = 1$ and $s = 2$, we get the *method of arc length transformation*.

3°. We can take $g = f/y$, which corresponds to $k = \infty$ in (1.14.4.15).

4°. For problems with non-monotonic blow-up solutions, a nonlocal transformation with $g = (1 + |f|)^{1/2}$ is more efficient than transformations with the functions of Item 2°; this regularizing function can be used for solutions having a pole of integer order at the blow-up point.

Remark 1.49. It follows from Items 1° and 2° that the method based on the hodograph transformation and the method of arc length transformation are special cases of the method based on a nonlocal transformation of general form.

Example 1.54. For the model problem (1.14.4.1), in which $f = y^2$, we take $g = f/y = y$ (see Item 3° above). By substituting these functions into (1.14.4.14), we arrive at the equivalent Cauchy problem

$$\begin{gathered} x'_\xi = \frac{1}{y}, \qquad y'_\xi = y \qquad (\xi > 0); \\ x(0) = 0, \qquad y(0) = a. \end{gathered} \tag{1.14.4.16}$$

Its exact solution is

$$x = \frac{1}{a}\bigl(1 - e^{-\xi}\bigr), \qquad y = ae^{\xi} \qquad (\xi \ge 0). \tag{1.14.4.17}$$

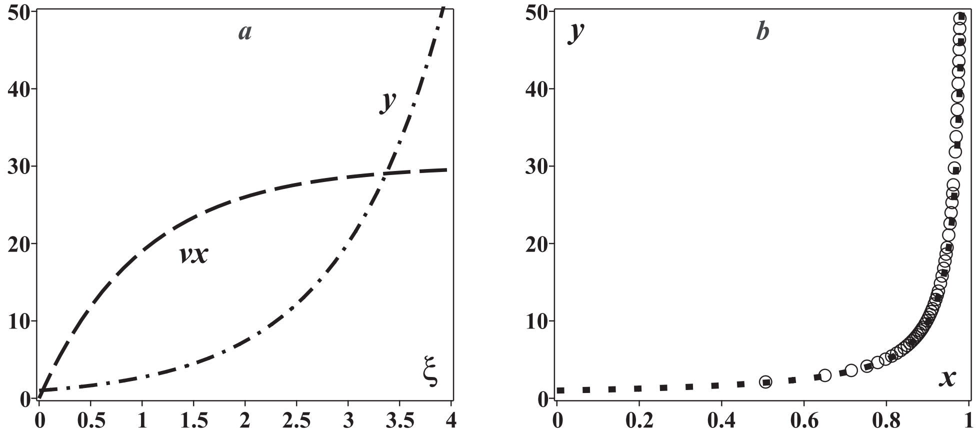

Figure 1.3: *a*, numerical solution $x = x(\xi)$, $y = y(\xi)$ of the Cauchy problem (1.14.4.16) with $a = 1$ ($\nu = 30$); *b*, exact solution (1.14.4.2) with $a = 1$, solid squares; numerical solution of the Cauchy problem (1.14.4.16), open circles.

This solution does have singularities. The function $x = x(\xi)$ monotonically increases for $\xi > 0$ and tends rapidly to the desired limit value $x_* = \lim_{\xi\to\infty} x(\xi) = 1/a$, while $y = y(\xi)$ monotonically exponentially increases with ξ.

Figure 1.3 shows a numerical solution of the Cauchy problem (1.14.4.16) in parametric form and compares the numerical solution with the exact solution (1.14.4.2).

Remark 1.50. The method based on the use of the special case of system (1.14.4.14) with $g = f/y$ (see Item 3°) is more efficient than the methods based on the hodograph transformation, arc length transformation, and differential variable $t = y'_x$.

► Method based on a special Rosenbrock scheme.

Another useful method for the numerical analysis of blow-up solutions is based on the *one-parameter Rosenbrock scheme*, which is defined by the formulas

$$y_{k+1} = y_k + h \operatorname{Re} \varphi_k, \qquad [1 - \alpha h f_y(x_{k+\frac{1}{2}}, y_k)]\varphi_k = f(x_{k+\frac{1}{2}}, y_k), \tag{1.14.4.18}$$

where α is a numerical (generally complex) parameter, f_y is the partial derivative of f with respect to y, $x_{k+\frac{1}{2}} = x_k + \frac{1}{2}h$, and $\operatorname{Re} \varphi_k$ is the real part of φ_k.

There is a *special complex scheme* from the family (1.14.4.18) that corresponds to $\alpha = \frac{1}{2}(1+i)$ with $i^2 = -1$,* which possesses some unique properties. The scheme is of second-order approximation in h; it is stable and monotonic on all linear problems, allows a simple generalization to systems of nonlinear ODEs (and PDEs), shows high reliability, and is suitable for stiff problems.

In explicit form, the special scheme (1.14.4.18) with $\alpha = \frac{1}{2}(1+i)$ is written as

$$y_{k+1} = y_k + h\frac{2f(x_{k+\frac{1}{2}}, y_k)[2 - hf_y(x_{k+\frac{1}{2}}, y_k)]}{[2 - hf_y(x_{k+\frac{1}{2}}, y_k)]^2 + h^2 f_y^2(x_{k+\frac{1}{2}}, y_k)}. \tag{1.14.4.19}$$

*This is a little-known scheme proposed by Rosenbrock (1963) and employed by Alshina, Kalitkin, and Koryakin (2005), whose results are used here (see also Kalitkin, Alshin, Alshina, and Rogov (2005)).

Example 1.55. Let us look at the more general differential equation than (1.14.4.1)

$$y'_x = by^\gamma. \tag{1.14.4.20}$$

For $b > 0$ and $\gamma > 1$, the exact solution of this equation has a pole:

$$y = A(x_* - x)^{-\beta}, \qquad A = (\beta/b)^\beta, \qquad \beta = \frac{1}{\gamma - 1} > 0. \tag{1.14.4.21}$$

If the solution is to satisfy the initial condition $y(0) = a$, the position of the pole is given by $x_* = (a/A)^{1-\gamma} = (a/A)^{-1/\beta}$.

For equation (1.14.4.20), the special scheme (1.14.4.21) becomes

$$y_{k+1} = y_k + 2hby_k^\gamma \frac{2 - hb\gamma y_k^{\gamma-1}}{(2 - hb\gamma y_k^{\gamma-1})^2 + (hb\gamma y_k^{\gamma-1})^2}. \tag{1.14.4.22}$$

Scheme (1.14.4.22) possesses the following properties:

Property 1. There is a value $y_* = [2/(hb\gamma)]^{\frac{1}{\gamma-1}}$ at which the numerical solution remains the same when it goes to the next step: $y_k = y_{k+1} = y_*$.

Property 2 (attraction property). If $y_{k+1} > y_*$, the increment of the function is negative ($y_{k+1} - y_k < 0$) and, conversely, if $y_{k+1} < y_*$, the increment of the function is positive ($y_{k+1} - y_k > 0$); this suggests that whatever y_k is, the special scheme (1.14.4.21) makes the next step toward the equilibrium $y_* = [2/(hb\gamma)]^{\frac{1}{\gamma-1}}$.

Property 3. If $1 < \gamma \le 2$, the numerical solution tends to its limit value y_* monotonically. For $\gamma > 2$, a nonmonotonicity near y_* is possible; however, each subsequent y_k comes closer and closer to y_*. On the whole, for $\gamma > 1$ and $b > 0$, the numerical solution obtained with the special Rosenbrock scheme tends to the limit y_*.

A solution obtained with the special scheme increases until it reaches the pole and then levels off at a constant value. The smaller the mesh increment the larger the height of the plateau. The scheme allows one to determine the position of the singular point x_* with a high accuracy. The article by Alshina, Kalitkin, and Koryakin (2005) also shows how to determine the degree of singularity β of the singular point numerically.

Remark 1.51. It is noteworthy that the qualitative behavior of a numerical blow-up solution to an equation of the form (1.14.4.20) with $b > 0$ and $\gamma > 1$ differs significantly between explicit and implicit Runge–Kutta schemes (explicit schemes of the first to fourth order were tested as well as the implicit Euler scheme). All explicit schemes give monotonically increasing solutions; the higher the order of approximations of the scheme, the faster the solution increases. Soon after the point x_* at which the solution has a pole is passed, arithmetic overflow occurs and further computation becomes impossible. This kind of qualitative behavior is extremely annoying, especially because it is uneasy to identify the cause of the overflow.

Implicit schemes have a different problem. At first, the solution increases and then, immediately before the pole, rapidly drops and becomes negative. In this situation, the calculation of the right-hand side of (1.14.4.20) for fractional γ becomes impossible, since raising a negative number in a fractional power is undefined.

Remark 1.52. Example 1.40 describes a much wider class of equations admitting blow-up solutions than (1.14.4.20).

Remark 1.53. Other numerical methods for blow-up solutions with their domains of applicability are discussed, for example, in Stuart and Floater (1990), Hirota and Ozawa (2006), and Dlamini and Khumalo (2012).

► Numerical solution of blow-up problems with logarithmic singularity.

There are blow-up problems that have a logarithmic singularity at the point x_*.

Example 1.56. Let us look at the model Cauchy problem for the separable ODE

$$y'_x = e^y \quad (x > 0), \qquad y(0) = a, \tag{1.14.4.23}$$

where $a > 0$. Its exact solution is

$$y = \ln \frac{e^a}{1 - e^a x}. \tag{1.14.4.24}$$

It has a logarithmic singularity at the point $x_* = e^{-a}$ and does not exist for $x > x_*$.

Problems with logarithmic singularities can usually be treated with the same methods as described above. Below are brief comments on the use of these methods for such problems.

Method based of the hodograph transformation. This method suggests that the original Cauchy problem (1.13.1.1)–(1.13.1.2) for $y = y(x)$ is replaced with the Cauchy problem (1.14.4.3) for $x = x(y)$.

Example 1.57. In the model problem (1.14.4.23), the hodograph transformation leads to the solution

$$x = e^{-a} - e^{-y},$$

which satisfies the initial condition $x(a) = 0$, does not have singularities, and monotonically increases for $y > a$ while tending to the desired limit value $x_* = \lim_{y\to\infty} x(y) = e^{-a}$.

Method based on the use of the differential variable $t = y'_x$. This method suggests the use of the new auxiliary variable $t = y'_x$ and replacement of the original problem (1.13.1.1)–(1.13.1.2) with the Cauchy problem for the system of two first-order equations (1.14.4.6) subject to the initial conditions (1.14.4.5).

Example 1.58. In the model problem (1.14.4.23), the introduction of the variable $t = y'_x$ leads to the following Cauchy problem for a system of two equations:

$$\begin{gathered} x'_t = e^{-y}/t, \quad y'_t = e^{-y}; \\ x(t_0) = 0, \quad y(t_0) = a, \quad t_0 = e^a. \end{gathered} \tag{1.14.4.25}$$

The exact solution of this problem is

$$x = \frac{1}{e^a} - \frac{1}{t}, \qquad y = \ln t \qquad (t \geq e^a).$$

It does not have singularities; the function $x = x(t)$ monotonically increases for $t > e^a$ and tends to the desired limit value $x_* = \lim_{t\to\infty} x(t) = e^{-a}$, while $y = y(t)$ increases monotonically and unboundedly with t.

Methods based on the arc length transformation or nonlocal transformations. These methods suggest that the original problem (1.13.1.1)–(1.13.1.2) is replaced with the equivalent Cauchy problem for the system of two first-order equations (1.14.4.9), (1.14.4.12), or (1.14.4.14).

Example 1.59. To the model problem (1.14.4.23) there corresponds the equivalent Cauchy problem (1.14.4.14) with $f(x, y) = e^y$ and $g = f/y = e^y/y$:

$$\begin{gathered} x'_t = y e^{-y}, \quad y'_t = y; \\ x(0) = 0, \quad y(0) = a. \end{gathered}$$

Its exact solution is

$$x = e^{-a} - \exp(-a e^t), \qquad y = a e^t \qquad (t \geq 0).$$

It does not have singularities; $x = x(t)$ monotonically increases for $t > 0$ and tends rapidly to the desired limit value $x_* = \lim_{t\to\infty} x(t) = e^{-a}$, while $y = y(t)$ increases monotonically and exponentially with t.

Method based on using the special Rosenbrock scheme. Scheme (1.14.4.19) can be used for the numerical analysis of blow-up solutions with logarithmic nonlinearity. Solutions obtained with this scheme level out (nonmonotonically) at $y_* = \ln(h/2)$, where h is the mesh increment.

Remark 1.54. The qualitative behavior of a numerical solution with a logarithmic nonlinearity is significantly different between explicit and implicit Runge–Kutta schemes in a similar way to that of solutions with a pole. Computation using explicit schemes results in overflow shortly after x_*, while computation based on implicit schemes is characterized by a sign change of the numerical solution.

1.14.5 Numerical Solution of Problems with Root Singularity

► Preliminary remarks. Solutions with a root singularity.

There are problems whose solutions exist, although limitedly, on a finite interval $x_0 \le x \le x_*$, where x_* is unknown in advance and $|y'_x(x_*)| = \infty$. In studying such problems, a practical question arises on how to determine, with numerical methods, the endpoint x_* as well as the solution near it.

Example 1.60. Let us look at the model Cauchy problem for the separable ODE

$$y'_x = -\frac{1}{2y} \quad (x > 0), \qquad y(0) = a, \tag{1.14.5.1}$$

where $a > 0$. Its exact solution is

$$y = \sqrt{a^2 - x}. \tag{1.14.5.2}$$

It is nonnegative and monotonically decreases from the initial value a at $x = 0$ to zero at $x_* = a^2$ and does not exist for $x > x_*$ (since the radicand becomes negative). Furthermore, very importantly, solution (1.14.5.2) has an infinite derivative at the finite point x_*.

If problem (1.14.4.1) is solved, for example, using the Euler method of polygonal lines, the resulting numerical solution will, at first, be positive and monotonically decreasing and then will become negative. The same qualitative behavior gives explicit Runge–Kutta schemes of high orders. The large errors of these methods near x_* are due to the infinite derivative at the endpoint x_* and the absence of solution for $x > x_*$.

In general, we will say that a solution has a *root singularity* at $x = x_*$ if the following approximate relation holds near this point:

$$y \approx A(x_* - x)^\mu, \qquad 0 < \mu < 1, \tag{1.14.5.3}$$

where A is some constant. For solutions with root singularities, we have $y'_x(x_*) = \infty$.

Example 1.61. The model equation (1.14.4.20) has solutions with a root singularity if the inequalities

$$b < 0, \qquad -\infty < \gamma < 0$$

hold. In this case, the *root singularity index* μ in (1.14.5.3) is linked to the parameter γ in (1.14.4.20) by the simple relation

$$\mu = \frac{1}{1 - \gamma}.$$

Below we briefly describe a few numerical methods for problems of the form (1.13.1.1)–(1.13.1.2) having solutions with a root singularity. To be specific, we assume that $f(x,y) \leq 0$ and $y_0 > 0$. Suppose that there are reasons to believe that the solution to the problem in question has a root singularity (for example, if there were strange problems with using explicit schemes).

► Method based on the hodograph transformation.

By making the hodograph transformation $x = x(y)$ followed by the change of variable $z = y_0 - y$, one reduces the Cauchy problem (1.14.4.3) to the form

$$x'_z = -\frac{1}{f(x, y_0 - z)} \quad (z > 0), \qquad x(0) = x_0. \tag{1.14.5.4}$$

The computation is carried out using, for example, a fourth-order explicit Runge–Kutta scheme. Problem (1.14.5.4) is solved starting from $z = 0$ and up until $z = y_0$. The endpoint of the existence domain is obtained as

$$x|_{z=y_0} = x_*. \tag{1.14.5.5}$$

Example 1.62. In the model problem (1.14.5.1), the hodograph transformation followed by the change of variable $z = a - y$ leads to the Cauchy problem

$$x'_z = 2(a - z) \quad (z > 0), \quad x(0) = 0.$$

Its exact solution is

$$x = 2az - z^2.$$

The function $x = x(z)$ does not have singularities, is infinitely differentiable, and increases monotonically for $0 \leq z < a$. The endpoint of the existence domain is determined by formula (1.14.5.5), $x_* = x|_{z=a} = a^2$.

Remark 1.55. This method is also suitable for higher-order equations when integrating Cauchy problems having solutions with a root singularity.

► Method based on the introduction of the new independent variable $t = -y'_x$.

Let $f(x,y) \leq 0$. We introduce the auxiliary variable $t = -y'_x$ and substitute it into problem (1.13.1.1)–(1.13.1.2) to obtain

$$f(x,y) + t = 0, \quad y'_x = -t \quad (t > t_0); \tag{1.14.5.6}$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = -f(x_0, y_0). \tag{1.14.5.7}$$

Then, taking advantage of the results of Section 1.14.3, we consider problem (1.14.3.5) with $F = f(x,y) + t$ and $G = -t$ to arrive at the Cauchy problem for the system of two first-order equations

$$x'_t = -\frac{1}{f_x - tf_y}, \quad y'_t = \frac{t}{f_x - tf_y} \quad (t > t_0) \tag{1.14.5.8}$$

subject to the initial conditions (1.14.5.7). Further, we solve problem (1.14.5.8), (1.14.5.7) numerically using, for example, the Runge–Kutta method (see Section 7.4.1 for relevant formulas). The resulting solution is a solution to the original problem (1.13.1.1)–(1.13.1.2) represented in parametric form. The endpoint of the existence domain of the solution, $x = x_*$, is determined numerically at sufficiently large t.

Example 1.63. In the model problem (1.14.5.1) with $a = 1$, the introduction of the variable $t = -y'_x$ leads to the following Cauchy problem for a system of equations:

$$x'_t = 2y^2/t, \quad y'_t = -2y^2 \quad (t > t_0);$$
$$x(t_0) = 0, \quad y(t_0) = 1, \quad t_0 = \tfrac{1}{2}.$$

Its exact solution is

$$x = 1 - \frac{1}{4t^2}, \quad y = \frac{1}{2t} \quad (t \geq \tfrac{1}{2}). \tag{1.14.5.9}$$

It does not have singularities; $x = x(t)$ monotonically increases for $t > \frac{1}{2}$ and tends to the desired limit value $x_* = \lim_{t\to\infty} x(t) = 1$, while $y = y(t)$ monotonically decreases with t so that $\lim_{t\to\infty} y(t) = 0$.

Example 1.64. In the more general Cauchy problem

$$y'_x = -y^{-k}, \quad y(0) = a,$$

where $a > 0$ and $k > 0$, the introduction of the new variable $t = -y'_x$ leads to a system of equations whose solution is given by

$$x = \frac{1}{k+1}\left(a^{k+1} - t^{-\frac{k+1}{k}}\right), \quad y = t^{-\frac{1}{k}} \quad (t \geq a^{-k}). \tag{1.14.5.10}$$

This solution has a similar qualitative behavior to that of solution (1.14.5.9).

► Method based on the use of an equivalent system of equations.

In problems having solutions $y = y(x)$ with a root singularity, the right-hand side of equation (1.13.1.1), equal to $f(x, y)$ and determining the derivative y'_x, tends to infinity as $x \to x_*$. The fact that $f(x, y)$ becomes infinite at an unknown finite value of the independent variable, x_*, is the main reason of failure of the standard numerical methods in solving problems whose solutions have a root singularity, just as in solving problems with blow-up solutions.

1°. This situation can be avoided if one replaces the original problem (1.13.1.1)–(1.13.1.2) with the equivalent Cauchy problem for the system of two first-order equations (1.14.4.9) or (1.14.4.12). It should be reminded that the computation must be carried out with respect to the new independent variable, s or τ, until $y = 0$, where the right-hand side of equation (1.13.1.1) becomes infinite.

Example 1.65. In the model problem (1.14.5.1) with $a = 1$, the equivalent problem for a system of equations (1.14.4.12) results in the parametric solution

$$x = \tau + \tfrac{1}{2}\sqrt{9 - 4\tau} - \tfrac{3}{2}, \quad y = \tfrac{1}{2}\sqrt{9 - 4\tau} - \tfrac{1}{2} \quad (0 \leq \tau \leq 2).$$

This solution satisfies the initial conditions $x(0) = 0$ and $y(0) = 1$ and does not have singularities for $0 \leq \tau \leq 2$. The function $y(\tau)$ monotonically increases and tends to the desired limit value $x_* = \lim_{\tau\to 2} x(\tau) = 1$.

2°. Apart from system (1.14.4.9) or (1.14.4.12), a number of other equivalent systems can be used to solve problems of the form (1.13.1.1)–(1.13.1.2) with a root singularity. In particular, if $f(x, y) < 0$ and $y(0) > 0$, the following Cauchy problem for a system of equations can be helpful:

$$x'_t = -\frac{y}{f(x, y)}, \quad y'_t = -y \quad (t > 0);$$
$$x(0) = x_0, \quad y(0) = y_0. \tag{1.14.5.11}$$

The unknown singular point $x = x_*$ of the solution to the original problem (1.13.1.1)–(1.13.1.2) becomes the known point at infinity $t = \infty$ of system (1.14.5.11), with $y = 0$ at $t = \infty$.

Example 1.66. To the model problem (1.14.5.1) there corresponds the equivalent Cauchy problem (1.14.5.11) with $f(x, y) = -(2y)^{-1}$:

$$x'_t = 2y^2, \quad y'_t = -y;$$
$$x(0) = 0, \quad y(0) = a.$$

Its exact solution is

$$x = a^2(1 - e^{-2t}), \quad y = ae^{-t} \qquad (t \geq 0).$$

This solution does not have singularities; $x = x(t)$ monotonically increases for $t > 0$ and tends rapidly to the desired limit value $x_* = \lim_{t\to\infty} x(t) = a^2$, while $y = y(t)$ monotonically exponentially decreases with t so that $\lim_{t\to\infty} y(t) = 0$.

► Method based on the use of a special Rosenbrock scheme.

The special Rosenbrock scheme (1.14.4.19) can also help in the solution of problems with root singularities. When used with a small step size, this scheme can cause oscillatory nonmonotonicity of the solution x_*, with the first local minimum, $y_{1\mathrm{m}}$ approximately determining the endpoint of the existence domain: $y(x_*) \approx y_{1\mathrm{m}}$. The article by Alshina, Kalitkin, and Koryakin (2005) describes a technique that allows finding the index of the root singularity numerically.

Remark 1.56. Numerical solutions obtained with explicit Runge–Kutta schemes intersects the tangent at the singular point x_* (coinciding with the asymptote of the derivative) at a $y > 0$ and, immediately after x_*, the computation breaks down to negative values. Computations using the implicit Euler scheme also fail resulting in negative values, after which the computation becomes impossible.

⊙ *Literature for Section 1.14*: H. H. Rosenbrock (1963), N. N. Kalitkin (1978), S. Moriguti, C. Okuno, R. Suekane, M. Iri, and K. Takeuchi (1979), M. Stuart and M. S. Floater (1990), U. M. Ascher and L. R. Petzold (1998), G. Acosta, G. Durán, and J. D. Rossi (2002), E. A. Alshina, N. N. Kalitkin, and P. V. Koryakin (2005), N. N. Kalitkin, A. B. Alshin, E. A. Alshina, and B. V. Rogov (2005), C. Hirota and R. Ozawa (2006), P. G. Dlamini and M. Khumalo (2012), A. Takayasu, K. Matsue, T. Sasaki, K. Tanaka, M. Mizuguchi, and S. Oishi (2017), A. D. Polyanin and A. I. Zhurov (2017b), A. D. Polyanin and I. K. Shingareva (2017a,b,c,d,e).

Chapter 2

Methods for Second-Order Linear Differential Equations

2.1 Homogeneous Linear Equations

2.1.1 Formulas for the General Solution. Wronskian Determinant

► General form of a homogeneous linear equation.

Consider a second-order homogeneous linear equation in the general form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{2.1.1.1}$$

The *trivial solution*, $y = 0$, is a particular solution of the homogeneous linear equation.

► Two particular solutions are known.

Let $y_1(x)$, $y_2(x)$ be a fundamental system of solutions (nontrivial linearly independent particular solutions) of equation (2.1.1.1). Then the general solution is given by

$$y = C_1 y_1(x) + C_2 y_2(x), \tag{2.1.1.2}$$

where C_1 and C_2 are arbitrary constants.

► One particular solution is known.

Let $y_1 = y_1(x)$ be any nontrivial particular solution of equation (2.1.1.1). Then its general solution can be represented as

$$y = y_1\left(C_1 + C_2\int\frac{e^{-F}}{y_1^2}\,dx\right), \qquad \text{where} \quad F = \int\frac{f_1}{f_2}\,dx. \tag{2.1.1.3}$$

► General solution of an equation of the canonical form.

Consider the equation

$$y''_{xx} + f(x)y = 0,$$

which is written in the canonical form; see Section 2.1.2 for the reduction of equations to this form. Let $y_1 = y_1(x)$ be any nontrivial partial solution of this equation. The general solution can be constructed by formula (2.1.1.3) with $F = 0$ or formula (2.1.1.2) in which

$$y_2(x) = y_1 \int \frac{[f(x)-1][y_1^2 - (y_1')^2]}{[y_1^2 + (y_1')^2]^2}\,dx + \frac{y_1'}{y_1^2 + (y_1')^2}.$$

Here the prime denotes differentiation with respect to x. The last formula is suitable where y_1 vanishes at some points.

► Special properties of some solutions.

1°. Suppose $y = C_1 f(x)[g(x)]^a + C_2 f(x)[g(x)]^b$ is the general solution of the homogeneous linear equation with $a \neq b$, where a and b are free parameters. Then the function $y = C_1 f(x)[g(x)]^a + C_2 f(x)[g(x)]^a \ln g(x)$ will be the general solution of this equation with $a = b$.

2°. Suppose a particular solution of a homogeneous linear equation is obtained in the closed form $y = [f(x)]^a$, with this formula valid for $f(x) \geq 0$. If the equation makes sense in a range of x where $f(x) < 0$, then the function $y = |f(x)|^a$ will be a particular solution of the equation in that range.

► Constant-coefficient linear equation.

The second-order constant-coefficient linear equation

$$y''_{xx} + ay'_x + by = 0 \qquad (2.1.1.4)$$

has the following fundamental system of solutions:

$$y_1(x) = \exp\left(-\tfrac{1}{2}ax\right)\sinh\left(\tfrac{1}{2}x\sqrt{a^2-4b}\right), \quad y_2(x) = \exp\left(-\tfrac{1}{2}ax\right)\cosh\left(\tfrac{1}{2}x\sqrt{a^2-4b}\right) \quad \text{if } a^2 > 4b;$$

$$y_1(x) = \exp\left(-\tfrac{1}{2}ax\right)\sin\left(\tfrac{1}{2}x\sqrt{4b-a^2}\right), \quad y_2(x) = \exp\left(-\tfrac{1}{2}ax\right)\cos\left(\tfrac{1}{2}x\sqrt{4b-a^2}\right) \quad \text{if } a^2 < 4b;$$

$$y_1(x) = \exp\left(-\tfrac{1}{2}ax\right), \quad y_2(x) = x\exp\left(-\tfrac{1}{2}ax\right) \quad \text{if } a^2 = 4b.$$

► Euler equation.

The *Euler equation*

$$x^2 y''_{xx} + axy'_x + by = 0$$

is reduced by the change of variable $x = ke^t$ $(k \neq 0)$ to the second-order constant-coefficient linear equation $y''_{tt} + (a-1)y'_t + by = 0$, see Eq. (2.1.1.4).

◆ *Solutions to some other second-order linear equations can be found in Section 14.1.*

► Wronskian determinant and Liouville's formula.

The *Wronskian determinant* (or *Wronskian*) is defined by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv y_1(y_2)'_x - y_2(y_1)'_x,$$

where $y_1(x)$, $y_2(x)$ is a fundamental system of solutions of equation (2.1.1.1).

Liouville's formula:

$$W(x) = W(x_0) \exp\left[-\int_{x_0}^{x} \frac{f_1(t)}{f_2(t)}\,dt\right].$$

2.1.2 Factorization and Some Transformations

► **Factorization.**

1°. Let $y_1 = y_1(x)$ be any nontrivial particular solution of equation (2.1.1.1). Then the equation can be factored as

$$L_2 L_1 y = 0, \tag{2.1.2.1}$$

where

$$L_1 = y_1 \frac{d}{dx} - y_1', \qquad L_2 = \frac{1}{y_1}\left(\frac{d}{dx} + \frac{f_1}{f_2}\right).$$

2°. Equation (2.1.1.1) also admits a more general factorization in the form (2.1.2.1) with

$$L_1 = \frac{1}{\psi}\left(\frac{d}{dx} - \frac{y_1'}{y_1}\right), \qquad L_2 = \psi\left(\frac{d}{dx} + \frac{y_1'}{y_1} + \frac{\psi'}{\psi} + \frac{f_1}{f_2}\right),$$

where $y_1 = y_1(x)$ is any nontrivial particular solution of the equation and $\psi = \psi(x)$ is an arbitrary function (the special case $\psi = 1/y_1$ coincides with Item 1°).

Remark 2.1. The factorization (2.1.2.1) of equation (2.1.1.1), with L_1 and L_2 being some first-order differential operators, is equivalent in complexity to seeking a nontrivial particular solution of the equation.

► **Reduction to the canonical form.**

1°. The substitution

$$y = u(x) \exp\left(-\frac{1}{2}\int \frac{f_1}{f_2}\,dx\right) \tag{2.1.2.2}$$

brings equation (2.1.1.1) to the canonical (or normal) form

$$u''_{xx} + f(x)u = 0, \qquad \text{where} \quad f = \frac{f_0}{f_2} - \frac{1}{4}\left(\frac{f_1}{f_2}\right)^2 - \frac{1}{2}\left(\frac{f_1}{f_2}\right)'_x. \tag{2.1.2.3}$$

2°. The substitution (2.1.2.2) is a special case of the more general transformation (φ is an arbitrary function)

$$x = \varphi(\xi), \quad y = u(\xi)\sqrt{|\varphi'_\xi(\xi)|}\, \exp\left(-\frac{1}{2}\int \frac{f_1(\varphi)}{f_2(\varphi)}\,d\varphi\right),$$

which also brings the original equation to the canonical form.

► **Reduction to the Riccati equation.**

The substitution $u = y'_x/y$ brings the second-order homogeneous linear equation (2.1.1.1) to the Riccati equation:

$$f_2(x)u'_x + f_2(x)u^2 + f_1(x)u + f_0(x) = 0,$$

which is discussed in Section 1.4.

► **Reduction to a constant-coefficient equation (a special case).**

Let $f_2 = 1$, $f_0 \neq 0$, and the condition

$$\frac{1}{|f_0|}\frac{d}{dx}\sqrt{|f_0|} + \frac{f_1}{\sqrt{|f_0|}} = a = \text{const}$$

be satisfied. Then the substitution $\xi = \int \sqrt{|f_0|}\,dx$ leads to a constant-coefficient linear equation,

$$y''_{\xi\xi} + ay'_\xi + y\,\text{sign}\,f_0 = 0.$$

⊙ *Literature for Section 2.1*: G. M. Murphy (1960), E. Kamke (1977), D. Zwillinger (1997), S. Yu. Dobrokhotov (1998), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

2.2 Nonhomogeneous Linear Equations

2.2.1 Existence Theorem. Kummer–Liouville Transformation

► **Existence and uniqueness theorem.**

A second-order nonhomogeneous linear equation has the form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x). \qquad (2.2.1.1)$$

EXISTENCE AND UNIQUENESS THEOREM. *On an open interval $a < x < b$, let the functions f_2, f_1, f_0, and g be continuous and $f_2 \neq 0$. Also let*

$$y(x_0) = A, \quad y'_x(x_0) = B$$

be arbitrary initial conditions, where x_0 is any point such that $a < x_0 < b$, and A and B are arbitrary prescribed numbers. Then a solution of equation (2.2.1.1) exists and is unique. This solutions is defined for all $x \in (a, b)$.

► **Kummer–Liouville transformation.**

The transformation

$$x = \alpha(t), \quad y = \beta(t)z + \gamma(t), \qquad (2.2.1.2)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are arbitrary sufficiently smooth functions ($\beta \not\equiv 0$), takes any linear differential equation for $y(x)$ to a linear equation for $z = z(t)$. In the special case $\gamma \equiv 0$, a homogeneous equation is transformed to a homogeneous one.

Special cases of transformation (2.2.1.2) are widely used to simplify second- and higher-order linear differential equations.

2.2.2 Formulas for the General Solution

► **Representation of the general solution as the sum of two solutions.**

The general solution of the nonhomogeneous linear equation (2.2.1.1) is the sum of the general solution of the corresponding homogeneous equation (2.1.1.1) and any particular solution of the nonhomogeneous equation (2.2.1.1).

► **Two particular solutions are known.**

Let $y_1 = y_1(x)$, $y_2 = y_2(x)$ be a fundamental system of solutions of the corresponding homogeneous equation, with $g \equiv 0$. Then the general solution of equation (2.2.1.1) can be represented as

$$y = C_1 y_1 + C_2 y_2 + y_2 \int y_1 \frac{g}{f_2} \frac{dx}{W} - y_1 \int y_2 \frac{g}{f_2} \frac{dx}{W}, \qquad (2.2.2.1)$$

where $W = y_1(y_2)'_x - y_2(y_1)'_x$ is the Wronskian determinant.

► **One particular solution is known.**

Given a nontrivial particular solution $y_1 = y_1(x)$ of the homogeneous equation (with $g \equiv 0$), a second particular solution $y_2 = y_2(x)$ can be calculated from the formula

$$y_2 = y_1 \int \frac{e^{-F}}{y_1^2}\, dx, \qquad \text{where} \quad F = \int \frac{f_1}{f_2}\, dx, \quad W = e^{-F}. \qquad (2.2.2.2)$$

Then the general solution of equation (2.2.1.1) can be constructed by (2.2.2.1).

► **A property of nonhomogeneous linear ODEs.**

Let $\bar{y}_1$ and $\bar{y}_2$ be respective solutions of the nonhomogeneous linear differential equations $\mathrm{L}\,[\bar{y}_1] = g_1(x)$ and $\mathrm{L}\,[\bar{y}_2] = g_2(x)$, which have the same left-hand side but different right-hand sides, where $\mathrm{L}\,[y]$ is the left-hand side of equation (2.2.1.1). Then the function $\bar{y} = \bar{y}_1 + \bar{y}_2$ is a solution of the equation $\mathrm{L}\,[\bar{y}] = g_1(x) + g_2(x)$.

⊙ *Literature for Section 2.2*: G. M. Murphy (1960), E. Kamke (1977), D. Zwillinger (1997), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

2.3 Representation of Solutions as a Series in the Independent Variable

2.3.1 Equation Coefficients are Representable in the Ordinary Power Series Form

Let us consider a homogeneous linear differential equation of the general form

$$y''_{xx} + f(x)y'_x + g(x)y = 0. \qquad (2.3.1.1)$$

Assume that the functions $f(x)$ and $g(x)$ are representable, in the vicinity of a point $x = x_0$, in the power series form,

$$f(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \qquad g(x) = \sum_{n=0}^{\infty} B_n (x - x_0)^n, \qquad (2.3.1.2)$$

on the interval $|x - x_0| < R$, where R stands for the minimum radius of convergence of the two series in (2.3.1.2). In this case, the point $x = x_0$ is referred to as an *ordinary point*, and equation (2.3.1.1) possesses two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n. \tag{2.3.1.3}$$

The coefficients a_n and b_n are determined by substituting the series (2.3.1.2) into equation (2.3.1.1) followed by extracting the coefficients of like powers of $(x - x_0)$.*

2.3.2 Equation Coefficients Have Poles at Some Point

Assume that the functions $f(x)$ and $g(x)$ are representable, in the vicinity of a point $x = x_0$, in the form

$$f(x) = \sum_{n=-1}^{\infty} A_n (x - x_0)^n, \qquad g(x) = \sum_{n=-2}^{\infty} B_n (x - x_0)^n, \tag{2.3.2.1}$$

on the interval $|x - x_0| < R$. In this case, the point $x = x_0$ is referred to as a *regular singular point*.

Let λ_1 and λ_2 be roots of the quadratic equation

$$\lambda^2 + (A_{-1} - 1)\lambda + B_{-2} = 0, \tag{2.3.2.2}$$

where A_{-1} and B_{-2} are the leading terms in formulas (2.3.2.1) at $x \to x_0$. There are three cases, depending on the values of the exponents of the singularity.

1°. *Case $\lambda_1 \neq \lambda_2$ and $\lambda_1 - \lambda_2$ is not an integer.*

Equation (2.3.1.1) has two linearly independent solutions of the form

$$\begin{aligned} y_1(x) &= |x - x_0|^{\lambda_1} \Bigl[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\Bigr], \\ y_2(x) &= |x - x_0|^{\lambda_2} \Bigl[1 + \sum_{n=1}^{\infty} b_n (x - x_0)^n\Bigr]. \end{aligned} \tag{2.3.2.3}$$

2°. *Case $\lambda_1 = \lambda_2 = \lambda$.*

Equation (2.3.1.1) possesses two linearly independent solutions:

$$\begin{aligned} y_1(x) &= |x - x_0|^{\lambda} \Bigl[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\Bigr], \\ y_2(x) &= y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda} \sum_{n=0}^{\infty} b_n (x - x_0)^n. \end{aligned} \tag{2.3.2.4}$$

3°. *Case $\lambda_1 = \lambda_2 + N$, where N is a positive integer.*

*Prior to that, the terms containing the same powers $(x - x_0)^k$, $k = 0, 1, \ldots$, should be collected.

Equation (2.3.1.1) has two linearly independent solutions of the form

$$
\begin{aligned}
y_1(x) &= |x-x_0|^{\lambda_1}\Big[1+\sum_{n=1}^{\infty} a_n(x-x_0)^n\Big],\\
y_2(x) &= k y_1(x)\ln|x-x_0| + |x-x_0|^{\lambda_2}\sum_{n=0}^{\infty} b_n(x-x_0)^n,
\end{aligned}
\tag{2.3.2.5}
$$

where k is a constant to be determined (it may be equal to zero). If $k \neq 0$, then we can set $k = 1$ without loss of generality.

To construct the solution in each of the three cases, the following procedure should be performed: substitute the above expressions of y_1 and y_2 into the original equation (2.3.1.1) and equate the coefficients of $(x-x_0)^n$ and $(x-x_0)^n \ln|x-x_0|$ for different values of n to obtain recurrence relations for the unknown coefficients. From these recurrence relations the solution sought can be found.

Example 2.1. The Bessel equation

$$
x^2 y''_{xx} + x y'_x + (x^2 - \nu^2) y = 0 \tag{2.3.2.6}
$$

is a special case of equation (2.3.1.1) with the functions of the form (2.3.2.1), where

$$
f(x) = \frac{1}{x}, \quad g(x) = -\frac{\nu^2}{x^2} + 1, \quad x_0 = 0.
$$

Therefore $A_{-1} = 1$ and $B_{-2} = -\nu^2$, and the quadratic equation (2.3.2.2) has the form

$$
\lambda^2 - \nu^2 = 0. \tag{2.3.2.7}
$$

The roots of the equation are $\lambda_1 = \nu$ and $\lambda_2 = -\nu$.

1°. If $\lambda_1 - \lambda_2 = 2\nu$ is not an arbitrary integer, then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.3).

2°. If $\nu = \lambda_1 = \lambda_2 = 0$, then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.4).

3°. If $\lambda_1 - \lambda_2 = 2\nu$ is an arbitrary integer, then there are two cases, depending on the values ν.

3.1. Case ν is an arbitrary integer (i.e., $\lambda_1 - \lambda_2$ is an even number). Then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.5) with $k = 1$.

3.2. Case $\nu = n + \frac{1}{2}$, where $n = 0, 1, 2, \dots$ (i.e., $\lambda_1 - \lambda_2$ is an odd number). Then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.5) with $k = 0$.

For more detailed information on solutions to the Bessel equation (2.3.2.6), see also Section 14.1.2 (Eq. 126) and Section S4.6.

⊙ *Literature for Section 2.3*: G. M. Murphy (1960), E. Kamke (1977), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003).

2.4 Asymptotic Solutions

This section presents asymptotic solutions, as $\varepsilon \to 0$ ($\varepsilon > 0$), of some second-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being real.

2.4.1 Equations Not Containing y'_x

► **Leading asymptotic terms.**

Consider the equation

$$\varepsilon^2 y''_{xx} - f(x)y = 0 \tag{2.4.1.1}$$

on a closed interval $a \le x \le b$.

Case 1. With the condition $f \neq 0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to 0$, are given by the formulas

$$y_1 = f^{-1/4} \exp\Bigl(-\frac{1}{\varepsilon}\int \sqrt{f}\,dx\Bigr), \qquad y_2 = f^{-1/4} \exp\Bigl(\frac{1}{\varepsilon}\int \sqrt{f}\,dx\Bigr) \qquad \text{if } f > 0,$$

$$y_1 = (-f)^{-1/4} \cos\Bigl(\frac{1}{\varepsilon}\int \sqrt{-f}\,dx\Bigr), \quad y_2 = (-f)^{-1/4} \sin\Bigl(\frac{1}{\varepsilon}\int \sqrt{-f}\,dx\Bigr) \quad \text{if } f < 0.$$

Case 2. Discuss the asymptotic solution of equation (2.4.1.1) in the vicinity of the point $x = x_0$, where function $f(x)$ vanishes, $f(x_0) = 0$ (such a point is referred to as a *transition point*). We assume that the function f can be presented in the form

$$f(x) = (x_0 - x)\psi(x), \qquad \text{where} \quad \psi(x) > 0.$$

In this case, the fundamental solutions, as $\varepsilon \to 0$, are described by three different formulas:

$$y_1 = \begin{cases} \dfrac{1}{|f(x)|^{1/4}} \sin\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_{x_0}^{x} \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x - x_0 \ge \delta, \\ \dfrac{\sqrt{\pi}}{[\varepsilon\psi(x_0)]^{1/6}} \operatorname{Ai}(z) & \text{if } |x - x_0| \le \delta, \\ \dfrac{1}{2[f(x)]^{1/4}} \exp\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_{x}^{x_0} \sqrt{f(x)}\,dx\Bigr] & \text{if } x_0 - x \ge \delta, \end{cases}$$

$$y_2 = \begin{cases} \dfrac{1}{|f(x)|^{1/4}} \cos\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_{x_0}^{x} \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x - x_0 \ge \delta, \\ \dfrac{\sqrt{\pi}}{[\varepsilon\psi(x_0)]^{1/6}} \operatorname{Bi}(z) & \text{if } |x - x_0| \le \delta, \\ \dfrac{1}{[f(x)]^{1/4}} \exp\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_{x}^{x_0} \sqrt{f(x)}\,dx\Bigr] & \text{if } x_0 - x \ge \delta, \end{cases}$$

where $\operatorname{Ai}(z)$ and $\operatorname{Bi}(z)$ are the Airy functions of the first and second kind, respectively (see Section S4.8), $z = \varepsilon^{-2/3}[\psi(x_0)]^{1/3}(x_0 - x)$, and $\delta = O(\varepsilon^{2/3})$.

► **Two-term asymptotic expansions.**

The two-term asymptotic expansions of the solution of equation (2.4.1.1) with $f > 0$, as $\varepsilon \to 0$, on a closed interval $a \le x \le b$, has the form

$$y_1 = f^{-1/4} \exp\Bigl(-\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{f}\,dx\Bigr)\Bigl\{1 - \varepsilon I(x) + O(\varepsilon^2)\Bigr\},$$

$$y_2 = f^{-1/4} \exp\Bigl(\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{f}\,dx\Bigr)\Bigl\{1 + \varepsilon I(x) + O(\varepsilon^2)\Bigr\}, \tag{2.4.1.2}$$

$$I(x) = \int_{x_0}^{x} \Bigl[\frac{1}{8}\frac{f''_{xx}}{f^{3/2}} - \frac{5}{32}\frac{(f'_x)^2}{f^{5/2}}\Bigr]\,dx,$$

where x_0 is an arbitrary number satisfying the inequality $a \le x_0 \le b$.

The asymptotic expansions of the fundamental system of solutions of equation (2.4.1.1) with $f<0$ are derived by separating the real and imaginary parts in either formula (2.4.1.2).

► Equations of the special form.

Consider the equation

$$\varepsilon^2 y''_{xx} - x^{m-2} f(x) y = 0 \qquad (2.4.1.3)$$

on a closed interval $a \le x \le b$, where $a<0$ and $b>0$, under the conditions that m is a positive integer and $f(x) \ne 0$. In this case, the leading term of the asymptotic solution, as $\varepsilon \to 0$, in the vicinity of the point $x=0$ is expressed in terms of a simpler model equation, which results from substituting the function $f(x)$ in equation (2.4.1.3) by the constant $f(0)$ (the solution of the model equation is expressed in terms of the Bessel functions of order $1/m$).

We specify below formulas by which the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2.4.1.3) with $a<x<0$ and $0<x<b$ are related (excluding a small vicinity of the point $x=0$). Three different cases can be extracted.

1°. Let m be an even integer and $f(x)>0$. Then,

$$y_1 = \begin{cases} [f(x)]^{-1/4} \exp\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x<0, \\ k^{-1}[f(x)]^{-1/4} \exp\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x>0, \end{cases}$$

$$y_2 = \begin{cases} [f(x)]^{-1/4} \exp\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x<0, \\ k[f(x)]^{-1/4} \exp\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x>0, \end{cases}$$

where $f=f(x)$, $k=\sin\Bigl(\dfrac{\pi}{m}\Bigr)$.

2°. Let m be an even integer and $f(x)<0$. Then,

$$y_1 = \begin{cases} |f(x)|^{-1/4} \cos\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x<0, \\ k^{-1}|f(x)|^{-1/4} \cos\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\Bigr] & \text{if } x>0, \end{cases}$$

$$y_2 = \begin{cases} |f(x)|^{-1/4} \cos\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\Bigr] & \text{if } x<0, \\ k|f(x)|^{-1/4} \cos\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x>0, \end{cases}$$

where $f=f(x)$, $k=\tan\Bigl(\dfrac{\pi}{2m}\Bigr)$.

3°. Let m be an odd integer. Then,

$$y_1 = \begin{cases} |f(x)|^{-1/4} \cos\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x<0, \\ \dfrac{1}{2}k^{-1}[f(x)]^{-1/4} \exp\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x>0, \end{cases}$$

$$y_2 = \begin{cases} |f(x)|^{-1/4} \cos\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\Bigr] & \text{if } x < 0, \\ k[f(x)]^{-1/4} \exp\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x > 0, \end{cases}$$

where $f = f(x)$, $k = \sin\Bigl(\dfrac{\pi}{2m}\Bigr)$.

► **Equation coefficients are dependent on ε.**

Consider an equation of the form

$$\varepsilon^2 y''_{xx} - f(x,\varepsilon)y = 0 \tag{2.4.1.4}$$

on a closed interval $a \le x \le b$ under the condition that $f \ne 0$. Assume that the following asymptotic relation holds:

$$f(x,\varepsilon) = \sum_{k=0}^{\infty} f_k(x)\varepsilon^k, \qquad \varepsilon \to 0.$$

Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2.4.1.4) are given by the formulas

$$y_1 = f_0^{-1/4}(x)\exp\Bigl[-\frac{1}{\varepsilon}\int\sqrt{f_0(x)}\,dx + \frac{1}{2}\int\frac{f_1(x)}{\sqrt{f_0(x)}}\,dx\Bigr]\bigl[1 + O(\varepsilon)\bigr],$$
$$y_2 = f_0^{-1/4}(x)\exp\Bigl[\frac{1}{\varepsilon}\int\sqrt{f_0(x)}\,dx + \frac{1}{2}\int\frac{f_1(x)}{\sqrt{f_0(x)}}\,dx\Bigr]\bigl[1 + O(\varepsilon)\bigr].$$

2.4.2 Equations Containing y'_x

► **Equations of a special form.**

1°. Consider an equation of the form

$$\varepsilon y''_{xx} + g(x)y'_x + f(x)y = 0$$

on a closed interval $0 \le x \le 1$. With $g(x) > 0$, the asymptotic solution of this equation, satisfying the boundary conditions $y(0) = C_1$ and $y(1) = C_2$, can be represented in the form

$$y = (C_1 - kC_2)\exp\bigl[-\varepsilon^{-1}g(0)x\bigr] + C_2\exp\Bigl[\int_x^1 \frac{f(x)}{g(x)}\,dx\Bigr] + O(\varepsilon),$$

where $k = \exp\Bigl[\displaystyle\int_0^1 \frac{f(x)}{g(x)}\,dx\Bigr]$.

2°. Now let us take a look at an equation of the form

$$\varepsilon^2 y''_{xx} + \varepsilon g(x)y'_x + f(x)y = 0 \tag{2.4.2.1}$$

on a closed interval $a \le x \le b$. Assume

$$D(x) \equiv [g(x)]^2 - 4f(x) \ne 0.$$

Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2.4.2.1), as $\varepsilon \to 0$, are expressed by

$$y_1 = |D(x)|^{-1/4} \exp\Bigl[-\frac{1}{2\varepsilon}\int \sqrt{D(x)}\,dx - \frac{1}{2}\int \frac{g'_x(x)}{\sqrt{D(x)}}\,dx\Bigr]\bigl[1+O(\varepsilon)\bigr],$$
$$y_2 = |D(x)|^{-1/4} \exp\Bigl[\frac{1}{2\varepsilon}\int \sqrt{D(x)}\,dx - \frac{1}{2}\int \frac{g'_x(x)}{\sqrt{D(x)}}\,dx\Bigr]\bigl[1+O(\varepsilon)\bigr].$$

► **Equations of the general form.**

The more general equation

$$\varepsilon^2 y''_{xx} + \varepsilon g(x,\varepsilon) y'_x + f(x,\varepsilon) y = 0$$

is reducible, with the aid of the substitution $y = w \exp\Bigl(-\dfrac{1}{2\varepsilon}\displaystyle\int g\,dx\Bigr)$, to an equation of the form (2.4.1.4),

$$\varepsilon^2 w''_{xx} + \bigl(f - \tfrac{1}{4}g^2 - \tfrac{1}{2}\varepsilon g'_x\bigr) w = 0,$$

which can be solved using the asymptotic formulas given above.

⊙ *Literature for Section 2.4*: W. Wasov (1965), F. W. J. Olver (1974), A. H. Nayfeh (1973, 1981), M. V. Fedoryuk (1993), A. D. Polyanin and V. F. Zaitsev (2003).

2.5 Boundary Value Problems. Green's Function

2.5.1 First, Second, Third, and Some Other Boundary Value Problems

We consider the second-order nonhomogeneous linear differential equation

$$f_2(x) y''_{xx} + f_1(x) y'_x + f_0(x) y = g(x) \qquad (2.5.1.1)$$

on a bounded interval $x_1 < x < x_2$. We assume that $f_2(x) \neq 0$.

► **First boundary value problem.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *first-type boundary conditions* (or *Dirichlet conditions*)

$$y = a_1 \quad \text{at} \quad x = x_1, \qquad y = a_2 \quad \text{at} \quad x = x_2. \qquad (2.5.1.2)$$

(The values of the unknown are prescribed at two distinct points x_1 and x_2.)

► **Second boundary value problem.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *second-type boundary conditions* (or *Neumann boundary conditions*)

$$y'_x = a_1 \quad \text{at} \quad x = x_1, \qquad y'_x = a_2 \quad \text{at} \quad x = x_2. \qquad (2.5.1.3)$$

(The values of the derivative of the unknown are prescribed at two distinct points x_1 and x_2.)

► **Third boundary value problem.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *third-type boundary conditions* (or *Robin boundary conditions*)

$$\begin{aligned} y'_x - k_1 y = a_1 \quad &\text{at} \quad x = x_1, \\ y'_x + k_2 y = a_2 \quad &\text{at} \quad x = x_2. \end{aligned} \tag{2.5.1.4}$$

► **Mixed boundary value problems.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *mixed-type boundary conditions*

$$y = a_1 \quad \text{at} \quad x = x_1, \qquad y'_x = a_2 \quad \text{at} \quad x = x_2. \tag{2.5.1.5}$$

(The unknown itself is prescribed at one point, and its derivative at another point.)

Other mixed boundary value problem: Find a solution of equation (2.5.1.1) satisfying the boundary conditions

$$y'_x = a_1 \quad \text{at} \quad x = x_1, \qquad y = a_2 \quad \text{at} \quad x = x_2. \tag{2.5.1.6}$$

Boundary conditions (2.5.1.2), (2.5.1.3), (2.5.1.4), (2.5.1.5) and (2.5.1.6) are called *homogeneous* if $a_1 = a_2 = 0$.

► **Problems with boundary conditions involving the values of the unknown (or/and its derivative) at both endpoints of the interval.**

Sometimes, one has to deal with problems whose boundary conditions involve the values of the unknown (or/and its derivative) at both ends of the interval.

Example 2.2. Here are two examples of such boundary conditions:

$$y(x_1) = a_1, \quad y(x_2) + k y'_x(x_1) = a_2$$

and

$$y(x_1) + k y(x_2) = a_1, \quad y'_x(x_2) = a_2.$$

► **Problems with a nonlocal condition.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying a boundary condition of the first kind at x_1 (see the first boundary condition in (2.5.1.2)) and the nonlocal condition

$$\int_{x_1}^{x_2} h(x) y(x)\, dx = b, \tag{2.5.1.7}$$

where $h(x)$ is a given function and b is a given number.

Condition (2.5.1.7) can be interpreted as a conservation law (with weight h) for the unknown. In particular, if

$$h(x) = \frac{1}{x_2 - x_1} = \text{const}$$

condition (2.5.1.7) defines the integral mean of the unknown.

The nonlocal condition (2.5.1.7) can be set together with a boundary condition of the second or third kind at one of the ends of the interval where the solution is sought.

► Boundary value problems with a degeneration at the boundary.

Let us look at the situation where the coefficient of the highest derivative in (2.5.1.1) becomes zero at the left endpoint:

$$f_2(x_1) = 0, \qquad f_1^2(x_1) + f_0^2(x_1) \neq 0.$$

In this case, one of the solutions to equation (2.5.1.1) may tend to infinity as $x \to x_1$; in order to establish this fact, one has to find the leading asymptotic terms of the fundamental system of solutions as $x \to x_1$). If one of the solutions is unbounded as $x \to x_1$, then for the problem to be well-posed, a boundedness condition for the solution needs to be set at the left endpoint:

$$|y| \neq \infty \quad \text{at} \quad x = x_1. \tag{2.5.1.8}$$

The boundary at the other end, $x = x_2$, can be any of the those listed above.

Example 2.3. Suppose the coefficients of equation (2.5.1.1) can be expanded in a Taylor series about $x = x_1$, so that

$$f_2(x) \simeq (x - x_1), \qquad f_1(x) \simeq b \neq 0, \qquad f_0(x) \simeq c \qquad (x \to x_1). \tag{2.5.1.9}$$

In view of (2.5.1.9), the leading asymptotic term of the (potentially) singular solution to equation (2.5.1.1) as $x \to x_1$ will be sought in the form

$$y \simeq (x - x_1)^\lambda. \tag{2.5.1.10}$$

(By virtue of the linearity of the equation, the solutions are determined up to a constant factor). Substituting (2.5.1.10) into ODE (2.5.1.1), taking into account (2.5.1.9), and dividing by $(x - x_1)^\lambda$, we obtain $\lambda(\lambda - 1 + b)(x - x_1)^{-1} + O(1) = 0$. For the left-hand side of this relation to be bounded as $x \to x_1$, we must set

$$\lambda(\lambda - 1 + b) = 0. \tag{2.5.1.11}$$

The zero root $\lambda = 0$ corresponds to a regular solution to equation (2.5.1.1), which does not have a singularity at $x = x_1$ and is expandable in a Taylor series in powers of $x - x_1$. The other root of the quadratic equation (2.5.1.11) is

$$\lambda = 1 - b. \tag{2.5.1.12}$$

If $b > 1$, then $\lambda < 0$; hence, the solution with the asymptotic behavior (2.5.1.10) is unbounded as $x \to x_1$. In this case, the boundedness condition (2.5.1.8) should be set at the left endpoint. If $b = 1$, equation (2.5.1.11) has a double root $\lambda = 0$, which determines a solution with a logarithmic singularity; in this case, a boundedness condition should also be set at the left endpoint of the interval $[x_1, x_2]$.

► Boundary value problems on an unbounded interval.

Consider equation (2.5.1.1) on the unbounded interval $x_1 < x < \infty$ (i.e., $x_2 = \infty$). Let one of the fundamental solutions of the equation tend to zero and be bounded as $x \to \infty$ and let the modulus of the other solution increase without bound. Then, a boundedness condition has to be set at the right end of the interval:

$$|y| \neq \infty \quad \text{as} \quad x \to \infty. \tag{2.5.1.13}$$

The boundary condition at the left endpoint, $x = x_1$, can be any of those listed previously.

Remark 2.2. If the bounded fundamental solution is monotonic for sufficiently large x, the equivalent condition

$$y'_x \to 0 \quad \text{as} \quad x \to \infty \tag{2.5.1.14}$$

can be used instead of (2.5.1.13).

Example 2.4. Let the coefficients of equation (2.5.1.1) be expandable in Taylor series as $x \to \infty$ and tend to constant quantities:

$$f_2(\infty) = a, \qquad f_1(\infty) = b, \qquad f_0(\infty) = c. \tag{2.5.1.15}$$

Then the qualitative behavior of the fundamental system of equations as $x \to \infty$ is determined by the roots of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0,$$

which is obtained by substituting $y = e^{\lambda x}$ into equation (2.5.1.1), whose coefficients are replaced with their leading asymptotic terms (2.5.1.15).

Condition (2.5.1.14) (or (2.5.1.13)) must be set if either (i) $ac < 0$ or (ii) $c = 0$ and $ab > 0$.

2.5.2 Simplification of Boundary Conditions. Self-Adjoint Form of Equations

► **Simplification of boundary conditions.**

Nonhomogeneous boundary conditions of the first-, second-, third-, and mixed kinds set at the endpoints of a bounded interval $[x_1, x_2]$ can be reduced to homogeneous ones by the change of variable

$$z = A_2 x^2 + A_1 x + A_0 + y,$$

with the constants A_2, A_1, and A_0 selected using the method of undetermined coefficients.

Table 2.1 gives examples of such transformations.

TABLE 2.1
Simple transformations of the form $z = A_2 x^2 + A_1 x + A_0 + y$ that lead to homogeneous boundary conditions ($x_1 \le x \le x_2$)

No	Problem	Boundary conditions	Transformation
1	First boundary value problem	$y = a_1$ at $x = x_1$ $y = a_2$ at $x = x_2$	$z = y - \dfrac{a_2 - a_1}{x_2 - x_1}(x - x_1) - a_1$
2	Second boundary value problem	$y'_x = a_1$ at $x = x_1$ $y'_x = a_2$ at $x = x_2$	$z = y + \dfrac{a_1 - a_2}{2(x_2 - x_1)}x^2 + \dfrac{a_2 x_1 - a_1 x_2}{x_2 - x_1}x$
3	Mixed boundary value problem	$y = a_1$ at $x = x_1$ $y'_x = a_2$ at $x = x_2$	$z = y - a_2 x + a_2 x_1 - a_1$
4	Mixed boundary value problem	$y'_x = a_1$ at $x = x_1$ $y = a_2$ at $x = x_2$	$z = y - a_1 x + a_1 x_2 - a_2$

► **Reduction of a bounded interval to a unit interval.**

The interval $x_1 \le x \le x_2$ on which a boundary problem is defined can be reduced with the change of variable $x = x_1 + (x_2 - x_1)\bar{x}$ to the unit interval $0 \le \bar{x} \le 1$. Homogeneous boundary conditions of the first-, second-, third-, and mixed kinds remain homogeneous under this transformation.

► **Self-adjoint form of equations.**

On multiplying by $p(x) = \exp\left[\int \frac{f_1(x)}{f_2(x)}\,dx\right]$, one reduces equation (2.5.1.1) to the self-adjoint form:

$$[p(x)y'_x]'_x + q(x)y = r(x), \tag{2.5.2.1}$$

where $q(x) = f_0(x)p(x)/f_2(x)$ and $r(x) = g(x)(x)p(x)/f_2(x)$.

Hence, without loss of generality, we can further deal with equation (2.5.2.1) instead of (2.5.1.1). We assume that the functions p, p'_x, q, and r are continuous on the interval $x_1 \le x \le x_2$, and p is positive.

2.5.3 Green's and Modified Green's Functions. Representation Solutions via Green's or Modified Green's Functions

► **Green's function. Linear problems for nonhomogeneous equations.**

A *Green's function* of the first boundary value problem for equation (2.5.2.1) with homogeneous boundary conditions (2.5.1.2) is a function of two variables $G(x,\xi)$ that satisfies the following conditions:

1°. $G(x,\xi)$ is continuous in x for fixed ξ, with $x_1 \le x \le x_2$ and $x_1 \le \xi \le x_2$.

2°. $G(x,\xi)$ is a solution of the homogeneous equation (2.5.2.1), with $r = 0$, for all $x_1 < x < x_2$ exclusive of the point $x = \xi$.

3°. $G(x,\xi)$ satisfies the homogeneous boundary conditions $G(x_1,\xi) = G(x_2,\xi) = 0$.

4°. The derivative $G'_x(x,\xi)$ has a jump of $1/p(\xi)$ at the point $x = \xi$, that is,

$$G'_x(x,\xi)\big|_{x\to\xi,\,x>\xi} - G'_x(x,\xi)\big|_{x\to\xi,\,x<\xi} = \frac{1}{p(\xi)}.$$

For the second, third, and mixed boundary value problems, the Green's function is defined likewise except that in 3° the homogeneous boundary conditions (2.5.1.3), (2.5.1.4), and (2.5.1.5), with $a_1 = a_2 = 0$, are adopted, respectively.

The solution of the nonhomogeneous equation (2.5.2.1) subject to appropriate homogeneous boundary conditions is expressed in terms of the Green's function as follows:*

$$y(x) = \int_{x_1}^{x_2} G(x,\xi)r(\xi)\,d\xi. \tag{2.5.3.1}$$

*The homogeneous boundary value problem—with $r(x) = 0$ and $a_1 = a_2 = 0$—is assumed to have only the trivial solution.

► **Representation of the Green's function in terms of particular solutions.**

1°. We consider the first boundary value problem. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be linearly independent particular solutions of the homogeneous equation (2.5.2.1), with $r=0$, that satisfy the conditions

$$y_1(x_1) = 0, \qquad y_2(x_2) = 0. \tag{2.5.3.2}$$

(Each of the solutions satisfies one of the homogeneous boundary conditions.)

The Green's function is expressed in terms of solutions of the homogeneous equation as follows:

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} & \text{for } x_1 \le x \le \xi, \\ \dfrac{y_1(\xi)y_2(x)}{p(\xi)W(\xi)} & \text{for } \xi \le x \le x_2, \end{cases} \tag{2.5.3.3}$$

where $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the Wronskian determinant.

2°. Formula (2.5.3.3) can also be used to construct Green's functions for the second, third, and mixed boundary value problems. To this end, one should find two linearly independent solutions, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, of the homogeneous equation with $r = 0$; the former satisfies the corresponding homogeneous boundary condition at $x=x_1$ and the latter satisfies the one at $x = x_2$ (see also the paragraph "Modified Green's function" below).

3°. The solution of the nonhomogeneous equation (2.5.2.1) subject to the second, third, and mixed homogeneous boundary conditions is also expressed in terms of an appropriate Green's function by formula (2.5.3.1).

4°. Table 2.2 contains the simplest examples of Green's functions $G(x,\xi)$ for some linear boundary value problems for ODEs of the form (2.5.2.1). In all these examples, $G(x,\xi) = G(\xi,x)$, and therefore the Green's function is specified only in the domain $x \le \xi$. For equations with the operator $L[y] = [f(x)y_x']_x'$, it is assumed that $f(x) > 0$ and $\varphi(x) = \displaystyle\int_0^x \frac{dt}{f(t)}$.

5°. Formula (2.5.3.3) can also be used to construct the Green's functions for boundary value problems when the equation has a singular point at the boundary (i.e., when $p(x)$ becomes zero at $x = x_1$ or/and $x = x_2$ or when $q(x)$ becomes infinite at these point). In such cases, the relevant boundary condition must be replaced with a boundedness condition at the singular point (see rows 7, 8, and 9 of Table 2.2 for examples).

► **Modified Green's function. Representation in terms of particular solutions.**

Now let us look at equation (2.5.1.1) subject to homogeneous boundary conditions of the general form

$$\begin{aligned} m_1 y_x' + k_1 y = 0 \quad &\text{at} \quad x = x_1, \\ m_2 y_x' + k_2 y = 0 \quad &\text{at} \quad x = x_2. \end{aligned} \tag{2.5.3.4}$$

With suitably selected coefficients k_n and m_n, these conditions cover the first, second, third, and mixed boundary conditions are special cases (see Section 2.5.1).

TABLE 2.2
Green's function for some boundary value problems for linear second-order ODEs $L[y] = r(x)$

No.	Differential operator, $L[y]$	Boundary conditions	Green's function, $G(x,\xi)$
1	y''_{xx}	$y(0) = y(a) = 0$	$x\left(\frac{\xi}{a} - 1\right)$
2	y''_{xx}	$y(0) = y'_x(a) = 0$	$-x$
3	y''_{xx}	$y'_x(0) = y(a) = 0$	$\xi - a$
4	y''_{xx}	$y(0) = 0,$ $y(a) + ky'_x(a) = 0$	$\frac{x(\xi - a - k)}{a + k}$
5	$y''_{xx} + k^2 y$	$y(0) = y(1) = 0$	$-\frac{\sin(kx)\sin[k(1-\xi)]}{k\sin k}$
6	$y''_{xx} - k^2 y$	$y(0) = y(1) = 0$	$-\frac{\sinh(kx)\sinh[k(1-\xi)]}{k\sinh k}$
7	$xy''_{xx} + y'_x \equiv (xy'_x)'_x$	$y(0) \neq \infty,\ y(a) = 0$	$\ln\frac{\xi}{a}$
8	$(xy'_x)'_x - \frac{n^2}{x}y$ (Bessel's operator)	$y(0) \neq \infty,\ y(a) = 0$	$-\frac{1}{2n}\left(\frac{x}{a}\right)^n + \frac{(x\xi)^n}{2na^{2n}}$ $(n = 1,\ 2,\ \ldots)$
9	$[(1-x^2)y'_x]'_x - \frac{n^2}{1-x^2}y$ (Legendre's operator)	$y(-1) \neq \infty,\ y(1) \neq \infty$	$-\frac{1}{2n}\left(\frac{1+x}{1-x}\cdot\frac{1-\xi}{1+\xi}\right)^{n/2}$ $(n = 1,\ 2,\ \ldots)$
10	$[f(x)y'_x]'_x$	$y(0) = y(a) = 0$	$-\varphi(x) + \frac{\varphi(x)\varphi(\xi)}{\varphi(a)}$
11	$[f(x)y'_x]'_x$	$y(0) = y'_x(a) = 0$	$-\varphi(x)$
12	$[f(x)y'_x]'_x$	$y(0) = 0,$ $y(a) + ky'_x(a) = 0$	$-\varphi(x) + \frac{f(a)\varphi(x)\varphi(\xi)}{f(a)\varphi(a) + k}$ $(k > 0)$

The solution of the nonhomogeneous equation (2.5.1.1) subject to homogeneous boundary conditions (2.5.3.4) is*

$$y(x) = \int_{x_1}^{x_2} \mathcal{G}(x,\xi)g(\xi)\,d\xi, \tag{2.5.3.5}$$

where $\mathcal{G}(x,\xi)$ is the *modified Green's function*

$$\mathcal{G}(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{f_2(\xi)W(\xi)} & \text{for } x_1 \le x \le \xi, \\ \dfrac{y_1(\xi)y_2(x)}{f_2(\xi)W(\xi)} & \text{for } \xi \le x \le x_2, \end{cases} \tag{2.5.3.6}$$

where $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are linearly independent particular solutions of the

*The homogeneous boundary value problem, with $g(x) = 0$, is assumed to have only the trivial solution.

homogeneous equation (2.5.1.1), with $g = 0$, that satisfy the conditions

$$\begin{aligned} m_1(y_1)'_x + k_1 y_1 = 0 \quad &\text{at} \quad x = x_1, \\ m_2(y_2)'_x + k_2 y_2 = 0 \quad &\text{at} \quad x = x_2, \end{aligned} \tag{2.5.3.7}$$

and $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the Wronskian determinant.

Example 2.5. Consider the equation

$$y''_{xx} + ay'_x = g(x) \tag{2.5.3.8}$$

with the homogeneous mixed boundary conditions

$$y(0) = 0, \qquad y'(1) = 0. \tag{2.5.3.9}$$

The general solution of equation (2.5.3.8) with $g(x) = 0$ is

$$y = C_1 + C_2 e^{-ax}, \tag{2.5.3.10}$$

where C_1 and C_2 are arbitrary constants. Linearly independent particular solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ that satisfy the homogeneous conditions $y_1(0) = 0$ and $y_2'(1) = 0$ are

$$y_1(x) = 1 - e^{-ax}, \qquad y_2(x) = 1. \tag{2.5.3.11}$$

Substituting (2.5.3.11) into (2.5.3.6) and taking into account that $f_2(x) = 1$, we find the modified Green's function

$$\mathcal{G}(x,\xi) = \begin{cases} -\dfrac{1}{a} e^{a\xi}(1 - e^{-ax}) & \text{for } x_1 \le x \le \xi, \\ -\dfrac{1}{a} e^{ax}(1 - e^{-a\xi}) & \text{for } \xi \le x \le x_2, \end{cases} \tag{2.5.3.12}$$

The solution to the boundary value problem (2.5.3.8)–(2.5.3.9) is defined by formulas (2.5.3.5) and (2.5.3.12).

⊙ *Literature for Section 2.5*: L. E. El'sgol'ts (1961), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2008).

2.6 Eigenvalue Problems

2.6.1 Sturm–Liouville Problem

Consider the second-order homogeneous linear differential equation

$$[p(x)y'_x]'_x + [\lambda s(x) - q(x)]y = 0 \tag{2.6.1.1}$$

subject to linear boundary conditions of the general form

$$\begin{aligned} \alpha_1 y'_x + \beta_1 y = 0 \quad &\text{at} \quad x = x_1, \\ \alpha_2 y'_x + \beta_2 y = 0 \quad &\text{at} \quad x = x_2. \end{aligned} \tag{2.6.1.2}$$

It is assumed that the functions p, p'_x, s, and q are continuous, and p and s are positive on an interval $x_1 \le x \le x_2$. It is also assumed that $|\alpha_1| + |\beta_1| > 0$ and $|\alpha_2| + |\beta_2| > 0$.

The *Sturm–Liouville problem*: Find the values λ_n of the parameter λ at which problem (2.6.1.1), (2.6.1.2) has a nontrivial solution. Such λ_n are called *eigenvalues* and the corresponding solutions $y_n = y_n(x)$ are called *eigenfunctions* of the Sturm–Liouville problem (2.6.1.1), (2.6.1.2).

2.6.2 General Properties of the Sturm–Liouville Problem (2.6.1.1), (2.6.1.2)

1°. There are infinitely (countably) many eigenvalues. All eigenvalues can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Moreover, $\lambda_n \to \infty$ as $n \to \infty$; hence, there can only be a finite number of negative eigenvalues. Each eigenvalue has multiplicity 1.

2°. The eigenfunctions are defined up to a constant factor. Each eigenfunction $y_n(x)$ has precisely $n-1$ zeros on the open interval (x_1, x_2).

3°. Any two eigenfunctions $y_n(x)$ and $y_m(x)$, $n \neq m$, are orthogonal with weight $s(x)$ on the interval $x_1 \le x \le x_2$:

$$\int_{x_1}^{x_2} s(x) y_n(x) y_m(x)\, dx = 0 \quad \text{if} \quad n \neq m.$$

4°. An arbitrary function $F(x)$ that has a continuous derivative and satisfies the boundary conditions of the Sturm–Liouville problem can be decomposed into an absolutely and uniformly convergent series in the eigenfunctions

$$F(x) = \sum_{n=1}^{\infty} F_n y_n(x),$$

where the Fourier coefficients F_n of $F(x)$ are calculated by

$$F_n = \frac{1}{\|y_n\|^2} \int_{x_1}^{x_2} s(x) F(x) y_n(x)\, dx, \qquad \|y_n\|^2 = \int_{x_1}^{x_2} s(x) y_n^2(x)\, dx.$$

5°. If the conditions

$$q(x) \ge 0, \quad \alpha_1 \beta_1 \le 0, \quad \alpha_2 \beta_2 \ge 0 \tag{2.6.2.1}$$

hold true, there are no negative eigenvalues. If $q \equiv 0$ and $\beta_1 = \beta_2 = 0$, the least eigenvalue is $\lambda_1 = 0$, to which there corresponds an eigenfunction $y_1 = \text{const}$. In the other cases where conditions (2.6.2.1) are satisfied, all eigenvalues are positive.

6°. The following asymptotic formula is valid for eigenvalues as $n \to \infty$:

$$\lambda_n = \frac{\pi^2 n^2}{\Delta^2} + O(1), \qquad \Delta = \int_{x_1}^{x_2} \sqrt{\frac{s(x)}{p(x)}}\, dx. \tag{2.6.2.2}$$

Sections 2.6.3 through 2.6.6 will describe special properties of the Sturm–Liouville problem that depend on the specific form of the boundary conditions.

Remark 2.3. Equation (2.6.1.1) can be reduced to the case where $p(x) \equiv 1$ and $s(x) \equiv 1$ by the change of variables

$$\zeta = \int \sqrt{\frac{s(x)}{p(x)}}\, dx, \qquad u(\zeta) = \big[p(x)s(x)\big]^{1/4} y(x).$$

In this case, the boundary conditions are transformed to boundary conditions of similar form.

Remark 2.4. The second-order linear equation

$$\varphi_2(x) y''_{xx} + \varphi_1(x) y'_x + [\lambda + \varphi_0(x)] y = 0$$

can be represented in the form of equation (2.6.1.1) where $p(x)$, $s(x)$, and $q(x)$ are given by

$$\begin{gathered} p(x) = \exp\biggl[\int \frac{\varphi_1(x)}{\varphi_2(x)}\, dx\biggr], \qquad s(x) = \frac{1}{\varphi_2(x)} \exp\biggl[\int \frac{\varphi_1(x)}{\varphi_2(x)}\, dx\biggr], \\ q(x) = -\frac{\varphi_0(x)}{\varphi_2(x)} \exp\biggl[\int \frac{\varphi_1(x)}{\varphi_2(x)}\, dx\biggr]. \end{gathered} \tag{2.6.2.3}$$

2.6.3 Problems with Boundary Conditions of the First Kind

Let us note some special properties of the Sturm–Liouville problem that is the first boundary value problem for equation (2.6.1.1) with the boundary conditions

$$y = 0 \quad \text{at} \quad x = x_1, \qquad y = 0 \quad \text{at} \quad x = x_2. \tag{2.6.3.1}$$

1°. For $n \to \infty$, the asymptotic relation (2.6.2.2) can be used to estimate the eigenvalues λ_n. In this case, the asymptotic formula

$$\frac{y_n(x)}{\|y_n\|} = \left[\frac{4}{\Delta^2 p(x)s(x)}\right]^{1/4} \sin\left[\frac{\pi n}{\Delta}\int_{x_1}^{x}\sqrt{\frac{s(x)}{p(x)}}\,dx\right] + O\Big(\frac{1}{n}\Big), \qquad \Delta = \int_{x_1}^{x_2}\sqrt{\frac{s(x)}{p(x)}}\,dx$$

holds true for the eigenfunctions $y_n(x)$.

2°. If $q \ge 0$, the following upper estimate holds for the least eigenvalue (*Rayleigh–Ritz principle*):

$$\lambda_1 \le \frac{\displaystyle\int_{x_1}^{x_2}\left[p(x)(z'_x)^2 + q(x)z^2\right]dx}{\displaystyle\int_{x_1}^{x_2} s(x)z^2\,dx}, \tag{2.6.3.2}$$

where $z = z(x)$ is any twice differentiable function that satisfies the conditions $z(x_1) = z(x_2) = 0$. The equality in (2.6.3.2) is attained if $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1. One can take $z = (x - x_1)(x_2 - x)$ or $z = \sin\left[\dfrac{\pi(x - x_1)}{x_2 - x_1}\right]$ in (2.6.3.2) to obtain specific estimates.

It is significant to note that the left-hand side of (2.6.3.2) usually gives a fairly precise estimate of the first eigenvalue (see Table 2.3).

TABLE 2.3
Example estimates of the first eigenvalue λ_1 in Sturm–Liouville problems with boundary conditions of the first kind $y(0) = y(1) = 0$ obtained using the Rayleigh–Ritz principle [the right-hand side of relation (2.6.3.2)]

Equation	Test function	λ_1, approximate	λ_1, exact
$y''_{xx} + \lambda(1 + x^2)^{-2}y = 0$	$z = \sin \pi x$	15.337	15.0
$y''_{xx} + \lambda(4 - x^2)^{-2}y = 0$	$z = \sin \pi x$	135.317	134.837
$[(1 + x)^{-1}y'_x]'_x + \lambda y = 0$	$z = \sin \pi x$	7.003	6.772
$\big(\sqrt{1 + x}\, y'_x\big)'_x + \lambda y = 0$	$z = \sin \pi x$	11.9956	11.8985
$y''_{xx} + \lambda(1 + \sin \pi x)y = 0$	$z = \sin \pi x$ $z = x(1 - x)$	$0.54105\,\pi^2$ $0.55204\,\pi^2$	$0.54032\,\pi^2$ $0.54032\,\pi^2$

3°. The extension of the interval $[x_1, x_2]$ leads to decreasing in eigenvalues.

4°. Let the inequalities

$$0 < p_{\min} \le p(x) \le p_{\max}, \quad 0 < s_{\min} \le s(x) \le s_{\max}, \quad 0 < q_{\min} \le q(x) \le q_{\max}$$

be satisfied. Then the following bilateral estimates hold:

$$\frac{p_{\min}}{s_{\max}} \frac{\pi^2 n^2}{(x_2 - x_1)^2} + \frac{q_{\min}}{s_{\max}} \le \lambda_n \le \frac{p_{\max}}{s_{\min}} \frac{\pi^2 n^2}{(x_2 - x_1)^2} + \frac{q_{\max}}{s_{\min}}.$$

5°. In engineering calculations for eigenvalues, the approximate formula

$$\lambda_n = \frac{\pi^2 n^2}{\Delta^2} + \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \frac{q(x)}{s(x)}\,dx, \qquad \Delta = \int_{x_1}^{x_2} \sqrt{\frac{s(x)}{p(x)}}\,dx \tag{2.6.3.3}$$

may be quite useful. This formula provides an exact result if $p(x)s(x) = \text{const}$ and $q(x)/s(x) = \text{const}$ (in particular, for constant equation coefficients, $p = p_0$, $q = q_0$, and $s = s_0$) and gives a correct asymptotic behavior of (2.6.2.2) for any $p(x)$, $q(x)$, and $s(x)$. In addition, relation (2.6.3.3) gives two correct leading asymptotic terms as $n \to \infty$ if $p(x) = \text{const}$ and $s(x) = \text{const}$ [and also if $p(x)s(x) = \text{const}$].

6°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi n}{x_2 - x_1} + \frac{1}{\pi n} Q(x_1, x_2) + O\Big(\frac{1}{n^2}\Big),$$

$$y_n(x) = \sin\frac{\pi n(x - x_1)}{x_2 - x_1} - \frac{1}{\pi n}\Big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\Big] \cos\frac{\pi n(x - x_1)}{x_2 - x_1} + O\Big(\frac{1}{n^2}\Big),$$

where

$$Q(u, v) = \frac{1}{2} \int_u^v q(x)\,dx. \tag{2.6.3.4}$$

7°. Let us consider the eigenvalue problem for the equation with a small parameter

$$y''_{xx} + [\lambda + \varepsilon q(x)]y = 0 \qquad (\varepsilon \to 0)$$

subject to the boundary conditions (2.6.3.1) with $x_1 = 0$ and $x_2 = 1$. We assume that $q(x) = q(-x)$.

This problem has the following eigenvalues and eigenfunctions:

$$\lambda_n = \pi^2 n^2 - \varepsilon A_{nn} + \frac{\varepsilon^2}{\pi^2} \sum_{k \ne n} \frac{A_{nk}^2}{n^2 - k^2} + O(\varepsilon^3), \quad A_{nk} = 2\int_0^1 q(x)\sin(\pi n x)\sin(\pi k x)\,dx;$$

$$y_n(x) = \sqrt{2}\,\sin(\pi n x) - \varepsilon \frac{\sqrt{2}}{\pi^2} \sum_{k \ne n} \frac{A_{nk}}{n^2 - k^2} \sin(\pi k x) + O(\varepsilon^2).$$

Here the summation is carried out over k from 1 to ∞. The next term in the expansion of y_n can be found in Nayfeh (1973).

2.6.4 Problems with Boundary Conditions of the Second Kind

Let us note some special properties of the Sturm–Liouville problem that is the second boundary value problem for equation (2.6.1.1) with the boundary conditions

$$y'_x = 0 \quad \text{at} \quad x = x_1, \qquad y'_x = 0 \quad \text{at} \quad x = x_2.$$

1°. If $q > 0$, the upper estimate (2.6.3.2) is valid for the least eigenvalue, with $z = z(x)$ being any twice-differentiable function that satisfies the conditions $z'_x(x_1) = z'_x(x_2) = 0$. The equality in (2.6.3.2) is attained if $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(n-1)}{x_2 - x_1} + \frac{1}{\pi(n-1)} Q(x_1, x_2) + O\Big(\frac{1}{n^2}\Big),$$
$$y_n(x) = \cos\frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + \frac{1}{\pi(n-1)}\Big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\Big]\sin\frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is given by (2.6.3.4).

2.6.5 Problems with Boundary Conditions of the Third Kind

We consider the third boundary value problem for equation (2.6.1.1) subject to condition (2.6.1.2) with $\alpha_1 = \alpha_2 = 1$. We assume that $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative.

The following asymptotic formulas hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(n-1)}{x_2 - x_1} + \frac{1}{\pi(n-1)}\big[Q(x_1, x_2) - \beta_1 + \beta_2\big] + O\Big(\frac{1}{n^2}\Big),$$
$$y_n(x) = \cos\frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + \frac{1}{\pi(n-1)}\Big\{(x_1 - x)\big[Q(x, x_2) + \beta_2\big] + (x_2 - x)\big[Q(x_1, x) - \beta_1\big]\Big\}\sin\frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is defined by (2.6.3.4).

2.6.6 Problems with Mixed Boundary Conditions

Let us note some special properties of the Sturm–Liouville problem that is the mixed boundary value problem for equation (2.6.1.1) with the boundary conditions

$$y'_x = 0 \quad \text{at} \quad x = x_1, \qquad y = 0 \quad \text{at} \quad x = x_2.$$

1°. If $q \ge 0$, the upper estimate (2.6.3.2) is valid for the least eigenvalue, with $z = z(x)$ being any twice-differentiable function that satisfies the conditions $z'_x(x_1) = 0$ and $z(x_2) = 0$. The equality in (2.6.3.2) is attained if $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(2n-1)}{2(x_2 - x_1)} + \frac{2}{\pi(2n-1)} Q(x_1, x_2) + O\Big(\frac{1}{n^2}\Big),$$
$$y_n(x) = \cos\frac{\pi(2n-1)(x-x_1)}{2(x_2 - x_1)} + \frac{2}{\pi(2n-1)}\Big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\Big]\sin\frac{\pi(2n-1)(x-x_1)}{2(x_2 - x_1)} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is defined by (2.6.3.4).

⊙ *Literature for Section 2.6*: L. Collatz (1963), E. Kamke (1977), A. G. Kostyuchenko and I. S. Sargsyan (1979), V. A. Marchenko (1986), B. M. Levitan and I. S. Sargsyan (1988), V. A. Vinokurov and V. A. Sadovnichii (2000), A. D. Polyanin (2002), A. D. Polyanin and V. F. Zaitsev (2003).

2.7 Theorems on Estimates and Zeros of Solutions

2.7.1 Theorem on Estimates of Solutions

Let $f_n(x)$ and $g_n(x)$ $(n = 1,\ 2)$ be continuous functions on the interval $a \le x \le b$.

THEOREM. *Let the following inequalities hold:*

$$0 \le f_1(x) \le f_2(x), \qquad 0 \le g_1(x) \le g_2(x).$$

If $y_n = y_n(x)$ are some solutions to the linear equations

$$y_n'' = f_n(x) y_n + g_n(x) \qquad (n = 1,\ 2)$$

and $y_1(a) \le y_2(a)$ and $y_1'(a) \le y_2'(a)$, then $y_1(x) \le y_2(x)$ and $y_1'(x) \le y_2'(x)$ on each interval $a \le x \le a_1$, where $y_2(x) > 0$.

2.7.2 Sturm Comparison Theorem on Zeros of Solutions

Consider the equation

$$[f(x) y']' + g(x) y = 0 \qquad (a \le x \le b), \tag{2.7.2.1}$$

where the function $f(x)$ is positive and continuously differentiable, and the function $g(x)$ is continuous.

COMPARISON THEOREM (STURM). *Let $y_n = y_n(x)$ be nonzero solutions of the linear equations*

$$[f_n(x) y_n']' + g_n(x) y_n = 0 \qquad (n = 1,\ 2)$$

and let the inequalities $f_1(x) \ge f_2(x) > 0$ and $g_1(x) \le g_2(x)$ hold. Then the function y_2 has at least one zero lying between any two adjacent zeros, x_1 and x_2, of the function y_1 (it is assumed that the identities $f_1 \equiv f_2$ and $g_1 \equiv g_2$ are not satisfied on any interval simultaneously).

COROLLARY 1. *If $g(x) \le 0$ or there exists a constant k_1 such that*

$$f(x) \ge k_1 > 0, \quad g(x) < k_1 \left(\frac{\pi}{b-a} \right)^2,$$

then every nontrivial solution to equation (2.7.2.1) has no more than one zero on the interval $[a, b]$.

COROLLARY 2. *If there exists a constant k_2 such that*

$$0 < f(x) \le k_2, \quad g(x) > k_2 \left(\frac{\pi m}{b-a} \right)^2, \quad \text{where} \quad m = 1,\ 2,\ \dots,$$

then every nontrivial solution to equation (2.7.2.1) has at least m zeros on the interval $[a, b]$.

2.7.3 Qualitative Behavior of Solutions as $x \to \infty$

Consider the equation

$$y'' + f(x)y = 0, \tag{2.7.3.1}$$

where $f(x)$ is a continuous function for $x \geq a$.

1°. For $f(x) \leq 0$, every nonzero solution has no more than one zero, and hence $y \neq 0$ for sufficiently large x.

If $f(x) \leq 0$ for all x and $f(x) \not\equiv 0$, then $y \equiv 0$ is the only solution bounded for all x.

2°. Suppose $f(x) \geq k^2 > 0$. Then every nontrivial solution $y(x)$ and its derivative $y'(x)$ have infinitely many zeros, with the distance between any adjacent zeros remaining finite.

If $f(x) \to k^2 > 0$ for $x \to \infty$ and $f' \geq 0$, then the solutions of the equation for large x behave similarly to those of the equation $y'' + k^2 y = 0$.

3°. Let $f(x) \to -\infty$ for $|x| \to \infty$. Then every nonzero solution has only finitely many zeros, and $|y'/y| \to \infty$ as $|x| \to \infty$. There are two linearly independent solutions, y_1 and y_2, such that $y_1 \to 0$, $y_1' \to 0$, $y_2 \to \infty$, and $y_2' \to -\infty$ as $x \to -\infty$, and there are two linearly independent solutions, $\bar{y}_1$ and $\bar{y}_2$, such that $\bar{y}_1 \to 0$, $\bar{y}_1' \to 0$, $\bar{y}_2 \to \infty$, and $\bar{y}_2' \to \infty$ as $x \to \infty$.

4°. If the function f in equation (2.7.3.1) is continuous, monotonic, and positive, then the amplitude of each solution decreases (resp., increases) as f increases (resp., decreases).

⊙ *Literature for Section 2.7*: E. Kamke (1977), A. D. Polyanin and A. V. Manzhirov (2007).

2.8 Numerical Methods

Linear problems can be solved using the numerical methods outlined in Section 3.8, which are designed for solving more complex, nonlinear problems. Due to their specific properties, linear problems are easier and more efficient to solve with special methods described below.

In the numerical methods discussed below, the second derivative is approximated with the following finite-difference expression:

$$y''_{xx} \approx \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2},$$

where $y_k = y(x_k)$, $x_k = x_0 + kh$ ($x_0 \leq x \leq x_*$), and h is the mesh increment.

2.8.1 Numerov's Method (Cauchy Problem)

The Cauchy problem for linear differential equations of the form

$$y''_{xx} + f(x)y = g(x) \tag{2.8.1.1}$$

can be solved using the recurrence formula

$$u_{k+1} = 2u_k - u_{k-1} + \left[-f_k y_k + g_k + \tfrac{1}{12}(g_{k+1} - 2g_k + g_{k-1})\right]h^2, \tag{2.8.1.2}$$

where

$$u_k = y_k\left(1 + \tfrac{1}{12} f_k h^2\right), \quad f_k = f(x_k), \quad g_k = g(x_k).$$

2.8.2 Modified Shooting Method (Boundary Value Problems)

Let us look at the linear boundary value problem defined by the equation

$$y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x) \tag{2.8.2.1}$$

and general homogeneous boundary conditions of the third kind

$$a_1 y'_x + b_1 y = 0 \quad \text{at} \quad x = 0, \tag{2.8.2.2}$$

$$a_2 y'_x + b_2 y = 0 \quad \text{at} \quad x = l. \tag{2.8.2.3}$$

We assume that a solution to problem (2.8.2.1)–(2.8.2.3) exists and is unique.

First, we find an auxiliary function $y_1 = y_1(x)$ that solves the first auxiliary Cauchy problem for the nonhomogeneous equation (2.8.2.1) with the initial conditions

$$y = a_1 \quad \text{at} \quad x = 0; \qquad y'_x = -b_1 \quad \text{at} \quad x = 0. \tag{2.8.2.4}$$

By virtue of (2.8.2.1), the function $y_1 = y_1(x)$ satisfies the left boundary condition (2.8.2.2). Then, we find an auxiliary function $y_0 = y_0(x)$ that solves the second auxiliary Cauchy problem for the homogeneous equation (2.8.2.1) with $g(x) = 0$ and the boundary conditions (2.8.2.4). By virtue of the linearity of the problem and homogeneous boundary conditions, the function $Cy_0(x)$ is also a solution to equation (2.8.2.1) satisfying the left boundary condition (2.8.2.2). Therefore, the solution of the original boundary value problem (2.8.2.1)–(2.8.2.3) can be sought as the sum

$$y(x) = y_1(x) + Cy_0(x). \tag{2.8.2.5}$$

The constant C is determined from the requirement that function (2.8.2.5) must satisfy the right boundary condition (2.8.2.3):

$$a_2 y'_1(l) + b_2 y_1(l) + C[a_2 y'_0(l) + b_2 y_0(l)] = 0. \tag{2.8.2.6}$$

Thus, solving the original boundary value problem is reduced to solving two auxiliary Cauchy problems; this can be done using, for example, the Runge–Kutta method (see Section 3.8). The case of nonhomogeneous boundary condition can be considered likewise.

Example 2.6. Let us look at the special case of equation (2.8.2.1)

$$y''_{xx} + f(x)y = g(x) \tag{2.8.2.7}$$

subject to the nonhomogeneous boundary conditions of the first kind

$$y = a \quad \text{at} \quad x = 0; \qquad y = b \quad \text{at} \quad x = l. \tag{2.8.2.8}$$

The mesh version of the above shooting method for this problem is as follows. By setting the initial values*

$$y_0^1 = a, \quad y_1^1 = \beta_1; \quad y_0^0 = 0, \quad y_1^0 = \beta_2, \tag{2.8.2.9}$$

we successively find $y_2^1, \dots, y_n^1$ and $y_2^0, \dots, y_n^0$ from the difference equations

$$\frac{y_{k+1}^1 - 2y_k^1 + y_{k-1}^1}{h^2} + f_k y_k^1 = g_k,$$

$$\frac{y_{k+1}^0 - 2y_k^0 + y_{k-1}^0}{h^2} + f_k y_k^0 = 0,$$

where $f_k = f(x_k)$, $g_k = g(x_k)$, and h is the mesh increment. Then, we find C from the equation $y_n^1 + Cy_n^0 = b$ and set $y_k = y_k^1 + Cy_k^0$; the function y_k is the desired solution.

*The numbers β_1 and $\beta_2 \neq 0$ can generally be any; in particular, we can set $\beta_1 = a$ and $\beta_2 = h$.

2.8.3 Sweep Method (Boundary Value Problems)

Below we outline the sweep method for the following system of difference equations:

$$A_k y_{k-1} - C_k y_k + B_k y_{k+1} = D_k, \qquad k = 1, \dots, n-1, \tag{2.8.3.1}$$

$$y_0 = \alpha y_1 + \beta, \qquad y_n = \gamma y_{n-1} + \delta. \tag{2.8.3.2}$$

Relation (2.8.3.1) approximates a linear differential equation, while relations (2.8.3.2) represent boundary conditions of the third kind (or the first kind if $\alpha = \gamma = 0$).

Provided that all of the conditions

$$A_k > 0,\ B_k > 0,\ C_k > 0;\ \ C_k \ge A_k + B_k;\ \ 0 \le \alpha < 1;\ \ 0 \le \gamma < 1 \tag{2.8.3.3}$$

hold true, problem (2.8.3.1)–(2.8.3.2) is solvable and has a unique solution.

Remark 2.5. Problem (2.8.2.7)–(2.8.2.8) is approximated by the difference equation (2.8.3.1) and boundary conditions (2.8.3.2) with

$$A_k = B_k = 1,\ \ C_k = 2 - h^2 f_k,\ \ D_k = h^2 g_k,\ \ \alpha = \gamma = 0,\ \ \beta = a,\ \ \delta = b.$$

We will look for numbers α_k and β_k, called *sweep coefficients*, such that for all $k = 1,\ 2,\ \dots,\ n$ the relations

$$y_{k-1} = \alpha_k y_k + \beta_k \tag{2.8.3.4}$$

hold. Substituting (2.8.3.4) into (2.8.3.1) yields

$$(A_k\alpha_k - C_k)y_k + B_k y_{k+1} + A_k\beta_k - D_k = 0.$$

By expressing y_k in terms of y_{k+1} using formula (2.8.3.4), we obtain

$$\big[(A_k\alpha_k - C_k)\alpha_{k+1} + B_k\big]y_{k+1} + \big[(A_k\alpha_k - C_k)\beta_{k+1} + A_k\beta_k - D_k\big] = 0.$$

Equating the expressions in square brackets with zero for all $k = 1,\ 2,\ \dots,\ n-1$, we arrive at recurrence relations to determine the coefficients α_{k+1} and β_{k+1} once $\alpha = \alpha_1$ and $\beta = \beta_1$ are known (*forward sweep*):

$$\alpha_{k+1} = \frac{B_k}{C_k - A_k\alpha_k}, \qquad \beta_{k+1} = \frac{A_k\beta_k - D_k}{C_k - A_k\alpha_k}. \tag{2.8.3.5}$$

If conditions (2.8.3.3) hold, the numerators in formulas (2.8.3.5) are positive and $0 \le \alpha_k < 1$.

From formula (2.8.3.4) with $k = n$ and the second boundary condition in (2.8.3.2) we find the last value of the unknown:

$$y_n = \frac{\gamma\beta_n + \delta}{1 - \gamma\alpha_n}, \tag{2.8.3.6}$$

where $1 - \gamma\alpha_n > 0$. Now, by formula (2.8.3.4), we can successively determine the unknowns y_{k-1} with $k = n,\ n-1,\ \dots,\ 1$ (*backward sweep*).

Remark 2.6. In the above sweep method, the coefficients are first determined starting from the left boundary condition and then the solution is recovered from right to left by formula (2.8.3.4). Quite similarly, the reverse scheme can be used where the coefficients are first determined starting from the right boundary condition and then the solution is recovered with the sweep from left to right.

2.8.4 Method of Accelerated Convergence in Eigenvalue Problems

Consider the Sturm–Liouville problem for the second-order nonhomogeneous linear equation

$$[f(x)y'_x]'_x + [\lambda g(x) - h(x)]y = 0 \qquad (2.8.4.1)$$

with linear homogeneous boundary conditions of the first kind

$$y(0) = y(1) = 0. \qquad (2.8.4.2)$$

It is assumed that the functions f, f'_x, g, h are continuous and $f > 0$, $g > 0$.

First, using the Rayleigh–Ritz principle, one finds an upper estimate for the first eigenvalue λ_1^0 [this value is determined by the right-hand side of relation (2.6.3.2)]. Then, one solves numerically the Cauchy problem for the auxiliary equation

$$[f(x)y'_x]'_x + [\lambda_1^0 g(x) - h(x)]y = 0 \qquad (2.8.4.3)$$

with the boundary conditions

$$y(0) = 0, \qquad y'_x(0) = 1. \qquad (2.8.4.4)$$

The function $y(x, \lambda_1^0)$ satisfies the condition $y(x_0, \lambda_1^0) = 0$, where $x_0 < 1$. The criterion of closeness of the exact and approximate solutions, λ_1 and λ_1^0, has the form of the inequality $|1 - x_0| \le \delta$, where δ is a sufficiently small given constant. If this inequality does not hold, one constructs a refinement for the approximate eigenvalue on the basis of the formula

$$\lambda_1^1 = \lambda_1^0 - \varepsilon_0 f(1)\frac{[y'_x(1)]^2}{\|y\|^2}, \qquad \varepsilon_0 = 1 - x_0, \qquad (2.8.4.5)$$

where $\|y\|^2 = \int_0^1 g(x)y^2(x)\,dx$. Then the value λ_1^1 is substituted for λ_1^0 in the Cauchy problem (2.8.4.3)–(2.8.4.4). As a result, a new solution y and a new point x_1 are found; and one has to check whether the criterion $|1 - x_1| \le \delta$ holds. If this inequality is violated, one refines the approximate eigenvalue by means of the formula

$$\lambda_1^2 = \lambda_1^1 - \varepsilon_1 f(1)\frac{[y'_x(1)]^2}{\|y\|^2}, \qquad \varepsilon_1 = 1 - x_1, \qquad (2.8.4.6)$$

and repeats the above procedure.

Remark 2.7. Formulas of the type (2.8.4.5) are obtained by a perturbation method based on a transformation of the independent variable x (see Section 3.6.3). If $x_n > 1$, the functions f, g, and h are smoothly extended to the interval $(1, \xi]$, where $\xi \ge x_n$.

Remark 2.8. The algorithm described above possesses the property of accelerated convergence, $\varepsilon_{n+1} \sim \varepsilon_n^2$, which ensures that the relative error of the approximate solution becomes 10^{-4} to 10^{-8} after two or three iterations for $\varepsilon_0 \sim 0.1$. This method is quite effective for high-precision calculations, is fail-safe, and guarantees against accumulation of roundoff errors.

Remark 2.9. In a similar way, one can compute subsequent eigenvalues λ_m, $m = 2, 3, \ldots$ (to that end, a suitable initial approximation λ_m^0 should be chosen).

Remark 2.10. A similar computation scheme can also be used in the case of boundary conditions of the second and the third kinds, periodic boundary conditions, etc. (see the reference below).

Example 2.7. The eigenvalue problem for the equation

$$y''_{xx} + \lambda(1 + x^2)^{-2} y = 0$$

with the boundary conditions (2.8.4.2) admits an exact analytic solution and has eigenvalues $\lambda_1 = 15$, $\lambda_2 = 63, \ldots, \lambda_n = 16n^2 - 1$.

According to the Rayleigh–Ritz principle, formula (2.6.3.2) for $z = \sin(\pi x)$ yields the approximate value $\lambda_1^0 = 15.33728$. The solution of the Cauchy problem (2.8.4.3)–(2.8.4.4) with $f(x) = 1$, $g(x) = \lambda(1 + x^2)^{-2}$, $h(x) = 0$ yields $x_0 = 0.983848$, $1 - x_0 = 0.016152$, $\|y\|^2 = 0.024585$, $y'_x(x_0) = -0.70622822$.

The first iteration for the first eigenvalue is determined by (2.8.4.5) and results in the value $\lambda_1^1 = 14.99245$ with the relative error $\Delta\lambda/\lambda_1^1 = 5 \times 10^{-4}$.

The second iteration results in $\lambda_1^2 = 14.999986$ with the relative error $\Delta\lambda/\lambda_1^2 < 10^{-6}$.

Example 2.8. Consider the eigenvalue problem for the equation

$$(\sqrt{1 + x}\, y'_x)'_x + \lambda y = 0$$

with the boundary conditions (2.8.4.2).

The Rayleigh–Ritz principle yields $\lambda_1^0 = 11.995576$. The next two iterations result in the values $\lambda_1^1 = 11.898578$ and $\lambda_1^2 = 11.898458$. For the relative error we have $\Delta\lambda/\lambda_1^2 < 10^{-5}$.

2.8.5 Well-Conditioned and Ill-Conditioned Problems

Numerical methods can only be applied to *well-conditioned* linear problems, in which small perturbations in the initial data (or the right-hand side of the equation, which determines its nonhomogeneity) lead to small changes in the solution. Otherwise, when the problem is *ill-conditioned*, small perturbations in the initial data (or the right-hand side of the equation) or equivalent small errors of the numerical method can significantly distort the solution.

Example 2.9. Let us look at the linear second-order ordinary differential equation

$$y''_{xx} + (1 + a)y'_x + ay = 0 \tag{2.8.5.1}$$

subject to the initial conditions

$$y(0) = 1, \quad y'_x(0) = -1, \tag{2.8.5.2}$$

where a is a free parameter ($a \neq 1$).

The solution of problem (2.8.5.1)–(2.8.5.2) is

$$y = e^{-x}. \tag{2.8.5.3}$$

Now let us suppose that the boundary conditions of equation (2.8.5.1) are slightly changed:

$$y(0) = 1 + \varepsilon, \quad y'_x(0) = -1, \tag{2.8.5.4}$$

where ε is a small positive number.

The solution of problem (2.8.5.1), (2.8.5.4) is

$$y_\varepsilon = \left(1 - \frac{a\varepsilon}{1 - a}\right) e^{-x} + \frac{\varepsilon}{1 - a} e^{-ax}. \tag{2.8.5.5}$$

Solution (2.8.5.5) behaves qualitatively differently depending on the value of the parameter a. Consider the possible situations.

If $a > 0$, solution (2.8.5.5) decays exponentially as $x \to \infty$. The difference between solutions (2.8.5.3) and (2.8.5.5) vanishes as $\varepsilon \to 0$ for all $x \geq 0$. If $a = 0$, the difference between solutions (2.8.5.3) and (2.8.5.5) is a small constant quantity equal to ε. If $a \geq 0$, problem (2.8.5.1)–(2.8.5.2) is well-conditioned.

Remark 2.11. For $0 < a < 1$ and fixed $\varepsilon > 0$, the second term in solution (2.8.5.5), proportional to e^{-ax}, dominates as $x \to \infty$ and the relative disturbance, $|y_\varepsilon - y|/y$, due to the small perturbation in the initial conditions, tends to infinity.

If $a < 0$, solution (2.8.5.5) increases without bound as $x \to \infty$. In this case, for any $\varepsilon > 0$, solutions (2.8.5.3) and (2.8.5.5) diverge indefinitely far apart as $x \to \infty$. If $a < 0$, problem (2.8.5.1)–(2.8.5.2) is ill-conditioned.

Remark 2.12. It is easy to show that for $a < 0$, the solution to the equation

$$y''_{xx} + (1 + a)y'_x + ay = \varepsilon \qquad (\varepsilon \ll 1)$$

subject to the initial conditions (2.8.5.2) increases indefinitely as $x \to \infty$. This means that for $a < 0$, problem (2.8.5.1)–(2.8.5.2) is ill-conditioned with respect to perturbations of the right-hand side.

For more details about iteration and numerical methods, see the list of references given below.

⊙ *Literature for Section 2.8*: J. D. Lambert (1973), N. N. Kalitkin (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), W. E. Schiesser (1994), L. F. Shampine (1994), L. D. Akulenko and S. V. Nesterov (1996, 1997, 2005), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and A. V. Manzhirov (2007), S. C. Chapra and R. P. Canale (2010).

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Chapter 3

Methods for Second-Order Nonlinear Differential Equations

3.1 General Concepts. Cauchy Problem. Uniqueness and Existence Theorems

3.1.1 Equations Solved for the Derivative. General Solution

A *second-order ordinary differential equation* solved for the highest derivative has the form

$$y''_{xx} = f(x, y, y'_x). \tag{3.1.1.1}$$

A *solution of a differential equation* is a function $y(x)$ that, when substituted into the equation, turns it into an identity. The *general solution of a differential equation* is the set of all its solutions.

The general solution of this equation depends on two arbitrary constants, C_1 and C_2. In some cases, the general solution can be written in explicit form, $y = \varphi(x, C_1, C_2)$, but more often implicit or parametric forms of the general solution are encountered.

3.1.2 Cauchy Problem. Existence and Uniqueness Theorem

Cauchy problem: Find a solution of equation (3.1.1.1) satisfying the *initial conditions*

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1. \tag{3.1.2.1}$$

(At a point $x = x_0$, the value of the unknown function, y_0, and its derivative, y_1, are prescribed.)

EXISTENCE AND UNIQUENESS THEOREM. *Let $f(x, y, z)$ be a continuous function in all its arguments in a neighborhood of a point (x_0, y_0, y_1) and let f have bounded partial derivatives f_y and f_z in this neighborhood, or the Lipschitz condition is satisfied: $|f(x, y, z) - f(x, \bar{y}, \bar{z})| \le K\big(|y - \bar{y}| + |z - \bar{z}|\big)$, where K is some positive number. Then a solution of equation (3.1.1.1) satisfying the initial conditions (3.1.2.1) exists and is unique.*

⊙ *Literature for Section 3.1*: E. L. Ince (1956), G. M. Murphy (1960), L. E. El'sgol'ts (1961), P. Hartman (1964), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and Rota (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997),

C. Chicone (1999), G. A. Korn and T. M. Korn (2000), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

3.2 Some Transformations. Equations Admitting Reduction of Order

3.2.1 Equations Not Containing y or x Explicitly. Related Equations

► **Equations not containing y explicitly.**

In the general case, an equation that does not contain y implicitly has the form

$$F(x, y'_x, y''_{xx}) = 0. \tag{3.2.1.1}$$

Such equations remain unchanged under an arbitrary translation of the dependent variable: $y \to y + \text{const}$. The substitution $y'_x = z(x)$, $y''_{xx} = z'_x(x)$ brings (3.2.1.1) to a first-order equation: $F(x, z, z'_x) = 0$.

► **Equations not containing x explicitly (autonomous equations).**

In the general case, an equation that does not contain x implicitly has the form

$$F(y, y'_x, y''_{xx}) = 0. \tag{3.2.1.2}$$

Such equations remain unchanged under an arbitrary translation of the independent variable: $x \to x + \text{const}$. Using the substitution $y'_x = w(y)$, where y plays the role of the independent variable, and taking into account the relations $y''_{xx} = w'_x = w'_y y'_x = w'_y w$, one can reduce (3.2.1.2) to a first-order equation: $F(y, w, ww'_y) = 0$.

Example 3.1. Consider the autonomous equation

$$y''_{xx} = f(y),$$

which often arises in the theory of heat and mass transfer and combustion. The change of variable $y'_x = w(y)$ leads to a separable first-order equation: $ww'_y = f(y)$. Integrating yields

$$w^2 = 2F(w) + C_1, \quad \text{where} \quad F(w) = \int f(w)\,dw.$$

where C_1 is an arbitrary constant. Solving for w and returning to the original variable, we obtain the separable equation $y'_x = \pm\sqrt{2F(w) + C_1}$. Its general solution is expressed as

$$\int \frac{dy}{\sqrt{2F(w) + C_1}} = \pm x + C_2,$$

or

$$\left[\int \frac{dy}{\sqrt{F(w) + c_1}}\right]^2 = 2(x + c_2)^2,$$

where C_2, c_1, and c_2 are arbitrary constants.

Remark 3.1. The equation $y''_{xx} = f(y + ax^2 + bx + c)$ is reduced by the change of variable $u = y + ax^2 + bx + c$ to an autonomous equation, $u''_{xx} = f(u) + 2a$.

▶ Related equations.

Consider equations of the form

$$F(ax+by, y'_x, y''_{xx}) = 0.$$

Such equations are invariant under simultaneous translations of the independent and dependent variables in accordance with the rule $x \to x + bc$, $y \to y - ac$, where c is an arbitrary constant.

For $b = 0$, see equation (3.2.1.1). For $b \neq 0$, the substitution $bw = ax + by$ leads to equation (3.2.1.2): $F(bw, w'_x - a/b, w''_{xx}) = 0$.

3.2.2 Homogeneous Equations

▶ Equations homogeneous in the independent variable.

The *equations homogeneous in the independent variable* remain unchanged under scaling of the independent variable, $x \to \alpha x$, where α is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$F(y, xy'_x, x^2 y''_{xx}) = 0. \qquad (3.2.2.1)$$

The substitution $z(y) = xy'_x$ leads to a first-order equation: $F(y, z, zz'_y - z) = 0$.

▶ Equations homogeneous in the dependent variable.

The *equations homogeneous in the dependent variable* remain unchanged under scaling of the variable sought, $y \to \alpha y$, where α is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$F(x, y'_x/y, y''_{xx}/y) = 0. \qquad (3.2.2.2)$$

The substitution $z(x) = y'_x/y$ leads to a first-order equation: $F(x, z, z'_x + z^2) = 0$.

▶ Equations homogeneous in both variables.

The *equations homogeneous in both variables* are invariant under simultaneous scaling (dilatation) of the independent and dependent variables, $x \to \alpha x$ and $y \to \alpha y$, where α is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$F(y/x, y'_x, xy''_{xx}) = 0. \qquad (3.2.2.3)$$

The transformation $t = \ln|x|$, $w = y/x$ leads to an autonomous equation

$$F(w, w'_t + w, w''_{tt} + w'_t) = 0,$$

see Section 3.2.1.

Example 3.2. The homogeneous equation

$$xy''_{xx} - y'_x = f(y/x)$$

is reduced by the transformation $t = \ln|x|$, $w = y/x$ to the autonomous form: $w''_{tt} = f(w) + w$. For the solution of this equation, see Example 3.1 in Section 3.2.1 (the function on the right-hand side has to be changed there).

3.2.3 Generalized Homogeneous Equations

► **Equations of a special form.**

The *generalized homogeneous equations* remain unchanged under simultaneous scaling of the independent and dependent variables in accordance with the rule $x \to \alpha x$ and $y \to \alpha^k y$, where α is an arbitrary nonzero number and k is some number. Such equations can be written in the form

$$F(x^{-k}y, x^{1-k}y'_x, x^{2-k}y''_{xx}) = 0. \qquad (3.2.3.1)$$

The transformation $t = \ln x$, $w = x^{-k}y$ leads to an autonomous equation (see Section 3.2.1):

$$F\big(w, w'_t + kw, w''_{tt} + (2k-1)w'_t + k(k-1)w\big) = 0.$$

► **Equations of the general form.**

The most general form of representation of generalized homogeneous equations is as follows:

$$\mathcal{F}(x^n y^m, xy'_x/y, x^2 y''_{xx}/y) = 0. \qquad (3.2.3.2)$$

The transformation $z = x^n y^m$, $u = xy'_x/y$ brings this equation to the first-order equation

$$\mathcal{F}\big(z, u, z(mu+n)u'_z - u + u^2\big) = 0.$$

Remark 3.2. For $m \neq 0$, equation (3.2.3.2) is equivalent to equation (3.2.3.1) in which $k = -n/m$. To the particular values $n = 0$ and $m = 0$ there correspond equations (3.2.2.1) and (3.2.2.2) homogeneous in the independent and dependent variables, respectively. For $n = -m \neq 0$, we have an equation homogeneous in both variables, which is equivalent to equation (3.2.2.3).

3.2.4 Equations Invariant under Scaling–Translation Transformations

► **Equations of the fist type.**

The equations of the form

$$F(e^{\lambda x}y, e^{\lambda x}y'_x, e^{\lambda x}y''_{xx}) = 0 \qquad (3.2.4.1)$$

remain unchanged under simultaneous translation and scaling of variables, $x \to x + \alpha$ and $y \to \beta y$, where $\beta = e^{-\alpha\lambda}$ and α is an arbitrary number. The substitution $w = e^{\lambda x}y$ brings (3.2.4.1) to an autonomous equation: $F(w,\, w'_x - \lambda w,\, w''_{xx} - 2\lambda w'_x + \lambda^2 w) = 0$ (see Section 3.2.1).

► **Equations of the first type. Alterative representation.**

The equation

$$F(e^{\lambda x}y^n, y'_x/y, y''_{xx}/y) = 0 \qquad (3.2.4.2)$$

is invariant under the simultaneous translation and scaling of variables, $x \to x + \alpha$ and $y \to \beta y$, where $\beta = e^{-\alpha\lambda/n}$ and α is an arbitrary number. The transformation $z = e^{\lambda x}y^n$, $w = y'_x/y$ brings (3.2.4.2) to a first-order equation: $F\big(z, w, z(nw+\lambda)w'_z + w^2\big) = 0$.

► **Equations of the second type.**

The equation

$$F(x^n e^{\lambda y}, xy'_x, x^2 y''_{xx}) = 0 \tag{3.2.4.3}$$

is invariant under the simultaneous scaling and translation of variables, $x \to \alpha x$ and $y \to y + \beta$, where $\alpha = e^{-\beta\lambda/n}$ and β is an arbitrary number. The transformation $z = x^n e^{\lambda y}$, $w = xy'_x$ brings (3.2.4.3) to a first-order equation: $F\big(z, w, z(\lambda w + n)w'_z - w\big) = 0$.

3.2.5 Exact Second-Order Equations

The second-order equation

$$F(x, y, y'_x, y''_{xx}) = 0 \tag{3.2.5.1}$$

is said to be exact if it is the total differential of some function, $F = \varphi'_x$, where $\varphi = \varphi(x, y, y'_x)$. If equation (3.2.5.1) is exact, then we have a first-order equation for y:

$$\varphi(x, y, y'_x) = C, \tag{3.2.5.2}$$

where C is an arbitrary constant.

If equation (3.2.5.1) is exact, then $F(x, y, y'_x, y''_{xx})$ must have the form

$$F(x, y, y'_x, y''_{xx}) = f(x, y, y'_x)y''_{xx} + g(x, y, y'_x). \tag{3.2.5.3}$$

Here f and g are expressed in terms of φ by the formulas

$$f(x, y, y'_x) = \frac{\partial\varphi}{\partial y'_x}, \qquad g(x, y, y'_x) = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}y'_x. \tag{3.2.5.4}$$

By differentiating (3.2.5.4) with respect to x, y, and $p = y'_x$, we eliminate the variable φ from the two formulas in (3.2.5.4). As a result, we have the following test relations for f and g:

$$\begin{gathered} f_{xx} + 2pf_{xy} + p^2 f_{yy} = g_{xp} + pg_{yp} - g_y, \\ f_{xp} + pf_{yp} + 2f_y = g_{pp}. \end{gathered} \tag{3.2.5.5}$$

Here the subscripts denote the corresponding partial derivatives.

If conditions (3.2.5.5) hold, then equation (3.2.5.1) with F of (3.2.5.3) is exact. In this case, we can integrate the first equation in (3.2.5.4) with respect to $p = y'_x$ to determine $\varphi = \varphi(x, y, y'_x)$:

$$\varphi = \int f(x, y, p)\,dp + \psi(x, y), \tag{3.2.5.6}$$

where $\psi(x, y)$ is an arbitrary function of integration. This function is determined by substituting (3.2.5.6) into the second equation in (3.2.5.4).

Example 3.3. The left-hand side of the equation

$$yy''_{xx} + (y'_x)^2 + 2axyy'_x + ay^2 = 0 \tag{3.2.5.7}$$

can be represented in the form (3.2.5.3), where $f = y$ and $g = p^2 + 2axyp + ay^2$. It is easy to verify that conditions (3.2.5.5) are satisfied. Hence, equation (3.2.5.7) is exact. Using (3.2.5.6), we obtain

$$\varphi = yp + \psi(x, y). \tag{3.2.5.8}$$

Substituting this expression into the second equation in (3.2.5.4) and taking into account the relation $g=p^2+2axyp+ay^2$, we find that $2axyp+ay^2=\psi_x+p\psi_y$. Since $\psi=\psi(x,y)$, we have $2axy=\psi_y$ and $ay^2=\psi_x$. Integrating yields $\psi=axy^2+\text{const}$. Substituting this expression into (3.2.5.8) and taking into account relation (3.2.5.2), we find a first integral of equation (3.2.5.7):

$$yp+axy^2=C_1, \qquad \text{where} \quad p=y'_x.$$

Setting $w=y^2$, we arrive at the first-order linear equation $w'_x+2axw=2C_1$, which is easy to integrate. Thus, we find the solution of the original equation in the form:

$$y^2=2C_1\exp\bigl(-ax^2\bigr)\int\exp\bigl(ax^2\bigr)\,dx+C_2\exp\bigl(-ax^2\bigr).$$

3.2.6 Nonlinear Equations Involving Linear Homogeneous Differential Forms

Consider the nonlinear differential equation

$$F\bigl(x,\mathrm{L}_1[y],\mathrm{L}_2[y]\bigr)=0, \tag{3.2.6.1}$$

where the $\mathrm{L}_n[y]$ are linear homogeneous differential forms,

$$\mathrm{L}_n[y]=\sum_{m=0}^{2}\varphi_{\mathrm{m}}^{(n)}(x)y_x^{(m)}, \qquad n=1,\,2.$$

Let $y_0=y_0(x)$ be a common particular solution of the two linear equations

$$\mathrm{L}_1[y_0]=0, \qquad \mathrm{L}_2[y_0]=0.$$

Then the substitution

$$w=\psi(x)\bigl[y_0(x)y'_x-y'_0(x)y\bigr] \tag{3.2.6.2}$$

with an arbitrary function $\psi(x)$ reduces by one the order of equation (3.2.6.1).

Example 3.4. Consider the second-order nonlinear equation

$$y''_{xx}=f(x)g(xy'_x-y).$$

It can be represented in the form (3.2.6.1) with

$$F(x,u,w)=w-f(x)g(u), \qquad u=\mathrm{L}_1[y]=xy'_x-y, \qquad w=\mathrm{L}_2[y]=y''_{xx}.$$

The linear equations $\mathrm{L}_n[y]=0$ are

$$xy'_x-y=0, \qquad y''_{xx}=0.$$

These equations have a common particular solution $y_0=x$. Therefore, the substitution $w=xy'_x-y$ (see formula (3.2.6.2) with $\psi(x)=1$) leads to a first-order equation with separable variables:

$$w'_x=xf(x)g(w).$$

For the solution of this equation, see Section 1.2.1.

3.2.7 Reduction of Quasilinear Equations to the Normal Form

Consider the quasilinear equation

$$y''_{xx} + f(x)y'_x + g(x)y = \Phi(x,y) \tag{3.2.7.1}$$

with linear left-hand side and nonlinear right-hand side. Let $y_1(x)$ and $y_2(x)$ form a fundamental system of solutions of the truncated linear equation corresponding to $\Phi \equiv 0$. The transformation

$$\xi = \frac{y_2(x)}{y_1(x)}, \qquad u = \frac{y}{y_1(x)} \tag{3.2.7.2}$$

brings equation (3.2.7.1) to the normal form:

$$u''_{\xi\xi} = \Psi(\xi,u), \qquad \text{where} \quad \Psi(\xi,u) = \frac{y_1^3(x)}{W^2(x)}\Phi\big(x, y_1(x)u\big).$$

Here, $W(x) = y_1y_2' - y_2y_1'$ is the Wronskian of the truncated equation; and the variable x must be expressed in terms of ξ using the first relation in (3.2.7.2).

Transformation (3.2.7.2) is convenient for the simplification and classification of equations having the form (3.2.7.1) with $\Phi(x,y) = h(x)y^k$, thus reducing the number of functions from three to one: $\{f, g, h\} \Longrightarrow \{0, 0, h_1\}$.

Example 3.5. Consider the equation

$$y''_{xx} - y'_x = e^{2x} f(y). \tag{3.2.7.3}$$

A fundamental system of solutions of the truncated linear equation with $f(y) \equiv 0$ are $y_1(x) = 1$ and $y_2(x) = e^x$. The transformation

$$\xi = e^x, \qquad u = y$$

brings equation (3.2.7.3) to the normal form:

$$u''_{xx} = f(u).$$

For solution of this autonomous equation, see Example 3.1 in Section 3.2.1.

3.2.8 Equations Defined Parametrically and Differential-Algebraic Equations

► **Preliminary remarks.**

In fluid dynamics, one often employs *von Mises* or *Crocco type transformations* to lower the order of boundary layer equations (and also some reduced equations that follow from the *Navier–Stokes equations*). Such transformations use suitable first- or second-order partial derivatives as new independent variables. The resulting equations sometimes admit exact solutions that are represented in implicit or parametric form. This leads to the problem: how to obtain exact solutions of the original hydrodynamic equations using these intermediate solutions.

To solve this problem, one has to be able to solve nonlinear ordinary differential equations defined parametrically. Due to their unusual form, such non-classical ODEs have been given very little attention.

► General form of equations defined parametrically. Some examples.

In general, second-order ordinary differential equations defined parametrically are defined by two coupled equations of the form

$$F_1(x, y, y'_x, y''_{xx}, t) = 0, \qquad F_2(x, y, y'_x, y''_{xx}, t) = 0, \tag{3.2.8.1}$$

where $y = y(x)$ is an unknown function, $t = t(x)$ is a functional parameter, $F_1(\dots)$ and $F_2(\dots)$ are given functions of their arguments. Below we consider two cases.

1°. *Degenerate case.* We assume that the derivative y''_{xx} can be eliminated from the equations (3.2.8.1) and the resulting equation can be solved for y'_x to obtain $y'_x = F(x, y, t)$. Using this expression, we eliminate the first derivative from one of the equations (3.2.8.1) to get $F_3(x, y, y''_{xx}, t) = 0$ and then solve this equation for y''_{xx}. The outlined procedure reduces the original equation (3.2.8.1) to the *canonical form*

$$y'_x = F(x, y, t), \qquad y''_{xx} = G(x, y, t). \tag{3.2.8.2}$$

Note that parametrically defined nonlinear differential equations (3.2.8.2) form a special class of coupled *differential-algebraic equations*.

Below we give a description of a method for integrating such equations and list a few simple equations of this kind whose general solutions can be obtained in parametric form; we deal with the general case where the parameter t cannot be eliminated from the equations (3.2.8.2).

On differentiating the first equation in (3.2.8.2) with respect to t, we obtain $(y'_x)'_t = F_x x'_t + F_y y'_t + F_t$. Taking into account the relations $y'_t = F x'_t$ and $(y'_x)'_t = x'_t y''_{xx}$, we find that

$$x'_t y''_{xx} = F_x x'_t + F F_y x'_t + F_t. \tag{3.2.8.3}$$

Eliminating the second derivative y''_{xx} with the help of equation (3.2.8.2), we arrive at the first-order equation

$$(G - F_x - F F_y) x'_t = F_t. \tag{3.2.8.4}$$

Taking into account that $y'_t = F x'_t$, we rewrite (3.2.8.4) in the form

$$(G - F_x - F F_y) y'_t = F F_t. \tag{3.2.8.5}$$

Equations (3.2.8.4) and (3.2.8.5) represent a system of first-order equations for $x = x(t)$ and $y = y(t)$. If we manage to solve this system, we thus obtain a solution to the original equation (3.2.8.2) in parametric form.

In some cases, it may be more convenient to use one of the equations (3.2.8.4) or (3.2.8.5) and the first equation (3.2.8.2).

Remark 3.3. With the above manipulations, isolated solutions may be lost, which satisfy the relation $G - F_x - F F_y = 0$ (this issue requires a further analysis).

Let us look at two special cases.

1°. If

$$G = F_x + F F_y + a(t) b(x) F_t,$$

where $a(t)$, $b(x)$, and $F = F(x,y,t)$ are arbitrary functions, the variables in equation (3.2.8.4) separate, thus resulting in the solution

$$\int b(x)\,dx = \int \frac{dt}{a(t)} + C_1$$

with C_1 is an arbitrary constant.

2°. If

$$G = F_x + FF_y + a(t)b(y)FF_t,$$

where $a(t)$, $b(y)$, and $F = F(x,y,t)$ are arbitrary functions, the variables in equation (3.2.8.5) separate, thus resulting in the solution

$$\int b(y)\,dy = \int \frac{dt}{a(t)} + C_1$$

with C_1 is an arbitrary constant.

Below are a few simple equations of the form (3.2.8.2) whose general solution can be obtained in parametric form.

Example 3.6. Consider the second-order parametric ODE

$$y'_x = \varphi(t), \qquad y''_{xx} = \psi(t), \tag{3.2.8.6}$$

where t is the parameter, while $\varphi(t)$ and $\psi(t)$ are given, sufficiently arbitrary functions.

In this case,

$$F = \varphi(t), \qquad G = \psi(t).$$

Substituting these expressions into (3.2.8.4) gives the equation $\psi(t)x'_t = \varphi'_t(t)$, whose general solution is

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \tag{3.2.8.7}$$

where C_1 is an arbitrary constant. Expression (3.2.8.7) together with the first equation (3.2.8.6) represent a first-order parametric ODE of the form (1.8.3.7) with

$$f(t) = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \qquad g(t) = \varphi(t). \tag{3.2.8.8}$$

Substituting (3.2.8.8) into (1.8.3.9) yields the general solution to ODE (3.2.8.6) in parametric form:

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \qquad y = \int \frac{\varphi(t)\varphi'_t(t)}{\psi(t)}\,dt + C_2, \tag{3.2.8.9}$$

where C_1 and C_2 are arbitrary constants.

Example 3.7. Consider equation (3.2.8.2) with

$$F = f(x)g(y)h(t), \qquad G = f^2(x)g(y)g'_y(y)h^2(t) - f'_x(x)g(y)\lambda(t), \tag{3.2.8.10}$$

where $f(x)$, $g(y)$, $h(t)$, and $\lambda(t)$ are arbitrary functions. Equation (3.2.8.4) now becomes

$$f'_x(x)\big[h(t) + \lambda(t)\big]x'_t = -f(x)h'_t(t), \tag{3.2.8.11}$$

and its general solution is expressed as

$$f(x) = C_1E(t), \qquad E(t) = \exp\biggl[-\int \frac{h'_t(t)\,dt}{h(t) + \lambda(t)}\biggr], \tag{3.2.8.12}$$

where C_1 is an arbitrary constant. Substituting expressions (3.2.8.10) and (3.2.8.12) into the first equation (3.2.8.2), we arrive at the separable first-order equation

$$y'_t = -g(y)\frac{f^2(x)}{f'_x(x)}\frac{h(t)h'_t(t)}{h(t)+\lambda(t)}, \tag{3.2.8.13}$$

in which x must be expressed via t using the integral (3.2.8.12).

In particular, if $f(x)=x$, the general solution to equation (3.2.8.13) is

$$\int \frac{dy}{g(y)} = -C_1^2 \int \frac{h(t)h'_t(t)E^2(t)}{h(t)+\lambda(t)}\,dt + C_2. \tag{3.2.8.14}$$

Formulas (3.2.8.12) and (3.2.8.14), where C_1 and C_2 are arbitrary constants, define the general solution to equation (3.2.8.2), (3.2.8.10) with $f(x)=x$.

Example 3.8. Consider a special case of equation (3.2.8.2) with

$$G = F_x + FF_y, \tag{3.2.8.15}$$

where $F=F(x,y,t)$ is an arbitrary function. In this case, the expressions in parentheses in (3.2.8.4) and (3.2.8.5) vanish and equation (3.2.8.2) admits the first integral

$$y'_x = F(x,y,C_1),$$

where C_1 is an arbitrary constant. In addition, there is a singular solution which is described by the parametric first-order equation

$$y'_x = F(x,y,t), \qquad F_t(x,y,t)=0.$$

2°. *Degenerate case.* Suppose one of the two equations in (3.2.8.1) does not contain derivatives. If the other equation can be solved for y''_{xx}, we obtained a parametrically defined equation of the form

$$F(x,y,t)=0, \qquad y''_{xx} = G(x,y,y'_x,t). \tag{3.2.8.16}$$

By differentiation of the first relation, such equations can be reduced to a nonlinear system of second-order equations. Without writing out this system, we give an example of such an equation whose solution can be obtained in parametric form.

Example 3.9. Consider the following second-order ODE defined parametrically:

$$y=\varphi(t), \qquad y''_{xx} = \psi(t). \tag{3.2.8.17}$$

Its solution is sought in the parametric form

$$x = \int f(t)\,dt + A, \qquad y=\varphi(t). \tag{3.2.8.18}$$

The derivatives are expressed as

$$y'_x = \frac{y'_t}{x'_t} = \frac{\varphi'_t}{f}, \qquad y''_{xx} = (y'_x)'_x = \frac{(y'_x)'_t}{x'_t} = \frac{(\varphi'_t/f)'_t}{f}. \tag{3.2.8.19}$$

By comparing the second derivatives in (3.2.8.17) and (3.2.8.19), we obtain a first-order equation for $f=f(t)$:

$$(\varphi'_t/f)'_t = \psi f. \tag{3.2.8.20}$$

The differentiation with respect to t in (3.2.8.20) results in a Bernoulli equation, whose general solution is expressed as

$$f(t) = \pm\varphi'_t(t)\left[2\int \psi(t)\varphi'_t(t)\,dt + B\right]^{-1/2}, \tag{3.2.8.21}$$

where B is an arbitrary constant. Formulas (3.2.8.18) and (3.2.8.21) define the general solution to to equation (3.2.8.17) in parametric form.

► **Reduction of standard differential equations to parametric differential equations**

A standard second-order ODE of the form

$$y''_{xx} = G(x, y, y'_x) \tag{3.2.8.22}$$

can be represented as a parametric ODE defined by two relations

$$\begin{aligned} y'_x &= t, \\ y''_{xx} &= G(x, y, t). \end{aligned} \tag{3.2.8.23}$$

This equation is a special case of equation (3.2.8.2) with $F(x, y, t) = t$; it can be reduced to the standard system of first-order ODEs

$$\begin{aligned} G(x, y, t)\, x'_t &= 1, \\ G(x, y, t)\, y'_t &= t. \end{aligned} \tag{3.2.8.24}$$

This system is obtained by substituting $F = t$ into equations (3.2.8.4)–(3.2.8.5).

System (3.2.8.24) is useful for the numerical solution of blow-up Cauchy problems or problems with a root singularity, in which the solution $y = y(x)$ or its derivative become infinite at a finite value $x = x_*$ (the value x_* is unknown in advance and has to be determined in the solution of the problem). In such and similar problems, the critical value $x = x_*$ for equation (3.2.8.22) corresponds to $t \to \pm\infty$ for system (3.2.8.24). For how one can use system (3.2.8.24) for the numerical integration of equations of the form (3.2.8.22) in blow-up problems, see Section 3.8.7.

⊙ *Literature for Section 3.2*: E. L. Ince (1956), G. M. Murphy (1960), L. E. El'sgol'ts (1961), P. Hartman (1964), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and Rota (1978), M. Tenenbaum and H. Pollard (1985), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), R. Grimshaw (1991), M. Braun (1993), D. Zwillinger (1997), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007), A. D. Polyanin (2016), A. D. Polyanin and A. I. Zhurov (2016a, 2016b).

3.3 Boundary Value Problems. Uniqueness and Existence Theorems. Nonexistence Theorems

◆ *Nonlinear boundary value problems for ODEs are much more complex for mathematical analysis than initial value problems. This is because initial value problems (with well-behaved functions) have unique solutions (i.e., are "well-posed"), whereas boundary value problems (even with well-behaved functions) may have one solution, several solutions, or no solution at all. This section highlights characteristic features of different classes of nonlinear boundary value problem, states useful theorems on existence or nonexistence of solutions, and discusses examples of specific problems having nonunique solutions.*

3.3.1 Uniqueness and Existence Theorems for Boundary Value Problems

► **Preliminary remarks.**

► **First boundary value problems. Existence theorems.**

We will be looking at boundary value problems for second-order nonlinear differential equations of the form

$$y''_{xx} = f(x, y, y'_x) \tag{3.3.1.1}$$

defined on the unit interval $0 \le x \le 1$ (as shown in Section 2.5.2, any finite interval for the independent variable can reduced to a unit interval) and subject to the first-type boundary conditions*

$$y(0) = A, \qquad y(1) = B. \tag{3.3.1.2}$$

EXISTENCE THEOREMS. *The first boundary value problem* (3.3.1.1)–(3.3.1.2) *has at least one solution if the function* $f = f(x, y, z)$ *is continuous in the domain* $\Omega = \{0 \le x \le 1,\ -\infty < y, z < \infty\}$ *and any of the following four assumptions holds:*

1. $f(x, y, z)$ *is bounded;*

2. For sufficiently large $|y|$, *the inequality* $f(x, y, z) < k|y|$ *holds, where* $k < \sqrt{3\pi^3} \approx 9.645$*;*

3. $\dfrac{f(x, y, z)}{|y| + |z|} \to 0$ *uniformly on the interval* $0 \le x \le 1$ *as* $|y| + |z| \to \infty$*; in addition, on each finite interval,* f *satisfies the Lipschitz condition*

$$|f(x, y, z) - f(x, \bar{y}, \bar{z})| \le K|y - \bar{y}| + L|z - \bar{z}|, \tag{3.3.1.3}$$

where K *and* L *are some positive numbers (Lipschitz constants);*

4. f *satisfies the Lipschitz condition* (3.3.1.3) *and has the form* $f = \varphi(x, y) + \psi(x, y, z)$, *where* φ *is continuous and monotonically increasing with respect to* y, *and* $\dfrac{\psi(x, y, z)}{|y| + |z|} \to 0$ *uniformly on the interval* $0 \le x \le 1$ *as* $|y| + |z| \to \infty$.

UNIQUENESS AND EXISTENCE THEOREMS.

1. Let the function $f = f(x, y, z)$ *be continuous in the domain* $\Omega = \{0 \le x \le 1,\ -\infty < y, z < \infty\}$ *and satisfy the Lipschitz condition* (3.3.1.3)*. Then problem* (3.3.1.1)–(3.3.1.2) *has one and only one solution if the inequality* $\frac{1}{8}K + \frac{1}{2}L < 1$ *holds, where* K *and* L *are Lipschitz constants.*

2. Let the function $f = f(x, y, z)$ *be continuous in the domain* $\Omega_N = \{0 \le x \le 1,\ -N \le y \le N,\ -4N \le z \le 4N\}$ *and satisfy the Lipschitz condition* (3.3.1.3) *in* Ω_N. *In addition, let*

$$m = \max_{0 \le x \le 1} |f(x, 0, 0)|, \qquad M = \max_{x, y, z \in \Omega_N} |f(x, y, z)|.$$

Then if

$$\alpha = \tfrac{1}{8}K + \tfrac{1}{2}L < 1$$

*First-, second-, third-, and mixed-type boundary conditions for second-order nonlinear differential equations are stated in exactly the same way as for linear equations; see Section 2.5.1.

and any of the two inequalities

$$\text{(i)}\quad m \le 8N(1-\alpha),$$
$$\text{(ii)}\quad M \le 8N$$

hold, then problem (3.3.1.1)–(3.3.1.2) *has one and only one solution* $y = y(x)$ *such that*

$$|y| \le N, \qquad |y'_x| \le 4N \qquad (0 \le x \le 1).$$

Remark 3.4. Under certain conditions, the unique solution to problem (3.3.1.1)–(3.3.1.2) can be obtained with Picard's method of successive approximations by solving the equations

$$y''_n = f(x, y_{n-1}, y'_{n-1}),$$

where each y_n is chosen so as to satisfy the boundary conditions (3.3.1.2); the desired solution is $y = \lim_{n\to\infty} y_n$. For the iterative process to converge, it suffices that the Lipschitz conditions (3.3.1.3) hold.

EXISTENCE THEOREMS (FOR EQUATIONS OF A SPECIAL FORM). *The first boundary value problem*

$$y''_{xx} = f(x,y); \qquad y(0) = A, \quad y(1) = B \tag{3.3.1.4}$$

has at least one solution if $f = f(x,y)$ *is continuous in the domain* $\Omega = \{0 \le x \le 1,\ -\infty < y < \infty\}$ *and any of the following two assumptions holds:*

1. *The function* f *is monotonically increasing (nondecreasing) with respect to* y *and satisfies the Lipschitz condition* $|f(x,y) - f(x,\bar y)| \le K\big|y - \bar y|$ *on each finite interval (or if* f_y *is bounded on each finite interval).*
2. *If* $A = B = 0$ *and the inequality*

$$\int_0^y f(x,t)\,dt \ge -c_1 y^2 - c_0$$

holds, where $c_0 \ge 0$ *and* $0 < c_1 < \frac{1}{2}\pi^2$.

Remark 3.5. Problem (3.3.1.4) has a unique solution if $f = f(x,y)$ is continuous in the domain Ω and satisfies the Lipschitz condition with the Lipschitz constant $K < \pi^2$.

▶ First boundary value problems. Lower and upper solution. Nagumo theorem.

Definition 1. Twice differentiable functions $u = u(x)$ and $v = v(x)$ are said to be *a lower and an upper solution to the boundary value problem* (3.3.1.4) if the following inequalities hold:

$$\begin{gathered} u''_{xx} - f(x,u) \ge 0 \quad \text{at} \quad 0 < x < 1; \\ v'_{xx} - f(x,v) \le 0 \quad \text{at} \quad 0 < x < 1; \\ u(0) \le A \le v(0), \quad u(1) \le B \le v(1). \end{gathered} \tag{3.3.1.5}$$

Here, $u(0) = \lim_{x\to 0} u(x)$; the values $v(0)$, $u(1)$, and $v(1)$ are defined likewise.

NAGUMO-TYPE THEOREM (FOR EQUATIONS OF A SPECIAL FORM). *Let the boundary value problem* (3.3.1.4) *have a lower solution* $u = u(x)$ *and an upper solution* $v = v(x)$, *with* $u(x) \le v(x)$ *for* $0 \le x \le 1$. *In addition, let* $f(x,y)$ *be continuous and satisfy the Lipschitz condition on* $0 \le x \le 1$ *with* $u(x) \le y \le v(x)$. *Then there exists a solution* $y = y(x)$ *to problem* (3.3.1.4) *satisfying the inequalities*

$$u(x) \le y \le v(x) \qquad (0 \le x \le 1). \tag{3.3.1.6}$$

This theorem allows one to effectively determine the domain of existence of solutions to some classes of nonlinear boundary value problems. The linear functions $u = C_1 + D_1 x$ and $v = C_2 + D_2 x$ can be used as lower and upper solutions, with the coefficients C_i and D_i chosen so as to satisfy the inequalities (3.3.1.5).

Example 3.10. Consider the first boundary value problem for the Emden–Fowler equation

$$y''_{xx} = x^n y^m; \qquad y(0) = A, \quad y(1) = B. \tag{3.3.1.7}$$

Let $n \ge 0$, $m > 1$, $A \ge 0$, and $B > 0$. In this case, $u(x) \equiv 0$ is a lower solution. Any constant C such that $C \ge \max[A, B]$ can be taken to be the upper solution, $v(x) = C$. Then, by the Nagumo-type theorem, there is a nonnegative solution to the boundary value problem (3.3.1.7) satisfying the inequalities

$$0 \le y(x) \le \max[A, B].$$

Example 3.11. Consider the first boundary value problem for the equation with a cubic nonlinearity

$$y''_{xx} = y[y + g(x)][y - h(x)]; \qquad y(0) = A, \quad y(1) = B, \tag{3.3.1.8}$$

where $g(x) > 0$ and $h(x) > 0$ are continuous functions in the domain $0 \le x \le 1$.

Let $A \ge 0$ and $B > 0$. In this case, $u(x) \equiv 0$ is a lower solution. Let $h_{\max} = \max\limits_{0 \le x \le 1} h(x)$.

We will show that any constant C such that $C \ge \max[A, B, h_{\max}]$ can be taken as the upper solution, $v(x) = C$. Indeed, we have $f(x,v) \ge 0$ and, therefore, $v''_{xx} - f(x,v) \le 0$. Then, by the Nagumo-type theorem, there exists a nonnegative solution to the boundary value problem (3.3.1.8) satisfying the inequalities

$$0 \le y(x) \le \max[A, B, h_{\max}]. \tag{3.3.1.9}$$

The estimate (3.3.1.9) can be improved. To this end, the lower solution can be taken in the form $u = \delta > 0$, where $\delta \le \min[A, B, h_{\min}]$ with $h_{\min} = \min\limits_{0 \le x \le 1} h(x)$. The upper solution will be left unchanged. It follows that there exists a nonnegative solution to the boundary value problem (3.3.1.8) satisfying the inequalities

$$\min[A, B, h_{\min}] \le y(x) \le \max[A, B, h_{\max}].$$

Definition 2. The function $f(x,y,z)$ will be said to belong to the *class of Nagumo functions* on a set $(x,y) \in D$ if there is a positive continuous function $\varphi(z)$ satisfying the following two conditions:

(i) $|f(x,y,z)| \le \varphi(|z|)$ for all $(x,y) \in D$ and $-\infty < z < \infty$;

(ii) $\displaystyle\int_0^\infty \frac{z\,dz}{\varphi(z)} = \infty$.

NAGUMO THEOREM. *Let* $u(x)$ *be a lower solution and* $v(x)$ *an upper solution to the first boundary value problem* (3.3.1.1)–(3.3.1.2) *such that*

1. *The inequality* $u(x) < v(x)$ *holds for* $0 \le x \le 1$.

2. *The function* $f(x,y,z)$ *belongs to the class of Nagumo functions on the set* $D = \{0 \le x \le 1,\ u(x) < y < v(x)\}$.

3. The function $f(x,y,z)$ is continuous in x and continuously differentiable with respect to y and z in the domain $0 \le x \le 1$, $u(x) < y < v(x)$, $-\infty < z < \infty$.

Then there exists at least one twice continuously differentiable solution $y = y(x)$ to problem (3.3.1.1)–(3.3.1.2) satisfying the inequalities

$$u(x) < y < v(x) \qquad (0 \le x \le 1).$$

► **Third boundary value problems.**

Let us consider the equation (3.3.1.1) with third-type boundary conditions

$$\alpha_0 y - \alpha_1 y'_x = A \quad \text{at} \quad x = 0, \tag{3.3.1.10}$$

$$\beta_0 y - \beta_1 y'_x = B \quad \text{at} \quad x = 1, \tag{3.3.1.11}$$

where α_0, α_1, β_0, and β_1 are nonnegative constants with $\alpha_0 + \alpha_1 > 0$, $\beta_0 + \beta_1 > 0$, and $\alpha_0 + \beta_0 > 0$.

EXISTENCE AND UNIQUENESS THEOREM. *There exists a unique solution $y = y(x)$ of the boundary value problem (3.3.1.1), (3.3.1.11) if the following conditions hold:*

1. The function $f(x,y,z)$ is continuous on the set $\Omega = \{0 \le x < \infty,\ -\infty < y, z < \infty\}$.

2. There exists an $M > 0$ such that $|f(x,y,z_2) - f(x,y,z_1)| \le M|z_2 - z_1|$, on Ω.

3. The function $f(x,y,z)$ is nondecreasing with respect to y on the set Ω.

3.3.2 Reduction of Boundary Value Problems to Integral Equations. Integral Identity. Jentzch Theorem

► **Reduction of boundary value problems to integral equations.**

We will be looking at boundary value problems for second-order nonlinear differential equations of the form*

$$y''_{xx} + \lambda f(x, y, y'_x) = 0 \tag{3.3.2.1}$$

with parameter λ and homogeneous boundary conditions of a different kind on the unit interval $0 \le x \le 1$.

Assuming $f(x, y(x), y'_x(x))$ to be a known function of x and using formula (2.5.3.1) with $r(x) = -\lambda f(x, y(x), y'_x(x))$ as well as suitable Green's functions for the operator $L[y] = y''_{xx}$ (see the first four rows of Table 2.2 with $a = 1$), we can represent boundary value problems for equation (3.3.2.1) subject to boundary conditions of the first or mixed kind as a nonlinear integral equation with constant limits of integration:

$$y(x) = \lambda \int_0^1 |G(x,\xi)| f(\xi, y(\xi), y'_\xi(\xi))\, d\xi. \tag{3.3.2.2}$$

The modulus of the Green's function is used to stress that the kernel of the integral operator is positive.

Table 3.1 lists a few Green's functions $|G(x,\xi)|$, which appear in the integral equation (3.3.2.2), for several boundary value problems on the unit interval $0 \le x \le 1$. Note that Table 3.1 contains a new Green's function (for the third boundary value problem) as compared to Table 2.2.

*Note that equations (3.3.1.1) and (3.3.2.1) differ in form.

TABLE 3.1
Kernel of the integral operator $G(x,\xi) = |G(x,\xi)|$ appearing on the right-hand side of equation (3.3.2.2) for some boundary value problems ($0 \le x \le 1$, $0 \le \xi \le 1$)

No.	Boundary value problem	Boundary conditions	Green's function, $G(x,\xi)$
1	First	$y(0) = y(1) = 0$	$x(1-\xi)$ if $x \le \xi$ $\xi(1-x)$ if $\xi \le x$
2	Mixed	$y(0) = y'_x(1) = 0$	x if $x \le \xi$ ξ if $\xi \le x$
3	Mixed	$y'_x(0) = y(1) = 0$	$1-\xi$ if $x \le \xi$ $1-x$ if $\xi \le x$
4	Mixed	$y(0) = 0$, $y(1) + ky'_x(1) = 0$ (with $k \ne -1$)	$\dfrac{x(k+1-\xi)}{k+1}$ if $x \le \xi$ $\dfrac{\xi(k+1-x)}{k+1}$ if $\xi \le x$
5	Third	$\alpha y(0) - \beta y'_x(0) = 0$, $\gamma y(1) + \delta y'_x(1) = 0$ (with $\alpha, \beta, \gamma, \delta \ge 0$ and $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$)	$\dfrac{1}{\rho}(\beta + \alpha x)(\gamma + \delta - \gamma\xi)$ if $x \le \xi$ $\dfrac{1}{\rho}(\beta + \alpha\xi)(\gamma + \delta - \gamma x)$ if $\xi \le x$

POSITIVE PROPERTY SOLUTIONS. If $\lambda > 0$ and $f > 0$ (f can be zero at isolated points $x = x_k$) and a boundary value problem for the nonlinear ODE (3.3.2.1) from Table 3.1 has a solution, then the right-hand side of the integral equation (3.3.2.2) is positive, and hence, the desired function $y = y(x)$ (on the left-hand side) is positive in the domain $0 < x < 1$.

► Integral identity.

Let us multiply the differential equation (3.3.2.1) by a test function $u = u(x)$ and then integrate with respect to x from 0 to 1 while using the identity $uy''_{xx} = (uy'_x)'_x - (yu'_x)'_x + yu''_{xx}$ to obtain

$$u(1)y'_x(1) - y(1)u'_x(1) - u(0)y'_x(0) + y(0)u'_x(0) + \int_0^1 y(x)u''_{xx}(x)\,dx + \lambda \int_0^1 u(x)f(x, y(x), y'_x(x))\,dx = 0. \qquad (3.3.2.3)$$

By choosing different test functions $u = u(x)$ in (3.3.2.3), we will be analyzing important qualitative features of some nonlinear boundary value problems in subsequent paragraphs.

► Properties of integral equations with positive kernel. Jentzch theorem.

A number σ is called a *characteristic value* of the linear integral equation

$$u(x) - \sigma \int_a^b K(x,t)u(t)\,dt = f(x)$$

if there exist nontrivial solutions of the corresponding homogeneous equation (with $f(x) \equiv 0$). The nontrivial solutions themselves are called the *eigenfunctions* of the integral equation corresponding to the characteristic value σ.

A kernel $K(x,t)$ of an integral operator $I[u] = \int_a^b K(x,\xi)u(\xi)\,d\xi$ is said to be *positive definite* if for all functions $\varphi(x)$ that are not identically zero we have

$$\int_a^b \int_a^b K(x,\xi)\varphi(x)\varphi(\xi)\,dx\,d\xi > 0,$$

and the above quadratic functional vanishes for $\varphi(x) = 0$ only. Such a kernel has positive characteristic values only. It is allowed that the kernel may vanish at isolated points (on a set of zero measure) of the domain $a \le x, t \le b$.

GENERALIZED JENTZCH THEOREM. *If a continuous or polar kernel $K(x,t)$ is positive, then its characteristic values σ_0 with the smallest modulus is positive and simple, and the corresponding eigenfunction $u_0(x)$ does not change sign on the interval $a \le x \le b$.*

3.3.3 Theorem on Nonexistence of Solutions to the First Boundary Value Problem. Theorems on Existence of Two Solutions

► **Theorem on nonexistence of solutions to the first boundary value problem.**

KEY ASSUMPTIONS:

1°. Let $\lambda > 0$ and $f(x,y,z) > 0$ be a continuous function in the domain $0 < x < 1$, $-\infty < y, z < \infty$ (f can be zero at finitely many isolated points $x = x_k$).

2°. Suppose that Assumption 1 holds and the function appearing in equation (3.3.2.1) possesses the property

$$f(x,y,z) > ay, \quad \text{where} \quad a > 0, \;\; y > 0. \tag{3.3.3.1}$$

Consider the nonlinear boundary value problem for equation (3.3.2.1) with the homogeneous boundary conditions of the first kind

$$y(0) = 0, \qquad y(1) = 0. \tag{3.3.3.2}$$

We assume that the problem has at least one solution. Let us take

$$u(x) = \sin(\pi x) \tag{3.3.3.3}$$

to be the test function, which possesses the properties

$$u(0) = u(1) = 0, \qquad u(x) > 0 \;\text{ for }\; 0 < x < 1, \qquad u''_{xx}(x) = -\pi^2 u(x). \tag{3.3.3.4}$$

By virtue of conditions (3.3.3.2) and (3.3.3.4), the first line of the integral identity (3.3.2.3) is zero. Using the last relation from (3.3.3.4), we rewrite (3.3.2.3) in the form

$$\begin{aligned}\int_0^1 y(x)u''_{xx}(x)\,dx + \lambda\int_0^1 u(x)f(x,y(x),y'_x(x))\,dx&\\ = \int_0^1 u(x)\big[\lambda f(x,y(x),y'_x(x)) - \pi^2 y(x)\big]\,dx = 0.&\end{aligned} \tag{3.3.3.5}$$

Using the key assumptions above, we obtain the estimate

$$\int_0^1 u(x)\big[\lambda f(x,y(x),y'_x(x)) - \pi^2 y\big]\,dx > \int_0^1 (\lambda a - \pi^2)u(x)y(x)\,dx. \tag{3.3.3.6}$$

Since $u(x)$ and $y(x)$ are both positive on $0 < x < 1$ (see the positive property solutions at the end of Section 3.3.2 and (3.3.3.4)), the second integral in (3.3.3.6) must also be positive, provided that $\lambda > \pi^2/a$. On the other hand, if the first integral in (3.3.3.6) is zero, the second integral must be negative. This contradiction, obtained under the assumption that the problem has a solution, allows us to state the following theorem.

NONEXISTENCE THEOREM (FIRST BOUNDARY VALUE PROBLEM). *If the key assumptions (see the beginning of this section) are valid and λ is a sufficiently large number such that*

$$\lambda > \pi^2/a, \tag{3.3.3.7}$$

the first boundary value problem for equation (3.3.2.1) *subject to the boundary conditions* (3.3.3.2) *does not have solutions.*

Examples of mixed boundary value problems that do not have solutions can be found in Section 3.3.4.

► On the evaluation of the constant a appearing in condition (3.3.3.1).

Let us look at the nonlinear boundary value problem for the autonomous equation

$$y''_{xx} + \lambda f(y) = 0 \tag{3.3.3.8}$$

subject to the boundary conditions of the first kind (3.3.3.2). Note that equation (3.3.3.8) coincides, up to notation, with the autonomous equation considered in Example 3.1, which admits order reduction and is easy to integrate.

We assume that the conditions

$$f > 0 \;\text{ for }\; -\infty < y < \infty, \qquad f'_y \ge 0 \;\text{ for }\; y \ge 0, \qquad \lim_{y\to\infty} f'_y = \infty$$

hold. The constant a appearing in (3.3.3.1) can be evaluated as

$$a = \min_{0\le y<\infty} \frac{f(y)}{y}. \tag{3.3.3.9}$$

Differentiating $f(y)/y$ with respect to y yields an algebraic (transcendental) equation for the minimum point y°:

$$f(y^\circ) - y^\circ f'_y(y^\circ) = 0. \tag{3.3.3.10}$$

Then a can be found using either formula

$$a = \frac{f(y^\circ)}{y^\circ} \quad \text{or} \quad a = f'_y(y^\circ). \tag{3.3.3.11}$$

Example 3.12. In the first boundary value problem for the equation

$$y''_{xx} + \lambda(\alpha + \beta|y|^k) = 0, \qquad \alpha, \beta > 0, \;\; k > 1,$$

subject to the boundary conditions (3.3.3.2), the constant a appearing in (3.3.3.1) is found as

$$a = \beta k\left[\frac{\alpha}{\beta(k-1)}\right]^{\frac{k-1}{k}}.$$

► Theorems on existence of two solutions for the first boundary value problem.

Let us look at the nonlinear boundary value problem with homogeneous boundary conditions of the first kind

$$y''_{xx} + f(x,y) = 0 \quad (0 < x < 1); \qquad y(0) = y(1) = 0. \tag{3.3.3.12}$$

Let the function $f(x,y) \geq 0$ be continuous in the domain $\Omega = \{0 \leq x \leq 1,\ 0 \leq y < \infty\}$ and let $f(x,y) \not\equiv 0$ on any subinterval of $0 \leq x \leq 1$ for $y > 0$. We use the notation: $\|y\| = \sup_{0 \leq x \leq 1} |y(x)|$.

ERBE–HU–WANG THEOREM 1 (A SPECIAL CASE). *Suppose the following two assumptions are valid:*

1. The limits relations

$$\lim_{y\to 0} \min_{0\leq x\leq 1} \frac{f(x,y)}{y} = \lim_{y\to\infty} \min_{0\leq x\leq 1} \frac{f(x,y)}{y} = \infty \tag{3.3.3.13}$$

hold.

2. There is a constant $p > 0$ such that

$$f(x,y) \leq 6p \quad \text{for} \quad 0 \leq x \leq 1,\ 0 \leq y \leq p. \tag{3.3.3.14}$$

Then the first boundary value problem (3.3.3.12) *has at least two positive solutions, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, such that*

$$0 < \|y_1\| < p < \|y_2\|.$$

Example 3.13. Consider the first boundary value problem

$$y''_{xx} + 1 + y^2 = 0 \quad (0 < x < 1); \qquad y(0) = y(1) = 0. \tag{3.3.3.15}$$

Condition (3.3.3.13) for this equation holds. Condition (3.3.3.14) becomes

$$1 + y^2 \leq 6p \quad \text{for} \quad 0 \leq y \leq p.$$

The maximum allowed value of p is determined from the quadratic equation $p^2 - 6p + 1 = 0$, which gives $p_\text{m} = 3 + 2\sqrt{2} \approx 5.828$. Hence, by virtue of the Erbe–Hu–Wang theorem (see above), problem (3.3.3.12) has at least two positive solutions y_1 and y_2 such that $0 < \|y_1\| < p_\text{m} < \|y_2\|$.

ERBE–HU–WANG THEOREM 2 (A SPECIAL CASE). *Let the following two assumptions be valid:*

1. The limits relations

$$\lim_{y\to 0} \max_{0\leq x\leq 1} \frac{f(x,y)}{y} = \lim_{y\to\infty} \max_{0\leq x\leq 1} \frac{f(x,y)}{y} = 0 \tag{3.3.3.16}$$

hold.

2. There is a constant $q > 0$ such that

$$f(x,y) \geq \tfrac{32}{3}q \quad \text{for} \quad \tfrac{1}{4} \leq x \leq \tfrac{3}{4},\ \tfrac{1}{4}q \leq y \leq q. \tag{3.3.3.17}$$

Then the boundary value problem (3.3.3.12) *has at least two positive solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ such that*

$$0 < \|y_1\| < q < \|y_2\|.$$

Remark 3.6. The above Erbe–Hu–Wang theorems are special cases of more general theorems for boundary value problems of the third kind, which are stated below in Section 3.3.7.

3.3.4 Examples of Existence, Nonuniqueness, and Nonexistence of Solutions to First Boundary Value Problems

Below we exemplify the above qualitative features of nonlinear boundary value problems with boundary conditions of the first kind by looking at a few specific problems admitting exact analytical solutions.

► A nonlinear boundary value problem arising in combustion theory.

Example 3.14. Consider the nonlinear boundary value problem described by the equation

$$y''_{xx} + \lambda e^y = 0 \tag{3.3.4.1}$$

subject to the homogeneous boundary conditions of the first kind (3.3.3.2). Equation (3.3.4.1) arises in combustion theory, when the Frank-Kamenetskii approximation is used for the kinetic function, with y denoting dimensionless excess temperature, x dimensionless distance, and $\lambda \geq 0$ is the dimensionless rate of reaction. Equation (3.3.4.1) is a special case of equation (3.3.3.8).

Let us analyze the qualitative features of problem (3.3.4.1), (3.3.3.2) for different values of the determining parameters λ, which is assumed positive.

Equation (3.3.4.1) is a special case of the autonomous second-order equation considered in Example 3.1, which admits order reduction and is easy to integrate. With $\lambda > 0$, the general solution to equation (3.3.4.1) is

$$y = \ln\left[\frac{2c^2}{\lambda \cosh^2(cx+b)}\right], \tag{3.3.4.2}$$

where b and c are arbitrary constants. From the boundary conditions (3.3.3.2), we obtain a system of transcendental equations for b and c,

$$2c^2 = \lambda \cosh^2 b, \quad 2c^2 = \lambda \cosh^2(c+b),$$

which is convenient to rewrite in the equivalent form

$$\lambda = \frac{8b^2}{\cosh^2 b}, \qquad c = -2b. \tag{3.3.4.3}$$

The first equation serves to determine b, after which the evaluation of c is elementary.

The function $p(b) = 8b^2/\cosh^2 b$ is positive if $b \neq 0$, it tends to zero as $b \to 0$ and $b \to \infty$, and it has the only maximum equal to $\lambda_{\rm f}^* = \max p(b) = 3.5138$. It follows that if

$$\lambda > \lambda_{\rm f}^*,$$

the first equation in (3.3.4.3) has no solution; hence, the original boundary value problem (3.3.4.1), (3.3.3.2) has no solution either (the critical value $\lambda = \lambda_{\rm f}^*$ corresponds to a thermal explosion). For $0 < \lambda < \lambda_{\rm f}^*$, the first equation in (3.3.4.3) has two distinct positive roots, b_1 and b_2, which generate two different solutions of the original boundary value problem (3.3.4.1), (3.3.3.2). When $\lambda = \lambda_{\rm f}^*$, the roots b_1 and b_2 become the same, $b_1 = b_2 = b_{\rm f}^* \approx 1.1997$, to give a single solution to the original problem.

Let us assess the accuracy of the critical value $\lambda_{\rm f}^*$ by using the above theorem on nonexistence of solutions to the first boundary value problem. In this case, $f(x, y, y'_x) = e^y$. It is not difficult to show that $e^y \geq ey$ for $y > 0$, which suggests that $a = e$. Substituting this value into (3.3.3.7) gives an approximate estimate for the boundary of the nonexistence domain with respect to the parameter λ:

$$\lambda > \lambda_{\rm f}^{\rm ap} = \pi^2/e \approx 3.6311.$$

This value, provided by the nonexistence theorem, differs from the exact value $\lambda_{\rm f}^*$ by only 3.3% (which is a very high accuracy for a qualitative analysis).

Now let us estimate the boundaries of the existence domain for the two solutions using Erbe–Hu–Wang theorem 1. The first condition of the theorem, (3.3.3.13), clearly holds, since

$$\lim_{y\to 0}(e^y/y) = \lim_{y\to\infty}(e^y/y) = \infty.$$

The second condition, (3.3.3.14), can be rewritten in the form

$$\lambda \le 6pe^{-y} \quad \text{for} \quad 0 \le y \le p.$$

It follows that $\lambda \le 6pe^{-p}$. The left-hand side of this inequality attains a maximum at $p = 1$; hence, the condition $\lambda \le 6/e \approx 2.207$ must hold to ensure that the two solutions exist. This estimate is lower than the exact boundary of the existence domain of two solutions by 37.2%.

Remark 3.7. The second boundary value problem for equation (3.3.4.1) subject to the boundary conditions $y'_x(0) = y'_x(1) = 0$ for any $\lambda > 0$ has no solution. This is easy to see from the general solution (3.3.4.2).

► A problem on an electron beam passing between two electrodes.

Example 3.15. Consider the autonomous equation

$$y''_{xx} = \lambda y^{-1/2} \qquad (0 < x < 1) \tag{3.3.4.4}$$

subject to the nonhomogeneous boundary conditions

$$y(0) = 1, \quad y(1) = 1. \tag{3.3.4.5}$$

The following notation is used here: y is dimensionless potential, x is dimensionless distance, and $\lambda \ge 0$ is dimensionless electric current density (Zinchenko, 1958).

Remark 3.8. Problem (3.3.4.4)–(3.3.4.5) is quite interesting because it can be reduced, with the change of variable $u = 1 - y$, to a problem of the form (3.3.3.8), (3.3.3.2) for which the conditions of the theorems stated in Section 3.3.3 do not hold.

Problem (3.3.4.4)–(3.3.4.5) is symmetric about the mid-point $x = 1/2$. Therefore, it reaches a maximum at $x = 1/2$, with $y'_x(1/2) = 0$. With this in mind, we integrate equation (3.3.4.4) multiplied by $2y'_x$ from x to $1/2$ to obtain

$$(y'_x)^2 = 4\lambda\bigl(\sqrt{y} - C\bigr), \tag{3.3.4.6}$$

where $C = \sqrt{y}|_{x=1/2}$ is an arbitrary constant. Integrating again from x to $1/2$ and rearranging, we arrive at the solution in implicit form

$$(\sqrt{y} - C)(\sqrt{y} + 2C)^2 = \tfrac{9}{64}\lambda(2x - 1)^2. \tag{3.3.4.7}$$

Formula (3.3.4.7) describes a family of third-order curves with respect to $\sqrt{y}$. The constant C depends on λ and satisfies the cubic equation

$$(1 - C)(1 + 2C)^2 = \tfrac{9}{64}\lambda, \tag{3.3.4.8}$$

which is obtained by inserting the boundary conditions (3.3.4.5) into equation (3.3.4.7) (both boundary conditions result in the same equation for C).

Since $\lambda \ge 0$, it follows from equation (3.3.4.8) that $C \le 1$. On the other hand, from the first integral (3.3.4.6) we get $C = \sqrt{y}_{\min} \ge 0$. In the range $0 \le C \le 1$, the maximum of the left-hand side of equation (3.3.4.8) is attained at $C = \frac{1}{2}$, which gives $\lambda_{\max} = \frac{128}{9} \approx 14.22$. Hence, problem (3.3.4.4)–(3.3.4.5) has no solution for $\lambda > \lambda_{\max}$.

A more detailed analysis of the curve (3.3.4.7) shows that three different situations are possible depending on the value of λ:

(i) If $0 \le \lambda < \lambda_*$ ($\lambda_* = \frac{64}{9} \approx 7.11$), problem (3.3.4.4)–(3.3.4.5) has only one solution, which corresponds to the only root of the cubic equation (3.3.4.8) in the domain $\frac{\sqrt{3}}{2} < C \le 1$. For small λ, equation (3.3.4.8) provides the asymptotic behavior

$$C = 1 - \tfrac{1}{64}\lambda + o(\lambda) \qquad (\lambda \to 0).$$

(ii) If $\lambda_* \le \lambda < 2\lambda_* = \lambda_{\max} \approx 14.22$, problem (3.3.4.4)–(3.3.4.5) has two solutions, which correspond to two distinct roots of the cubic equation (3.3.4.8) in the domain $0 \le C \le \frac{\sqrt{3}}{2} \approx 0.866$. For the upper curve, which has a physical meaning (the other solution has no physical meaning), the value of C gradually decreases as λ increases. When $\lambda = \lambda_{\max}$, which corresponds to $C_{1,2} = \frac{1}{2}$, the two solutions become the same.

(iii) If $\lambda > \lambda_{\max}$, the problem has no solution.

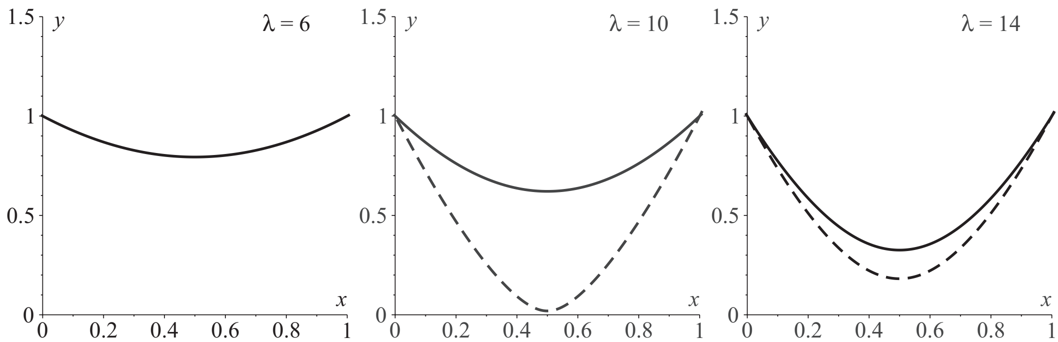


Figure 3.1: Solutions to problem (3.3.4.4)–(3.3.4.5) for different values of λ.

Figure 3.1 displays solutions to problem (3.3.4.4)–(3.3.4.5) for different values of the parameter: $\lambda = 6,\ 10,\ 14$; the dashed lines correspond to the second solution (which has no physical meaning). Figure 3.2 shows the dependence of the roots $C_{1,2}$ of the cubic equation (3.3.4.8) on the parameter λ (the root C_1 corresponds to the solution having a physical meaning).

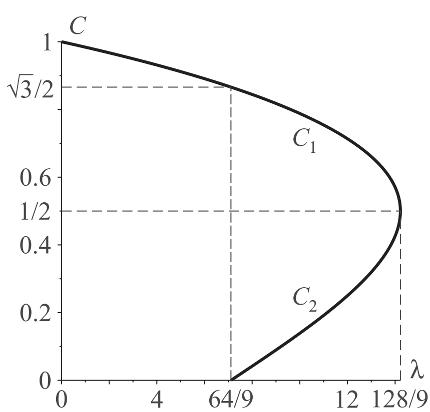


Figure 3.2: Dependence of the roots of the cubic equation (3.3.4.8) on the parameter λ (the root C_1 corresponds to the solution having a physical meaning).

Remark 3.9. For $\lambda < 0$, the boundary value problem (3.3.4.4)–(3.3.4.5) has no solution.

▶ **A model boundary value problem with the modulus of the unknown.**

Example 3.16. Consider the nonlinear boundary value problem

$$y''_{xx} + k^2|y| = 0 \qquad (0 < x < a); \tag{3.3.4.9}$$

$$y(0) = 0, \quad y(a) = -b, \tag{3.3.4.10}$$

where a, b, and k are all positive numbers.

Depending on the sign of y, the nonlinear equation (3.3.4.9) reduces to two linear equations, $y''_{xx} \pm k^2 y = 0$, whose solutions are expressed in terms of trigonometric and hyperbolic functions. For $ak > \pi$, problem (3.3.4.9) has two solutions:

$$y_1 = -\frac{b}{\sinh(ka)} \sinh(kx); \tag{3.3.4.11}$$

$$y_2 = \begin{cases} \dfrac{b}{\sinh(ka-\pi)} \sin(kx) & \text{if } 0 \le x \le \pi/k, \\ -\dfrac{b}{\sinh(ka-\pi)} \sinh(kx-\pi) & \text{if } \pi/k \le x \le a. \end{cases} \tag{3.3.4.12}$$

Here, $y_1 = y_1(x)$ is a monotonically decreasing function such that $y_1(x) \le 0$. The function $y_2 = y_2(x)$ monotonically increases for $0 \le x < \pi/(2k)$, attains a maximum at $x = \pi/(2k)$ and monotonically decreases for $\pi/(2k) < x \le a$. It is positive for $0 < x < \pi/k$, becomes zero at $x = \pi/k$, and is negative for $x > \pi/k$. For all $0 < x < a$, the inequality $y_2 > y_1$ holds.

◆ *See also Section 8.3.3.*

3.3.5 Theorems on Nonexistence of Solutions for the Mixed Problem. Theorems on Existence of Two Solutions

▶ **Theorems on nonexistence of solutions for the mixed problem.**

Let us look at the nonlinear boundary value problem for equation (3.3.2.1) subject to homogeneous mixed boundary conditions of the form

$$y'_x(0) = 0, \quad y(1) = 0. \tag{3.3.5.1}$$

It is assumed to have at least one solution.

Suppose that the key assumptions stated at the beginning of Section 3.3.3 are valid. This means that the function appearing in equation (3.3.2.1) has the property (3.3.3.1). Just as previously, we use the integral identity (3.3.2.3). We take

$$u(x) = \cos\left(\tfrac{\pi}{2}x\right) \tag{3.3.5.2}$$

as the test function; it possesses the properties

$$u'_x(0) = u(1) = 0, \quad u(x) > 0 \ \text{ for } \ 0 < x < 1, \quad u''_{xx}(x) = -\tfrac{1}{4}\pi^2 u(x). \tag{3.3.5.3}$$

By virtue of conditions (3.3.5.1) and (3.3.5.3), the first line of the integral identity (3.3.2.3) is zero. Now using the last relation in (3.3.5.3), we rewrite (3.3.2.3) in the form

$$\int_0^1 y(x)u''_{xx}(x)\,dx + \lambda \int_0^1 u(x) f(x, y(x), y'_x(x))\,dx$$
$$= \int_0^1 u(x)\big[\lambda f(x, y(x), y'_x(x)) - \tfrac{1}{4}\pi^2 y(x)\big]\,dx = 0. \tag{3.3.5.4}$$

In view of inequality (3.3.3.1), it follows that

$$\int_0^1 u(x)\big[\lambda f(x,y(x),y'_x(x)) - \tfrac{1}{4}\pi^2 y\big]\,dx > \int_0^1 (\lambda a - \tfrac{1}{4}\pi^2)u(x)y(x)\,dx. \qquad (3.3.5.5)$$

Since $u(x)$ and $y(x)$ are both positive on $0<x<1$ (see positive property solutions at the end of Section 3.3.2 and (3.3.3.4)), the second integral in (3.3.5.5) must be positive, provided that $\lambda > \frac{1}{4}\pi^2/a$. On the other hand, the first integral in (3.3.5.5) is zero, suggesting that the second integral must be negative. This contradiction, obtained under the assumption that the problem has at least one solution, allows one to state the following theorem.

NONEXISTENCE THEOREM 1 (MIXED BOUNDARY VALUE PROBLEM). *If the key assumptions from Section 3.3.3 are valid and λ is a sufficiently large number such that*

$$\lambda > \tfrac{1}{4}\pi^2/a, \qquad (3.3.5.6)$$

the mixed boundary value problem for equation (3.3.2.1) *with the boundary conditions* (3.3.5.1) *has no solution.*

See Section 3.3.6 for examples of mixed boundary value problems having no solution.

► Generalization of nonexistence theorem 1 for the mixed problem.

Suppose that the function $f(x,y,z)$ appearing in (3.3.2.1) satisfies the inequality

$$f(x,y,z) \ge \varphi(x)y \qquad (0<x<1,\ y>0), \qquad (3.3.5.7)$$

where $\varphi(x) > 0$ is a continuous function.

To be specific, we will consider a boundary value problem for equation (3.3.2.1) subject to the mixed boundary conditions

$$y'_x(0) = y(1) = 0. \qquad (3.3.5.8)$$

The problem is assumed to have at least one solution. Let us impose conditions on the test function $u(x)$ such that the first line of the integral identity (3.3.2.2) is zero. These are

$$u(0) = u'_x(1) = 0. \qquad (3.3.5.9)$$

As a result, equation (3.3.2.3) becomes

$$\int_0^1 y(x)u''_{xx}(x)\,dx + \lambda\int_0^1 u(x)f(x,y(x),y'_x(x))\,dx = 0. \qquad (3.3.5.10)$$

Let $u = u(x)$ satisfy the linear equation

$$u''_{xx} + \sigma\varphi(x)u = 0. \qquad (3.3.5.11)$$

where $\varphi(x)$ is the function appearing in inequality (3.3.5.7) and σ is some (spectral) parameter. The boundary value problem (3.3.5.11), (3.3.5.9) is equivalent to the integral equation

$$u(x) = \sigma\int_0^1 |G_2(x,\xi)|\varphi(\xi)u(\xi)\,d\xi, \qquad (3.3.5.12)$$

where $|G_2(x,\xi)|$ is the modulus of the Green's function shown in the second row of Table 3.1.

Since the kernel of the integral operator (3.3.5.12) is positive, it follows from the Jentzch theorem (see Section 3.3.2) that the least eigenvalue is positive, $\sigma_0 > 0$, and the corresponding eigenfunction $u_0(x)$ does not change its sign on $0 \le x \le 1$. In equations (3.3.5.10) and (3.3.5.11), we first set $u = u_0(x)$ and $\sigma = \sigma_0$ and then eliminate the second derivative $(u_0)''_{xx}$ from (3.3.5.10) with the help of (3.3.5.11). The resulting expression can be written as

$$\frac{\sigma_0}{\lambda} = \frac{\int_0^1 u_0(x) f(x, y(x), y'_x(x))\, dx}{\int_0^1 u_0(x)\varphi(x) y(x)\, dx}. \tag{3.3.5.13}$$

Since, by assumption, inequality (3.3.5.7) holds, it follows from (3.3.5.13) that $\sigma_0/\lambda \ge 1$. However, for sufficiently large $\lambda > \sigma_0$, this estimate cannot be ensured. For such values of λ, the boundary value problem (3.3.2.1), (3.3.2.1) surely has no solution. In the class of boundary value problems concerned, there is a critical value of the parameter, λ_*, that delimits the domains of existence and nonexistence of solutions. For $\lambda > \lambda_*$ with $\lambda_* < \sigma_0$, there are no solutions (σ_0 provides an upper estimate for the critical value λ_* beyond which there are no solutions).

These results allow us to state the following theorem on nonexistence of solutions to the mixed problem.

NONEXISTENCE THEOREM 2 (MIXED BOUNDARY VALUE PROBLEM). *If inequalities* (3.3.5.7) *hold and λ is sufficiently large, $\lambda > \lambda_* > 0$, the mixed boundary value problem for equation* (3.3.2.1) *subject to the boundary conditions* (3.3.5.8) *has no solution. The critical value satisfies the inequality $\lambda_* < \sigma_0$, where σ_0 is the least eigenvalue of the linear eigenvalue problem* (3.3.5.11), (3.3.5.9).

Remark 3.10. The nonexistence theorem can be elaborated further if the boundary value problem is for a nonlinear equation of the special form

$$y''_{xx} + \lambda\big[\varphi(x)g(y) + h(x,y,y'_x)\big] = 0$$

with the initial conditions (3.3.5.8). If the conditions

$$\varphi(x) > 0, \quad g(y) > 0, \quad \lim_{y\to\infty} g'_y(y) = \infty, \quad h(x,y,y'_x) \ge 0 \qquad (0 < x < 1,\ y > 0)$$

hold, the problem has no solution for sufficiently large $\lambda > \lambda_* > 0$.

► Theorems on existence of two solutions for the mixed boundary value problem.

Let us look at the nonlinear boundary value problem with homogeneous boundary conditions of the first kind

$$y''_{xx} + f(x,y) = 0 \quad (0 < x < 1); \qquad y(0) = y'_x(1) = 0. \tag{3.3.5.14}$$

Let the function $f(x,y) \ge 0$ be continuous in the domain $\Omega = \{0 \le x \le 1,\ 0 \le y < \infty\}$ and $f(x,y) \not\equiv 0$ on any subinterval of $0 \le x \le 1$ for $y > 0$. We use the notation: $\|y\| = \sup_{0\le x\le 1} |y(x)|$.

ERBE–HU–WANG THEOREM 1 (A SPECIAL CASE). *Let the following two assumptions hold:*

1. $\lim\limits_{y\to 0}\min\limits_{0\le x\le 1}\dfrac{f(x,y)}{y} = \lim\limits_{y\to\infty}\min\limits_{0\le x\le 1}\dfrac{f(x,y)}{y} = \infty.$
2. *There is a constant* $p > 0$ *such that*

$$f(x,y) \le 2p \quad \text{for} \quad 0 \le x \le 1,\ 0 \le y \le p.$$

Then the first boundary value problem (3.3.5.14) *has at least two positive solutions,* $y_1 = y_1(x)$ *and* $y_2 = y_2(x)$, *such that*

$$0 < \|y_1\| < p < \|y_2\|.$$

ERBE–HU–WANG THEOREM 2 (A SPECIAL CASE). *Let the following two assumptions hold:*

1. $\lim\limits_{y\to 0}\max\limits_{0\le x\le 1}\dfrac{f(x,y)}{y} = \lim\limits_{y\to\infty}\max\limits_{0\le x\le 1}\dfrac{f(x,y)}{y} = 0.$
2. *There is a constant* $q > 0$ *such that*

$$f(x,y) \ge \tfrac{32}{7}q \quad \text{for} \quad \tfrac{1}{4} \le x \le \tfrac{3}{4},\ \tfrac{1}{4}q \le y \le q.$$

Then boundary value problem (3.3.5.14) *has at least two positive solutions,* $y_1 = y_1(x)$ *and* $y_2 = y_2(x)$, *such that*

$$0 < \|y_1\| < q < \|y_2\|.$$

Remark 3.11. The above Erbe–Hu–Wang theorems are special cases of more general theorems for boundary value problems of the third kind, which are stated below in Section 3.3.7.

3.3.6 Examples of Existence, Nonuniqueness, and Nonexistence of Solutions to Mixed Boundary Value Problems

In this section, we exemplify the above qualitative features of nonlinear boundary value problems with mixed boundary conditions by looking at a few specific problems admitting exact analytical solutions.

► **Plane problem I arising in combustion theory (Frank-Kamenetskii approximation).**

Example 3.17. Consider a *one-dimensional problem on thermal explosion* in a plane channel described by equation (3.3.4.1) subject to the mixed boundary conditions (3.3.5.1):

$$y''_{xx} + \lambda e^y = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.1}$$

where $y = y(x)$ is dimensionless excess temperature.

We proceed from the general solution to the equation, which is given by formula (3.3.4.2). Using the boundary conditions, we get the equations for the constants b and c:

$$b = 0, \qquad \lambda = \frac{2c^2}{\cosh^2 c}. \tag{3.3.6.2}$$

The function $q(c) = 2c^2/\cosh^2 c$ is positive for $c \ne 0$, it tends to zero as $c \to 0$ and $c \to \infty$, and it has the only maximum equal to $\lambda^*_{\rm m} = \max q(c) \approx 0.8785$. Consequently, if

$$\lambda > \lambda^*_{\rm m},$$

the second equation in (3.3.6.2) has no solution; it follows that the original boundary value problem (3.3.6.1) has no solution either. For $0 < \lambda < \lambda^*_{\rm m}$, the second equation in (3.3.6.2) has two distinct roots, c_1 and c_2, which determine two different solutions of problem (3.3.6.1). If $\lambda = \lambda^*_{\rm m}$, the roots c_1 and c_2 merge to become one, $c_1 = c_2 = c^*_{\rm m} \approx 1.1997$, which corresponds to a single solution of the problem. The critical value $\lambda = \lambda^*_{\rm m}$ corresponds to a heat explosion.

By comparing the critical values of the parameter λ determining the boundary of a nonexistence domain for solutions to the first and mixed boundary value problems, we obtain the simple relation

$$\lambda^*_{\rm f} = 4\lambda^*_{\rm m}.$$

This relation is exact; it follows from the equation $p(b) = 4q(b)$, which is valid for all b.

The maximum value of the dimensionless excess temperature is attained at $x = 0$; it is given by the formula $y(0) = \ln(2c^2/\lambda)$, which is derived from (3.3.4.2) and (3.3.6.2). The critical values $\lambda^*_{\rm m}$ and $c^*_{\rm m}$ correspond to a thermal explosion. Substituting these values in the formula for the temperature at $x=0$ yields the critical temperature $y_*(0) \approx 1.1868$ leading to the thermal explosion.

Now let us assess the accuracy of the critical value $\lambda^{\rm ap}_{\rm f}$ provided by theorem 1 on nonexistence of solutions (see the previous section). In this case, $f(x, y, y'_x) = e^y$. So we have $e^y \ge ey$ for $y > 0$; hence, $a = e$. Substituting this value into (3.3.5.6) yields an approximate estimate for the boundary of nonexistence of solutions with respect to λ:

$$\lambda > \lambda^{\rm ap}_{\rm f} = \tfrac{1}{4}\pi^2/e \approx 0.9077.$$

One can see that, in this problem, the difference between $\lambda^{\rm ap}_{\rm f}$, estimated using the nonexistence theorem, and the exact value $\lambda^*_{\rm f}$ is just over 3%.

► Plane problem II arising in combustion theory (Arrhenius law-based model).

Example 3.18. Let us look at a more realistic model of thermal explosion than that considered in Example 3.17, in which the kinetic function describing heat release is now bounded and determined by the *Arrhenius law*. In terms of suitable dimensionless variables, the corresponding nonlinear boundary value problem is

$$y''_{xx} + \lambda \exp\Bigl(\frac{y}{1+\sigma y}\Bigr) = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.3}$$

where $\lambda \ge 0$ and $\sigma > 0$.

The general solution to the equation of (3.3.6.3) can be obtained by quadrature (e.g., using formulas from Example 3.1); however, this solution cannot be expressed in terms of elementary functions. In the limit case of $\sigma = 0$, problem (3.3.6.3) becomes (3.3.6.1).

It can be shown that, for $\sigma > 0$, problem (3.3.6.3) has at least one solution for any $\lambda \ge 0$. Furthermore, for sufficiently small σ, there is a domain of λ with three solutions (the curve $y_0 = y_0(\lambda)$, with $y_0 = y(0)$, has an S-shaped portion).

A numerical analysis of problem (3.3.6.3) shows that at $\sigma = 0.2$, there are two critical values, $\lambda^*_1 \approx 0.877$ and $\lambda^*_2 \approx 1.162$, called *hysteresis parameters*, such that

(i) there is only one solution for $0 < \lambda_1$ and $\lambda > \lambda_2$,
(ii) there are three solutions for $\lambda_1 < \lambda < \lambda_2$, and
(iii) there are two solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

► An axisymmetric problem arising in combustion theory (Frank-Kamenetskii approximation).

Example 3.19. Now consider the one-dimensional problem on thermal explosion in a cylindrical vessel described by the following equation and mixed boundary conditions:

$$y''_{xx} + \frac{1}{x}y'_x + \lambda e^y = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.4}$$

where x is a dimensional radial coordinate.

Problem (3.3.6.4) is solved explicitly in terms of elementary functions:

$$y = -2\ln\bigl[b + (1-b)x^2\bigr],$$
$$\lambda = 8b(1-b), \quad b = e^{-y_0/2}, \quad y_0 = y(0). \tag{3.3.6.5}$$

One can see that for $0 < \lambda < \lambda_{\rm m}^* = 2$, there are two solutions corresponding to two distinct values of y_0 (the solution having a physical meaning must satisfy the condition $0 \le y_0 < y_{\rm m}^* \approx 1.3863$). When $\lambda = \lambda_{\rm m}^*$, the two solution merge to become one. For $\lambda > \lambda_{\rm m}^*$, there are no solutions. The critical value $\lambda = \lambda_{\rm m}^* = 2$ corresponds to thermal explosion.

► A problem on bending of a flexible electrode in an electrostatic field.

Example 3.20. Consider the nonlinear boundary value problem on the interval $0 \le x \le 1$ with mixed boundary conditions

$$y''_{xx} + \frac{\lambda}{(1-y)^2} = 0; \qquad y'_x(0) = y(1) = 0. \tag{3.3.6.6}$$

This problem describes the shape of a flexible electrode bending under the action of electrostatic forces due to potential difference between electrodes, with y denoting dimensionless deflexion of the electrode, x denoting dimensionless distance, and λ being a dimensionless parameter proportional to the squared potential difference between electrodes.

The equation of (3.3.6.6) is a special case of the autonomous second-order equations considered in Example 3.1, which admits order reduction and so is easy to integrate. The solution to problem (3.3.6.6) can be written in implicit form as

$$x = \frac{1}{\varphi(a)}\left[\sqrt{(1-a)(a-y)} + (1-a)\ln\frac{\sqrt{1-y}+\sqrt{a-y}}{\sqrt{1-a}}\right],$$
$$\lambda = \frac{1}{2}(1-a)\varphi^2(a), \qquad \varphi(a) = \sqrt{a} + (1-a)\ln\frac{1+\sqrt{a}}{\sqrt{1-a}}, \tag{3.3.6.7}$$

where $a = y(0)$, $0 \le a < 1$, $0 \le y \le a$, and $\lambda > 0$.

The function $y = y(x)$ is convex; at $x = 0$, it has a maximum equal to a and monotonically decreases with x to zero at $x = 1$. An analysis of formula (3.3.6.7) shows that for $0 < \lambda < \lambda_{\rm m}^*$, there are two solutions corresponding to two distinct values of a; the physically realizable (stable) solution corresponds to $0 < a < y_{\rm m}^* \approx 0.3883$. When $\lambda = \lambda_{\rm m}^*$, the two solutions merge to become one. For $\lambda > \lambda_{\rm m}^*$, there are no solutions.

► A model problem having three solutions.

Example 3.21. Let us look at the nonlinear boundary value problem

$$y''_{xx} + \lambda\frac{\sinh(ky)}{\cosh^3(ky)} = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.8}$$

where $k > 0$ and $\lambda > 0$. The function $f(y) = \sinh(ky)/\cosh^3(ky)$ is nonmonotonic and it changes sign; it vanishes ar $y = 0$ and tends to zero as $y \to \pm\infty$. Its extrema are at the points $y_{\rm m} = \pm 0.6585/k$ and are equal to $f_{\rm m} = \pm\frac{2}{3\sqrt{3}} \approx \pm 0.3849$.

Problem (3.3.6.8) admits the trivial solution $y = 0$ for any λ.

If y is a solution to problem (3.3.6.8), then $-y$ is also a solution to the problem.

The positive solution is determined implicitly by the formula

$$\arcsin\left[\frac{\sinh(ky)}{\sinh(ka)}\right] = \frac{\pi}{2}(1-x), \qquad \lambda = \frac{\pi^2}{4k}\cosh^2(ka), \tag{3.3.6.9}$$

where $a = y(0) > 0$ and $y > 0$ for $0 < x < 1$. This solution can be represented in the explicit form

$$y = \frac{1}{k}\operatorname{arsinh}\left\{\sinh(ka)\sin\left[\frac{\pi}{2}(1-x)\right]\right\}, \qquad \lambda = \frac{\pi^2}{4k}\cosh^2(ka), \tag{3.3.6.10}$$

where $\operatorname{arsinh} z = \ln\left(z + \sqrt{z^2+1}\,\right)$.

The negative solution to problem (3.3.6.8) is given by formula (3.3.6.10) where a must be replaced with $-a$.

◆ *See also Section 8.3.3.*

3.3.7 Theorems on Existence of Two Solutions for the Third Boundary Value Problem

► Statement of the problem. Initial assumptions.

Consider a boundary value problem for the nonlinear equation

$$y''_{xx} + f(x,y) = 0 \qquad (0 < x < 1) \tag{3.3.7.1}$$

subject to the boundary conditions of the third kind

$$\begin{aligned} \alpha y(0) - \beta y'_x(0) &= 0, \\ \gamma y(1) + \delta y'_x(1) &= 0. \end{aligned} \tag{3.3.7.2}$$

The following conditions will be assumed to hold throughout this section:

(i) The function $f(x,y) \geq 0$ is continuous in the domain $\Omega = \{0 \leq x \leq 1,\ 0 \leq y < \infty\}$ with $f(x,y) \not\equiv 0$ on any subinterval of $0 \leq x \leq 1$ for $y > 0$.

(ii) The coefficients $\alpha, \beta, \gamma, \delta \geq 0$ and $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$.

► Erbe–Hu–Wang theorems on nonuniqueness of a solution to the boundary value problem.

THEOREM 1. *Let conditions (i) and (ii) hold and the following assumptions be valid:*

(iii) $\lim\limits_{y\to 0}\min\limits_{0\leq x\leq 1}\dfrac{f(x,y)}{y} = \lim\limits_{y\to\infty}\min\limits_{0\leq x\leq 1}\dfrac{f(x,y)}{y} = \infty.$

(iv) *There is a constant $p > 0$ such that*

$$f(x,y) \leq \eta p \quad \text{for} \quad 0 \leq x \leq 1,\ 0 \leq y \leq p,$$

where $\eta = \left[\int_0^1 G(\xi,\xi)\,d\xi\right]^{-1} = \frac{6\rho}{\alpha\gamma+3\alpha\delta+3\beta\gamma+6\beta\delta}$. (Here $G(x,\xi)$ is the Green's function for the equation $y''_{xx} = 0$ with respect to the boundary conditions (3.3.7.2); the expression of this Green's function can be found at the end of Table 3.1.)

Then the boundary value problem (3.3.7.1)–(3.3.7.2) has at least two positive solutions, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, such that

$$0 < \|y_1\| < p < \|y_2\|.$$

Here $\|y\| = \sup\limits_{0\leq x\leq 1}|y(x)|$.

THEOREM 2. *Let conditions (i) and (ii) hold and the following assumptions be valid:*
(v) $\lim_{y\to 0}\max_{0\le x\le 1}\dfrac{f(x,y)}{y}=\lim_{y\to\infty}\max_{0\le x\le 1}\dfrac{f(x,y)}{y}=0.$
(vi) *There is a constant* $q>0$ *such that*

$$f(x,y)\ge \mu q \quad\text{for}\quad \tfrac14\le x\le \tfrac34,\ \sigma q\le y\le q,$$

where $\mu=\bigl[\int_{1/4}^{3/4}G(\frac12,\xi)\,d\xi\bigr]^{-1}=\frac{32\rho}{3\alpha\gamma+7\alpha\delta+7\beta\gamma+16\beta\delta}$ *and* $\sigma=\min\bigl[\frac{\alpha+4\beta}{4(\alpha+\beta)},\frac{\gamma+4\delta}{4(\gamma+\delta)}\bigr]$.

Then the boundary value problem (3.3.7.1)–(3.3.7.2) *has at least two positive solutions,* $y_1=y_1(x)$ *and* $y_2=y_2(x)$*, such that*

$$0<\|y_1\|<q<\|y_2\|.$$

3.3.8 Boundary Value Problems for Linear Equations with Nonlinear Boundary Conditions

► Statements of problems. Solution procedure.

In this section, we consider a few boundary value problems for linear homogeneous second-order differential equations

$$y''_{xx}+f_1(x)y'_x+f_0(x)y=0 \tag{3.3.8.1}$$

subject to a nonlinear boundary condition

$$y'_x=\varphi(y)\quad\text{at}\quad x=x_1 \tag{3.3.8.2}$$

and a linear homogeneous boundary condition at $x=x_2$.

Such problems are solved successively in a few stages. First, one obtains the general solution to equation (3.3.8.1). Then, one finds a particular solution $y=\bar y(x)$, satisfying the boundary condition at the right end, $x=x_2$. Finally, one seeks the solution to the problem in the form

$$y=A\bar y(x), \tag{3.3.8.3}$$

where A is a constant determined from the algebraic (transcendental) equation

$$A\bar y'_x(x_1)=\varphi(A\bar y(x_1)), \tag{3.3.8.4}$$

obtained by substituting (3.3.8.3) into the nonlinear boundary condition at the left end (3.3.8.2).

► Qualitative features of some problems with nonlinear boundary conditions.

Solutions to boundary value problems for linear equations satisfying nonlinear boundary conditions can significantly differ from those satisfying linear boundary conditions.

Example 3.22. Consider a boundary value problem for a linear equation subject to a nonlinear boundary condition at $x=0$ and a homogeneous linear condition of the first kind at $x=a$:

$$y''_{xx}+k^2y=0; \tag{3.3.8.5}$$
$$y'_x=\varphi(y)\quad\text{at}\quad x=0,\qquad y=0\quad\text{at}\quad x=a. \tag{3.3.8.6}$$

The general solution of the linear equation with constant coefficients (3.3.8.5) is given by

$$y = C_1 \sin(kx) + C_2 \cos(kx), \tag{3.3.8.7}$$

where C_1 and C_2 are arbitrary constants. In order to find a particular solution $\bar{y}$ satisfying the second boundary condition (3.3.8.6), we can set $C_1 = -\cos(ak)$ and $C_2 = \sin(ak)$ in (3.3.8.7) to obtain

$$\bar{y} = -\cos(ak)\sin(kx) + \sin(ak)\cos(kx) = \sin[k(a-x)].$$

The solution to problem (3.3.8.5)–(3.3.8.6) is sought in the form

$$y = A\bar{y} = A\sin[k(a-x)]. \tag{3.3.8.8}$$

For any A, this solution satisfies equation (3.3.8.5) and the second boundary condition (3.3.8.6). Substituting (3.3.8.8) into the first boundary condition (3.3.8.6) yields an algebraic (or transcendental) equation for A:

$$Ak\cos(ak) + \varphi(A\sin(ak)) = 0. \tag{3.3.8.9}$$

Let us dwell on the first boundary condition (3.3.8.6) having a power-law nonlinearity

$$\varphi(y) = by^m. \tag{3.3.8.10}$$

Equation (3.3.8.9) then becomes

$$Ak\cos(ak) + bA^m \sin^m(ak) = 0. \tag{3.3.8.11}$$

For any $m > 0$, this equation has the trivial solution $A = 0$ (or $k = 0$). Let us look at different special cases.

$1°$. To get a linear boundary condition of the third kind, one should set $m = 1$ in (3.3.8.10)–(3.3.8.11). The corresponding eigenvalue problem gives solution (3.3.8.8) with A being an arbitrary constant and a and k linked to each other by the discrete relations

$$ak = \frac{\pi}{2} - \theta_0 + \pi n, \quad \theta_0 = \arctan\frac{b}{k}, \quad n = 0,\, 1,\, 2,\, \dots \tag{3.3.8.12}$$

To boundary conditions of the first and second kind there correspond the limit cases $b = \infty$ ($\theta_0 = \frac{\pi}{2}$) and $b = 0$ ($\theta_0 = 0$).

$2°$. In the case of a quadratic nonlinearity, with $m = 2$, equation (3.3.8.11) has a nontrivial solution

$$A = -\frac{k\cos(ak)}{b\sin^2(ak)}$$

for any a, b, and k ($abk \neq 0$)

$3°$. In the case of a cubic nonlinearity, $m = 3$, equation (3.3.8.11) can have two nontrivial solutions or no solutions at all depending on the sign of the expression $bk\tan(ak)$:

$$A_{1,2} = \pm\left[-\frac{k\cos(ak)}{b\sin^3(ak)}\right]^{1/2} \quad \text{if } \ bk\tan(ak) < 0,$$
$$\text{no nontrivial solutions} \quad \text{if } \ bk\tan(ak) > 0.$$

$4°$. In the case of a fractional nonlinearity with $m = \frac{1}{2}$, equation (3.3.8.11) can have one nontrivial solution or no solution at all depending on the sign of the expression $bk\tan(ak)$:

$$A = \frac{b^2\sin(ak)}{k^2\cos^2(ak)} \quad \text{if } \ bk\tan(ak) < 0,$$
$$\text{no nontrivial solutions} \quad \text{if } \ bk\tan(ak) > 00.$$

It is apparent from Items $1°$–$4°$ that the solutions to boundary value problems with linear and nonlinear boundary conditions can significantly differ from each other; specifically, in linear problems, for nontrivial solutions to exist, the parameters a and k must be connected with each other by discrete relations of the form (3.3.8.11) with A being an arbitrary number, while in nonlinear problems, a and k can change independently from each other, with A expressed via them (under certain conditions, several nontrivial solutions can exist or a nontrivial solution can be absent at all).

► **A problem of convective mass transfer with a heterogeneous chemical reaction.**

Consider the equation

$$y''_{xx} + ax^n y'_x = 0 \tag{3.3.8.13}$$

subject to the boundary condition

$$y'_x = -k\Phi(y) \quad \text{at} \quad x = 0, \qquad y \to 0 \quad \text{as} \quad x \to \infty. \tag{3.3.8.14}$$

Problem (3.3.8.13)–(3.3.8.14), written in terms of dimensionless variables, describes convective mass transfer about the critical point of a drop (for $n = 1$) or a solid particle (for $n = 2$) with a heterogeneous chemical reaction on the surface. In (3.3.8.13) and (3.3.8.14), y is concentration, $\Phi(y)$ is the kinetic function, satisfying the condition $\Phi(1) = 0$, k is the rate of chemical reaction, and a is a positive constant. For a reaction of order m, we have $\Phi(y) = (1-y)^m$. To the limit case $k \to \infty$ there corresponds the diffusion mode of the surface reaction with $y(0) = 1$.

The solution to equation (3.3.8.13) satisfying the second boundary condition (3.3.8.14) is given by

$$y = A\int_x^\infty \exp\Bigl(-\frac{a}{n+1}\xi^{n+1}\Bigr)\,d\xi, \tag{3.3.8.15}$$

where A is a constant. Substituting (3.3.8.15) into the first boundary condition (3.3.8.14) yields an algebraic (or transcendental) equation for A:

$$A = k\Phi(Ac), \tag{3.3.8.16}$$

where

$$c = \int_0^\infty \exp\Bigl(-\frac{a}{n+1}\xi^{n+1}\Bigr)\,d\xi = a^{-\frac{1}{n+1}}(n+1)^{-\frac{n}{n+1}}\Gamma\Bigl(\frac{1}{n+1}\Bigr)$$

and $\Gamma(z)$ is the gamma function. In particular, for a reaction with the fractional order $m = 1/2$, we have $\Phi(y) = (1-y)^{1/2}$; hence, the solution to equation (3.3.8.16) is $A = -\frac{1}{2}ck^2 + \sqrt{\frac{1}{4}c^2k^4 + k^2}$.

⊙ *Literature for Section 3.3*: K. Akô (1967, 1968), P. B. Bailey, L. F. Shampine, and P. E. Waltman (1968), J. Bebernes and R. Gaines (1968), E. Kamke (1977), L. K. Jackson and P. K. Palamides (1984), D. A. Frank-Kamenetskii (1987), V. F. Zaitsev and A. D. Polyanin (1993, 1994), L. H. Erbe, S. Hu, and H. Wang (1994), L. H. Erbe and H. Wang (1994), S.-H. Wang (1994), W.-C. Lian, F.-H. Wong, and C.-C. Yen (1996), P. Korman and Y. Li (1999, 2010), P. Korman, Y. Li, and T. Ouyang (2005), A. B. Vasil'eva and H. H. Nefedov (2006), S. I. Faddeev and V. V. Kogan (2008), G. L. Karakostas (2012).

3.4 Method of Regular Series Expansions with Respect to the Independent Variable. Padé Approximants

3.4.1 Method of Expansion in Powers of the Independent Variable

A solution of the Cauchy problem

$$y''_{xx} = f(x, y, y'_x), \tag{3.4.1.1}$$

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1 \tag{3.4.1.2}$$

can be sought in the form of a Taylor series in powers of the difference $(x-x_0)$, specifically:

$$y(x)=y(x_0)+y'_x(x_0)(x-x_0)+\frac{y''_{xx}(x_0)}{2!}(x-x_0)^2+\frac{y'''_{xxx}(x_0)}{3!}(x-x_0)^3+\cdots. \quad (3.4.1.3)$$

The first two coefficients $y(x_0)$ and $y'_x(x_0)$ in solution (3.4.1.3) are defined by the initial conditions (3.4.1.2). The values of the subsequent derivatives of y at the point $x = x_0$ are determined from equation (3.4.1.1) and its derivative equations (obtained by successive differentiation of the equation) taking into account the initial conditions (3.4.1.2). In particular, setting $x = x_0$ in (3.4.1.1) and substituting (3.4.1.2), we obtain the value of the second derivative:

$$y''_{xx}(x_0) = f(x_0, y_0, y_1). \quad (3.4.1.4)$$

Further, differentiating (3.4.1.1) yields

$$y'''_{xxx} = f_x(x, y, y'_x) + f_y(x, y, y'_x)y'_x + f_{y'_x}(x, y, y'_x)y''_{xx}. \quad (3.4.1.5)$$

On substituting $x = x_0$, the initial conditions (3.4.1.2), and the expression of $y''_{xx}(x_0)$ of (3.4.1.4) into the right-hand side of equation (3.4.1.5), we calculate the value of the third derivative:

$$y'''_{xxx}(x_0) = f_x(x_0, y_0, y_1) + f_y(x_0, y_0, y_1)y_1 + f(x_0, y_0, y_1)f_{y'_x}(x_0, y_0, y_1).$$

The subsequent derivatives of the unknown are determined likewise.

The thus obtained solution (3.4.1.3) can only be used in a small neighborhood of the point $x = x_0$.

Example 3.23. Consider the following Cauchy problem for a second-order nonlinear equation:

$$y''_{xx} = yy'_x + y^3; \quad (3.4.1.6)$$
$$y(0) = y'_x(0) = 1. \quad (3.4.1.7)$$

Substituting the initial values of the unknown and its derivative (3.4.1.7) into equation (3.4.1.6) yields the initial value of the second derivative:

$$y''_{xx}(0) = 2. \quad (3.4.1.8)$$

Differentiating equation (3.4.1.6) gives

$$y'''_{xxx} = yy''_{xx} + (y'_x)^2 + 3y^2y'_x. \quad (3.4.1.9)$$

Substituting here the initial values from (3.4.1.7) and (3.4.1.8), we obtain the initial condition for the third derivative:

$$y'''_{xxx}(0) = 6. \quad (3.4.1.10)$$

Differentiating (3.4.1.9) followed by substituting (3.4.1.7), (3.4.1.8), and (3.4.1.10), we find that

$$y''''_{xxxx}(0) = 24. \quad (3.4.1.11)$$

On substituting the initial data (3.4.1.7), (3.4.1.8), (3.4.1.10), and (3.4.1.11) into (3.4.1.3), we arrive at the Taylor series expansion of the solution about $x = 0$:

$$y = 1 + x + x^2 + x^3 + x^4 + \cdots. \quad (3.4.1.12)$$

This geometric series is convergent only for $|x| < 1$.

3.4.2 Padé Approximants

Suppose the $k+1$ leading coefficients in the Taylor series expansion of a solution to a differential equation about the point $x=0$ are obtained by the method presented in Section 3.4.1, so that

$$y_{k+1}(x) = a_0 + a_1 x + \cdots + a_k x^k. \tag{3.4.2.1}$$

The partial sum (3.4.2.1) pretty well approximates the solution at small x but is poor for intermediate and large values of x, since the series can be slowly convergent or even divergent. This is also related to the fact that $y_k \to \infty$ as $x \to \infty$, while the exact solution can well be bounded.

In many cases, instead of the expansion (3.4.2.1), it is reasonable to consider a Padé approximant $P_M^N(x)$, which is the ratio of two polynomials of degree N and M, specifically,

$$P_M^N(x) = \frac{A_0 + A_1 x + \cdots + A_N x^N}{1 + B_1 x + \cdots + B_M x^M}, \qquad \text{where} \quad N + M = k. \tag{3.4.2.2}$$

The coefficients $A_1, \ldots, A_N$ and $B_1, \ldots, B_M$ are selected so that the $k+1$ leading terms in the Taylor series expansion of (3.4.2.2) coincide with the respective terms of the expansion (3.4.2.1). In other words, the expansions (3.4.2.1) and (3.4.2.2) must be asymptotically equivalent as $x \to 0$.

In practice, one usually takes $N = M$ (the diagonal sequence). It often turns out that formula (3.4.2.2) pretty well approximates the exact solution on the entire range of x (for sufficiently large N).

Example 3.24. Consider the Cauchy problem (3.4.1.6)–(3.4.1.7) again. The Taylor series expansion of the solution about $x = 0$ has the form (3.4.1.12). This geometric series is convergent only for $|x| < 1$.

The diagonal sequence of Padé approximants corresponding to series (3.4.1.12) is

$$P_1^1(x) = \frac{1}{1-x}, \quad P_2^2(x) = \frac{1}{1-x}, \quad P_3^3(x) = \frac{1}{1-x}. \tag{3.4.2.3}$$

It is not difficult to verify that the function $y(x) = \dfrac{1}{1-x}$ is the exact solution of the Cauchy problem (3.4.1.6)–(3.4.1.7). Hence, in this case, the diagonal sequence of Padé approximants recovers the exact solution from only a few terms in the Taylor series.

Example 3.25. Consider the Cauchy problem for a second-order nonlinear equation:

$$y''_{xx} = 2yy'_x; \qquad y(0) = 0, \quad y'_x(0) = 1. \tag{3.4.2.4}$$

Following the method presented in Section 3.4.1, we obtain the Taylor series expansion of the solution to problem (3.4.2.4) in the form

$$y(x) = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \cdots. \tag{3.4.2.5}$$

The exact solution of problem (3.4.2.4) is given by $y(x) = \tan x$. Hence it has singularities at $x = \pm\frac{1}{2}(2n+1)\pi$. However, any finite segment of the Taylor series (3.4.2.5) does not have any singularities.

With series (3.4.2.5), we construct the diagonal sequence of Padé approximants:

$$P_2^2(x) = \frac{3x}{3-x^2}, \quad P_3^3(x) = \frac{x(x^2-15)}{3(2x^2-5)}, \quad P_4^4(x) = \frac{5x(21-2x^2)}{x^4 - 45x^2 + 105}. \tag{3.4.2.6}$$

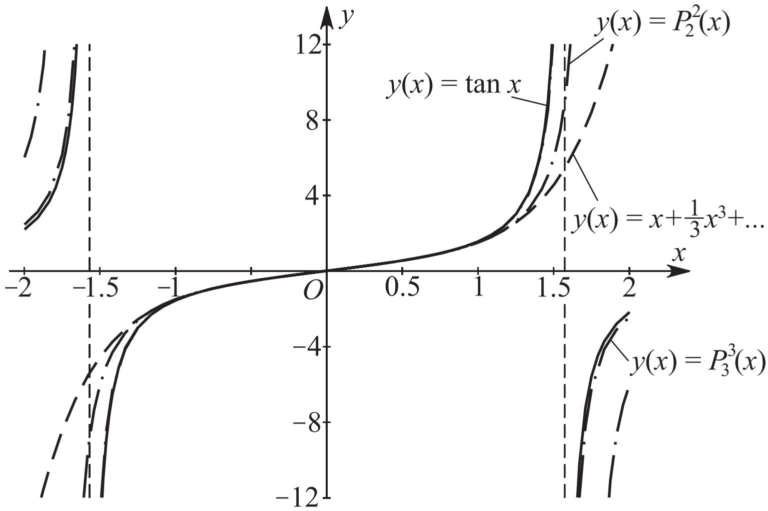


Figure 3.3: Comparison of the exact solution to problem (3.4.2.4) with the approximate truncated series solution (3.4.2.5) and associated Padé approximants (3.4.2.6).

These Padé approximants have singularities (at the points where the denominators vanish):

$$\begin{aligned} &x \simeq \pm 1.732 && \text{for } P_2^2(x),\\ &x \simeq \pm 1.581 && \text{for } P_3^3(x),\\ &x \simeq \pm 1.571 \text{ and } x \simeq \pm 6.522 && \text{for } P_4^4(x). \end{aligned}$$

It is apparent that the Padé approximants are attempting to recover the singularities of the exact solution at $x = \pm\pi/2$ and $x = \pm 3\pi/2$.

In Fig. 3.3, the solid line shows the exact solution of problem (3.4.2.4), the dashed line corresponds to the four-term Taylor series solution (3.4.2.5), and the dot-and-dash line depicts the Padé approximants (3.4.2.6). It is evident that the Padé approximant $P_4^4(x)$ gives an accurate numerical approximation of the exact solution on the interval $|x| \le 2$; everywhere the error is less than 1%, except for a very small neighborhood of the point $x = \pm\pi/2$ (the error is 1% for $x = \pm 1.535$ and 0.84% for $x = \pm 2$).

⊙ *Literature for Section 3.4*: A. H. Nayfeh (1973, 1981), G. A. Baker (Jr.) and P. Graves–Morris (1981), D. Zwillinger (1997), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

3.5 Movable Singularities of Solutions of Ordinary Differential Equations. Painlevé Equations

3.5.1 Preliminary Remarks. Singular Points of Solutions

► Fixed and movable singular points of solutions to ODEs.

Singular points of solutions to ordinary differential equations can be *fixed* or *movable*. The coordinates of fixed singular points remain the same for different solutions of an equation.*

*Solutions of linear ordinary differential equations can only have fixed singular points, and their positions are determined by the singularities of the equation coefficients.

The coordinates of movable singular points vary depending on the particular solution selected (i.e., they depend on the initial conditions).

Listed below are simple examples of first-order ordinary differential equations and their solutions having movable singularities:

Equation	*Solution*	*Solution's singularity type*
$y'_z = -y^2$	$y = 1/(z - z_0)$	movable pole
$y'_z = 1/y$	$y = 2\sqrt{z - z_0}$	algebraic branch point
$y'_z = e^{-y}$	$y = \ln(z - z_0)$	logarithmic branch point
$y'_z = -y \ln^2 y$	$y = \exp[1/(z - z_0)]$	essential singularity

Algebraic branch points, logarithmic branch points, and essential singularities are called *movable critical points*.

► Classification of second-order ODEs. Painlevé equations.

The Painlevé equations arise from the classification of the following second-order differential equations over the complex plane:

$$y''_{zz} = R(z, y, y'_z),$$

where $R = R(z, y, w)$ is a function rational in y and w and analytic in z. It was shown by P. Painlevé (1897–1902) and B. Gambier (1910) that all equations of this type whose solutions do not have movable critical points (but are allowed to have fixed singular points and movable poles) can be reduced to 50 classes of equations. Moreover, 44 classes out of them are integrable by quadrature or admit reduction of order. The remaining 6 equations are irreducible; these are known as the *Painlevé equations*, and their solutions are known as the *Painlevé transcendental functions* or *Painlevé transcendents*.

Remark 3.12. The Painlevé equations are sometimes referred to as the Painlevé transcendents, but in this section this term will be used only for their solutions.

The canonical forms of the Painlevé equations are given below in Sections 3.5.2 through 3.5.7. Solutions of the first, second, and fourth Painlevé equations have movable poles (no fixed singular points). Solutions of the third and fifth Painlevé equations have two fixed logarithmic branch points, $z = 0$ and $z = \infty$. Solutions of the sixth Painlevé transcendent have three fixed logarithmic branch points, $z = 0$, $z = 1$, and $z = \infty$.

It is significant that the Painlevé equations often arise in mathematical physics.

3.5.2 First Painlevé Equation

► Form of the first Painlevé equation. Solutions in the vicinity of a movable pole.

The *first Painlevé equation* has the form

$$y''_{zz} = 6y^2 + z. \qquad (3.5.2.1)$$

The solutions of the first Painlevé equation are single-valued functions of z.

The solutions of equation (3.5.2.1) can be presented, in the vicinity of a movable pole z_p, in terms of the series

$$y = \frac{1}{(z-z_\mathrm{p})^2} + \sum_{n=2}^{\infty} a_n (z-z_\mathrm{p})^n,$$
$$a_2 = -\tfrac{1}{10} z_\mathrm{p}, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = C, \quad a_5 = 0, \quad a_6 = \tfrac{1}{300} z_\mathrm{p}^2,$$

where z_p and C are arbitrary constants; the coefficients a_j ($j \ge 7$) are uniquely defined in terms of z_p and C.

Remark 3.13. The first Painlevé equation (3.5.2.1) is invariant under scaling of variables, $z = \lambda \bar{z}$, $y = \lambda^3 \bar{y}$, where $\lambda^5 = 1$.

▶ Solutions in the form of a Taylor series.

In a neighborhood of a fixed point $z = z_0$, the solution of the Cauchy problem for the first Painlevé equation (3.5.2.1) can be represented by the Taylor series (see Section 3.4.1):

$$y = A + B(z-z_0) + \tfrac{1}{2}(6A^2 + z_0)(z-z_0)^2 + \tfrac{1}{6}(12AB+1)(z-z_0)^3$$
$$+ \tfrac{1}{2}(6A^3 + B^2 + Az_0)(z-z_0)^4 + \cdots,$$

where A and B are initial data of the Cauchy problem, so that $y|_{z=z_0} = A$ and $y'_z|_{z=z_0} = B$.

Remark 3.14. The solutions of the Cauchy problems for the second and fourth Painlevé equations can be expressed likewise (fixed singular points should be excluded from consideration for the remaining Painlevé equations).

▶ Asymptotic formulas and some properties.

1°. There are solutions of equation (3.5.2.1) such that

$$y(x) = -\bigl(\tfrac{1}{6}|x|\bigr)^{1/2} + a_1 |x|^{-1/8} \sin\bigl[\phi(x) - b_1\bigr] + o\bigl(|x|^{-1/8}\bigr) \quad \text{as} \quad x \to -\infty, \tag{3.5.2.2}$$

where

$$\phi(x) = (24)^{1/4}\bigl(\tfrac{4}{5}|x|^{5/4} - \tfrac{5}{8} a_1^2 \ln|x|\bigr),$$

and a_1 and b_1 are some constants (there are also solutions such that $a_1 = 0$).

2°. For given initial conditions $y(0) = 0$ and $y'_x(0) = k$, with k real, $y(x)$ has at least one pole on the real axis. There are two special values, $k_1 \approx -0.45143$ and $k_2 \approx 1.85185$, such that:

(a) If $k < k_1$, then $y(x) > 0$ for $x_\mathrm{p} < x < 0$, where x_p is the first pole on the negative real axis.

(b) If $k_1 < k < k_2$ then $y(x)$ oscillates about, and is asymptotic to, $-\bigl(\tfrac{1}{6}|x|\bigr)^{1/2}$ as $x \to -\infty$ (see formula (3.5.2.2)).

(c) If $k_2 < k$ then $y(x)$ changes sign once, from positive to negative, as x passes from x_p to 0.

3°. For large values of $|z| \to \infty$, the following asymptotic formula holds:

$$y \sim z^{1/2} \wp\left(\tfrac{4}{5} z^{5/4} - a_2;\ 12,\ b_2\right),$$

where the elliptic Weierstrass function $\wp(\zeta; 12, b_2)$ is defined implicitly by the integral

$$\zeta = \int \frac{d\wp}{\sqrt{4\wp^3 - 12\wp - b_2}},$$

and a_2 and b_2 are some constants.

3.5.3 Second Painlevé Equation

► **Form of the 2nd Painlevé equation. Solutions in the vicinity of a movable pole.**

The *second Painlevé equation* has the form

$$y''_{zz} = 2y^3 + zy + \alpha. \tag{3.5.3.1}$$

The solutions of the second Painlevé equation are single-valued functions of z.

The solutions of equation (3.5.3.1) can be represented, in the vicinity of a movable pole z_p, in terms of the series

$$y = \frac{m}{z - z_\mathrm{p}} + \sum_{n=1}^{\infty} b_n (z - z_\mathrm{p})^n,$$

$$b_1 = -\tfrac{1}{6} m z_\mathrm{p},\ \ b_2 = -\tfrac{1}{4}(m + \alpha),\ \ b_3 = C,\ \ b_4 = \tfrac{1}{72} z_\mathrm{p}(m + 3\alpha),$$
$$b_5 = \tfrac{1}{3024}\left[(27 + 81\alpha^2 - 2z_\mathrm{p}^3)m + 108\alpha - 216 C z_\mathrm{p}\right],$$

where z_p and C are arbitrary constants, $m = \pm 1$, and the coefficients b_n ($n \ge 6$) are uniquely defined in terms of z_p and C.

► **Relations between solutions. Bäcklund transformations.**

For fixed α, denote the solution by $y(z, \alpha)$. Then the following relation holds:

$$y(z, -\alpha) = -y(z, \alpha), \tag{3.5.3.2}$$

while the solutions $y(z, \alpha)$ and $y(z, \alpha - 1)$ are related by the Bäcklund transformations:

$$\begin{aligned} y(z, \alpha - 1) &= -y(z, \alpha) + \frac{2\alpha - 1}{2y'_z(z, \alpha) - 2y^2(z, \alpha) - z}, \\ y(z, \alpha) &= -y(z, \alpha - 1) - \frac{2\alpha - 1}{2y'_z(z, \alpha - 1) + 2y^2(z, \alpha - 1) + z}. \end{aligned} \tag{3.5.3.3}$$

Therefore, in order to study the general solution of equation (3.5.3.1) with arbitrary α, it is sufficient to construct the solution for all α out of the band $0 \le \operatorname{Re}\alpha < \frac{1}{2}$.

Three solutions corresponding to α and $\alpha \pm 1$ are related by the rational formulas

$$y_{\alpha+1} = -\frac{(y_{\alpha-1} + y_\alpha)(4y_\alpha^3 + 2zy_\alpha + 2\alpha + 1) + (2\alpha - 1)y_\alpha}{2(y_{\alpha-1} + y_\alpha)(2y_\alpha^2 + z) + 2\alpha - 1},$$

where y_α stands for $y(z, \alpha)$.

The solutions $y(z,\alpha)$ and $y(z,-\alpha-1)$ are related by the Bäcklund transformations:

$$y(z,-\alpha-1) = y(z,\alpha) + \frac{2\alpha+1}{2y'_z(z,\alpha)+2y^2(z,\alpha)+z},$$
$$y(z,\alpha) = y(z,-\alpha-1) - \frac{2\alpha+1}{2y'_z(z,-\alpha-1)+2y^2(z,-\alpha-1)+z}.$$

► **Rational particular solutions.**

For $\alpha=0$, equation (3.5.3.1) has the trivial solution $y=0$. Taking into account this fact and relations (3.5.3.2) and (3.5.3.3), we find that the second Painlevé equation with $\alpha=\pm1,\ \pm2,\ \ldots$ has the rational particular solutions

$$y(z,\pm1) = \mp\frac{1}{z}, \quad y(z,\pm2) = \pm\Bigl(\frac{1}{z}-\frac{3z^2}{z^3+4}\Bigr),$$
$$y(z,\pm3) = \pm\biggl[\frac{3z^2}{z^3+4}-\frac{6z^2(z^3+10)}{z^6+20z^3-80}\biggr],$$
$$y(z,\pm4) = \pm\biggl[-\frac{1}{z}+\frac{6z^2(z^3+10)}{z^6+20z^3-80}-\frac{9z^5(z^3+40)}{z^9+60z^6+11200}\biggr], \quad \ldots$$

► **Solutions in terms of Bessel functions.**

For $\alpha=\frac{1}{2}$, equation (3.5.3.1) admits the one-parameter family of solutions:

$$y(z,\tfrac{1}{2}) = -\frac{w'_z}{w}, \qquad \text{where} \quad w=\sqrt{z}\,\Bigl[C_1J_{1/3}\bigl(\tfrac{\sqrt{2}}{3}z^{3/2}\bigr)+C_2Y_{1/3}\bigl(\tfrac{\sqrt{2}}{3}z^{3/2}\bigr)\Bigr]. \quad (3.5.3.4)$$

(Here the function w is a solution of the Airy equation, $w''_{zz}+\frac{1}{2}zw=0$.)

It follows from (3.5.3.2)–(3.5.3.4) that the second Painlevé equation for all $\alpha=n+\frac{1}{2}$ with $n=0,\ \pm1,\ \pm2,\ \ldots$ has a one-parameter family of solutions that can be expressed in terms of Bessel functions.

► **Asymptotic formulas and some properties with $\alpha=0$.**

1°. Any nontrivial real solution of (3.5.3.1) with $\alpha=0$ that satisfies the boundary condition

$$y\to0 \quad \text{as} \quad x\to+\infty$$

is asymptotic to $k\,\mathrm{Ai}(x)$, for some nonzero real k, where Ai denotes the Airy function (see Section S.4.8).

Conversely, for any nonzero real k, there is a unique solution $y_k(x)$ of (3.5.3.1) with $\alpha=0$ that is asymptotic to $k\,\mathrm{Ai}(x)$ as $x\to+\infty$. The asymptotic behavior of this solution as $x\to-\infty$ depends on $|k|$; three possible situations are highlighted below.

If $|k|<1$, then

$$y_k(x) = b|x|^{-1/4}\sin\bigl[\phi(x)-c\bigr]+o\bigl(|x|^{-1/4}\bigr) \quad \text{as} \quad x\to-\infty,$$

where

$$\phi(x)=\tfrac{2}{3}|x|^{3/2}-\tfrac{3}{4}b^2\ln|x|, \qquad b=-\tfrac{1}{\pi}\ln(1-k^2),$$

with c is a real constant.

If $|k| = 1$, then

$$y_k(x) \sim \left(\tfrac{1}{2}|x|\right)^{1/2} \operatorname{sign} k \quad \text{as} \quad x \to -\infty.$$

If $|k| < 1$, then $y_k(x)$ has a pole at a finite point $x = x_{\rm p}$, dependent on k, and

$$y_k(x) \sim \frac{\operatorname{sign} k}{x - x_{\rm p}} \quad \text{at} \quad x \to x_{\rm p}^{+}.$$

2°. Replacement of y by iy in (3.5.3.1) with $\alpha = 0$ gives the modified second Painlevé equation

$$y''_{zz} = -2y^3 + zy. \tag{3.5.3.5}$$

Any nontrivial real solution of (3.5.3.5) satisfies

$$y(x) = b|x|^{-1/4} \sin\bigl[\phi(x) - c\bigr] + O\bigl(|x|^{-5/4}\bigr) \quad \text{as} \quad x \to -\infty,$$

where

$$\phi(x) = \tfrac{2}{3}|x|^{3/2} + \tfrac{3}{4}b^2 \ln|x|,$$

with $b \neq 0$ and c are real constants.

3.5.4 Third Painlevé Equation

► Form of the third Painlevé equation.

The *third Painlevé equation* has the form

$$y''_{zz} = \frac{(y'_z)^2}{y} - \frac{y'_z}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}. \tag{3.5.4.1}$$

In terms of the new independent variable ζ defined by $z = e^{\zeta}$, the solutions of the transformed equation will be single-valued functions of ζ.

In some special cases, equation (3.5.4.1) can be integrated by quadrature.

If $\gamma\delta \neq 0$ in (3.5.4.1), then set $\gamma = 1$ and $\delta = -1$, without loss of generality, by rescaling y and z if necessary. If $\gamma = 0$ and $\alpha\delta \neq 0$ in (3.5.4.1), then set $\alpha = 1$ and $\delta = -1$, without loss of generality. Lastly, if $\delta = 0$ and $\beta\gamma \neq 0$, then set $\beta = -1$ and $\gamma = 1$, without loss of generality.

► Rational particular solutions.

Let $y = y(z, \alpha, \beta, \gamma, \delta)$ be a solution of equation (3.5.4.1). Then special rational solutions of the third Painlevé equation are

$$\begin{aligned} y(z, \mu, -\mu k^2, \lambda, -\lambda k^4) &= k, \\ y(z, 0, -\mu, 0, \mu k) &= kz, \\ y(z, 2k+3, -2k+1, 1, -1) &= \frac{z+k}{z+k+1}, \end{aligned}$$

where k, λ, and μ are arbitrary constants.

In the general case $\gamma\delta \neq 0$, we may set $\gamma = 1$ and $\delta = -1$. Then equation (3.5.4.1) has rational solutions iff

$$\alpha \pm \beta = 4n,$$

where n is integers. These solutions have the form $y = P_m(z)/Q_m(z)$, where $P_m(z)$ and $Q_m(z)$ are polynomials of degree m, with no common zeros. For examples see Milne et al. (1997) and Clarkson (2003).

► Elementary nonrational particular solutions I.

Elementary nonrational solutions of equation (3.5.4.1) are

$$\begin{aligned} y(z,\mu,0,0,-\mu k^3) &= kz^{1/2}, \\ y(z,0,-2k,0,4k\mu-\lambda^2) &= z(k\ln^2 z+\lambda z+\mu), \\ y(z,\nu^2\lambda,0,\nu^2(\lambda^2-4k\mu),0) &= \frac{z^{\nu-1}}{kz^{2\nu}+\lambda z^{\nu}+\mu}, \end{aligned}$$

where k, λ, μ, and ν are arbitrary constants.

Let $\gamma = 0$ and $\alpha\delta \neq 0$. In this case we assume $\alpha = 1$ and $\delta = -1$ (without loss of generality). Then equation (3.5.4.1) has algebraic solution iff

$$\beta = 2n, \qquad n \in \mathbb{Z}.$$

These are rational solutions in $\zeta = z^{1/3}$ of the form

$$y = P_{n^2+1}(\zeta)/Q_{n^2}(\zeta),$$

where $P_{n^2+1}(\zeta)$ and $Q_{n^2}(\zeta)$ are polynomials of degree n^2+1 and n^2, respectively, with no common zeros. Similar results hold when $\delta = 0$ and $\beta\gamma \neq 0$.

► Elementary nonrational particular solutions II.

In some special cases, equation (3.5.4.1) can be integrated by quadrature. Rewrite equation (3.5.4.1) in the form of integro-differential relations in two ways:

$$\Big(\frac{y'_\zeta}{y}\Big)^2+\Big(\frac{\delta}{y^2}-\gamma y^2\Big)e^{2\zeta}+2\Big(\frac{\beta}{y}-\alpha y\Big)e^{\zeta}=2\int\Big[\Big(\frac{\delta}{y^2}-\gamma y^2\Big)e^{2\zeta}+\Big(\frac{\beta}{y}-\alpha y\Big)e^{\zeta}\Big]\,d\zeta; \tag{3.5.4.2}$$

$$\frac{y'_\zeta}{y}=\int\Big[\Big(\frac{\delta}{y^2}+\gamma y^2\Big)e^{2\zeta}+\Big(\frac{\beta}{y}+\alpha y\Big)e^{\zeta}\Big]\,d\zeta, \qquad z=e^{\zeta}. \tag{3.5.4.3}$$

It is obvious from (3.5.4.2) that for $\alpha = \beta = \gamma = \delta = 0$, the general solution has the form: $y = C_1 z^{C_2}$.

Adding (3.5.4.3) multiplied by 2 to (3.5.4.2), we obtain

$$\Big(\frac{y'_\zeta}{y}\Big)^2+2\frac{y'_\zeta}{y}+\Big(\frac{\delta}{y^2}-\gamma y^2\Big)e^{2\zeta}+2\Big(\frac{\beta}{y}-\alpha y\Big)e^{\zeta}=4\int\frac{\delta e^{2\zeta}+\beta e^{\zeta}y}{y^2}\,d\zeta. \tag{3.5.4.4}$$

Subtracting (3.5.4.3) times 2 from (3.5.4.2) yields

$$\Big(\frac{y'_\zeta}{y}\Big)^2-2\frac{y'_\zeta}{y}+\Big(\frac{\delta}{y^2}-\gamma y^2\Big)e^{2\zeta}+2\Big(\frac{\beta}{y}-\alpha y\Big)e^{\zeta}=-4\int(\gamma e^{2\zeta}y^2+\alpha e^{\zeta}y)\,d\zeta. \tag{3.5.4.5}$$

Substituting $\delta=\beta=0$ into equation (3.5.4.4) and $\gamma=\alpha=0$ into equation (3.5.4.5), we arrive at

$$\left(\frac{y'_\zeta}{y}\right)^2+2\frac{y'_\zeta}{y}-2\alpha ye^\zeta-\gamma y^2e^{2\zeta}=C_1, \tag{3.5.4.6}$$

$$\left(\frac{y'_\zeta}{y}\right)^2-2\frac{y'_\zeta}{y}+\frac{\delta}{y^2}e^{2\zeta}+\frac{2\beta}{y}e^\zeta=C_2. \tag{3.5.4.7}$$

Equations (3.5.4.6) and (3.5.4.7) are integrable by elementary functions. Substituting $y=e^{-\zeta}/v$ into (3.5.4.6), we obtain an autonomous equation:

$$(v'_\zeta)^2=2\alpha v+\gamma+(1+C_1)v^2. \tag{3.5.4.8}$$

As a result, we find:

$$y=\begin{cases}\dfrac{2\alpha}{z(\alpha^2\ln^2 z+2\alpha C\ln z+C^2-\gamma)} & \text{if } C_1=-1,\ \beta=\delta=0;\\[2ex] \dfrac{1}{z\left(\sqrt{\gamma}\ln z+C\right)} & \text{if } C_1=-1,\ \alpha=\beta=\delta=0;\\[2ex] \dfrac{z^{m-1}}{C_2z^{2m}+K_1z^m+K_2} & \text{if } C_1\neq-1,\ \beta=\delta=0,\end{cases}$$

where $C_2\neq0$, $K_1=-\dfrac{\alpha}{C_1+1}$, $K_2=\dfrac{\alpha^2-\gamma(1+C_1)}{4C_2(1+C_1)^2}$, $m^2=1+C_1$.

Accordingly, equation (3.5.4.7) is reduced to equation (3.5.4.8) with the substitution $y=ve^\zeta$.

If $\beta=-\alpha$ and $\delta=-\gamma$, the substitution $y=e^{-iw}$ brings equation (3.5.4.1) to the following form: $w''_{zz}+\dfrac{1}{z}w'_z=\dfrac{2\alpha}{z}\sin w+2\gamma\sin 2w$.

► A solution in terms of Bessel functions.

Any solution of the Riccati equation

$$y'_z=ky^2+\frac{\alpha-k}{kz}y+c, \tag{3.5.4.9}$$

where $k^2=\gamma$, $c^2=-\delta$, $k\beta+c(\alpha-2k)=0$, is a solution of equation (3.5.4.1). Substituting $z=\lambda\tau$, $y=-\dfrac{u'_z}{ku}$, where $\lambda^2=\dfrac{1}{kc}$, into (3.5.4.9), we obtain a linear equation

$$u''_{\tau\tau}+\frac{k-\alpha}{k\tau}u'_\tau+u=0,$$

whose general solution is expressed in terms of Bessel functions:

$$u=\tau^{\frac{\alpha}{2k}}\Big[C_1J_{\frac{\alpha}{2k}}(\tau)+C_2Y_{\frac{\alpha}{2k}}(\tau)\Big].$$

► Asymptotic formulas and some properties.

Let $\alpha=-\beta=2\nu$ ($\nu\in\mathbb{R}$) and $\gamma=-\delta=1$. Then

$$y(x)-1\sim -c_12^{-2\nu}\Gamma\left(\nu+\tfrac12\right)x^{-(2\nu+1)/2}e^{-2x}\quad\text{as}\quad x\to+\infty, \tag{3.5.4.10}$$

where c_1 is an arbitrary constant such that $-1/\pi < c_1 < 1/\pi$, and

$$y(x) \sim c_2 x^{\sigma} \quad \text{at} \quad x \to 0, \tag{3.5.4.11}$$

where c_2 and σ are constants such that $c_2 \neq 0$ and $|\operatorname{Re}\sigma| < 1$. The connection formulas relating (3.5.4.10) and (3.5.4.11) are

$$\sigma = \frac{2}{\pi}\arcsin(\pi c_1), \qquad c_2 = 2^{-2\sigma}\frac{\Gamma^2\left(\frac{1}{2}-\frac{1}{2}\sigma\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma+\nu\right)}{\Gamma^2\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\sigma+\nu\right)}.$$

3.5.5 Fourth Painlevé Equation

► Form of the fourth Painlevé equation. Solutions in the vicinity of a movable pole.

The *fourth Painlevé equation* has the form

$$y''_{zz} = \frac{(y'_z)^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}. \tag{3.5.5.1}$$

The solutions of the fourth Painlevé equation are single-valued functions of z.

The Laurent-series expansion of the solution of equation (3.5.5.1) in the vicinity of a movable pole $z_{\rm p}$ is given by

$$y = \frac{m}{z-z_{\rm p}} - z_{\rm p} - \frac{m}{3}(z_{\rm p}^2 + 2\alpha - 4m)(z-z_{\rm p}) + C(z-z_{\rm p})^2 + \sum_{j=3}^{\infty} a_j(z-z_{\rm p})^j,$$

where $m = \pm 1$; $z_{\rm p}$ and C are arbitrary constants; and the a_j ($j \geq 3$) are uniquely defined in terms of α, β, $z_{\rm p}$, and C.

Remark 3.15. Equation (3.5.5.1) is invariant under the transformation $y = \lambda\bar{y}$, $z = \lambda\bar{z}$, $\alpha = \bar{\alpha}\lambda^2$, $\beta = \bar{\beta}$, where $\lambda^4 = 1$.

► Rational particular solutions.

Let $y = y(z,\alpha,\beta)$ be a solution of equation (3.5.5.1). Then special rational solutions of the fourth Painlevé equation are

$$y_1(z,\pm 2,-2) = \pm 1/z, \qquad y_2(z,0,-2) = -2z, \qquad y_3(z,0,-\tfrac{2}{9}) = -\tfrac{2}{3}z.$$

There are also three more complex families of solutions of equation (3.5.5.1) of the form

$$\begin{aligned}
y_1(z,\alpha_1,\beta_1) &= P_{1,n-1}(z)/Q_{1,n}(z),\\
y_2(z,\alpha_2,\beta_2) &= -2z + \left[P_{2,n-1}(z)/Q_{2,n}(z)\right],\\
y_3(z,\alpha_3,\beta_3) &= -\tfrac{2}{3}z + \left[P_{3,n-1}(z)/Q_{3,n}(z)\right],
\end{aligned}$$

where $P_{j,n-1}(z)$ and $Q_{j,n}(z)$ are polynomials of degrees $n-1$ and n, respectively, with no common zeros.

Some rational particular solutions:

$$y(z,-m,-2(m-1)^2) = -\frac{H'_{m-1}(z)}{H_{m-1}(z)}, \qquad m = 1,\,2,\,3,\,\ldots,$$
$$y(z,-m,-2(m+1)^2) = -2z + \frac{H'_m(z)}{H_m(z)}, \qquad m = 0,\,1,\,2,\,\ldots,$$

where $H_m(z)$ are the Hermite polynomials.

In general, equation (3.5.5.1) has rational solutions iff either

$$\alpha = m, \qquad \beta = -2(1+2n-m)^2,$$

or

$$\alpha = m, \qquad \beta = -2(\tfrac{1}{3}+2n-m)^2,$$

with $m, n \in \mathbb{Z}$.

► Relation between solutions of two equations. Bäcklund transformations.

Two solutions of equation (3.5.5.1) corresponding to different values of the parameters α and β are related to each other by the Bäcklund transformations:

$$\widetilde{y} = \frac{1}{2sy}(y'_z - q - 2szy - sy^2), \qquad q^2 = -2\beta,$$
$$y = -\frac{1}{2s\widetilde{y}}(\widetilde{y}'_z - p + 2sz\widetilde{y} + s\widetilde{y}^2), \qquad p^2 = -2\widetilde{\beta},$$
$$2\beta = -(\widetilde{\alpha}s - 1 - \tfrac{1}{2}p)^2, \qquad 4\alpha = -2s - 2\widetilde{\alpha} - 3sp,$$

where $y = y(z,\alpha,\beta)$, $\widetilde{y} = \widetilde{y}(z,\widetilde{\alpha},\widetilde{\beta})$, and s is an arbitrary parameter.

► A solution in terms of solutions of the Riccati equation.

If the condition

$$\beta = -2(1+\epsilon\alpha)^2 \quad \text{with} \quad \epsilon = \pm 1$$

is satisfied, then every solution of the Riccati equation

$$y'_z = \epsilon y^2 + 2\epsilon zy - 2(1+\epsilon\alpha) \tag{3.5.5.2}$$

is simultaneously a solution of the fourth Painlevé equation (3.5.5.1). The general solution of equation (3.5.5.2) can be expressible in terms of parabolic cylinder functions.

For $\alpha = 1$ and $\epsilon = -1$, equation (3.5.5.2) has a solution

$$y = \frac{2\exp(-z^2)}{\sqrt{\pi}\,(C - \operatorname{erfc} z)},$$

where C is an arbitrary constant and erfc z is the complementary error function.

Remark 3.16. In general, equation (3.5.5.2) has solutions expressible in terms of parabolic cylinder functions iff either

$$\beta = -2(2n+1+\epsilon\alpha)^2 \quad \text{or} \quad \beta = -2n^2,$$

with $n \in \mathbb{Z}$ and $\epsilon = \pm 1$.

► **Symmetric forms.**

Let

$$f_1' + f_1(f_2 - f_3) + 2\mu_1 = 0,$$
$$f_2' + f_2(f_3 - f_1) + 2\mu_2 = 0,$$
$$f_3' + f_3(f_1 - f_2) + 2\mu_2 = 0,$$

where μ_1, μ_2, μ_3 are constants, f_1, f_2, f_3 are functions of z (the prime denotes differentiation with respect to z), with

$$\mu_1 + \mu_2 + \mu_3 = 1,$$
$$f_1 + f_2 + f_3 = -2z.$$

Then the function $y = f_1(z)$ satisfies equation (3.5.5.1) with

$$\alpha = \mu_3 - \mu_2, \qquad \beta = -2\mu_1^2.$$

3.5.6 Fifth Painlevé Equation

► **Form of the fifth Painlevé equation. Relations between solutions.**

The *fifth Painlevé equation* has the form

$$y''_{zz} = \frac{3y - 1}{2y(y-1)}(y'_z)^2 - \frac{y'_z}{z} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{z} + \frac{\delta y(y+1)}{y-1}. \qquad (3.5.6.1)$$

If we pass on to the new independent variable $z = e^{\zeta}$, the solutions are single-valued functions of ζ.

Solutions of the fifth Painlevé equation (3.5.6.1) corresponding to different values of parameters are related by:

$$y(z, \alpha, \beta, \gamma, \delta) = y(-z, \alpha, \beta, -\gamma, \delta),$$
$$y(z, \alpha, \beta, \gamma, \delta) = \frac{1}{y(z, -\beta, -\alpha, -\gamma, \delta)}.$$

► **Rational particular solutions.**

Let $y = y(z, \alpha, \beta, \gamma, \delta)$ be a solution of equation (3.5.6.1). Then special rational solutions of the fourth Painlevé equation are

$$y(z, \tfrac{1}{2}, -\tfrac{1}{2}\mu^2, 2k - k\mu, -\tfrac{1}{2}k^2) = kz + \mu,$$
$$y(z, \tfrac{1}{2}, k^2\mu, 2k\mu, \mu) = k/(k + z),$$
$$y(z, \tfrac{1}{8}, -\tfrac{1}{8}, -k\mu, \mu) = (k + z)/(k - z),$$

where k and μ are arbitrary constants.

If $\delta \neq 0$ in (3.5.6.1), then set $\delta = 1/2$, without loss of generality. In this case the fifth Painlevé equation has a rational solution iff one of the following holds with $m, n \in \mathbb{Z}$ and $\epsilon \pm 1$:

(a) $\alpha = \frac{1}{2}(m+\epsilon\gamma)^2$ and $\beta = -\frac{1}{2}n^2$, where $n > 0$, $n+m$ is odd, and $\alpha \neq 0$ when $|m| < n$.

(b) $\alpha = \frac{1}{2}n^2$ and $\beta = -\frac{1}{2}(m+\epsilon\gamma)^2$, where $n > 0$, $n+m$ is odd, and $\beta \neq 0$ when $|m| < n$.

(c) $\alpha = \frac{1}{2}a^2$, $\beta = -\frac{1}{2}(a+n)^2$, and $\gamma = m$, where $n+m$ is even, and a is an arbitrary constant.

(d) $\alpha = \frac{1}{2}(b+n)^2$, $\beta = -\frac{1}{2}b^2$, and $\gamma = m$, where $n+m$ is even, and b is an arbitrary constant.

(e) $\alpha = \frac{1}{8}(2m+1)^2$, $\beta = -\frac{1}{8}(2n+1)^2$, and $\gamma \notin= \mathbb{Z}$. These rational solutions have the form

$$y = \lambda z + \mu + \frac{P_{n-1}(z)}{Q_n(z)},$$

where $P_{n-1}(z)$ and $Q_n(z)$ are polynomials of degrees $n-1$ and n, respectively, with no common zeros.

► Elementary nonrational particular solutions.

Elementary nonrational solutions of the fifth Painlevé equation are

$$y(z, \mu, -\tfrac{1}{8}, -k^2\mu, 0) = 1 + kz^{1/2},$$
$$y(z, 0, 0, \mu, -\tfrac{1}{2}\mu^2) = k\exp(\mu z),$$

where k and μ are arbitrary constants.

Equation (3.5.6.1), with $\delta = 0$, has algebraic solutions if either

$$\alpha = \tfrac{1}{2}\mu^2, \quad \beta = -\tfrac{1}{8}(2n-1)^2, \quad \gamma = -1,$$

or

$$\alpha = \tfrac{1}{8}(2n-1)^2, \quad \beta = -\tfrac{1}{2}\mu^2, \quad \gamma = 1,$$

with $n \in \mathbb{Z}$. These are rational solutions in $\zeta = z^{1/2}$ of the form

$$y = P_{n^2-n+1}(\zeta)/Q_{n^2-n}(\zeta),$$

where $P_{n^2-n+1}(\zeta)$ and $Q_{n^2-n}(\zeta)$ are polynomials of degrees n^2-n+1 and n^2-n, respectively, with no common zeros.

► Cases when the fifth Painlevé equation are solvable by quadrature.

1°. Equation (3.5.6.1), with $\gamma = \delta = 0$, has a first integral

$$z^2(y'_z)^2 = (y-1)^2(2\alpha y^2 + Cy - 2\beta),$$

which is solvable by quadrature (C is an arbitrary constant).

2°. On setting $z = e^t$ in (3.5.6.1), we obtain

$$y''_{tt} = \frac{3y-1}{2y(y-1)}(y'_t)^2 + (y-1)^2\left(\alpha y + \frac{\beta}{y}\right) + \gamma y e^t + \frac{\delta y(y+1)}{y-1}e^{2t}. \qquad (3.5.6.2)$$

If $\gamma = \delta = 0$, equation (3.5.6.2) is reduced, by integration, to a first-order autonomous equation:

$$y'_t = (y-1)\sqrt{2\alpha y^2 + Cy - 2\beta},$$

which is readily integrable by quadrature.

► **Solutions in terms of Whittaker functions.**

If $\delta \neq 0$ in (3.5.6.1), then set $\delta = 1/2$, without loss of generality. Then the fifth Painlevé equation has solutions expressible in terms of Whittaker functions, only in the following three cases:

$$(a) \quad a + b + \epsilon_3 \gamma = 2n + 1, \tag{3.5.6.3}$$

$$(b) \quad a = n, \tag{3.5.6.4}$$

$$(c) \quad b = n, \tag{3.5.6.5}$$

where $n \in \mathbb{Z}$, $a = \epsilon_1 \sqrt{2\alpha}$, and $b = \epsilon_2 \sqrt{-2\beta}$, with $\epsilon_j = \pm 1$, $j = 1, 2, 3$, independently.

In the case when $n = 0$ in (3.5.6.3), any solution of the Riccati equation

$$zy'_z = ay^2 + (b - a + \epsilon_3 z)y - b \tag{3.5.6.6}$$

is simultaneously a solution of the fifth Painlevé equation (3.5.6.1). If $a \neq 0$, then equation (3.5.6.6) has the solution

$$y = -z\phi'_z(z)/\phi(z),$$

where

$$\phi(z) = \zeta^{-k} \exp\bigl(\tfrac{1}{2}\zeta\bigr)\bigl[C_1 M_{k,\mu}(\zeta) + C_2 W_{k,\mu}(\zeta)\bigr],$$

with $\zeta = \epsilon_3 z$, $k = \frac{1}{2}(a - b + 1)$, $\mu = \frac{1}{2}(a + b)$; C_1 and C_2 are arbitrary constants, and $M_{k,\mu}(\zeta)$ and $W_{k,\mu}(\zeta)$ are Whittaker functions.

3.5.7 Sixth Painlevé Equation

► **Form of the sixth Painlevé equation. Relations between solutions.**

The *sixth Painlevé equation* has the form

$$y''_{zz} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y'_z)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y'_z + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right]. \tag{3.5.7.1}$$

In equation (3.5.7.1), the points $z = 0$, $z = 1$, and $z = \infty$ are fixed logarithmic branch points.

Solutions of the sixth Painlevé equation (3.5.7.1) corresponding to different values of parameters are related by:

$$y(z, -\beta, -\alpha, \gamma, \delta) = \frac{1}{y\left(\dfrac{1}{z}, \alpha, \beta, \gamma, \delta\right)},$$

$$y(z, -\beta, -\gamma, \alpha, \delta) = 1 - \frac{1}{y\left(\dfrac{1}{1-z}, \alpha, \beta, \gamma, \delta\right)},$$

$$y\left(z, -\beta, -\alpha, -\delta + \frac{1}{2}, -\gamma + \frac{1}{2}\right) = \frac{z}{y(z, \alpha, \beta, \gamma, \delta)}.$$

The successive application of these relations yields 24 equations of the form (3.5.7.1) with different values of parameters related by known transformations.

► Rational particular solutions.

Let $y = y(z, \alpha, \beta, \gamma, \delta)$ be a solution of equation (3.5.7.1). Then special rational solutions of the sixth Painlevé equation are

$$
\begin{aligned}
y(z, \mu, -k^2\mu, \tfrac{1}{2}, -\tfrac{1}{2} - \mu(k-1)^2) &= kz,\\
y(z, 0, 0, 2, 0) &= kz^2,\\
y(z, 0, 0, \tfrac{1}{2}, -\tfrac{3}{2}) &= k/z,\\
y(z, 0, 0, 2, -4) &= k/z^2,\\
y(z, \tfrac{1}{2}(k+\mu)^2, -\tfrac{1}{2}, \tfrac{1}{2}(\mu-1)^2, \tfrac{1}{2}k(2-k)) &= z/(k+\mu z),
\end{aligned}
$$

where k and μ are arbitrary constants.

In the general case, the sixth Painlevé equation has rational solutions if

$$
\begin{gathered}
a + b + c + d = 2n + 1, \qquad n \in \mathbb{Z},\\
a = \epsilon_1\sqrt{2\alpha}, \quad b = \epsilon_2\sqrt{-2\beta}, \quad c = \epsilon_3\sqrt{2\gamma}, \quad d = \epsilon_4\sqrt{1-2\delta},
\end{gathered}
$$

where $\epsilon_j = \pm 1$, $j = 1, 2, 3, 4$, independently, and at least one of numbers a, b, c or d is an integer.

► Solutions in terms of the elliptic function.

1°. If $\alpha = \beta = \gamma = \delta = 0$, the general solution of equation (19) has the form:

$$y = E(C_1\omega_1 + C_2\omega_2,\ z),$$

where $E(u, z)$ is the elliptic function, defined by the integral

$$u = \int_0^E \frac{dy}{\sqrt{y(y-1)(y-z)}}, \tag{3.5.7.2}$$

with periods $2\omega_1$ and $2\omega_2$, which are functions of z.

2°. If $\alpha = \beta = \gamma = 0$, $\delta = \frac{1}{2}$, the general solution of equation (3.5.7.1) has the form:

$$y = E(w + C_1\omega_1 + C_2\omega_2,\ z),$$

where $w \not\equiv 0$ is any particular solution of the linear equation

$$z(z-1)w''_{zz} + (2z-1)w'_z + \tfrac{1}{4}w = 0$$

and $E(u, z)$ is the elliptic function defined by formula (3.5.7.2).

► Solutions in terms of hypergeometric functions.

Equation (3.5.7.1) has solutions expressible in terms of hypergeometric functions iff

$$
\begin{gathered}
a + b + c + d = 2n + 1, \qquad n \in \mathbb{Z},\\
a = \epsilon_1\sqrt{2\alpha}, \quad b = \epsilon_2\sqrt{-2\beta}, \quad c = \epsilon_3\sqrt{2\gamma}, \quad d = \epsilon_4\sqrt{1-2\delta},
\end{gathered}
\tag{3.5.7.3}
$$

with $\epsilon_j = \pm 1$, $j = 1,\ 2,\ 3$, independently.

If $n=1$ in (3.5.7.3), then every solution of the Riccati equation

$$z(z-1)y'_z = ay^2 + [(b+c)z - a - c]y - bz, \tag{3.5.7.4}$$

is simultaneously a solution of equation (3.5.7.1). If $a \neq 0$, then (3.5.7.4) has the solution

$$y = \frac{\zeta - 1}{a}\frac{\phi'_\zeta(\zeta)}{\phi(\zeta)}, \qquad \zeta = \frac{1}{1-z},$$

where

$$\phi(\zeta) = C_1 F(b, -a, b+c; \zeta) + C_2 \zeta^{1-b-c} F(1-a-b-c, 1-c, 2-b-c; \zeta),$$

C_1 and C_2 are arbitrary constants, and $F(\alpha, \beta, \gamma; \zeta)$ is the hypergeometric function.

◆ *For more details about Painlevé equations (including of some illustrative figures of Painlevé transcendental functions), see the list of references given below.*

⊙ *Literature for Section 3.5*: P. Painlevé (1900), B. Gambier (1910), V. V. Golubev (1950), A. S. Fokas and M. J. Ablowitz (1982), A. R. Its and V. Yu. Novokshenov (1986), V. I. Gromak and N. A. Lukashevich (1990), R. Conte (1999), A. R. Chowdhury (2000), V. F. Zaitsev and A. D. Polyanin (2001), V. I. Gromak (2002), P. A. Clarkson (2003, 2006), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007), F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (2010).

3.6 Perturbation Methods of Mechanics and Physics

3.6.1 Preliminary Remarks. Summary Table of Basic Methods

Perturbation methods are widely used in nonlinear mechanics and theoretical physics for solving problems that are described by differential equations with a small parameter ε. The primary purpose of these methods is to obtain an approximate solution that would be equally suitable at all (small, intermediate, and large) values of the independent variable as $\varepsilon \to 0$.

Equations with a small parameter can be classified according to the following:

(i) the order of the equation remains the same at $\varepsilon = 0$;

(ii) the order of the equation reduces at $\varepsilon = 0$.

For the first type of equations, solutions of related problems* are sufficiently smooth (little varying as ε decreases). The second type of equation is said to be degenerate at $\varepsilon = 0$, or singularly perturbed. In related problems, thin boundary layers usually arise whose thickness is significantly dependent on ε; such boundary layers are characterized by high gradients of the unknown.

All perturbation methods have a limited domain of applicability; the possibility of using one or another method depends on the type of equations or problems involved. The most commonly used methods are summarized in Table 3.2 (the method of regular series expansions is set out in Section 3.6.2). In subsequent paragraphs, additional remarks and specific examples are given for some of the methods. In practice, one usually confines oneself to a few leading terms of the asymptotic expansion.

In many problems of nonlinear mechanics and theoretical physics, the independent variable is dimensionless time t. Therefore, in this subsection we use the conventional t $(0 \le t < \infty)$ instead of x.

*Further on, we assume that the initial and/or boundary conditions are independent of the parameter ε.

TABLE 3.2
Perturbation methods of nonlinear mechanics and theoretical physics
(the third column gives n leading asymptotic terms with respect to the small parameter ε).

Method name	Examples of problems solved by the method	Form of the solution sought	Additional conditions and remarks
Method of scaled parameters ($0\le t<\infty$)	One looks for periodic solutions of the equation $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$; see also Section 3.6.3	$y(t)=\sum_{k=0}^{n-1}\varepsilon^k y_k(z)$, $t=z\Bigl(1+\sum_{k=1}^{n-1}\varepsilon^k\omega_k\Bigr)$	Unknowns: y_k and ω_k; $y_{k+1}/y_k=O(1)$; secular terms are eliminated through selection of the constants ω_k
Method of strained coordinates ($0\le t<\infty$)	Cauchy problem: $y'_t=f(t,y,\varepsilon)$; $y(t_0)=y_0$ (f is of a special form); see also the problem in the method of scaled parameters	$y(t)=\sum_{k=0}^{n-1}\varepsilon^k y_k(z)$, $t=z+\sum_{k=1}^{n-1}\varepsilon^k\varphi_k(z)$	Unknowns: y_k and φ_k; $y_{k+1}/y_k=O(1)$, $\varphi_{k+1}/\varphi_k=O(1)$
Averaging method ($0\le t<\infty$)	Cauchy problem: $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$, $y(0)=y_0$, $y'_t(0)=y_1$; for more general problems see Section 3.6.4	$y=a(t)\cos\varphi(t)$, the amplitude a and phase φ are governed by the equations $\frac{da}{dt}=-\frac{\varepsilon}{\omega_0}f_s(a)$, $\frac{d\varphi}{dt}=\omega_0-\frac{\varepsilon}{a\omega_0}f_c(a)$	Unknowns: a and φ; $f_s=\frac{1}{2\pi}\int_0^{2\pi}\sin\varphi F\,d\varphi$, $f_c=\frac{1}{2\pi}\int_0^{2\pi}\cos\varphi F\,d\varphi$, $F=f(a\cos\varphi,-a\omega_0\sin\varphi)$
Krylov–Bogolyubov–Mitropolskii method ($0\le t<\infty$)	One looks for periodic solutions of the equation $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$; Cauchy problem for this and other equations	$y=a\cos\varphi+\sum_{k=1}^{n-1}\varepsilon^k y_k(a,\varphi)$, a and φ are determined by the equations $\frac{da}{dt}=\sum_{k=1}^{n}\varepsilon^k A_k(a)$, $\frac{d\varphi}{dt}=\omega_0+\sum_{k=1}^{n}\varepsilon^k\Phi_k(a)$	Unknowns: y_k, A_k, Φ_k; y_k are 2π-periodic functions of φ; the y_k are assumed not to contain $\cos\varphi$
Method of two-scale expansions ($0\le t<\infty$)	Cauchy problem: $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$, $y(0)=y_0$, $y'_t(0)=y_1$; for boundary value problems see Section 3.6.5	$y=\sum_{k=0}^{n-1}\varepsilon^k y_k(\xi,\eta)$, where $\xi=\varepsilon t$, $\eta=t\Bigl(1+\sum_{k=2}^{n-1}\varepsilon^k\omega_k\Bigr)$, $\frac{d}{dt}=\varepsilon\frac{\partial}{\partial\xi}+(1+\varepsilon^2\omega_2+\cdots)\frac{\partial}{\partial\eta}$	Unknowns: y_k and ω_k; $y_{k+1}/y_k=O(1)$; secular terms are eliminated through selection of ω_k
Multiple scales method ($0\le t<\infty$)	One looks for periodic solutions of the equation $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$; Cauchy problem for this and other equations	$y=\sum_{k=0}^{n-1}\varepsilon^k y_k$, where $y_k=y_k(T_0,T_1,\dots,T_n)$, $T_k=\varepsilon^k t$, $\frac{d}{dt}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\cdots+\varepsilon^n\frac{\partial}{\partial T_n}$	Unknowns: y_k; $y_{k+1}/y_k=O(1)$; for $n=1$, this method is equivalent to the averaging method
Method of matched asymptotic expansions ($0\le x\le b$)	Boundary value problem: $\varepsilon y''_{xx}+f(x,y)y'_x=g(x,y)$, $y(0)=y_0$, $y(b)=y_b$ (f assumed positive); for other problems see Section 3.6.6	Outer expansion: $y=\sum_{k=0}^{n-1}\sigma_k(\varepsilon)y_k(x)$, $O(\varepsilon)\le x\le b$; inner expansion ($z=x/\varepsilon$): $\widetilde{y}=\sum_{k=0}^{n-1}\widetilde{\sigma}_k(\varepsilon)\widetilde{y}_k(z)$, $0\le x\le O(\varepsilon)$	Unknowns: y_k, $\widetilde{y}_k$, σ_k, $\widetilde{\sigma}_k$; $y_{k+1}/y_k=O(1)$, $\widetilde{y}_{k+1}/\widetilde{y}_k=O(1)$; the procedure of matching expansions is used: $y(x\to 0)=\widetilde{y}(z\to\infty)$
Method of composite expansions ($0\le x\le b$)	Boundary value problem: $\varepsilon y''_{xx}+f(x,y)y'_x=g(x,y)$, $y(0)=y_0$, $y(b)=y_b$ (f assumed positive); boundary value problems for other equations	$y=Y(x,\varepsilon)+\widetilde{Y}(z,\varepsilon)$, $Y=\sum_{k=0}^{n-1}\sigma_k(\varepsilon)Y_k(x)$, $\widetilde{Y}=\sum_{k=0}^{n-1}\widetilde{\sigma}_k(\varepsilon)\widetilde{Y}_k(z)$, $z=\dfrac{x}{\varepsilon}$; here, $\widetilde{Y}_k\to 0$ as $z\to\infty$	Unknowns: Y_k, $\widetilde{Y}_k$, σ_k, $\widetilde{\sigma}_k$; $Y(b,\varepsilon)=y_b$, $Y(0,\varepsilon)+\widetilde{Y}(0,\varepsilon)=y_0$; two forms for the equation (in terms of x and z) are used to obtain solutions

3.6.2 Method of Regular (Direct) Expansion in Powers of the Small Parameter

We consider an equation of general form with a parameter ε:

$$y''_{tt} + f(t, y, y'_t, \varepsilon) = 0. \tag{3.6.2.1}$$

We assume that the function f can be represented as a series in powers of ε:

$$f(t, y, y'_t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_n(t, y, y'_t). \tag{3.6.2.2}$$

Solutions of the Cauchy problem and various boundary value problems for equation (3.6.2.1) with $\varepsilon \to 0$ are sought in the form of a power series expansion:

$$y = \sum_{n=0}^{\infty} \varepsilon^n y_n(t). \tag{3.6.2.3}$$

One should substitute (3.6.2.3) into equation (3.6.2.1) taking into account (3.6.2.2). Then the functions f_n are expanded into a power series in the small parameter and the coefficients of like powers of ε are collected and equated to zero to obtain a system of equations for y_n:

$$y''_0 + f_0(t, y_0, y'_0) = 0, \tag{3.6.2.4}$$

$$y''_1 + F(t, y_0, y'_0)y'_1 + G(t, y_0, y'_0)y_1 + f_1(t, y_0, y'_0) = 0, \tag{3.6.2.5}$$

$$F = \frac{\partial f_0}{\partial y'}, \qquad G = \frac{\partial f_0}{\partial y}.$$

Here only the first two equations are written out. The prime denotes differentiation with respect to t. To obtain the initial (or boundary) conditions for y_n, the expansion (3.6.2.3) is taken into account.

The success in the application of this method is primarily determined by the possibility of constructing a solution of equation (3.6.2.4) for the leading term y_0. It is significant to note that the other terms y_n with $n \geq 1$ are governed by linear equations with homogeneous initial conditions.

Example 3.26. The *Duffing equation*

$$y''_{tt} + y + \varepsilon y^3 = 0 \tag{3.6.2.6}$$

with initial conditions

$$y(0) = a, \quad y'_t(0) = 0$$

describes the motion of a cubic oscillator, i.e., oscillations of a point mass on a nonlinear spring. Here, y is the deviation of the point mass from the equilibrium and t is dimensionless time.

For $\varepsilon \to 0$, an approximate solution of the problem is sought in the form of the asymptotic expansion (3.6.2.3). We substitute (3.6.2.3) into equation (3.6.2.6) and initial conditions and expand in powers of ε. On equating the coefficients of like powers of the small parameter to zero, we obtain the following problems for y_0 and y_1:

$$y''_0 + y_0 = 0, \qquad y_0 = a, \quad y'_0 = 0;$$

$$y''_1 + y_1 = -y_0^3, \qquad y_1 = 0, \quad y'_1 = 0.$$

The solution of the problem for y_0 is given by

$$y_0 = a\cos t.$$

Substituting this expression into the equation for y_1 and taking into account the identity $\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$, we obtain

$$y_1'' + y_1 = -\tfrac{1}{4}a^3(\cos 3t + 3\cos t), \qquad y_1 = 0, \quad y_1' = 0.$$

Integrating yields

$$y_1 = -\tfrac{3}{8}a^3 t\sin t + \tfrac{1}{32}a^3(\cos 3t - 3\cos t).$$

Thus the two-term solution of the original problem is given by

$$y = a\cos t + \varepsilon a^3\bigl[-\tfrac{3}{8}t\sin t + \tfrac{1}{32}(\cos 3t - 3\cos t)\bigr] + O(\varepsilon^2).$$

Remark 3.17. The term $t\sin t$ causes $y_1/y_0 \to \infty$ as $t \to \infty$. For this reason, the solution obtained is unsuitable at large times. It can only be used for $\varepsilon t \ll 1$; this results from the condition of applicability of the expansion, $y_0 \gg \varepsilon y_1$.

This circumstance is typical of the method of regular series expansions with respect to the small parameter; in other words, the expansion becomes unsuitable at large values of the independent variable. This method is also inapplicable if the expansion (3.6.2.3) begins with negative powers of ε. Methods that allow avoiding the above difficulties are discussed below in Sections 3.6.3 through 3.6.5.

Remark 3.18. Growing terms as $t \to \infty$, like $t\sin t$, that narrow down the domain of applicability of asymptotic expansions are called *secular*.

3.6.3 Method of Scaled Parameters (Lindstedt–Poincaré Method)

We illustrate the characteristic features of the method of scaled parameters with a specific example (the transformation of the independent variable we use here as well as the form of the expansion are specified in the first row of Table 3.2).

Example 3.27. Consider the Duffing equation (3.6.2.6) again. On performing the change of variable

$$t = z(1 + \varepsilon\omega_1 + \cdots),$$

we have

$$y''_{zz} + (1 + \varepsilon\omega_1 + \cdots)^2(y + \varepsilon y^3) = 0. \tag{3.6.3.1}$$

The solution is sought in the series form

$$y = y_0(z) + \varepsilon y_1(z) + \cdots.$$

Substituting it into equation (3.6.3.1) and matching the coefficients of like powers of ε, we arrive at the following system of equations for two leading terms of the series:

$$y_0'' + y_0 = 0, \tag{3.6.3.2}$$

$$y_1'' + y_1 = -y_0^3 - 2\omega_1 y_0, \tag{3.6.3.3}$$

where the prime denotes differentiation with respect to z.

The general solution of equation (3.6.3.2) is given by

$$y_0 = a\cos(z + b), \tag{3.6.3.4}$$

where a and b are constants of integration. Taking into account (3.6.3.4) and rearranging terms, we reduce equation (3.6.3.3) to

$$y_1'' + y_1 = -\tfrac{1}{4}a^3\cos\bigl[3(z+b)\bigr] - 2a\bigl(\tfrac{3}{8}a^2 + \omega_1\bigr)\cos(z+b). \tag{3.6.3.5}$$

For $\omega_1 \neq -\frac{3}{8}a^2$, the particular solution of equation (3.6.3.5) contains a secular term proportional to $z\cos(z+b)$. In this case, the condition of applicability of the expansion $y_1/y_0 = O(1)$ (see the first row and the last column of Table 3.2) cannot be satisfied at sufficiently large z. For this condition to be met, one should set

$$\omega_1 = -\tfrac{3}{8}a^2. \tag{3.6.3.6}$$

In this case, the solution of equation (3.6.3.5) is given by

$$y_1 = \tfrac{1}{32}a^3\cos\bigl[3(z+b)\bigr]. \tag{3.6.3.7}$$

Subsequent terms of the expansion can be found likewise.

With (3.6.3.4), (3.6.3.6), and (3.6.3.7), we obtain a solution of the Duffing equation in the form

$$y = a\cos(\omega t+b) + \tfrac{1}{32}\varepsilon a^3\cos\bigl[3(\omega t+b)\bigr] + O(\varepsilon^2),$$
$$\omega = \bigl[1 - \tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2)\bigr]^{-1} = 1 + \tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2).$$

3.6.4 Averaging Method (Van der Pol–Krylov–Bogolyubov Scheme)

► Averaging method for equations of a special form.

1°. The averaging method involved two stages. First, the second-order nonlinear equation

$$y''_{tt} + \omega_0^2 y = \varepsilon f(y, y'_t) \tag{3.6.4.1}$$

is reduced with the transformation

$$y = a\cos\varphi, \quad y'_t = -\omega_0 a\sin\varphi, \qquad \text{where} \quad a = a(t), \quad \varphi = \varphi(t),$$

to an equivalent system of two first-order differential equations:

$$\begin{aligned} a'_t &= -\frac{\varepsilon}{\omega_0} f(a\cos\varphi, -\omega_0 a\sin\varphi)\sin\varphi,\\ \varphi'_t &= \omega_0 - \frac{\varepsilon}{\omega_0 a} f(a\cos\varphi, -\omega_0 a\sin\varphi)\cos\varphi. \end{aligned} \tag{3.6.4.2}$$

The right-hand sides of equations (3.6.4.2) are periodic in φ, with the amplitude a being a slow function of time t. The amplitude and the oscillation character are changing little during the time the phase φ changes by 2π.

At the second stage, the right-hand sides of equations (3.6.4.2) are being averaged with respect to φ. This procedure results in an approximate system of equations:

$$\begin{aligned} a'_t &= -\frac{\varepsilon}{\omega_0} f_s(a),\\ \varphi'_t &= \omega_0 - \frac{\varepsilon}{\omega_0 a} f_c(a), \end{aligned} \tag{3.6.4.3}$$

where

$$f_s(a) = \frac{1}{2\pi}\int_0^{2\pi} \sin\varphi\, f(a\cos\varphi, -\omega_0 a\sin\varphi)\, d\varphi,$$
$$f_c(a) = \frac{1}{2\pi}\int_0^{2\pi} \cos\varphi\, f(a\cos\varphi, -\omega_0 a\sin\varphi)\, d\varphi.$$

System (3.6.4.3) is substantially simpler than the original system (3.6.4.2)—the first equation in (3.6.4.3), for the oscillation amplitude a, is a separable equation and, hence, can readily be integrated; then the second equation in (3.6.4.3), which is linear in φ, can also be integrated.

Note that the Krylov–Bogolyubov–Mitropolskii method (the fourth row in Table 3.2) generalizes the above approach and allows obtaining subsequent asymptotic terms as $\varepsilon \to 0$.

► **General scheme of the averaging method.**

Below we outline the general scheme of the averaging method. We consider the second-order nonlinear equation with a small parameter:

$$y''_{tt} + g(t, y, y'_t) = \varepsilon f(t, y, y'_t). \tag{3.6.4.4}$$

Equation (3.6.4.4) should first be transformed to the equivalent system of equations

$$\begin{aligned} y'_t &= u, \\ u'_t &= -g(t, y, u) + \varepsilon f(t, y, u). \end{aligned} \tag{3.6.4.5}$$

Suppose the general solution of the "truncated" system (3.6.4.5), with $\varepsilon = 0$, is known:

$$y_0 = \varphi(t, C_1, C_2), \qquad u_0 = \psi(t, C_1, C_2), \tag{3.6.4.6}$$

where C_1 and C_2 are constants of integration. Taking advantage of the method of variation of constants, we pass from the variables y, u in (3.6.4.5) to Lagrange's variables x_1, x_2 according to the formulas

$$y = \varphi(t, x_1, x_2), \qquad u = \psi(t, x_1, x_2), \tag{3.6.4.7}$$

where φ and ψ are the same functions that define the general solution of the "truncated" system (3.6.4.6). Transformation (3.6.4.7) enables the reduction of system (3.6.4.5) to the *standard form*

$$\begin{aligned} x'_1 &= \varepsilon F_1(t, x_1, x_2), \\ x'_2 &= \varepsilon F_2(t, x_1, x_2). \end{aligned} \tag{3.6.4.8}$$

Here the prime denotes differentiation with respect to t and

$$F_1 = \frac{\varphi_2 f(t, \varphi, \psi)}{\varphi_2 \psi_1 - \varphi_1 \psi_2}, \quad F_2 = -\frac{\varphi_1 f(t, \varphi, \psi)}{\varphi_2 \psi_1 - \varphi_1 \psi_2}; \quad \varphi_k = \frac{\partial \varphi}{\partial x_k}, \quad \psi_k = \frac{\partial \psi}{\partial x_k},$$
$$\varphi = \varphi(t, x_1, x_2), \quad \psi = \psi(t, x_1, x_2).$$

It is noteworthy that system (3.6.4.8) is equivalent to the original equation (3.6.4.4). The unknowns x_1 and x_2 are slow functions of time.

As a result of averaging, system (3.6.4.8) is replaced by a simpler, approximate autonomous system of equations:

$$\begin{aligned} x'_1 &= \varepsilon \mathcal{F}_1(x_1, x_2), \\ x'_2 &= \varepsilon \mathcal{F}_2(x_1, x_2), \end{aligned} \tag{3.6.4.9}$$

where

$$\mathcal{F}_k(x_1, x_2) = \frac{1}{T}\int_0^T F_k(t, x_1, x_2)\, dt \qquad \text{if } F_k \text{ is a } T\text{-periodic function of } t;$$

$$\mathcal{F}_k(x_1, x_2) = \lim_{T\to\infty} \frac{1}{T}\int_0^T F_k(t, x_1, x_2)\, dt \quad \text{if } F_k \text{ is not periodic in } t.$$

Remark 3.19. The averaging method is applicable to equations (3.6.4.1) and (3.6.4.4) with non-smooth right-hand sides.

Remark 3.20. The averaging method has rigorous mathematical substantiation. There is also a procedure that allows finding subsequent asymptotic terms. For this procedure, e.g., see the books by Bogolyubov and Mitropolskii (1974), Zhuravlev and Klimov (1988), and Arnold, Kozlov, and Neishtadt (1993).

3.6.5 Method of Two-Scale Expansions (Cole–Kevorkian Scheme)

► **Method of two-scale expansions for a specific example (Van der Pol equation).**

We illustrate the characteristic features of the method of two-scale expansions with a specific example. Thereafter we outline possible generalizations and modifications of the method.

Example 3.28. Consider the *Van der Pol equation*

$$y''_{tt} + y = \varepsilon(1 - y^2)y'_t. \tag{3.6.5.1}$$

The solution is sought in the form (see the fifth row in Table 3.2):

$$\begin{aligned} &y = y_0(\xi, \eta) + \varepsilon y_1(\xi, \eta) + \varepsilon^2 y_2(\xi, \eta) + \cdots, \\ &\xi = \varepsilon t, \quad \eta = \bigl(1 + \varepsilon^2 \omega_2 + \cdots\bigr)t. \end{aligned} \tag{3.6.5.2}$$

On substituting (3.6.5.2) into (3.6.5.1) and on matching the coefficients of like powers of ε, we obtain the following system for two leading terms:

$$\frac{\partial^2 y_0}{\partial \eta^2} + y_0 = 0, \tag{3.6.5.3}$$

$$\frac{\partial^2 y_1}{\partial \eta^2} + y_1 = -2\frac{\partial^2 y_0}{\partial \xi \partial \eta} + (1 - y_0^2)\frac{\partial y_0}{\partial \eta}. \tag{3.6.5.4}$$

The general solution of equation (3.6.5.3) is given by

$$y_0 = A(\xi)\cos\eta + B(\xi)\sin\eta. \tag{3.6.5.5}$$

The dependence of A and B on the slow variable ξ is not being established at this stage.

We substitute (3.6.5.5) into the right-hand side of equation (3.6.5.4) and perform elementary manipulations to obtain

$$\begin{aligned} \frac{\partial^2 y_1}{\partial \eta^2} + y_1 = {} & \bigl[-2B'_\xi + \tfrac{1}{4}B(4 - A^2 - B^2)\bigr]\cos\eta + \bigl[2A'_\xi - \tfrac{1}{4}A(4 - A^2 - B^2)\bigr]\sin\eta \\ & + \tfrac{1}{4}(B^3 - 3A^2B)\cos 3\eta + \tfrac{1}{4}(A^3 - 3AB^2)\sin 3\eta. \end{aligned} \tag{3.6.5.6}$$

The solution of this equation must not contain unbounded terms as $\eta \to \infty$; otherwise the necessary condition $y_1/y_0 = O(1)$ is not satisfied. Therefore the coefficients of $\cos\eta$ and $\sin\eta$ must be set equal to zero:

$$\begin{aligned} -2B'_\xi + \tfrac{1}{4}B(4 - A^2 - B^2) &= 0, \\ 2A'_\xi - \tfrac{1}{4}A(4 - A^2 - B^2) &= 0. \end{aligned} \tag{3.6.5.7}$$

Equations (3.6.5.7) serve to determine $A = A(\xi)$ and $B = B(\xi)$. We multiply the first equation in (3.6.5.7) by $-B$ and the second by A and add them together to obtain

$$r'_\xi - \tfrac{1}{8}r(4 - r^2) = 0, \qquad \text{where} \quad r^2 = A^2 + B^2. \tag{3.6.5.8}$$

The integration by separation of variables yields

$$r^2 = \frac{4r_0^2}{r_0^2 + (4 - r_0^2)e^{-\xi}}, \tag{3.6.5.9}$$

where r_0 is the initial oscillation amplitude.

On expressing A and B in terms of the amplitude r and phase φ, we have $A = r\cos\varphi$ and $B = -r\sin\varphi$. Substituting these expressions into either of the two equations in (3.6.5.7) and using (3.6.5.8), we find that $\varphi'_\xi = 0$ or $\varphi = \varphi_0 = \text{const}$. Therefore the leading asymptotic term can be represented as

$$y_0 = r(\xi)\cos(\eta + \varphi_0),$$

where $\xi = \varepsilon t$ and $\eta = t$, and the function $r(\xi)$ is determined by (3.6.5.9).

► **General scheme of the method of two-scale expansions.**

The method of two-scale expansions can also be used for solving boundary value problems where the small parameter appears together with the highest derivative as a factor (such problems for $0 \le x \le a$ are indicated in the seventh row of Table 3.2 and in Section 3.6.6). In the case where a boundary layer arises near the point $x = 0$ (and its thickness has an order of magnitude of ε), the solution is sought in the form

$$y = y_0(\xi, \eta) + \varepsilon y_1(\xi, \eta) + \varepsilon^2 y_2(\xi, \eta) + \cdots,$$
$$\xi = x, \quad \eta = \varepsilon^{-1}\bigl[g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots\bigr],$$

where the functions $y_k = y_k(\xi, \eta)$ and $g_k = g_k(x)$ are to be determined. The derivative with respect to x is calculated in accordance with the rule

$$\frac{d}{dx} = \frac{\partial}{\partial \xi} + \eta'_x \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{1}{\varepsilon}\bigl(g'_0 + \varepsilon g'_1 + \varepsilon^2 g'_2 + \cdots\bigr)\frac{\partial}{\partial \eta}.$$

Additional conditions are imposed on the asymptotic terms in the domain under consideration; namely, $y_{k+1}/y_k = O(1)$ and $g_{k+1}/g_k = O(1)$ for $k = 0,\ 1,\ \ldots$, and $g_0(x) \to x$ as $x \to 0$.

Remark 3.21. The two-scale method is also used to solve problems that arise in mechanics and physics and are described by partial differential equations.

3.6.6 Method of Matched Asymptotic Expansions

► **Method of matched asymptotic expansions for a specific example.**

We illustrate the characteristic features of the method of matched asymptotic expansions with a specific example (the form of the expansions is specified in the seventh row of Table 3.2). Thereafter we outline possible generalizations and modifications of the method.

Example 3.29. Consider the linear boundary value problem

$$\varepsilon y''_{xx} + y'_x + f(x)y = 0, \tag{3.6.6.1}$$
$$y(0) = a, \quad y(1) = b, \tag{3.6.6.2}$$

where $0 < f(0) < \infty$.

At $\varepsilon = 0$ equation (3.6.6.1) degenerates; the solution of the resulting first-order equation

$$y'_x + f(x)y = 0 \tag{3.6.6.3}$$

cannot meet the two boundary conditions (3.6.6.2) simultaneously. It can be shown that the condition at $x = 0$ has to be omitted in this case (a boundary layer arises near this point).

The leading asymptotic term of the outer expansion, $y = y_0(x) + O(\varepsilon)$, is determined by equation (3.6.6.3). The solution of (3.6.6.3) that satisfies the second boundary condition in (3.6.6.2) is given by

$$y_0(x) = b\exp\biggl[\int_x^1 f(\xi)\,d\xi\biggr]. \tag{3.6.6.4}$$

We seek the leading term of the inner expansion, in the boundary layer adjacent to the left boundary, in the following form (see the seventh row and third column in Table 3.2):

$$\widetilde{y} = \widetilde{y}_0(z) + O(\varepsilon), \quad z = x/\varepsilon, \tag{3.6.6.5}$$

where z is the extended variable. Substituting (3.6.6.5) into (3.6.6.1) and extracting the coefficient of ε^{-1}, we obtain

$$\widetilde{y}_0'' + \widetilde{y}_0' = 0, \tag{3.6.6.6}$$

where the prime denotes differentiation with respect to z. The solution of equation (3.6.6.6) that satisfies the first boundary condition in (3.6.6.2) is given by

$$\widetilde{y}_0 = a - C + Ce^{-z}. \tag{3.6.6.7}$$

The constant of integration C is determined from the condition of matching the leading terms of the outer and inner expansions:

$$y_0(x \to 0) = \widetilde{y}_0(z \to \infty). \tag{3.6.6.8}$$

Substituting (3.6.6.4) and (3.6.6.7) into condition (3.6.6.8) yields

$$C = a - be^{\langle f \rangle}, \qquad \text{where} \quad \langle f \rangle = \int_0^1 f(x)\,dx. \tag{3.6.6.9}$$

Taking into account relations (3.6.6.4), (3.6.6.5), (3.6.6.7), and (3.6.6.9), we represent the approximate solution in the form

$$y = \begin{cases} be^{\langle f \rangle} + \bigl(a - be^{\langle f \rangle}\bigr)e^{-x/\varepsilon} & \text{for } 0 \le x \le O(\varepsilon), \\ b\exp\Bigl[\int_x^1 f(\xi)\,d\xi\Bigr] & \text{for } O(\varepsilon) \le x \le 1. \end{cases} \tag{3.6.6.10}$$

It is apparent that inside the thin boundary layer, whose thickness is proportional to ε, the solution rapidly changes by a finite value, $\Delta = be^{\langle f \rangle} - a$.

To determine the function y on the entire interval $x \in [0, 1]$ using formula (3.6.6.10), one has to "switch" at some intermediate point $x = x_0$ from one part of the solution to the other. Such switching is not convenient and, in practice, one often resorts to a *composite solution* instead of using the double formula (3.6.6.10). In similar cases, a composite solution is defined as

$$y = y_0(x) + \widetilde{y}_0(z) - A, \qquad A = \lim_{x\to 0} y_0(x) = \lim_{z\to\infty} \widetilde{y}_0(z).$$

In the problem under consideration, we have $A = be^{\langle f \rangle}$ and hence the composite solution becomes

$$y = \bigl(a - be^{\langle f \rangle}\bigr)e^{-x/\varepsilon} + b\exp\Bigl[\int_x^1 f(\xi)\,d\xi\Bigr].$$

For $\varepsilon \ll x \le 1$, this solution transforms to the outer solution $y_0(x)$ and for $0 \le x \ll \varepsilon$, to the inner solution, thus providing an approximate representation of the unknown over the entire domain.

► General scheme of the method of matched asymptotic expansions. Some remarks.

We now consider an equation of the general form

$$\varepsilon y''_{xx} = F(x, y, y'_x) \tag{3.6.6.11}$$

subject to boundary conditions (3.6.6.2).

For the leading term of the outer expansion $y = y_0(x) + \cdots$, we have the equation

$$F(x, y_0, y_0') = 0.$$

In the general case, when using the method of matched asymptotic expansions, the position of the boundary layer and the form of the inner (extended) variable have to be determined in the course of the solution of the problem.

First we assume that the boundary layer is located near the left boundary. In (3.6.6.11), we make a change of variable $z = x/\delta(\varepsilon)$ and rewrite the equation as

$$y''_{zz} = \frac{\delta^2}{\varepsilon} F\Big(\delta z, y, \frac{1}{\delta} y'_z\Big). \tag{3.6.6.12}$$

The function $\delta = \delta(\varepsilon)$ is selected so that the right-hand side of equation (3.6.6.12) has a nonzero limit value as $\varepsilon \to 0$, provided that z, y, and y'_z are of the order of 1.

Example 3.30. For $F(x, y, y'_x) = -kx^\lambda y'_x + y$, where $0 \le \lambda < 1$, the substitution $z = x/\delta(\varepsilon)$ brings equation (3.6.6.11) to

$$y''_{zz} = -\frac{\delta^{1+\lambda}}{\varepsilon} kz^\lambda y'_z + \frac{\delta^2}{\varepsilon} y.$$

In order that the right-hand side of this equation has a nonzero limit value as $\varepsilon \to 0$, one has to set $\delta^{1+\lambda}/\varepsilon = 1$ or $\delta^{1+\lambda}/\varepsilon = \text{const}$, where const is any positive number. It follows that $\delta = \varepsilon^{\frac{1}{1+\lambda}}$.

The leading asymptotic term of the inner expansion in the boundary layer, $y = \widetilde{y}_0(z) + \cdots$, is determined by the equation $\widetilde{y}''_0 + kz^\lambda \widetilde{y}'_0 = 0$, where the prime denotes differentiation with respect to z.

If the position of the boundary layer is selected incorrectly, the outer and inner expansions cannot be matched. In this situation, one should consider the case where an arbitrary boundary layer is located on the right (this case is reduced to the previous one with the change of variable $x = 1 - z$). In Example 3.30 above, the boundary layer is on the left if $k > 0$ and on the right if $k < 0$.

There is a procedure for matching subsequent asymptotic terms of the expansion (see the seventh row and last column in Table 3.2). In its general form, this procedure can be represented as

inner expansion of the outer expansion (y*-expansion for* $x \to 0$)

$=$ *outer expansion of the inner expansion* ($\widetilde{y}$*-expansion for* $z \to \infty$).

Remark 3.22. The method of matched asymptotic expansions can also be applied to construct periodic solutions of singularly perturbed equations (e.g., in the problem of relaxation oscillations of the Van der Pol oscillator).

Remark 3.23. Two boundary layers can arise in some problems (e.g., in cases where the right-hand side of equation (3.6.6.11) does not explicitly depend on y'_x).

Remark 3.24. The method of matched asymptotic expansions is also used for solving equations (in semi-infinite domains) that do not degenerate at $\varepsilon = 0$. In such cases, there are no boundary layers; the original variable is used in the inner domain, and an extended coordinate is introduced in the outer domain.

Remark 3.25. The method of matched asymptotic expansions is successfully applied for the solution of various problems in mathematical physics that are described by partial differential equations; in particular, it plays an important role in the theory of heat and mass transfer and in hydrodynamics.

⊙ *Literature for Section 3.6*: M. Van Dyke (1964), G. D. Cole (1968), G. E. O. Giacaglia (1972), A. H. Nayfeh (1973, 1981), N. N. Bogolyubov and Yu. A. Mitropolskii (1974), J. Kevorkian and J. D. Cole (1981, 1996), P. A. Lagerstrom (1988), V. Ph. Zhuravlev and D. M. Klimov (1988), J. A. Murdock (1991), V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt (1993), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

3.7 Galerkin Method and Its Modifications (Projection Methods)

3.7.1 Approximate Solution for a Boundary Value Problem

► **Approximate solution is linear with respect to unknown coefficients.**

Consider a boundary value problem for the equation

$$\mathfrak{F}[y] - f(x) = 0 \tag{3.7.1.1}$$

with linear homogeneous boundary conditions* at the points $x = x_1$ and $x = x_2$ ($x_1 \le x \le x_2$). Here, $\mathfrak{F}$ is a linear or nonlinear differential operator of the second order (or a higher order operator); $y = y(x)$ is the unknown function and $f = f(x)$ is a given function. It is assumed that $\mathfrak{F}[0] = 0$.

Let us choose a sequence of linearly independent functions (called *basis functions*)

$$\varphi = \varphi_n(x) \qquad (n = 1,\, 2,\, \ldots,\, N) \tag{3.7.1.2}$$

satisfying the same boundary conditions as $y = y(x)$. According to all methods that will be considered below, an approximate solution of equation (3.7.1.1) is sought as a linear combination

$$y_N = \sum_{n=1}^{N} A_n \varphi_n(x), \tag{3.7.1.3}$$

with the unknown coefficients A_n to be found in the process of solving the problem.

The finite sum (3.7.1.3) is called an *approximation function*. The remainder term R_N obtained after the finite sum has been substituted into the left-hand side of equation (3.7.1.1),

$$R_N = \mathfrak{F}[y_N] - f(x). \tag{3.7.1.4}$$

If the remainder R_N is identically equal to zero, then the function y_N is the exact solution of equation (3.7.1.1). In general, $R_N \not\equiv 0$.

► **General form of an approximate solution.**

Instead of the approximation function (3.7.1.3), which is linear in the unknown coefficients A_n, one can look for a more general form of the approximate solution:

$$y_N = \Phi(x, A_1, \ldots, A_N), \tag{3.7.1.5}$$

where $\Phi(x, A_1, \ldots, A_N)$ is a given function (based on experimental data or theoretical considerations suggested by specific features of the problem) satisfying the boundary conditions for any values of the coefficients $A_1, \ldots, A_N$.

*For second-order ODEs, nonhomogeneous boundary conditions can be reduced to homogeneous ones by the change of variable $z = A_2x^2 + A_1x + A_0 + y$ (the constants A_2, A_1, and A_0 are selected using the method of undetermined coefficients).

3.7.2 Galerkin Method. General Scheme

In order to find the coefficients A_n in (3.7.1.3), consider another sequence of linearly independent functions

$$\psi = \psi_k(x) \qquad (k = 1,\, 2,\, \dots,\, N). \tag{3.7.2.1}$$

Let us multiply both sides of (3.7.1.4) by ψ_k and integrate the resulting relation over the region $V = \{x_1 \le x \le x_2\}$, in which we seek the solution of equation (3.7.1.1). Next, we equate the corresponding integrals to zero (for the exact solutions, these integrals are equal to zero). Thus, we obtain the following system of linear algebraic equations for the unknown coefficients A_n:

$$\int_{x_1}^{x_2} \psi_k R_N\, dx = 0 \qquad (k = 1,\, 2,\, \dots,\, N). \tag{3.7.2.2}$$

Relations (3.7.2.2) mean that the approximation function (3.7.1.3) satisfies equation (3.7.1.1) "on the average" (i.e., in the integral sense) with weights ψ_k. Introducing the scalar product $\langle g, h\rangle = \int_{x_1}^{x_2} gh\, dx$ of arbitrary functions g and h, we can consider equations (3.7.2.2) as the condition of orthogonality of the remainder R_N to all weight functions ψ_k.

The Galerkin method can be applied not only to boundary value problems, but also to eigenvalue problems (in the latter case, one takes $f = \lambda y$ and seeks eigenfunctions y_n, together with eigenvalues λ_n).

Mathematical justification of the Galerkin method for specific boundary value problems can be found in the literature listed at the end of Section 3.7. Below we describe some other methods that are in fact special cases of the Galerkin method.

Remark 3.26. Most often, one takes suitable sequences of polynomials or trigonometric functions as $\varphi_n(x)$ in the approximation function (3.7.1.3).

3.7.3 Bubnov–Galerkin, Moment, and Least Squares Methods

▸ Bubnov–Galerkin method.

The sequences of functions (3.7.1.2) and (3.7.2.1) in the Galerkin method can be chosen arbitrarily. In the case of equal functions,

$$\varphi_k(x) = \psi_k(x) \qquad (k = 1,\, 2,\, \dots,\, N), \tag{3.7.3.1}$$

the method is often called the *Bubnov–Galerkin method*.

▸ Moment method.

2°. The *moment method* is the Galerkin method with the weight functions (3.7.2.1) being powers of x,

$$\psi_k = x^k. \tag{3.7.3.2}$$

► **Least squares method.**

Sometimes, the functions ψ_k are expressed in terms of φ_k by the relations

$$\psi_k = \mathfrak{F}[\varphi_k] \qquad (k = 1,\, 2,\, \dots),$$

where $\mathfrak{F}$ is the differential operator of equation (3.7.1.1). This version of the Galerkin method is called the *least squares method*.

3.7.4 Collocation Method

In the collocation method, one chooses a sequence of points x_k, $k = 1, \dots, N$, and imposes the condition that the remainder (3.7.1.4) be zero at these points,

$$R_N = 0 \quad \text{at} \quad x = x_k \qquad (k = 1, \dots, N). \tag{3.7.4.1}$$

When solving a specific problem, the points x_k, at which the remainder R_N is set equal to zero, are regarded as most significant. The number of collocation points N is taken equal to the number of the terms of the series (3.7.1.3). This enables one to obtain a complete system of algebraic equations for the unknown coefficients A_n (for linear boundary value problems, this algebraic system is linear).

Note that the collocation method is a special case of the Galerkin method with the sequence (3.7.2.1) consisting of the Dirac delta functions:

$$\psi_k = \delta(x - x_k).$$

In the collocation method, there is no need to calculate integrals, and this essentially simplifies the procedure of solving nonlinear problems (although usually this method yields less accurate results than other modifications of the Galerkin method).

Example 3.31. Consider the boundary value problem for the linear variable-coefficient second-order ordinary differential equation

$$y''_{xx} + g(x)y - f(x) = 0 \tag{3.7.4.2}$$

subject to the boundary conditions of the first kind

$$y(-1) = y(1) = 0. \tag{3.7.4.3}$$

Assume that the coefficients of equation (3.7.4.2) are smooth even functions, so that $f(x) = f(-x)$ and $g(x) = g(-x)$. We use the collocation method for the approximate solution of problem (3.7.4.2)–(3.7.4.3).

1°. Take the polynomials

$$y_n(x) = x^{2n-2}(1 - x^2), \qquad n = 1,\, 2,\, \dots\, N,$$

as the basis functions; they satisfy the boundary conditions (3.7.4.3), $y_n(\pm 1) = 0$.

Let us consider three collocation points

$$x_1 = -\sigma, \quad x_2 = 0, \quad x_3 = \sigma \qquad (0 < \sigma < 1) \tag{3.7.4.4}$$

and confine ourselves to two basis functions ($N = 2$), so that the approximation function is taken in the form

$$y(x) = A_1(1 - x^2) + A_2 x^2 (1 - x^2). \tag{3.7.4.5}$$

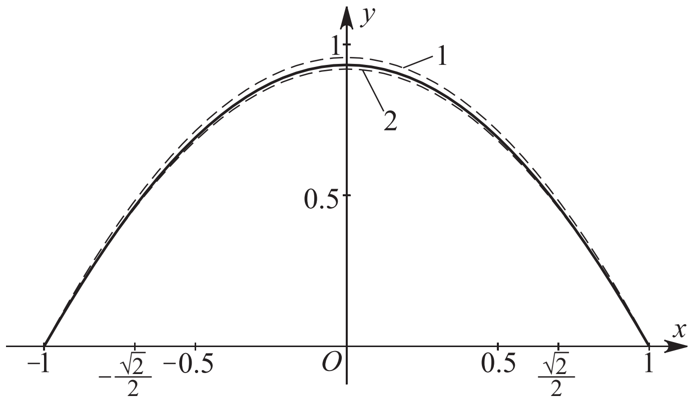


Figure 3.4: Comparison of the numerical solution of problem (3.7.4.2), (3.7.4.3), (3.7.4.7) with the approximate analytical solution (3.7.4.5), (3.7.4.8) obtained with the collocation method.

Substituting (3.7.4.5) in the left-hand side of equation (3.7.4.2) yields the remainder

$$R(x) = A_1\big[-2 + (1 - x^2)g(x)\big] + A_2\big[2 - 12x^2 + x^2(1 - x^2)g(x)\big] - f(x).$$

It must vanish at the collocation points (3.7.4.4). Taking into account the properties $f(\sigma) = f(-\sigma)$ and $g(\sigma) = g(-\sigma)$, we obtain two linear algebraic equations for the coefficients A_1 and A_2:

$$\begin{aligned} A_1\big[-2 + g(0)\big] + 2A_2 - f(0) &= 0 \qquad \text{(at } x = 0\text{)},\\ A_1\big[-2 + (1 - \sigma^2)g(\sigma)\big] + A_2\big[2 - 12\sigma^2 + \sigma^2(1 - \sigma^2)g(\sigma)\big] - f(\sigma) &= 0 \qquad \text{(at } x = \pm\sigma\text{)}. \end{aligned} \tag{3.7.4.6}$$

2°. To be specific, let us take the following functions entering equation (3.7.4.2):

$$f(x) = -1, \quad g(x) = 1 + x^2. \tag{3.7.4.7}$$

On solving the corresponding system of algebraic equations (3.7.4.6), we find the coefficients

$$A_1 = \frac{\sigma^4 + 11}{\sigma^4 + 2\sigma^2 + 11}, \qquad A_2 = -\frac{\sigma^2}{\sigma^4 + 2\sigma^2 + 11}. \tag{3.7.4.8}$$

In Fig. 3.4, the solid line depicts the numerical solution to problem (3.7.4.2)–(3.7.4.3), with the functions (3.7.4.7), obtained by the shooting method (see Section 3.8.5). The dashed lines 1 and 2 show the approximate solutions obtained by the collocation method using the formulas (3.7.4.5), (3.7.4.8) with $\sigma = \frac{1}{2}$ (equidistant points) and $\sigma = \frac{\sqrt{2}}{2}$ (Chebyshev points, see Section 4.5), respectively. It is evident that both cases provide good coincidence of the approximate and numerical solutions; the use of Chebyshev points gives a more accurate result.

Remark 3.27. The theorem of convergence of the collocation method for linear boundary value problems is given in Section 4.5, where nth-order differential equations are considered.

3.7.5 Method of Partitioning the Domain

The domain $V = \{x_1 \le x \le x_2\}$ is split into N subdomains: $V_k = \{x_{k1} \le x \le x_{k2}\}$, $k = 1, \ldots, N$. In this method, the weight functions are chosen as follows:

$$\psi_k(x) = \begin{cases} 1 & \text{for } x \in V_k, \\ 0 & \text{for } x \notin V_k. \end{cases}$$

The subdomains V_k are chosen according to the specific properties of the problem under consideration and can generally be arbitrary (the union of all subdomains V_k may differ from the domain V, and some V_k and V_m may overlap).

3.7.6 Least Squared Error Method

Sometimes, in order to find the coefficients A_n of the approximation function (3.7.1.3), one uses the least squared error method based on the minimization of the functional:

$$\Phi = \int_{x_1}^{x_2} R_N^2\, dx \to \min. \qquad (3.7.6.1)$$

For given functions φ_n in (3.7.1.3), the integral Φ is a function with respect to the coefficients A_n. The corresponding necessary conditions of minimum in (3.7.6.1) have the form

$$\frac{\partial \Phi}{\partial A_n} = 0 \qquad (n = 1, \dots, N).$$

This is a system of algebraic (transcendental) equations for the coefficients A_n.

⊙ *Literature for Section 3.7*: L. V. Kantorovich and V. I. Krylov (1962), M. A. Krasnosel'skii, G. M. Vainikko, et al. (1969), B. A. Finlayson (1972), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

3.8 Iteration and Numerical Methods

3.8.1 Method of Successive Approximations (Cauchy Problem)

The method of successive approximations is implemented in two steps. First, the Cauchy problem

$$y''_{xx} = f(x, y, y'_x) \qquad \text{(equation)}, \qquad (3.8.1.1)$$

$$y(x_0) = y_0, \quad y'_x(x_0) = y'_0 \qquad \text{(initial conditions)} \qquad (3.8.1.2)$$

is reduced to an equivalent system of integral equations by the introduction of the new variable $u(x) = y'_x$. These integral equations have the form

$$u(x) = y'_0 + \int_{x_0}^{x} f\bigl(t, y(t), u(t)\bigr)\, dt, \qquad y(x) = y_0 + \int_{x_0}^{x} u(t)\, dt. \qquad (3.8.1.3)$$

Then the solution of system (3.8.1.3) is sought by means of successive approximations defined by the following recurrence formulas:

$$u_{n+1}(x) = y'_0 + \int_{x_0}^{x} f\bigl(t, y_n(t), u_n(t)\bigr)\, dt, \quad y_{n+1}(x) = y_0 + \int_{x_0}^{x} u_n(t)\, dt; \qquad n = 0,\ 1,\ 2,\ \dots$$

As the initial approximation, one can take $y_0(x) = y_0$ and $u_0(x) = y'_0$.

Remark 3.28. If the right-hand side of equation (3.8.1.1) is independent of the derivative, i.e., $f(x, y, y'_x) = f(x, y)$, the equation can simply be differentiated twice taking into account the initial conditions without reducing it to system (3.8.1.3). In doing so, we arrive at the integral equation

$$y = y_0 + y'_0 x + \int_0^x (x - t) f\bigl(t, y(t)\bigr)\, dt.$$

The solution of problem (3.8.1.1)–(3.8.1.1) (3.8.1.3) is sought using successive approximations defined by the recurrence formulas

$$y_{n+1}(x) = y_0 + y_0' x + \int_0^x (x-t) f\bigl(t, y_n(t)\bigr)\, dt; \qquad n = 0,\,1,\,2,\,\dots$$

As the initial approximation, one can take $y_0(x) = y_0 + y_0' x$.

3.8.2 Runge–Kutta Method (Cauchy Problem)

For the numerical integration of the Cauchy problem (3.8.1.1)–(3.8.1.2), one often uses the Runge–Kutta method of the fourth-order approximation.

Let the mesh increment h be sufficiently small. We introduce the following notation:

$$x_k = x_0 + kh, \quad y_k = y(x_k), \quad y_k' = y_x'(x_k), \quad f_k = f(x_k, y_k, y_k'); \qquad k = 0,\,1,\,2,\,\dots$$

The desired values y_k and y_k' are successively found by the formulas

$$\begin{aligned} y_{k+1} &= y_k + h y_k' + \tfrac{1}{6} h^2 (\varphi_1 + \varphi_2 + \varphi_3), \\ y_{k+1}' &= y_k' + \tfrac{1}{6} h (\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4), \end{aligned}$$

where

$$\begin{aligned} \varphi_1 &= f\bigl(x_k,\, y_k,\, y_k'\bigr), \\ \varphi_2 &= f\bigl(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h y_k',\, y_k' + \tfrac{1}{2}h\varphi_1\bigr), \\ \varphi_3 &= f\bigl(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h y_k' + \tfrac{1}{4}h^2\varphi_1,\, y_k' + \tfrac{1}{2}h\varphi_2\bigr), \\ \varphi_4 &= f\bigl(x_k + h,\, y_k + h y_k' + \tfrac{1}{2}h^2\varphi_2,\, y_k' + h\varphi_3\bigr). \end{aligned}$$

In practice, the step Δx is determined in the same way as for first-order equations (see Remark 1.32 in Section 1.13.1).

3.8.3 Reduction to a System of Equations (Cauchy Problem)

The Cauchy problem (3.8.1.1)–(3.8.1.2) for a single second-order equation can be reduced with the new variable $z = y_x'$ to the Cauchy problem for a system of two first-order equations:

$$\begin{aligned} &y_x' = z, \quad z_x' = f(x,y,z) \quad \text{(equations)}, \\ &y(x_0) = y_0, \quad z(x_0) = y_0' \quad \text{(initial conditions)}. \end{aligned}$$

This problem can be numerically integrated using the methods described in Section 7.4.

3.8.4 Predictor–Corrector Methods (Cauchy Problem)

► **Second-order equation of the general form.**

We look at equation (3.8.1.1).

(i) *Predictor step.* With the values at x_{k-3}, x_{k-2}, x_{k-1}, and x_k, one uses the formula

$$\widetilde{y}_{k+1}' = y_{k-3}' + \tfrac{4}{3} h (2f_k - f_{k-1} + 2f_{k-2})$$

to compute an initial guess value of the derivative at x_{k+1}.

(ii) *Corrector step.* One improves the initial guess by computing the value of y and its derivative at x_{k+1} using the formulas

$$y_{k+1} = y_{k-1} + \tfrac{1}{3}h(\widetilde{y}'_{k+1} + 4y'_k + y'_{k-1}),$$
$$y'_{k+1} = y'_{k-1} + \tfrac{1}{3}h(\widetilde{f}_{k+1} + 4f_k + f_{k-1}),$$

where $\widetilde{f}_{k+1} = f(x_{k+1}, y_{k+1}, \widetilde{y}'_{k+1})$.

► **Second-order equation of a special form.**

If the right-hand side of equation (3.8.1.1) is independent of the derivative, i.e., $f = f(x, y)$, one can use the predictor formula

$$\widetilde{y}_{k+1} = 2y_{k-1} - y_{k-3} + \tfrac{4}{3}h^2(f_k + f_{k-1} + f_{k-2})$$

and then use *Stoermer's rule* as the corrector:

$$y_{k+1} = 2y_k - y_{k-1} + \tfrac{1}{12}h^2(\widetilde{f}_{k+1} + 10f_k + f_{k-1}).$$

3.8.5 Shooting Method (Boundary Value Problems)

The key idea of the shooting method is to reduce the solution of the original boundary value problem for a given differential equation to multiple solutions of auxiliary Cauchy problems for the same differential of equation.

► **Boundary problems with first, second, third and mixed boundary conditions.**

1°. Suppose that one deals with a boundary value problem, in the domain $x_1 \le x \le x_2$, for equation (3.8.1.1) subject to the simple boundary conditions of the first kind

$$y(x_1) = a, \quad y(x_2) = b, \tag{3.8.5.1}$$

where a and b are given numbers.

Let us look at an auxiliary Cauchy problem for equation (3.8.1.1) with the initial conditions

$$y(x_1) = a, \quad y'_x(x_1) = \lambda. \tag{3.8.5.2}$$

For any λ, the solution to this Cauchy problem satisfies the first boundary condition in (3.8.5.1) at the point $x = x_1$ (the solution can be obtained by the Runge–Kutta method or any other suitable numerical method). The original problem will be solved if we find a value $\lambda = \lambda_*$ such that the solution $y = y(x, \lambda_*)$ coincides at the point $x = x_2$ with the value required by the second boundary condition in (3.8.5.1):

$$y(x_2, \lambda_*) = b.$$

First, we set an arbitrary number $\lambda = \lambda_1$ (e.g., $\lambda_1 = 0$) and solve the Cauchy problem (3.8.1.1), (3.8.5.2) numerically. The solution results in the number

$$\Delta_1 = y(x_2, \lambda_1) - b. \tag{3.8.5.3}$$

Then, we choose a different value $\lambda = \lambda_2$ and compute

$$\Delta_2 = y(x_2, \lambda_2) - b. \qquad (3.8.5.4)$$

Suppose that λ_2 has been chosen so that Δ_1 and Δ_2 have different signs (perhaps, a few tries will be required to choose a suitable λ_2). By virtue of the continuity of the solution in λ, the desired value λ_* will lie between λ_1 and λ_2. Then, we set, for example, $\lambda_3 = \frac{1}{2}(\lambda_1 + \lambda_2)$ and solve the Cauchy problem to obtain Δ_3. Out of the two previous values λ_j ($j = 1,\ 2$), we keep the one for which Δ_j and Δ_3 have different signs. The desired λ_* will be between the λ_j and λ_3. Further, by setting $\lambda_4 = \frac{1}{2}(\lambda_j + \lambda_3)$, we find Δ_4 and so on. The process is repeated until we find λ_* with a required accuracy.

Remark 3.29. The above algorithm can be improved by using, instead of bisections, the following formulas:

$$\lambda_3 = \frac{|\Delta_2|\lambda_1 + |\Delta_1|\lambda_2}{|\Delta_2| + |\Delta_1|}, \quad \lambda_4 = \frac{|\Delta_3|\lambda_j + |\Delta_j|\lambda_3}{|\Delta_3| + |\Delta_j|}, \quad \ldots$$

2°. Table 3.3 lists the initial conditions that should be used in the auxiliary Cauchy problem to numerically solve boundary value problems for the second-order equation (3.8.1.1) with different linear boundary conditions at the left endpoint. The parameter λ in the Cauchy problem is selected so as to satisfy the boundary condition at the right endpoint.

TABLE 3.3
Initial conditions in the auxiliary Cauchy problem used to solve boundary value problems by the shooting method ($x_1 \le x \le x_2$)

No	Problem	Boundary condition at the left end	Initial conditions
1	First boundary value problem	$y(x_1) = a$	$y(x_1) = a,\ y'_x(x_1) = \lambda$
2	Second boundary value problem	$y'_x(x_1) = a$	$y(x_1) = \lambda,\ y'_x(x_1) = a$
3	Third boundary value problem	$y'_x(x_1) - ky(x_1) = a$	$y(x_1) = \lambda,\ y'_x(x_1) = a + k\lambda$

Importantly, nonlinear boundary value problems can have one solution, several solutions, or no solutions at all (see Examples 3.14 and 3.17, which illustrate all these scenarios based on exact analyses of two one-parameter problems from combustion theory). Therefore, special care is required when treating nonlinear problems; after finding a suitable $\lambda = \lambda_1$, one should look for other possible allowable values in a wider range of λ. If one fails to find a suitable λ_1, one should consider the possibility that the problem may simply have no solution.

► Problems with more complex linear or nonlinear boundary conditions.

In a similar way, one constructs the solution of the boundary value problem with nonlinear boundary conditions of the form

$$y'_x = \varphi(y) \quad \text{at} \quad x = x_1, \qquad (3.8.5.5)$$
$$\psi(y, y'_x) = 0 \quad \text{at} \quad x = x_2. \qquad (3.8.5.6)$$

The first boundary condition is a generalization of a linear nonhomogeneous boundary condition of the third kind. This condition can arise, for example, in mass transfer problems with a heterogeneous reaction, where $g(y)$ defines the rate of the chemical reaction. The second boundary condition is quite general.

Consider an auxiliary Cauchy problem for equation (3.8.1.1) with the initial conditions

$$y(x_1) = \lambda, \qquad y'_x(x_1) = \varphi(\lambda). \tag{3.8.5.7}$$

For any λ, the solution to this Cauchy problem will satisfy the first boundary condition (3.8.5.5).

We set an arbitrary value $\lambda = \lambda_1$ and solve the Cauchy problem (3.8.1.1), (3.8.5.5) numerically to obtain the number

$$\Delta_1 = \psi(y, y'_x)|_{\lambda=\lambda_1,\, x=x_2}. \tag{3.8.5.8}$$

Then we set a different value $\lambda = \lambda_2$ and compute

$$\Delta_2 = \psi(y, y'_x)|_{\lambda=\lambda_2,\, x=x_2}. \tag{3.8.5.9}$$

We assume that λ_2 is chosen so that Δ_1 and Δ_2 have different signs. The desired value $\lambda = \lambda_*$, for which the boundary condition (3.8.5.6) is satisfied exactly, will lie between λ_1 and λ_2. The subsequent procedure of numerical solution coincides with that outlined above for equation (3.8.1.1) with the simple linear boundary conditions of the first kind (3.8.5.1).

Remark 3.30. In a similar way, one can solve the boundary value problem described by equation (3.8.1.1), boundary condition (3.8.5.5), and the nonlocal linear condition

$$\int_{x_1}^{x_2} h(x)y(x)\,dx = c, \tag{3.8.5.10}$$

where $h(x)$ is a given function and c is a given number. To this end, one solves the Cauchy problem (3.8.1.1), (3.8.5.5) numerically with two different values $\lambda = \lambda_1$ and $\lambda = \lambda_2$ such that

$$\Delta_1 = \int_{x_1}^{x_2} h(x)y(x,\lambda_1)\,dx - c \quad \text{and} \quad \Delta_2 = \int_{x_1}^{x_2} h(x)y(x,\lambda_2)\,dx - c$$

have different signs. The subsequent procedure of numerical solution completely coincides with the one outlined above for equation (3.8.1.1) with the boundary conditions of the first kind (3.8.5.1).

Remark 3.31. One should bear in mind that the boundary value problem (3.8.1.1), (3.8.5.5), (3.8.5.6) can have two or more solutions, corresponding to different values λ_{*i}.

In a similar way, one can solve the boundary value problem described by equation (3.8.1.1), boundary condition (3.8.5.5), and the general nonlocal nonlinear condition

$$\int_{x_1}^{x_2} \Phi(x, y(x))\,dx = c,$$

where $\Phi(x, y)$ is a given function. In particular, this condition with the quadratic function $\Phi(x, y) = y^2$, independent of x, represents a normalization condition (which arises, for example, in quantum mechanics).

3.8.6 Numerical Methods for Problems with Equations Defined Implicitly or Parametrically

► **Numerical solution of the Cauchy problem for parametrically defined equations.**

In this paragraph, we outline the ideas of two numerical methods for solving the Cauchy problem for the second-order equation represented in parametric form using two relations (see Section 3.2.8)

$$y'_x = F(x, y, t), \qquad y''_{xx} = G(x, y, t) \tag{3.8.6.1}$$

with the initial conditions (3.8.1.2).

First method. We start directly from equations (3.8.6.1). Consider two auxiliary Cauchy problems

$$y'_x = F(x, y, t), \qquad y(x_0) = y_0 \qquad \text{(first problem)}; \tag{3.8.6.2}$$

$$y''_{xx} = G(x, y, t), \qquad y(x_0) = y_0, \ \ y'_x(x_0) = y'_0 \qquad \text{(second problem)}. \tag{3.8.6.3}$$

Let $y_F = y_F(x, t)$ and $y_G = y_G(x, t)$ be their respective solutions. Introduce the difference of the two solutions

$$\Delta(x, t) = y_G(x, t) - y_F(x, t). \tag{3.8.6.4}$$

Now we fix a value of the parameter, $t = t_k$, and find numerical solutions $y_F(x, t_k)$ and $y_G(x, t_k)$ using, for example, the Runge–Kutta method. Further, by varying x, we find an x_k at which the right-hand side of equation (3.8.6.4) vanishes: $\Delta(x_k, t_k) = 0$. To this x_k there corresponds the value of the desired function $y_k = y_F(x_k, t_k) = y_G(x_k, t_k)$. Thus, to each t_k there corresponds a point (x_k, y_k) in the (x, y) plane at which the curves $y_F = y_F(x, t_k)$ and $y_G = y_G(x, t_k)$ intersect. On taking another value of the parameter, $t = t_{k+1}$, we find a new point (x_{k+1}, y_{k+1}). The combination of discrete points (x_k, y_k) with $k = 0,\ 1,\ 2,\ \ldots$ defines an approximation to the solution $y = y(x)$ of the original problem (3.8.6.1), (3.8.1.2).

The initial value $t = t_0$ is determined from the algebraic (or transcendental) equation

$$y'_0 = F(x_0, y_0, t_0), \tag{3.8.6.5}$$

where x_0, y_0, and y'_0 are the values appearing in the initial conditions (3.8.6.2)–(3.8.6.2), obtained from (3.8.1.2)).

Second method. With the method outlined in Section 3.2.8, we reduce the parametric equation (3.8.6.1) to a standard system of first-order differential equations for $x = x(t)$ and $y = y(t)$ (see equations (3.2.8.4) and (3.2.8.5)):

$$x'_t = \frac{F_t}{G - F_x - FF_y}, \qquad y'_t = \frac{FF_t}{G - F_x - FF_y}. \tag{3.8.6.6}$$

Suppose that $G - F_x - FF_y \neq 0$. Then system (3.8.6.6) subject to the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0, \tag{3.8.6.7}$$

where t_0 is found from the algebraic (or transcendental) of equation (3.8.6.5), is solved numerically using, for example, the Runge–Kutta method (see Section 7.4.1 for relevant formulas). This solution will also solve the original parametric problem (3.8.6.1), (3.8.1.2).

Remark 3.32. In general, the algebraic (or transcendental) equation (3.8.6.5) can have several different roots, in which case the original problem (3.8.6.1), (3.8.1.2) will have the same number of different solutions.

► **First boundary value problem. Numerical solution procedure.**

Let us look at the first boundary value problem for the parametric second-order ODE (3.8.6.1) in the range $x_1 \le x \le x_2$ with the boundary conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2. \tag{3.8.6.8}$$

Below we present the main idea of a numerical procedure to solve this kind of problem.

Consider two auxiliary Cauchy problems for the equation (see the second equation in Eq. (3.8.6.1))

$$y''_{xx} = G(x, y, t) \tag{3.8.6.9}$$

subject to the initial conditions

$$y(x_1) = y_1, \quad y'_x(x_1) = F(x_1, y_1, t) \qquad \text{(problem 1);} \tag{3.8.6.10}$$
$$y(x_2) = y_2, \quad y'_x(x_2) = F(x_2, y_2, t) \qquad \text{(problem 2).} \tag{3.8.6.11}$$

By choosing a specific value of the parameter, $t = t_k$, we solve the auxiliary Cauchy problems numerically (e.g., by the Runge–Kutta method) to obtain $y^1 = y^1(x, t_k)$ and $y^2 = y^2(x, t_k)$, respectively (the superscripts indicate the problem number). To any t_k there corresponds a point (x_k, y_k) in the (x, y) plane at which the curves corresponding to the solutions $y^1 = y^1(x, t_k)$ and $y^2 = y^2(x, t_k)$ intersect. By choosing a different value, t_{k+1}, we find another point, (x_{k+1}, y_{k+1}). The discrete set of points (x_k, y_k) with $k = 0,\ 1,\ 2,\ \dots$ defines an approximation to the solution $y = y(x)$ of the original boundary value problem (3.8.6.1), (3.8.6.8).

► **Numerical integration of equations defined implicitly.**

Let us look at the Cauchy problem for the implicit equation

$$y'_x = F(x, y, y''_{xx}) \tag{3.8.6.12}$$

subject to the initial condition (3.8.1.2).

The substitution $y''_{xx} = t$ reduces equation (3.8.6.12) to the parametric equation

$$y'_x = F(x, y, t), \qquad y''_{xx} = t \tag{3.8.6.13}$$

with the initial conditions (3.8.1.2).

Problem (3.8.6.13), (3.8.1.2) is a special case of problem (3.8.6.1), (3.8.1.2) in which $G(x, y, t) = t$, and hence, it can be solved with the numerical methods described previously.

► **Differential-algebraic equations.**

Parametrically defined nonlinear differential equations of the form (3.8.6.1) are a special class of coupled (DAEs for short). Numerical methods for DAEs other than those discussed above can be found in the books by Hairer, Lubich, and Roche (1989), Schiesser (1994), Hairer and Wanner (1996), Brenan, Campbell, and Petzold (1996), Ascher and Petzold (1998), and Rabier and Rheinboldt (2002).

3.8.7 Numerical Solution Blow-Up Problems*

► Preliminary remarks. Blow-up solutions with a power-law singularity.

Below, we will be concerned with *blow-up problems*, whose solution tends to infinity as the independent variable approaches a finite value $x = x_*$, which is unknown in advance. The important question arises as to how one can determine the singular point x_* with numerical methods.

Example 3.32. Consider the model Cauchy problem for the nonlinear second-order ODE

$$y''_{xx} = 2y^3 \quad (x > 0), \qquad y(0) = 1, \quad y'_x(0) = 1. \tag{3.8.7.1}$$

Its exact solution is given by

$$y = \frac{1}{1-x} \tag{3.8.7.2}$$

and has a power-law singularity (a pole) at $x_* = 1$. For $x > x_*$, there is no solution.

If one solves problem (3.8.7.1) using, for example, explicit Runge–Kutta methods of different order of accuracy, one obtains a numerical solution which is positive, monotonically increases, and exists for arbitrarily large x_k. From the form of the solution, one cannot conclude that the exact solution has a pole (it appears that the exact solution rapidly increases and exists for any $x > 0$). Note that the standard explicit schemes do not work well either is similar situations.

Below we outline a few numerical methods for blow-up problems. We assume that the preliminary numerical (or analytical) analysis has caused a suspicion that the problem may have a blow-up solution.

► Method based on the hodograph transformation.

For monotonic blow-up solutions, having made the hodograph transformation, we can solve the Cauchy problem for $x = x(y)$ rather than $y = y(x)$. Since $y_x = 1/x'_y$ and $y''_{xx} = -x''_{yy}/(x'_y)^3$, problem (3.8.1.1)–(3.8.1.2) becomes

$$\begin{gathered} x''_{yy} = -(x'_y)^3 f(x, y, 1/x'_y) \quad (y > y_0), \\ x(y_0) = x_0, \qquad x'_y(y_0) = 1/y'_0. \end{gathered} \tag{3.8.7.3}$$

The computation can be carried out using, for example, the explicit fourth-order Runge–Kutta scheme. For sufficiently large y, we find the asymptote $x = x_*$ numerically.

Example 3.33. The hodograph transformation reduces the model problem (3.8.7.1) to

$$x''_{yy} = -2y^3 (x'_y)^3 \quad (y > 1); \qquad x(1) = 0, \quad x'_y(1) = 1.$$

The solution of this problem is given by

$$x = 1 - \frac{1}{y};$$

it does not have singularities and monotonically increases for $y > 1$ and tends to the desired limit value $x_* = \lim_{y\to\infty} x(y) = 1$.

*Prior to reading this section, the reader should refer to Section 1.14.4, which discusses blow-up problems for first-order equations.

▶ Method based on the use of the differential variable $t = y'_x$.

First, assuming the inequalities $f(x, y, y'_x) > 0$ for $y > y_0 > 0$ and $y'_x > y'_0 > 0$ to hold, we rewrite problem (3.8.1.1)–(3.8.1.2) in the parametric form

$$y'_x = t \quad y''_{xx} = f(x, y, t) \quad (t > t_0); \tag{3.8.7.4}$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = y'_0. \tag{3.8.7.5}$$

Then, relying on the results of Section 3.8.6, we change to system (3.8.6.6) with $F = t$ and $G = f(x, y, t)$ to arrive at the Cauchy problem for a system of two first-order equations

$$x'_t = \frac{1}{f(x, y, t)}, \quad y'_t = \frac{t}{f(x, y, t)} \quad (t > t_0) \tag{3.8.7.6}$$

subject to the initial conditions (3.8.7.5) . Further, we solve problem (3.8.7.6), (3.8.7.5) numerically using, for example, the Runge–Kutta methods (see Section 7.4.1 for relevant formulas). The resulting solution is also a solution to the original problem (3.8.1.1)–(3.8.1.2) in parametric form. The boundary of the existence domain, $x = x_*$, is determined numerically for sufficiently large t.

Example 3.34. In the model problem (3.8.7.1), the introduction of the auxiliary variable $t = y'_x$ followed by the substitution of $f(x, y, t) = 2y^3$ into (3.8.7.4)–(3.8.7.6) results in the Cauchy problem for a system of two equations

$$x'_t = \frac{1}{2y^3}, \quad y'_t = \frac{t}{2y^3} \quad (t > 1);$$

$$x(1) = 0, \quad y(1) = 1 \quad (t_0 = 1).$$

The exact solution to this problem is

$$x = 1 - \frac{1}{\sqrt{t}}, \quad y = \sqrt{t} \quad (t \geq 1).$$

It does not have singularities; the function $x = x(t)$ monotonically increases for $t > 1$ and tends to the desired limit value $x_* = \lim_{t\to\infty} x(t) = 1$, while $y = y(t)$ monotonically increases without bound.

▶ Method based on nonlocal transformations. Monotonic blow-up solutions.

First, equation (3.8.1.1) can be represented as a system of two equations

$$y'_x = t, \quad t'_x = f(x, y, t),$$

and then we introduce a nonlocal variable of general form by the formula

$$\xi = \int_{x_0}^{x} g(x, y, t)\, dx, \quad y = y(x), \quad t = t(x), \tag{3.8.7.7}$$

where $g = g(x, y, t)$ is a *regularizing function* which can be varied. As a result, the Cauchy problem (3.8.1.1)–(3.8.1.2) can be transformed to the following equivalent problem for an autonomous system of three equations:

$$x'_\xi = \frac{1}{g(x, y, t)}, \quad y'_\xi = \frac{t}{g(x, y, t)}, \quad t'_\xi = \frac{f(x, y, t)}{g(x, y, t)} \quad (\xi > 0);$$
$$x(0) = x_0, \quad y(0) = y_0, \quad t(0) = y_1. \tag{3.8.7.8}$$

With a suitably chosen function $g = g(x, y, t)$ (subject to not-very-restrictive conditions), the Cauchy problem (3.8.7.8) can be numerically integrated using standard numerical methods, without the fear of getting blow-up solutions.

Here are a few possible ways of how the regularizing function g in system (3.8.7.8) can be chosen.

1°. The special case $g = t$ is equivalent to the hodograph transformation with an additional translation in the dependent variable, which gives $\xi = y - y_0$.

2°. We can take $g = \bigl(c + |t|^s + |f|^s\bigr)^{1/s}$ with $c \ge 0$ and $s > 0$. The case $c = 1$ and $s = 2$ corresponds to the method of arc length transformation.

3°. By taking $g = f$ in (3.8.7.8), after the integration of the third equation, we arrive at system (3.8.7.6). It follows that the method based on the nonlocal transformation (3.8.7.7) is a generalization of the method based on the differential variable.

4°. Also, we can take $g = f/y$, $g = f/t$, or $g = t/y$ (in the last two cases, system (3.8.7.8) is simplified, since one of its equations is directly integrated).

Remark 3.33. It follows from Items 1°, 2°, and 3° that the method based on the hodograph transformation, the method of arc length transformation, and the method based on the differential variable are special cases of the method based on a nonlocal transformation of general form.

Remark 3.34. One does not have to compute integrals of the form (3.8.7.7) to apply nonlocal transformations.

Example 3.35. For the test problem (3.8.7.1), in which $f = 2y^3$, we set $g = t/y$ (see Item 4° with $g = t/y$). Substituting these functions into (3.8.7.8), we arrive at the Cauchy problem

$$\begin{gathered} x'_\xi = \frac{y}{t}, \quad y'_\xi = y, \quad t'_\xi = \frac{2y^4}{t} \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a, \quad t(0) = a^2. \end{gathered} \tag{3.8.7.9}$$

The exact solution of this problem is

$$x = \frac{1}{a}\bigl(1 - e^{-\xi}\bigr), \quad y = ae^{\xi}, \quad t = a^2 e^{2\xi}.$$

One can see that the unknown $x = x(\xi)$ exponentially tends to the asymptotic value $x = x_* = 1/a$ as $\xi \to \infty$.

Figure 3.5 displays a numerical solution of the Cauchy problem (3.8.7.9) in parametric form and compares the numerical solution with the exact solution (3.8.7.2).

Remark 3.35. The method based on the use of the special case of system (3.8.7.8) with $g = t/y$ (see Item 4° with $g = t/y$ above) is more efficient as compared to the methods based on the hodograph transformation, arc length transformation, and differential variable.

► Problems with non-monotonic blow-up solutions.

For problems with non-monotonic blow-up solutions, it is reasonable to choose regularizing functions of the form

$$g = G(|t|, |f|), \tag{3.8.7.10}$$

where $f = f(x, y, t)$ is the right-hand side of equation (3.8.1.1) and $t = y'_x$. We impose the following conditions on the function $G = G(u, v)$:

$$G > 0; \quad G_u \ge 0, \quad G_v \ge 0, \quad G \to \infty \ \text{ as } \ u + v \to \infty, \tag{3.8.7.11}$$

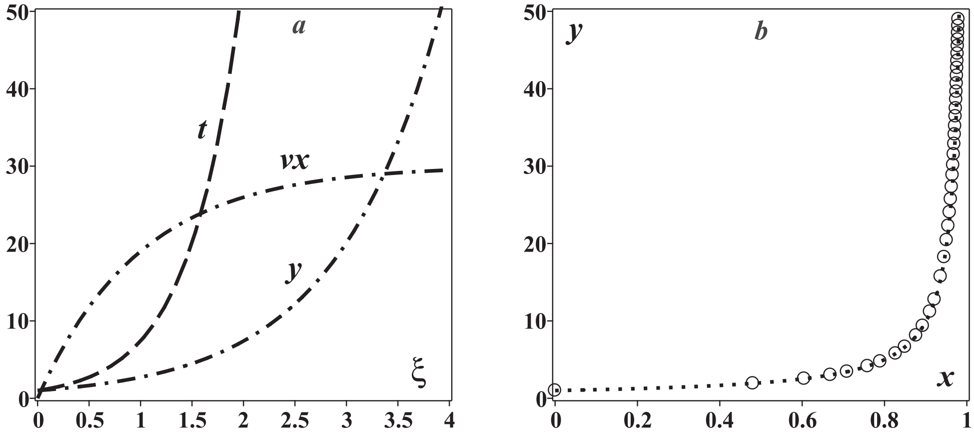


Figure 3.5: (a) numerical solution $t = t(\xi)$, $x = x(\xi)$, $y = y(\xi)$ of the Cauchy problem (3.8.7.9) with $a = 1$ ($\nu = 30$); (b) exact solution (3.8.7.2) with $a = 1$, solid dots; numerical solution of the Cauchy problem (3.8.7.9), open circles.

where $u \geq 0$, $v \geq 0$. By selecting a suitable function G, we can ensure that the Cauchy problem (3.8.7.8) has no blow-up singularity on the half-line $0 \leq \xi < \infty$; this problem can be solved by applying standard fixed-step numerical methods.

Example 3.36. Consider a three-parameter Cauchy problem for the nonlinear second-order autonomous ODE:

$$y''_{xx} - 3yy'_x - 2\lambda y'_x + y^3 + 2\lambda y^2 + (\beta^2 + \lambda^2)y = 0; \tag{3.8.7.12}$$

$$y(0) = b\beta, \qquad y'_x(0) = 2b\beta\lambda + b^2\beta^2. \tag{3.8.7.13}$$

The exact solution of the problem is

$$y = \frac{b[\lambda\sin(\beta x) + \beta\cos(\beta x)]}{e^{-\lambda x} - b\sin(\beta x)}. \tag{3.8.7.14}$$

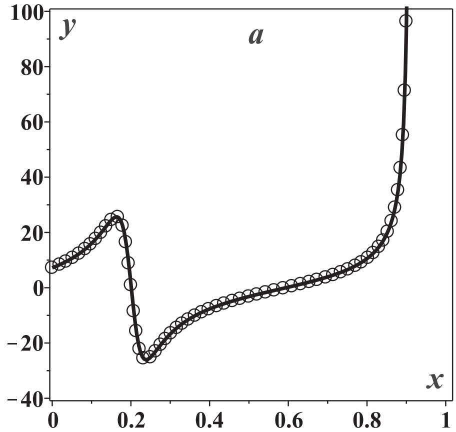
This solution can change the sign and, for certain values of the parameters, is a non-monotonic blow-up solution.

For problem (3.8.7.12)–(3.8.7.13), we choose a regularizing function in the form $g = (1+|t|)^{1/3}$. Substituting it into (3.8.7.8), we arrive at the Cauchy problem

$$x'_\xi = \frac{1}{(1+|t|+|f|)^{1/3}}, \quad y'_\xi = \frac{t}{(1+|t|+|f|)^{1/3}}, \quad t'_\xi = \frac{f}{(1+|t|+|f|)^{1/3}}; \tag{3.8.7.15}$$
$$x(0) = 0, \quad y(0) = b\beta, \quad t(0) = 2b\beta\lambda + b^2\beta^2,$$

where $f = 3yt + 2\lambda t - y^3 - 2\lambda y^2 - (\beta^2 + \lambda^2)y$.

The numerical solutions of problem (3.8.7.15) obtained using two sets of parameters, $b = 0.9$, $\beta = 8$, $\lambda = 0.3$ and $b = 0.5$, $\beta = 5$, $\lambda = 0.1$, and the fourth-order Runge–Kutta method with the fixed step size $h = 0.01$ are shown by open circles in Fig. 3.6a and Fig. 3.6b. For this step size, the maximum difference between the exact solution (3.8.7.14) and the numerical solution of the Cauchy problem for system (3.8.7.15) at $y = 50$ was found to be 0.0002500% for the first set of parameters and 0.0011033% for the second set. The solution for the first set of parameters exists in a finite region $0 \leq x < x_* = 0.9112959$, while that for the second set of parameters displays a pronounced non-monotonic sawtooth behavior with six local maxima and exists in a finite region $0 \leq x < x_* = 7.7730738$ (see Fig. 3.6b).



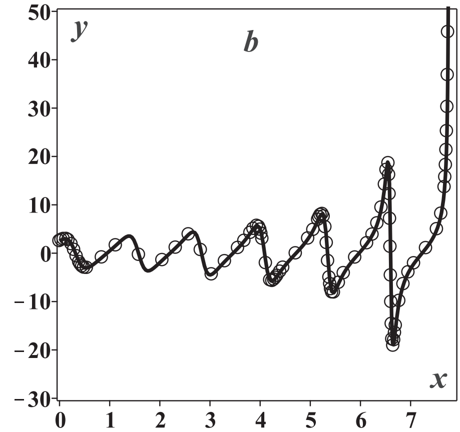


Figure 3.6: The exact solution (3.8.7.14) of the original problem (3.8.7.12)–(3.8.7.13) (solid line) and the numerical solution of the transformed problem (3.8.7.15) (open circles) for two sets of parameters: (a) $b = 0.9$, $\beta = 8$, $\lambda = 0.3$ and (b) $b = 0.5$, $\beta = 5$, $\lambda = 0.1$.

Table 3.4 compares the efficiency of various functions g used in the numerical integration of the transformed problem (3.8.7.8) in order to solve the original problem (3.8.7.12)–(3.8.7.13) with $b = 0.5$, $\beta = 5$, and $\lambda = 0.1$. The maximum allowed error was set to be 0.01% at $y = 100$. The main integration parameters (largest interval $0 \le \xi \le \xi_{\max}$, step size h, and number of grid points N) used to achieve the required accuracy are specified in the table.

TABLE 3.4

A comparison of the efficiency of various regularizing functions g in the transformed problem (3.8.7.8), used for the numerical solution of the original problem (3.8.7.12)–(3.8.7.13), with the prescribed maximum error 0.01% at $y = 100$, for $b = 0.5$, $\beta = 5$, and $\lambda = 0.1$ ($x_* = 7.7730738$)

No.	Regularizing function	$\xi_{\max}$	Step size h	N
1	$g = (1 + t^2)^{1/2}$	274.050	0.0029000000	94,500
2	$g = (1 + t^2 + f^2)^{1/2}$	11,742.300	0.1800000000	65,235
3	$g = (1 + \lvert t\rvert)^{1/2}$	35.764	0.0029593683	12,085
4	$g = \frac{1}{2}(1 + \lvert t\rvert)^{1/3} + \frac{1}{2}(1 + \lvert f\rvert)^{1/3}$	28.442	0.0090899000	3,129
5	$g = (1 + \lvert t\rvert + \lvert f\rvert)^{1/3}$	39.702	0.0185090000	2,145

⊙ *Literature for Section 3.8*: M. Abramowitz and I. A. Stegun (1964), S. K. Godunov and V. S. Ryaben'kii (1973), J. D. Lambert (1973), H. B. Keller (1976), N. S. Bakhvalov (1977), N. N. Kalitkin (1978), S. Moriguti, C. Okuno, R. Suekane, M. Iri, and K. Takeuchi (1979), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), M. Stuart and M. S. Floater (1990), W. E. Schiesser (1994), V. F. Zaitsev and A. D. Polyanin (1993), L. F. Shampine (1994), K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), J. R. Dormand (1996), E. Hairer and G. Wanner (1996), D. Zwillinger (1997), U. M. Ascher and L. R. Petzold (1998), G. A. Korn and T. M. Korn (2000), G. Acosta, G. Durán, and J. D. Rossi (2002), P. J. Rabier and W. C. Rheinboldt (2002), A. D. Polyanin and V. F. Zaitsev (2003), H. J. Lee and W. E. Schiesser (2004), A. D. Polyanin and A. V. Manzhirov (2007), S. C. Chapra and R. P. Canale (2010), M. Mizuguchi, and S. Oishi (2017), A. D. Polyanin and A. I. Zhurov (2017b), A. D. Polyanin and I. K. Shingareva (2017a,b,c,d,e).

Chapter 4

Methods for Linear ODEs of Arbitrary Order

4.1 Linear Equations with Constant Coefficients

4.1.1 Homogeneous Linear Equations. General Solution

An nth-order homogeneous linear equation with constant coefficients has the general form

$$y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y_x' + a_0 y = 0. \qquad (4.1.1.1)$$

The general solution of this equation is determined by the roots of the characteristic equation

$$P(\lambda) = 0, \qquad \text{where} \quad P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0. \qquad (4.1.1.2)$$

The following cases are possible:

1°. All roots λ_1, λ_2, ..., λ_n of the characteristic equation (4.1.1.2) are real and distinct. Then the general solution of the homogeneous linear differential equation (4.1.1.1) has the form

$$y = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x) + \cdots + C_n \exp(\lambda_n x).$$

2°. There are m equal real roots $\lambda_1 = \lambda_2 = \cdots = \lambda_m$ ($m \le n$), and the other roots are real and distinct. In this case, the general solution is given by

$$y = \exp(\lambda_1 x)(C_1 + C_2 x + \cdots + C_m x^{m-1}) \\ + C_{m+1} \exp(\lambda_{m+1} x) + C_{m+2} \exp(\lambda_{m+2} x) + \cdots + C_n \exp(\lambda_n x).$$

3°. There are m equal complex conjugate roots $\lambda = \alpha \pm i\beta$ ($2m \le n$), and the other roots are real and distinct. In this case, the general solution is

$$y = \exp(\alpha x)\cos(\beta x)(A_1 + A_2 x + \cdots + A_m x^{m-1}) \\ + \exp(\alpha x)\sin(\beta x)(B_1 + B_2 x + \cdots + B_m x^{m-1}) \\ + C_{2m+1} \exp(\lambda_{2m+1} x) + C_{2m+2} \exp(\lambda_{2m+2} x) + \cdots + C_n \exp(\lambda_n x),$$

where $A_1, \ldots, A_m$, $B_1, \ldots, B_m$, $C_{2m+1}, \ldots, C_n$ are arbitrary constants.

4°. In the general case, where there are r different roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ of multiplicities $m_1, m_2, \ldots, m_r$, respectively, the left-hand side of the characteristic equation (4.1.1.2) can be represented as the product

$$P(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_r)^{m_r},$$

where $m_1 + m_2 + \cdots + m_r = n$. The general solution of the original equation is given by the formula

$$y = \sum_{k=1}^{r} \exp(\lambda_k x)(C_{k,0} + C_{k,1}x + \cdots + C_{k,m_k-1}x^{m_k-1}),$$

where $C_{k,l}$ are arbitrary constants.

If the characteristic equation (4.1.1.2) has complex conjugate roots, then in the above solution, one should extract the real part on the basis of the relation $\exp(\alpha \pm i\beta) = e^{\alpha}(\cos\beta \pm i\sin\beta)$.

Example 4.1. Find the general solution of the linear third-order equation

$$y''' + ay'' - y' - ay = 0.$$

Its characteristic equation is $\lambda^3 + a\lambda^2 - \lambda - a = 0$, or, in factorized form,

$$(\lambda + a)(\lambda - 1)(\lambda + 1) = 0.$$

Depending on the value of the parameter a, three cases are possible.

1. Case $a \neq \pm 1$. There are three different roots, $\lambda_1 = -a$, $\lambda_2 = -1$, and $\lambda_3 = 1$. The general solution of the differential equation is expressed as $y = C_1e^{-ax} + C_2e^{-x} + C_3e^{x}$.

2. Case $a = 1$. There is a double root, $\lambda_1 = \lambda_2 = -1$, and a simple root, $\lambda_3 = 1$. The general solution of the differential equation has the form $y = (C_1 + C_2x)e^{-x} + C_3e^{x}$.

3. Case $a = -1$. There is a double root, $\lambda_1 = \lambda_2 = 1$, and a simple root, $\lambda_3 = -1$. The general solution of the differential equation is expressed as $y = (C_1 + C_2x)e^{x} + C_3e^{-x}$.

Example 4.2. Consider the linear fourth-order equation

$$y''''_{xxxx} - y = 0.$$

Its characteristic equation, $\lambda^4 - 1 = 0$, has four distinct roots, two real and two pure imaginary,

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = i, \quad \lambda_4 = -i.$$

Therefore the general solution of the equation in question has the form (see Item 3°)

$$y = C_1e^{x} + C_2e^{-x} + C_3\sin x + C_4\cos x.$$

4.1.2 Nonhomogeneous Linear Equations. General and Particular Solutions

1°. An nth-order nonhomogeneous linear equation with constant coefficients has the general form

$$y_x^{(n)} + a_{n-1}y_x^{(n-1)} + \cdots + a_1y'_x + a_0y = f(x). \qquad (4.1.2.1)$$

The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation with $f(x) \equiv 0$ (see Section 4.1.1) and any particular solution of the nonhomogeneous equation (4.1.2.1).

If all the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation (4.1.1.2) are different, equation (4.1.2.1) has the general solution:

$$y = \sum_{\nu=1}^{n} C_\nu e^{\lambda_\nu x} + \sum_{\nu=1}^{n} \frac{e^{\lambda_\nu x}}{P'_\lambda(\lambda_\nu)} \int f(x) e^{-\lambda_\nu x}\, dx \qquad (4.1.2.2)$$

(for complex roots, the real part should be taken).

In the general case, if the characteristic equation (4.1.1.2) has multiple roots, the solution to equation (4.1.2.1) can be constructed using formula (4.2.2.2).

2°. Table 4.1 lists the forms of particular solutions corresponding to some special forms of functions on the right-hand side of the linear nonhomogeneous equation.

TABLE 4.1

Forms of particular solutions of the constant-coefficient nonhomogeneous linear equation $y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y'_x + a_0 y = f(x)$ that correspond to some special forms of the function $f(x)$

Form of the function $f(x)$	Roots of the characteristic equation $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$	Form of a particular solution $y = \widetilde{y}(x)$
$P_m(x)$	Zero is not a root of the characteristic equation (i.e., $a_0 \neq 0$)	$\widetilde{P}_m(x)$
	Zero is a root of the characteristic equation (multiplicity r)	$x^r \widetilde{P}_m(x)$
$P_m(x)e^{\alpha x}$ (α is a real constant)	α is not a root of the characteristic equation	$\widetilde{P}_m(x)e^{\alpha x}$
	α is a root of the characteristic equation (multiplicity r)	$x^r \widetilde{P}_m(x)e^{\alpha x}$
$P_m(x)\cos\beta x + Q_n(x)\sin\beta x$	$i\beta$ is not a root of the characteristic equation	$\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x$
	$i\beta$ is a root of the characteristic equation (multiplicity r)	$x^r[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]$
$[P_m(x)\cos\beta x + Q_n(x)\sin\beta x]e^{\alpha x}$	$\alpha + i\beta$ is not a root of the characteristic equation	$[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]e^{\alpha x}$
	$\alpha + i\beta$ is a root of the characteristic equation (multiplicity r)	$x^r[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]e^{\alpha x}$

Notation: P_m and Q_n are polynomials of degrees m and n with given coefficients; $\widetilde{P}_m$, $\widetilde{P}_\nu$, and $\widetilde{Q}_\nu$ are polynomials of degrees m and ν whose coefficients are determined by substituting the particular solution into the basic equation; $\nu = \max(m, n)$; and α and β are real numbers, $i^2 = -1$.

3°. Consider the Cauchy problem for equation (4.1.2.1) subject to the homogeneous initial conditions

$$y(0) = y'_x(0) = \cdots = y_x^{(n-1)}(0) = 0. \qquad (4.1.2.3)$$

Let $y(x)$ be the solution of problem (4.1.2.1), (4.1.2.3) for arbitrary $f(x)$ and let $u(x)$ be the solution of the auxiliary, simpler problem (4.1.2.1), (4.1.2.3) with $f(x) \equiv 1$, so that $u(x) = y(x)|_{f(x)\equiv 1}$. Then the formula

$$y(x) = \int_0^x f(t)u'_x(x-t)\,dt$$

holds. It is called the *Duhamel integral*.

⊙ *Literature for Section 4.1*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), N. M. Matveev (1967), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

4.2 Linear Equations with Variable Coefficients

4.2.1 Homogeneous Linear Equations. General Solution. Order Reduction. Liouville Formula

► **Structure of the general solution.**

The general solution of the nth-order homogeneous linear differential equation

$$f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = 0 \qquad (4.2.1.1)$$

has the form

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x). \qquad (4.2.1.2)$$

Here $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ is a fundamental system of solutions (the y_k are linearly independent particular solutions, $y_k \not\equiv 0$); $C_1, C_2, \ldots, C_n$ are arbitrary constants.

► **Utilization of particular solutions for reducing the order of the equation.**

1°. Let $y_1 = y_1(x)$ be a nontrivial particular solution of equation (4.2.1.1). The substitution

$$y = y_1(x)\int z(x)\,dx$$

results in a linear equation of order $n-1$ for the function $z(x)$.

2°. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent solutions of equation (4.2.1.1). The substitution

$$y = y_1\int y_2w\,dx - y_2\int y_1w\,dx$$

results in a linear equation of order $n-2$ for $w(x)$.

3°. Suppose that m linearly independent solutions $y_1(x)$, $y_2(x)$, ..., $y_m(x)$ of equation (4.2.1.1) are known. Then one can reduce the order of the equation to $n-m$ by successive application of the following procedure. The substitution $y = y_m(x)\int z(x)\,dx$ leads to an equation of order $n-1$ for the function $z(x)$ with known linearly independent solutions:

$$z_1 = \left(\frac{y_1}{y_m}\right)'_x, \quad z_2 = \left(\frac{y_2}{y_m}\right)'_x, \quad \ldots, \quad z_{m-1} = \left(\frac{y_{m-1}}{y_m}\right)'_x.$$

The substitution $z = z_{m-1}(x)\int w(x)\,dx$ yields an equation of order $n-2$. Repeating this procedure m times, we arrive at a homogeneous linear equation of order $n-m$.

► **Wronskian determinant and Liouville formula.**

The *Wronskian determinant* (or simply, *Wronskian*) is the function defined as

$$W(x) = \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}, \tag{4.2.1.3}$$

where $y_1(x), \ldots, y_n(x)$ is a fundamental system of solutions of the homogeneous equation (4.2.1.1); $y_k^{(m)}(x) = \dfrac{d^m y_k}{dx^m}$, $m = 1, \ldots, n-1$; $k = 1, \ldots, n$.

The following *Liouville formula* holds:

$$W(x) = W(x_0)\exp\left[-\int_{x_0}^{x} \frac{f_{n-1}(t)}{f_n(t)}\,dt\right].$$

4.2.2 Nonhomogeneous Linear Equations. General Solution. Superposition Principle

► **Construction of the general solution.**

1°. The general nonhomogeneous nth-order linear differential equation has the form

$$f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y_x' + f_0(x)y = g(x). \tag{4.2.2.1}$$

The general solution of the nonhomogeneous equation (4.2.2.1) can be represented as the sum of its particular solution and the general solution of the corresponding homogeneous equation (4.2.1.1).

2°. Let $y_1(x), \ldots, y_n(x)$ be a fundamental system of solutions of the homogeneous equation (4.2.1.1), and let $W(x)$ be the Wronskian determinant (4.2.1.3). Then the general solution of the nonhomogeneous linear equation (4.2.2.1) can be represented as

$$y = \sum_{\nu=1}^{n} C_\nu y_\nu(x) + \sum_{\nu=1}^{n} y_\nu(x) \int \frac{W_\nu(x)\,dx}{f_n(x)W(x)}, \tag{4.2.2.2}$$

where $W_\nu(x)$ is the determinant of the matrix (4.2.1.3) in which the νth column is replaced by the column vector with the elements $0, 0, \ldots, 0, g$.

► **Superposition principle.**

The solution of a nonhomogeneous linear equation

$$\mathbf{L}[y] = \sum_{k=1}^{m} g_k(x), \qquad \mathbf{L}[y] \equiv f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y_x' + f_0(x)y$$

is determined by adding together the solutions,

$$y = \sum_{k=1}^{m} y_k,$$

of m (simpler) equations,

$$\mathbf{L}[y_k] = g_k(x), \qquad k = 1,\, 2,\, \ldots,\, m,$$

corresponding to respective nonhomogeneous terms in the original equation.

► **Euler equation.**

1°. The nonhomogeneous Euler equation has the form

$$x^n y_x^{(n)} + a_{n-1} x^{n-1} y_x^{(n-1)} + \cdots + a_1 x y'_x + a_0 y = f(x).$$

The substitution $x = be^t$ ($b \neq 0$) leads to a constant-coefficient linear equation of the form (4.1.2.1).

2°. Particular solutions of the homogeneous Euler equation [with $f(x) \equiv 0$] are sought in the form $y = x^k$. If all k are real and distinct, its general solution is expressed as

$$y(x) = C_1 |x|^{k_1} + C_2 |x|^{k_2} + \cdots + C_n |x|^{k_n}.$$

Remark 4.1. To a pair of complex conjugate values $k = \alpha \pm i\beta$ there corresponds a pair of particular solutions: $y = |x|^\alpha \sin(\beta |x|)$ and $y = |x|^\alpha \cos(\beta |x|)$.

4.2.3 Nonhomogeneous Linear Equations. Cauchy Problem. Reduction to Integral Equations

► **Cauchy problem. Cauchy formula.**

Let $y(x,\sigma)$ be the solution to the Cauchy problem for the homogeneous equation (4.2.1.1) with nonhomogeneous initial conditions at $x = \sigma$:

$$y(\sigma) = y'_x(\sigma) = \cdots = y_x^{(n-2)}(\sigma) = 0, \quad y_x^{(n-1)}(\sigma) = 1,$$

where σ is an arbitrary parameter. Then a particular solution of the nonhomogeneous linear equation (4.2.2.1) with homogeneous boundary conditions

$$y(x_0) = y'_x(x_0) = \cdots = y_x^{(n-1)}(x_0) = 0$$

is given by the *Cauchy formula*

$$\bar{y}(x) = \int_{x_0}^{x} y(x,\sigma) \frac{g(\sigma)}{f_n(\sigma)}\, d\sigma.$$

► **Reduction of the Cauchy problem for ODEs to integral equations.**

1°. Integral equations play an important role in the theory of ordinary differential equations. The reduction of Cauchy and boundary value problems to integral equations allows for the application of iteration and finite-difference methods of solving integral equations. These methods are, as a rule, substantially simpler than those used for solving differential equations. Moreover, many delicate proofs and qualitative results of the theory of differential equations have been obtained by the investigation of the corresponding integral equations.

2°. Consider the Cauchy problem for nth order ODE (4.2.2.1) with the homogeneous initial conditions at the point $x = a$:

$$y(a) = y'_x(a) = \cdots = y_x^{(n-1)}(a) = 0. \qquad (4.2.3.1)$$

Introducing a new unknown function by

$$y(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} u(t)\, dt \qquad (4.2.3.2)$$

and differentiating (4.2.3.2) n times, we get

$$\begin{aligned} &y_x^{(k)}(x) = \frac{1}{(n-k-1)!} \int_a^x (x-t)^{n-k-1} u(t)\, dt, \qquad k = 1, \ldots, n-1; \\ &y_x^{(n)}(x) = u(x). \end{aligned} \qquad (4.2.3.3)$$

Obviously, the function (4.2.3.2) satisfies the initial conditions (4.2.3.1). By substituting (4.2.3.3) into the left-hand side of equation (4.2.2.1), we obtain

$$f_n(x)u(x) + \int_a^x K(x,t)u(t)\, dt = g(x), \qquad (4.2.3.4)$$

where

$$K(x,t) = f_{n-1}(x) + f_{n-2}(x)\frac{x-t}{1!} + \cdots + f_0(x)\frac{(x-t)^{n-1}}{(n-1)!}. \qquad (4.2.3.5)$$

Thus, the Cauchy problem (4.2.2.1)–(4.2.3.1) has been reduced to the integral equation (4.2.3.4)–(4.2.3.5), which is a *Volterra equation of the second kind*. Finding the function $u(x)$ from (4.2.3.4) and using formula (4.2.3.2) we obtain the desired solution $y(x)$.

The solution of the integral equation (4.2.3.4) can be obtained using, for example, the method of successive approximations with the recurrence relation

$$u_{m+1}(x) + \frac{1}{f_n(x)} \int_a^x K(x,t)u_m(t)\, dt = \frac{g(x)}{f_n(x)}, \qquad (4.2.3.6)$$

where $m = 0,\ 1,\ 2,\ \ldots$ The function $u_0(x) = 0$ can be taken as the zeroth approximation; then $u_1(x) = g(x)/f_n(x)$.

For more efficient numerical methods for integral equations of the form (4.2.3.4), see the book by Polyanin & Manzhirov (2008).

Remark 4.2. The Cauchy problem for equation (4.2.2.1) with nonhomogeneous boundary conditions

$$y(a) = b_0, \quad y'_x(a) = b_1, \quad \dots, \quad y_x^{(n-1)}(a) = b_{n-1}$$

can be reduced to a Cauchy problem with homogeneous boundary conditions for another function $w(x)$ with the help of the substitution

$$y(x) = w(x) + \sum_{k=1}^{n-1} b_k \frac{(x-a)^k}{k!}.$$

⊙ *Literature for Section 4.2*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), N. M. Matveev (1967), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007, 2008).

4.3 Laplace Transform and the Laplace Integral. Applications to Linear ODEs

4.3.1 Laplace Transform and the Inverse Laplace Transform

▶ Laplace transform.

The *Laplace transform* of an arbitrary (complex-valued) function $f(x)$ of a real variable x ($x \geq 0$) is defined by

$$\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\, dx, \qquad (4.3.1.1)$$

where $p = s + i\sigma$ is a complex variable.

The Laplace transform exists for any continuous or piecewise-continuous function satisfying the condition $|f(x)| < Me^{\sigma_0 x}$ with some $M > 0$ and $\sigma_0 \geq 0$. In the following, σ_0 often means the greatest lower bound of the possible values of σ_0 in this estimate; this value is called the *growth exponent* of the function $f(x)$.

For any $f(x)$, the transform $\widetilde{f}(p)$ is defined in the half-plane $\operatorname{Re} p > \sigma_0$ and is analytic there.

For brevity, we shall write formula (4.3.1.1) as follows:

$$\widetilde{f}(p) = \mathfrak{L}\{f(x)\}.$$

▶ Inverse Laplace transform.

Given the transform $\widetilde{f}(p)$, the function $f(x)$ can be found by means of the inverse Laplace transform

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(p) e^{px}\, dp, \qquad i^2 = -1, \qquad (4.3.1.2)$$

where the integration path is parallel to the imaginary axis and lies to the right of all singularities of $\widetilde{f}(p)$, which corresponds to $c > \sigma_0$.

The integral in inversion formula (4.3.1.2) is understood in the sense of the Cauchy principal value:

$$\int_{c-i\infty}^{c+i\infty} \widetilde{f}(p) e^{px}\, dp = \lim_{\omega\to\infty} \int_{c-i\omega}^{c+i\omega} \widetilde{f}(p) e^{px}\, dp.$$

In the domain $x < 0$, formula (4.3.1.2) gives $f(x) \equiv 0$.

Formula (4.3.1.2) holds for continuous functions. If $f(x)$ has a (finite) jump discontinuity at a point $x = x_0 > 0$, then the left-hand side of (4.3.1.2) is equal to $\frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$ at this point (for $x_0 = 0$, the first term in the square brackets must be omitted).

For brevity, we write the Laplace inversion formula (4.3.1.2) as follows:

$$f(x) = \mathfrak{L}^{-1}\{\widetilde{f}(p)\}.$$

There are tables of direct and inverse Laplace transforms (see Sections S3.1 and S3.2, which are handy in solving linear differential and integral equations.

4.3.2 Main Properties of the Laplace Transform. Inversion Formulas for Some Functions

► Main properties of the Laplace transform.

1°. The main properties of the correspondence between functions and their Laplace transforms are gathered in Table 4.2.

2°. The Laplace transforms of some functions are listed in Table 4.3; for more detailed tables see Section S3.1 and the list of references at the end of this section.

TABLE 4.2
Main properties of the Laplace transform

No.	Function	Laplace transform	Operation
1	$af_1(x) + bf_2(x)$	$a\widetilde{f}_1(p) + b\widetilde{f}_2(p)$	Linearity
2	$f(x/a),\ a > 0$	$a\widetilde{f}(ap)$	Scaling
3	$f(x - a)$, $f(\xi) \equiv 0$ for $\xi < 0$	$e^{-ap}\widetilde{f}(p)$	Shift of the argument
4	$x^n f(x);\ n = 1, 2, \dots$	$(-1)^n \widetilde{f}^{(n)}_p(p)$	Differentiation of the transform
5	$\dfrac{1}{x} f(x)$	$\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	Integration of the transform
6	$e^{ax} f(x)$	$\widetilde{f}(p - a)$	Shift in the complex plane
7	$f'_x(x)$	$p\widetilde{f}(p) - f(+0)$	Differentiation
8	$f^{(n)}_x(x)$	$p^n\widetilde{f}(p) - \sum\limits_{k=1}^{n} p^{n-k} f^{(k-1)}_x(+0)$	Differentiation
9	$x^m f^{(n)}_x(x),\ m = 1, 2, \dots$	$(-1)^m \dfrac{d^m}{dp^m}\Big[p^n\widetilde{f}(p) - \sum\limits_{k=1}^{n} p^{n-k} f^{(k-1)}_x(+0)\Big]$	Differentiation
10	$\dfrac{d^n}{dx^n}\big[x^m f(x)\big],\ m \ge n$	$(-1)^m p^n \dfrac{d^m}{dp^m}\widetilde{f}(p)$	Differentiation
11	$\displaystyle\int_0^x f(t)\,dt$	$\dfrac{\widetilde{f}(p)}{p}$	Integration
12	$\displaystyle\int_0^x f_1(t) f_2(x - t)\,dt$	$\widetilde{f}_1(p)\widetilde{f}_2(p)$	Convolution

TABLE 4.3
The Laplace transforms of some functions

No.	Function, $f(x)$	Laplace transform, $\widetilde{f}(p)$	Remarks
1	1	$1/p$	
2	x^n	$\dfrac{n!}{p^{n+1}}$	$n=1,\ 2,\ \ldots$
3	x^a	$\Gamma(a+1)p^{-a-1}$	$a>-1$
4	e^{-ax}	$(p+a)^{-1}$	
5	$x^a e^{-bx}$	$\Gamma(a+1)(p+b)^{-a-1}$	$a>-1$
6	$\sinh(ax)$	$\dfrac{a}{p^2-a^2}$	
7	$\cosh(ax)$	$\dfrac{p}{p^2-a^2}$	
8	$\ln x$	$-\dfrac{1}{p}(\ln p+\mathcal{C})$	$\mathcal{C}=0.5772\ldots$ is the Euler constant
9	$\sin(ax)$	$\dfrac{a}{p^2+a^2}$	
10	$\cos(ax)$	$\dfrac{p}{p^2+a^2}$	
11	$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{x}}\right)$	$\dfrac{1}{p}\exp\bigl(-a\sqrt{p}\bigr)$	$a\ge 0$
12	$J_0(ax)$	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(x)$ is the Bessel function

► **Inverse transforms of rational functions.**

Consider the important case in which the transform is a rational function of the form

$$\widetilde{f}(p)=\frac{R(p)}{Q(p)}, \tag{4.3.2.1}$$

where $Q(p)$ and $R(p)$ are polynomials in the variable p and the degree of $Q(p)$ exceeds that of $R(p)$.

Assume that the zeros of the denominator are simple, i.e.,

$$Q(p)\equiv \text{const}\,(p-\lambda_1)(p-\lambda_2)\ldots(p-\lambda_n).$$

Then the inverse transform can be determined by the formula

$$f(x)=\sum_{k=1}^{n}\frac{R(\lambda_k)}{Q'(\lambda_k)}\exp(\lambda_k x), \tag{4.3.2.2}$$

where the primes denote the derivatives.

If $Q(p)$ has multiple zeros, i.e.,

$$Q(p)\equiv \text{const}\,(p-\lambda_1)^{s_1}(p-\lambda_2)^{s_2}\ldots(p-\lambda_m)^{s_m},$$

then

$$f(x)=\sum_{k=1}^{m}\frac{1}{(s_k-1)!}\lim_{p\to s_k}\frac{d^{s_k-1}}{dp^{s_k-1}}\bigl[(p-\lambda_k)^{s_k}\widetilde{f}(p)e^{px}\bigr]. \tag{4.3.2.3}$$

Example 4.3. The transform

$$\widetilde{f}(p) = \frac{b}{p^2 - a^2} \qquad (a, b \text{ real numbers})$$

can be represented as the fraction (4.3.2.1) with $R(p) = b$ and $Q(p) = (p - a)(p + a)$. The denominator $Q(p)$ has two simple roots, $\lambda_1 = a$ and $\lambda_2 = -a$. Using formula (4.3.2.2) with $n = 2$ and $Q'(p) = 2p$, we obtain the inverse transform in the form

$$f(x) = \frac{b}{2a}e^{ax} - \frac{b}{2a}e^{-ax} = \frac{b}{a}\sinh(ax).$$

Example 4.4. The transform

$$\widetilde{f}(p) = \frac{b}{p^2 + a^2} \qquad (a, b \text{ real numbers})$$

can be written as the fraction (4.3.2.1) with $R(p) = b$ and $Q(p) = (p - ia)(p + ia)$, $i^2 = -1$. The denominator $Q(p)$ has two simple pure imaginary roots, $\lambda_1 = ia$ and $\lambda_2 = -ia$. Using formula (4.3.2.2) with $n = 2$, we find the inverse transform:

$$f(x) = \frac{b}{2ia}e^{iax} - \frac{b}{2ia}e^{-iax} = -\frac{bi}{2a}\bigl[\cos(ax) + i\sin(ax)\bigr] + \frac{bi}{2a}\bigl[\cos(ax) - i\sin(ax)\bigr] = \frac{b}{a}\sin(ax).$$

Example 4.5. The transform

$$\widetilde{f}(p) = ap^{-n},$$

where n is a positive integer, can be written as the fraction (.2.2.1) with $R(p) = a$ and $Q(p) = p^n$. The denominator $Q(p)$ has one root of multiplicity n, $\lambda_1 = 0$. By formula (.2.2.3) with $m = 1$ and $s_1 = n$, we find the inverse transform:

$$f(x) = \frac{a}{(n-1)!}x^{n-1}.$$

◆ *Detailed tables of inverse Laplace transforms can be found in Section S3.2.*

4.3.3 Limit Theorems. Representation of Inverse Transforms as Convergent Series and Asymptotic Expansions

▶ **Limit theorems.**

THEOREM 1. *Let $0 \le x < \infty$ and $\widetilde{f}(p) = \mathfrak{L}\{f(x)\}$ be the Laplace transform of $f(x)$. If a limit of $f(x)$ as $x \to 0$ exists, then*

$$\lim_{x\to 0} f(x) = \lim_{p\to\infty}\bigl[p\widetilde{f}(p)\bigr].$$

THEOREM 2. *If a limit of $f(x)$ as $x \to \infty$ exists, then*

$$\lim_{x\to\infty} f(x) = \lim_{p\to 0}\bigl[p\widetilde{f}(p)\bigr].$$

▶ **Representation of inverse transforms as convergent series.**

THEOREM 1. *Suppose the transform $\widetilde{f}(p)$ can be expanded into series in negative powers of p,*

$$\widetilde{f}(p) = \sum_{n=1}^{\infty}\frac{a_n}{p^n},$$

convergent for $|p| > R$, where R is an arbitrary positive number; note that the transform tends to zero as $|p| \to \infty$. Then the inverse transform can be obtained by the formula

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1},$$

where the series on the right-hand side is convergent for all x.

THEOREM 2. *Suppose the transform $\widetilde{f}(p)$, $|p| > R$, is represented by an absolutely convergent series,*

$$\widetilde{f}(p) = \sum_{n=0}^{\infty} \frac{a_n}{p^{\lambda_n}}, \tag{4.3.3.1}$$

where $\{\lambda_n\}$ is any positive increasing sequence, $0 < \lambda_0 < \lambda_1 < \cdots \to \infty$. Then it is possible to proceed termwise from series (4.3.3.1) to the following inverse transform series:

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\lambda_n)} x^{\lambda_n - 1}, \tag{4.3.3.2}$$

where $\Gamma(\lambda)$ is the Gamma function. Series (4.3.3.2) is convergent for all real and complex values of x other than zero (if $\lambda_0 \geq 1$, the series is convergent for all x).

► Representation of inverse transforms as asymptotic expansions as $x \to \infty$.

1°. Let $p = p_0$ be a singular point of the Laplace transform $\widetilde{f}(p)$ with the greatest real part (it is assumed there is only one such point). If $\widetilde{f}(p)$ can be expanded near $p = p_0$ into an absolutely convergent series,

$$\widetilde{f}(p) = \sum_{n=0}^{\infty} c_n (p - p_0)^{\lambda_n} \qquad (\lambda_0 < \lambda_1 < \cdots \to \infty) \tag{4.3.3.3}$$

with arbitrary λ_n, then the inverse transform $f(x)$ can be expressed in the form of the asymptotic expansion

$$f(x) \sim e^{p_0 x} \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\lambda_n)} x^{-\lambda_n - 1} \quad \text{as} \quad x \to \infty. \tag{4.3.3.4}$$

The terms corresponding to nonnegative integer λ_n must be omitted from the summation, since $\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \cdots = \infty$.

2°. If the transform $\widetilde{f}(p)$ has several singular points, $p_1, \ldots, p_m$, with the same greatest real part, $\operatorname{Re} p_1 = \cdots = \operatorname{Re} p_m$, then expansions of the form (4.3.3.3) should be obtained for each of these points and the resulting expressions must be added together.

► Post–Widder formula.

In applications, one can find $f(x)$ if the Laplace transform $\widetilde{f}(t)$ on the real semiaxis is known for $t = p \geq 0$. To this end, one uses the Post–Widder formula

$$f(x) = \lim_{n\to\infty} \left[\frac{(-1)^n}{n!} \left(\frac{n}{x}\right)^{n+1} \widetilde{f}_t^{(n)}\left(\frac{n}{x}\right) \right]. \tag{4.3.3.5}$$

Approximate inversion formulas are obtained by taking sufficiently large positive integer n in (4.3.3.5) instead of passing to the limit.

4.3.4 Solution of the Cauchy Problem for Constant-Coefficient Linear ODEs. Applications to Integro-Differential Equations

► **Cauchy problem for constant-coefficient linear ODEs.**

Consider the Cauchy problem for equation (4.1.2.1) with arbitrary initial conditions

$$y(0) = y_0, \quad y'_x(0) = y_1, \quad \dots, \quad y_x^{(n-1)}(0) = y_{n-1}, \tag{4.3.4.1}$$

where $y_0, y_1, \dots, y_{n-1}$ are given constants.

Problem (4.1.2.1), (4.3.4.1) can be solved using the Laplace transform based on the formulas (for details, see Section 4.3.1)

$$\widetilde{y}(p) = \mathfrak{L}\{y(x)\}, \quad \widetilde{f}(p) = \mathfrak{L}\{f(x)\}, \quad \text{where} \quad \mathfrak{L}\{f(x)\} \equiv \int_0^\infty e^{-px} f(x)\,dx.$$

To this end, let us multiply equation (4.1.2.1) by e^{-px} and then integrate with respect to x from zero to infinity. Taking into account the differentiation rule

$$\mathfrak{L}\{y_x^{(n)}(x)\} = p^n \widetilde{y}(p) - \sum_{k=1}^{n} p^{n-k} y_x^{(k-1)}(+0)$$

and the initial conditions (4.3.4.1), we arrive at a linear algebraic equation for the transform $\widetilde{y}(p)$:

$$P(p)\widetilde{y}(p) - Q(p) = \widetilde{f}(p), \tag{4.3.4.2}$$

where

$$P(p) = p^n + a_{n-1}p^{n-1} + \dots + a_1 p + a_0, \qquad Q(p) = b_{n-1}p^{n-1} + \dots + b_1 p + b_0,$$
$$b_k = y_{n-k-1} + a_{n-1}y_{n-k-2} + \dots + a_{k+2}y_1 + a_{k+1}y_0, \qquad k = 0,\, 1,\, \dots,\, n-1.$$

The polynomial $P(p)$ coincides with the characteristic polynomial (4.1.1.2) at $\lambda = p$.

The solution of equation (4.3.4.2) is given by the formula

$$\widetilde{y}(p) = \frac{\widetilde{f}(p) + Q(p)}{\widetilde{P}(p)}. \tag{4.3.4.3}$$

On applying the Laplace inversion formula (4.3.1.2) to (4.3.4.3), we obtain a solution to problem (4.1.2.1), (4.3.4.1) in the form

$$y(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\widetilde{f}(p) + Q(p)}{\widetilde{P}(p)} e^{px}\,dp. \tag{4.3.4.4}$$

Since the transform $\widetilde{y}(p)$ (4.3.4.3) is a rational function, the inverse Laplace transform (4.3.4.4) can be obtained using the formulas from Section 4.3.2 or the tables of Section S3.2.

Remark 4.3. In practice, the solution method for the Cauchy problem based on the Laplace transform leads to the solution faster than the direct application of general formulas like (4.1.2.2), where one has to determine the coefficients $C_1, \dots, C_n$.

Example 4.6. Consider the following Cauchy problem for a homogeneous fourth-order equation:

$$y''''_{xxxx} + a^4 y = 0; \qquad y(0) = y'_x(0) = y'''_{xxx}(0) = 0, \quad y''_{xx}(0) = b.$$

Using the Laplace transform reduces this problem to a linear algebraic equation for the $\widetilde{y}(p)$: $(p^4 + a^4)\widetilde{y}(p) - bp = 0$. It follows that

$$\widetilde{y}(p) = \frac{bp}{p^4 + a^4}.$$

In order to invert this expression, let us use the table of inverse Laplace transforms (see Section S3.2.2, row 52) and take into account that a constant multiplier can be taken outside the transform operator to obtain the solution to the original Cauchy problem in the form

$$y(x) = \frac{b}{a^2} \sin\Bigl(\frac{ax}{\sqrt{2}}\Bigr) \sinh\Bigl(\frac{ax}{\sqrt{2}}\Bigr).$$

► Cauchy problem for integro-differential equations.

The Laplace transform can also be effective in solving some linear integro-differential equations. This is illustrated below with a specific example:

Example 4.7. Consider the Cauchy problem for the linear integro-differential equation

$$\frac{dy}{dx} + \int_0^x K(x-t)y(t)\,dt = f(x) \qquad (0 \le x < \infty) \tag{4.3.4.5}$$

with the initial condition

$$y = a \quad \text{at} \quad x = 0. \tag{4.3.4.6}$$

Multiply equation (4.3.4.5) by e^{-px} and then integrate with respect to x from zero to infinity. Using properties 7 and 12 of the Laplace transform (Table 4.2) and taking into account the initial condition (4.3.4.6), we obtain a linear algebraic equation for the transform $\widetilde{y}(p)$:

$$p\widetilde{y}(p) - a + \widetilde{K}(p)\widetilde{y}(p) = \widetilde{f}(p).$$

It follows that

$$\widetilde{y}(p) = \frac{\widetilde{f}(p) + a}{p + \widetilde{K}(p)}.$$

By the inversion formula (4.3.1.2), the solution to the original problem (4.3.4.5)–(4.3.4.6) is found in the form

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\widetilde{f}(p) + a}{p + \widetilde{K}(p)} e^{px}\,dp, \qquad i^2 = -1. \tag{4.3.4.7}$$

Consider the special case of $a = 0$ and $K(x) = \cos(bx)$. From row 10 of Table 4.3 it follows that $\widetilde{K}(p) = \dfrac{p}{p^2 + b^2}$. Rearrange the integrand of (4.3.4.7):

$$\frac{\widetilde{f}(p)}{p + \widetilde{K}(p)} = \frac{p^2 + b^2}{p(p^2 + b^2 + 1)}\widetilde{f}(p) = \Bigl(\frac{1}{p} - \frac{1}{p(p^2 + b^2 + 1)}\Bigr)\widetilde{f}(p).$$

In order to invert this expression, let us use the convolution theorem (see formula 16 of Section S3.2.1) as well as formulas 1 and 28 for the inversion of rational functions, Section S3.2.2. As a result, we arrive at the solution in the form

$$y(x) = \int_0^x \frac{b^2 + \cos\bigl(t\sqrt{b^2 + 1}\bigr)}{b^2 + 1} f(x-t)\,dt.$$

4.3.5 Solution of Linear Equations with Polynomial Coefficients Using the Laplace Transform

► **Solution of equations using the Laplace transform. General description.**

1°. Some classes of equations (4.2.1.1) or (4.2.2.1) with polynomial coefficients

$$f_k(x) = \sum_{m=0}^{s_k} a_{km} x^m$$

may be solved using the Laplace transform (see Sections 4.3.1, 4.3.2, and S3.1). To this end, one uses the following formula for the Laplace transform of the product of a power function and a derivative of the unknown function:

$$\mathfrak{L}\left\{x^m y_x^{(n)}(x)\right\} = (-1)^m \frac{d^m}{dp^m}\left[p^n \widetilde{y}(p) - \sum_{k=1}^{n} p^{n-k} y_x^{(k-1)}(+0)\right]. \qquad (4.3.5.1)$$

The right-hand side contains initial data $y_x^{(m)}(+0)$, $m = 0,\ 1,\ \ldots,\ n-1$ (specified in the Cauchy problem). As a result, one arrives at a linear ordinary differential equation, with respect to p, for the transform $\widetilde{y}(p)$; the order of this equation is equal to $\max\limits_{1\le k\le n}\{s_k\}$, the highest degree of the polynomials that determine the equation coefficients. In some cases, the equation for $\widetilde{y}(p)$ turns out to be simpler than the initial equation for $y(x)$ and can be solved in closed form. The desired function $y(x)$ is found by inverting the transform $\widetilde{y}(p)$ using the formulas from Section 4.3.2 or the tables from Section S3.2.

► **Application to the Laplace equation.**

Consider the *Laplace equation*

$$(a_n + b_n x) y_x^{(n)} + (a_{n-1} + b_{n-1} x) y_x^{(n-1)} + \cdots + (a_1 + b_1 x) y'_x + (a_0 + b_0 x) y = 0, \quad (4.3.5.2)$$

whose coefficients are linear functions of the independent variable x. The application of the Laplace transform, in view of formulas (4.3.5.1), brings it to a linear first-order ordinary differential equation for the transform $\widetilde{y}(p)$.

Example 4.8. Consider a special case of equation (4.3.5.2):

$$x y''_{xx} + y'_x + a x y = 0. \qquad (4.3.5.3)$$

Denote $y(0) = y_0$ and $y'_x(0) = y_1$. Let us apply the Laplace transform to this equation using formulas (4.3.5.1). On rearrangement, we obtain a linear first-order equation for $\widetilde{y}(p)$:

$$-(p^2\widetilde{y} - y_0 p - y_1)'_p + (p\widetilde{y} - y_0) - a\widetilde{y}'_p = 0 \qquad \Longrightarrow \qquad (p^2 + a)\widetilde{y}'_p + p\widetilde{y} = 0.$$

Its general solution is expressed as

$$\widetilde{y} = \frac{C}{\sqrt{p^2 + a}}, \qquad (4.3.5.4)$$

where C is an arbitrary constant. Applying the inverse Laplace transform to (4.3.5.4) and taking into account formulas 19 and 20 from Section S3.2.3, we find a solution to the original equation (4.3.5.3):

$$y(x) = \begin{cases} C J_0(x\sqrt{a}\,) & \text{if } a > 0, \\ C I_0(x\sqrt{-a}\,) & \text{if } a < 0, \end{cases} \qquad (4.3.5.5)$$

where $J_0(x)$ is the Bessel function of the first kind and $I_0(x)$ is the modified Bessel function of the first kind.

In this case, only one solution (4.3.5.5) has been obtained. This is due to the fact that the other solution goes to infinity as $x \to 0$, and hence formula (4.3.5.1) cannot be applied to it; this formula is only valid for finite initial values of the function and its derivatives.

4.3.6 Solution of Linear Equations with Polynomial Coefficients Using the Laplace Integral

► Solution of equations using the Laplace integral. General description.

Solutions to linear differential equations with polynomial coefficients can sometimes be represented as a *Laplace integral* in the form

$$y(x) = \int_{\mathcal{K}} e^{px} u(p)\, dp. \tag{4.3.6.1}$$

For now, no assumptions are made about the domain of integration $\mathcal{K}$; it could be a segment of the real axis or a curve in the complex plane.

Let us exemplify the usage of the Laplace integral (4.3.6.1) by considering equation (4.3.5.2). It follows from (4.3.6.1) that

$$y_x^{(k)}(x) = \int_{\mathcal{K}} e^{px} p^k u(p)\, dp,$$
$$xy_x^{(k)}(x) = \int_{\mathcal{K}} x e^{px} p^k u(p)\, dp = \Big[e^{px} p^k u(p)\Big]_{\mathcal{K}} - \int_{\mathcal{K}} e^{px} \frac{d}{dp}\Big[p^k u(p)\Big]\, dp.$$

Substituting these expressions into (4.3.5.2) yields

$$\int_{\mathcal{K}} e^{px} \left\{ \sum_{k=0}^{n} a_k p^k u(p) - \sum_{k=0}^{n} b_k \frac{d}{dp}\Big[p^k u(p)\Big] \right\} dp + \sum_{k=0}^{n} b_k \Big[e^{px} p^k u(p)\Big]_{\mathcal{K}} = 0. \tag{4.3.6.2}$$

This equation is satisfied if the expression in braces vanishes, thus resulting in a linear first-order ordinary differential equation for $u(p)$:

$$u(p) \sum_{k=0}^{n} a_k p^k - \frac{d}{dp}\left[u(p) \sum_{k=0}^{n} b_k p^k\right] = 0. \tag{4.3.6.3}$$

The remaining term in (4.3.6.2) must also vanish:

$$\left[\sum_{k=0}^{n} b_k e^{px} p^k u(p)\right]_{\mathcal{K}} = 0. \tag{4.3.6.4}$$

This condition can be met by appropriately selecting the path of integration $\mathcal{K}$. Consider the example below to illustrate the aforesaid.

► Application to the second-order Laplace equation of the special form.

Consider the linear variable-coefficient second-order equation

$$xy''_{xx} + (x + a + b)y'_x + ay = 0 \qquad (a > 0,\ b > 0), \tag{4.3.6.5}$$

that is a special case of equation (4.3.5.2) with $n=2$, $a_2=0$, $a_1=a+b$, $a_0=a$, $b_2=b_1=1$, and $b_0=0$. On substituting these values into (4.3.6.3), we arrive at an equation for $u(p)$:

$$p(p+1)u'_p - [(a+b-2)p + a - 1]u = 0.$$

Its solution is given by

$$u(p) = p^{a-1}(p+1)^{b-1}. \tag{4.3.6.6}$$

It follows from condition (4.3.6.4), in view of formula (4.3.6.6), that

$$\Big[e^{px}(p+p^2)u(p)\Big]_\alpha^\beta = \Big[e^{px}p^a(p+1)^b\Big]_\alpha^\beta = 0, \tag{4.3.6.7}$$

where a segment of the real axis, $\mathcal{K}=[\alpha,\beta]$, has been chosen to be the path of integration. Condition (4.3.6.7) is satisfied if we set $\alpha=-1$ and $\beta=0$. Consequently, one of the solutions to equation (4.3.6.5) has the form

$$y(x) = \int_{-1}^{0} e^{px}p^{a-1}(p+1)^{b-1}\,dp. \tag{4.3.6.8}$$

Remark 4.4. If a is noninteger, it is necessary to separate the real and imaginary parts in (4.3.6.8) to obtain real solutions.

Remark 4.5. By setting $\alpha=-\infty$ and $\beta=0$ in (4.3.6.7), one can find a second solution to equation (4.3.6.5) (at least for $x>0$).

⊙ *Literature for Section 4.3*: G. Doetsch (1950, 1956, 1974), H. Bateman and A. Erdélyi (1954), G. M. Murphy (1960), V. A. Ditkin and A. P. Prudnikov (1965), J. W. Miles (1971), F. Oberhettinger and L. Badii (1973), E. Kamke (1977), W. R. LePage (1980), R. Bellman and R. Roth (1984), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992*a,b*), M. Ya. Antimirov (1993), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

4.4 Asymptotic Solutions of Linear Equations

This section presents asymptotic solutions, as $\varepsilon\to 0$ ($\varepsilon>0$), of some higher-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being real.

4.4.1 Fourth-Order Linear Differential Equations

▶ **Binomial equation.**

1°. Consider the equation

$$\varepsilon^4 y''''_{xxxx} - f(x)y = 0$$

on a closed interval $a\le x\le b$. With the condition $f>0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon\to 0$, are given by the formulas

$$y_1 = [f(x)]^{-3/8}\exp\biggl\{-\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\},\quad y_2 = [f(x)]^{-3/8}\exp\biggl\{\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\},$$

$$y_3 = [f(x)]^{-3/8}\cos\biggl\{\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\},\qquad y_4 = [f(x)]^{-3/8}\sin\biggl\{\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\}.$$

► **Trinomial equation.**

Now consider the "biquadratic" equation

$$\varepsilon^4 y''''_{xxxx} - 2\varepsilon^2 g(x) y''_{xx} - f(x)y = 0. \tag{4.4.1.1}$$

Introduce the notation

$$D(x) = [g(x)]^2 + f(x).$$

In the range where the conditions $f(x) \neq 0$ and $D(x) \neq 0$ are satisfied, the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (4.4.1.1) are described by the formulas

$$y_k = [\lambda_k(x)]^{-1/2} [D(x)]^{-1/4} \exp\biggl\{\frac{1}{\varepsilon}\int \lambda_k(x)\,dx - \frac{1}{2}\int \frac{[\lambda_k(x)]'_x}{\sqrt{D(x)}}\,dx\biggr\}; \quad k = 1,\ 2,\ 3,\ 4,$$

where

$$\lambda_1(x) = \sqrt{g(x) + \sqrt{D(x)}}, \qquad \lambda_2(x) = -\sqrt{g(x) + \sqrt{D(x)}},$$
$$\lambda_3(x) = \sqrt{g(x) - \sqrt{D(x)}}, \qquad \lambda_4(x) = -\sqrt{g(x) - \sqrt{D(x)}}.$$

4.4.2 Higher-Order Linear Differential Equations

► **Binomial equation.**

Consider an equation of the form

$$\varepsilon^n y_x^{(n)} - f(x)y = 0$$

on a closed interval $a \le x \le b$. Assume that $f \neq 0$. Then the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to 0$, are given by

$$y_m = \bigl[f(x)\bigr]^{-\frac{1}{2} + \frac{1}{2n}} \exp\biggl\{\frac{\omega_m}{\varepsilon}\int \bigl[f(x)\bigr]^{\frac{1}{n}}\,dx\biggr\}\bigl[1 + O(\varepsilon)\bigr],$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are roots of the equation $\omega^n = 1$:

$$\omega_m = \cos\Bigl(\frac{2\pi m}{n}\Bigr) + i\sin\Bigl(\frac{2\pi m}{n}\Bigr), \qquad m = 1,\ 2,\ \ldots,\ n.$$

► **More complex equation.**

Now consider an equation of the form

$$\varepsilon^n y_x^{(n)} + \varepsilon^{n-1} f_{n-1}(x) y_x^{(n-1)} + \cdots + \varepsilon f_1(x) y'_x + f_0(x)y = 0 \tag{4.4.2.1}$$

on a closed interval $a \le x \le b$. Let $\lambda_m = \lambda_m(x)$ $(m = 1, 2, \ldots, n)$ be the roots of the characteristic equation

$$P(x, \lambda) \equiv \lambda^n + f_{n-1}(x)\lambda^{n-1} + \cdots + f_1(x)\lambda + f_0(x) = 0.$$

Let all the roots of the characteristic equation be different on the interval $a \le x \le b$, i.e., the conditions $\lambda_m(x) \neq \lambda_k(x)$, $m \neq k$, are satisfied, which is equivalent to the fulfillment of the conditions $P_\lambda(x,\lambda_m) \neq 0$. Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (4.4.2.1), as $\varepsilon \to 0$, are given by

$$y_m = \exp\biggl\{\frac{1}{\varepsilon}\int \lambda_m(x)\,dx - \frac{1}{2}\int [\lambda_m(x)]'_x \frac{P_{\lambda\lambda}\bigl(x,\lambda_m(x)\bigr)}{P_\lambda\bigl(x,\lambda_m(x)\bigr)}\,dx\biggr\},$$

where

$$P_\lambda(x,\lambda) \equiv \frac{\partial P}{\partial\lambda} = n\lambda^{n-1} + (n-1)f_{n-1}(x)\lambda^{n-2} + \cdots + 2\lambda f_2(x) + f_1(x),$$
$$P_{\lambda\lambda}(x,\lambda) \equiv \frac{\partial^2 P}{\partial\lambda^2} = n(n-1)\lambda^{n-2} + (n-1)(n-2)f_{n-1}(x)\lambda^{n-3} + \cdots + 6\lambda f_3(x) + 2f_2(x).$$

⊙ *Literature for Section 4.4*: W. Wasov (1965), M. V. Fedoryuk (1993), A. D. Polyanin and V. F. Zaitsev (2003).

4.5 Collocation Method

4.5.1 Statement of the Problem. Approximate Solution

1°. Consider the linear boundary value problem defined by the equation

$$\mathrm{L}y \equiv y^{(n)}_x + f_{n-1}(x)y^{(n-1)}_x + \cdots + f_1(x)y'_x + f_0(x)y = g(x), \qquad -1 < x < 1, \quad (4.5.1.1)$$

and the boundary conditions

$$\sum_{j=0}^{n-1}\bigl[\alpha_{ij}y^{(j)}_x(-1) + \beta_{ij}y^{(j)}_x(1)\bigr] = 0, \qquad i = 1,\ldots,n. \qquad (4.5.1.2)$$

2°. We seek an approximate solution to problem (4.5.1.1)–(4.5.1.2) in the form

$$y_m(x) = A_1\varphi_1(x) + A_2\varphi_2(x) + \cdots + A_m\varphi_m(x),$$

where $\varphi_k(x)$ is a polynomial of degree $n+k-1$ that satisfies the boundary conditions (4.5.1.2). The coefficients A_k are determined by the linear system of algebraic equations

$$\bigl[\mathrm{L}y_m - g(x)\bigr]_{x=x_i} = 0, \qquad i = 1,\ldots,m, \qquad (4.5.1.3)$$

with *Chebyshev nodes* $x_i = \cos\Bigl(\dfrac{2i-1}{2m}\pi\Bigr)$, $i = 1,\,\ldots,\,m$.

4.5.2 Convergence Theorem

THEOREM. *Let the functions $f_j(x)$ ($j = 0,\ldots,n-1$) and $g(x)$ be continuous on the interval $[-1,1]$ and let the boundary value problem (4.5.1.1)–(4.5.1.2) have a unique solution, $y(x)$. Then there exists an m_0 such that system (4.5.1.3) is uniquely solvable for $m \geq m_0$; and the limit relations*

$$\max_{-1\le x\le 1}\bigl|y^{(k)}_m(x) - y^{(k)}(x)\bigr| \leq cE_m\bigl(y^{(n)}\bigr) \to 0, \qquad k = 0,\,1,\,\ldots,\,n-1;$$
$$\biggl\{\int_{-1}^{1}\frac{\bigl|y^{(n)}_m(x) - y^{(n)}(x)\bigr|^2}{\sqrt{1-x^2}}\,dx\biggr\}^{1/2} \leq cE_m\bigl(y^{(n)}\bigr) \to 0$$

hold for $m \to \infty$. *Here* $c = \text{const}$ *and*

$$E_m(v) = \min_{b_0,\dots,b_{m-1}} \max_{-1\le x\le 1} \left| v(x) - \sum_{j=0}^{m-1} b_j x^j \right|.$$

Remark 4.6. A similar result holds true if the nodes are roots of some orthogonal polynomials with some weight function. If the nodes are equidistant, the method diverges.

⊙ *Literature for Section 4.5*: R. D. Russell and L. F. Shampine (1972), C. de Boor and B. Swartz (1993), Mathematical Encyclopedia (1979, p. 951), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 5

Methods for Nonlinear ODEs of Arbitrary Order

5.1 General Concepts. Cauchy Problem. Uniqueness and Existence Theorems

5.1.1 Equations Solved for the Derivative. General Solution

▶ **Equations solved for the highest derivative. Structure of the general solution.**

An *nth-order differential equation solved for the highest derivative* has the form

$$y_x^{(n)} = f(x, y, y_x', \dots, y_x^{(n-1)}). \tag{5.1.1.1}$$

A *solution of a differential equation* is a function $y(x)$ that, when substituted into the equation, turns it into an identity. The *general solution of a differential equation* is the set of all its solutions.

The general solution of this equation depends on n arbitrary constants $C_1, \dots, C_n$. In some cases, the general solution can be written in explicit form as

$$y = \varphi(x, C_1, \dots, C_n). \tag{5.1.1.2}$$

▶ **Cauchy problem. Existence and uniqueness theorem.**

The *Cauchy problem*: find a solution of equation (5.1.1.1) with the *initial conditions*

$$y(x_0) = y_0, \quad y_x'(x_0) = y_0^{(1)}, \quad \dots, \quad y_x^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{5.1.1.3}$$

(At a point x_0, the values of the unknown function $y(x)$ and all its derivatives of orders $\le n-1$ are prescribed.)

EXISTENCE AND UNIQUENESS THEOREM. *Let the function $f(x, y, z_1, \dots, z_{n-1})$ be continuous in all its arguments in a neighborhood of the point $(x_0, y_0, y_0^{(1)}, \dots, y_0^{(n-1)})$ and have bounded derivatives with respect to y, z_1, ..., z_{n-1} in this neighborhood. Then a solution of equation (5.1.1.1) satisfying the initial conditions (5.1.1.3) exists and is unique.*

5.1.2 Some Transformations

▶ Construction of a differential equation by a given general solution.

Suppose a general solution (5.1.1.2) of an unknown nth-order ordinary differential equation is given. The equation corresponding to the general solution can be obtained by eliminating the arbitrary constants $C_1, \ldots, C_n$ from the identities

$$\begin{aligned} &y = \varphi(x, C_1, \ldots, C_n),\\ &y'_x = \varphi'_x(x, C_1, \ldots, C_n),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &y^{(n)}_x = \varphi^{(n)}_x(x, C_1, \ldots, C_n), \end{aligned}$$

obtained by differentiation from formula (5.1.1.2).

▶ Reduction of an nth-order equation to a system of n first-order equations.

The differential equation (5.1.1.1) is equivalent to the following system of n first-order equations:

$$y'_1 = y_2, \quad y'_2 = y_3, \quad \ldots, \quad y'_{n-1} = y_n, \quad y'_n = f(x, y_1, y_2, \ldots, y_n), \qquad (5.1.2.1)$$

where the notation $y_1 \equiv y$ is adopted.

The initial conditions (5.1.1.3) for equation (5.1.1.1) become the initial conditions

$$y_1(x_0) = y_0, \quad y_2(x_0) = y^{(1)}_0, \quad \ldots, \quad y_n(x_0) = y^{(n-1)}_0 \qquad (5.1.2.2)$$

for system (5.1.2.1).

Remark 5.1. For the numerical integration of equation (5.1.1.1) and system (5.1.2.1), see Sections 7.4.2 and 5.4.1.

⊙ *Literature for Section 5.1*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), N. M. Matveev (1967), I. G. Petrovskii (1970), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

5.2 Equations Admitting Reduction of Order

5.2.1 Equations Not Containing y or x Explicitly

▶ Equations not containing $y, y'_x, \ldots, y^{(k)}_x$ explicitly.

An equation that does not explicitly contain the unknown function and its derivatives up to order k inclusive can generally be written as

$$F\bigl(x, y^{(k+1)}_x, \ldots, y^{(n)}_x\bigr) = 0 \qquad (1 \le k+1 < n). \qquad (5.2.1.1)$$

Such equations are invariant under arbitrary translations of the unknown function, $y \to y + \text{const}$ (the form of such equations is also preserved under the transformation $u(x) = y + a_k x^k + \cdots + a_1 x + a_0$, where the a_m are arbitrary constants). The substitution $z(x) = y^{(k+1)}_x$ reduces (5.2.1.1) to an equation whose order is by $k+1$ lower than that of the original equation, $F\bigl(x, z, z'_x, \ldots, z^{(n-k-1)}_x\bigr) = 0$.

► **Equations not containing x explicitly (autonomous equations).**

An equation that does not explicitly contain x has in the general form

$$F\big(y, y'_x, \dots, y_x^{(n)}\big) = 0. \tag{5.2.1.2}$$

Such equations are invariant under arbitrary translations of the independent variable, $x \to x + \text{const}$. The substitution $y'_x = w(y)$ (where y plays the role of the independent variable) reduces by one the order of an autonomous equation. Higher derivatives can be expressed in terms of w and its derivatives with respect to the new independent variable, $y''_{xx} = ww'_y$, $y'''_{xxx} = w^2 w''_{yy} + w(w'_y)^2, \dots$

► **Related equations.**

Equations of the form

$$F\big(ax + by, y'_x, \dots, y_x^{(n)}\big) = 0$$

are invariant under simultaneous translations of the independent variable and the unknown function, $x \to x + bc$ and $y \to y - ac$, where c is an arbitrary constant.

For $b = 0$, see equation (5.2.1.1). For $b \neq 0$, the substitution $w(x) = y + (a/b)x$ leads to an autonomous equation of the form (5.2.1.2).

5.2.2 Homogeneous Equations

► **Equations homogeneous in the independent variable.**

Equations homogeneous in the independent variable are invariant under scaling of the independent variable, $x \to \alpha x$, where α is an arbitrary constant ($\alpha \neq 0$). In general, such equations can be written in the form

$$F\big(y, xy'_x, x^2 y''_{xx}, \dots, x^n y_x^{(n)}\big) = 0.$$

The substitution $z(y) = xy'_x$ reduces by one the order of this equation.

► **Equations homogeneous in the unknown function.**

Equations homogeneous in the unknown function are invariant under scaling of the unknown function, $y \to \alpha y$, where α is an arbitrary constant ($\alpha \neq 0$). Such equations can be written in the general form

$$F\big(x, y'_x/y, y''_{xx}/y, \dots, y_x^{(n)}/y\big) = 0.$$

The substitution $z(x) = y'_x/y$ reduces by one the order of this equation.

► **Equations homogeneous in both variables.**

Equations homogeneous in both variables are invariant under simultaneous scaling (dilatation) of the independent and dependent variables, $x \to \alpha x$ and $y \to \alpha y$, where α is an arbitrary constant ($\alpha \neq 0$). Such equations can be written in the general form

$$F\big(y/x, y'_x, xy''_{xx}, \dots, x^{n-1} y_x^{(n)}\big) = 0.$$

The transformation $t = \ln|x|$, $w = y/x$ leads to an autonomous equation considered in Section 5.2.1 (see Eq. (5.2.1.2)).

5.2.3 Generalized Homogeneous Equations

► Equations of a special form.

Generalized homogeneous equations (*equations homogeneous in the generalized sense*) are invariant under simultaneous scaling of the independent variable and the unknown function, $x \to \alpha x$ and $y \to \alpha^k y$, where $\alpha \neq 0$ is an arbitrary constant and k is a given number. Such equations can be written in the general form

$$F\big(x^{-k}y, x^{1-k}y'_x, \dots, x^{n-k}y^{(n)}_x\big) = 0.$$

The transformation $t = \ln x$, $w = x^{-k}y$ leads to an autonomous equation considered in Section 5.2.1 (see Eq. (5.2.1.2)).

► Equations of the general form.

The most general form of generalized homogeneous equations is

$$\mathcal{F}\big(x^n y^m, xy'_x/y, \dots, x^n y^{(n)}_x/y\big) = 0.$$

The transformation $z = x^n y^m$, $u = xy'_x/y$ reduces the order of this equation by one.

5.2.4 Equations Invariant under Scaling-Translation Transformations

► Equations of the first type.

The equations of the form

$$F\big(e^{\lambda x}y^m,\ y'_x/y,\ y''_{xx}/y,\ \dots,\ y^{(n)}_x/y\big) = 0$$

are invariant under the simultaneous translation and scaling of variables, $x \to x + \alpha$ and $y \to \beta y$, where $\beta = \exp(-\alpha\lambda/m)$ and α is an arbitrary constant. The transformation $z = e^{\lambda x}y^m$, $w = y'_x/y$ leads to an equation of order $n - 1$.

► Equations of the second type.

The equations of the form

$$F\big(x^m e^{\lambda y},\ xy'_x,\ x^2 y''_{xx},\ \dots,\ x^n y^{(n)}_x\big) = 0$$

are invariant under the simultaneous scaling and translation of variables, $x \to \alpha x$ and $y \to y + \beta$, where $\alpha = \exp(-\beta\lambda/m)$ and β is an arbitrary constant. The transformation $z = x^m e^{\lambda y}$, $w = xy'_x$ leads to an equation of order $n - 1$.

5.2.5 Other Equations

► Equations of the form $F\big(x, xy'_x - y, y''_{xx}, \dots, y^{(n)}_x\big) = 0$.

The substitution $w(x) = xy'_x - y$ reduces the order of this equation by one.

This equation is a special case of the equation

$$F\big(x, xy'_x - my, y^{(m+1)}_x, \dots, y^{(n)}_x\big) = 0, \qquad \text{where} \quad m = 1,\ 2,\ \dots,\ n-1. \qquad (5.2.5.1)$$

The substitution $w(x) = xy'_x - my$ reduces by one the order of equation (5.2.5.1).

► **Nonlinear equations involving linear homogeneous differential forms.**

Consider the nonlinear differential equation

$$F\big(x, \mathrm{L}_1[y], \ldots, \mathrm{L}_k[y]\big) = 0, \qquad (5.2.5.2)$$

where the $\mathrm{L}_s[y]$ are linear homogeneous differential forms,

$$\mathrm{L}_s[y] = \sum_{m=0}^{n_s} \varphi_m^{(s)}(x) y_x^{(m)}, \qquad s = 1, \ldots, k.$$

Let $y_0 = y_0(x)$ be a common particular solution of the linear equations

$$\mathrm{L}_s[y_0] = 0 \qquad (s = 1, \ldots, k).$$

Then the substitution

$$w = \psi(x)\big[y_0(x)y'_x - y'_0(x)y\big] \qquad (5.2.5.3)$$

with an arbitrary function $\psi(x)$ reduces by one the order of equation (5.2.5.2).

Example 5.1. Consider the third-order equation

$$y'''_{xxx} = y + f(y'_x - y).$$

It can be represented in the form (5.2.5.2) with

$$k = 2, \quad F(x, u, w) = u - f(w), \quad u = \mathrm{L}_1[y] = y'''_{xxx} - y, \quad w = \mathrm{L}_2[y] = y'_x - y.$$

The linear equations $\mathrm{L}_k[y] = 0$ are

$$y'''_{xxx} - y = 0, \quad y'_x - y = 0.$$

These equations have a common particular solution $y_0 = e^x$. Therefore, the substitution $w = y'_x - y$ (see formula (5.2.5.3) with $\psi(x) = e^{-x}$) leads to a second-order autonomous equation: $w''_{xx} + w'_x + w = f(w)$.

Example 5.2. Consider the other third-order equation

$$xy'''_{xxx} = f(xy'_x - 2y).$$

It can be represented in the form (5.2.5.2) with

$$k = 2, \quad F(x, u, w) = xu - f(w), \quad u = \mathrm{L}_1[y] = y'''_{xxx}, \quad w = \mathrm{L}_2[y] = xy'_x - 2y.$$

The linear equations $\mathrm{L}_k[y] = 0$ are

$$y'''_{xxx} = 0, \quad xy'_x - 2y = 0.$$

These equations have a common particular solution $y_0 = x^2$. Therefore, the substitution $w = xy'_x - 2y$ (see formula (5.2.5.3) with $\psi(x) = 1/x$) leads to a second-order autonomous equation: $w''_{xx} = f(w)$. For the solution of this equation, see Example 3.1 in Section 3.2.1.

Example 5.3. The $2n$th-order equation

$$y_x^{(2n)} = f(x, y''_{xx} - y) + y \qquad (5.2.5.4)$$

can be represented in the form (5.2.5.2) with

$$k = 2, \quad F(x, u, w) = u - f(x, w), \quad \mathrm{L}_1[y] = y_x^{(2n)} - y, \quad \mathrm{L}_2[y] = y''_{xx} - y.$$

Consider the linear equations

$$\mathrm{L}_1[y] \equiv y_x^{(2n)} - y = 0, \quad \mathrm{L}_2[y] \equiv y''_{xx} - y = 0. \qquad (5.2.5.5)$$

There are two cases.

1°. Equations (5.2.5.5) have a common particular solution, $y_0 = e^x$. Therefore, the substitution $w = y'_x - y$ (see formula (5.2.5.3) with $\varphi(x) = e^{-x}$) takes Eq. (5.2.5.4) to an $(n-1)$st-order equation.

2°. Equations (5.2.5.5) also have another common particular solution, $y_0 = e^{-x}$. Therefore, the substitution $w = y'_x + y$ (see formula (5.2.5.3) with $\varphi(x) = e^x$) leads Eq. (5.2.5.4) to an $(n-1)$st-order equation.

Both of the above cases can be combined together. Specifically, the substitution $u = y''_{xx} - y$ reduces Eq. (5.2.5.4) to an $(n-2)$nd-order equation.

In particular, a fourth-order equation of the form

$$y_x^{(4)} = f(y''_{xx} - y) + y$$

can be reduced with the substitution $u = y''_{xx} - y$ to the second-order autonomous equation $u''_{xx} = f(u) - u$, whose general solution can be represented in implicit form (see Example 3.1).

⊙ *Literature for Section 5.2*: G. M. Murphy (1960), N. M. Matveev (1967), E. Kamke (1977), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

5.3 Method for Construction of Solvable Equations of General Form

5.3.1 Description of the Method

Consider a function

$$y = f(x, C_1, C_2, \dots, C_{n+1}) \tag{5.3.1.1}$$

depending on $n+1$ free parameters C_k. Differentiating relation (5.3.1.1) n times, we obtain the following sequence of equations:

$$y_x^{(k)} = f_x^{(k)}(x, C_1, C_2, \dots, C_{n+1}), \qquad k = 1,\ 2,\ \dots,\ n. \tag{5.3.1.2}$$

Treating relations (5.3.1.1), (5.3.1.2) as an algebraic (transcendental) system of equations for the parameters C_1, C_2, $\dots$, C_{n+1} and solving this system, we obtain

$$C_k = \varphi_k\bigl(x, y, y'_x, \dots, y_x^{(n)}\bigr), \qquad k = 1,\ 2,\ \dots,\ n+1. \tag{5.3.1.3}$$

Consider a general nth-order equation of the form

$$F(\varphi_1, \varphi_2, \dots, \varphi_{n+1}) = 0, \tag{5.3.1.4}$$

where F is an arbitrary function of $(n+1)$ variables and $\varphi_k = \varphi_k\bigl(x, y, y'_x, \dots, y_x^{(n)}\bigr)$ are the functions from (5.3.1.3). Equation (5.3.1.4) is satisfied by the function (5.3.1.1), where the $(n+1)$ arbitrary parameters C_1, C_2, $\dots$, C_{n+1} are related by a single constraint:

$$F(C_1, C_2, \dots, C_{n+1}) = 0.$$

Remark 5.2. Equation (5.3.1.4) may also have singular solutions depending on a smaller number of arbitrary constants. In order to examine these solutions, one should differentiate equation (5.3.1.4); see Example 5.4.

Remark 5.3. Instead of (5.3.1.4), one can consider a more general equation

$$F(\psi_1, \psi_2, \dots, \psi_{n+1}) = 0, \qquad \text{where} \quad \psi_k = \psi_k(\varphi_1, \varphi_2, \dots, \varphi_{n+1}).$$

Remark 5.4. The original expression (5.3.1.1) can be specified in an implicit form.

Remark 5.5. The original expression (5.3.1.1) can be written as an mth-order differential equation ($m < n$) with $n-m+1$ free parameters C_k. The solution of the nth-order differential equation obtained in this way can be expressed in terms of the solution of an mth-order differential equation (see Example 5.7).

5.3.2 Illustrative Examples

Example 5.4. Consider the function

$$y = -C_1 e^{-x} + C_2. \tag{5.3.2.1}$$

By differentiation we obtain

$$y'_x = C_1 e^{-x}. \tag{5.3.2.2}$$

Let us solve equations (5.3.2.1)–(5.3.2.2) for the parameters C_1 and C_2. We have

$$C_1 = e^x y'_x, \quad C_2 = y'_x + y.$$

Using the above method, we construct an equation in accordance with (5.3.1.4):

$$F\big(e^x y'_x,\, y'_x + y\big) = 0. \tag{5.3.2.3}$$

This equation admits a solution of the form (5.3.2.1) with constants C_1 and C_2 related by the constraint $F(C_1, C_2) = 0$.

Singular solution. Differentiating equation (7) with respect to x, we get

$$(y''_{xx} + y'_x)(e^x F_u + F_v) = 0, \tag{5.3.2.4}$$

where the subscripts u and v indicate the respective partial derivatives of the function $F = F(u, v)$. Equating the first factor in (5.3.2.4) to zero, we obtain solution (5.3.2.1). Equating the second factor to zero, we obtain an expression which, combined with equation (5.3.2.3), yields a singular solution in parametric form:

$$F(u, v) = 0, \quad e^x F_u + F_v = 0, \qquad \text{where} \quad u = e^x t, \quad v = t + y.$$

One should eliminate $t = y'_x$ from these expressions.

Example 5.5. Consider the function

$$y = C_1 x^2 + C_2 x + C_3. \tag{5.3.2.5}$$

Differentiating this function twice, we get

$$\begin{aligned} y'_x &= 2C_1 x + C_2, \\ y''_{xx} &= 2C_1. \end{aligned} \tag{5.3.2.6}$$

Solving (5.3.2.5)–(5.3.2.6) for the parameters C_k, we find that

$$C_1 = \tfrac{1}{2} y''_{xx}, \quad C_2 = y'_x - x y''_{xx}, \quad C_3 = y - x y'_x + \tfrac{1}{2} x^2 y''_{xx}.$$

These relations lead to a second-order equation of general form:

$$F\big(\tfrac{1}{2} y''_{xx},\, y'_x - x y''_{xx},\, y - x y'_x + \tfrac{1}{2} x^2 y''_{xx}\big) = 0,$$

which has a solution of the type (5.3.2.5) with the three constants C_1, C_2, and C_3 related by the constraint $F(C_1, C_2, C_3) = 0$.

Example 5.6. In Example 5.5, one can choose the functions ψ_k of the form (see Remark 5.3)

$$\psi_1 = 2\varphi_1, \quad \psi_2 = -\varphi_2, \quad \psi_3 = 4\varphi_1\varphi_3 - \varphi_2^2,$$

where $\varphi_1 = \frac{1}{2}y''_{xx}$, $\varphi_2 = y'_x - xy''_{xx}$, $\varphi_3 = y - xy'_x + \frac{1}{2}x^2 y''_{xx}$. As a result, we obtain the differential equation:

$$\mathcal{F}\big(y''_{xx},\, xy''_{xx} - y'_x,\, 2yy''_{xx} - (y'_x)^2\big) = 0.$$

Its solution is given by (5.3.2.5) with three constants C_1, C_2, and C_3 related by a single constraint $\mathcal{F}(2C_1, -C_2, 4C_1C_3 - C_3^2) = 0$.

Example 5.7. Consider the autonomous equation

$$y''_{xx} = C_1 y^{-a} + C_2. \tag{5.3.2.7}$$

Its solution can be represented in implicit form (see Example 3.1 and Eq. 14.9.1.1). Differentiating (5.3.2.7), we obtain

$$y'''_{xxx} = -aC_1 y^{-a-1} y'_x. \tag{5.3.2.8}$$

Let us solve equations (5.3.2.7)–(5.3.2.8) for the parameters C_1 and C_2:

$$C_1 = -y^{a+1}\frac{y'''_{xxx}}{ay'_x}, \quad C_2 = y''_{xx} + y\frac{y'''_{xxx}}{ay'_x}.$$

Taking $\psi_1 = -a\varphi_1$ and $\psi_2 = a\varphi_2$ (see Remark 5.3), we obtain the equation:

$$F\left(y^{a+1}\frac{y'''_{xxx}}{y'_x},\, y\frac{y'''_{xxx}}{y'_x} + ay''_{xx}\right) = 0.$$

This equation is satisfied by the solutions of a second-order autonomous equation of the form (5.3.2.7), where the constants C_1 and C_2 are related by the constraint $F(-aC_1, aC_2) = 0$.

⊙ *Literature for Section 5.3*: A. D. Polyanin and V. F. Zaitsev (2003).

5.4 Numerical Integration of n-order Equations

5.4.1 Numerical Solution of the Cauchy Problem for n-order ODEs

The Cauchy problem for the nth-order equation (5.1.1.1) subject to the initial conditions (5.1.1.3) is solved numerically in two stages. First, equation (5.1.1.1) is reduced to the equivalent system of n first-order equations (5.1.2.1) with the initial conditions (5.1.2.2). In the second stage, the resulting system (5.1.2.1) is integrated numerically with standard methods outlined in Section 7.4.2.

5.4.2 Numerical Solution of Equations Defined Implicitly or Parametrically

► **Numerical integration of equations defined parametrically.**

Below we describe a numerical method for solving the Cauchy problem for the n-order equation represented in parametric form by two relations

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(m-1)}, t), \quad y_x^{(n)} = G(x, y, y'_x, \dots, y_x^{(n-1)}, t), \quad m < n, \tag{5.4.2.1}$$

subject to the initial conditions (5.1.1.3), with t being a functional parameter.

We start directly from equations (5.4.2.1). Consider two auxiliary Cauchy problems

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(m-1)}, t),$$
$$y(x_0) = y_0,\ \ y'_x(x_0) = y_0^{(1)},\ \ \dots,\ \ y_x^{(m-1)}(x_0) = y_0^{(m-1)} \ \text{(1st problem)}; \tag{5.4.2.2}$$
$$y_x^{(n)} = G(x, y, y'_x, \dots, y_x^{(n-1)}, t),$$
$$y(x_0) = y_0,\ \ y'_x(x_0) = y_0^{(1)},\ \ \dots,\ \ y_x^{(n-1)}(x_0) = y_0^{(n-1)} \ \text{(2nd problem)}. \tag{5.4.2.3}$$

Let $y_F = y_F(x,t)$ and $y_G = y_G(x,t)$ denote their respective solutions. Introduce the difference of these solutions

$$\Delta(x,t) = y_G(x,t) - y_F(x,t). \tag{5.4.2.4}$$

By fixing a value of the parameter, $t = t_k$, we compute the solutions $y_F(x, t_k)$ and $y_G(x, t_k)$ using, for example, the Runge–Kutta method. Further, by varying x, we find a x_k at which the right-hand side of equation (5.4.2.3) becomes zero: $\Delta(x_k, t_k) = 0$. To this x_k there corresponds the value $y_k = y_F(x_k, t_k) = y_G(x_k, t_k)$. Thus, to each t_k there corresponds a point (x_k, y_k) in the (x, y) plane. By choosing a different value t_{k+1}, we find a new point (x_{k+1}, y_{k+1}). The combination of discrete points (x_k, y_k) with $k = 0,\ 1,\ 2,\ \dots$ determines an approximation to the solution $y = y(x)$ of the original problem (5.4.2.1), (5.1.1.3).

The initial value of the parameter, $t = t_0$, is determined from the algebraic (or transcendental) equation

$$y_0^{(m)} = F(x_0, y_0, y_0^{(1)}, \dots, y_0^{(m-1)}, t_0), \tag{5.4.2.5}$$

where x_0, y_0, $y_0^{(1)}$, $\dots$, $y_0^{(m)}$ are the values appearing in the initial conditions (5.4.2.2)–(5.4.2.3), obtained from (5.1.1.3)).

Remark 5.6. In general, the algebraic (or transcendental) equation (5.4.2.5) can have one, two, or more different roots, in which case the original problem (5.4.2.1), (5.1.1.3) will have the same number of different solutions.

► Numerical integration of equations defined implicitly.

Consider the Cauchy problem for the implicitly defined equation

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(n-1)}, y_x^{(n)}), \qquad m < n \tag{5.4.2.6}$$

subject to the initial conditions (5.1.1.3).

The substitution $y_x^{(n)} = t$ reduces equation (5.4.2.6) to the parametric equation

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(n-1)}, t), \qquad y_x^{(n)} = t \tag{5.4.2.7}$$

with the initial conditions (5.1.1.3).

Problem (5.4.2.7), (5.1.1.3) is a special case of problem (5.4.2.1), (5.1.1.3) in which $G(\dots) = t$; hence, the above method is suitable for its solution.

⊙ *Literature for Section 5.4*: N. N. Kalitkin (1978), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), W. E. Schiesser (1994), L. F. Shampine (1994), K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), J. R. Dormand (1996), E. Hairer and G. Wanner (1996), D. Zwillinger (1997), U. M. Ascher and L. R. Petzold (1998), G. A. Korn and T. M. Korn (2000), P. J. Rabier and W. C. Rheinboldt (2002), S. C. Chapra and R. P. Canale (2010), A. D. Polyanin (2016b).

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Chapter 6

Methods for Linear Systems of ODEs

6.1 Systems of Linear Constant-Coefficient Equations

6.1.1 Systems of First-Order Linear Homogeneous Equations. General Solution

1°. In general, a homogeneous linear system of constant-coefficient first-order ordinary differential equations has the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n, \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n, \end{aligned} \tag{6.1.1.1}$$

where a prime stands for the derivative with respect to x. In the sequel, all the coefficients a_{ij} of the system are assumed to be real numbers.

The homogeneous system (6.1.1.1) has the trivial particular solution $y_1 = y_2 = \dots = y_n = 0$.

Superposition principle for a homogeneous system: any linear combination of particular solutions of system (6.1.1.1) is also a solution of this system.

The general solution of the system of differential equations (6.1.1.1) is the sum of its n linearly independent (nontrivial) particular solutions multiplied by an arbitrary constant.

System (6.1.1.1) can be reduced to a single homogeneous linear constant-coefficient nth-order equation; see Section 7.1.3.

2°. For brevity (and clearness), system (6.1.1.1) is conventionally written in vector-matrix form:

$$\mathbf{y}' = \mathbf{a}\mathbf{y}, \tag{6.1.1.2}$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)^{\mathrm{T}}$ is the column vector of the unknowns and $\mathbf{a} = (a_{ij})$ is the matrix of the equation coefficients. The superscript T denotes the transpose of a matrix or a vector. So, for example, a row vector is converted into a column vector:

$$(y_1, y_2)^{\mathrm{T}} \equiv \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The right-hand side of equation (6.1.1.2) is the product of the $n \times n$ square matrix **a** by the $n \times 1$ matrix (column vector) **y**.

Let $\mathbf{y}_k = (y_{k1}, y_{k2}, \dots, y_{kn})^{\mathrm{T}}$ be linearly independent particular solutions* of the homogeneous system (6.1.1.1), where $k = 1, 2, \dots, n$; the first subscript in $y_{km} = y_{km}(x)$ denotes the number of the solution and the second subscript indicates the component of the vector solution. Then the general solution of the homogeneous system (6.1.1.2) is expressed as

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + \dots + C_n\mathbf{y}_n. \tag{6.1.1.3}$$

A method for the construction of particular solutions that can be used to obtain the general solution by formula (6.1.1.3) is presented below.

6.1.2 Systems of First-Order Linear Homogeneous Equations. Particular Solutions

Particular solutions to system (6.1.1.1) are determined by the roots of the characteristic equation

$$\Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) \equiv \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}. \tag{6.1.2.1}$$

The following cases are possible:

1°. Let $\lambda = \lambda_k$ be a simple real root of the characteristic equation (6.1.2.1). The corresponding particular solution of the homogeneous linear system of equations (6.1.1.1) has the exponential form

$$y_1 = A_1 e^{\lambda x}, \quad y_2 = A_2 e^{\lambda x}, \quad \dots, \quad y_n = A_n e^{\lambda x}, \tag{6.1.2.2}$$

where the coefficients $A_1, A_2, \dots, A_n$ are determined by solving the associated homogeneous system of algebraic equations obtained by substituting expressions (6.1.2.2) into the differential equation (6.1.1.1) and dividing by $e^{\lambda x}$:

$$\begin{aligned} &(a_{11}-\lambda)A_1 + a_{12}A_2 + \dots + a_{1n}A_n = 0, \\ &a_{21}A_1 + (a_{22}-\lambda)A_2 + \dots + a_{2n}A_n = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}A_1 + a_{n2}A_2 + \dots + (a_{nn}-\lambda)A_n = 0. \end{aligned} \tag{6.1.2.3}$$

The solution of this system is unique to within a constant factor.

If all roots of the characteristic equation $\lambda_1, \lambda_2, \dots, \lambda_n$ are real and distinct, then the general solution of system (6.1.1.1) has the form

$$\begin{aligned} &y_1 = C_1A_{11}e^{\lambda_1 x} + C_2A_{12}e^{\lambda_2 x} + \dots + C_nA_{1n}e^{\lambda_n x}, \\ &y_2 = C_1A_{21}e^{\lambda_1 x} + C_2A_{22}e^{\lambda_2 x} + \dots + C_nA_{2n}e^{\lambda_n x}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &y_n = C_1A_{n1}e^{\lambda_1 x} + C_2A_{n2}e^{\lambda_2 x} + \dots + C_nA_{nn}e^{\lambda_n x}, \end{aligned} \tag{6.1.2.4}$$

*This means that the condition $\det|y_{mk}(x)| \neq 0$ holds.

where $C_1, C_2, \ldots, C_n$ are arbitrary constants. The second subscript in A_{mk} indicates a coefficient corresponding to the root λ_k.

2°. For each simple complex root, $\lambda = \alpha \pm i\beta$, of the characteristic equation (6.1.2.1), the corresponding particular solution is obtained in the same way as in the simple real root case; the associated coefficients $A_1, A_2, \ldots, A_n$ in (6.1.2.2) will be complex. Separating the real and imaginary parts in (6.1.2.2) results in two real particular solutions to system (6.1.1.1); the same two solutions are obtained if one takes the complex conjugate root, $\bar{\lambda} = \alpha - i\beta$.

3°. Let λ be a real root of the characteristic equation (6.1.2.1) of multiplicity m. The corresponding particular solution of system (6.1.1.1) is sought in the form

$$y_1 = P_m^1(x)e^{\lambda x}, \quad y_2 = P_m^2(x)e^{\lambda x}, \quad \ldots, \quad y_n = P_m^n(x)e^{\lambda x}, \tag{6.1.2.5}$$

where the $P_m^k(x) = \sum\limits_{i=0}^{m-1} B_{ki}x^i$ are polynomials of degree $m-1$. The coefficients of these polynomials result from the substitution of expressions (6.1.2.5) into equations (6.1.1.1); after dividing by $e^{\lambda x}$ and collecting like terms, one obtains n equations, each representing a polynomial equated to zero. By equating the coefficients of all resulting polynomials to zero, one arrives at a linear algebraic system of equations for the coefficients B_{ki}; the solution to this system will contain m free parameters.

4°. For a multiple complex, $\lambda = \alpha + i\beta$, of multiplicity m, the corresponding particular solution is sought, just as in the case of a multiple real root, in the form (6.1.2.5); here the coefficients B_{ki} of the polynomials $P_m^k(x)$ will be complex. Finally, in order to obtain real solutions of the original system (6.1.1.1), one separates the real and imaginary parts in formulas (6.1.2.5), thus obtaining two particular solutions with m free parameters each. The two solutions correspond to the complex conjugate roots $\lambda = \alpha \pm i\beta$.

5°. In the general case, where the characteristic equation (6.1.2.1) has simple and multiple, real and complex roots (see Items 1°–4°), the general solution to system (6.1.1.1) is obtained as the sum of all particular solutions multiplied by arbitrary constants.

Example 6.1. Consider the homogeneous system of two linear differential equations

$$y_1' = y_1 + 4y_2,$$
$$y_2' = y_1 + y_2.$$

The associated characteristic equation,

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0,$$

has two distinct real roots:

$$\lambda_1 = 3, \qquad \lambda_2 = -1.$$

The system of algebraic equations (6.1.2.3) for solution coefficients becomes

$$\begin{aligned} (1-\lambda)A_1 + 4A_2 &= 0, \\ A_1 + (1-\lambda)A_2 &= 0. \end{aligned} \tag{6.1.2.6}$$

Substituting the first root, $\lambda = 3$, into system (6.1.2.6) yields $A_1 = 2A_2$. We can set $A_1 = 2$ and $A_2 = 1$, since the solution is determined to within a constant factor. Thus the first particular solution of the homogeneous system of linear ordinary differential equations (6.1.2.6) has the form

$$y_1 = 2e^{3x}, \qquad y_2 = e^{3x}. \tag{6.1.2.7}$$

The second particular solution, corresponding to $\lambda = -1$, is found in the same way:

$$y_1 = -2e^{-x}, \quad y_2 = e^{-x}. \tag{6.1.2.8}$$

The sum of the two particular solutions (6.1.2.7), (6.1.2.8) multiplied by arbitrary constants, C_1 and C_2, gives the general solution to the original homogeneous system of linear ordinary differential equations:

$$y_1 = 2C_1e^{3x} - 2C_2e^{-x}, \quad y_2 = C_1e^{3x} + C_2e^{-x}.$$

Example 6.2. Consider the system of ordinary differential equations

$$\begin{aligned} y_1' &= -y_2, \\ y_2' &= 2y_1 + 2y_2. \end{aligned} \tag{6.1.2.9}$$

The characteristic equation,

$$\begin{vmatrix} -\lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0$$

has complex conjugate roots:

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i.$$

The algebraic system (6.1.2.3) for the complex coefficients A_1 and A_2 becomes

$$\begin{aligned} -\lambda A_1 - A_2 &= 0, \\ 2A_1 + (2-\lambda)A_2 &= 0. \end{aligned}$$

With $\lambda = 1 + i$, one nonzero solution is given by $A_1 = 1$ and $A_2 = -1 - i$. The corresponding complex solution to system (6.1.2.9) has the form

$$y_1 = e^{(1+i)x}, \quad y_2 = (-1-i)e^{(1+i)x}.$$

Separating the real and imaginary parts, taking into account the formulas

$$\begin{aligned} e^{(1+i)x} &= e^x(\cos x + i\sin x) = e^x\cos x + ie^x\sin x, \\ (-1-i)e^{(1+i)x} &= -(1+i)e^x(\cos x + i\sin x) = e^x(\sin x - \cos x) - ie^x(\sin x + \cos x), \end{aligned}$$

and making linear combinations from them, one arrives at the general solution to the original system (6.1.2.9):

$$\begin{aligned} y_1 &= C_1e^x\cos x + C_2e^x\sin x, \\ y_2 &= C_1e^x(\sin x - \cos x) - C_2e^x(\sin x + \cos x). \end{aligned}$$

Remark 6.1. Systems of two homogeneous linear constant-coefficient second-order differential equations are treated in detail in Section 6.1.8.

6.1.3 Nonhomogeneous Systems of Linear First-Order Equations

1°. In general, a nonhomogeneous linear system of constant-coefficient first-order ordinary differential equations has the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(x), \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(x). \end{aligned} \tag{6.1.3.1}$$

For brevity, the conventional vector notation will also be used:

$$\mathbf{y}' = \mathbf{a}\mathbf{y} + \mathbf{f}(x),$$

where $\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_n(x))^{\mathrm{T}}$.

The general solution of this system is the sum of the general solution to the corresponding homogeneous system with $f_k(x) \equiv 0$ [see system (6.1.1.1)] and any particular solution of the nonhomogeneous system (6.1.3.1).

System (6.1.3.1) can also be reduced to a single nonhomogeneous linear constant-coefficient nth-order equation; see Section 7.1.3.

2°. Let $\mathbf{y}_m = (D_{m1}(x), D_{m1}(x), \dots, D_{mn}(x))^{\mathrm{T}}$ represent particular solutions to the homogeneous linear system of constant-coefficient first-order ordinary differential equations (6.1.1.1) that satisfy the special initial conditions

$$y_m(0) = 1, \quad y_k(0) = 0 \quad \text{if} \quad k \neq m; \quad m,\, k = 1,\, \dots,\, n.$$

Then the general solution to the nonhomogeneous system (6.1.3.1) is expressed as

$$y_k(x) = \sum_{m=1}^{n} \int_0^x f_m(t) D_{mk}(x-t)\,dt + \sum_{m=1}^{n} C_m D_{mk}(x), \qquad k = 1, \dots, n. \tag{6.1.3.2}$$

Alternatively, the general solution to the nonhomogeneous linear system of equations (6.1.3.1) can be obtained using the formulas from Section 6.2.2.

The solution of the Cauchy problem for the nonhomogeneous system (6.1.3.1) with arbitrary initial conditions,

$$y_1(0) = y_1^\circ, \quad y_2(0) = y_2^\circ, \quad \dots, \quad y_n(0) = y_n^\circ, \tag{6.1.3.3}$$

is determined by formulas (6.1.3.2) with $C_m = y_m^\circ$, $m = 1, \dots, n$.

6.1.4 Homogeneous Linear Systems of Higher-Order Differential Equations

An arbitrary system of homogeneous linear systems of constant-coefficient ordinary differential equations consists of n equations, each representing a linear combination of unknowns, y_k, and their derivatives, y_k', y_k'', $\dots$, $y_k^{(m_k)}$, $k = 1,\, 2,\, \dots,\, n$.

The general solution of such systems is a linear combination of particular solutions multiplied by arbitrary constants. In total, such a system has $m_1 + m_2 + \dots + m_n$ linearly independent particular solutions (the system is assumed to be consistent and nondegenerate, so that the constituent equations are linearly independent).

Particular solutions of the system are sought in the form (6.1.2.2). On substituting these expressions into the differential equations and dividing by $e^{\lambda x}$, one obtains a homogeneous linear algebraic system for coefficients A_1, A_2, $\dots$, A_n. For this system to have nontrivial solutions, the determinant of the system must vanish. This results in an algebraic equation for the exponent λ; in physics, this equation is called a *dispersion equation*. To different roots of the dispersion equation there correspond different particular solutions of the original system of equations. For simple real and complex-conjugate roots, the procedure of finding particular solutions is the same as in the case of a linear system of first-order equations (6.1.1.1).

Example 6.3. Consider the linear system of constant-coefficient second-order equations

$$\begin{aligned} y_1'' + y_2' + ay_2 &= 0, \\ y_2'' + y_1' + ay_1 &= 0. \end{aligned} \tag{6.1.4.1}$$

Particular solutions are sought in the form

$$y_1 = A_1 e^{\lambda x}, \quad y_2 = A_2 e^{\lambda x}. \tag{6.1.4.2}$$

Substituting (6.1.4.2) into (6.1.4.1) yields a homogeneous linear algebraic system for the coefficients A_1 and A_2:

$$\begin{aligned} \lambda^2 A_1 + (\lambda + a)A_2 &= 0, \\ (\lambda + a)A_1 + \lambda^2 A_2 &= 0. \end{aligned} \tag{6.1.4.3}$$

For this system to have nontrivial solutions, its determinant must vanish. This results in the dispersion equation

$$\begin{vmatrix} \lambda^2 & \lambda + a \\ \lambda + a & \lambda^2 \end{vmatrix} = \lambda^4 - (\lambda + a)^2 = 0.$$

Its roots are

$$\lambda_{1,2} = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + a}, \quad \lambda_{3,4} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - a}. \tag{6.1.4.4}$$

Let us confine ourselves to the simplest case of $-\frac{1}{4} < a < \frac{1}{4}$, where all roots of the dispersion equation are real and distinct. It follows from the system of algebraic equations (6.1.4.3) that $A_1 = \lambda + a$ and $A_2 = -\lambda^2$, where $\lambda = \lambda_n$. Substituting these values into (6.1.4.2) yields the particular solutions $y_{1n} = (\lambda_n + a)e^{\lambda_n x}$, $y_{2n} = -\lambda_n^2 e^{\lambda_n x}$ $(n = 1,\, 2,\, 3,\, 4)$. A linear combination of the particular solutions gives the general solution of system (6.1.4.1):

$$\begin{aligned} y_1 &= C_1(\lambda_1 + a)e^{\lambda_1 x} + C_2(\lambda_2 + a)e^{\lambda_2 x} + C_3(\lambda_3 + a)e^{\lambda_3 x} + C_4(\lambda_4 + a)e^{\lambda_4 x}, \\ y_2 &= -C_1\lambda_1^2 e^{\lambda_1 x} - C_2\lambda_2^2 e^{\lambda_2 x} - C_3\lambda_3^2 e^{\lambda_3 x} - C_4\lambda_4^2 e^{\lambda_4 x}, \end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, and the roots λ_n are determined by formulas (6.1.4.4).

Remark 6.2. Section 6.1.7 (see Item 2°) presents a method for the solution of systems of arbitrary homogeneous linear constant-coefficients ordinary differential equations using the Laplace transform.

6.1.5 Normal Coordinates and Natural Oscillations

Small undamped oscillations of mechanical or electrical systems are often described by a system of n linear constant-coefficient ordinary differential equations of the second order

$$\sum_{k=1}^{n} \bigl(b_{jk} y_k'' + a_{jk} y_k\bigr) = 0 \qquad (j = 1,\, 2,\, \dots,\, n). \tag{6.1.5.1}$$

Both matrices, (a_{jk}) and (b_{jk}), are symmetric and positive definite; in addition, they have the property that the characteristic equation, obtained by substituting a solution of the form (6.1.2.2) into (6.1.5.1), has $2n$ different nonzero pure imaginary roots: $\pm i\omega_1$, $\pm i\omega_2$, …, $\pm i\omega_n$.

System (6.1.5.1) can be simplified with so-called *normal coordinates* $\bar{y}_1$, $\bar{y}_2$, …, $\bar{y}_n$ using a linear transformation of the form

$$y_k = \sum_{m=1}^{n} c_{km} \bar{y}_m \qquad (k = 1,\, 2,\, \dots,\, n), \tag{6.1.5.2}$$

with the coefficients c_{km} chosen so as to reduce both matrices, (a_{jk}) and (b_{jk}), to a diagonal form simultaneously. As a result, system (6.1.5.1) becomes

$$\bar{y}''_m + \omega_m^2 \bar{y}_m = 0 \qquad (m = 1,\ 2,\ \ldots,\ n), \tag{6.1.5.3}$$

where all of the equations are isolated and independent of one another. The general solution of system (6.1.5.3) can be written as

$$\bar{y}_m = A_m \cos(\omega_m x) + B_m \sin(\omega_m x) \qquad (m = 1,\ 2,\ \ldots,\ n), \tag{6.1.5.4}$$

where A_m and B_m are arbitrary constants.

◆ *Very often, normal coordinates have a clear physical meaning. For details of the method for determining the coefficients c_{km} in the transformation (6.1.5.2), see, for example, the handbooks by Korn & Korn (2000) and Polyanin & Cheroutsan (2011).*

Example 6.4. Let us look at the system

$$\begin{aligned} y_1'' + \omega^2 y_1 + \sigma^2 (y_1 - y_2) &= 0, \\ y_2'' + \omega^2 y_2 - \sigma^2 (y_1 - y_2) &= 0. \end{aligned} \tag{6.1.5.5}$$

It is not difficult to show that

$$\bar{y}_1 = y_1 + y_2, \qquad \bar{y}_2 = y_1 - y_2$$

are normal coordinates for this system. In terms of the normal coordinates, the system becomes

$$\bar{y}_1'' + \omega^2 y_1 = 0, \qquad \bar{y}_2'' + (\omega^2 + 2\sigma^2) y_2 = 0. \tag{6.1.5.6}$$

Under the initial conditions

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_1'(0) = y_2'(0) = 0,$$

which are equivalent to

$$\bar{y}_1(0) = \bar{y}_2(0) = 1, \quad \bar{y}_1'(0) = \bar{y}_2'(0) = 0$$

in the normal coordinates, the solution of (6.1.5.6) is

$$\bar{y}_1 = \cos(\omega x), \qquad \bar{y}_2 = \cos\bigl(\sqrt{\omega^2 + 2\sigma^2}\, x\bigr).$$

Consequently, the solution of the original system (6.1.5.5) is expressed as

$$\begin{gathered} y_1 = \tfrac{1}{2}(\bar{y}_1 + \bar{y}_2) = \cos(px)\cos(qx), \qquad y_2 = \tfrac{1}{2}(\bar{y}_1 - \bar{y}_2) = \sin(px)\sin(qx), \\ p = \tfrac{1}{2}\bigl(\sqrt{\omega^2 + 2\sigma^2} + \omega\bigr), \qquad q = \tfrac{1}{2}\bigl(\sqrt{\omega^2 + 2\sigma^2} - \omega\bigr). \end{gathered}$$

6.1.6 Nonhomogeneous Higher-Order Linear Systems. D'Alembert's Method

Consider the system of two linear constant-coefficient mth-order differential equations

$$\begin{aligned} y_1^{(m)} &= a_{11} y_1 + a_{12} y_2 + f_1(x), \\ y_2^{(m)} &= a_{21} y_1 + a_{22} y_2 + f_2(x). \end{aligned} \tag{6.1.6.1}$$

Let us multiply the second equation of system (6.1.6.1) by k and add it termwise to the first equation to obtain, after rearrangement,

$$(y_1 + k y_2)^{(m)} = (a_{11} + k a_{21}) \left(y_1 + \frac{a_{12} + k a_{22}}{a_{11} + k a_{21}} y_2 \right) + f_1(x) + k f_2(x). \tag{6.1.6.2}$$

Let us take the constant k so that $\dfrac{a_{12}+ka_{22}}{a_{11}+ka_{21}}=k$, which results in a quadratic equation for k:

$$a_{21}k^2+(a_{11}-a_{22})k-a_{12}=0. \qquad (6.1.6.3)$$

In this case, (6.1.6.2) is a nonhomogeneous linear constant-coefficient equation for $z=y_1+ky_2$:

$$z^{(m)}=(a_{11}+ka_{21})z+f_1(x)+kf_2(x).$$

Integrating this equation yields

$$y_1+ky_2=C_1\varphi_1(x,k)+\cdots+C_m\varphi_m(x,k)+\psi(x,k).$$

It follows that if the roots of the quadratic equation (6.1.6.3) are distinct, we have two relations,

$$\begin{aligned}y_1+k_1y_2&=C_1\varphi_1(x,k_1)+\cdots+C_m\varphi_m(x,k_1)+\psi(x,k_1),\\ y_1+k_2y_2&=C_{m+1}\varphi_1(x,k_2)+\cdots+C_{2m}\varphi_m(x,k_2)+\psi(x,k_2),\end{aligned}$$

which represent a linear algebraic system of equations for the functions y_1 and y_2.

Remark 6.3. The above method for the solution of system (6.1.6.1) is known as *D Alembert's method*. The quantity $z=y_1+ky_2$ in the above reasoning gives an example of an integrable combination (see Section 7.2.1).

Remark 6.4. The more complicated system where y_1 and y_2 on the right-hand side are replaced by the derivatives of the same order, $y_1^{(n)}$ and $y_2^{(n)}$, can be treated likewise.

Remark 6.5. System (6.1.6.1) can be solved using the Laplace transform (see Section 6.1.7).

6.1.7 Usage of the Laplace Transform for Solving Linear Systems of Equations

1°. To solve the Cauchy problem for the nonhomogeneous linear system of differential equations (6.1.3.1) with the initial conditions (6.1.3.3), one can use the Laplace transform, based on the following formulas (for details, see Section 4.3):

$$\widetilde{y}_k(p)=\mathfrak{L}\{y_k(x)\},\quad \widetilde{f}_k(p)=\mathfrak{L}\{f_k(x)\},\quad \text{where}\quad \mathfrak{L}\{f(x)\}\equiv\int_0^\infty e^{-px}f(x)\,dx.$$

To this end, one should multiply each equation in (6.1.3.1) by e^{-px} and then integrate with respect to x from zero to infinity. In view of the differentiation rule $\mathfrak{L}\{y'_k(x)\}=p\widetilde{y}_k(p)-y_k(0)$ and the initial conditions (6.1.3.3), one arrives at a nonhomogeneous linear system of algebraic equations for the transforms $\widetilde{y}_k(p)$:

$$\begin{aligned}(a_{11}-p)\widetilde{y}_1+a_{12}\widetilde{y}_2+\cdots+a_{1n}\widetilde{y}_n&=-\widetilde{f}_1(p)-y_1^\circ,\\ a_{21}\widetilde{y}_1+(a_{22}-p)\widetilde{y}_2+\cdots+a_{2n}\widetilde{y}_n&=-\widetilde{f}_2(p)-y_2^\circ,\\ \cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\\ a_{n1}\widetilde{y}_1+a_{n2}\widetilde{y}_2+\cdots+(a_{nn}-p)\widetilde{y}_n&=-\widetilde{f}_n(p)-y_n^\circ.\end{aligned} \qquad (6.1.7.1)$$

The solution of this system is obtained by Kramer's rule and is given by

$$\widetilde{y}_k=\frac{\Delta_k(p)}{\Delta(p)};\qquad k=1,\dots,n, \qquad (6.1.7.2)$$

where $\Delta(p)$ is the determinant of the basic matrix of system (6.1.7.1), coinciding with the determinant in (6.1.2.1) with $\lambda = p$, and $\Delta_k(p)$ is the determinant of the matrix obtained from the basic matrix by replacing its kth column with the column of free terms of system (6.1.7.1).

On applying the Laplace inversion formula (see Section 4.3.1) to (6.1.7.2), one obtains a solution to the Cauchy problem (6.1.3.1), (6.1.3.3) in the form

$$y_k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Delta_k(p)}{\Delta(p)} e^{px}\,dp; \qquad k = 1, \dots, n.$$

The formulas from Section 4.3.2 and tables from Section S3.2 can be used to find the inverse Laplace transform of the function $\Delta_k(p)/\Delta(p)$.

2°. The Laplace transform is also suitable for the solution of systems of second- and higher-order ordinary differential equations with constant coefficients.

Example 6.5. Consider the Cauchy problem for the nonhomogeneous linear system of constant-coefficient second-order differential equations

$$\sum_{k=1}^{n} \bigl(a_{mk} y_k'' + b_{mk} y_k' + c_{mk} y_k\bigr) = f_m(x), \qquad m = 1,\, 2,\, \dots,\, n,$$

subject to the initial conditions

$$y_k(0) = \alpha_k, \quad y_k'(0) = \beta_k, \qquad k = 1,\, 2,\, \dots,\, n.$$

The Laplace transform reduces this problem to a linear system of algebraic equations for the transform $\widetilde{y}_k(p)$:

$$\sum_{k=1}^{n} \bigl(a_{mk} p^2 + b_{mk} p + c_{mk}\bigr)\widetilde{y}_k(p) = \widetilde{f}_m(p) + \sum_{k=1}^{n} \bigl[(a_{mk} p + b_{mk})\alpha_k + \beta_k\bigr], \qquad m = 1,\, 2,\, \dots,\, n.$$

The solution to this system can be obtained using Kramer's rule. By applying then the inverse Laplace transform to the resulting expressions of $\widetilde{y}_k(p)$, one obtains the solution to the Cauchy problem.

6.1.8 Classification of Equilibrium Points of Two-Dimensional Linear Systems

► Two linear constant-coefficient coupled equations. Characteristic equation.

Let us study the behavior of solutions near the equilibrium (also called stationary, steady-state, or fixed) point $x = y = 0$ for the system of two homogeneous linear constant-coefficient equations

$$\begin{aligned} x_t' &= a_{11} x + a_{12} y, \\ y_t' &= a_{21} x + a_{22} y. \end{aligned} \qquad (6.1.8.1)$$

By convention, for clearness and convenience of interpretation of the results, t will be used to designate the independent variable and will be treated as time. A solution $x = x(t)$, $y = y(t)$ of system (6.1.8.1) plotted in the plane x, y (the phase plane) is called a (phase) trajectory of the system.

A solution to system (6.1.8.1) will be sought in the form

$$x = k_1 e^{\lambda t}, \qquad y = k_2 e^{\lambda t}. \tag{6.1.8.2}$$

On substituting (6.1.8.2) into (6.1.8.1), one obtains the characteristic equation for the exponent λ:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{6.1.8.3}$$

The coefficients k_1 and k_2 are found as

$$k_1 = Ca_{12}, \qquad k_2 = C(\lambda - a_{11}), \tag{6.1.8.4}$$

where C is an arbitrary constant. To two different roots of the quadratic equation (6.1.8.3) there correspond two pairs of coefficients (6.1.8.4).

▶ Discriminant of the characteristic equation. Classification of equilibrium points.

Denote the discriminant of the quadratic equation (6.1.8.3) by

$$D = (a_{11} - a_{22})^2 + 4a_{12}a_{21}. \tag{6.1.8.5}$$

Three situations are possible.

1°. If $D > 0$, the roots of the characteristic equation (6.1.8.3) are real and distinct ($\lambda_1 \neq \lambda_2$):

$$\lambda_{1,2} = \tfrac{1}{2}(a_{11} + a_{22}) \pm \tfrac{1}{2}\sqrt{D}.$$

The general solution of system (6.1.8.1) is expressed as

$$\begin{aligned} x &= C_1 a_{12} e^{\lambda_1 t} + C_2 a_{12} e^{\lambda_2 t}, \\ y &= C_1(\lambda_1 - a_{11}) e^{\lambda_1 t} + C_2(\lambda_2 - a_{11}) e^{\lambda_2 t}, \end{aligned} \tag{6.1.8.6}$$

where C_1 and C_2 are arbitrary constants. For $C_1 = 0$, $C_2 \neq 0$ and $C_2 = 0$, $C_1 \neq 0$, the trajectories in the phase plane x, y are straight lines. Four cases are possible here.

1.1. Two negative real roots, $\lambda_1 < 0$ and $\lambda_2 < 0$. This corresponds to $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. The equilibrium point is asymptotically stable and all trajectories starting within a small neighborhood of the origin tend to the origin as $t \to \infty$. To $C_1 = 0$, $C_2 \neq 0$ and $C_2 = 0$, $C_1 \neq 0$ there correspond straight lines passing through the origin. Fig. 6.1*a* depicts the arrangement of the phase trajectories near an equilibrium point called a *stable node* (or a *sink*). The direction of motion along the trajectories with increasing t is shown by arrows.

1.2. $\lambda_1 > 0$ and $\lambda_2 > 0$. This corresponds to $a_{11} + a_{22} > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. The phase trajectories in the vicinity of the equilibrium point have the same pattern as in the preceding case; however, the trajectories go in the opposite direction, away from the equilibrium point; see Fig. 6.1*b*. An equilibrium point of this type is called an *unstable node* (or a *source*).

1.3. $\lambda_1 > 0$ and $\lambda_2 < 0$ (or $\lambda_1 < 0$ and $\lambda_2 > 0$). This corresponds to $a_{11}a_{22} - a_{12}a_{21} < 0$. In this case, the equilibrium point is also unstable, since the trajectory (6.1.8.6) with $C_2 = 0$ goes beyond a small neighborhood of the origin as t increases. If $C_1C_2 \neq 0$, then the

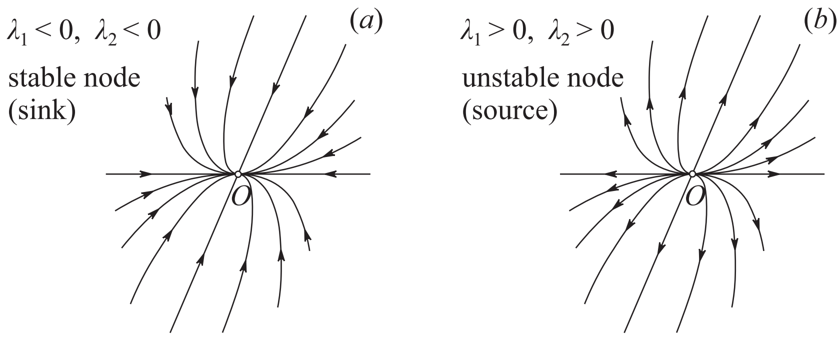

Figure 6.1: Phase trajectories of a system of differential equations near an equilibrium point of the node type.

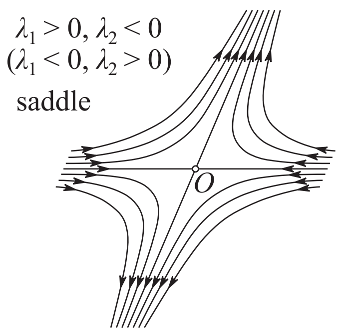

Figure 6.2: Phase trajectories of a system of differential equations near an equilibrium point of the saddle type.

trajectories leave the neighborhood of the origin as $t \to -\infty$ and $t \to \infty$. An equilibrium point of this type is called a *saddle* (or a *hyperbolic point*); see Fig. 6.2.

1.4. $\lambda_1 = 0$ and $\lambda_2 = a_{11} + a_{22} \neq 0$. This corresponds to $a_{11}a_{22} - a_{12}a_{21} = 0$. The general solution of system (6.1.8.1) is expressed as

$$x = C_1 a_{12} + C_2 a_{12} e^{(a_{11}+a_{22})t},$$
$$y = -C_1 a_{11} + C_2 a_{22} e^{(a_{11}+a_{22})t}, \tag{6.1.8.7}$$

where C_1 and C_2 are arbitrary constants. By eliminating time t from (6.1.8.7), one obtains a family of parallel lines defined by the equation $a_{22}x - a_{12}y = a_{12}(a_{11} + a_{22})C_1$. To $C_2 = 0$ in (6.1.8.7) there corresponds a one-parameter family of equilibrium points that lie on the straight line $a_{11}x + a_{12}y = 0$.

(i) If $\lambda_2 < 0$, then the trajectories approach the equilibrium point lying as $t \to \infty$; see Fig. 6.3. The equilibrium point $x = y = 0$ is stable (or neutrally stable)—there is no asymptotic stability.

(ii) If $\lambda_2 > 0$, the trajectories have the same pattern as in case (i), but they go, as $t \to \infty$, in the opposite direction, away from the equilibrium point. The point $x = y = 0$ is unstable.

2°. If $D < 0$, the characteristic equation (6.1.8.3) has complex-conjugate roots:

$$\lambda_{1,2} = \alpha \pm i\beta, \qquad \alpha = \tfrac{1}{2}(a_{11} + a_{22}), \quad \beta = \tfrac{1}{2}\sqrt{|D|}, \quad i^2 = -1.$$

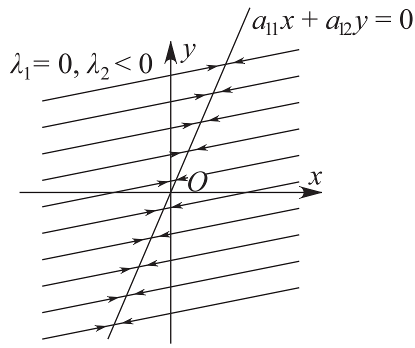

Figure 6.3: Phase trajectories of a system of differential equations near a set of equilibrium points located on a straight line.

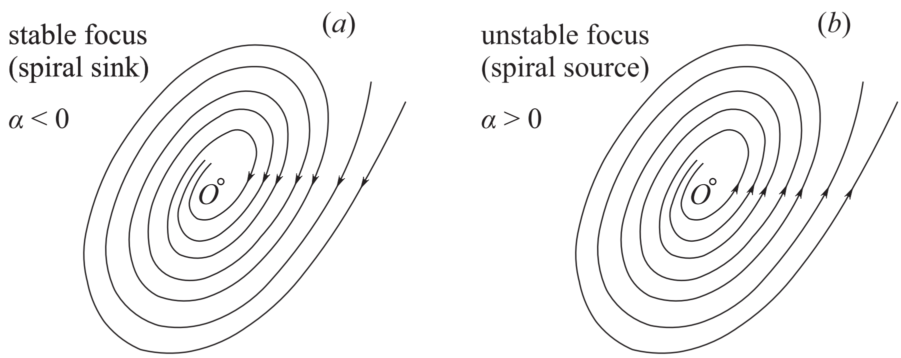

Figure 6.4: Phase trajectories of a system of differential equations near an equilibrium point of the focus type.

The general solution of system (6.1.8.1) has the form

$$x = e^{\alpha t}[C_1 \cos(\beta t) + C_2 \sin(\beta t)],$$
$$y = e^{\alpha t}[C_1^* \cos(\beta t) + C_2^* \sin(\beta t)], \qquad (6.1.8.8)$$

where C_1 and C_2 are arbitrary constants, and C_1^* and C_2^* are defined by linear combinations of C_1 and C_2. The following cases are possible.

2.1. For $\alpha < 0$, the trajectories in the phase plane are spirals asymptotically approaching the origin of coordinates (the equilibrium point) as $t \to \infty$; see Fig. 6.4*a*. Therefore the equilibrium point is asymptotically stable and is called a *stable focus* (also a *stable spiral point* or a *spiral sink*). A focus is characterized by the fact that the tangent to a trajectory changes its direction all the way to the equilibrium point.

2.2. For $\alpha > 0$, the phase trajectories are also spirals, but unlike the previous case they spiral away from the origin as $t \to \infty$; see Fig. 6.4*b*. Therefore such an equilibrium point is called an *unstable focus* (also an *unstable spiral point* or a *spiral source*).

2.3. At $\alpha = 0$, the phase trajectories are closed curves, containing the equilibrium point inside (see Fig. 6.5). Such an equilibrium point is called a *center*. A center is a stable equilibrium point. Note that there is no asymptotic stability in this case.

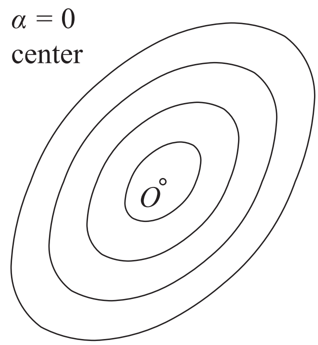

Figure 6.5: Phase trajectories of a system of differential equations near an equilibrium point of the center type.

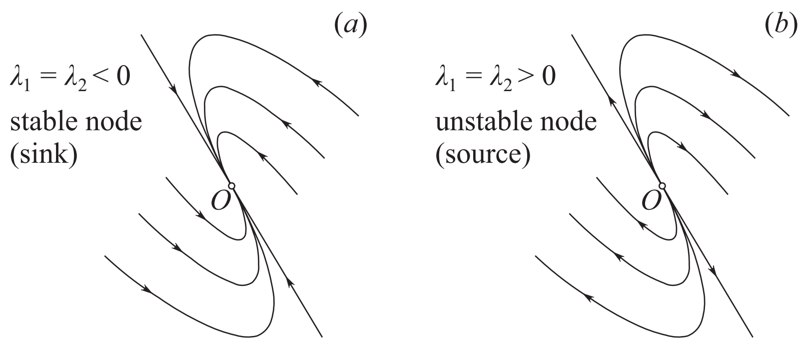

Figure 6.6: Phase trajectories of a system of differential equations near an equilibrium point of the node type in the case of a double root, $\lambda_1 = \lambda_2$.

3°. If $D = 0$, the characteristic equation (6.1.8.3) has a double real root, $\lambda_1 = \lambda_2 = \frac{1}{2}(a_{11} + a_{22})$. The following cases are possible.

3.1. If $\lambda_1 = \lambda_2 = \lambda < 0$, the general solution of system (6.1.8.1) has the form

$$x = a_{12}(C_1 + C_2 t)e^{\lambda t},$$
$$y = [(\lambda - a_{11})C_1 + C_2 + C_2(\lambda - a_{11})t]e^{\lambda t}, \tag{6.1.8.9}$$

where C_1 and C_2 are arbitrary constants.

Since there is a rapidly decaying factor, $e^{\lambda t}$, all trajectories tend to the equilibrium point as $t \to \infty$; see Fig. 6.6*a*. To $C_2 = 0$ there corresponds a straight line in the phase plane x, y. The equilibrium point is asymptotically stable and is called a *stable node* (a *sink*). Such a node is in intermediate position between a node from Item 1.1 and a focus from Item 2.1.

3.2. If $\lambda_1 = \lambda_2 = \lambda > 0$, the general solution of system (6.1.8.1) is determined by formulas (6.1.8.9). The phase trajectories are similar to those from Item 3.1, but they go in the opposite direction, as $t \to \infty$, rapidly away from the equilibrium point. Such an equilibrium point is called an *unstable node* (a *source*); see Fig. 6.6*b*.

3.3. If $\lambda_1 = \lambda_2 = 0$, which corresponds to

$$a_{11} + a_{22} = 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} = 0$$

simultaneously, the general solution of system (6.1.8.1) is obtained by substituting $\lambda = 0$ into (6.1.8.9) and has the form

$$x = a_{12}C_1 + a_{12}C_2 t,$$
$$y = C_2 - a_{11}C_1 - a_{11}C_2 t.$$

For $a_{12} \neq 0$ all trajectories are parallel straight lines. As $t \to \pm\infty$, the trajectories go away from the origin. The equilibrium point is unstable.

For clearness, the classification results for equilibrium points of systems of two linear constant-coefficient differential equations (6.1.8.1) are summarized in Table 6.1.

TABLE 6.1
Classification of equilibrium points for systems of constant-coefficient equations (6.1.8.1); the symbols $\circ$ and $*$ indicate stable and unstable equilibrium points, respectively, where not clearly stated

Discriminant, formula (6.1.8.5)	Roots of quadratic equation (6.1.8.3), λ_1 and λ_2	Conditions for coefficients a_{ij} of homogeneous liner ordinary differential equations (6.1.8.1)	Type of equilibrium points or shape of phase trajectories
$D>0$	$\lambda_1<0$, $\lambda_2<0$, $\lambda_1\neq\lambda_2$ $\lambda_1>0$, $\lambda_2>0$, $\lambda_1\neq\lambda_2$ roots have unlike signs $\lambda_1=0$, $\lambda_2<0$ $\lambda_1=0$, $\lambda_2>0$	$a_{11}+a_{22}<0$, $a_{11}a_{22}-a_{12}a_{21}>0$ $a_{11}+a_{22}>0$, $a_{11}a_{22}-a_{12}a_{21}>0$ $a_{11}a_{22}-a_{12}a_{21}<0$ $a_{11}+a_{22}<0$, $a_{11}a_{22}-a_{12}a_{21}=0$ $a_{11}+a_{22}>0$, $a_{11}a_{22}-a_{12}a_{21}=0$	stable node unstable node saddle* parallel lines$^\circ$ parallel lines*
$D<0$	$\lambda_{1,2}=\alpha\pm i\beta$, $\alpha>0$ $\lambda_{1,2}=\alpha\pm i\beta$, $\alpha<0$ $\lambda_{1,2}=\pm i\beta$, imaginary roots	$a_{11}+a_{22}<0$, $(a_{11}-a_{22})^2+4a_{12}a_{21}<0$ $a_{11}+a_{22}>0$, $(a_{11}-a_{22})^2+4a_{12}a_{21}<0$ $a_{11}+a_{22}=0$, $a_{11}a_{22}-a_{12}a_{21}>0$	stable focus unstable focus center$^\circ$
$D=0$	$\lambda_1=\lambda_2<0$ $\lambda_1=\lambda_2>0$ $\lambda_1=\lambda_2=0$	$a_{11}+a_{22}<0$, $(a_{11}-a_{22})^2+4a_{12}a_{21}=0$ $a_{11}+a_{22}>0$, $(a_{11}-a_{22})^2+4a_{12}a_{21}=0$ $a_{11}+a_{22}=0$, $a_{11}a_{22}-a_{12}a_{21}=0$	stable node unstable node saddle* parallel lines*

Remark 6.6. For general definitions of a stable and an unstable equilibrium point, see Section 7.3.1.

⊙ *Literature for Section 6.1*: G. M. Murphy (1960), V. A. Ditkin and A. P. Prudnikov (1965), G. A. Korn and T. M. Korn (2000), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and A. V. Manzhirov (2007).

6.2 Systems of Linear Variable-Coefficient Equations

6.2.1 Homogeneous Systems of Linear First-Order Equations

► **Superposition principle for a homogeneous system.**

In general, a homogeneous linear system of variable-coefficient first-order ordinary differential equations has the form

$$\begin{aligned} y_1' &= f_{11}(x)y_1 + f_{12}(x)y_2 + \cdots + f_{1n}(x)y_n,\\ y_2' &= f_{21}(x)y_1 + f_{22}(x)y_2 + \cdots + f_{2n}(x)y_n,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ y_n' &= f_{n1}(x)y_1 + f_{n2}(x)y_2 + \cdots + f_{nn}(x)y_n, \end{aligned} \qquad (6.2.1.1)$$

where the prime denotes a derivative with respect to x. It is assumed further on that the functions $f_{ij}(x)$ are continuous of an interval $a \le x \le b$ (intervals are allowed with $a = -\infty$ or/and $b = +\infty$).

Any homogeneous linear system of the form (6.2.1.1) has the trivial particular solution $y_1 = y_2 = \dots = y_n = 0$.

Superposition principle for a homogeneous system: any linear combination of particular solutions to system (6.2.1.1) is also a solution to this system.

► Wronskian determinant. General solution of the homogeneous system.

Let

$$\mathbf{y}_k = (y_{k1}, y_{k2}, \dots, y_{kn})^{\mathrm{T}}, \qquad y_{km} = y_{km}(x); \qquad k,\ m = 1,\ 2,\ \dots,\ n \tag{6.2.1.2}$$

be nontrivial particular solutions of the homogeneous system of equations (6.2.1.1). Solutions (6.2.1.2) are linearly independent if the *Wronskian determinant* is nonzero:

$$W(x) \equiv \begin{vmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{vmatrix} \neq 0. \tag{6.2.1.3}$$

If condition (6.2.1.3) is satisfied, the general solution of the homogeneous system (6.2.1.1) is expressed as

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + \dots + C_n\mathbf{y}_n, \tag{6.2.1.4}$$

where C_1, C_2, …, C_n are arbitrary constants. The vector functions $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ in (6.2.1.4) are called *fundamental solutions* of system (6.2.1.1).

► Liouville formula.

Suppose condition (6.2.1.3) is met. Then the *Liouville formula*

$$W(x) = W(x_0)\exp\left[\int_{x_0}^{x}\left(\sum_{s=1}^{n} f_{ss}(t)\right)dt\right]$$

holds.

COROLLARY. *Particular solutions (6.2.1.2) are linearly independent on the interval $[a, b]$ if and only if there exists a point $x_0 \in [a, b]$ such that the Wronskian determinant is nonzero at x_0: $W(x_0) \neq 0$.*

► Reduction of the number of unknowns.

Suppose a nontrivial particular solution of system (6.2.1.1),

$$\mathbf{y}_1 = (u_1, u_2, \dots, u_n)^{\mathrm{T}}, \qquad u_m = u_m(x), \qquad m = 1,\ 2,\ \dots,\ n,$$

is known. Then the number of unknowns can be reduced by one. To this end, one considers the auxiliary homogeneous linear system of $n-1$ equations

$$z_k' = \sum_{q=2}^{n} \left[f_{kq}(x) - \frac{u_k(x)}{u_1(x)} f_{1q}(x) \right] z_q, \qquad k = 2, \ldots, n. \tag{6.2.1.5}$$

Let

$$\mathbf{z}_p = (z_{p2}, z_{p3}, \ldots, z_{pn})^{\mathrm{T}}, \qquad z_{mk} = z_{mk}(x); \qquad p = 2, \ldots, n,$$

be a fundamental system of solutions to system (6.2.1.5) and let

$$\mathbf{Z}_p = (0, z_{p2}, z_{p3}, \ldots, z_{pn})^{\mathrm{T}}, \qquad F_p(x) = \int \left(\frac{1}{u_1(x)} \sum_{s=2}^{n} f_{1s}(x) z_{ps}(x) \right) dx, \qquad p = 2, \ldots, n,$$

the vector $\mathbf{Z}_p$ having an additional component compared with $\mathbf{z}_p$. Then the vector functions $\mathbf{y}_p = F_p(x)\mathbf{y}_1 + \mathbf{z}_p$ ($p = 2, \ldots, n$) together with $\mathbf{y}_1$ form a fundamental system of solutions to the initial homogeneous system of equations (6.2.1.1).

6.2.2 Nonhomogeneous Systems of Linear First-Order Equations

► Nonhomogeneous linear systems. Existence and uniqueness theorem.

In general, a nonhomogeneous linear system of variable-coefficient first-order differential equations has the form

$$\begin{aligned} y_1' &= f_{11}(x)y_1 + f_{12}(x)y_2 + \cdots + f_{1n}(x)y_n + g_1(x), \\ y_2' &= f_{21}(x)y_1 + f_{22}(x)y_2 + \cdots + f_{2n}(x)y_n + g_2(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' &= f_{n1}(x)y_1 + f_{n2}(x)y_2 + \cdots + f_{nn}(x)y_n + g_n(x). \end{aligned} \tag{6.2.2.1}$$

Alternatively, the system can be written in the short vector-matrix notation as

$$\mathbf{y}' = \mathbf{f}(x)\mathbf{y} + \mathbf{g}(x),$$

with $\mathbf{f}(x) = \bigl(f_{ij}(x)\bigr)$ being the matrix of equation coefficients and $\mathbf{g}(x) = \bigl(g_1(x), g_2(x), \ldots, g_n(x)\bigr)^{\mathrm{T}}$ being the vector function defining the nonhomogeneous part of the equations.

EXISTENCE AND UNIQUENESS THEOREM. *Let the functions $f_{ij}(x)$ and $g_i(x)$ be continuous on an interval $a < x < b$. Then, for any set of values x°, y_1°, …, y_n°, where $a < x^\circ < b$, there exists a unique solution $y_1 = y_1(x)$, …, $y_n = y_n(x)$ satisfying the initial conditions*

$$y_1(x^\circ) = y_1^\circ, \quad \ldots, \quad y_n(x^\circ) = y_n^\circ,$$

and this solution is defined on the whole interval $a < x < b$.

► General solution.

Let

$$\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)^{\mathrm{T}}, \qquad \bar{y}_k = \bar{y}_k(x); \qquad k = 1,\ 2,\ \ldots,\ n,$$

be a particular solution to the nonhomogeneous system of equations (6.2.2.1). The general solution of this system is the sum of the general solution of the corresponding homogeneous system (6.2.1.1), which corresponds to $g_k(x) \equiv 0$ in (6.2.2.1), and any particular solution of the nonhomogeneous system (6.2.2.1), or

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + \cdots + C_n\mathbf{y}_n + \bar{\mathbf{y}}, \qquad (6.2.2.2)$$

where $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent solutions of the homogeneous system (6.2.1.1).

► A particular solution.

Given a fundamental system of solutions $y_{km}(x)$ (6.2.1.2) of the homogeneous system (6.2.1.1), a particular solution of the nonhomogeneous system (6.2.2.1) is found as

$$\bar{y}_k = \sum_{m=1}^{n} y_{mk}(x) \int \frac{W_m(x)}{W(x)}\,dx, \qquad k = 1,\ 2,\ \ldots,\ n,$$

where $W_m(x)$ is the determinant obtained by replacing the mth row in the Wronskian determinant (6.2.1.3) by the row of free terms, $g_1(x),\ g_2(x),\ \ldots,\ g_n(x)$, of equation (6.2.2.1). The general solution of the nonhomogeneous system (6.2.2.1) is given by (6.2.2.2).

► Superposition principle for a nonhomogeneous system.

A particular solution, $\mathbf{y} = \bar{\mathbf{y}}$, of the nonhomogeneous system of linear differential equations,

$$\mathbf{y}' = \mathbf{f}(x)\mathbf{y} + \sum_{k=1}^{m} \mathbf{g}_k(x),$$

is given by the sum

$$\mathbf{y} = \sum_{k=1}^{m} \mathbf{y}_k,$$

where the $\mathbf{y}_k$ are particular solutions of m (simpler) systems of equations

$$\mathbf{y}_k' = \mathbf{f}(x)\mathbf{y}_k + \mathbf{g}_k(x), \qquad k = 1,\ 2,\ \ldots,\ m,$$

corresponding to individual nonhomogeneous terms of the original system.

6.2.3 Euler System of Ordinary Differential Equations

► Euler system of ODEs. Reduction to a constant-coefficient linear system.

A homogeneous Euler system is a homogeneous linear system of ordinary differential equations composed by linear combinations of the following terms:

$$y_k, \quad xy_k', \quad x^2y_k'', \quad \ldots, \quad x^{m_k}y_k^{(m_k)}; \qquad k = 1,\ 2,\ \ldots,\ n.$$

Such a system is invariant under scaling in the independent variable (i.e., it preserves its form under the change of variable $x \to \alpha x$, where α is any nonzero number). A nonhomogeneous Euler system contains additional terms, given functions.

The substitution $x = be^t$ ($b \neq 0$) brings an Euler system, both homogeneous and nonhomogeneous, to a constant-coefficient linear system of equations.

Example 6.6. In general, a nonhomogeneous Euler system of second-order equations has the form

$$\sum_{k=1}^{n}\left(a_{mk}x^2\frac{d^2y_k}{dx^2}+b_{mk}x\frac{dy_k}{dx}+c_{mk}y_k\right)=f_m(x),\qquad m=1,\,2,\,\dots,\,n. \tag{6.2.3.1}$$

The substitutions $x=\pm e^t$ bring this system to a constant-coefficient linear system,

$$\sum_{k=1}^{n}\left[a_{mk}\frac{d^2y_k}{dt^2}+(b_{mk}-a_{mk})\frac{dy_k}{dt}+c_{mk}y_k\right]=f_m(\pm e^t),\qquad m=1,\,2,\,\dots,\,n,$$

which can be solved using, for example, the Laplace transform (see Example 6.5 from Section 6.1.7).

► Particular solutions.

Particular solutions to a homogeneous Euler system (for system (6.2.3.1), corresponding to $f_m(x)\equiv 0$) are sought in the form of power functions:

$$y_1=A_1x^\sigma,\quad y_2=A_2x^\sigma,\quad \dots,\quad y_n=A_nx^\sigma, \tag{6.2.3.2}$$

where the coefficients A_1, A_2, …, A_n are determined by solving the associated homogeneous system of algebraic equations obtained by substituting expressions (6.2.3.2) into the differential equations of the system in question and dividing by x^σ. Since the system is homogeneous, for it to have nontrivial solutions, its determinant must vanish. This results in a dispersion equation for the exponent σ.

⊙ *Literature for Section 6.2*: G. M. Murphy (1960), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 7

Methods for Nonlinear Systems of ODEs

7.1 Solutions and First Integrals. Uniqueness and Existence Theorems

7.1.1 Systems Solved for the Derivative. A Solution and the General Solution

We will be dealing with a system of first-order ordinary differential equations solved for the derivatives

$$y'_k = f_k(x, y_1, \dots, y_n), \qquad k = 1, \dots, n. \tag{7.1.1.1}$$

Throughout the current chapter, the prime denotes a derivative with respect to the independent variable x (unless otherwise stated).

A set of numbers $x, y_1, \dots, y_n$ is convenient to treat as a point in the $(n+1)$-dimensional space.

For brevity, system (7.1.1.1) is conventionally written in vector form:

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}),$$

where $\mathbf{y}$ and $\mathbf{f}$ are the vectors defined as $\mathbf{y} = (y_1, \dots, y_n)^T$ and $\mathbf{f} = (f_1, \dots, f_n)^T$.

A *solution* (also an *integral* or an *integral curve*) *of a system of differential equations* (7.1.1.1) is a set of functions $y_1 = y_1(x), \dots, y_n = y_n(x)$ such that, when substituted into all equations (7.1.1.1), they turn them into identities. The *general solution of a system of differential equations* is the set of all its solutions. In the general case, the general solution of system (7.1.1.1) depends on n arbitrary constants.

7.1.2 Existence and Uniqueness Theorems

EXISTENCE THEOREM (PEANO). *Let the functions $f_k(x, y_1, \dots, y_n)$ ($k = 1, \dots, n$) be continuous in a domain G of the $(n+1)$-dimensional space of the variables $x, y_1, \dots, y_n$. Then there is at least one integral curve passing through every point $M(x^\circ, y_1^\circ, \dots, y_n^\circ)$ in G. Each of such curves can be extended on both ends up to the boundary of any closed domain completely belonging to G and containing the point M inside.*

Remark 7.1. If there is more than one integral curve passing through the point M, there are infinitely many integral curves passing through M.

UNIQUENESS THEOREM. *There is a unique integral curve passing through the point* $M(x^\circ, y_1^\circ, \dots, y_n^\circ)$ *if the functions* f_k *have partial derivatives with respect to all* y_m*, continuous in* $x, y_1, \dots, y_n$ *in the domain* G*, or if each function* f_k *in* G *satisfies the Lipschitz condition:*

$$|f_k(x, \bar{y}_1, \dots, \bar{y}_n) - f_k(x, y_1, \dots, y_n)| \le A \sum_{m=1}^{n} |\bar{y}_m - y_m|,$$

where A *is some positive number.*

7.1.3 Reduction of Systems of Equations to a Single Equation or to an Autonomous System of Equations

► **Reduction of systems of equations to a single equation.**

Suppose the right-hand sides of equations (7.1.1.1) are n times differentiable in all variables. Then system (7.1.1.1) can be reduced to a single nth-order equation. Indeed, using the chain rule, let us differentiate the first equation of system (7.1.1.1) with respect to x to get

$$y_1'' = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial y_1} y_1' + \dots + \frac{\partial f_1}{\partial y_n} y_n'. \qquad (7.1.3.1)$$

Then change the first derivatives y_k' in (7.1.3.1) to $f_k(x, y_1, \dots, y_n)$ [the right-hand sides of equations (7.1.1.1)] to obtain

$$y_1'' = F_2(x, y_1, \dots, y_n), \qquad (7.1.3.2)$$

where $F_2(x, y_1, \dots, y_n) \equiv \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_1}{\partial y_1} f_1 + \dots + \dfrac{\partial f_1}{\partial y_n} f_n$. Now differentiate equation (7.1.3.2) with respect to x and replace the first derivatives y_k' on the right-hand side of the resulting equation by f_k. As a result, we obtain

$$y_1''' = F_3(x, y_1, \dots, y_n),$$

where $F_3(x, y_1, \dots, y_n) \equiv \dfrac{\partial F_2}{\partial x_1} + \dfrac{\partial F_2}{\partial y_1} f_1 + \dots + \dfrac{\partial F_2}{\partial y_n} f_n$. Repeating this procedure as many times as required, one arrives at the following system of equations:

$$\begin{aligned} y_1' &= F_1(x, y_1, \dots, y_n), \\ y_1'' &= F_2(x, y_1, \dots, y_n), \\ &\dots\dots\dots\dots\dots\dots \\ y_1^{(n)} &= F_n(x, y_1, \dots, y_n), \end{aligned}$$

where

$$\begin{aligned} F_1(x, y_1, \dots, y_n) &\equiv f_1(x, y_1, \dots, y_n), \\ F_{k+1}(x, y_1, \dots, y_n) &\equiv \frac{\partial F_k}{\partial x_1} + \frac{\partial F_k}{\partial y_1} f_1 + \dots + \frac{\partial F_k}{\partial y_n} f_n. \end{aligned}$$

Expressing $y_2, y_3, \dots, y_n$ from the $n-1$ first equations of this system in terms of x, y_1, y_1', $\dots$, $y_1^{(n-1)}$ and then substituting the resulting expressions into the last equation of system (7.1.1.1), one finally arrives at an nth-order equation:

$$y_1^{(n)} = \Phi(x, y_1, y_1', \dots, y_1^{(n-1)}). \tag{7.1.3.3}$$

Remark 7.2. If (7.1.1.1) is a linear system of first-order differential equations, then (7.1.3.3) is a linear nth-order equation.

Remark 7.3. Any equation of the form (7.1.3.3) can be reduced to a system on n first-order equations (see the end of Section 5.1.2).

► **Reduction of the nonautonomous system of equations to an autonomous system of equations.**

In general, the nonautonomous system (7.1.1.1), consisting of n equations, can be reduced to the autonomous system

$$x'_\xi = 1, \qquad (y_k)'_\xi = f_k(x, y_1, \dots, y_n), \qquad k = 1, \dots, n, \tag{7.1.3.4}$$

consisting of $n+1$ equations.

7.1.4 First Integrals. Using Them to Reduce System Dimension

1°. A relation of the form

$$\Psi(x, y_1, \dots, y_n) = C, \tag{7.1.4.1}$$

where C is an arbitrary constant, is called a *first integral* of system (7.1.1.1) if its left-hand side Φ, generally not identically constant, is turned into a constant by any particular solution, $y_1, \dots, y_n$, of system (7.1.1.1). In the sequel, we consider only continuously differentiable functions $\Psi(x, y_1, \dots, y_n)$ in a given domain of variation of its arguments.

THEOREM. *An expression of the form (7.1.4.1) is a first integral of system (7.1.1.1) if and only if the function $\Psi = \Psi(x, y_1, \dots, y_n)$ satisfies the relation*

$$\frac{\partial \Psi}{\partial x} + \sum_{k=1}^{n} \frac{\partial \Psi}{\partial y_k} f_k(x, y_1, \dots, y_n) = 0.$$

This relation may be treated as a first-order partial differential equation for Ψ.

Different first integrals of system (7.1.1.1) are called *independent* if the Jacobian of their left-hand sides is nonzero.

System (7.1.1.1) admits n independent first integrals if the conditions of the uniqueness theorem from Section 7.1.2 are met.

2°. Given a first integral (7.1.4.1) of system (7.1.1.1), it may be treated as an implicit specification of one of the unknowns. Solving (7.1.4.1), for example, for y_n yields $y_n = G(x, y_1, \dots, y_{n-1})$. Substituting this expression into the first $n-1$ equations of system (7.1.1.1), one obtains a system in $n-1$ variables with one arbitrary constant.

Likewise, given m independent first integrals of system (7.1.1.1),

$$\Psi_k(x, y_1, \dots, y_n) = C_k, \quad k = 1, \dots, m \quad (m < n),$$

the system may be reduced to a system of $n-m$ first-order equations in $n-m$ unknowns.

7.2 Integrable Combinations. Autonomous Systems of Equations

7.2.1 Integrable Combinations

► Systems of first-order ordinary differential equations.

In some cases, first integrals of systems of differential equations may be obtained by finding *integrable combinations*. An integrable combination is a differential equation that is easy to integrate and is a consequence of the equations of the system under consideration. Most commonly, an integrable combination is an equation of the form

$$d\,\Psi(x, y_1, \dots, y_n) = 0 \tag{7.2.1.1}$$

or an equation reducible by a change of variables to one of the integrable types of equations in one unknown.

Example 7.1. Consider the nonlinear system

$$ay_1' = (b-c)y_2y_3, \quad by_2' = (c-a)y_1y_3, \quad cy_3' = (a-b)y_1y_2, \tag{7.2.1.2}$$

where a, b, and c are some constants. Such systems arise in the theory of motion of a rigid body.

Let us multiply the first equation by y_1, the second by y_2, and the third by y_3 and add together to obtain

$$ay_1y_1' + by_2y_2' + cy_3y_3' = 0 \quad \Longrightarrow \quad d(ay_1^2 + by_2^2 + cy_3^2) = 0.$$

Integrating yields a first integral:

$$ay_1^2 + by_2^2 + cy_3^2 = C_1. \tag{7.2.1.3}$$

Now multiply the first equation of the system by ay_1, the second by by_2, and the third by cy_3 and add together to obtain

$$a^2y_1y_1' + b^2y_2y_2' + c^2y_3y_3' = 0 \quad \Longrightarrow \quad d(a^2y_1^2 + b^2y_2^2 + c^2y_3^2) = 0.$$

Integrating yields another first integral:

$$a^2y_1^2 + b^2y_2^2 + c^2y_3^2 = C_2. \tag{7.2.1.4}$$

If the case $a = b = c$, where system (7.2.1.2) can be integrated directly, does not take place, the above two first integrals (7.2.1.3) and (7.2.1.4) are independent. Hence, using them, one can express y_2 and y_3 in terms of y_1 and then substitute the resulting expressions into the first equation of system (7.2.1.2). As a result, one arrives at a single separable first-order differential equation for y_1.

In this example, the integrable combinations have the form (7.2.1.1).

Example 7.2. A specific example of finding an integrable combination reducible with a change of variables to a simpler, integrable linear equation in one unknown can be found in Section 6.1.6.

► Systems of second-order ordinary differential equations.

In relatively few cases, integrals for systems of second-order ordinary differential equations can be found. Let us look at a few examples.

Example 7.3. Consider the *Ermakov system*

$$y''_{xx} + a(x)y = y^{-3}f(z/y), \tag{7.2.1.5}$$
$$z''_{xx} + a(x)z = z^{-3}f(y/z), \tag{7.2.1.6}$$

where $a(x)$, $f(\xi)$, and $g(\eta)$ are arbitrary functions.

Multiplying (7.2.1.5) by z and (7.2.1.6) by $-y$, adding the results together, and using the idenity $zy''_{xx} - yz''_{xx} = (zy'_x - yz'_x)'_x$, we obtain

$$d(zy'_x - yz'_x) = [zy^{-3}f(z/y) - yz^{-3}g(y/x)]\,dx.$$

Multiplying this relation by $(zy'_x - yz'_x)$ and integrating with respect to x, we find that

$$\frac{1}{2}(zy'_x - yz'_x)^2 = \int (zy'_x - yz'_x)zy^{-3}f(z/y)\,dx - \int (zy'_x - yz'_x)yz^{-3}g(y/x)\,dx + C,$$

where C is an arbitrary constant. Using the change of variable $\xi = z/y$ in the first integral and $\eta = y/z$ in the second integral, we arrive at the conservation law

$$\frac{1}{2}(zy'_x - yz'_x)^2 = \int^{z/y} \xi f(\xi)\,d\xi - \int^{y/z} \eta g(\eta)\,d\eta + C,$$

which is independent of $a(x)$.

Remark 7.4. System (7.2.1.5)–(7.2.1.6) admits a class of exact solutions of the form

$$y = y(x), \qquad z = ky(x),$$

where k is a root of the algebraic (or transcendental) equation $f(k) = k^2 g(1/k)$ (to distinct roots there correspond different solutions) and $y = y(x)$ is a solution to the Ermakov (Yermakov) equation $y''_{xx} + a(x)y = f(k)y^{-3}$ (its general solution is expressed in terms of the solution to the truncated linear equation with $f \equiv 0$, see Eq. 14.9.1.2).

7.2.2 Autonomous Systems and Their Reduction to Systems of Lower Dimension

1°. A system of equations is called *autonomous* if the right-hand sides of the equations do not depend explicitly on x. In general, such systems have the form

$$y'_k = f_k(y_1, \dots, y_n), \qquad k = 1, \dots, n. \tag{7.2.2.1}$$

If $\mathbf{y}(x)$ is a solution of the autonomous system (7.2.2.1), then the function $\mathbf{y}(x + C)$, where C is an arbitrary constant, is also a solution of this system.

A point $\mathbf{y}^\circ = (y_1^\circ, \dots, y_n^\circ)$ is called an *equilibrium point* (or a *stationary point*) of the autonomous system (7.2.2.1) if

$$f_k(y_1^\circ, \dots, y_n^\circ) = 0, \qquad k = 1, \dots, n.$$

To an equilibrium point there corresponds a special, simplest particular solution when all unknowns are constant:

$$y_1 = y_1^\circ, \quad \dots, \quad y_n = y_n^\circ, \qquad k = 1, \dots, n.$$

2°. Any n-dimensional autonomous system of the form (7.2.2.1) can be reduced to an $(n-1)$-dimensional system of equations independent of x. To this end, one should select one of the equations and divide the other $n - 1$ equations of the system by it.

Example 7.4. The autonomous system of two first-order equations

$$y'_x = f_1(y, z), \qquad z'_x = f_2(y, z) \tag{7.2.2.2}$$

is reduced by dividing the first equation by the second to a single equation for $y = y(z)$:

$$y'_z = \frac{f_1(y, z)}{f_2(y, z)}. \tag{7.2.2.3}$$

If the general solution of equation (7.2.2.3) is obtained in the form

$$y = \varphi(z, C_1), \tag{7.2.2.4}$$

then $z = z(x)$ is found in implicit form from the second equation in (7.2.2.2) by quadrature:

$$\int \frac{dz}{f_2(\varphi(z, C_1), z)} = x + C_2. \tag{7.2.2.5}$$

Formulas (7.2.2.4)–(7.2.2.5) determine the general solution of system (7.2.2.2) with two arbitrary constants, C_1 and C_2.

Remark 7.5. The dependent variables y and z in the autonomous system (7.2.2.2) are often called *phase variables*; the plane y, z they form is called a *phase plane*, which serves to display integral curves of equation (7.2.2.3).

⊙ *Literature for Section 7.2*: E. Kamke (1977), J. R. Ray and J. L. Reid (1979), A. D. Polyanin and A. V. Manzhirov (2007).

7.3 Elements of Stability Theory

7.3.1 Lyapunov Stability. Asymptotic Stability. Unstable Solutions

1°. In many applications, time t plays the role of the independent variable, and the associated system of differential equations is conventionally written in the following notation:

$$x'_k = f_k(t, x_1, \dots, x_n), \qquad k = 1, \dots, n. \tag{7.3.1.1}$$

Here the $x_k = x_k(t)$ are unknown functions that may be treated as coordinates of a moving point in an n-dimensional space.

Let us supply system (7.3.1.1) with initial conditions

$$x_k = x_k^\circ \quad \text{at} \quad t = t^\circ \qquad (k = 1, \dots, n). \tag{7.3.1.2}$$

Denote by

$$x_k = \varphi_k(t; t^\circ, x_1^\circ, \dots, x_n^\circ), \qquad k = 1, \dots, n, \tag{7.3.1.3}$$

the solution of system (7.3.1.1) with the initial conditions (7.3.1.2).

A solution (7.3.1.3) of system (7.3.1.1) is called *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$|x_k^\circ - \widetilde{x}_k^\circ| < \delta, \qquad k = 1, \dots, n, \tag{7.3.1.4}$$

then the following inequalities hold for $t^\circ \le t < \infty$:

$$\bigl|\varphi_k(t; t^\circ, x_1^\circ, \dots, x_n^\circ) - \varphi_k(t; t^\circ, \widetilde{x}_1^\circ, \dots, \widetilde{x}_n^\circ)\bigr| < \varepsilon, \qquad k = 1, \dots, n.$$

Any solution which is not stable is called *unstable*. Solution (7.3.1.3) is called unperturbed and the solution $\varphi_k(t; t^\circ, \widetilde{x}_1^\circ, \dots, \widetilde{x}_n^\circ)$ is called perturbed. Geometrically, Lyapunov stability means that the trajectory of the perturbed solution stays at all times $t \ge t^\circ$ within a small neighborhood of the associated unperturbed solution.

2°. A solution (7.3.1.3) of system (7.3.1.1) is called *asymptotically stable* if it is Lyapunov stable and, in addition, with inequalities (7.3.1.4) met, satisfies the conditions

$$\lim_{t\to\infty}\bigl|\varphi_k(t;t^\circ,x_1^\circ,\dots,x_n^\circ)-\varphi_k(t;t^\circ,\widetilde{x}_1^\circ,\dots,\widetilde{x}_n^\circ)\bigr|=0,\qquad k=1,\,\dots,\,n. \tag{7.3.1.5}$$

3°. In stability analysis, it is normally assumed, without loss of generality, that $t^\circ=x_1^\circ=\dots=x_n^\circ=0$ (this can be achieved by shifting each of the variables by a constant value). Further, with the changes of variables

$$z_k=x_k-\varphi_k(t;t^\circ,x_1^\circ,\dots,x_n^\circ)\qquad (k=1,\,\dots,\,n),$$

the stability analysis of any solution is reduced to that of the zero solution $z_1=\dots=z_n=0$.

7.3.2 Theorems of Stability and Instability by First Approximation

► Statement of the problem.

In studying stability of the trivial solution $x_1=\dots=x_n=0$ of system (7.3.1.1) the following method is often employed. The right-hand sides of the equations are approximated by the principal (linear) terms of the expansion into Taylor series about the equilibrium point:

$$f_k(t,x_1,\dots,x_n)\approx a_{k1}(t)x_1+\dots+a_{kn}(t)x_n,$$
$$a_{km}(t)=\frac{\partial f_k}{\partial x_m}\bigg|_{x_1=\dots=x_n=0},\qquad k=1,\,\dots,\,n.$$

Then a stability analysis of the resulting simplified, linear system is performed. The question arises: Is it possible to draw correct conclusions about the stability of the original nonlinear system (7.3.1.1) from the analysis of the linearized system? Two theorems stated below give a partial answer to this question.

► Stability by first approximation.

THEOREM (STABILITY BY FIRST APPROXIMATION). *Suppose in the system*

$$x_k'=a_{k1}x_1+\dots+a_{kn}x_n+\psi_k(t,x_1,\dots,x_n),\qquad k=1,\dots,n, \tag{7.3.2.1}$$

the functions ψ_k are defined and continuous in a domain $t\ge 0$, $|x_k|\le b$ $(k=1,\dots,n)$ and, in addition, the inequality

$$\sum_{k=1}^n|\psi_k|\le A\sum_{k=1}^n|x_k| \tag{7.3.2.2}$$

holds for some constant A. In particular, this implies that $\psi_k(t,0,\dots,0)=0$, and therefore

$$x_1=\dots=x_n=0 \tag{7.3.2.3}$$

is a solution of system (7.3.2.1). Suppose further that

$$\frac{\sum_{k=1}^n|\psi_k|}{\sum_{k=1}^n|x_k|}\to 0\quad\text{as}\quad\sum_{k=1}^n|x_k|\to 0\quad\text{and}\quad t\to\infty, \tag{7.3.2.4}$$

and the real parts of all roots of the characteristic equation

$$\det|a_{ij}-\lambda\delta_{ij}|=0,\qquad \delta_{ij}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j\end{cases}\tag{7.3.2.5}$$

are negative. Then solution (7.3.2.3) is stable.

Remark 7.6. Necessary and sufficient conditions for the real parts of all roots of the characteristic equation (7.3.2.5) to be negative are established by Hurwitz's theorem, which allows avoiding its solution.

Remark 7.7. In the above system, the a_{ij}, x_k, and ψ_k may be complex valued.

► Instability by first approximation.

THEOREM (INSTABILITY BY FIRST APPROXIMATION). *(instability by first approximation). Suppose conditions (7.3.2.2) and (7.3.2.4) are met and the conditions for the functions ψ_k from the previous theorem are also met. If at least one root of the characteristic equation (7.3.2.5) has a positive real part, then the equilibrium point (7.3.2.3) of system (7.3.2.1) is unstable.*

Example 7.5. Consider the following two-dimensional system of the form (7.3.2.1) with real coefficients:

$$\begin{aligned} x'_t &= a_{11}x+a_{12}y+\psi_1(t,x,y),\\ y'_t &= a_{21}x+a_{22}y+\psi_2(t,x,y).\end{aligned}\tag{7.3.2.6}$$

We assume that the functions ψ_1 and ψ_2 satisfy conditions (7.3.2.2) and (7.3.2.4).

The characteristic equation of the linearized system (obtained by setting $\psi_1=\psi_2=0$) is given by

$$\lambda^2-b\lambda+c=0,\quad \text{where}\quad b=a_{11}+a_{22},\quad c=a_{11}a_{22}-a_{12}a_{21}.\tag{7.3.2.7}$$

1. Using the theorem of stability by first approximation and examining the roots of the quadratic equation (7.3.2.7), we obtain two sufficient stability conditions for system (7.3.2.6):

$$\begin{aligned} &b<0,\quad 0<\tfrac14 b^2<c \quad \text{(complex roots with negative real part)};\\ &b<0,\quad 0<c<\tfrac14 b^2 \quad \text{(negative real roots)}.\end{aligned}$$

The two conditions can be combined into one:

$$b<0,\quad c>0,\qquad \text{or}\qquad a_{11}+a_{22}<0,\quad a_{11}a_{22}-a_{12}a_{21}>0.$$

These inequalities define the second quadrant in the plane b, c; see Fig. 7.1.

2. Using the theorem of instability by first approximation and examining the roots of the quadratic equation (7.3.2.7), we get three sufficient instability conditions for system (7.3.2.6):

$$\begin{aligned} &b>0,\quad 0<\tfrac14 b^2<c \quad \text{(complex roots with positive real part)};\\ &b>0,\quad 0<c<\tfrac14 b^2 \quad \text{(positive real roots)};\\ &c<0,\quad b \text{ is any} \quad \text{(real roots with different signs)}.\end{aligned}$$

The first two conditions can be combined into one:

$$b>0,\quad c>0,\qquad \text{or}\qquad a_{11}+a_{22}>0,\quad a_{11}a_{22}-a_{12}a_{21}>0.$$

The domain of instability of system (7.3.2.6) covers the first, third, and fourth quadrants in the plane b, c (shaded in Fig. 7.1).

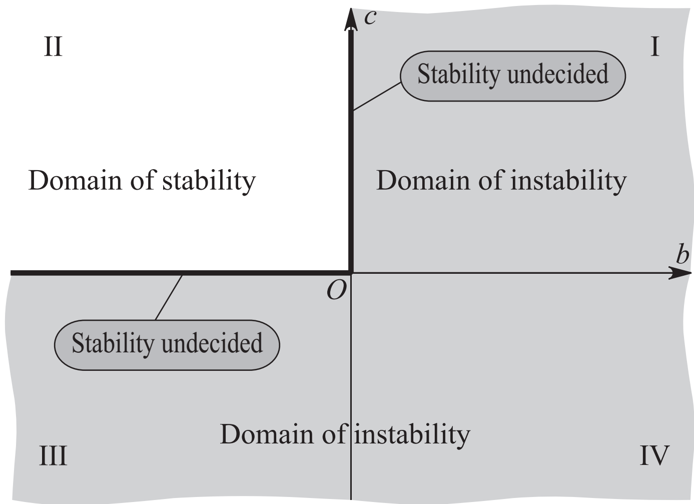

Figure 7.1: Domains of stability and instability of the trivial solution of system (7.3.2.6).

3. The conditions obtained above in Items 1 and 2 do not cover the whole domain of variation of the parameters a_{ij}. Stability or instability is not established for the boundary of the second quarter (shown by thick solid line in Fig. 7.1). This corresponds to the cases

$$b = 0, \quad c \geq 0 \quad \text{(two pure imaginary or two zero roots);}$$
$$c = 0, \quad b \leq 0 \quad \text{(one zero root and one negative real or zero root).}$$

Specific examples of such systems are considered below in Section 7.3.3.

Remark 7.8. When the conditions of Item 1 or 2 hold, the phase trajectories of the nonlinear system (7.3.2.6) have the same qualitative arrangement in a neighborhood of the equilibrium point $x = y = 0$ as that of the phase trajectories of the linearized system (with $\psi_2 = \psi_1 = 0$). A detailed classification of equilibrium points of linear systems with associated arrangements of the phase trajectories can be found in Section 6.1.8.

7.3.3 Lyapunov Function. Theorems of Stability and Instability

► **Lyapunov function.**

In the cases where the theorems of stability and instability by first approximation fail to resolve the issue of stability for a specific system of nonlinear differential equations, more subtle methods must be used. Such methods are considered below.

A *Lyapunov function* for system of equations (7.3.1.1) is a differentiable function $V = V(x_1, \ldots, x_n)$ such that

$$1) \quad V > 0 \quad \text{if} \quad \sum_{k=1}^{n} x_k^2 \neq 0, \qquad V = 0 \quad \text{if} \quad x_1 = \cdots = x_n = 0;$$

$$2) \quad \frac{dV}{dt} = \sum_{k=1}^{n} f_k(t, x_1, \ldots, x_n) \frac{\partial V}{\partial x_k} \leq 0 \quad \text{for} \quad t \geq 0.$$

Remark 7.9. The derivative with respect to t in the definition of a Lyapunov function is taken along an integral curve of system (7.3.1.1).

► Theorems of stability and instability.

THEOREM (STABILITY, LYAPUNOV). *Let system (7.3.1.1) have the trivial solution* $x_1 = x_2 = \cdots = x_n = 0$. *This solution is stable if there exists a Lyapunov function for the system.*

THEOREM (ASYMPTOTIC STABILITY, LYAPUNOV). *Let system (7.3.1.1) have the trivial solution* $x_1 = \cdots = x_n = 0$. *This solution is asymptotically stable if there exists a Lyapunov function satisfying the additional condition*

$$\frac{dV}{dt} \le -\beta < 0 \quad \textit{with} \quad \sum_{k=1}^{n} x_k^2 \ge \varepsilon_1 > 0, \quad t \ge \varepsilon_2 \ge 0,$$

where ε_1 *and* ε_2 *are any positive numbers.*

Example 7.6. Let us perform a stability analysis of the two-dimensional system

$$x'_t = -ay - x\varphi(x,y), \qquad y'_t = bx - y\psi(x,y),$$

where $a > 0$, $b > 0$, $\varphi(x,y) \ge 0$, and $\psi(x,y) \ge 0$ (φ and ψ are continuous functions).

A Lyapunov function will be sought in the form $V = Ax^2 + By^2$, where A and B are constants to be determined. The first condition characterizing a Lyapunov function will be satisfied automatically if $A > 0$ and $B > 0$ (it will be shown later that these inequalities do hold). To verify the second condition, let us compute the derivative:

$$\begin{aligned}\frac{dV}{dt} &= f_1(x,y)\frac{\partial V}{\partial x} + f_2(x,y)\frac{\partial V}{\partial y} = -2Ax[ay + x\varphi(x,y)] + 2By[bx - y\psi(x,y)]\\ &= 2(Bb - Aa)xy - 2Ax^2\varphi(x,y) - 2By^2\psi(x,y).\end{aligned}$$

Setting here $A = b > 0$ and $B = a > 0$ (thus satisfying the first condition), we obtain the inequality

$$\frac{dV}{dt} = -2bx^2\varphi(x,y) - 2ay^2\psi(x,y) \le 0.$$

This means that the second condition characterizing a Lyapunov function is also met. Hence, the trivial solution of the system in question is stable.

Example 7.7. Let us perform a stability analysis for the trivial solution of the nonlinear system

$$x'_t = -xy^2, \qquad y'_t = yx^4.$$

Let us show that the $V(x,y) = x^4 + y^2$ is a Lyapunov function for the system. Indeed, both conditions are satisfied:

$$\begin{aligned}&1) \quad x^4 + y^2 > 0 \quad \text{if} \quad x^2 + y^2 \ne 0, \qquad V(0,0) = 0 \quad \text{if} \quad x = y = 0;\\ &2) \quad \frac{dV}{dt} = -4x^4y^2 + 2x^4y^2 = -2x^4y^2 \le 0.\end{aligned}$$

Hence the trivial solution of the system is stable.

Remark 7.10. No stability analysis of the systems considered in Examples 7.6 and 7.7 is possible based on the theorem of stability by first approximation.

THEOREM (INSTABILITY, CHETAEV). *Suppose there exists a differentiable function* $W = W(x_1, \ldots, x_n)$ *that possesses the following properties:*

1. In an arbitrarily small domain R *containing the origin of coordinates, there exists a subdomain* $R_+ \subset R$ *in which* $W > 0$, *with* $W = 0$ *on part of the boundary of* R_+ *in* R.

2. The condition

$$\frac{dW}{dt} = \sum_{k=1}^{n} f_k(t, x_1, \dots, x_n) \frac{\partial W}{\partial x_k} > 0$$

holds in R_+ and, moreover, in the domain of the variables where $W \geq \alpha > 0$, the inequality $\dfrac{dW}{dt} \geq \beta > 0$ holds.

Then the trivial solution $x_1 = \dots = x_n = 0$ of system (7.3.1.1) is unstable.

Example 7.8. Perform a stability analysis of the nonlinear system

$$x'_t = y^3 \varphi(x, y, t) + x^5, \qquad y'_t = x^3 \varphi(x, y, t) + y^5,$$

where $\varphi(x, y, t)$ is an arbitrary continuous function.

Let us show that the $W = x^4 - y^4$ satisfies the conditions of the Chetaev theorem. We have:

1. $W > 0$ for $|x| > |y|$, $\quad W = 0$ for $|x| = |y|$.
2. $\dfrac{dW}{dt} = 4x^3[y^3 \varphi(x, y, t) + x^5] - 4y^3[x^3 \varphi(x, y, t) + y^5] = 4(x^8 - y^8) > 0$ for $|x| > |y|$.

Moreover, if $W \geq \alpha > 0$, we have $\dfrac{dW}{dt} = 4\alpha(x^4 + y^4) \geq 4\alpha^2 = \beta > 0$. It follows that the equilibrium point $x = y = 0$ of the system in question is unstable.

7.4 Numerical Integration

7.4.1 Systems of Two Equations

► Preliminary remarks.

The majority of the numerical methods for single first-order equations discussed in Section 1.13 generate analogous numerical methods for solving systems of first-order equations (7.1.1.1).

We illustrate this with the Cauchy problem described by the system of first-order differential equations

$$y'_x = f(x, y, z), \qquad z'_x = g(x, y, z) \tag{7.4.1.1}$$

with the initial conditions

$$y(x_0) = y_0, \qquad z(x_0) = z_0. \tag{7.4.1.2}$$

It is required to find $y = y(x)$ and $z = z(x)$.

► Method of Euler polygonal lines.

The unknowns are calculated successively by the formulas

$$y_{k+1} = y_k + hf(x_k, y_k, z_k), \qquad z_{k+1} = z_k + hg(x_k, y_k, z_k),$$

where

$$x_k = x_0 + kh, \qquad y_k = y(x_k), \qquad z_k = z(x_k), \qquad k = 0,\ 1,\ 2,\ \dots .$$

The Euler method is the simplest explicit method of the first-order approximation (with respect to the step size h).

► Modified Euler method.

The *modified Euler method* is more accurate than the method of Euler polygonal lines. One first calculates the intermediate quantities

$$x_{k+\frac{1}{2}} = x_k + \tfrac{1}{2}h, \qquad y_{k+\frac{1}{2}} = y_k + \tfrac{1}{2}hf(x_k, y_k, z_k), \qquad z_{k+\frac{1}{2}} = z_k + \tfrac{1}{2}hg(x_k, y_k, z_k)$$

and then finds y_{k+1} and z_{k+1} by the formulas

$$y_{k+1} = y_k + hf\bigl(x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}, z_{k+\frac{1}{2}}\bigr), \qquad z_{k+1} = z_k + hg\bigl(x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}, z_{k+\frac{1}{2}}\bigr).$$

The modified Euler method is of the second order of accuracy.

► Runge–Kutta method of the fourth-order approximation.

The unknown values y_k and z_k are successively found by the formulas

$$y_{k+1} = y_k + \tfrac{1}{6}h(\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4), \qquad z_{k+1} = z_k + \tfrac{1}{6}h(\psi_1 + 2\psi_2 + 2\psi_3 + \psi_4),$$

where

$$\begin{aligned}
&\varphi_1 = f(x_k,\, y_k,\, z_k), \qquad \psi_1 = g(x_k,\, y_k,\, z_k),\\
&\varphi_2 = f(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_1,\, z_k + \tfrac{1}{2}h\psi_1),\\
&\psi_2 = g(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_1,\, z_k + \tfrac{1}{2}h\psi_1),\\
&\varphi_3 = f(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_2,\, z_k + \tfrac{1}{2}h\psi_2),\\
&\psi_3 = g(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_2,\, z_k + \tfrac{1}{2}h\psi_2),\\
&\varphi_4 = f(x_k + h,\, y_k + h\varphi_3,\, z_k + h\psi_3),\\
&\psi_4 = g(x_k + h,\, y_k + h\varphi_3,\, z_k + h\psi_3).
\end{aligned}$$

This scheme is convenient because the step size h can be changed (reduced if the unknowns change rapidly or increased otherwise) starting from any k. In practice, the choice of the step size h can be controlled using the following simple technique. For each k, one calculates the parameters

$$\theta_1 = \left|\frac{\varphi_2 - \varphi_3}{\varphi_1 - \varphi_2}\right|, \qquad \theta_2 = \left|\frac{\psi_2 - \psi_3}{\psi_1 - \psi_2}\right|.$$

If θ_i $(i = 1,\ 2)$ are of the order of a few hundredths of unity, the calculations are continued with the same step size. If they are over one tenth, the step size should be decreased. If they are less that one hundredth, the step size can be increased to speed up the calculations.

► Numerical integration of problems with blow-up solutions.

In problems having a blow-up solution,* the right-hand side of at least one of the equations (7.4.1.1), which determines the derivative y'_x (or/and z'_x), tends to infinity as $x \to x_*$. When either or both of the functions $f(x, y, z)$ and $g(x, y, z)$ become infinite at a finite value of the independent variable, x_*, unknown in advance, we see the main reason why standard numerical methods fail to provide an acceptable solution for such problems.

*Refer to Section 1.14.4 for details.

Autonomous systems of equations. Consider the Cauchy problem for the autonomous system of equations of general form, whose right-hand side is independent explicitly of x,

$$y'_x = f(y, z), \qquad z'_x = g(y, z) \qquad (x > x_0), \tag{7.4.1.3}$$

with the initial conditions (7.4.1.2).

Let us look at the equivalent autonomous system of equations

$$y'_t = \frac{f(y, z)}{\sqrt{f^2(y, z) + g^2(y, z)}}, \qquad z'_t = \frac{g(y, z)}{\sqrt{f^2(y, z) + g^2(y, z)}} \qquad (t > t_0) \tag{7.4.1.4}$$

with the initial conditions

$$y(t_0) = y_0, \qquad z(t_0) = z_0. \tag{7.4.1.5}$$

The initial value t_0 can be chosen arbitrarily (in particular, it is often convenient to set $t_0 = 0$).

Suppose we have found a solution $y = y(t)$, $z = z(t)$ to the Cauchy problem (7.4.1.4)–(7.4.1.5). Then the formulas

$$\begin{gathered} x = x(t), \qquad y = y(t), \qquad z = z(t), \\ x(t) = x_0 + \int_{t_0}^{t} \frac{d\tau}{\sqrt{f^2(y(\tau), z(\tau)) + g^2(y(\tau), z(\tau))}} \end{gathered} \tag{7.4.1.6}$$

determine a solution to the original problem (7.4.1.3) in parametric form.

Unlike the original system (7.4.1.2), the right-hand sides of system (7.4.1.3) do not have singularities, since the derivatives are always bounded: $|y'_t| \le 1$ and $|z'_t| \le 1$ (recall that, for blow-up solutions, at least one of the derivatives y'_x or z'_x tends to infinity as $x \to x_*$).

A numerical solution to problem (7.4.1.4)–(7.4.1.5) can be obtained using, for example, the Runge–Kutta method (see above). The desired value x_*, determining the point of singularity of the problem, is found by calculating the integral in (7.4.1.6): $x_* = \lim_{t\to\infty} x(t)$.

This method allows for various modifications and generalizations. For example, system (7.4.1.4) can be replaced with the autonomous system

$$y'_t = \frac{f(y, z)}{|f(y, z)| + |g(y, z)|}, \qquad z'_t = \frac{g(y, z)}{|f(y, z)| + |g(y, z)|} \qquad (t > t_0). \tag{7.4.1.7}$$

The modulus sign in the denominators is used for generality, to ensure that system (7.4.1.7) can be used for the numerical solution of problems with root singularities even when f and g have different signs.

If a solution $y = y(t)$, $z = z(t)$ to the Cauchy problem (7.4.1.7), (7.4.1.5) has been found, the formulas

$$\begin{gathered} x = x(t), \qquad y = y(t), \qquad z = z(t), \\ x(t) = x_0 + \int_{t_0}^{t} \frac{d\tau}{|f(y(\tau), z(\tau))| + g|(y(\tau), z(\tau))|} \end{gathered} \tag{7.4.1.8}$$

define a solution to the original problem (7.4.1.3), (7.4.1.2) in parametric form.

The right-hand sides of system (7.4.1.7) do not have singularities, since the derivatives are always bounded: $|y'_t| \le 1$ and $|z'_t| \le 1$. The desired point of singularity is determined by calculating the integral in (7.4.1.8), $x_* = \lim_{t\to\infty} x(t)$.

Example 7.9. Consider the model Cauchy problem for the autonomous system of equations

$$y'_x = 1, \quad z'_x = z^2 \qquad (x > 0); \\ y(0) = 0, \quad z(0) = 1. \tag{7.4.1.9}$$

The problems has the exact solution

$$y = x, \quad z = \frac{1}{1-x}, \tag{7.4.1.10}$$

which only exists on a bounded interval, $0 \le x < x_* = 1$, and corresponds to a blow-up mode. As $x \to x_*$, we have $z \to \infty$ and $z'_x \to \infty$.

Instead of system (7.4.1.9), we will solve the special case of system 7.4.1.7 with $f(y,z) = 1$ and $g(y,z) = z^2$:

$$y'_t = \frac{1}{1+z^2}, \quad z'_t = \frac{z^2}{1+z^2} \qquad (t > 0); \\ y(t=0) = 0, \quad z(t=0) = 1. \tag{7.4.1.11}$$

The old independent variable x is expressed in terms of t as

$$x = \int_0^t \frac{d\tau}{1+z^2(\tau)}. \tag{7.4.1.12}$$

The solution of problem (7.4.1.11) followed by the computation of the integral (7.4.1.12) allows us to find a solution to the original problem (7.4.1.9) in parametric form

$$x = 1 + \tfrac{1}{2}t - \tfrac{1}{2}\sqrt{t^2+4}, \quad y = 1 + \tfrac{1}{2}t - \tfrac{1}{2}\sqrt{t^2+4}, \quad z = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{t^2+4}. \tag{7.4.1.13}$$

One can see that solution (7.4.1.13) exists for all $0 \le t < \infty$ and does not have singularities (unlike solution (7.4.1.10)). The functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ all monotonically increase with t; moreover, the limit relations $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = x_* = 1$ hold.

Nonautonomous systems of equations. In general, the Cauchy problem for nonautonomous systems of two equations (7.4.1.1) subject to the initial conditions (7.4.1.2) reduces the autonomous system of three equations

$$x'_\xi = 1, \quad y'_\xi = f(x,y,z), \quad z'_\xi = g(x,y,z) \tag{7.4.1.14}$$

with the initial conditions

$$x(\xi_0) = x_0, \quad y(\xi_0) = y_0, \quad z(\xi_0) = z_0, \tag{7.4.1.15}$$

where the initial value of the additional variable can be taken in the form $\xi_0 = 1$.

The numerical solution of the blow-up problem (7.4.1.14)–(7.4.1.15) is carried out using the method described in Section 7.4.2.

► Numerical integration of problems with root singularity.

Systems (7.4.1.4) and (7.4.1.7) can also be used for the numerical analysis of Cauchy problems of the form (7.4.1.3), (7.4.1.2) having solutions with a root singularity.

Example 7.10. Consider the model Cauchy problem for the autonomous system equation

$$y'_x = 1, \quad z'_x = -\frac{1}{2z} \qquad (x > 0); \\ y(0) = 0, \quad z(0) = 1. \tag{7.4.1.16}$$

It follows from the second initial condition that $z = z(x)$ is positive and decreases with x. It is fairly easy to verify that problem (7.4.1.16) admits the exact solution with a root singularity

$$y = x, \qquad z = \sqrt{1-x}, \tag{7.4.1.17}$$

which only exists on a bounded interval, $0 \le x < x_* = 1$, since the radicand in (7.4.1.17) becomes negative for $x > x_*$. As $x \to x_*$, we have $|z'_x| \to \infty$.

For numerical solution, instead of system (7.4.1.16), we will use the special case of system 7.4.1.7 with $f(y,z) = 1$ and $g(y,z) = -(2z)^{-1}$:

$$\begin{gathered} y'_t = \frac{2z}{1+2z}, \qquad z'_t = -\frac{1}{1+2z} \qquad (t>0); \\ y(t=0) = 0, \qquad z(t=0) = 1. \end{gathered} \tag{7.4.1.18}$$

A solution to problem (7.4.1.18) is sought in the domain $z > 0$, where $|-z| = z$; it must stop at $z = 0$, when the denominator of the right-hand side of the second equation in (7.4.1.16) becomes zero.

The old independent variable x is expressed in terms of the new variable t as

$$x = 2\int_0^t \frac{z(\tau)\,d\tau}{1+2z(\tau)}. \tag{7.4.1.19}$$

The solution of problem (7.4.1.18) followed by the computation of the integral (7.4.1.19) allows us to find a solution to the original problem (7.4.1.16) in parametric form:

$$x = t + \tfrac{1}{2}\sqrt{9-4t} - \tfrac{3}{2}, \qquad y = t + \tfrac{1}{2}\sqrt{9-4t} - \tfrac{3}{2}, \qquad z = \tfrac{1}{2}\sqrt{9-4t} - \tfrac{1}{2}. \tag{7.4.1.20}$$

Solution (7.4.1.20) only exists in a bounded domain, $0 \le t < 2$, since $z(2) = 0$ (recall that the solution is sought in the domain $z > 0$), and does not have singularities in this domain (unlike solution (7.4.1.17)). The functions $x = x(t)$ and $y = y(t)$ both monotonically increase with t; moreover, the relations $\lim_{t\to 2} x(t) = \lim_{t\to 2} y(t) = x_* = 1$ hold. The function $z = z(t)$ monotonically decreases with t and vanishes at $t = 2$.

7.4.2 Systems Involving Three or More Equations

► Form of the system.

Consider the system of first-order equations of general form

$$y'_m = f_m(x, y_1, y_2, \ldots, y_n), \qquad m = 1,\ 2,\ \ldots,\ n \tag{7.4.2.1}$$

subject to the initial conditions

$$y_m(x_0) = y_0^m \quad \text{with} \quad m = 1,\ 2,\ \ldots,\ n. \tag{7.4.2.2}$$

► Method of Euler polygonal lines.

The unknown quantities are calculated successively by the formulas

$$y_{k+1}^m = y_k^m + hf_m(x_k, y_k^1, y_k^2, \ldots, y_k^{n-1}), \qquad m = 1,\ 2,\ \ldots,\ n,$$

where

$$x_k = x_0 + kh, \qquad y_k^m = y_m(x_k), \qquad k = 0,\ 1,\ 2,\ \ldots\ .$$

► **Modified Euler method.**

First, one computes the intermediate quantities

$$x_{k+\frac{1}{2}} = x_k + \tfrac{1}{2}h, \qquad y^m_{k+\frac{1}{2}} = y^m_k + \tfrac{1}{2}hf_m(x_k, y^1_k, y^2_k, \dots, y^{n-1}_k).$$

Then, one finds the values y^m_{k+1} by the formulas

$$y^m_{k+1} = y^m_k + hf_m\bigl(x_{k+\frac{1}{2}}, y^1_{k+\frac{1}{2}}, y^2_{k+\frac{1}{2}}, \dots, y^{n-1}_{k+\frac{1}{2}}\bigr).$$

► **Fourth-order Runge–Kutta method.**

The unknown values y^m_k are successively found by the formulas

$$y^m_{k+1} = y^m_k + \tfrac{1}{6}h(\varphi^m_1 + 2\varphi^m_2 + 2\varphi^m_3 + \varphi^m_4), \qquad m = 1,\, 2,\, \dots,\, n,$$

where

$$\begin{aligned}
\varphi^m_1 &= f_m(x_k, y^1_k, y^2_k, \dots, y^{n-1}_k),\\
\varphi^m_2 &= f_m(x_k + \tfrac{1}{2}h,\ y^1_k + \tfrac{1}{2}h\varphi^1_1,\ y^2_k + \tfrac{1}{2}h\varphi^2_1,\ \dots,\ y^n_k + \tfrac{1}{2}h\varphi^n_1),\\
\varphi^m_3 &= f_m(x_k + \tfrac{1}{2}h,\ y^1_k + \tfrac{1}{2}h\varphi^1_2,\ y^2_k + \tfrac{1}{2}h\varphi^2_2,\ \dots,\ y^n_k + \tfrac{1}{2}h\varphi^n_2),\\
\varphi^m_4 &= f_m(x_k + h,\ y^1_k + h\varphi^1_3,\ y^2_k + h\varphi^2_3,\ \dots,\ y^n_k + h\varphi^n_3).
\end{aligned}$$

► **A system of special type resulting from a single nth-order ODE.**

Let us look at the system of first-order equations of the special form

$$\begin{aligned}
&y'_1 = y_2, \quad y'_2 = y_3, \quad \dots, \quad y'_{n-1} = y_n,\\
&y'_n = f(x, y_1, y_2, \dots, y_n),
\end{aligned} \tag{7.4.2.3}$$

which is obtained from the single nth-order ODE

$$y^{(n)}_x = f(x, y, y'_x, \dots, y^{(n-1)}_x),$$

with $y \equiv y_1$.

System (7.4.2.3) is a special case of system (7.4.2.1) with

$$\begin{aligned}
f_m(x, y_1, y_2, \dots, y_n) &\equiv y_{m+1}, \qquad m = 1,\, 2,\, \dots,\, n-1,\\
f_n(x, y_1, y_2, \dots, y_n) &\equiv f(x, y_1, y_2, \dots, y_n).
\end{aligned}$$

Hence, it is solvable using the numerical methods described previously in Section 7.4.2.

► **Numerical integration of problems with blow-up solutions.**

Autonomous systems of equations. Consider the Cauchy problem for the autonomous system of equations of general form, whose right-hand side is independent explicitly of x,

$$\frac{dy_m}{dx} = f_m(y_1, \dots, y_n), \qquad m = 1, \dots, n \qquad (x > x_0), \tag{7.4.2.4}$$

with the initial conditions (7.4.2.2).

In problems having blow-up solutions, the right-hand side of a least one of the equations (7.4.2.4) tends to infinity as $x \to x_*$, with x_* unknown in advance.

Instead of (7.4.2.4), we will be looking at the equivalent autonomous system of equations

$$\frac{dy_m}{dt} = \frac{f_m(y_1, \dots, y_n)}{\sqrt{\sum_{j=1}^n f_j^2(y_1, \dots, y_n)}}, \qquad m = 1, \dots, n, \qquad (t > t_0) \tag{7.4.2.5}$$

with the initial conditions

$$y_m(t_0) = y_0^m \quad \text{with} \quad m = 1,\ 2,\ \dots,\ n. \tag{7.4.2.6}$$

The initial value t_0 can be chosen arbitrarily (in particular, it is often convenient to set $t_0 = 0$).

Suppose a solution $y_m = y_m(t)$ $(m = 1, \dots, n)$ to the Cauchy problem (7.4.2.5), (7.4.2.6) has been found. Then the formulas

$$y_m = y_m(t), \qquad m = 1, \dots, n, \qquad x = x_0 + \int_{t_0}^t \frac{d\tau}{\sqrt{\sum_{j=1}^n f_j^2(y_1(\tau), \dots, y_n(\tau))}}$$

define a solution to the original problem (7.4.2.4), (7.4.2.2) in parametric form.

Unlike system (7.4.2.4), the right-hand sides of system (7.4.2.5) do not have singularities, since the derivatives are all bounded: $|(y_m)'_t| \le 1$ $(m = 1, \dots, n)$; recall that, for blow-up solutions, at least one of the derivatives $(y_m)'_t$ tends to infinity as $x \to x_*$.

A numerical solution to problem (7.4.2.5)–(7.4.2.6) can be obtained using, for example, the Runge–Kutta method (see above).

This presented method admits various modifications and generalizations. For example, instead of (7.4.2.5), one uses the following autonomous system for numerical solution:

$$\frac{dy_m}{dt} = \frac{f_m(y_1, \dots, y_n)}{\sum_{j=1}^n |f_j(y_1, \dots, y_n)|}, \qquad m = 1, \dots, n, \qquad (t > t_0) \tag{7.4.2.7}$$

If a solution $y_m = y_m(t)$ $(m = 1, \dots, n)$ to the Cauchy problem (7.4.2.7), (7.4.2.6) has been obtained, the formulas

$$y_m = y_m(t), \qquad m = 1, \dots, n, \qquad x = x_0 + \int_{t_0}^t \frac{d\tau}{\sum_{j=1}^n |f_j(y_1(\tau), \dots, y_n(\tau))|}$$

define a solution to the original problem (7.4.2.4), (7.4.2.2) in parametric form.

The right-hand sides of system (7.4.2.7) do not have singularities, since the derivatives are all bounded: $|(y_m)'_t| \le 1$ $(m = 1, \dots, n)$.

Nonautonomous systems of equations. In general, the Cauchy problem for the nonautonomous system of n equations (7.4.2.1) subject to the initial conditions (7.4.2.2) is first reduced to an autonomous system of $n + 1$ equation (see Section 7.1.3). Then, one constructs a numerical solution to one of the two equivalent auxiliary systems described above.

► **Numerical integrations of problem having solutions with root singularity.**

The autonomous systems (7.4.2.5) and (7.4.2.7) can also be used for the numerical analysis of Cauchy problems of the form (7.4.2.4), (7.4.2.2) having solutions with a root singularity (see Section 7.4.1 for systems of two equations).

The nonautonomous system of n equations of general form (7.4.2.1) subject to the initial conditions (7.4.2.2) is first reduced to an autonomous system consisting of $n+1$ equations (see Section 7.1.3) and then replaced with a suitable equivalent autonomous system discussed above.

► **Differential-algebraic equations.**

Systems of differential-algebraic equations (DAEs for short) are systems in which one or more dependent variables occur without their derivatives. Numerical methods for the solution of DAEs can be found in the books by Hairer, Lubich, and Roche (1989), Schiesser (1994), Hairer and Wanner (1996), Brenan, Campbell, and Petzold (1996), Ascher and Petzold (1998), and Rabier and Rheinboldt (2002).

⊙ *Literature for Section 7.4*: N. S. Bakhvalov (1977), N. N. Kalitkin (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), W. E. Schiesser (1994), U. M. Ascher and L. R. Petzold (1998), G. A. Korn and T. M. Korn (2000), H. J. Lee and W. E. Schiesser (2004).

Chapter 8

Elements of Bifurcation Theory

8.1 Dynamical Systems. Rough and Nonrough Systems

8.1.1 Bifurcation. Dynamical Systems. Phase Portrait

► **Preliminary remarks.**

The term *bifurcation* is generally used to denote different qualitative structural changes or transformations of various objects when some parameters characterizing the object change. Mathematical bifurcation theory deals with changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, the solutions of a family of dynamical systems, and the solutions of a family of boundary value problems.

In the theory of dynamical systems, a bifurcation is a qualitative change in the properties of a system of differential equations due to an indefinitely small change in its parameters. The theory of nonlinear boundary value problems studies bifurcations associated with branching of solutions (multiplicity of solutions) or nonexistence of solution at certain values of the parameters of the problem.

► **Dynamical systems described by ODEs. Phase portrait.**

Dynamical systems with finitely many variables are described by autonomous systems of first-order ordinary differential equations

$$\mathbf{x}'_t = \mathbf{f}(\mathbf{x}, \mathbf{a}), \qquad (8.1.1.1)$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is the vector of unknowns, t is time, $\mathbf{a} = (a_1, \ldots, a_m)$ is the vector of parameters, and $\mathbf{f} = (f_1, \ldots, f_n)$ is a given vector function whose components depend on the unknowns and parameters.

System (8.1.1.1) is associated with an n-dimensional *phase space*, whose coordinate axes measure the values of the variables $x_1, \ldots, x_n$, known as the *phase variables*. A change in the state of system (8.1.1.1) in time corresponds to the motion of a point in the phase space along a line called the *phase trajectory*. A combination of phase trajectories forms a *phase portrait of the dynamical system*.

Phase trajectories of a dynamical system are described by a system of ODEs for the phase variables consisting of $n - 1$ equations obtained from (8.1.1.1) by eliminating t.

The system for the phase variables has the form $dx_m/dx_1 = f_m/f_1$ with $m = 2, \dots, n$ and, by the existence and uniqueness theorems for systems of ODEs (see Section 7.1.2), it has a unique solution, provided that the initial data are not selected at stationary points. Hence an important consequence follows that phase trajectories cannot intersect at regular points. The impossibility of self-intersections and the existence of invariant manifolds largely determines the structure of a phase portrait.

8.1.2 Topologically Equivalent Systems. Rough and Nonrough Systems

► **Topologically equivalent dynamical systems.**

The set of properties of a dynamical system that remain unchanged under a continuous deformation of its phase portrait determine the system's local *topological* (*qualitative*) *structure*.

A dynamical system I in a domain $U_1 \in \mathbb{R}^n$ is said to be *topologically equivalent to a dynamical system* II in a domain $U_2 \in \mathbb{R}^n$ if there is a one-to-one mapping $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{F}(U_1) = U_2$, between them such that the mapping and its inverse are both continuous, with $\mathcal{F}$ taking trajectories of the first system from U_1 into trajectories of the second system from U_2 while preserving the time course. The phase portraits of topologically equivalent systems are also called topologically equivalent. Note that the definition involves domains U_1 and U_2 from $\mathbb{R}^n$, which must meet the only condition: they cannot shrink indefinitely. This defines *local equivalence*.

► **Rough and nonrough dynamical systems.**

The theory of bifurcations of dynamical systems studies changes in the qualitative picture of decomposition of the phase space depending on changes of a parameter (or a few parameters).

Let $\bar{\mathbf{a}} \in A_m$, where A_m is some domain of the m-dimensional Euclidean space. If there is a $\delta > 0$ such that the phase portraits of system (8.1.1.1) at $\mathbf{a} = \bar{\mathbf{a}}$ and any $\mathbf{a} \in A$ satisfying the condition $\|\mathbf{a} - \bar{\mathbf{a}}\| < \delta$ are topologically equivalent, then system (8.1.1.1) is said to be *rough* at $\mathbf{a} = \bar{\mathbf{a}}$. A rough system is a system whose qualitative pattern of motions remains unchanged under sufficiently small changes in parameters. Conservative systems are not rough: for example, oscillations of a perfect frictionless pendulum are periodic (do not decay), whereas periodicity is violated when there is even indefinitely small friction.

If at $\mathbf{a} = \bar{\mathbf{a}}$, system (8.1.1.1) is not rough, the vector $\bar{\mathbf{a}}$ is called a bifurcation set of parameter values. The parameter, a change in which causes a bifurcation, is called a critical parameter (*bifurcation parameter*), while the value at which the bifurcation occurs is called a critical value. A point in the parametric space at which bifurcation occurs is called a *bifurcation point*. A bifurcation point can be a source from which several solutions (stable or unstable) may come out. The oscillation of a critical parameter about a critical point causes a *hysteresis* (uncertainty) of the solution properties. A bifurcation point that is a source of only stable solutions is called an *attracting point* (or an *attractor*).

Example 8.1. Consider the Malthus model

$$x'_t = ax, \tag{8.1.2.1}$$

which describes the dynamics of population quantity $x(t)$ under unlimited resources. The solution satisfying the initial condition $x(0)=x_0$ is $x(t)=x_0\exp(at)$. It is clear that if $x_0>0$, the population increases indefinitely for any positive value of a and dies out for any negative a ($x(t)\to 0$ with time). Hence, the solutions of close equations (corresponding to close values of the parameter) will be qualitatively different near $a=0$. Therefore, a bifurcations of equation (8.1.2.1) occurs when a changes its sign (the equilibrium $x=0$ changes from stable to unstable or vice versa).

⊙ *Literature for Section 8.1*: A. A. Andronov, E. A. Leontovich, and I. I. Gordon (1971), J. E. Marsden and M. McCracken (1976), V. I. Arnold (1978), J. Guckenheimer and P. Holms (1983), A. D. Bazykin (1985), R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky (1988), S. Wiggins (1988), G. M. Zaslavsky (1988), J. D. Crawford (1991), G. Iooss and D. D. Joseph (1997), V. I. Arnold and V. S. Afraimovich (1999), T. Puu (2000), Y. A. Kuznetsov (2004), E. M. Izhikevich (2007), A. S. Bratus, A. S. Novozhilov, and A. P. Platonov (2009), A. A. Andronov, A. A. Vitt, and S. E. Khaikin (2011), A. Yu. Alexandrov, A. V. Platonov, V. N. Starkov, and N. A. Stepenko (2016), E. G. Wiens (2016).

8.2 Bifurcations of Second-Order Dynamical Systems

8.2.1 Second-Order Dynamical Systems. Rough and Nonrough Systems

► Classification of singular points of a linearized system.

Consider the autonomous system consisting of two differential equations

$$\begin{aligned} x'_t &= P(x,y,\mathbf{a}),\\ y'_t &= Q(x,y,\mathbf{a}). \end{aligned} \tag{8.2.1.1}$$

Let us fix the set of parameters. Let (x_0,y_0) be an equilibrium point of system (8.2.1.1) for the selected set of parameters. Denote

$$\Delta=\begin{vmatrix} P_x(x_0,y_0,\mathbf{a}) & P_y(x_0,y_0,\mathbf{a})\\ Q_x(x_0,y_0,\mathbf{a}) & Q_y(x_0,y_0,\mathbf{a}) \end{vmatrix},\qquad \sigma=P_x(x_0,y_0,\mathbf{a})+Q_y(x_0,y_0,\mathbf{a}).$$

Then the characteristic polynomial of the matrix of system (8.2.1.1) linearized about the point (x_0,y_0) has the form

$$\varphi(\lambda)=\lambda^2-\sigma\lambda+\Delta. \tag{8.2.1.2}$$

Depending on the values of Δ and σ, the singular point (x_0,y_0) is classified as follows:

(a) $\Delta>0$, $\sigma^2-4\Delta\ge 0$: the point is a node (if $\sigma^2-4\Delta=0$, the node is degenerate of dicritical);

(b) $\Delta<0$: saddle point;

(c) $\Delta>0$, $\sigma^2-4\Delta<0$, $\sigma\ne 0$: focus (spiral point);

(d) $\Delta>0$, $\sigma=0$: complex focus or center (depending on the nonlinear terms);

(e) $\Delta=0$: complex (multiple) equilibrium.

In cases (a), (b), and (c), there are no roots with zero real part among the roots of the characteristic polynomial (8.2.1.2). In case (d), there is a pair of purely imaginary complex conjugate roots. In case (e), at least one of the roots is zero. If $\Delta=0$, the equilibrium can share features of a node, saddle point, and focus; for example, it can be a complex node, complex saddle, saddle-node, etc. The equilibrium is rough if it falls into one of the cases (a), (b), or (c).

► **Rough limit cycles.**

Suppose that, for fixed values of the parameters, system (8.2.1.1) has a closed trajectory (limit cycle). Such a trajectory corresponds to a periodic solution $x(t)$, $y(t)$. Let T denote the period of this solution; that is, $x(t+T) = x(t)$ and $y(t+T) = y(t)$ for all t. The number

$$h = \frac{1}{T}\int_0^T \big[P_x(x(t), y(t), \mathbf{a}) + Q_y(x(t), y(t), \mathbf{a})\big]\,dt$$

is called the *characteristic index* of the closed trajectory. If $h \neq 0$, the limit cycle is called rough; it is unstable for $h > 0$ and stable for $h < 0$.

► **Rough and nonrough dynamical systems. Bifurcation values.**

THEOREM ON ROUGHNESS OF DYNAMICAL SYSTEMS. *System* (8.2.1.1) *is rough in a closed domain G for a fixed set of values of the parameters if and only if the system has neither nonrough equilibria or limit cycles nor separatrices coming from a saddle to a saddle.*

In general, the parameter space is divided into domains of rough systems separated by surfaces of nonrough systems. Bifurcation theory studies changes of the qualitative picture of a dynamical system when its parameters change continuously. In mechanical systems, stationary motions (such as equilibria or relative equilibria) often depend on parameters. The values of parameters at which the number of equilibria changes are bifurcation values. The curves or surfaces representing equilibrium manifolds in the state space or parameter space are called bifurcation curves or bifurcation surfaces. As a parameter passes its bifurcation value, the equilibrium properties usually change. Bifurcations of equilibria can be accompanied with the birth or death of periodic or more complex motions.

Nonrough systems are classified by their degree of nonroughness depending on the order of the terms that can turn a nonrough system (8.2.1.1) into another nonrough system with a topologically nonequivalent phase portrait. Conservative systems may be treated as having an infinite degree of nonroughness. These can only have simple equilibria such as a center of a saddle point; closed trajectories in conservative systems cannot be isolated but occupy entire domains.

8.2.2 Bifurcations in Systems of the First Degree of Nonroughness

► **Systems of the first degree of nonroughness with one parameter.**

Consider the simplest bifurcations arising in autonomous second-order systems of the first degree of nonroughness with a single parameter ($m = 1$). Suppose a value of the parameter, $a = \bar{a}$, is bifurcation and all sufficiently close values $a \neq \bar{a}$ correspond to rough systems. Let the qualitative structures of the rough systems be different for $a < \bar{a}$ and $a > \bar{a}$. We assume that $a = \bar{a}$ corresponds to a system of the first degree of nonroughness; that is, there is only one of the nonrough special elements: a complex focus of the first order, a saddle-node, a double limit cycle, a separatrix going from one saddle point to another, or a separatrix forming a loop about a saddle point (for which $\sigma \neq 0$).

Below we describe the simplest bifurcations in second-order dynamical systems with one or more parameters.

► Bifurcations of a complex focus of the first order.

Two types of bifurcations of a complex focus of the first order (*Andronov–Hopf bifurcations*) are possible:

1. For all $a < \bar{a}$ (sufficiently close to $\bar{a}$), there is a rough stable focus with no closed trajectories in its neighborhood. For $a = \bar{a}$, the focus becomes a stable complex focus of the first order. As a passes the bifurcation value $\bar{a}$, with $a > \bar{a}$, the focus becomes rough and unstable, with a single stable limit cycle arising about it, which gets bigger as a increases.

2. For $a < \bar{a}$, there is a rough stable focus surrounded by an unstable limit cycle. As $a \to \bar{a} - 0$, the limit cycle shrinks and merges with the equilibrium at $a = \bar{a}$, becoming an unstable complex focus of the first order. For $a > \bar{a}$, the focus becomes rough and unstable, with no closed trajectories about it.

Example 8.2. Let us look at an example revealing the first type of bifurcations. Suppose the equations (8.2.1.1) are

$$\begin{aligned} x'_t &= ax - 2y - x(x^2 + y^2),\\ y'_t &= 2x + ay - y(x^2 + y^2), \end{aligned} \tag{8.2.2.1}$$

where a is a scaler parameter. The origin of coordinates is an equilibrium of system (8.2.2.1) for any a. The linearized system about this equilibrium is

$$\begin{aligned} x'_t &= ax - 2y,\\ y'_t &= 2x + ay. \end{aligned}$$

The eigenvalues of the matrix of this linear system are $\lambda_{1,2} = a \pm 2i$. It follows that the point $(0, 0)$ is a stable focus for $a < 0$ and unstable focus for $a > 0$. At $a = 0$, the eigenvalues are purely imaginary ($\Delta > 0$, $\sigma = 0$) and the equilibrium is a center. The bifurcation value is $\bar{a} = 0$.

For the nonlinear system (8.2.2.1), we get the following: (i) for $a < 0$, the phase portrait remains the same, (ii) at $a = 0$, the point $(0, 0)$ becomes an asymptotically stable complex focus, and (iii) for $a > 0$, a stable limit cycle arises. Figure 8.1 displays the integral curves passing through the points $(1, 0)$ and $(-1, 0)$ for different values of a, with the last graph also showing the integral curves passing through $(1, 1)$ and $(-1, -1)$.

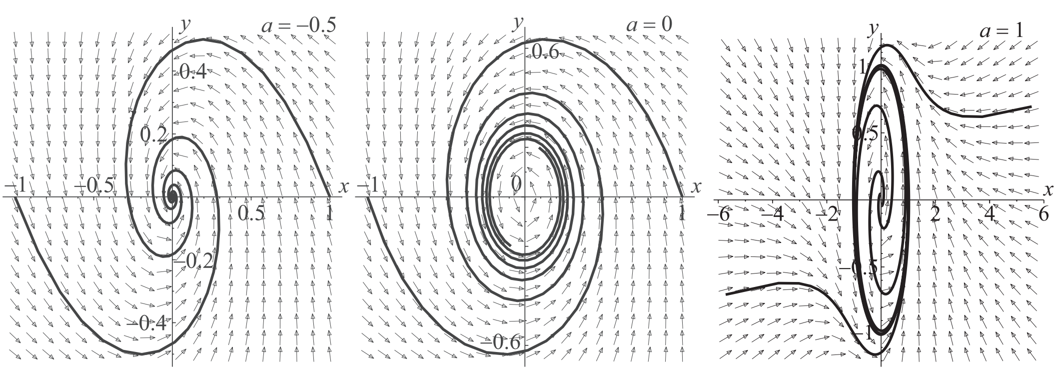

Figure 8.1: Phase portraits of system (8.2.2.1) for $a = -0.5$, 0, and 1.

► Bifurcation of a double saddle-node.

Two types of bifurcation are possible:

1. For $a < \bar{a}$, there are no equilibria; at $a = \bar{a}$, a saddle-node arises; and for $a > \bar{a}$, the saddle-node splits into a rough saddle and a rough node.

2. For $a < \bar{a}$, there are two rough points of equilibrium, a saddle and a node; at $a = \bar{a}$, these merge to form a saddle-node, which disappears for $a > \bar{a}$.

Example 8.3. Consider a system corresponding to the first type of bifurcation:

$$x'_t = xy - a,\quad y'_t = x - y. \tag{8.2.2.2}$$

The bifurcation value is $\bar{a} = 0$. For $a > 0$, system (8.2.2.2) has two points of equilibrium: a saddle point $(\sqrt{a}, \sqrt{a})$ and a stable node $(-\sqrt{a}, -\sqrt{a})$. At $a = 0$, these merge at the point $(0, 0)$ to form a saddle-node. For $a < 0$, the system has no equilibria.

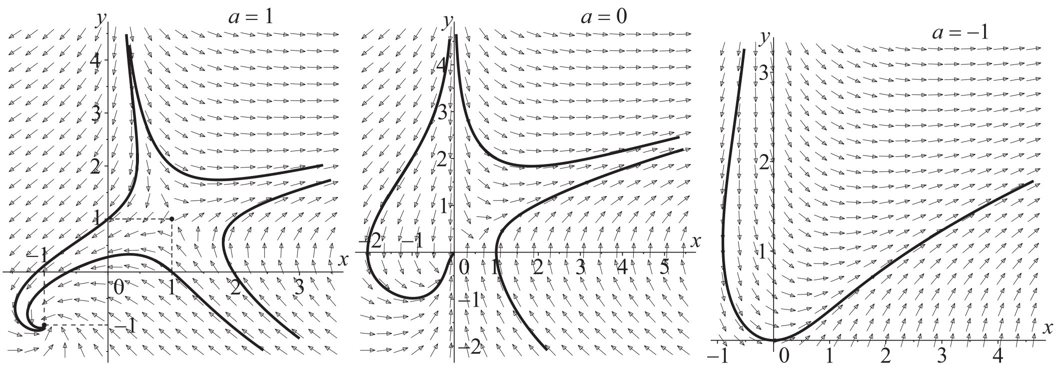

Figure 8.2: Phase portraits of system (8.2.2.2) for $a = 1$, 0, and -1.

Figure 8.2 displays the integral curves passing through (i) the regular points $(1, 0)$, $(1, 2)$, $(2, 1)$, and $(0, 1)$ as well as the singular points $(-1, -1)$ and $(1, 1)$ at $a = 1$, (ii) the regular points $(1, 0)$, $(1, 2)$, and $(-1, 2)$ and singular point $(0, 0)$ at $a = 0$, and (iii) the point $(0, 0)$ at $a = -1$.

Example 8.4. Consider the dynamical system

$$x'_t = x - \frac{xy}{1 + \alpha x} - \beta x^2,\quad y'_t = -\gamma y\left(1 - \frac{x}{1 + \alpha x}\right), \tag{8.2.2.3}$$

which is used to model the interaction between predator and prey populations taking into account prey competition and predator saturation; Bazykin (1985) and Alexandrov, Platonov, Starkov, and Stepenko (2016).

If $\beta = 0$, prey competition is not considered. In the first quadrant, an equilibrium only arises for $\alpha < 1$; it is always unstable. For $\alpha > 1$, there is no nontrivial equilibrium (the predator population is doomed to die out). Therefore, prey competition is a stabilizing factor; if it is too weak, stability may be lost and self-oscillations may arise.

The equilibria in system (8.2.2.3) are determined by isoclines on which $x'_t = 0$ and $y'_t = 0$, that is, by the lines $x = 1/(1 - \alpha)$ and $y = (1 + \alpha x)(1 - \beta x)$. The condition for the isoclines to intersect in the first quadrant is $\alpha + \beta < 1$. Let us select the following regions on the parametric plane (α, β) (see Fig. 8.3):

I. $\alpha + \beta > 1$.
II. $\alpha + \beta < 1$, $\beta > \alpha(1-\alpha)/(1+\alpha)$.
III. $\beta < \alpha(1-\alpha)/(1+\alpha)$.

In region II, there is a nontrivial stable equilibrium. At the boundary between regions II and III, the equilibrium loses stability and, in region III, a stable limit cycle arises. The qualitative behavior of system (8.2.2.3) is independent of the values of γ.

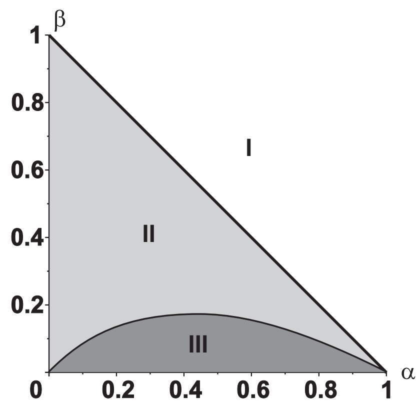

Figure 8.3: Regions determined by equilibrium isoclines of system (8.2.2.3).

► Bifurcation of a limit cycle.

Two types of bifurcation are also possible here:

1. For $a < \bar{a}$, there is a region with no limit cycle; at $a = \bar{a}$, a double limit cycle arises, which for $a > \bar{a}$, splits into two rough limit cycles, stable and unstable.

2. For $a < \bar{a}$, there are two rough limit cycles, stable and unstable; at $a = \bar{a}$, they merge to form a double limit cycle, which further disappears for $a > \bar{a}$.

Example 8.5. Consider a dynamical system corresponding to the first type of bifurcation:

$$\begin{aligned} x'_t &= y, \\ y'_t &= -x + \mu(ay + by^3 - cy^5). \end{aligned} \tag{8.2.2.4}$$

It arises in modeling power generation processes in vacuum tubes (Andronov, Leontovich, and Gordon (1971)). Here, μ is a small parameter, while b and c are positive constants. The bifurcation value is $\bar{a} = -0.25$.

Figure 8.4 shows phase portraits at $\mu = 1/10$, $b = 3/4$, and $c = 5/8$ as well as $a = -0.5$, -0.25, and -0.15. The first and second graphs present the integral curves passing through the point $(0.5, 0.5)$. The third graph displays the integral curves passing through the points $(0.3, 0.3)$, an unstable limit cycle, $(0.63, 0.63)$, a stable limit cycle (shown by a solid line), and the points $(0.1, 0.1)$, $(0.5, 0.5)$, and $(1.2, 1.15)$ shown by a dashed line.

⊙ *Literature for Section 8.2*: A. A. Andronov, E. A. Leontovich, and I. I. Gordon (1971), J. E. Marsden and M. McCracken (1976), V. I. Arnold (1978), J. Guckenheimer and P. Holms (1983), A. D. Bazykin (1985), R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky (1988), S. Wiggins (1988), G. M. Zaslavsky (1988), J. D. Crawford (1991), G. Iooss and D. D. Joseph (1997), V. I. Arnold and V. S. Afraimovich (1999), T. Puu (2000), Y. A. Kuznetsov (2004), E. M. Izhikevich (2007), A. S. Bratus, A. S. Novozhilov, and A. P. Platonov (2009), A. A. Andronov, A. A. Vitt, and S. E. Khaikin (2011), A. Yu. Alexandrov, A. V. Platonov, V. N. Starkov, and N. A. Stepenko (2016), E. G. Wiens (2016).

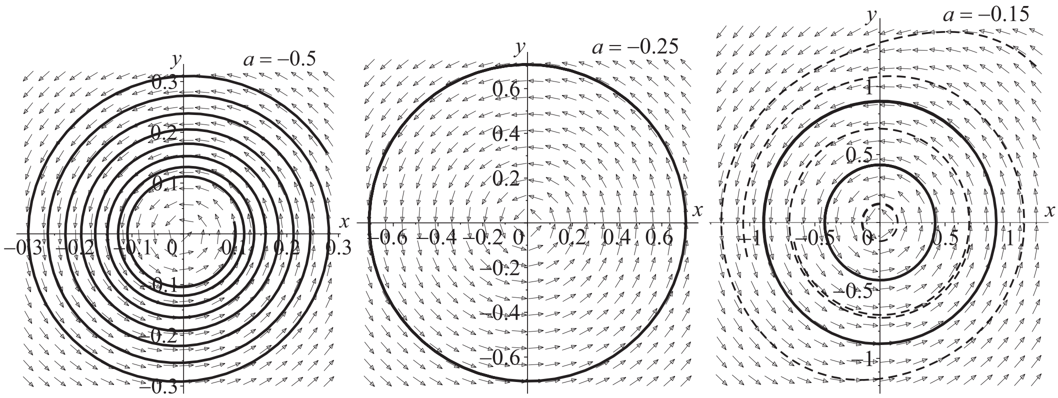

Figure 8.4: Phase portraits of system (8.2.2.4) for $a = -0.5$, -0.25, and -0.15.

8.3 Bifurcations of Solutions to Boundary Value Problems

8.3.1 Bifurcations of Solutions to Linear Boundary Value Problems

Let us look at the linear eigenvalue problem

$$\mathrm{L}[u] = \lambda u, \qquad \Gamma[u] = 0, \tag{8.3.1.1}$$

where L is a linear differential operator, λ is a real number, and $\Gamma[u] = 0$ is a symbolic representation of different linear boundary conditions (see Section 2.5 for the most common types of boundary conditions).

For any λ, problem (8.3.1.1) has the trivial solution $u = 0$. Suppose that there is a set of eigenvalues $\lambda_1 < \lambda_2 < \cdots$ and the corresponding normalized eigenfunctions u_1, u_2, ... such that $\mathrm{L}u_j = \lambda_j u_j$, $\|u_j\| = 1$, $j = 1, 2, \ldots$. Then, if c is an arbitrary real number, problem (8.2.2.3) has other solutions given by

$$u = cu_j, \quad j = 1, 2, \ \ldots. \tag{8.3.1.2}$$

It is clear that, for each eigenvalue λ_j, the solution $u = 0$ splits into two branches: the $u = 0$ branch and the (8.3.1.2) branch. It follows that $\lambda = \lambda_j$ are bifurcation points of problem (8.3.1.1).

8.3.2 Bifurcations in Solutions to Nonlinear Boundary Value Problems

► **Analysis of bifurcations in boundary value problems by linearization of equations.**

Let (8.3.1.1) be a linearization of some nonlinear eigenvalue problem. Then the solution of the linear problems determines bifurcation points of solutions to the nonlinear problem.

Example 8.6. Consider a thin rod with clamped ends, lying in the xz-plane, and prescribed displacements of the ends in the x-direction. The shape of the rod is described by two functions,

$u = u(x)$ and $w = w(x)$, which denote the dimensionless displacements along the x and z axes, respectively. These functions satisfy the following differential equations and boundary conditions:

$$w''_{xx} + \lambda w = 0, \qquad 0 < x < 1, \tag{8.3.2.1}$$
$$u'_x + \tfrac{1}{2}(w'_x)^2 = -\beta\lambda, \qquad 0 < x < 1, \tag{8.3.2.2}$$
$$w(0) = w(1) = 0, \tag{8.3.2.3}$$
$$u(0) = -u(1) = c > 0. \tag{8.3.2.4}$$

The constant λ in (8.3.2.1) and (8.3.2.2) is proportional to the axial stress in the rod. The positive constant c is proportional to the displacement of either end and is unknown. The quantity $\beta > 0$ is a given constant.

First, let us look at the linearized problem by neglecting the rem $\frac{1}{2}(w'_x)^2$ in (8.3.2.2). We have

$$u'_x = -\beta\lambda, \quad 0 < x < 1.$$

The solution to this equation satisfying the boundary condition (8.3.2.4) is

$$u = c(1 - 2x), \quad c = \tfrac{1}{2}\beta\lambda. \tag{8.3.2.5}$$

Problem (8.3.2.1), (8.3.2.3) has the following solution:

$$\begin{aligned} &w = 0, && \lambda \text{ is an arbitrary constant;} \\ &w = A_n w_n(x) \equiv A_n \sin n\pi x, && \lambda = \lambda_n \equiv (n\pi)^2, \ \ n = 1, 2, \ldots \end{aligned} \tag{8.3.2.6}$$

where A_n are arbitrary constants. It follows from (8.3.2.5) that, for $c = c_n \equiv \frac{1}{2}\beta\lambda_n$, the rod buckles and acquires the shape defined by formulas (8.3.2.5) and (8.3.2.6) with undetermined amplitude A_n. For $c \neq c_n$, the rod remains rectilinear.

Now look at the nonlinear problem (8.3.2.1)–(8.3.2.4). The solutions to problem (8.3.2.1), (8.3.2.3) remain the same, (8.3.2.6). The solution of equation (8.3.2.2) with $\lambda = \lambda_n$ satisfying the first boundary condition (8.3.2.4) is given by

$$u = u_n(x) \equiv c - \beta\lambda_n \left(1 + \frac{A_n^2}{4\beta}\right) x - \frac{n\pi A_n^2}{8} \sin(2n\pi x). \tag{8.3.2.7}$$

Substituting this expression into the second boundary condition (8.3.2.3) yields

$$c = c_n \left(1 + \frac{A_n^2}{4\beta}\right), \quad c_n = \frac{1}{2}\beta\lambda_n. \tag{8.3.2.8}$$

Relation (8.3.2.8) expresses the link between the contraction at the end and the response of the rod. It is clear that for $c < c_1$, the solution is unique (zero) and stable. For $c > c_1$, the uniqueness is violated: for any c from the interval $c_n < c < c_{n+1}$, there are $2n + 1$ solutions (with each new point c_n, two more solutions are added, one corresponding to a positive A_n and the other corresponding to a negative A_n; see Fig. 8.5). The solutions branch out from the unbuckled shape $A_n = 0$ at the points c_n. Hence, the solution of the linear problem determines the bifurcation points of the nonlinear problem. Figure 8.5 displays the parabolas (8.3.2.8) with $\beta = 1$ and $c_n = 1, 2, 3$.

Example 8.7. For a thin rod of unit length lying in the xz-plane, suppose that one of its ends, $x = 0$, is fixed, while the other end, $x = 1$, lies freely on the x-axis. Both ends can turn freely about the y-axis. The rod is loaded by a given axial compressive stress. The shape of the rod is determined by the function $\psi = \psi(x)$, the angle between the central line of the deformed rod and the x-axis, as well as the functions $u = u(x)$ and $w = w(x)$, the displacements along the x- and z-axis, respectively. The shape is determined from the boundary value problem of inextensible elasticity

$$\psi''_{xx} + \lambda \sin\psi = 0, \qquad 0 < x < 1, \tag{8.3.2.9}$$
$$\psi'_x(0) = \psi'_x(1) = 0, \tag{8.3.2.10}$$
$$u'_x = \cos\psi - 1, \quad w'_x = \sin\psi, \qquad 0 < x < 1, \tag{8.3.2.11}$$
$$u(0) = w(0) = w(1) = 0. \tag{8.3.2.12}$$

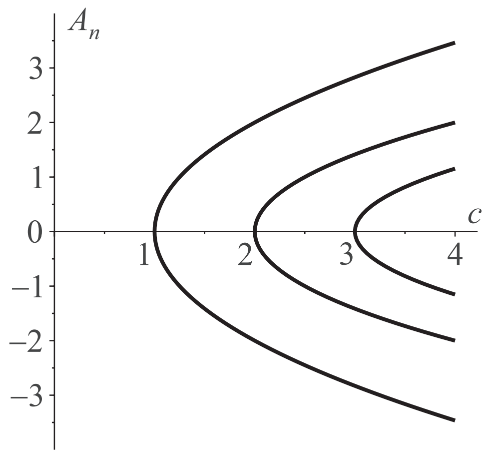

Figure 8.5: Bifurcation points: intersections of the parabolas (8.3.2.8) with the abscissa axis; $\beta = 1$ and $c_n = 1, 2, 3$.

The constant λ is proportional to the load applied. The linearization of equation (8.3.2.9), (8.3.2.11) about $\psi = 0$ leads to the eigenvalue problem

$$\begin{aligned} &\psi''_{xx} + \lambda\psi = 0, \quad 0 < x < 1, \quad \psi'_x(0) = \psi'_x(1) = 0, \\ &u'_x = 0, \qquad\qquad w'_x = \psi, \qquad u(0) = w(0) = w(1) = 0. \end{aligned} \tag{8.3.2.13}$$

This problem has the following solutions:

$$\begin{aligned} &\lambda = \lambda_n \equiv (n\pi)^2, \qquad n = 1,\, 2,\, \ldots, \\ &\psi = A_n \cos n\pi x, \quad u = 0, \quad w = \frac{A_n}{n\pi} \sin n\pi x, \end{aligned} \tag{8.3.2.14}$$

where A_n are arbitrary constants.

It is clear that if ψ, u, w is a solution at some λ, then $\pm\psi + 2n\pi$, u, $\pm w$ are also solutions for any integer n and the same λ (either upper or lower signs are take simultaneously). Therefore, without loss of generality, we can assume that $\lambda \geq 0$ and set

$$\psi(0) = \alpha, \qquad 0 \leq \alpha \leq \pi. \tag{8.3.2.15}$$

A first integral of equation (8.3.2.9), in view of (8.3.2.10) and (8.3.2.15), has the form

$$(\psi'_x)^2 = 2\lambda(\cos\psi - \cos\alpha). \tag{8.3.2.16}$$

From the continuity of the function $\psi = \psi(x)$, condition (8.3.2.15), and nonnegativity of the right-hand side of (8.3.2.16) it follows that $|\psi| \leq \alpha$.

If $0 < \lambda < \lambda_1$, then the constants $\psi = 0$ and $\psi = \pi$ are unique solutions of problem (8.3.2.9)–(8.3.2.12) satisfying condition (8.3.2.15).

For $\lambda \geq \lambda_1$, we introduce the new variable $\varphi = \varphi(x)$ defined by the relations

$$k \sin\varphi = \sin\frac{\psi}{2}, \quad k = \sin\frac{\alpha}{2}. \tag{8.3.2.17}$$

Then, it follows from (8.3.2.16) that

$$\mu\frac{dx}{d\varphi} = (1 - k^2 \sin^2\varphi)^{-1/2}, \quad \mu = \sqrt{\lambda}. \tag{8.3.2.18}$$

The range of $\varphi(x)$ is determined by the equalities $\sin\varphi(0) = 1$ and $\sin^2\varphi(1) = 1$, which follow from (8.3.2.10), (8.3.2.16), and (8.3.2.17); hence,

$$\varphi(0) = \varphi_p \equiv \frac{4p+1}{2}\pi, \qquad p = 0, \pm1, \pm2, \ldots, \tag{8.3.2.19}$$

$$\varphi(1) = \varphi_q \equiv \frac{2p+1}{2}\pi, \qquad q = 0, \pm1, \pm2, \ldots \tag{8.3.2.20}$$

Integrating (8.3.2.18) using condition (8.3.2.19), we obtain the implicit solution

$$\mu x = \int_{\varphi_p}^{\varphi(x)} (1 - k^2\sin^2\varphi)^{-1/2}\, d\varphi, \quad p = 0, \pm1, \pm2, \ldots, \tag{8.3.2.21}$$

which involves Jacobi's elliptic integral of the first kind. The integrand in (8.3.2.21) is periodic with period π; it reaches a maximum, equal to $(1-k^2)^{-1/2}$, at $\varphi = \varphi_q$ and a minimum, equal to 1, at $\varphi = n\pi$, $n = 0, \pm1, \ldots$. By setting $x = 1$ in (8.3.2.21), we get

$$\mu = \int_{\varphi_p}^{\varphi_q} (1 - k^2\sin^2\varphi)^{-1/2}\, d\varphi, \quad p,\ q = 0, \pm1, \pm2, \ldots \tag{8.3.2.22}$$

The integrals on the right-hand side of (8.3.2.22) computed over one period are equal to $2K$, with

$$K(k) = \int_0^{\pi/2} (1 - k^2\sin^2\varphi)^{-1/2}\, d\varphi$$

being the complete elliptic integral of the first kind. The integrals in (8.3.2.22) are taken from any φ_p to any φ_q and, hence, are expressed as

$$\mu = \mu_m \equiv 2mK(k), \quad m = 1,\ 2,\ \ldots\ .$$

Thus, for any m we get a characteristic of the rod, that is, a relation between the loading parameter μ and deformation measure $k = \sin\frac{\alpha}{2}$. Since $K(0) = \frac{1}{2}\pi$, each curve branches out from $k = 0$, $\mu = \mu_m(0) = m\pi$, which represent square roots of the eigenvalues of the linear problem (8.3.2.13); this means that a linear eigenvalue problem defines bifurcation points of a nonlinear problem.

8.3.3 Bifurcation Analysis of Boundary Value Problems without Linearizing Equations

► **A mixed boundary value problem. Bifurcation diagrams. Turning points.**

Consider the mixed boundary value problem

$$y''_{xx} + \lambda f(y) = 0; \qquad y'_x(0) = 0, \quad y(1) = 0 \tag{8.3.3.1}$$

with $\lambda > 0$ and $f(y) > 0$.

We use the notation $\alpha = y|_{x=0}$. It can be shown that $y > 0$ and $y'_x \le 0$ $(0 \le x \le 1)$.

The parameter α can be treated as free; its value uniquely determines a value of λ and a solution $y = y(x)$ (by the uniqueness theorem for initial value problems). Hence, any solution of problem (8.3.3.1) can be associated with a point of a curve in the (α, λ) plane. It is customary to refer to these curves as *bifurcation diagrams*. The shape of any bifurcation diagram is determined by the *turning points* (singular points).

The general solution to problem (8.3.3.1) can be represented in implicit form as

$$\left[\int_0^y \frac{d\tau}{\sqrt{F(\alpha) - F(\tau)}}\right]^2 = 2\lambda(1-x)^2, \qquad F(y) = \int_0^y f(y)\, dy, \tag{8.3.3.2}$$

with λ related to α by

$$\lambda = \frac{1}{2}J^2(\alpha), \qquad J(\alpha) = \int_0^\alpha \frac{d\tau}{\sqrt{F(\alpha)-F(\tau)}}. \tag{8.3.3.3}$$

Formulas (8.3.3.2)–(8.3.3.3) follow from the results of Example 3.1.

The function $\lambda = \lambda(\alpha)$ (bifurcation diagram) passes the origin of coordinates $\lambda(0) = 0$; extrema of this function determine turning points. A necessary condition for the existence of an extremum is the equality $\lambda'_\alpha = 0$. Since $\lambda'_\alpha = JJ'_\alpha$, where $J = J(\alpha)$, the necessary condition becomes $J'_\alpha = 0$.

The integral with the variable upper limit $J(\alpha)$ has an integrable singularity (the denominator of the integrand vanishes at $\tau = \alpha$). Using the identity

$$\frac{1}{\sqrt{F(\alpha)-F(\tau)}} \equiv -2\frac{d}{d\tau}\left[\frac{1}{f(\tau)}\sqrt{F(\alpha)-F(\tau)}\right] - 2\frac{f'_\tau(\tau)}{f^2(\tau)}\sqrt{F(\alpha)-F(\tau)},$$

we rewrite the function $J(\alpha)$ in the form

$$J(a) = \frac{2\sqrt{F(\alpha)}}{f(0)} - 2\int_0^\alpha \frac{f'_\tau(\tau)}{f^2(\tau)}\sqrt{F(\alpha)-F(\tau)}\,d\tau. \tag{8.3.3.4}$$

The integral in (8.3.3.4) now has no singularity, which makes it more convenient for numerical calculations.

The following theorem holds.

THEOREM (KORMAN–LI–OUYANG). *A solution of the problem* (8.3.3.1) *with the maximal value* $\alpha = y(0)$ *is singular if and only if*

$$\sqrt{F(\alpha)}\int_0^\alpha \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau = 2. \tag{8.3.3.5}$$

Example 8.8. For a plane problem of combustion theory, one should set $f(y) = e^y$ in (8.3.3.1). Then, $F(\tau) = e^\tau - 1$. After computing the integral, condition (8.3.3.5) leads to the transcendental equation

$$\zeta \tanh \zeta = 1, \qquad \text{where} \qquad \zeta = (1 - e^{-\alpha})^{-1/2}.$$

Numerical analysis gives $\zeta \approx 1.19968$ and $\alpha \approx 1.1868$, which coincides with the results obtained in Example 3.17). The critical value $\alpha \approx 1.1868$ corresponds to a thermal explosion; the problem has no solution for greater α.

► **Extension to the case of a nonhomogeneous boundary condition.**

Consider the mixed boundary value problem

$$y''_{xx} + \lambda f(y) = 0; \qquad y'_x(0) = 0, \quad y(1) = \beta, \tag{8.3.3.6}$$

which differs from (8.3.3.1) in the more general second boundary condition.

If, as before, we use the notation $\alpha = y|_{x=0}$, the solution to problem (8.3.3.6) can be represented in the implicit form

$$\left[\int_\beta^y \frac{d\tau}{\sqrt{F(\alpha)-F(\tau)}}\right]^2 = 2\lambda(1-x)^2, \qquad F(y) = \int_0^y f(y)\,dy, \tag{8.3.3.7}$$

with λ related to a and b by

$$\lambda = \frac{1}{2}J^2(\alpha,\beta), \qquad J(\alpha,\beta) = \int_\beta^\alpha \frac{d\tau}{\sqrt{F(\alpha)-F(\tau)}}. \tag{8.3.3.8}$$

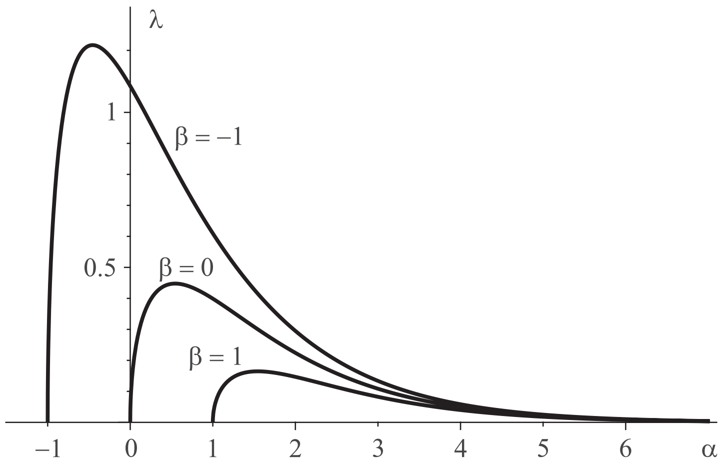

Figure 8.6: Bifurcation curves described by formula (8.3.3.9) at $\beta = -1,\ 0,\ 1$.

Example 8.9. For $f(y) = e^y$, the bifurcation diagram corresponding to problem (8.3.3.6) is expressed as

$$\lambda = \tfrac{1}{2} e^{-\alpha} \bigl[2 \ln\bigl(e^{\alpha/2} + \sqrt{e^{\alpha} - e^{\beta}}\bigr) - \beta\bigr]^2. \tag{8.3.3.9}$$

In the special case $b = 0$, this formula defines the bifurcation diagram for a plane problem of combustion theory (see Example 3.17).

Figure. 8.6 displays the bifurcation curves described by formula (8.3.3.9) at $\beta = -1,\ 0,\ 1$.

► A first boundary value problem. Reduction to a mixed boundary value problem.

Consider the first boundary value problem on the interval $[x_1, x_2]$ with equal boundary values

$$y''_{xx} + \lambda f(y) = 0; \qquad y(x_1) = \beta, \quad y(x_2) = \beta. \tag{8.3.3.10}$$

The substitution

$$z = \frac{2}{x_2 - x_1} x - \frac{x_2 + x_1}{x_2 - x_1}, \qquad y = \bar{y} \tag{8.3.3.11}$$

reduces problem (8.3.3.10) to a problem on the interval $[-1, 1]$:

$$\bar{y}''_{zz} + \bar{\lambda} f(\bar{y}) = 0; \qquad \bar{y}(-1) = \beta, \quad \bar{y}(1) = \beta; \qquad \bar{\lambda} = \tfrac{1}{4}(x_2 - x_1)^2 \lambda. \tag{8.3.3.12}$$

The solution to the first boundary value problem (8.3.3.12) is an even function, $\bar{y}(x) = \bar{y}(-x)$; in the domain $[0, 1]$, it coincides with the solution to a mixed problem of the form (8.3.3.6):

$$\bar{y}''_{zz} + \bar{\lambda} f(\bar{y}) = 0; \qquad \bar{y}'_z(0) = 0, \quad \bar{y}(1) = \beta; \qquad \bar{\lambda} = \tfrac{1}{4}(x_2 - x_1)^2 \lambda.$$

Therefore, the corresponding bifurcation diagram is described by formula (8.3.3.8), in which λ must be replaced with $\bar{\lambda}$.

◆ *See also Sections 3.3.3–3.3.7.*

⊙ *Literature for Section 8.3*: J. B. Keller (1960), J. Keller and S. Antman (1969), E. L. Reiss (1969), T. Laetsch (1970), E. L. Reiss and B. J. Matkowsky (1971), S.-H. Wang (1994, 2007), P. Korman and Y. Li (1999, 2010), P. Korman, Y. Li, and T. Ouyang (2005), P. Korman (2006).

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Chapter 9

Elementary Theory of Using Invariants for Solving Equations

This chapter describes a simple scheme for the analysis of mathematical equations which relies on using invariants and makes it possible to simplify algebraic equations, reduce the order of ordinary differential equations (or integrate them), and find exact solutions of nonlinear partial differential equations. Invariants are constructed by searching for transformations that preserve the form of the equations; the notions and complex techniques of symmetry analysis (see Chapter 9) are not used here. Numerous examples of solving specific differential equations are given. It is significant that even with the simplest linear transformations of translation and scaling, as well as their compositions, the number of solvable ordinary differential equations (or those admitting order reduction) that can be described in a unified way is more than those discussed in the overwhelming majority of available textbooks. For nonlinear equations of mathematical physics, this approach makes it possible to find all of the most common invariant solutions. To use this simple method, one does not have to have a strong mathematical background — what is required is to be able to solve simple algebraic equations (and system of equations) and differentiate. To distinguish it from the classical group analysis method, the approach presented in this chapter will be called the *method of invariants*.

9.1 Introduction. Symmetries. General Scheme of Using Invariants for Solving Mathematical Equations

9.1.1 Symmetries. Transformations Preserving the Form of Equations. Invariants

Symmetries of mathematical equations are understood as transformations that preserve the form of equations. Given below are examples of specific equations that remain unchanged under some simple transformations.

Example 9.1. Consider the biquadratic equation

$$x^4 + ax^2 + 1 = 0. \qquad (9.1.1.1)$$

The change of variable

$$x = -\bar{x}$$

results in exactly the same equation

$$\bar{x}^4 + a\bar{x}^2 + 1 = 0.$$

This means that equation (9.1.1.1) preserves its form under the transformation $x = -\bar{x}$.

Two other transformations

$$x = \pm\frac{1}{\widetilde{x}}$$

also preserve the form of equation (9.1.1.1), since multiplying by $\widetilde{x}^4$ gives

$$\widetilde{x}^4 + a\widetilde{x}^2 + 1 = 0.$$

Example 9.2. The form of the differential equation

$$y''_{xx} - y'_x = 0 \tag{9.1.1.2}$$

will not change if we make any of the transformations

$$\begin{aligned} &x = \bar{x} + a, \quad &&y = \bar{y} &&(a \text{ is any number}); \\ &x = \bar{x}, &&y = \bar{y} + b &&(b \text{ is any number}); \\ &x = \bar{x}, &&y = c\bar{y} &&(c \text{ is any nonzero number}), \end{aligned}$$

since we obtain exactly the same equation

$$\bar{y}''_{\bar{x}\bar{x}} - \bar{y}'_{\bar{x}} = 0$$

for each of the three transformations.

It will be shown below that transformations preserving the form of equations enable us to “multiply” solutions.

An *invariant of a transformation* is a nonconstant function that remains unchanged under the action of the transformation. Invariants of transformations can depend on the independent and dependent variables and their derivatives (when we deal with differential equations). To clarify the concept of an invariant that preserves its form under a transformation, we consider a few simple examples.

Example 9.3. The transformation of simultaneous translation in two coordinate axes

$$x = \bar{x} + a, \quad y = \bar{y} + a,$$

where a is any number, has the invariant

$$I = y - x = \bar{y} - \bar{x}.$$

If x is the independent variable and y is the dependent one, then the derivatives

$$I_2 = y'_x = \bar{y}'_{\bar{x}}, \quad I_3 = y''_{xx} = \bar{y}''_{\bar{x}\bar{x}}, \quad \dots$$

are also invariants of the transformation.

Example 9.4. The transformation of uniform scaling in two coordinate axes

$$x = a\bar{x}, \quad y = a\bar{y},$$

where a is any nonzero number, has the invariant

$$I_1 = \frac{y}{x} = \frac{\bar{y}}{\bar{x}}.$$

If x is the independent variable and y is the dependent one, then there are also more complicated invariants that depend on derivatives and remain unchanged under the transformation. Examples are

$$I_2 = y'_x = \bar{y}'_{\bar{x}}, \quad I_3 = xy''_{xx} = \bar{x}\bar{y}''_{\bar{x}\bar{x}}, \quad \dots .$$

9.1.2 General Scheme of Using Invariants for Solving Mathematical Equations

Displayed below is a schematic diagram for the analysis of mathematical equations which is based on searching for transformations that preserve the form of equations followed by changing, in the equations, from the original variable to new ones—invariants of the transformations.

Original equation

$\Downarrow$

Searching for a transformation preserving the equation

$\Downarrow$

Determining invariants that remain unchanged under the transformation

$\Downarrow$

Rewriting the original equation in terms of invariants

Figure 9.1: General scheme of using invariants for solving mathematical equations.

Once the above steps have been completed, the equation is often simplified and reduced to a solvable form. It is important to note that the above scheme can successfully be applied to various types of mathematical equations (see Sections 9.2 and 9.3 below).

For better understanding and learning of the ideas of how to use invariants or solving mathematical equations, we follow the approach "from simple to complex," first parenting results for algebraic equations, then for ordinary differential equations, and finally for nonlinear partial differential equations.

⊙ *Literature for Section 9.1*: A. D. Polyanin (2008), A. D. Polyanin and V. F. Zaitsev (2012).

9.2 Algebraic Equations and Systems of Equations

9.2.1 Algebraic Equations with Even Powers

Consider the algebraic equation

$$a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + a_{2n-4}x^{2n-4} + \cdots + a_4x^4 + a_2x^2 + a_0 = 0, \qquad (9.2.1.1)$$

which only contains even powers of x. A biquadratic equation is a special case of equation (9.2.1.1) with $n = 2$.

The change of variable

$$x = -\bar{x} \qquad (9.2.1.2)$$

leads to exactly the same equation for $\bar{x}$; equation (9.2.1.1) is said to be invariant under transformation (9.2.1.2). It follows that if $x = x_1$ is a solution of equation (9.2.1.1), then $x = -x_1$ is also a solution of this equation.

By squaring (9.2.1.2), we get a simple algebraic function that is left unchanged by transformation (9.2.1.2):

$$x^2 = \bar{x}^2. \qquad (9.2.1.3)$$

This function is an invariant of transformation (9.2.1.2). By taking the invariant (9.2.1.3) as the new variable, $z = x^2$, we can represent equation (9.2.1.1) of degree $2n$ as an equation of degree n:

$$a_{2n}z^n + a_{2n-2}z^{n-1} + a_{2n-4}z^{n-2} + \cdots + a_4z^2 + a_2z + a_0 = 0.$$

Thus, in this case, the change from the original variable x to the invariant $z = x^2$ of transformation (9.2.1.3) enables us to simplify the original equation—its degree has been halved.

9.2.2 Reciprocal Equations

► Reciprocal equations of even degree.

A *reciprocal* (*palindromic*) *polynomial equation* of even degree has the form

$$a_0x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0 \qquad (a_0 \neq 0). \tag{9.2.2.1}$$

The left-hand side of this equation is called a *reciprocal polynomial* or *palindromic polynomial*.

The change of variable

$$x = \frac{1}{\bar{x}} \tag{9.2.2.2}$$

transforms (9.2.2.1) into exactly the same equation (after multiplication by $\bar{x}^{2n}$). It follows that if $x = x_1$ is a root of equation (9.2.2.1), then $x = 1/x_1$ is also a root of the equation.

The simplest reciprocal equation is a quadratic equation

$$a_0x^2 + a_1x + a_0 = 0.$$

Dividing it by x and grouping the first and last terms together, we get

$$a_0\Big(x + \frac{1}{x}\Big) + a_1 = 0.$$

The result is convenient to rewrite as a first-degree equation

$$a_0z + a_1 = 0,$$

where

$$z = x + \frac{1}{x} = \bar{x} + \frac{1}{\bar{x}} \tag{9.2.2.3}$$

is the simplest *invariant of transformation* (9.2.2.2).

THEOREM (FOR THE RECIPROCAL EQUATION OF EVEN DEGREE). *In the general case, the reciprocal equation* (9.2.2.1) *of even degree* $2n$ *can be simplified with substitution* (9.2.2.3), *resulting in an algebraic equation of degree* n.

Example 9.5. Consider the quartic reciprocal equation

$$ax^4 + bx^3 + cx^2 + bx + a = 0.$$

Dividing it by x^2 and grouping terms, we get

$$a\Big(x^2 + \frac{1}{x^2}\Big) + b\Big(x + \frac{1}{x}\Big) + c = 0. \tag{9.2.2.4}$$

Taking into account that

$$\Big(x + \frac{1}{x}\Big)^2 = x^2 + 2 + \frac{1}{x^2} \quad\Longrightarrow\quad x^2 + \frac{1}{x^2} = z^2 - 2$$

and using the change of variable (9.2.2.3), which is an invariant of transformation (9.2.2.2), we reduce (9.2.2.4) to the quadratic equation

$$az^2 + bz + c - 2a = 0.$$

► Reciprocal equations of odd degree.

THEOREM (FOR THE RECIPROCAL EQUATION OF ODD DEGREE). *In the general case, a reciprocal equation of odd degree*

$$P_{2n+1}(x)=0, \quad \text{where} \quad P_{2n+1}(x)\equiv a_0x^{2n+1}+a_1x^{2n}+a_2x^{2n-1}+\dots+a_2x^2+a_1x+a_0,$$

has a root $x=-1$*, and the left-hand side can be represented as*

$$P_{2n+1}(x)=(x+1)Q_{2n}(x),$$

where $Q_{2n}(x)$ *is a reciprocal polynomial of degree* $2n$*.*

Example 9.6. The cubic reciprocal equation

$$ax^3+bx^2+bx+a=0$$

can be represented in the form

$$(x+1)[ax^2+(b-a)x+a]=0.$$

It follows from Theorem 2 that a reciprocal equation of degree $2n+1$ can be reduced, by dividing by $(x+1)$ and introducing the new variable (9.2.2.3), to an algebraic equation of degree n.

► Generalized reciprocal equations of even degree.

A *generalized reciprocal polynomial equation* of even degree has the form

$$\begin{aligned} a_0x^{2n}+a_1x^{2n-1}+\dots+a_{n-1}x^{n+1}+a_nx^n \\ +\lambda a_{n-1}x^{n-1}+\lambda^2a_{n-2}x^{n-2}+\dots+\lambda^{n-1}a_1x+\lambda^na_0=0 \qquad (a_0\neq 0). \end{aligned} \tag{9.2.2.5}$$

The first $n+1$ terms (written in the first row) coincide with the respective terms of the reciprocal equation (9.2.2.1) and the remaining terms (in the second row) differ from the respective terms of equation (9.2.2.1) by factors λ^m. In the special case $\lambda=1$, equation (9.2.2.5) coincides with (9.2.2.1).

It is not difficult to verify that the transformation

$$x=\frac{\lambda}{\bar{x}} \tag{9.2.2.6}$$

leaves equation (9.2.2.5) unchanged, and the simplest invariant of transformation (9.2.2.6) is written as

$$z=x+\frac{\lambda}{x}=\bar{x}+\frac{\lambda}{\bar{x}}. \tag{9.2.2.7}$$

The introduction of the new variable (9.2.2.7) reduces (9.2.2.5) to an equation of degree n.

Example 9.7. Consider the quartic equation

$$ax^4+bx^3+cx^2-bx+a=0,$$

which is a special case of equation (9.2.2.5) with $n=2$ and $\lambda=-1$. The change of variable

$$z=x-\frac{1}{x}$$

leads to the quadratic equation

$$az^2+bz+2a+c=0.$$

9.2.3 Systems of Algebraic Equations Symmetric with Respect to Permutation of Arguments

A bivariate polynomial $P(x, y)$ is called *symmetric* if it does not change after the permutation of its arguments: $P(x, y) = P(y, x)$.

Remark 9.1. In terms of transformations, a symmetric polynomial is defined as a polynomial that is left unchanged by the transformation $x = \bar{y}$, $y = \bar{x}$.

The simplest symmetric polynomials

$$u = x + y, \quad w = xy \tag{9.2.3.1}$$

are called *elementary*. These polynomials are the simplest algebraic invariants to the permutation of arguments. Any symmetric bivariate polynomial can be uniquely expressed in terms of the elementary polynomials.

For the solution of systems of two algebraic equations

$$P(x, y) = 0, \quad Q(x, y) = 0,$$

where P and Q are symmetric polynomials, it is helpful to use the elementary symmetric polynomials (9.2.3.1) as the new variables. Such systems possess the following property: if $x = x_0$, $y = y_0$ is a solution to the system, then $x = y_0$, $y = x_0$ is also a solution.

Example 9.8. Consider the nonlinear system of algebraic equations

$$\begin{aligned} x^2 + axy + y^2 &= b, \\ x^4 + cx^2y^2 + y^4 &= d. \end{aligned} \tag{9.2.3.2}$$

It remains unchanged under the permutation of the variables.

In (9.2.3.2), by changing from x and y to the variables (9.2.3.1) and taking into account the formulas

$$\begin{aligned} x^2 + y^2 &= (x + y)^2 - 2xy = u^2 - 2w, \\ x^4 + y^4 &= (x^2 + y^2)^2 - 2x^2y^2 = (u^2 - 2w)^2 - 2w^2 = u^4 - 4u^2w + 2w^2, \end{aligned}$$

we obtain

$$\begin{aligned} u^2 - (a - 2)w &= b, \\ u^4 - 4u^2w + (c + 2)w^2 &= d. \end{aligned} \tag{9.2.3.3}$$

Eliminating u, we arrive at the quadratic equation

$$(a^2 + c - 2)w^2 - 2abw + b^2 - d = 0.$$

The further procedure of finding solutions is straightforward and omitted here.

Example 9.9. Consider the nonlinear system of algebraic equations

$$\begin{aligned} x^2 + y^2 &= a, \\ x^3 + y^3 &= b. \end{aligned} \tag{9.2.3.4}$$

Changing to the variables (9.2.3.1) and taking into account that $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$, we get

$$\begin{aligned} u^2 - 2w &= a, \\ u^3 - 3uw &= b. \end{aligned}$$

Eliminating w yields the cubic equation

$$u^3 - 3au + 2b = 0. \qquad (9.2.3.5)$$

Note that the straightforward elimination of y from system (9.2.3.4) results in a much more complex equation of degree 6:

$$(a - x^2)^3 = (b - x^3)^2 \quad \Longrightarrow \quad 2x^6 - 3ax^4 - 2bx^3 + 3a^2x^2 + b^2 - a^3 = 0.$$

⊙ *Literature for Section 9.2*: N. A. Kudryashov (1998), V. G. Boltyanskii and N. Ya. Vilenkin (2002), A. D. Polyanin (2008), A. D. Polyanin and V. F. Zaitsev (2012).

9.3 Ordinary Differential Equations

9.3.1 Transformations Preserving the Form of Equations. Invariants

An ordinary differential equation

$$F(x, y, y'_x, \ldots, y_x^{(n)}) = 0 \qquad (9.3.1.1)$$

is said to be *invariant under an invertible transformation*

$$x = \varphi(\bar{x}, \bar{y}), \quad y = \psi(\bar{x}, \bar{y}) \qquad (9.3.1.2)$$

if the substitution of (9.3.1.2) into (9.3.1.1) leads to exactly the same equation

$$F(\bar{x}, \bar{y}, \bar{y}'_{\bar{x}}, \ldots, \bar{y}_{\bar{x}}^{(n)}) = 0. \qquad (9.3.1.3)$$

The function F is the same in both equations (9.3.1.1) and (9.3.1.3).

Transformations that preserve the form of an equation can be used to “multiply” its solutions. Indeed, suppose

$$y = g(x) \qquad (9.3.1.4)$$

is a particular solution to equation (9.3.1.1). Since equation (9.3.1.1) is left the same by the change of variables (9.3.1.2), then

$$\bar{y} = g(\bar{x}) \qquad (9.3.1.5)$$

is a solution to the transformed equation (9.3.1.3). In (9.3.1.5), changing back to the old variables using (9.3.1.2) (the relations must be solved for $\bar{x}$ and $\bar{y}$), we obtain a solution to equation (9.3.1.1) that differs, in general, from the original solution (9.3.1.4).

Example 9.10. The third-order equation

$$y'''_{xxx} - y'_x = 0 \qquad (9.3.1.6)$$

has a particular solution

$$y = e^x.$$

The transformation

$$x = \bar{x} + a, \quad y = \bar{y} + b \qquad (9.3.1.7)$$

leaves the equation unchanged, and therefore the transformed equation $\bar{y}'''_{\bar{x}\bar{x}\bar{x}} - \bar{y}'_{\bar{x}} = 0$ has a solution $\bar{y} = e^{\bar{x}}$. Inserting the old variables, by inverting the formulas (9.3.1.7), we obtain a new solution to equation (9.3.1.6):

$$y = Ae^x + b, \quad A = e^{-a},$$

which involves two arbitrary constants A and b.

A function $I(x, y, y'_x)$, other than a constant, is called an *invariant of transformation* (9.3.1.2) if it remains unchanged under the transformation:

$$I(x, y, y'_x) = I(\bar{x}, \bar{y}, \bar{y}'_{\bar{x}}).$$

Remark 9.2. If $I = I(x, y, y'_x)$ is an invariant of transformation (9.3.1.2), then $\Psi(I)$, where Ψ is an arbitrary function, is also an invariant of the transformation.

9.3.2 Order Reduction Procedure for Equations with $n \geq 2$ (Reduction to Solvable Form with $n = 1$)

Let us now consider in more detail the scheme outlined in Section M28.1 for using invariants in the analysis of mathematical equations as applied to ordinary differential equations.

Given an nth-order equation (9.3.1.1), one should seek, at the first stage, a transformation

$$x = \varphi(\bar{x}, \bar{y}; a), \qquad y = \psi(\bar{x}, \bar{y}; a) \tag{9.3.2.1}$$

that preserves the form of the equation. Transformation (9.3.2.1) must depend on a *single free parameter* $a \in [a_1, a_2]$; the original equation (9.3.1.1) is independent of this parameter.

At the second stage, for second- and higher-order equations ($n \geq 2$), one constructs two functionally independent invariants of transformation (9.3.2.1):

$$I_1 = I_1(x, y), \qquad I_2 = I_2(x, y, y'_x). \tag{9.3.2.2}$$

At the third stage, the invariants (9.3.2.2) are taken as the new variable for equation (9.3.1.1) and the transformation

$$u = I_2, \quad z = I_1, \quad u = u(z)$$

is performed. This results in an $(n-1)$st-order equation, so that the order of the original equation is reduced by one.

For first-order equations ($n = 1$), one should make the change of variable

$$z = I_1, \quad z = z(x),$$

in (9.3.1.1) at the third stage. This results in a solvable (separable) equation.

9.3.3 Simple Transformations. Invariant Determination Procedure

In what follows, we will only use the simplest transformations

$$\begin{aligned} &x = \bar{x} + A, \quad y = \bar{y} + B \qquad (\textit{translation}); \\ &x = A\bar{x}, \qquad\ \ y = B\bar{y} \qquad\quad (\textit{scaling}) \end{aligned}$$

and their compositions

$$x = A_1\bar{x} + B_1, \quad y = A_2\bar{y} + B_2. \tag{9.3.3.1}$$

Then the derivatives satisfy linear relations:

$$y'_x = \frac{A_2}{A_1}\bar{y}'_{\bar{x}}, \quad y''_{xx} = \frac{A_2}{A_1^2}\bar{y}''_{\bar{x}\bar{x}}, \quad y^{(n)}_x = \frac{A_2}{A_1^n}\bar{y}^{(n)}_{\bar{x}}. \tag{9.3.3.2}$$

The transformation coefficients A_1, A_2, B_1, and B_2 are determined from the invariance condition for the equation; these coefficients must depend on a single free parameter a.

The following statement is true. Suppose transformation (9.3.3.1) preserves the form of an equation that has a particular solution (9.3.1.4). Then

$$y = B_2 + A_2 g\left(\frac{x - B_1}{A_1}\right)$$

is also a solution of the equation.

The first invariant I_1 is obtained by eliminating a from (9.3.3.1). The second invariant I_2 is obtained by eliminating a from one of the relations in (9.3.3.1) and the first relation in (9.3.3.2).

9.3.4 Analysis of Some Ordinary Differential Equations. Useful Remarks

Example 9.11. The second-order equation

$$y''_{xx} = F(x, y'_x), \tag{9.3.4.1}$$

which does not involve y explicitly, remains unchanged under an arbitrary translation in the dependent variable: $y \Longrightarrow y + a$ (which corresponds to $y = \bar{y} + a$), where a is a free parameter. Moreover, out of the three variables x, y, and y'_x, two remain unchanged:

$$x, \quad y'_x.$$

These are invariants of equation (9.3.4.1); hence, $I_1 = x$ and $I_2 = y'_x$. These can be taken as the new variables:

$$u = y'_x, \quad z = x, \quad u = u(z).$$

As a result, we arrive at a first-order equation: $u'_x = F(x, u)$.

Example 9.12. The autonomous second-order equation

$$y''_{xx} = F(y, y'_x), \tag{9.3.4.2}$$

which does not involve x explicitly, remains unchanged under an arbitrary translation in the independent variable: $x \Longrightarrow x + a$, where a is a free parameter. Out of the three variables x, y, and y'_x, two remain unchanged:

$$y, \quad y'_x.$$

These are invariants of equation (9.3.4.2); hence, $I_1 = y$ and $I_2 = y'_x$. We choose them as the new variables:

$$u = y'_x, \quad z = y, \quad u = u(z).$$

This results in a first-order equation: $uu'_y = F(y, u)$.

Example 9.13. The nonlinear second-order equation

$$y''_{xx} = yF\left(x, \frac{y'_x}{y}\right) \tag{9.3.4.3}$$

does not change if the dependent variable is scaled: $y \Longrightarrow ay$. Two combinations out of the three variables x, y, and y'_x do not change:

$$x, \quad \frac{y'_x}{y}.$$

These are invariants of equation (9.3.4.3) and can be taken as the new variables:

$$u = \frac{y'_x}{y}, \quad z = x, \quad u = u(z).$$

Differentiating u linear x yields

$$u'_x = \frac{y''_{xx}}{y} - \left(\frac{y'_x}{y}\right)^2 = \frac{y''_{xx}}{y} - u^2.$$

Using this relation to eliminate y''_{xx} in (9.3.4.3), one arrives at the first-order equation

$$u'_x = F(x, u) - u^2.$$

Remark 9.3. For $F(x, u) = g(x) + f(x)u$, the original equation (9.3.4.3) is a general linear homogeneous second-order equation. With the above transformation, it is reduced to a first-order equation with a quadratic nonlinearity.

Example 9.14. Consider the nonlinear second-order equation

$$yy''_{xx} - (y'_x)^2 = ky^3 e^{\lambda x}. \tag{9.3.4.4}$$

We look for an invariant transformation of the form

$$x = \bar{x} + b, \quad y = a\bar{y}. \tag{9.3.4.5}$$

Substituting (9.3.4.5) into (9.3.4.4) and canceling by a, we obtain

$$\bar{y}\bar{y}''_{\bar{x}\bar{x}} - (\bar{y}'_{\bar{x}})^2 = ae^{\lambda b} k\bar{y}^3 e^{\lambda \bar{x}}.$$

Requiring that this equation coincide with (9.3.4.4), we get the relation for determining the parameter b:

$$ae^{\lambda b} = 1 \quad \Longrightarrow \quad b = -\frac{1}{\lambda}\ln a. \tag{9.3.4.6}$$

The parameter a remains free.

Substituting (9.3.4.6) into (9.3.4.5) and eliminating a from the second relation with the aid of the first relation, we obtain

$$y = \bar{y}e^{\lambda(\bar{x} - x)} \quad \Longrightarrow \quad ye^{\lambda x} = \bar{y}e^{\lambda \bar{x}}.$$

Hence,

$$I_1 = ye^{\lambda x} \tag{9.3.4.7}$$

is an invariant of transformation (9.3.4.5)–(9.3.4.6). Another invariant can be found by calculating the derivative

$$y'_x = a\bar{y}'_{\bar{x}}.$$

Eliminating a with the aid of the second relation in (9.3.4.5) yields

$$\frac{y'_x}{y} = \frac{\bar{y}'_{\bar{x}}}{\bar{y}} = I_2. \tag{9.3.4.8}$$

To reduce the order of the original equation, one should take the invariants (9.3.4.7)–(9.3.4.8) as the new variables:

$$z = e^{\lambda x} y, \quad u = \frac{y'_x}{y}, \quad u = u(z). \tag{9.3.4.9}$$

On the one hand,

$$u'_x = \frac{y''_{xx}}{y} - \left(\frac{y'_x}{y}\right)^2 = \frac{y''_{xx}}{y} - u^2; \tag{9.3.4.10}$$

on the other hand,

$$u'_x = u_z z'_x = (\lambda e^{\lambda x} y + e^{\lambda x} y'_x) u'_z = \left(\lambda z + e^{\lambda x} y \frac{y'_x}{y}\right) u'_z = (\lambda z + zu) u'_z. \qquad (9.3.4.11)$$

Equating (9.3.4.10) with (9.3.4.11), we get

$$\frac{y''_{xx}}{y} - u^2 = (\lambda z + zu) u'_z \quad \Longrightarrow \quad \frac{y''_{xx}}{y} = u^2 + (\lambda z + zu) u'_z.$$

Inserting this into (9.3.4.4) and performing elementary rearrangements, we arrive at a separable first-order equation:

$$(\lambda + u) u'_z = k.$$

Remark 9.4. The more general, nonlinear second-order equation

$$y''_{xx} = yF\left(e^{\lambda x} y, \frac{y'_x}{y}\right)$$

has similar properties. Transformation (9.3.4.9) reduces it to the first-order equation

$$u^2 + (\lambda z + zu) u'_z = F(z, u).$$

Example 9.15. Now consider the nonlinear first-order equation

$$y'_x = f\left(\frac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2}\right). \qquad (9.3.4.12)$$

Its left-hand side remains unchanged under transformations of the form

$$x = a\bar{x} + b, \quad y = a\bar{y} + c, \qquad (9.3.4.13)$$

where a, b, and c are arbitrary constants. Substituting (9.3.4.13) into the argument of the right-hand side function of (9.3.4.12) gives

$$\frac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2} = \frac{a(\alpha_1 \bar{x} + \beta_1 \bar{y}) + \alpha_1 b + \beta_1 c + \gamma_1}{a(\alpha_2 \bar{x} + \beta_2 \bar{y}) + \alpha_2 b + \beta_2 c + \gamma_2}. \qquad (9.3.4.14)$$

For equation (9.3.4.12) to be invariant under transformation (9.3.4.13), one must set

$$\alpha_1 b + \beta_1 c + \gamma_1 = a\gamma_1, \quad \alpha_2 b + \beta_2 c + \gamma_2 = a\gamma_2 \qquad (9.3.4.15)$$

in (9.3.4.14). These relations can be viewed as a system of two linear algebraic equations for the coefficients b and c; the coefficient a remains arbitrary. Thus, equation (9.3.4.12) is invariant under transformation (9.3.4.13) subject to conditions (9.3.4.15). The argument of the right-hand side function of (9.3.4.12) is an invariant of the transformation. Therefore, the change of variable

$$z = \frac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2}, \quad \text{where} \quad z = z(x), \qquad (9.3.4.16)$$

should be made in equation (9.3.4.12). Solving (9.3.4.16) for y, differentiating with respect to x, substituting y'_x by $f(z)$, which follows from (9.3.4.12) and (9.3.4.16), and performing elementary rearrangements, one arrives at the separable equation

$$\frac{(\alpha_2\beta_1 - \alpha_1\beta_2)x + \beta_1\gamma_2 - \beta_2\gamma_1}{(\beta_2 z - \beta_1)^2} z'_x = f(z) + \frac{\alpha_2 z - \alpha_1}{\beta_2 z - \beta_1}.$$

Table 9.1 lists some second-order ordinary differential equations that admit order reduction by using the simplest invariant transformations. For first-order equations, where $F(u, v, w)$ is independent of the third argument, the equations listed in Table M28.1 can be solved by changing from y to the new dependent variable $z = I_1(x, y)$, where I_1 is the first invariant.

The results presented in Table 9.1 are easy to generalize to nonlinear equations of arbitrary order.

TABLE 9.1
Some second-order ordinary differential equations that admit order reduction, and invariants

Equation	Invariant transformation	First invariant	Second invariant
$F(x, y'_x, y''_{xx})=0$	$y=\bar{y}+a$	$I_1=x$	$I_2=y'_x$
$F(y, y'_x, y''_{xx})=0$	$x=\bar{x}+a$	$I_1=y$	$I_2=y'_x$
$F(\alpha x+\beta y, y'_x, y''_{xx})=0$	$x=\bar{x}+a\beta,\ y=\bar{y}-a\alpha$	$I_1=\alpha x+\beta y$	$I_2=y'_x$
$F\Big(x, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\Big)=0$	$y=a\bar{y}$	$I_1=x$	$I_2=\frac{y'_x}{y}$
$F(y, xy'_x, x^2y''_{xx})=0$	$x=a\bar{x}$	$I_1=y$	$I_2=xy'_x$
$F\Big(e^{\lambda x}y, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\Big)=0$	$x=\bar{x}-\frac{1}{\lambda}\ln a,\ y=a\bar{y}$	$I_1=e^{\lambda x}y$	$I_2=\frac{y'_x}{y}$
$F(e^{\lambda x}y, e^{\lambda x}y'_x, e^{\lambda x}y''_{xx})=0$	$x=\bar{x}-\frac{1}{\lambda}\ln a,\ y=a\bar{y}$	$I_1=e^{\lambda x}y$	$I_2=e^{\lambda x}y'_x$
$F(xe^{\lambda y}, xy'_x, x^2y''_{xx})=0$	$x=a\bar{x},\ y=\bar{y}-\frac{1}{\lambda}\ln a$	$I_1=xe^{\lambda y}$	$I_2=xy'_x$
$F(x^ky, x^{k+1}y'_x, x^{k+2}y''_{xx})=0$	$x=a\bar{x},\ y=a^{-k}\bar{y}$	$I_1=x^ky$	$I_2=x^{k+1}y'_x$
$F\Big(x^ny^m, \frac{xy'_x}{y}, \frac{x^2y''_{xx}}{y}\Big)=0$	$x=a^m\bar{x},\ y=a^{-n}\bar{y}$	$I_1=x^ny^m$	$I_2=\frac{xy'_x}{y}$

Remark 9.5. The above method of invariants for the analysis of ordinary differential equations takes advantage of the ideas of the group analysis method but is much simpler. To learn how to apply the former method, one should only be able to solve simple algebraic equations (and systems) and differentiate, whereas the application of the group analysis method requires, at intermediate stages, the solution of partial differential equations (which leads beyond the standard courses of ordinary differential equations). Other advantages of the simple method of invariants described include the fact that there is no need to introduce new concepts, which are abundant in the group analysis method, and that the number of solvable ordinary differential equations (or those admitting order reduction) describable in a unified way is more than those discussed in the overwhelming majority of available textbooks.

◆ *The book by Polyanin and Zaitsev (2012, Section 28.4) gives examples of using the elementary theory of invariants to construct exact solutions of nonlinear partial differential equations.*

⊙ *Literature for Section 9.3*: G. W. Bluman and J. D. Cole (1974), P. J. Olver (1995), N. A. Kudryashov (1998), N. H. Ibragimov (1994, 1999), P. E. Hydon (2000), V. G. Boltyanskii and N. Ya. Vilenkin (2002), A. D. Polyanin and V. F. Zaitsev (2003, 2012), A. D. Polyanin (2008).

Chapter 10

Methods for the Construction of Particular Solutions

10.1 Two Problems on Searching for Particular Solutions to ODEs with Parameters

10.1.1 Preliminary Remarks. Traveling Wave Solutions

► **Preliminary remarks.**

In the theory of ordinary differential equations, it is customary to deal with methods† that allow one to find general solutions. However, methods for seeking particular solutions to nonlinear ODEs receive practically no attention. This hinders the development of related methods of the theory of partial differential equations for finding exact solutions to nonlinear PDEs that can be expressed in terms of elementary function, special functions or quadratures.

► **Traveling wave solutions for nonlinear PDEs and their relation to ODEs.**

The overwhelming majority of nonlinear equations of mathematical physics are of partial differential equations of the form

$$\Phi(w, w_z, w_t, w_{zz}, w_{zt}, w_{tt}, \dots) = 0, \qquad (10.1.1.1)$$

which do not explicitly involve the independent variable; for simplicity, we consider equations with two independent variables, t and z, where t can be treated as time or a space coordinate.

In general, equation (10.1.1.1) admits solutions of the traveling wave type:

$$w = y(x), \quad x = a_1 z + a_2 t, \qquad (10.1.1.2)$$

where a_1 and a_2 are arbitrary constants. Substituting (10.1.1.2) into (10.1.1.1) yields the ordinary differential equation

$$\Phi(y, a_1 y'_x, a_2 y'_x, a_1^2 y''_{xx}, a_1 a_2 y''_{xx}, a_2^2 y''_{xx}, \dots) = 0. \qquad (10.1.1.3)$$

† Here and henceforth, we discuss exact methods for the integration of differential equations.

Thus, the ordinary differential equation (10.1.1.3) describes exact solutions to the special type of partial differential equations (10.1.1.1). Since the traveling wave solutions (10.1.1.2) are the most common type of exact solution to nonlinear equations of mathematical physics, it is of great importance to be able to find solutions to relevant ordinary differential equations.

Apart from the free parameters a_1 and a_2, equation (10.1.1.3) can often involve other parameters, which can also vary within certain ranges. In particular, for equations of the form (10.1.1.1), which can be represented in the divergence form (as a conservation law)

$$\begin{gathered}\frac{\partial}{\partial t}\Phi_1+\frac{\partial}{\partial z}\Phi_2=0,\\ \Phi_i=\Phi_i(w,w_z,w_t,w_{zz},w_{zt},w_{tt},\dots),\quad i=1,\,2,\end{gathered}\tag{10.1.1.4}$$

searching for traveling wave solutions (10.1.1.2) leads to the ordinary differential equation

$$\begin{gathered}a_2\Phi_1+a_1\Phi_2+a_3=0,\\ \Phi_i=\Phi_i(y,\,a_1y'_x,a_2y'_x,a_1^2y''_{xx},a_1a_2y''_{xx},a_2^2y''_{xx},\dots),\end{gathered}\tag{10.1.1.5}$$

involving three arbitrary constants: a_1, a_2, and a_3.

Importantly, methods of generalized and functional separation of variables reduce nonlinear PDEs to ODEs or systems of ODEs, which can include many free parameters that do not appear in the original equation. For relevant examples, see the literature cited at the end of the current section.

10.1.2 Two Problems for ODEs with Parameters. Conditional Capacity of Exact Solutions.

► Two problems for ODEs describing exact solutions to PDEs.

It follows from the above that there are a large number of equations in mathematical physics whose solutions can be expressed in terms of ordinary differential equations*

$$F(x,y,y'_x,\dots,y_x^{(n)};a_1,\dots,a_k)=0,\tag{10.1.2.1}$$

containing a set of free parameters a_i ($i=1,\dots,k$), which are not involved in the original partial differential equation. Below are two fundamentally different problems arising in dealing with equation (10.1.2.1).

PROBLEM 1. Find the values of the parameters a_i at which the general solution of equation (10.1.2.1) is possible (here and henceforth, we mean solutions that can be expressed in terms of elementary or special functions).

PROBLEM 2. Find the values of the parameters a_i at which the partial (exact) solutions of equation (10.1.2.1) are possible.

► Conditional capacity of exact solutions to nonlinear PDEs.

For a comparative analysis of the results of solving problems 1 and 2, the following definitions will be useful.

*The form of these solutions can differ from (10.1.1.2).

Definition. The conditional capacity of an exact solution of a nonlinear PDE is equal to the number of arbitrary constants involved in the solution but not the original equation. The conditional capacity of a solution will be denoted "cc."

The practical sense of this definition is clear: the more arbitrary constants are involved in a solution, the more important, interesting, and valuable the solution is (the generality of a solution is determined by the number of arbitrary constants involved).

In problem 1, the general solution to the corresponding ordinary differential equation (10.1.2.1) can be obtained in closed form in only relatively few specific values of the parameters a_i or under certain limitations; in the latter case, there will be fewer free parameters, $a_1, \dots, a_p$, and the other parameters, $a_{p+1}, \dots, a_k$, will be dependent on them. The conditional capacities of such solutions is calculated as

$$\mathsf{cc}_1 = p + n, \tag{10.1.2.2}$$

where n is the order of equation (10.1.2.1).

In problem 2, one often manages to obtain an exact solution to the ordinary differential equation (10.1.2.1) under fewer constraints on the parameters a_i, suggesting that more free parameters, $a_1, \dots, a_q$, will remain than in problem 1 ($q \geq p$). In addition, the exact solution itself can depend on m constants of integration, with $m \leq n$. The conditional capacity of such solutions is evaluated as

$$\mathsf{cc}_2 = q + m. \tag{10.1.2.3}$$

By comparing formulas (10.1.2.2) and (10.1.2.3), one can see that the conditional capacity of particular solutions to problem 2 can be lower than, equal to, or higher than that of general solutions to problem 1. This suggests that solutions to problems 1 and 2 are, in general, equally important with respect to the analysis of the original nonlinear PDEs.

Problem 1 is classical; it is solved using well-developed methods of integration of ordinary differential equations.

Problem 2 is nonclassical; solution methods for this problem have not yet been sufficiently well developed, which is primarily because problem 2 has not received much attention from the specialists in the area of ordinary differential equations. In the literature, there are relatively few methods for solving such problems, which, in addition, often have a very narrow area of application (these methods are most frequently used to treat autonomous equations with power-law nonlinearity).

► Two problems for ODEs with parameters.

The statements of problems 1 and 2 above can be arrived at from completely different considerations, without taking into account any relations between ordinary differential equations. For example, one can treat the ordinary differential equation (10.1.2.1) as dependent on physical-chemical constants a_i, which play an important role in applications and can vary within wide ranges. In this case, the role of the constants of integration, appearing in the general or particular solution to the equation, and the role of the physical-chemical constants a_i can be treated as equal; often, finding a particular solution to a wide class of equations can be much more useful than finding the general solution to a narrow class of equations.

Subsequent sections outline methods for constructing particular solutions to nonlinear ordinary differential equation with variable parameters without going into the physical or other meaning of these parameters.

⊙ *Literature for Section 10.1*: A. D. Polyanin and V. F. Zaitsev (2003, 2012), V. A. Galaktionov and S. R. Svirshchevskii (2006), A. D. Polyanin (2016).

10.2 Method of Undetermined Coefficients and Its Special Cases

10.2.1 General Description of the Method of Undetermined Coefficients

In general, the method of undetermined coefficients as applied to linear or nonlinear ordinary differential equations suggests particular solutions should be sought in a preset form dependent on a set of free (undetermined) parameters. On substituting the solution structure into the equation, one selects the values of the parameters so as to satisfy the equation exactly. Particular solutions are usually sought in the form of a finite sum

$$y = \sum_{k=0}^{n} a_k \varphi_k(x) \tag{10.2.1.1}$$

where $\varphi_k(x)$ are given elementary functions and a_k are free (undetermined) parameters.

Most frequently, solutions are constructed using special cases of formulas (10.2.1.1):

$$y = \sum_{k=0}^{n} a_k \varphi^k(\lambda x) \quad \text{or} \quad y = \sum_{k=0}^{n} a_k \varphi^{m_k}(\lambda x). \tag{10.2.1.2}$$

These are based on a single generating function $\varphi(z)$, which is present by the researcher. The constants n, a_k, m_k, and λ are to be determined; the second formula in (10.2.1.2) can include negative powers m_k. As $\varphi(z)$ in (10.2.1.2), one usually takes power-law, exponential, hyperbolic, or trigonometric functions (see Sections 10.2.2 and 10.2.3).

The determination of the constants n, a_k, m_k, and λ in (10.2.1.2) can often be simplified with modern computer algebra systems such as Maple or Mathematica, which allow one to perform a lot of cumbersome analytical calculations.

Remark 10.1. Seeking solutions using the first formula in (10.2.1.2) is equivalent to carrying out two consecutive actions: (i) performing the change of variable $\xi = \varphi(\lambda x)$ in the original ODE and (ii) searching for a solution to the transformed equation in the truncated series form $y = \sum_{k=0}^{n} a_k \xi^k$. This approach is technically simpler than the direct substitution of the first formula (10.2.1.2) into the original equation.

One may succeed in searching for particular solutions in a more general form than (10.2.1.1):

$$y = \Phi(x; a_0, \dots, a_n), \tag{10.2.1.3}$$

where $\Phi(x; a_0, \dots, a_n)$ is a given function and $a_0, \dots, a_n$ are free parameters.

A considerable limitation of such direct methods is that solutions are sought in explicit form, while the overwhelming majority of known general solutions to nonlinear equations are in implicit or parametric form (this follows from a statistical analysis of the results presented in the present handbook).

Most frequently, the method of undetermined coefficients is used to seek particular solutions to linear nonhomogeneous ODEs with constant coefficients. Table 4.1 lists recommended solution structures for such equations for special forms of the right-hand side (in particular, if the right-hand side of the equation is a polynomial, solutions are sought in the polynomial form).

Remark 10.2. The special cases of the method of undetermined coefficients discussed below in Sections 10.2.2 and 10.2.3 have become very common in searching for exact traveling-wave solutions to nonlinear partial differential equations (such solutions are described by ODEs following from the original PDEs).

10.2.2 Power-Law, Tanh-Coth, and Sine-Cosine Methods

▶ Methods based on power-law functions.

1°. *Power-law function method.* The main idea of the method is the assumption that a particular solution of the ODE can be expressed in terms of power-law functions, which corresponds to $\varphi_k(x) = x^{p_k}$ in (10.2.1.1), with the exponents p_k to be determined. In the special case $p_k = k$, such a solution will be a polynomial of degree n.

Example 10.1. Consider the generalized Emden–Fowler equation

$$y''_{xx} = Ax^n y^m (y'_x)^l. \tag{10.2.2.1}$$

Its particular solution will be sought in the form of a power-law function

$$y = ax^p. \tag{10.2.2.2}$$

Substituting (10.2.2.2) into (10.2.2.1) yields

$$ap(p-1)x^{p-2} = Aa^m (ap)^l x^{n+mp+l(p-1)}.$$

For this equation to be satisfied identically, one must set

$$p - 2 = n + mp + l(p-1), \qquad ap(p-1) = Aa^m (ap)^l.$$

On solving this system for a and p, we arrive at the constants determining solution (10.2.2.2):

$$p = \frac{n-l+2}{1-m-l}, \qquad a = \left(\frac{p-1}{Ap^{l-1}}\right)^{\frac{1}{m+l-1}}$$

with $n-l+2 \neq 0$, $m+l-1 \neq 0$, and $n+m+1 \neq 0$. Furthermore, for $l > 0$, there is a degenerate solution (10.2.2.2) with $p = 0$ and any a.

2°. *A modification.* The following fact may be useful in searching for particular solutions.

PROPOSITION. Suppose one deals with a nonlinear differential equation for $y = y(x)$, which has been reduced with a change of variable $y = f(x, w)$ to the equation

$$\Phi(w''_{xx}, \dots, w^{(n)}_x) + (b_1 x^2 + b_2 x + b_3)(w''_{xx})^2 + b_4 w w''_{xx} + (b_5 x^2 + b_6 x + b_7) w''_{xx}$$
$$+ b_8 (w'_x)^2 + (b_9 x + b_{10}) w'_x + b_{11} w + b_{12} x^2 + b_{13} x + b_{14} = 0, \tag{10.2.2.3}$$

where b_i are some constants. Then the original equation admits solutions of the form

$$y = f(x, Ax^2 + Bx + C), \tag{10.2.2.4}$$

where A, B, and C are determined from a system of three algebraic equations (omitted here).

Remark 10.3. The above remains valid also for equations of the form (10.2.2.3), where $b_k = b_k(w''_{xx}, \dots, w^{(n)}_x)$ are arbitrary functions. The functions Φ and b_k can, in addition, depend on the combination $2ww''_{xx} - (w'_x)^2$.

Example 10.2. Consider the second-order nonautonomous equation

$$y''_{xx} = ay^m + (b_2x^2 + b_1x + b_0)y^n, \qquad m \neq n. \tag{10.2.2.5}$$

Let us see for which values of the parameters a, b_j, m, and n this equation admits solutions of the form (10.2.2.4).

Let us make the change of variable $y = w^p$, with the exponent p to be determined, and multiply the result by w^{2-p} to obtain

$$pww''_{xx} + p(p-1)(w'_x)^2 - aw^{(m-1)p+2} - (b_2x^2 + b_1x + b_0)w^{(n-1)p+2} = 0. \tag{10.2.2.6}$$

For equation (10.2.2.6) to fall in the class of equations (10.2.2.3), one must set

$$(m-1)p + 2 = 1, \qquad (n-1)p + 2 = 0.$$

This results in the relation between the exponents m and n and the desired expression of p:

$$n = 2m - 1, \qquad p = \frac{1}{1-m} \qquad (m \neq 1 \text{ is an arbitrary}). \tag{10.2.2.7}$$

(The remaining parameters, a and b_j, remain arbitrary for now.) Thus, for $n = 2m - 1$, the change of variable $y = w^{\frac{1}{1-m}}$ reduces equation (10.2.2.5) to

$$ww''_{xx} + s(w'_x)^2 + a(m-1)w + (m-1)(b_2x^2 + b_1x + b_0) = 0, \qquad s = \frac{m}{1-m}. \tag{10.2.2.8}$$

An exact solution to this equation has the form of a quadratic polynomial:

$$w = Ax^2 + Bx + C \qquad (y = w^{\frac{1}{1-m}}). \tag{10.2.2.9}$$

Substituting (10.2.2.9) into (10.2.2.8) and rearranging, we obtain

$$[2(2s+1)A^2 + a(m-1)A + b_2(m-1)]x^2 + [2(2s+1)AB + a(m-1)B + b_1(m-1)]x \\ + [2A + a(m-1)]C + sB^2 + b_0(m-1) = 0.$$

By equating the coefficients of the different powers of x to zero, we get the algebraic system of equations

$$\begin{aligned} 2(2s+1)A^2 + a(m-1)A + b_2(m-1) &= 0, \\ 2(2s+1)AB + a(m-1)B + b_1(m-1) &= 0, \\ [2A + a(m-1)]C + sB^2 + b_0(m-1) &= 0, \end{aligned} \tag{10.2.2.10}$$

with $s = \frac{m}{1-m}$. The first quadratic equation of system (10.2.2.10) serves to determine A (it has two distinct roots in a wide range of the parameters a, b_2, and m). By multiplying the first equation in (10.2.2.10) by B and the second by $-A$ and add together to obtain the simple relation

$$b_2B = b_1A, \tag{10.2.2.11}$$

which allows us to express B via A, provided that $b_2 \neq 0$, to get $B = (b_1/b_2)A$. Now C is easily determined from the last equation in (10.2.2.10).

For the autonomous equation (10.2.2.5) with $b_1 = b_2 = 0$, system (10.2.2.10) has the solution

$$A = \frac{a(m-1)^2}{2(m+1)}, \quad B \text{ is an arbitrary constant;} \quad C = \frac{m+1}{2a}\left[\frac{B^2}{(m-1)^2} - \frac{b_0}{m}\right]. \tag{10.2.2.12}$$

Let us focus on the special case of equation (10.2.2.5) with

$$a = b_1 = b_2 = 0, \quad m = -1, \quad n = -3.$$

It follows from system (10.2.2.10) that

$$A \text{ and } B \text{ are arbitrary constants,} \quad C = \frac{1}{4A}(B^2 + 4b_0).$$

Thus, we have found that the second-order equation $y''_{xx} = b_0 y^{-3}$ has the exact solution $y = \sqrt{Ax^2 + Bx + \frac{1}{4A}(B^2 + 4b_0)}$ involving two arbitrary constants. The general solution to this equation consists of two branches: $y = \pm\sqrt{Ax^2 + Bx + \frac{1}{4A}(B^2 + 4b_0)}$.

Example 10.3. The equation with an exponential nonlinearity

$$y''_{xx} + ce^{\lambda y} y'_x = ae^{\lambda y} + (b_2 x^2 + b_1 x + b_0)e^{2\lambda y} \tag{10.2.2.13}$$

can be reduced with the change of variable $y = -\frac{1}{\lambda}\ln w$ to a special case of equation (10.2.2.3):

$$ww''_{xx} - (w'_x)^2 + cw'_x + a\lambda w + \lambda(b_2 x^2 + b_1 x + b_0) = 0.$$

Hence, equation (10.2.2.13) admits an exact solution of the form $y = -\frac{1}{\lambda}\ln(Ax^2 + Bx + C)$. In particular, the autonomous equation $y''_{xx} = ae^{\lambda y} + b_0 e^{2\lambda y}$, which is the special case of equation (10.2.2.13) with $c = b_1 = b_2 = 0$, has a particular solution

$$y = -\frac{1}{\lambda}\ln\left(\frac{1}{2}a\lambda x^2 + Bx + \frac{B^2}{2a\lambda} - \frac{b_0}{2a}\right),$$

where B is an arbitrary constant.

► Tanh-coth and sinh-cosh methods.

1°. *Tanh-coth method.* The main idea of the tanh-coth method is the assumption that a particular solution can be expressed in terms of the hyperbolic tangent or hyperbolic cotangent functions, which corresponds to $\varphi(z) = \tanh z$ or $\varphi(z) = \coth z$ in (10.2.1.2).

Example 10.4. Consider the second-order nonlinear differential equation

$$y''_{xx} + by - cy^3 = 0. \tag{10.2.2.14}$$

We seek particular solutions of the equation in the form

$$y = \sum_{k=0}^{n} a_k z^k, \qquad z = \tanh(\lambda x), \tag{10.2.2.15}$$

with a_k, λ, and n to be determined. Differentiating (10.2.2.15) twice and taking into account that $z'_x = \lambda/\cosh^2(\lambda x) = \lambda(1 - z^2)$, we obtain

$$\begin{aligned} y'_x &= y'_z z'_x = \left(\sum_{k=0}^{n} a_k k z^{k-1}\right)\lambda(1 - z^2) = \lambda\sum_{k=0}^{n} a_k k z^{k-1} - \lambda\sum_{k=0}^{n} a_k k z^{k+1}, \\ y''_{xx} &= (y'_x)_z z'_x = \lambda\left(\sum_{k=0}^{n} a_k k(k-1)z^{k-2} - \lambda\sum_{k=0}^{n} a_k k(k+1)z^k\right)\lambda(1 - z^2) \\ &= \lambda^2\sum_{k=0}^{n} a_k k(k-1)z^{k-2} - 2\lambda^2\sum_{k=0}^{n} a_k k^2 z^k + \lambda^2\sum_{k=0}^{n} a_k k(k+1)z^{k+2}. \end{aligned} \tag{10.2.2.16}$$

From (10.2.2.15) and (10.2.2.16) it follows that the terms in equation (10.2.2.14) are represented as:

y''_{xx} is a linear combination of different powers of z up to z^{n+2} inclusive,
y is a linear combination of different powers of z up to z^n inclusive,
y^3 is a linear combination of different powers of z up to z^{3n} inclusive.

For ODE (10.2.2.14) to be satisfied identically, the terms with highest power of z must be matched up. Hence, the equality $n+2=3n$ must hold, resulting in $n=1$.

Substituting formulas (10.2.2.15) and (10.2.2.16) with $n=1$ into (10.2.2.14) and rearranging, we arrive at a cubic equation for z:

$$a_1(2\lambda^2 - a_1^2 c)z^3 - 3a_0a_1^2cz^2 + a_1(b - 2\lambda^2 - 3a_0^2c)z + a_0(b - a_0^2c) = 0.$$

Equating the coefficients of the different powers of z to zero results in the overdetermined system of algebraic equations

$$a_1(2\lambda^2 - a_1^2c) = 0, \quad a_0a_1c = 0, \quad a_1(b - 2\lambda^2 - 3a_0^2c) = 0, \quad a_0(b - a_0^2c) = 0.$$

This system can be satisfied, for example, with $a_0 = 0$, $a_1 = \pm\sqrt{b/c}$, and $\lambda = \pm\sqrt{b/2}$. As a result, we get the following particular solutions to equation (10.2.2.14):

$$y = \pm\sqrt{b/c}\,\tanh\bigl(\sqrt{b/2}\,x\bigr). \tag{10.2.2.17}$$

Remark 10.4. Since equation (10.2.2.14) is invariant to the translation transformation $x \Longrightarrow x + \text{const}$, it also admits the solutions $y = \pm\sqrt{b/c}\,\tanh\bigl(\sqrt{b/2}\,x + s\bigr)$, where s is an arbitrary constant.

Remark 10.5. In a similar fashion, we can also obtain the following particular solutions to equation (10.2.2.14): $y = \pm\sqrt{b/c}\,\coth\bigl(\sqrt{b/2}\,x\bigr)$ and $y = \pm\sqrt{b/c}\,\coth\bigl(\sqrt{b/2}\,x + s\bigr)$.

2°. *Sinh-cosh method.* The sinh-cosh method is based on the assumption that a particular solution can be expressed in terms of the hyperbolic sine or hyperbolic cosine functions, and corresponds to $\varphi(z) = \sinh z$ or $\varphi(z) = \cosh z$ in (10.2.1.2).

Example 10.5. Consider the fourth-order nonlinear differential equation

$$y''''_{xxxx} = b_1[yy''_{xx} - (y'_x)^2] + b_2y + b_3. \tag{10.2.2.18}$$

We seek particular solutions to the equation in the form

$$y = a_0 + a_1\sinh(\lambda x). \tag{10.2.2.19}$$

Substituting (10.2.2.19) into (10.2.2.18) and rearranging taking into account the identity $\cosh^2 z - \sinh^2 z = 1$, we obtain

$$a_1(\lambda^4 - a_0b_1\lambda^2 - b_2)\sinh(\lambda x) + a_1^2b_1\lambda^2 - a_0b_2 - b_3 = 0.$$

For this equation to be satisfied identically for any x, one must set

$$\lambda^4 - a_0b_1\lambda^2 - b_2 = 0, \qquad a_1^2b_1\lambda^2 - a_0b_2 - b_3 = 0.$$

Solving these equations for a_0 and a_1 yields

$$a_0 = \frac{\lambda^4 - b_2}{b_1\lambda^2}, \qquad a_1 = \pm\frac{1}{b_1\lambda^2}\sqrt{b_1\lambda^4 + b_1b_3\lambda^2 - b_2^2}. \tag{10.2.2.20}$$

Formulas (10.2.2.19) and (10.2.2.20) define particular solutions of equation (10.2.2.18) involving one free parameter, λ, with the restrictions that $\lambda \neq 0$ and the radicand must be positive.

Remark 10.6. Likewise, one can obtain more general, two-parameter particular solutions to equation (10.2.2.18):

$$y = a_0 + a_1\sinh(\lambda x) + a_2\cosh(\lambda x),$$
$$a_0 = \frac{\lambda^4 - b_2}{b_1\lambda^2}, \qquad a_1 = \pm\frac{1}{b_1\lambda^2}\sqrt{a_2^2b_1^2\lambda^4 + b_1\lambda^4 + b_1b_3\lambda^2 - b_2^2},$$

where a_2 and λ are arbitrary constants.

► **Sine-cosine and tan-cot methods.**

1°. *Sine-cosine method.* The sine-cosine method is based on the assumption that a particular solution can be expressed in terms of the sine or cosine function, which corresponds to $\varphi(z) = \sin z$ or $\varphi(z) = \cos z$ in (10.2.1.2).

Example 10.6. Consider once again equation (10.2.2.18). We seek particular solutions of the form

$$y = a_0 + a_1 \sin(\lambda x). \tag{10.2.2.21}$$

Substituting (10.2.2.21) in (10.2.2.18) and rearranging while taking into account the identity $\cos^2 z + \sin^2 z = 1$, we obtain

$$a_1(\lambda^4 + a_0 b_1 \lambda^2 - b_2)\sin(\lambda x) + a_1^2 b_1 \lambda^2 - a_0 b_2 - b_3 = 0.$$

For this equation to be satisfied identically for any x, one must set

$$\lambda^4 + a_0 b_1 \lambda^2 - b_2 = 0, \qquad a_1^2 b_1 \lambda^2 - a_0 b_2 - b_3 = 0.$$

Solving these equations for a_0 and a_1 gives

$$a_0 = \frac{b_2 - \lambda^4}{b_1 \lambda^2}, \qquad a_1 = \pm\frac{1}{b_1 \lambda^2}\sqrt{b_2^2 + b_1 b_3 \lambda^2 - b_1 \lambda^4}. \tag{10.2.2.22}$$

Formulas (10.2.2.19) and (10.2.2.22) define particular solutions to equation (10.2.2.18) involving one free parameter, λ, with the restriction that $\lambda \neq 0$ and the radicand must be positive.

Remark 10.7. Since equation (10.2.2.18) is invariant to translation, $x \Longrightarrow x + \text{const}$, it also admits solutions of the form $y = a_0 + a_1 \sin(\lambda x + c)$, where a_0 and a_1 are given by (10.2.2.22), while c and λ are arbitrary constants.

2°. *Tan-cot method.* The tan-cot method is based on the assumption that a particular solution can be expressed in terms of the tangent or cotangent functions, which corresponds to $\varphi(z) = \tan z$ or $\varphi(z) = \cot z$ in (10.2.1.2).

Example 10.7. Consider equation (10.2.2.14) with $b < 0$ and $c > 0$. We seek particular solutions of the form $y = \sum_{k=0}^{n} a_k z^k$ with $z = \tan(\lambda x)$, where a_k, λ, and n are undetermined constants. Arguing in the same way as in Example 10.4, we find that $n = 1$ and $a_0 = 0$. As a result, we obtain the particular solutions

$$y = \pm\sqrt{-b/c}\,\tan(\sqrt{-b/2}\,x).$$

Remark 10.8. Since equation (10.2.2.14) is invariant to translation, $x \Longrightarrow x + \text{const}$, it also admits the solutions $y = \pm\sqrt{-b/c}\,\tan(\sqrt{-b/2}\,x + s)$, where s is an arbitrary constant.

10.2.3 Exp-Function, Q-Expansion and Related Methods

► **Exp-function method. The simplest version.**

In the simplest case, the exp-function method is based on the assumption that a particular solution can be expressed in terms of the exponential function, which corresponds to $\varphi(z) = \exp z$ in (10.2.1.2).

Importantly, the autonomous differential equation of arbitrary order with a quadratic nonlinearity

$$\sum_{k=1}^{n} S_k[w] w_x^{(k)} + b_0 = 0, \qquad S_k[w] = \sum_{s=0}^{k} b_{ks} w_x^{(s)}, \quad w_x^{(0)} = w \tag{10.2.3.1}$$

admits particular solutions of the form

$$w = A + Be^{\lambda x}, \tag{10.2.3.2}$$

provided that there is a single relation between the coefficients b_{ks} and b_0. The constants A and λ are to be determined, while B is arbitrary. Therefore, one should first try to reduce the equation by a change of variable $y = f(w)$ to an equation with a quadratic nonlinearity (10.2.3.1) and then look for its particular solutions of the form (10.2.3.1).

Example 10.8. Consider the equation

$$y''_{xx} + (a_1 + a_2 y^{m-1})y'_x = by + cy^m. \tag{10.2.3.3}$$

First, we make the change of variable $y = w^p$, with the exponent p to be determined. Then, on multiplying the result by w^{2-p}, we get

$$pww''_{xx} + p(p-1)(w'_x)^2 + p(a_1 w + a_2 w^{(m-1)p+1})w'_x = bw^2 + cw^{(m-1)p+2}.$$

In order to obtain an equation with a quadratic nonlinearity, one must set $p = \frac{1}{1-m}$. Thus, the change of variable $y = w^{\frac{1}{1-m}}$ reduces equation (10.2.3.3) to the form

$$ww''_{xx} + s(w'_x)^2 + a_1 ww'_x + a_2 w'_x + b(m-1)w^2 + c(m-1)w = 0, \qquad s = \frac{m}{1-m}. \tag{10.2.3.4}$$

Substituting (10.2.3.2) into (10.2.3.4) and rearranging, we obtain

$$B^2[(s+1)\lambda^2 + a_1\lambda + b(m-1)]E^2 + B\{A[\lambda^2 + a_1\lambda + 2b(m-1)] + a_2\lambda + c(m-1)\}E$$
$$+ (m-1)A(Ab+c) = 0, \qquad E = e^{\lambda x}, \qquad s = \frac{m}{1-m}.$$

Equating the coefficients of the various powers of E to zero results in the algebraic system of equations

$$\begin{aligned} (s+1)\lambda^2 + a_1\lambda + b(m-1) &= 0, \\ A[\lambda^2 + a_1\lambda + 2b(m-1)] + a_2\lambda + c(m-1) &= 0, \\ A(Ab+c) &= 0. \end{aligned} \tag{10.2.3.5}$$

The trivial cases of $B = 0$ (constant solution) and $m = 1$ (linear equation) have been discarded.

The first quadratic equation in system (10.2.3.5) serves to determine λ (in a wide range of the parameters a_1, b, and m, it has two distinct roots). From the last equation in (10.2.3.5) one can see that there are two possibilities, $A = 0$ and $A \neq 0$, which need to be treated separately.

1. In the degenerate case of $A = 0$ and $a_2 \neq 0$, we get the solution

$$w = Be^{\lambda x}, \qquad \lambda = \frac{c(1-m)}{a_2},$$

which exists under the condition that

$$a_2 c^2 + a_1 a_2 c - a_2^2 b = 0.$$

2. In the nondegenerate case $A \neq 0$, the first and third equations in (10.2.3.5) give the parameters of two particular solutions (10.2.3.2):

$$A = -\frac{c}{b}, \qquad \lambda_{1,2} = \frac{1}{2}(m-1)\Big(a_1 \pm \sqrt{a_1^2 + 4b}\Big).$$

The second equation of system (10.2.3.5), with $A = -c/b$ and the value of λ_1 (or $A = -c/b$ and λ_2) inserted, determines the relationship between the coefficients of the original equation (10.2.3.3) required for the existence of such a solution (the relationship is omitted).

► **Q-expansion and logistic function methods.**

1°. There is a more complex method based on the usage of exponential functions suggesting that particular solutions to autonomous equations with polynomial nonlinearity should be sought in the form

$$y=\sum_{k=0}^{n}a_kQ^k(\lambda x),\qquad Q(z)=\frac{1}{1+Ce^z},\tag{10.2.3.6}$$

where C is an arbitrary constant and the remaining constants, a_k, λ, and n, are to be determined.

Expression (10.2.3.6) is substituted into the ODE of interest and then, after multiplying by Q^{-n} and matching the coefficients of like powers of e^{kz}, one arrives at a system of algebraic equations for the unknowns a_k, λ, and n as well as the coefficients involved in the equation of interest.

The representation of solutions in form (10.2.3.6) with $C=\lambda=1$ constitutes the *Q-expansion method* and corresponds to $\varphi(z)=(1+e^z)^{-1}$ in the first formula in (10.2.1.2). In the special case $C=1$ and $\lambda=-1$, the function $Q(z)=(1+e^{-x})^{-1}$ appearing in the solution is called a *logistic function* (or the *sigmoid function*).

The function $Q(z)$ in (10.2.3.6) can be represented equivalently in terms of hyperbolic functions as follows:

$$Q(z)=\frac{1}{1+e^{z+z_0}}=\frac{1}{2}\left[1-\tanh\left(\frac{z+z_0}{2}\right)\right],\quad z_0=\ln C,\quad \text{if }\ C>0;\tag{10.2.3.7}$$

$$Q(z)=\frac{1}{1-e^{z+z_0}}=\frac{1}{2}\left[1-\coth\left(\frac{z+z_0}{2}\right)\right],\quad z_0=\ln|C|,\quad \text{if }\ C<0.\tag{10.2.3.8}$$

The comparison of formulas (10.2.3.6), (10.2.3.7), and (10.2.3.6), (10.2.3.8) with formula (10.2.1.2) at $\varphi(z)=\tanh z$ and $\varphi(z)=\coth z$ shows that the current modification of the exp-function method allows one to cover all the solutions that can be obtained using the tanh-coth methods. Furthermore, the representation of solutions in the form (10.2.3.6) is more compact and is simpler as it does not require the knowledge of hyperbolic functions or relations between them.

2°. The current method admits an alternative and more economical usage based on the fact that the function $Q=Q(z)$ is the general solution of the Bernoulli equation

$$Q'_z=Q^2-Q.\tag{10.2.3.9}$$

Differentiating (10.2.3.9) with respect to z and eliminating Q'_z with the help of (10.2.3.9), we find successively

$$\begin{aligned}Q''_{zz}&=2QQ'_z-Q'_z=2Q^3-3Q^2+Q,\\ Q'''_{zzz}&=(6Q^2-6Q+1)Q'_z=6Q^4-12Q^3+7Q^2-Q,\\ Q''''_{zzzz}&=(24Q^3-36Q^2+14Q-1)Q'_z=24Q^5-60Q^4+50Q^3-15Q^2+Q.\end{aligned}\tag{10.2.3.10}$$

In a similar fashion, we we can obtain the representation of the derivative $Q^{(k)}_z$ as a polynomial $P_{k+1}(Q)$.

Using formulas (10.2.3.9)–(10.2.3.10), we obtain

$$
\begin{aligned}
y'_x &= \lambda y'_z = \lambda\biggl(\sum_{k=0}^{n} a_k k Q^{k-1}\biggr) Q'_z = \lambda \sum_{k=0}^{n} a_k k Q^{k+1} - \lambda \sum_{k=0}^{n} a_k k Q^k,\\
y''_{xx} &= \lambda^2\biggl(\sum_{k=0}^{n} a_k k(k+1) Q^k - \sum_{k=0}^{n} a_k k^2 Q^{k-1}\biggr) Q'_z \\
&= \lambda^2 \sum_{k=0}^{n} a_k k(k+1) Q^{k+2} - \lambda^2 \sum_{k=0}^{n} a_k k(2k+1) Q^{k+1} + \lambda^2 \sum_{k=0}^{n} a_k k^2 Q^k.
\end{aligned}
\tag{10.2.3.11}
$$

In a similar fashion, we can express the derivative $y_x^{(k)}$ in terms of a polynomial $\widetilde{P}_{n+k}(Q)$.

The degree n of the polynomial (10.2.3.6) is obtained as follows. We replace the terms of the ODE under consideration by the rule

$$
y_x^{(k)} \Longrightarrow Q^{n+k}, \quad y^m \Longrightarrow Q^{nm} \quad (k, m = 0,\, 1,\, \dots) \tag{10.2.3.12}
$$

and then match up the two (or more) terms with the largest powers of Q. As a result, we obtain a simple equation for n. We can use this technique if n is a positive integer. In the case of noninteger n, we have to use a transformation of the solution $y = y(x)$. For example, if we obtain $n = \frac{1}{m}$, where m is an integer, we can transform the solution as $y = u^m$, where $u = u(x)$ is the new function.

On determining n, we substitute (10.2.3.6) into the differential equation of interest and replace the derivatives $Q_z^{(k)}$ with the expressions (10.2.3.9)–(10.2.3.10) to obtain an algebraic equation for Q. Equating the coefficients of this equation to zero results in an algebraic system for the coefficients a_k and λ.

Example 10.9. Consider the nonlinear second-order equation with a quadratic nonlinearity*

$$
y''_{xx} + b y'_x + c(y - y^3) = 0. \tag{10.2.3.13}
$$

We look for solutions to equation (10.2.3.13) as the sum (10.2.3.6). On replacing the terms of the equation by the rule (10.2.3.12), we equate the exponents of the highest-order terms in Q (which have the correspondence $y''_{xx} \Longrightarrow Q^{n+2}$ and $y^2 \Longrightarrow Q^{3n}$) to obtain $n = 1$.

Using formulas (10.2.3.6) and (10.2.3.11) with $n = 1$, we get

$$
\begin{aligned}
y &= a_0 + a_1 Q,\\
y'_x &= \lambda a_1 (Q^2 - Q),\\
y''_{xx} &= \lambda^2 a_1 (2Q^3 - 3Q^2 + Q).
\end{aligned}
$$

Inserting these expressions into (10.2.3.13) yields a polynomial of degree 3 in Q. Equating its coefficients of the different powers of Q to zero and dividing by $a_1 \neq 0$ and $c \neq 0$, we arrive at the algebraic system equations

$$
2\lambda^2 - c a_1^2 = 0, \tag{10.2.3.14}
$$

$$
3\lambda^2 - b\lambda + 3 a_0 a_1 c = 0, \tag{10.2.3.15}
$$

$$
\lambda^2 - b\lambda + c - 3 a_0^2 c = 0, \tag{10.2.3.16}
$$

$$
a_0 (1 - a_0^2) = 0. \tag{10.2.3.17}
$$

*This equation arises when one looks for exact solutions to the nonlinear Burgers–Huxley PDE $u_t = u_{\xi\xi} + c(u - u^3)$ in the form of a traveling wave, $u = y(x)$ with $x = \xi - bt$, where b is an arbitrary constant.

Subtracting equation (10.2.3.16) from (10.2.3.15) and eliminating λ^2 with the help of (10.2.3.14), we obtain the relation between a_0 and a_1:

$$3a_0^2 + 3a_0a_1 + a_1^2 - 1 = 0. \tag{10.2.3.18}$$

The cubic equation (10.2.3.17) has three roots: $a_0 = 0$, $a_0 = 1$, and $a_0 = -1$. To each a_0 there correspond two roots of the quadratic equation (10.2.3.18) for a_1, and to each a_1 there correspond two roots of equation (10.2.3.14). Substituting a_0, a_1, and λ into equation (10.2.3.15) (or (10.2.3.16)), we arrive at the relationship between the coefficients b and c that ensures the existence of a solution. To sum up, there are the following possibilities:

$$\begin{aligned}
&\text{(i)} && a_0 = 0, && a_1 = 1, && \lambda = \pm\sqrt{c/2}, && b = \pm 3\sqrt{c/2};\\
&\text{(ii)} && a_0 = 0, && a_1 = -1, && \lambda = \pm\sqrt{c/2}, && b = \pm 3\sqrt{c/2};\\
&\text{(iii)} && a_0 = 1, && a_1 = -1, && \lambda = \pm\sqrt{c/2}, && b = \mp 3\sqrt{c/2};\\
&\text{(iv)} && a_0 = 1, && a_1 = -2, && \lambda = \pm\sqrt{2c}, && b = 0;\\
&\text{(v)} && a_0 = -1, && a_1 = 1, && \lambda = \pm\sqrt{c/2}, && b = \mp 3\sqrt{c/2};\\
&\text{(vi)} && a_0 = -1, && a_1 = 2, && \lambda = \pm\sqrt{2c}, && b = 0.
\end{aligned}$$

It follows from equation (10.2.3.13) that if y is a solution, then $-y$ is also a solution. The above formulas for the coefficients determine three pairs of solutions of the form $y = a_0 + \dfrac{a_1}{1 + Ce^{\lambda x}}$, which differ in sign.

$3°$. Another convenient technique to seek particular solutions of the form (10.2.3.6) is based on using the change of variable $\xi = \dfrac{1}{1 + Ce^{\lambda x}}$ in the equation of interest followed by representing solutions in the form of finite power series in ξ: $y = \sum_{k=0}^{n} a_k \xi^k$ (see Remark 10.1).

► Solutions in the form of the ratio of exponential polynomials.

Particular solutions to ODEs can also be sought in the form of the ratio of exponential polynomials

$$y(x) = \frac{\sum_{k=-r}^{s} a_k e^{kz}}{\sum_{j=-p}^{q} b_j e^{jz}} = \frac{a_{-r}e^{-rz} + \cdots + a_s e^{sz}}{b_{-p}e^{-pz} + \cdots + b_q e^{qz}}, \qquad z = \lambda x, \tag{10.2.3.19}$$

where r, s, p and q are unknown positive integers to be determined and a_k, b_j, and λ are unknown constants. Symbolic computations with computer algebra systems (such as Maple or Mathematica) can often be very helpful in searching for such solutions.

For example, in the special case $p = q = r = s = 1$, (10.2.3.19) becomes

$$y(x) = \frac{a_{-1}e^{-z} + a_0 + a_1 e^z}{b_{-1}e^{-z} + b_0 + b_1 e^z}, \qquad z = \lambda x.$$

By substituting this expression into the ODE and by matching the coefficients of like powers of e^{kz}, we generate the system of algebraic equations for the unknowns a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, and λ as well as the coefficients involved in the equation.

⊙ *Literature for Section 10.2*: W. Malfliet (1992), W. Malfliet and W. Hereman (1996), E. Fan (2000), A. M. Wazwaz (2004, 2007a, 2007b, 2008), D.-S. Wang, Y.-J. Ren, and H.-Q. Zhang (2005), J.-H. He and X.-H. Wu (2006), A. Ebaid (2007), J.-H. He and M. A. Abdou (2007), S. Zhang (2007), A. Bekir (2008), L. Wazzan (2009), N. A. Kudryashov (2010b, 2012, 2013, 2015), E. J. Parkes (2010), N. A. Kudryashov and D. I. Sinelshchikov (2012), A. D. Polyanin and V. F. Zaitsev (2012), N. A. Kudryashov and A. S. Zakharchenko (2014).

10.3 Method of Differential Constraints

10.3.1 Preliminary Remarks. First-Order Differential Constraints and Their Applications

The main idea of the method is that exact solutions to a complex (nonintegrable) equation are sought by jointly analyzing this equation and an auxiliary simpler (integrable) equation, called a *differential constraint*.*

The order of a differential constraint is the order of the highest derivative involved. Usually, the order of the differential constraint is less than that of the equation; first-order differential constraints are simplest and most common. The equation and differential constraint must involve a set of free parameters (or even arbitrary functions) whose values are chosen by ensuring that the equation and the constraint are consistent. After the consistency analysis, all solutions obtained by integrating the differential constraint will be simultaneously solutions to the original equation. The method makes it possible to find particular solutions to the original equation for some values of the determining parameters.

For simplicity, we first consider autonomous ordinary differential equations of the form

$$F(y, y'_x, \dots, y_x^{(n)}; \mathbf{a}) = 0, \tag{10.3.1.1}$$

which do not involve the independent variable x explicitly and depend on a vector of free parameters $\mathbf{a} = \{a_1, \dots, a_k\}$. For equations (10.3.1.1), one should take first-order differential constraints in the autonomous form

$$G(y, y'_x; \mathbf{b}) = 0, \tag{10.3.1.2}$$

dependent on a vector of free parameters $\mathbf{b} = \{b_1, \dots, b_s\}$.

By differentiating relation (10.3.1.2) successively several times, one can express higher-order derivatives in terms of y and y'_x: $y_x^{(k)} = \varphi_k(y, y'_x; \mathbf{b})$. Substituting these expressions into the original equation (10.3.1.1), one arrives at a first-order equation

$$\mathcal{F}(y, y'_x; \mathbf{a}, \mathbf{b}) = 0. \tag{10.3.1.3}$$

By eliminating the derivative y'_x from (10.3.1.2) and (10.3.1.3), one obtains an algebraic/transcendental equation

$$P(y; \mathbf{a}, \mathbf{b}) = 0. \tag{10.3.1.4}$$

Further, one looks for the values of $\mathbf{a}$ and $\mathbf{b}$ at which equation (10.3.1.4) is satisfied identically for any y (this may result in some restrictions on the components of the vector $\mathbf{a}$). After this, one expresses the vector $\mathbf{b}$ in terms of $\mathbf{a}$, so that $\mathbf{b} = \mathbf{b}(\mathbf{a})$, and substitutes it back into the differential constraint (10.3.1.2) to obtain a first-order ordinary differential equation

$$g(y, y'_x; \mathbf{a}) = 0 \qquad (g = G|_{\mathbf{b}=\mathbf{b}(\mathbf{a})}). \tag{10.3.1.5}$$

*The ideas of this method as applied to searching for exact solutions to nonlinear PDEs were first put forward by Yanenko (1964). The studies by Galaktionov (1994), Olver and Vorob'ev (1996), Andreev, Kaptsov, Pukhnachov, and Rodionov (1998), Kaptsov and Verevkin (2003), Polyanin and Zaitsev (2004, 2012), Polyanin, Zaitsev, and Zhurov (2005) give a number of nontrivial examples of how to use this method to construct exact solutions (other than traveling wave solutions) to different nonlinear PDEs of mathematical physics.

This equation is consistent with the original equation (10.3.1.1); in other words, the original equation is a consequence of equation (10.3.1.5) and, therefore, inherits all of its solutions. Finally, by solving for the derivative, equation (10.3.1.5) is reduced to a separable equation, which is integrated to obtain a general solution. The general solution of equation (10.3.1.5) is also an exact solution of the original equation (10.3.1.1).

Remark 10.9. If a first-order differential constraint is defined in explicit form, $y'_x = h(y; \mathbf{b})$, the successive differentiation enables one to express the higher-order derivatives in terms of y, so that

$$y''_{xx} = (y'_x)'_y y'_x = hh'_y, \quad y'''_{xxx} = (y''_{xx})'_y y'_x = h(hh'_y)'_y, \quad \ldots$$

Using these expressions and the differential constraint to eliminate the derivatives from (10.3.1.1), one immediately arrives at an algebraic/transcendental equation of the form (10.3.1.4).

Remark 10.10. Instead of y'_x, one can eliminate the dependent variable y from (10.3.1.2) and (10.3.1.3) to obtain an algebraic/transcendental equation for the derivative: $Q(y'_x; \mathbf{a}, \mathbf{b}) = 0$.

The structure of the nonlinearity of the differential constraint (10.3.1.2) can often be taken to be similar to that of the original equation (10.3.1.1) so as to have different determining parameters. This will be illustrated below by specific examples of second-, third-, fourth-, and higher-order equations.

Example 10.10. Consider the second-order ordinary differential equation with a power-law nonlinearity

$$y''_{xx} - cy'_x = ay + by^n, \tag{10.3.1.6}$$

which arises in the theory of chemical reactors, combustion theory, and mathematical biology.*

Let us supplement equation (10.3.1.6) with the first-order differential constraint

$$y'_x = \alpha y + \beta y^m, \tag{10.3.1.7}$$

which is a separable equation and is easy to integrate. The form of the right-hand side of (10.3.1.7) has been chosen to be similar to that of the original equation (10.3.1.6).

The equation and differential constraint involve seven parameters: a, b, c, n, m, α, and β. The further analysis aims at determining the parameters α, β, and m of the differential constraint so as to express them in terms of a, b, c, and n. Simultaneously, restrictions on the equation parameters will be found.

Differentiating (10.3.1.7) and replacing the first derivative with the right-hand side of (10.3.1.7), we get

$$\begin{aligned} y''_{xx} &= (\alpha + m\beta y^{m-1}) y'_x = (\alpha + m\beta y^{m-1})(\alpha y + \beta y^m) \\ &= \alpha^2 y + \alpha\beta(m+1) y^m + m\beta^2 y^{2m-1}. \end{aligned} \tag{10.3.1.8}$$

Eliminating the first and second derivatives from (10.3.1.6) using (10.3.1.7) and (10.3.1.8) and rearranging, we obtain

$$(\alpha^2 - \alpha c - a)y + \beta[\alpha(m+1) - c]y^m + m\beta^2 y^{2m-1} - by^n = 0.$$

For this equation to hold for all y, one must set

$$\begin{aligned} &\alpha^2 - \alpha c - a = 0, \\ &\alpha(m+1) - c = 0, \\ &2m - 1 = n, \\ &m\beta^2 - b = 0. \end{aligned} \tag{10.3.1.9}$$

*Equations (10.3.1.6) and (10.3.1.11) describe traveling-wave solutions of the Kolmogorov–Petrovskii–Piskunov PDE, $u_t = u_{\xi\xi} - f(u)$, for some forms of the kinetic function $f(u)$. In this case, we have $u = y(x)$ with $x = \xi + ct$.

If conditions (10.3.1.9) hold, then solutions to equation (10.3.1.7) are also solutions to the more complex equation (10.3.1.6). The determining system of four equations (10.3.1.9) contains seven parameters a, b, c, n, m, α, and β. The three parameters b, c, and n of the original equation can be regarded as arbitrary and the other parameters are expressed as follows:

$$a = -\frac{2c^2(n+1)}{(n+1)^2}, \quad m = \frac{n+1}{2}, \quad \alpha = \frac{2c}{n+3}, \quad \beta = \pm\sqrt{\frac{2b}{n+1}}. \tag{10.3.1.10}$$

It is apparent that for equations (10.3.1.6) and (10.3.1.7) to be consistent, the original equation parameter a must be connected with two other parameters, c and n. In this case, two families of parameters (10.3.1.10) of equation (10.3.1.7) can be identified that determine two different one-parameter solutions to equations (10.3.1.6) and (10.3.1.7); recall that equation (10.3.1.7) is separable and is easy to integrate.

Example 10.11. The second-order equation with an exponential nonlinearity

$$y''_{xx} - cy'_x = a + be^{\lambda y} \tag{10.3.1.11}$$

can be investigated in a similar manner. The equation will be considered in conjunction with the first-order differential constraint

$$y'_x = \alpha + \beta e^{\mu y}. \tag{10.3.1.12}$$

The analysis shows that three parameters of the original equation, b, c, and λ, can be regarded as arbitrary and the other parameters are expressed as

$$a = -\frac{2c^2}{\lambda}, \quad \alpha = \frac{2c}{\lambda}, \quad \beta = \pm\sqrt{\frac{2b}{\lambda}}, \quad \mu = \frac{\lambda}{2}. \tag{10.3.1.13}$$

It is apparent that for equations (10.3.1.11) and (10.3.1.12) to be consistent, the parameter a must be related in a certain way to two other parameters of the equation, c and λ. In this case, two families of parameters (10.3.1.13) of the differential constraint (10.3.1.12) can be identified, which determine two different one-parameter solutions to equations (10.3.1.11) and (10.3.1.12). Equation (10.3.1.12) is separable and is easy to integrate.

Example 10.12. Consider the nonlinear third-order equation

$$y'''_{xxx} = ay^4 + by^2 + c \tag{10.3.1.14}$$

in conjunction with the first-order differential constraint

$$y'_x = \alpha y^2 + \beta. \tag{10.3.1.15}$$

Using (10.3.1.15), we find the derivatives

$$y''_{xx} = 2\alpha y y'_x = 2\alpha y(\alpha y^2 + \beta) = 2\alpha^3 y^3 + 2\alpha\beta y,$$
$$y'''_{xxx} = (6\alpha^2 y^2 + 2\alpha\beta)y'_x = (6\alpha^2 y^2 + 2\alpha\beta)(\alpha y^2 + \beta) = 6\alpha^3 y^4 + 8\alpha^2\beta y^2 + 2\alpha\beta^2.$$

For the last equation to coincide with (10.3.1.14), the relations

$$a = 6\alpha^3, \quad b = 8\alpha^2\beta, \quad c = 2\alpha\beta^2$$

must hold. On solving the first two equations for α and β and substituting the resulting expressions into the last equation, we obtain

$$\alpha = \left(\frac{a}{6}\right)^{1/3}, \quad \beta = \left(\frac{a}{6}\right)^{-2/3}\frac{b}{8}, \quad c = \frac{3b^2}{16a}. \tag{10.3.1.16}$$

It follows that with this c, the third-order equation (10.3.1.14) has a particular solution resulting from solving the first-order separable equation (10.3.1.15) whose parameters are connected with those of the original equation by the first two relations in (10.3.1.16).

Example 10.13. Consider the nonlinear fourth-order equation

$$y''''_{xxxx} = ay^n + by^{2n+3} \tag{10.3.1.17}$$

in conjunction with the first-order differential constraint

$$(y'_x)^2 = \alpha y^m + \beta. \tag{10.3.1.18}$$

Differentiating (10.3.1.18), we get the derivatives

$$\begin{aligned} y''_{xx} &= \tfrac{1}{2}\alpha m y^{m-1} \qquad \text{(after canceling by } y'_x\text{)},\\ y'''_{xxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y'_x,\\ y''''_{xxxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y''_{xx} + \tfrac{1}{2}\alpha m(m-1)(m-2)y^{m-3}(y'_x)^2\\ &= \tfrac{1}{2}\alpha\beta m(m-1)(m-2)y^{m-3} + \tfrac{1}{4}\alpha^2 m(m-1)(3m-4)y^{2m-3}. \end{aligned} \tag{10.3.1.19}$$

Comparing the right-hand side of (10.3.1.17) and that of the last equation in (10.3.1.19) enables us to draw the following conclusions about the consistency of (10.3.1.17) and (10.3.1.19).

1°. For any values of the parameters of the original equation (10.3.1.17) except for $n \neq -1$, -2, -3, $-\frac{5}{3}$ and $b \neq 0$, one can calculate the parameters of the differential constraint (10.3.1.18) by the formulas

$$m = n+3, \quad \alpha = \pm 2\sqrt{\frac{b}{(n+2)(n+3)(3n+5)}}, \quad \beta = \frac{2a}{\alpha(n+1)(n+2)(n+3)}.$$

2°. For $b = 0$ and $n = -\frac{5}{3}$, we have

$$m = \frac{4}{3}, \quad \beta = -\frac{27a}{4\alpha}, \quad \alpha \neq 0 \text{ is an arbitrary constant.}$$

In this case, the solution to equation (10.3.1.18) will depend on two arbitrary constants (α plays the role of an additional constant of integration).

Remark 10.11. For $b = 0$ and $n = -\frac{5}{3}$, one can find the general solution of equation (10.3.1.17) (see Eq. 1 in Section 16.2.1).

Example 10.14. The fourth-order equation with an exponential nonlinearity

$$y''''_{xxxx} = ae^{\lambda y} + b^{2\lambda y} \tag{10.3.1.20}$$

can be analyzed using the differential constraint

$$(y'_x)^2 = \alpha e^{\lambda y} + \beta. \tag{10.3.1.21}$$

Analysis shows that for any values of the parameters of the original equation (10.3.1.20) satisfying the condition $b\lambda > 0$, two families of parameters of the differential constraint (10.3.1.21) can be found using the formulas

$$\alpha = \pm\frac{a}{\lambda^2}\left(\frac{3\lambda}{b}\right)^{1/2}, \quad \beta = \pm\frac{2}{\lambda}\left(\frac{b}{3\lambda}\right)^{1/2}.$$

Here, one takes either the upper or lower signs simultaneously.

Example 10.15. The nonlinear nth-order equation

$$y_x^{(n)} = ae^{\lambda y}$$

admits the first-order differential constraint

$$y'_x = \alpha e^{\mu y}.$$

The successive differentiation of the differential constraint gives $y_x^{(n)} = \alpha^n \mu^{n-1}(n-1)!\, e^{n\mu y}$. Comparing this expression with the equation yields $\lambda = n\mu$ and $a = \alpha^n\mu^{n-1}(n-1)!$, or

$$\mu = \frac{\lambda}{n}, \quad \alpha = \left[\frac{an^{n-1}}{\lambda^{n-1}(n-1)!}\right]^{1/n}.$$

10.3.2 Differential Constraints of Arbitrary Order. General Consistency Method for Two Equations

In general, a differential constraint is an ordinary differential equation of arbitrary order. Therefore, it is necessary to be able to analyze overdetermined systems of two ordinary differential equations for consistency. Outlined below is the general algorithm for the analysis of such systems.

1°. First, let us consider two ordinary differential equations of the same order

$$F_1(x, y, y'_x, \dots, y_x^{(n)}) = 0, \tag{10.3.2.1}$$

$$F_2(x, y, y'_x, \dots, y_x^{(n)}) = 0; \tag{10.3.2.2}$$

here and henceforth, it is assumed that the equations depend on free parameters, which are omitted for brevity. We eliminate the highest derivative (by solving one of the equations for $y_x^{(n)}$ and substituting the resulting expression into the other equation) to obtain the $(n-1)$st-order equation

$$G_1(x, y, y'_x, \dots, y_x^{(n-1)}) = 0. \tag{10.3.2.3}$$

Differentiating (10.3.2.3) with respect to x and eliminating the derivative $y_x^{(n)}$ from the resulting equation using either of the equations (10.3.2.1) and (10.3.2.2), one arrives at another $(n-1)$st-order equation

$$G_2(x, y, y'_x, \dots, y_x^{(n-1)}) = 0. \tag{10.3.2.4}$$

Thus, the analysis of two nth-order equations (10.3.2.1) and (10.3.2.2) is reduced to the analysis of two $(n-1)$st-order equations (10.3.2.3) and (10.3.2.4). By reducing the order of equations in a similar manner further, one ultimately arrives at a single algebraic/transcendental equation (since two first-order differential equations are reducible to a single algebraic equation). The analysis of the resulting algebraic equation presents no fundamental difficulties and is performed in the same way as previously in Section 10.3.1 for the case of a first-order differential constraint.

2°. Suppose there are two ordinary differential equations having different orders:

$$F_1(x, y, y'_x, \dots, y_x^{(n)}) = 0, \tag{10.3.2.5}$$

$$F_2(x, y, y'_x, \dots, y_x^{(m)}) = 0, \tag{10.3.2.6}$$

with $m < n$. Then, by differentiating (10.3.2.6) $n-m$ times, one reduces system (10.3.2.5)–(10.3.2.6) to a system of the form (10.3.2.1)–(10.3.2.2), in which both equations have the same order n.

Example 10.16. Consider the fourth-order equation with a quadratic nonlinearity

$$y''''_{xxxx} = a(y''_{xx})^2 - by^2 + c \tag{10.3.2.7}$$

in conjunction with the second-order differential constraint

$$y''_{xx} = \alpha y + \beta. \tag{10.3.2.8}$$

Differentiating (10.3.2.8) twice gives $y''''_{xxxx} = \alpha^2 y + \alpha\beta$. Using this expression and the differential constraint (10.3.2.8) to eliminate the derivatives from (10.3.2.7), one arrives at a quadratic equation for y, which is satisfied identically if the conditions

$$a\alpha^2 - b = 0, \quad \alpha - 2a\beta = 0, \quad c = \alpha\beta - a\beta^2$$

hold. Two parameters of the original equation, a and b, can be regarded as arbitrary and the third parameter, c, with coefficients of differential constraint (10.3.2.8) are expressed in terms of them as follows:

$$c = \frac{b}{4a^2}, \quad \alpha = \pm\sqrt{\frac{b}{a}}, \quad \beta = \pm\frac{1}{2a}\sqrt{\frac{b}{a}}.$$

Example 10.17. The equation of order mn with a quadratic nonlinearity

$$y_x^{(mn)} = a[y_x^{(n)}]^2 + byy_x^{(n)} + cy_x^{(n)} + dy^2 + ky + p \qquad (m \text{ is positive integer}),$$

which generalizes equation (10.3.2.7), can be investigated using the nth-order differential constraint

$$y_x^{(n)} = \alpha y + \beta.$$

3°. The general autonomous second-order differential constraint

$$y''_{xx} = f(y)$$

is equivalent to the autonomous first-order differential constraint

$$(y'_x)^2 = F(y),$$

where $F(y) = 2\int f(y)\,dy + C$ and C is an arbitrary constant. This is proved by differentiating the latter relation and comparing with the original differential constraint.

With this in mind, the second-order differential constraint (10.3.2.8) in Example 10.16 could be replaced by the first-order constraint $(y'_x)^2 = \alpha y^2 + 2\beta y + \gamma$, where γ is an extra free parameter. However, the differential constraint (10.3.2.8) is linear and is easy to integrate.

4°. In principle, any differential constraint of arbitrary order (10.3.2.6) can be replaced by a suitable first-order differential constraint. Indeed, the above algorithm for successive order reduction of system (10.3.2.5)–(10.3.2.6) leads, in the nondegenerate case, to a system of first-order equations, one of which can be treated as a first-order differential constraint.

10.3.3 Using Point Transformations in Combination with the Method of Differential Constraints

► **General description of the solution-seeking procedure.**

1°. In some cases, it is first useful to reduce the ODE of interest, with a point transformation, to another equation (simpler or more convenient for investigation), which can then be analyzed using a suitable differential constraint. With this approach, solutions to the autonomous equation (10.3.1.1) are sought in the form

$$y = G(w; \mathbf{b}), \tag{10.3.3.1}$$

where G is a given function and $w = w(x)$ is a function satisfying the first-order differential equation (the differential constraint)

$$H(w, w'_x; \mathbf{c}) = 0. \tag{10.3.3.2}$$

The functions G and H in (10.3.3.1) and (10.3.3.2) depend on the vectors of free parameters $\mathbf{b}$ and $\mathbf{c}$.

The introduction of the new variable w defined by relation (10.3.3.1) reduces equation (10.3.1.1) to a new ODE with one differential constraint (10.3.3.2); this creates the standard situation discussed in Section 10.3.1).

► Examples of constructing particular solutions.

Example 10.18. Let us look at the second-order ordinary differential equation with a power-law nonlinearity of arbitrary degree

$$y''_{xx} - cy'_x = ay + by^n + dy^{2n-1}, \tag{10.3.3.3}$$

which generalizes equation (10.3.1.6) to the case of $d \neq 0$.

We choose the linking dependence (10.3.3.1) in the power-law form

$$y = w^p, \tag{10.3.3.4}$$

with the exponent p to be determined. Substituting (10.3.3.4) into (10.3.3.3) and multiplying by w^{2-p}, we obtain

$$pww''_{xx} + p(p-1)(w'_x)^2 - cpww'_x = aw^2 + bw^{k+1} + dw^{2k}, \quad k = p(n-1)+1. \tag{10.3.3.5}$$

Let us discuss a few possibilities for choosing p that allow one to find exact solutions.

Case 1. Suppose that

$$p = \frac{1}{1-n} \quad (k=0). \tag{10.3.3.6}$$

The change of variable (10.3.3.4), (10.3.3.6) converts the original equation (10.3.3.3) into the equation with a quadratic nonlinearity

$$ww''_{xx} + \frac{n}{1-n}(w'_x)^2 - cww_x = a(1-n)w^2 + b(1-n)w + d(1-n), \tag{10.3.3.7}$$

which is more convenient for analysis.

1.1. Let us supplement equation (10.3.3.7) with the linear differential constraint

$$w'_x = \alpha w + \beta. \tag{10.3.3.8}$$

We use this relation to eliminate the derivatives in (10.3.3.7) to obtain a quadratic equation for w, which is satisfied identically if the conditions (determining system of algebraic equations)

$$\frac{1}{1-n}\alpha^2 - c\alpha = a(1-n), \quad \frac{1+n}{1-n}\alpha\beta - c\beta = b(1-n), \quad n\beta^2 = d(1-n)^2$$

hold. The first and last equations give two pairs of solutions each,

$$\alpha_{1,2} = \frac{1}{2}(1-n)\bigl(c \pm \sqrt{c^2+4a}\,\bigr), \quad \beta_{1,2} = \pm\frac{1}{1-n}\sqrt{\frac{d}{n}}, \tag{10.3.3.9}$$

which are then substituted into the second equation. As a result, for each pair α_i, β_j we obtain one constraint (not written out here) that connects the parameters a, b, c, d, and n.

It is apparent from (10.3.3.9) that for $a = 0$, equation (10.3.3.7) admits a simple, degenerate first-order differential constraint

$$w'_x = \beta = \text{const}.$$

1.2. For $c = 0$, equation (10.3.3.7) can be supplemented with the differential constraint

$$(w'_x)^2 = \alpha w^2 + \beta w + \gamma. \tag{10.3.3.10}$$

A simple analysis shows that the constraint coefficients in (10.3.3.10) are expressed in terms of the equation coefficients in (10.3.3.3) as follows:

$$\alpha = \frac{a(1-n)^2}{2-n}, \quad \beta = b(1-n)^2, \quad \gamma = \frac{d(1-n)^2}{n}.$$

Remark 10.12. For $a = c = 0$, equation (10.3.3.7) admits a solution of the form $w = Ax^2 + Bx + C$.

Remark 10.13. For $n = -1/3$, a particular solution of equation (10.3.3.7) can be obtained with the help of the differential constraint

$$w'_x = \alpha w + \beta\sqrt{w} + \gamma.$$

Case 2. Suppose that

$$p = \frac{1}{n-1} \qquad (k = 2). \tag{10.3.3.11}$$

The change of variable (10.3.3.4), (10.3.3.11) converts the original equation (10.3.3.3) into the equation with a nonlinearity of fourth degree

$$ww''_{xx} + \frac{2-n}{n-1}(w'_x)^2 - cww_x = a(n-1)w^2 + b(n-1)w^3 + d(n-1)w^4, \tag{10.3.3.12}$$

We look for particular solutions to equation (10.3.3.7) using the quadratic differential constraint

$$w'_x = \alpha w^2 + \beta w + \gamma. \tag{10.3.3.13}$$

We use this relation to eliminate the derivatives from (10.3.3.12) to obtain an equation of fourth degree for w. Equating its coefficients to zero results in the algebraic system

$$\begin{aligned}
&n\alpha^2 = d(n-1)^2,\\
&\frac{(n+1)\alpha\beta}{n-1} - c\alpha = b(n-1),\\
&\frac{\beta^2 + 2\alpha\gamma}{n-1} - c\beta = a(n-1),\\
&\gamma\left(\frac{3-n}{n-1}\beta - c\right) = 0,\\
&(2-n)\gamma^2 = 0.
\end{aligned} \tag{10.3.3.14}$$

The cases $\gamma = 0$ and $n = 2$ need to be considered; these correspond to solutions of the last equation.

2.1. For $\gamma = 0$, we determine the original coefficients and a particular solution using the first, second, and fourth equations of (10.3.3.14) as well as the differential constraint (10.3.3.13):

$$\alpha = \pm(n-1)\sqrt{\frac{d}{n}}, \quad \beta = \frac{n-1}{n+1}\left(c \pm b\sqrt{\frac{n}{d}}\right), \quad \gamma = 0, \quad w = -\frac{\beta}{\alpha + Ce^{-\beta x}}, \tag{10.3.3.15}$$

where C is an arbitrary constant. The third equation of (10.3.3.14) defines a necessary relation between the coefficients of the equation of interest:

$$a = \frac{1}{(n+1)^2}\left(c \pm b\sqrt{\frac{n}{d}}\right)\left(-nc \pm b\sqrt{\frac{n}{d}}\right).$$

Either the upper or lower signs must be taken in the above formulas.

2.2. For $n = 2$, the coefficients of the differential constraint are determined from the first, third, and fourth equations of (10.3.3.14):

$$\alpha = \pm\sqrt{d/2}, \quad \beta = c, \quad \gamma = \pm\frac{a}{\sqrt{2d}}. \tag{10.3.3.16}$$

The second equation defined the relation between the equation coefficients: $b = \pm c\sqrt{2d}$ (either the upper or lower signs must be taken in all formulas). The desired solution is determined by integrating the separable equation (10.3.3.13) taking into account (10.3.3.16).

Case 3. On setting $k = 3$ in (10.3.3.5), we can look for a solution in the form $w = a_0 + a_1 Q + a_2 Q^2$ with $Q'_x = b_2 Q^2 + b_1 Q + b_0$.

Example 10.19. Consider the second-order ordinary differential equation with an exponential nonlinearity

$$y''_{xx} - cy'_x = a + be^{\lambda y} + de^{2\lambda y}. \qquad (10.3.3.17)$$

The linking dependence (10.3.3.1) will be taken in the logarithmic form

$$y = \frac{k}{\lambda} \ln w, \qquad (10.3.3.18)$$

with the coefficient k to be determined. Substituting (10.3.3.18) into (10.3.3.17) and multiplying by λw^2, we obtain

$$kww''_{xx} - k(w'_x)^2 - ckww'_x = a\lambda w^2 + b\lambda w^{k+2} + d\lambda w^{2k+2}. \qquad (10.3.3.19)$$

Let us look at a few possibilities of choosing the coefficient k that allow us to find exact solutions.

Case $k=-1$. The substitution (10.3.3.18) with $k=-1$ converts the original equation (10.3.3.17) into the equation with a quadratic nonlinearity

$$ww''_{xx} - (w'_x)^2 - cww_x = -a\lambda w^2 - b\lambda w - d, \qquad (10.3.3.20)$$

which only differs from equation (10.3.3.7) in coefficients. The differential constraint (10.3.3.8) allows one to find a particular solution to equation (10.3.3.20) (details are omitted).

Remark 10.14. For $a = c = 0$, equation (10.3.3.20) admits a solution of the form $w = Ax^2 + Bx + C$.

Case $k = 1$. The substitution (10.3.3.18) with $k = 1$ converts the original equation (10.3.3.17) into the equation with a quartic nonlinearity

$$ww''_{xx} - (w'_x)^2 - cww'_x = a\lambda w^2 + b\lambda w^3 + d\lambda w^4, \qquad (10.3.3.21)$$

which only differs from equation (10.3.3.12) in coefficients. The differential constraint (10.3.3.13) allows one to find a particular solution to equation (10.3.3.21) (details are omitted).

► Modification of the solution-seeking procedure.

If the differential constraint (10.3.3.2) can be solved for the derivative,

$$w'_x = h(w; \mathbf{b}), \qquad (10.3.3.22)$$

then a different order of actions may be more convenient.

By differentiating relation (10.3.3.22) repeatedly and expressing the derivatives via w, one obtains relations of the form $w^{(k)}_x = \varphi_k(w; \mathbf{b})$. Then, on substituting (10.3.3.1) into the equation of interest, one eliminates the derivatives with the help of (10.3.3.22).

Most frequently, one uses differential constraints of the form (10.3.3.22). These represent separable Riccati or Bernoulli equations.

Example 10.20. Example 10.9 considered previously demonstrates the work of this method for the original equation taken in the form (10.2.3.13), relation (10.3.3.1) taken in the form of a finite sum (10.2.3.6) with $w \equiv Q$, and the differential constraint (10.3.3.22) taken in the form of the Bernoulli equation (10.2.3.9).

10.3.4 Using Several Differential Constraints. G'/G-Expansion Method and Simplest Equation Method

► Using several differential constraints.

In some situations, the equation under study is supplemented with several differential constraints containing additional unknown constants. To be specific, let us return to the nth-order autonomous equation (10.3.1.1) and supplement it with two first-order differential constraints

$$y = G(w, w'_x; \mathbf{b}), \tag{10.3.4.1}$$

$$H(w, w'_x; \mathbf{c}) = 0, \tag{10.3.4.2}$$

where $\mathbf{b}$ and $\mathbf{c}$ are vectors of free parameters. On substituting (10.3.4.1) into (10.3.1.1), one obtains an $(n+1)$st-order equation for $w = w(x)$:

$$F_1(w, w'_x, \dots, w_x^{(n+1)}; \mathbf{b}, \mathbf{c}) = 0. \tag{10.3.4.3}$$

This equation in conjunction with the differential constraint (10.3.4.2) is analyzed with the method outlined in Sections 10.3.1 and 10.2.3. There is an insignificant distinction that the order of equation (10.3.4.3) is higher than that of the original equation (10.3.1.1).

Remark 10.15. The differential constraints (10.3.4.1) and (10.3.4.2) can involve higher derivatives of w with respect to x (see below).

► G'/G-expansion method.

With the G'/G-expansion method, one looks for particular solutions to autonomous equations using two differential constraints (10.3.4.1) and (10.3.4.2) of the special form

$$y = \sum_{k=0}^{n} b_k \left(\frac{w'_x}{w}\right)^k, \tag{10.3.4.4}$$

$$w''_{xx} - c_1 w'_x - c_0 w = 0. \tag{10.3.4.5}$$

Remark 10.16. In the original papers by Wang, Li, and Zhang (2008) and subsequent publications, the notation $w = G$ was used.

It was shown by Kudryashov (2010) that searching for particular solutions of an ODE with the G'/G-expansion method based on the differential constraints (10.3.4.4)–(10.3.4.5) with $c_1^2 + 4c_0 > 0$ leads to the same results as the tanh method (see Section 10.2.2). For $c_1^2 + 4c_0 < 0$, the G'/G-expansion method is equivalent to the tan method.

For specific examples of how to use the G'/G-expansion method, see the list of literature cited at end of this section.

► Simplest equation method.

The simplest equation method, devised by Kudryashov (2005) for seeking particular solutions, is equivalent to using two differential constraints. In his papers, Kudryashov chose the differential constraint (10.3.4.2) in one of the following three forms:

$$w_x + w^2 - c_1 w - c_2 = 0, \tag{10.3.4.6}$$

$$(w'_x)^2 - 4w^3 - c_1 w^2 - c_2 w - c_3 = 0, \tag{10.3.4.7}$$

$$(w'_x)^2 - w^4 - c_1 w^3 - c_2 w^2 - c_3 w - c_4 = 0. \tag{10.3.4.8}$$

The differential constraint (10.3.4.1) was chosen from the class of functions

$$y = \sum_{k=0}^{K} b_{1k} w^k + w'_x \sum_{l=0}^{L} b_{2l} w^l + \sum_{m=1}^{M} b_{3m} \left(\frac{w'_x}{w}\right)^m. \tag{10.3.4.9}$$

For the differential constraint (10.3.4.6), it was assumed that $K = M$ and $b_{2l} = 0$ ($l = 1, \dots, L$) in (10.3.4.9). This resulted in a number of new exact solutions to nonlinear second-, third-, and fourth-order equations.

Remark 10.17. All equations (10.3.4.6)–(10.3.4.8) are reduced to separable equations, whose solutions are expressed in terms of elementary functions or/and integrals of elementary functions. A solution to (10.3.4.7) can be expressed through the Weierstrass function $\wp = \wp(z, g_2, g_3)$. A solution to equation (10.3.4.8) is expressed through the Jacobi elliptic function.

For specific examples of how to use the simplest equation method, see the articles cited below.

⊙ *Literature for Section 10.3*: N. N. Yanenko (1994), V. A. Galaktionov (1994), P. J. Olver and E. M. Vorob'ev (1996), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1998), O. V. Kaptsov and I. V. Verevkin (2003), A. D. Polyanin and V. F. Zaitsev (2004, 2012), A. D. Polyanin, V. F. Zaitsev, and A. I. Zhurov (2005), N. A. Kudryashov (2005, 2008, 2010a, 2010b, 2014), A. Bekir (2008), N. A. Kudryashov and N. V. Loguinova (2008), M. L. Wang, X. Li, J. Zhang (2008), J. Zhang, X. Wei, Y. Lu (2008), H. Zhang (2009), E. M. E. Zayed (2009), E. M. E. Zayed and K. A. Gepreel (2009), N. K. Vitanov and Z. I. Dimitrova (2010), N. K. Vitanov, Z. I. Dimitrova, and H. Kantz (2010), A. D. Polyanin (2016).

Chapter 11

Group Methods for ODEs

11.1 Lie Group Method. Point Transformations

11.1.1 Local One-Parameter Lie Group of Transformations. Invariance Condition

► **Preliminary remarks.**

The Lie group method for ordinary differential equations presents a routine procedure that allows obtaining the following:

(i) transformations under which differential equations are invariant (such transformations bring the given equation to itself);

(ii) new variables (dependent and independent) in which differential equations become considerably simpler (so that the resulting equation can be completely integrated or has a lower order than the original equation).

Remark 11.1. The Lie group method for ordinary differential equations may be treated as a significant extension of the method outlined in Section 9.3.

► **Local one-parameter Lie group of transformations. Infinitesimal operator.**

Here, we examine transformations of the ordinary differential equation

$$y_x^{(n)} = F\big(x, y, \dots, y_x^{(n-1)}\big). \tag{11.1.1.1}$$

Consider the set of transformations

$$\mathrm{T}_\varepsilon = \begin{cases} \bar{x} = \varphi(x, y, \varepsilon), & \bar{x}|_{\varepsilon=0} = x, \\ \bar{y} = \psi(x, y, \varepsilon), & \bar{y}|_{\varepsilon=0} = y, \end{cases} \tag{11.1.1.2}$$

where φ, ψ are smooth functions of their arguments and ε is a real parameter. The set T_ε is called a *continuous one-parameter Lie group of point transformations* if, for any ε_1 and ε_2, the following relation holds:

$$\mathrm{T}_{\varepsilon_1} \circ \mathrm{T}_{\varepsilon_1} = \mathrm{T}_{\varepsilon_1+\varepsilon_2}, \tag{11.1.1.3}$$

i.e., consecutive application of two transformations of the form (11.1.1.1) with parameters ε_1 and ε_2 is equivalent to a single transformation of the same form with parameter $\varepsilon_1 + \varepsilon_2$.

In what follows, we consider local continuous one-parameter Lie groups of point transformations (briefly called point groups) corresponding to an infinitesimal transformation (11.1.1.2) for $\varepsilon \to 0$. Taylor's expansion of $\bar{x}$ and $\bar{y}$ in (11.1.1.2) with respect to the parameter ε about $\varepsilon = 0$ yields:

$$\bar{x} \simeq x + \xi(x, y)\varepsilon, \quad \bar{y} \simeq y + \eta(x, y)\varepsilon, \tag{11.1.1.4}$$

where

$$\xi(x, y) = \frac{\partial \varphi(x, y, \varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \qquad \eta(x, y) = \frac{\partial \psi(x, y, \varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0}.$$

At each point (x, y), the vector (ξ, η) is tangent to the curve described by the transformed points $(\bar{x}, \bar{y})$.

S. LIE THEOREM. *Let the functions φ and ψ satisfy the group property* (11.1.1.3) *and allow the expansion* (11.1.1.4). *Then, these are solutions to the system of first-order ordinary differential equations (known as the Lie equations)*

$$\frac{d\varphi}{d\varepsilon} = \xi(\varphi, \psi), \qquad \frac{d\psi}{d\varepsilon} = \eta(\varphi, \psi) \tag{11.1.1.5}$$

subject to the initial conditions (11.1.1.2). *Conversely, for any smooth vector field* (ξ, η), *a solution to the Cauchy problem* (11.1.1.5), (11.1.1.2)), *which exists and is unique, satisfies the group property* (11.1.1.3).

The first-order linear differential operator

$$\mathrm{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{11.1.1.6}$$

corresponding to the infinitesimal transformation (11.1.1.4), is called the *infinitesimal operator* (or *infinitesimal generator*) of the group.

By definition, the *universal invariant* (briefly, invariant) of the group (11.1.1.2) and the operator (11.1.1.6) is a function $I_0(x, y)$, satisfying the condition

$$I_0(\bar{x}, \bar{y}) = I_0(x, y).$$

Taylor's expansion with respect to the small parameter ε yields the following linear partial differential equation for I_0:

$$\mathrm{X}I_0 = \xi(x, y)\frac{\partial I_0}{\partial x} + \eta(x, y)\frac{\partial I_0}{\partial y} = 0. \tag{11.1.1.7}$$

► Prolonged operator. Invariance condition and mth-order differential invariant.

Equation (11.1.1.1) will be treated as a relation for $n + 2$ variables x, y, y'_x, ..., $y^{(n)}_x$ with the differential constraints

$$y^{(k+1)}_x = \frac{dy^{(k)}}{dx}. \tag{11.1.1.8}$$

The space of these $n + 2$ variables is called the space of nth prolongation; and in order to work with differential equations, one has to define the action of operator (11.1.1.6) on the "new" variables $y'_x, \dots, y_x^{(n)}$, taking into account the differential constraints (11.1.1.8). For example, let us calculate the infinitesimal transformation of the first derivative. We have

$$\frac{d\bar{y}}{d\bar{x}} = \frac{D_x(y+\eta\varepsilon)}{D_x(x+\xi\varepsilon)} \simeq \frac{y'_x + (\eta_x + \eta_y y'_x)\varepsilon}{1 + (\xi_x + \xi_y y'_x)\varepsilon},$$

$$D_x = \frac{\partial}{\partial x} + y'_x \frac{\partial}{\partial y} + y''_{xx} \frac{\partial}{\partial y'_x} + \cdots,$$

where D_x is called the *operator of total derivative*. Expanding the right-hand side into a power series with respect to the parameter ε and preserving the first-order terms, we obtain

$$\bar{y}'_{\bar{x}} \simeq y'_x + \zeta_1(x, y, y'_x)\varepsilon,$$

where

$$\zeta_1 = \eta_x + (\eta_y - \xi_x)y'_x - \xi_y (y'_x)^2 = D_x(\eta) - y'_x D_x(\xi).$$

The action of the group on higher-order derivatives is determined by the recurrence formula:

$$\zeta_{k+1} = D_x(\zeta_k) - y_x^{(k+1)} D_x(\xi).$$

To a prolonged group there corresponds a *prolonged operator*:

$$\underset{n}{\mathrm{X}} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \sum_{k=1}^{n} \zeta_k\bigl(x, y, y', \dots, y_x^{(k)}\bigr)\frac{\partial}{\partial y_x^{(k)}}. \qquad (11.1.1.9)$$

The ordinary differential equation (11.1.1.1) admits the group (11.1.1.2) if

$$\underset{n}{\mathrm{X}}\bigl[y_x^{(n)} - F\bigl(x, y, y'_x, \dots, y_x^{(n-1)}\bigr)\bigr]\big|_{y_x^{(n)}=F} = 0. \qquad (11.1.1.10)$$

Relation (11.1.1.10) is called the *invariance condition*.

Remark 11.2. The invariant I_0, which is a solution of equation (11.1.1.7), also satisfies the equation $\underset{n}{\mathrm{X}} I_0 = 0$.

By definition, an *mth-order differential invariant* of the operator X is a function $I_m = I_m\bigl(x, y, y'_x, \dots, y_x^{(m)}\bigr)$, satisfying the linear partial differential equation $\underset{m}{\mathrm{X}} I_m = 0$ with the operator $\underset{m}{\mathrm{X}}$ defined by (11.1.1.9).

► Inverse problem.

In solving different modeling problems, it is required to construct a model equation that satisfies certain a priori conditions, for example, symmetry laws. Two different statements of the problem are possible here.

1°. Suppose there is a preset symmetry defined by the operator (11.1.1.6), with the coordinates $\xi(x,y)$ and $\eta(x,y)$ defined explicitly as specific functions. It is required to compute a universal invariant $I_0(x,y)$ and a first differential invariant $I_1(x,y,y'_x)$ of the operator (11.1.1.6). The class of nth-order equations that admit the operator (11.1.1.6) is given by the formula

$$\Phi\left(I_0, I_1, \frac{dI_1}{dI_0}, \dots, \frac{d^{n-1}I_1}{d^{n-1}I_0}\right) = 0.$$

Thus, problem 1° is always solvable as long as two first invariants of the operator are known, with this being also valid for nonpoint operators.

2°. Suppose there is a preset symmetry defined by the class of operators (11.1.1.6), with the coordinates $\xi(x,y)$ and $\eta(x,y)$ being arbitrary functions. A solution to the inverse problem is a class of equations of a given order that admit an arbitrary operator of the form (11.1.1.6). The universal method is the use of the similarity principle of one-parameter Lie groups of point transformations. Since any autonomous equation

$$y_x^{(n)} = \mathcal{F}\bigl(y, y'_x,\ \dots,\ y_x^{(n-1)}\bigr)$$

admits translations along the x-axis (i.e., operator $\mathrm{X} = \partial_x$), the arbitrary invertible point transformation $x = f(t,u)$, $y = g(t,u)$ produces a general class on nth-order equations with two-functional arbitrariness that admit a certain point operator. See Section 11.1.2 for examples.

11.1.2 Group Analysis of Second-Order Equations

► Structure of an admissible operator for second-order equations.

For second-order nonlinear equations

$$y''_{xx} = F(x, y, y'_x), \tag{11.1.2.1}$$

the invariance condition (11.1.1.10) is written in the form

$$\begin{aligned}\eta_{xx} + (2\eta_{xy} - \xi_{xx})y'_x + (\eta_{yy} - 2\xi_{xy})(y'_x)^2 - \xi_{yy}(y'_x)^3& \\ = (2\xi_x - \eta_y + 3\xi_y y'_x)F + \xi F_x + \eta F_y + [\eta_x + (\eta_y - \xi_x)y'_x - \xi_y(y'_x)^2]F_{y'_x}&,\end{aligned}$$

where $F = F(x, y, y'_x)$. This condition is in fact a second-order partial differential equation for two unknown functions $\xi(x,y)$ and $\eta(x,y)$. Since the unknown functions do not depend on the derivative y'_x, this equation can be represented (after F has been expanded in a power series with respect to y'_x, unless it is already a polynomial) in the form

$$\sum_{k=0}^{\infty} \Phi_k (y'_x)^k = 0, \tag{11.1.2.2}$$

with the Φ_k independent of y'_x. In order to ensure that condition (10) holds identically, one should set $\Phi_k = 0$, $k = 0,\ 1,\ \dots$ Thus, the invariance condition for a second-order equation can be "split" and represented as a system of equations (whose number can generally be infinite).

► Illustrative examples.

Example 11.1. If $F = F(x,y)$, i.e., the right-hand side of equation (11.1.2.1) does not depend on y'_x, then the determining equation can be "split" and represented as the system:

$$\begin{aligned}&\xi_{yy} = 0,\\ &\eta_{yy} - 2\xi_{xy} = 0,\\ &2\eta_{xy} - \xi_{xx} - 3F(x,y)\xi_y = 0,\\ &\eta_{xx} + (\eta_y - 2\xi_x)F(x,y) - F_x(x,y)\xi - F_y(x,y)\eta = 0.\end{aligned}$$

From the first two equations we find that

$$\xi = a(x)y + b(x), \quad \eta = a'(x)y^2 + c(x)y + d(x),$$

where $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are arbitrary functions. Substituting these expressions into the third and the fourth equations, we get

$$\begin{aligned}&3a''y + 2c' - b'' - 3F(x,y)a = 0,\\&a'''y^2 + c''y + d'' + (c - 2b')F - (ay + b)F_x - (a'y^2 + cy + d)F_y = 0.\end{aligned} \qquad (11.1.2.3)$$

In what follows, it is assumed that the function $F(x,y)$ is nonlinear with respect to the second argument. Then from the first equation in (11.1.2.3), we find that $a = 0$ and $c = \frac{1}{2}b' + \alpha$, where α is an arbitrary constant. The second equation in (11.1.2.3) becomes

$$\tfrac{1}{2}b'''y + d'' + \bigl(\alpha - \tfrac{3}{2}b'\bigr)F - bF_x - \bigl[\bigl(\tfrac{1}{2}b' + \alpha\bigr)y + d\bigr]F_y = 0. \qquad (11.1.2.4)$$

Equation (11.1.2.4) enables us to solve two different problems.

1°. If the function $F(x,y)$ is given, then, splitting equation (11.1.2.4) with respect to powers of y (the unknown functions b and d are independent of y), we obtain a new system, from which b, d, and α can be found; i.e., we ultimately obtain an admissible operator.

2°. Assuming that the functions b, d and the constant α are known but arbitrary, one can regard relation (11.1.2.4) as an equation for the unknown function $F(x,y)$. Solving this equation, we obtain a class of equations admitting a point operator. Thus, problem 2° is stated as an inverse problem.

Example 11.2. Let $F(x,y) = Ax^n y^m$, i.e., we are dealing with the Emden–Fowler equation $y''_{xx} = Ax^n y^m$. Then equation (11.1.2.4) becomes

$$\tfrac{1}{2}b'''y + d'' + \bigl(\alpha - \tfrac{3}{2}b'\bigr)Ax^n y^m - bnAx^{n-1}y^m - \bigl[\bigl(\tfrac{1}{2}b' + \alpha\bigr)y + d\bigr]mAx^n y^{m-1} = 0.$$

This relation must be satisfied identically by any function $y = y(x)$, and therefore, the coefficients of different powers of y must be equal to zero. As a result, we obtain a new system whose structure essentially depends on the value of m.

1°. It was assumed above that $F(x,y)$ is nonlinear in its second argument, and therefore, $m \neq 0$ and $m \neq 1$. Let $m \neq 2$. Then the system has the form:

$$\begin{aligned}&d'' = 0,\\&b''' = 0,\\&d = 0,\\&\bigl[\alpha(1-m) - \tfrac{1}{2}(3-m)b'\bigr]x - nb = 0.\end{aligned}$$

It follows that $d = 0$ and $b(x) = b_2x^2 + b_1x + b_0$, and the last equation of the system can be written in the form

$$(m + n + 3)b_2x^2 + \bigl[\tfrac{1}{2}(m + 2n + 3)b_1 + \alpha(m-1)\bigr]x + nb_0 = 0. \qquad (11.1.2.5)$$

To ensure relation (11.1.2.5), we equate all coefficients of this quadratic trinomial to zero to obtain

$$(m + n + 3)b_2 = 0, \quad \tfrac{1}{2}(m + 2n + 3)b_1 + \alpha(m-1) = 0, \quad nb_0 = 0. \qquad (11.1.2.6)$$

Analysis of system (11.1.2.6) yields solutions of the determining system corresponding to three different operators:

$$\begin{aligned}&\mathrm{X}_1 = (m-1)x\partial_x - (n+2)y\partial_y && \text{if } n \text{ and } m \text{ are arbitrary},\\&\mathrm{X}_2 = \partial_x && \text{if } n = 0,\\&\mathrm{X}_3 = x^2\partial_x + xy\partial_y && \text{if } m + n + 3 = 0.\end{aligned}$$

2°. Let $m = 2$. Then equation (11.1.2.4) becomes

$$d'' + \bigl(\tfrac{1}{2}b''' - 2Adx^n\bigr)y - \bigl[\bigl(\tfrac{5}{2}b' + \alpha\bigr)x + nb\bigr]Ax^{n-1}y^2 = 0.$$

Equating the term d'' and the coefficient of y in parentheses to zero, we get

$$d(x) = d_1 x + d_0,$$
$$b(x) = \frac{4ad_1 x^{n+4}}{(n+2)(n+3)(n+4)} + \frac{4ad_0 x^{n+3}}{(n+1)(n+2)(n+3)} + b_2 x^2 + b_1 x + b_0,$$

where $n \neq -1, -2, -3, -4$. The expression in square brackets (the coefficient of y^2) can be split with respect to powers of x and we obtain an algebraic system which, to within nonzero coefficients, has the form:

$$\begin{aligned}
&(7n+20)d_1 = 0,\\
&(7n+15)d_0 = 0,\\
&(n+5)b_2 = 0,\\
&(2n+5)b_1 + 2\alpha = 0,\\
&nb_0 = 0.
\end{aligned}$$

The last three equations coincide with the corresponding equations of system (11.1.2.6), whose solutions are already known. The first two equations yield two cases of prolongation of the admissible group:

$$\mathrm{X}_1 = 343Ax^{8/7}\partial_x + 4\bigl(49Ax^{1/7}y - 3x\bigr)\partial_y \quad \text{if } n = -\tfrac{20}{7},$$
$$\mathrm{X}_2 = 343Ax^{6/7}\partial_x + 3\bigl(49Ax^{-1/7}y + 4\bigr)\partial_y \quad \text{if } n = -\tfrac{15}{7}.$$

Example 11.3. Let us look at the inverse problem of Item 2° in Example 11.1. The solution of equation (11.1.2.4) is

$$F = b^{-3/2}E\left\{\Phi(u) + \int\left[\frac{1}{2}bb'''(u+V) + b^{1/2}d''E^{-1}\right]dx\right\}, \tag{11.1.2.7}$$

where $b(x)$ and $d(x)$ are arbitrary functions, Φ is an arbitrary function of its argument,

$$u = b^{-1/2}E^{-1}y - V, \quad V = \int b^{-3/2}dE^{-1}\,dx, \quad E = \exp\left(\alpha\int\frac{dx}{b(x)}\right),$$

and α is an arbitrary constant. The integral in formula (11.1.2.7) can be expressed in terms of V and E as

$$F = b^{-3/2}E\left[\Phi(u) + \alpha^2 V\right] + \frac{2bb'' - (b')^2}{4b^2}y + \frac{2bd' - b'd + 2\alpha d}{2b^2}.$$

A similar method is always used to solve the inverse problem for the equation of arbitrary (nth) order

$$y_x^{(n)} = \mathcal{F}\bigl(x, y, y'_x, \ldots, y_x^{(n-2)}\bigr),$$

provided that the right-hand side $\mathcal{F}$ does not contain the derivative $y_x^{(n-1)}$.

Example 11.4. Consider the problem from the previous example for the general second-order equation

$$y''_{xx} = F(x, y, y'_x). \tag{11.1.2.8}$$

Obviously, the most general class of equations admitting the translation group along the $\widetilde{x}$-axis is a subclass of autonomous equations from the class (11.1.2.8); specifically,

$$\widetilde{y}''_{\widetilde{x}\widetilde{x}} = F(\widetilde{y}, \widetilde{y}'_{\widetilde{x}}). \tag{11.1.2.9}$$

The translation operator $\mathrm{X} = \partial_{\widetilde{x}}$ can be converted into any operator of the form $\mathrm{X} = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y$ by the point transformation

$$\widetilde{x} = \varphi(x,y), \quad \widetilde{y} = \psi(x,y), \quad \varphi_x\psi_y - \varphi_y\psi_x \neq 0. \tag{11.1.2.10}$$

It is clear that the substitution of (11.1.2.10) into equation (11.1.2.9) results in a subclass of *all* equations (11.1.2.8) admitting a point operator:

$$(\varphi_x\psi_y-\psi_x\varphi_y)y''_{xx}+(\varphi_y\psi_{yy}-\psi_y\varphi_{yy})(y'_x)^3+(\varphi_x\psi_{yy}-\psi_x\varphi_{yy}+ \\ +2\varphi_y\psi_{xy}-2\psi_y\varphi_{xy})(y'_x)^2+(\varphi_y\psi_{xx}-\psi_y\varphi_{xx}+2\varphi_x\psi_{xy}-2\psi_x\varphi_{xy})y'_x+ \\ +\varphi_x\psi_{xx}-\psi_x\varphi_{xx}=(\varphi_x+\varphi_y y'_x)^3F\left(\psi,\frac{\psi_x+\psi_y y'_x}{\varphi_x+\varphi_y y'_x}\right).$$

11.1.3 Utilization of Local Groups for Reducing the Order of Equations and Their Integration

Suppose that an ordinary differential equation (11.1.1.1) admits an infinitesimal operator X of the form (11.1.1.6). Then the order of the equation can be reduced by one. Below we describe two methods for reducing the order of ODEs.

► **First method for reducing the order of equations.**

The transformation

$$t=f(x,y),\qquad u=g(x,y),\tag{11.1.3.1}$$

with f and g ($g\not\equiv 0$) being arbitrary particular solutions of the first-order linear partial differential equations

$$\begin{aligned}\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y}&=k,\\ \xi(x,y)\frac{\partial g}{\partial x}+\eta(x,y)\frac{\partial g}{\partial y}&=0,\end{aligned}\tag{11.1.3.2}$$

reduces equation (11.1.1.1) to an autonomous equation (the constant $k\neq 0$ can be chosen arbitrarily). The function $g=g(x,y)$ is a universal invariant of the operator X.

Suppose that the general solution of the characteristic equation

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}$$

has the form

$$U(x,y)=C,$$

where C is an arbitrary constant. Then the general solutions of equations (11.1.3.2) are given by (see A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002):

$$f=k\int\frac{dx}{\bar\xi(x,U)}+\Psi_1(U),$$
$$g=\Psi_2(U),\qquad U=U(x,y),$$

where $\Psi_1(U)$ and $\Psi_2(U)$ are arbitrary functions, $\bar\xi\big(x,U(x,y)\big)\equiv\xi(x,y)$, and U in the integral is regarded as a parameter.

Example 11.5. The Emden–Fowler equation $y''_{xx}=Ax^{-15/7}y^2$ admits the operator (cf. the operator X_2 in Item 2° of Example 11.2):

$$X=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y},\qquad\text{where}\qquad \xi(x,y)=343Ax^{6/7},\ \ \eta(x,y)=147Ax^{-1/7}y+12.$$

Equations (11.1.3.2) for $k = 49A$ admit the particular solutions

$$f = x^{1/7}, \qquad g = x^{-3/7}y + \frac{6}{49A}x^{-2/7}.$$

Solving (11.1.3.1) for x and y, we obtain the transformation

$$x = t^7, \qquad y = t^3u - \frac{6}{49A}t,$$

which reduces the original equation to the autonomous equation

$$u''_{tt} = 49Au^2,$$

which can easily be integrated by quadrature.

► Second method for reducing the order of equations.

Suppose that we know two invariants of the admissible operator X:

$$I_0 = I_0(x, y) \qquad \text{(universal invariant)}, \tag{11.1.3.3}$$

$$I_1 = I_1(x, y, y'_x) \qquad \text{(first differential invariant)}. \tag{11.1.3.4}$$

Then the second differential invariant can be found by differentiation,

$$I_2(x, y, y'_x, y''_{xx}) = \frac{dI_1}{dI_0}, \tag{11.1.3.5}$$

where $dI_m = (D_xI_m)\,dx$. Using (11.1.3.4)–(11.1.3.5), let us eliminate the derivatives y'_x and y''_{xx} from the original equation and take into account relation (11.1.3.3). Thus we obtain the first-order equation:

$$\frac{dI_1}{dI_0} = G(I_0, I_1).$$

Example 11.6. The Emden–Fowler equation $y''_{xx} = Ax^{-6}y^3$ admits an operator whose first prolongation has the form:

$$\underset{1}{\mathrm{X}} = x^2\partial_x + xy\partial_y + (y - xy')\partial_{y'}.$$

This operator admits the invariants:

$$I_0 = y/x, \qquad I_1 = xy'_x - y, \tag{11.1.3.6}$$

which form an integral basis of the first-order linear partial differential equation

$$x^2\frac{\partial I}{\partial x} + xy\frac{\partial I}{\partial y} + (y - xy')\frac{\partial I}{\partial y'} = 0.$$

Using (11.1.3.5) and (11.1.3.6), we find the second invariant:

$$I_2 = \frac{dI_1}{dI_0} = \frac{x^3y''_{xx}}{xy'_x - y}. \tag{11.1.3.7}$$

Let us express the unknown function and its derivatives from (11.1.3.6)–(11.1.3.7) to obtain

$$y = ux, \quad y'_x = \frac{ux + w}{x}, \quad y''_{xx} = \frac{ww'_u}{x^3}, \qquad \text{where} \quad u = I_0, \;\; w = I_1.$$

Substituting these expressions into the original equation, we see that the variable x is canceled and the equation takes the form

$$ww'_u = Au^3,$$

i.e., it becomes a first-order separable equation.

11.1.4 Seeking Particular Solutions

Particular solutions can be sought using Marius Sophus Lie's method, since the admitted group converts a solution of the equation into another solution. Therefore, a particular solution that is not invariant under this group generates a one-parameter family of particular solutions. Under the same condition, a particular solution to a first-order equation generates the general solution as a function of the group parameter a, which can be treated as an arbitrary constant.

Example 11.7. Consider the equation

$$y''_{xx} = Ax^{-15/7}y^2. \tag{11.1.4.1}$$

Its general solution is written in parametric form as

$$x = \alpha C_1^7\tau^7, \quad y = \beta C_1\tau(\tau^2\wp \mp 1), \qquad A = \pm\frac{6}{49}\alpha^{1/7}\beta^{-1}, \tag{11.1.4.2}$$

where $\wp$ is the Weierstrass function (see also 14.3.1.20). Equation (11.1.4.1) can be integrated by the classical group method, since it admits a two-dimensional point Lie algebra with operators

$$\mathrm{X}_1 = 7x\partial_x + y\partial_y, \quad \mathrm{X}_2 = 343x^{6/7}\partial_x + 3\left(49Ax^{-1/7}y + 4\right)\partial_y.$$

The finite-group of transformations for the operator X_2 is given by

$$\widetilde{x} = \left(49Aa + x^{1/7}\right)^7, \quad \widetilde{y} = \frac{49^2A^2\left(49Ay + 6x^{1/7}\right)}{x^{3/7}}\left(a + \frac{x^{1/7}}{49A}\right)^3 - 6\left(a + \frac{x^{1/7}}{49A}\right). \tag{11.1.4.3}$$

The Emden–Fowler equation, with (11.1.4.1) being its special case, has a particular solution in the form of a power-law function:

$$y_0 = -\frac{6}{49A}x^{1/7}. \tag{11.1.4.4}$$

Solution (11.1.4.4) is invariant under the group determined by the operator X_1. However, it turns out that the solution is also invariant under the transformations (11.1.4.3). Consequently, solution (11.1.4.4) is unsuitable for multiplication.

Example 11.8. Note that equation (11.1.4.1) has the trivial (zero) solution $y = 0$. It is also invariant under the operator X_1, but not under the operator X_2. This allows us to construct a one-parameter family of particular solutions to equation (11.1.4.1) by applying transformation (11.1.4.3) to y:

$$y = \frac{6}{49A}\left(\frac{x^{3/7}}{\left(49AC + x^{1/7}\right)^2} - x^{1/7}\right). \tag{11.1.4.5}$$

The group parameter a has been replaced with the arbitrary constant C. It is noteworthy that the family of algebraic functions (11.1.4.5) appears in the general solution (11.1.4.2), which is not obvious.

Thus, there is a simple algorithm for seeking families of particular solutions to broad classes of ordinary differential equations admitting a certain operator. As initial solutions, it is the easiest to use simple solutions such as constants, which are easy to verify.

If a differential equation has the zero solution ($y = 0$) and admits the operator

$$\mathrm{X} = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

the condition $\eta \neq 0$ must hold for the solution to be suitable for multiplication to obtain a one-parameter family of particular solutions.

Remark 11.3. Unfortunately, the above condition is not sufficient for the construction of a nontrivial family of solutions. Indeed, any equation that does not involve the variable y explicitly admits the operator $\mathrm{X} = \partial_y$. However, the zero solution (if any) generates the family $y = C$, which is too obvious and does not make much sense.

Example 11.9. Consider the equation

$$(y-x)y''_{xx} - (y'_x)^2 + (xy - x^2 - 2)y'_x - xy = 0.$$

It has the trivial solution and admits the operator

$$\mathrm{X} = (y-x)^{-1}\partial_y.$$

The finite group of transformations for the operator X has the form

$$\widetilde{x} = x, \quad \widetilde{y} = x - \sqrt{(y-x)^2 + a}.$$

On applying this transformation to y, we arrive at the nontrivial family of particular solutions

$$y = x - \sqrt{x^2 + C}.$$

⊙ *Literature for Section 11.1*: G. W. Bluman and J. D. Cole (1974), L. V. Ovsiannikov (1982), J. M. Hill (1982), P. J. Olver (1986), G. W. Bluman and S. Kumei (1989), H. Stephani (1989), N. H. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2003), V. F. Zaitsev and L. V. Linchuk (2009, 2014).

11.2 Contact and Bäcklund Transformations. Formal Operators. Factorization Principle

11.2.1 Contact Transformations

► **Continuous one-parameter Lie group of tangential transformations.**

The set of transformations

$$\mathrm{T}_\varepsilon = \begin{cases} \bar{x} = \varphi(x, y, y'_x, \varepsilon), & \bar{x}|_{\varepsilon=0} = x, \\ \bar{y} = \psi(x, y, y'_x, \varepsilon), & \bar{y}|_{\varepsilon=0} = y, \\ \bar{y}'_{\bar{x}} = \chi(x, y, y'_x, \varepsilon), & \bar{y}'_{\bar{x}}|_{\varepsilon=0} = y'_x \end{cases} \qquad (11.2.1.1)$$

(here, φ, ψ, χ are smooth functions of their arguments and ε is a real parameter) is called a *continuous one-parameter Lie group of tangential transformations* (or simply, a *tangential* or *contact group*) if $\mathrm{T}_{\varepsilon_1} \circ \mathrm{T}_{\varepsilon_2} = \mathrm{T}_{\varepsilon_1+\varepsilon_2}$, i.e., if successive application of transformations (11.2.1.1) with parameters ε_1 and ε_2 is equivalent to the same transformation with parameter $\varepsilon_1 + \varepsilon_2$. The transformed derivative $\bar{y}'_{\bar{x}}$ depends only on the first derivative y'_x and does not depend on the second derivative. Thus, the functions φ and ψ in (11.2.1.1) cannot be arbitrary but are related by (see Section 1.9.1):

$$\frac{\partial\psi}{\partial y'_x}\Big(\frac{\partial\varphi}{\partial x} + y'_x\frac{\partial\varphi}{\partial y}\Big) - \frac{\partial\varphi}{\partial y'_x}\Big(\frac{\partial\psi}{\partial x} + y'_x\frac{\partial\psi}{\partial y}\Big) = 0,$$

where the function χ is defined by

$$\chi = \frac{\partial\psi}{\partial y'_x}\Big/\frac{\partial\varphi}{\partial y'_x}.$$

► **Infinitesimal operator. Invariance condition.**

Proceeding as in Section 11.1.1, we consider the Taylor expansions of $\bar{x}$, $\bar{y}$, and $\bar{y}'_{\bar{x}}$ in (11.2.1.1) with respect to the parameter ε about $\varepsilon=0$, preserving only the first-order terms. We have

$$\bar{x} \simeq x+\xi(x,y,y'_x)\varepsilon, \quad \bar{y} \simeq y+\eta(x,y,y'_x)\varepsilon, \quad \bar{y}'_{\bar{x}} \simeq y'_x+\zeta(x,y,y'_x)\varepsilon,$$

where

$$\xi(x,y,y'_x)=\frac{\partial\varphi(x,y,y'_x,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}, \qquad \eta(x,y,y'_x)=\frac{\partial\psi(x,y,y'_x,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0},$$
$$\zeta(x,y,y'_x)=\frac{\partial\chi(x,y,y'_x,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}.$$

On the other hand,

$$\bar{y}'_{\bar{x}} \equiv \frac{d\bar{y}}{d\bar{x}}=\frac{D_x(y+\eta\varepsilon)}{D_x(x+\xi\varepsilon)} \simeq \frac{y'_x+(\eta_x+\eta_y y'_x+\eta_{y'_x}y''_{xx})\varepsilon}{1+(\xi_x+\xi_y y'_x+\xi_{y'_x}y''_{xx})\varepsilon}. \tag{11.2.1.2}$$

Expanding (11.2.1.2) with respect to ε and requiring that ζ be independent of y''_{xx}, we find that the three functions ξ, η, and ζ are expressed in terms of a single function $W(x,y,y'_x)$ as follows:

$$\xi=-\frac{\partial W}{\partial y'_x}, \qquad \eta=W-y'_x\frac{\partial W}{\partial y'_x}, \qquad \zeta=\frac{\partial W}{\partial x}+y'_x\frac{\partial W}{\partial y}. \tag{11.2.1.3}$$

To an infinitesimal tangential transformation (11.2.1.1) there corresponds the infinitesimal operator:

$$\mathrm{X}=\xi(x,y,y'_x)\frac{\partial}{\partial x}+\eta(x,y,y'_x)\frac{\partial}{\partial y}+\zeta(x,y,y'_x)\frac{\partial}{\partial y'_x} \tag{11.2.1.4}$$

whose coordinates satisfy relations (11.2.1.3).

The action of the group on higher derivatives is determined by the recurrence formula:

$$\zeta_{k+1}=D_x(\zeta_k)-y_x^{(k+1)}D_x(\xi),$$

where $\zeta_1=\zeta$. The invariance condition and the algorithm of finding tangential operators (11.2.1.4) admitted by ordinary differential equations are similar to those for point operators. The only difference is that the coordinates of the tangential operator depend on the first derivative; therefore, the determining equation can be split and reduced to a system only in the case of equations whose order is greater or equal to three.

Remark 11.4. There are no tangential transformations of finite order $k>1$ other than prolonged point transformations and contact transformations [these transformations are described by formulas similar to (11.2.1.1) and, in addition to y'_x, $\bar{y}'_{\bar{x}}$, contain higher derivatives of up to order k inclusive].

11.2.2 Bäcklund Transformations. Formal Operators and Nonlocal Variables

► **Lie–Bäcklund groups. Operator in canonical form.**

1°. If the coordinates of the infinitesimal operator are allowed to depend on the derivatives of arbitrary (up to infinity) orders, we obtain Lie–Bäcklund groups (of tangential transformations of infinite order). However, on the manifold determined by an ordinary differential equation, all higher derivatives are expressed through finitely many lower derivatives, as dictated by the structure of the equation itself and the differential relations obtained from the equation. The substitution of the right-hand side of equation (11.2.1.1) into an infinite series with derivatives usually results in very cumbersome formulas hardly suitable for practical calculations. For this reason, the Lie–Bäcklund groups are widely used only for the investigation of partial differential equations, whereas in the case of ordinary differential equations, a more effective approach is that based on the canonical form of an operator and the notion of a formal operator.

2°. The canonical form $\widetilde{\mathrm{X}}$ is defined by the relation

$$\widetilde{\mathrm{X}} = \mathrm{X} - \xi(x,y)D_x = [\eta(x,y) - \xi(x,y)y'_x]\frac{\partial}{\partial y},$$

where $\mathrm{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ is the infinitesimal operator of the group [see formula (11.1.1.6) in Section 11.1.1], and D_x is the operator of total derivative. The operators X and $\widetilde{\mathrm{X}}$ are equivalent in the sense that if one of them is admissible for the equation, then the other is also admissible (the operator of total derivative is admissible for any ordinary differential equation). The function $I_0(x,y) \equiv x$ is an invariant of any operator in canonical form.

The action of the group on higher order derivatives for an operator in canonical form is determined by the simple recurrence formula $\widetilde{\zeta}_{k+1} = D_x(\widetilde{\zeta}_k)$. The order of an equation that admits an operator in canonical form can be reduced on the basis of the algorithm described in Section 11.1.3 (see Paragraph "Second method").

► **Formal operators and nonlocal variables.**

By definition, a formal operator is an infinitesimal operator of the form

$$X = \Phi\,\partial_y, \qquad (11.2.2.1)$$

where the function Φ depends on x, y, y'_x, …, $y^{(k)}_x$ (with k smaller than the order of the equation under investigation) and auxiliary variables whose definition involves the symbol of indefinite integral, for instance,

$$\int \zeta(x,y,y'_x)\,dx$$

(the integration is with respect to the variable x which is involved both explicitly and implicitly, through the dependence of y on x). Such auxiliary variables are called nonlocal, in contrast to the coordinates of the prolonged space defined pointwise. The nonlocal variables depend on derivatives of arbitrarily high order, for instance,

$$\int y\,dx = \sum_{m=0}^{\infty}(-1)^m\frac{x^{m+1}}{(m+1)!}y^{(m)}_x.$$

This formula is obtained by successive integration by parts of its left-hand side. Thus, a nonlocal variable can be represented as an infinite formal series; and this enables us to express the coordinates of the Lie–Bäcklund operator in concise form.

A formal operator is a far-reaching generalization of an operator in canonical form. The function $I_0(x,y) \equiv x$ is an invariant of the formal operator (11.2.2.1) for any Φ.

When solving the direct problem, one usually prescribes the nonlocal operator in the general form

$$\mathrm{X} = \Big[\eta_1 \exp\Big(\int \zeta\, dx\Big) + \eta_2\Big]\partial_y \quad \text{or} \quad \mathrm{X} = \Big(\eta_1 \int \zeta\, dx + \eta_2\Big)\partial_y, \tag{11.2.2.2}$$

and then, in order to find an admissible operator, one uses a search algorithm similar to that described in Section 11.1.2. The coordinates of the prolonged operator are found by the formulas $\zeta_k = D_x(\zeta_{k-1})$, where $\zeta_0 = \Phi$. In contrast to the method of finding a point operator, in the present case, there are three unknown functions (η_1, η_2, ζ); and the splitting procedure to obtain a system can be realized with respect to all “independent” variables, in particular, the nonlocal variables.

Suppose that the differential equation

$$y_x^{(n)} = F\big(x, y, y'_x, \ldots, y_x^{(n-1)}\big) \tag{11.2.2.3}$$

can be written in new variables $x = I_0$, $z = I_1(x, y, y'_x)$, z'_x, z''_{xx}, $\ldots$, $z_x^{(n-1)}$, where I_0 and I_1 are invariants of an admissible operator of the form (11.2.2.1). Then the coordinate Φ of this operator satisfies the equation

$$\frac{\partial I_1}{\partial y}\Phi + \frac{\partial I_1}{\partial y'_x} D_x[\Phi] = 0,$$

which is an analogue of a linear ordinary differential equation for a function of several variables, since it involves the total derivative of the unknown function (exact differential equation). Its solution has the form:

$$\Phi = \exp\Big(-\int \frac{\partial I_1/\partial y}{\partial I_1/\partial y'_x}\, dx\Big), \tag{11.2.2.4}$$

where the integral is taken with respect to x involved explicitly and implicitly (through the dependence of y, y'_x, …on x), which means that this representation of an operator through a nonlocal variable is most universal. The function (11.2.2.4) generates a nonlocal exponential operator of the form (11.2.2.1) [the class of nonlocal exponential operators is specified by the first expression in (11.2.2.2) with $\eta_2 \equiv 0$].

THEOREM 1. *Any first-order equation*

$$y'_x = F(x, y) \tag{11.2.2.5}$$

admits a ***unique*** *formal operator (up to identical transformations on the manifold (11.2.2.5)) with coordinate explicitly independent of derivatives*

$$X = \exp\left(\int \frac{\partial F}{\partial y}\, dx\right)\partial_y. \tag{11.2.2.6}$$

Indeed, the invariance condition for equation (11.2.2.5) is

$$\underset{1}{X}\big[y'_x - F(x,y)\big]\Big|_{[F]} = 0.$$

It follows that

$$D_x[\Phi] - \frac{\partial F}{\partial y}\Phi = 0,$$

and hence,

$$\Phi = \exp\left(\int \frac{\partial F}{\partial y}\,dx\right).$$

Remark 11.5. It follows from Theorem 1 that equation (11.2.2.5) is integrable by quadrature if

$$\frac{\partial F}{\partial y} = D_x\Phi(x,y)\Big|_{y'_x=F(x,y)}, \tag{11.2.2.7}$$

where $\Phi(x,y)$ is some function. Indeed, if condition (11.2.2.7) holds, (11.2.2.6) is a point operator.

Example 11.10. The equation

$$y''_{xx} = 0$$

admits two Lie–Bäcklund operators:

$$\mathrm{X}_1 = \xi(x,y,y'_x)D_x, \quad \mathrm{X}_2 = \sum_{m=0}^{\infty} D_x^m\Big[yg(y-xy'_x,y'_x) + h(y-xy'_x,y'_x)\Big]\frac{\partial}{\partial y_x^{(m)}},$$

where ξ, g, h are arbitrary functions of their variables. The first operator is trivial (the operator of total derivative is admissible for any differential equation), while the second operator determines the maximal group of contact transformations admitted by the equation under consideration.

► **Construction methods for Lie–Bäcklund operators admitted by ODEs.**

A Lie–Bäcklund operator admitted by an ordinary differential equation can by found by three methods:

(i) in the form of an infinite formal series;

(ii) by passing to an equivalent system of ordinary first-order differential equations:

$$y'_1 = y_2, \quad y'_2 = y_3, \quad \dots, \quad y'_n = F(x,y_1,y_2,\dots,y_n),$$

and finding an admissible point group;

(iii) by its representation as a formal operator whose coordinates depend on nonlocal variables (the general form of the operator is chosen by the investigator).

In all cases, the search algorithm amounts to solving the determining system which is constructed by a procedure similar to that of Section 11.1. From the standpoint of simplicity and the possibility of integrating equations, the third method seems to be the most effective if one takes into account that an equation admitting an operator can be written in terms of new variables—invariants of the admissible operator—as a new ordinary differential equation whose order is by one less than that of the original equation.

11.2.3 Factorization Principles

► **Factorization principle: a special case.**

The use of formal operators enables us to formulate universal principles for reducing the order of an equation, independently of the specific structure of the operator (it can be a point operator, a tangential or nonlocal operator, or a Lie–Bäcklund operator).

THEOREM 1. *An arbitrary* n*th-order differential equation* (11.2.2.3) *can be factorized to a system of special form*

$$z_x^{(n-1)} = G\bigl(x, z, z'_x, \dots, z_x^{(n-2)}\bigr), \quad z = H(x, y, y'_x), \tag{11.2.3.1}$$

if and only if equation (11.2.2.3) *admits the nonlocal exponential operator:*

$$\mathrm{X} = \exp\Bigl(-\int \frac{H_y}{H_{y'_x}}\,dx\Bigr)\frac{\partial}{\partial y}. \tag{11.2.3.2}$$

The function $H(x, y, y'_x)$ is the first differential invariant of the operator (11.2.3.2). Therefore, having found an admissible operator (11.2.3.2) of the form

$$\mathrm{X} = \Phi\,\frac{\partial}{\partial y}, \qquad \Phi \equiv \exp\Bigl[\int Q(x, y, y'_x)\,dx\Bigr], \tag{11.2.3.3}$$

we can calculate H by solving the first-order linear partial differential equation

$$H_y + QH_{y'_x} = 0.$$

The function $Q(x, y, y'_x)$ is found as a solution of the determining system obtained by "splitting" the invariance condition for operator (11.2.3.3):

$$\underset{n}{\mathrm{X}}\bigl[y_x^{(n)} - F\bigl(x, y, y'_x, \dots, y_x^{(n-1)}\bigr)\bigr]\big|_{y_x^{(n)}=F} = 0,$$

where

$$\underset{n}{\mathrm{X}} = \sum_{k=0}^{n} \Phi_k \frac{\partial}{\partial y_x^{(k)}}, \qquad \Phi_k = D_x\Phi_{k-1}, \qquad \Phi_0 = \Phi,$$
$$D_x = \frac{\partial}{\partial x} + y'_x\frac{\partial}{\partial y} + y''_{xx}\frac{\partial}{\partial y'_x} + \cdots.$$

Theorem 1 generalizes the classical Lie algorithm, which is restricted to the case of unconditional solvability of the second equation of system (11.2.3.1). On the other hand, the introduction of the factor system (11.2.3.1) allows for two more cases, since the first equation is independent of y. These cases are the following:

1°. The first equation of system (11.2.3.1) allows for the reduction of the order or is solvable.

2°. The first equation of system (11.2.3.1) has some special properties, for instance, admits a fundamental system of solutions.

Example 11.11. The equation

$$y''_{xx} = f(x)y + g'_x(x)y^{-1} - [g(x)]^2 y^{-3} \tag{11.2.3.4}$$

for arbitrary functions $f(x)$ and $g(x)$ is the only equation of the form (its uniqueness is to within a Kummer–Liouville equivalence transformation; see Section 2.2.1)

$$y''_{xx} = F(x, y)$$

admitting the nonlocal exponential operator:

$$\mathrm{X} = \exp\Bigl(\int \zeta\,dx\Bigr)\eta\frac{\partial}{\partial y} = \exp\Bigl[\int\Bigl(\zeta + \frac{\eta_x + \eta_y y'_x}{\eta}\Bigr)dx\Bigr]\frac{\partial}{\partial y}, \qquad \eta = \eta(x, y), \quad \zeta = \zeta(x, y).$$

The second prolongation of the operator X has the form:

$$\underset{2}{\mathrm{X}} = \exp\Bigl(\int \zeta\,dx\Bigr)\bigl\{\eta\partial_y + (\eta_x + \eta_y y'_x + \eta\zeta)\partial_{y'_x}$$
$$+ \bigl[\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + (2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y)y'_x + \eta_{yy}(y'_x)^2 + \eta_y y''_{xx}\bigr]\partial_{y''_{xx}}\bigr\}.$$

Applying this operator to the equation $y''_{xx} = F(x,y)$ and replacing all instances of y''_{xx} by $F = F(x,y)$, we obtain the invariance condition in the form:

$$\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + \eta_y F - \eta F_y + (2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y)y'_x + \eta_{yy}(y'_x)^2 = 0.$$

Splitting this relation with respect to powers of the "independent" variable y'_x, we obtain the following system of three equations for the functions η, ζ, and F:

$$\begin{aligned}
&\eta_{yy} = 0,\\
&2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y = 0,\\
&\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + \eta_y F - \eta F_y = 0.
\end{aligned}$$

From the first two equations it follows that

$$\begin{aligned}
\eta &= a(x)y + b(x),\\
\zeta &= -\frac{aa'y^2 + 2a'by + c(x)}{(ay+b)^2},
\end{aligned}$$

where $a = a(x)$, $b = b(x)$, and $c = c(x)$ are arbitrary functions. The third equation can be treated as a first-order linear differential equation for the unknown function $F = F(x,y)$:

$$\frac{dF}{dy} - \frac{\eta_y}{\eta}F = \frac{1}{\eta}(\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \eta\zeta^2).$$

Substituting the above expressions of η and ζ into this relation and integrating the result, we obtain

$$F(x,y) = (ay+b)f(x) + \frac{[aa'' - 2(a')^2]b - (ab'' - 2a'b')a}{a^3}$$
$$- \frac{[aa'' - 3(a')^2]b^2 + 2aa'bb' - (ac' - 2a'c)a}{2a^3(ay+b)} - \frac{(a'b^2 - ac)^2}{4a^3(ay+b)^3},$$

where $f(x)$ is an arbitrary function.

The differential invariant z of the operator X satisfies the linear partial differential equation

$$\eta\frac{\partial z}{\partial y} + (\eta_x + \eta_y y'_x + \eta\zeta)\frac{\partial z}{\partial y'_x} = 0$$

(obtained after the division by $\exp(\int \zeta\,dx)$). Substituting the above η and ζ into this equation, we pass to the characteristic equation

$$\frac{dw}{dy} = \frac{aw}{ay+b} + \frac{2aa'y^2 + (3a'b + ab')y + bb' + c}{(ay+b)},$$

where $w = y'_x$. Integrating this equation, we find the differential invariant:

$$z = \frac{y'_x}{ay+b} - \frac{a'b - ab'}{a^2(ay+b)} + \frac{a'b^2 - ac}{2a^2(ay+b)^2}.$$

Having calculated the derivative z'_x, one can find y''_{xx} and, taking into account the known structure of the function $F(x,y)$, one obtains the factorization of the original equation:

$$\begin{aligned}
&z'_x + az^2 + (a'/a)z = f,\\
&(ay+b)y'_x = (ay+b)^2 z + a^{-2}(a'b - ab')(ay+b) - \tfrac{1}{2}a^{-2}(a'b^2 - ac).
\end{aligned}$$

An equivalence transformation of the form $ay + b \to y$, combined with the corresponding transformation of the independent variable and changed notation, yields:

$$\begin{aligned} &z'_x + z^2 = f(x), \\ &y'_x = zy + g(x)y^{-1}. \end{aligned} \tag{11.2.3.5}$$

The first equation of system (11.2.3.5) is a Riccati equation. Its general solution can be represented in terms of a fundamental system of solutions of the "truncated" linear equation:

$$y''_{xx} = f(x)y, \tag{11.2.3.6}$$

which coincides with (11.2.3.4) for $g \equiv 0$. The second equation of system (11.2.3.5) is a Bernoulli equation. It can be integrated by quadrature for an arbitrary function $z = z(x, C)$. Therefore, the general solution of equation (11.2.3.4) can be expressed in terms of a fundamental system of solutions of the linear equation (11.2.3.6). Note that in the general case, equation (11.2.3.4) admits no point groups.

Theorem 1 can be made more general. Let the second-order ordinary differential equation

$$y''_{xx} = F(x, y, y'_x), \tag{11.2.3.7}$$

admit the exponential nonlocal operator

$$\mathrm{X} = \bigl[\xi(x, y, y'_x)\partial_x + \eta(x, y, y'_x)\partial_y\bigr]\Omega, \quad \Omega = \exp\left(\int \zeta(x, y, y'_x)\,dx\right). \tag{11.2.3.8}$$

To describe *all* equations of the form (11.2.3.7) admitting factorization, it suffices to consider the operator (11.2.3.8). Equation (11.2.3.7) is then factorized to the system

$$\begin{aligned} &\dot{u}_t = G(t, u), \\ &u(t) = H_1(x, y, y'_x), \\ &t = H_0(x, y), \end{aligned} \tag{11.2.3.9}$$

where H_0 and H_1 are invariants of the operator (11.2.3.8). The last two equations in (11.2.3.9) are essentially one equation determining the function $u(t)$ in parametric form.

▶ Factorization principle: the general case.

If an operator admitted by equation (11.2.1.1) has no differential invariants of the first-order, then it is possible to apply the general factorization principle.

THEOREM 2. *An arbitrary nth-order differential equation* (11.2.1.1) *can be factorized to the system of special structure*

$$\begin{aligned} &z_x^{(n-k)} = G\bigl(x, z, z'_x, \dots, z_x^{(n-k-1)}\bigr), \\ &z = H\bigl(x, y, y'_x, \dots, y_x^{(k)}\bigr), \qquad \frac{\partial z}{\partial y_x^{(k)}} \neq 0, \end{aligned} \tag{11.2.3.10}$$

provided that equation (11.2.2.3) *admits a formal operator of the form* (11.2.2.1) *for which* $H\bigl(x, y, y'_x, \dots, y_x^{(k)}\bigr)$ *is a lower-order differential invariant on the manifold defined by* (11.2.2.3)*. The coordinate Φ of this operator satisfies the linear equation with total derivatives:*

$$\Phi\frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \dots + D_x^{(k)}[\Phi]\frac{\partial z}{\partial y^{(k)}} = 0. \tag{11.2.3.11}$$

Equation (11.2.3.11) plays a crucial role in both the direct and inverse problems. It can be regarded as an equation for the determination of the coordinate of the canonical operator (if one knows the invariant z). It can also be regarded as an equation for the determination of an invariant (if one knows the coordinate Φ). In the latter case, this is a first-order partial differential equation.

Example 11.12. The third-order nonlinear equation

$$yy'''_{xxx} + \bigl(y''_{xx}\bigr)^2 - y'_x y''_{xx} - f(x)y^2 = 0 \tag{11.2.3.12}$$

admits two operators

$$\mathrm{X}_1 = y\,\partial_y, \quad \mathrm{X}_2 = \Bigl(y\int y^{-2}\,dx\Bigr)\partial_y, \tag{11.2.3.13}$$

which can be found with the help of the direct algorithm, if the structure of the operator is specified by the second expression in (11.2.2.2). The first operator, X_1, is the usual point operator of scaling (the original equation is homogeneous) and provides the usual reduction of order of equation (11.2.3.12) by one. The second operator, X_2, is nonlocal.

Let us construct differential invariants of the operator X_2. To this end, we should solve the equations:

$$\begin{aligned} &\Phi\frac{\partial I_1}{\partial y} + D_x[\Phi]\frac{\partial I_1}{\partial y'_x} = 0, \quad \Phi = y\int y^{-2}\,dx,\\ &\Phi\frac{\partial I_2}{\partial y} + D_x[\Phi]\frac{\partial I_2}{\partial y'_x} + D_x^2[\Phi]\frac{\partial I_2}{\partial y''_{xx}} = 0. \end{aligned} \tag{11.2.3.14}$$

After differentiation with respect to Φ, the first equation in (11.2.3.14) becomes

$$\Bigl(y\int y^{-2}\,dx\Bigr)\frac{\partial I_1}{\partial y} + \Bigl[y^{-1} + \Bigl(y'_x\int y^{-2}\,dx\Bigr)\Bigr]\frac{\partial I_1}{\partial y'_x} = 0.$$

Let us show that this equation admits no solutions depending only on x, y, y'_x, and $\partial I_1/\partial y'_x \neq 0$, i.e., there are no first-order differential invariants. The nonlocal expression $\int y^{-2}\,dx$ depends on derivatives of arbitrarily high orders and can be regarded as an independent quantity. Therefore, the first equation (11.2.3.14) can be split and we obtain the system:

$$y\frac{\partial I_1}{\partial y} + y'_x\frac{\partial I_1}{\partial y'_x} = 0, \qquad y^{-1}\frac{\partial I_1}{\partial y'_x} = 0.$$

It follows that $\partial I_1/\partial y'_x = 0$.

Let us find a second-order differential invariant. After differentiation with respect to Φ, the second equation in (11.2.3.14) becomes

$$\Bigl(y\int y^{-2}\,dx\Bigr)\frac{\partial I_2}{\partial y} + \Bigl[y^{-1} + \Bigl(y'_x\int y^{-2}\,dx\Bigr)\Bigr]\frac{\partial I_2}{\partial y'_x} + \Bigl(y''_{xx}\int y^{-2}\,dx\Bigr)\frac{\partial I_2}{\partial y''_{xx}} = 0.$$

Splitting this equation with respect to the nonlocal variable $\int y^{-2}\,dx$, we find that $\partial I_2/\partial y'_x = 0$. In the remaining equation, the nonlocal variable is canceled,

$$y\frac{\partial I_2}{\partial y} + y''_{xx}\frac{\partial I_2}{\partial y''_{xx}} = 0.$$

It follows that $I_2 = z = y''_{xx}/y$, and equation (11.2.3.12) is factorized to the system

$$\begin{aligned} &z'_x + z^2 = f(x),\\ &y''_{xx} - yz = 0. \end{aligned}$$

► **Applications to third-order ODEs.**

Consider the nonlocal nonexponential operator

$$\mathrm{X} = \big(\xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y\big) \int \zeta(x,y,y'_x,y''_{xx})\,dx \qquad (11.2.3.15)$$

(see the previous example). It can be used to factorize the third-order ODE

$$y'''_{xxx} = F(x,y,y'_x,y''_{xx}). \qquad (11.2.3.16)$$

It is clear that the universal invariant of the operator (11.2.3.15) coincides with the invariant of the point operator

$$\mathrm{X}_0 = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y. \qquad (11.2.3.17)$$

The following theorem holds.

THEOREM 1. *The third-order ODE (11.2.3.16) admitting the nonlocal operator (11.2.3.15) always admits the point operator (11.2.3.17).*

Consequently, one needs to look for the nonlocal operator (11.2.3.15) for equation (11.2.3.16) only if the equation possesses a point symmetry. It may seem that this fact reduces the value of operators of the form (11.2.3.15). In fact, there are a large number of equations (in particular, third-order equations) for which no operators are known other than a single point operator. Therefore, the presence of at least one operator, even though a nonlocal one, admitted by the equation can significantly facilitate the integration and investigation of the original equation.

1°. *Preliminary remarks*. Consider the problem of seeking a class of third-order equations admitting a nonlocal nonexponential operator of the form

$$\mathrm{X} = \eta(x,y,y'_x)\left(\int \zeta(x,y,y'_x)\,dx\right)\partial_y. \qquad (11.2.3.18)$$

By virtue of Theorem 1, we can restrict ourselves to the class of autonomous equations, thus setting $\eta \equiv y'$ and looking for an operator in the form

$$\mathrm{X} = y'_x\left(\int \zeta(x,y,y'_x)\,dx\right)\partial_y. \qquad (11.2.3.19)$$

Then, we can find all such classes of equations by applying an arbitrary point transformation.

Let us find the prolongation of the operator (11.2.3.19). Let I denote the nonlocal variable:

$$I = \int \zeta(x,y,y'_x)\,dx.$$

Since $\widetilde{\eta} = y'_x I$, we get

$$\begin{aligned}
\widetilde{\eta}_1 &= D_x\widetilde{\eta} = y''_{xx}I + y'_x\zeta,\\
\widetilde{\eta}_2 &= D_x^2\widetilde{\eta} = y'''_{xxx}I + 2y''_{xx}\zeta + y'_x\big(\zeta_x + \zeta_y y'_x + \zeta_{y'_x}y''_{xx}\big),\\
\widetilde{\eta}_3 &= D_x^3\widetilde{\eta} = y_x^{(4)}I + 3y'''_{xxx}\zeta + 3y''_{xx}\big(\zeta_x + \zeta_y y'_x + \zeta_{y'_x}y'_x\big) + y'_x\big[\zeta_{xx} + 2\zeta_{xy}y'_x\\
&\quad + \zeta_{yy}(y'_x)^2 + 2\zeta_{xy'_x}y''_{xx} + 2\zeta_{yy'_x}y'_x y''_{xx} + \zeta_{y'_x y'_x}\big(y''_{xx}\big)^2 + \zeta_y y''_{xx} + \zeta_{y'_x}y'''_{xxx}\big].
\end{aligned}$$

2°. *Equations of the form* $y'''_{xxx} = F(y)$. The invariance condition is written in the form

$$\left(\widetilde{\eta}_3 - \widetilde{\eta}\frac{\partial F}{\partial y}\right)\bigg|_{y'''_{xxx}=F(y)} = 0.$$

Replacing y'''_{xxx} and $y^{(4)}_x$ with the help of the original equation and its differential consequence $y^{(4)}_x = y'_x F'(y)$ and splitting the remaining expression into powers of y''_{xx} and I, we arrive at a system that only has the trivial solution ($F \equiv 0$ or F is any but $\zeta \equiv 0$). Therefore, the following statement holds.

THEOREM 2. *There is no equation of the form* $y'''_{xxx} = F(y)$, *other than the trivial equation, admitting a nonlocal operator of the form (11.2.3.18).*

3°. *Equations of the form* $y'''_{xxx} = F(y, y'_x)$. Now consider the autonomous third-order equation without the second derivative

$$y'''_{xxx} = F(y, y'_x), \tag{11.2.3.20}$$

admitting the nonlocal nonexponential operator (11.2.3.19). The invariance condition is written as

$$\left(\widetilde{\eta}_3 - \widetilde{\eta}\frac{\partial F}{\partial y} - \widetilde{\eta}_1\frac{\partial F}{\partial y'_x}\right)\bigg|_{y'''_{xxx}=F(y,y'_x)} = 0.$$

The determining system has the form

$$\begin{aligned}
&3\zeta_{y'_x} + y'_x\zeta_{y'_x y'_x} = 0,\\
&3\zeta_x + 2y'_x(\zeta_{xy'_x} + 2\zeta_y + \zeta_{yy'_x}y'_x) = 0,\\
&(3\zeta + y'_x\zeta_{y'_x})F - y'_x\zeta F_{y'_x} + y'_x\zeta_{xx} + 2\zeta_{xy}(y'_x)^2 + \zeta_{yy}(y'_x)^3 = 0.
\end{aligned}$$

THEOREM 3. *Equation (11.2.3.20) admits the nonlocal nonexponential operator (11.2.3.19) if and only if the right-hand side has the form*

$$F(y, y'_x) = y'_x\bigl[C(y'_x)^2 + G(y)\bigr]H(y) - \frac{1}{2C}G''(y)y'_x, \tag{11.2.3.21}$$

with

$$\zeta(x, y, y'_x) = C + \frac{G(y)}{(y'_x)^2}, \tag{11.2.3.22}$$

where $G(y)$ *and* $H(y)$ *are arbitrary functions and* $C \neq 0$ *is an arbitrary constant.*

Remark 11.6. The value $C = 0$ is possible only if $G''(y) \equiv 0$. However, in this case, the original equation is trivial and easy to integrate.

The operator (11.2.3.19) has no first differential invariant (more precisely, it has no invariant dependent on the first derivative alone). To compute the second differential invariant of the operator

$$\mathrm{X} = y'_x\left[\int\left(C + \frac{G(y)}{(y'_x)^2}\right)dx\right]\partial_y, \tag{11.2.3.23}$$

we have to solve the equation

$$\widetilde{\eta}\frac{\partial\Phi}{\partial y} + \widetilde{\eta}_1\frac{\partial\Phi}{\partial y'_x} + \widetilde{\eta}_2\frac{\partial\Phi}{\partial y''_{xx}} = 0.$$

Inserting the coordinates of the operator (11.2.3.23) and splitting the equation in the nonlocal variable I, we obtain the system of two equations

$$y'_x\left[C+\frac{G(y)}{(y'_x)^2}\right]\frac{\partial\Phi}{\partial y'_x}+\big(2Cy''_{xx}+G'(y)\big)\frac{\partial\Phi}{\partial y''_{xx}}=0,$$
$$y'_x\frac{\partial\Phi}{\partial y}+y''_{xx}\frac{\partial\Phi}{\partial y'_x}+\left[y'_x\Big(C(y'_x)^2+G(y)\Big)H(y)-\frac{1}{2C}G''(y)y'_x\right]\frac{\partial\Phi}{\partial y''_{xx}}=0. \qquad (11.2.3.24)$$

Note that in the second equation, the derivative y'''_{xxx} is replaced with the right-hand side of equation (11.2.3.21); that is, the invariant is sought *on the manifold* of solutions of the original equation. The solution of the first equation in (11.2.3.24) is

$$\Omega\left(y,\frac{2Cy''_{xx}+G'(y)}{C(y'_x)^2+G(y)}\right). \qquad (11.2.3.25)$$

Inserting (11.2.3.25) into the second equation of the system results in a first-order linear partial differential equation for Ω:

$$\frac{\partial\Omega}{\partial y}+\left[H(y)-2C\omega^2\right]\frac{\partial\Omega}{\partial\omega}=0, \qquad (11.2.3.26)$$

where ω is the second argument of the function Ω. The equation in characteristics for (11.2.3.26) is a canonical Riccati equation, which is always reduced to a second-order linear equation. In a large number of cases, the solution to (11.2.3.26) can be expressed in closed form (in terms of elementary or special functions). The actual representation significantly depends on the function $H(y)$.

Example 11.13. If $H(y)=y^k$ or $H(y)=e^y$, the second differential invariant is expressed in terms of Bessel functions. In addition, in the case of the power-law function, we get a special Riccati equation and if the fraction $\frac{k+3}{k+2}$ is a half-integer, the second differential invariant is an elementary function. For example, if $k=0$, we get

$$\Omega=\sqrt{2C}y-\operatorname{arth}\left(\frac{2Cy''_{xx}+G'(y)}{\sqrt{2C}\,(C(y'_x)^2+G(y))}\right).$$

By direct verification, one can see that $\Omega'_x=0$ by virtue of the original equation $\Omega'_x=0$, suggesting the factorization

$$\Omega'_x\Big|_{y'''_{xxx}=F}=0,$$
$$\Omega=\sqrt{2C}y-\operatorname{arth}\left(\frac{2Cy''_{xx}+G'(y)}{\sqrt{2C}\,(C(y'_x)^2+G(y))}\right).$$

Thus, the function Ω is an *autonomous first integral* of the original equation, while the symmetry is analogous to variational symmetry (see Section 11.3).

Example 11.14. The equation

$$y'''_{xxx}=f(x)y+y^{-1}(y''_{xx})^2+y^{-4}(y^3y'_x+2A)y''_{xx}+A^2y^{-7}$$

can be factorized to the system

$$z'_x=z^2+f(x),$$
$$z=\frac{y''_{xx}}{y}+Ay^{-4}.$$

The first equation is a Riccati equation and the second one is the Ermakov equation; therefore, the solution to the original equation is here uniquely determined by two fundamental systems of solutions of two linear second-order equations. Note that this system can be factorized further, since the Ermakov equation always admits an exponential nonlocal operator.

Apparently, it makes sense to perform a *test for a nonexponential operator* for any third- or higher-order equation possessing a point symmetry. This test allows one to find an additional nonexponential nonlocal operator admitted by the equation in order to facilitate its subsequent integration or seek an unobvious first integral.

⊙ *Literature for Section 11.2*: R. L. Anderson and N. H. Ibragimov (1979), O. N. Pavlovskii and G. N. Yakovenko (1982), N. H. Ibragimov (1985), P. J. Olver (1986), V. F. Zaitsev (2001), A. D. Polyanin and V. F. Zaitsev (2003), V. F. Zaitsev and L. V. Linchuk (2014).

11.3 First Integrals (Conservation Laws)

11.3.1 Algorithm of Finding First Integrals of ODEs

A function $P = P\big(x, y, y'_x, \ldots, y_x^{(n-1)}\big)$ is called a *first integral* (*conservation law*) of the ordinary differential equation

$$y_x^{(n)} = F\big(x, y, \ldots, y_x^{(n-1)}\big) \tag{11.3.1.1}$$

if the total derivative of the function P along the trajectories of equation (11.3.1.1) is zero or, equivalently, if

$$D_x[P] \equiv M\big(x, y, y'_x, \ldots, y_x^{(n-1)}\big)\big[y_x^{(n)} - F\big(x, y, \ldots, y_x^{(n-1)}\big)\big] = 0, \tag{11.3.1.2}$$

where M is an integrating factor. From this definition it is clear that $M = \frac{\partial P}{\partial y_x^{(n-1)}}$.

The algorithm of finding a first integral is similar to that of finding an admissible operator. It is necessary to prescribe the desired structure of the first integral (or the integrating factor) and substitute it into the determining equation (11.3.1.2). Subsequent splitting with respect to lower derivatives (assumed to be independent variables) leads to the determining system.

Remark 11.7. An arbitrary function of first integrals is also a first integral of the same equation. Therefore, having found a first integral depending on $(y'_x)^k$, one has to make sure that it is nontrivial; i.e., it cannot be represented as the product of first integrals depending on lower powers of the derivative.

Remark 11.8. If the equation has k functionally independent first integrals, then its order can be reduced by k by successively excluding higher derivatives (see Example 11.15).

▶ Direct method.

Rewriting equation (11.3.1.2) in expanded form, we get

$$P_x + y'_x P_y + y''_x P_{y'_x} + \ldots + y_x^{(n)} P_{y_x^{(n-1)}} = -FM + y_x^{(n)} M. \tag{11.3.1.3}$$

Substituting here the value of M gives the equation

$$P_x + y'_x P_y + \ldots + y_x^{(n-1)} P_{y_x^{n-2}} + F P_{y_x^{(n-1)}} = 0. \tag{11.3.1.4}$$

No general solution to this homogeneous linear partial differential equation with respect to x, y, y', …, $y^{(n-1)}$ can usually be obtained (as this equation is equivalent to the original one). However, it is quite likely that its particular solutions can be found with the splitting method by imposing certain conditions on the form of the desired first integral. For example, this can be achieved by assuming that $P(x, y, y'_x, \dots, y_x^{(n-1)})$ is linear in the highest derivative,

$$P = R\big(x, y, y'_x, \dots, y_x^{(n-2)}\big) y_x^{(n-1)} + Q\big(x, y, y'_x, \dots, y_x^{(n-2)}\big), \qquad (11.3.1.5)$$

or quadratic in the highest derivative,

$$P = R\big(x, y, y'_x, \dots, y_x^{(n-2)}\big)\big(y_x^{(n-1)}\big)^2 + Q\big(x, y, y'_x, \dots, y_x^{(n-2)}\big) y_x^{(n-1)} + S\big(x, y, y'_x, \dots, y_x^{(n-2)}\big). \qquad (11.3.1.6)$$

Substituting the structure of the first integral into (11.3.1.4) and splitting the resulting equation in powers of $y_x^{(n-2)}$, we obtain the determining system, whose solution gives us the desired first integral.

Example 11.15. The equation

$$y''''_{xxxx} = Ay^{-5/3} \qquad (11.3.1.7)$$

admits three first integrals:

$$\begin{aligned} P_1 &= y'_x y'''_{xxx} - \tfrac{1}{2}(y''_{xx})^2 + \tfrac{3}{2}Ay^{-2/3}, \\ P_2 &= xP_1 - \tfrac{3}{2}yy'''_{xxx} + \tfrac{1}{2}y'_x y''_{xx}, \\ P_3 &= xP_2 - \tfrac{1}{2}x^2 P_1 + \tfrac{3}{2}yy''_{xx} - (y'_x)^2. \end{aligned}$$

Equating these expressions to independent constants C_1, C_2, C_3 and eliminating y'''_{xxx} and y''_{xx}, we obtain a first-order equation (see 4.2.1.1).

► Factorization method.

Let the nth-order equation (11.2.1.1) admit a (nonlocal) infinitesimal operator and let the factor system (11.2.3.10) have the form

$$\begin{aligned} &z'_x = 0, \\ &z = H\big(x, y, y'_x, \dots, y_x^{(n-1)}\big), \qquad \frac{\partial z}{\partial y_x^{(n-1)}} \neq 0. \end{aligned} \qquad (11.3.1.8)$$

Then the function $H\big(x, y, y'_x, \dots, y_x^{(n-1)}\big)$ is a first integral equation (11.2.1.1) and, simultaneously, a differential invariant of the admitted operator by virtue of (11.2.1.1).

► Other methods.

There are methods for finding an integrating factor for ODEs, which generalize the well-known approach to first-order equations. These use high-order Euler operators (see the paragraph on Nöther's theorem) and lead to results that are fundamentally the same as those of the direct method (see the literature for the present chapter).

11.3.2 Applications to Second-Order ODEs

For second-order equations

$$y''_{xx} = F(x, y, y'_x), \tag{11.3.2.1}$$

the determining equation (11.3.1.2) can be written in the form

$$\frac{\partial P}{\partial x} + y'_x \frac{\partial P}{\partial y} + F(x, y, y'_x) \frac{\partial P}{\partial y'_x} = 0. \tag{11.3.2.2}$$

In this case, one can solve the direct problem (find P for the given equation), as well as the inverse problem (find possible F for the given structure of the first integral).

Example 11.16. Let us find all equations of the form

$$y''_{xx} = F(x, y) \tag{11.3.2.3}$$

admitting a first integral that is quadratic with respect to the first derivative:

$$P = R(x, y)(y'_x)^2 + S(x, y)y'_x + Q(x, y).$$

Then the left-hand side of the determining equation (11.3.2.2) is a cubic polynomial with respect to y'_x. The procedure of splitting with respect to powers of y'_x yields the system of four equations:

$$\begin{aligned}
&R_y = 0,\\
&R_x + S_y = 0,\\
&S_x + Q_y + 2RF = 0,\\
&Q_x + SF = 0.
\end{aligned}$$

The solution of this system for F is given by:

$$\begin{aligned}
F(x, y) &= R^{-3/2}\Psi(z) + \frac{1}{2}R^{-2}\Big[\Big(RR''_{xx} - \frac{1}{2}R'^2_x\Big)y - R\varphi'_x + \frac{1}{2}R'_x\varphi\Big],\\
z &= R^{-1/2}y + \frac{1}{2}\int \varphi R^{-3/2}\,dx,
\end{aligned}$$

where $\Psi = \Psi(z)$, $R = R(x)$, and $\varphi = \varphi(x)$ are arbitrary functions. The class of equations obtained is essentially a solution to the inverse problem for equation (11.3.2.3), having a quadratic first integral, which is expressed as follows:

$$P = R(y'_x)^2 - (R'_x y - \varphi)y'_x + \frac{1}{4}R^{-1}(R'_x)^2y^2 - \frac{1}{2}R^{-1}R'_x\varphi y + \frac{1}{4}R^{-1}\varphi^2 - 2\int \Psi(z)dz.$$

Example 11.17. Consider the equation

$$y''_{xx} = Axy^{-1/2}.$$

Let us find its first integral, which is a cubic polynomial with respect to the first derivative:

$$P = R(x, y)(y'_x)^3 + S(x, y)(y'_x)^2 + Q(x, y)y'_x + U(x, y).$$

In this case, the left-hand side of the determining equation (11.3.2.2) is a fourth-order polynomial in y'_x, and hence the determining system consists of five equations:

$$\begin{aligned}
&R_y = 0,\\
&R_x + S_y = 0,\\
&S_x + Q_y + 3Axy^{-1/2}R = 0,\\
&Q_x + U_y + 2Axy^{-1/2}S = 0,\\
&U_x + Axy^{-1/2}Q = 0.
\end{aligned}$$

Solving this system, we obtain the first integral in the form:

$$P = (y'_x)^3 - 6Axy^{1/2}y'_x + 4Ay^{3/2} + 2A^2x^3.$$

The factorization method allows one to formulate, for second-order equations, a more rigorous result than in the general case. If equation (11.2.3.7) is factorized to system (11.2.3.9), then two cases of order reduction are possible:

(i) If the second (inner) equation is integrable, we obtain the classical method of order reduction, with the only difference that the application of exponential nonlocal operators provides a significant generalization of the results obtained using point operators (the generalization of the Ermakov equation is a good example).

(ii) If the first (outer) equation is integrable, we obtain a first integral of the original equation; this approach does not have classical analogues.

Example 11.18. The class of equations

$$y''_{xx} + \frac{\Psi\big(\sqrt{x^2+2ay}-x\big)}{\sqrt{x^2+2ay}} = 0,$$

where Ψ is an arbitrary function of its argument, admits the operator

$$X = \Big[a\partial_x + \big(\sqrt{x^2+2ay}-x\big)\partial_y\Big]\exp\int\frac{dx}{\sqrt{x^2+2ay}}.$$

Substituting its invariants

$$y'_x = u(t), \quad \sqrt{x^2+2ay}-x = t \qquad (11.3.2.4)$$

yields the first-order equation

$$\dot u_t = \frac{\Psi(t)}{au-t},$$

which is reduced, with the transformation $au - t = -w(t)$, to an Abel equation of the second kind

$$w\dot w_t - w = a\Psi(t). \qquad (11.3.2.5)$$

If the general solution to equation (11.3.2.5) is known, the first integral of the original equation is obtained in the (parametric) form (11.3.2.4).

11.3.3 Lie–Bäcklund Symmetries Generated by First Integrals

▶ Theorems on symmetries of first integrals.

1°. First, we note an important property of symmetries of differential equations: if an equation, having a first integral P, admits an operator X, the application of the operator X to the first integral P generates a first integral again (which can possibly be trivial). The following four cases are possible:

$$\begin{aligned}
&1.\quad \mathrm{X}(P) = 0,\\
&2.\quad \mathrm{X}(P) = C, \quad C = \text{const},\\
&3.\quad \mathrm{X}(P) = F(P),\\
&4.\quad \mathrm{X}(P) = P_1.
\end{aligned}$$

The second case gives a trivial first integral, while the third case gives the already known first integral; these cases are of no interest. The first case signifies that the *first integral inherits the symmetry of the original equation*, while the fourth case gives a new first integral, which is functionally independent of the already known ones. These two cases allow us to reduce the order of the original equation by two.

2°. Since any first integral of an ODE on the manifold of its solutions is a constant, the point operator admitted by the equation is essentially an infinite-dimensional Lie–Bäcklund algebra.

THEOREM 1. *Let the equation*

$$y^{(n)} = \mathcal{F}\big(x, y, y_x, \ldots, y_x^{(n-1)}\big) \tag{11.3.3.1}$$

admit a Lie algebra L_k *with basis* $\{\mathrm{X}_\alpha\}$, $\mathrm{X}_\alpha = \eta_\alpha \partial_y$, $\alpha = 1, \ldots, k$, *and have* s *independent first integrals* $\{P_\sigma\}$, $\sigma = 1, \ldots, s$ $(s \leqslant n)$. *Then, equation (11.3.3.1) admits an infinite-dimensional Lie–Bäcklund algebra with operator*

$$\mathrm{X}_B = \left(\sum_{\alpha=1}^{k} \eta_\alpha F_\alpha\right) \partial_y, \tag{11.3.3.2}$$

where F_α $(\alpha = 1, \ldots, k)$ *are arbitrary functions of* s *arguments* $P_1, \ldots, P_s$.

Example 11.19. Equation (11.3.2.5) admits a three-dimensional point Lie algebra defined by the operators

$$\mathrm{L}_3\colon \quad \mathrm{X}_1 = y'_x \partial_y, \quad \mathrm{X}_2 = \left(x y'_x - \frac{3}{2} y\right) \partial_y, \quad \mathrm{X}_3 = \left(\frac{1}{2} x^2 y'_x - \frac{3}{2} x y\right) \partial_y.$$

Hence, the equation also admits the infinite-dimensional Lie–Bäcklund algebra defined by the operator

$$\mathrm{X}_B = \left[y' F_1 + \left(x y'_x - \frac{3}{2} y\right) F_2 + \left(\frac{1}{2} x^2 y'_x - \frac{3}{2} x y\right) F_3\right] \partial_y, \tag{11.3.3.3}$$

where $F_i = F_i(P_1, P_2, P_3)$, $i = 1, 2, 3$, are arbitrary functions and P_1, P_2, and P_3 are first integrals of equation (11.3.2.5).

It follows from the theorem that the knowledge of *one* lowest symmetry and *one* first integral suffices to obtain an infinite-dimensional Lie–Bäcklund algebra.

Example 11.20. The equation $y'''_{xxx} = A x y^{-5/4}$ admits the Lie–Bäcklund algebra defined by the operator $\mathrm{X}_B = [(9 x y'_x - 16 y) F(P)] \partial_y$, where $P = y (y''_{xx})^2 - \frac{1}{2} (y'_x)^2 y''_{xx} - 2 A x y^{-1/4} y'_x + \frac{8}{3} A y^{3/4}$ is a first integral of the equation.

Example 11.21. The equation $y'''_{xxx} = A y^{-1}$ admits the Lie–Bäcklund algebra defined by the operator $\mathrm{X}_B = [y'_x F_1(P) + (2 x y'_x - 3y) F_2(P)] \partial_y$, where $P = y y''_{xx} - \frac{1}{2} (y'_x)^2 - A x$ is a first integral of the equation.

Remark 11.9. Theorem 1 does not guarantee the completeness of the Lie–Bäcklund algebra obtained.

It is well known how an operator transforms when differential substitutions are used (in particular, nonlocal can arise in order reduction); however, this is not so obvious with first integrals. Some of the lowest symmetries may seem to disappear when a first integral is used. Below we show that this disappearance is only apparent. Let us look at the symmetry properties of first integrals.

THEOREM 2. *Let equation (11.3.3.1) admit the Lie–Bäcklund algebra defined by the operator (11.3.3.2). Then, for any* $P_\nu \in \{P_\sigma\}$, $\sigma = 1, \ldots, s$, *the equation*

$$P_\nu = C_\nu \tag{11.3.3.4}$$

1) has $s - 1$ *first integrals* $\{\widetilde{P}_\sigma\}$, $\sigma = 1, \ldots, s$, $\sigma \neq \nu$, *where* $\widetilde{P}_\sigma = P_\sigma|_{P_\nu = C_\nu}$;

2) admits the Lie–Bäcklund algebra defined by an operator of the form (11.3.3.2) with an arbitrariness of no less than $k-1$ *functions of* $s-1$ *variables* $\{\widetilde{P}_\sigma\}$.

To construct the algebra admitted by first integrals, we take advantage of the property mentioned at the beginning of this paragraph: the action of any admissible operator on a first integral gives a first integral again (which may be trivial). Let us denote $\mathrm{X}_\alpha[P_\sigma]=Q_{\alpha\sigma}$ and construct the operator

$$\mathrm{X}_B = \left(\sum_{\alpha=1}^{k} \eta_\alpha \widetilde{F}_\alpha\right) \partial_y,$$

where $\widetilde{F}_\alpha$, $\alpha = 1, \ldots, k$, are arbitrary functions of $s-1$ arguments $\{\widetilde{P}_\sigma\}$, $\sigma \neq \nu$. The invariance condition is written as

$$\widetilde{\mathrm{X}}_B[P_\nu - C_\nu]\Big|_{P_\nu = C_\nu} = \sum_{\alpha=1}^{k} \widetilde{F}_\alpha \mathrm{X}_\alpha[P_\nu]\Big|_{P_\nu=C_\nu} = \sum_{\alpha=1}^{k} \widetilde{F}_\alpha Q_{\alpha\nu}\Big|_{P_\nu=C_\nu} = 0. \qquad (11.3.3.5)$$

Since the arguments of the functions $\widetilde{F}_\alpha$ are the first integrals $\widetilde{P}_\sigma$, and the quantities $Q_{\alpha\nu}$ are also first integrals (or constants), the last equality in (11.3.3.5) allows us to express any function $\widetilde{F}_\beta$ in terms of the others (provided that $Q_{\beta\nu} \neq 0$). The admissible operator is

$$\mathrm{X}_B = \left[\sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{k} \left(\eta_\alpha - \frac{Q_{\alpha\nu}}{Q_{\beta\nu}} \eta_\beta\right) \widetilde{F}_\alpha\right] \partial_y, \quad Q_{\beta\nu} \neq 0.$$

Remark 11.10. The arbitrariness of k functions of $s-1$ variables is achieved only if $Q_{\alpha\nu} = 0$ for all $\alpha = 1, \ldots, k$, that is, if and only if equation (11.3.3.4) admits all $\{X_\alpha\}$.

Example 11.22. The equation $P_1 = C_1$, where P_1 is an integral of equation (11.3.2.5), so that

$$y'''_{xxx} = \frac{1}{2}(y'_x)^{-1}(y''_{xx})^2 - \frac{3}{2}Ay^{-2/3}(y'_x)^{-1} + C_1(y'_x)^{-1}, \qquad (11.3.3.6)$$

admits the Lie–Bäcklund operator

$$\widetilde{\mathrm{X}}_B = \left\{ y'_x \widetilde{F}_1(\widetilde{P}_2, \widetilde{P}_3) + \left[\left(xy'_x - \frac{3}{2}y\right)\widetilde{P}_2 + C_1\left(\frac{1}{2}x^2y'_x - \frac{3}{2}xy\right)\right] \widetilde{F}_3(\widetilde{P}_2, \widetilde{P}_3)\right\} \partial_y,$$

where

$$\widetilde{P}_2 = \frac{3}{4}y(y'_x)^{-1}(y''_{xx})^2 - \frac{1}{2}y'_x y''_{xx} - \frac{9}{4}Ay^{1/3}(y'_x)^{-1} + \frac{3}{2}C_1 y(y'_x)^{-1} - C_1 x,$$
$$\widetilde{P}_3 = \frac{3}{4}xy(y'_x)^{-1}(y''_{xx})^2 - \frac{1}{2}(xy'_x - 3y)y''_{xx} - \frac{9}{4}Axy^{1/3}(y'_x)^{-1} + \frac{3}{2}C_1 xy(y'_x)^{-1} - \frac{3}{2}C_1 x^2$$

are first integrals of equation (11.3.3.6).

► Nöther's theorem.

Definition 1. The operator

$$E_n = \sum_{i=0}^{n} (-D_x)^i \partial_{y_x^{(i)}} = \partial_y - D_x \partial_{y'_x} + D^2 \partial_{y''_{xx}} - \cdots + (-D_x)^n \partial_{y_x^{(n)}}, \qquad (11.3.3.7)$$

where

$$D = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots + y^{(n)}\partial_{y^{(n-1)}} + \cdots,$$

is called an Euler operator of order n.

Let us look at the functional

$$S[y(x)] = \int\limits_V L(x, y, y'_x, \dots, y_x^{(n)})\, dx. \qquad (11.3.3.8)$$

Searching for extremals of a variation problem for the functional (11.3.3.8) is known to reduce to solving the Euler–Lagrange equation

$$E_n\big[L(x, y, y'_x, \dots, y_x^{(n)})\big] = 0. \qquad (11.3.3.9)$$

What is of interest is the case where, among all symmetry groups admitted by the Euler–Lagrange equation (11.3.3.9), there are also groups admitted by the Lagrangian $L(x, y, y'_x, \dots, y_x^{(n)})$. Such symmetry groups are known as *variational* or *Nötherian*; they play a major role in physics and mathematics, since they are closely related to conservation laws. Obviously, the order of equation (11.3.3.9) is $2n$.

Definition 2. The differential equation

$$F(x, y, y'_x, \dots, y_x^{(n)}) = 0, \qquad \frac{\partial F}{\partial_{y_x^{(n)}}} \not\equiv 0, \qquad (11.3.3.10)$$

has a variational formulation if its solutions $y = \Theta(x)$ in the domain V coincide with extremals of the functional (11.3.3.8).

THEOREM 1. *A $2n$th-order differential equation has a variational formulation if and only if it coincides with the Euler–Lagrange equation of a Lagrangian $L(x, y, y'_x, \dots, y_x^{(n)})$, so that*

$$F(x, y, y'_x, \dots, y_x^{(2n)}) = E_m[L(x, y, y'_x, \dots, y_x^{(m)})]. \qquad (11.3.3.11)$$

Remark 11.11. Only even-order equations can have a variational formulation.

Definition 3. The functional $S[y(x)]$ admits an infinitesimal operator X if its Lagrangian L is invariant under transformations of the group defined by the operator. The admitted group is called variational.

THEOREM 2. *If G is a group of variational symmetries of the function (11.3.3.8), then it is also a group of symmetries of the Euler–Lagrange equation $E_n(L) = 0$. In this case, the equation is said to admit the variational (Nötherian) symmetry.*

THEOREM 3. *Every admitted Nötherian group allows one to reduce the order of the Euler–Lagrange equation by two.*

The algorithm of order reduction for the Euler–Lagrange equation using a Nötherian operator is quite clear. However, the implementation of this algorithm as applied to differential equations causes certain difficulties: first, one has to find a suitable Lagrangian $L\big(x, y, y'_x, \dots, y_x^{(n)}\big)$, but there is no algorithm for obtaining it. The algorithm allowing order reduction for the equation by two without finding a Lagrangian relies on the following theorem.

EMMY NÖTHER'S THEOREM. *A symmetry of an even-order equation is Nötherian if the coordinate of its infinitesimal operator (in canonical form) coincides, up to a constant factor, with an integrating factor of a first integral of the equation. Furthermore, this operator makes the first integral identically zero.*

► Analogues of variational symmetries. Inverse problems.

Variational symmetry is only defined for even-order equations. All attempts to introduce a Hamiltonian structure for odd-order ODEs have failed so far (in terms of integrability). Over time, the impression has been formed that there is no similar symmetry structure for odd-order equations. However, this is not so. The following simple third-order equation is a counterexample:

$$y'''_{xxx} = 2yy'_x. \qquad (11.3.3.12)$$

It is autonomous and has an autonomous first integral

$$y''_{xx} = y^2 + C. \qquad (11.3.3.13)$$

This means that the symmetry of equation (11.3.3.12) is perfectly analogous to variational symmetry in the sense that the first integral (11.3.3.13) inherits it and allows the order of the original equation to be reduced by two.

We will be looking for classes of third-order equations possessing an analogue of variational symmetry or, equivalently, classes of equations admitting a point operator and a first integral inheriting this symmetry. From the viewpoint of integrability, this property is a direct analogue of Nötherian symmetry. The solution of this inverse problem is directly reduced to the simultaneous solution of *three* complicated determining systems: the invariance condition for the original equation with respect to an arbitrary point symmetry, the existence condition for a first integral with a set structure, and the invariance condition for this first integral with respect to the same point symmetry. Except for the simplest cases, the implementation of this algorithm is extremely difficult. Therefore, a different strategy will be followed; specifically, we will use the principle of similarity of one-parameter point groups in the plane and solve the problem for a selected simple symmetry (e.g., for an autonomous equation admitting the operator $\mathrm{X} = \partial_x$), while indenting to extend the obtained result to an arbitrary point symmetry.

1°. Let us compute point groups of equivalence for some subclasses of third-order equations. Obviously, the group of equivalence for the whole class of third-order equations is the set of arbitrary invertible point transformations

$$y = f(t,u), \qquad x = g(t,u) \qquad (11.3.3.14)$$

with a nonzero Jacobian

$$D = \begin{vmatrix} f_t & g_t \\ f_u & g_u \end{vmatrix} = f_t g_u - f_u g_t \not\equiv 0. \qquad (11.3.3.15)$$

Assuming t in (11.3.3.14) to be the independent variable, we write out the formulas for the transformation of the derivatives:

$$y'_x = \frac{f_t + f_u \dot{u}_t}{g_t + g_u \dot{u}_t}, \qquad (11.3.3.16)$$

$$\begin{aligned} y''_{xx} = \big[(g_t f_u - g_u f_t)\ddot{u}_{tt} + (g_u f_{uu} - g_{uu} f_u)(\dot{u}_t)^3 + \\ + (g_t f_{uu} - g_{uu} f_t + 2g_u f_{tu} - 2g_{tu} f_u)(\dot{u}_t)^2 + \\ + (g_u f_{tt} - g_{tt} f_u + 2g_t f_{tu} - 2g_{tu} f_t)\dot{u}_t + \\ + g_t f_{tt} - g_{tt} f_t\big](g_t + g_u \dot{u}_t)^{-3}, \end{aligned} \qquad (11.3.3.17)$$

$$
\begin{aligned}
y'''_{xxx} = \big\{&(g_t + g_u\dot u_t)(f_u g_t - f_t g_u)\dddot u_{ttt} + 3g_u(f_t g_u - f_u g_t)(\ddot u_{tt})^2 + \\
&+ 3\big[g_t(f_{uu}g_u - f_u g_{uu}) - g_u(f_{tu}g_u - f_u g_{tu}) - g_{uu}(f_u g_t - f_t g_u)\big](\dot u_t)^2\ddot u_{tt} + \\
&+ 3\big[g_t(f_{uu}g_t - f_t g_{uu}) - g_u(f_{tt}g_u - f_u g_{tt}) - 3g_{tu}(f_u g_t - f_t g_u)\big]\dot u_t\ddot u_{tt} + \\
&+ 3\big[g_t(f_{tu}g_t - f_t g_{tu}) - g_u(f_{tt}g_t - f_t g_{tt}) - g_{tt}(f_u g_t - f_t g_u)\big]\ddot u_{tt} + \\
&+ \big[g_u(f_{uuu}g_u - f_u g_{uuu}) - 3g_{uu}(f_{uu}g_u - f_u g_{uu})\big](\dot u_t)^5 + \\
&+ \big[g_t(f_{uuu}g_u - f_u g_{uuu}) + g_u(f_{uuu}g_t - f_t g_{uuu} + 3f_{tuu}g_u - 3f_u g_{tuu}) - \\
&- 6g_{tu}(f_{uu}g_u - f_u g_{uu}) - 3g_{uu}(f_{uu}g_t - f_t g_{uu} + 2f_{tu}g_u - 2f_u g_{tu})\big](\dot u_t)^4 + \\
&+ \big[g_t(f_{uuu}g_t - f_t g_{uuu} + 3f_{uut}g_u - 3f_u g_{uut}) + \\
&+ 3g_u(f_{ttu}g_u - f_u g_{ttu} + f_{tuu}g_t - f_t g_{tuu}) - 6g_{ut}(f_{uu}g_t - f_t g_{uu} + 2f_{ut}g_u - 2f_u g_{tu}) - \\
&- 3g_{uu}(f_{tt}g_u - f_u g_{tt} + 2f_{ut}g_t - 2f_t g_{ut}) - 3g_{tt}(f_{uu}g_u - f_u g_{uu})\big](\dot u_t)^3 + \\
&+ \big[g_u(f_{ttt}g_u - f_u g_{ttt} + 3f_{ttu}g_t - 3f_t g_{ttu}) + \\
&+ 3g_t(f_{ttu}g_u - f_u g_{ttu} + f_{tuu}g_t - f_t g_{tuu}) - 6g_{tu}(f_{tt}g_u - f_u g_{tt} + 2f_{tu}g_t - 2f_t g_{ut}) - \\
&- 3g_{tt}(f_{uu}g_t - f_t g_{uu} + 2f_{tu}g_u - 2f_u g_{tu}) - 3g_{uu}(f_{tt}g_t - f_t g_{tt})\big](\dot u_t)^2 + \\
&+ \big[g_u(g_t f_{ttt} - f_t g_{ttt}) + g_t(g_u f_{ttt} - f_u g_{ttt} + 3g_t f_{ttu} - 3f_t g_{ttu}) - \\
&- 3g_{tt}(g_u f_{tt} - f_u g_{tt} + 2g_t f_{tu} - 2f_t g_{tu}) - 6g_{tu}(g_t f_{tt} - f_t g_{tt})\big]\dot u_t + \\
&+ g_t(g_t f_{ttt} - g_{ttt}f_t) + 3g_{tt}(f_t g_{tt} - f_{tt}g_t)\big\}(g_t + g_u\dot u_t)^{-5}. \qquad (11.3.3.18)
\end{aligned}
$$

Now, in order to find equivalence groups on a given subclass, we must find conditions for the form of the subclass to be preserved by using relations (11.3.3.16)–(11.3.3.18). First, let us find the equivalence group on the class of equations not involving the intermediate derivatives explicitly,

$$y'''_{xxx} = F(x, y). \qquad (11.3.3.19)$$

So we look for transformations of the form

$$y'''_{xxx} = F(x, y) \longrightarrow \dddot u_{ttt} = G(t, u).$$

To this end, we require that the expression of the third derivative of the transformed variable (11.3.3.18) does not contain intermediate derivatives. It is clear that a necessary condition for that is $g_u \equiv 0$, or $g = g(t)$. By splitting expression (11.3.3.18) in powers of $\ddot u$ and $\dot u$, we obtain a determining system for the transformation elements (functions f and g):

$$
\begin{aligned}
&g_u = 0,\\
&f_{uu} = 0,\\
&g_t f_{tu} - g_{tt} f_u = 0,\\
&3g_t^2 f_{ttu} - g_t g_{ttt} f_u - 6g_t g_{tt} f_{tu} + 3g_{tt}^2 f_u = 0.
\end{aligned} \qquad (11.3.3.20)
$$

Solving the first three equations gives

$$
\begin{aligned}
x &= g(t),\\
y &= Cg'(t)u + h(t),
\end{aligned} \qquad (11.3.3.21)
$$

where $g(t)$ and $h(t)$ are arbitrary functions of t and C is an arbitrary constant ($C \neq 0$). In view of (11.3.3.21), the last equation in (11.3.3.20) becomes

$$2(g')^2 g''' - 3g'(g'')^2 = 0.$$

Its solution is

$$g = \frac{C_1}{t + C_2} + C_3.$$

Hence, the equivalence group for the class (11.3.3.19) consists of transformations of the form

$$x = \frac{C_1}{t + C_2} + C_3, \quad y = \frac{C_4 u}{(t + C_2)^2} + h(t), \tag{11.3.3.22}$$

where $C_1, \dots, C_4$ are arbitrary constants and $h(t)$ is an arbitrary function of t. Very similar arguments lead to exactly the same result for the subclass of equations not involving the first derivative:

$$y'''_{xxx} = F(x, y, y''_{xx}). \tag{11.3.3.23}$$

THEOREM 1. *An arbitrary equivalence point transformation for the subclasses of third-order equations (11.3.3.19) and (11.3.3.23) has the form (11.3.3.22).*

For the subclass

$$y'''_{xxx} = F(x, y, y'_x), \tag{11.3.3.24}$$

the equivalence group is much wider and has a functional arbitrariness. The last equation in system (11.3.3.20) disappears and the solution to the system becomes (11.3.3.21).

THEOREM 2. *An arbitrary equivalence point transformation for the subclasses of third-order equations (11.3.3.24) has the form (11.3.3.21).*

2°. Let us focus on the problem, stated in the previous section, for the subclass of third-order equations not involving intermediate derivatives, i.e., subclass (11.3.3.19). In view of the known equivalence group, first integrals can be sought for the simplest autonomous third-order equation

$$y'''_{xxx} = F(y). \tag{11.3.3.25}$$

1. There is an autonomous first integral linear in the derivative y''_{xx}:

$$P = R(y, y'_x) y'_x + Q(y, y'_x). \tag{11.3.3.26}$$

THEOREM 3. *There is a nontrivial equation (11.3.3.25), with $F(y) \neq 0$, having an autonomous first integral of the form (11.3.3.26).*

Remark 11.12. Theorem 3 does not prevent equation (11.3.3.25) from having linear first integrals. A counterexample is the nontrivial equation

$$y'''_{xxx} = y^{-1}.$$

It has the linear first integral

$$P = y y''_{xx} - \frac{1}{2}(y'_x)^2 - x,$$

which is however not autonomous.

2. Now let us look, for equation (11.3.3.25), at the autonomous first integral quadratic in y''_{xx}:

$$P = R(y, y'_x)(y''_{xx})^2 + Q(y, y'_x) y''_{xx} + S(y, y'_x). \tag{11.3.3.27}$$

THEOREM 4. *The equation*

$$y'''_{xxx} = (ay^2 + by + c)^{-5/4}, \tag{11.3.3.28}$$

where a, b, and c are arbitrary constants, is the only equation from class (11.3.3.25) that has an autonomous first integral quadratic in the second derivative.

3. Autonomous first integrals of equation (11.3.3.25) cubic the second derivative.

THEOREM 5. *The equation*

$$y'''_{xxx} = (ay + b)^{-5/2} \tag{11.3.3.29}$$

is the only equation from class (11.3.3.25) that has a first integral cubic in the second derivative.

3°. Let us focus on inverse problems for the subclass (11.3.3.24). In this case, there are equations that have a linear first integral. An example is equation (11.3.3.24) with

$$F = \frac{R''(y'_x)^3 - 2S'y'_x}{2R}, \tag{11.3.3.30}$$

where R and S are arbitrary functions of y. The first integral is given by

$$P = Ry''_{xx} - \frac{1}{2}R'(y'_x)^2 + S.$$

We will now look for subclasses having a quadratic first integral (11.3.3.27). Applying the direct method results in the determining system

$$\begin{aligned} &R_{y'_x} = 0,\\ &R_y y'_x + Q_{y'_x} = 0,\\ &Q_y y'_x + S_{y'_x} = -2RF,\\ &S_y y'_x = -QF. \end{aligned} \tag{11.3.3.31}$$

From the third and forth equations of system (11.3.3.31), we obtain the consistency condition

$$\left(\frac{T}{y'_x} - \frac{1}{2}R_y y'_x\right) F_{y'_x} - 2RF_y = \left(\frac{T}{(y'_x)^2} + \frac{5}{2}R_y\right) F - \frac{1}{2}R_{yyy}(y'_x)^3 + T_{yy}y'_x, \tag{11.3.3.32}$$

which is a linear nonhomogeneous partial differential equation for $F(y, y')$. Solving this equation leads to the following statement.

THEOREM 6. *The subclass of equations*

$$y'''_{xxx} = R^{-3/2}y'_x\Phi(u) + \frac{2RR'' - (R')^2}{8R^2}(y'_x)^3 - \frac{2RT' - R'T}{4R^2}y'_x, \tag{11.3.3.33}$$

where $u = R^{-1/2}(y'_x)^2 + \int TR^{-3/2}\,dy$, R and T are arbitrary functions of y, and Φ is an arbitrary function of u, is the only subclass of equations of class (11.3.3.24) having a quadratic first integral in y''_{xx}:

$$P = R(y'')^2 + \left[-\frac{1}{2}R'(y'_x)^2 + T(y)\right] y''_{xx} + \frac{1}{16}\frac{(R')^2}{R}(y'_x)^4 - \int \Phi(u)\,du - \frac{R'T}{4R}(y'_x)^2 + \frac{1}{4}\frac{T^2}{R}. \tag{11.3.3.34}$$

Formula (11.3.3.33) gives *all* right-hand sides of equation (11.3.3.24) that possess the given property; in the special case $T \equiv 0$, it becomes much simpler and

$$F = R^{-5/4}\Phi_1\left(R^{-1/4}y'_x\right) + \frac{2RR'' - (R')^2}{8R^2}(y'_x)^3. \qquad (11.3.3.35)$$

Remark 11.13. Of course, formula (11.3.3.33) contains (11.3.3.30), in which case the quadratic first integral is a quadratic form of the linear first integral.

⊙ *Literature for Section 11.3*: P. J. Olver (1986), G. W. Bluman and S. C. Anco (2002), A. D. Polyanin and V. F. Zaitsev (2003), N. H. Ibragimov (2010), V. F. Zaitsev and H. N. Huan (2013, 2014), V. F. Zaitsev and L. V. Linchuk (2014, 2015).

11.4 Underdetermined Equations

11.4.1 Preliminary Remarks

Consider the differential relation

$$y_x^{(n)} = F\big(x, y, y'_x, \ldots, w, w'_x, \ldots, w_x^{(n)}\big), \qquad (11.4.1.1)$$

where $y(x)$ and $w(x)$ are some (unknown) smooth functions of the independent variable x. Relation (11.4.1.1) can be treated as an underdetermined differential equation or as a differential constraint between y and w.

Underdetermined ordinary differential equations and systems of such equations arise when one searches for exact solutions to nonlinear partial differential equations with the methods of generalized or functional separation of variables as the original PDEs are reduced to an underdetermined system of ODEs. Monge seems to have been the first to consider such systems when he was working on his geometric theory of PDEs (this is why such equations are sometimes referred to as *Monge equations*). Below is an example that illustrates such an ODE resulting from seeking generalized separable solutions to unsteady Navier–Stokes equations.

Example 11.23. Consider the first-order equation

$$yw'_x - wy'_x + k(y^2 + w^2) = 0 \qquad (11.4.1.2)$$

which relates y and w. We change to the new variables

$$y = \rho\cos\xi, \quad w = -\rho\sin\xi,$$

where $\rho = \rho(x)$ and $\xi = \xi(x)$. As a result, we get the simple equation $\xi'_x = k$. It follows that

$$y = \rho(x)\cos(kx + C), \quad w = -\rho(x)\sin(kx + C), \qquad (11.4.1.3)$$

where $\rho = \rho(x)$ is an arbitrary function and C is an arbitrary constant.

Interestingly, if $w = w(x)$ in (11.4.1.2) was treated as a given function, the equation would be a Riccati equation for $y = y(x)$, whose general solution would be much more difficult to obtain. In this case, the solution is given either in implicit form or by two relations

$$y^2 + w^2(x) = \rho^2(x), \quad w(x)/y = -\tan(kx + C),$$

which follow from (11.4.1.3).

11.4.2 Factorization Principle

Let X be a linear operator of the form

$$\mathrm{X} = \Phi\left(x, y, w, y'_x, w'_x, \ldots\right)\frac{\partial}{\partial y} + \Psi\left(x, y, w, y'_x, w'_x, \ldots\right)\frac{\partial}{\partial w}. \qquad (11.4.2.1)$$

Operator (11.4.2.1) is a Lie–Bäcklund operator in the space of variables (x, y, w). The functions Φ and Ψ, dependent on arbitrarily high-order derivatives, are called the coordinates of this operator.

The operator

$$\underset{k}{\mathrm{X}} = \sum_{i=0}^{k}\left[D_x^i(\Phi)\frac{\partial}{\partial y_x^{(i)}} + D_x^i(\Psi)\frac{\partial}{\partial w^{(i)}}\right], \qquad (11.4.2.2)$$

is called the kth prolongation of operator (11.4.2.1). Here, D_x is the total derivative operator defined by the formal series

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} y_x^{(i+1)}\frac{\partial}{\partial y_x^{(i)}} + \sum_{i=0}^{\infty} w_x^{(i+1)}\frac{\partial}{\partial w_x^{(i)}}.$$

The Lie–Bäcklund operator (11.4.2.1) is admitted by equation (11.4.1.1) if

$$\underset{n}{\mathrm{X}}\big[y_x^{(n)} - F\big(x, y, w, y'_x, w'_x, \ldots, y_x^{(n-1)}, w_x^{(n-1)}, w_x^{(n)}\big)\big]\big|_{[y_x^{(n)}=F]} = 0. \qquad (11.4.2.3)$$

The transformation $J_k = J_k(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)})$ is called a kth-order differential invariant of operator (11.4.2.1) by virtue of equation (11.4.1.1) if

$$\underset{k}{\mathrm{X}}[J_k]\big|_{[y_x^{(n)}=F]} = 0 \qquad (11.4.2.4)$$

and $\left|\partial J_k/\partial y_x^{(k)}\right| + \left|\partial J_k/\partial w_x^{(k)}\right| \neq 0$.

Let the underdetermined differential equation (11.4.1.1) admit operator (11.4.2.1) and let $\mathbf{J}_k$ denote the set of functionally independent invariants of order not higher that k of operator (11.4.2.1) by virtue of equation (11.4.1.1). The universal invariant is $x = J_0 \in \mathbf{J}_k$ for any k.

By definition, equation (11.4.1.1) is factorized to the system

$$\begin{aligned} &G\big(x, z_1, z'_{1_x}, \ldots, z_{1_x}^{(m_1)}, z_2, z'_{2_x}, \ldots, z_{2_x}^{(m_2)}\big) = 0,\\ &z_i = H_i\big(x, y, y'_x, \ldots, y_x^{(r_1)}, w, w'_x, \ldots, w_x^{(r_2)}\big), \quad i = 1, 2, \end{aligned} \qquad (11.4.2.5)$$

with $r_1, r_2, m_1, m_2 < n$ or the system

$$\begin{aligned} &G\big(x, z, z'_x, \ldots, z_x^{(m)}\big) = 0,\\ &z = H\big(x, y, y'_x, \ldots, y_x^{(n-m)}, w, w'_x, \ldots, w_x^{(n-m)}\big), \end{aligned} \qquad (11.4.2.6)$$

with $0 < m < n$, if system (11.4.2.5) or (11.4.2.6) is a consequence of equation (11.4.1.1) (in the sense that if $y = y(x)$ and $w = w(x)$ satisfy equation (11.4.1.1), they also satisfy system (11.4.2.5) or (11.4.2.6)). These systems are called factor systems.

A factor system is a kind of Russian nesting doll in which the first equation is either an ordinary differential equation of order $< n$ (system (11.4.2.6)) or an underdetermined differential equation of a reduced order (system (11.4.2.5)). The remaining equations of system (11.4.2.5) and (11.4.2.6) also have a simpler structure than the original equation.

THEOREM 1. *Let the underdetermined differential equation (11.4.1.1) admit a Lie–Bäcklund operator (11.4.2.1) having a low invariant*

$$z = H\big(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)}\big) \in \mathbf{J}_k \quad (k \leqslant n).$$

1. *If* $D_x^{n-k}(z)\big|_{[y_x^{(n)}=F]} \in \mathbf{J}_{n-1}$, *then equation (11.4.1.1) is factorized to the system of two equations*

$$\begin{aligned} &z = H\big(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)}\big),\\ &z_x^{(n-k)} = G\big(x, z, \ldots, z_x^{(n-k-1)}\big). \end{aligned}$$

2. *If* $D_x^{n-k}(z)\big|_{[y_x^{(n)}=F]} \in \mathbf{J}_n \setminus \mathbf{J}_{n-1}$ *and* $z^* = H^*\big(x, y, w, \ldots, y_x^{(n)}, w_x^{(n)}\big)$ *is such that* $z^* \in \mathbf{J}_n \setminus \mathbf{J}_{n-1}$ *and the mappings* $x, z^{(0)}, \ldots, z_x^{(n-k)}, z^*$ *are functionally independent, then equation (11.4.1.1) reduces to the system of three equations*

$$\begin{aligned} &z = H\big(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)}\big),\\ &z^* = H^*\big(x, y, w, \ldots, y_x^{(n)}, w_x^{(n)}\big),\\ &z^* = G\big(x, z, \ldots, z_x^{(n-k)}\big). \end{aligned}$$

THEOREM 2. *Let the underdetermined differential equation (11.4.1.1) admit a formal operator (11.4.2.1) having two low invariants*

$$z_i = H_i\big(x, y, w, \ldots, y_x^{(k_i)}, w_x^{(k_i)}\big) \in \mathbf{J}_{k_i}, \quad i = 1, 2,$$

of order $k_i \leqslant n$. *Then*

1) *if* $D_x^{n-k_1}(z_1)\big|_{[y_x^{(n)}=F]} \in \mathbf{J}_{n-1}\big|_{[y_x^{(n)}=F]}$, *then equation (11.4.1.1) can be represented as the factor system*

$$\begin{aligned} &z_1 = H_1\big(x, y, w, \ldots, y_x^{(k_1)}, w_x^{(k_1)}\big),\\ &z_2 = H_2\big(x, y, w, \ldots, y_x^{(k_2)}, w_x^{(k_2)}\big),\\ &z_1^{(n-k_1)} = G\big(x, z_1, \ldots, z_{1_x}^{(n-k_1-1)}, z_2, \ldots, z_{2_x}^{(n-k_2-1)}\big), \end{aligned}$$

for $k_2 < n$,

$$\begin{aligned} &z_1 = H_1\big(x, y, w, \ldots, y_x^{(k_1)}, w_x^{(k_1)}\big),\\ &z_1^{(n-k_1)} = G\big(x, z_1, \ldots, z_{1_x}^{(n-k_1-1)}\big), \end{aligned}$$

for $k_2 = n$;

2) *if* $D_x^{n-k_1}(z_1)\big|_{[y^{(n)}=F]}$ *and* $D_x^{n-k_2}(z_2)\big|_{[y^{(n)}=F]} \in \mathbf{J}_n \setminus \mathbf{J}_{n-1}$, *then equation (11.4.1.1) is factorized to the system*

$$\begin{aligned} &z_1 = H_1\big(x, y, w, \ldots, y_x^{(k_1)}, w_x^{(k_1)}\big),\\ &z_2 = H_2\big(x, y, w, \ldots, y_x^{(k_2)}, w_x^{(k_2)}\big),\\ &z_1^{(n-k_1)} = G\big(x, z_1, \ldots, z_{1_x}^{(n-k_1-1)}, z_2, \ldots, z_{2_x}^{(n-k_2)}\big). \end{aligned}$$

THEOREM 3. *Let the underdetermined differential equation of order* n *(11.4.1.1)*

1) *reduce to the ordinary differential equation*

$$z_1^{(n-k_1)} = G\big(x, z_1, z'_{1_x}, \ldots, z_{1_x}^{(m_1)}\big)$$

with the substitution

$$z_1 = H_1\big(x, y(x), w(x), \ldots, y_x^{(k_1)}(x), w_x^{(k_1)}(x)\big), \quad k_1 < n,$$

where $0 \leqslant m_1 < n - k_1$*;*

2) *reduce to the underdetermined differential equation*

$$z_{1_x}^{(n-k_1)} = G\big(x, z_1, z'_{1_x}, \ldots, z_{1_x}^{(m_1)}, z_2, z'_{2_x}, \ldots, z_{2_x}^{(m_2)}\big)$$

with a substitution of the form

$$\begin{aligned} z_1 &= H_1\big(x, y(x), w(x), \ldots, y_x^{(k_1)}(x), w_x^{(k_1)}(x)\big), \qquad k_1 < n, \\ z_2 &= H_2\big(x, y(x), w(x), \ldots, y_x^{(k_2)}(x), w_x^{(k_2)}(x)\big), \qquad k_2 < n, \\ &\frac{\partial z_1}{\partial y_x^{(k_1)}} \neq 0, \quad -1 < m_1 < n - k_1, \quad 0 \leqslant m_2 \leqslant n - k_2. \end{aligned}$$

Then the original equation admits a formal operator (11.4.2.1) such that all z_i *(*$i = 1, 2$*) are its invariants:* $z_i \in \mathbf{J}_{k_i}$.

11.4.3 Some Technical Elements. Examples

Consider the first-order underdetermined differential equation linear in the derivatives

$$y'_x + G(x, y, w) w'_x + F(x, y, w) = 0 \tag{11.4.3.1}$$

and let us look for a point infinitesimal operator admissible by equation (11.4.3.1) and having the form

$$\mathrm{X} = \xi(x, y, w)\,\partial_x + \eta(x, y, w)\,\partial_y + \zeta(x, y, w)\,\partial_w. \tag{11.4.3.2}$$

Note that, unlike first-order ODEs, the invariance condition for underdetermined differential equations can be split into a system, because the independent variable w'_x arises.

The determining system consists of three equations, with the first one (the coefficient of $(w'_x)^2$) satisfied identically and the other two expressed as

$$\begin{aligned} &\eta_w - \eta_y G + G(\zeta_w - \zeta_y G) + F(\xi_w - \xi_y G) + \xi G_x + \eta G_y + \zeta G_w = 0, \\ &\eta_x - \eta_y F + G(\zeta_x - \zeta_y F) + F(\xi_x - \xi_y F) + \xi F_x + \eta F_y + \zeta F_w = 0. \end{aligned}$$

Let us require that the functions $I_1 = x$ and $I_2 = z = H(x, y, w)$ are invariants of operator (11.4.3.2), so that the second invariant is independent of derivatives. To this end, we rewrite operator (11.4.3.2) in the canonical form

$$\widetilde{\mathrm{X}} = (\eta - \xi y'_x)\,\partial_y + (\zeta - \xi w'_x)\,\partial_w$$

and solve the characteristic equation

$$\frac{dy}{\eta - \xi y'_x} = \frac{dw}{\zeta - \xi w'_x}$$

or, by virtue of the original equation (11.4.3.1),

$$\frac{dy}{\eta + \xi(Gw'_x + F)} = \frac{dw}{\zeta - \xi w'_x}.$$

This equation must have the integral $H(x, y, w) = C$, so that $H_y\,dy + H_w\,dw = 0$. As a result, the characteristic equation becomes

$$H_w(\xi w'_x - \zeta) + H_y[\eta + \xi(Gw'_x + F)] = 0.$$

After splitting with respect to the independent variable w'_x, this equation gives two conditions

$$H_w = GH_y, \qquad \eta = \xi F - \zeta G.$$

Substituting the second condition in both determining equations gives another condition relating F and G:

$$F_w + FG_y - F_yG - G_x = 0.$$

This allows us to uniquely determine the coefficients of equation (11.4.3.1) in terms of the invariant H:

$$F = \frac{H_x}{H_y} + \frac{1}{H_y}\Phi(x, H), \qquad G = \frac{H_w}{H_y}. \tag{11.4.3.3}$$

Thus, equation (11.4.3.1) is factorized to the system

$$\begin{aligned} &z'_x + \Phi(x, z) = 0,\\ &z = H(x, y, w), \end{aligned} \tag{11.4.3.4}$$

if the coefficients of equation (11.4.3.1) satisfy conditions (11.4.3.3); furthermore, as one can easily see, these conditions are necessary and sufficient.

If the first equation of system (11.4.3.4) has been integrated, the underdetermined differential equation (11.4.3.1) reduces to a functional (not differential) equation, the second equation of system (11.4.3.4).

It should be stressed once again that the obtained result is absolutely *independent* of whether there are additional constraints between the variables y and w. One should only bear in mind that if there is such a constraint, the quantity w' must be replaced with its value in terms of y and its derivatives.

Example 11.24. The underdetermined first-order differential equation

$$y'_x + G(x)w'_x + f(x)y + g(x)w + h(x) = 0$$

with $g(x) = G'(x) + f(x)G(x)$ admits the infinitesimal operator

$$\mathrm{X} = G(x)\partial_y - \partial_w$$

and is factorized to the system

$$\begin{aligned} &z'_x + f(x)z + h(x) = 0,\\ &z = y + G(x)w. \end{aligned}$$

11.4.4 On Second-Order Equations

This section will look thoroughly into some of the results following from the general theorems 1 to 3 as applied to second-order underdetermined differential equations.

THEOREM 4. *For the canonical infinitesimal operator*

$$\hat{\mathrm{X}} = \big[\eta_1(x,y,w) - \xi(x,y,w)y'_x\big]\partial_y + \big[\eta_2(x,y,w) - \xi(x,y,w)w'_x\big]\partial_w, \qquad (11.4.4.1)$$

admitted by the underdetermined second-order differential equation

$$y''_{xx} = F(x,y,w,y'_x,w'_x,w''_{xx}), \qquad (11.4.4.2)$$

to possess first-order differential invariants, it is necessary that

1) *for $\xi \neq 0$, the equation be linear in the highest derivative of w, so that*

$$F = f_1 w''_{xx} + f_2, \quad F_{w''_{xx}} \neq 0, \quad f_i = f_i(x,y,w,y'_x,w'_x), \quad i = 1,2, \qquad (11.4.4.3)$$

or

$$F = g_1 w'_x + g_2, \quad F_{w'_x} \neq 0, \quad g_i = g_i(x,y,w,y'_x), \quad i = 1,2, \qquad (11.4.4.4)$$

with the last condition being also sufficient for the class of equations

$$y''_{xx} = F(x,y,w,y'_x,w'_x),$$

if the relation

$$\begin{aligned} &\eta_1 g_{1x} + (2\eta_1 - \xi y'_x)y'_x g_{1y} + \eta_2 y'_x g_{1w} - \eta_2 {g_1}^2 + \\ &\quad + \Big[\eta_{1x} + (\eta_{1y} - \xi_x)y'_x - \xi_y {y'_x}^2\Big] y'_x g_{1y'_x} + \Big(\eta_{1x} + \eta_{2w} y'_x + \xi_y {y'_x}^2\Big) g_1 + \\ &\qquad + (\eta_1 - \xi y'_x)(g_{1y'_x} g_2 - g_1 g_{2y'_x} - g_{2w}) - \xi g_1 g_2 = 0 \qquad (11.4.4.5) \end{aligned}$$

holds;

2) *for $\xi = 0$, such invariants always exist.*

However, as mentioned previously, what is important is not only the existence of differential invariants but also the dimensionality of the invariant basis admitted by the operator, since it affects the structure of the system to which the original equation is reduced.

If the coordinate ξ in the operator (11.4.4.1) is zero, the invariant basis consists of two universal invariants, including $J_0 = x$, and one first-order differential invariant. If the equation has the form (11.4.4.4) and condition (11.4.4.5) holds, the dimensionality of the invariant basis admitted by operator (11.4.4.1) equals two, as the basis consists of one invariant of the zeroth order $J_0 = x$ and one first-order differential invariant. This case is remarkable because the factorization of the underdetermined differential equation (11.4.4.1) reduces it to a first-order ordinary differential equation. If the structure of the original equation satisfies condition (11.4.4.4), the invariant basis contains one universal invariant $J_0 = x$ and no more than two first-order differential invariants; in addition, the following theorem holds.

THEOREM 5. *For the canonical infinitesimal operator (11.4.4.1), with $\xi \neq 0$, admitted by equation (11.4.4.2) to have two different first-order differential invariants, it is necessary and sufficient that the right-hand side of equation (11.4.4.2) have the form (11.4.4.3) and the functions f_1 and f_2 satisfy the relations*

$$f_1 = \frac{\eta_1 - \xi y'_x}{\eta_2 - \xi w'_x},$$
$$f_1 f_{2y'_x} + f_{2w'_x} - 2D(f_1)\Big|_{y''_{xx}=f_1 w''_{xx}+f_2} = 0.$$

Significant restrictions on the structure of invariants of point operators lead one to consider the Lie–Bäcklund operator; however, the algorithm for finding an admissible operator becomes more complicated. In general, such an operator can be written as

$$\mathrm{X} = \exp\left(\int \zeta_1\, dx\right)\partial_y + \exp\left(\int \zeta_2\, dx\right)\partial_w, \qquad (11.4.4.6)$$

where ζ_1 and ζ_2 can depend on x, y, and w and their derivatives of any order. For simplicity, we will give the case $\zeta_i = \zeta_i(x, y, w, y'_x, w'_x)$, $i = 1, 2$, a detailed consideration. Let us write out the determining equation for the underdetermined second-order differential equation

$$y''_{xx} = F(x, y, w, y'_x, w'_x, w''_{xx}) \qquad (11.4.4.7)$$

and operator (11.4.4.6):

$$\begin{aligned}&\big[\zeta_{1x} + y'_x\zeta_{1y} + w'\zeta_{1w} + +F\zeta_{1y'_x} + w''_{xx}\zeta_{1w'_x} + {\zeta_1}^2 - \zeta_1 F_{y'_x} - F_y\big] -\\ &\qquad - \big[(\zeta_{2x} + y'_x\zeta_{2y} + w'_x\zeta_{2w} + F\zeta_{2y'_x} + w''_{xx}\zeta_{2w'_x} + {\zeta_2}^2)F_{w''_{xx}} +\\ &\qquad\qquad + F_{w'_x}\zeta_2 + F_w\big]\exp\left\{\int(\zeta_2 - \zeta_1)\, dx\right\} = 0. \qquad (11.4.4.8)\end{aligned}$$

In splitting equation (11.4.4.8), we must take into account the structure of the nonlocal factor $\exp\{\int(\zeta_2 - \zeta_1)\, dx\}$. If the integrand is a total derivative of some function, the further reasoning is similar to that used in constructing the determining system for a point operator. Otherwise, if the integrand is not a total derivative, equation (11.4.4.8) should first be split with respect to the nonlocal variable. The invariant basis admitted by the operators found in the latter case can be chosen so that each invariant depends on either x, y, y' or x, w, w'. The structure of the invariants of the basis affects the type of factorization of equation (11.4.4.7).

Example 11.25. The underdetermined second-order differential equation

$$\begin{aligned}y''_{xx} = C{w'_x}^2 + (\psi_1 y + \psi_2)y'_x + (\chi_1 w + \chi_2)w'_x + \tfrac{1}{2}({\psi_1}' + \psi_1\alpha - \psi_1\psi_2)y^2 +\\ + (\alpha' + \alpha^2 - \psi_2\alpha)y + h(x, w), \quad (11.4.4.9)\end{aligned}$$

where ψ_1, ψ_2, χ_1, χ_2, and α are sufficiently smooth functions of x and $C \in \mathbb{R}$, admits, under the condition that $2Cw'_x + \chi_1 w + \chi_2 \neq 0$, the Lie–Bäcklund operator

$$\mathrm{X} = \exp\left[\int(\psi_1 y + \alpha)dx\right]\partial_y + \exp\left[-\int \frac{\chi_2 w'_x + h_{2w}}{2Cw'_x + \chi_2 w + \chi_3}\, dx\right]\partial_w.$$

Apart from the universal invariant x, this operator has two low invariants (two first-order differential invariants) using which, one can factorize the original equation to the system

$$\begin{aligned} z_1 &= y'_x - \frac{1}{2}\psi_1 y^2 - \alpha y, \\ z_2 &= C{w'_x}^2 + \chi_1 w w'_x + \chi_2 w' + h, \\ {z_1}'_x &= (\psi_2 - \alpha) z_1 + z_2. \end{aligned} \tag{11.4.4.10}$$

The outer equation in this system is an underdetermined first-order differential equation.

THEOREM 6. *Equation (11.4.4.7) is factorized to the system*

$$\begin{aligned} u &= J_1^1(x, y, y'_x), \\ v &= J_1^2(x, w, w'_x), \\ G&(x, u, v, v', u') = 0, \end{aligned}$$

where

$$\frac{\partial J_1^1(x, y, y'_x)}{\partial y'_x} \neq 0, \qquad \frac{\partial J_1^2(x, w, w'_x)}{\partial w'_x} \neq 0,$$

if and only if it admits operator (11.4.4.6) whose structural components ζ_1 *and* ζ_2 *satisfy the system*

$$\begin{aligned} &\zeta_{1x} + y'\zeta_{1y} + F\zeta_{1y'_x} + (\zeta_1 - F_{y'_x})\zeta_1 - F_y = 0, \\ &F_{w''_{xx}}\zeta_{2x} + F_{w''_{xx}} w'_x \zeta_{2w} + F_{w''_{xx}} w''_{xx} \zeta_{2w'_x} + (F_{w'_x} + F_{w''_{xx}}\zeta_2)\zeta_2 + F_w = 0. \end{aligned}$$

The nature of the admitted operator itself may suggest that, on the manifold in question, the operator has only one first-order differential invariant and one universal invariant x. Then, according to Item 1 of Theorem 1, the original equation reduces to a system of two equations. It follows that the outer equation is surely a first-order ordinary differential equation; on solving this equation, we are guaranteed to reduce the order of the original equation by one.

Example 11.26. The underdetermined second-order differential equation

$$y y''_{xx} + w''_{xx} + (y'_x)^2 + (w'_x)^2 + (y y'_x + w) w'_x = 0, \tag{11.4.4.11}$$

admits the nonlocal operator

$$\mathrm{X} = \frac{\partial}{\partial w} - \left[y^{-1} \int (y y'_x + w'_x + w)\, dx \right] \frac{\partial}{\partial y}.$$

The factor system has the form

$$\begin{aligned} z &= e^w (y y'_x + w'_x + w - 1), \\ z'_x &= 0. \end{aligned}$$

Integrating the last equation yields a first integral of the original equation:

$$e^w (y y'_x + w'_x + w - 1) = C. \tag{11.4.4.12}$$

It follows that any underdetermined second-order differential equation (11.4.4.11) can be reduced to the underdetermined first-order equation (11.4.4.12).

Example 11.27. Let us look at the underdetermined differential equation

$$y''_{xx} = C(y'_x)^2 + (\psi_1 y - C\psi_2 w + \psi_3)y'_x + \psi_2 w'_x + H_1 w + H_0,$$

where $C \neq 0$, ψ_1, ψ_2, and ψ_3 are sufficiently smooth functions of x, $\psi_2 \neq 0$, and H_1 and H_0 are given by

$$\begin{aligned} H_0 &= -\frac{1}{C^2{\alpha_1}^2}\Big\{\alpha_2 \exp(Cy) + \big[C(\psi_1 y + \psi_3) + \psi_1\big](C\alpha_3 + {\alpha_1}')\alpha_1 \\ &\quad + \big[C({\psi_1}' y + {\psi_3}') + {\psi_1}'\big]{\alpha_1}^2 + C({\alpha_1}''\alpha_1 + C{\alpha_1}'\alpha_3 + C\alpha_1{\alpha_3}' + C^2{\alpha_3}^2)\Big\}, \\ H_1 &= \frac{\psi_2{\alpha_1}' + {\psi_2}'\alpha_1 + C\psi_2\alpha_3}{\alpha_1}, \end{aligned}$$

with $\alpha_i = \alpha_i(x)$, $i = 1, 2, 3$, and $\alpha_1 \neq 0$. Using the classical algorithm for solving a direct problem, we find a family of admissible point operators whose canonical form is

$$\hat{\mathrm{X}} = (\eta_1 - \xi y'_x)\partial_y + (\eta_2 - \xi w'_x)\partial_w,$$

and the coordinates are

$$\begin{aligned} \xi &= \alpha_1, \\ \eta_1 &= g\exp(Cy) + \alpha_3, \\ \eta_2 &= \frac{(g\exp(Cy) + \alpha_3)C\psi_2 w - N - (H_1 w + H_0)\alpha_1}{\psi_2}, \end{aligned}$$

where N is given by

$$\begin{aligned} N = \frac{1}{\alpha_1}\big\{\big[(\psi_1 y + \psi_3)\alpha_1 g + {\alpha_1}' g - \alpha_1 g' + Cg\alpha_3\big]\exp(Cy) \\ + (\psi_1 y + \psi_3)\alpha_1\alpha_3 + {\alpha_1}'\alpha_3 - \alpha_1{\alpha_3}' + C{\alpha_3}^2\big\}, \end{aligned}$$

and $g = g(x)$. The basis of the zeroth- and first-order invariants consists (regardless of g and, in particular, for $g \equiv 0$) of two functions: the universal invariant x and one differential invariant. Therefore, the equation is factorized to the system

$$\begin{aligned} z &= \frac{C^2\alpha_1(y'_x - \psi_2 w) + C\psi_1\alpha_1 y + (\psi_1 + C\psi_3)\alpha_1 + C(C\alpha_3 + \alpha'_1)}{C^2\alpha_1\exp(Cy)}, \\ z'_x &+ \frac{\alpha'_1 + C\alpha_3}{\alpha_1}z + \frac{\alpha_2}{C^2\alpha_1^2} = 0. \end{aligned}$$

The second equation involves only one dependent variable, z, with respect to which it is a first-order linear differential equation, which is always solvable.

The admissible Lie algebra is infinite-dimensional, consisting of a one-dimensional subalgebra L_1 and an infinite-dimensional subalgebra L_∞ defined by the operators

$$\begin{aligned} \hat{\mathrm{X}}_1 = (\alpha_3 - \alpha_1 y'_x)\partial_y + \bigg[\frac{(C\psi_2 w - \psi_1 y - \psi_3)\alpha_1\alpha_3 - \alpha'_1\alpha_3 + \alpha_1\alpha'_3 - C\alpha_3^2}{\psi_2\alpha_1} - \\ - \frac{(H_1 w + H_0)\alpha_1}{\psi_2} - \alpha_1 w'_x\bigg]\partial_w, \end{aligned}$$

and

$$\hat{\mathrm{X}}_\infty = g\exp(Cy)\partial_y + \frac{\big[(C\psi_2 w - \psi_1 y - \psi_3)\alpha_1 g - \alpha'_1 g + \alpha_1 g' - Cg\alpha_3\big]\exp(Cy)}{\psi_2\alpha_1}\partial_w.$$

The factorization obtained using the operator $\hat{\mathrm{X}}_1$ is specified above. The invariant basis of the second operator, $\hat{\mathrm{X}}_\infty$, consists, unlike $\hat{\mathrm{X}}_1$, of three invariants: two universal invariants and one differential invariant. The system to which the original equation is reduced has the form

$$
\begin{aligned}
&z_1 = \frac{Cgy'_x + g'}{Cg\exp(Cy)},\\
&z_2 = \frac{C\alpha_1 g(\psi_1 y - C\psi_2 w + \psi_3) + \psi_1\alpha_1 g + C^2\alpha_3 g - C\alpha_1 g' + C\alpha'_1 g}{C^2\psi_2\alpha_1 g\exp(Cy)},\\
&z'_1 + \psi_2 z'_2 + \frac{C\psi_2\alpha_3 + \psi_2\alpha'_1 + \psi'_2\alpha_1}{\alpha_1} z_1 + \frac{C\alpha_3 + \alpha'_1}{\alpha_1} z_2 + \frac{\alpha_2}{C^2\alpha_1^2} = 0.
\end{aligned}
$$

With the change of variable $z = z_1 + \psi_2 z_2$, we can reduce the original equation to a first-order ordinary differential equation, which is obtained by using the operator $\hat{\mathrm{X}}_1$.

⊙ *Literature for Section 11.4*: L. V. Linchuk (2001), V. I. Elkin (2009, 2010), A. D. Polyanin and V. F. Zaitsev (2012), V. F. Zaitsev and L. V. Linchuk (2014), A. D. Polyanin and A. I. Zhurov (2016c).

Chapter 12

Discrete-Group Methods

12.1 Discrete Group Method for Point Transformations

12.1.1 Classes of ODEs with Parameters. Discrete Group of Point Transformations

Consider transformations of the class of ordinary differential equations

$$y_x^{(n)} = F\bigl(x, y, \dots, y_x^{(n-1)}, \mathbf{a}\bigr), \tag{12.1.1.1}$$

whose elements are uniquely defined by a vector of essential parameters $\mathbf{a}$.

Any set of invertible transformations

$$x = f(t, u), \quad y = g(t, u) \qquad (f_t g_u - f_u g_t \neq 0), \tag{12.1.1.2}$$

mapping each equation of class (12.1.1.1) into some (other) equation of the same class

$$u_t^{(n)} = F\bigl(t, u, \dots, u_t^{(n-1)}, \mathbf{b}\bigr), \tag{12.1.1.3}$$

and containing the identical transformation is called a *discrete point group of transformations* admitted by the class (12.1.1.1). Transformation (12.1.1.2) maps any solution of equation (12.1.1.1) to a solution of equation (12.1.1.3). Therefore, knowing the discrete group of transformations for some class of equations and having a set of solvable equations of this class, one can construct new solvable cases.

Point transformations (12.1.1.2) can be found by a direct method—namely, if one substitutes an arbitrary transformation of the form (12.1.1.2) into equation (12.1.1.1) and imposes condition (12.1.1.3), one arrives at a determining equation containing partial derivatives up to order n of the unknown functions f and g and having variable coefficients depending on x, y, y'_x, $\dots$, $y_x^{(n-1)}$. Since the functions f and g do not depend on the derivatives, the determining equation can be "split" with respect to the "independent" variables y'_x, $\dots$, $y_x^{(n-1)}$, and we obtain an overdetermined system which is nonlinear, in contrast to that obtained by the Lie method (see Section 11.1.1).

12.1.2 Illustrative Examples

Example 12.1. For second-order equations

$$y''_{xx} = F(x, y, y'_x, \mathbf{a}), \tag{12.1.2.1}$$

the substitution of (12.1.1.2) into (12.1.2.1) yields

$$\begin{aligned}(f_t g_u - g_t f_u)u''_{tt} &+ (f_u g_{uu} - g_u f_{uu})(u'_t)^3 \\ &+ (f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut})(u'_t)^2 \\ &+ (f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut})u'_t + f_t g_{tt} - g_t f_{tt} \\ &= (f_t + f_u u'_t)^3 F\Big(f, g, \frac{g_t + g_u u'_t}{f_t + f_u u'_t}, \mathbf{a}\Big). \end{aligned} \tag{12.1.2.2}$$

Let us require that the transformed equation (12.1.2.2) belong to the class (12.1.2.1), i.e.,

$$u''_{tt} = F(t, u, u'_t, \mathbf{b}). \tag{12.1.2.3}$$

Condition (12.1.2.3) imposed on the determining equation (12.1.2.2), i.e., the replacement of u''_{tt} by the right-hand side of equation (12.1.2.3), leads us to the relation

$$\begin{aligned}(f_t g_u - g_t f_u)F(t, u, u'_t, \mathbf{b}) &+ (f_u g_{uu} - g_u f_{uu})(u'_t)^3 \\ &+ (f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut})(u'_t)^2 \\ &+ (f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut})u'_t + f_t g_{tt} - g_t f_{tt} \\ &= (f_t + f_u u'_t)^3 F\Big(f, g, \frac{g_t + g_u u'_t}{f_t + f_u u'_t}, \mathbf{a}\Big), \end{aligned} \tag{12.1.2.4}$$

which contains the "independent" variable u'_t. Expanding the function F into a series in powers of u'_t, we can represent (12.1.2.4) in the form

$$\sum_{k=0}^{\infty} P_k\big(x, y, [f], [g]\big)(u'_t)^k = 0, \tag{12.1.2.5}$$

where the symbols $[f]$ and $[g]$ indicate dependence on the functions f, g and their partial derivatives involved in (12.1.2.4). The sum in (12.1.2.5) is finite if F is a polynomial with respect to the third variable [for a polynomial of degree $n \geq 4$, both sides of the equation must be first multiplied by $(f_t + f_u u'_t)^{n-3}$]. Condition (12.1.2.5) is satisfied if the following equations hold:

$$P_k = 0, \quad k = 0,\ 1,\ 2,\ \ldots$$

Example 12.2. Consider a special case of equation (12.1.2.1) with the right-hand side independent of the derivative y'_x:

$$y''_{xx} = F(x, y, \mathbf{a}). \tag{12.1.2.6}$$

Relation (12.1.2.4) has the form:

$$\begin{aligned}(f_t g_u - g_t f_u)F(t, u, \mathbf{b}) + (f_u g_{uu} - g_u f_{uu})(u'_t)^3 + (f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut})(u'_t)^2 \\ + (f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut})u'_t + f_t g_{tt} - g_t f_{tt} = (f_t + f_u u'_t)^3 F(f, g, \mathbf{a}).\end{aligned}$$

In this case, the sum (12.1.2.5) is finite and the determining system has the form:

$$\begin{aligned}&f_u g_{uu} - g_u f_{uu} = f_u^3 F(f, g, \mathbf{a}),\\ &f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut} = 3f_t f_u^2 F(f, g, \mathbf{a}),\\ &f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut} = 3f_t^2 f_u F(f, g, \mathbf{a}),\\ &f_t g_{tt} - g_t f_{tt} + (f_t g_u - g_t f_u)F(t, u, \mathbf{b}) = f_t^3 F(f, g, \mathbf{a}).\end{aligned} \tag{12.1.2.7}$$

It can be shown that for $f_t f_u g_t g_u \neq 0$, solving system (12.1.2.7) is equivalent to solving the original equation (12.1.2.6).

Consider the case $f_u = 0$. In this case, the first equation of the system holds identically and the system becomes

$$\begin{aligned}&f'_t g_{uu} = 0,\\ &g_u f_{tt} - 2f_t g_{ut} = 0,\\ &f_t g_{tt} - g_t f_{tt} + f_t g_u F(t, u, \mathbf{b}) = f_t^3 F(f, g, \mathbf{a}).\end{aligned} \tag{12.1.2.8}$$

Since $f'_t \neq 0$, the first two equations yield

$$g(t,u) = T(t)u + \Theta(t), \quad f'_t = C[T(t)]^2. \tag{12.1.2.9}$$

Substituting (12.1.2.9) into the last equation of system (12.1.2.8) and "splitting" the resulting relation with respect to powers of the "independent" variable u, we obtain a new system of (ordinary) differential equations. Solving this system, we find the unknown functions T and Θ, and finally, the desired discrete group of transformations. In order to give calculation details, one has to know the specific structure of the function $F(x,y)$, for in the general case it was only shown that any discrete point group of transformations of equation (12.1.2.6) for $f_u = 0$ consists of Kummer–Liouville transformations (12.1.2.9).

Example 12.3. Consider the generalized Emden–Fowler equation:

$$y''_{xx} = Ax^n y^m (y'_x)^l. \tag{12.1.2.10}$$

Here, $\mathbf{a} = \{n,\, m,\, l\}$ is the vector of essential parameters, and A is an unessential parameter (it can be made equal to unity by scaling the independent variable and the unknown function).

1°. First, we note that equation (12.1.2.10) admits a discrete group of transformations determined by the hodograph transformation, i.e., by passing to the inverse function:

$$x = u, \quad y = t, \qquad \text{where} \quad u = u(t). \tag{12.1.2.11}$$

This transformation is a consequence of the invariance of equation (12.1.2.10) with respect to the transformation $x \longleftrightarrow y$, $n \longleftrightarrow m$, $l \longleftrightarrow 3-l$, $A \longleftrightarrow -A$ (note that the hodograph transformation changes the sign of the unessential parameter A). Denoting the transformation (12.1.2.11) by $\mathcal{F}$, let us schematically represent its action on the parameters of the equation as follows:

$$\{n,\, m,\, l\} \;\leftarrow - - \rightarrow\; \{m,\, n,\, 3-l\} \qquad \textit{transformation } \mathcal{F}. \tag{12.1.2.12}$$

Double application of the transformation $\mathcal{F}$ yields the original equation.

2°. For $l = 0$, equation (12.1.2.10) is of the class (12.1.2.5), and the last equation of system (11) becomes

$$[TT''_{tt} - 2(T'_t)^2]u + T\Theta''_{tt} - 2T'_t\Theta'_t + BT^2 t^\nu u^\mu = AC^2(Tu+\Theta)^m f^n, \tag{12.1.2.13}$$

where ν, μ, and B are the parameters of the transformed equation $u''_{tt} = Bt^\nu u^\mu$, and

$$f(t) = C\int [T(t)]^2 dt.$$

Let $m,\ \mu \neq 0,\ 1,\ 2$. Then relation (12.1.2.13) is possible only if $\Theta(t) \equiv 0$. Splitting with respect to powers of u leads us to the system:

$$\begin{aligned} &TT''_{tt} - 2(T'_t)^2 = 0,\\ &Bt^\nu = AC^2 T^{m+3} f^n. \end{aligned} \tag{12.1.2.14}$$

By integration we find that $T = t^{-1}$, $f = t^{-1}$ (to within unessential coefficients). Thus, we arrive at the transformation

$$x = t^{-1}, \quad y = t^{-1}u, \qquad \text{where} \quad u = u(t). \tag{12.1.2.15}$$

Denoting the transformation (12.1.2.15) by $\mathcal{H}$, let us schematically represent its action on the parameters of the equation:

$$\{n,\, m,\, 0\} \;\longleftrightarrow\; \{-n-m-3,\, m,\, 0\} \qquad \textit{transformation } \mathcal{H}. \tag{12.1.2.16}$$

Double application of the transformation $\mathcal{H}$ yields the original equation.

3°. Let $l=0$ and $m=2$. Then, $\mu=2$ and the splitting procedure for equation (12.1.2.13) yields the system of three equations:

$$TT''_{tt}-2(T'_t)^2=2AC^2T^6\Theta f^n,$$
$$T\Theta''_{tt}-2T'_t\Theta'_t=AC^2T^5\Theta^2 f^n,$$
$$Bt^\nu=AC^2T^5f^n.$$

Its solution gives us the transformation

$x=t^r,\quad y=t^ku+\alpha t^s$ transformation of the variables, $u=u(t)$;

$\{n,\,2,\,0\}\longleftrightarrow\{\nu,\,2,\,0\}$ transformation of the vector of essential parameters;

where we use the notation:

$$r=(8n^2+40n+49)^{-1/2},\quad k=\frac{r-1}{2},\quad \nu=\frac{1}{2}[r(2n+5)-5],$$
$$s=-r(n+2),\quad \alpha=\frac{(n+2)(n+3)}{A}. \tag{12.1.2.17}$$

Example 12.4. Likewise, for the more general class of equations

$$y''_{xx}=f(x)g(y)h(y'_x)$$

we find two transformations of the variables:

$\mathcal{F}:\{f,\,g,\,h\}\quad\leftarrow - - -\rightarrow\quad\{g,\,f,\,-(y'_x)^3h(1/y'_x)\}$ transformation (12.1.2.11);

$\mathcal{H}:\{f,\,y^m,\,1\}\quad\longleftrightarrow\quad\{t^{-m-3}f(t^{-1}),\,y^m,\,1\}$ transformation (12.1.2.15).

⊙ *Literature for Section 12.1*: V. F. Zaitsev and A. D. Polyanin (1993, 1994), A. D. Polyanin and V. F. Zaitsev (2003).

12.2 Discrete Group Method Based on RF-Pairs

12.2.1 General Description of the Method. First and Second RF-Pairs

The direct method (see Section 12.1) is unsuitable for finding nonpoint transformations of second-order equations (i.e., transformations containing derivatives), since the determining equation cannot be split into equations forming an overdetermined system. Therefore, instead of searching for Bäcklund transformations in the form of arbitrary functions $x=f(t,u,u'_t)$, $y=g(t,u,u'_t)$, one uses the superposition of some "standard" transformation containing the derivative and a point transformation which can be found by the direct method. The "standard" dependence on the derivative can be introduced by means of an RF-pair, which amounts to a transformation of successively increasing and decreasing the order of the equation (this transformation is not equivalent to the identity transformation). An additional point-transformation is necessary, since the equation obtained by an RF-pair is usually outside the original class.

1°. Suppose that any equation of the original class can be solved for the independent variable x:

$$F(y,y'_x,y''_{xx})=x.$$

Termwise differentiation of this equation with respect to x yields the following autonomous equation:

$$\frac{\partial F}{\partial y}y'_x+\frac{\partial F}{\partial y'_x}y''_{xx}+\frac{\partial F}{\partial y''_{xx}}y'''_{xxx}=1,$$

whose order can be reduced with the substitution $y'_x=z(y)$. This pair of transformations is called a *first RF-pair*.

2°. Suppose that any equation of the original class can be solved for the dependent variable y:

$$F(x, y'_x, y''_{xx}) = y.$$

Then, termwise differentiation of this equation with respect to x brings us to the following equation which does not explicitly contain y:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y'_x} y''_{xx} + \frac{\partial F}{\partial y''_{xx}} y'''_{xxx} = y'_x.$$

The order of this equation can be reduced by means of the substitution $y'_x = z(x)$. This pair of transformations is called a *second RF-pair*.

Table 12.1 lists the main Bäcklund transformations for second-order differential equations, which are useful in conjunction with point transformations in searching for an equation of a given class.

TABLE 12.1
Main Bäcklund transformations for second-order differential equations

No.	Original equation	Algebraically transformed (equivalent) equation, differentiated w.r.t. x	New variables	Resulting equation
1	$F(x, y, y'_x, y''_{xx}) = 0$	$\Phi(y, y'_x, y''_{xx}) = x$	$w(y) = y'_x$	$w \frac{d}{dy} \Phi(y, w, w w'_y) = 1$
2	$F(x, y, y'_x, y''_{xx}) = 0$	$\Phi(x, y'_x, y''_{xx}) = y$	$w(x) = y'_x$	$\frac{d}{dx} \Phi(x, w, w'_x) = w$
3	$F\Bigl(x^n y^m, x^k y^s, \frac{x y'_x}{y}, \frac{x^2 y''_{xx}}{y}\Bigr) = 0$	$\frac{1}{x^k y^s} \Phi\Bigl(x^n y^m, \frac{x y'_x}{y}, \frac{x^2 y''_{xx}}{y}\Bigr) = 1$	$z = x^n y^m$, $w = \frac{x y'_x}{y}$	$z(mw + n) \frac{d\Phi}{dz} = (sw + k)\Phi$, where $\Phi = \Phi(z, w, v)$, $v = z(mw + n) w'_z + w^2 - w$
4	$F\bigl(x^n e^{\alpha y}, x^m e^{\beta y}, x y'_x, x^2 y''_{xx}\bigr) = 0$	$\frac{1}{x^m e^{\beta y}} \Phi\bigl(x^n e^{\alpha y}, x y'_x, x^2 y''_{xx}\bigr) = 1$	$z = x^n e^{\alpha y}$, $w = x y'_x$	$z(\alpha w + n) \frac{d\Phi}{dz} = (\beta w + m)\Phi$, where $\Phi = \Phi(z, w, v)$, $v = z(\alpha w + n) w'_z - w$
5	$F\bigl(e^{\alpha x} y^n, e^{\beta x} y^m, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\bigr) = 0$	$\frac{1}{e^{\beta x} y^m} \Phi\Bigl(e^{\alpha x} y^n, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\Bigr) = 1$	$z = e^{\alpha x} y^n$, $w = \frac{y'_x}{y}$	$z(nw + \alpha) \frac{d\Phi}{dz} = (mw + \beta)\Phi$, where $\Phi = \Phi(z, w, v)$, $v = z(nw + \alpha) w'_z + w^2$

Remark 12.1. To look for equations of a given class listed in Table 12.1, one can use the Bäcklund transformations in conjunction with point transformations and contact transformations described in Section 1.9; see Example 12.8 for a similar combination of transformations as well as the Legendre transformation.

12.2.2 Illustrative Examples

Example 12.5. Consider transformations of the class of generalized Emden–Fowler equations:

$$y''_{xx} = A x^n y^m (y'_x)^l. \tag{12.2.2.1}$$

This class will be briefly denoted by the vector of essential parameters $\{n,\, m,\, l\}$. Application of the first RF-pair transforms this equation to

$$z''_{yy} = (l-1) z^{-1} (z'_y)^2 + m y^{-1} z'_y + n A^{\frac{1}{n}} y^{\frac{m}{n}} z^{\frac{l-n-1}{n}} (z'_y)^{\frac{n-1}{n}}. \tag{12.2.2.2}$$

Now we have to find a point transformation that maps class (12.2.2.2) into class (12.2.2.1) (with another vector of parameters):

$$u''_{tt} = Bt^{\nu} u^{\mu} (u'_y)^{\lambda}. \tag{12.2.2.3}$$

Note that in this case, the desired transformation does not map the given class into itself as in Section 12.1, but is a mapping of the equation classes (12.2.2.2) $\longrightarrow$ (12.2.2.1). Nevertheless, the method for finding transformations

$$y = f(t,u), \quad z = g(t,u) \qquad (f_t g_u - f_u g_t \neq 0)$$

is completely the same and involves solving the determining equation:

$$\begin{aligned}(f_t g_u - g_t f_u) B t^{\nu} u^{\mu} (u'_t)^{\lambda} &+ (f_u g_{uu} - g_u f_{uu})(u'_t)^3 + (f_t g_{uu} - g_t f_{uu} + 2 f_u g_{ut} - 2 g_u f_{ut})(u'_t)^2 \\ &+ (f_u g_{tt} - g_u f_{tt} + 2 f_t g_{ut} - 2 g_t f_{ut}) u'_t + f_t g_{tt} - g_t f_{tt} = \frac{l-1}{g} (f_t + f_u u'_t)(g_t + g_u u'_t)^2 \\ &+ \frac{m}{f} (f_t + f_u u'_t)^2 (g_t + g_u u'_t) + n A^{\frac{1}{n}} f^{\frac{m}{n}} g^{\frac{l-n-1}{n}} (f_t + f_u u'_t)^{\frac{2n+1}{n}} (g_t + g_u u'_t)^{\frac{n-1}{n}}.\end{aligned} \tag{12.2.2.4}$$

Following the procedure set out in Section 12.1, we omit the general case $f_t f_u g_t g_u \neq 0$ and consider transformations for which at least one of the above partial derivatives is zero.

1°. *Case* $f_u = 0$, $g_t = 0$. Equation (12.2.2.4) has the form

$$\begin{aligned}B f_t g_u t^{\nu} u^{\mu} (u'_t)^{\lambda} + f_t g_{uu} (u'_t)^2 - g_u f_{tt} u'_t &= \frac{l-1}{g} f_t (g_u)^2 (u_t)^2 \\ &+ \frac{m}{f} (f_t)^2 g_u u'_t + n A^{\frac{1}{n}} f^{\frac{m}{n}} g^{\frac{l-n-1}{n}} (f_t)^{\frac{2n+1}{n}} (g_u)^{\frac{n-1}{n}} (u_t)^{\frac{n-1}{n}}.\end{aligned}$$

and for $n \neq 0, -1, \lambda \neq 1, 2$ can easily be solved by splitting,

$$f = t^{\frac{1}{m+1}}, \qquad g = u^{\frac{1}{l-2}}.$$

As a result, using an RF-pair, we obtain:

$$\begin{aligned}&x = (u'_t)^{\frac{1}{n}}, \quad y = t^{\frac{1}{m+1}}, \quad y'_x = u^{\frac{1}{2-l}} && \text{transformation of variables;} \\ &\{n, m, l\} \longmapsto \left\{-\frac{m}{m+1}, \frac{1}{l-2}, \frac{n-1}{n}\right\} && \text{transformation of parameters,}\end{aligned} \tag{12.2.2.5}$$

where $u = u(t)$.

2°. *Case* $f_t = 0$, $g_u = 0$. Similar calculations bring us to the formulas:

$$\begin{aligned}&x = (u'_t)^{-\frac{1}{n}}, \quad y = u^{\frac{1}{m+1}}, \quad y'_x = t^{\frac{1}{2-l}} && \text{transformation of variables;} \\ &\{n, m, l\} \longmapsto \left\{\frac{1}{1-l}, -\frac{n}{n+1}, \frac{2m+1}{m}\right\} && \text{transformation of parameters.}\end{aligned} \tag{12.2.2.6}$$

Transformation (12.2.2.6) can be obtained by successive application of transformation (12.2.2.5) and the hodograph transformation $\mathcal{F}$ (see Example 12.3, Item 1°).

The inverse transformations have a similar structure. For instance, the inverse of transformation (12.2.2.5) can be written (after changing notation) as follows:

$$x = u^{\frac{1}{n+1}}, \quad y = (u'_t)^{-\frac{1}{m}}, \quad y'_x = t^{\frac{1}{1-l}}, \qquad \text{where} \quad u = u(t). \tag{12.2.2.7}$$

Denoting the transformation (12.2.2.7) by $\mathcal{G}$, let us schematically represent its action on the parameters of the equation:

$$\{n, m, l\} \longmapsto \left\{\frac{1}{1-l}, -\frac{n}{n+1}, \frac{2m+1}{m}\right\} \qquad \textit{transformation } \mathcal{G}. \tag{12.2.2.8}$$

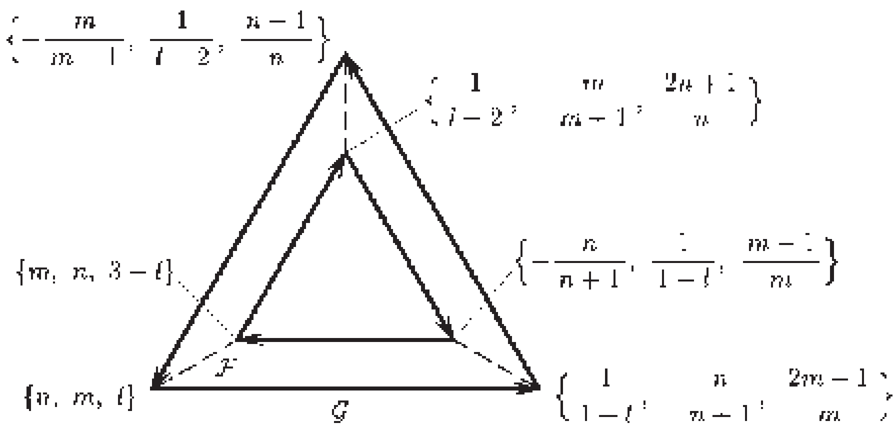

Figure 12.1: Parameters of the original and the transformed equations of the form (12.2.2.1) are obtained by superposition of the transformations $\mathcal{G}$ and $\mathcal{F}$.

Applying the transformation $\mathcal{G}$ three times, we obtain the original equation.

It can be shown that all transformations which can be found from equation (12.2.2.4), without additional restrictions on the parameters of the original and the transformed equations, are obtained by superposition of the transformations $\mathcal{G}$ and $\mathcal{F}$ (see Example 12.3, Item 1°), which form a group of order 6. The parameters of these equations are given in Figure 12.1.

Example 12.6. Suppose that $l = 0$ in equation (12.2.2.1). Then, on the class of Emden–Fowler equations

$$y''_{xx} = Ax^n y^m \qquad \text{(briefly denoted by } \{n, m, 0\}\text{)}, \tag{12.2.2.9}$$

one can define the transformation $\mathcal{H}$ (see Example 12.3, Item 2°). Therefore, in this case, the group considered in the previous example is prolonged to a group of order 12 (see Figure 12.2).

This prolongation takes place each time the third component of the parameter vector becomes equal to zero. This happens, for instance, if $n = 1$ in equation (12.2.2.9). In this case, the order of the group is equal to 24 (see Figure 12.3).

Example 12.7. The class of second-order equations

$$y''_{xx} = f(x)g(y)h(y'_x) \tag{12.2.2.10}$$

admits a discrete group of transformations similar to that for the generalized Emden–Fowler equation. Most simply, this group can be obtained by inverting the transformation (12.2.2.6). Thus, we seek the parameters of the transformation as functions of a single variable,

$$x = \varphi(u'_t), \quad y = \psi(u), \quad y'_x = \chi(t).$$

Introducing a point generator $\mathcal{F}$ (see Example 12.3, Item 1°), we find a discrete group of transformations relating the equations shown in Figure 12.4. The functions $f_1(x_1)$, $g_1(y_1)$, $h_1(y'_{x_1})$ determine the original equation, while the corresponding functions for the transformed equations, $f_k(x_k)$, $g_k(y_k)$, $h_k(y'_{x_k})$ with $k = 2, 3$, are determined by the parametric formulas:

$$\begin{aligned} f_2(x_2) &= w_1, & x_2 &= \int \frac{dw_1}{h_1(w_1)},\\ g_2(y_2) &= \frac{1}{f_1(x_1)}, & y_2 &= \int f_1(x_1)\,dx,\\ h_2(w_2) &= -\frac{1}{[g_1(y_1)]^3}\frac{dg_1}{dy_1}, & w_2 &= \frac{1}{g_1(y_1)} \end{aligned} \tag{12.2.2.11}$$

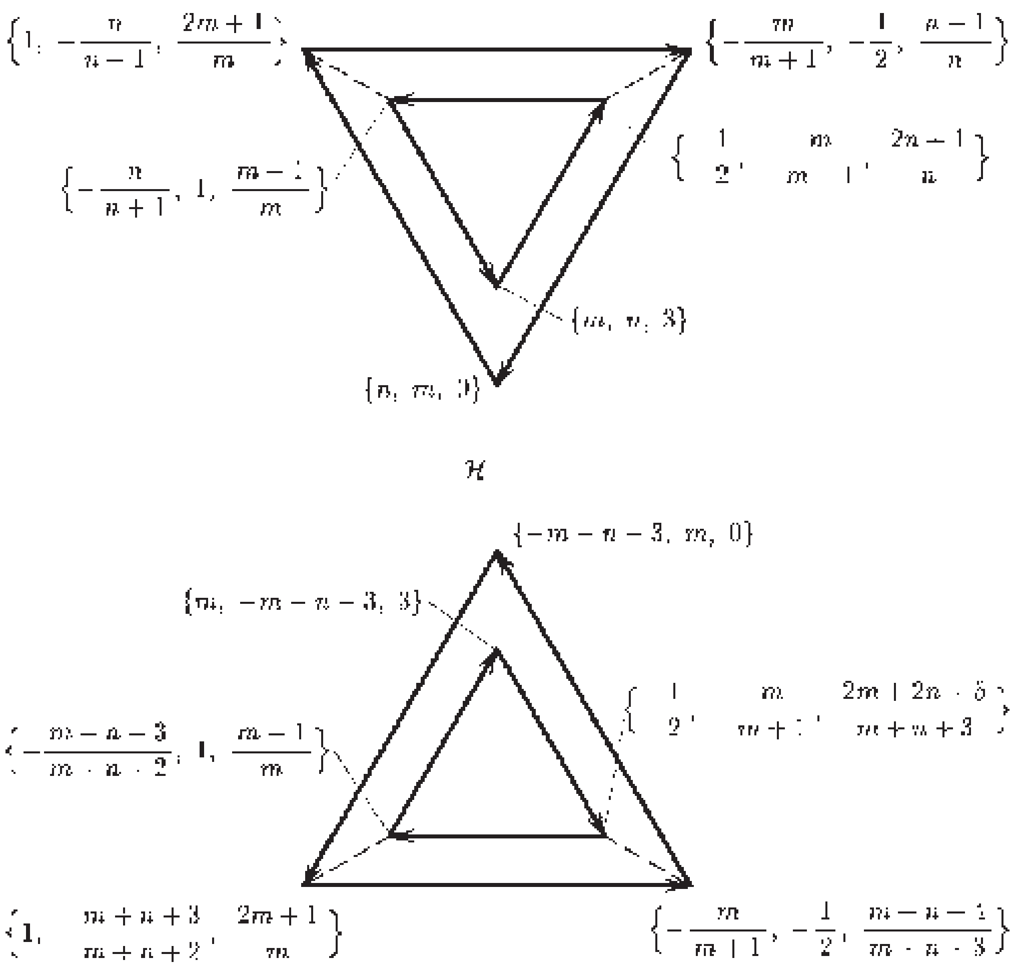

Figure 12.2: Parameters of the original equation (12.2.2.9) and the transformed generalized Emden–Fowler equations of the form (12.2.2.1) are obtained by superposition of the transformations $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{H}$.

and

$$
\begin{aligned}
f_3(x_3) &= \frac{1}{g_1(y_1)}, & x_3 &= \int g_1(y_1)\,dy_1,\\
g_3(y_3) &= \frac{1}{w_1}, & y_3 &= \int \frac{w_1\,dw_1}{h_1(w_1)},\\
h_3(w_3) &= \frac{df_1}{dx_1}, & w_3 &= f_1(x_1),
\end{aligned}
\qquad (12.2.2.12)
$$

where $w_k = y'_{x_k}$, $k = 1, 2, 3$.

The above example enables us to eliminate “singular points” of the group of transformations defined by (12.2.2.7) for $n = -1$, $m = -1$, $l = 1, 2$. For these values of the parameters, the form (12.2.2.10) and the transformations (12.2.2.11), (12.2.2.12) should be used.

⊙ *Literature for Section 12.2*: V. F. Zaitsev and A. D. Polyanin (1993, 1994), A. D. Polyanin and V. F. Zaitsev (2003), V. F. Zaitsev, L. V. Linchuk, and A. V. Flegontov (2014).

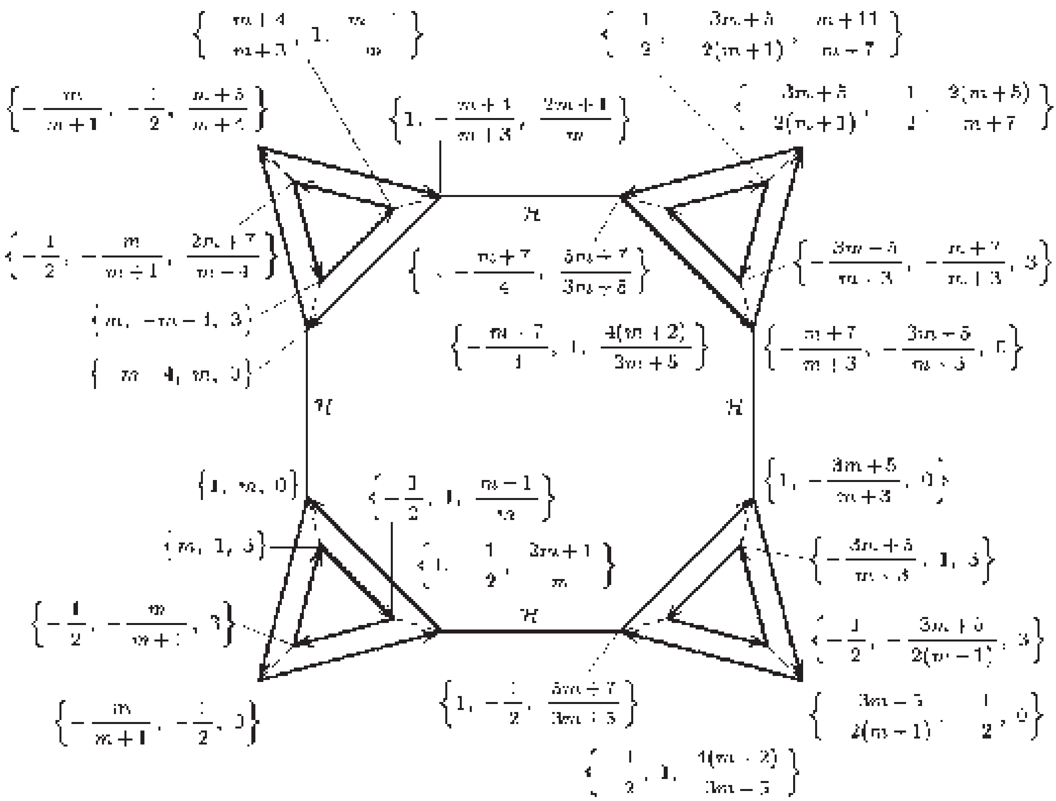

Figure 12.3: Parameters of the original equation (12.2.2.9) with $n = 1$ and the transformed Emden–Fowler equations of the form (12.2.2.1) are obtained by superposition of the transformations $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{H}$.

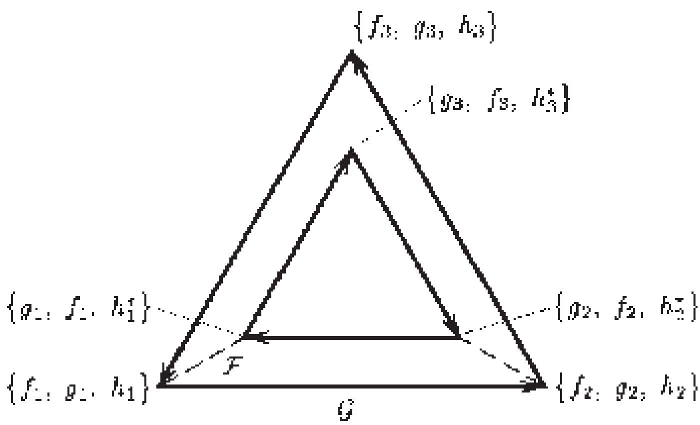

Figure 12.4: Parameters of the original and the transformed equations of the form (12.2.2.10) are obtained by superposition of the transformations $\mathcal{G}$ and $\mathcal{F}$.

12.3 Discrete Group Method Based on the Inclusion Method

To construct generators admitted by a certain class of ODEs, one can take advantage of the inclusion method. It was used above (Example 12.7) to extend the group admitted by the class of generalized Emden–Fowler equations to a wider class (12.2.2.10). Let us look at the reverse situation: suppose we study a class of equations DE_1, admitting a discrete group G_1, which is enclosed in another class, DE_2, admitting a known group G_2 such that part of its generators are not contained in G_1. Then, there is a possibility (not guaranteed) that some combination of the generators of G_2 will be closed on class DE_1.

Example 12.8. It is clear that the class of generalized Emden–Fowler equations is enclosed in the four-parameter class

$$y''_{xx} = Ax^n y^m (y'_x)^l \bigl(xy'_x - y\bigr)^k,$$

whose element will be denoted $[n, m, l, k]$. The generators $\mathcal{F}$ and $\mathcal{H}$ are closed on this class, but the generator $\mathcal{G}$ is not. However, an additional generator $\mathcal{L}$ can be introduced using the tangential Legendre transformation:

$$\begin{aligned} &\mathcal{F}\colon \quad x = u, \quad y = t, && [n, m, l, k] \longrightarrow [m, n, 3-l-k, k],\\ &\mathcal{H}\colon \quad x = 1/t, \quad y = u/t, && [n, m, l, k] \longrightarrow [-n-m-3, m, k, l],\\ &\mathcal{L}\colon \quad x = u'_t, \quad y = tu'_t - u, && [n, m, l, k] \longrightarrow [-l, -k, -n, -m]. \end{aligned}$$

The structure of the group becomes obvious if we use the minimal group code (minimal basis of the group) and introduce the new generator $\mathcal{P} = \mathcal{HL}$:

$$\mathcal{P}\colon \quad x = \frac{1}{u'_t}, \quad y = \frac{tu'_t - u}{u'_t}, \qquad [n, m, l, k] \longrightarrow [-k, -l, n+m+3, -m].$$

Then $\mathcal{P}^6 = E$ and we obtain a group of order 12.

Obviously, for $k=0$ and $m+n+3=0$, the graph of the group will have two new vertices, which correspond to generalized Emden–Fowler equations, with the transformation $\mathcal{P}^3 \equiv \mathcal{Q}$ defining a new partial generator on the class concerned:

$$\mathcal{Q}\colon \quad x = -\frac{u'_t}{tu'_t - u}, \quad y = \frac{1}{tu'_t - u}, \qquad \{-n-m-3, m, l\} \longrightarrow \{-l, l-3, m+3\}.$$

⊙ *Literature for Section 12.3*: V. F. Zaitsev and A. D. Polyanin (1993, 1994), A. D. Polyanin and V. F. Zaitsev (2003), O. V. Zaitsev and Z. N. Khakimova (2014), V. F. Zaitsev, L. V. Linchuk, and A. V. Flegontov (2014).

Part II

Exact Solutions of Ordinary Differential Equations

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Chapter 13

First-Order Ordinary Differential Equations

13.1 Simplest Equations with Arbitrary Functions Integrable in Closed Form

◆ *No special cases of equations 13.1.1–13.1.5 for specific functions f, f_0, f_1, f_n, and g are discussed in this book; such cases can readily be recognized by the appearance of equations investigated, and the solution can be obtained using the general formulas given in Section 13.1.*

13.1.1 Equations of the Form $y'_x = f(x)$

Solution:[†] $y = \int f(x)\,dx + C.$

13.1.2 Equations of the Form $y'_x = f(y)$

Solution: $x = \int \dfrac{dy}{f(y)} + C.$

Particular solutions: $y = A_k$, where A_k are roots of the algebraic (transcendental) equation $f(A_k) = 0$.

13.1.3 Separable Equations $y'_x = f(x)g(y)$

Solution: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$

Particular solutions: $y = A_k$, where A_k are roots of the algebraic (transcendental) equation $g(A_k) = 0$.

Remark 13.1. The equation of the form $f_1(x)g_1(y)y'_x = f_2(x)g_2(y)$ is reduced to the form 13.1.3 by dividing both sides by $f_1 g_1$.

[†]Hereinafter we shall often use the term "solution" to mean "general solution."

13.1.4 Linear Equation $g(x)y'_x = f_1(x)y + f_0(x)$

Solution:

$$y = Ce^F + e^F \int e^{-F} \frac{f_0(x)}{g(x)}\,dx, \quad \text{where} \quad F(x) = \int \frac{f_1(x)}{g(x)}\,dx.$$

13.1.5 Bernoulli Equation $g(x)y'_x = f_1(x)y + f_n(x)y^n$

Here, n is an arbitrary number. The substitution $w(x) = y^{1-n}$ leads to a linear equation: $g(x)w'_x = (1-n)f_1(x)w + (1-n)f_n(x)$.
Solution:

$$y^{1-n} = Ce^F + (1-n)e^F \int e^{-F} \frac{f_n(x)}{g(x)}\,dx, \quad \text{where} \quad F(x) = (1-n)\int \frac{f_1(x)}{g(x)}\,dx.$$

13.1.6 Homogeneous Equation $y'_x = f(y/x)$

The substitution $u(x) = y/x$ leads to a separable equation: $xu'_x = f(u) - u$.
Solution: $\displaystyle\int \frac{du}{f(u)-u} = \ln|x| + C$.
Particular solutions: $y = A_k x$, where A_k are roots of the algebraic (transcendental) equation $A_k - f(A_k) = 0$.

13.2 Riccati Equation $g(x)y'_x = f_2(x)y^2 + f_1(x)y + f_0(x)$

13.2.1 Preliminary Remarks

For $f_2 \equiv 0$, we obtain a linear equation (see Section 13.1.4); and for $f_0 \equiv 0$, we have a Bernoulli equation (see Section 13.1.5 with $n = 2$), whose solutions were given previously. Below we discuss equations with $f_0 f_2 \not\equiv 0$.

1°. Given a particular solution $y_0 = y_0(x)$ of the Riccati equation, the general solution can be written as:

$$y = y_0(x) + \Phi(x)\left[C - \int \Phi(x)\frac{f_2(x)}{g(x)}\,dx\right]^{-1}, \quad \Phi(x) = \exp\left\{\int \left[2f_2(x)y_0(x) + f_1(x)\right]\frac{dx}{g(x)}\right\}.$$

To the particular solution $y_0(x)$ there corresponds $C = \infty$.

Often only particular solutions will be given for the specific equations presented below in Sections 13.2.2–13.2.8. The general solutions of these equations can be obtained by the above formulas.

2°. The substitution

$$u(x) = \exp\left(-\int \frac{f_2}{g} y\,dx\right)$$

reduces the Riccati equation to a second-order linear equation:

$$f_2 g^2 u''_{xx} + g\left[f_2 g'_x - g(f_2)'_x - f_1 f_2\right]u'_x + f_0 f_2^2 u = 0.$$

The latter often may be easier to solve than the original Riccati equation. Specific second-order linear equations are outlined in Section 13.2.

13.2.2 Equations Containing Power Functions

► **Equations of the form** $g(x)y'_x = f_2(x)y^2 + f_0(x)$.

1. $y'_x = ay^2 + bx + c.$

For $b = 0$, we have a separable equation of the form 13.1.2. For $b \neq 0$, the substitution $bt = bx + c$ leads to an equation of the form 13.2.2.4: $y'_t = ay^2 + bt$.

2. $y'_x = y^2 - a^2x^2 + 3a.$

Particular solution: $y_0 = ax - x^{-1}$.

3. $y'_x = y^2 + ax^2 + bx + c.$

This is a special case of equation 13.2.2.27 with $\alpha = 0$ and $\beta = 0$.

4. $y'_x = ay^2 + bx^n.$

Special Riccati equation, n is an arbitrary number.

Solution: $y = -\dfrac{1}{a}\dfrac{w'_x}{w}$, where $w(x) = \sqrt{x}\left[C_1 J_{\frac{1}{2k}}\left(\frac{1}{k}\sqrt{ab}\,x^k\right) + C_2 Y_{\frac{1}{2k}}\left(\frac{1}{k}\sqrt{ab}\,x^k\right)\right]$, $k = \frac{1}{2}(n+2)$; $J_m(z)$ and $Y_m(z)$ are Bessel functions, $n \neq -2$. For the case $n = -2$, see equation 13.2.2.13.

5. $y'_x = y^2 + anx^{n-1} - a^2x^{2n}.$

Particular solution: $y_0 = ax^n$.

6. $y'_x = ay^2 + bx^{2n} + cx^{n-1}.$

For the case $n = -1$, see equation 13.2.2.13. For $n \neq -1$, the transformation $\xi = \dfrac{1}{n+1}x^{n+1}$, $\eta = yx^{-n}$ leads to an equation of the form 13.2.2.38: $\xi\eta'_\xi + a\xi\eta^2 + \dfrac{n}{n+1}\eta = b\xi + \dfrac{c}{n+1}$.

7. $y'_x = ax^ny^2 + bx^{-n-2}.$

Solution: $\sqrt{ab}\ln x = \displaystyle\int \frac{du}{u^2 + \beta u + 1} + C$, where $u = \sqrt{\dfrac{a}{b}}\,x^{n+1}y$, $\beta = \dfrac{n+1}{\sqrt{ab}}$.

8. $y'_x = ax^ny^2 + bx^m.$

1°. For $n \neq -1$, the substitution $\xi = x^{n+1}$ leads to a Riccati equation of the form 13.2.2.4: $y'_\xi = \dfrac{a}{n+1}y^2 + \dfrac{b}{n+1}\xi^{\frac{m-n}{n+1}}$.

2°. For $n = -1$ and $m \neq -1$, the transformation $\zeta = x^{m+1}$, $w = -1/y$ leads to a Riccati equation of the form 13.2.2.4: $w'_\zeta = \dfrac{b}{m+1}w^2 + \dfrac{a}{m+1}\zeta^{-1}$.

3°. For $n = m = -1$, the original equation is a separable equation. In this case we have the solution: $\ln|x| = \displaystyle\int \frac{dy}{ay^2 + b} + C$.

9. $y'_x = y^2 + k(ax+b)^n(cx+d)^{-n-4}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $u = \dfrac{1}{\Delta}[(cx+d)^2y + c(cx+d)]$, where $\Delta = ad - bc$, leads to an equation of the form 13.2.2.4: $u'_\xi = u^2 + k\Delta^{-2}\xi^n$.

10. $y'_x = ax^n y^2 + bmx^{m-1} - ab^2 x^{n+2m}$.

Particular solution: $y_0 = bx^m$.

11. $y'_x = (ax^{2n} + bx^{n-1})y^2 + c$.

The substitution $y = -1/w$ leads to an equation of the form 13.2.2.6: $w'_x = cw^2 + ax^{2n} + bx^{n-1}$.

12. $(a_2 x + b_2)(y'_x + \lambda y^2) + a_0 x + b_0 = 0$.

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.108: $(a_2 x + b_2)u''_{xx} + \lambda(a_0 x + b_0)u = 0$.

13. $x^2 y'_x = ax^2 y^2 + b$.

Solution: $y = \dfrac{\lambda}{x} - x^{2a\lambda}\Big(\dfrac{ax}{2a\lambda + 1}x^{2a\lambda} + C\Big)^{-1}$, where λ is a root of the quadratic equation $a\lambda^2 + \lambda + b = 0$.

14. $x^2 y'_x = x^2 y^2 - a^2 x^4 + a(1 - 2b)x^2 - b(b + 1)$.

Particular solution: $y_0 = ax + bx^{-1}$.

15. $x^2 y'_x = ax^2 y^2 + bx^n + c$.

The substitution $w = xy + A$, where A is a root of the quadratic equation $aA^2 - A + c = 0$, leads to an equation of the form 13.2.2.35: $xw'_x = aw^2 + (1 - 2aA)w + bx^n$.

16. $x^2 y'_x = x^2 y^2 + ax^{2m}(bx^m + c)^n + \frac{1}{4}(1 - n^2)$.

The transformation $\xi = bx^m + c$, $w = \dfrac{1}{bm}x^{1-m}y + \dfrac{1-m}{2bm}x^{-m}$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + a(bm)^{-2}\xi^n$.

17. $(c_2 x^2 + b_2 x + a_2)(y'_x + \lambda y^2) + a_0 = 0$.

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.179: $(c_2 x^2 + b_2 x + a_2)u''_{xx} + \lambda a_0 u = 0$.

18. $x^4 y'_x = -x^4 y^2 - a^2$.

Solution: $y = \dfrac{1}{x} + \dfrac{a}{x^2}\tan\Big(\dfrac{a}{x} + C\Big)$.

19. $ax^2(x - 1)^2(y'_x + \lambda y^2) + bx^2 + cx + s = 0$.

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.218: $ax^2(x - 1)^2 u''_{xx} + \lambda(bx^2 + cx + s)u = 0$.

20. $(ax^2 + bx + c)^2(y'_x + y^2) + A = 0$.

The substitution $y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.234: $(ax^2 + bx + c)^2 u''_{xx} + Au = 0$.

21. $x^{n+1} y'_x = ax^{2n} y^2 + cx^m + d$.

The substitution $w = x^n y + A$, where A is a root of the quadratic equation $aA^2 - nA + d = 0$, leads to an equation of the form 13.2.2.35: $xw'_x = aw^2 + (n - 2aA)w + cx^m$.

22. $(ax^n + b)y'_x = by^2 + ax^{n-2}.$

Particular solution: $y_0 = -1/x$.

23. $(ax^n + bx^m + c)(y'_x - y^2) + an(n-1)x^{n-2} + bm(m-1)x^{m-2} = 0.$

Particular solution: $y_0 = -\dfrac{anx^{n-1} + bmx^{m-1}}{ax^n + bx^m + c}$.

► **Other equations.**

24. $y'_x = ay^2 + by + cx + k.$

The substitution $y = -\dfrac{w'_x}{aw}$ leads to a second-order linear equation of the form 14.1.2.12: $w''_{xx} = bw'_x - a(cx + k)w$.

25. $y'_x = y^2 + ax^n y + ax^{n-1}.$

Particular solution: $y_0 = -1/x$.

26. $y'_x = y^2 + ax^n y + bx^{n-1}.$

The substitution $y = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.45: $u''_{xx} - ax^n u'_x + bx^{n-1}u = 0$.

27. $y'_x = y^2 + (\alpha x + \beta)y + ax^2 + bx + c.$

The substitution $y = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.31: $u''_{xx} - (\alpha x + \beta)u'_x + (ax^2 + bx + c)u = 0$.

28. $y'_x = y^2 + ax^n y - abx^n - b^2.$

Particular solution: $y_0 = b$.

29. $y'_x = -(n+1)x^n y^2 + ax^{n+m+1}y - ax^m.$

Particular solution: $y_0 = x^{-n-1}$.

30. $y'_x = ax^n y^2 + bx^m y + bcx^m - ac^2x^n.$

Particular solution: $y_0 = -c$.

31. $y'_x = ax^n y^2 - ax^n(bx^m + c)y + bmx^{m-1}.$

Particular solution: $y_0 = bx^m + c$.

32. $y'_x = -anx^{n-1}y^2 + cx^m(ax^n + b)y - cx^m.$

Particular solution: $y_0 = (ax^n + b)^{-1}$.

33. $y'_x = ax^n y^2 + bx^m y + ckx^{k-1} - bcx^{m+k} - ac^2x^{n+2k}.$

Particular solution: $y_0 = cx^k$.

34. $xy'_x = ay^2 + by + cx^{2b}.$

The transformation $t = x^b$, $w = x^{-b}y$ leads to a separable equation: $bw'_t = aw^2 + c$.

35. $xy'_x = ay^2 + by + cx^n.$

The transformation $\xi = x^b$, $\eta = yx^{-b}$ leads to the special Riccati equation of the form 13.2.2.4: $\eta'_\xi = \frac{a}{b}\eta^2 + \frac{c}{b}\xi^m$, where $m = \frac{n}{b} - 2$.

36. $xy'_x = ay^2 + (n + bx^n)y + cx^{2n}.$

The substitution $y = wx^n$ leads to a separable equation: $w'_x = x^{n-1}(aw^2 + bw + c)$.

37. $xy'_x = xy^2 + ay + bx^n.$

The substitution $y = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.67: $xu''_{xx} - au'_x + bx^n u = 0$.

38. $xy'_x + a_3xy^2 + a_2y + a_1x + a_0 = 0.$

The substitution $a_3y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.64: $xu''_{xx} + a_2u'_x + a_3(a_1x + a_0)u = 0$.

39. $xy'_x = ax^ny^2 + by + cx^{-n}.$

The substitution $w = yx^n$ leads to a separable equation: $xw'_x = aw^2 + (b + n)w + c$.

40. $xy'_x = ax^ny^2 + my - ab^2x^{n+2m}.$

Particular solution: $y_0 = bx^m$.

41. $xy'_x = x^{2n}y^2 + (m - n)y + x^{2m}.$

Solution: $y = x^{m-n}\tan\Bigl(\frac{x^{n+m}}{n+m} + C\Bigr)$.

42. $xy'_x = ax^ny^2 + by + cx^m.$

The transformation $\xi = x^{n-b}$, $\eta = yx^b$ leads to a special Riccati equation of the form 13.2.2.4: $(n + b)\eta'_\xi = a\eta^2 + c\xi^k$, where $k = \frac{m - n - 2b}{n + b}$.

43. $xy'_x = ax^{2n}y^2 + (bx^n - n)y + c.$

For $n = 0$, this is a separable equation. For $n \neq 0$, the solution is:

$$n\int\frac{dw}{aw^2 + bw + c} = x^n + C, \qquad \text{where} \quad w = yx^n.$$

44. $xy'_x = ax^{2n+m}y^2 + (bx^{n+m} - n)y + cx^m.$

The substitution $w = yx^n$ leads to a separable equation: $w'_x = x^{n+m-1}(aw^2 + bw + c)$.

45. $(a_2x + b_2)(y'_x + \lambda y^2) + (a_1x + b_1)y + a_0x + b_0 = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.108: $(a_2x + b_2)u''_{xx} + (a_1x + b_1)u'_x + \lambda(a_0x + b_0)u = 0$.

46. $(ax + c)y'_x = \alpha(ay + bx)^2 + \beta(ay + bx) - bx + \gamma.$

The substitution $t = ay + bx$ leads to a first-order linear equation with respect to $x = x(t)$: $(\alpha at^2 + \beta at + \gamma a + bc)x'_t = ax + c$.

47. $\boldsymbol{2x^2y'_x = 2y^2 + xy - 2a^2x.}$

Particular solution: $y_0 = a\sqrt{x}$.

48. $\boldsymbol{2x^2y'_x = 2y^2 + 3xy - 2a^2x.}$

Particular solution: $y_0 = a\sqrt{x} - \frac{1}{2}x$.

49. $\boldsymbol{x^2y'_x = ax^2y^2 + bxy + c.}$

The substitution $w = xy$ leads to a separable equation: $xw'_x = aw^2 + (b+1)w + c$.

50. $\boldsymbol{x^2y'_x = cx^2y^2 + (ax^2 + bx)y + \alpha x^2 + \beta x + \gamma.}$

The substitution $cy = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.139: $x^2u''_{xx} - x(ax+b)u'_x + c(\alpha x^2 + \beta x + \gamma)u = 0$.

51. $\boldsymbol{x^2y'_x = ax^2y^2 + bxy + cx^n + s.}$

The substitution $ay = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.132: $x^2u''_{xx} - bxu'_x + a(cx^n + s)u = 0$.

52. $\boldsymbol{x^2y'_x = ax^2y^2 + bxy + cx^{2n} + sx^n.}$

The substitution $ay = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.133: $x^2u''_{xx} - bxu'_x + ax^n(cx^n + s)u = 0$.

53. $\boldsymbol{x^2y'_x = cx^2y^2 + (ax^n + b)xy + \alpha x^{2n} + \beta x^n + \gamma.}$

The substitution $cy = -u'_x/u$ leads to a second-order linear equation of the form 14.1.2.146: $x^2u''_{xx} - (ax^n + b)xu'_x + c(\alpha x^{2n} + \beta x^n + \gamma)u = 0$.

54. $\boldsymbol{x^2y'_x = (\alpha x^{2n} + \beta x^n + \gamma)y^2 + (ax^n + b)xy + cx^2.}$

The substitution $y = -1/w$ leads to an equation of the form 13.2.2.53: $x^2w'_x = cx^2w^2 - (ax^n + b)xw + \alpha x^{2n} + \beta x^n + \gamma$.

55. $\boldsymbol{(x^2 - 1)y'_x + \lambda(y^2 - 2xy + 1) = 0.}$

The substitution $y = \dfrac{2\lambda - 1}{\lambda}x + \dfrac{1-\lambda}{\lambda}\dfrac{1}{u(x)}$ leads to an equation of the same form:

$$(x^2 - 1)u'_x + (\lambda - 1)(u^2 - 2xu + 1) = 0.$$

If $\lambda = n$ is a positive integer, then by using the above substitution, the original equation can be reduced to an equation of the same form in which $\lambda = 1$, i.e., to an equation of the form 13.2.2.58 with $a = 1$, $b = -1$.

56. $\boldsymbol{(ax^2 + b)y'_x + \alpha y^2 + \beta xy + \frac{b}{\alpha}(a + \beta) = 0.}$

Particular solution: $y_0 = -\dfrac{a+\beta}{\alpha}x$.

57. $\boldsymbol{(ax^2 + b)y'_x + \alpha y^2 + \beta xy + \gamma = 0.}$

The substitution $y = -\dfrac{a+\beta}{\alpha}x - \dfrac{1}{u(x)}$ leads to an equation of the same form: $(ax^2+b)u'_x + \Big(\gamma - \dfrac{a+\beta}{\alpha}b\Big)u^2 + (2a+\beta)xu + \alpha = 0$.

58. $(ax^2+b)y'_x+y^2-2xy+(1-a)x^2-b=0.$

Solution: $y=x+\Big(\int\frac{dx}{ax^2+b}+C\Big)^{-1}.$

59. $(ax^2+bx+c)y'_x=y^2+(2\lambda x+b)y+\lambda(\lambda-a)x^2+\mu.$

Particular solutions: $y_0=-\lambda x+A$, where $A=\frac{1}{2}\big(-b\pm\sqrt{b^2-4\mu-4\lambda c}\big)$.

60. $(ax^2+bx+c)y'_x=y^2+(ax+\mu)y-\lambda^2x^2+\lambda(b-\mu)x+\lambda c.$

Particular solution: $y_0=\lambda x$.

61. $(a_2x^2+b_2x+c_2)y'_x=y^2+(a_1x+b_1)y-\lambda(\lambda+a_1-a_2)x^2+\lambda(b_2-b_1)x+\lambda c_2.$

Particular solution: $y_0=\lambda x$.

62. $(a_2x^2+b_2x+c_2)y'_x=y^2+(a_1x+b_1)y+a_0x^2+b_0x+c_0.$

Let λ and β be roots of the system of the quadratic equations

$$\lambda^2+\lambda(a_1-a_2)+a_0=0,\quad \beta^2+\beta b_1+c_0-\lambda c_2=0,$$

where the first equation is solved independently (in the general case there are four roots). If some roots satisfy the condition $2\lambda\beta+\lambda b_1+\beta a_1+b_0-\lambda b_2=0$, the original equation possesses a particular solution: $y_0=\lambda x+\beta$.

63. $(x-a)(x-b)y'_x+y^2+k(y+x-a)(y+x-b)=0.$

To the case $k=0$ there corresponds a separable equation. To $k=-1$ there corresponds a linear equation. For $k\neq-1$ and $k\neq0$, with the aid of the substitution $ku(x)=y+k(y+x)$, we obtain the general solution:

$$\frac{y+k(y+x-a)}{y+k(y+x-b)}\Big(\frac{x-a}{x-b}\Big)^k=C\qquad\text{if } a\neq b,$$
$$\frac{1}{y+k(y+x-a)}+\frac{1}{x-a}=C\qquad\text{if } a=b.$$

64. $(c_2x^2+b_2x+a_2)(y'_x+\lambda y^2)+(b_1x+a_1)y+a_0=0.$

The substitution $\lambda y=u'_x/u$ leads to a second-order linear equation of the form 14.1.2.179: $(c_2x^2+b_2x+a_2)u''_{xx}+(b_1x+a_1)u'_x+\lambda a_0u=0$.

65. $x^3y'_x=ax^3y^2+(bx^2+c)y+sx.$

The substitution $ay=-u'_x/u$ leads to a second-order linear equation of the form 14.1.2.183: $x^3u''_{xx}-(bx^2+c)u'_x+asxu=0$.

66. $x^3y'_x=ax^3y^2+x(bx+c)y+\alpha x+\beta.$

The substitution $ay=-u'_x/u$ leads to a second-order linear equation of the form 14.1.2.186: $x^3u''_{xx}-x(bx+c)u'_x+a(\alpha x+\beta)u=0$.

67. $x(x^2+a)(y'_x+\lambda y^2)+(bx^2+c)y+sx=0.$

The substitution $\lambda y=u'_x/u$ leads to a second-order linear equation of the form 14.1.2.190: $x(x^2+a)u''_{xx}+(bx^2+c)u'_x+\lambda sxu=0$.

68. $x^2(x+a)(y'_x + \lambda y^2) + x(bx+c)y + \alpha x + \beta = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.194: $x^2(x+a)u''_{xx} + x(bx+c)u'_x + \lambda(\alpha x + \beta)u = 0$.

69. $(ax^2 + bx + c)(xy'_x - y) - y^2 + x^2 = 0.$

Solution: $\ln\left|\dfrac{y-x}{y+x}\right| = C + 2\displaystyle\int \frac{dx}{ax^2+bx+c}$.

70. $x^2(x^2+a)(y'_x + \lambda y^2) + x(bx^2+c)y + s = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.219: $x^2(x^2+a)u''_{xx} + x(bx^2+c)u'_x + \lambda s u = 0$.

71. $a(x^2-1)^2(y'_x + \lambda y^2) + bx(x^2-1)y + cx^2 + dx + s = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.227: $a(x^2-1)^2u''_{xx} + bx(x^2-1)u'_x + \lambda(cx^2+dx+s)u = 0$.

72. $x^{n+1}y'_x = ax^{2n}y^2 + bx^n y + cx^m + d.$

The substitution $w = x^n y + A$, where A is a root of the quadratic equation $aA^2 - (b+n)A + d = 0$, leads to an equation of the form 13.2.2.35: $xw'_x = aw^2 + (n+b-2aA)w + cx^m$.

73. $x(ax^k + b)y'_x = \alpha x^n y^2 + (\beta - anx^k)y + \gamma x^{-n}.$

The transformation $t = x^n y$, $z = x^{-k}$ leads to a separable equation: $[\alpha t^2 + (\beta + bn)t + \gamma]z'_t = -k(bz+a)$.

74. $x^2(ax^n - 1)(y'_x + \lambda y^2) + (px^n + q)xy + rx^n + s = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second-order linear equation of the form 14.1.2.254: $x^2(ax^n-1)u''_{xx} + (px^n+q)xu'_x + \lambda(rx^n+s)u = 0$.

75. $(ax^n + bx^m + c)y'_x = cy^2 - bx^{m-1}y + ax^{n-2}.$

Particular solution: $y_0 = -1/x$.

76. $(ax^n + bx^m + c)y'_x = ax^{n-2}y^2 + bx^{m-1}y + c.$

Particular solution: $y_0 = x$.

77. $(ax^n + bx^m + c)y'_x = \alpha x^k y^2 + \beta x^s y - \alpha\lambda^2 x^k + \beta\lambda x^s.$

Particular solution: $y_0 = -\lambda$.

78. $(ax^n + bx^m + c)(xy'_x - y) + sx^k(y^2 - \lambda x^2) = 0.$

Particular solutions: $y_0 = \pm x\sqrt{\lambda}$.

13.2.3 Equations Containing Exponential Functions

► **Equations with exponential functions.**

1. $y'_x = ay^2 + be^{\lambda x}.$

The substitution $t = e^{\lambda x}$ leads to an equation of the form 13.2.2.35: $\lambda t y'_t = ay^2 + bt$.

2. $y'_x = y^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

3. $y'_x = \sigma y^2 + a + be^{\lambda x} + ce^{2\lambda x}$.

The substitution $\sigma y = -u'_x/u$ leads to a second-order linear equation of the form 14.1.3.5: $u''_{xx} + \sigma(a + be^{\lambda x} + ce^{2\lambda x})u = 0$.

4. $y'_x = \sigma y^2 + ay + be^x + c$.

The substitution $\sigma y = -u'_x/u$ leads to a second-order linear equation of the form 14.1.3.10: $u''_{xx} - au'_x + \sigma(be^x + c)u = 0$.

5. $y'_x = y^2 + by + a(\lambda - b)e^{\lambda x} - a^2 e^{2\lambda x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

6. $y'_x = y^2 + ae^{\lambda x}y - abe^{\lambda x} - b^2$.

Particular solution: $y_0 = b$.

7. $y'_x = y^2 + ae^{2\lambda x}(e^{\lambda x} + b)^n - \frac{1}{4}\lambda^2$.

The transformation $\xi = e^{\lambda x} + b$, $w = \dfrac{1}{\lambda}\Big(e^{-\lambda x}y - \dfrac{\lambda}{2}e^{-\lambda x}\Big)$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + a\lambda^{-2}\xi^n$.

8. $y'_x = y^2 + ae^{8\lambda x} + be^{6\lambda x} + ce^{4\lambda x} - \lambda^2$.

The transformation $\xi = e^{2\lambda x}$, $w = e^{-2\lambda x}\Big(\dfrac{y}{2\lambda} - \dfrac{1}{2}\Big)$ leads to an equation of the form 13.2.2.3: $w'_\xi = w^2 + (2\lambda)^{-2}(a\xi^2 + b\xi + c)$.

9. $y'_x = ae^{kx}y^2 + be^{sx}, \qquad k \neq 0$.

The substitution $t = e^{kx}$ leads to an equation of the form 13.2.2.4: $ky'_t = ay^2 + bt^{s-k}$.

10. $y'_x = be^{\mu x}y^2 + a\lambda e^{\lambda x} - a^2 be^{(\mu+2\lambda)x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

11. $y'_x = ae^{\lambda x}y^2 + by + ce^{-\lambda x}$.

The substitution $z = e^{\lambda x}y$ leads to a separable equation: $z'_x = az^2 + (b + \lambda)z + c$.

12. $y'_x = ae^{\mu x}y^2 + \lambda y - ab^2 e^{(\mu+2\lambda)x}$.

Particular solution: $y_0 = be^{\lambda x}$.

13. $y'_x = e^{\lambda x}y^2 + ae^{\mu x}y + a\lambda e^{(\mu-\lambda)x}$.

Particular solution: $y_0 = -\lambda e^{-\lambda x}$.

14. $y'_x = -\lambda e^{\lambda x}y^2 + ae^{\mu x}y - ae^{(\mu-\lambda)x}$.

Particular solution: $y_0 = e^{-\lambda x}$.

15. $y'_x = ae^{\mu x}y^2 + abe^{(\lambda+\mu)x}y - b\lambda e^{\lambda x}$.

Particular solution: $y_0 = -be^{\lambda x}$.

16. $y'_x = ae^{kx}y^2 + by + ce^{sx} + de^{-kx}.$

The substitution $t = e^{kx}$ leads to an equation of the form 13.2.2.51: $kt^2y'_t = at^2y^2 + bty + ct^{(k+s)/k} + d.$

17. $y'_x = ae^{(2\lambda+\mu)x}y^2 + [be^{(\lambda+\mu)x} - \lambda]y + ce^{\mu x}.$

The substitution $w = e^{\lambda x}y$ leads to a separable equation: $w'_x = e^{(\lambda+\mu)x}(aw^2 + bw + c).$

18. $y'_x = ae^{kx}y^2 + by + ce^{knx} + de^{k(2n+1)x}.$

The substitution $t = e^{kx}$ leads to an equation of the form 13.2.2.52: $kt^2y'_t = at^2y^2 + bty + ct^{n+1} + dt^{2(n+1)}.$

19. $y'_x = e^{\mu x}(y - be^{\lambda x})^2 + b\lambda e^{\lambda x}.$

Particular solution: $y_0 = be^{\lambda x}.$

20. $(ae^{\lambda x} + be^{\mu x} + c)y'_x = y^2 + ke^{\nu x}y - m^2 + kme^{\nu x}.$

Particular solution: $y_0 = -m.$

21. $(ae^{\lambda x} + be^{\mu x} + c)(y'_x - y^2) + a\lambda^2e^{\lambda x} + b\mu^2e^{\mu x} = 0.$

Particular solution: $y_0 = -\dfrac{a\lambda e^{\lambda x} + b\mu e^{\mu x}}{ae^{\lambda x} + be^{\mu x} + c}.$

► **Equations with power and exponential functions.**

22. $y'_x = y^2 + axe^{\lambda x}y + ae^{\lambda x}.$

Particular solution: $y_0 = -1/x.$

23. $y'_x = ae^{\lambda x}y^2 + be^{-\lambda x}.$

Solution: $\displaystyle\int \frac{dz}{az^2 + \lambda z + b} = x + C,$ where $z = e^{\lambda x}y.$

24. $y'_x = ae^{\lambda x}y^2 + bnx^{n-1} - ab^2e^{\lambda x}x^{2n}.$

Particular solution: $y_0 = bx^n.$

25. $y'_x = e^{\lambda x}y^2 + ax^ny + a\lambda x^ne^{-\lambda x}.$

Particular solution: $y_0 = -\lambda e^{-\lambda x}.$

26. $y'_x = -\lambda e^{\lambda x}y^2 + ax^ne^{\lambda x}y - ax^n.$

Particular solution: $y_0 = e^{-\lambda x}.$

27. $y'_x = ae^{\lambda x}y^2 - abx^ne^{\lambda x}y + bnx^{n-1}.$

Particular solution: $y_0 = bx^n.$

28. $y'_x = ax^ny^2 + b\lambda e^{\lambda x} - ab^2x^ne^{2\lambda x}.$

Particular solution: $y_0 = be^{\lambda x}.$

29. $y'_x = ax^ny^2 + \lambda y - ab^2x^ne^{2\lambda x}.$

Particular solution: $y_0 = be^{\lambda x}.$

30. $y'_x = ax^n y^2 - abx^n e^{\lambda x} y + b\lambda e^{\lambda x}.$

Particular solution: $y_0 = be^{\lambda x}$.

31. $y'_x = -(k+1)x^k y^2 + ax^{k+1} e^{\lambda x} y - ae^{\lambda x}.$

Particular solution: $y_0 = x^{-k-1}$.

32. $y'_x = ax^n y^2 - ax^n(be^{\lambda x} + c)y + b\lambda e^{\lambda x}.$

Particular solution: $y_0 = be^{\lambda x} + c$.

33. $y'_x = ax^n e^{2\lambda x} y^2 + (bx^n e^{\lambda x} - \lambda)y + cx^n.$

The substitution $w = e^{\lambda x} y$ leads to a separable equation: $w'_x = x^n e^{\lambda x}(aw^2 + bw + c)$.

34. $y'_x = ae^{\lambda x}(y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

35. $xy'_x = ae^{\lambda x} y^2 + ky + ab^2 x^{2k} e^{\lambda x}.$

Solution: $y = bx^k \tan\Bigl(ab \int x^{k-1} e^{\lambda x}\, dx + C\Bigr)$.

36. $xy'_x = ax^{2n} e^{\lambda x} y^2 + (bx^n e^{\lambda x} - n)y + ce^{\lambda x}.$

Solution: $\displaystyle\int \frac{dw}{aw^2 + bw + c} = \int x^{n-1} e^{\lambda x}\, dx + C$, where $w = x^n y$.

37. $y'_x = y^2 + 2a\lambda x e^{\lambda x^2} - a^2 e^{2\lambda x^2}.$

Particular solution: $y_0 = ae^{\lambda x^2}$.

38. $y'_x = ae^{-\lambda x^2} y^2 + \lambda xy + ab^2.$

Solution: $y = be^{\lambda x^2/2} \tan\Bigl(ab \int e^{-\lambda x^2/2}\, dx + C\Bigr)$.

39. $y'_x = ax^n y^2 + \lambda xy + ab^2 x^n e^{\lambda x^2}.$

Solution: $y = be^{\lambda x^2/2} \tan\Bigl(ab \int x^n e^{\lambda x^2/2}\, dx + C\Bigr)$.

40. $x^4(y'_x - y^2) = a + b\exp(k/x) + c\exp(2k/x).$

The transformation $\xi = 1/x$, $w = -x^2 y - x$ leads to a Riccati equation of the form 13.2.3.3: $w'_\xi = w^2 + a + be^{k\xi} + ce^{2k\xi}$.

13.2.4 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine and cosine.**

1. $y'_x = y^2 - a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh^2(\lambda x).$

Particular solution: $y_0 = a\cosh(\lambda x)$.

2. $y'_x = y^2 + a\sinh(\beta x)y + ab\sinh(\beta x) - b^2.$

Particular solution: $y_0 = -b$.

3. $y'_x = y^2 + ax\sinh^m(bx)y + a\sinh^m(bx).$

Particular solution: $y_0 = -1/x$.

4. $y'_x = \lambda\sinh(\lambda x)y^2 - \lambda\sinh^3(\lambda x).$

Particular solution: $y_0 = \cosh(\lambda x)$.

5. $y'_x = [a\sinh^2(\lambda x) - \lambda]y^2 - a\sinh^2(\lambda x) + \lambda - a.$

Particular solution: $y_0 = \coth(\lambda x)$.

6. $[a\sinh(\lambda x) + b]y'_x = y^2 + c\sinh(\mu x)\,y - d^2 + cd\sinh(\mu x).$

Particular solution: $y_0 = -d$.

7. $[a\sinh(\lambda x) + b](y'_x - y^2) + a\lambda^2\sinh(\lambda x) = 0.$

Particular solution: $y_0 = -\dfrac{a\lambda\cosh(\lambda x)}{a\sinh(\lambda x) + b}$.

8. $y'_x = \alpha y^2 + \beta + \gamma\cosh x.$

The transformation $x = 2t$, $\alpha y = -u'_x/u$ leads to the modified Mathieu equation 2.1.4.9: $u''_{tt} - (a - 2q\cosh 2t)u = 0$, where $a = -4\alpha\beta$, $q = 2\alpha\gamma$.

9. $y'_x = y^2 + a\cosh(\beta x)y + ab\cosh(\beta x) - b^2.$

Particular solution: $y_0 = -b$.

10. $y'_x = y^2 + ax\cosh^m(bx)y + a\cosh^m(bx).$

Particular solution: $y_0 = -1/x$.

11. $y'_x = [a\cosh^2(\lambda x) - \lambda]y^2 + a + \lambda - a\cosh^2(\lambda x).$

Particular solution: $y_0 = \tanh(\lambda x)$.

12. $2y'_x = [a - \lambda + a\cosh(\lambda x)]y^2 + a + \lambda - a\cosh(\lambda x).$

Particular solution: $y_0 = \tanh\left(\tfrac{1}{2}\lambda x\right)$.

13. $y'_x = y^2 - \lambda^2 + a\cosh^n(\lambda x)\sinh^{-n-4}(\lambda x).$

The transformation $\xi = \coth(\lambda x)$, $w = -\dfrac{1}{\lambda}\sinh^2(\lambda x)y - \sinh(\lambda x)\cosh(\lambda x)$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + \lambda^{-2}\xi^n$.

14. $y'_x = a\sinh(\lambda x)y^2 + b\sinh(\lambda x)\cosh^n(\lambda x).$

The transformation $\xi = \cosh(\lambda x)$, $w = \dfrac{a}{\lambda}y$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

15. $y'_x = a\cosh(\lambda x)y^2 + b\cosh(\lambda x)\sinh^n(\lambda x).$

The transformation $\xi = \sinh(\lambda x)$, $w = \dfrac{a}{\lambda}y$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

16. $[a\cosh(\lambda x)+b]y'_x = y^2 + c\cosh(\mu x)\,y - d^2 + cd\cosh(\mu x).$

Particular solution: $y_0 = -d$.

17. $[a\cosh(\lambda x)+b](y'_x - y^2) + a\lambda^2\cosh(\lambda x) = 0.$

Particular solution: $y_0 = -\dfrac{a\lambda\sinh(\lambda x)}{a\cosh(\lambda x)+b}$.

► **Equations with hyperbolic tangent and cotangent.**

18. $y'_x = y^2 + a\lambda - a(a+\lambda)\tanh^2(\lambda x).$

Particular solution: $y_0 = a\tanh(\lambda x)$.

19. $y'_x = y^2 + 3a\lambda - \lambda^2 - a(a+\lambda)\tanh^2(\lambda x).$

Particular solution: $y_0 = a\tanh(\lambda x) - \lambda\coth(\lambda x)$.

20. $y'_x = y^2 + ax\tanh^m(bx)y + a\tanh^m(bx).$

Particular solution: $y_0 = -1/x$.

21. $[a\tanh(\lambda x)+b]y'_x = y^2 + c\tanh(\mu x)\,y - d^2 + cd\tanh(\mu x).$

Particular solution: $y_0 = -d$.

22. $y'_x = y^2 + a\lambda - a(a+\lambda)\coth^2(\lambda x).$

Particular solution: $y_0 = a\coth(\lambda x)$.

23. $y'_x = y^2 - \lambda^2 + 3a\lambda - a(a+\lambda)\coth^2(\lambda x).$

Particular solution: $y_0 = a\coth(\lambda x) - \lambda\tanh(\lambda x)$.

24. $y'_x = y^2 + ax\coth^m(bx)y + a\coth^m(bx).$

Particular solution: $y_0 = -1/x$.

25. $[a\coth(\lambda x)+b]y'_x = y^2 + c\coth(\mu x)\,y - d^2 + cd\coth(\mu x).$

Particular solution: $y_0 = -d$.

26. $y'_x = y^2 - 2\lambda^2\tanh^2(\lambda x) - 2\lambda^2\coth^2(\lambda x).$

Particular solution: $y_0 = \lambda\tanh(\lambda x) + \lambda\coth(\lambda x)$.

27. $y'_x = y^2 + a\lambda + b\lambda - 2ab - a(a+\lambda)\tanh^2(\lambda x) - b(b+\lambda)\coth^2(\lambda x).$

Particular solution: $y_0 = a\tanh(\lambda x) + b\coth(\lambda x)$.

13.2.5 Equations Containing Logarithmic Functions

► **Equations of the form** $g(x)y'_x = f_2(x)y^2 + f_0(x)$.

1. $y'_x = a(\ln x)^n y^2 + bmx^{m-1} - ab^2x^{2m}(\ln x)^n.$

Particular solution: $y_0 = bx^m$.

2. $xy'_x = ay^2 + b\ln x + c.$

The substitution $x = e^t$ leads to an equation of the form 13.2.2.1: $y'_t = ay^2 + bt + c$.

3. $xy'_x = ay^2 + b\ln^k x + c\ln^{2k+2} x.$

The substitution $t = \ln x$ leads to an equation of the form 13.2.2.6 with $k = n - 1$: $y'_t = ay^2 + bt^k + ct^{2k+2}$.

4. $xy'_x = xy^2 - a^2 x\ln^2(\beta x) + a.$

Particular solution: $y_0 = a\ln(\beta x)$.

5. $xy'_x = xy^2 - a^2 x\ln^{2k}(\beta x) + ak\ln^{k-1}(\beta x).$

Particular solution: $y_0 = a\ln^k(\beta x)$.

6. $xy'_x = ax^n y^2 + b - ab^2 x^n \ln^2 x.$

Particular solution: $y_0 = b\ln x$.

7. $x^2 y'_x = x^2 y^2 + a\ln^2 x + b\ln x + c.$

The transformation $\xi = \ln x$, $w = xy + \frac{1}{2}$ leads to an equation of the form 13.2.2.3: $w'_\xi = w^2 + a\xi^2 + b\xi + c - \frac{1}{4}$.

8. $x^2 y'_x = x^2 y^2 + a(b\ln x + c)^n + \frac{1}{4}.$

The transformation $\xi = b\ln x + c$, $w = \dfrac{x}{b}y + \dfrac{1}{2b}$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + ab^{-2}\xi^n$.

9. $x^2\ln(ax)(y'_x - y^2) = 1.$

Particular solution: $y_0 = -[x\ln(ax)]^{-1}$.

► **Equations of the form** $g(x)y'_x = f_2(x)y^2 + f_1(x)y'_x + f_0(x)$.

10. $y'_x = y^2 + a\ln(\beta x)y - ab\ln(\beta x) - b^2.$

Particular solution: $y_0 = b$.

11. $y'_x = y^2 + ax\ln^m(bx)y + a\ln^m(bx).$

Particular solution: $y_0 = -1/x$.

12. $y'_x = ax^n y^2 - abx^{n+1}\ln x\, y + b\ln x + b.$

Particular solution: $y_0 = bx\ln x$.

13. $y'_x = -(n+1)x^n y^2 + ax^{n+1}(\ln x)^m y - a(\ln x)^m.$

Particular solution: $y_0 = x^{-n-1}$.

14. $y'_x = a(\ln x)^n y^2 - abx(\ln x)^{n+1} y + b\ln x + b.$

Particular solution: $y_0 = bx\ln x$.

15. $y'_x = a(\ln x)^k (y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

16. $y'_x = a(\ln x)^n y^2 + b(\ln x)^m y + bc(\ln x)^m - ac^2(\ln x)^n.$

Particular solution: $y_0 = -c$.

17. $xy'_x = (ay + b\ln x)^2.$

Solution: $\ln x = \int \frac{dz}{az^2 + b} + C$, where $z = ay + b\ln x$.

18. $xy'_x = a\ln^m(\lambda x)y^2 + ky + ab^2x^{2k}\ln^m(\lambda x).$

Solution: $y = bx^k \tan\Big[ab\int x^{k-1}\ln^m(\lambda x)\,dx + C\Big]$.

19. $xy'_x = ax^n(y + b\ln x)^2 - b.$

Solution: $\dfrac{1}{y + b\ln x} + \dfrac{a}{n}x^n = C$.

20. $xy'_x = ax^{2n}(\ln x)y^2 + (bx^n\ln x - n)y + c\ln x.$

Solution: $\int \frac{dw}{aw^2 + bw + c} = \int x^{n-1}\ln x\,dx + C$, where $w = x^n y$.

21. $x^2y'_x = a^2x^2y^2 - xy + b^2\ln^n x.$

The substitution $a^2y = -u'_x/u$ leads to a second-order linear equation of the form 14.1.5.27: $x^2u''_{xx} + xu'_x + (ab)^2\ln^n x\,u = 0$.

22. $(a\ln x + b)y'_x = y^2 + c(\ln x)^n y - \lambda^2 + \lambda c(\ln x)^n.$

Particular solution: $y_0 = -\lambda$.

23. $(a\ln x + b)y'_x = (\ln x)^n y^2 + cy - \lambda^2(\ln x)^n + c\lambda.$

Particular solution: $y_0 = -\lambda$.

13.2.6 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y'_x = \alpha y^2 + \beta + \gamma\sin(\lambda x).$

The substitution $2t = 2\lambda x + \pi$ leads to an equation of the form 13.2.6.14: $\lambda y'_t = \alpha y^2 + \beta + \gamma\cos t$.

2. $y'_x = y^2 - a^2 + a\lambda\sin(\lambda x) + a^2\sin^2(\lambda x).$

Particular solution: $y_0 = -a\cos(\lambda x)$.

3. $y'_x = y^2 + \lambda^2 + c\sin^n(\lambda x + a)\sin^{-n-4}(\lambda x + b).$

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{\sin^2(\lambda x + b)}{\sin(b - a)}\Big[\dfrac{y}{\lambda} + \cot(\lambda x + b)\Big]$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + A\xi^n$, where $A = c[\lambda\sin(b - a)]^{-2}$.

4. $y'_x = y^2 + a\sin(\beta x)y + ab\sin(\beta x) - b^2.$

Particular solution: $y_0 = -b$.

5. $y'_x = y^2 + ax\sin^m(bx)y + a\sin^m(bx).$

Particular solution: $y_0 = -1/x$.

6. $y'_x = \lambda\sin(\lambda x)y^2 + \lambda\sin^3(\lambda x).$

Particular solution: $y_0 = -\cos(\lambda x)$.

7. $2y'_x = [\lambda + a - a\sin(\lambda x)]y^2 + \lambda - a - a\sin(\lambda x).$

Particular solution: $y_0 = \tan\bigl(\frac{1}{2}\lambda x + \frac{1}{4}\pi\bigr)$.

8. $y'_x = [\lambda + a\sin^2(\lambda x)]y^2 + \lambda - a + a\sin^2(\lambda x).$

Particular solution: $y_0 = -\cot(\lambda x)$.

9. $y'_x = -(k+1)x^k y^2 + ax^{k+1}(\sin x)^m y - a(\sin x)^m.$

Particular solution: $y_0 = x^{-k-1}$.

10. $y'_x = a\sin^k(\lambda x + \mu)(y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

11. $xy'_x = a\sin^m(\lambda x)y^2 + ky + ab^2x^{2k}\sin^m(\lambda x).$

Solution: $y = bx^k\tan\Bigl[ab\int x^{k-1}\sin^m(\lambda x)\,dx + C\Bigr]$.

12. $[a\sin(\lambda x) + b]y'_x = y^2 + c\sin(\mu x)\,y - d^2 + cd\sin(\mu x).$

Particular solution: $y_0 = -d$.

13. $[a\sin(\lambda x) + b](y'_x - y^2) - a\lambda^2\sin(\lambda x) = 0.$

Particular solution: $y_0 = -\dfrac{a\lambda\cos(\lambda x)}{a\sin(\lambda x) + b}$.

► **Equations with cosine.**

14. $y'_x = \alpha y^2 + \beta + \gamma\cos x.$

The transformation $x = 2t$, $\alpha y = -u'_x/u$ leads to a Mathieu equation of the form 14.1.6.29: $u''_{tt} + (a - 2q\cos 2t)u = 0$, where $a = 4\alpha\beta$, and $q = -2\alpha\gamma$.

15. $y'_x = y^2 - a^2 + a\lambda\cos(\lambda x) + a^2\cos^2(\lambda x).$

Particular solution: $y_0 = a\sin(\lambda x)$.

16. $y'_x = y^2 + \lambda^2 + c\cos^n(\lambda x + a)\cos^{-n-4}(\lambda x + b).$

The substitution $\lambda x = \lambda z - \frac{\pi}{2}$ leads to an equation of the form 13.2.6.3: $y'_z = y^2 + \lambda^2 + c\sin^n(\lambda z + a)\sin^{-n-4}(\lambda z + b)$.

17. $y'_x = y^2 + a\cos(\beta x)y + ab\cos(\beta x) - b^2.$

Particular solution: $y_0 = -b$.

18. $y'_x = y^2 + ax\cos^m(bx)y + a\cos^m(bx).$

Particular solution: $y_0 = -1/x$.

19. $y'_x = \lambda \cos(\lambda x) y^2 + \lambda \cos^3(\lambda x)$.

Particular solution: $y_0 = \sin(\lambda x)$.

20. $2y'_x = [\lambda + a + a\cos(\lambda x)]y^2 + \lambda - a + a\cos(\lambda x)$.

Particular solution: $y_0 = \tan\left(\frac{1}{2}\lambda x\right)$.

21. $y'_x = [\lambda + a\cos^2(\lambda x)]y^2 + \lambda - a + a\cos^2(\lambda x)$.

Particular solution: $y_0 = \tan(\lambda x)$.

22. $y'_x = -(k+1)x^k y^2 + ax^{k+1}(\cos x)^m y - a(\cos x)^m$.

Particular solution: $y_0 = x^{-k-1}$.

23. $y'_x = a\cos^k(\lambda x + \mu)(y - bx^n - c)^2 + bnx^{n-1}$.

Particular solution: $y_0 = bx^n + c$.

24. $xy'_x = a\cos^m(\lambda x)y^2 + ky + ab^2 x^{2k}\cos^m(\lambda x)$.

Solution: $y = bx^k \tan\left[ab\int x^{k-1}\cos^m(\lambda x)\,dx + C\right]$.

25. $[a\cos(\lambda x) + b]y'_x = y^2 + c\cos(\mu x)\,y - d^2 + cd\cos(\mu x)$.

Particular solution: $y_0 = -d$.

26. $[a\cos(\lambda x) + b](y'_x - y^2) - a\lambda^2\cos(\lambda x) = 0$.

Particular solution: $y_0 = \dfrac{a\lambda\sin(\lambda x)}{a\cos(\lambda x) + b}$.

► **Equations with tangent.**

27. $y'_x = y^2 + a\lambda + a(\lambda - a)\tan^2(\lambda x)$.

Particular solution: $y_0 = a\tan(\lambda x)$.

28. $y'_x = y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\tan^2(\lambda x)$.

Particular solution: $y_0 = a\tan(\lambda x) - \lambda\cot(\lambda x)$.

29. $y'_x = ay^2 + b\tan x\,y + c$.

The substitution $ay = -u'_x/u$ leads to a second-order linear equation of the form 14.1.6.53: $u''_{xx} - b\tan x\,u'_x + acu = 0$.

30. $y'_x = ay^2 + 2ab\,\tan x\,y + b(ab-1)\tan^2 x$.

The substitution $u = y + b\tan x$ leads to a separable equation of the form 13.1.2: $u'_x = au^2 + b$.

31. $y'_x = y^2 + a\tan(\beta x)y + ab\tan(\beta x) - b^2$.

Particular solution: $y_0 = -b$.

32. $y'_x = y^2 + ax\tan^m(bx)y + a\tan^m(bx)$.

Particular solution: $y_0 = -1/x$.

33. $y'_x = -(k+1)x^k y^2 + ax^{k+1}(\tan x)^m y - a(\tan x)^m.$

Particular solution: $y_0 = x^{-k-1}$.

34. $y'_x = a\tan^n(\lambda x)y^2 - ab^2\tan^{n+2}(\lambda x) + b\lambda\tan^2(\lambda x) + b\lambda.$

Particular solution: $y_0 = b\tan(\lambda x)$.

35. $y'_x = a\tan^k(\lambda x + \mu)(y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

36. $xy'_x = a\tan^m(\lambda x)y^2 + ky + ab^2x^{2k}\tan^m(\lambda x).$

Solution: $y = bx^k \tan\Big[ab\int x^{k-1}\tan^m(\lambda x)\,dx + C\Big]$.

37. $[a\tan(\lambda x) + b]y'_x = y^2 + k\tan(\mu x)\,y - d^2 + kd\tan(\mu x).$

Particular solution: $y_0 = -d$.

► **Equations with cotangent.**

38. $y'_x = y^2 + a\lambda + a(\lambda - a)\cot^2(\lambda x).$

Particular solution: $y_0 = -a\cot(\lambda x)$.

39. $y'_x = y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\cot^2(\lambda x).$

Particular solution: $y_0 = \lambda\tan(\lambda x) - a\cot(\lambda x)$.

40. $y'_x = y^2 - 2a\cot(ax)y + b^2 - a^2.$

Particular solution: $y_0 = a\cot(ax) - b\cot(bx)$.

41. $y'_x = y^2 + a\cot(\beta x)y + ab\cot(\beta x) - b^2.$

Particular solution: $y_0 = -b$.

42. $y'_x = y^2 + ax\cot^m(bx)y + a\cot^m(bx).$

Particular solution: $y_0 = -1/x$.

43. $y'_x = -(k+1)x^k y^2 + ax^{k+1}(\cot x)^m y - a(\cot x)^m.$

Particular solution: $y_0 = x^{-k-1}$.

44. $y'_x = a\cot^k(\lambda x + \mu)(y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

45. $xy'_x = a\cot^m(\lambda x)y^2 + ky + ab^2x^{2k}\cot^m(\lambda x).$

Solution: $y = bx^k \tan\Big[ab\int x^{k-1}\cot^m(\lambda x)\,dx + C\Big]$.

46. $[a\cot(\lambda x) + b]y'_x = y^2 + c\cot(\mu x)\,y - d^2 + cd\cot(\mu x).$

Particular solution: $y_0 = -d$.

► **Equations containing combinations of trigonometric functions.**

47. $y'_x = y^2 + \lambda^2 + c\sin^n(\lambda x)\cos^{-n-4}(\lambda x)$.

This is a special case of equation 13.2.6.3 with $a = 0$ and $b = \pi/2$.

48. $y'_x = a\sin(\lambda x)y^2 + b\sin(\lambda x)\cos^n(\lambda x)$.

The transformation $\xi = \cos(\lambda x)$, $w = -\dfrac{a}{\lambda}y$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

49. $y'_x = \lambda\sin(\lambda x)y^2 + a\cos^n(\lambda x)y - a\cos^{n-1}(\lambda x)$.

Particular solution: $y_0 = 1/\cos(\lambda x)$.

50. $y'_x = a\cos(\lambda x)y^2 + b\cos(\lambda x)\sin^n(\lambda x)$.

The transformation $\xi = \sin(\lambda x)$, $w = \dfrac{a}{\lambda}y$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

51. $y'_x = \lambda\sin(\lambda x)y^2 + ax^n\cos(\lambda x)y - ax^n$.

Particular solution: $y_0 = 1/\cos(\lambda x)$.

52. $\sin^{n+1}(2x)y'_x = ay^2\sin^{2n}x + b\cos^{2n}x$.

The substitution $z = y\tan^n x$ leads to a separable equation: $2^n\sin(2x)z'_x = az^2 + n2^{n+1}z + b$.

53. $y'_x = y^2 - y\tan x + a(1-a)\cot^2 x$.

Particular solution: $y_0 = -a\cot x$.

54. $y'_x = y^2 - my\tan x + b^2\cos^{2m}x$.

Solution: $y = -b\cos^m x\cot\Big(b\displaystyle\int\cos^m x\,dx + C\Big)$.

55. $y'_x = y^2 + my\cot x + b^2(\sin x)^{2m}$.

Solution: $y = -b\sin^m x\cot\Big(b\displaystyle\int\sin^m x\,dx + C\Big)$.

56. $y'_x = y^2 - 2\lambda^2\tan^2(\lambda x) - 2\lambda^2\cot^2(\lambda x)$.

Particular solution: $y_0 = \lambda\cot(\lambda x) - \lambda\tan(\lambda x)$.

57. $y'_x = y^2 + \lambda a + \lambda b + 2ab + a(\lambda - a)\tan^2(\lambda x) + b(\lambda - b)\cot^2(\lambda x)$.

Particular solution: $y_0 = a\tan(\lambda x) - b\cot(\lambda x)$.

58. $y'_x = y^2 - \frac{1}{2}\lambda^2 - \frac{3}{4}\lambda^2\tan^2(\lambda x) + a\cos^2(\lambda x)\sin^n(\lambda x)$.

The transformation $\xi = \sin(\lambda x)$, $w = \dfrac{y}{\lambda\cos(\lambda x)} + \dfrac{\sin(\lambda x)}{2\cos^2(\lambda x)}$ leads to an equation of the form 13.2.2.4: $w'_\xi = w^2 + a\lambda^{-2}\xi^n$.

59. $y'_x = \lambda\sin(\lambda x)y^2 + a\sin(\lambda x)y - a\tan(\lambda x)$.

Particular solution: $y_0 = 1/\cos(\lambda x)$.

13.2.7 Equations Containing Inverse Trigonometric Functions

► **Equations containing arcsine.**

1. $y'_x = y^2 + \lambda(\arcsin x)^n y - a^2 + a\lambda(\arcsin x)^n.$

Particular solution: $y_0 = -a$.

2. $y'_x = y^2 + \lambda x(\arcsin x)^n y + \lambda(\arcsin x)^n.$

Particular solution: $y_0 = -1/x$.

3. $y'_x = -(k+1)x^k y^2 + \lambda(\arcsin x)^n(x^{k+1}y - 1).$

Particular solution: $y_0 = x^{-k-1}$.

4. $y'_x = \lambda(\arcsin x)^n y^2 + ay + ab - b^2\lambda(\arcsin x)^n.$

Particular solution: $y_0 = -b$.

5. $y'_x = \lambda(\arcsin x)^n y^2 - b\lambda x^m(\arcsin x)^n y + bmx^{m-1}.$

Particular solution: $y_0 = bx^m$.

6. $y'_x = \lambda(\arcsin x)^n y^2 + \beta m x^{m-1} - \lambda\beta^2 x^{2m}(\arcsin x)^n.$

Particular solution: $y_0 = \beta x^m$.

7. $y'_x = \lambda(\arcsin x)^n(y - ax^m - b)^2 + amx^{m-1}.$

Particular solution: $y_0 = ax^m + b$.

8. $xy'_x = \lambda(\arcsin x)^n y^2 + ky + \lambda b^2 x^{2k}(\arcsin x)^n.$

Solution: $y = bx^k \tan\Big[\lambda b \int x^{k-1}(\arcsin x)^n\,dx + C\Big]$.

9. $xy'_x = (ax^{2n}y^2 + bx^n y + c)(\arcsin x)^m - ny.$

The substitution $z = x^n y$ leads to a separable equation:

$$z'_x = x^{n-1}(\arcsin x)^m(az^2 + bz + c).$$

► **Equations containing arccosine.**

10. $y'_x = y^2 + \lambda(\arccos x)^n y - a^2 + a\lambda(\arccos x)^n.$

Particular solution: $y_0 = -a$.

11. $y'_x = y^2 + \lambda x(\arccos x)^n y + \lambda(\arccos x)^n.$

Particular solution: $y_0 = -1/x$.

12. $y'_x = -(k+1)x^k y^2 + \lambda(\arccos x)^n(x^{k+1}y - 1).$

Particular solution: $y_0 = x^{-k-1}$.

13. $y'_x = \lambda(\arccos x)^n y^2 + ay + ab - b^2\lambda(\arccos x)^n.$

Particular solution: $y_0 = -b$.

14. $y'_x = \lambda(\arccos x)^n y^2 - b\lambda x^m(\arccos x)^n y + bmx^{m-1}$.

Particular solution: $y_0 = bx^m$.

15. $y'_x = \lambda(\arccos x)^n y^2 + \beta m x^{m-1} - \lambda\beta^2 x^{2m}(\arccos x)^n$.

Particular solution: $y_0 = \beta x^m$.

16. $y'_x = \lambda(\arccos x)^n(y - ax^m - b)^2 + amx^{m-1}$.

Particular solution: $y_0 = ax^m + b$.

17. $xy'_x = \lambda(\arccos x)^n y^2 + ky + \lambda b^2 x^{2k}(\arccos x)^n$.

Solution: $y = bx^k \tan\Bigl[\lambda b \int x^{k-1}(\arccos x)^n\,dx + C\Bigr]$.

18. $xy'_x = (ax^{2n}y^2 + bx^n y + c)(\arccos x)^m - ny$.

The substitution $z = x^n y$ leads to a separable equation:

$$z'_x = x^{n-1}(\arccos x)^m(az^2 + bz + c).$$

► Equations containing arctangent.

19. $y'_x = y^2 + \lambda(\arctan x)^n y - a^2 + a\lambda(\arctan x)^n$.

Particular solution: $y_0 = -a$.

20. $y'_x = y^2 + \lambda x(\arctan x)^n y + \lambda(\arctan x)^n$.

Particular solution: $y_0 = -1/x$.

21. $y'_x = -(k+1)x^k y^2 + \lambda(\arctan x)^n(x^{k+1}y - 1)$.

Particular solution: $y_0 = x^{-k-1}$.

22. $y'_x = \lambda(\arctan x)^n y^2 + ay + ab - b^2\lambda(\arctan x)^n$.

Particular solution: $y_0 = -b$.

23. $y'_x = \lambda(\arctan x)^n y^2 - b\lambda x^m(\arctan x)^n y + bmx^{m-1}$.

Particular solution: $y_0 = bx^m$.

24. $y'_x = \lambda(\arctan x)^n y^2 + bmx^{m-1} - \lambda b^2 x^{2m}(\arctan x)^n$.

Particular solution: $y_0 = bx^m$.

25. $y'_x = \lambda(\arctan x)^n(y - ax^m - b)^2 + amx^{m-1}$.

Particular solution: $y_0 = ax^m + b$.

26. $xy'_x = \lambda(\arctan x)^n y^2 + ky + \lambda b^2 x^{2k}(\arctan x)^n$.

Solution: $y = bx^k \tan\Bigl[\lambda b \int x^{k-1}(\arctan x)^n\,dx + C\Bigr]$.

27. $xy'_x = (ax^{2n}y^2 + bx^n y + c)(\arctan x)^m - ny$.

The substitution $z = x^n y$ leads to a separable equation:

$$z'_x = x^{n-1}(\arctan x)^m(az^2 + bz + c).$$

► **Equations containing arccotangent.**

28. $y'_x = y^2 + \lambda(\operatorname{arccot} x)^n y - a^2 + a\lambda(\operatorname{arccot} x)^n.$

Particular solution: $y_0 = -a$.

29. $y'_x = y^2 + \lambda x(\operatorname{arccot} x)^n y + \lambda(\operatorname{arccot} x)^n.$

Particular solution: $y_0 = -1/x$.

30. $y'_x = -(k+1)x^k y^2 + \lambda(\operatorname{arccot} x)^n (x^{k+1}y - 1).$

Particular solution: $y_0 = x^{-k-1}$.

31. $y'_x = \lambda(\operatorname{arccot} x)^n y^2 + ay + ab - b^2\lambda(\operatorname{arccot} x)^n.$

Particular solution: $y_0 = -b$.

32. $y'_x = \lambda(\operatorname{arccot} x)^n y^2 - b\lambda x^m(\operatorname{arccot} x)^n y + bmx^{m-1}.$

Particular solution: $y_0 = bx^m$.

33. $y'_x = \lambda(\operatorname{arccot} x)^n y^2 + bmx^{m-1} - \lambda b^2 x^{2m}(\operatorname{arccot} x)^n.$

Particular solution: $y_0 = bx^m$.

34. $y'_x = \lambda(\operatorname{arccot} x)^n (y - ax^m - b)^2 + amx^{m-1}.$

Particular solution: $y_0 = ax^m + b$.

35. $xy'_x = \lambda(\operatorname{arccot} x)^n y^2 + ky + \lambda b^2 x^{2k}(\operatorname{arccot} x)^n.$

Solution: $y = bx^k \tan\Big[\lambda b \int x^{k-1}(\operatorname{arccot} x)^n\,dx + C\Big]$.

36. $xy'_x = (ax^{2n}y^2 + bx^n y + c)(\operatorname{arccot} x)^m - ny.$

The substitution $z = x^n y$ leads to a separable equation:

$$z'_x = x^{n-1}(\operatorname{arccot} x)^m(az^2 + bz + c).$$

13.2.8 Equations with Arbitrary Functions

◆ *Notation:* $f = f(x)$ *and* $g = g(x)$ *are arbitrary functions;* a, b, n, *and* λ *are arbitrary parameters.*

► **Equations containing arbitrary functions (but not containing their derivatives).**

1. $y'_x = y^2 + fy - a^2 - af.$

Particular solution: $y_0 = a$.

2. $y'_x = fy^2 + ay - ab - b^2 f.$

Particular solution: $y_0 = b$.

3. $y'_x = y^2 + xfy + f.$

Particular solution: $y_0 = -1/x$.

4. $y'_x = fy^2 - ax^n fy + anx^{n-1}$.

Particular solution: $y_0 = ax^n$.

5. $y'_x = fy^2 + anx^{n-1} - a^2x^{2n}f$.

Particular solution: $y_0 = ax^n$.

6. $y'_x = -(n+1)x^n y^2 + x^{n+1}fy - f$.

Particular solution: $y_0 = x^{-n-1}$.

7. $xy'_x = fy^2 + ny + ax^{2n}f$.

Solution: $$y = \begin{cases} \sqrt{a}\,x^n \tan\Bigl(\sqrt{a}\displaystyle\int x^{n-1}f\,dx + C\Bigr) & \text{if } a > 0, \\ \sqrt{|a|}\,x^n \tanh\Bigl(-\sqrt{|a|}\displaystyle\int x^{n-1}f\,dx + C\Bigr) & \text{if } a < 0. \end{cases}$$

8. $xy'_x = x^{2n}fy^2 + (ax^n f - n)y + bf$.

The substitution $z = x^n y$ leads to a separable equation: $z'_x = x^{n-1}f(x)(z^2 + az + b)$.

9. $y'_x = fy^2 + gy - a^2 f - ag$.

Particular solution: $y_0 = a$.

10. $y'_x = fy^2 + gy + anx^{n-1} - ax^n g - a^2 fx^{2n}$.

Particular solution: $y_0 = ax^n$.

11. $y'_x = fy^2 - ax^n gy + anx^{n-1} + a^2x^{2n}(g - f)$.

Particular solution: $y_0 = ax^n$.

12. $y'_x = ae^{\lambda x}y^2 + ae^{\lambda x}fy + \lambda f$.

Particular solution: $y_0 = -\dfrac{\lambda}{a}e^{-\lambda x}$.

13. $y'_x = fy^2 - ae^{\lambda x}fy + a\lambda e^{\lambda x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

14. $y'_x = fy^2 + a\lambda e^{\lambda x} - a^2e^{2\lambda x}f$.

Particular solution: $y_0 = ae^{\lambda x}$.

15. $y'_x = fy^2 + \lambda y + ae^{2\lambda x}f$.

Solution: $$y = \begin{cases} \sqrt{a}\,e^{\lambda x} \tan\Bigl(\sqrt{a}\displaystyle\int e^{\lambda x}f\,dx + C\Bigr) & \text{if } a > 0, \\ \sqrt{|a|}\,e^{\lambda x} \tanh\Bigl(-\sqrt{|a|}\displaystyle\int e^{\lambda x}f\,dx + C\Bigr) & \text{if } a < 0. \end{cases}$$

16. $y'_x = fy^2 - f(ae^{\lambda x} + b)y + a\lambda e^{\lambda x}$.

Particular solution: $y_0 = ae^{\lambda x} + b$.

17. $y'_x = e^{\lambda x}fy^2 + (af - \lambda)y + be^{-\lambda x}f$.

The substitution $z = e^{\lambda x}y$ leads to a separable equation: $z'_x = f(x)(z^2 + az + b)$.

18. $y'_x = fy^2 + gy + a\lambda e^{\lambda x} - ae^{\lambda x}g - a^2e^{2\lambda x}f.$

Particular solution: $y_0 = ae^{\lambda x}$.

19. $y'_x = fy^2 - ae^{\lambda x}gy + a\lambda e^{\lambda x} + a^2e^{2\lambda x}(g - f).$

Particular solution: $y_0 = ae^{\lambda x}$.

20. $y'_x = fy^2 + 2a\lambda xe^{\lambda x^2} - a^2fe^{2\lambda x^2}.$

Particular solution: $y_0 = ae^{\lambda x^2}$.

21. $y'_x = fy^2 + \lambda xy + afe^{\lambda x^2}.$

Solution: $y = \begin{cases} \sqrt{a}\, e^{\lambda x^2/2} \tan\Bigl(\sqrt{a} \displaystyle\int e^{\lambda x^2/2} f\, dx + C\Bigr) & \text{if } a > 0, \\ \sqrt{|a|}\, e^{\lambda x^2/2} \tanh\Bigl(-\sqrt{|a|} \displaystyle\int e^{\lambda x^2/2} f\, dx + C\Bigr) & \text{if } a < 0. \end{cases}$

22. $y'_x = fy^2 - a\tanh^2(\lambda x)(af + \lambda) + a\lambda.$

Particular solution: $y_0 = a\tanh(\lambda x)$.

23. $y'_x = fy^2 - a\coth^2(\lambda x)(af + \lambda) + a\lambda.$

Particular solution: $y_0 = a\coth(\lambda x)$.

24. $y'_x = fy^2 - a^2f + a\lambda\sinh(\lambda x) - a^2f\sinh^2(\lambda x).$

Particular solution: $y_0 = a\cosh(\lambda x)$.

25. $xy'_x = fy^2 + a - a^2f(\ln x)^2.$

Particular solution: $y_0 = a\ln x$.

26. $xy'_x = f(y + a\ln x)^2 - a.$

Solution: $\dfrac{1}{y + a\ln x} + \displaystyle\int \frac{f(x)}{x}\, dx = C.$

27. $y'_x = fy^2 - ax\ln x fy + a\ln x + a.$

Particular solution: $y_0 = ax\ln x$.

28. $y'_x = -a\ln x\, y^2 + af(x\ln x - x)y - f.$

Particular solution: $y_0 = \dfrac{1}{a(x\ln x - x)}$.

29. $y'_x = \lambda\sin(\lambda x)y^2 + f\cos(\lambda x)y - f.$

Particular solution: $y_0 = 1/\cos(\lambda x)$.

30. $y'_x = fy^2 - a^2f + a\lambda\sin(\lambda x) + a^2f\sin^2(\lambda x).$

Particular solution: $y_0 = -a\cos(\lambda x)$.

31. $y'_x = fy^2 - a^2f + a\lambda\cos(\lambda x) + a^2f\cos^2(\lambda x).$

Particular solution: $y_0 = a\sin(\lambda x)$.

32. $y'_x = fy^2 - a\tan^2(\lambda x)(af - \lambda) + a\lambda.$

Particular solution: $y_0 = a\tan(\lambda x)$.

33. $y'_x = fy^2 - a\cot^2(\lambda x)(af - \lambda) + a\lambda.$

Particular solution: $y_0 = -a\cot(\lambda x)$.

► **Equations containing arbitrary functions and their derivatives.**

34. $y'_x = y^2 - f^2 + f'_x.$

Particular solution: $y_0 = f$.

35. $y'_x = fy^2 - fgy + g'_x.$

Particular solution: $y_0 = g$.

36. $y'_x = -f'_x y^2 + fgy - g.$

Particular solution: $y_0 = 1/f$.

37. $y'_x = g(y-f)^2 + f'_x.$

Particular solution: $y_0 = f$.

38. $y'_x = \dfrac{f'_x}{g} y^2 - \dfrac{g'_x}{f}.$

Particular solution: $y_0 = -g/f$.

39. $f^2 y'_x - f'_x y^2 + g(y-f) = 0.$

Particular solution: $y_0 = f$.

40. $y'_x = f'_x y^2 + ae^{\lambda x} fy + ae^{\lambda x}.$

Particular solution: $y_0 = -1/f$.

41. $y'_x = fy^2 + g'_x y + afe^{2g}.$

Solution: $$y = \begin{cases} \sqrt{a}\, e^g \tan\Bigl(\sqrt{a} \displaystyle\int fe^g\,dx + C\Bigr) & \text{if } a > 0, \\ \sqrt{|a|}\, e^g \tanh\Bigl(-\sqrt{|a|} \displaystyle\int fe^g\,dx + C\Bigr) & \text{if } a < 0. \end{cases}$$

42. $y'_x = y^2 - \dfrac{f''_{xx}}{f}.$

Particular solution: $y_0 = -f'_x/f$.

13.2.9 Some Transformations

◆ *Notation:* f, g, *and* h *are arbitrary composite functions of their argument, which is written in parentheses following the function name (the argument is a function of* x*).*

1. $y'_x = y^2 + a^2 f(ax+b).$

The transformation $\xi = ax + b,\ u = y/a$ leads to the equation $u'_\xi = u^2 + f(\xi)$.

2. $y'_x = y^2 + x^{-4} f(1/x).$

The transformation $\xi = 1/x,\ w = -x^2 y - x$ leads to the equation $w'_\xi = w^2 + f(\xi)$.

3. $\boldsymbol{y'_x = y^2 + \dfrac{1}{(cx+d)^4} f\Big(\dfrac{ax+b}{cx+d}\Big).}$

The transformation

$$\xi = \frac{ax+b}{cx+d}, \quad w = \frac{1}{\Delta}\big[(cx+d)^2 y + c(cx+d)\big], \quad \text{where} \quad \Delta = ad - bc,$$

leads to a simpler equation: $w'_\xi = w^2 + \Delta^{-2} f(\xi)$.

4. $\boldsymbol{x^2 y'_x = x^4 f(x) y^2 + 1.}$

The substitution $u = -\dfrac{1}{x^2 y} - \dfrac{1}{x}$ leads to the equation $u'_x = u^2 + f(x)$.

5. $\boldsymbol{x^2 y'_x = x^2 y^2 + x^{2n} f(ax^n + b) + \frac{1}{4}(1 - n^2).}$

The transformation $\xi = ax^n + b$, $w = \dfrac{1}{an} x^{1-n} y + \dfrac{1-n}{2an} x^{-n}$ leads to a simpler equation: $w'_\xi = w^2 + (an)^{-2} f(\xi)$.

6. $\boldsymbol{y'_x = f(x) y^2 + g(x) y + h(x).}$

The substitution $y = -1/w$ leads to an equation of the same form: $w'_x = h(x)w^2 - g(x)w + f(x)$.

7. $\boldsymbol{y'_x = y^2 + e^{2\lambda x} f(e^{\lambda x}) - \frac{1}{4}\lambda^2.}$

The transformation $\xi = e^{\lambda x}$, $u = \dfrac{1}{\lambda} e^{-\lambda x} y - \dfrac{1}{2} e^{-\lambda x}$ leads to a simpler equation: $u'_\xi = u^2 + \lambda^{-2} f(\xi)$.

8. $\boldsymbol{y'_x = y^2 - \dfrac{\lambda^2}{4} + \dfrac{e^{2\lambda x}}{(ce^{\lambda x}+d)^4} f\Big(\dfrac{ae^{\lambda x}+b}{ce^{\lambda x}+d}\Big).}$

The transformation

$$\xi = \frac{ae^{\lambda x}+b}{ce^{\lambda x}+d}, \quad w = \frac{(ce^{\lambda x}+d)^2}{\Delta \lambda e^{\lambda x}} y + \frac{c^2 e^{2\lambda x} - d^2}{2\Delta e^{\lambda x}}, \quad \text{where} \quad \Delta = ad - bc,$$

leads to a simpler equation: $w'_\xi = w^2 + (\Delta\lambda)^{-2} f(\xi)$.

9. $\boldsymbol{y'_x = y^2 - \lambda^2 + \sinh^{-4}(\lambda x) f\big(\coth(\lambda x)\big).}$

The transformation $\xi = \coth(\lambda x)$, $w = -\lambda^{-1} \sinh^2(\lambda x) y - \frac{1}{2}\sinh(2\lambda x)$ leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

10. $\boldsymbol{y'_x = y^2 - \lambda^2 + \cosh^{-4}(\lambda x) f\big(\tanh(\lambda x)\big).}$

The transformation $\xi = \tanh(\lambda x)$, $w = \lambda^{-1} \cosh^2(\lambda x) y + \frac{1}{2}\sinh(2\lambda x)$ leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

11. $\boldsymbol{x^2 y'_x = x^2 y^2 + f(a \ln x + b) + \frac{1}{4}.}$

The transformation $\xi = a \ln x + b$, $w = \dfrac{1}{a} xy + \dfrac{1}{2a}$ leads to a simpler equation: $w'_\xi = w^2 + a^{-2} f(\xi)$.

12. $y'_x = y^2 + \lambda^2 + \sin^{-4}(\lambda x) f(\cot(\lambda x)).$

The transformation $\xi = \cot(\lambda x)$, $w = -\sin^2(\lambda x)\Big[\dfrac{y}{\lambda} + \cot(\lambda x)\Big]$ leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

13. $y'_x = y^2 + \lambda^2 + \cos^{-4}(\lambda x) f\big(\tan(\lambda x)\big).$

The transformation $\xi = \tan(\lambda x)$, $w = \cos^2(\lambda x)\Big[\dfrac{y}{\lambda} - \tan(\lambda x)\Big]$ leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

14. $y'_x = y^2 + \lambda^2 + \sin^{-4}(\lambda x + b) f\left(\dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}\right).$

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{\sin^2(\lambda x + b)}{\sin(b - a)}\Big[\dfrac{y}{\lambda} + \cot(\lambda x + b)\Big]$ leads to a simpler equation: $w'_\xi = w^2 + [\lambda \sin(b - a)]^{-2} f(\xi)$.

13.3 Abel Equations of the Second Kind

13.3.1 Equations of the Form $yy'_x - y = f(x)$

► **Preliminary remarks. Classification tables.**

For the sake of convenience, listed in Tables 13.1–13.4 are all the Abel equations discussed in Section 13.3. Tables 13.1–13.3 classify Abel equations in which the functions f are of the same form; Table 13.8 gives other Abel equations. In Table 13.1, equations are arranged in accordance with the growth of the parameter m. In Table 13.2, equations are arranged in accordance with the growth of the parameter p. In Table 13.3, equations are arranged in accordance with the growth of the parameter s. The rightmost column of the tables indicates the equation numbers where the corresponding solutions are written out.

TABLE 13.1

Solvable Abel equations of the form $yy'_x - y = sx + Ax^m$, A is an arbitrary parameter

m	s	Equation	m	s	Equation
arbitrary	$-\dfrac{2(m+1)}{(m+3)^2}$	13.3.1.10	-1	0	13.3.1.16
-7	$15/4$	13.3.1.56	$-1/2$	$-2/9$	13.3.1.26
-4	6	13.3.1.54	$-1/2$	$-4/25$	13.3.1.22
$-5/2$	12	13.3.1.47	$-1/2$	0	13.3.1.32
-2	0	13.3.1.33	$-1/2$	20	13.3.1.55
-2	2	13.3.1.19	0	arbitrary	13.3.1.2
$-5/3$	$-3/16$	13.3.1.30	0	0	13.3.1.1
$-5/3$	$-9/100$	13.3.1.23	$1/2$	$-12/49$	13.3.1.53
$-5/3$	$63/4$	13.3.1.48	2	$-6/25$	13.3.1.45
$-7/5$	$-5/36$	13.3.1.27	2	$6/25$	13.3.1.46

TABLE 13.2

Solvable Abel equations of the form $yy'_x - y = sx + \alpha A x^p + \beta A^2 x^q$, A is an arbitrary parameter

p	q	s	α	β	Equation
-1	-3	arbitrary	1	-1	13.3.1.5
-1	-3	$\dfrac{2m+1}{4m^2}$	1	-1	13.3.1.13
-1	-3	0	1	-1	13.3.1.7
$-3/5$	$-7/5$	$-5/36$	arbitrary	arbitrary	13.3.1.62
$-5/11$	$-13/11$	$-33/196$	$286A/3$	$-770A/9$	13.3.1.69
$-1/3$	$-5/3$	$-3/16$	arbitrary	arbitrary	13.3.1.61
$-1/3$	$-5/3$	$-3/16$	3	-12	13.3.1.40
$-1/3$	$-5/3$	$-3/16$	5	-12	13.3.1.15
$-1/3$	$-5/3$	$15/4$	6	-3	13.3.1.60
$-1/5$	$-4/5$	$-10/49$	$13A/5$	$-7A/20$	13.3.1.68
0	$-1/2$	$-2/9$	arbitrary	arbitrary	13.3.1.3
2	3	$4/9$	2	2	13.3.1.14

Given below in this section are all solvable Abel equations known so far. The equations are arranged into groups, in which all solutions are expressed in terms of the same functions. Notation is given before each group.

In most cases the solutions are presented in parametric form:

$$x = F_1(\tau, C), \qquad y = F_2(\tau, C),$$

where τ is the parameter and C is an arbitrary constant.

► Solvable equations and their solutions.

1. $yy'_x - y = A.$

Solution: $x = y - A \ln|y + A| + C.$

2. $yy'_x - y = Ax + B, \qquad A \neq 0.$

Solution in parametric form:

$$x = C \exp\left(-\int \frac{\tau\,d\tau}{\tau^2 - \tau - A}\right) - \frac{B}{A}, \qquad y = C\tau \exp\left(-\int \frac{\tau\,d\tau}{\tau^2 - \tau - A}\right).$$

3. $yy'_x - y = -\frac{2}{9}x + A + Bx^{-1/2}.$

1°. Solution in parametric form with $A > 0$:

$$x = a\left[\frac{(2k-1)C\tau^k - (k-2)\tau - k - 1}{C\tau^k + \tau + 1}\right]^2, \qquad y = -6a\frac{(k-1)^2 C\tau^{k+1} + k^2 C\tau^k + \tau}{C\tau^k + \tau + 1},$$

where $A = \frac{2}{3}a(k^2 - k + 1)$, $B = \frac{2}{3}a^{3/2}(2k-1)(k-2)(k+1)$.

TABLE 13.3

Solvable Abel equations of the form $yy'_x - y = sx + \sigma A(\alpha x^{1/2} + \beta A + \gamma A^2 x^{-1/2})$,
A is an arbitrary parameter

s	σ	α	β	γ	Equation
arbitrary $\neq 0$	arbitrary	0	arbitrary	0	13.3.1.2
$\dfrac{2(m-1)}{(m-3)^2}$	$\dfrac{2}{(m-3)^2}$	$m(m+3)$	$4m^2+3m+9$	$3m(m+3)$	13.3.1.12
$-1/4$	$1/4$	1	5	3	13.3.1.17
$-30/121$	$3/242$	21	35	6	13.3.1.29
$-12/49$	arbitrary	arbitrary	0	0	13.3.1.53
$-12/49$	$1/98$	25	41	10	13.3.1.25
$-12/49$	$6/49$	1	8	5	13.3.1.38
$-12/49$	$2/49$	5	34	15	13.3.1.24
$-12/49$	$4/49$	-10	27	10	13.3.1.31
$-12/49$	$1/49$	5	262	65	13.3.1.52
$-12/49$	$6/49$	-3	23	12	13.3.1.28
$-12/49$	$2/49$	1	166	55	13.3.1.58
$-12/49$	1	$3/49+3B$	$12/49-15B/2$	$15/196+75B/16$	13.3.1.64
$-6/25$	$2/25$	2	19	6	13.3.1.20
$-6/25$	$6/25$	2	7	4	13.3.1.39
$-28/121$	$2/121$	5	106	15	13.3.1.51
$-2/9$	arbitrary	0	arbitrary	arbitrary	13.3.1.3
$-2/9$	arbitrary	0	0	arbitrary	13.3.1.26
$-2/9$	6	0	1	2	13.3.1.11
$-10/49$	$2/49$	4	61	12	13.3.1.57
$-4/25$	arbitrary	0	0	arbitrary	13.3.1.22
$-4/25$	$1/50$	7	49	6	13.3.1.59
0	arbitrary	0	0	arbitrary	13.3.1.32
0	1	1	2	arbitrary	13.3.1.36
0	$n+2$	1	$2(n+2)$	$(n+1)(n+3)$	13.3.1.34
0	$n+2$	1	$2(n+2)$	$2n+3$	13.3.1.35
0	1	-1	2	0	13.3.1.37
0	2	1	4	3	13.3.1.4
0	arbitrary	0	arbitrary	0	13.3.1.1
2	2	-10	19	30	13.3.1.50
2	2	10	31	30	13.3.1.49
20	arbitrary	0	0	arbitrary	13.3.1.55

TABLE 13.4
Other solvable Abel equations of the form $yy'_x - y = f(x)$

Function $f(x)$	Equation
$Ax^{k-1} - kBx^k + kB^2x^{2k-1}$	13.3.1.6 (particular solution)
$Ax^2 - \frac{9}{625}A^{-1}$	13.3.1.44
$\frac{3}{4}x - \frac{3}{2}Ax^{1/3} + \frac{3}{4}A^2x^{-1/3} - \frac{27}{625}A^4x^{-5/3}$	13.3.1.66
$-\frac{6}{25}x + \frac{7}{5}Ax^{1/3} + \frac{31}{3}A^2x^{-1/3} - \frac{100}{3}A^4x^{-5/3}$	13.3.1.67
$-\frac{6}{25}x + ax^{1/3} + b + cx^{-1/3} + dx^{-2/3}$ (coefficients a, b, c, and d are related by an equality)	13.3.1.65
$-\frac{21}{100}x + \frac{7}{9}A^2\bigl(123x^{-1/7} + 280\,Ax^{-5/7} - 400\,A^2x^{-9/7}\bigr)$	13.3.1.70
$\dfrac{k}{\sqrt{Ax^2 + Bx + C}}$	13.3.1.63
$\dfrac{A}{\sqrt{x^2 + 4A}}$	13.3.1.18
$-\dfrac{3}{32}x + \dfrac{9a^2 - 6x^2}{64\sqrt{x^2 + a^2}}$	13.3.1.43
$\dfrac{3}{8}x + \dfrac{6x^2 + 5a^2}{16\sqrt{x^2 + a^2}}$	13.3.1.21
$\dfrac{3}{8}x + \dfrac{6x^2 + 9A}{16\sqrt{x^2 + A}}$	13.3.1.41
$\dfrac{9}{32}x + \dfrac{30x^2 + 33A}{64\sqrt{x^2 + A}}$	13.3.1.42
$A + B\exp(-2x/A)$	13.3.1.8
$A[\exp(2x/A) - 1]$	13.3.1.9
$a^2\lambda e^{2\lambda x} - a(b\lambda + 1)e^{\lambda x} + b$	13.3.1.73 (particular solution)
$a^2\lambda e^{2\lambda x} + a\lambda x e^{\lambda x} + be^{\lambda x}$	13.3.1.74 (particular solution)
$2a^2\lambda\sin(2\lambda x) + 2a\sin(\lambda x)$	13.3.1.75 (particular solution)

2°. Solution in parametric form with $A < 0$:

$$x = \xi[2\lambda e^{-\lambda\tau} - (C_1\lambda - 3C_2\omega)\sin\omega\tau - (3C_1\omega + C_2\lambda)\cos\omega\tau]^2,$$
$$y = 6\xi\bigl\{(C_1^2 + C_2^2)\omega^2 - [C_1(\lambda^2 - \omega^2) - 2C_2\omega\lambda]e^{-\lambda\tau}\sin\omega\tau$$
$$- [2C_1\omega\lambda + C_2(\lambda^2 - \omega^2)]e^{-\lambda\tau}\cos\omega\tau\bigr\},$$

where

$\xi = a(e^{-\lambda\tau} + C_1 \sin\omega\tau + C_2\cos\omega\tau)^{-2}, \quad A = -\tfrac{2}{3}a(3\omega^2 - \lambda^2), \quad B = \tfrac{4}{9}a^{3/2}\lambda(9\omega^2 + \lambda^2).$

3°. For the case $A = 0$, see equation 13.3.1.26.

4. $yy'_x - y = 2A(x^{1/2} + 4A + 3A^2x^{-1/2}).$

Solution in parametric form:

$$x = \tfrac{1}{4}a(3 \pm 2\tau L_\pm)^2, \quad y = \pm a L_\pm (R_\pm^2 L_\pm + \tau), \qquad A = -\tfrac{1}{2}a^{1/2},$$

where

$$L_+ = \begin{cases} \displaystyle\int \frac{d\tau}{1+\tau^2} = \arctan\tau - C, & R_+ = \sqrt{1+\tau^2}, \\ \displaystyle\int \frac{d\tau}{\tau^2 - 1} = \frac{1}{2}\ln\left|\frac{\tau-1}{\tau+1}\right| - C, & R_+ = \sqrt{\tau^2 - 1}, \end{cases}$$

$$L_- = \int \frac{d\tau}{1-\tau^2} = \frac{1}{2}\ln\left|\frac{1+\tau}{1-\tau}\right| - C, \qquad R_- = \sqrt{1-\tau^2}.$$

5. $yy'_x - y = Ax + Bx^{-1} - B^2x^{-3}.$

Solution in parametric form:

$$x = \left(\frac{V}{W}\right)^{-1/2}, \quad y = (\tau+1)\left(\frac{V}{W}\right)^{-1/2} - B\left(\frac{V}{W}\right)^{1/2}.$$

Here,

$$V = \begin{cases} (\tau^2 + \tau - A)\exp\left(\dfrac{2}{\sqrt{-\Delta}}\arctan\dfrac{2\tau+1}{\sqrt{-\Delta}}\right) & \text{if } \Delta < 0, \\ (\tau^2 + \tau - A)\exp\left(-\dfrac{2}{2\tau+1}\right) & \text{if } \Delta = 0, \\ (\tau^2 + \tau - A)\left(\dfrac{2\tau+1-\sqrt{\Delta}}{2\tau+1+\sqrt{\Delta}}\right)^{\frac{1}{\sqrt{\Delta}}} & \text{if } \Delta > 0, \end{cases}$$

$$W = \begin{cases} C + 2B\displaystyle\int \exp\left(\dfrac{2}{\sqrt{-\Delta}}\arctan\dfrac{2\tau+1}{\sqrt{-\Delta}}\right)d\tau & \text{if } \Delta < 0, \\ C + 2B\displaystyle\int \exp\left(-\dfrac{2}{2\tau+1}\right)d\tau & \text{if } \Delta = 0, \\ C + 2B\displaystyle\int \left(\dfrac{2\tau+1-\sqrt{\Delta}}{2\tau+1+\sqrt{\Delta}}\right)^{\frac{1}{\sqrt{\Delta}}} d\tau & \text{if } \Delta > 0, \end{cases}$$

where $\Delta = 4A + 1$.

6. $yy'_x - y = Ax^{k-1} - kBx^k + kB^2x^{2k-1}.$

Particular solution: $y_0 = x - Bx^k - \dfrac{A}{kB}$.

7. $yy'_x - y = Ax^{-1} - A^2x^{-3}.$

Solution in parametric form:

$$x = a\tau^{-1}(\tau - \ln|1+\tau| - C)^{1/2},$$

$$y = a\left[\frac{1+\tau}{\tau}(\tau - \ln|1+\tau| - C)^{1/2} - \frac{1}{2}\tau(\tau - \ln|1+\tau| - C)^{-1/2}\right],$$

where $A = a^2/2$.

8. $yy'_x - y = A + Be^{-2x/A}$.

Solution in parametric form:

$$x = A\ln\left|\frac{\sqrt{\tau^2+AB}}{A\ln\left|\tau+\sqrt{\tau^2+AB}\right|+C}\right|, \qquad y = \tau\frac{A\ln\left|\tau+\sqrt{\tau^2+AB}\right|+C}{\sqrt{\tau^2+AB}} - A.$$

9. $yy'_x - y = A(e^{2x/A} - 1)$.

Solution in parametric form:

$$x = A\ln\left|\frac{\tau^2+1}{\tau}(\arctan\tau - C)\right|, \qquad y = \frac{A}{\tau}\bigl[\tau + (\tau^2-1)(\arctan\tau - C)\bigr].$$

◆ *In the solutions of equations 10–15, the following notation is used:*

$$E_{m,l} = \int (1\pm\tau^{m+1})^{\frac{1}{l-2}}\,d\tau - C, \qquad E_m = E_{m,0} = \int (1\pm\tau^{m+1})^{-1/2}\,d\tau - C,$$
$$R_m = \sqrt{1\pm\tau^{m+1}}, \qquad F_m = R_m E_m - \tau.$$

10. $yy'_x - y = -\dfrac{2(m+1)}{(m+3)^2}x + Ax^m$.

Solution in parametric form:

$$x = \frac{m+3}{m-1}a\tau E_m^{\frac{2}{m-1}}, \qquad y = aE_m^{\frac{2}{m-1}}\left(R_m E_m + \frac{2}{m-1}\tau\right),$$

where $A = \pm\dfrac{m+1}{2}\left(\dfrac{m-1}{m+3}\right)^{m+1} a^{1-m}$.

11. $yy'_x - y = -\frac{2}{9}x + 6A^2(1 + 2Ax^{-1/2}), \qquad A > 0$.

Solution in parametric form:

$$x = A^2R^{-4}E^{-2}(R^2E \pm 6\tau^{1/2})^2, \qquad y = -12A^2R^{-4}E^{-2}(R^2E - 2\tau),$$

where $E = E_{-1/2,\,3/2}$, $R = R_{-1/2}$.

12. $yy'_x - y = \dfrac{2(m-1)}{(m-3)^2}x$
$$+\frac{2A}{(m-3)^2}\bigl[m(m+3)x^{1/2} + (4m^2+3m+9)A + 3m(m+3)A^2x^{-1/2}\bigr].$$

Solution in parametric form:

$$x = \frac{a}{(m-3)^2}\tau^{-2}[(m-3)R_mE_m + 3\tau]^2,$$
$$y = \frac{a}{m-3}\tau^{-2}E_m[\pm(m-1)\tau^{m+1}E_m - 2E_m + 2\tau R_m],$$

where $A = -\dfrac{a^{1/2}}{m-3}$.

13. $yy'_x - y = \dfrac{2m+1}{4m^2}x + Ax^{-1} - A^2x^{-3}.$

Solution in parametric form:

$$x = \frac{E^{1/2}}{a\tau^{1/2}R_m^2}, \quad y = \frac{\tau - \left[1 \mp (2m+1)\tau^{m+1}\right]R_m^2 E}{2am\tau^{1/2}R_m^2 E^{1/2}} \quad \text{with} \quad a^2 = -2mA, \quad E = E_{m,\,3/2}.$$

14. $yy'_x - y = \frac{4}{9}x + 2Ax^2 + 2A^2x^3.$

Solution in parametric form:

$$x = \frac{1}{3A}\tau^{-1}F_3, \qquad y = \frac{1}{9A}\tau^{-2}E_3(\tau R_3 - E_3 \pm \tau^4 E_3).$$

15. $yy'_x - y = -\dfrac{3}{16}x + 5Ax^{-1/3} - 12A^2x^{-5/3}.$

Solution in parametric form:

$$x = a\tau^{1/2}E^{-3/2}F^{3/2}, \qquad y = \tfrac{1}{4}a\tau^{1/2}E^{-3/2}F^{-1/2}(F^2 - 2\tau F - \tau^{-2/3}E^2),$$

where $A = \frac{1}{24}a^{4/3}$, $E = E_{-5/3}$, $F = F_{-5/3}$.

◆ *In the solutions of equations 16–18, the following notation is used:*

$$f = \int \exp(\mp\tau^2)\,d\tau - C, \qquad g = 2\tau\left[\int \exp(\mp\tau^2)\,d\tau - C\right] \pm \exp(\mp\tau^2).$$

16. $yy'_x - y = Ax^{-1}.$

Solution in parametric form:

$$x = af^{-1}\exp(\mp\tau^2), \quad y = af^{-1}\left[\exp(\mp\tau^2) \pm 2\tau f\right], \qquad \text{where} \quad A = \mp 2a^2.$$

17. $yy'_x - y = -\frac{1}{4}x + \frac{1}{4}A(x^{1/2} + 5A + 3A^2x^{-1/2}).$

Solution in parametric form:

$$x = \tfrac{1}{16}a\left[3 \pm 8\tau f\exp(\pm\tau^2)\right]^2, \qquad y = af\exp(\pm\tau^2)\left[(2\tau^2 \pm 1)f\exp(\pm\tau^2) \pm \tau\right],$$

where $A = \frac{1}{4}\sqrt{a}$.

18. $yy'_x - y = \pm\dfrac{2a^2}{\sqrt{x^2 \pm 8a^2}}.$

Solution in parametric form:

$$x = \pm a(fg)^{-1}(g^2 \mp 2f^2), \qquad y = a(fg)^{-1}[\exp(\mp\tau^2)g - 2f^2].$$

◆ *In the solutions of equations 19–21, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\bigl|C\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr)\bigr|, \qquad R = \sqrt{\frac{\tau+1}{\tau}},$$
$$F = 1 - \sqrt{\frac{\tau+1}{\tau}}\ln\bigl|C\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr)\bigr|.$$

19. $yy'_x - y = 2x + Ax^{-2}$.

Solution in parametric form:

$$x = \tfrac{1}{3}aE^{-2/3}\tau, \quad y = aE^{-2/3}\big(\tfrac{2}{3}\tau - RE\big), \qquad \text{where} \quad A = -\tfrac{243}{2}a^3.$$

20. $yy'_x - y = -\frac{6}{25}x + \frac{2}{25}A(2x^{1/2} + 19A + 6A^2x^{-1/2})$.

Solution in parametric form:

$$x = a\tau^{-2}(5RE - 3\tau)^2, \quad y = 5a\tau^{-3}E[(2\tau + 3)E - 2\tau^2 R], \qquad \text{where} \quad A = -\sqrt{a}.$$

21. $yy'_x - y = \frac{3}{8}x + \frac{3}{8}\sqrt{x^2 + a^2} - \dfrac{a^2}{16\sqrt{x^2 + a^2}}$.

Solution in parametric form:

$$x = \frac{a}{2\sqrt{2}}\,\frac{E^2 - 2\tau^2 F}{\tau E\sqrt{F}}, \qquad y = \frac{a}{4\sqrt{2}}\,\frac{4\tau F^2 - E^2}{\tau E\sqrt{F}}.$$

◆ *In the solutions of equations 22–25, the following notation is used:*

$$P_2 = \pm(\tau^2 - 1), \qquad P_3 = \tau^3 - 3\tau + C, \qquad P_4 = \pm(\tau^4 - 6\tau^2 + 4C\tau - 3).$$

22. $yy'_x - y = -\frac{4}{25}x + Ax^{-1/2}$.

Solution in parametric form:

$$x = 5aP_2^2P_3^{-4/3}, \quad y = 4aP_3^{-4/3}(P_2^2 - \tau P_3), \qquad \text{where} \quad A = \pm\tfrac{4}{5}a\sqrt{5a}.$$

23. $yy'_x - y = -\frac{9}{100}x + Ax^{-5/3}$.

Solution in parametric form:

$$x = 10aP_3^{3/2}P_4^{-9/8}, \quad y = 9aP_3^{-1/2}P_4^{-9/8}(P_3^2 - P_2P_4), \qquad \text{where} \quad A = \pm 9a^2(10a)^{2/3}.$$

24. $yy'_x - y = -\frac{12}{49}x + \frac{2}{49}A(5x^{1/2} + 34A + 15A^2x^{-1/2})$.

Solution in parametric form:

$$x = aP_2^{-4}(14\tau P_3 - 9P_2^2)^2, \quad y = 28aP_2^{-4}P_3(4\tau^2 P_3 - 3\tau P_2^2 \mp P_2P_3), \quad \text{where } A = -3\sqrt{a}.$$

25. $yy'_x - y = -\frac{12}{49}x + \frac{1}{98}A(25x^{1/2} + 41A + 10A^2x^{-1/2})$.

Solution in parametric form:

$$x = aP_3^{-4}(21P_2P_4 - 16P_3^2)^2, \quad y = 21aP_3^{-4}P_4(9P_2^2P_4 \mp P_4^2 - 8P_2P_3^2) \ \text{ with } \ A = -8\sqrt{a}.$$

◆ *In the solutions of equations 26–29, the following notation is used:*

$$S_1 = \exp(\sqrt{3}\,\tau) + C\sin\tau, \quad S_2 = 2\exp(\sqrt{3}\,\tau) - C\sin\tau + \sqrt{3}\,C\cos\tau,$$
$$S_3 = 2\exp(\sqrt{3}\,\tau) - C\sin\tau - \sqrt{3}\,C\cos\tau, \quad S_4 = 4S_1S_3 - S_2^2.$$

26. $yy'_x - y = -\frac{2}{9}x + Ax^{-1/2}$.

Solution in parametric form:

$$x = 3aS_1^{-2}S_2^2, \quad y = 2aS_1^{-2}(S_2^2 - 2S_1S_3), \qquad \text{where} \quad A = 16(3a)^{3/2}.$$

27. $\boldsymbol{yy'_x - y = -\frac{5}{36}x + Ax^{-7/5}}$.

Solution in parametric form:

$$x = 48aS_1^{5/2}S_4^{-5/4}, \quad y = 5aS_1^{-1/2}S_4^{-5/4}(8S_1^3 - S_2S_4), \quad \text{where} \quad A = (48a)^{2/5}a^2.$$

28. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + \frac{6}{49}A(-3x^{1/2} + 23A + 12A^2x^{-1/2})}$.

Solution in parametric form:

$$x = aS_2^{-4}(7S_1S_3 - 2S_2^2)^2, \quad y = -7aS_1S_2^{-4}(4S_1^2S_2 - 4S_1S_3^2 + S_2^2S_3), \quad \text{where} \quad A = \sqrt{a}/2.$$

29. $\boldsymbol{yy'_x - y = -\frac{30}{121}x + \frac{3}{242}A(21x^{1/2} + 35A + 6A^2x^{-1/2})}$.

Solution in parametric form:

$$x = aS_1^{-6}(11S_2S_4 - 64S_1^3)^2, \quad y = -11aS_1^{-6}S_4(S_4^2 - 5S_2^2S_4 + 32S_1^3S_2) \quad \text{with} \quad A = -32\sqrt{a}.$$

◆ *In the solutions of equations 30 and 31, the following notation is used:*

$$T_1 = \tanh(\tau + C) + \tan\tau, \quad T_2 = \tanh(\tau + C) - \tan\tau,$$
$$\theta_1 = \cosh\tau - \sin(\tau + C), \quad \theta_2 = \sinh\tau + \cos(\tau + C), \quad \theta_3 = \sinh\tau - \cos(\tau + C).$$

30. $\boldsymbol{yy'_x - y = -\frac{3}{16}x + Ax^{-5/3}}$.

1°. Solution in parametric form with $A < 0$:

$$x = 8aT_1^{-3/2}, \quad y = 3aT_1^{-3/2}(2 - T_1T_2), \quad \text{where} \quad A = -12a^{8/3}.$$

2°. Solution in parametric form with $A > 0$:

$$x = 4a\theta_1^{3/2}\theta_2^{-3/2}, \quad y = 3a\theta_1^{-1/2}\theta_2^{-3/2}(\theta_1^2 - \theta_2\theta_3), \quad \text{where} \quad A = 3a^2(4a)^{2/3}.$$

31. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + \frac{4}{49}A(-10x^{1/2} + 27A + 10A^2x^{-1/2})}$.

1°. Solution in parametric form with $A < 0$:

$$x = a(10 - 7T_1T_2)^2, \quad y = 7aT_1(T_1^3 + 3T_1T_2^2 - 4T_2), \quad \text{where} \quad A = -2\sqrt{a}.$$

2°. Solution in parametric form with $A > 0$:

$$x = a\theta_1^{-4}(7\theta_2\theta_3 - 5\theta_1^2), \quad y = -7a\theta_1^{-4}\theta_2(\theta_2^3 - 3\theta_2\theta_3^2 + 2\theta_1^2\theta_3), \quad \text{where} \quad A = \sqrt{a}.$$

◆ *In the solutions of equations 32–43, the following notation is used:*

$$Z_\nu = \begin{cases} C_1J_\nu(\tau) + C_2Y_\nu(\tau) & \text{for the upper sign,} \\ C_1I_\nu(\tau) + C_2K_\nu(\tau) & \text{for the lower sign,} \end{cases}$$
$$f_\nu = \tau(Z_\nu)'_\tau + \nu Z_\nu, \quad Z = Z_{1/3}, \quad U_1 = \tau Z'_\tau + \tfrac{1}{3}Z,$$
$$U_2 = U_1^2 \pm \tau^2Z^2, \quad U_3 = \pm\tfrac{2}{3}\tau^2Z^3 - 2U_1U_2,$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are Bessel functions, and $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

Remark 13.2. *The solutions of equations 32–43 contain only the ratio Z'_τ/Z, where the prime denotes differentiation with respect to τ. Therefore, for symmetry, function Z is defined in terms of two "arbitrary" constants C_1 and C_2 (instead, we can set, for instance, $C_1 = 1$ and $C_2 = C$).*

32. $yy'_x - y = Ax^{-1/2}$**.**

Solution in parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_1^2, \quad y = a\tau^{-4/3} Z^{-2} U_2, \qquad \text{where} \quad A = \mp\tfrac{1}{3} a^{3/2}.$$

33. $yy'_x - y = Ax^{-2}$**.**

Solution in parametric form:

$$x = 2a\tau^{4/3} Z^2 U_2^{-1}, \quad y = \pm 3a\tau^{-2/3} Z^{-1} U_2^{-1} U_3, \qquad \text{where} \quad A = -36a^3.$$

34. $yy'_x - y = A(n+2)[x^{1/2} + 2(n+2)A + (n+1)(n+3)A^2 x^{-1/2}]$**.**

Solution in parametric form:

$$x = aZ_\nu^{-2}[f_\nu - (\nu+1)Z_\nu]^2, \quad y = aZ_\nu^{-2}(f_\nu^2 - 2\nu Z_\nu f_\nu \pm \tau^2 Z_\nu^2),$$

where $A = \nu\sqrt{a}$, $\nu = \dfrac{1}{n+2}$.

35. $yy'_x - y = A(n+2)[x^{1/2} + 2(n+2)A + (2n+3)A^2 x^{-1/2}]$**.**

Solution in parametric form:

$$x = af_\nu^{-2}[\tau^2 Z_\nu \pm (2-\nu) f_\nu]^2, \quad y = \pm a\tau^2 f_\nu^{-2}[f_\nu^2 + 2(1-\nu)Z_\nu f_\nu \pm \tau^2 Z_\nu^2],$$

where $A = \mp\nu\sqrt{a}$, $\nu = \dfrac{1}{n+2}$.

36. $yy'_x - y = Ax^{1/2} + 2A^2 + Bx^{-1/2}$**.**

Solution in parametric form:

$$x = A^2 Z_\nu^{-2}(\tau Z'_\nu - Z_\nu)^2, \quad y = A^2 Z_\nu^{-2}[\tau^2 (Z'_\nu)^2 - (\nu^2 \mp \tau^2) Z_\nu^2],$$

where $B = (1-\nu^2)A^3$ and the prime denotes differentiation with respect to τ.

37. $yy'_x - y = 2A^2 - Ax^{1/2}$**.**

Solution in parametric form:

$$x = a(Z'_0)^{-2}(\tau Z_0 \pm 2Z'_0)^2, \quad y = \pm a\tau (Z'_0)^{-2}[\tau (Z'_0)^2 + 2Z_0 Z'_0 \pm \tau Z_0^2],$$

where $A = \sqrt{a}$ and the prime denotes differentiation with respect to τ.

38. $yy'_x - y = -\frac{12}{49}x + \frac{6}{49}A(x^{1/2} + 8A + 5A^2 x^{-1/2})$**.**

Solution in parametric form:

$$x = 3aU_1^{-4}(5U_1^2 - 7\tau^2 Z^2)^2, \quad y = 28a\tau^2 Z^2 U_1^{-4}(3\tau^2 Z^2 - ZU_1 - 3U_1^2),$$

where $A = 2\sqrt{3a}$; Z and U_1 are expressed in term of modified Bessel functions.

39. $yy'_x - y = -\frac{6}{25}x + \frac{6}{25}A(2x^{1/2} + 7A + 4A^2x^{-1/2}).$

Solution in parametric form:

$$x = a\tau^{-4}Z^{-6}(U_1U_2 - 2U_3)^2, \quad y = 5a\tau^{-4}Z^{-6}U_2(U_2^2 - U_1U_3), \quad \text{where} \quad A = -\sqrt{a}/2.$$

40. $yy'_x - y = -\frac{3}{16}x + 3Ax^{-1/3} - 12A^2x^{-5/3}.$

Solution in parametric form:

$$x = \frac{af_{3/2}^{3/2}}{\tau^{3/2}Z_{3/2}^{3/2}}, \quad y = \frac{3a}{4}\frac{f_{3/2}^2 - 2Z_{3/2}f_{3/2} - \tau^2 Z_{3/2}^2}{\tau^{3/2}Z_{3/2}^{3/2}f_{3/2}^{1/2}},$$

where $Z_{3/2}$ and $f_{3/2}$ are expressed in terms of modified Bessel functions; $A = \frac{1}{8}a^{4/3}$.

41. $yy'_x - y = \frac{3}{8}x + \frac{3}{8}\sqrt{x^2 \pm b^2} \pm \frac{3b^2}{16\sqrt{x^2 \pm b^2}}.$

Solution in parametric form:

$$x = -\tfrac{1}{4}a\tau^{-1}Z^{-3/2}U_1^{-1/2}U_2^{-1}(2\tau^2Z^3U_1 \mp 3U_2^2),$$
$$y = \mp\tfrac{1}{8}a\tau^{-1}Z^{-3/2}U_1^{-1/2}U_2^{-1}(3U_2^2 - 12U_1^2U_2 \pm 4\tau^2Z^3U_1),$$

where $b^2 = \frac{3}{2}a^2$.

42. $yy'_x - y = \frac{9}{32}x + \frac{15}{32}\sqrt{x^2 \mp b^2} \mp \frac{3b^2}{64\sqrt{x^2 \mp b^2}}.$

Solution in parametric form:

$$x = -\tfrac{1}{2}a\tau^{-1}Z^{-3/2}U_2^{-3/2}U_3^{-1/2}(2\tau^2Z^3U_3 \pm 3U_2^3),$$
$$y = \pm\tfrac{1}{4}a\tau^{-1}Z^{-3/2}U_2^{-3/2}U_3^{-1/2}(3U_2^3 \mp \tau^2Z^3U_3 - 3U_3^2),$$

where $b^2 = 6a^2$.

43. $yy'_x - y = -\frac{3}{32}x - \frac{3}{32}\sqrt{x^2 + a^2} + \frac{15a^2}{64\sqrt{x^2 + a^2}}.$

Solution in parametric form:

$$x = \tfrac{1}{2}aU_2^{-3/2}U_3^{-1}(U_3^2 - U_2^3), \quad y = \tfrac{1}{24}aU_2^{-3/2}U_3^{-1}(3U_3^2 - 12U_2^3 \pm 4\tau^2Z^3U_3).$$

◆ *In the solutions of equations 44–52, the following notation is used:*

$$E_1 = \tau^3\sqrt{\pm(4\wp^3 - 1)} + 3\tau^2\wp \mp 1, \quad E_2 = \tau^2\wp \mp 1,$$
$$E_3 = \sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2, \quad E_4 = \tau\sqrt{\pm(4\wp^3 - 1)} + 2\wp.$$

Here, the function $\wp = \wp(\tau)$ *is given implicitly as follows:* $\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$ *The upper sign in this formula corresponds to the classical elliptic Weierstrass function* $\wp = \wp(\tau + C_2,\, 0,\, 1)$.

44. $yy'_x - y = Ax^2 - \frac{9}{625}A^{-1}$.

Solution in parametric form:

$$x = 5a\left(\tau^2\wp \mp \tfrac{1}{2}\right), \quad y = a\tau^2 E_4, \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

45. $yy'_x - y = -\frac{6}{25}x + Ax^2$.

Solution in parametric form:

$$x = 5a\tau^2\wp, \quad y = a\tau^2 E_4, \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

46. $yy'_x - y = \frac{6}{25}x + Ax^2$.

Solution in parametric form:

$$x = 5aE_2, \quad y = a\tau^2 E_4, \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

47. $yy'_x - y = 12x + Ax^{-5/2}$.

Solution in parametric form:

$$x = a\wp^{-6/7}E_3^{-4/7}, \quad y = a\wp^{-6/7}E_3^{-4/7}(14\wp^2 E_4 - 3), \qquad \text{where} \quad A = \mp 147a^{7/2}.$$

48. $yy'_x - y = \frac{63}{4}x + Ax^{-5/3}$.

Solution in parametric form:

$$x = 2aE_3^{3/2}E_4^{-9/8}, \quad y = aE_3^{-1/2}E_4^{-9/8}(9E_3^2 \mp 16\wp E_4^2), \qquad \text{where} \quad A = -\tfrac{128}{3}a^2(2a)^{2/3}.$$

49. $yy'_x - y = 2x + 2A(10x^{1/2} + 31A + 30A^2x^{-1/2})$.

Solution in parametric form:

$$x = a\wp^{-2}\left[\tau\sqrt{\pm(4\wp^3 - 1)} - 3\wp\right]^2, \quad y = -2a\tau\wp^{-2}\left[\wp\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^3 \pm \tau\right],$$

where $A = \sqrt{a}$.

50. $yy'_x - y = 2x + 2A(-10x^{1/2} + 19A + 30A^2x^{-1/2})$.

Solution in parametric form:

$$x = aE_2^{-2}(E_1 - 6E_2)^2, \quad y = -2aE_2^{-2}(\pm 6E_2^3 - E_1^2 + 7E_1E_2), \qquad \text{where} \quad A = -\sqrt{a}.$$

51. $yy'_x - y = -\frac{28}{121}x + \frac{2}{121}A(5x^{1/2} + 106A + 15A^2x^{-1/2})$.

Solution in parametric form:

$$x = a(22\wp^2 E_4 - 5)^2, \quad y = \pm 44a\wp^2 E_3(7\wp E_3 \mp 2\tau), \qquad \text{where} \quad A = \pm 2\sqrt{a}.$$

52. $yy'_x - y = -\frac{12}{49}x + \frac{1}{49}A(5x^{1/2} + 262A + 65A^2x^{-1/2})$.

Solution in parametric form:

$$x = aE_3^{-4}(28\wp E_4^2 \mp 15E_3^2)^2, \quad y = 56aE_3^{-4}E_4^2(6\wp E_2 + E_4), \qquad \text{where} \quad A = \mp 3\sqrt{a}.$$

◆ *In the solutions of equations 53–60, the following notation is used:*

$$I = \int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - 1)}} \quad (\textit{incomplete elliptic integral of the second kind}),$$

$$R = \sqrt{\pm(4\tau^3 - 1)}, \; I_1 = \tau(2I \mp \tau^{-1}R + C), \; I_2 = \tau^{-1}(RI_1 - 1), \; I_3 = 4\tau I_1^2 \mp I_2^2.$$

53. $yy'_x - y = -\frac{12}{49}x + Ax^{1/2}.$

Solution in parametric form:

$$x = 7a\tau^2(I+C)^{-4}, \quad y = -2a(I+C)^{-4}(IR+CR-2\tau^2), \qquad \text{where} \quad A = \pm\tfrac{12}{49}\sqrt{7a}.$$

54. $yy'_x - y = 6x + Ax^{-4}.$

Solution in parametric form:

$$x = a\tau^{-3/5}I_1^{-2/5}, \quad y = a\tau^{-3/5}I_1^{-2/5}(5RI_1 - 2), \qquad \text{where} \quad A = \mp 150a^5.$$

55. $yy'_x - y = 20x + Ax^{-1/2}.$

Solution in parametric form:

$$x = aI_1^{-4/3}I_2^2, \quad y = -4aI_1^{-4/3}(I_2^2 \mp 9\tau I_1^2), \qquad \text{where} \quad A = \pm 108a^{3/2}.$$

56. $yy'_x - y = \frac{15}{4}x + Ax^{-7}.$

Solution in parametric form:

$$x = aI_1^{1/2}I_3^{-3/8}, \quad y = \tfrac{1}{2}aI_1^{-3/2}I_3^{-3/8}(I_2I_3 - 3I_1^2), \qquad \text{where} \quad A = \pm\tfrac{3}{4}a^8.$$

57. $yy'_x - y = -\frac{10}{49}x + \frac{2}{49}A(4x^{1/2} + 61A + 12A^2x^{-1/2}).$

Solution in parametric form:

$$x = a(7RI_1 - 3)^2, \quad y = 14aI_1[\pm(10\tau^3 - 1)I_1 - R], \qquad \text{where} \quad A = \sqrt{a}.$$

58. $yy'_x - y = -\frac{12}{49}x + \frac{2}{49}A(x^{1/2} + 166A + 55A^2x^{-1/2}).$

Solution in parametric form:

$$x = aI_2^{-4}(42\tau I_1^2 \mp 5I_2^2)^2, \quad y = \mp 84aI_1^2I_2^{-4}(3\tau I_2^2 + I_2 \mp 12\tau^2 I_1^2), \qquad \text{where} \quad A = \pm\sqrt{a}.$$

59. $yy'_x - y = -\frac{4}{25}x + \frac{1}{50}A(7x^{1/2} + 49A + 6A^2x^{-1/2}).$

Solution in parametric form:

$$x = aI_1^{-4}(5I_2I_3 - 16I_1^2)^2, \quad y = -5aI_1^{-4}I_3(\pm 3I_3^2 - I_2^2I_3 + 8I_1^2I_3), \qquad \text{where} \quad A = 8\sqrt{a}.$$

60. $yy'_x - y = \frac{15}{4}x + 6Ax^{-1/3} - 3A^2x^{-5/3}.$

Solution in parametric form:

$$x = 2a\tau^{3/2}I_1^{3/2}I_2^{-3/4}, \quad y = a\tau^{-1/2}I_1^{-1/2}I_2^{-3/4}(2\tau I_2^2 + I_2 - 3\tau^2 I_1^2) \quad \text{with} \quad A = -\tfrac{1}{3}a(2a)^{1/3}.$$

61. $yy'_x - y = -\frac{3}{16}x + Ax^{-1/3} + Bx^{-5/3}.$

The substitution $x = \tau^{-3/2}$ leads to an equation

$$yy'_\tau = -\tfrac{3}{2}\tau^{-5/2}y + \tfrac{9}{32}\tau^{-4} - \tfrac{3}{2}A\tau^{-2} - \tfrac{3}{2}B,$$

coincident with equation 13.3.3.13 when $n = -1/2$, $c = 0$, $b = 3/4$, $d = 3A/2$, $a^2 = -3B$.

62. $yy'_x - y = -\frac{5}{36}x + Ax^{-3/5} - Bx^{-7/5}, \qquad B > 0.$

The transformation $x = \left(w - \frac{1}{3}\sqrt{\tau} + A/B\right)^{-5/4}$, $y = \frac{5}{6}x + \left(\frac{5}{3}B\right)^{1/2}\sqrt{\tau}\,x^{1/5}$ leads to an equation of the form 13.3.1.3:

$$ww'_\tau - w = -\frac{2}{9}\tau + \frac{2A}{3B} + \left(\frac{5}{27B}\right)^{1/2}\frac{1}{\sqrt{\tau}}.$$

63. $yy'_x - y = k(Ax^2 + Bx + C)^{-1/2}.$

The transformation $x = \dfrac{4(b_2w^2 + b_1w + b_0)}{4A - w^2}$, $y = \xi + \dfrac{4(b_2w^2 + b_1w + b_0)}{4A - w^2}$, where parameters b_2, b_1, and b_0 are found from the relations $B = 4Ab_2 - b_0$ and $C = b_1^2 - 4b_0b_2$, leads to a Riccati equation:

$$\pm kw'_\xi = (-\tfrac{1}{4}\xi + b_2)w^2 + b_1w + A\xi + b_0.$$

For $C > 0$, we can set $b_2 = 0$, $b_1 = \sqrt{C}$, and $b_0 = -B$.

In books by Zaitsev & Polyanin (1993, 1994) it is shown that the original equation is reducible to the degenerate hypergeometric equation.

64. $yy'_x - y = -\frac{12}{49}x + 3A(\frac{1}{49} + B)x^{1/2}$
$\qquad + 3A^2(\frac{4}{49} - \frac{5}{2}B) + \frac{15}{4}A^3(\frac{1}{49} + \frac{5}{4}B)x^{-1/2}.$

The substitution $x = (\xi^2 + \frac{5}{4}A)^2$ leads to an equation of the form 13.3.3.13 with $n = 3$, $a = 4/7$, $c = 0$, $b = A$, and $d = 12A(\frac{2}{7} - B)$:

$$yy'_\xi = (4\xi^2 + 5A)\xi y - \left[\tfrac{48}{49}\xi^4 + 12A\left(\tfrac{2}{7} - B\right)\xi^2 + 3A^2\right]\xi^3.$$

65. $yy'_x - y = -\frac{6}{25}x$
$\qquad + \frac{4}{75}B^2\left[(2-A)x^{1/3} - \frac{3}{2}B(2A+1) + B^2(1-3A)x^{-1/3} - AB^3x^{-2/3}\right].$

The transformation $x = w^{-3}$, $y = \left[\xi + \dfrac{B^2(3 - 2Bw)}{5w(Bw+1)}\right]\left(w + \dfrac{1}{B}\right)^2 w^{-2}$ leads to a Riccati equation:

$$\left(2\xi^2 - \tfrac{2}{5}B^3\xi + \tfrac{4}{25}AB^6\right)w'_\xi = B\xi w^2 + \left(\xi - \tfrac{2}{5}B^3\right)w + \tfrac{3}{5}B^2.$$

66. $yy'_x - y = \frac{3}{4}x - \frac{3}{2}Ax^{1/3} + \frac{3}{4}A^2x^{-1/3} - \frac{27}{625}A^4x^{-5/3}.$

The transformation

$$x = A^{3/2}f^{-3/2}, \quad y = 3A^{3/2}\left(\xi f^2 - \tfrac{3}{25}f^2 - \tfrac{1}{2}f + \tfrac{1}{2}\right)f^{-3/2}, \quad \text{where} \quad f = w\left(\xi^2 - \tfrac{6}{25}\xi\right)^{-1},$$

leads to an equation of the form 13.3.1.46: $ww'_\xi - w = \frac{6}{25}\xi - \xi^2$.

67. $yy'_x - y = -\frac{6}{25}x + \frac{7}{5}Ax^{1/3} + \frac{31}{3}A^2x^{-1/3} - \frac{100}{3}A^4x^{-5/3}.$

Denote $A = \frac{7}{100}a$ and perform the transformation

$$x = \xi^{3/2}, \quad y = \tfrac{7}{20}\left(w + \tfrac{8}{7}\xi - \tfrac{3}{5}a - \tfrac{7}{50}a^2\xi^{-1}\right)\sqrt{\xi}, \quad \text{where} \quad \xi = z - \tfrac{3}{10}a.$$

As a result we obtain an equation of the form 13.3.4.30 with $n = \frac{1}{7}$, $c = -\frac{3}{10}a$:

$$\left[\left(z - \tfrac{3}{10}a\right)w + \tfrac{8}{7}z^2 - \tfrac{9}{7}az + \tfrac{1}{7}a^2\right]w'_z = -\tfrac{1}{2}w^2 + 2zw.$$

68. $yy'_x - y = -\frac{10}{49}x + \frac{13}{5}A^2x^{-1/5} - \frac{7}{20}A^3x^{-4/5}$.

Denote $A = 8a^{-2}$. The transformation

$$\sqrt{\tau} = a^3\left(\tfrac{5}{112}x^{3/5} - \tfrac{1}{16}yx^{-2/5}\right), \qquad w = \tfrac{4}{7}\tau + x^{-3/5} - \tfrac{39}{42}a^2$$

leads to an equation of the form 13.3.1.64 with $B = -1/49$:

$$ww'_\tau - w = -\tfrac{12}{49}\tau + \tfrac{39}{98}a^2 - \tfrac{15}{784}a^3\tau^{-1/2}.$$

69. $yy'_x - y = -\frac{33}{196}x + \frac{286}{3}A^2x^{-5/11} - \frac{770}{9}A^3x^{-13/11}$.

Denote $A = \frac{8}{5}a^{-2}$. The transformation

$$\sqrt{\tau} = \tfrac{15}{448}a^3\left(x^{8/11} - \tfrac{14}{11}yx^{-3/11}\right), \qquad w = \tfrac{3}{7}\tau + x^{-8/11} - \tfrac{39}{56}a^2$$

leads to an equation of the form 13.3.1.64 with $B = -1/49$:

$$ww'_\tau - w = -\tfrac{12}{49}\tau + \tfrac{39}{98}a^2 - \tfrac{15}{784}a^3\tau^{-1/2}.$$

70. $yy'_x - y = -\frac{21}{100}x + \frac{7}{9}A^2(123x^{-1/7} + 280Ax^{-5/7} - 400A^2x^{-9/7})$.

Denote $A = 1/a$. The transformation

$$x = \xi^{-7/4}, \quad y = \tfrac{35}{3}a^{-2}\left(w + 4\xi + \tfrac{7}{5}a + \tfrac{3}{50}a^2\xi^{-1}\right)\xi^{-3/4}, \qquad \text{where} \quad \xi = z - \tfrac{21}{20}a,$$

leads to an equation of the form 13.3.4.30 with $n = 3$, $c = -\frac{21}{20}a$:

$$\left[\left(z - \tfrac{21}{20}a\right)w + 4z^2 - 7az + 3a^2\right]w'_z = \tfrac{3}{4}w^2 + 2zw.$$

71. $yy'_x - y = ax + bx^m$.

1°. For $m \neq 3$, the transformation

$$\tau = B^2\left[(m-3)\frac{y}{x} + 1\right]^2, \quad w = 2(m-3)B^2\left(bx^{m-1} - \frac{y^2}{x^2} + \frac{y}{x} + a\right)$$

leads to an equation

$$ww'_\tau - w = \frac{2(m-1)}{(m-3)^2}\left\{\tau - mB\tau^{1/2} + [2m-3-a(m-3)^2]B^2 + [2-m+a(m-3)^2]B^3\tau^{-1/2}\right\}.$$

2°. Let $m \neq 1$ and $a > -1/4$. Denote

$$a = -\frac{(n+2)(n+m+1)}{(2n+m+3)^2}, \qquad \text{where} \quad n = n_{1,2} = \frac{1}{2}\left(\pm\frac{m-1}{\sqrt{1+4a}} - m - 3\right).$$

Then the transformation

$$x = \xi^{\frac{n+2}{m-1}}w, \quad y = \frac{m-1}{2n+m+3}\xi^{\frac{n+2}{m-1}}\left(\xi w'_\xi + \frac{n+2}{m-1}w\right), \qquad n = n_{1,2}$$

reduce the original equation to the classical Emden–Fowler equation $w''_{\xi\xi} = A\xi^n w^m$, where $A = b\left(\dfrac{2n+m+3}{m-1}\right)^2$, which is discussed below in Section 14.3.

72. $yy'_x - y = -\dfrac{m+1}{(m+2)^2}x + Ax^{2m+1} + Bx^{3m+1}.$

Denote $A = -\dfrac{am}{2(m+2)^2b^2}$, $B = -\dfrac{m^2}{2(m+2)^3b^2}$. The transformation

$$\sqrt{\tau} = -\frac{(m+2)^2}{m}byx^{-m-1} + \frac{m+2}{m}bx^{-m}, \quad w = \frac{2(m+1)}{m+2}\tau + x^m + \frac{m+2}{m}a$$

leads to the equation

$$ww'_\tau - w = \frac{2m(m+1)}{(m+2)^2}\tau + a + b\tau^{-1/2}$$

(see Table 13.3 with $\alpha = 0$ in Section 13.3.1).

73. $yy'_x - y = a^2\lambda e^{2\lambda x} - a(b\lambda + 1)e^{\lambda x} + b.$

Particular solution: $y_0 = ae^{\lambda x} - b.$

74. $yy'_x - y = a^2\lambda e^{2\lambda x} + a\lambda x e^{\lambda x} + be^{\lambda x}.$

Particular solution: $y_0 = ae^{\lambda x} + x + \dfrac{b}{a\lambda}.$

75. $yy'_x - y = 2a^2\lambda\sin(2\lambda x) + 2a\sin(\lambda x).$

Particular solution: $y_0 = -2a\sin(\lambda x).$

76. $yy'_x - y = a^2 f'_x f''_{xx} - \dfrac{(f+b)^2}{(f'_x)^3}f''_{xx}, \qquad f = f(x).$

Particular solutions: $y_1 = af'_x + \dfrac{f+b}{f'_x}$, $y_2 = -af'_x + \dfrac{f+b}{f'_x}.$

13.3.2 Equations of the Form $yy'_x = f(x)y + 1$

1. $yy'_x = (ax+b)y + 1.$

The substitution $\xi = y - \frac{1}{2}ax^2 - bx$ leads to a Riccati equation with respect to $x = x(\xi)$: $x'_\xi = \frac{1}{2}ax^2 + bx + \xi.$

2. $yy'_x = (ax+b)^{-2}y + 1.$

The substitution $a\xi = -(ax+b)^{-1}$ leads to an equation of the form 13.3.1.33: $yy'_\xi = y + (a\xi)^{-2}.$

3. $yy'_x = \left(a - \dfrac{1}{ax}\right)y + 1.$

The substitution $\xi = y - ax$ leads to a Bernoulli equation: $\xi x'_\xi + a\xi x + a^2x^2 = 0.$

4. $yy'_x = (ax+b)^{-1/2}y + 1.$

The substitution $z = \dfrac{2}{a}(ax+b)^{1/2}$ leads to an equation of the form 13.3.1.2: $yy'_z = y + \frac{1}{2}az.$

5. $yy'_x = 3(ax^{3/2} + 8x)^{-1/2}y + 1.$

The substitution $z = 12a^{-1}(ax^{1/2} + 8)^{1/2}$ leads to an equation of the form 13.3.1.10 with $m = 3$: $yy'_z = y - \frac{2}{9}z + \frac{1}{5184}a^2z^3.$

6. $yy'_x = (ax^{-2/3} - \frac{2}{3}a^{-1}x^{-1/3})y + 1.$

The transformation $x = a^{3/2}w^3$, $y = \xi - w^2$ leads to a Riccati equation: $3a^{3/2}\xi w'_\xi = \xi - w^2$.

7. $yy'_x = ae^{\lambda x}y + 1.$

The substitution $\xi = \dfrac{a}{\lambda}e^{\lambda x}$ leads to an equation of the form 13.3.1.16: $yy'_\xi = y + (\lambda\xi)^{-1}$.

8. $yy'_x = (ae^{\lambda x} + be^{-\lambda x})y + 1.$

The transformation $\xi = y + \dfrac{b}{\lambda}e^{-\lambda x} - \dfrac{a}{\lambda}e^{\lambda x}$, $w = e^{\lambda x}$ leads to a Riccati equation: $w'_\xi = aw^2 + \lambda\xi w - b$.

9. $yy'_x = ay\cosh x + 1.$

This is a special case of equation 13.3.3.75 with $b = 0$ and $c = 1$.

10. $yy'_x = ay\sinh x + 1.$

This is a special case of equation 13.3.3.76 with $b = 0$ and $c = 1$.

11. $yy'_x = a\cos(\lambda x)\,y + 1.$

The transformation $x = -\dfrac{2}{\lambda}\arctan\dfrac{4u}{\lambda}$, $y = \tau - \dfrac{8au}{16u^2 + \lambda^2}$ leads to a Riccati equation: $u'_\tau = -2\tau u^2 + au - \frac{1}{8}\lambda^2\tau$.

12. $yy'_x = a\sin(\lambda x)\,y + 1.$

The substitution $x = \xi + \dfrac{\pi}{2\lambda}$ leads to a similar equation of the form 13.3.2.11: $yy'_\xi = a\cos(\lambda\xi)\,y + 1$.

13.3.3 Equations of the Form $yy'_x = f_1(x)y + f_0(x)$

► **Preliminary remarks.**

With the aid of the substitution $\xi = \int f_1(x)\,dx$, these equations are reducible to the form

$$yy'_\xi = y + f(\xi), \qquad \text{where} \quad f(\xi) = f_0(x)/f_1(x), \tag{1}$$

and by means of the substitution $z = \int f_0(x)\,dx$, they can be reduced to the form

$$yy'_z = g(z)y + 1, \qquad \text{where} \quad g(z) = f_1(x)/f_0(x). \tag{2}$$

Specific equations of the form (1) and (2) are outlined in Sections 13.3.1 and 13.3.2, respectively.

► **Solvable equations and their solutions.**

1. $yy'_x = (ax + 3b)y + cx^3 - abx^2 - 2b^2x.$

The substitution $y = x^2t + bx$ leads to a linear equation with respect to $x = x(t)$: $(-2t^2 + at + c)x'_t = tx + b$.

2. $yy'_x = (3ax+b)y - a^2x^3 - abx^2 + cx.$

The substitution $y = xw + ax^2$ leads to a Bernoulli equation with respect to $x = x(w)$: $(-w^2 + bw + c)x'_w = wx + ax^2$.

3. $2yy'_x = (7ax+5b)y - 3a^2x^3 - 2cx^2 - 3b^2x.$

This is a special case of equation 13.3.3.11 with $m = 3/2$, $k = 1/2$.

4. $yy'_x = [(3-m)x - 1]y + (m-1)(x^3 - x^2 - ax).$

The transformation $x = w/z$, $y = -z^{m-1} + x^2 - x - a$ leads to an equation $ww'_z = w + az + z^m$ whose solvable cases are outlined in Section 13.3.1 (see Table 13.1).

5. $yy'_x + x(ax^2+b)y + x = 0.$

The substitution $z = -\frac{1}{2}x^2$ leads to an equation of the form 13.3.2.1 with respect to $y = y(z)$: $yy'_z = (-2az+b)y + 1$.

6. $yy'_x + a(1-x^{-1})y = a^2.$

Solution in parametric form:

$$x = -(\tau + e^{-\tau}E^{-1} + \ln E), \quad y = -a(\tau + e^{-\tau}E^{-1}), \qquad \text{where} \quad E = \int \frac{e^{-\tau}\,d\tau}{\tau} + C.$$

7. $yy'_x - a(1-bx^{-1})y = a^2b.$

Solution in parametric form:

$$x = \tfrac{1}{2}b\exp(\mp\tau^2)f^{-1}, \quad y = \mp\tfrac{1}{2}abf^{-1}\bigl[2\tau^2 f \pm \exp(\mp\tau^2)\bigr],$$

where $f = \displaystyle\int \exp(\mp\tau^2)\,\frac{d\tau}{\tau} + C$.

8. $yy'_x = x^{n-1}[(1+2n)x + an]y - nx^{2n}(x+a).$

The transformation $x = \dfrac{w}{z}$, $y = -\dfrac{1}{z^n} + x^{n+1} + ax^n$ leads to a separable equation: $w'_z = w^{-n} - a$.

9. $yy'_x = a(x-nb)x^{n-1}y + c[x^2 - (2n+1)bx + n(n+1)b^2]x^{2n-1}.$

The substitution $\xi = ax^n\Bigl(\dfrac{x}{n+1} - b\Bigr)$ leads to an Abel equation of the form 13.3.1.2: $yy'_\xi = y + (n+1)ca^{-2}\xi$.

10. $yy'_x = [a(2n+k)x^k + b]x^{n-1}y + (-a^2nx^{2k} - abx^k + c)x^{2n-1}.$

The substitution $y = x^n(w + ax^k)$ leads to a Bernoulli equation with respect to $x = x(w)$: $(nw^2 - bw - c)x'_w = -wx - ax^{k+1}$.

11. $yy'_x = [a(2m+k)x^{2k} + b(2m-k)]x^{m-k-1}y$
$$- (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}.$$

The transformation $z = x^k$, $y = x^m(t + ax^k + bx^{-k})$ leads to a Riccati equation with respect to $z = z(t)$:

$$(-mt^2 + 2abm - c)z'_t = akz^2 + ktz + bk. \tag{1}$$

The substitution $z = \dfrac{mt^2 + c_0}{ak}\dfrac{w'_t}{w}$, where $c_0 = c - 2abm$, reduces equation (1) to a second-order linear equation:

$$(mt^2 + c_0)^2 w''_{tt} + (2m + k)t(mt^2 + c_0)w'_t + abk^2 w = 0. \tag{2}$$

The transformation $\xi = \dfrac{t}{\sqrt{t^2 + (c_0/m)}}$, $u = (1 - \xi^2)^{\mu/2} w$, where $\mu = -\dfrac{m + k}{2m}$, brings equation (2) to the Legendre equation 2.1.2.226:

$$(1 - \xi^2)u''_{\xi\xi} - 2\xi u'_\xi + [\nu(\nu + 1) - \mu^2(1 - \xi^2)^{-1}]u = 0,$$

where ν is a root of the quadratic equation $\nu^2 + \nu + \dfrac{m^2 - k^2}{4m^2} - \dfrac{abk^2}{mc_0} = 0.$

12. $\boldsymbol{yy'_x = [(m + 2l - 3)x + n - 2l + 3]x^{-l}y + [(m + l - 1)x^2 + (n - m - 2l + 3)x - n + l - 2]x^{1-2l}.}$

The transformation $x = \dfrac{\xi}{w}w'_\xi$, $y = A\xi^{n-l+2}w^{m+l-1} - x^{2-l} + x^{1-l}$ leads to the generalized Emden–Fowler equation $w''_{\xi\xi} = A\xi^n w^m (w'_\xi)^l$, which is discussed in Section 14.5.

13. $\boldsymbol{yy'_x = [a(2n + 1)x^2 + cx + b(2n - 1)]x^{n-2}y - (na^2x^4 + acx^3 + dx^2 + bcx + nb^2)x^{2n-3}.}$

Here, a, b, c, d, and n are arbitrary numbers.

The substitution $y = x^n t + ax^{n+1} + bx^{n-1}$ leads to a Riccati equation with respect to $x = x(t)$: $(-nt^2 + ct - d + 2nab)x'_t = ax^2 + tx + b$.

14. $\boldsymbol{yy'_x = [a(n - 1)x + b(2\lambda + n)]x^{\lambda-1}(ax + b)^{-\lambda-2}y - [anx + b(\lambda + n)]x^{2\lambda-1}(ax + b)^{-2\lambda-3}.}$

The substitution $y = \left[\dfrac{1}{w} + \dfrac{1}{x^n(ax + b)}\right]x^{\lambda+n}(ax + b)^{-\lambda}$ leads to an equation of the form 13.3.4.5: $(w + ax^{n+1} + bx^n)w'_x = [anx^n + b(\lambda + n)x^{n-1}]w$.

15. $\boldsymbol{yy'_x - a\big[(m - 1)x + 1\big]x^{-1}y = a^2x^{-1}(mx + 1)(x - 1).}$

Solution in parametric form:

$$x = \frac{(m - 1)(\tau^{m+1} + 1)}{\tau}E + \ln\Big(\frac{\tau^{m+1} + 1}{\tau}E\Big), \qquad y = a\big[1 + (m\tau^m - \tau^{-1})E\big],$$

where $E = \displaystyle\int \frac{d\tau}{\tau^{m+1} + 1} + C.$

16. $\boldsymbol{yy'_x - a(1 - bx^{-1/2})y = a^2bx^{-1/2}.}$

Solution in parametric form:

$$x = \mp b^2\tau^2 Z^{-2}(Z'_\tau)^2, \qquad y = \pm ab^2\tau^2 Z^{-2}\big[(Z'_\tau)^2 \pm Z^2\big],$$

where

$$Z = \begin{cases} C_1 J_0(\tau) + C_2 Y_0(\tau) & \text{for the upper sign,} \\ C_1 I_0(\tau) + C_2 K_0(\tau) & \text{for the lower sign,} \end{cases}$$

$J_0(\tau)$ and $Y_0(\tau)$ are Bessel functions, and $I_0(\tau)$ and $K_0(\tau)$ are modified Bessel functions.

17. $yy'_x = 3(ax+b)^{-1/3}x^{-5/3}y + 3(ax+b)^{-2/3}x^{-7/3}.$

The substitution $w = \dfrac{1}{xy} + \dfrac{1}{3}\Big(\dfrac{ax+b}{x}\Big)^{1/3}$ leads to a separable equation for $w = w(x)$: $w'_x = x^{-1/3}(ax+b)^{-2/3}(\frac{1}{9}a - 3w^3)$.

18. $3yy'_x = (-7Ax + 6s - 2\lambda)x^{-1/3}y + 6(\lambda sx - 1)x^{-2/3}$
$$+\,2(Ax+5\lambda)(-Ax+3s+4\lambda)x^{1/3}, \qquad A = \lambda s(3s+4\lambda).$$

The transformation $x = (\xi + \lambda s)^{-1}$, $y = (w + 4\lambda + 3s - Ax)x^{2/3}$ leads to an equation of the form 13.3.4.10 with $a = 1/3$: $[(\xi+\lambda s)w + (4\lambda+3s)\xi]w'_\xi = \frac{2}{3}w^2 + 2(3\lambda+s)w + 2\xi$.

◆ *In the solutions of equations 19 and 20, the following notation is used:*

$$f = \sqrt{\tau(\tau+1)} - \ln\big|C\big(\sqrt{\tau} + \sqrt{\tau+1}\,\big)\big|, \qquad g = 1 - \sqrt{\frac{\tau+1}{\tau}}\,\ln\big|C\big(\sqrt{\tau} + \sqrt{\tau+1}\,\big)\big|.$$

19. $yy'_x + \frac{1}{2}a(6x-1)x^{-1}y = -\frac{1}{2}a^2(x-1)(4x-1)x^{-1}.$

Solution in parametric form:

$$x = \tau f^{-2}g^2, \qquad y = a(1 - \tau f^{-2}g^2 - \tau^2 f^{-2}g).$$

20. $yy'_x - \frac{1}{2}a(1+2bx^{-2})y = \frac{1}{16}a^2(3x+4bx^{-1}).$

Solution in parametric form:

$$x = c\tau^{-1}fg^{-1/2}, \qquad y = -\tfrac{1}{4}ac\tau^{-1}f^{-1}g^{-1/2}(f^2 - 4\tau g^2), \qquad b = -c^2.$$

◆ *In the solutions of equations 21–23, the following notation is used:*

$$E = \exp(3\tau), \quad S_1 = E + C\sin(\sqrt{3}\,\tau), \quad S_2 = 2E - C\sin(\sqrt{3}\,\tau) + \sqrt{3}C\cos(\sqrt{3}\,\tau),$$
$$S_3 = 2S_1(S_2)'_\tau - (S_1)'_\tau S_2 - S_1S_2, \quad S_4 = 2S_1(S_3)'_\tau - 5(S_1)'_\tau S_3 + S_1S_3.$$

21. $yy'_x + \frac{1}{14}a(13x-20)x^{-9/7}y = -\frac{3}{14}a^2(x-1)(x-8)x^{-11/7}.$

Solution in parametric form:

$$x = 64S_1^3S_2S_3^{-2}, \qquad y = a(4S_1)^{-6/7}S_2^{-2/7}S_3^{-10/7}(S_3^2 - 64S_1^3S_2 + 7S_2^2S_3).$$

22. $yy'_x + \frac{5}{56}a(23x-16)x^{-9/7}y = -\frac{3}{56}a^2(x-1)(25x-32)x^{-11/7}.$

Solution in parametric form:

$$x = -\tfrac{256}{25}S_1^3S_3^{-3}S_4, \qquad y = a\big(\tfrac{256}{25}S_1^3S_4\big)^{-2/7}S_3^{-15/7}\big(S_3^3 + 7S_4^2 + \tfrac{256}{25}S_1^3S_4\big).$$

23. $yy'_x + \frac{1}{26}a(19x+85)x^{-18/13}y = -\frac{3}{26}a^2(x-1)(x+25)x^{-23/13}.$

Solution in parametric form:

$$x = -25S_3^3S_4^{-2}, \qquad y = a(25S_3^3)^{-5/13}S_4^{-16/13}(S_4^2 + 25S_3^3 - 208S_1^3S_4).$$

◆ *In the solutions of equations 24–27, the following notation is used:*

$$T_1 = \cosh(\tau+C)\cos\tau, \qquad T_2 = \tanh(\tau+C) + \tan\tau, \quad T_3 = \tanh(\tau+C) - \tan\tau,$$
$$\theta_1 = \cosh\tau - \sin(\tau+C), \quad \theta_2 = \sinh\tau + \cos(\tau+C), \quad \theta_3 = \sinh\tau - \cos(\tau+C),$$
$$T_4 = 3T_2T_3, \quad \theta_4 = 3\theta_2\theta_3 - 2\theta_1^2.$$

24. $yy'_x + \frac{1}{15}a(13x-18)x^{-7/5}y = -\frac{4}{15}a^2(x-1)(x-6)x^{-9/5}.$

1°. Solution in parametric form with $A<0$:

$$x = -12T_1^{-3}T_2, \qquad y = a(12T_2)^{-2/5}T_1^{-9/5}(T_1^3 - 5T_1T_2^2 + 12T_2).$$

2°. Solution in parametric form with $A>0$:

$$x = 6\theta_1^2\theta_2^{-3}\theta_3, \qquad y = a(6\theta_1^2\theta_3)^{-2/5}\theta_2^{-9/5}(\theta_2^3 + 5\theta_2\theta_3^2 - 6\theta_1^2\theta_3).$$

25. $yy'_x + \frac{1}{2}a(5x+1)x^{-1/2}y = a^2(1-x^2).$

Solution in parametric form:

$$x = T_1^2T_2^{-2}, \qquad y = -aT_2^{-3}(T_1^3 - T_1T_2^2 + 4T_2).$$

26. $yy'_x + \frac{3}{35}a(19x-14)x^{7/5}y = -\frac{4}{35}a^2(x-1)(9x-14)x^{9/5}.$

1°. Solution in parametric form with $A<0$:

$$x = -\tfrac{28}{9}T_1^4T_3, \qquad y = a\big(\tfrac{28}{9}T_1^6T_3\big)^{-2/5}\big(T_1^4 - \tfrac{5}{9}T_3^2 + \tfrac{28}{9}T_3\big).$$

2°. Solution in parametric form with $A>0$:

$$x = \tfrac{14}{9}\theta_1^2\theta_2^4\theta_4, \qquad y = \big(\tfrac{14}{9}\theta_1^2\theta_4\big)^{-2/5}\big(\theta_2^4 + \tfrac{5}{9}\theta_4^2 - \tfrac{14}{9}\theta_1^2\theta_4\big).$$

27. $yy'_x + \frac{3}{10}a(3x+7)x^{-13/10}y = -\frac{1}{5}a^2(x-1)(x+9)x^{-8/5}.$

1°. Solution in parametric form with $A<0$:

$$x = 9T_1^4T_3^{-2}, \qquad y = a(9T_1^4)^{-3/10}T_3^{-7/5}(T_3^2 - 20T_3 - 9T_1^4).$$

2°. Solution in parametric form with $A>0$:

$$x = -\tfrac{9}{2}\theta_2^4\theta_4^2, \qquad y = -4a\big(\tfrac{9}{2}\theta_2^4\big)^{-3/10}\theta_4^{-7/5}\big(\theta_4^2 - 5\theta_1^2\theta_4 + \tfrac{9}{2}\theta_2^4\big).$$

◆ *In the solutions of equations 28–30, the following notation is used:*

$$P_2 = \pm(\tau^2-1), \quad P_3 = \tau^3 - 3\tau + C, \quad P_4 = \pm(\tau^4 - 6\tau^2 + 4C\tau - 3),$$
$$P_6 = \pm(\tau^6 - 15\tau^4 + 20C\tau^3 - 45\tau^2 + 12C\tau - 8C^2 + 27).$$

28. $yy'_x + \frac{1}{10}a(7x-12)x^{-7/5}y = -\frac{1}{10}a^2(x-1)(x-16)x^{-9/5}.$

Solution in parametric form:

$$x = \pm 16P_2P_3^2P_4^{-2}, \qquad y = a(16P_2P_3^2P_4^3)^{-2/5}(P_4^2 \pm 15P_2^2P_4 \mp 16P_2P_3^2).$$

29. $yy'_x + \frac{3}{20}a(13x-8)x^{-7/5}y = -\frac{1}{20}a^2(x-1)(27x-32)x^{-9/5}.$

Solution in parametric form:

$$x = \tfrac{32}{27}P_3^2P_4^{-3}P_6, \qquad y = \pm a(3P_4)^{-9/5}P_3^{-4/5}P_6^{-2/5}\big(\tfrac{5}{4}P_6^2 \mp 8P_3^2P_6 \pm \tfrac{27}{4}P_4^3\big).$$

30. $yy'_x + \frac{3}{14}a(3x+11)x^{-10/7}y = -\frac{1}{14}a^2(x-1)(x-27)x^{-13/7}.$

Solution in parametric form:

$$x = \mp 27P_4^3P_6^{-2}, \qquad y = \mp a(3P_4)^{-9/7}P_6^{-8/7}(P_6^2 \mp 28P_3^2P_6 \pm 27P_4^3).$$

◆ *In the solutions of equations 31–38, the following notation is used:*

$$I=\int\frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3-1)}}\quad\text{(incomplete elliptic integral of the second kind)},$$

$$R=\sqrt{\pm(4\tau^3-1)},\qquad I_1=\tau(2I\mp C^{-1}R+C),\qquad I_2=\tau^{-1}(RI_1-1),$$

$$I_3=4\tau I_1^2\mp I_2^2,\qquad I_4=I_2I_3-8I_1^2,\qquad I_5=2R(I+C)-\tau^2.$$

31. $\boldsymbol{yy'_x-\frac{1}{2}a(x+1)x^{-7/4}y=\frac{1}{4}a^2(x-1)(3x+5)x^{-5/2}.}$

Solution in parametric form:

$$x=\tfrac{1}{6}\tau^{-3}I_1^{-1}R,\qquad y=\tfrac{1}{3}a\big(\tfrac{1}{6}\tau R\big)^{-3/4}I_1^{-1/4}\big[(11\tau^3-2)I_1-\tfrac{1}{2}R\big].$$

The lower sign is taken in the notation adopted.

32. $\boldsymbol{yy'_x-\frac{1}{2}a(x+1)x^{-7/4}y=\frac{1}{4}a^2(x-1)(x+5)x^{-5/2}.}$

Solution in parametric form:

$$x=-\tfrac{1}{3}\tau^{-2}I_1^2I_2,\qquad y=-\tfrac{1}{3}a(\tau I_1)^{-1/2}\big(\tfrac{1}{3}I_2\big)^{-3/4}(2\tau I_2^2+I_2-3\tau^2I_1^2).$$

The lower sign is taken in the notation adopted.

33. $\boldsymbol{yy'_x-\frac{1}{14}a(4x+3)x^{-8/7}y=-\frac{1}{14}a^2(x-1)(16x+5)x^{-9/7}.}$

Solution in parametric form:

$$x=\pm\tfrac{3}{16}I_1^{-2}I_2^{-1}I_3^2,\qquad y=-\tfrac{1}{16}a\big(\tfrac{13}{16}I_1^{12}I_2^6I_3^2\big)^{-1/7}(3I_3^2\pm7I_2^2I_3\mp16I_1^2I_2).$$

34. $\boldsymbol{yy'_x+\frac{1}{6}a(13x-3)x^{-2/3}y=-\frac{1}{6}a^2(x-1)(5x-1)x^{-1/3}.}$

Solution in parametric form:

$$x=\mp I_3^3I_4^{-2},\qquad y=-aI_3I_4^{-8/3}(I_3^3\pm I_4^2\pm4I_1^2I_4).$$

35. $\boldsymbol{yy'_x-\frac{1}{28}a(8x-1)x^{-8/7}y=\frac{1}{28}a^2(x-1)(32x+3)x^{-9/7}.}$

Solution in parametric form:

$$x=\mp I_1^{-2}I_3^3I_4^{-1},\qquad y=-\tfrac{1}{32}a\big(\tfrac{3}{32}I_1^{12}I_3^3I_4^6\big)^{-1/7}(3I_3^3\pm7I_4^2\pm32I_1^2I_4).$$

36. $\boldsymbol{yy'_x-a(5x-4)x^{-4}y=a^2(x-1)(3x-1)x^{-7}.}$

Solution in parametric form:

$$x=\pm\tfrac{1}{6}\tau^{-1}(I+C)^{-1}R,\qquad y=36a(I+C)^2R^{-3}[(1\pm2\tau^3)(I+C)-\tau^2R].$$

37. $\boldsymbol{yy'_x-\frac{2}{5}a(3x-10)x^{-4}y=\frac{1}{5}a^2(x-1)(8x-5)x^{-7}.}$

Solution in parametric form:

$$x=\pm\tfrac{5}{24}\tau^{-1}(I+C)^{-2}I_5,\qquad y=\pm\tfrac{576}{125}a(I+C)^4I_5^{-3}[I_5^2+5\tau^2I_5\mp\tau^3(I+C)^2].$$

38. $\boldsymbol{yy'_x+\frac{1}{42}a(39x-4)x^{-9/7}y=-\frac{1}{42}a^2(x-1)(9x-16)x^{-11/7}.}$

Solution in parametric form:

$$x=\pm16\tau^3(I+C)^2I_5^{-2},\qquad y=\tfrac{1}{3}a[16\tau^3(I+C)^2I_5^5]^{-2/7}[3I_5^2+7\tau^2I_5\mp48\tau^3(I+C)^2].$$

◆ *In the solutions of equations 39–41, the following notation is used:*

$$f=\int \exp(\mp\tau^2)\,d\tau+C, \qquad g=2\tau f\pm\exp(\mp\tau^2).$$

39. $yy'_x+a(x-2)x^{-1}y=2a^2(x-1)x^{-1}$.

Solution in parametric form:

$$x=\tau\exp(\mp\tau^2)f, \qquad y=a\bigl[1\mp 2\tau^2-\tau\exp(\mp\tau^2)f^{-1}\bigr].$$

40. $yy'_x+a(3x-2)x^{-1}y=-2a^2(x-1)^2x^{-1}$.

Solution in parametric form:

$$x=\tfrac{1}{2}\exp(\mp\tau^2)f^{-2}g, \qquad y=\tfrac{1}{2}a\bigl[2-\exp(\mp\tau^2)f^{-2}g\mp f^{-2}g^{-2}\bigr].$$

41. $yy'_x+a(1-bx^{-2})y=a^2bx^{-1}$.

Solution in parametric form:

$$x=\frac{1}{\sqrt{\mp 2b}}fg^{-1}, \qquad y=\frac{a}{\sqrt{\mp 8b}}f^{-1}g^{-1}\bigl[g\exp(\mp\tau^2)-2f^2\bigr].$$

◆ *In the solutions of equations 42–52, the following notation is used:*

$$\begin{aligned}
E_1&=\tau^3\sqrt{\pm(4\wp^3-1)}+3\tau^2\wp\mp 1, & E_2&=\tau^2\wp\mp 1,\\
E_3&=\sqrt{\pm(4\wp^3-1)}\pm 2\tau\wp^2, & E_4&=\tau\sqrt{\pm(4\wp^3-1)}+2\wp,\\
E_5&=\tau^3\sqrt{\pm(4\wp^3-1)}-4\tau^2\wp\pm 6, & E_6&=\tau\sqrt{\pm(4\wp^3-1)}-\wp.
\end{aligned}$$

Here the function $\wp=\wp(\tau)$ *is defined implicitly as follows:* $\tau=\displaystyle\int\frac{d\wp}{\sqrt{\pm(4\wp^3-1)}}-C$. *The upper sign in the above relations corresponds to the classical Weierstrass elliptic function* $\wp=\wp(\tau+C,\,0,\,1)$.

42. $yy'_x-\frac{1}{4}a(3x-4)x^{-5/2}y=\frac{1}{4}a^2(x-1)(x+2)x^{-4}$.

Solution in parametric form:

$$x=\tfrac{2}{3}\tau^{-2}\wp^{-2}E_6, \qquad y=-\tfrac{1}{2}a\tau\bigl(\tfrac{2}{3}E_6^3\bigr)^{-1/2}(E_6^2+2\wp E_6-3\tau^2\wp^3).$$

The upper sign is taken in the notation adopted.

43. $yy'_x+\frac{1}{30}a(33x+2)x^{-6/5}y=-\frac{1}{30}a^2(x-1)(9x-4)x^{-7/5}$.

Solution in parametric form:

$$x=4\tau^2\wp^3E_6^{-2}, \qquad y=\pm\tfrac{1}{3}a(4\tau^2\wp^3E_6^8)^{-1/5}(3E_6^2+5\wp E_6\mp 12\tau^2\wp^3).$$

44. $yy'_x-\frac{1}{8}a(x-8)x^{-5/2}y=-\frac{1}{8}a^2(x-1)(3x-4)x^{-4}$.

Solution in parametric form:

$$x=\tfrac{4}{3}E_1E_2^{-2}, \qquad y=\tfrac{1}{4}a\bigl(\tfrac{4}{3}E_1^3\bigr)^{-1/2}(E_1^2-4E_1E_2+3E_2^3).$$

The lower sign is taken in the notation adopted.

45. $yy'_x + \frac{1}{30}a(17x+18)x^{-22/15}y = -\frac{1}{30}a^2(x-1)(x+4)x^{-29/15}.$

Solution in parametric form:

$$x = \pm 4E_1^{-2}E_2^3, \quad y = \pm aE_1^{-16/15}(4E_2)^{-7/15}(E_1^2 - 5E_1E_2 \mp 4E_2^3).$$

46. $yy'_x - \frac{1}{13}a(6x-13)x^{-5/2}y = -\frac{1}{26}a^2(x-1)(x-13)x^{-4}.$

Solution in parametric form:

$$x = \tfrac{13}{6}E_2^{-2}E_5, \quad y = -\tfrac{1}{13}a\big(\tfrac{13}{6}E_5^3\big)^{-1/2}(2E_5^2 \pm 13E_2E_5 - 6E_2^3).$$

The upper sign is taken in the notation adopted.

47. $yy'_x + \frac{1}{30}a(24x+11)x^{27/20}y = -\frac{1}{60}a^2(x-1)(9x+1)x^{-17/10}.$

Solution in parametric form:

$$x = 4E_2^3E_5^{-2}, \quad y = \tfrac{1}{3}a(4E_2^3)^{-7/20}E_5^{-13/10}(3E_5^2 + 20E_2E_5 - 12E_2^3).$$

The upper sign is taken in the notation adopted.

48. $yy'_x - \frac{2}{5}a(3x+2)x^{-8/5}y = \frac{1}{5}a^2(x-1)(8x+1)x^{-11/5}.$

Solution in parametric form:

$$x = \mp\tfrac{1}{3}\wp^{-1}E_3^{-2}E_4, \quad y = \mp a(3\wp E_3^2)^{-2/5}E_4^{-3/5}(3\wp E_3^2 \mp \wp^2E_4^2 \pm E_4).$$

49. $yy'_x - \frac{6}{5}a(4x+1)x^{-7/5}y = \frac{1}{5}a^2(x-1)(27x+8)x^{-9/5}.$

Solution in parametric form:

$$x = -E_2E_3^2E_4^{-3}, \quad y = aE_2^{-2/5}E_3^{-3/5}E_4^{-9/5}(E_4^3 + E_2E_3^2 - 10E_2^2).$$

50. $yy'_x + \frac{3}{10}a(13x-3)x^{-4/5}y = -\frac{1}{10}a^2(x-1)(27x-7)x^{-3/5}.$

Solution in parametric form:

$$x = 2E_2^2E_4^{-3}, \quad y = a(4E_2^{-1}E_4^9)^{-2/5}(2E_4^3 - 5E_2E_3^2 + 4E_2^2).$$

51. $yy'_x - \frac{1}{5}a(x+4)x^{-8/5}y = \frac{1}{5}a^2(x-1)(3x+7)x^{-11/5}.$

Solution in parametric form:

$$x = \tfrac{1}{3}E_3^{-1}\wp^{-3}\sqrt{\pm(4\wp^3-1)},$$
$$y = -\tfrac{1}{6}aE_3^{-2/5}\Big[\tfrac{1}{3}\wp^2\sqrt{\pm(4\wp^3-1)}\Big]^{-3/5}\Big[14\wp^3E_3 + 2\sqrt{\pm(4\wp^3-1)}\Big].$$

52. $yy'_x - a(2x-1)x^{-5/2}y = \frac{1}{2}a^2(x-1)(3x+1)x^{-4}.$

Solution in parametric form:

$$x = \tfrac{1}{6}\tau^{-1}\wp^{-2}\sqrt{4\wp^3-1}, \quad y = -a\big[6\tau\big(\sqrt{4\wp^3-1}\,\big)^{-3}\big]^{1/2}\big(\wp\sqrt{4\wp^3-1} + 2\tau\wp^3 - 2\tau\big).$$

The upper sign is taken in the notation adopted.

◆ *In the solutions of equations 53–55, the following notation is used:*

$$Q_1 = \pm\tau^2 + C\tau - 1, \quad Q_2 = \tau^2 \pm 1, \quad Q_3 = \tau^3 \pm 3\tau + C.$$

53. $yy'_x + \frac{1}{5}a(x-6)x^{-7/5}y = \frac{2}{5}a^2(x-1)(x+4)x^{-9/5}.$

Solution in parametric form:

$$x = \pm 3\tau Q_2^2 Q_3^{-1}, \quad y = a(3\tau Q_2^2)^{-2/5} Q_3^{-3/5}\bigl[(1 \pm 5\tau^2)Q_3 \mp 3\tau Q_2^2\bigr].$$

54. $yy'_x + \frac{1}{5}a(21x+19)x^{-7/5}y = -\frac{2}{5}a^2(x-1)(9x-4)x^{-9/5}.$

Solution in parametric form:

$$x = \pm Q_1 Q_2^2 Q_3^{-2}, \quad y = \pm a Q_1^{-2/5} Q_2^{-4/5} Q_3^{-6/5}(Q_3^2 \mp Q_1 Q_2^2 \pm Q_1^2).$$

55. $yy'_x - 3ax^{-7/4}y = \frac{1}{4}a^2(x-1)(x-9)x^{-5/2}.$

Solution in parametric form:

$$x = Q_1^{-2} Q_3^2, \quad y = -a Q_1^{-1/2} Q_3^{-3/2}(Q_3^2 + Q_1 Q_2^2 - Q_1^2).$$

The lower sign is taken in the notation adopted.

◆ *In the solutions of equations 56 and 57, the following notation is used:*

$$h_k = \int \tau^{\frac{k}{k-2}} \exp(\mp\tau^2)\,d\tau + C.$$

56. $yy'_x - a\bigl[(k+1)x - 1\bigr]x^{-2}y = a^2(k+1)(x-1)x^{-2}.$

Solution in parametric form:

$$x = \frac{2}{k+1}\tau^{-\frac{2}{k+1}} \exp(\pm\tau^2) h_\beta, \quad y = a\Bigl[\frac{k+1}{2}\tau^{\frac{2}{k+1}} \exp(\mp\tau^2) h_\beta^{-1} \pm (k+1)\tau^2 - 1\Bigr],$$

where $\beta = \dfrac{k-1}{k}$.

57. $yy'_x - a[(k-2)x + 2k - 3]x^{-k}y = a^2(k-2)(x-1)^2 x^{1-2k}.$

Solution in parametric form:

$$x = \mp 2\tau^{\frac{2}{2-k}} \exp(\pm\tau^2) h_k,$$
$$y = a(\mp 2h_k)^{1-k} \exp\bigl[\mp(k-2)\tau^2\bigr]\Bigl[\tau^{\frac{2(1-k)}{2-k}} \exp(\mp\tau^2) + \Bigl(\frac{2}{2-k} \pm 4\tau^2\Bigr) h_k\Bigr].$$

58. $yy'_x - \frac{1}{2}a[(4k-7)x - 4k + 5]x^{-k}y = \frac{1}{2}a^2(2k-3)(x-1)^2 x^{1-2k}.$

Solution in parametric form:

$$x = (\tau Z)^2 U^{-2}, \quad y = \tfrac{1}{2}a(\tau Z)^{-3/5} U^{-7/5}(2U^2 + 5ZU - \tau^2 Z^2),$$

where $Z = C_1 I_\nu(\tau) + C_2 K_\nu(\tau)$, $U = \tau Z'_\tau + \nu Z$, $\nu = \dfrac{1-k}{3-2k}$; $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

◆ *In the solutions of equations 59 and 60, the following notation is used:*

$$N_n = \int \frac{d\tau}{\tau^n + a} + C.$$

59. $\boldsymbol{yy'_x - \big[(2n-1)x - an\big]x^{-n-1}y = n(x-a)x^{-2n}.}$

1°. Solution in parametric form:

$$x = \tau N_n^{-1}, \qquad y = \tau^{-n} N_n^{n-1}[(\tau^n + a)N_n - \tau].$$

2°. Particular solution: $y_0 = (a-x)x^{-n}$.

60. $\boldsymbol{yy'_x - \big[(n+1)x - an\big]x^{n-1}(x-a)^{-n-2}y = nx^{2n}(x-a)^{-2n-3}.}$

1°. Solution in parametric form:

$$x = \tau N_n^{-1}, \qquad y = (\tau - aN_n)^{-n-1}[\tau - (\tau^n + a)N_n].$$

2°. Particular solution: $y_0 = -x^n(x-a)^{-n-1}$.

61. $\boldsymbol{yy'_x - a\big[(2k-3)x + 1\big]x^{-k}y = a^2(k-2)[(k-1)x+1]x^{2(1-k)}.}$

Solution in parametric form:

$$x = \frac{\tau + 1}{(1-k)(2-k)} E^{k-2}\Big[(1-k)(2k-3)(\tau+1)^{\frac{1}{1-k}} + (2-k)E\Big],$$
$$y = \frac{a}{1-k}\Big[\tau E - (1-k)(\tau+1)^{\frac{2-k}{1-k}}\Big],$$

where $E = \displaystyle\int \frac{1}{\tau}(\tau+1)^{\frac{1}{1-k}}\,d\tau + C.$

◆ *In the solutions of equations 62–66, the following notation is used:*

$$R_n = \sqrt{1 \pm \tau^{n+1}}, \qquad E_n = \int \big(1 \pm \tau^{n+1}\big)^{-1/2}\,d\tau + C,$$
$$F_n = R_n E_n - \tau, \qquad E_{nk} = \int \big(1 \pm \tau^{n+1}\big)^{\frac{1}{k-2}}\,d\tau + C.$$

62. $\boldsymbol{yy'_x - a\big[(n+2k-3)x + 3 - 2k\big]x^{-k}y}$
$\boldsymbol{= a^2\big[(n+k-1)x^2 - (n+2k-3)x + k - 2\big]x^{1-2k}.}$

Solution in parametric form:

$$x = \tau^{-1} E_{nk} R_n^{\frac{2}{2-k}}, \qquad y = a\tau^{k-2} E_{nk}^{1-k}\bigg(\pm\frac{n+1}{2-k}\tau^{n+1}E_{nk} - E_{nk}R_n^2 + \tau R_n^{\frac{2(1-k)}{2-k}}\bigg).$$

63. $\boldsymbol{yy'_x - \frac{a}{n}\big[(n+2)x - 2\big]x^{-\frac{2n+1}{n}}y = \frac{a^2}{n}\big[(n+1)x^2 - 2x - n + 1\big]x^{-\frac{3n+2}{n}}.}$

Solution in parametric form:

$$x = \pm 2E_n^{-1}R_n, \qquad y = \pm 2^{-\frac{n+1}{n}} aE_n^{\frac{1}{n}} R_n^{-\frac{n+1}{n}}(E_n \pm 2R_n).$$

64. $yy'_x - \frac{a}{n}\Big(\frac{n+4}{n+2}x - 2\Big)x^{-\frac{2n+1}{n}}y$
$$= \frac{a^2}{n(n+2)}\big[2x^2 + (n^2+n-4)x - (n-1)(n+2)\big]x^{-\frac{3n+2}{n}}.$$

Solution in parametric form:

$$x = 2b\tau^{-n}E_n^{-2}F_n, \quad y = a(2b)^{-\frac{1}{n}}E_n^{\frac{2}{n}}F_n^{-\frac{1+n}{n}}\Big(-\frac{n}{n+2}F_n^2 - \tau F_n + \frac{1}{bn}\tau^{n+1}E_n^2\Big),$$

where $b = \pm\dfrac{n+2}{n+1}$.

65. $yy'_x + \frac{a}{n+3}\Big(\frac{3n+5}{2}x + \frac{n-1}{n+1}\Big)x^{-\frac{n+4}{n+3}}y$
$$= -\frac{a^2}{2(n+3)}\Big[(n+1)x^2 - \frac{n^2+2n+5}{n+1}x + \frac{4}{n+1}\Big]x^{-\frac{n+5}{n+3}}.$$

Solution in parametric form:

$$x=\tau^{n+1}E_n^2F_n^{-2}, \quad y=a\tau^{-\frac{n+1}{n+3}}E_n^{-\frac{2}{n+3}}F_n^{-\frac{2(n+2)}{n+3}}\Big[F_n^2-\tau^{n+1}E_n^2+(\pm1)^{\frac{1}{n+3}}\frac{n+3}{n+1}\tau F_n\Big].$$

66. $yy'_x - a\Big(\frac{n+2}{n} + bx^n\Big)y = -\frac{a^2}{n}x\Big(\frac{n+1}{n} + bx^n\Big).$

Solution in parametric form:

$$x = c\tau E_{m1}^{1/n}, \quad y = acE_{n1}^{1/n}\Big(E_{n1}R_n^2 + \frac{1}{n}\tau\Big).$$

67. $yy'_x = (ae^x + b)y + ce^{2x} - abe^x - b^2.$

The transformation $x = \ln w$, $y = tw + b$ leads to a linear equation: $(-t^2 + at + c)w'_t = tw + b$.

68. $yy'_x = [a(2\mu+\lambda)e^{\lambda x} + b]e^{\mu x}y + (-a^2\mu e^{2\lambda x} - abe^{\lambda x} + c)e^{2\mu x}.$

The substitution $z = e^x$ leads to an equation of the form 13.3.3.10:

$$yy'_z = [a(2\mu+\lambda)z^\lambda + b]z^{\mu-1}y + (-a^2\mu z^{2\lambda} - abz^\lambda + c)z^{2\mu-1}.$$

69. $yy'_x = (ae^{\lambda x} + b)y + c[a^2e^{2\lambda x} + ab(\lambda x+1)e^{\lambda x} + b^2\lambda x].$

The substitution $\xi = \dfrac{a}{\lambda}e^{\lambda x} + bx$ leads to an equation of the form 13.3.1.2: $yy'_\xi = y + c\lambda\xi$.

70. $yy'_x = e^{\lambda x}(2a\lambda x + a + b)y - e^{2\lambda x}(a^2\lambda x^2 + abx + c).$

The substitution $y = e^{\lambda x}(\xi + ax)$ leads to a linear equation with respect to $x = x(\xi)$: $(-\lambda\xi^2 + b\xi - c)x'_\xi = ax + \xi$.

71. $yy'_x = e^{ax}(2ax^2 + 2x + b)y + e^{2ax}(-ax^4 - bx^2 + c).$

The substitution $y = e^{ax}(\xi + x^2)$ leads to a Riccati equation with respect to $x = x(\xi)$: $(-a\xi^2 + b\xi + c)x'_\xi = x^2 + \xi$.

72. $yy'_x + a(1+2bx)e^{bx}y = -a^2bx^2e^{2bx}$.

Solution in parametric form:

$$x = \frac{2}{b}\exp(\pm\tau^2)f, \quad y = -\frac{a}{2b}\tau^{-2}\bigl[2\tau^2\exp(\pm\tau^2)f \pm 1\bigr]\exp[2\exp(\pm\tau^2)f],$$

where $f = \int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C$.

73. $yy'_x - a\bigl[1+2n+2n(n+1)x\bigr]e^{(n+1)x}y = -a^2n(n+1)(1+nx)xe^{2(n+1)x}$.

Solution in parametric form:

$$x = \Bigl(2n\tau^nE + \frac{1}{n+1}\Bigr)\exp\bigl[(n+1)\tau^nE\bigr], \quad y = a\tau^n\Bigl(\frac{\tau}{1\pm\tau^n} + nE\Bigr)\exp\bigl[(n+1)\tau^nE\bigr],$$

where $E = \int (1\pm\tau^{n+1})^{-1/2}\,d\tau + C$.

74. $yy'_x + a(1+2bx^{1/2})\exp(2bx^{1/2})y = -a^2bx^{3/2}\exp(4bx^{1/2})$.

Solution in parametric form:

$$x = c\tau^{-4}Z^{-2}U^2, \quad y = -ac\tau^{-4}Z^{-2}(U^2 \pm \tau^2Z^2)\exp(-2b\tau^{-2}Z^{-1}U).$$

Here,

$$b = (\mp c)^{-1/2}, \quad U = \tau Z'_\tau + Z, \quad Z = \begin{cases} C_1J_1(\tau) + C_2Y_1(\tau) & \text{for the upper sign,} \\ C_1I_1(\tau) + C_2K_1(\tau) & \text{for the lower sign,} \end{cases}$$

where $J_1(\tau)$ and $Y_1(\tau)$ are Bessel functions, and $I_1(\tau)$ and $K_1(\tau)$ are modified Bessel functions.

75. $yy'_x = (a\cosh x + b)y - ab\sinh x + c$.

The transformation $t = y - a\sinh x$, $\xi = e^x$ leads to a Riccati equation: $2(bt+c)\xi'_t = a\xi^2 + 2t\xi - a$.

76. $yy'_x = (a\sinh x + b)y - ab\cosh x + c$.

The transformation $t = y - a\cosh x$, $\xi = e^x$ leads to a Riccati equation: $2(bt+c)\xi'_t = a\xi^2 + 2t\xi + a$.

77. $yy'_x = (2\ln x + a + 1)y + x(-\ln^2 x - a\ln x + b)$.

The transformation $x = e^w$, $y = (\xi + w)e^w$ leads to a linear equation: $(-\xi^2 + a\xi + b)w'_\xi = w + \xi$.

78. $yy'_x = (2\ln^2 x + 2\ln x + a)y + x(-\ln^4 x - a\ln^2 x + b)$.

The transformation $x = e^w$, $y = (z + w^2)e^w$ leads to a Riccati equation: $(-z^2 + az + b)w'_z = w^2 + z$.

79. $yy'_x = ax\cos(\lambda x^2)\,y + x$.

The substitution $z = \frac{1}{2}x^2$ leads to an Abel equation of the form 13.3.2.11: $yy'_z = a\cos(2\lambda z)y + 1$.

80. $yy'_x = ax\sin(\lambda x^2)\,y + x$.

The substitution $z = \frac{1}{2}x^2$ leads to an Abel equation of the form 13.3.2.12: $yy'_z = a\sin(2\lambda z)y + 1$.

13.3.4 Equations of the Form $[g_1(x)y + g_0(x)]y'_x = f_2(x)y^2 + f_1(x)y + f_0(x)$

► **Preliminary remarks.**

With the aid of the substitution

$$w = \Big(y + \frac{g_0}{g_1}\Big)E, \qquad \text{where} \quad E = \exp\Big(-\int \frac{f_2}{g_1}\,dx\Big), \tag{1}$$

these equations are reducible to a simpler form:

$$ww'_x = F_1(x)w + F_0(x), \tag{2}$$

where

$$F_1 = \left[\frac{d}{dx}\Big(\frac{g_0}{g_1}\Big) + \frac{f_1}{g_1} - 2\frac{g_0 f_2}{g_1^2}\right]E, \qquad F_0 = \Big(\frac{f_0}{g_1} - \frac{g_0 f_1}{g_1^2} + \frac{g_0^2 f_2}{g_1^3}\Big)E^2.$$

Specific Abel equations of the form (2) are outlined in Sections 13.3.1–13.3.3. In the degenerate cases with $F_0 \equiv 0$ or $F_1 \equiv 0$, the variables in Eq. (2) are separable.

► **Solvable equations and their solutions.**

1. $(Ay + Bx + a)y'_x + By + kx + b = 0.$

Solution: $Ay^2 + kx^2 + 2(Bxy + ay + bx) = C.$

2. $(y + ax + b)y'_x = \alpha y + \beta x + \gamma.$

The substitution $y = u - ax - b$ leads to the equation

$$uu'_x = (a + \alpha)u + (\beta - a\alpha)x + \gamma - b\alpha$$

which is separable with $a = -\alpha$. For $a \neq -\alpha$, the substitution $u = (a + \alpha)w$ leads to an equation of the form 13.3.1.1 or 13.3.1.2:

$$ww'_x = w + \Delta^{-2}(\beta - a\alpha)x + \Delta^{-2}(\gamma - b\alpha), \qquad \text{where} \quad \Delta = a + \alpha.$$

3. $(y + akx^2 + bx + c)y'_x = -ay^2 + 2akxy + my + k(k + b - m)x + s.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[-az^2 + (m - k)z + s - ck]x'_z = akx^2 + (b + k)x + z + c.$$

4. $(y + Ax^n + a)y'_x + nAx^{n-1}y + kx^m + b = 0.$

Solution: $y^2 + \dfrac{2k}{m+1}x^{m+1} + 2(Ax^n y + ay + bx) = C.$

5. $(y + ax^{n+1} + bx^n)y'_x = (anx^n + cx^{n-1})y.$

The substitution $y = x^n(w - b)$ leads to a Bernoulli equation with respect to $x = x(w)$: $[-nw^2 + (bn + c)w - bc]x'_w = wx + ax^2.$

6. $xyy'_x = ay^2 + by + cx^n + s.$

The transformation $\xi = x^{-a}$, $w = -\dfrac{a}{b}x^{-a}y$ leads to an equation $ww'_\xi = w + A\xi + B\xi^m$, where $A = -ab^{-2}s$, $B = -ab^{-2}c$, $m = (a - n)/a$ (see Section 13.3.1).

7. $xyy'_x = -ny^2 + a(2n+1)xy + by - a^2nx^2 - abx + c.$

The substitution $y = w + ax$ leads to a Bernoulli equation with respect to $x = x(w)$: $(-nw^2 + bw + c)x'_w = wx + ax^2$.

8. $2xyy'_x = (1-n)y^2 + [a(2n+1)x + 2n - 1]y - a^2nx^2 - bx - n.$

The transformation $x = \xi^2$, $y = \xi t + a\xi^2 + 1$ leads to a Riccati equation:

$$(-nt^2 + 2an - b)\xi'_t = a\xi^2 + t\xi + 1.$$

9. $(Axy - Aky + Bx - Bk)y'_x = Cy^2 + Dxy + (B - Dk)y.$

The transformation $x = w + k$, $y = \xi w$ leads to a linear equation with respect to $w = w(x)$: $[(C-A)\xi^2 + D\xi]w'_\xi = A\xi w + B$.

10. $[(3ax + \lambda s)y + (4\lambda + 3s)x]y'_x = 2ay^2 + 2(3\lambda + s)y + 2x.$

The substitution $w = ay^2 + (3\lambda + s)y + x$ leads to an Abel equation of the form 13.3.3.3: $2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y$, where $b = s + 2\lambda$, $c = \frac{1}{2}a(13\lambda + 6s)$.

11. $[(4ax + \lambda s)y + (4\lambda + 3s)x]y'_x = \frac{3}{2}ay^2 + 2(3\lambda + s)y + 2x.$

The substitution $w = \frac{3}{4}ay^2 + (3\lambda + s)y + x$ leads to an Abel equation of the form 13.3.3.3: $2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y$, where $b = s + 2\lambda$, $c = \frac{1}{8}a(60\lambda + 25s)$.

12. $(2Axy + ay + bx + c)y'_x = Ay^2 + Ak^2x^2 + my + k(ak + b - m)x + s.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (m - ak)z + s - ck]x'_z = 2Akx^2 + (2Az + ak + b)x + az + c.$$

13. $\left[2xy + (1-m)Ay - \frac{2(m+1)}{m+3}x\right]y'_x = \frac{1-m}{2}y^2 + \frac{m-1}{m+3}y + x.$

The substitution $w = \dfrac{1-m}{2}y^2 + \dfrac{m-1}{m+3}y + x$ leads to an equation of the form 13.3.3.4:

$$ww'_y = [(3-m)y - 1]w + (m-1)(y^3 - y^2 - ay), \qquad \text{where} \quad a = A - 2(m+1)(m+3)^{-2}.$$

14. $x(2ay + bx)y'_x = a(2-m)y^2 + b(1-m)xy + cx^2 + Ax^{m+2}.$

The transformation $z = y/x$, $w = -Ax^m + amz^2 + bmz - c$ leads to a separable equation: $ww'_z = m(2az + b)(amz^2 + bmz - c)$.

15. $(xy + x^2 + a)y'_x = y^2 + xy + b.$

Solution: $(x+y)^2 + a + b = C(bx - ay)^2$.

16. $(2Axy + Bx^2 + b)y'_x = Ay^2 + k(Ak + B)x^2 + c.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$(Az^2 + c - bk)x'_z = (2Ak + B)x^2 + 2Azx + b.$$

17. $(Axy + Bx^2 + kx)y'_x = Dy^2 + Exy + Fx^2 + ky.$

The substitution $y = xz$ leads to a linear equation with respect to $x = x(z)$:

$$[(D-A)z^2 + (E-B)z + F]x'_z = (Az + B)x + k.$$

18. $(Axy + Bx^2 + kx)y'_x = Ay^2 + Bxy + (Ab + k)y + Bbx + bk.$

This is a special case of equation 13.3.4.22. Solutions: $y = Cx - b$ and $Ay + Bx + k = 0$.

19. $\boldsymbol{(2Axy + Bx^2 + kx)y'_x = Ay^2 + Cxy + Dx^2 + ky - C\beta x - A\beta^2 - k\beta.}$

The substitution $y = \xi x + \beta$ leads to a linear equation with respect to $x = x(\xi)$:

$$[-A\xi^2 + (C - B)\xi + D]x'_\xi = (2A\xi + B)x + 2A\beta + k.$$

20. $\boldsymbol{(Axy + Bx^2 + kx)y'_x = Ay^2 + Cxy + Dx^2 + (k - A\beta)y - C\beta x - k\beta.}$

The substitution $y = \xi x + \beta$ leads to a linear equation with respect to $x = x(\xi)$: $[(C - B)\xi + D]x'_\xi = (A\xi + B)x + A\beta + k$.

21. $\boldsymbol{(Axy + Aky + Bx^2 + Bkx)y'_x = Cy^2 + Dxy + k(D - B)y.}$

The transformation $x = w - k,\ y = \xi w$ leads to a linear equation with respect to $w = w(\xi)$: $[(C - A)\xi^2 + (D - B)\xi]w'_\xi = (A\xi + B)w - kB$.

22. $\boldsymbol{(Axy + Bx^2 + a_1x + b_1y + c_1)y'_x = Ay^2 + Bxy + a_2x + b_2y + c_2.}$

Jacobi equation.

1°. With the help of the transformation $x = \bar{x} + \alpha$, $y = \bar{y} + \beta$, where α and β are the parameters which are determined by solving the algebraic system

$$A\alpha\beta + B\alpha^2 + a_1\alpha + b_1\beta + c_1 = 0, \qquad A\beta^2 + B\alpha\beta + a_2\alpha + b_2\beta + c_2 = 0,$$

we obtain the equation

$$(A\bar{x}\bar{y} + B\bar{x}^2 + \bar{a}_1\bar{x} + \bar{b}_1\bar{y})\bar{y}'_{\bar{x}} = A\bar{y}^2 + B\bar{x}\bar{y} + \bar{a}_2\bar{x} + \bar{b}_2\bar{y},$$

where $\bar{a}_1 = 2B\alpha + A\beta + a_1,\ \bar{a}_2 = B\beta + a_2,\ \bar{b}_1 = A\alpha + b_1,\ b_2 = 2A\beta + B\alpha + b_2$. The transformation $z = \bar{y}/\bar{x},\ \zeta = 1/\bar{x}$ leads to a linear equation:

$$[\bar{b}_1z^2 + (\bar{a}_1 - \bar{b}_2)z - \bar{a}_2]\zeta'_z = (\bar{b}_1z + \bar{a}_1)\zeta + Az + B.$$

2°. The original equation can be also rewritten in the form

$$(xy'_x - y)(n_3x + m_3y + k_3) - y'_x(n_1x + m_1y + k_1) + n_2x + m_2y + k_2 = 0.$$

The solution of this equation in parametric form can be obtained from the solution of the following system of constant coefficient linear differential equations:

$$\begin{aligned}(x_1)'_t &= n_1x_1 + m_1x_2 + k_1x_3,\\ (x_2)'_t &= n_2x_1 + m_2x_2 + k_2x_3,\\ (x_3)'_t &= n_3x_1 + m_3x_2 + k_3x_3,\end{aligned}$$

using the formulas $x(t) = x_1/x_3$ and $y(t) = x_2/x_3$.

3°. In the homogeneous coordinates $x = x_1/x_3$ and $y = x_2/x_3$, Equation 2° becomes

$$\begin{vmatrix} dx_1 & dx_2 & dx_3\\ x_1 & x_2 & x_3\\ a_x & b_x & c_x \end{vmatrix} = 0, \qquad (13.3.4.1)$$

where a_x, b_x, and c_x are linear forms of the homogeneous coordinates $a_x = a_1x_1 + a_2x_2 + a_3x_3$, $b_x = b_1x_1 + b_2x_2 + b_3x_3$, and $c_x = c_1x_1 + c_2x_2 + c_3x_3$. The coefficients of these forms depend on those of the original equations:

$$\begin{aligned} &a_1 = -n_1 + k_3 + k, && a_2 = -m_1, && a_3 = -k_1,\\ &b_1 = -n_2, && b_2 = -m_2 + k_3 + k, && b_3 = -k_2,\\ &c_1 = -n_3, && c_2 = -m_3, && c_3 = k, \end{aligned}$$

where k is an arbitrary constant. To return from the homogeneous coordinates to the original ones, x and y, it suffices to set $x_3 = 1$.

Equation (13.3.4.1) has linear particular integrals of the form

$$\sum u_i x_i \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0, \qquad (13.3.4.2)$$

where u_1, u_2, and u_3 are some constants satisfying the linear system

$$\begin{cases} (a_1 - \lambda)u_1 + b_1u_2 + c_1u_3 = 0,\\ a_2u_1 + (b_2 - \lambda)u_2 + c_2u_3 = 0,\\ a_3u_1 + b_3u_2 + (c_3 - \lambda)u_3 = 0, \end{cases} \qquad (13.3.4.3)$$

whose nonzero solutions are found from the cubic equation

$$\begin{vmatrix} a_1 - \lambda & b_1 & c_1\\ a_2 & b_2 - \lambda & c_2\\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0. \qquad (13.3.4.4)$$

Since a cubic equation always has at least one real root, the Jacobi equation has at least one linear particular integral (13.3.4.2).

Suppose we have found one real root of equation (13.3.4.4); the corresponding solution to system (13.3.4.3) is the integral curve $u_1x_1 + u_2x_2 + u_3x_3 = 0$. If $u_3 \neq 0$, the changes of variables $x_1 = \xi_1$, $x_2 = \xi_2$, and $u_1x_1 + u_2x_2 + u_3x_3 = \xi_3$ (if, for example, $u_3 = 0$ but $u_1 \neq 0$, we set $u_x = \xi_1$, $x_2 = \xi_2$, and $x_3 = \xi_3$) lead to the equations

$$\begin{vmatrix} d\xi_1 & d\xi_2 & d\xi_3\\ \xi_1 & \xi_2 & \xi_3\\ \alpha_\xi & \beta_\xi & \gamma_\xi \end{vmatrix} = 0.$$

It is clear that $\gamma_\xi = \gamma_3\xi_3$. Changing to the Cartesian coordinates and setting $\xi_1 = \xi$, $\xi_2 = \eta$, and $\xi_3 = 1$, we obtain an equation of the form (1.2.2.2) (see also Section 13.1.6),

$$(\gamma_3\eta - \beta_1\xi - \beta_2\eta - \beta_3)\,d\xi - (\gamma_3\xi - \alpha_1\xi - \alpha_2\eta - \alpha_3)\,d\eta = 0,$$

which is integrable by quadrature. In some special cases, the form of the general solution can be given in more detail.

1. All roots of equation (13.3.4.4) are real and distinct; then we get three integral straight lines of the Jacobi equation

$$\begin{aligned} u_x &\equiv u_1x_1 + u_2x_2 + u_3x_3,\\ v_x &\equiv v_1x_1 + v_2x_2 + v_3x_3,\\ w_x &\equiv w_1x_1 + w_2x_2 + w_3x_3. \end{aligned} \qquad (13.3.4.5)$$

We take these straight lines to be the axes of the new trilinear coordinate system (ξ_1, ξ_2, ξ_3), so that $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$ are solutions in this coordinate system:

$$\begin{vmatrix} d\xi_1 & d\xi_2 & d\xi_3 \\ \xi_1 & \xi_2 & \xi_3 \\ a_\xi & b_\xi & c_\xi \end{vmatrix} = 0. \tag{13.3.4.6}$$

Substituting $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$ and requiring that (13.3.4.6) is satisfied, we find that $a_\xi = a_1\xi_1$, $b_\xi = b_2\xi_2$, and $c_\xi = c_3\xi_3$ in the new coordinates; hence, equation (13.3.4.6) becomes

$$\begin{vmatrix} d\xi_1 & d\xi_2 & d\xi_3 \\ \xi_1 & \xi_2 & \xi_3 \\ a_1\xi_1 & b_2\xi_2 & c_3\xi_3 \end{vmatrix} = \begin{vmatrix} \frac{d\xi_1}{\xi_1} & \frac{d\xi_2}{\xi_2} & \frac{d\xi_3}{\xi_3} \\ 1 & 1 & 1 \\ a_1 & b_2 & c_3 \end{vmatrix} = 0,$$

or

$$(c_3 - b_2)\frac{d\xi_1}{\xi_1} + (a_1 - c_3)\frac{d\xi_2}{\xi_2} + (b_2 - a_1)\frac{d\xi_3}{\xi_3} = 0.$$

Its general integral is

$$\xi_1^{c_3 - b_2}\xi_2^{a_1 - c_3}\xi_3^{b_2 - a_1} = C,$$

Inserting here the straight lines (13.3.4.5) and setting $x_3 = 1$, we obtain the solution in the Cartesian coordinates

$$(u_1x + u_2y + u_3)^\alpha(v_1x + v_2y + v_3)^\beta(w_1x + w_2y + w_3)^\gamma = C, \tag{13.3.4.7}$$

where $\alpha + \beta + \gamma \equiv 0$.

Example 13.1. Let us look at the equation

$$(7x + 8y + 5)\,dx - (7x + 8y)\,dy + 5(x - y)(y\,dx - x\,dy) = 0.$$

In the homogeneous coordinates, it becomes

$$\begin{vmatrix} d\xi_1 & d\xi_2 & d\xi_3 \\ \xi_1 & \xi_2 & \xi_3 \\ -7x_1 - 8x_2 & -7x_1 - 8x_2 - 5x_3 & 5x_1 - 5x_2 \end{vmatrix} = 0.$$

The equation for λ is $\lambda^3 + 15\lambda^2 - 25\lambda - 375 = 0$; its roots are $\lambda_1 = -15$, $\lambda_2 = -5$, and $\lambda_3 = 5$. As a result, we get

$$u_x = 2x_1 + 3x_2 + x_3, \quad v_x = -x_1 + x_2 + x_3, \quad w_x = x_1 - x_2 + x_3.$$

Substituting the variables u, v, and w determined by these formulas into the equation yields $\frac{du}{u} - 2\frac{dv}{v} + \frac{dw}{w} = 0$. Integrating this relation gives $uw = Cv^2$. Hence, in terms of the original variables, we get $(2x + 3y + 1)(x - y + 1) = C(-x + y + 1)^2$.

To analyze the other special cases, we introduce the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

2. Suppose equation (13.3.4.4) has three real roots with two of them being multiple, $\lambda_2 = \lambda_3$. Then the form of the general solution depends on the quantity

$$\rho = \operatorname{rank}(\mathbf{A} - \lambda_1\mathbf{E}) + \operatorname{rank}(\mathbf{A} - \lambda_2\mathbf{E}),$$

where $\mathbf{E}$ is the 3×3 unit matrix.

2a. $\rho = 3$. The matrix $\mathbf{A}$ has three eigenvectors and the solution of the original equation is written in the form (13.3.4.7).

2b. $\rho < 3$. In this case, the matrix $\mathbf{A}$ has less than three eigenvectors, which leads us to the general case 1.

3. Finally, if all three roots of equation (13.3.4.4) coincide, the form of the solution is determined by value of $\rho = \operatorname{rank}(\mathbf{A} - \lambda\mathbf{E})$.

3a. $\rho = 2$. Again, this leads to the general case 1.

3b. $\rho = 1$. In this case, instead of an integral straight line, we get a bundle of integral curves, since one equation is insufficient for determining two ratios, u_1/u_3 and u_2/u_3. Expressing u_3 in terms of u_1 and u_2 and substituting into the equation $u_x = 0$, we obtain the equation of the bundle, which contains one significant constant and, hence, is a general integral of the Jacobi equation.

Example 13.2. In the homogeneous coordinates, the equation

$$(14x + 13y + 6)\,dx + (4x + 5y + 3)\,dy + (7x + 5y)(y\,dx - x\,dy) = 0$$

becomes

$$\begin{vmatrix} dx_1 & dx_2 & dx_3 \\ x_1 & x_2 & x_3 \\ 4x_1 + 5x_2 + 3x_3 & -14x_1 - 13x_2 - 6x_3 & 7x_1 + 5x_2 \end{vmatrix} = 0.$$

The equation for λ is $\lambda^3 + 9\lambda^2 + 27\lambda + 27 = 0$. It has three identical roots: $\lambda_1 = \lambda_2 = \lambda_3 = -3$. The equation of the bundle is $u_1(x_1 - x_3) + u_2(x_2 + 2x_3) = 0$. The general integral of the original equation is $x - 1 = C(y + 2)$.

⊙ *Literature for equation* 13.3.4.22: V. V. Stepanov (1958), V. F. Zaitsev and L. V. Linchuk (2015).

23. $\boldsymbol{(Axy + Bx^2 + ay + bx + c)y'_x = kAxy + kBx^2 + my + k(ak + b - m)x + s.}$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[(m - ak)z + s - ck]x'_z = (Ak + B)x^2 + (Az + ak + b)x + az + c.$$

24. $\boldsymbol{(2Axy + Bx^2 + ay + bx + c)y'_x = Ay^2 + k(Ak + B)x^2 + aky + bkx + s.}$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$(Az^2 + s - ck)x'_z = (2Ak + B)x^2 + (2Az + ak + b)x + az + c.$$

25. $\boldsymbol{(2Axy - Akx^2 + ay + bx + c)y'_x = Ay^2 + my + k(ak + b - m)x + s.}$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (m - ak)z + s - ck]x'_z = Akx^2 + (2Az + ak + b)x + az + c.$$

26. $\boldsymbol{(2Axy + Bx^2 + ay - akx + b)y'_x = Ay^2 + k(Ak + B)x^2 + my - mkx + s.}$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (m - ak)z + s - bk]x'_z = (2Ak + B)x^2 + 2Azx + az + c.$$

27. $\boldsymbol{(2Axy + Bx^2 + ay + bx + c)y'_x = Ay^2 + k(Ak + B)x^2 + by + ak^2x + s.}$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (b - ak)z + s - ck]x'_z = (2Ak + B)x^2 + (2Az + ak + b)x + az + c.$$

28. $[Axy + Bx^2 + (k-1)Aay - (Abk + Ba)x]y'_x$
$= Ay^2 + Bxy - (Ab + Bak)y + (k-1)Bbx.$

This is a special case of equation 13.3.4.22. Solution in parametric form:

$$x = \frac{at + ACt^k}{t + C}, \qquad y = \frac{bt - BCt^k}{t + C}.$$

The solution can be presented in implicit form as well:

$$C^k(Ay + Bx)^k + [A(b-y) + B(a-x)]^{k-1}(ay - bx) = 0.$$

29. $[(ax + c)y + (1-n)x^2 + (2n-1)x - n]y'_x = 2ay^2 + 2xy.$

The substitution $w = ay + x$ leads to an equation of the form 13.3.4.8:

$$2yww'_y = (1-n)w^2 + [a(2n+1)y + 2n - 1]w - a^2ny^2 - by - n, \qquad \text{where} \quad b = (2n-1)a - c.$$

30. $[(x + c)y + (n+1)x^2 - a(2n+1)x + a^2n]y'_x = \dfrac{2n}{3n-1}y^2 + 2xy.$

The transformation $z = \dfrac{3n-1}{n-1}\dfrac{1}{y}$, $w = \dfrac{3n-1}{n-1}\dfrac{x}{y} + \dfrac{n}{n-1}$ leads to an equation of the form 13.3.4.8:

$$2zww'_z = (1-n)w^2 + [a(2n+1)z + 2n - 1]w - a^2nz^2 - bz - n, \quad b = \frac{(3n-1)c + an(2n+1)}{n-1}.$$

31. $x(2axy + b)y'_x = -a(m+3)xy^2 - b(m+2)y + cx^m.$

The transformation $z = xy$, $w = -cx^{m+1} + a(m+1)x^2y^2 + b(m+1)xy$ leads to a separable equation: $ww'_z = (m+1)^2(2az + b)(az^2 + bz)$.

32. $[(a_2x^2 + a_1x + a_0)y + b_2x^2 + b_1x + b_0]y'_x = c_2y^2 + c_1y + c_0.$

This is a Riccati equation with respect to $x = x(y)$.

The substitution $x = -\dfrac{c_2y^2 + c_1y + c_0}{a_2y + b_2}\dfrac{w'_y}{w}$ leads to a second-order linear equation:

$$f_2w''_{yy} - [(f_2)'_y + f_1f_2]w'_y + f_0f_2^2w = 0, \qquad \text{where} \quad f_i = \frac{a_iy + b_i}{c_2y^2 + c_1y + c_0}; \quad i = 1,\ 2,\ 3.$$

33. $[(12a^2x^2 - 7ax + 1)y + 4cx^2 - 5bx]y'_x = -2x(3a^2y^2 + 2cy + 3b^2).$

The substitution $w = x(3a^2y^2 + 2cy + 3b^2)$ leads to an Abel equation of the form 13.3.3.3: $2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y$.

34. $x[(m-1)(Ax + B)y + m(Dx^2 + Ex + F)]y'_x$
$= [A(1-n)x - Bn]y^2 + [D(2-n)x^2 + E(1-n)x - Fn]y.$

Solution: $Axy + Dx^2 + Ex + By + F = Cx^ny^m$.

35. $x^2(2axy + b)y'_x = -4ax^2y^2 - 3bxy + cx^2 + k.$

The transformation $z = xy$, $w = 2ax^2y^2 + 2bxy - cx^2 - k$ leads to a separable equation: $ww'_z = 2(2az + b)(2az^2 + 2bz - k)$.

36. $(xy + ax^n + bx^2)y'_x = y^2 + cx^n + bxy.$

The transformation $t = y/x$, $z = x^{n-2}$ leads to a first-order linear equation: $(c - at)z'_t = (n-2)(az + t + b)$.

37. $x(2ax^n y + b)y'_x = -a(3n+m)x^n y^2 - b(2n+m)y + Ax^m + cx^{-n}.$

The transformation $z = x^n y$, $w = -Ax^{n+m} + (n+m)(az^2 + bz) - c$ leads to a separable equation: $ww'_z = (n+m)^2(2az+b)\Bigl(az^2 + bz - \dfrac{c}{n+m}\Bigr)$.

38. $yy'_x = -ny^2 + a(2n+1)e^x y + by - a^2 n e^{2x} - abe^x + c.$

The transformation $x = \ln\xi$, $y = w + a\xi$ leads to a Bernoulli equation with respect to $\xi = \xi(w)$: $(-nw^2 + bw + c)\xi'_w = w\xi + a\xi^2$.

13.3.5 Some Types of First- and Second-Order Equations Reducible to Abel Equations of the Second Kind

◆ *Notation:* f, g, h, p, φ, ψ, Φ, F, *and* G *are arbitrary functions of their arguments.*

► **Quasi-homogeneous equations.**

1°. Let us consider a quasi-homogeneous equation of the form

$$f(x^\nu y)x^{\nu+1}y'_x + g(x^\nu y) + Ax^\lambda = 0.$$

In the special case $\lambda = 0$ this equation is homogeneous.

The transformation $z = x^\nu y$, $w = Ax^\lambda + g(z) - \nu z f(z)$ leads to an Abel equation:

$$ww'_z = [-(\lambda+\nu)f + g'_z - \nu z f'_z]w + \lambda f(g - \nu z f).$$

2°. A quasi-homogeneous equation of the form

$$f(x^\nu y)x^{\nu+1}y'_x + g(x^\nu y) + x^\lambda[h(x^\nu y)x^{\nu+1}y'_x + p(x^\nu y)] = 0$$

can be reduced by the transformation $z = x^\nu y$, $\zeta = x^{-\lambda}$ to an Abel equation:

$$\bigl\{[g(z) - \nu z f(z)]\zeta + p(z) - \nu z h(z)\bigr\}\zeta'_z = \lambda f(z)\zeta^2 + \lambda h(z)\zeta.$$

► **Equations of the theory of chemical reactors and the combustion theory.**

In the theory of chemical reactors and the combustion theory, one encounters equations of the form

$$y''_{xx} - ay'_x = f(y).$$

The substitution $w(y) = y'_x/a$ leads to the Abel equation $ww'_y - w = a^{-2}f(y)$, whose solvable cases are given in Section 13.3.1.

► **Equations of the theory of nonlinear oscillations.**

1°. Let us consider equations of the theory of nonlinear oscillations of the form

$$y''_{xx} + \varphi(y)y'_x + y = 0.$$

The substitution $z(y) = y'_x$ leads to the Abel equation

$$zz'_y + \varphi(y)z + y = 0, \tag{1}$$

which is reduced, with the aid of the substitution $\tau = \frac{1}{2}(a - y^2)$, to the following form:

$$zz'_\tau = g(\tau)z + 1, \qquad \text{where} \quad g(\tau) = \pm\frac{\varphi\bigl(\pm\sqrt{a-2\tau}\,\bigr)}{\sqrt{a-2\tau}}. \tag{2}$$

Specific cases of Eq. (2) are outlined in Section 13.3.2.

2°. An equation of the theory of nonlinear oscillations of the form

$$y''_{xx} + \Phi(y'_x) + y = 0$$

can be reduced by the transformation $z = y'_x$, $w = -y - \Phi(y'_x)$ to an Abel equation of the form (1):

$$ww'_z + \Phi'_z(z)w + z = 0.$$

► **Second-order homogeneous equations of various types.**

1°. A homogeneous equation with respect to the independent variable has the form

$$x^2 y''_{xx} = xg(y)y'_x + f(y).$$

The substitution $w(y) = xy'_x$ leads to an Abel equation: $ww'_y = [g(y) + 1]w + f(y)$.

2°. A generalized homogeneous equation

$$xy''_{xx} = g(yx^k)y'_x + x^{-k-1}f(yx^k)$$

can be reduced by the transformation $t = yx^k$, $u = x^k(xy'_x + ky)$ to an Abel equation:

$$uu'_t = [g(t) + 2k + 1]u + f(t) - ktg(t) - k(k+1)t.$$

To the Emden–Fowler equation, discussed in Section 14.3, there correspond $g(t) = 0$, $f(t) = At^m$, and $k = \frac{n+2}{m-1}$.

3°. A generalized homogeneous equation

$$y''_{xx} = x^\alpha y^\beta F\Bigl(\frac{x}{y}y'_x\Bigr) + yx^{-2}G\Bigl(\frac{x}{y}y'_x\Bigr)$$

can be reduced by the transformation $\eta = \dfrac{x}{y}y'_x$, $w = x^{\alpha+2}y^{\beta-1}$ to an Abel equation:

$$[F(\eta)w + G(\eta) + \eta - \eta^2]w'_\eta = [(\beta-1)\eta + \alpha + 2]w.$$

To the generalized Emden–Fowler equation, discussed in Section 14.5, there correspond $\alpha = n - l$, $\beta = m + l$, $F(\eta) = A\eta^l$, and $G(\eta) = 0$.

► **Second-order equations invariant under some transformations.**

1°. An equation invariant under "dilatation–translation" transformation has the form

$$y''_{xx} = x^\alpha e^{\beta y} f(xy'_x) + x^{-2} g(xy'_x).$$

The transformation $\zeta = xy'_x,\ u = x^{\alpha+2} e^{\beta y}$ leads to an Abel equation:

$$[f(\zeta)u + g(\zeta) + \zeta]u'_\zeta = (\beta\zeta + \alpha + 2)u.$$

2°. An equation invariant under "translation–dilatation" transformation has the form

$$y''_{xx} = e^{\alpha x} y^\beta f\Big(\frac{y'_x}{y}\Big) + yg\Big(\frac{y'_x}{y}\Big).$$

The transformation $\xi = y'_x/y,\ w = e^{\alpha x} y^{\beta-1}$ leads to an Abel equation:

$$[f(\xi)w + g(\xi) - \xi^2]w'_\xi = [(\beta - 1)\xi + \alpha]w.$$

13.4 Equations Containing Polynomial Functions of y

13.4.1 Abel Equations of the First Kind $y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$

► **Preliminary remarks.**

1°. If $y_0 = y_0(x)$ is a particular solution of the equation in question, the substitution $y - y_0 = 1/w$ reduces it to an Abel equation of the second kind:

$$ww'_x = -(3f_3y_0^2 + 2f_2y_0 + f_1)w^2 - (3f_3y_0 + f_2)w - f_3,$$

which is discussed in Section 13.3. For $f_0(x) \equiv 0$, we can choose $y_0 \equiv 0$ as a particular solution.

2°. The transformation

$$\xi = \int f_3E^2\,dx, \quad u = \Big(y + \frac{f_2}{3f_3}\Big)E^{-1}, \quad \text{where} \quad E = \exp\Big[\int\Big(f_1 - \frac{f_2^2}{3f_3}\Big)\,dx\Big],$$

brings the original equation to the normal form:

$$u'_\xi = u^3 + \Phi(\xi), \quad \text{where} \quad \Phi = \frac{1}{f_3E^3}\Big[f_0 + \frac{1}{3}\frac{d}{dx}\Big(\frac{f_2}{f_3}\Big) - \frac{f_1f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}\Big].$$

► **Solvable equations and their solutions.**

1. $\boldsymbol{y'_x = ay^3 + bx^{-3/2}}$.

This is a special case of equation 13.4.1.9 with $n = -1/2$.

2. $\boldsymbol{y'_x = -y^3 + 3a^2x^2y - 2a^3x^3 + a}$.

The substitution $y = 1/u + ax$ leads to an Abel equation of the form 13.3.2.1: $uu'_x = 3axu + 1$.

3. $y'_x = -y^3 + (ax+b)y^2$.

The substitution $y = -1/u$ leads to an Abel equation of the form 13.3.2.1: $uu'_x = (ax+b)u + 1$.

4. $y'_x = -y^3 + (ax+b)^{-2}y^2$.

The substitution $y = -1/u$ leads to an Abel equation of the form 13.3.2.2: $uu'_x = (ax+b)^{-2}u + 1$.

5. $y'_x = -y^3 + (ax+b)^{-1/2}y^2$.

The substitution $y = -1/u$ leads to an Abel equation of the form 13.3.2.4: $uu'_x = (ax+b)^{-1/2}u + 1$.

6. $y'_x = ay^3 + 3abxy^2 - b - 2ab^3x^3$.

This is a special case of equation 13.4.1.10 with $n = 0$ and $m = 1$.

7. $y'_x = axy^3 + by^2$.

The substitution $u = xy$ leads to a separable equation: $xu'_x = au^3 + bu^2 + u$.

8. $y'_x = axy^3 + 3abx^2y^2 - b - 2ab^3x^4$.

This is a special case of equation 13.4.1.10 with $n = m = 1$.

9. $y'_x = ax^{2n+1}y^3 + bx^{-n-2}$.

The substitution $w = yx^{n+1}$ leads to a separable equation: $xw'_x = aw^3 + (n+1)w + b$.

For $a = -\frac{1}{3}(n+1)A^{-2}$ and $b = \frac{2}{3}A(n+1)$, the solution is written in parametric form:

$$x = \exp\left(\frac{F}{n+1}\right), \quad y = -A\left(1 + \frac{1}{\tau}\right)\exp(-F), \qquad \text{where} \quad F = \tau - \tfrac{1}{3}\ln\left|\tau + \tfrac{1}{3}\right| + C.$$

10. $y'_x = ax^ny^3 + 3abx^{n+m}y^2 - bmx^{m-1} - 2ab^3x^{n+3m}$.

The substitution $w = y + bx^m$ leads to a Bernoulli equation: $w'_x = ax^nw^3 - 3ab^2x^{n+2m}w$.

11. $y'_x = ax^ny^3 + 3abx^{n+m}y^2 + cx^ky - 2ab^3x^{n+3m} + bcx^{m+k} - bmx^{m-1}$.

The substitution $u = y + bx^m$ leads to a Bernoulli equation: $u'_x = ax^nu^3 + (cx^k - 3ab^2x^{n+2m})u$.

12. $9y'_x = -x^m(ax^{1-m} + b)^{2\lambda+1}y^3 - x^{-2m}(9a + 2 + 9bmx^{m-1})(ax^{1-m} + b)^{-\lambda-2}$.

For $\lambda = \dfrac{1}{3a(1-m)}$, the substitution $y = \left(\dfrac{3}{w} + \dfrac{1}{ax + bx^m}\right)\left(ax^{1-m} + b\right)^{-\lambda}$ leads to the Abel equation $ww'_x = w + ax + bx^m$, which is discussed in Section 13.3.1.

13. $xy'_x = ax^4y^3 + (bx^2 - 1)y + cx$.

The substitution $w = xy$ leads to a separable equation: $w'_x = x(aw^3 + bw + c)$.

14. $xy'_x = ay^3 + 3abx^ny^2 - bnx^n - 2ab^3x^{3n}$.

The substitution $w = y + bx^n$ leads to a Bernoulli equation: $w'_x = ax^{-1}w^3 - 3ab^2x^{2n-1}w$.

15. $xy'_x = ax^{2n+1}y^3 + (bx-n)y + cx^{1-n}.$

The substitution $w = yx^n$ leads to a separable equation: $w'_x = aw^3 + bw + c$.

16. $xy'_x = ax^{n+2}y^3 + (bx^n - 1)y + cx^{n-1}.$

The substitution $w = xy$ leads to a separable equation: $w'_x = x^{n-1}(aw^3 + bw + c)$.

17. $x^2y'_x = y^3 - 3a^2x^4y + 2a^3x^6 + 2ax^3.$

The transformation $x = 1/\xi$, $y = ax^2 + 1/w$ leads to an equation of the form 13.3.2.2: $ww'_\xi = 3a\xi^{-2}w + 1$.

18. $y'_x = -(ax + bx^m)y^3 + y^2.$

The substitution $y = -1/w$ leads to an equation $ww'_x = w + ax + bx^m$, which is discussed in Section 13.3.1.

19. $y'_x = (Ax^2 + Bx + C)^{-1/2}y^3 + y^2.$

The substitution $y = -1/w$ leads to an Abel equation of the form 13.3.1.63: $ww'_x = w - (Ax^2 + Bx + C)^{-1/2}$.

20. $y'_x = -x^{-16/9}\left(ax - \frac{6}{25}\right)^{34/9}y^3 + \frac{2}{27}\left(9ax - \frac{2}{25}\right)x^{-11/18}\left(ax - \frac{6}{25}\right)^{-61/18}.$

Solution in parametric form:

$$x = \pm\frac{6}{25a}\tau^2\wp, \quad y = \mp\frac{125a}{108}(aE_1)^{-25/18}\tau^{7/9}\wp^{7/18}E_1^{-1}E_2^{-1}(18\wp E_1 \pm 5E_2),$$

where

$$E_1 = \tau^2\wp \mp 1, \quad E_2 = \tau\sqrt{\pm(4\wp^3 - 1)} + 2\wp.$$

The function $\wp = \wp(\tau)$ is defined implicitly by $\tau = \displaystyle\int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C$. The upper sign in the formulas corresponds to the classical Weierstrass elliptic function $\wp = \wp(\tau + C,\, 0,\, 1)$.

21. $y'_x = -y^3 + ae^{\lambda x}y^2.$

The substitution $y = -1/w$ leads to an Abel equation of the form 13.3.2.7: $ww'_x = ae^{\lambda x}w + 1$.

22. $y'_x = -y^3 + 3a^2e^{2\lambda x}y - 2a^3e^{3\lambda x} + a\lambda e^{\lambda x}.$

The substitution $y = \dfrac{1}{w} + ae^{\lambda x}$ leads to an Abel equation of the form 13.3.2.7: $ww'_x = 3ae^{\lambda x}w + 1$.

23. $y'_x = -\frac{1}{3}\lambda^{-1}e^{2\lambda x}y^3 + \frac{2}{3}\lambda^2e^{-\lambda x}.$

Solution in parametric form:

$$x = \frac{F}{\lambda}, \quad y = -\lambda\left(1 + \frac{1}{\tau}\right)e^{-F}, \qquad \text{where} \quad F = \tau - \tfrac{1}{3}\ln\left|\tau + \tfrac{1}{3}\right| + C.$$

24. $y'_x = ae^{2\lambda x}y^3 + be^{\lambda x}y^2 + cy + de^{-\lambda x}.$

The substitution $y = ue^{-\lambda x}$ leads to a separable equation: $u'_x = au^3 + bu^2 + (c + \lambda)u + d$.

25. $y'_x = ae^{\lambda x}y^3 + 3abe^{\lambda x}y^2 + cy - 2ab^3e^{\lambda x} + bc.$

The substitution $u = y + b$ leads to a Bernoulli equation: $u'_x = ae^{\lambda x}u^3 + (c - 3ab^2e^{\lambda x})u$.

26. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 - 2ab^3e^{(\lambda+3\mu)x} - b\mu e^{\mu x}.$

The substitution $u = y + be^{\mu x}$ leads to a Bernoulli equation: $u'_x = ae^{\lambda x}u^3 - 3ab^2e^{(\lambda+2\mu)x}u$.

27. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 + 2ab^2e^{(\lambda+2\mu)x}y - b\mu e^{\mu x}.$

The substitution $u = y + be^{\mu x}$ leads to a Bernoulli equation: $u'_x = ae^{\lambda x}u^3 - ab^2e^{(\lambda+2\mu)x}u$.

28. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 + \mu y - 2ab^3e^{(\lambda+3\mu)x}.$

The substitution $u = y + be^{\mu x}$ leads to a Bernoulli equation: $u'_x = ae^{\lambda x}u^3 + \left[\mu - 3ab^2e^{(\lambda+2\mu)x}\right]u$.

29. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 + [(3ab^2 + c)e^{(\lambda+2\mu)x} + s]y + b(ab^2 + c)e^{(\lambda+3\mu)x} + b(s - \mu)e^{\mu x}.$

The substitution $u = y + be^{\mu x}$ leads to a Bernoulli equation: $u'_x = ae^{\lambda x}u^3 + [ce^{(\lambda+2\mu)x} + s]u$.

30. $y'_x = [a + b\exp(2x/a)]y^3 + y^2.$

The substitution $y = -1/u$ leads to an equation of the form 13.3.1.8: $uu'_x = u - a - b\exp(2x/a)$.

31. $y'_x = -\frac{2}{3}ax^{-1}\exp(2ax^2)y^3 + (1 - \frac{4}{3}ax^2)\exp(-ax^2).$

The substitution $y = \left(\dfrac{1}{2au} + x\right)\exp(-ax^2)$ leads to an equation of the form 13.3.1.16: $uu'_x = u + (6ax)^{-1}$.

32. $y'_x = -a\exp(2ax^3)y^3 + (1 - 2ax^3)\exp(-ax^3).$

The transformation $\xi = x^2$, $y = \left(\dfrac{2}{3au} + x\right)\exp(-ax^3)$ leads to an equation of the form 13.3.1.32: $uu'_\xi = u \pm 2(9a)^{-1}\xi^{-1/2}$.

33. $y'_x = -ax^{-2}\exp(2ax^3)y^3 + 2x(1 - ax^3)\exp(-ax^3).$

The substitution $y = \left(\dfrac{1}{3au} + x^2\right)\exp(-ax^3)$ leads to an equation of the form 13.3.1.33: $uu'_x = u + (9a)^{-1}x^{-2}$.

34. $y'_x = -\frac{2}{9}a^{-1}x^{-1/2}\exp(2ax^{3/2})y^3 + \frac{3}{4}ax^{-1/2}(2ax^{3/2} - 1)\exp(-ax^{3/2}).$

Solution in parametric form:

$$x = b\tau^{-4/3}Z^{-2}U_1^2, \quad y = -\frac{1}{b}\tau^{-2/3}Z^{-1}U_2^{-1}(\tau^2Z^3 \mp U_1U_2)\exp\left(\pm\tfrac{2}{3}\tau^{-2}Z^{-3}U_1^3\right),$$

where

$$a = \mp\tfrac{2}{3}b^{-3/2}, \quad U_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \quad U_2 = U_1^2 \pm \tau^2Z^2,$$

$$Z = \begin{cases} C_1J_{1/3}(\tau) + C_2Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1I_{1/3}(\tau) + C_2K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

35. $y'_x = -ax^{-3}(x^2-a)^{4/3}\exp\Big(\frac{x^2}{3a}\Big)y^3 + \frac{1}{27a^2}x^2(x^2-a)^{-13/6}(2x^4-9ax^2+27a^2)\exp\Big(-\frac{x^2}{6a}\Big).$

Solution in parametric form:

$$x = \frac{\sqrt{2af}}{\tau}, \quad y = -\frac{\sqrt{2f}}{3a^{2/3}\tau^{2/3}(2f-\tau^2)^{7/6}}\,\frac{4(\tau+1)f^2+4\tau^2 f-3\tau^4}{2(\tau+1)f-\tau^2}\exp\Big(-\frac{f}{3\tau^2}\Big),$$

where $f = \tau - \ln|1+\tau| + C$.

36. $y'_x = ay^3 + b\cosh(\lambda x)y^2.$

The transformation $t = \dfrac{1}{y} + \dfrac{b}{\lambda}\sinh(\lambda x)$, $w = e^{\lambda x}$ leads to a Riccati equation: $2aw'_t = bw^2 - 2\lambda tw - b$.

37. $y'_x = ay^3 + b\sinh(\lambda x)y^2.$

The transformation $t = \dfrac{1}{y} + \dfrac{b}{\lambda}\cosh(\lambda x)$, $w = e^{\lambda x}$ leads to a Riccati equation: $2aw'_t = bw^2 - 2\lambda tw + b$.

38. $y'_x = -y^3 + 3a^2\cosh^2 x\, y - 2a^3\cosh^3 x + a\sinh x.$

The substitution $y = a\cosh x + 1/w$ leads to an Abel equation of the form 13.3.2.9: $ww'_x = 3a\cosh x\, w + 1$.

39. $y'_x = -y^3 + 3a^2\sinh^2 x\, y - 2a^3\sinh^3 x + a\cosh x.$

The substitution $y = a\sinh x + 1/w$ leads to an Abel equation of the form 13.3.2.10: $ww'_x = 3a\sinh x\, w + 1$.

40. $y'_x = -y^3 + a\cos(\lambda x)y^2.$

The substitution $y = -1/u$ leads to an Abel equation of the form 13.3.2.11: $uu'_x = a\cos(\lambda x)u + 1$.

41. $y'_x = -y^3 + a\sin(\lambda x)y^2.$

The substitution $y = -1/u$ leads to an Abel equation of the form 13.3.2.12: $uu'_x = a\sin(\lambda x)u + 1$.

42. $y'_x = -y^3 + 3a^2\cos^2(\lambda x)y + a\lambda\sin(\lambda x) + 2a^3\cos^3(\lambda x).$

The substitution $y = -a\cos(\lambda x) + 1/w$ leads to an Abel equation of the form 13.3.2.11: $ww'_x = -3a\cos(\lambda x)w + 1$.

43. $y'_x = -y^3 + 3a^2\sin^2(\lambda x)y + a\lambda\cos(\lambda x) - 2a^3\sin^3(\lambda x).$

The substitution $y = a\sin(\lambda x) + 1/w$ leads to an Abel equation of the form 13.3.2.12: $ww'_x = 3a\sin(\lambda x)w + 1$.

◆ *In equations 44–47, the following notation is used:* $f = f(x)$, $g = g(x)$, $h = h(x)$.

44. $y'_x = afy^3 + \Big(bfg^2 + \frac{g'_x}{g}\Big)y + cfg^3.$

The substitution $y = gw$ leads to a separable equation: $w'_x = fg^2(aw^3 + bw + c)$.

45. $y'_x = fy^3 + 3fhy^2 + (g + 3fh^2)y + fh^3 + gh - h'_x.$

The substitution $w = y + h(x)$ leads to a Bernoulli equation: $w'_x = g(x)w + f(x)w^3$.

46. $y'_x = \dfrac{g'_x}{f^2(ag+b)^3}y^3 + \dfrac{f'_x}{f}y + fg'_x.$

Solution: $\displaystyle\int \frac{dw}{w^3 - aw + 1} + C = \frac{1}{a}\ln|ag+b|$, where $w = \dfrac{y}{f(ag+b)}$.

47. $y'_x = (y-f)(y-g)\Big(y - \dfrac{af+bg}{a+b}\Big)h + \dfrac{y-g}{f-g}f'_x + \dfrac{y-f}{g-f}g'_x.$

Solution:

$$|y-f|^a|y-g|^b\left|y - \frac{af+bg}{a+b}\right|^{-a-b} = C\exp\Big[\frac{ab}{a+b}\int (f-g)^2 h\,dx\Big].$$

13.4.2 Equations of the Form $(A_{22}y^2+A_{12}xy+A_{11}x^2+A_0)y'_x = B_{22}y^2+B_{12}xy+B_{11}x^2+B_0$

► **Preliminary remarks. Some transformations.**

1°. For $A_{22}=0$, this is an Abel equation (see Section 13.3.4). For $B_{11}=0$, this is an Abel equation with respect to $x = x(y)$.

2°. The transformation $z = y/x$, $\zeta = x^{-2}$ leads to an Abel equation of the second kind:

$$[(A_0z - B_0)\zeta + A_{22}z^3 + (A_{12} - B_{22})z^2 + (A_{11} - B_{12})z - B_{11}]\zeta'_z = 2A_0\zeta^2 + 2(A_{22}z^2 + A_{12}z + A_{11})\zeta.$$

3°. The transformation $x = \bar{x} + \alpha$, $y = \bar{y} + \beta$, where α and β are parameters, which are determined by solving the second-order algebraic system

$$A_{22}\beta^2 + A_{12}\alpha\beta + A_{11}\alpha^2 + A_0 = 0, \quad B_{22}\beta^2 + B_{12}\alpha\beta + B_{11}\alpha^2 + B_0 = 0,$$

leads to the equation

$$[A_{22}\bar{y}^2 + A_{12}\bar{x}\bar{y} + A_{11}\bar{x}^2 + (2A_{22}\beta + A_{12}\alpha)\bar{y} + (2A_{11}\alpha + A_{12}\beta)\bar{x})]\bar{y}'_{\bar{x}} = B_{22}\bar{y}^2 + B_{12}\bar{x}\bar{y} + B_{11}\bar{x}^2 + (2B_{22}\beta + B_{12}\alpha)\bar{y} + (2B_{11}\alpha + B_{12}\beta)\bar{x}.$$

The transformation $\xi = \bar{y}/\bar{x}$, $w = 1/\bar{x}$ reduces this equation to an Abel equation of the second kind:

$$\{[a_2\xi^2 + (a_1 - b_2)\xi - b_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi = (a_2\xi + a_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w,$$

where $a_1 = 2A_{11}\alpha + A_{12}\beta$, $b_1 = 2B_{11}\alpha + B_{12}\beta$, $a_2 = 2A_{22}\beta + A_{12}\alpha$, and $b_2 = 2B_{22}\beta + B_{12}\alpha$.

4°. The substitution $y = t + \varepsilon x$, where parameter ε is determined by solving the cubic equation

$$(A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\varepsilon - B_{22}\varepsilon^2 - B_{12}\varepsilon - B_{11} = 0,$$

leads to an Abel equation of the second kind with respect to $x = x(t)$:

$$[Qtx + (B_{22} - A_{22}\varepsilon)t^2 + B_0 - A_0\varepsilon]x'_t = (A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})x^2 + (2A_{22}\varepsilon + A_{12})tx + A_{22}t^2 + A_0,$$

where $Q = 2B_{22}\varepsilon + B_{12} - \varepsilon(2A_{22}\varepsilon + A_{12})$.

► **Solvable equations and their solutions.**

1. $(Ay^2 + x^2)y'_x = -2xy + Bx^2 + a.$

Solution: $Ay^3 - Bx^3 + 3(x^2y - ax) = C.$

2. $(Ay^2 + Bx^2 - a^2B)y'_x = Cy^2 + 2Bxy.$

The transformation $x = w + a$, $y = \xi w$ leads to a linear equation: $(-A\xi^3 + C\xi^2 + B\xi)w'_\xi = (A\xi^2 + B)w + 2aB.$

3. $(Ay^2 + Bxy + Cx^2)y'_x = Dy^2 + Exy + Fx^2.$

Homogeneous equation. The substitution $z = y/x$ leads to a separable equation: $xz'_x = (Az^2 + Bz + C)^{-1}[-Az^3 + (D - B)z^2 + (E - C)z + F].$

4. $(Ay^2 - 2Akxy + Bkx^2)y'_x = -By^2 + 2Bkxy - Ak^3x^2 + a.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[-(Ak + B)z^2 + a]x'_z = k(B - Ak)x^2 + Az^2.$$

5. $(Ay^2 + 2Bxy + Ak^2x^2)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + a.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z^2 + a]x'_z = 2k(Ak + B)x^2 + 2(Ak + B)zx + Az^2.$$

6. $(Ay^2 + Bxy + Cx^2 + a)y'_x = Aky^2 + Bkxy + Ckx^2 + b.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$(b - ak)x'_z = (Ak^2 + Bk + C)x^2 + (2Ak + B)zx + Az^2 + a.$$

7. $(Ay^2 + 2Bxy + Dx^2 + a)y'_x = -By^2 - 2Dxy + Ex^2 + b.$

Solution: $Ay^3 - Ex^3 + 3(Bxy^2 + Dx^2y + ay - bx) = C.$

8. $(Ay^2 - 2Axy + Bx^2 + A - B)y'_x = -Ay^2 + 2Bxy - Bx^2 + A - B.$

This is a special case of equation 13.4.2.21 with $a = 1$ and $b = 1$.

9. $(Ay^2 + 2Axy + Bx^2 + A - B)y'_x = Ay^2 + 2Bxy + Bx^2 - A + B.$

This is a special case of equation 13.4.2.21 with $a = 1$ and $b = -1$.

10. $(Ay^2 - 4Axy + Bx^2 + 4A - B)y'_x = -2Ay^2 + 2Bxy - 2Bx^2 + 8A - 2B.$

This is a special case of equation 13.4.2.21 with $a = 1$ and $b = 2$.

11. $(Ay^2 + 4Axy + Bx^2 + 4A - B)y'_x = 2Ay^2 + 2Bxy + 2Bx^2 - 8A + 2B.$

This is a special case of equation 13.4.2.21 with $a = 1$ and $b = -2$.

12. $(Ay^2 - 6Axy + Bx^2 + 9A - B)y'_x = -3Ay^2 + 2Bxy - 3Bx^2 + 27A - 3B.$

This is a special case of equation 13.4.2.21 with $a = 1$ and $b = 3$.

13. $(Ay^2 + 6Axy + Bx^2 + 9A - B)y'_x = 3Ay^2 + 2Bxy + 3Bx^2 - 27A + 3B.$

This is a special case of equation 13.4.2.21 with $a = 1$ and $b = -3$.

14. $2(Ay^2 - Axy + Bx^2 + A - 4B)y'_x = -Ay^2 + 4Bxy - Bx^2 + A - 4B.$

This is a special case of equation 13.4.2.21 with $a = 2$ and $b = 1$.

15. $2(Ay^2 + Axy + Bx^2 + A - 4B)y'_x = Ay^2 + 4Bxy + Bx^2 - A + 4B.$

This is a special case of equation 13.4.2.21 with $a = 2$ and $b = -1$.

16. $(ay^2 - 2bxy + ax^2 + ab^2 - a^3)y'_x = -by^2 + 2axy - bx^2 + b^3 - a^2b.$

This is a special case of equation 13.4.2.21 with $A = 1$ and $B = 1$.

17. $(ay^2 - 2bxy - ax^2 + ab^2 + a^3)y'_x = -by^2 - 2axy + bx^2 + b^3 + a^2b.$

This is a special case of equation 13.4.2.21 with $A = 1$ and $B = -1$.

18. $(ay^2 - 2bxy + 2ax^2 + ab^2 - 2a^3)y'_x = -by^2 + 4axy - 2bx^2 + b^3 - 2a^2b.$

This is a special case of equation 13.4.2.21 with $A = 1$ and $B = 2$.

19. $(ay^2 - 2bxy - 2ax^2 + ab^2 + 2a^3)y'_x = -by^2 - 4axy + 2bx^2 + b^3 + 2a^2b.$

This is a special case of equation 13.4.2.21 with $A = 1$ and $B = -2$.

20. $(Ay^2 + Bxy + Cx^2 + a)y'_x$
$$= Dy^2 + k(2Ak + B - 2D)xy + k(-Ak^2 + Dk + C)x^2 + b.$$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[(D - Ak)z^2 + b - ak]x'_z = (Ak^2 + Bk + C)x^2 + (2Ak + B)zx + Az^2 + a.$$

21. $(aAy^2 - 2bAxy + aBx^2 + ab^2A - a^3B)y'_x$
$$= -bAy^2 + 2aBxy - bBx^2 + b^3A - a^2bB.$$

The transformation $x = w + a,\ y = \xi w + b$ leads to a linear equation:

$$(-aA\xi^3 + bA\xi^2 + aB\xi - bB)w'_\xi = (aA\xi^2 - 2bA\xi + aB)w + 2a^2B - 2b^2A.$$

13.4.3 Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x$

► **Preliminary remarks.**

1°. For $A_{22} = 0$, this is an Abel equation (see Section 13.3.4). For $B_{11} = 0$ this is an Abel equation with respect to $x = x(y)$.

2°. The transformation $\xi = y/x,\ w = 1/x$ leads to an Abel equation of the second kind:

$$\{[A_2\xi^2 + (A_1 - B_2)\xi - B_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi$$
$$= (A_2\xi + A_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w.$$

3°. In Item 3° of Section 13.4.4, another transformation is given which reduces the original equation to an Abel equation of the second kind.

4°. Dynamical systems of the second-order

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

which describe the behavior of simplest Lagrangian and Hamiltonian systems in mechanics, are often reduced to equations of the type in question if

$$\begin{aligned} P(x, y) &= f(x, y)(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2 y + A_1 x), \\ Q(x, y) &= f(x, y)(B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2 y + B_1 x), \end{aligned} \tag{2}$$

where $f = f(x, y)$ is an arbitrary function.

In particular, dynamical systems (1) with functions (2) and $f \equiv 1$ arise in analyzing complex equilibrium states. In this case, the functions P and Q are substituted by their Taylor-series expansions in the vicinity of the equilibrium state $x = y = 0$ with the first- and second-order terms retained.

Whenever a solution of the ordinary differential equation

$$(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2 y + A_1 x)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2 y + B_1 x$$

is obtained in parametric form, $x = x(u, C_1)$, $y = y(u, C_1)$, the corresponding solution of system (1), (2) is determined by

$$x = x(u, C_1), \quad y = y(u, C_1), \quad t = \int \frac{x'_u\, du}{P\bigl(x(u, C_1),\, y(u, C_1)\bigr)} + C_2.$$

The last relation defines an implicit dependence of the parameter u on t, $u = u(t, C_1, C_2)$, and makes it possible to establish, with the aid of the first two formulas, the dependence of x and y on t.

► **Solvable equations and their solutions.**

1. $\boldsymbol{(y^2 - x^2 + ay)y'_x = y^2 - x^2 + ax.}$

Solution in parametric form:

$$x = at + C|t|^{-1}e^{4t}, \qquad y = -at + C|t|^{-1}e^{4t}.$$

2. $\boldsymbol{(y^2 - x^2 + ay)y'_x = 2y^2 - 2xy + ay.}$

Solution in parametric form:

$$x = t + Ct^2 e^{a/t}, \qquad y = Ct^2 e^{a/t}.$$

3. $\boldsymbol{(y^2 - x^2 + ay - ax)y'_x = y^2 - x^2 - ay + ax.}$

Solution in parametric form:

$$x = at + Ce^{2t}, \qquad y = -at + Ce^{2t}.$$

4. $\boldsymbol{(y^2 - x^2 + ay + 2ax)y'_x = y^2 - x^2 + 2ay + ax.}$

Solution in parametric form:

$$x = -at + C|t|^3 e^{4t}, \qquad y = at + C|t|^3 e^{4t}.$$

5. $(y^2 - x^2 + ay + 2ax)y'_x = 2xy - 2x^2 + ay + 2ax.$

Solution in parametric form:

$$x = t + Ct^{-2}e^{-a/t}, \quad y = -2t + Ct^{-2}e^{-a/t}.$$

6. $(y^2 - x^2 + ay - 2ax)y'_x = 4y^2 - 6xy + 2x^2 + ay - 2ax.$

Solution in parametric form:

$$x = \tfrac{1}{3}t + C|t|^{2/3}e^{a/t}, \quad y = \tfrac{2}{3}t + C|t|^{2/3}e^{a/t}.$$

7. $(y^2 - x^2 + ay + 3ax)y'_x = -y^2 + 4xy - 3x^2 + ay + 3ax.$

Solution in parametric form:

$$x = \tfrac{1}{2}t + C|t|^{-1}e^{-a/t}, \quad y = -\tfrac{3}{2}t + C|t|^{-1}e^{-a/t}.$$

8. $(y^2 - xy + ay + ax)y'_x = xy - x^2 + ay + ax.$

Solution in parametric form:

$$x = -t + C|t|^{-1}e^{a/t}, \quad y = t + C|t|^{-1}e^{a/t}.$$

9. $(y^2 - xy + ay + ax)y'_x = y^2 - xy + 2ay.$

Solution in parametric form:

$$x = -at + Ct^2e^t, \quad y = Ct^2e^t.$$

10. $(y^2 - xy + ay - 2ax)y'_x = 3y^2 - 5xy + 2x^2 + ay - 2ax.$

Solution in parametric form:

$$x = \tfrac{1}{2}t + C|t|^{1/2}e^{a/t}, \quad y = t + C|t|^{1/2}e^{a/t}.$$

11. $(y^2 + xy - 2x^2 + ay + ax)y'_x = y^2 + xy - 2x^2 + 2ax.$

Solution in parametric form:

$$x = at + Ct^{-2}e^{9t}, \quad y = -2at + Ct^{-2}e^{9t}.$$

12. $(y^2 + xy - 2x^2 + ay + ax)y'_x = 2y^2 - xy - x^2 + ay + ax.$

Solution in parametric form:

$$x = t + C|t|^3e^{a/t}, \quad y = -t + C|t|^3e^{a/t}.$$

13. $(y^2 + xy - 2x^2 + ay - ax)y'_x = y^2 + xy - 2x^2 - 2ay + 2ax.$

Solution in parametric form:

$$x = at + Ce^{3t}, \quad y = -2at + Ce^{3t}.$$

14. $(y^2 + xy - 2x^2 + ay - 2ax)y'_x = 5y^2 - 7xy + 2x^2 + ay - 2ax.$

Solution in parametric form:

$$x = \tfrac{1}{4}t + C|t|^{3/4}e^{a/t}, \quad y = \tfrac{1}{2}t + C|t|^{3/4}e^{a/t}.$$

15. $(y^2 - 2xy + x^2 + ay)y'_x = ay.$

Solution: $x = y + \dfrac{a}{C - \ln|y|}.$

16. $(y^2 - 2xy + x^2 + ay + ax)y'_x = -y^2 + 2xy - x^2 + ay + ax.$

Solution in parametric form:

$$x = -\frac{a}{2\ln|t|} + Ct, \quad y = \frac{a}{2\ln|t|} + Ct.$$

17. $(y^2 - 2xy + x^2 + ay + 2ax)y'_x = -2(y^2 - 2xy + x^2) + ay + 2ax.$

Solution in parametric form:

$$x = -\frac{a}{3\ln|t|} + Ct, \quad y = \frac{2a}{3\ln|t|} + Ct.$$

18. $(y^2 - 2xy + x^2 + ay - 2ax)y'_x = 2(y^2 - 2xy + x^2) + ay - 2ax.$

Solution in parametric form:

$$x = \frac{a}{\ln|t|} + Ct, \quad y = \frac{2a}{\ln|t|} + Ct.$$

19. $(y^2 + 2xy + x^2 + ay + 2ax)y'_x = -y^2 - 2xy - x^2 + 2ay + ax.$

Solution in parametric form:

$$x = C^2\left(t^{1/3} + \frac{4t^2}{5a}\right) + Ct, \quad y = -C^2\left(t^{1/3} + \frac{4t^2}{5a}\right) + Ct, \qquad a \neq 0.$$

20. $(y^2 + 2xy + x^2 + ay - ax)y'_x = -y^2 - 2xy - x^2 + ay - ax.$

Solution in parametric form:

$$x = C^3\sqrt{1 - \frac{4t^3}{3a}} + C^2 t, \quad y = -C^3\sqrt{1 - \frac{4t^3}{3a}} + C^2 t, \qquad a \neq 0.$$

21. $(y^2 + 2xy + x^2 + ay - 2ax)y'_x = -y^2 - 2xy - x^2 - 2ay + ax.$

Solution in parametric form:

$$x = C^2\left(t^3 + \frac{4t^2}{a}\right) + Ct, \quad y = -C^2\left(t^3 + \frac{4t^2}{a}\right) + Ct, \qquad a \neq 0.$$

22. $(y^2 + 2xy - 3x^2 + ay + ax)y'_x = 3y^2 - 2xy - x^2 + ay + ax.$

Solution in parametric form:

$$x = \tfrac{1}{2}t + Ct^2 e^{a/t}, \quad y = -\tfrac{1}{2}t + Ct^2 e^{a/t}.$$

23. $(y^2 + 2xy - 3x^2 + ay + ax)y'_x = y^2 + 2xy - 3x^2 - ay + 3ax.$

Solution in parametric form:

$$x = at + C|t|^{-1}e^{8t}, \quad y = -3at + C|t|^{-1}e^{8t}.$$

24. $(y^2 + 2xy - 3x^2 + ay + 2ax)y'_x = y^2 + 2xy - 3x^2 + 3ax.$

Solution in parametric form:

$$x = at + C|t|^{-3}e^{16t}, \quad y = -3at + C|t|^{-3}e^{16t}.$$

25. $(y^2 - x^2 + ay + bx)y'_x = y^2 - x^2 + by + ax.$

Solution in parametric form:

$$x = (a-b)t + C|t|^{-\frac{a+b}{a-b}} e^{4t}, \quad y = (b-a)t + C|t|^{-\frac{a+b}{a-b}} e^{4t}, \qquad a \neq b.$$

26. $(y^2 - xy + ay + bx)y'_x = y^2 - xy + (a+b)y.$

Solution in parametric form:

$$x = -bt + C|t|^{\frac{a+b}{b}} e^{t}, \quad y = C|t|^{\frac{a+b}{b}} e^{t}, \qquad b \neq 0.$$

27. $(y^2 + xy - 2x^2 + ay + bx)y'_x = y^2 + xy - 2x^2 + (b-a)y + 2ax.$

Solution in parametric form:

$$x = (2a-b)t + C|t|^{-\frac{a+b}{2a-b}} e^{9t}, \quad y = 2(b-2a)t + C|t|^{-\frac{a+b}{2a-b}} e^{9t}, \qquad b \neq 2a.$$

28. $(y^2 - 2xy + x^2 + ay - abx)y'_x = b(y^2 - 2xy + x^2) + ay - abx.$

Solution in parametric form:

$$x = \frac{a}{b-1}\frac{1}{\ln|t|} + Ct, \quad y = \frac{ab}{b-1}\frac{1}{\ln|t|} + Ct, \qquad b \neq 1.$$

29. $(y^2 + 2xy - 3x^2 + ay + bx)y'_x = y^2 + 2xy - 3x^2 + (b-2a)y + 3ax.$

Solution in parametric form:

$$x = (3a-b)t + C|t|^{-\frac{a+b}{3a-b}} e^{16t}, \quad y = 3(b-3a)t + C|t|^{-\frac{a+b}{3a-b}} e^{16t}, \qquad b \neq 3a.$$

30. $(y^2 - 3xy + 2x^2 + ay + bx)y'_x = y^2 - 3xy + 2x^2 + (3a+b)y - 2ax.$

Solution in parametric form:

$$x = (2a+b)t + C|t|^{\frac{a+b}{2a+b}} e^{-t}, \quad y = 2(2a+b)t + C|t|^{\frac{a+b}{2a+b}} e^{-t}, \qquad b \neq -2a.$$

31. $(y^2 + 3xy - 4x^2 + ay + bx)y'_x = y^2 + 3xy - 4x^2 + (b-3a)y + 4ax.$

Solution in parametric form:

$$x = (4a-b)t + C|t|^{-\frac{a+b}{4a-b}} e^{25t}, \quad y = 4(b-4a)t + C|t|^{-\frac{a+b}{4a-b}} e^{25t}, \qquad b \neq 4a.$$

32. $[y^2 + Axy - (A+1)x^2 + by - 2bx]y'_x$
$= (A+4)y^2 - (A+6)xy + 2x^2 + by - 2bx.$

Solution in parametric form:

$$x = \frac{t}{A+3} + C|t|^{\frac{A+2}{A+3}} e^{b/t}, \quad y = \frac{2t}{A+3} + C|t|^{\frac{A+2}{A+3}} e^{b/t}, \qquad A \neq -3.$$

33. $(y^2 - 2Axy + A^2x^2 + by - bx)y'_x = Ay^2 - 2A^2xy + A^3x^2 + by - bx.$

Solution in parametric form:

$$x = C^3\sqrt{1 + \frac{2(A-1)}{3b}t^3} + C^2 t, \quad y = AC^3\sqrt{1 + \frac{2(A-1)}{3b}t^3} + C^2 t, \qquad b \neq 0.$$

34. $[y^2-2Axy+(2A-1)x^2+by-Abx]y'_x=(2-A)y^2-2xy+Ax^2+by-Abx.$

Solution in parametric form:

$$x=\frac{t}{1-A}+Ct^2e^{b/t},\quad y=\frac{At}{1-A}+Ct^2e^{b/t},\qquad A\neq 1.$$

35. $(y^2-2Axy+A^2x^2+ay+bx)y'_x$
$=A(y^2-2Axy+A^2x^2)+(aA+a+b)y-aAx.$

Solution in parametric form:

$$x=C^2\left[t^{\frac{aA+b}{a+b}}+\frac{(1-A)^2}{(2-A)a+b}t^2\right]+Ct,\quad y=AC^2\left[t^{\frac{aA+b}{a+b}}+\frac{(1-A)^2}{(2-A)a+b}t^2\right]+Ct,$$

where $a+b\neq 0$ and $(2-A)a+b\neq 0$.

36. $[y^2-(A+2)xy+(A+1)x^2+by-Abx]y'_x=-Axy+Ax^2+by-Abx.$

Solution in parametric form:

$$x=\frac{t}{1-A}+C|t|^Ae^{(A-1)b/t},\quad y=\frac{At}{1-A}+C|t|^Ae^{(A-1)b/t},\qquad A\neq 1.$$

37. $[Ay^2+xy-(A+1)x^2+by+bx]y'_x=(A+1)y^2-xy-Ax^2+by+bx.$

Solution in parametric form:

$$x=t+C|t|^{2A+1}e^{b/t},\qquad y=-t+C|t|^{2A+1}e^{b/t}.$$

38. $(Ay^2+Bxy+Cx^2+kx)y'_x=Dy^2+Exy+Fx^2+ky.$

The substitution $y=xz$ leads to a linear equation with respect to $x=x(z)$:

$$[-Az^3+(D-B)z^2+(E-C)z+F]x'_z=(Az^2+Bz+C)x+k.$$

39. $(Ay^2+Bxy+Cx^2-\alpha By-\alpha Cx)y'_x=Dy^2+Exy+\alpha(C-E)y.$

The transformation $x=w+\alpha$, $y=\xi w$ leads to a linear equation:

$$[-A\xi^3+(D-B)\xi^2+(E-C)\xi]w'_\xi=(A\xi^2+B\xi+C)w+\alpha C.$$

40. $(Ay^2+2Bxy+Ak^2x^2+ay+bx)y'_x=By^2+2Ak^2xy+Bk^2x^2+by+ak^2x.$

This is a special case of equation 13.4.3.57 with $C=Ak^2$.

41. $(Ay^2+2Bxy+Ak^2x^2+ay+bx)y'_x=By^2+2Ak^2xy+Bk^2x^2+aky+bkx.$

This is a special case of equation 13.4.3.62 with $C=Ak^2$.

42. $(Ay^2+2Bxy+Ak^2x^2+ay-akx)y'_x$
$=By^2+2Ak^2xy+Bk^2x^2+my-mkx.$

This is a special case of equation 13.4.3.61 with $C=Ak^2$.

43. $(Ay^2+2Bxy-Bkx^2+ay+bx)y'_x=By^2+2Ak^2xy-Ak^3x^2+by+ak^2x.$

This is a special case of equation 13.4.3.58 with $m=b$.

44. $(Ay^2+2Bxy-Bkx^2+ay+bx)y'_x=By^2+2Ak^2xy-Ak^3x^2+aky+bkx.$

This is a special case of equation 13.4.3.62 with $C=-Bk$.

45. $(Ay^2+2Bxy-Bkx^2+ay-akx)y'_x$
$$=By^2+2Ak^2xy-Ak^3x^2+my-mkx.$$

This is a special case of equation 13.4.3.61 with $C=-Bk$.

46. $(Ay^2+2Akxy+Cx^2+ay+bx)y'_x=Aky^2+2Ak^2xy+Ckx^2+by+ak^2x.$

This is a special case of equation 13.4.3.57 with $B=Ak$.

47. $(Ay^2+2Akxy+Cx^2+ay+bx)y'_x=Aky^2+2Ak^2xy+Ckx^2+aky+bkx.$

This is a special case of equation 13.4.3.62 with $B=Ak$.

48. $(Ay^2+2Akxy+Cx^2+ay-akx)y'_x$
$$=Aky^2+2Ak^2xy+Ckx^2+my-mkx.$$

This is a special case of equation 13.4.3.61 with $B=Ak$.

49. $(Ay^2-2Akxy+Bkx^2+ay+bx)y'_x$
$$=-By^2+2Bkxy-Ak^3x^2+by+ak^2x.$$

This is a special case of equation 13.4.3.59 with $m=b$.

50. $(Ay^2-2Akxy+Bkx^2+ay+bx)y'_x$
$$=-By^2+2Bkxy-Ak^3x^2+aky+bkx.$$

This is a special case of equation 13.4.3.59 with $m=ak$.

51. $[y^2+2Axy+A^2x^2+(A-1)By-2ABx]y'_x$
$$=-A(y^2+2Axy+A^2x^2)-(A^2+1)By+A(A-1)Bx.$$

Solution in parametric form ($A\neq 2,\ B\neq 0$):

$$x=C^2\left[t^A+\frac{A+1}{(A-2)B}t^2\right]+Ct,\qquad y=-AC^2\left[t^A+\frac{A+1}{(A-2)B}t^2\right]+Ct.$$

52. $[y^2-2Axy+A^2x^2+(B-1)ky+(A-B)kx]y'_x$
$$=A(y^2-2Axy+A^2x^2)+(AB-1)ky-A(B-1)kx.$$

Solution in parametric form ($B\neq 2,\ k\neq 0$):

$$x=C^2\left[t^B-\frac{A-1}{(B-2)k}t^2\right]+Ct,\qquad y=AC^2\left[t^B-\frac{A-1}{(B-2)k}t^2\right]+Ct.$$

53. $[2y^2-(A+3)xy+(A+1)x^2+By-ABx]y'_x$
$$=(A+1)y^2-(3A+1)xy+2Ax^2+By-ABx.$$

Solution in parametric form:

$$x=\frac{t}{1-A}+C|t|^{-1}e^{-B/t},\quad y=\frac{At}{1-A}+C|t|^{-1}e^{-B/t},\qquad A\neq 1.$$

54. $[2y^2-(3A+1)xy+(3A-1)x^2+By-ABx]y'_x$
$=(3-A)y^2-(A+3)xy+2Ax^2+By-ABx.$

Solution in parametric form:

$$x=\frac{t}{1-A}+C|t|^3e^{B/t},\quad y=\frac{At}{1-A}+C|t|^3e^{B/t},\qquad A\neq 1.$$

55. $[A(y^2-2xy+x^2)-A(A-B)y+B(A-B)x]y'_x$
$=B(y^2-2xy+x^2)-A(A-B)y+B(A-B)x.$

Solution in parametric form:

$$x=\frac{A}{\ln|t|}+Ct,\quad y=\frac{B}{\ln|t|}+Ct.$$

56. $(Ay^2+Bxy+Cx^2+ay+bx)y'_x$
$=Aky^2+Bkxy+Ckx^2+ny+(ak+b-n)x.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$(n-ak)zx'_z=(Ak^2+Bk+C)x^2+[(2Ak+B)z+ak+b]x+Az^2+az.$$

57. $(Ay^2+2Bxy+Cx^2+ay+bx)y'_x$
$=By^2+2Ak^2xy+k(-Ak^2+Bk+C)x^2+by+ak^2x.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z+b-ak]zx'_z=(Ak^2+2Bk+C)x^2+[2(Ak+B)z+ak+b]x+Az^2+az.$$

58. $(Ay^2+2Bxy-Bkx^2+ay+bx)y'_x$
$=By^2+2Ak^2xy-Ak^3x^2+my+k(ak+b-m)x.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z+m-ak]zx'_z=(Ak^2+Bk)x^2+[2(Ak+B)z+ak+b]x+Az^2+az.$$

59. $(Ay^2-2Akxy+Bkx^2+ay+bx)y'_x$
$=-By^2+2Bkxy-Ak^3x^2+my+k(ak+b-m)x.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[-(Ak+B)z+m-ak]zx'_z=k(B-Ak)x^2+(ak+b)x+Az^2+az.$$

60. $(Ay^2+2Bxy+Ak^2x^2+ay+bx)y'_x$
$=By^2+2Ak^2xy+Bk^2x^2+my+k(ak+b-m)x.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z+m-ak]zx'_z=2k(Ak+B)x^2+[2(Ak+B)z+ak+b]x+Az^2+az.$$

61. $(Ay^2+2Bxy+Cx^2+ay-akx)y'_x$
$=By^2+2Ak^2xy+k(-Ak^2+Bk+C)x^2+my-mkx.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z^2+m-ak]zx'_z=(Ak^2+2Bk+C)x^2+2(Ak+B)zx+Az^2+az.$$

62. $(Ay^2 + 2Bxy + Cx^2 + ay + bx)y'_x$
$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + aky + bkx.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$(B - Ak)z^2x'_z = (Ak^2 + 2Bk + C)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

63. $\{(A-1)y^2 + [2 - A(k+1)]xy + (Ak-1)x^2 + By - Bkx\}y'_x$
$= (A-k)y^2 + [2k - A(k+1)]xy + (A-1)kx^2 + By - Bkx.$

Solution in parametric form:

$$x = \frac{t}{1-k} + C|t|^A e^{B/t}, \quad y = \frac{kt}{1-k} + C|t|^A e^{B/t}, \qquad k \neq 1.$$

64. $[A(\alpha y^2 + \beta xy + \gamma x^2) + (2\alpha - A^2\sigma)y + (\beta - AB\sigma)x]y'_x$
$+ B(\alpha y^2 + \beta xy + \gamma x^2) + (\beta - AB\sigma)y + (2\gamma - B^2\sigma)x = 0.$

Solution: $\alpha y^2 + \beta xy + \gamma x^2 - A\sigma y - B\sigma x + \sigma = C\exp(-Ay - Bx)$.

65. $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x$
$= B_{22}y^2 + k(2A_{22}k + A_{12} - 2B_{22})xy + k(-A_{22}k^2 + B_{22}k + A_{11})x^2$
$+ B_2y + k(A_2k + A_1 - B_2)x.$

The substitution $y = z + kx$ leads to a Riccati equation with respect to $x = x(z)$:

$$[(B_{22} - A_{22}k)z + B_2 - A_2k]zx'_z = (A_{22}k^2 + A_{12}k + A_{11})x^2 + [(2A_{22}k + A_{12})z + A_2k + A_1]x + A_{22}z^2 + A_2z.$$

◆ *In equations 66–70, the following notation is used:* $\Delta = Ab - aB \neq 0$, $\delta = Ab + aB$.

66. $(Aa^2y^2 - 2Aabxy + Ab^2x^2 - \Delta Aay + \Delta aBx)y'_x$
$= a^2By^2 - 2aBbxy + Bb^2x^2 - \Delta Aby + \Delta Bbx.$

Solution in parametric form:

$$x = \frac{A}{\ln|t|} + aCt, \quad y = \frac{B}{\ln|t|} + bCt.$$

67. $[kAa^2y^2 - k\delta axy + kaBbx^2 - lAay + (laB - \Delta)x]y'_x$
$= kAaby^2 - k\delta bxy + kBb^2x^2 - (lAb + \Delta)y + lBbx.$

Solution in parametric form:

$$x = At + aC|t|^{l+1}e^{k\Delta t}, \quad y = Bt + bC|t|^{l+1}e^{k\Delta t}.$$

68. $[kAa^2y^2 - a(k\delta - \Delta)xy + b(kaB - \Delta)x^2 + lAay - laBx]y'_x$
$= a(kBb + \Delta)y^2 - b(k\delta + \Delta)xy + kBb^2x^2 + lAby - lBbx.$

Solution in parametric form:

$$x = At + aC|t|^{k+1}\exp\Bigl(\frac{l}{\Delta t}\Bigr), \quad y = Bt + bC|t|^{k+1}\exp\Bigl(\frac{l}{\Delta t}\Bigr).$$

69. $(kA^3y^2 - 2kA^2Bxy + kAB^2x^2 - a^2y + abx)y'_x$
$$= kA^2By^2 - 2kAB^2xy + kB^3x^2 - aby + b^2x.$$

Solution in parametric form:

$$x = AC^3\sqrt{\tfrac{2}{3}k\Delta t^3 + 1} + aC^2t, \qquad y = BC^3\sqrt{\tfrac{2}{3}k\Delta t^3 + 1} + bC^2t.$$

70. $\big[kA^3y^2 - 2kA^2Bxy + kAB^2x^2 + lAay - (lAb + \Delta)x\big]y'_x$
$$= kA^2By^2 - 2kAB^2xy + kB^3x^2 + (laB - \Delta)y - lBbx.$$

Solution in parametric form ($l \neq 1$):

$$x = AC^2\Big(t^{l+1} + \frac{k\Delta}{l-1}t^2\Big) + aCt, \qquad y = BC^2\Big(t^{l+1} + \frac{k\Delta}{l-1}t^2\Big) + bCt.$$

13.4.4 Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x + A_0)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x + B_0$

► **Preliminary remarks. Some transformations.**

1°. With $A_{22} = 0$, this is an Abel equation (see Section 13.3.4). With $B_{11} = 0$, this is an Abel equation with respect to $x = x(y)$.

See Section 13.4.2 for the case $A_2 = A_1 = B_2 = B_1 = 0$.

See Section 13.4.3 for the case $A_0 = B_0 = 0$.

2°. The transformation $x = \bar{x} + \alpha$, $y = \bar{y} + \beta$, where α and β are parameters, which are determined by solving the second-order algebraic system

$$A_{22}\beta^2 + A_{12}\alpha\beta + A_{11}\alpha^2 + A_2\beta + A_1\alpha + A_0 = 0,$$
$$B_{22}\beta^2 + B_{12}\alpha\beta + B_{11}\alpha^2 + B_2\beta + B_1\alpha + B_0 = 0,$$

leads to the equation

$$(A_{22}\bar{y}^2 + A_{12}\bar{x}\bar{y} + A_{11}\bar{x}^2 + a_2\bar{y} + a_1\bar{x})\bar{y}'_{\bar{x}} = B_{22}\bar{y}^2 + B_{12}\bar{x}\bar{y} + B_{11}\bar{x}^2 + b_2\bar{y} + b_1\bar{x}, \quad (1)$$

where

$$a_2 = 2A_{22}\beta + A_{12}\alpha + A_2, \qquad a_1 = 2A_{11}\alpha + A_{12}\beta + A_1,$$
$$b_2 = 2B_{22}\beta + B_{12}\alpha + B_2, \qquad b_1 = 2B_{11}\alpha + B_{12}\beta + B_1.$$

The transformation $\xi = \bar{y}/\bar{x}$, $w = 1/\bar{x}$ reduces Eq. (1) to an Abel equation of the second kind:

$$\{[a_2\xi^2 + (a_1 - b_2)\xi - b_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi$$
$$= (a_2\xi + a_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w.$$

3°. The substitution $y = z + \varepsilon x$, where the parameter ε is determined by solving the cubic equation

$$(A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\varepsilon - B_{22}\varepsilon^2 - B_{12}\varepsilon - B_{11} = 0,$$

leads to an Abel equation of the second kind with respect to $x = x(z)$:

$$[(Qz + R)x + (B_{22} - A_{22}\varepsilon)z^2 + (B_2 - A_2\varepsilon)z + B_0 - A_0\varepsilon]x'_z$$
$$= (A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})x^2 + [(2A_{22}\varepsilon + A_{12})z + A_2\varepsilon + A_1]x + A_{22}z^2 + A_2z + A_0,$$

where $Q = 2B_{22}\varepsilon + B_{12} - \varepsilon(2A_{22}\varepsilon + A_{12})$, $R = B_2\varepsilon + B_1 - \varepsilon(A_2\varepsilon + A_1)$.

► **Solvable equations and their solutions.**

1. $(ax+by+c)^2 y'_x = (\alpha x+\beta y+\gamma)^2.$

This is a special case of equation 13.7.1.6 with $f(z)=z^{-2}$.

2. $(Ay^2+Bxy-\alpha By+kx-\alpha k)y'_x = Cy^2+Dxy+(k-\alpha D)y.$

The transformation $x=w+\alpha,\ y=w\xi$ leads to a linear equation:

$$[-A\xi^3+(C-B)\xi^2+D\xi]w'_\xi = (A\xi^2+B\xi)w+k.$$

3. $(Ay^2+2Axy+Bx^2+A-B)y'_x = Ay^2+2Bxy+Dx^2+2(B-D)x+D-A.$

The transformation $x=w+1,\ y=\xi w-1$ leads to a linear equation:

$$(-A\xi^3-A\xi^2+B\xi+D)w'_\xi = (A\xi^2+2A\xi+B)w+2(B-A).$$

4. $(Ay^2-2Axy+Bx^2+A-B)y'_x = -Ay^2+2Bxy+Cx^2+2(B+C)x+A+C.$

The transformation $x=w-1,\ y=\xi w-1$ leads to a linear equation:

$$(-A\xi^3+A\xi^2+B\xi+C)w'_\xi = (A\xi^2-2A\xi+B)w+2(A-B).$$

5. $(Ay^2+2Axy+Bx^2+A-B)y'_x = Ay^2+2Bxy+Cx^2+2(C-B)x-A+C.$

The transformation $x=w-1,\ y=\xi w+1$ leads to a linear equation:

$$(-A\xi^3-A\xi^2+B\xi+C)w'_\xi = (A\xi^2+2A\xi+B)w+2(A-B).$$

6. $(Ay^2-2Axy+Bx^2+A-B)y'_x = -Ay^2+2Bxy+Cx^2-2(B+C)x+A+C.$

The transformation $x=w+1,\ y=\xi w+1$ leads to a linear equation:

$$(-A\xi^3+A\xi^2+B\xi+C)w'_\xi = (A\xi^2-2A\xi+B)w+2(B-A).$$

7. $(Ay^2-2Axy+Bx^2+A-B)y'_x$
$= Cy^2+2Bxy+Dx^2-2(A+C)y-2(B+D)x+2A+C+D.$

The transformation $x=w+1,\ y=\xi w+1$ leads to a linear equation:

$$[-A\xi^3+(2A+C)\xi^2+B\xi+D]w'_\xi = (A\xi^2-2A\xi+B)w+2(B-A).$$

8. $(2Ay^2-2Axy+Bx^2+2A-4B)y'_x$
$= -Ay^2+2Bxy+Dx^2-2(B+2D)x+A+4D.$

This is a special case of equation 13.4.4.34 with $\alpha=2$, $\beta=1$, and $C=-A$.

9. $(Ay^2+4Axy+Bx^2+4A-B)y'_x$
$= 2Ay^2+2Bxy+Cx^2-2(C-2B)x+C-8A.$

The transformation $x=w+1,\ y=\xi w-2$ leads to a linear equation:

$$(-A\xi^3-2A\xi^2+B\xi+C)w'_\xi = (A\xi^2+4A\xi+B)w+2B-8A.$$

10. $(Ay^2-4Axy+Bx^2+4A-B)y'_x$
$= -2Ay^2+2Bxy+Cx^2-2(2B+C)x+8A+C.$

The transformation $x=w+1,\ y=\xi w+2$ leads to a linear equation:

$$(-A\xi^3+2A\xi^2+B\xi+C)w'_\xi = (A\xi^2-4A\xi+B)w+2B-8A.$$

11. $(Ay^2 + 4Axy + Bx^2 + 4A - B)y'_x$
$= Cy^2 + 2Bxy + 2Bx^2 + 4(C - 2A)y + 2B + 4C - 16A.$

The transformation $x = w + 1,\ y = \xi w - 2$ leads to a linear equation:

$$[-A\xi^3 + (C - 4A)\xi^2 + B\xi + 2B]w'_\xi = (A\xi^2 + 4A\xi + B)w + 2B - 8A.$$

12. $(2Ay^2 + 2Axy + Bx^2 + 2A - 4B)y'_x$
$= Ay^2 + 2Bxy + Dx^2 + 2(B - 2D)x + 4D - A.$

This is a special case of equation 13.4.4.34 with $\alpha = 2$, $\beta = -1$, and $C = A$.

13. $(2Ay^2 + 2Axy - Bx^2 + 2A + 4B)y'_x$
$= Ay^2 - 2Bxy - Dx^2 + 2(B - 2D)x - A - 4D.$

This is a special case of equation 13.4.4.34 with $\alpha = -2$, $\beta = 1$, and $C = -A$.

14. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx + m)y'_x$
$= By^2 + 2Ak^2xy + Bk^2x^2 + by + ak^2x + s.$

This is a special case of equation 13.4.4.27 with $C = Ak^2$.

15. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx + m)y'_x$
$= By^2 + 2Ak^2xy + Bk^2x^2 + aky + bkx + s.$

This is a special case of equation 13.4.4.32 with $C = Ak^2$.

16. $(Ay^2 + 2Bxy + Ak^2x^2 + ay - akx + b)y'_x$
$= By^2 + 2Ak^2xy + Bk^2x^2 + my - mkx + s.$

This is a special case of equation 13.4.4.31 with $C = Ak^2$.

17. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx + c)y'_x$
$= By^2 + 2Ak^2xy - Ak^3x^2 + by + ak^2x + s.$

This is a special case of equation 13.4.4.28 with $m = b$.

18. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx + m)y'_x$
$= By^2 + 2Ak^2xy - Ak^3x^2 + aky + bkx + s.$

This is a special case of equation 13.4.4.32 with $C = -Bk$.

19. $(Ay^2 + 2Bxy - Bkx^2 + ay - akx + b)y'_x$
$= By^2 + 2Ak^2xy - Ak^3x^2 + my - mkx + s.$

This is a special case of equation 13.4.4.31 with $C = -Bk$.

20. $(Ay^2 + 2Akxy + Cx^2 + ay + bx + m)y'_x$
$= Aky^2 + 2Ak^2xy + Ckx^2 + by + ak^2x + s.$

This is a special case of equation 13.4.4.27 with $B = Ak$.

21. $(Ay^2 + 2Akxy + Cx^2 + ay + bx + m)y'_x$
$= Aky^2 + 2Ak^2xy + Ckx^2 + aky + bkx + s.$

This is a special case of equation 13.4.4.32 with $B = Ak$.

22. $(Ay^2+2Akxy+Cx^2+ay-akx+b)y'_x$
$$=Aky^2+2Ak^2xy+Ckx^2+my-mkx+s.$$

This is a special case of equation 13.4.4.31 with $B=Ak$.

23. $(Ay^2-2Akxy+Bkx^2+ay+bx+c)y'_x$
$$=-By^2+2Bkxy-Ak^3x^2+by+ak^2x+s.$$

This is a special case of equation 13.4.4.29 with $m=b$.

24. $(Ay^2-2Akxy+Bkx^2+ay+bx+c)y'_x$
$$=-By^2+2Bkxy-Ak^3x^2+aky+bkx+s.$$

This is a special case of equation 13.4.4.29 with $m=ak$.

25. $(Ay^2+2Bxy+Cx^2-2A\beta y+kx+A\beta^2)y'_x$
$$=By^2+Exy+Fx^2+ky-E\beta x-B\beta^2-k\beta.$$

The substitution $w=y-\beta$ leads to an equation of the form 13.4.3.38:

$$(Aw^2+2Bxw+Cx^2+\bar{k}x)w'_x=Bw^2+Exw+Fx^2+\bar{k}w,\quad\text{where}\quad\bar{k}=k+2B\beta.$$

26. $(Ay^2+Bxy+Cx^2+ay+bx+m)y'_x$
$$=Aky^2+Bkxy+Ckx^2+ny+k(ak+b-n)x+s.$$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(n-ak)z+s-mk]x'_z=(Ak^2+Bk+C)x^2+[(2Ak+B)z+ak+b]x+Az^2+az+m.$$

27. $(Ay^2+2Bxy+Cx^2+ay+bx+m)y'_x$
$$=By^2+2Ak^2xy+k(-Ak^2+Bk+C)x^2+by+ak^2x+s.$$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z^2+(b-ak)z+s-mk]x'_z$$
$$=(Ak^2+2Bk+C)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+m.$$

28. $(Ay^2+2Bxy-Bkx^2+ay+bx+c)y'_x$
$$=By^2+2Ak^2xy-Ak^3x^2+my+k(ak+b-m)x+s.$$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z^2+(m-ak)z+s-ck]x'_z=(Ak^2+Bk)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+c.$$

29. $(Ay^2-2Akxy+Bkx^2+ay+bx+c)y'_x$
$$=-By^2+2Bkxy-Ak^3x^2+my+k(ak+b-m)x+s.$$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[-(Ak+B)z^2+(m-ak)z+s-ck]x'_z=k(B-Ak)x^2+(ak+b)x+Az^2+az+c.$$

30. $(Ay^2+2Bxy+Ak^2x^2+ay+bx+c)y'_x$
$$=By^2+2Ak^2xy+Bk^2x^2+my+k(ak+b-m)x+s.$$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z^2+(m-ak)z+s-ck]x'_z=2k(Ak+B)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+c.$$

31. $(Ay^2+2Bxy+Cx^2+ay-akx+b)y'_x$
$= By^2+2Ak^2xy+k(-Ak^2+Bk+C)x^2+my-mkx+s.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z^2+(m-ak)z+s-bk]x'_z=(Ak^2+2Bk+C)x^2+2(Ak+B)zx+Az^2+az+b.$$

32. $(Ay^2+2Bxy+Cx^2+ay+bx+m)y'_x$
$= By^2+2Ak^2xy+k(-Ak^2+Bk+C)x^2+aky+bkx+s.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B-Ak)z^2+s-mk]x'_z=(Ak^2+2Bk+C)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+m.$$

33. $[A(\alpha y^2+\beta xy+\gamma x^2)+(A\delta+2\alpha)y+(A\varepsilon+\beta)x+A\sigma+\delta]y'_x$
$+B(\alpha y^2+\beta xy+\gamma x^2)+(B\delta+\beta)y+(B\varepsilon+2\gamma)x+B\sigma+\varepsilon=0.$

Solution: $\alpha y^2+\beta xy+\gamma x^2+\delta y+\varepsilon x+\sigma=C\exp(-Ay-Bx)$.

34. $(\alpha Ay^2-2\beta Axy+Bx^2+\alpha\beta^2A-\alpha^2B)y'_x=Cy^2+2Bxy$
$+Dx^2-2\beta(\beta A+C)y-2(\alpha D+\beta B)x+\alpha^2D+\beta^2(2\beta A+C).$

The transformation $x=w+\alpha,\ y=\xi w+\beta$ leads to a linear equation:

$$[-\alpha A\xi^3+(2\beta A+C)\xi^2+B\xi+D]w'_\xi=(\alpha A\xi^2-2\beta A\xi+B)w+2(\alpha B-\beta^2A).$$

35. $(A_{22}y^2+A_{12}xy+A_{11}x^2+A_2y+A_1x+A_0)y'_x$
$=B_{22}y^2+k(2A_{22}k+A_{12}-2B_{22})xy+k(-A_{22}k^2+B_{22}k+A_{11})x^2$
$+B_2y+k(A_2k+A_1-B_2)x+B_0.$

The substitution $y=z+kx$ leads to a Riccati equation with respect to $x=x(z)$:

$$[(B_{22}-A_{22}k)z^2+(B_2-A_2k)z+B_0-A_0k]x'_z$$
$$=(A_{22}k^2+A_{12}k+A_{11})x^2+[(2A_{22}k+A_{12})z+A_2k+A_1]x+A_{22}z^2+A_2z+A_0.$$

36. $(A_{22}y^2+A_{12}xy+A_{11}x^2+A_2y+A_1x+A_0)y'_x$
$=B_{22}y^2+B_{12}xy+B_{11}x^2+B_2y+B_1x+B_0.$

Here, A_{ij}, B_{ij}, and A_1 are arbitrary parameters, and the other parameters are defined by the relations:

$$A_2=-A_{12}\alpha-2A_{22}\beta,$$
$$A_0=-A_{11}\alpha^2+A_{22}\beta^2-A_1\alpha,$$
$$B_2=(2A_{11}-B_{12})\alpha+(A_{12}-2B_{22})\beta+A_1,$$
$$B_1=-2B_{11}\alpha-B_{12}\beta,$$
$$B_0=B_{11}\alpha^2+(B_{12}-2A_{11})\alpha\beta+(B_{22}-A_{12})\beta^2-A_1\beta$$

(α, β are arbitrary parameters).

The transformation $x=w+\alpha,\ y=\xi w+\beta$ leads to a linear equation:

$$[-A_{22}\xi^3+(B_{22}-A_{12})\xi^2+(B_{12}-A_{11})\xi+B_{11}]w'_\xi=(A_{22}\xi^2+A_{12}\xi+A_{11})w+k,$$

where $k=2A_{11}\alpha+A_{12}\beta+A_1$.

13.4.5 Equations of the Form $(A_3y^3 + A_2xy^2 + A_1x^2y + A_0x^3 + a_1y + a_0x)y'_x = B_3y^3 + B_2xy^2 + B_1x^2y + B_0x^3 + b_1y + b_0x$

1. $(y^3 - x^2y + ay + bx)y'_x = xy^2 - x^3 + by + ax.$

Solution in parametric form $(b \neq 0)$:

$$x = C^{-1}t|t|^{\frac{a-b}{2b}}e^{-t} + \tfrac{1}{2}bC|t|^{-\frac{a-b}{2b}}e^t, \qquad y = C^{-1}t|t|^{\frac{a-b}{2b}}e^{-t} - \tfrac{1}{2}bC|t|^{-\frac{a-b}{2b}}e^t.$$

2. $(y^3 - xy^2 - x^2y + x^3 + ay)y'_x = -y^3 + xy^2 + x^2y - x^3 + ax.$

Solution in parametric form:

$$x = C^{-1}\operatorname{sign}t\,e^{-1/t} + \tfrac{1}{8}aC|t|e^{1/t}, \qquad y = C^{-1}\operatorname{sign}t\,e^{-1/t} - \tfrac{1}{8}aC|t|e^{1/t}.$$

3. $(y^3 + xy^2 - x^2y - x^3 + ay + bx)y'_x = -y^3 - xy^2 + x^2y + x^3 + by + ax.$

Solution in parametric form $(a \neq -b)$:

$$x = t + C|t|^{\frac{b-a}{b+a}}\exp\Bigl(-\frac{4t^2}{a+b}\Bigr), \qquad y = t - C|t|^{\frac{b-a}{b+a}}\exp\Bigl(-\frac{4t^2}{a+b}\Bigr).$$

4. $(y^3 + xy^2 - 2x^2y + 2ay + ax)y'_x = -y^3 + xy^2 + 4x^2y - 4x^3 - ay + 4ax.$

Solution in parametric form:

$$x = C^{-1}t^{-1}e^{-1/t} + \tfrac{1}{3}aCt^2e^{1/t}, \qquad y = C^{-1}t^{-1}e^{-1/t} - \tfrac{2}{3}aCt^2e^{1/t}.$$

5. $(y^3 + xy^2 - 5x^2y + 3x^3 + ay + ax)y'_x = -3y^3 - 3xy^2 + 15x^2y - 9x^3 - ay + 3ax.$

Solution in parametric form:

$$x = C^{-1}\operatorname{sign}t\,e^{-1/t} + \tfrac{1}{32}aC|t|e^{1/t}, \qquad y = C^{-1}\operatorname{sign}t\,e^{-1/t} - \tfrac{3}{32}aC|t|e^{1/t}.$$

6. $(y^3 + 2xy^2 - x^2y - 2x^3 + 2ay + ax)y'_x = 2xy^2 + 2x^2y - 4x^3 - ay + 4ax.$

Solution in parametric form:

$$x = C^{-1}t|t|e^{-1/t} - \tfrac{1}{3}aC|t|^{-1}e^{1/t}, \qquad y = C^{-1}t|t|e^{-1/t} + \tfrac{2}{3}aC|t|^{-1}e^{1/t}.$$

7. $(y^3 - 3x^2y + 2x^3 + 2ay + ax)y'_x = -2y^3 + 6x^2y - 4x^3 - ay + 4ax.$

Solution in parametric form:

$$x = C^{-1}\operatorname{sign}t\,e^{-1/t} + \tfrac{1}{9}aC|t|e^{1/t}, \qquad y = C^{-1}\operatorname{sign}t\,e^{-1/t} - \tfrac{2}{9}aC|t|e^{1/t}.$$

8. $(y^3 + 3xy^2 - 4x^3 + ay + bx)y'_x = -2y^3 - 6xy^2 + 8x^3 + (b-a)y + 2ax.$

Solution in parametric form $(a \neq -b)$:

$$x = t + C|t|^{\frac{b-2a}{b+a}}\exp\Bigl[-\frac{27\,t^2}{2(a+b)}\Bigr], \qquad y = t - 2C|t|^{\frac{b-2a}{b+a}}\exp\Bigl[-\frac{27\,t^2}{2(a+b)}\Bigr].$$

9. $(y^3 + 3xy^2 - x^2y - 3x^3 + ay + bx)y'_x$
$= -y^3 + xy^2 + 9x^2y - 9x^3 - (2a - b)y + 3ax.$

Solution in parametric form $(a \neq b)$:

$$x = C^{-1}t|t|^{-\frac{3a-b}{2(a-b)}}e^{-t} - \tfrac{1}{16}(a-b)C|t|^{\frac{3a-b}{2(a-b)}}e^{t},$$
$$y = C^{-1}t|t|^{-\frac{3a-b}{2(a-b)}}e^{-t} + \tfrac{3}{16}(a-b)C|t|^{\frac{3a-b}{2(a-b)}}e^{t}.$$

10. $(y^3+3xy^2+3x^2y+x^3-ay+ax)y'_x = -y^3-3xy^2-3x^2y-x^3-ay+ax.$

Solution in parametric form:

$$x = Ct \pm \frac{C^2}{\sqrt{a}}\sqrt{2t^4+1}, \qquad y = Ct \mp \frac{C^2}{\sqrt{a}}\sqrt{2t^4+1}.$$

11. $(y^3+3xy^2+3x^2y+x^3+ay+bx)y'_x = -y^3-3xy^2-3x^2y-x^3+by+ax.$

Solution in parametric form $(b \neq -2a)$:

$$x = Ct + C^3\Big(|t|^{\frac{b-a}{a+b}} + \frac{4t^3}{2a+b}\Big), \qquad y = Ct - C^3\Big(|t|^{\frac{b-a}{a+b}} + \frac{4t^3}{2a+b}\Big).$$

12. $(y^3-4xy^2+4x^2y+ay-ax)y'_x = 3y^3-14xy^2+20x^2y-8x^3+2ay-2ax.$

Solution in parametric form:

$$x = t + Ct^2\exp\Big(-\frac{a}{2t^2}\Big), \qquad y = t + 2Ct^2\exp\Big(-\frac{a}{2t^2}\Big).$$

13. $(y^3 - 4xy^2 + 5x^2y - 2x^3 + 2ay - 3ax)y'_x$
$= 2y^3 - 8xy^2 + 10x^2y - 4x^3 + 3ay - 4ax.$

Solution in parametric form:

$$x = C^{-1}\operatorname{sign}t\,e^{-1/t} + aC|t|e^{1/t}, \qquad y = C^{-1}\operatorname{sign}t\,e^{-1/t} + 2aC|t|e^{1/t}.$$

14. $(y^3 - 5xy^2 + 7x^2y - 3x^3 + ay - 2ax)y'_x$
$= 3y^3 - 15xy^2 + 21x^2y - 9x^3 + 2ay - 3ax.$

Solution in parametric form:

$$x = C^{-1}\operatorname{sign}t\,e^{-1/t} + \tfrac{1}{8}aC|t|e^{1/t}, \qquad y = C^{-1}\operatorname{sign}t\,e^{-1/t} + \tfrac{3}{8}aC|t|e^{1/t}.$$

15. $(y^3 - 5xy^2 + 8x^2y - 4x^3 + ay + bx)y'_x$
$= 2y^3 - 10xy^2 + 16x^2y - 8x^3 + (3a+b)y - 2ax.$

Solution in parametric form $(a \neq -b)$:

$$x = t + C|t|^{\frac{2a+b}{a+b}}\exp\Big[\frac{t^2}{2(a+b)}\Big], \qquad y = t + 2C|t|^{\frac{2a+b}{a+b}}\exp\Big[\frac{t^2}{2(a+b)}\Big].$$

16. $(y^3 + 5xy^2 + 3x^2y - 9x^3 + ay + bx)y'_x$
$= -3y^3 - 15xy^2 - 9x^2y + 27x^3 + (b-2a)y + 3ax.$

Solution in parametric form $(a \neq -b)$:

$$x = t + C|t|^{\frac{b-3a}{b+a}}\exp\Big(-\frac{32\,t^2}{a+b}\Big), \qquad y = t - 3C|t|^{\frac{b-3a}{b+a}}\exp\Big(-\frac{32\,t^2}{a+b}\Big).$$

17. $(y^3 - 6xy^2 + 11x^2y - 6x^3 + ay + bx)y'_x$
$= 2y^3 - 11xy^2 + 18x^2y - 9x^3 + (4a+b)y - 3ax.$

Solution in parametric form $(a \neq -\frac{1}{2}b)$:

$$x = C^{-1}t|t|^{-\frac{3a+b}{4a+2b}}e^{-t} - (a + \tfrac{1}{2}b)C|t|^{\frac{3a+b}{4a+2b}}e^{t},$$
$$y = C^{-1}t|t|^{-\frac{3a+b}{4a+2b}}e^{-t} - 3(a + \tfrac{1}{2}b)C|t|^{\frac{3a+b}{4a+2b}}e^{t}.$$

18. $(y^3 - 6xy^2 + 12x^2y - 8x^3 - ay + ax)y'_x$
$= 2y^3 - 12xy^2 + 24x^2y - 16x^3 - ay + ax.$

Solution in parametric form $(a > 0)$:

$$x = Ct \pm \frac{C^2}{2\sqrt{a}}\sqrt{2t^4+1}, \qquad y = Ct \pm \frac{C^2}{\sqrt{a}}\sqrt{2t^4+1}.$$

19. $(2y^3 - 3xy^2 + x^2y + ay + bx)y'_x = y^3 - xy^2 + (a+b)y.$

Solution in parametric form $(a \neq -2b)$:

$$x = C^{-1}t|t|^{-\frac{b}{a+2b}}e^{-t} + (a+2b)C|t|^{\frac{b}{a+2b}}e^{t}, \qquad y = C^{-1}t|t|^{-\frac{b}{a+2b}}e^{-t}.$$

20. $(2y^3 + 3xy^2 - 3x^2y - 2x^3 + ay + bx)y'_x$
$= -y^3 + 3xy^2 + 6x^2y - 8x^3 - (a-b)y + 2ax.$

Solution in parametric form $(a \neq 2b)$:

$$x = C^{-1}t|t|^{-\frac{2a-b}{a-2b}}e^{-t} - \tfrac{1}{27}(a-2b)C|t|^{\frac{2a-b}{a-2b}}e^{t},$$
$$y = C^{-1}t|t|^{-\frac{2a-b}{a-2b}}e^{-t} + \tfrac{2}{27}(a-2b)C|t|^{\frac{2a-b}{a-2b}}e^{t}.$$

21. $(2y^3 - 9xy^2 + 13x^2y - 6x^3 + ay + bx)y'_x$
$= 3y^3 - 13xy^2 + 18x^2y - 8x^3 + (3a+b)y - 2ax.$

Solution in parametric form $(a \neq -\frac{2}{3}b)$:

$$x = C^{-1}t|t|^{-\frac{2a+b}{3a+2b}}e^{-t} - (3a+2b)C|t|^{\frac{2a+b}{3a+2b}}e^{t},$$
$$y = C^{-1}t|t|^{-\frac{2a+b}{3a+2b}}e^{-t} - 2(3a+2b)C|t|^{\frac{2a+b}{3a+2b}}e^{t}.$$

22. $(3y^3 - xy^2 - 3x^2y + x^3 + ay)y'_x = -y^3 + 3xy^2 + x^2y - 3x^3 + ax.$

Solution in parametric form:

$$x = C^{-1}t^{-1}e^{-1/t} + \tfrac{1}{8}aCt^2e^{1/t}, \qquad y = C^{-1}t^{-1}e^{-1/t} - \tfrac{1}{8}aCt^2e^{1/t}.$$

23. $(3y^3 + xy^2 - 3x^2y - x^3 + ay)y'_x = y^3 + 3xy^2 - x^2y - 3x^3 + ax.$

Solution in parametric form:

$$x = C^{-1}t|t|e^{-1/t} - \tfrac{1}{8}aC|t|^{-1}e^{1/t}, \qquad y = C^{-1}t|t|e^{-1/t} + \tfrac{1}{8}aC|t|^{-1}e^{1/t}.$$

24. $(xy^2 - 2kx^2y + k^2x^3 + ay - ax)y'_x = y^3 - 2kxy^2 + k^2x^2y + kay - kax.$

Solution in parametric form $(k \neq 1)$:

$$x = t + C|t| \exp\Bigl[-\frac{a}{2(k-1)t^2}\Bigr], \quad y = t + kC|t| \exp\Bigl[-\frac{a}{2(k-1)t^2}\Bigr].$$

25. $(y^3 - 3kxy^2 + 3k^2x^2y - k^3x^3 - ay + ax)y'_x$
$= ky^3 - 3k^2xy^2 + 3k^3x^2y - k^4x^3 - ay + ax.$

Solution in parametric form $(a > 0,\ k \neq 1)$:

$$x = Ct \pm C^2\frac{k-1}{2\sqrt{a}}\sqrt{2t^2+1}, \quad y = Ct \pm kC^2\frac{k-1}{2\sqrt{a}}\sqrt{2t^2+1}.$$

26. $(y^3 - 3kxy^2 + 3k^2x^2y - k^3x^3 + ay + bx)y'_x$
$= ky^3 - 3k^2xy^2 + 3k^3x^2y - k^4x^3 + [(k+1)a + b]y - kax.$

Solution in parametric form $(b \neq \frac{1}{2}a(k-3),\ k \neq 1)$:

$$x = Ct + C^3\Bigl[|t|^{\frac{ka+b}{a+b}} + \frac{(k-1)^3}{(k-3)a-2b}t^3\Bigr], \quad y = Ct + kC^3\Bigl[|t|^{\frac{ka+b}{a+b}} + \frac{(k-1)^3}{(k-3)a-2b}t^3\Bigr].$$

27. $[y^3 - (k+2)xy^2 + (2k+1)x^2y - kx^3 + 2ay - (k+1)ax]y'_x$
$= ky^3 - k(k+2)xy^2 + k(2k+1)x^2y - k^2x^3 + (k+1)ay - 2kax.$

Solution in parametric form $(k \neq 1)$:

$$x = C^{-1}\operatorname{sign} t\, e^{-1/t} + \frac{aC}{(k-1)^2}|t|e^{1/t}, \quad y = C^{-1}\operatorname{sign} t\, e^{-1/t} + \frac{akC}{(k-1)^2}|t|e^{1/t}.$$

28. $[y^3 - (k+2)xy^2 - k(k-4)x^2y + k^2(k-2)x^3 + ay - ax]y'_x$
$= (2k-1)y^3 - k(4k-1)xy^2 + k^2(2k+1)x^2y - k^3x^3 + kay - kax.$

Solution in parametric form $(k \neq -1)$:

$$x = t + Ct^2 \exp\Bigl[-\frac{a}{2(k-1)^2t^2}\Bigr], \quad y = t + kCt^2 \exp\Bigl[-\frac{a}{2(k-1)^2t^2}\Bigr].$$

29. $[y^3 - (2k+1)xy^2 + k(k+2)x^2y - k^2x^3 + ay + bx]y'_x$
$= ky^3 - k(2k+1)xy^2 + k^2(k+2)x^2y - k^3x^3 + [(k+1)a + b]y - kax.$

Solution in parametric form $(a \neq -b,\ k \neq 1)$:

$$x = t + C|t|^{\frac{ka+b}{a+b}} \exp\Bigl[\frac{(k-1)^3}{2(a+b)}t^2\Bigr], \quad y = t + kC|t|^{\frac{ka+b}{a+b}} \exp\Bigl[\frac{(k-1)^3}{2(a+b)}t^2\Bigr].$$

30. $(Ay^3 + xy^2 - Ax^2y - x^3 + ay + bx)y'_x = y^3 + Axy^2 - x^2y - Ax^3 + by + ax.$

1°. Solution in parametric form with $b \neq 0$:

$$x = C^{-1}t|t|^{\frac{a-b}{2b}}|t+1|^{\frac{bA-a}{2b}} - \tfrac{1}{4}bC|t|^{\frac{b-a}{2b}}|t+1|^{\frac{a-bA}{2b}},$$
$$y = C^{-1}t|t|^{\frac{a-b}{2b}}|t+1|^{\frac{bA-a}{2b}} + \tfrac{1}{4}bC|t|^{\frac{b-a}{2b}}|t+1|^{\frac{a-bA}{2b}}.$$

2°. Solution in parametric form with $b=0$:

$$x=C^{-1}t|t|^{\frac{A-1}{2}}e^{-1/t}-\tfrac{1}{8}aC|t|^{\frac{1-A}{2}}e^{1/t},\qquad y=C^{-1}t|t|^{\frac{A-1}{2}}e^{-1/t}+\tfrac{1}{8}aC|t|^{\frac{1-A}{2}}e^{1/t}.$$

31. $(A_3y^3+A_2xy^2+A_1x^2y+A_0x^3+ax)y'_x$
$\qquad = B_3y^3+B_2xy^2+B_1x^2y+B_0x^3+ay.$

This is a special case of equation 13.7.1.13 with $R_m(x,y)=a$. The transformation $t=y/x$, $u=x^2$ leads to a linear equation:

$$[P_B(t)-tP_A(t)]u'_t=2P_A(t)u+2a,$$

where $P_A(t)=A_3t^3+A_2t^2+A_1t+A_0$ and $P_B(t)=B_3t^3+B_2t^2+B_1t+B_0$.

32. $[Ay^3+(A+2)xy^2-(A-4)x^2y-(A-2)x^3+ay-ax]y'_x$
$\qquad =-(A-2)y^3-(A-4)xy^2+(A+2)x^2y+Ax^3-ay+ax.$

Solution in parametric form:

$$x=t+C|t|^{1-A}\exp\Bigl(\frac{a}{8t^2}\Bigr),\qquad y=t-C|t|^{1-A}\exp\Bigl(\frac{a}{8t^2}\Bigr).$$

33. $[Ay^3+3(A+1)xy^2+12x^2y-4(A-3)x^3+ay-ax]y'_x$
$\qquad =-(2A-3)y^3-6(A-2)xy^2+12x^2y+8Ax^3-2ay+2ax.$

Solution in parametric form:

$$x=t+C|t|^{1-A}\exp\Bigl(\frac{a}{18t^2}\Bigr),\qquad y=t-2C|t|^{1-A}\exp\Bigl(\frac{a}{18t^2}\Bigr).$$

13.5 Equations of the Form $f(x,y)y'_x=g(x,y)$ Containing Arbitrary Parameters

13.5.1 Equations Containing Power Functions

► **Equations of the form $y'_x=f(x,y)$.**

1. $y'_x=A\sqrt{y}+Bx^{-1/2}.$

The substitution $w=2A^{-1}\sqrt{y}$ leads to an Abel equation of the form 13.3.1.32: $ww'_x=w+2BA^{-2}x^{-1/2}$.

2. $y'_x=A\sqrt{y}+Bx^{-1}.$

Let $A=\pm2a^{-1}\sqrt{b}$, $B=\mp4b$ $(b>0)$. Solution in parametric form:

$$x=af(\tau),\quad y=b[2\tau\pm f(\tau)]^2,\qquad \text{where}\quad f(\tau)=\exp(\mp\tau^2)\Bigl[\int\exp(\mp\tau^2)\,d\tau+C\Bigr]^{-1}.$$

3. $y'_x=A\sqrt{y}+Bx^{-2}.$

The substitution $w=2A^{-1}\sqrt{y}$ leads to an Abel equation of the form 13.3.1.33: $ww'_x=w+2BA^{-2}x^{-2}$.

4. $y'_x = a\sqrt{y} + bx + cx^m$.

The substitution $w = 2a^{-1}\sqrt{y}$ leads to the Abel equation $ww'_x = w + 2a^{-2}(bx + cx^m)$, which is discussed in Section 13.3.1 (see Table 13.1).

5. $y'_x = ay^n + bx^{\frac{n}{1-n}}$.

Solution: $\displaystyle\int \frac{dw}{aw^n + \frac{1}{1-n}w + b} = \ln|x| + C$, where $w = yx^{\frac{1}{n-1}}$.

6. $y'_x = Ay^s - Bx^k$.

The transformation $x = (w'_z)^{1/k}$, $y = \lambda(w/z)^{1/s}$, where $\lambda = (B/A)^{1/s}$, leads to the generalized Emden–Fowler equation:

$$w''_{zz} = -\frac{\lambda k}{sB} z^{-\frac{1}{s}} w^{\frac{1-s}{s}} (w'_z)^{\frac{k-1}{k}},$$

which is discussed in Section 14.5 (in the classification table, one should search for the equations satisfying the condition $n + m + 1 = 0$).

7. $y'_x = (ax + by + c)^n$.

This is a special case of equation 13.7.1.1 with $f(\xi) = \xi^n$.

8. $y'_x = ax^{m-n-nm}y^n + bx^m$.

Solution:

$$\int \frac{dw}{w^n - \lambda w + 1} + C = b\left(\frac{a}{b}\right)^{1/n} \ln|x|, \quad \text{where} \quad w = \left(\frac{a}{b}\right)^{1/n} yx^{-m-1}, \ \lambda = \frac{m+1}{b}\left(\frac{b}{a}\right)^{1/n}.$$

9. $y'_x = ax^{n-1}y^{m+1} + bx^{nk-1}y^{mk+1}$.

This is a generalized homogeneous equation of the form 13.7.1.3 with $f(\xi) = a\xi + b\xi^k$.

10. $y'_x = ax^k y\sqrt{y} + bx^m y + cx^s\sqrt{y}$.

This is a special case of equation 13.7.1.4 with $f(x) = ax^k$, $g(x) = bx^m$, $h(x) = cx^s$, and $n = 1/2$.

11. $y'_x = ax^k y^{1+n} + bx^m y + cx^s y^{1-n}$.

This is a special case of equation 13.7.1.4 with $f(x) = ax^k$, $g(x) = bx^m$, and $h(x) = cx^s$.

12. $y'_x = x^{n-1}y^{1-m}(ax^n + by^m)^k$.

This is a special case of equation 13.7.1.7 with $f(\xi) = \xi^k$.

► **Other equations.**

13. $xy'_x = ay + b\sqrt{y^2 + cx^2}$.

The substitution $w = y/x$ leads to a separable equation: $xw'_x = (a-1)w + b\sqrt{w^2 + c}$.

14. $xy'_x = y + ax^{n-m}y^m + bx^{n-k}y^k$.

The substitution $y = xw$ leads to a separable equation: $w'_x = x^{n-2}(aw^m + bw^k)$.

15. $(ay^n + bx)y'_x = 1.$

Solution: $x = e^{by}\Big(C + a\int y^n e^{-by}\,dy\Big).$

16. $x(xy^n + a)y'_x + by = 0.$

Solution: $nb - a = x\big(Cy^{a/b} + y^n\big).$

17. $x(ay^m + m)y'_x = y\big[bx^{n(\lambda-1)}y^{m\lambda} - n\big].$

This is a special case of equation 13.7.1.16 with $f(\xi) = a\xi$, $g(\xi) = 1$, $h(\xi) = b\xi^\lambda$, and $k = n$.

18. $(ax^n + bx^2 + cxy)y'_x = kx^n + bxy + cy^2.$

The transformation $t = y/x,\ z = x^{n-2}$ leads to a linear equation with respect to $z = z(t)$: $(k - at)z'_t = (n-2)(az + b + ct)$.

19. $(ay^n + bx^2 + cxy)y'_x = ky^n + bxy + cy^2.$

The transformation $t = y/x,\ z = x^{n-2}$ leads to a linear equation with respect to $z = z(t)$: $t^n(k - at)z'_t = (n-2)(at^n z + b + ct)$.

20. $(ax^n + by^n + x)y'_x = \alpha x^k y^{n-k} + \beta x^m y^{n-m} + y.$

The transformation $t = y/x,\ z = x^{n-1}$ leads to a linear equation:

$$(\alpha t^{n-k} + \beta t^{n-m} - bt^{n+1} - at)z'_t = (n-1)(bt^n + a)z + n - 1.$$

21. $(ax^n + by^n + Ax^2 + Bxy)y'_x = \alpha x^k y^{n-k} + \beta x^m y^{n-m} + Axy + By^2.$

The transformation $t = y/x,\ z = x^{n-2}$ leads to a linear equation:

$$(\alpha t^{n-k} + \beta t^{n-m} - bt^{n+1} - at)z'_t = (n-2)(bt^n + a)z + (n-2)(Bt + A).$$

22. $[(ax + by)^n + bx]y'_x = c(ax + by)^m - ax.$

This is a special case of equation 13.7.1.14 with $f(\xi) = \xi^n$, $g(\xi) = 1$, and $h(\xi) = c\xi^m$.

23. $[(ax + by)^n + by]y'_x = c(ax + by)^m - ay.$

This is a special case of equation 13.7.1.15 with $f(\xi) = \xi^n$, $g(\xi) = 1$, and $h(\xi) = c\xi^m$.

24. $(\alpha x + \beta y + \gamma)^n y'_x = (ax + by + c)^n.$

This is a special case of equation 13.7.1.6 with $f(\xi) = \xi^n$.

25. $(ax^n + by^m)y'_x = x^{n-1}y^{1-m}.$

This is a special case of equation 13.7.1.7 with $f(\xi) = 1/\xi$.

26. $(ay^m + bx^n + s)y'_x + \alpha x^k + bnx^{n-1}y + \beta = 0.$

Solution:

$$a\varphi(y) + \alpha\psi(x) + bx^n y + sy + \beta x = C,$$

where $\varphi(y) = \begin{cases} \dfrac{y^{m+1}}{m+1} & \text{if } m \neq -1, \\ \ln|y| & \text{if } m = -1, \end{cases}$ $\quad \psi(x) = \begin{cases} \dfrac{x^{k+1}}{k+1} & \text{if } k \neq -1, \\ \ln|x| & \text{if } k = -1. \end{cases}$

27. $(ax^2y^n + bxy^m + cy^k)y'_x = \alpha y^p + \beta y^q + \gamma.$

This is a Riccati equation with respect to $x = x(y)$.

28. $(ax^n y^m + x)y'_x = bx^k y^{n+m-k} + y.$

The transformation $t = y/x$, $z = x^{n+m-1}$ leads to a linear equation: $t^m(bt^{n-k} - at)z'_t = (n+m-1)(at^m z + 1)$.

29. $x(ax^n y^m + \alpha)y'_x + y(bx^n y^m + \beta) = 0.$

Solution: $\dfrac{(y^a x^b)^A}{A} + \dfrac{(y^\alpha x^\beta)^B}{B} = C$, where $A = \dfrac{m\beta - n\alpha}{a\beta - b\alpha}$, $B = \dfrac{mb - na}{a\beta - b\alpha}$.

30. $x(anx^k y^{n+k} + s)y'_x + y(bmx^{m+k}y^k + s) = 0.$

Solution: $aky^n + bkx^m - s(xy)^{-k} = C$.

31. $(ax^n y^m + Ax^2 + Bxy)y'_x = bx^k y^{n+m-k} + Axy + By^2.$

The transformation $t = y/x$, $z = x^{n+m-2}$ leads to a linear equation: $t^m(bt^{n-k} - at)z'_t = (n+m-2)(at^m z + Bt + A)$.

32. $(amx^n y^{m-1} + by^k)y'_x + anx^{n-1}y^m + cx^s = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = by^k$ and $g(x) = cx^s$.

33. $(ax^n y^m + bxy^k)y'_x = \alpha y^s + \beta.$

This is a Bernoulli equation with respect to $x = x(y)$ (see Section 13.1.5).

34. $x(ax^{n-k}y^m + m)y'_x = y(bx^{\lambda n-k}y^{\lambda m} - n).$

This is a special case of equation 13.7.1.16 with $f(\xi) = a\xi$, $g(\xi) = 1$, and $h(\xi) = b\xi^\lambda$.

35. $x(ax^n y^{m-k} + m)y'_x = y(bx^{\lambda n}y^{\lambda m-k} - n).$

This is a special case of equation 13.7.1.17 with $f(\xi) = a\xi$, $g(\xi) = 1$, and $h(\xi) = b\xi^\lambda$.

36. $(ax^{n+1}y^{m-1} + bx^{nk+1}y^{mk-1})y'_x = cx^{ns}y^{ms}.$

This is a special case of equation 13.7.1.3 with $f(\xi) = c\xi^s(a\xi + b\xi^k)^{-1}$.

37. $(ax^n + by^m)^k y'_x = cx^{n-1}y^{1-m}.$

This is a special case of equation 13.7.1.7 with $f(\xi) = c\xi^{-k}$.

38. $\left(e_1\dfrac{x+a}{r_1^3} + e_2\dfrac{x-a}{r_2^3}\right)y'_x - y\left(\dfrac{e_1}{r_1^3} + \dfrac{e_2}{r_2^3}\right) = 0,$

where $r_1^2 = (x+a)^2 + y^2,\quad r_2^2 = (x-a)^2 + y^2.$

This is the equation of force lines corresponding to the Coulomb law in electricity.

Solution: $e_1\dfrac{x+a}{r_1} + e_2\dfrac{x-a}{r_2} = C$.

39. $xy'_x - y = (ax^k + by^k)(yy'_x + x).$

This is a special case of equation 13.7.1.24 with $f(u) = u^{k/2}$ and $g(v,w) = av^k + bw^k$.

40. $xy'_x - y = (ax^k + by^k)(yy'_x - x).$

This is a special case of equation 13.7.1.25 with $f(u) = u^{k/2}$ and $g(v,w) = av^k + bw^k$.

41. $yy'_x + x = (ax^k + by^k)(xy'_x - y).$

This is a special case of equation 13.7.1.24 with $f(u) = u^{-k/2}$ and $g(v,w) = (av^k + bw^k)^{-1}$.

13.5.2 Equations Containing Exponential Functions

▶ **Equations with exponential functions.**

1. $y'_x = ae^{\lambda y} + b.$

Solution: $y = -\dfrac{1}{\lambda}\ln\Bigl(Ce^{-b\lambda x} - \dfrac{a}{b}\Bigr).$

2. $y'_x = ae^{y} + be^{x}.$

Solution: $y = be^{x} - \ln\Bigl[C - a\displaystyle\int \exp(be^{x})\,dx\Bigr].$

3. $y'_x = Ae^{y+ax} - a.$

This is a special case of equation 13.7.1.2 with $f(\xi) = Ae^{\xi}$, $n = 1$, and $b = 0$.

4. $y'_x = ae^{\nu x+\lambda y} + be^{\mu x}.$

This is a special case of equation 13.7.2.5 with $f(x) = ae^{\nu x}$ and $g(x) = be^{\mu x}$.

5. $y'_x = ae^{\nu x+\lambda y} + be^{\mu x-\lambda y}.$

This is a special case of equation 13.7.2.8 with $f(x) = ae^{\nu x}$, $g(x) = 0$, and $h(x) = be^{\mu x}$.

6. $y'_x = ae^{2\alpha x-\beta y} + be^{\alpha x} + ce^{\alpha x-\beta y}.$

This is a special case of equation 13.7.2.9 with $f(\xi) = \xi + c$.

7. $y'_x = ae^{\alpha x+\lambda y} + be^{\beta x} + ce^{\gamma x-\lambda y}.$

The substitution $w = e^{\lambda y}$ leads to a Riccati equation: $w'_x = a\lambda e^{\alpha x}w^2 + b\lambda e^{\beta x}w + c\lambda e^{\gamma x}$.

8. $(ae^{y} + be^{x})y'_x = 1.$

Solution: $x = ae^{y} - \ln\Bigl[C - b\displaystyle\int \exp(ae^{y})\,dy\Bigr].$

9. $(be^{\alpha y} + c)y'_x = e^{ax+by} - ae^{\alpha y}.$

This is a special case of equation 13.7.2.13 with $f(\xi) = c$, $g(\xi) = 1$, and $h(\xi) = e^{\xi}$.

10. $(ae^{y+\beta x} + b)y'_x = c.$

The substitution $w(x) = y + \beta x$ leads to a separable equation: $w'_x = \beta + c(ae^{w} + b)^{-1}$.

11. $(ae^{\alpha x} + be^{\beta y})y'_x = e^{\alpha x-\beta y}.$

This is a special case of equation 13.7.2.9 with $f(\xi) = \xi^{-1}$.

12. $(e^{\alpha x+\gamma y} + a\beta)y'_x + be^{\nu x+\beta y} + a\alpha = 0.$

This is a special case of equation 13.7.2.15 with $f(y) = e^{\gamma y}$ and $g(x) = be^{\nu x}$.

► **Equations with power and exponential functions.**

13. $y'_x = ae^y + bx^n.$

1°. Solution in parametric form with $n \neq -1$:

$$x = \tau^{\frac{1}{n+1}}, \quad y = \frac{b}{n+1}\tau - \ln\Big[C - \frac{a}{n+1}\int \tau^{-\frac{n}{n+1}} \exp\Big(\frac{b\tau}{n+1}\Big)\,d\tau\Big].$$

2°. Solution in parametric form with $n = -1$, $b \neq -1$:

$$x = e^\tau, \quad y = -\ln\Big(Ce^{-b\tau} - \frac{a}{b+1}e^\tau\Big).$$

3°. Solution in parametric form with $n = -1$, $b = -1$:

$$x = e^\tau, \quad y = -\tau - \ln(C - a\tau).$$

14. $y'_x = ay^{-1} + be^x.$

Solution in parametric form:

$$x = \ln(AE^{-1}) \mp \tau^2, \quad y = B[2 \pm \exp(\mp\tau^2)E^{-1}],$$

where $a = \mp 2B^2,\ b = \pm A^{-1}B,\ E = \displaystyle\int \exp(\mp\tau^2)\,d\tau + C.$

15. $y'_x = ae^{\nu x + \lambda y} + bx^n.$

This is a special case of equation 13.7.2.5 with $f(x) = ae^{\nu x}$ and $g(x) = bx^n$.

16. $y'_x = ax^n e^{\lambda y} + be^{\nu x}.$

This is a special case of equation 13.7.2.5 with $f(x) = ax^n$ and $g(x) = be^{\nu x}$.

17. $y'_x = ax^n e^{\lambda y} + bx^m.$

This is a special case of equation 13.7.2.5 with $f(x) = ax^n$ and $g(x) = bx^m$.

18. $y'_x = ax^n e^{\lambda y} + bx^m e^{-\lambda y}.$

This is a special case of equation 13.7.2.8 with $f(x) = ax^n$, $g(x) = 0$, and $h(x) = bx^m$.

19. $y'_x = c(y + ae^{\lambda x} + b)^n - a\lambda e^{\lambda x}.$

This is a special case of equation 13.7.2.10 with $f(\xi) = c\xi^n$.

20. $y'_x = (ae^y + bx^{-k})^{1/k}.$

Solution in parametric form:

$$x = \exp\Big\{\tau - \frac{1}{k}\big[f(\tau) + C\big]\Big\}, \quad y = f(\tau) + C, \quad \text{where} \quad f(\tau) = \int \frac{k\,d\tau}{k(b + ae^{k\tau})^{-1/k} + 1}.$$

21. $y'_x = (ay^k + be^x)^{1/k}.$

Solution in parametric form:

$$x = f(\tau) + C, \quad y = \exp\Big\{\tau + \frac{1}{k}\big[f(\tau) + C\big]\Big\}, \quad \text{where} \quad f(\tau) = \int \frac{k\,d\tau}{k(a + be^{-k\tau})^{1/k} - 1}.$$

22. $y'_x = ax^{n-1}e^{\lambda n y} + bx^{m-1}e^{\lambda m y}.$

This is a special case of equation 13.7.2.2 with $f(\xi) = a\xi^{n-1} + b\xi^{m-1}$.

23. $y'_x = ax^{n-1}e^{\alpha y} + bx^{nm-1}e^{\alpha my}$.

This is a special case of equation 13.7.2.4 with $f(\xi) = a\xi + b\xi^m$.

24. $y'_x = ae^{\lambda nx}y^{n+1} + be^{-\lambda x}$.

This is a special case of equation 13.7.2.1 with $f(\xi) = a\xi^{n+1} + b$.

25. $y'_x = ae^{\alpha x}y^{m+1} + be^{\alpha nx}y^{nm+1}$.

This is a special case of equation 13.7.2.3 with $f(\xi) = a\xi + b\xi^n$.

26. $y'_x = ae^{\lambda nx}y^{n+1} + be^{\lambda mx}y^{m+1}$.

This is a special case of equation 13.7.2.1 with $f(\xi) = a\xi^{n+1} + b\xi^{m+1}$.

27. $y'_x = ax^n y^k + bx^n e^{\alpha x}y^{k+1} - \alpha y$.

This is a special case of equation 13.7.2.7 with $f(x) = x^n$, $g(\xi) = a + b\xi$, and $m = 1$.

28. $y'_x = ae^{\lambda x}y^{1+n} + be^{\mu x}y + ce^{\nu x}y^{1-n}$.

This is a special case of equation 13.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu x}$, and $h(x) = ce^{\nu x}$.

29. $y'_x = ae^{\lambda x}y^{1+n} + be^{\mu x}y + cx^m y^{1-n}$.

This is a special case of equation 13.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu x}$, and $h(x) = cx^m$.

30. $y'_x = ax^k y^{1+n} + be^{\lambda x}y + cx^m y^{1-n}$.

This is a special case of equation 13.7.1.4 with $f(x) = ax^k$, $g(x) = be^{\lambda x}$, and $h(x) = cx^m$.

31. $y'_x = ae^{\lambda x}y^{1+n} + bx^m y + ce^{\mu x}y^{1-n}$.

This is a special case of equation 13.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = bx^m$, and $h(x) = ce^{\mu x}$.

32. $y'_x = ae^{\lambda x}y^{1+n} + bx^m y + cx^k y^{1-n}$.

This is a special case of equation 13.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = bx^m$, and $h(x) = cx^k$.

33. $xy'_x = ax^{n+k}e^y + bx^{nm+k}e^{my} - n$.

This is a special case of equation 13.7.2.6 with $f(x) = x^{k-1}$ and $g(\xi) = a\xi + b\xi^m$.

34. $(by + \lambda)y'_x = ce^{ax+by} - ay$.

This is a special case of equation 13.7.1.15 with $f(\xi) = \lambda$, $g(\xi) = 1$, and $h(\xi) = ce^{\xi}$.

35. $xyy'_x = ax^n e^y - ny$.

This is a special case of equation 13.7.2.11 with $f(\xi) = a\xi$ and $\alpha = 1$.

36. $xy^2y'_x = ax^n e^y - ny^2$.

This is a special case of equation 13.7.2.12 with $f(\xi) = a\xi$ and $\alpha = 1$.

37. $(ay^n + be^x)y'_x = 1.$

1°. Solution in parametric form with $n \neq -1$:

$$x = \frac{a}{n+1}\tau - \ln\Big[C - \frac{b}{n+1}\int \tau^{-\frac{n}{n+1}} \exp\Big(\frac{a\tau}{n+1}\Big)\,d\tau\Big], \qquad y = \tau^{\frac{1}{n+1}}.$$

2°. Solution in parametric form with $n = -1$ and $a \neq -1$:

$$x = -\ln\Big(Ce^{-a\tau} - \frac{b}{a+1}e^{\tau}\Big), \qquad y = e^{\tau}.$$

3°. Solution in parametric form with $n = -1$ and $a = -1$:

$$x = -\tau - \ln(C - b\tau), \qquad y = e^{\tau}.$$

38. $(ae^y + bx)y'_x = 1.$

Solution in implicit form: $x = \begin{cases} Ce^{by} + \dfrac{a}{1-b}e^y & \text{if } b \neq 1, \\ e^y(C + ay) & \text{if } b = 1. \end{cases}$

39. $(ae^y + bx^2)y'_x = 1.$

Solutions in parametric form:

$$x = -\frac{1}{2b}\tau(\ln Z)'_\tau, \quad y = \ln\Big(\frac{\tau^2}{4ab}\Big), \qquad Z = C_1J_0(\tau) + C_2Y_0(\tau)$$

and

$$x = -\frac{1}{2b}\tau(\ln Z)'_\tau, \quad y = \ln\Big(-\frac{\tau^2}{4ab}\Big), \qquad Z = C_1I_0(\tau) + C_2K_0(\tau),$$

where $J_0(\tau)$ and $Y_0(\tau)$ are Bessel functions, and $I_0(\tau)$ and $K_0(\tau)$ are modified Bessel functions.

40. $(ae^y + bx^{-1})y'_x = 1.$

Let $a = \pm A/B$, $b = \mp 2A^2$. Solution in parametric form:

$$x = A\big[2\tau \pm \exp(\mp\tau^2)f(\tau)\big], \quad y = \ln\big[Bf(\tau)\big] \mp \tau^2,$$

where $f(\tau) = \Big[\displaystyle\int \exp(\mp\tau^2)\,d\tau + C\Big]^{-1}.$

41. $(e^{ax+by} + bx)y'_x = ce^{ax+by} - ax.$

This is a special case of equation 13.7.1.14 with $f(\xi) = e^\xi$, $g(\xi) = 1$, and $h(\xi) = ce^\xi$.

42. $(e^{ax+by} + by)y'_x = ce^{ax+by} - ay.$

This is a special case of equation 13.7.1.15 with $f(\xi) = e^\xi$, $g(\xi) = 1$, and $h(\xi) = ce^\xi$.

43. $(ae^{\alpha x}y^m + b)y'_x = y.$

This is a special case of equation 13.7.2.3 with $f(\xi) = (a\xi + b)^{-1}$.

44. $(e^{\alpha x}y^n + a\beta)y'_x + be^{\nu x + \beta y} + a\alpha = 0.$

This is a special case of equation 13.7.2.15 with $f(y) = y^n$ and $g(x) = be^{\nu x}$.

45. $(e^{\alpha x}y^n + a\beta)y'_x + bx^m e^{\beta y} + a\alpha = 0.$

This is a special case of equation 13.7.2.15 with $f(y) = y^n$ and $g(x) = bx^m$.

46. $(e^{\alpha x}y^m + mx)y'_x = y(be^{\alpha nx}y^{nm} - \alpha x).$

This is a special case of equation 13.7.2.17 with $f(\xi) = \xi$, $g(\xi) = 1$, and $h(\xi) = b\xi^n$.

47. $x(x^n e^{\alpha y} + \alpha y)y'_x = bx^{nm}e^{\alpha my} - ny.$

This is a special case of equation 13.7.2.16 with $f(\xi) = \xi$, $g(\xi) = 1$, and $h(\xi) = b\xi^m$.

48. $(ax^n e^{\lambda y} + bxe^{\mu y})y'_x = e^{\nu y}.$

This is a Bernoulli equation with respect to $x = x(y)$ (see Section 13.1.5).

49. $(ax^n e^{\lambda y} + bxy^m)y'_x = e^{\mu y}.$

This is a Bernoulli equation with respect to $x = x(y)$.

50. $(ax^n y^m + bxe^{\lambda y})y'_x = y^k.$

This is a Bernoulli equation with respect to $x = x(y)$.

51. $(ax^n y^m + bxy^k)y'_x = e^{\lambda y}.$

This is a Bernoulli equation with respect to $x = x(y)$.

52. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + ce^{\lambda x} = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b$ and $g(x) = ce^{\lambda x}$.

53. $(amx^n y^{m-1} + be^{\lambda y})y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = be^{\lambda y}$ and $g(x) = c$.

54. $(amx^n y^{m-1} + by^k)y'_x + anx^{n-1}y^m + ce^{\lambda x} = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = by^k$ and $g(x) = ce^{\lambda x}$.

55. $[(ax + by)^n + be^{\alpha x}]y'_x = c(ax + by)^m - ae^{\alpha x}.$

This is a special case of equation 13.7.2.14 with $f(\xi) = \xi^n$, $g(x) = 1$, and $h(\xi) = c\xi^m$.

56. $[(ax + by)^n + be^{\alpha y}]y'_x = c(ax + by)^m - ae^{\alpha y}.$

This is a special case of equation 13.7.2.13 with $f(\xi) = \xi^n$, $g(x) = 1$, and $h(\xi) = c\xi^m$.

13.5.3 Equations Containing Hyperbolic Functions

1. $y'_x = a\cosh(\lambda y) + b\cosh(\nu x).$

This is a special case of equation 13.7.2.18 with $f(x) = 0$, $g(x) = a$, and $h(x) = b\cosh(\nu x)$.

2. $y'_x = a\sinh(\lambda y) + b\sinh(\nu x).$

This is a special case of equation 13.7.2.18 with $f(x) = a$, $g(x) = 0$, and $h(x) = b\sinh(\nu x)$.

3. $y'_x = ax^n\cosh(\lambda y) + bx^m.$

This is a special case of equation 13.7.2.18 with $f(x) = 0$, $g(x) = ax^n$, and $h(x) = bx^m$.

4. $y'_x = ax^n \sinh(\lambda y) + bx^m.$

This is a special case of equation 13.7.2.18 with $f(x) = ax^n$, $g(x) = 0$, and $h(x) = bx^m$.

5. $y'_x = ay^{1+n} + by + c\sinh(\lambda x)y^{1-n}.$

This is a special case of equation 13.7.1.4 with $f(x) = a$, $g(x) = b$, and $h(x) = c\sinh(\lambda x)$.

6. $y'_x = ay^{1+n} + b\sinh(\lambda x)y + cy^{1-n}.$

This is a special case of equation 13.7.1.4 with $f(x) = a$, $g(x) = b\sinh(\lambda x)$, and $h(x) = c$.

7. $y'_x = y\cosh x\,(ay^{nm}\sinh^{n-1}x + by^m).$

This is a special case of equation 13.7.2.22 with $f(\xi) = a\xi^n + b\xi$.

8. $y'_x = y\sinh x\,(ay^{nm}\cosh^{n-1}x + by^m).$

This is a special case of equation 13.7.2.24 with $f(\xi) = a\xi^n + b\xi$.

9. $xy'_x = (ax^n\cosh y + b)\coth y.$

This is a special case of equation 13.7.2.25 with $f(\xi) = a\xi + b$.

10. $xy'_x = (ax^n\sinh y + b)\tanh y.$

This is a special case of equation 13.7.2.23 with $f(\xi) = a\xi + b$.

11. $(ay^m\cosh x + b)y'_x = y^{m+1}\sinh x.$

This is a special case of equation 13.7.2.24 with $f(\xi) = \xi(a\xi + b)^{-1}$.

12. $(ay^m\sinh x + b)y'_x = y^{m+1}\cosh x.$

This is a special case of equation 13.7.2.22 with $f(\xi) = \xi(a\xi + b)^{-1}$.

13. $(ax^n + bx\cosh^m y)y'_x = y^k.$

This is a Bernoulli equation with respect to $x = x(y)$ (see Section 13.1.5).

14. $(ax^n + bx\tanh^m y)y'_x = y^k.$

This is a Bernoulli equation with respect to $x = x(y)$.

15. $(ax^n + bx\cosh^m y)y'_x = \cosh^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

16. $(ax^n + bx\tanh^m y)y'_x = \tanh^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

17. $(amx^ny^{m-1} + b)y'_x + anx^{n-1}y^m + c\sinh^k(\lambda x) = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b$ and $g(x) = c\sinh^k(\lambda x)$.

18. $(amx^ny^{m-1} + b)y'_x + anx^{n-1}y^m + c\tanh^k(\lambda x) = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b$ and $g(x) = c\tanh^k(\lambda x)$.

19. $(ax^ny^m + bx)y'_x = \cosh^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

20. $(ax^n y^m + bx)y'_x = \tanh^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

21. $(ax^n \cosh^m y + bx)y'_x = \sinh^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

22. $(ax^n \tanh^m y + bx)y'_x = y^k.$

This is a Bernoulli equation with respect to $x = x(y)$.

23. $(amx^n y^{m-1} + b\sinh^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b\sinh^k y$ and $g(x) = c$.

24. $(amx^n y^{m-1} + b\tanh^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b\tanh^k y$ and $g(x) = c$.

13.5.4 Equations Containing Logarithmic Functions

1. $y'_x = y(\alpha x + m\ln y + \beta).$

This is a special case of equation 13.7.2.3 with $f(\xi) = \ln\xi + \beta$.

2. $y'_x = ax^{kn-1}y^{km+1}(n\ln x + m\ln y).$

This is a special case of equation 13.7.1.3 with $f(\xi) = a\xi^k\ln\xi$.

3. $y'_x = ax^n y\ln^2 y + bx^m y\ln y + cx^k y.$

This is a special case of equation 13.7.3.1 with $f(x) = ax^n$, $g(x) = bx^m$, and $h(x) = cx^k$.

4. $xy'_x = (\alpha y + n\ln x)^m + \beta.$

This is a special case of equation 13.7.2.4 with $f(\xi) = \ln^m\xi + \beta$.

5. $xy'_x = y(n\ln x + m\ln y).$

This is a special case of equation 13.7.1.3 with $f(\xi) = \ln\xi$.

6. $mxy'_x = ax^s y^k(n\ln x + m\ln y) - ny.$

This is a special case of equation 13.7.1.5 with $f(x) = \dfrac{a}{m}x^{s-1}$ and $g(\xi) = \ln\xi$.

7. $(x^a + b)y'_x = yx^{a-1} + c(\ln y - \ln x).$

This is a special case of equation 13.7.1.12 with $f(\xi) = b$, $g(\xi) = c\ln\xi$, and $h(\xi) = 1$.

8. $x(\alpha y + \beta)y'_x = n\ln x + (\alpha - n)y.$

This is a special case of equation 13.7.2.16 with $f(\xi) = \beta$, $g(\xi) = 1$, and $h(\xi) = \ln\xi$.

9. $x(a + mx^k)y'_x = y(bn\ln x + bm\ln y - nx^k).$

This is a special case of equation 13.7.1.16 with $f(\xi) = a$, $g(\xi) = 1$, and $h(\xi) = b\ln\xi$.

10. $x(a + my^k)y'_x = y(bn\ln x + bm\ln y - ny^k).$

This is a special case of equation 13.7.1.17 with $f(\xi) = a$, $g(\xi) = 1$, and $h(\xi) = b\ln\xi$.

11. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c\ln^k(\lambda x) = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b$ and $g(x) = c\ln^k(\lambda x)$.

12. $(a\ln y + bx)y'_x = 1.$

Solution: $x = e^{by}\left(\frac{a}{b}\int \frac{e^{-by}}{y}\,dy + C\right) - \frac{a}{b}\ln y.$

13. $x(\ln y)y'_x = y(ax^{nk}y^k + bx^n y) - ny\ln y.$

This is a special case of equation 13.7.3.7 with $f(\xi) = a\xi^k + b\xi$ and $m = 1$.

14. $x(a + m\ln y)y'_x = y(bx^n y^m - n\ln y + c).$

This is a special case of equation 13.7.3.9 with $f(\xi) = a$, $g(\xi) = 1$, and $h(\xi) = b\xi + c$.

15. $(ax^n + bx\ln^m y)y'_x = \ln^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

16. $x(ax^n y^m + m\ln x)y'_x = y(bx^{nk}y^{mk} - n\ln x).$

This is a special case of equation 13.7.3.10 with $f(\xi) = a\xi$, $g(\xi) = 1$, and $h(\xi) = b\xi^k$.

17. $x(ax^n y^m + m\ln y)y'_x = y(bx^{nk}y^{mk} - n\ln y).$

This is a special case of equation 13.7.3.9 with $f(\xi) = a\xi$, $g(\xi) = 1$, and $h(\xi) = b\xi^k$.

18. $(amx^n y^{m-1} + b\ln^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b\ln^k y$ and $g(x) = c$.

19. $(ax^n\ln^m y + bx)y'_x = \ln^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

20. $(ax^n\ln^m y + bx\ln^k y)y'_x = y^s.$

This is a Bernoulli equation with respect to $x = x(y)$.

13.5.5 Equations Containing Trigonometric Functions

1. $y'_x = \alpha\cos(ay) + \beta\cos(bx).$

This is a special case of equation 13.7.4.11 with $f(x) = \alpha$, $g(x) = 0$, and $h(x) = \beta\cos(bx)$.

2. $y'_x = \sin(ax)\cos(by) + \cos(ax)\sin(by).$

This is a special case of equation 13.7.1.1 with $f(\xi) = \sin\xi$ and $c = 0$.

3. $y'_x = a\tan(bxy).$

The solution is given by the relation:

$$\int_0^w \exp\left(\tfrac{1}{2}t^2\right)\cos\left(\sqrt{ab}\,xt\right)dt = C\exp\left(\tfrac{1}{2}abx^2\right), \qquad \text{where} \quad w = y\sqrt{\frac{b}{a}}.$$

4. $y'_x = bx^n\cos(ay) + cx^m.$

This is a special case of equation 13.7.4.11 with $f(x) = bx^n$, $g(x) = 0$, and $h(x) = cx^m$.

5. $y'_x = bx^n \sin(ay) + cx^m.$

This is a special case of equation 13.7.4.11 with $f(x) = 0$, $g(x) = bx^n$, and $h(x) = cx^m$.

6. $y'_x = y \cos x \, (ay^{nm} \sin^{n-1} x + by^m).$

This is a special case of equation 13.7.4.4 with $f(\xi) = a\xi^n + b\xi$.

7. $y'_x = y \sin x \, (ay^{nm} \cos^{n-1} x + by^m).$

This is a special case of equation 13.7.4.3 with $f(\xi) = a\xi^n + b\xi$.

8. $y'_x = a\dfrac{\sin^2 y}{\cos^2 x} + b\dfrac{\cos^2 y}{\sin^2 x}.$

This is a special case of equation 13.7.4.14 with $f(\xi) = a\xi + b\xi^{-1}$.

9. $y'_x = ay^{1+n} + by + c\sin(\lambda x)y^{1-n}.$

This is a special case of equation 13.7.1.4 with $f(x) = a$, $g(x) = b$, and $h(x) = c\sin(\lambda x)$.

10. $y'_x = ay^{1+n} + b\sin(\lambda x)y + cy^{1-n}.$

This is a special case of equation 13.7.1.4 with $f(x) = a$, $g(x) = b\sin(\lambda x)$, and $h(x) = c$.

11. $xy'_x + a\sin(bx + cy) = 0.$

The substitution $w = x\tan\dfrac{bx + cy}{2}$ leads to a Riccati equation of the form 13.2.2.35 with $n = 2$: $2xw'_x - bw^2 + 2(ac - 1)w - bx^2 = 0$.

12. $xy'_x = ax^2 \tan(by) + y.$

The substitution $y = xw$ leads to an equation of the form 13.5.5.3: $w'_x = a\tan(bxw)$.

13. $xy'_x = ax^n \cos^2 y + b\cos y \sin y.$

This is a special case of equation 13.7.4.8 with $f(\xi) = \frac{1}{2}(a\xi + b)$.

14. $xy'_x = ax^n \sin^2 y + b\cos y \sin y.$

This is a special case of equation 13.7.4.7 with $f(\xi) = \frac{1}{2}(a\xi + b)$.

15. $xy'_x = ax^m \sin^k y \cos^{2-k} y - n\sin 2y.$

This is a special case of equation 13.7.4.18 with $f(x) = ax^{m-2nk-1}$ and $g(\xi) = \xi^k$.

16. $(1 + \tan^2 y)y'_x = a\tan^{m+1} y + b\tan y + cx^n \tan^{1-m} y.$

This is a special case of equation 13.7.4.19 with $f(x) = a$, $g(x) = b$, and $h(x) = cx^n$.

17. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c\sin^k(\lambda x) = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b$ and $g(x) = c\sin^k(\lambda x)$.

18. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c\tan^k(\lambda x) = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b$ and $g(x) = c\tan^k(\lambda x)$.

19. $(ax^n y^m + bx)y'_x = \cos^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$ (see Section 13.1.5).

20. $(ax^n y^m + bx)y'_x = \tan^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

21. $(ay^m \cos x + b)y'_x = y^{m+1} \sin x.$

This is a special case of equation 13.7.4.3 with $f(\xi) = \xi(a\xi + b)^{-1}$.

22. $(ay^m \sin x + b)y'_x = y^{m+1} \cos x.$

This is a special case of equation 13.7.4.4 with $f(\xi) = \xi(a\xi + b)^{-1}$.

23. $(ax^n + bx \cos^m y)y'_x = y^k.$

This is a Bernoulli equation with respect to $x = x(y)$.

24. $(ax^n + bx \cos^m y)y'_x = \cos^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

25. $(amx^n y^{m-1} + b\cos^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b\cos^k y$ and $g(x) = c$.

26. $(ax^n \cos^m y + bx)y'_x = \cos^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

27. $(ax^n + bx \tan^m y)y'_x = y^k.$

This is a Bernoulli equation with respect to $x = x(y)$.

28. $(ax^n + bx \tan^m y)y'_x = \tan^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

29. $(amx^n y^{m-1} + b\tan^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 13.7.1.19 with $f(y) = b\tan^k y$ and $g(x) = c$.

30. $(ax^n \tan^m y + bx)y'_x = \tan^k(\lambda y).$

This is a Bernoulli equation with respect to $x = x(y)$.

13.5.6 Equations Containing Combinations of Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. $y'_x = ax^n e^{\lambda y} + b\ln^m x.$

This is a special case of equation 13.7.2.5 with $f(x) = ax^n$ and $g(x) = b\ln^m x$.

2. $y'_x = a\ln^n(\nu x)e^{\lambda y} + bx^m.$

This is a special case of equation 13.7.2.5 with $f(x) = a\ln^n(\nu x)$ and $g(x) = bx^m$.

3. $y'_x = ae^{\lambda y}(\lambda y + \ln x)^m.$

This is a special case of equation 13.7.2.2 with $f(\xi) = a\ln^m \xi$.

4. $y'_x = ae^{-\lambda x}(\lambda x + \ln y)^m.$

This is a special case of equation 13.7.2.1 with $f(\xi) = a\ln^m \xi$.

5. $y'_x = ay\ln^2 y + by\ln y + ce^{\lambda x}y.$

This is a special case of equation 13.7.3.1 with $f(x) = a$, $g(x) = b$, and $h(x) = ce^{\lambda x}$.

6. $y'_x = ay\ln^2 y + be^{\lambda x}y\ln y + cy.$

This is a special case of equation 13.7.3.1 with $f(x) = a$, $g(x) = be^{\lambda x}$, and $h(x) = c$.

7. $y'_x = ae^y \sin x + b\tan x.$

This is a special case of equation 13.7.5.6 with $f(\xi) = a\xi + b$.

8. $y'_x = (ae^x \sin y + b)\tan y.$

This is a special case of equation 13.7.5.4 with $f(\xi) = a\xi + b$.

9. $y'_x = ae^x \sin^2 y + be^{-x}\cos^2 y.$

This is a special case of equation 13.7.5.8 with $f(\xi) = \frac{1}{2}(a\xi + b/\xi)$.

10. $y'_x = a\cos^n(\mu x)e^{\lambda y} + bx^m.$

This is a special case of equation 13.7.2.5 with $f(x) = a\cos^n(\mu x)$ and $g(x) = bx^m$.

11. $y'_x = ax^n e^{\lambda y} + b\cos^m(\mu x).$

This is a special case of equation 13.7.2.5 with $f(x) = ax^n$ and $g(x) = b\cos^m(\mu x)$.

12. $y'_x = ax^n e^{\lambda y} + b\tan^m(\mu x).$

This is a special case of equation 13.7.2.5 with $f(x) = ax^n$ and $g(x) = b\tan^m(\mu x)$.

13. $y'_x = a\tan^n(\mu x)e^{\lambda y} + bx^m.$

This is a special case of equation 13.7.2.5 with $f(x) = a\tan^n(\mu x)$ and $g(x) = bx^m$.

14. $y'_x = Ae^{\lambda x}\cos(ay) + Be^{\mu x}\sin(ay) + Ae^{\lambda x}.$

The substitution $w = \tan(\frac{1}{2}ay)$ leads to a linear equation: $w'_x = aBe^{\mu x}w + aAe^{\lambda x}$.

15. $y'_x = a\sin(\mu x)\sinh(\lambda y) + b\cos(\mu x)\cosh(\lambda y).$

This is a special case of equation 13.7.2.18 with $f(x) = a\sin(\mu x)$, $g(x) = b\cos(\mu x)$, and $h(x) = 0$.

16. $y'_x = ay\ln^2 y + by\ln y + c\sin^n(\lambda x)y.$

This is a special case of equation 13.7.3.1 with $f(x) = a$, $g(x) = b$, and $h(x) = c\sin^n(\lambda x)$.

17. $(1 + \tan^2 y)y'_x = a\tan^{1+m} y + b\tan y + ce^{\lambda x}\tan^{1-m} y.$

This is a special case of equation 13.7.4.19 with $f(x) = a$, $g(x) = b$, and $h(x) = ce^{\lambda x}$.

18. $(ae^x \cos y + b)y'_x = \cot y.$

This is a special case of equation 13.7.5.5 with $f(\xi) = (a\xi + b)^{-1}$.

19. $(ae^x \sin y + b)y'_x = \tan y.$

This is a special case of equation 13.7.5.4 with $f(\xi) = (a\xi + b)^{-1}$.

20. $(ae^y \cos x + b)y'_x = \tan x.$

This is a special case of equation 13.7.5.6 with $f(\xi) = (a\xi + b)^{-1}$.

21. $(ae^y \sin x + b)y'_x = \cot x.$

This is a special case of equation 13.7.5.7 with $f(\xi) = (a\xi + b)^{-1}$.

22. $(e^{\alpha x}y^n + a\beta)y'_x + be^{\beta y} \ln^m x + a\alpha = 0.$

This is a special case of equation 13.7.2.15 with $f(y) = y^n$ and $g(x) = b \ln^m x$.

23. $(e^{\alpha x}y^n + a\beta)y'_x + be^{\beta y} \cos^m x + a\alpha = 0.$

This is a special case of equation 13.7.2.15 with $f(y) = y^n$ and $g(x) = b \cos^m x$.

24. $(e^{\alpha x} \cos^n y + a\beta)y'_x + be^{\beta y} \cos^m(\lambda x) + a\alpha = 0.$

This is a special case of equation 13.7.2.15 with $f(y) = \cos^n y$ and $g(x) = b \cos^m(\lambda x)$.

13.6 Equations of the Form $F(x, y, y'_x) = 0$ Containing Arbitrary Parameters

13.6.1 Equations of the Second Degree in y'_x

▶ Equations of the form $f(x, y)(y'_x)^2 = g(x, y)$.

1. $(y'_x)^2 = ay + bx^2.$

See equation 13.6.3.43.

2. $(y'_x)^2 = y + ax^2 + bx + c.$

The substitution $w = 2\sqrt{y + ax^2 + bx + c}$ leads to an Abel equation of the form 13.3.1.2: $ww'_x - w = 4ax + 2b$.

3. $(y'_x)^2 = ay^3 + by + c.$

Solution: $x = C \pm \int (ay^3 + by + c)^{-1/2}\, dy$.

4. $(y'_x)^2 = ay + b\sqrt{x}.$

See equation 13.6.3.26.

5. $(y'_x)^2 = ay + b\sqrt{x} + c, \qquad a \neq 0.$

The substitution $aw = 2\sqrt{ay + b\sqrt{x} + c}$ leads to an Abel equation of the form 13.3.1.32: $ww'_x - w = ba^{-2}x^{-1/2}$.

6. $(y'_x)^2 = y + ax^{m+1} - \dfrac{m+1}{2(m+3)^2}x^2 + b.$

The substitution $w = 2\Big[y + ax^{m+1} - \dfrac{m+1}{2(m+3)^2}x^2 + b\Big]^{1/2}$ leads to an Abel equation of the form 13.3.1.10: $ww'_x - w = -\dfrac{2(m+1)}{(m+3)^2}x + 2a(m+1)x^m$.

7. $(y'_x)^2 = \lambda y + ax^2 + bx^{m+1} + c.$

For $\lambda \neq 0$, the substitution $\lambda w = 2(\lambda y + ax^2 + bx^{m+1} + c)^{1/2}$ leads to the Abel equation $ww'_x - w = 4a\lambda^{-2}x + 2b\lambda^{-2}(m+1)x^m$, which is outlined in Section 13.3.1 (see Table 13.1).

Special cases of the original equation are equations 13.6.1.1–13.6.1.6.

8. $x(y'_x)^2 = axy + b.$

See equation 13.6.3.32.

9. $x(y'_x)^2 = axy + bx + c, \qquad a \neq 0.$

The substitution $aw = 2\sqrt{ay + b + cx^{-1}}$ leads to an Abel equation of the form 13.3.1.33: $ww'_x - w = -2ca^{-2}x^{-2}$.

10. $y^2(y'_x)^2 = ax^2y^2 + b.$

See equation 13.6.3.34.

11. $y^2(y'_x)^2 = ax^{-2/5}y^2 + b.$

See equation 13.6.3.28.

12. $(ay^2 + bx)(y'_x)^2 = 1.$

See equation 13.6.3.44.

13. $(ax^2 + by)(y'_x)^2 = x^2y.$

See equation 13.6.3.46.

14. $(axy + b)(y'_x)^2 = y.$

See equation 13.6.3.33.

15. $xy^2(y'_x)^2 = ay^2 + bx.$

See equation 13.6.3.45.

16. $(ax^2y^2 + b)(y'_x)^2 = x^2.$

See equation 13.6.3.35.

17. $(a\sqrt{y} + bx)(y'_x)^2 = 1.$

See equation 13.6.3.27.

18. $(ax^2y^{3/5} + by)(y'_x)^2 = x^2y.$

See equation 13.6.3.29.

19. $(y'_x)^2 = ae^y + b.$

See equation 13.6.3.3 with $k = 2$.

20. $(y'_x)^2 = a + be^x.$

See equation 13.6.3.4 with $k = 2$.

21. $(y'_x)^2 = ay^2 + be^x.$

See equation 13.6.3.8 with $k = 2$.

22. $x^2(y'_x)^2 = ax^2e^y + b.$

See equation 13.6.3.9 with $k = 2$.

23. $(ae^y + bx^2)(y'_x)^2 = 1.$

See equation 13.6.3.9 with $k = -2$.

24. $(ae^x y^2 + b)(y'_x)^2 = y^2.$

See equation 13.6.3.8 with $k = -2$.

25. $(y'_x)^2 = ay + b\ln x.$

See equation 13.6.3.13.

26. $(y'_x)^2 = \lambda y + a\ln x + b, \qquad \lambda \neq 0.$

The substitution $\lambda w = 2\sqrt{\lambda y + a\ln x + b}$ leads to an Abel equation of the form 13.3.1.16: $ww'_x - w = 2a\lambda^{-2}x^{-1}$.

27. $(a\ln y + bx)(y'_x)^2 = 1.$

See equation 13.6.3.14.

► **Equations of the form** $f(x, y)(y'_x)^2 = g(x, y)y'_x + h(x, y)$.

28. $(y'_x)^2 + ay'_x + by = 0.$

Solution in parametric form:

$$bx = -2t - a\ln t + C, \qquad by = -t^2 - at.$$

29. $(y'_x)^2 + ayy'_x = bx + c.$

We differentiate the equation with respect to x, take y as the independent variable, and assume $\xi = y'_x$ to obtain a linear equation with respect to $y = y(\xi)$:

$$(a\xi^2 - b)y'_\xi + a\xi y + 2\xi^2 = 0.$$

30. $(y'_x)^2 + axy'_x + by + cx^2 = 0.$

The transformation $x = e^t$, $y = x^2 u$ leads to an autonomous equation: $\left(u'_t + 2u + \frac{1}{2}a\right)^2 = \frac{1}{4}a^2 - c - bu$. Having extracted the root and carried over the terms $2u + \frac{1}{2}a$ from the left-hand side to the right-hand side, we obtain a separable equation of the form 13.1.2.

31. $y = xy'_x + ax^2 + b(y'_x)^2 + cy'_x + d, \qquad a \neq 0.$

Differentiating with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at an Abel equation of the form 13.3.1.2: $ww'_t - w = -4abt - 2ac$.

32. $(y'_x)^2 + (ax + b)y'_x - ay + c = 0, \qquad a \neq 0.$

Solutions: $y = (ax + b)C + aC^2 + ca^{-1}$ and $4ay = 4c - (ax + b)^2$.

33. $(y'_x)^2 + (ay + bx)y'_x + abxy = 0.$

This equation can be factorized: $(y'_x + ay)(y'_x + bx) = 0$. Therefore, the solutions are: $y = Ce^{-ax}$ and $y = -\frac{1}{2}bx^2 + C$.

34. $(y'_x)^2 + ax^2 y'_x + bxy = 0.$

The transformation $z = \ln x$, $u = yx^{-3}$ leads to an equation independent implicitly of z: $(u'_z)^2 + (a + 6u)u'_z + (3a + b + 9u)u = 0$. Rewriting the latter equation to solve for u'_z, we obtain a separable equation of the form 13.1.2.

35. $a(y'_x)^2 - yy'_x - x = 0.$

Solution in parametric form:

$$x = \frac{t}{\sqrt{t^2+1}}\bigl[C + a\ln\bigl(t + \sqrt{t^2+1}\,\bigr)\bigr], \qquad y = at - \frac{x}{t}.$$

36. $x(y'_x)^2 - ayy'_x + b = 0.$

1°. For $a \neq 1$, the solution in parametric form is written as:

$$x = Ct^k + \frac{b}{2a-1}t^2, \quad aty = xt^2 + b, \qquad \text{where} \quad k = \frac{1}{a-1}.$$

2°. For $a = 1$, the solution is: $y = Cx + b/C$. There are two singular solutions: $y = \pm 2\sqrt{bx}$.

37. $x(y'_x)^2 + ayy'_x + bx = 0.$

1°. For $a \neq -1$, the solution in parametric form is written as:

$$x = Ct|(a+1)t^2 + b|^{-\frac{a+2}{2(a+1)}}, \qquad y = -\frac{x}{at}(t^2 + b).$$

There are two singular solutions: $y = \pm x\sqrt{-b/(a+1)}$.

2°. For $a = -1$, the solution in parametric form is written as:

$$x = Ct\exp\Bigl(-\frac{t^2}{2b}\Bigr), \qquad y = x\Bigl(t + \frac{b}{t}\Bigr).$$

38. $x(y'_x)^2 - yy'_x + ay = 0.$

Solution in parametric form:

$$x = C(t-a)\exp(-t/a), \qquad y = Ct^2\exp(-t/a).$$

There is a singular solution: $y = 0$.

39. $x(y'_x)^2 - yy'_x + ax^2 y'_x + by'_x + c = 0, \qquad a \neq 0.$

We divide the equation by y'_x and differentiate with respect to x. Passing to the new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at an Abel equation of the form 13.3.1.33: $ww'_t - w = act^{-2}$.

40. $y(y'_x)^2 + axy'_x + by = 0.$

Solution in parametric form:

$$axt + y(b + t^2) = 0, \quad Cy(t^2 + a + b)^m = t^{b/(a+b)}, \qquad \text{where} \quad m = \frac{a+2b}{2(a+b)}.$$

There two singular solutions $y = \pm x\sqrt{-a-b}$ corresponding to the limit $C \to \infty$. In addition, $y = 0$ is also a singular solution.

41. $\boldsymbol{x(y'_x)^2 + (a - y)y'_x + b = 0.}$

Solutions: $C(Cx - y + a) + b = 0$ and $(y - a)^2 = 4bx$.

42. $\boldsymbol{ax(y'_x)^2 + (bx - ay + k)y'_x - by = 0.}$

Solution: $y = Cx + \dfrac{kC}{aC + b}$. In addition, there is a singular solution which can be written in parametric form as:

$$x = -\frac{bk}{(at + b)^2}, \qquad y = xt + \frac{kt}{at + b}.$$

43. $\boldsymbol{ax(y'_x)^2 - (ay + bx - a - b)y'_x + by = 0.}$

Differentiating with respect to x and factorizing, we obtain

$$(2axy'_x - ay - bx + a + b)y''_{xx} = 0.$$

Equating both factors to zero and integrating, we arrive at the solutions:

$$y = Cx + \frac{C(a + b)}{aC - b} \quad \text{and} \quad (ay + bx - a - b)^2 - 4abxy = 0.$$

44. $\boldsymbol{x(y'_x)^2 + ayy'_x + bx^n y^m = 0.}$

The substitution $x = e^t$ leads to an equation of the form 13.6.1.68: $(y'_t)^2 + ayy'_t + be^{(n+1)t}y^m = 0$.

45. $\boldsymbol{x^2(y'_x)^2 - (2xy + a)y'_x + y^2 = 0.}$

Solutions: $y = aC^2x + aC$ and $y = -\frac{1}{4}ax^{-1}$.

46. $\boldsymbol{ax^2(y'_x)^2 - 2axyy'_x + y^2 - a(a - 1)x^2 = 0.}$

Solutions: $y \pm \sqrt{y^2 + ax^2} = Cx^{1+k}$, where $k = \sqrt{(a - 1)/a}$.

47. $\boldsymbol{(a^2 - 1)x^2(y'_x)^2 + 2xyy'_x - y^2 + a^2x^2 = 0.}$

Solution in parametric form:

$$x = C(t^2 + 1)^{-1/2}\bigl(t + \sqrt{t^2 + 1}\bigr)^{-1/a}, \qquad y = xt + ax\sqrt{t^2 + 1}.$$

48. $\boldsymbol{x^2(y'_x)^2 + (ax^2y^3 + b)y'_x + aby^3 = 0.}$

The equation can be factorized: $(y'_x + ay^3)(x^2y'_x + b) = 0$. Equating each of the factors to zero, we obtain the solutions: $y = \pm(2ax + C)^{-1/2}$ and $y = b/x + C$.

49. $\boldsymbol{(x^2 - a)(y'_x)^2 - 2xyy'_x - x^2 = 0.}$

Solving for y, differentiating with respect to x, and setting $w(x) = y'_x$, we obtain a factorized equation: $(xw'_x - w)(x^2w^2 + x^2 - aw^2) = 0$. Equating each of the factors to zero, we arrive at the solutions:

$$y = \frac{1}{2C}(x^2 - a - C^2) \quad \text{and} \quad y^2 + x^2 = a \ \ (y \neq 0).$$

50. $\boldsymbol{(x^2 - a^2)(y'_x)^2 + 2xyy'_x + y^2 = 0.}$

The equation can be factorized: $(xy'_x + ay'_x + y)(xy'_x - ay'_x + y) = 0$. Equating each of the factors to zero, we obtain the solutions: $(x + a)y = C$ and $(x - a)y = C$.

51. $\boldsymbol{(x^2+a)(y'_x)^2-2xyy'_x+y^2+b=0.}$

Differentiating with respect to x, we obtain a factorized equation: $[(x^2+a)y'_x-xy]y''_{xx}=0$. Therefore, the solutions of the original equation are:

$$y=C_1x+C_2, \quad \text{where } aC_1^2+C_2^2+b=0; \qquad bx^2+ay^2+ab=0.$$

52. $\boldsymbol{x^3(y'_x)^2+x^2yy'_x+a=0.}$

Solutions: $Cxy=C^2x+a$ and $xy^2=4a$.

53. $\boldsymbol{axy(y'_x)^2-(ay^2+bx^2+k)y'_x+bxy=0.}$

This differential equation represents an equation of curvature lines of a surface defined by the relation $Ax^2+By^2+Cz^2=1$, where $a=AB(C-B)$, $b=AB(A-C)$, $k=C(B-A)$.

Solutions:

$$(aC-b)y^2=C(aC-b)x^2-kC \quad \text{and} \quad ay^2=bx^2\pm 2x\sqrt{-bk}-k.$$

54. $\boldsymbol{y^2(y'_x)^2+2axyy'_x+(1-a)y^2+ax^2+(a-1)b=0.}$

Solutions:

$$y^2+ax^2-b=(a-1)(x+C)^2 \quad \text{and} \quad y^2+ax^2-b=0.$$

55. $\boldsymbol{(a-b)y^2(y'_x)^2-2bxyy'_x+ay^2-bx^2-ab=0.}$

Solutions:

$$x^2+y^2=Cx+b-\frac{a-b}{4a}C^2 \quad \text{and} \quad (a-b)y^2-bx^2=(a-b)b.$$

56. $\boldsymbol{(ay-x^2)(y'_x)^2+2xyy'_x-y^2=0.}$

Solution: $(Cy+x)^2=4ay$.

57. $\boldsymbol{(y^2-a^2x^2)(y'_x)^2+2xyy'_x+(1-a^2)x^2=0.}$

Solution in parametric form:

$$x=\frac{Ct}{\sqrt{t^2+1}}, \qquad y=aC-\frac{C}{\sqrt{t^2+1}}.$$

58. $\boldsymbol{(ay-bx)^2[a^2(y'_x)^2+b^2]-k^2(ay'_x+b)^2=0.}$

We solve the equation for $ay-bx$ and differentiate with respect to x. Setting $w(x)=y'_x$, we obtain a factorized equation with respect to $w(x)$: $(aw-b)[(a^2w^2+b^2)^{3/2}\pm abkw'_x]=0$. Equating each of the factors to zero and integrating, we arrive at the solutions:

$$(bx-C)^2+(ay-C)^2=k^2 \quad \text{and} \quad ay-bx=\pm k\sqrt{2}.$$

59. $\boldsymbol{(xy'_x+a)^2-2ay+x^2=0, \qquad a\neq 0.}$

The substitution $2ay-x^2=u^2$ leads to the equation $xuu'_x-a(u-a)+x^2=0$. Further assuming $u-a=xw(x)$, we obtain $(xw+a)w'_x+w^2+1=0$. Taking w to be the independent variable, we arrive at a first-order linear equation whose solution is: $x=(w^2+1)^{-1/2}\big[C-a\ln\big(w+\sqrt{w^2+1}\,\big)\big]$.

60. $(xy'_x + ny)^2 + ax^{n+1}y'_x + b = 0.$

Solution: $y = \dfrac{C^2n^2 + b}{an}x^{-n} + C.$

61. $(xy'_x + ny)^2 - ax^{2n+2}(y'_x)^2 - b = 0.$

Solutions: $y = Cx^{-n} \pm \sqrt{aC^2 + bn^{-2}}.$

62. $(yy'_x + x)^2 = a^2(x^2 + y^2)^k[(y'_x)^2 + 1].$

This equation splits into two equations of the form 13.8.1.4 with $f(u) = \pm au^{k/2}$:

$$yy'_x + x = \pm a(x^2 + y^2)^{k/2}\sqrt{(y'_x)^2 + 1}.$$

63. $ax^{2n}(xy'_x + my)^2 + bx^{2m}(xy'_x + ny)^2 = 1.$

Solution: $y = C_1x^{-n} + C_2x^{-m}$. Here, the constants C_1 and C_2 are related by the constraint $(aC_1^2 + bC_2^2)(n - m)^2 = 1$.

64. $(xy'_x - y)^2 = a^2(x^2 + y^2)^k(yy'_x + x)^2.$

This equation splits into two equations of the form 13.7.1.20 with $f(u) = \pm au^{k/2}$:

$$xy'_x - y = \pm a(x^2 + y^2)^{k/2}(yy'_x + x).$$

65. $(xy'_x - y)^2 = a^2(x^2 + y^2)^k[(y'_x)^2 + 1].$

This equation splits into two equations of the form 13.8.1.3 with $f(u) = \pm au^{k/2}$:

$$xy'_x - y = \pm a(x^2 + y^2)^{k/2}\sqrt{(y'_x)^2 + 1}.$$

66. $(xy'_x + y + 2ax)^2 = 4(xy + ax^2 + b).$

The substitution $u = xy + ax^2 + b$ leads to a separable equation: $u'_x = \pm 2\sqrt{u}$.

67. $(a_2x + b_2y + c_2)(y'_x)^2 + (a_1x + b_1y + c_1)y'_x + a_0x + b_0y + c_0 = 0.$

The Legendre transformation $x = u'_t$, $y = tu'_t - u$ $(y'_x = t)$ leads to a linear equation:

$$[f(t) + tg(t)]u'_t = g(t)u + h(t),$$

where $f(t) = a_2t^2 + a_1t + a_0$, $g(t) = b_2t^2 + b_1t + b_0$, and $h(t) = -c_2t^2 - c_1t - c_0$.

68. $(y'_x)^2 + ayy'_x + be^{\lambda x}y^m = 0.$

1°. With $m \neq 2$, solving for y'_x and performing the substitution $w = e^{\lambda x}y^{m-2}$, we arrive at a separable equation: $w'_x = \lambda w + \dfrac{2-m}{2}\big(a \pm \sqrt{a^2 - 4bw}\,\big)w$ (see Section 13.1.2).

2°. With $m = 2$, solving the original equation for y'_x, we obtain a separable equation: $2y'_x = y\big(-a \pm \sqrt{a^2 - 4be^{\lambda x}}\,\big)$.

69. $y'_x + y = ae^{3x}(y'_x - y)^2.$

Solution: $y = \frac{1}{2}aC^2e^x - \frac{1}{2}Ce^{-x}$.

70. $y'_x = (y'^2_x - y^2)(ae^x + be^{-x}).$

This is a special case of equation 13.8.1.56 with $f(u) = u$.

71. $ae^{2\lambda x}(y'_x+\beta y)^2+be^{2\beta x}(y'_x+\lambda y)^2=1.$

Solution: $y=C_1e^{-\lambda x}+C_2e^{-\beta x}$. Here, the constants C_1 and C_2 are related by the constraint $(aC_1^2+bC_2^2)(\beta-\lambda)^2=1$.

72. $y'_x=a(y'^2_x-y^2)\cosh x.$

This is a special case of equation 13.8.1.58 with $f(u)=au$.

73. $y'_x=a(y'^2_x-y^2)\sinh x.$

This is a special case of equation 13.8.1.59 with $f(u)=au$.

74. $a(y'_x\cosh x-y\sinh x)^2+b(y'_x\sinh x-y\cosh x)^2=1.$

Solution: $y=C_1\sinh x+C_2\cosh x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^2+bC_2^2=1$.

75. $(y'_x)^2-xyy'_x+y^2\ln(ay)=0.$

Solutions: $ay=\exp(Cx-C^2)$ and $ay=\exp(\frac{1}{4}x^2)$.

76. $y'_x=a(y'^2_x+y^2)\cos x.$

This is a special case of equation 13.8.1.64 with $f(u)=au$.

77. $y'_x=b(y'^2_x+y^2)\sin x.$

This is a special case of equation 13.8.1.65 with $f(u)=u$ and $a=0$.

78. $a(y'_x\cos x+y\sin x)^2+b(y'_x\sin x-y\cos x)^2=1.$

Solution: $y=C_1\sin x+C_2\cos x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^2+bC_2^2=1$.

79. $a(y'_x\cosh x-y\sinh x)^2+b[(y'_x)^2-y^2]+c=0.$

Solution: $y=C_1\sinh x+C_2\cosh x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^2+b(C_1^2-C_2^2)+c=0$.

80. $a(y'_x\cos x+y\sin x)^2+b[(y'_x)^2+y^2]+c=0.$

Solution: $y=C_1\sin x+C_2\cos x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^2+b(C_1^2+C_2^2)+c=0$.

13.6.2 Equations of the Third Degree in y'_x

► **Equations of the form** $f(x,y)(y'_x)^3=g(x,y)y'_x+h(x,y)$.

1. $(y'_x)^3+ax+by+c=0.$

This is a special case of equation 13.8.1.9 with $f(w)=w^3$.

2. $(ax+by+c)(y'_x)^3=\alpha x+\beta y+\gamma.$

Dividing both sides by $ax+by+c$ and raising to the power 1/3, we finally arrive at an equation of the form 13.7.1.6 with $f(w)=w^{-1/3}$.

3. $a(y'_x)^3 + by'_x = x.$

This is a special case of equation 13.8.1.7 with $f(w) = aw^3 + bw$.

4. $a(y'_x)^3 + by'_x = y.$

This is a special case of equation 13.8.1.8 with $f(w) = aw^3 + bw$.

5. $a(y'_x)^3 + xy'_x = y.$

This is a special case of equation 13.8.1.10 with $f(w) = aw^3$.

6. $a(y'_x)^3 + bxy'_x = y.$

This is a special case of equation 13.8.1.11 with $f(w) = bw$ and $g(x) = aw^3$.

7. $(y'_x)^3 - axy'_x + x^3 = 0, \qquad a \neq 0.$

Solution in parametric form:

$$x = \frac{at}{t^3 + 1}, \qquad y = C + \frac{a^2}{6}\,\frac{4t^3 + 1}{(t^3 + 1)^2}.$$

8. $(y'_x)^3 - axyy'_x + 2ay^2 = 0.$

Differentiating with respect to x and eliminating y, we obtain a factorized equation with respect to $w(x) = y'_x$: $[2(w'_x)^2 - axw'_x + aw](9w - ax^2) = 0$. Equating each of the factors to zero and integrating, we find the solutions: $y = \frac{1}{4}aC(x - C)^2$ and $y = \frac{1}{27}ax^3$.

9. $ax(y'_x)^3 + by'_x = y.$

This is a special case of equation 13.8.1.11 with $f(w) = aw^3$ and $g(w) = bw$.

10. $ax^{3/2}(y'_x)^3 + 2xy'_x = y.$

Solution: $y = 2C\sqrt{x} + aC^3$.

11. $ax^n(y'_x)^3 + xy'_x = y.$

This is a special case of equation 13.8.1.15 with $f(w) = aw^3$.

12. $ae^{3x}(y'_x)^3 + b(y'_x + y) + c = 0.$

Solution: $y = Ce^{-x} + (aC^3 - c)b^{-1}$.

► **Equations of the form** $f(x, y)(y'_x)^3 = g(x, y)(y'_x)^2 + h(x, y)y'_x + r(x, y).$

13. $a(y'_x)^3 + b(y'_x)^2 = x.$

This is a special case of equation 13.8.1.7 with $f(w) = aw^3 + bw^2$.

14. $a(y'_x)^3 + b(y'_x)^2 = y.$

This is a special case of equation 13.8.1.8 with $f(w) = aw^3 + bw^2$.

15. $(y'_x)^3 + a(y'_x)^2 + by + abx + d = 0.$

Solution in parametric form:

$$2bx = -3t^2 + 2at - 2a^2 \ln(t + a) + C, \qquad by = -abx - t^3 - at^2 - d.$$

In addition, there is a singular solution: $y = -ax - d/b$.

16. $a(y'_x)^3 + b(y'_x)^2 + cy'_x = y + d.$

Solution in parametric form:

$$x = C + \tfrac{3}{2}at^2 + 2bt + c\ln|t|, \qquad y = at^3 + bt^2 + ct - d.$$

17. $a(y'_x)^3 + bx(y'_x)^2 = y.$

This is a special case of equation 13.8.1.11 with $f(w) = bw^2$ and $g(w) = aw^3$.

18. $ax(y'_x)^3 + b(y'_x)^2 = y.$

This is a special case of equation 13.8.1.11 with $f(w) = aw^3$ and $g(w) = bw^2$.

19. $(x^2 - a^2)(y'_x)^3 + bx(x^2 - a^2)(y'_x)^2 + y'_x + bx = 0.$

The equation can be factorized: $(y'_x + bx)[(y'_x)^2(x^2 - a^2) + 1] = 0$, whence we find the solutions: $y = -\frac{1}{2}bx^2 + C$ and $y = \pm\arcsin(x/a) + C$.

20. $a(y'_x + y)^3 + be^{3x}(y'_x)^3 + c = 0.$

Solution: $y = C_1 - C_2e^{-x}$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^3 + bC_2^3 + c = 0$.

21. $(xy'_x - y)^3 + ay + bx = 0.$

This is a special case of equation 13.8.1.16 with $f(w) = 1$, $g(w) = a$, $h(w) = b$, and $n = 3$.

22. $(xy'_x - y)^3 + ayy'_x + bx = 0.$

This is a special case of equation 13.8.1.16 with $f(w) = 1$, $g(w) = aw$, $h(w) = b$, and $n = 3$.

23. $(xy'_x - y)^3 + axy'_x + by = 0.$

This is a special case of equation 13.8.1.16 with $f(w) = 1$, $g(w) = b$, $h(w) = aw$, and $n = 3$.

24. $a(y'_x \cosh x - y\sinh x)^3 + b(y'_x \sinh x - y\cosh x)^3 + c = 0.$

Solution: $y = C_1\sinh x - C_2\cosh x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^3 + bC_2^3 + c = 0$.

25. $a(y'_x \cosh x - y\sinh x)^3 + b(y'^2_x - y^2) + c = 0.$

Solution: $y = C_1\sinh x + C_2\cosh x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^3 + b(C_1^2 - C_2^2) + c = 0$.

26. $y'_x(y'_x \cosh x - y\sinh x)^3 = b(y'^2_x - y^2)\cosh x.$

This is a special case of equation 13.6.4.21 with $n = 3$ and $m = 1$.

27. $y'_x(y'_x \sinh x - y\cosh x)^3 = b(y'^2_x - y^2)\sinh x.$

This is a special case of equation 13.6.4.22 with $n = 3$ and $m = 1$.

28. $a(y'_x \cos x + y\sin x)^3 + b(y'_x \sin x - y\cos x)^3 + c = 0.$

Solution: $y = C_1\sin x - C_2\cos x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^3 + bC_2^3 + c = 0$.

29. $a(y'_x \cos x + y \sin x)^3 + b(y'^2_x + y^2) + c = 0.$

This is a special case of equation 13.6.4.24 with $n = 3$ and $k = 1$.

30. $y'_x(y'_x \cos x + y \sin x)^3 = b(y'^2_x + y^2) \cos x.$

This is a special case of equation 13.6.4.25 with $n = 3$ and $m = 1$.

31. $y'_x(y'_x \sin x - y \cos x)^3 = b(y'^2_x + y^2) \sin x.$

This is a special case of equation 13.6.4.26 with $n = 3$ and $m = 1$.

13.6.3 Equations of the Form $(y'_x)^k = f(y) + g(x)$

► **Some transformations.**

$1°$. In the general case, the equation

$$(y'_x)^k = f(y) + g(x) \tag{1}$$

can be reduced with the aid of the transformation $t = \int [g(x)]^{1/k}\,dx$, $u = \int [f(y)]^{-1/k}\,dy$ to the same form

$$(u'_t)^k = F(u) + G(t), \tag{2}$$

where functions $F = F(u)$ and $G = G(t)$ are defined parametrically by the following formulas:

$$F(u) = \frac{1}{f(y)}, \quad u = \int [f(y)]^{-1/k}\,dy,$$
$$G(t) = \frac{1}{g(x)}, \quad t = \int [g(x)]^{1/k}\,dx.$$

$2°$. Taking y as the independent variable, we obtain from Eq. (1) an equation of the same class for $x = x(y)$:

$$(x'_y)^{-k} = g(x) + f(y).$$

$3°$. The equation

$$y'_x = a\sqrt{y} + g(x) \qquad \big(k = 1,\ \ f = a\sqrt{y}\,\big)$$

can be reduced with the aid of the substitution $w(x) = 2a^{-1}\sqrt{y}$ to the Abel equation $ww'_x - w = 2a^{-2}g(x)$, which is outlined in Section 13.3.1.

$4°$. The equation

$$y'_x = y^{-1} + g(x) \qquad (k = 1,\ \ f = y^{-1})$$

is an alternative form of representation of the Abel equation $yy'_x = g(x)y + 1$, which is outlined in Section 13.3.2.

$5°$. The equation

$$y'_x = ay^s + g(x) \qquad \big(k = 1,\ \ f = ay^s\big)$$

can be reduced, with the aid of the substitution $aw = y - \int g(x)\,dx$ followed by raising both sides of the equation to the power of $1/s$, to an equation of the class in question:

$$(w'_x)^{1/s} = aw + \int g(x)\,dx.$$

6°. The equation

$$(y'_x)^2 = ay + g(x) \qquad (k = 2, \;\; f = ay, \;\; a \neq 0)$$

can be reduced with the aid of the substitution $aw = 2\sqrt{ay + g(x)}$ to an Abel equation of the second kind:

$$ww'_x = w + \varphi(x), \qquad \text{where} \quad \varphi = 2a^{-2}g'_x(x),$$

which is outlined in Section 13.3.1.

7°. The equation

$$(y'_x)^{1/2} = ay + g(x) \qquad (k = 1/2, \;\; f = ay)$$

can be reduced by squaring both sides and performing the substitution $z = ay + g(x)$ to the Riccati equation:

$$z'_x = az^2 + g'_x.$$

For some specific functions $g = g(x)$, the solutions of the latter equation are given in Section 13.2.

8°. The equation

$$(y'_x)^{1/2} = ay^{1/2} + g(x) \qquad (k = 1/2, \;\; f = ay^{1/2})$$

can be reduced by squaring both sides and performing the substitution $y = \exp(a^2x)\xi^2$ to an Abel equation of the second kind:

$$\xi\xi'_x = a\exp(-\tfrac{1}{2}a^2x)g\xi + \tfrac{1}{2}\exp(-a^2x)g^2$$

(see Section 13.3.3).

9°. The equation

$$(y'_x)^{-1/2} = f(y) + ax \qquad (k = -1/2, \;\; g = ax)$$

can be reduced by squaring both sides and performing the substitution $v = f(y) + ax$ to a Riccati equation:

$$v'_y = av^2 + f'_y.$$

For some specific functions $f = f(y)$, the solutions of the latter equation are given in Section 13.2.

► **Classification tables and exact solutions.**

For the sake of convenience, Tables 13.5–13.9 given below list all the equations outlined in Section 13.6.3. The five tables classify the equations in which functions f and g are of the same form. The right-most column of a table indicates the numbers of the equations where the corresponding solutions are given. After the tables follow the equations — they are arranged into groups so that the solutions of the equations within each group are expressed in terms of the functions indicated before the groups as a notation list.

TABLE 13.5
Solvable equations of the form $(y'_x)^k = Ay^s + Bx^r$

k	s	r	Equation	k	s	r	Equation
arbitrary	arbitrary ($s \neq k$)	$\frac{ks}{k-s}$	13.6.3.7	-1	1	-1	13.6.3.23
arbitrary	arbitrary	0	13.6.3.1	-1	1	$1/2$	13.6.3.42
arbitrary ($k \neq -1,\ 1$)	$\frac{k}{1-k}$	$-\frac{k}{1+k}$	13.6.3.6	$-1/2$	arbitrary ($s \neq -1,\ 0$)	1	13.6.3.17
arbitrary	0	arbitrary	13.6.3.2	$-1/2$	-1	1	13.6.3.38
arbitrary	1	1	13.6.3.5	$1/2$	1	arbitrary ($r \neq -1,\ 0$)	13.6.3.16
-2	-1	-2	13.6.3.46	$1/2$	1	-1	13.6.3.37
-2	-1	1	13.6.3.33	1	-1	-2	13.6.3.20
-2	$-2/5$	-2	13.6.3.29	1	-1	$-1/2$	13.6.3.39
-2	$1/2$	1	13.6.3.27	1	-1	1	13.6.3.22
-2	2	-2	13.6.3.35	1	$1/2$	-2	13.6.3.30
-2	2	1	13.6.3.44	1	$1/2$	-1	13.6.3.11
-1	arbitrary ($s \neq 0$)	1	13.6.3.10	1	$1/2$	$-1/2$	13.6.3.24
-1	arbitrary ($s \neq -2,\ 0$)	2	13.6.3.15	1	$1/2$	1	13.6.3.41
-1	-2	-1	13.6.3.21	2	-2	-1	13.6.3.45
-1	-2	$1/2$	13.6.3.31	2	-2	$-2/5$	13.6.3.28
-1	-2	2	13.6.3.36	2	-2	2	13.6.3.34
-1	-1	$1/2$	13.6.3.12	2	1	-1	13.6.3.32
-1	$-1/2$	-1	13.6.3.40	2	1	$1/2$	13.6.3.26
-1	$-1/2$	$1/2$	13.6.3.25	2	1	2	13.6.3.43

1. $\boldsymbol{(y'_x)^k = Ay^s + B.}$

Solution: $x = \displaystyle\int (Ay^s + B)^{-1/k}\, dy + C.$

2. $\boldsymbol{(y'_x)^k = A + Bx^r.}$

Solution: $y = \displaystyle\int (A + Bx^r)^{1/k}\, dx + C.$

TABLE 13.6
Solvable equations of the form
$(y'_x)^k = Ae^y + Bx^r$

k	r	Equation
arbitrary	$-k$	13.6.3.9
arbitrary	0	13.6.3.3
-1	-1	13.5.2.40
-1	1	13.5.2.38
-1	2	13.5.2.39
$-1/2$	1	13.6.3.19
1	arbitrary	13.5.2.13

TABLE 13.7
Solvable equations of the form
$(y'_x)^k = Ay^s + Be^x$

k	s	Equation
arbitrary	k	13.6.3.8
arbitrary	0	13.6.3.4
-1	arbitrary	13.5.2.37
$1/2$	1	13.6.3.18
1	-1	13.5.2.14

TABLE 13.8
Solvable equations of the form
$(y'_x)^k = Ae^y + Be^x$

k	Equation
-1	13.5.2.8
1	13.5.2.2

TABLE 13.9
Solvable equations containing
logarithmic functions

Form of equation	Equation
$(y'_x)^{-2} = A\ln y + Bx$	13.6.3.14
$(y'_x)^{-1} = A\ln y + Bx$	13.5.4.12
$(y'_x)^2 = Ay + B\ln x$	13.6.3.13

3. $\boldsymbol{(y'_x)^k = Ae^y + B.}$

Solution: $x = \int (Ae^y + B)^{-1/k}\,dy + C.$

4. $\boldsymbol{(y'_x)^k = A + Be^x.}$

Solution: $y = \int (A + Be^x)^{1/k}\,dx + C.$

5. $\boldsymbol{(y'_x)^k = Ay + Bx.}$

Solution in parametric form:

$$x = \int (A\tau^{1/k} + B)^{-1}\,d\tau + C, \qquad y = \frac{1}{A}\left[\tau - B\int (A\tau^{1/k} + B)^{-1}\,d\tau - BC\right].$$

6. $\boldsymbol{(y'_x)^k = Ay^{\frac{k}{1-k}} + Bx^{-\frac{k}{1+k}}, \qquad |k| \neq 1.}$

Solution in parametric form:

$$x = a\left(\int \frac{d\tau}{\gamma\tau^{1/k} + \beta} + C\right)^{\frac{k+1}{k}}, \qquad y = b\left(\tau - \beta\int \frac{d\tau}{\gamma\tau^{1/k} + \beta} - \beta C\right)^{\frac{k-1}{k}},$$

where $A = a^{-\frac{k}{1+k}} b^{-\frac{k}{1-k}} \beta B$, $B = a^{\frac{k}{1+k}} \left[\frac{b(k-1)}{a(k+1)}\gamma\right]^k$.

7. $(y'_x)^k = Ay^s + Bx^{\frac{ks}{k-s}}, \qquad k \neq s.$

Solution in parametric form:

$$x = C^{k-s} \exp\biggl\{\int \biggl[\frac{k-s}{s}\Bigl(A + B\tau^{\frac{ks}{s-k}}\Bigr)^{\frac{1}{k}} - \tau\biggr]^{-1} d\tau\biggr\},$$

$$y = C^k \biggl\{\tau \exp \int \biggl[\frac{k-s}{s}\Bigl(A + B\tau^{\frac{ks}{s-k}}\Bigr)^{\frac{1}{k}} - \tau\biggr]^{-1} d\tau\biggr\}^{\frac{k}{k-s}}.$$

8. $(y'_x)^k = Ay^k + Be^x.$

Solution in parametric form:

$$x = \int \Bigl[(A + Be^{-k\tau})^{1/k} - \frac{1}{k}\Bigr]^{-1} d\tau + C,$$

$$y = \exp\biggl\{\tau + \frac{1}{k}\int \Bigl[(A + Be^{-k\tau})^{1/k} - \frac{1}{k}\Bigr]^{-1} d\tau + \frac{C}{k}\biggr\}.$$

9. $(y'_x)^k = Ae^y + Bx^{-k}.$

Solution in parametric form:

$$x = \exp\biggl\{\tau - \frac{1}{k}\int \Bigl[(B + Ae^{k\tau})^{-1/k} + \frac{1}{k}\Bigr]^{-1} d\tau - \frac{C}{k}\biggr\},$$

$$y = \int \Bigl[(B + Ae^{k\tau})^{-1/k} + \frac{1}{k}\Bigr]^{-1} d\tau + C.$$

10. $(y'_x)^{-1} = Ay^s + Bx.$

Solution: $x = e^{By}\Bigl(A\int y^s e^{-By}\,dy + C\Bigr).$

◆ *In the solutions of equations 11–14, the following notation is used:*

$$F = \Bigl[\int \exp(\mp\tau^2)\,d\tau + C\Bigr]^{-1}.$$

11. $y'_x = Ay^{1/2} + Bx^{-1}.$

Solution in parametric form:

$x = aF\exp(\mp\tau^2), \quad y = b[2\tau \pm F\exp(\mp\tau^2)]^2, \qquad$ where $\quad A = \pm 2a^{-1}b^{1/2}, \; B = \mp 4b.$

12. $(y'_x)^{-1} = Ay^{-1} + Bx^{1/2}.$

Solution in parametric form:

$x = a[2\tau \pm F\exp(\mp\tau^2)]^2, \quad y = bF\exp(\mp\tau^2), \qquad$ where $\quad A = \mp 4a, \; B = \pm 2a^{1/2}b^{-1}.$

13. $(y'_x)^2 = Ay + B\ln x.$

Solution in parametric form:

$$x = aF\exp(\mp\tau^2), \qquad y = b\bigl\{[2\tau \pm F\exp(\mp\tau^2)]^2 \pm 4\ln(aF) - 4\tau^2\bigr\},$$

where $A = 4a^{-2}b, \; B = \mp 4bA.$

14. $(y'_x)^{-2} = A \ln y + Bx.$

Solution in parametric form:

$$x = a\{[2\tau \pm F \exp(\mp\tau^2)]^2 \pm 4\ln(bF) - 4\tau^2\}, \qquad y = bF\exp(\mp\tau^2),$$

where $A = \mp 4aB,\ B = 4ab^{-2}$.

◆ *In the solutions of equations 15–19, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \text{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \text{for the lower sign,} \end{cases}$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are Bessel functions, and $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

Remark 13.3. *The solutions of equations 15–19 contain only the ratio $Z'_\tau/Z = (\ln Z)'_\tau$. Therefore, for the sake of symmetric appearance, two "arbitrary" constants C_1 and C_2 are indicated in the definition of function Z (instead, we can set, for instance, $C_1 = 1$ and $C_2 = C$).*

15. $(y'_x)^{-1} = Ay^s + Bx^2, \qquad s \neq -2,\ s \neq 0.$

Solution in parametric form:

$$x = a\tau^{-2\nu}\bigl[\tau(\ln Z)'_\tau + \nu\bigr], \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{s+2},\ A = \mp\dfrac{s+2}{2}ab^{-1-s},\ B = -\dfrac{s+2}{2}a^{-1}b^{-1}.$

16. $(y'_x)^{1/2} = Ay + Bx^r, \qquad r \neq -1,\ r \neq 0.$

Solution in parametric form:

$$x = a\tau^{2\nu}, \qquad y = b\tau^{2\nu}\left[\tau(\ln Z)'_\tau + \nu \pm \frac{r+1}{2r}\tau^2\right],$$

where $\nu = \dfrac{1}{r+1},\ A = b^{-1}\left[-\dfrac{(r+1)b}{2a}\right]^{1/2},\ B = \mp\dfrac{r+1}{2r}a^{-r}bA.$

17. $(y'_x)^{-1/2} = Ay^s + Bx, \qquad s \neq -1,\ s \neq 0.$

Solution in parametric form:

$$x = a\tau^{2\nu}\left[\tau(\ln Z)'_\tau + \nu \pm \frac{s+1}{2s}\tau^2\right], \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{s+1},\ A = \mp\dfrac{s+1}{2s}ab^{-s}B,\ B = a^{-1}\left[-\dfrac{(s+1)a}{2b}\right]^{1/2}.$

18. $(y'_x)^{1/2} = Ay + Be^x.$

Solution in parametric form:

$$x = \ln(a\tau^2), \qquad y = b\bigl[\tau(\ln Z)'_\tau \pm \tfrac{1}{2}\tau^2\bigr],$$

where $\nu = 0,\ A = b^{-1}\bigl(-\tfrac{1}{2}b\bigr)^{1/2},\ B = \mp\tfrac{1}{2}a^{-1}bA.$

19. $(y'_x)^{-1/2} = Ae^y + Bx.$

Solution in parametric form:

$$x = a\big[\tau(\ln Z)'_\tau \pm \tfrac{1}{2}\tau^2\big], \qquad y = \ln(b\tau^2),$$

where $\nu = 0$, $A = \mp\frac{1}{2}ab^{-1}B$, $B = a^{-1}\big(-\frac{1}{2}a\big)^{1/2}$.

◆ *In the solutions of equations 20–35, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

where $J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions;

$$U_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \qquad U_2 = U_1^2 \pm \tau^2 Z^2, \qquad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1U_2.$$

Remark 13.4. *The solutions of equations 20–35 contain only the ratio $Z'_\tau/Z = (\ln Z)'_\tau$. Therefore, for the sake of symmetric appearance, two "arbitrary" constants C_1 and C_2 are indicated in the definition of function Z (instead, we can set, for instance, $C_1 = 1$ and $C_2 = C$).*

20. $y'_x = Ay^{-1} + Bx^{-2}.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_2, \quad y = b\tau^{-2/3}Z^{-1}U_2^{-1}U_3, \qquad \text{where} \quad A = 2a^{-1}b^2, \ B = \mp\tfrac{2}{3}ab.$$

21. $(y'_x)^{-1} = Ay^{-2} + Bx^{-1}.$

Solution in parametric form:

$$x = a\tau^{-2/3}Z^{-1}U_2^{-1}U_3, \quad y = b\tau^{-4/3}Z^{-2}U_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}ab, \ B = 2a^2b^{-1}.$$

22. $y'_x = Ay^{-1} + Bx.$

Solution in parametric form:

$$x = a\tau^{-2/3}Z^{-1}U_1, \quad y = b\tau^{-4/3}Z^{-2}U_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-1}b^2, \ B = 2a^{-2}b.$$

23. $(y'_x)^{-1} = Ay + Bx^{-1}.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_2, \quad y = b\tau^{-2/3}Z^{-1}U_1, \qquad \text{where} \quad A = 2a^{-2}b, \ B = \mp\tfrac{2}{3}a^2b^{-1}.$$

24. $y'_x = Ay^{1/2} + Bx^{-1/2}.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_1^2, \quad y = b\tau^{-8/3}Z^{-4}U_2^2, \qquad \text{where} \quad A = 2a^{-1}b^{1/2}, \ B = \mp\tfrac{2}{3}a^{-1/2}b.$$

25. $(y'_x)^{-1} = Ay^{-1/2} + Bx^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-8/3}Z^{-4}U_2^2, \quad y = b\tau^{-4/3}Z^{-2}U_1^2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}ab^{-1/2}, \ B = 2a^{1/2}b^{-1}.$$

26. $(y'_x)^2 = Ay + Bx^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_1^2, \quad y = b\tau^{-8/3}Z^{-4}\big(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1\big),$$

where $A = 4a^{-2}b$, $B = \mp\frac{4}{3}a^{-1/2}bA$.

27. $(y'_x)^{-2} = Ay^{1/2} + Bx.$

Solution in parametric form:

$$x = a\tau^{-8/3}Z^{-4}(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1), \qquad y = b\tau^{-4/3}Z^{-2}U_1^2,$$

where $A = \mp\frac{4}{3}ab^{-1/2}B,\ B = 4ab^{-2}$.

28. $(y'_x)^2 = Ay^{-2} + Bx^{-2/5}.$

Solution in parametric form:

$$x = a\tau^{-5/3}Z^{-5/2}U_1^{5/2}, \qquad y = b\tau^{-4/3}Z^{-2}(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1)^{1/2},$$

where $A = \mp\frac{4}{3}a^{-2/5}b^2B,\ B = \frac{16}{25}a^{-8/5}b^2$.

29. $(y'_x)^{-2} = Ay^{-2/5} + Bx^{-2}.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1)^{1/2}, \qquad y = b\tau^{-5/3}Z^{-5/2}U_1^{5/2},$$

where $A = \frac{16}{25}a^2b^{-8/5},\ B = \mp\frac{4}{3}a^2b^{-2/5}A$.

30. $y'_x = Ay^{1/2} + Bx^{-2}.$

Solution in parametric form:

$$x = a\tau^{4/3}Z^2U_2^{-1}, \quad y = b\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2, \qquad \text{where} \quad A = \pm\tfrac{4}{3}a^{-1}b^{1/2},\ B = -4ab.$$

31. $(y'_x)^{-1} = Ay^{-2} + Bx^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2, \quad y = b\tau^{4/3}Z^2U_2^{-1}, \qquad \text{where} \quad A = -4ab,\ B = \pm\tfrac{4}{3}a^{1/2}b^{-1}.$$

32. $(y'_x)^2 = Ay + Bx^{-1}.$

Solution in parametric form:

$$x = a\tau^{4/3}Z^2U_2^{-1}, \quad y = b\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3), \qquad \text{where} \quad A = \tfrac{16}{9}a^{-2}b,\ B = 4abA.$$

33. $(y'_x)^{-2} = Ay^{-1} + Bx.$

Solution in parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3), \quad y = b\tau^{4/3}Z^2U_2^{-1}, \qquad \text{where} \quad A = 4abB,\ B = \tfrac{16}{9}ab^{-2}.$$

34. $(y'_x)^2 = Ay^{-2} + Bx^2.$

Solution in parametric form:

$$x = a\tau^{2/3}ZU_2^{-1/2}, \quad y = b\tau^{-2/3}Z^{-1}U_2^{-1}(U_3^2 - 4U_2^3)^{1/2},$$

where $A = 4a^2b^2B,\ B = \frac{16}{9}a^{-4}b^2$.

35. $(y'_x)^{-2} = Ay^2 + Bx^{-2}$**.**

Solution in parametric form:

$$x = a\tau^{-2/3} Z^{-1} U_2^{-1} (U_3^2 - 4U_2^3)^{1/2}, \quad y = b\tau^{2/3} Z U_2^{-1/2},$$

where $A = \frac{16}{9} a^2 b^{-4}$, $B = 4a^2 b^2 A$.

◆ *In the solutions of equations 36–46, the following notation is used:*

$$R = \begin{cases} C_1 \tau^{\nu} + C_2 \tau^{-\nu} & \text{for the upper sign,} \\ C_1 \sin(\nu \ln \tau) + C_2 \cos(\nu \ln \tau) & \text{for the lower sign,} \\ C_1 \ln \tau + C_2 & \text{for } \nu = 0, \end{cases}$$

$$Q = \begin{cases} (1+\nu) C_1 \tau^{\nu} + (1-\nu) C_2 \tau^{-\nu} & \text{for the upper sign,} \\ (C_1 - \nu C_2) \sin(\nu \ln \tau) + (C_2 + \nu C_1) \cos(\nu \ln \tau) & \text{for the lower sign,} \\ C_1 \ln \tau + C_1 + C_2 & \text{for } \nu = 0. \end{cases}$$

Remark 13.5. *The expressions of R and Q contain two "arbitrary" constants C_1 an C_2. One of them can be fixed to set it equal to any nonzero number (for example, we can set $C_2 = \pm 1$), while the other constant remains arbitrary.*

36. $(y'_x)^{-1} = Ay^{-2} + Bx^2$**.**

Solution in parametric form:

$$x = a\tau^{-2} R^{-1} Q, \quad y = b\tau^2, \quad \nu = \sqrt{|1 - 4AB|},$$

where $A = -\dfrac{1 \mp \nu^2}{2} ab$, $B = -\frac{1}{2} a^{-1} b^{-1}$.

37. $(y'_x)^{1/2} = Ay + Bx^{-1}$**.**

Solution in parametric form:

$$x = a\tau^2, \quad y = b\tau^{-2}\Big(R^{-1} Q - \frac{1 \mp \nu^2}{2}\Big), \quad \text{where} \quad A = b^{-1}\Big(-\frac{b}{2a}\Big)^{1/2}, \quad B = \frac{1 \mp \nu^2}{2} abA.$$

38. $(y'_x)^{-1/2} = Ay^{-1} + Bx$**.**

Solution in parametric form:

$$x = a\tau^{-2}\Big(R^{-1} Q - \frac{1 \mp \nu^2}{2}\Big), \quad y = b\tau^2, \quad \text{where} \quad A = \frac{1 \mp \nu^2}{2} abB, \quad B = a^{-1}\Big(-\frac{a}{2b}\Big)^{1/2}.$$

39. $y'_x = Ay^{-1} + Bx^{-1/2}$**.**

Solution in parametric form:

$$x = a\tau^2 R^2, \quad y = b\tau Q, \quad \text{where} \quad A = (-1 \pm \nu^2)\frac{b^2}{2a}, \quad B = a^{-1/2} b.$$

40. $(y'_x)^{-1} = Ay^{-1/2} + Bx^{-1}$**.**

Solution in parametric form:

$$x = a\tau Q, \quad y = b\tau^2 R^2, \quad \text{where} \quad A = ab^{-1/2}, \quad B = (-1 \pm \nu^2)\frac{b^2}{2a}.$$

41. $y'_x = Ay^{1/2} + Bx.$

Solution in parametric form:

$$x = a\tau R, \quad y = b\tau^2 Q^2, \qquad \text{where} \quad A = 2(-1 \pm \nu^2)a^{-1}b^{1/2}, \quad B = 4a^{-2}b.$$

42. $(y'_x)^{-1} = Ay + Bx^{1/2}.$

Solution in parametric form:

$$x = a\tau^2 Q^2, \quad y = b\tau R, \qquad \text{where} \quad A = 4ab^{-2}, \quad B = 2(-1 \pm \nu^2)a^{1/2}b^{-1}.$$

43. $(y'_x)^2 = Ay + Bx^2.$

Solution in parametric form:

$$x = a\tau R, \quad y = b\tau^2[Q^2 - (-1 \pm \nu^2)R^2], \qquad \text{where} \quad A = 16a^{-2}b, \; B = (-1 \pm \nu^2)a^{-2}bA.$$

44. $(y'_x)^{-2} = Ay^2 + Bx.$

Solution in parametric form:

$$x = a\tau^2[Q^2 - (-1 \pm \nu^2)R^2], \quad y = b\tau R, \qquad \text{where} \quad A = (-1 \pm \nu^2)ab^{-2}B, \; B = 16ab^{-2}.$$

45. $(y'_x)^2 = Ay^{-2} + Bx^{-1}.$

Solution in parametric form:

$$x = a\tau^2 R^2, \; y = b\tau[Q^2 - (-1 \pm \nu^2)R^2]^{1/2}, \quad \text{where} \; A = (-1 \pm \nu^2)a^{-1}b^2 B, \; B = a^{-1}b^2.$$

46. $(y'_x)^{-2} = Ay^{-1} + Bx^{-2}.$

Solution in parametric form:

$$x = a\tau[Q^2 - (-1 \pm \nu^2)R^2]^{1/2}, \; y = b\tau^2 R^2, \quad \text{where} \; A = a^2 b^{-1}, \; B = (-1 \pm \nu^2)a^2 b^{-1}A.$$

13.6.4 Other Equations

► **Equations containing algebraic and power functions with respect to y'_x.**

1. $y = xy'_x + ax^2 + b\sqrt{y'_x} + c, \qquad a \neq 0.$

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at an Abel equation of the form 13.3.1.32: $ww'_t - w = -abt^{-1/2}$.

2. $xy'_x - x = (ax + by)\sqrt{(y'_x)^2 + 1}.$

This is a special case of equation 13.8.1.5 with $f(u, v) = au + bv$.

3. $yy'_x + x = (ax + by)\sqrt{(y'_x)^2 + 1}.$

This is a special case of equation 13.8.1.6 with $f(u, v) = au + bv$.

4. $y = xy'_x + a(y'_x)^n.$

Solution: $y = Cx + aC^n$. In addition, there is a singular solution: $y = Ax^{\frac{n}{n-1}}$, where $aA^{n-1}n^n = -(n-1)^{n-1}$, $n \neq 1$.

5. $y = xy'_x + ax^n(y'_x)^m.$

This is a special case of equation 13.8.1.15 with $f(w) = aw^m$.

6. $y = ax^n(y'_x)^{2n} + 2xy'_x.$

This is a special case of equation 13.8.1.51 with $f(w) = aw^n$.

7. $y = xy'_x + ax^2 + b(y'_x)^2 + c(y'_x)^{m+1} + d, \qquad a \neq 0.$

Differentiating the equation with respect to x and passing to the new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at the Abel equation $ww'_t - w = -4abt - 2ac(m+1)t^m$, whose solvable cases are outlined in Section 13.3.1.

8. $a(y'_x)^n + b(y'_x)^m = x.$

1°. Solution in parametric form with $n \neq -1$ and $m \neq -1$:

$$x = at^n + bt^m, \qquad y = C + \frac{an}{n+1}t^{n+1} + \frac{bm}{m+1}t^{m+1}.$$

2°. Solution in parametric form with $n = -1$ and $m \neq -1$:

$$x = \frac{a}{t} + bt^m, \qquad y = C + a\ln|t| + \frac{bm}{m+1}t^{m+1}.$$

9. $a(y'_x)^n + b(y'_x)^m = y.$

1°. Solution in parametric form with $n \neq 1$ and $m \neq 1$:

$$x = C + \frac{an}{n-1}t^{n-1} + \frac{bm}{m-1}t^{m-1}, \qquad y = at^n + bt^m.$$

2°. Solution in parametric form with $n = 1$ and $m \neq 1$:

$$x = C + a\ln|t| + \frac{bm}{m-1}t^{m-1}, \qquad y = at + bt^m.$$

10. $ax(y'_x)^n + by(y'_x)^m + c(y'_x)^k = 0.$

This is a special case of equation 13.8.1.12 with $f(u) = au^n$, $g(u) = bu^m$, and $h(u) = cu^k$.

11. $y'_x = ax^n(xy'_x - y)^m.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ $(y'_x = t)$ leads to a separable equation: $t = aw^m(w'_t)^n$.

1°. Solution in parametric form with $m \neq -n$ and $n \neq -1$:

$$x = \left(\frac{t}{a}\right)^{\frac{1}{n}}\left[\frac{m+n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}} + C\right]^{-\frac{m}{m+n}},$$

$$y = \left[\frac{1-m}{1+n}t\left(\frac{t}{a}\right)^{\frac{1}{n}} - C\right]\left[\frac{m+n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}} + C\right]^{-\frac{m}{m+n}}.$$

2°. Solution in parametric form with $m = -n$ and $n \neq -1$:

$$x = C\left(\frac{t}{a}\right)^{\frac{1}{n}}\exp\left[\frac{n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}}\right], \qquad y = C\left[t\left(\frac{t}{a}\right)^{\frac{1}{n}} - 1\right]\exp\left[\frac{n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}}\right].$$

3°. Solution in parametric form with $m \neq -n$ and $n = -1$:

$$x = \frac{a}{t}\left[a(1-m)\ln|t| + C\right]^{\frac{m}{1-m}}, \qquad y = tx - \left[a(1-m)\ln|t| + C\right]^{\frac{1}{1-m}}.$$

4°. Solution with $m = 1$ and $n = -1$: $y = Cx^{\frac{a}{a-1}}$.

12. $y'_x + y = ae^{kx}(y'_x - y)^{k-1}.$

Solution: $y = \frac{1}{2}aC^{k-1}e^x - \frac{1}{2}Ce^{-x}$.

13. $ax^{kn+k}(y'_x)^k + b(xy'_x + ny)^m + c = 0.$

Solution: $y = -C_1x^{-n} + C_2$. Here, the constants C_1 and C_2 are related by the constraint $a(C_1n)^k + b(C_2n)^m + c = 0$.

14. $a_1x^{n_1m_1}(xy'_x + n_2y)^{m_1} + a_2x^{n_2m_2}(xy'_x + n_1y)^{m_2} + b = 0, \qquad n_1 \neq n_2.$

Solution: $y = \dfrac{C_1}{n_2 - n_1}x^{-n_1} - \dfrac{C_2}{n_2 - n_1}x^{-n_2}$. Here, the constants C_1 and C_2 are related by the constraint $a_1C_1^{m_1} + a_2C_2^{m_2} + b = 0$.

15. $y'_x(y'_x + ax)^n + b(y'^2_x + 2ay)^m + c = 0.$

The contact transformation $X = y'_x + ax$, $Y = (y'_x)^2 + 2ay$, $Y'_X = 2y'_x$, where $Y = Y(X)$, leads to a separable equation: $Y'_X = -2X^{-n}(bY^m + c)$.

The inverse contact transformation: $x = \frac{1}{2}a^{-1}(2X - Y'_X)$, $y = \frac{1}{8}a^{-1}[4Y - (Y'_X)^2]$, $y'_x = \frac{1}{2}Y'_X$.

16. $a(y'_x + y)^n + be^{kx}(y'_x)^k + c = 0.$

Solution: $y = C_1 - C_2e^{-x}$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^n + bC_2^k + c = 0$.

17. $ae^{kx}(y'_x - y)^k + b[(y'_x)^2 - y^2]^n + c = 0.$

Solution: $y = \frac{1}{2}C_1e^x - \frac{1}{2}C_2e^{-x}$. Here, the constants C_1 and C_2 are related by the constraint $aC_2^k + b(C_1C_2)^n + c = 0$.

18. $ae^{k\beta x}(y'_x + \gamma y)^k + be^{n\gamma x}(y'_x + \beta y)^n + c = 0, \qquad \beta \neq \gamma.$

Solution: $y = \dfrac{C_1}{\gamma - \beta}e^{-\beta x} - \dfrac{C_2}{\gamma - \beta}e^{-\gamma x}$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^k + bC_2^n + c = 0$.

19. $a(y'_x \cosh x - y \sinh x)^n + b(y'_x \sinh x - y \cosh x)^k + c = 0.$

Solution: $y = C_1 \sinh x - C_2 \cosh x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^n + bC_2^k + c = 0$.

20. $a(y'_x \cosh x - y \sinh x)^n + b(y'^2_x - y^2)^k + c = 0.$

Solution: $y = C_1 \sinh x + C_2 \cosh x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^n + b(C_1^2 - C_2^2)^k + c = 0$.

21. $y'_x(y'_x \cosh x - y \sinh x)^n = b(y'^2_x - y^2)^m \cosh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \cosh x - y \sinh x$, $Y'_X = \frac{1}{2}\cosh x\,(y'_x)^{-1}$ leads to a separable equation: $2bX^mY'_X = Y^n$.

22. $y'_x(y'_x \sinh x - y \cosh x)^n = b(y'^2_x - y^2)^m \sinh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \sinh x - y \cosh x$, $Y'_X = \frac{1}{2}\sinh x\,(y'_x)^{-1}$ leads to a separable equation: $2bX^mY'_X = Y^n$.

23. $a(y'_x \cos x + y \sin x)^n + b(y'_x \sin x - y \cos x)^k + c = 0.$

Solution: $y = C_1 \sin x - C_2 \cos x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^n + bC_2^k + c = 0$.

24. $a(y'_x \cos x + y \sin x)^n + b(y'^2_x + y^2)^k + c = 0.$

Solution: $y = C_1 \sin x + C_2 \cos x$. Here, the constants C_1 and C_2 are related by the constraint $aC_1^n + b(C_1^2 + C_2^2)^k + c = 0$.

25. $y'_x(y'_x \cos x + y \sin x)^n = b(y'^2_x + y^2)^m \cos x.$

The contact transformation $X = (y'_x)^2 + y^2$, $Y = y'_x \cos x + y \sin x$, $Y'_X = \frac{1}{2} \cos x (y'_x)^{-1}$ leads to a separable equation: $2bX^m Y'_X = Y^n$.

26. $y'_x(y'_x \sin x - y \cos x)^n = b(y'^2_x + y^2)^m \sin x.$

The contact transformation $X = (y'_x)^2 + y^2$, $Y = y'_x \sin x - y \cos x$, $Y'_X = \frac{1}{2} \sin x (y'_x)^{-1}$ leads to a separable equation: $2bX^m Y'_X = Y^n$.

► **Equations containing exponential and other functions with respect to y'_x.**

27. $x = a \exp(\lambda y'_x) + b \exp(\mu y'_x).$

This is a special case of equation 13.8.1.7 with $f(w) = a \exp(\lambda w) + b \exp(\mu w)$.

28. $y = a \exp(\lambda y'_x) + b \exp(\mu y'_x).$

This is a special case of equation 13.8.1.8 with $f(w) = a \exp(\lambda w) + b \exp(\mu w)$.

29. $y = xy'_x + ax^n \exp(\lambda y'_x).$

This is a special case of equation 13.8.1.15 with $f(w) = a \exp(\lambda w)$.

30. $y = ax \exp(\lambda y'_x) + b \exp(\mu y'_x).$

This is a special case of equation 13.8.1.11 with $f(w) = a \exp(\lambda w)$ and $g(w) = b \exp(\mu w)$.

31. $x = a \sinh(\lambda y'_x) + b \sinh(\mu y'_x).$

This is a special case of equation 13.8.1.7 with $f(w) = a \sinh(\lambda w) + b \sinh(\mu w)$.

32. $y = a \sinh(\lambda y'_x) + b \sinh(\mu y'_x).$

This is a special case of equation 13.8.1.8 with $f(w) = a \sinh(\lambda w) + b \sinh(\mu w)$.

33. $y = xy'_x + ax^n \sinh^m(\lambda y'_x).$

This is a special case of equation 13.8.1.15 with $f(w) = a \sinh^m(\lambda w)$.

34. $y = ax \sinh(\lambda y'_x) + b \sinh(\mu y'_x).$

This is a special case of equation 13.8.1.11 with $f(w) = a \sinh(\lambda w)$ and $g(w) = b \sinh(\mu w)$.

35. $x = a \cosh(\lambda y'_x) + b \cosh(\mu y'_x).$

This is a special case of equation 13.8.1.7 with $f(w) = a \cosh(\lambda w) + b \cosh(\mu w)$.

36. $y = a \cosh(\lambda y'_x) + b \cosh(\mu y'_x).$

This is a special case of equation 13.8.1.8 with $f(w) = a \cosh(\lambda w) + b \cosh(\mu w)$.

37. $y = xy'_x + ax^n \cosh^m(\lambda y'_x).$

This is a special case of equation 13.8.1.15 with $f(w) = a\cosh^m(\lambda w)$.

38. $y = ax\cosh(\lambda y'_x) + b\cosh(\mu y'_x).$

This is a special case of equation 13.8.1.11 with $f(w) = a\cosh(\lambda w)$ and $g(w) = b\cosh(\mu w)$.

39. $\ln y'_x + xy'_x + ay + b = 0.$

1°. Solution in parametric form with $a \neq 0$, $a \neq -1$:

$$x = \frac{1}{at} + Ct^{-\frac{1}{a+1}}, \qquad y = -\frac{1}{a}(xt + \ln t + b).$$

2°. Solution in parametric form with $a = 0$:

$$x = -\frac{\ln t + b}{t}, \qquad y = C + (b-1)\ln t + \frac{1}{2}(\ln t)^2.$$

3°. Solutions with $a = -1$:

$$y = Cx + \ln C + b \quad \text{and} \quad y = \ln(-1/x) + b - 1.$$

40. $x + \ln y'_x + a(y'_x + y)^k + b = 0.$

This is a special case of equation 13.8.1.41 with $F(u, w) = \ln u + aw^k + b$.

41. $y = xy'_x + ax^2 + b\ln y'_x + c, \qquad a \neq 0.$

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at an Abel equation of the form 13.3.1.16: $ww'_t - w = -2abt^{-1}$.

42. $y = xy'_x + ax^n \ln^m(\lambda y'_x).$

This is a special case of equation 13.8.1.15 with $f(w) = a\ln^m(\lambda w)$.

43. $x = a\sin(\lambda y'_x) + b\sin(\mu y'_x).$

This is a special case of equation 13.8.1.7 with $f(w) = a\sin(\lambda w) + b\sin(\mu w)$.

44. $y = a\sin(\lambda y'_x) + b\sin(\mu y'_x).$

This is a special case of equation 13.8.1.8 with $f(w) = a\sin(\lambda w) + b\sin(\mu w)$.

45. $y = xy'_x + ax^n \sin^m(\lambda y'_x).$

This is a special case of equation 13.8.1.15 with $f(w) = a\sin^m(\lambda w)$.

46. $y = ax\sin(\lambda y'_x) + b\sin(\mu y'_x).$

This is a special case of equation 13.8.1.11 with $f(w) = a\sin(\lambda w)$ and $g(w) = b\sin(\mu w)$.

47. $x = a\cos(\lambda y'_x) + b\cos(\mu y'_x).$

This is a special case of equation 13.8.1.7 with $f(w) = a\cos(\lambda w) + b\cos(\mu w)$.

48. $y = a\cos(\lambda y'_x) + b\cos(\mu y'_x).$

This is a special case of equation 13.8.1.8 with $f(w) = a\cos(\lambda w) + b\cos(\mu w)$.

49. $y = xy'_x + ax^n \cos^m(\lambda y'_x)$.

This is a special case of equation 13.8.1.15 with $f(w) = a\cos^m(\lambda w)$.

50. $y = ax\cos(\lambda y'_x) + b\cos(\mu y'_x)$.

This is a special case of equation 13.8.1.11 with $f(w) = a\cos(\lambda w)$ and $g(w) = b\cos(\mu w)$.

51. $y = xy'_x + ax^n \tan^m(\lambda y'_x)$.

This is a special case of equation 13.8.1.15 with $f(w) = a\tan^m(\lambda w)$.

13.7 Equations of the Form $f(x,y)y'_x = g(x,y)$ Containing Arbitrary Functions

◆ *Notation: f, g, and h are arbitrary composite functions whose argument, indicated after the function name, can depend on both x and y.*

13.7.1 Equations Containing Power Functions

1. $y'_x = f(ax + by + c)$.

In the case $b = 0$, we have an equation of the form 13.1.1. If $b \neq 0$, the substitution $u(x) = ax + by + c$ leads to a separable equation: $u'_x = bf(u) + a$.

2. $y'_x = f(y + ax^n + b) - anx^{n-1}$.

The substitution $u = y + ax^n + b$ leads to a separable equation: $u'_x = f(u)$.

3. $y'_x = \frac{y}{x} f(x^n y^m)$.

Generalized homogeneous equation. The substitution $z = x^n y^m$ leads to a separable equation: $xz'_x = nz + mzf(z)$.

4. $y'_x = f(x)y^{1+n} + g(x)y + h(x)y^{1-n}$.

The substitution $w = y^n$ leads to a Riccati equation: $w'_x = nf(x)w^2 + ng(x)w + nh(x)$.

5. $y'_x = -\frac{n}{m}\frac{y}{x} + y^k f(x) g(x^n y^m)$.

The substitution $z = x^n y^m$ leads to a separable equation: $z'_x = mx^{\frac{n-nk}{m}} f(x) z^{\frac{k+m-1}{m}} g(z)$.

6. $y'_x = f\left(\frac{ax + by + c}{\alpha x + \beta y + \gamma}\right)$.

1°. For $\Delta = a\beta - b\alpha \neq 0$, the transformation $x = u + \frac{b\gamma - c\beta}{\Delta}$, $y = v(u) + \frac{c\alpha - a\gamma}{\Delta}$ leads to an equation:

$$v'_u = f\left(\frac{au + bv}{\alpha u + \beta v}\right).$$

Dividing both the numerator and denominator of the fraction on the right-hand side by u, we obtain a homogeneous equation of the form 13.1.6.

2°. For $\Delta = 0$ and $b \neq 0$, the substitution $v(x) = ax + by + c$ leads to a separable equation of the form 13.1.2:

$$v'_x = a + bf\left(\frac{bv}{\beta v + b\gamma - c\beta}\right).$$

3°. For $\Delta = 0$ and $\beta \neq 0$, the substitution $v(x) = \alpha x + \beta y + \gamma$ also leads to a separable equation:

$$v'_x = \alpha + \beta f\left(\frac{bv + c\beta - b\gamma}{\beta v}\right).$$

7. $\boldsymbol{y'_x = x^{n-1} y^{1-m} f(ax^n + by^m).}$

The substitution $w = ax^n + by^m$ leads to a separable equation: $w'_x = x^{n-1}[an + bmf(w)]$.

8. $\boldsymbol{y^n y'_x + ax^n + g(x) f(y^{n+1} + ax^{n+1}) = 0.}$

The substitution $u = y^{n+1} + ax^{n+1}$ leads to a separable equation: $u'_x + (n+1)g(x)f(u) = 0$.

9. $\boldsymbol{[x^n f(y) + xg(y)]y'_x = h(y).}$

This is a Bernoulli equation with respect to $x = x(y)$ (see Section 13.1.5).

10. $\boldsymbol{[x^2 + xf(y) + g(y)]y'_x = h(y).}$

This is a Riccati equation with respect to $x = x(y)$ (see Section 13.2).

11. $\boldsymbol{y'_x = [f(x)y + g(x)]\sqrt{(y-a)(y-b)}.}$

The substitution $u^2 = (y-a)/(y-b)$ leads to a Riccati equation:

$$\pm 2u'_x = [bf(x) + g(x)]u^2 - af(x) - g(x).$$

12. $\boldsymbol{\left[f\left(\frac{y}{x}\right) + x^a h\left(\frac{y}{x}\right)\right]y'_x = g\left(\frac{y}{x}\right) + yx^{a-1}h\left(\frac{y}{x}\right).}$

The substitution $y = xt$ leads to a Bernoulli equation with respect to $x = x(t)$: $[g(t) - tf(t)]x'_t = f(t)x + h(t)x^{a+1}$.

13. $\boldsymbol{[P_n(x,y) + xR_m(x,y)]y'_x = Q_n(x,y) + yR_m(x,y).}$

Darboux equation. Here, P_n and Q_n are homogeneous polynomials of order n, and R_m is a homogeneous polynomial of order m. Dividing the Darboux equation by x^n leads to an equation of the form 13.7.1.12.

14. $\boldsymbol{[f(ax+by) + bxg(ax+by)]y'_x = h(ax+by) - axg(ax+by).}$

The substitution $t = ax + by$ leads to a linear equation with respect to $x = x(t)$: $[af(t) + bh(t)]x'_t = bg(t)x + f(t)$.

15. $\boldsymbol{[f(ax+by) + byg(ax+by)]y'_x = h(ax+by) - ayg(ax+by).}$

The substitution $t = ax + by$ leads to a linear equation with respect to $y = y(t)$: $[af(t) + bh(t)]y'_t = -ag(t)y + h(t)$.

16. $\boldsymbol{x[f(x^n y^m) + mx^k g(x^n y^m)]y'_x = y[h(x^n y^m) - nx^k g(x^n y^m)].}$

The transformation $t = x^n y^m$, $z = x^{-k}$ leads to a linear equation with respect to $z = z(t)$: $t[nf(t) + mh(t)]z'_t = -kf(t)z - kmg(t)$.

17. $x[f(x^n y^m) + my^k g(x^n y^m)]y'_x = y[h(x^n y^m) - ny^k g(x^n y^m)].$

The transformation $t = x^n y^m$, $z = y^{-k}$ leads to a linear equation with respect to $z = z(t)$: $t[nf(t) + mh(t)]z'_t = -kh(t)z + kng(t)$.

18. $x[sf(x^n y^m) - mg(x^k y^s)]y'_x = y[ng(x^k y^s) - kf(x^n y^m)].$

The transformation $t = x^n y^m$, $w = x^k y^s$ leads to a separable equation: $tf(t)w'_t = wg(w)$.

19. $[f(y) + amx^n y^{m-1}]y'_x + g(x) + anx^{n-1}y^m = 0.$

Solution: $\int f(y)\,dy + \int g(x)\,dx + ax^n y^m = C$.

20. $xy'_x - y = f(x^2 + y^2)(yy'_x + x).$

Setting $x = r(t)\cos t$, $y = r(t)\sin t$ and integrating, we obtain a solution in implicit form: $t = \int r^{-1} f(r^2)\,dr + C$.

⊙ *Literature*: G. W. Bluman, J. D. Cole (1974, page 100).

21. $xy'_x - y = f(x^2 - y^2)(yy'_x - x).$

Setting $x = r(t)\cosh t$, $y = r(t)\sinh t$ and integrating, we obtain a solution in implicit form: $t = -\int r^{-1} f(r^2)\,dr + C$.

22. $[xf(x^2 + y^2) + yg(x^2 + y^2) + h(x^2 + y^2)](yy'_x + x) = xy'_x - y.$

The transformation $x = r\cos\varphi$, $y = r\sin\varphi$ leads to an equation of the form 13.7.4.11 with respect to $\varphi = \varphi(r)$: $\varphi'_r = f(r^2)\cos\varphi + g(r^2)\sin\varphi + r^{-1}h(r^2)$.

23. $[xf(x^2 - y^2) + yg(x^2 - y^2) + h(x^2 - y^2)](yy'_x - x) = xy'_x - y.$

1°. For $x > y$, the transformation $x = r\cosh\varphi$, $y = r\sinh\varphi$ leads to an equation of the form 13.7.2.18 with respect to $\varphi = \varphi(r)$: $\varphi'_r = -f(r^2)\cosh\varphi - g(r^2)\sinh\varphi - r^{-1}h(r^2)$.

2°. For $x < y$, the transformation $x = z\sinh\psi$, $y = z\cosh\psi$ leads to an equation of the form 13.7.2.18 with respect to $\psi = \psi(z)$: $\psi'_z = -f(-z^2)\sinh\psi - g(-z^2)\cosh\psi - z^{-1}h(-z^2)$.

24. $xy'_x - y = f(x^2 + y^2)\,g\left(\dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}}\right)(yy'_x + x).$

Setting $x = r(t)\cos t$, $y = r(t)\sin t$ and integrating, we obtain the solution:

$$\int \frac{dt}{g(\cos t, \sin t)} = \int \frac{f(r^2)}{r}\,dr + C.$$

25. $xy'_x - y = f(x^2 - y^2)\,g\left(\dfrac{x}{\sqrt{x^2 - y^2}}, \dfrac{y}{\sqrt{x^2 - y^2}}\right)(yy'_x - x).$

Setting $x = r(t)\cosh t$, $y = r(t)\sinh t$ and integrating, we obtain the solution:

$$\int \frac{dt}{g(\cosh t, \sinh t)} + \int \frac{f(r^2)}{r}\,dr = C.$$

26. $f(x,y)y'_x + g(x,y) = 0, \qquad \text{where} \quad \dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y}.$

Exact differential equation.

Solution: $\displaystyle\int_{y_0}^{y} f(x_0,t)\,dt + \int_{x_0}^{x} g(t,y)\,dt = C$, where x_0 and y_0 are arbitrary numbers.

13.7.2 Equations Containing Exponential and Hyperbolic Functions

1. $y'_x = e^{-\lambda x} f(e^{\lambda x} y).$

The substitution $u = e^{\lambda x} y$ leads to a separable equation: $u'_x = f(u) + \lambda u$.

2. $y'_x = e^{\lambda y} f(e^{\lambda y} x).$

The substitution $u = e^{\lambda y} x$ leads to a separable equation: $xu'_x = \lambda u^2 f(u) + u$.

3. $y'_x = y f(e^{\alpha x} y^m).$

This equation is invariant under the "translation–dilatation" transformation. The substitution $z = e^{\alpha x} y^m$ leads to a separable equation: $z'_x = \alpha z + m z f(z)$.

4. $y'_x = \dfrac{1}{x} f(x^n e^{\alpha y}).$

This equation is invariant under the "dilatation–translation" transformation. The substitution $z = x^n e^{\alpha y}$ leads to a separable equation: $xz'_x = nz + \alpha z f(z)$.

5. $y'_x = f(x) e^{\lambda y} + g(x).$

The substitution $u = e^{-\lambda y}$ leads to a linear equation: $u'_x = -\lambda g(x) u - \lambda f(x)$.

6. $y'_x = -\dfrac{n}{x} + f(x) g(x^n e^y).$

The substitution $z = x^n e^y$ leads to a separable equation: $z'_x = f(x) z g(z)$.

7. $y'_x = -\dfrac{\alpha}{m} y + y^k f(x) g(e^{\alpha x} y^m).$

The substitution $z = e^{\alpha x} y^m$ leads to a separable equation:

$$z'_x = m \exp\Bigl[\frac{\alpha}{m}(1-k)x\Bigr] f(x) z^{\frac{k+m-1}{m}} g(z).$$

8. $y'_x = f(x) e^{\lambda y} + g(x) + h(x) e^{-\lambda y}.$

The substitution $u = e^{\lambda y}$ leads to a Riccati equation: $u'_x = \lambda f(x) u^2 + \lambda g(x) u + \lambda h(x)$.

9. $y'_x = e^{\alpha x - \beta y} f(a e^{\alpha x} + b e^{\beta y}).$

The substitution $w = a e^{\alpha x} + b e^{\beta y}$ leads to a separable equation: $w'_x = e^{\alpha x}[a\alpha + b\beta f(w)]$.

10. $y'_x = f(y + a e^{\lambda x} + b) - a\lambda e^{\lambda x}.$

The substitution $w = y + a e^{\lambda x} + b$ leads to a separable equation: $w'_x = f(w)$.

11. $y'_x = -\dfrac{n}{\alpha x} + \dfrac{f(x^n e^{\alpha y})}{xy}.$

The substitution $t = x^n e^{\alpha y}$ leads to a linear equation with respect to $y = y(t)$: $\alpha^2 t f(t) y'_t = -ny + \alpha f(t)$.

12. $y'_x = -\dfrac{n}{\alpha x} + \dfrac{f(x^n e^{\alpha y})}{xy^2}.$

The substitution $t = x^n e^{\alpha y}$ leads to a Riccati equation: $\alpha^2 t f(t) y'_t = -ny^2 + \alpha f(t)$.

13. $[f(ax+by) + be^{\alpha y} g(ax+by)]y'_x = h(ax+by) - ae^{\alpha y} g(ax+by).$

The transformation $t = ax+by$, $z = e^{-\alpha y}$ leads to a linear equation with respect to $z = z(t)$: $[af(t) + bh(t)]z'_t = -\alpha h(t) z + \alpha a g(t)$.

14. $[f(ax+by) + be^{\alpha x} g(ax+by)]y'_x = h(ax+by) - ae^{\alpha x} g(ax+by).$

The transformation $t = ax+by$, $z = e^{-\alpha x}$ leads to a linear equation with respect to $z = z(t)$: $[af(t) + bh(t)]z'_t = -\alpha f(t) z - \alpha b g(t)$.

15. $[e^{\alpha x} f(y) + a\beta]y'_x + e^{\beta y} g(x) + a\alpha = 0.$

Solution: $\displaystyle\int e^{-\beta y} f(y)\,dy + \int e^{-\alpha x} g(x)\,dx - ae^{-\alpha x - \beta y} = C.$

16. $x[f(x^n e^{\alpha y}) + \alpha y g(x^n e^{\alpha y})]y'_x = h(x^n e^{\alpha y}) - nyg(x^n e^{\alpha y}).$

The substitution $t = x^n e^{\alpha y}$ leads to a linear equation with respect to $y = y(t)$: $t[nf(t) + \alpha h(t)]y'_t = -ng(t)y + h(t)$.

17. $[f(e^{\alpha x} y^m) + mxg(e^{\alpha x} y^m)]y'_x = y[h(e^{\alpha x} y^m) - \alpha x g(e^{\alpha x} y^m)].$

The substitution $t = e^{\alpha x} y^m$ leads to a linear equation with respect to $x = x(t)$: $t[\alpha f(t) + mh(t)]x'_t = mg(t)x + f(t)$.

18. $y'_x = f(x)\sinh(\lambda y) + g(x)\cosh(\lambda y) + h(x).$

The substitution $u = e^{\lambda y}$ leads to a Riccati equation: $2u'_x = \lambda(f+g)u^2 + 2\lambda h u + \lambda(g-f)$.

19. $y'_x = f(x)\sinh^2(\lambda y) + g(x)\cosh^2(\lambda y) + h(x)\sinh(2\lambda y) + s(x).$

The substitution $w = \tanh(\lambda y)$ leads to a Riccati equation: $w'_x = \lambda(f+s)w^2 + 2\lambda h w + \lambda(g-s)$.

20. $y'_x = f(x)\sinh\big(\lambda y + g(x)\big).$

The substitution $w = \lambda y + g(x)$ leads to an equation of the form 13.7.2.18: $w'_x = \lambda f(x)\sinh w + g'_x(x)$.

21. $y'_x = f(x)\cosh\big(\lambda y + g(x)\big).$

The substitution $w = \lambda y + g(x)$ leads to an equation of the form 13.7.2.18: $w'_x = \lambda f(x)\cosh w + g'_x(x)$.

22. $y'_x = y\coth x\, f(y^m \sinh x).$

The transformation $t = \sinh x$, $z = y^m$ leads to an equation of the form 13.7.1.3: $tz'_t = mzf(tz)$.

23. $y'_x = x^{-1}\tanh y\, f(x^n \sinh y).$

The transformation $t = x^n$, $z = \sinh y$ leads to an equation of the form 13.7.1.3: $ntz'_t = zf(tz)$.

24. $y'_x = y \tanh x\, f(y^m \cosh x)$.

The substitution $t = \cosh x$ leads to an equation of the form 13.7.1.3: $ty'_t = yf(ty^m)$.

25. $y'_x = x^{-1} \coth y\, f(x^n \cosh y)$.

The substitution $z = \cosh y$ leads to an equation of the form 13.7.1.3: $xz'_x = zf(x^n z)$.

13.7.3 Equations Containing Logarithmic Functions

1. $y'_x = f(x)y \ln^2 y + g(x)y \ln y + h(x)y$.

The substitution $u = \ln y$ leads to a Riccati equation: $u'_x = f(x)u^2 + g(x)u + h(x)$.

2. $y'_x = x^{-1}y^{m+1} f(y^m \ln x)$.

The substitution $t = \ln x$ leads to an equation of the form 13.7.1.3: $y'_t = \dfrac{y}{t}[ty^m f(ty^m)]$.

3. $y'_x = x^{-n-1} y f(x^n \ln y)$.

The substitution $z = \ln y$ leads to an equation of the form 13.7.1.3: $z'_x = \dfrac{z}{x}\dfrac{f(x^n z)}{x^n z}$.

4. $y'_x = x^{-1} e^y f(e^y \ln x)$.

The substitution $t = \ln x$ leads to an equation of the form 13.7.2.4: $y'_t = \dfrac{1}{t}[te^y f(te^y)]$.

5. $y'_x = ye^{-x} f(e^x \ln y)$.

The substitution $z = \ln y$ leads to an equation of the form 13.7.2.3: $z'_x = z\dfrac{f(e^x z)}{e^x z}$.

6. $y'_x = -nx^{-1} y \ln y + yf(x)g(x^n \ln y)$.

The substitution $w(x) = x^n \ln y$ leads to a separable equation: $w'_x = x^n f(x)g(w)$.

7. $y'_x = -\dfrac{n}{m}\dfrac{y}{x} + \dfrac{yf(x^n y^m)}{x \ln y}$.

The transformation $t = x^n y^m$, $z = \ln y$ leads to a linear equation with respect to $z = z(t)$: $m^2 tf(t)z'_t = -nz + mf(t)$.

8. $y'_x = -\dfrac{n}{m}\dfrac{y}{x} + \dfrac{yf(x^n y^m)}{x(\ln y)^2}$.

The transformation $t = x^n y^m$, $z = \ln y$ leads to a Riccati equation: $m^2 tf(t)z'_t = -nz^2 + mf(t)$.

9. $x[f(x^n y^m) + m \ln y\, g(x^n y^m)]y'_x = y[h(x^n y^m) - n \ln y\, g(x^n y^m)]$.

The transformation $t = x^n y^m$, $z = \ln y$ leads to a linear equation with respect to $z = z(t)$: $t[nf(t) + mh(t)]z'_t = -ng(t)z + h(t)$.

10. $x[f(x^n y^m) + m \ln x\, g(x^n y^m)]y'_x = y[h(x^n y^m) - n \ln x\, g(x^n y^m)]$.

The transformation $t = x^n y^m$, $z = \ln x$ leads to a linear equation with respect to $z = z(t)$: $t[nf(t) + mh(t)]z'_t = mg(t)z + f(t)$.

13.7.4 Equations Containing Trigonometric Functions

1. $y'_x = y^{m+1} \sin x \, F(y^m \cos x)$.

This is an equation of the type 13.7.4.3 with $f(\xi) = \xi F(\xi)$.

2. $y'_x = y^{m+1} \cos x \, F(y^m \sin x)$.

This is an equation of the type 13.7.4.4 with $f(\xi) = \xi F(\xi)$.

3. $y'_x = y \tan x \, f(y^m \cos x)$.

The substitution $t = \cos x$ leads to an equation of the form 13.7.1.3: $ty'_t = -yf(ty^m)$.

4. $y'_x = y \cot x \, f(y^m \sin x)$.

The substitution $t = \sin x$ leads to an equation of the form 13.7.1.3: $ty'_t = yf(ty^m)$.

5. $y'_x = x^{-1} \tan y \, f(x^n \sin y)$.

The transformation $t = x^n$, $z = \sin y$ leads to an equation of the form 13.7.1.3: $ntz'_t = zf(tz)$.

6. $y'_x = x^{-1} \cot y \, f(x^n \cos y)$.

The transformation $t = x^n$, $z = \cos y$ leads to an equation of the form 13.7.1.3: $ntz'_t = -zf(tz)$.

7. $y'_x = x^{-1} \sin 2y \, f(x^n \tan y)$.

The transformation $t = x^n$, $z = \tan y$ leads to an equation of the form 13.7.1.3: $ntz'_t = 2zf(tz)$.

8. $y'_x = x^{-1} \sin 2y \, f(x^n \cot y)$.

The transformation $t = x^n$, $z = \cot y$ leads to an equation of the form 13.7.1.3: $ntz'_t = -2zf(tz)$.

9. $y'_x = \dfrac{y}{\sin 2x} f(y^m \tan x)$.

The substitution $t = \tan x$ leads to an equation of the form 13.7.1.3: $2ty'_t = yf(ty^m)$.

10. $y'_x = \dfrac{y}{\sin 2x} f(y^m \cot x)$.

The substitution $t = \cot x$ leads to an equation of the form 13.7.1.3: $2ty'_t = -yf(ty^m)$.

11. $y'_x = f(x) \cos(ay) + g(x) \sin(ay) + h(x)$.

The substitution $u = \tan(ay/2)$ leads to a Riccati equation: $2u'_x = a(h-f)u^2 + 2agu + a(f+h)$.

12. $y'_x = f(x) \cos^2(ay) + g(x) \sin^2(ay) + h(x) \sin(2ay) + s(x)$.

The substitution $u = \tan(ay)$ leads to a Riccati equation: $u'_x = a(g+s)u^2 + 2ahu + a(f+s)$.

13. $y'_x = f(y + a \tan x) - a \tan^2 x$.

The substitution $u = y + a \tan x$ leads to a separable equation: $u'_x = a + f(u)$.

14. $y'_x = \dfrac{\sin 2y}{\sin 2x} f(\tan x \tan y).$

The transformation $t = \tan x$, $z = \tan y$ leads to an equation of the form 13.7.1.3: $tz'_t = zf(tz)$.

15. $y'_x = \cot x \tan y \, f(\sin x \sin y).$

The transformation $t = \sin x$, $z = \sin y$ leads to an equation of the form 13.7.1.3: $tz'_t = zf(tz)$.

16. $y'_x = -\cot x \tan y + \dfrac{f(x)}{\cos y} g(\sin x \sin y).$

The substitution $w(x) = \sin x \sin y$ leads to a separable equation: $w'_x = \sin x \, f(x) g(w)$.

17. $y'_x = -\dfrac{\sin 2y}{\sin 2x} + \cos^2 y \, f(x) g(\tan x \tan y).$

The substitution $w(x) = \tan x \tan y$ leads to a separable equation: $w'_x = \tan x \, f(x) g(w)$.

18. $y'_x = -nx^{-1} \sin 2y + \cos^2 y \, f(x) g(x^{2n} \tan y).$

The substitution $w(x) = x^{2n} \tan y$ leads to a separable equation: $w'_x = x^{2n} f(x) g(w)$.

19. $(1 + \tan^2 y) y'_x = f(x) \tan^{m+1} y + g(x) \tan y + h(x) \tan^{1-m} y.$

The substitution $u = \tan^m y$ leads to a Riccati equation: $u'_x = mf(x)u^2 + mg(x)u + mh(x)$.

20. $y'_x = f(x) \sin \bigl(\lambda y + g(x)\bigr).$

The substitution $w = \lambda y + g(x)$ leads to an equation of the form 13.7.4.11: $w'_x = \lambda f(x) \sin w + g'_x(x)$.

21. $y'_x = f(x) \cos \bigl(\lambda y + g(x)\bigr).$

The substitution $w = \lambda y + g(x)$ leads to an equation of the form 13.7.4.11: $w'_x = \lambda f(x) \cos w + g'_x(x)$.

22. $y'_x = f(x) \sin^2 \bigl(\lambda y + g(x)\bigr).$

The substitution $w = \lambda y + g(x)$ leads to an equation of the form 13.7.4.12: $w'_x = \lambda f(x) \sin^2 w + g'_x(x)$.

23. $y'_x = f(x) \cos^2 \bigl(\lambda y + g(x)\bigr).$

The substitution $w = \lambda y + g(x)$ leads to an equation of the form 13.7.4.12: $w'_x = \lambda f(x) \cos^2 w + g'_x(x)$.

13.7.5 Equations Containing Combinations of Exponential, Logarithmic, and Trigonometric Functions

1. $y'_x = -\sin 2y + \cos^2 y \, f(x) g(e^{2x} \tan y).$

The substitution $w(x) = e^{2x} \tan y$ leads to a separable equation: $w'_x = e^{2x} f(x) g(w)$.

2. $y'_x = \dfrac{F(e^x \cos y)}{e^x \sin y}.$

This is an equation of the type 13.7.5.5 with $f(\xi) = F(\xi)/\xi$.

3. $y'_x = e^y \cos x\, F(e^y \sin x)$.

This is an equation of the type 13.7.5.7 with $f(\xi) = \xi F(\xi)$.

4. $y'_x = \tan y\, f(e^x \sin y)$.

The substitution $z = \sin y$ leads to an equation of the form 13.7.2.3: $z'_x = zf(e^x z)$.

5. $y'_x = \cot y\, f(e^x \cos y)$.

The substitution $z = \cos y$ leads to an equation of the form 13.7.2.3: $z'_x = -zf(e^x z)$.

6. $y'_x = \tan x\, f(e^y \cos x)$.

The substitution $t = \cos x$ leads to an equation of the form 13.7.2.4: $ty'_t = -f(te^y)$.

7. $y'_x = \cot x\, f(e^y \sin x)$.

The substitution $t = \sin x$ leads to an equation of the form 13.7.2.4: $ty'_t = f(te^y)$.

8. $y'_x = \sin 2y\, f(e^x \tan y)$.

The substitution $z = \tan x$ leads to an equation of the form 13.7.2.3: $z'_x = 2zf(e^x z)$.

9. $y'_x = \sin 2y\, f(e^x \cot y)$.

The substitution $z = \cot x$ leads to an equation of the form 13.7.2.3: $z'_x = -2zf(e^x z)$.

10. $y'_x = \dfrac{F(e^x \sin y)}{e^x \cos y}$.

This is an equation of the type 13.7.5.4 with $f(\xi) = F(\xi)/\xi$.

11. $y'_x = e^y \sin x\, F(e^y \cos x)$.

This is an equation of the type 13.7.5.6 with $f(\xi) = \xi F(\xi)$.

12. $y'_x = \dfrac{f(e^y \tan x)}{\sin 2x}$.

The substitution $t = \tan x$ leads to an equation of the form 13.7.2.4: $2ty'_t = f(te^y)$.

13. $y'_x = \dfrac{f(e^y \cot x)}{\sin 2x}$.

The substitution $t = \cot x$ leads to an equation of the form 13.7.2.4: $2ty'_t = -f(te^y)$.

14. $y'_x = e^{-\lambda x} f(\lambda x + \ln y)$.

The substitution $u = \lambda x + \ln y$ leads to a separable equation: $u'_x = e^{-u} f(u) + \lambda$.

15. $y'_x = e^{\lambda y} f(\lambda y + \ln x)$.

The substitution $u = \lambda y + \ln x$ leads to a separable equation: $xu'_x = \lambda e^u f(u) + 1$.

13.8 Equations Not Solved for the Derivative and Equations Defined Parametrically

13.8.1 Equations Not Solved for the Derivative Containing Arbitrary Functions

► **Arguments of arbitrary functions depend on x and y.**

1. $(y'_x)^2 + [f(x) + g(x)]y'_x + f(x)g(x) = 0.$

The equation can be factorized: $[y'_x + f(x)][y'_x + g(x)] = 0$, i.e., it falls into two simpler equations $y'_x + f(x) = 0$ and $y'_x + g(x) = 0$. Therefore, the solutions are:

$$y + \int f(x)\,dx = C \quad \text{and} \quad y + \int g(x)\,dx = C.$$

2. $(y'_x)^2 + 2fyy'_x + gy^2 = (g - f^2)\exp\Big(-2\int_a^x f\,dx\Big).$

Here, $f = f(x)$, $g = g(x)$. Solution:

$$y = \begin{cases} \exp\Big(-\int_a^x f\,dx\Big)\sin\Big(\int_a^x \sqrt{g - f^2}\,dx + C\Big) & \text{if } g > f^2, \\ C\exp\Big(-\int_a^x f\,dx\Big) & \text{if } g \equiv f^2, \\ \exp\Big(-\int_a^x f\,dx\Big)\cosh\Big(\int_a^x \sqrt{f^2 - g}\,dx + C\Big) & \text{if } g < f^2. \end{cases}$$

3. $xy'_x - y = f(x^2 + y^2)\sqrt{(y'_x)^2 + 1}.$

Raising the equation to the second power and applying the transformation $x = r(t)\cos t$, $y = r(t)\sin t$, one arrives at the relation $r^4 = f^2(r^2)[(r'_t)^2 + r^2]$. Solving it for r'_t yields a separable equation: $f(r^2)r'_t = \pm r\sqrt{r^2 - f^2(r^2)}$.

4. $yy'_x + x = f(x^2 + y^2)\sqrt{(y'_x)^2 + 1}.$

Raising the equation to the second power and applying the transformation $x = r(t)\cos t$, $y = r(t)\sin t$, one arrives at the relation $r^2(r'_t)^2 = f^2(r^2)[(r'_t)^2 + r^2]$. Solving it for r'_t yields a separable equation: $r'_t = \pm\dfrac{rf(r^2)}{\sqrt{r^2 - f^2(r^2)}}$.

⊙ *Literature*: G. W. Bluman, J. D. Cole (1974, page 100).

5. $xy'_x - y = \sqrt{x^2 + y^2}\, f\bigg(\dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}}\bigg)\sqrt{(y'_x)^2 + 1}.$

Raising the equation to the second power and applying the transformation $x = r(t)\cos t$, $y = r(t)\sin t$, one arrives at the relation $r^2 = f^2(\cos t, \sin t)[(r'_t)^2 + r^2]$. Solving it for r'_t yields a separable equation: $f(\cos t, \sin t)r'_t = \pm r\sqrt{1 - f^2(\cos t, \sin t)}$.

6. $yy'_x + x = \sqrt{x^2 + y^2}\, f\bigg(\dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}}\bigg)\sqrt{(y'_x)^2 + 1}.$

Raising the equation to the second power and applying the transformation $x = r(t)\cos t$, $y = r(t)\sin t$, one arrives at the relation $r^2(r'_t)^2 = f^2(\cos t, \sin t)[(r'_t)^2 + r^2]$. Solving it for r'_t yields a separable equation: $\sqrt{1 - f^2(\cos t, \sin t)}\, r'_t = \pm rf(\cos t, \sin t)$.

▶ Argument of arbitrary functions is y'_x.

7. $x = f(y'_x)$.

Solution in parametric form:

$$x = f(t), \qquad y = \int t f'_t(t)\,dt + C.$$

8. $y = f(y'_x)$.

Solution in parametric form:

$$x = \int f'_t(t)\frac{dt}{t} + C, \qquad y = f(t).$$

9. $f(y'_x) + ax + by + s = 0$.

Solution in parametric form:

$$x = C - \int \frac{f'_t(t)\,dt}{a + bt}, \qquad by = -ax - s - f(t).$$

In addition, there is a particular solution $y = \alpha x + \beta$, where α and β are determined by solving the system of two algebraic equations:

$$a + b\alpha = 0, \qquad f(\alpha) + b\beta + s = 0.$$

10. $y = xy'_x + f(y'_x)$.

The Clairaut equation. Solution: $y = Cx + f(C)$.

In addition, there is a singular solution, which may be written in the parametric form as:

$$x = -f'_t(t), \qquad y = -tf'_t(t) + f(t).$$

11. $y = xf(y'_x) + g(y'_x)$.

The Lagrange–d'Alembert equation. For the case $f(t) = t$, see equation 13.8.1.10. Having differentiated with respect to x, we arrive at a linear equation with respect to $x = x(t)$, where $t = y'_x$: $[t - f(t)]x'_t = f'_t(t)x + g'_t(t)$. See also 1.8.1.12.

12. $xf(y'_x) + yg(y'_x) + h(y'_x) = 0$.

The Legendre transformation $X = y'_x$, $Y = xy'_x - y$, $Y'_X = x$ leads to a linear equation: $[f(X) + Xg(X)]Y'_X - g(X)Y + h(X) = 0$.

Inverse transformation: $x = Y'_X$, $y = XY'_X - Y$, $y'_x = X$.

13. $y = x^2 f(y'_x) + xg(y'_x) + h(y'_x)$.

Having differentiated with respect to x, we arrive at an Abel equation with respect to $x = x(t)$, where $t = y'_x$:

$$\left[2f(t)x + g(t) - t\right]x'_t = -f'_t(t)x^2 - g'_t(t)x - h'_t(t)$$

(see Section 13.3.4).

14. $\boldsymbol{x = y^2 f(y'_x) + yg(y'_x) + h(y'_x).}$

Having differentiated with respect to x, we arrive at an Abel equation with respect to $y = y(t)$, where $t = y'_x$:

$$\big[2tf(t)y + tg(t) - 1\big]y'_t = -tf'_t(t)y^2 - tg'_t(t)y - th'_t(t)$$

(see Section 13.3.4).

15. $\boldsymbol{y = x^n f(y'_x) + xy'_x.}$

Differentiating with respect to x and denoting $t = y'_x$, we obtain a Bernoulli equation for $x = x(t)$: $nf(t)x'_t - f'_t(t)x - x^{2-n} = 0$.

16. $\boldsymbol{(xy'_x - y)^n f(y'_x) + yg(y'_x) + xh(y'_x) = 0.}$

The Legendre transformation $x = u'_t$, $y = tu'_t - u$ $(y'_x = t)$ leads to a Bernoulli equation: $[tg(t) + h(t)]u'_t = g(t)u - f(t)u^n$.

► Arguments of arbitrary functions are linear with respect to y'_x.

17. $\boldsymbol{y = a(y'_x)^2 + f(x - 2ay'_x).}$

Solution: $y = f(C) + \dfrac{1}{4a}(x - C)^2$. In addition, there is a singular solution, which can be represented in parametric form as:

$$x = t + 2af'_t(t), \qquad y = f(t) + a[f'_t(t)]^2.$$

18. $\boldsymbol{y'_x = f(y'_x + ax) + g(y'_x + ax)(y'^2_x + 2ay).}$

The contact transformation $X = y'_x + ax$, $Y = \frac{1}{2}(y'_x)^2 + ay$, $Y'_X = y'_x$, where $Y = Y(X)$, leads to a linear equation: $Y'_X = 2g(X)Y + f(X)$.

Inverse transformation: $x = a^{-1}(X - Y'_X)$, $y = \frac{1}{2}a^{-1}[2Y - (Y'_X)^2]$, $y'_x = Y'_X$.

19. $\boldsymbol{y'_x = f(y'_x + ax)(y'^2_x + 2ay) + g(y'_x + ax)(y'^2_x + 2ay)^k.}$

The contact transformation $X = y'_x + ax$, $Y = \frac{1}{2}(y'_x)^2 + ay$, $Y'_X = y'_x$, where $Y = Y(X)$, leads to a Bernoulli equation: $Y'_X = 2f(X)Y + 2^k g(X)Y^k$.

Inverse transformation: $x = a^{-1}(X - Y'_X)$, $y = \frac{1}{2}a^{-1}[2Y - (Y'_X)^2]$, $y'_x = Y'_X$.

20. $\boldsymbol{x^n e^{\alpha y} = f(xy'_x).}$

The substitution $y = \ln w$ leads to an equation of the form 13.8.1.32: $x^n w^\alpha = f(xw'_x/w)$.

21. $\boldsymbol{x = f(y'_x)g(xy'_x - y).}$

The Legendre transformation $X = y'_x$, $Y = xy'_x - y$, $Y'_X = x$ leads to a separable equation: $Y'_X = f(X)g(Y)$.

Inverse transformation: $x = Y'_X$, $y = XY'_X - Y$, $y'_x = X$.

22. $\boldsymbol{xf(xy'_x - y)y'_x + xg(xy'_x - y)(y'_x)^k = a.}$

The modified Legendre transformation $X = xy'_x - y$, $Y = y'_x$, $Y'_X = 1/x$ leads to a Bernoulli equation: $aY'_X = f(X)Y + g(X)Y^k$.

Inverse transformation: $x = (Y'_X)^{-1}$, $y = Y(Y'_X)^{-1} - X$, $y'_x = Y$.

23. $x = f(xy'_x + y)g(x^2 y'_x)$.

The contact transformation $X = xy'_x + y$, $Y = x^2 y'_x$, $Y'_X = x$, where $Y = Y(X)$, leads to a separable equation: $Y'_X = f(X)g(Y)$.

Inverse transformation: $x = Y'_X$, $y = X - Y(Y'_X)^{-1}$, $y'_x = Y(Y'_X)^{-2}$.

24. $x = f(xy'_x + y) + x^2 y'_x g(xy'_x + y)$.

The contact transformation $X = xy'_x + y$, $Y = x^2 y'_x$, $Y'_X = x$, where $Y = Y(X)$, leads to a linear equation: $Y'_X = g(X)Y + f(X)$.

Inverse transformation: $x = Y'_X$, $y = X - Y(Y'_X)^{-1}$, $y'_x = Y(Y'_X)^{-2}$.

25. $xy'_x f(xy'_x + y) + x^3 (y'_x)^2 g(xy'_x + y) = a$.

The contact transformation $X = xy'_x + y$, $Y = x^2 y'_x$, $Y'_X = x$, where $Y = Y(X)$, leads to a Bernoulli equation: $aY'_X = f(X)Y + g(X)Y^2$.

Inverse transformation: $x = Y'_X$, $y = X - Y(Y'_X)^{-1}$, $y'_x = Y(Y'_X)^{-2}$.

26. $f(yy'_x + x) = y^2 (y'^2_x + 1)$.

Setting $u(x) = yy'_x + x$ and differentiating with respect to x, we obtain

$$u'_x [f'_u(u) - 2u + 2x] = 0. \qquad (1)$$

Equating the first factor to zero, after integrating we find $y^2 = -(x - C)^2 + B$. Substituting the latter into the original equation yields $B = f(C)$. As a result we obtain the solution: $y^2 = f(C) - (x - C)^2$.

There is also a singular solution that corresponds to equating the second factor of (1) to zero. This solution in parametric form is written as:

$$x = u - \tfrac{1}{2} f'_u(u), \qquad y^2 = f(u) - \tfrac{1}{4}[f'_u(u)]^2.$$

27. $yy'_x = f(yy'_x - x) + y^2 (y'^2_x - 1) g(yy'_x - x)$.

The contact transformation $x = YY'_X - X$, $y = Y[(Y'_X)^2 - 1]^{1/2}$, $y'_x = Y'_X[(Y'_X)^2 - 1]^{-1/2}$, where $Y = Y(X)$, leads to the equation $YY'_X = f(X) + Y^2 g(X)$, which is linear in $W = Y^2$.

Inverse transformation: $X = yy'_x - x$, $Y = -y[(y'_x)^2 - 1]^{1/2}$, $Y'_X = -y'_x[(y'_x)^2 - 1]^{-1/2}$.

28. $y'_x = \dfrac{1}{x^2} f\Big(y'_x + \dfrac{y}{x}\Big) + \Big(y'_x - \dfrac{y}{x}\Big) g\Big(y'_x + \dfrac{y}{x}\Big)$.

The contact transformation $X = y'_x + y/x$, $Y = x^2 (y'_x)^2 - y^2$, $Y'_X = 2x^2 y'_x$ leads to a linear equation: $XY'_X = 2g(X)Y + 2Xf(X)$.

Inverse transformation:

$$x = \pm \frac{1}{X}\sqrt{XY'_X - Y}, \quad y = \pm \frac{XY'_X - 2Y}{2\sqrt{XY'_X - Y}}, \quad y'_x = \frac{X^2 Y'_X}{2(XY'_X - Y)}.$$

29. $x^{a-1} f\Big(y'_x - a\dfrac{y}{x}\Big) + \Big(y'_x - \dfrac{y}{x}\Big) g\Big(y'_x - a\dfrac{y}{x}\Big) = b$.

For $a \neq 1$, the contact transformation $X = y'_x - ay/x$, $Y = x^{1-a} y'_x - x^{-a} y$, $Y'_X = x^{1-a}$ leads to a linear equation: $bY'_X = g(X)Y + f(X)$.

Inverse transformation:

$$x = (Y'_X)^{\frac{1}{1-a}}, \quad y = \frac{1}{1-a}(XY'_X - Y)(Y'_X)^{\frac{a}{1-a}}, \quad y'_x = \frac{XY'_X - aY}{(1-a)Y'_X}.$$

30. $x^{a+1} f\left(y'_x + a\frac{y}{x}\right) = g(x^{a+1}y'_x - x^a y).$

For $a \neq -1$, the contact transformation $X = y'_x + ay/x,\ Y = x^{a+1}y'_x - x^a y,\ Y'_X = x^{a+1}$ leads to a separable equation: $f(X)Y'_X = g(Y)$.

Inverse transformation:

$$x = (Y'_X)^{\frac{1}{a+1}}, \quad y = \frac{1}{a+1}(XY'_X - Y)(Y'_X)^{-\frac{a}{a+1}}, \quad y'_x = \frac{XY'_X + aY}{(a+1)Y'_X}.$$

31. $e^{\alpha x} y^n = f(y'_x/y).$

The substitution $x = \ln t$ leads to an equation of the form 13.8.1.32: $t^\alpha y^n = f(ty'_t/y)$.

32. $x^n y^m = f(xy'_x/y).$

We pass to a new variable $w(x) = xy'_x/y$, divide both sides of the equation by $x^n y^m$, and differentiate with respect to x. As a result we arrive at a separable equation: $xf'_w(w)w'_x = (mw+n)f(w)$.

Solution in parametric form:

$$\ln|x| = \int \frac{f'_w(w)\,dw}{(mw+n)f(w)} + C, \qquad x^n y^m = f(w).$$

In addition, there are singular solutions $y = A_k x^{-n/m}$, where A_k are roots of the algebraic equation $A_k^m - f(-n/m) = 0$.

33. $y'_x = e^x f(e^x y'_x) - y.$

The contact transformation $X = e^x y'_x,\ Y = y'_x + y,\ Y'_X = e^{-x}$, where $Y = Y(X)$, leads to a separable equation: $YY'_X = f(X)$.

Inverse transformation: $x = -\ln Y'_X,\ y = Y - XY'_X,\ y'_x = XY'_X$.

34. $e^x f(e^x y'_x) - g(y'_x + y) = 0.$

The contact transformation $X = e^x y'_x,\ Y = y'_x + y,\ Y'_X = e^{-x}$, where $Y = Y(X)$, leads to a separable equation: $g(Y)Y'_X = -f(X)$.

Inverse transformation: $x = -\ln Y'_X,\ y = Y - XY'_X,\ y'_x = XY'_X$.

35. $f(e^x y'_x) + g(e^x y'_x)(y'_x + y) = ae^{-x}.$

The contact transformation $X = e^x y'_x,\ Y = y'_x + y,\ Y'_X = e^{-x}$, where $Y = Y(X)$, leads to a linear equation: $aY'_X = g(X)Y + f(X)$.

Inverse transformation: $x = -\ln Y'_X,\ y = Y - XY'_X,\ y'_x = XY'_X$.

36. $f(e^x y'_x) - g(y'_x + y) = x.$

This equation can be rewritten in the form 13.8.1.34:

$$e^x F(e^x y'_x) - G(y'_x + y) = 0, \qquad \text{where} \quad f(u) = -\ln F(u), \quad g(v) = -\ln G(v).$$

37. $y(y'_x \sin x - y\cos x) = y'_x f(y'_x \cos x + y\sin x)$.

The contact transformation

$$X = \frac{1}{\sqrt{(y'_x)^2 + y^2}}, \quad Y = \frac{y'_x \cos x + y\sin x}{\sqrt{(y'_x)^2 + y^2}}, \quad Y'_X = \frac{y}{y'_x}(y'_x \sin x - y\cos x)$$

leads to the homogeneous equation: $Y'_X = f(Y/X)$.

38. $F(x^{n+1}y'_x,\ xy'_x + ny) = 0$.

Solution: $y = C_1 x^{-n} + C_2$. Here, the constants C_1 and C_2 are related by the constraint $F(-C_1 n, C_2 n) = 0$.

39. $F(x^2 y'_x + 2xy,\ x^3 y'_x + x^2 y) = 0$.

Solution: $y = C_1 x^{-1} + C_2 x^{-2}$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1, -C_2) = 0$.

The singular solution can be represented in parametric form as:

$$F(u,v) = 0, \quad F_u(u,v) + xF_v(u,v) = 0, \qquad \text{where} \quad u = x^2 t + 2xy, \quad v = x^3 t + x^2 y.$$

The subscripts u and v denote the respective partial derivatives, and t is the parameter.

40. $F(x^{n+1}y'_x + mx^n y,\ x^{m+1}y'_x + nx^m y) = 0$.

Solution: $y = C_1 x^{-n} + C_2 x^{-m}$. Here, the constants C_1 and C_2 are related by the constraint $F\big(C_1(m-n), C_2(n-m)\big) = 0$.

41. $F(e^x y'_x,\ y'_x + y) = 0$.

Solution: $y = C_1 e^{-x} + C_2$. Here, the constants C_1 and C_2 are related by the constraint $F(-C_1, C_2) = 0$.

42. $F(e^{\alpha x} y'_x + \beta e^{\alpha x} y,\ e^{\beta x} y'_x + \alpha e^{\beta x} y) = 0$.

Solution: $y = C_1 e^{-\alpha x} + C_2 e^{-\beta x}$. Here, the constants C_1 and C_2 are related by the constraint $F\big(C_1(\beta - \alpha), C_2(\alpha - \beta)\big) = 0$.

43. $F(y'_x \cosh x - y\sinh x,\ y'_x \sinh x - y\cosh x) = 0$.

Solution: $y = C_1 \sinh x + C_2 \cosh x$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1,\ -C_2) = 0$.

44. $F(xy'_x,\ y - xy'_x \ln x) = 0$.

Solution: $y = C_1 \ln x + C_2$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1, C_2) = 0$.

45. $F\Big(\frac{y'_x}{y},\ \ln y - \frac{x}{y} y'_x\Big) = 0$.

Solution: $y = C_1 \exp(C_2 x)$. Here, the constants C_1 and C_2 are related by the constraint $F(C_2, \ln C_1) = 0$.

46. $F\Big(\frac{xy'_x}{y},\ \ln y - \frac{xy'_x}{y} \ln x\Big) = 0$.

Solution: $y = C_1 x^{C_2}$. Here, the constants C_1 and C_2 are related by the constraint $F(C_2, \ln C_1) = 0$.

47. $F(y'_x\cos x + y\sin x,\ y'_x\sin x - y\cos x) = 0.$

Solution: $y = C_1\sin x + C_2\cos x$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1,\ -C_2) = 0$.

48. $F\Big(\frac{y'_x}{\varphi'_x},\ y - \frac{\varphi}{\varphi'_x}y'_x\Big) = 0, \qquad \varphi = \varphi(x).$

Solution: $y = C_1\varphi(x) + C_2$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1, C_2) = 0$.

49. $F\Big(\frac{\psi'_x y - \psi y'_x}{\varphi\psi'_x - \psi\varphi'_x},\ \frac{\varphi'_x y - \varphi y'_x}{\varphi\psi'_x - \psi\varphi'_x}\Big) = 0, \qquad \varphi = \varphi(x), \quad \psi = \psi(x).$

Solution: $y = C_1\varphi(x) + C_2\psi(x)$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1, -C_2) = 0$.

The singular solution can be represented in parametric form as:

$$F(u,v)=0, \quad \psi F_u(u,v)+\varphi F_v(u,v)=0, \quad \text{where } u=\frac{\psi'_x y - t\psi}{\varphi\psi'_x - \psi\varphi'_x}, \quad v=\frac{\varphi'_x y - t\varphi}{\varphi\psi'_x - \psi\varphi'_x}.$$

The subscripts u and v denote the respective partial derivatives, and t is the parameter.

50. $F\big(\varphi_x + \varphi_y y'_x,\ \varphi - x(\varphi_x + \varphi_y y'_x)\big) = 0.$

Here, $\varphi = \varphi(x,y)$, $\varphi_x = \frac{\partial\varphi}{\partial x}$, $\varphi_y = \frac{\partial\varphi}{\partial y}$. Differentiating with respect to x, we obtain

$$(\varphi_x + \varphi_y y'_x)'_x(F_u - xF_v) = 0,$$

where $F_u = \frac{\partial F}{\partial u}$ and $F_v = \frac{\partial F}{\partial v}$ are partial derivatives of function $F(u,v)$. Equating the first factor to zero, we find the solution:

$$\varphi(x,y) = Cx + A, \qquad \text{where} \quad F(C,A) = 0.$$

It remains to be checked whether the equation $F_u - xF_v = 0$ possesses any solutions and which of them satisfy the original equation.

► Arguments of arbitrary functions are nonlinear with respect to y'_x.

51. $y = f(xy'^2_x) + 2xy'_x.$

Solution: $[y - f(C)]^2 = 4Cx$.

52. $y = 2a(y'_x)^3 + f(x - 3ay'^2_x).$

This is a special case of equation 13.8.1.72 with $n = 3$.

Solution: $y = f(C) + 2a\Big(\frac{x-C}{3a}\Big)^{3/2}$. In addition, there is the following singular solution written in parametric form:

$$x = t + 3a[f'_t(t)]^2, \qquad y = f(t) + 2a[f'_t(t)]^3.$$

53. $y'_x = f(ay'^2_x - bx) + (2ay'^3_x - 3by)g(ay'^2_x - bx), \qquad b \neq 0.$

The contact transformation $X = a(y'_x)^2 - bx$, $Y = 2a(y'_x)^3 - 3by$, $Y'_X = 3y'_x$ leads to a linear equation: $Y'_X = 3g(X)Y + 3f(X)$.

Inverse transformation: $x = \frac{1}{9}b^{-1}[a(Y'_X)^2 - 9X]$, $y = \frac{1}{81}b^{-1}[2a(Y'_X)^3 - 27Y]$, $y'_x = \frac{1}{3}Y'_X$.

54. $y'_x = f(y'_x + ax)g(\frac{1}{2}y'^2_x + ay).$

The contact transformation $X = y'_x + ax$, $Y = \frac{1}{2}(y'_x)^2 + ay$, $Y'_X = y'_x$, where $Y = Y(X)$, leads to a separable equation: $Y'_X = f(X)g(Y)$.

Inverse transformation: $x = a^{-1}(X - Y'_X)$, $y = \frac{1}{2}a^{-1}[2Y - (Y'_X)^2]$, $y'_x = Y'_X$.

55. $y'_x(y'_x - y) = f(y'^2_x - y^2).$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = e^x(y'_x - y)$, $Y'_X = \frac{1}{2}e^x(y'_x)^{-1}$ leads to a linear (separable) equation: $2f(X)Y'_X = Y$.

56. $y'_x = f(y'^2_x - y^2)(ae^x + be^{-x}).$

The contact transformation

$$X = (y'_x)^2 - y^2, \quad Y = y'_x(ae^x + be^{-x}) - y(ae^x - be^{-x}), \quad Y'_X = \tfrac{1}{2}(ae^x + be^{-x})(y'_x)^{-1}$$

leads to a separable equation: $2f(X)Y'_X = 1$.

57. $y'_x = e^x f(y'^2_x - y^2)g(e^x y'_x - e^x y).$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = e^x(y'_x - y)$, $Y'_X = \frac{1}{2}e^x(y'_x)^{-1}$ leads to a separable equation: $2f(X)g(Y)Y'_X = 1$.

58. $y'_x = f(y'^2_x - y^2)\cosh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \cosh x - y \sinh x$, $Y'_X = \frac{1}{2}\cosh x (y'_x)^{-1}$ leads to a separable equation: $2f(X)Y'_X = 1$.

59. $y'_x = f(y'^2_x - y^2)\sinh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \sinh x - y \cosh x$, $Y'_X = \frac{1}{2}\sinh x (y'_x)^{-1}$ leads to a separable equation: $2f(X)Y'_X = 1$.

60. $y'_x = f(y'^2_x - y^2)(a\cosh x + b\sinh x).$

The contact transformation

$$X = (y'_x)^2 - y^2, \quad Y = y'_x(a\cosh x + b\sinh x) - y(a\sinh x + b\cosh x),$$
$$Y'_X = \frac{a\cosh x + b\sinh x}{2y'_x}$$

leads to a separable equation: $2f(X)Y'_X = 1$.

61. $y'_x = f(y'^2_x - y^2)g(y'_x \cosh x - y\sinh x)\cosh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \cosh x - y \sinh x$, $Y'_X = \frac{1}{2}\cosh x (y'_x)^{-1}$ leads to a separable equation: $2f(X)g(Y)Y'_X = 1$.

62. $y'_x = f(y'^2_x - y^2)g(y'_x \sinh x - y\cosh x)\sinh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \sinh x - y \cosh x$, $Y'_X = \frac{1}{2}\sinh x (y'_x)^{-1}$ leads to a separable equation: $2f(X)g(Y)Y'_X = 1$.

63. $y'_x f(y'^2_x - y^2) + ay'^2_x \cosh x - ayy'_x \sinh x = \cosh x.$

The contact transformation $X = (y'_x)^2 - y^2$, $Y = y'_x \cosh x - y \sinh x$, $Y'_X = \frac{1}{2}\cosh x (y'_x)^{-1}$ leads to a linear equation: $2Y'_X = aY + f(X)$.

64. $y'_x = f(y'^2_x + y^2)\cos x.$

This is a special case of equation 13.8.1.65 with $a = 1$ and $b = 0$.

65. $y'_x = f(y'^2_x + y^2)(a\cos x + b\sin x).$

The contact transformation

$$X = (y'_x)^2 + y^2, \quad Y = y'_x(a\cos x + b\sin x) + y(a\sin x - b\cos x),$$
$$Y'_X = \tfrac{1}{2}(a\cos x + b\sin x)(y'_x)^{-1}$$

leads to a separable equation: $2f(X)Y'_X = 1$.

66. $y'_x = f(y'^2_x + y^2)g(y'_x\cos x + y\sin x)\cos x.$

The contact transformation $X = (y'_x)^2 + y^2$, $Y = y'_x\cos x + y\sin x$, $Y'_X = \frac{1}{2}\cos x(y'_x)^{-1}$ leads to a separable equation: $2f(X)g(Y)Y'_X = 1$.

67. $y'_x f(y'^2_x + y^2) + ay'^2_x\cos x + ayy'_x\sin x = \cos x.$

The contact transformation $X = (y'_x)^2 + y^2$, $Y = y'_x\cos x + y\sin x$, $Y'_X = \frac{1}{2}\cos x(y'_x)^{-1}$ leads to a linear equation: $2Y'_X = aY + f(X)$.

68. $y'_x = f(y'^2_x + y^2)g(y'_x\sin x - y\cos x)\sin x.$

The contact transformation $X = (y'_x)^2 + y^2$, $Y = y'_x\sin x - y\cos x$, $Y'_X = \frac{1}{2}\sin x(y'_x)^{-1}$ leads to a separable equation: $2f(X)g(Y)Y'_X = 1$.

69. $x^2y'_x = f\Bigl(y'_x + \dfrac{y}{x}\Bigr)g(x^2y'^2_x - y^2).$

The contact transformation $X = y'_x + y/x$, $Y = x^2(y'_x)^2 - y^2$, $Y'_X = 2x^2y'_x$ leads to a separable equation: $Y'_X = 2f(X)g(Y)$.

Inverse transformation:

$$x = \pm\frac{1}{X}\sqrt{XY'_X - Y}, \quad y = \pm\frac{XY'_X - 2Y}{2\sqrt{XY'_X - Y}}, \quad y'_x = \frac{X^2Y'_X}{2(XY'_X - Y)}.$$

70. $y'_x = f(ay'^2_x - bx)g(2ay'^3_x - 3by), \qquad b \neq 0.$

The contact transformation $X = a(y'_x)^2 - bx$, $Y = 2a(y'_x)^3 - 3by$, $Y'_X = 3y'_x$ leads to a separable equation: $Y'_X = 3f(X)g(Y)$.

Inverse transformation: $x = \frac{1}{9}b^{-1}[a(Y'_X)^2 - 9X]$, $y = \frac{1}{81}b^{-1}[2a(Y'_X)^3 - 27Y]$, $y'_x = \frac{1}{3}Y'_X$.

71. $y = f(xy'^n_x) + \dfrac{n}{n-1}xy'_x.$

Solution: $y = f(C^n) + \dfrac{nC}{n-1}x^{\frac{n-1}{n}}$.

72. $y = a(n-1)(y'_x)^n + f\bigl(x - an(y'_x)^{n-1}\bigr).$

Differentiating with respect to x, we obtain a factorized equation:

$$\bigl[1 - an(n-1)(y'_x)^{n-2}y''_{xx}\bigr]\bigl[y'_x - f'_t(t)\bigr] = 0, \tag{1}$$

where $t=x-an(y'_x)^{n-1}$. Equate the first factor to zero and integrate the obtained equation. Substituting the expression obtained into the original equation, we find the solution:

$$y = f(C) + a(n-1)\Big(\frac{x-C}{an}\Big)^{\frac{n}{n-1}}.$$

Equating the second factor in (1) to zero, we have another solution that can be written in parametric form as:

$$x = t + an[f'_t(t)]^{n-1}, \quad y = f(t) + a(n-1)[f'_t(t)]^n.$$

73. $\boldsymbol{y'_x = f\big(a(y'_x)^k - bx\big)g\big(ak(y'_x)^{k+1} - b(k+1)y\big).}$

The contact transformation ($ab \neq 0$, $k \neq -1$)

$$X = a(y'_x)^k - bx, \quad Y = ak(y'_x)^{k+1} - b(k+1)y, \quad Y'_X = (k+1)y'_x$$

leads to a separable equation: $Y'_X = (k+1)f(X)g(Y)$.

Inverse transformation:

$$x = \frac{a(Y'_X)^k}{b(k+1)^k} - \frac{X}{b}, \quad y = \frac{ak(Y'_X)^{k+1}}{b(k+1)^{k+2}} - \frac{Y}{b(k+1)}, \quad y'_x = \frac{Y'_X}{k+1}.$$

74. $\boldsymbol{y'_x = f\big(a(y'_x)^k - bx\big) + \big(ak(y'_x)^{k+1} - b(k+1)y\big)g\big(a(y'_x)^k - bx\big).}$

The contact transformation ($ab \neq 0$, $k \neq -1$)

$$X = a(y'_x)^k - bx, \quad Y = ak(y'_x)^{k+1} - b(k+1)y, \quad Y'_X = (k+1)y'_x$$

leads to a linear equation: $Y'_X = (k+1)g(X)Y + (k+1)f(X)$.

Inverse transformation:

$$x = \frac{a(Y'_X)^k}{b(k+1)^k} - \frac{X}{b}, \quad y = \frac{ak(Y'_X)^{k+1}}{b(k+1)^{k+2}} - \frac{Y}{b(k+1)}, \quad y'_x = \frac{Y'_X}{k+1}.$$

75. $\boldsymbol{F\big(yy'_x + ax,\ y\sqrt{y'^2_x + a}\,\big) = 0.}$

Solution: $y^2 = -ax^2 + 2C_1x + C_2$. Here, the constants C_1 and C_2 are related by the constraint

$$F\big(C_1,\ \sqrt{C_1^2 + aC_2}\,\big) = 0 \quad \text{if } y > 0,$$
$$F\big(C_1,\ -\sqrt{C_1^2 + aC_2}\,\big) = 0 \quad \text{if } y < 0.$$

76. $\boldsymbol{F(e^x y'_x - e^x y,\ y'^2_x - y^2) = 0.}$

Solution: $y = C_1e^x + C_2e^{-x}$. Here, the constants C_1 and C_2 are related by the constraint $F(-2C_2, -4C_1C_2) = 0$.

77. $\boldsymbol{F(y'_x \cosh x - y \sinh x,\ y'^2_x - y^2) = 0.}$

Solution: $y = C_1 \sinh x + C_2 \cosh x$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1,\ C_1^2 - C_2^2) = 0$.

78. $\boldsymbol{F(y'_x \cos x + y \sin x,\ y'^2_x + y^2) = 0.}$

Solution: $y = C_1 \sin x + C_2 \cos x$. Here, the constants C_1 and C_2 are related by the constraint $F(C_1,\ C_1^2 + C_2^2) = 0$.

13.8.2 Some Transformations of Equations Not Solved for the Derivative

1. $x = f(y, y'_x)$**.**

Substituting $t = y'_x$ and differentiating both sides of the equation with respect to x, we obtain an equation with respect to $y = y(t)$:

$$[1 - tf_y(y,t)]y'_t = tf_t(y,t), \qquad \text{where} \quad f_t = \tfrac{\partial f}{\partial t}, \quad f_y = \tfrac{\partial f}{\partial y}.$$

If $y = y(t)$ is the solution of the latter equation, the solution of the original equation can be represented in parametric form as:

$$x = f(y(t),t), \quad y = y(t).$$

2. $y = f(x, y'_x)$**.**

Differentiating with respect to x and setting $t = y'_x$, we obtain an equation with respect to $x = x(t)$:

$$[t - f_x(x,t)]x'_t = f_t(x,t), \qquad \text{where} \quad f_t = \tfrac{\partial f}{\partial t}, \quad f_x = \tfrac{\partial f}{\partial x}.$$

If $x = x(t)$ is the solution of the latter equation, the solution of the original equation can be represented in parametric form as:

$$x = x(t), \quad y = f(x(t),t).$$

3. $x^n y^m = f\Big(x^k y^s, \dfrac{xy'_x}{y}\Big)$**.**

Set $z = x^k y^s$ and $w = \dfrac{xy'_x}{y}$. Divide both sides of the equation by $x^n y^m$ and differentiate with respect to x. As a result we arrive at the following equation with respect to $w = w(z)$:

$$z(sw + k)(f_z + f_w w'_z) = (mw + n)f, \qquad \text{where} \quad f = f(z,w),$$

which is usually simpler than the original equation, since it is readily solved for the derivative. If $w = w(z)$ is the solution of the equation obtained, the solution of the original equation is written in parametric form as:

$$x^k y^s = z, \quad x^n y^m = f(z, w(z)).$$

4. $x^n e^{\alpha y} = f(x^m e^{\beta y},\ xy'_x)$**.**

The substitution $y = \ln u$ leads to an equation of the form 13.8.2.3:

$$x^n u^\alpha = f\big(x^m u^\beta,\ xu'_x/u\big).$$

5. $e^{\alpha x} y^n = f(e^{\beta x} y^m,\ y'_x/y)$**.**

The substitution $x = \ln t$ leads to an equation of the form 13.8.2.3: $t^\alpha y^n = f\big(t^\beta y^m,\ ty'_t/y\big)$.

6. $f(x,\ xy'_x - y,\ y'_x) = 0$**.**

The Legendre transformation $x = u'_t$, $y = tu'_t - u$ $(y'_x = t)$, where $u = u(t)$, leads to the equation $f(u'_t, u, t) = 0$. Inverse transformation: $t = y'_x$, $u = xy'_x - y$, $u'_t = x$.

7. $(y'_x)^2 = \lambda y + f(x).$

For $\lambda \neq 0$, the transformation $\lambda w = 2\sqrt{\lambda y + f(x)}$ leads to an Abel equation of the second kind,

$$ww'_x = w + \varphi(x), \qquad \text{where} \quad \varphi = 2\lambda^{-2} f'_x(x),$$

which is outlined in Section 13.3.1 for specific functions φ.

8. $y = xy'_x + ax^2 + f(y'_x), \qquad a \neq 0.$

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w = -2ax$, we arrive at an Abel equation of the second kind,

$$ww'_t = w + \varphi(t), \qquad \text{where} \quad \varphi = -2af'_t(t),$$

which is outlined in Section 13.3.1 for specific functions φ.

◆ *For information about contact transformations, see Section 1.9.*

13.8.3 Equations Defined Parametrically Containing Arbitrary Functions

1. $x = f(t), \quad y'_x = g(t).$

General solution in parametric form:

$$x = f(t), \qquad y = \int f'_t(t)g(t)\,dt + C,$$

where C is an arbitrary constant.

2. $y = f(t), \quad y'_x = g(t).$

General solution in parametric form:

$$x = \int \frac{f'_t(t)}{g(t)}\,dt + C, \qquad y = f(t),$$

where C is an arbitrary constant.

3. $y = f(t)x, \quad y'_x = g(t).$

General solution in parametric form:

$$x = C\exp\left[\int \frac{f'_t(t)\,dt}{g(t) - f(t)}\right], \qquad y = Cf(t)\exp\left[\int \frac{f'_t(t)\,dt}{g(t) - f(t)}\right],$$

where C is an arbitrary constant.

4. $y = f(t)x + g(t), \quad y'_x = h(t).$

General solution in parametric form:

$$x = C\varphi(t) + \varphi(t)\int \frac{g'_t(t)\,dt}{\varphi(t)[h(t) - f(t)]}, \qquad y = f(t)x + g(t),$$

where C is an arbitrary constant and $\varphi(t) = \exp\left[\int \frac{f'_t(t)\,dt}{h(t) - f(t)}\right]$.

5. $\boldsymbol{y = a(t)x + b(t), \quad y'_x = c(t)x + d(t).}$

Differentiating the first equation with respect to x, taking into account that $y'_x = y'_t/x'_t$, and eliminating y'_x with the help of the second equation, we arrive at an Abel equation of the second kind for $x = x(t)$:

$$(cx + d - a)x'_t = a'_t x + b'_t.$$

6. $\boldsymbol{y = f(t)x^k - g(t), \quad y'_x = kf(t)x^{k-1}.}$

General solution:

$$y = f(C)x^k - g(C),$$

where C is an arbitrary constant.

Singular solution in parametric form:

$$x = \left[\frac{g'_t(t)}{f'_t(t)}\right]^{1/k}, \qquad y = f(t)\frac{g'_t(t)}{f'_t(t)} - g(t).$$

7. $\boldsymbol{y = f(t)e^{\lambda(t)x}, \quad y'_x = g(t)e^{\lambda(t)x}.}$

General solution in parametric form:

$$x = CE(t) + E(t)\int \frac{f'_t(t)\,dt}{E(t)[g(t) - f(t)\lambda(t)]}, \qquad y = f(t)e^{\lambda(t)x},$$

$$E(t) = \exp\left[\int \frac{f(t)\lambda'_t(t)\,dt}{g(t) - f(t)\lambda(t)}\right],$$

where C is an arbitrary constant.

8. $\boldsymbol{y = f(x) + g(t), \quad y'_x = f'_x(x) + h(t).}$

General solution in parametric form:

$$x = \int \frac{g'_t(t)}{h(t)}\,dt + C, \qquad y = f(x) + g(t),$$

where C is an arbitrary constant.

9. $\boldsymbol{y = f(t)x^{k+1} + g(t)x, \quad y'_x = (k+1)f(t)x^k + h(t).}$

This equation can be solved using the second technique described in Section 1.8.3. We get

$$dy = \big[(k+1)fx^k + g\big]\,dx + \big(f'_t x^{k+1} + g'_t x\big)\,dt,$$
$$dy = \big[(k+1)fx^k + h\big]\,dx.$$

The first relation is a differential consequence of the first equation (for brevity, the arguments of the functions f, g, and h are omitted). Eliminating dy, we arrive at a Bernoulli equation for $x = x(t)$:

$$(h - g)x'_t = f'_t x^{k+1} + g'_t x.$$

Integrating yields

$$x = \left[C\varphi + k\varphi\int \frac{f'_t\,dt}{\varphi(g - h)}\right]^{-1/k}, \qquad \varphi = \exp\left(k\int \frac{g'_t\,dt}{g - h}\right),$$

where C is an arbitrary constant. This formula and the expression $y = f(t)x^{k+1} + g(t)x$ determine the general solution to the original equation in parametric form.

10. $\boldsymbol{y = f(t)e^{\lambda x} + g(t), \quad y'_x = \lambda f(t)e^{\lambda x} + h(t).}$

This equation can be solved using the second technique described in Section 1.8.3. We get

$$dy = \lambda f e^{\lambda x}\,dx + \bigl(f'_t e^{\lambda x} + g'_t\bigr)\,dt,$$
$$dy = \bigl(\lambda f e^{\lambda x} + h\bigr)\,dx.$$

The first relation is a differential consequence of the first equation (for brevity, the arguments of the functions f, g, and h are omitted). Eliminating dy, we arrive at an equation for $x = x(t)$:

$$hx'_t = f'_t e^{\lambda x} + g'_t.$$

The substitution $u = e^{-\lambda x}$ reduces it to a linear equation. Integrating yields

$$x = \int \frac{g'_t}{h}\,dt - \frac{1}{\lambda}\ln\biggl(C - \lambda \int \frac{f'_t}{Eh}\,dt\biggr), \qquad E = \exp\biggl(-\lambda \int \frac{g'_t}{h}\,dt\biggr),$$

where C is an arbitrary constant. This formula and the expression $y = f(t)e^{\lambda x} + g(t)$ determine the general solution to the original equation in parametric form.

11. $\boldsymbol{y = f(t)g(x) + h(t), \quad y'_x = f(t)g_x(x).}$

General solution:

$$y = f(C)g(x) + h(C),$$

where C is an arbitrary constant.

Also, there is a singular solution, which is defined parametrically as

$$g(x) = -\frac{h'_t(t)}{f'_t(t)}, \qquad y = -\frac{f(t)h'_t(t)}{f'_t(t)} + h(t).$$

12. $\boldsymbol{y = f(x,t), \quad y'_x = f_x(x,t).}$

General solution:

$$y = f(x, C),$$

where C is an arbitrary constant.

13. $\boldsymbol{y = f(x)g(x,y,t), \quad y'_x = h(x)g(x,y,t).}$

General solution:

$$y = C\exp\biggl[\int \frac{h(x)}{f(x)}\,dx\biggr],$$

where C is an arbitrary constant.

The dependence $t = t(x)$ is defined implicitly by the equation $y = f(x)g(x,y,t)$.

14. $\boldsymbol{f(y) = g(x,y,t), \quad y'_x = h(y)g(x,y,t).}$

General solution:

$$x = \int \frac{dy}{f(y)h(y)} + C,$$

where C is an arbitrary constant.

The dependence $t = t(y)$ is defined implicitly by the equation $f(y) = g(x,y,t)$.

15. $\boldsymbol{y = f_1(x) f_2(y) g(x, y, t), \quad y'_x = h_1(x) h_2(y) g(x, y, t).}$

General solution:

$$\int \frac{f_2(y)\,dy}{y h_2(y)} = \int \frac{h_1(x)}{f_1(x)}\,dx + C,$$

where C is an arbitrary constant. The dependence $t = t(x, y)$ is defined implicitly by the equation $y = f_1(x) f_2(y) g(x, y, t)$.

Chapter 14

Second-Order Ordinary Differential Equations

14.1 Linear Equations

14.1.1 Representation of the General Solution through a Particular Solution

A homogeneous linear equation of the second order has the general form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution ($y_0 \not\equiv 0$) of this equation. Then the general solution of equation (1) can be found from the formula:

$$y = y_0\Big(C_1 + C_2 \int \frac{e^{-F}}{y_0^2}\,dx\Big), \qquad \text{where} \quad F = \int \frac{f_1}{f_2}\,dx. \tag{2}$$

For specific equations described below in 14.1.2–14.1.9, often only particular solutions are given, while the general solutions can be obtained with formula (2) (see also Section 2.1.1).

Remark 14.1. Only homogeneous equations are considered in Sections 14.1.2 through 14.1.8; the solutions of the corresponding nonhomogeneous equations can be obtained using formulas in Section 2.2.2.

14.1.2 Equations Containing Power Functions

► **Equations of the form** $y''_{xx} + f(x)y = 0$.

1. $y''_{xx} + ay = 0$.

Equation of free oscillations.

$$\text{Solution:} \quad y = \begin{cases} C_1 \sinh(x\sqrt{|a|}\,) + C_2 \cosh(x\sqrt{|a|}\,) & \text{if } a < 0, \\ C_1 + C_2 x & \text{if } a = 0, \\ C_1 \sin(x\sqrt{a}\,) + C_2 \cos(x\sqrt{a}\,) & \text{if } a > 0. \end{cases}$$

2. $y''_{xx} - (ax+b)y = 0, \qquad a \neq 0.$

The substitution $\xi = a^{-2/3}(ax+b)$ leads to the Airy equation:

$$y''_{\xi\xi} - \xi y = 0, \tag{1}$$

which often arises in various applications. The solution of equation (1) can be written as:

$$y = C_1 \operatorname{Ai}(\xi) + C_2 \operatorname{Bi}(\xi),$$

where $\operatorname{Ai}(\xi)$ and $\operatorname{Bi}(\xi)$ are the Airy functions of the first and second kind, respectively.

The Airy functions admit the following integral representation:

$$\operatorname{Ai}(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + \xi t\right) dt, \quad \operatorname{Bi}(\xi) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\tfrac{1}{3}t^3 + \xi t\right) + \sin\left(\tfrac{1}{3}t^3 + \xi t\right)\right] dt.$$

The Airy functions can be expressed in terms of the Bessel functions and the modified Bessel functions of order $1/3$ by the relations:

$$\operatorname{Ai}(\xi) = \tfrac{1}{3}\sqrt{\xi}\left[I_{-1/3}(z) - I_{1/3}(z)\right], \quad \operatorname{Ai}(-\xi) = \tfrac{1}{3}\sqrt{\xi}\left[J_{-1/3}(z) + J_{1/3}(z)\right],$$
$$\operatorname{Bi}(\xi) = \sqrt{\tfrac{1}{3}\xi}\left[I_{-1/3}(z) + I_{1/3}(z)\right], \quad \operatorname{Bi}(-\xi) = \sqrt{\tfrac{1}{3}\xi}\left[J_{-1/3}(z) - J_{1/3}(z)\right],$$

where $z = \frac{2}{3}\xi^{3/2}$.

For large values of ξ, the leading terms of the asymptotic expansions of the Airy functions are:

$$\operatorname{Ai}(\xi) = \frac{1}{2\sqrt{\pi}}\xi^{-1/4}\exp(-z), \quad \operatorname{Ai}(-\xi) = \frac{1}{\sqrt{\pi}}\xi^{-1/4}\sin\left(z + \frac{\pi}{4}\right),$$
$$\operatorname{Bi}(\xi) = \frac{1}{\sqrt{\pi}}\xi^{-1/4}\exp(z), \qquad \operatorname{Bi}(-\xi) = \frac{1}{\sqrt{\pi}}\xi^{-1/4}\cos\left(z + \frac{\pi}{4}\right).$$

The Airy equation (1) is a special case of equation 14.1.2.7 with $a = n = 1$.

3. $y''_{xx} - (a^2x^2 + a)y = 0.$

Particular solution: $y_0 = \exp\left(\frac{1}{2}ax^2\right)$.

4. $y''_{xx} - (ax^2 + b)y = 0.$

The Weber equation (two canonical forms of the equation correspond to $a = \pm\frac{1}{4}$).

1°. The transformation $z = x^2\sqrt{a}$, $u = e^{z/2}y$ leads to the degenerate hypergeometric equation 14.1.2.70: $zu''_{zz} + \left(\dfrac{1}{2} - z\right)u'_z - \dfrac{1}{4}\left(\dfrac{b}{\sqrt{a}} + 1\right)u = 0.$

2°. For $a = k^2 > 0$, $b = -(2n+1)k$, where $n = 1,\ 2,\ \ldots$, there is a solution of the form:

$$y = \exp\left(-\tfrac{1}{2}kx^2\right)H_n\left(\sqrt{k}\,x\right), \qquad k > 0,$$

where $H_n(z) = (-1)^n \exp\left(z^2\right)\dfrac{d^n}{dz^n}\exp\left(-z^2\right)$ is the Hermite polynomial of order n.

See also Section S3.12.

⊙ *Literature*: H. Bateman and A. Erdélyi (1953, Vol. 2), M. Abramowitz and I. A. Stegun (1964).

5. $y''_{xx} + a^3x(2-ax)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}a^2x^2 + ax\bigr)$.

6. $y''_{xx} - (ax^2 + bx + c)y = 0.$

The substitution $\xi = x + \dfrac{b}{2a}$ leads to an equation of the form 14.1.2.4:

$$y''_{\xi\xi} - \Bigl(a\xi^2 + c - \frac{b^2}{4a}\Bigr)y = 0.$$

7. $y''_{xx} - ax^n y = 0.$

1°. For $n=-2$, this is the Euler equation 14.1.2.123, while for $n=-4$, this is the equation 14.1.2.211 (in both cases the solution is expressed in terms of elementary function).

2°. Assume $2/(n+2) = 2m+1$, where m is an integer. Then the solution is:

$$y = \begin{cases} x(x^{1-2q}D)^{m+1}\Bigl[C_1\exp\Bigl(\dfrac{\sqrt{a}}{q}x^q\Bigr) + C_2\exp\Bigl(-\dfrac{\sqrt{a}}{q}x^q\Bigr)\Bigr] & \text{if } m \ge 0,\\ (x^{1-2q}D)^{-m}\Bigl[C_1\exp\Bigl(\dfrac{\sqrt{a}}{q}x^q\Bigr) + C_2\exp\Bigl(-\dfrac{\sqrt{a}}{q}x^q\Bigr)\Bigr] & \text{if } m < 0,\end{cases}$$

where $D = \dfrac{d}{dx}$, $q = \dfrac{n+2}{2} = \dfrac{1}{2m+1}$.

3°. For any n, the solution is expressed in terms of the Bessel functions and modified Bessel functions of the first or second kind (see 14.1.2.126 and 14.1.2.127):

$$y = \begin{cases} C_1\sqrt{x}\,J_{\frac{1}{2q}}\Bigl(\dfrac{\sqrt{-a}}{q}x^q\Bigr) + C_2\sqrt{x}\,Y_{\frac{1}{2q}}\Bigl(\dfrac{\sqrt{-a}}{q}x^q\Bigr) & \text{if } a < 0,\\ C_1\sqrt{x}\,I_{\frac{1}{2q}}\Bigl(\dfrac{\sqrt{a}}{q}x^q\Bigr) + C_2\sqrt{x}\,K_{\frac{1}{2q}}\Bigl(\dfrac{\sqrt{a}}{q}x^q\Bigr) & \text{if } a > 0,\end{cases}$$

where $q = \frac{1}{2}(n+2)$.

8. $y''_{xx} - a(ax^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(\dfrac{a}{n+1}x^{n+1}\Bigr)$.

9. $y''_{xx} - ax^{n-2}(ax^n + n + 1)y = 0.$

Particular solution: $y_0 = x\exp\bigl(ax^n/n\bigr)$.

10. $y''_{xx} + (ax^{2n} + bx^{n-1})y = 0.$

The substitution $\xi = x^{n+1}$ leads to a linear equation of the form 14.1.2.108: $(n+1)^2\xi y''_{\xi\xi} + n(n+1)y'_\xi + (a\xi + b)y = 0$.

► Equations of the form $y''_{xx} + f(x)y'_x + g(x)y = 0$

11. $y''_{xx} + ay'_x + by = 0.$

Second-order constant coefficient linear equation. In physics this equation is called an *equation of damped vibrations.*

$$\text{Solution: } y = \begin{cases} \exp\bigl(-\frac{1}{2}ax\bigr)\bigl[C_1\exp\bigl(\frac{1}{2}\lambda x\bigr) + C_2\exp\bigl(-\frac{1}{2}\lambda x\bigr)\bigr] & \text{if } \lambda^2 = a^2 - 4b > 0,\\ \exp\bigl(-\frac{1}{2}ax\bigr)\bigl[C_1\sin\bigl(\frac{1}{2}\lambda x\bigr) + C_2\cos\bigl(\frac{1}{2}\lambda x\bigr)\bigr] & \text{if } \lambda^2 = 4b - a^2 > 0,\\ \exp\bigl(-\frac{1}{2}ax\bigr)\bigl(C_1x + C_2\bigr) & \text{if } a^2 = 4b.\end{cases}$$

12. $y''_{xx} + ay'_x + (bx + c)y = 0.$

1°. Solution with $b \neq 0$:

$$y = \exp\bigl(-\tfrac{1}{2}ax\bigr)\sqrt{\xi}\,\bigl[C_1 J_{1/3}\bigl(\tfrac{2}{3}\sqrt{b}\,\xi^{3/2}\bigr) + C_2 Y_{1/3}\bigl(\tfrac{2}{3}\sqrt{b}\,\xi^{3/2}\bigr)\bigr], \qquad \xi = x + \frac{4c - a^2}{4b},$$

where $J_{1/3}(z)$ and $Y_{1/3}(z)$ are Bessel functions.

2°. For $b = 0$, see equation 14.1.2.11.

13. $y''_{xx} + ay'_x - (bx^2 + c)y = 0.$

The substitution $y = w\exp\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$ leads to a linear equation of the form 14.1.2.108: $w''_{xx} + (2\sqrt{b}\,x + a)w'_x + (a\sqrt{b}\,x - c + \sqrt{b})w = 0$.

14. $y''_{xx} + ay'_x + b(-bx^2 + ax + 1)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)$.

15. $y''_{xx} + ay'_x + bx(-bx^3 + ax + 2)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\tfrac{1}{3}bx^3\bigr)$.

16. $y''_{xx} + ay'_x + b(-bx^{2n} + ax^n + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{n+1}x^{n+1}\Bigr)$.

17. $y''_{xx} + ay'_x + b(-bx^{2n} - ax^n + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{n+1}x^{n+1} - ax\Bigr)$.

18. $y''_{xx} + xy'_x + (n + 1)y = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Solution: $y = \dfrac{d^n}{dx^n}\Bigl\{\exp\bigl(-\tfrac{1}{2}x^2\bigr)\Bigl[C_1 + C_2\displaystyle\int \exp\bigl(\tfrac{1}{2}x^2\bigr)\,dx\Bigr]\Bigr\}$.

19. $y''_{xx} - 2xy'_x + 2ny = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Solution: $y = \exp(x^2)\dfrac{d^n}{dx^n}\Bigl\{\exp(-x^2)\Bigl[C_1 + C_2\displaystyle\int \exp(x^2)\,dx\Bigr]\Bigr\}$.

For $C_1 = (-1)^n$ and $C_2 = 0$, this solution defines the Hermite polynomials.

20. $y''_{xx} + axy'_x + by = 0.$

Solution:

$$y = C_1\Phi\bigl(\tfrac{1}{2}a^{-1}b,\ \tfrac{1}{2},\ -\tfrac{1}{2}ax^2\bigr) + C_2\Psi\bigl(\tfrac{1}{2}a^{-1}b,\ \tfrac{1}{2},\ -\tfrac{1}{2}ax^2\bigr),$$

where $\Phi(a, b; x)$ and $\Psi(a, b; x)$ are the degenerate hypergeometric functions (see equation 14.1.2.70 and Section S4.9).

21. $y''_{xx} + axy'_x + bxy = 0.$

Solution:

$$y = e^{-bx/a}\bigl[C_1\Phi\bigl(\tfrac{1}{2}a^{-3}b^2,\ \tfrac{1}{2},\ -\tfrac{1}{2}a\xi^2\bigr) + C_2\Psi\bigl(\tfrac{1}{2}a^{-3}b^2,\ \tfrac{1}{2},\ -\tfrac{1}{2}a\xi^2\bigr)\bigr], \quad \xi = x - 2a^{-2}b,$$

where $\Phi(a, b; x)$ and $\Psi(a, b; x)$ are the degenerate hypergeometric functions (see equation 14.1.2.70 and Section S4.9).

22. $y''_{xx} + axy'_x + (bx + c)y = 0.$

This is a special case of equation 14.1.2.108 with $a_2 = b_1 = 0$ and $b_2 = 1$.

23. $y''_{xx} + 2axy'_x + (bx^4 + a^2x^2 + cx + a)y = 0.$

This is a special case of equation 14.1.2.49 with $n = 1$ and $m = 2$.

24. $y''_{xx} + (ax + b)y'_x + ay = 0.$

Particular solution: $y_0 = \exp\bigl(-\tfrac{1}{2}ax^2 - bx\bigr)$.

25. $y''_{xx} + (ax + b)y'_x - ay = 0.$

Particular solution: $y_0 = ax + b$.

26. $y''_{xx} + (ax + b)y'_x + c(ax + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

27. $y''_{xx} + (ax + 2b)y'_x + (abx - a + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

28. $y''_{xx} + (ax + b)y'_x + (cx + d)y = 0.$

This is a special case of equation 14.1.2.108 with $a_2 = 0$ and $b_2 = 1$.

29. $y''_{xx} + (ax + b)y'_x + c[(a - c)x^2 + bx + 1]y = 0.$

Particular solution: $y_0 = \exp\bigl(-\tfrac{1}{2}cx^2\bigr)$.

30. $y''_{xx} + 2(ax + b)y'_x + (a^2x^2 + 2abx + c)y = 0.$

The substitution $u = y\exp\bigl(\tfrac{1}{2}ax^2 + bx\bigr)$ leads to a constant coefficient linear equation of the form 14.1.2.1: $u''_{xx} + (c - a - b^2)u = 0$.

31. $y''_{xx} + (ax + b)y'_x + (\alpha x^2 + \beta x + \gamma)y = 0.$

The substitution $y = u\exp(sx^2)$, where s is a root of the quadratic equation $4s^2 + 2as + \alpha = 0$, leads to an equation of the form 14.1.2.108:

$$u''_{xx} + [(a + 4s)x + b]u'_x + [(\beta + 2bs)x + \gamma + 2s]u = 0.$$

32. $y''_{xx} + (ax + b)y'_x + c(-cx^{2n} + ax^{n+1} + bx^n + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{n+1}x^{n+1}\Bigr)$.

33. $y''_{xx} + a(x^2 - b^2)y'_x - a(x + b)y = 0.$

Particular solution: $y_0 = x - b$.

34. $y''_{xx} + (ax^2 + b)y'_x + c(ax^2 + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

35. $y''_{xx} + (ax^2 + 2b)y'_x + (abx^2 - ax + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

36. $y''_{xx} + (2x^2 + a)y'_x + (x^4 + ax^2 + 2x + b)y = 0.$

The substitution $u = y\exp\bigl(\frac{1}{3}x^3\bigr)$ leads to a constant coefficient linear equation of the form 14.1.2.11: $u''_{xx} + au'_x + bu = 0$.

37. $y''_{xx} + (ax^2 + bx)y'_x + (\alpha x^2 + \beta x + \gamma)y = 0.$

1°. This is a special case of equation 14.1.2.146 with $n = 1$.

2°. Let $\alpha = 0$, $\beta = 3a$, $\gamma = 2b$. Particular solution: $y_0 = x\exp\bigl(-\frac{1}{3}ax^3 - \frac{1}{2}bx^2\bigr)$.

38. $y''_{xx} + (abx^2 + bx + 2a)y'_x + a^2(bx^2 + 1)y = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

39. $y''_{xx} + (ax^2 + bx + c)y'_x + x(abx^2 + bc + 2a)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{3}ax^3 - cx\bigr)$.

40. $y''_{xx} + (ax^2 + bx + c)y'_x + (abx^3 + acx^2 + b)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}bx^2 - cx\bigr)$.

41. $y''_{xx} + (ax^3 + 2b)y'_x + (abx^3 - ax^2 + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

42. $y''_{xx} + (ax^3 + bx)y'_x + 2(2ax^2 + b)y = 0.$

Particular solution: $y_0 = x\exp\bigl(-\frac{1}{4}ax^4 - \frac{1}{2}bx^2\bigr)$.

43. $y''_{xx} + (abx^3 + bx^2 + 2a)y'_x + a^2(bx^3 + 1)y = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

44. $y''_{xx} + ax^n y'_x = 0.$

This equation is encountered in the theory of diffusion boundary layer.

Solution: $y = C_1 + C_2 \int \exp\Bigl(-\dfrac{ax^{n+1}}{n+1}\Bigr)\,dx.$

45. $y''_{xx} + ax^n y'_x + bx^{n-1}y = 0.$

For $n = -1$, we obtain the Euler equation 14.1.2.123. For $n \neq -1$, the substitution $z = x^{n+1}$ leads to an equation of the form 14.1.2.108: $(n+1)^2 zy''_{zz} + (n+1)(az+n)y'_z + by = 0$.

46. $y''_{xx} + 2ax^n y'_x + a(ax^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

47. $y''_{xx} + ax^n y'_x + (bx^{2n} + cx^{n-1})y = 0.$

The substitution $\xi = x^{n+1}$ leads to a linear equation of the form 14.1.2.108: $(n+1)^2\xi y''_{\xi\xi} + (n+1)(a\xi + n)y'_\xi + (b\xi + c)y = 0$.

48. $y''_{xx} + ax^n y'_x - b(ax^{n+m} + bx^{2m} + mx^{m-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(\dfrac{b}{m+1}x^{m+1}\Bigr)$.

49. $y''_{xx} + 2ax^n y'_x + (a^2x^{2n} + bx^{2m} + anx^{n-1} + cx^{m-1})y = 0.$

The substitution $w = y\exp\Big(\dfrac{a}{n+1}x^{n+1}\Big)$ leads to a linear equation of the form 14.1.2.10: $w''_{xx} + (bx^{2m} + cx^{m-1})w = 0$.

50. $y''_{xx} + (ax^n + b)y'_x + c(ax^n + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

51. $y''_{xx} + (ax^n + 2b)y'_x + (abx^n - ax^{n-1} + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

52. $y''_{xx} + (abx^n + bx^{n-1} + 2a)y'_x + a^2(bx^n + 1)y = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}$.

53. $y''_{xx} + (abx^n + 2bx^{n-1} - a^2x)y'_x + a(abx^n + bx^{n-1} - a^2x)y = 0.$

Particular solution: $y_0 = (ax + 2)e^{-ax}$.

54. $y''_{xx} + x^n[ax^2 + (ac + b)x + bc]y'_x - x^n(ax + b)y = 0.$

Particular solution: $y_0 = x + c$.

55. $y''_{xx} + (ax^n + bx^m)y'_x - (ax^{n-1} + bx^{m-1})y = 0.$

Particular solution: $y_0 = x$.

56. $y''_{xx} + (ax^n + bx^m)y'_x + (anx^{n-1} + bmx^{m-1})y = 0.$

Integrating yields a first-order linear equation: $y'_x + (ax^n + bx^m)y = C$.

57. $y''_{xx} + (ax^n + bx^m)y'_x + [a(n+1)x^{n-1} + b(m+1)x^{m-1}]y = 0.$

Particular solution: $y_0 = x\exp\Big(-\dfrac{a}{n+1}x^{n+1} - \dfrac{b}{m+1}x^{m+1}\Big)$.

58. $y''_{xx} + (ax^n + bx^m)y'_x + c(ax^n + bx^m - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

59. $y''_{xx} + (ax^n + bx^m)y'_x + [abx^{m+n} + b(m+1)x^{m-1} - ax^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Big(-\dfrac{b}{m+1}x^{m+1}\Big)$.

60. $y''_{xx} + (ax^n + bx^m + c)y'_x + (abx^{m+n} + bcx^m + anx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{a}{n+1}x^{n+1} - cx\Big)$.

► **Equations of the form** $(ax + b)y''_{xx} + f(x)y'_x + g(x)y = 0.$

61. $xy''_{xx} + \frac{1}{2}y'_x + ay = 0.$

Solution: $y = \begin{cases} C_1\cos\sqrt{4ax} + C_2\sin\sqrt{4ax} & \text{if } ax > 0, \\ C_1\cosh\sqrt{4|ax|} + C_2\sinh\sqrt{4|ax|} & \text{if } ax < 0. \end{cases}$

62. $xy''_{xx} + ay'_x + by = 0.$

1°. The solution is expressed in terms of Bessel functions:

$$y = x^{\frac{1-a}{2}}\big[C_1 J_\nu\big(2\sqrt{bx}\,\big) + C_2 Y_\nu\big(2\sqrt{bx}\,\big)\big], \qquad \text{where} \quad \nu = |1-a|.$$

2°. For $a = \frac{1}{2}(2n+1)$, where $n = 0,\ 1,\ \ldots$, the solution is:

$$y = \begin{cases} C_1 \dfrac{d^n}{dx^n}\cos\sqrt{4bx} + C_2 \dfrac{d^n}{dx^n}\sin\sqrt{4bx} & \text{if } bx > 0, \\ C_1 \dfrac{d^n}{dx^n}\cosh\sqrt{4|bx|} + C_2 \dfrac{d^n}{dx^n}\sinh\sqrt{4|bx|} & \text{if } bx < 0. \end{cases}$$

63. $xy''_{xx} + ay'_x + bxy = 0.$

1°. The solution is expressed in terms of Bessel functions:

$$y = x^{\frac{1-a}{2}}\big[C_1 J_\nu\big(\sqrt{b}\,x\big) + C_2 Y_\nu\big(\sqrt{b}\,x\big)\big], \qquad \text{where} \quad \nu = \tfrac{1}{2}|1-a|.$$

2°. For $a = 2n$, where $n = 1,\ 2,\ \ldots$, the solution is:

$$y = \begin{cases} C_1\Big(\dfrac{1}{x}\dfrac{d}{dx}\Big)^n \cos\big(x\sqrt{b}\,\big) + C_2\Big(\dfrac{1}{x}\dfrac{d}{dx}\Big)^n \sin\big(x\sqrt{b}\,\big) & \text{if } b > 0, \\ C_1\Big(\dfrac{1}{x}\dfrac{d}{dx}\Big)^n \cosh\big(x\sqrt{-b}\,\big) + C_2\Big(\dfrac{1}{x}\dfrac{d}{dx}\Big)^n \sinh\big(x\sqrt{-b}\,\big) & \text{if } b < 0. \end{cases}$$

64. $xy''_{xx} + ay'_x + (bx+c)y = 0.$

This is a special case of equation 14.1.2.108 with $a_2 = 1$ and $a_1 = b_2 = 0$.

65. $xy''_{xx} + ny'_x + bx^{1-2n}y = 0.$

For $n = 1$, this is the Euler equation 14.1.2.123. For $n \neq 1$, the solution is:

$$y = \begin{cases} C_1 \sin\Big(\dfrac{\sqrt{b}}{n-1}x^{1-n}\Big) + C_2\cos\Big(\dfrac{\sqrt{b}}{n-1}x^{1-n}\Big) & \text{if } b > 0, \\ C_1 \exp\Big(\dfrac{\sqrt{-b}}{n-1}x^{1-n}\Big) + C_2\exp\Big(\dfrac{-\sqrt{-b}}{n-1}x^{1-n}\Big) & \text{if } b < 0. \end{cases}$$

66. $xy''_{xx} + (1-3n)y'_x - a^2n^2x^{2n-1}y = 0.$

Solution: $y = C_1(ax^n+1)\exp(-ax^n) + C_2(-ax^n+1)\exp(ax^n).$

67. $xy''_{xx} + ay'_x + bx^n y = 0.$

If $n = -1$ and $b = 0$, we have the Euler equation 14.1.2.123. If $n \neq -1$ and $b \neq 0$, the solution is expressed in terms of Bessel functions:

$$y = x^{\frac{1-a}{2}}\bigg[C_1 J_\nu\Big(\frac{2\sqrt{b}}{n+1}x^{\frac{n+1}{2}}\Big) + C_2 Y_\nu\Big(\frac{2\sqrt{b}}{n+1}x^{\frac{n+1}{2}}\Big)\bigg], \qquad \text{where} \quad \nu = \frac{|1-a|}{n+1}.$$

68. $xy''_{xx} + ay'_x + bx^n(-bx^{n+1} + a + n)y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{n+1}x^{n+1}\Big).$

69. $xy''_{xx} + axy'_x + ay = 0.$

Particular solution: $y_0 = xe^{-ax}$.

70. $xy''_{xx} + (b - x)y'_x - ay = 0.$

The degenerate hypergeometric equation.

1°. If $b \neq 0, -1, -2, -3, \dots$, Kummer's series is a particular solution:

$$\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

where $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$. If $b > a > 0$, this solution can be written in terms of a definite integral:

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1}\, dt,$$

where $\Gamma(z) = \displaystyle\int_0^{\infty} e^{-t} t^{z-1}\, dt$ is the gamma function.

Table S4.1 (see Section S4.9) gives some special cases where Φ is expressed in terms of simpler functions.

If b is not an integer, then the general solution has the form:

$$y = C_1 \Phi(a, b; x) + C_2 x^{1-b} \Phi(a - b + 1,\ 2 - b;\ x).$$

The function Φ possesses the properties:

$$\Phi(a, b; x) = e^x \Phi(b - a,\ b;\ -x); \qquad \frac{d^n}{dx^n} \Phi(a, b; x) = \frac{(a)_n}{(b)_n} \Phi(a + n,\ b + n;\ x).$$

The following asymptotic relations hold:

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left[1 + O\Big(\frac{1}{|x|}\Big)\right] \qquad \text{if } x \to +\infty,$$

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(b-a)} (-x)^{-a} \left[1 + O\Big(\frac{1}{|x|}\Big)\right] \quad \text{if } x \to -\infty.$$

2°. The following function is a solution of the degenerate hypergeometric equation:

$$\Psi(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \Phi(a, b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} \Phi(a - b + 1,\ 2 - b;\ x).$$

Calculate the limit as $b \to n$ (n is an integer) to obtain

$$\begin{aligned} \Psi(a, n+1; x) = \frac{(-1)^{n-1}}{n!\,\Gamma(a-n)} \Bigg\{ & \Phi(a, n{+}1; x) \ln x \\ & + \sum_{r=0}^{\infty} \frac{(a)_r}{(n+1)_r} \big[\psi(a+r) - \psi(1+r) - \psi(1+n+r)\big] \frac{x^r}{r!} \Bigg\} \\ & + \frac{(n-1)!}{\Gamma(a)} \sum_{r=0}^{n-1} \frac{(a-n)_r}{(1-n)_r} \frac{x^{r-n}}{r!}, \end{aligned}$$

where $n=0, 1, 2, \ldots$ (the last sum is omitted for $n=0$), $\psi(z)=[\ln\Gamma(z)]'_z$ is the logarithmic derivative of the gamma function:

$$\psi(1)=-\gamma, \quad \psi(n)=-\gamma+\sum_{k=1}^{n-1}k^{-1}, \quad \gamma=0.5772\ldots \text{ is the Euler constant.}$$

Table S4.2 (see Section S4.9) gives some special cases where Ψ is expressed in terms of simpler functions.

If b is a negative number, then the function Ψ can be expressed in terms of the one with a positive second argument using the relation

$$\Psi(a,b;x)=x^{1-b}\Psi(a-b+1,\ 2-b;\ x),$$

which holds for any value of x.

3°. For $b\neq 0, -1, -2, -3, \ldots$, the general solution of the degenerate hypergeometric equation can be written in the form:

$$y=C_1\Phi(a,b;x)+C_2\Psi(a,b;x),$$

while for $b=0, -1, -2, -3, \ldots$, it can be represented as:

$$y=x^{1-b}\big[C_1\Phi(a-b+1,\ 2-b;\ x)+C_2\Psi(a-b+1,\ 2-b;\ x)\big].$$

The functions Φ and Ψ are described in Section S4.9 in more detail; see also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 1).

71. $xy''_{xx}+(ax+b)y'_x+c[(a-c)x+b]y=0.$

Particular solution: $y_0=e^{-cx}$.

72. $xy''_{xx}+(2ax+b)y'_x+a(ax+b)y=0.$

Solution: $y=\begin{cases} e^{-ax}(C_1+C_2x^{1-b}) & \text{if } b\neq 1,\\ e^{-ax}(C_1+C_2\ln|x|) & \text{if } b=1.\end{cases}$

73. $xy''_{xx}+[(a+b)x+n+m]y'_x+(abx+an+bm)y=0.$

Here, n and m are positive integers; $a\neq b$ or $n\neq m$.

Solution: $y=C_1e^{-ax}\dfrac{d^{m-1}}{dx^{m-1}}\big[x^{-n}e^{(a-b)x}\big]+C_2e^{-bx}\dfrac{d^{n-1}}{dx^{n-1}}\big[x^{-m}e^{(b-a)x}\big].$

74. $xy''_{xx}+(ax+b)y'_x+(cx+d)y=0.$

This is a special case of equation 14.1.2.108.

75. $xy''_{xx}-(ax+1)y'_x-bx^2(bx+a)y=0.$

Particular solution: $y_0=\exp\big(-\frac{1}{2}bx^2\big)$.

76. $xy''_{xx}-(2ax+1)y'_x+(bx^3+a^2x+a)y=0.$

Solution: $y=e^{ax}\Big[C_1\sin\big(\frac{1}{2}x^2\sqrt{b}\big)+C_2\cos\big(\frac{1}{2}x^2\sqrt{b}\big)\Big].$

77. $xy''_{xx} + (ax + b)y'_x + cx(-cx^2 + ax + b + 1) = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}cx^2\bigr)$.

78. $xy''_{xx} - (2ax^2 + 1)y'_x + bx^3 y = 0.$

Solution: $y = C_1 \exp\bigl[\frac{1}{2}\bigl(a + \sqrt{a^2 - b}\,\bigr)x^2\bigr] + C_2 \exp\bigl[\frac{1}{2}\bigl(a - \sqrt{a^2 - b}\,\bigr)x^2\bigr]$.

79. $xy''_{xx} + (abx^2 + b - 5)y'_x + 2a^2(b - 2)x^3 y = 0.$

Particular solution: $y_0 = (ax^2 + 1)\exp(-ax^2)$.

80. $xy''_{xx} + (ax^2 + bx)y'_x - [acx^2 + (a + bc + c^2)x + b + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

81. $xy''_{xx} + (ax^2 + bx + 2)y'_x + by = 0.$

Particular solution: $y_0 = a + b/x$.

82. $xy''_{xx} + (ax^2 + bx + c)y'_x + (2ax + b)y = 0.$

Integrating, we obtain a first-order linear equation: $xy'_x + (ax^2 + bx + c - 1)y = C$.

83. $xy''_{xx} + (ax^2 + bx + c)y'_x + (c - 1)(ax + b)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

84. $xy''_{xx} + (ax^2 + bx + c)y'_x + (Ax^2 + Bx + C)y = 0.$

1°. Let $A = ak$, $B = k(b - k)$, $C = ck$, where k is an arbitrary number.
Particular solution: $y_0 = e^{-kx}$.

2°. Let $A = a(b + k)$, $B = a(c + 1) - k(b + k)$, $C = -ck$.
Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}ax^2 + kx\bigr)$.

3°. Let $A = a(b + k)$, $B = 2a - bk - k^2$, $C = b(c - 1) + k(c - 2)$.
Particular solution: $y_0 = x^{1-c}\exp\bigl(-\frac{1}{2}ax^2 + kx\bigr)$.

4°. Let $A = -ak$, $B = a(c - 1) - k(b + k)$, $C = b(c - 1) + k(c - 2)$.
Particular solution: $y_0 = x^{1-c}e^{kx}$.

85. $xy''_{xx} + (ax^2 + bx + 2)y'_x + (cx^2 + dx + b)y = 0.$

The substitution $w = xy$ leads to a linear equation of the form 14.1.2.108:

$$w''_{xx} + (ax + b)w'_x + (cx + d - a)w = 0.$$

86. $xy''_{xx} + (ax^3 + b)y'_x + a(b - 1)x^2 y = 0.$

Particular solution: $y_0 = x^{1-b}$.

87. $xy''_{xx} + x(ax^2 + b)y'_x + (3ax^2 + b)y = 0.$

Particular solution: $y_0 = x\exp\bigl(-\frac{1}{3}ax^3 - bx\bigr)$.

88. $xy''_{xx} + (ax^3 + bx^2 + 2)y'_x + bxy = 0.$

Particular solution: $y_0 = a + b/x$.

89. $xy''_{xx} + (abx^3 + bx^2 + ax - 1)y'_x + a^2bx^3 y = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}$.

90. $xy''_{xx} + (ax^3 + bx^2 + cx + d)y'_x + (d-1)(ax^2 + bx + c)y = 0.$

Particular solution: $y_0 = x^{1-d}$.

91. $xy''_{xx} + ax^n y'_x + (abx^n - ax^{n-1} - b^2x + 2b)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

92. $xy''_{xx} + (ax^n + 2)y'_x + ax^{n-1}y = 0.$

Particular solution: $y_0 = x^{-1}$.

93. $xy''_{xx} + (x^n + 1 - n)y'_x + bx^{2n-1}y = 0.$

1°. For $b \neq \frac{1}{4}$, the solution has the form: $y = C_1 \exp(\beta_1 x^n) + C_2 \exp(\beta_2 x^n)$. Here, β_1 and β_2 are roots of the quadratic equation: $n^2\beta^2 + n\beta + b = 0$.

2°. For $b = \frac{1}{4}$, the solution has the form: $y = (C_1 + C_2 x^n)\exp\left(-\frac{1}{2}n^{-1}x^n\right)$.

94. $xy''_{xx} + (ax^n + b)y'_x + anx^{n-1}y = 0.$

Particular solution: $y_0 = x^{1-b}\exp(-ax^n/n)$.

95. $xy''_{xx} + (ax^n + b)y'_x + a(b-1)x^{n-1}y = 0.$

Particular solution: $y_0 = x^{1-b}$.

96. $xy''_{xx} + (ax^n + b)y'_x + a(b + n - 1)x^{n-1}y = 0.$

Particular solution: $y_0 = \exp(-ax^n/n)$.

97. $xy''_{xx} + (ax^n + b)y'_x + c(ax^n - cx + b)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

98. $xy''_{xx} + (abx^n + b - 3n + 1)y'_x + a^2n(b-n)x^{2n-1}y = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n)$.

99. $xy''_{xx} + (ax^n + b)y'_x + (cx^{2n-1} + dx^{n-1})y = 0.$

This is a special case of equation 14.1.2.146 with $\gamma = 0$.

100. $xy''_{xx} + (ax^n + bx^{n-1} + 2)y'_x + bx^{n-2}y = 0.$

Particular solution: $y_0 = a + b/x$.

101. $xy''_{xx} + (ax^n + bx)y'_x + (abx^n + anx^{n-1} - b)y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

102. $xy''_{xx} + (abx^n + bx^{n-1} + ax - 1)y'_x + a^2bx^n y = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}$.

103. $xy''_{xx} + (ax^n + bx^m + c)y'_x + (c-1)(ax^{n-1} + bx^{m-1})y = 0.$

Particular solution: $y_0 = x^{1-c}$.

104. $xy''_{xx} + (abx^{n+m} + anx^n + bx^m + 1 - 2n)y'_x + a^2bnx^{2n+m-1}y = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n)$.

105. $(x+a)y''_{xx}+(bx+c)y'_x+by=0.$

Particular solution: $y_0=\exp\Bigl(-\int\frac{bx+c-1}{x+a}\,dx\Bigr).$

106. $(a_1x+a_0)y''_{xx}+(b_1x+b_0)y'_x-mb_1y=0.$

If $m=1,\,2,\,3,\,\dots$, a polynomial of order m in x is a particular solution of the equation, which can be represented as: $y_0=\sum_{k=0}^{m}\Bigl(-\frac{1}{b_1}\Bigr)^k\bigl\{x^mIx^{-m-1}[(a_1x+a_0)D^2+b_0D]\bigr\}^kx^m$, where $D=\frac{d}{dx}$, $Ix^\nu=\frac{x^{\nu+1}}{\nu+1}$ with $\nu\neq-1$.

107. $(ax+b)y''_{xx}+s(cx+d)y'_x-s^2[(a+c)x+b+d]y=0.$

Particular solution: $y_0=e^{sx}$.

108. $(a_2x+b_2)y''_{xx}+(a_1x+b_1)y'_x+(a_0x+b_0)y=0.$

Let the function $\mathcal{J}(a,b;x)$ be an arbitrary solution of the degenerate hypergeometric equation $xy''_{xx}+(b-x)y'_x-ay=0$ (see 14.1.2.70), and the function $Z_\nu(x)$ be an arbitrary solution of the Bessel equation $x^2y''_{xx}+xy'_x+(x^2-\nu^2)y=0$ (see 14.1.2.126). The results of solving the original equation are presented in Table 14.1.

TABLE 14.1
Solutions of equation 14.1.2.108 for different values of the determining parameters

Solution: $y=e^{kx}w(z)$, where $z=\frac{x-\mu}{\lambda}$					
Constraints	k	λ	μ	w	Parameters
$a_2\neq0$, $a_1^2\neq4a_0a_2$	$\frac{\sqrt{D}-a_1}{2a_2}$	$-\frac{a_2}{2a_2k+a_1}$	$-\frac{b_2}{a_2}$	$\mathcal{J}(a,b;z)$	$a=B(k)/(2a_2k+a_1)$, $b=(a_2b_1-a_1b_2)a_2^{-2}$
$a_2=0$, $a_1\neq0$	$-\frac{a_0}{a_1}$	1	$-\frac{2b_2k+b_1}{a_1}$	$\mathcal{J}\bigl(a,\frac{1}{2};\beta z^2\bigr)$	$a=B(k)/(2a_1)$, $\beta=-a_1/(2b_2)$
$a_2\neq0$, $a_1^2=4a_0a_2$	$-\frac{a_1}{2a_2}$	a_2	$-\frac{b_2}{a_2}$	$z^{\nu/2}Z_\nu\bigl(\beta\sqrt{z}\bigr)$	$\nu=1-(2b_2k+b_1)a_2^{-1}$, $\beta=2\sqrt{B(k)}$
$a_2=a_1=0$, $a_0\neq0$	$-\frac{b_1}{2b_2}$	1	$\frac{b_1^2-4b_0b_2}{4a_0b_2}$	$z^{1/2}Z_{1/3}(\beta z^{3/2})$ see also 14.1.2.12	$\beta=\frac{2}{3}\Bigl(\frac{a_0}{b_2}\Bigr)^{1/2}$
Notation: $D=a_1^2-4a_0a_2$, $B(k)=b_2k^2+b_1k+b_0$					

109. $(x+\lambda)y''_{xx}+(ax^n+bx^m+c)y'_x+(anx^{n-1}+bmx^{m-1})y=0.$

Particular solution: $y_0=\exp\Bigl(-\int\frac{ax^n+bx^m+c-1}{x+\lambda}\,dx\Bigr).$

► **Equations of the form** $x^2y''_{xx}+f(x)y'_x+g(x)y=0$.

110. $x^2y''_{xx}+ay=0.$

This is a special case of equation 14.1.2.123. The substitution $x=e^t$ leads to a constant coefficient linear equation: $y''_{tt}-y'_t+ay=0$.

111. $x^2y''_{xx}+(ax+b)y=0.$

This is a special case of equation 14.1.2.132.

112. $x^2y''_{xx}+[a^2x^2-n(n+1)]y=0,\qquad n=0,\ 1,\ 2,\ \dots$

Solution: $yx^{n+1}=(x^3D)^n\Bigl(\dfrac{C_1\cos ax+C_2\sin ax}{x^{2n-1}}\Bigr)$, where $D=\dfrac{d}{dx}$.

113. $x^2y''_{xx}-[a^2x^2+n(n+1)]y=0,\qquad n=0,\ 1,\ 2,\ \dots$

Solution: $yx^{n+1}=(x^3D)^n\Bigl(\dfrac{C_1e^{ax}+C_2e^{-ax}}{x^{2n-1}}\Bigr)$, where $D=\dfrac{d}{dx}$.

114. $x^2y''_{xx}-(a^2x^2+2abx+b^2-b)y=0.$

Particular solution: $y_0=x^be^{ax}$.

115. $x^2y''_{xx}+(ax^2+bx+c)y=0.$

The substitution $y=x^\lambda u$, where λ is a root of the quadratic equation $\lambda^2-\lambda+c=0$, leads to an equation of the form 14.1.2.108: $xu''_{xx}+2\lambda u'_x+(ax+b)u=0$.

For $a=-\frac{1}{4}$, $b=k$, and $c=\frac{1}{4}-m^2$, the original equation is referred to as *Whittaker's equation*.

116. $x^2y''_{xx}-\bigl(ax^3+\frac{5}{16}\bigr)y=0.$

Particular solution: $y_0=x^{-1/4}\exp\bigl(\frac{2}{3}\sqrt{a}\,x^{3/2}\bigr)$.

117. $x^2y''_{xx}-[a^2x^4+a(2b-1)x^2+b(b+1)]y=0.$

Particular solution: $y_0=x^{-b}\exp\bigl(-\frac{1}{2}ax^2\bigr)$.

118. $x^2y''_{xx}+(ax^n+b)y=0.$

This is a special case of equation 14.1.2.132.

119. $x^2y''_{xx}-[a^2x^{2n}+a(2b+n-1)x^n+b(b-1)]y=0.$

Particular solution: $y_0=x^b\exp(ax^n/n)$.

120. $x^2y''_{xx}+(ax^{2n}+bx^n+c)y=0.$

This is a special case of equation 14.1.2.146.

121. $x^2y''_{xx}+\bigl(ax^{3n}+bx^{2n}+\frac{1}{4}-\frac{1}{4}n^2\bigr)y=0.$

The transformation $\xi=ax^n+b$, $w=yx^{\frac{n-1}{2}}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi}+(an)^{-2}\xi w=0$.

122. $x^2y''_{xx}+\bigl[ax^{2n}(bx^n+c)^m+\frac{1}{4}-\frac{1}{4}n^2\bigr]y=0.$

The transformation $\xi=bx^n+c$, $w=yx^{\frac{n-1}{2}}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi}+a(bn)^{-2}\xi^mw=0$.

123. $x^2y''_{xx} + axy'_x + by = 0.$

The Euler equation. Solution:

$$y = \begin{cases} |x|^{\frac{1-a}{2}}\left(C_1|x|^{\mu} + C_2|x|^{-\mu}\right) & \text{if } (1-a)^2 > 4b, \\ |x|^{\frac{1-a}{2}}\left(C_1 + C_2\ln|x|\right) & \text{if } (1-a)^2 = 4b, \\ |x|^{\frac{1-a}{2}}\left[C_1\sin(\mu\ln|x|) + C_2\cos(\mu\ln|x|)\right] & \text{if } (1-a)^2 < 4b, \end{cases}$$

where $\mu = \frac{1}{2}|(1-a)^2 - 4b|^{1/2}$.

124. $x^2y''_{xx} + xy'_x + \left[x^2 - \left(n + \frac{1}{2}\right)^2\right]y = 0, \qquad n = 0,\ 1,\ 2,\ \ldots$

This is a special case of equation 14.1.2.126.

Solution: $y = x^{n+1/2}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(C_1\frac{\sin x}{x} + C_2\frac{\cos x}{x}\right)\right].$

125. $x^2y''_{xx} + xy'_x - \left[x^2 + \left(n + \frac{1}{2}\right)^2\right]y = 0, \qquad n = 0,\ 1,\ 2,\ \ldots$

This is a special case of equation 14.1.2.127.

Solution: $y = x^{n+1/2}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(C_1\frac{e^x}{x} + C_2\frac{e^{-x}}{x}\right)\right].$

126. $x^2y''_{xx} + xy'_x + (x^2 - \nu^2)y = 0.$

The Bessel equation.

1°. Let ν be an arbitrary noninteger. Then the general solution is given by:

$$y = C_1J_\nu(x) + C_2Y_\nu(x), \tag{1}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind:

$$J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \qquad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}. \tag{2}$$

Solution (1) is denoted by $y = Z_\nu(x)$ which is referred to as the cylindrical function.

The cylindrical functions possess the following properties:

$$2\nu Z_\nu(x) = x[Z_{\nu-1}(x) + Z_{\nu+1}(x)],$$

$$\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \qquad \frac{d}{dx}[x^{-\nu}Z_\nu(x)] = -x^{-\nu}Z_{\nu+1}(x).$$

The functions $J_\nu(x)$ and $Y_\nu(x)$ can be expressed in terms of definite integrals (with $x > 0$):

$$\pi J_\nu(x) = \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu\int_0^\infty \exp(-x\sinh t - \nu t)\,dt,$$

$$\pi Y_\nu(x) = \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu)e^{-x\sinh t}\,dt.$$

2°. In the case $\nu = n + \frac{1}{2}$, where $n = 0,\ 1,\ 2,\ \ldots$, the Bessel functions are expressed in terms of elementary functions:

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\,x^{n+\frac{1}{2}}\left(-\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x}, \qquad J_{-n-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\,x^{n+\frac{1}{2}}\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x},$$

$$Y_{n+\frac{1}{2}}(x) = (-1)^{n+1}J_{-n-\frac{1}{2}}(x).$$

3°. Let $\nu = n$ be an arbitrary integer. The following relations hold:

$$J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x).$$

The solution is given by formula (1) in which the function $J_n(x)$ is obtained by substituting $\nu = n$ into formula (2), while $Y_n(x)$ is found by taking the limit as $\nu \to n$ and for nonnegative n becomes

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln \frac{x}{2} - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{2}{x}\right)^{n-2k}$$
$$- \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{\psi(k+1) + \psi(n+k+1)}{k!\,(n+k)!},$$

where $\psi(1) = -\mathcal{C}$, $\psi(n) = -\mathcal{C} + \sum\limits_{k=1}^{n-1} k^{-1}$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant, $\psi(x) = [\ln \Gamma(x)]'_x$ is the logarithmic derivative of the gamma function.

For nonnegative integer n and large x, we can write

$$\sqrt{\pi x}\, J_{2n}(x) = (-1)^n (\cos x + \sin x) + O(x^{-2}),$$
$$\sqrt{\pi x}\, J_{2n+1}(x) = (-1)^{n+1} (\cos x - \sin x) + O(x^{-2}).$$

The function $J_n(x)$ can be expressed in terms of a definite integral:

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin t - nt)\, dt; \qquad n = 0, 1, 2, \ldots$$

The Bessel functions are described in Section S4.6 in more detail; see also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 2).

127. $x^2 y''_{xx} + x y'_x - (x^2 + \nu^2) y = 0.$

The modified Bessel equation. It can be reduced to equation 14.1.2.126 by means of the substitution $x = i\bar{x}$ $(i^2 = -1)$.

Solution:

$$y = C_1 I_\nu(x) + C_2 K_\nu(x),$$

where $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions of the first and second kind:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma(\nu + k + 1)}, \qquad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}.$$

The modified Bessel function $I_\nu(x)$ can be expressed in terms of the Bessel function:

$$I_\nu(x) = e^{-\pi\nu i/2} J_\nu(x e^{\pi i/2}), \quad i^2 = -1.$$

The case $\nu = n + \frac{1}{2}$, where $n = 0, 1, 2, \ldots$, is given in 14.1.2.125.

If $\nu = n$ is a nonnegative integer, we have

$$K_n(x) = (-1)^{n+1} I_n(x) \ln \frac{x}{2} + \frac{1}{2} \sum_{m=0}^{n-1} (-1)^m \left(\frac{x}{2}\right)^{2m-n} \frac{(n-m-1)!}{m!}$$
$$+ \frac{1}{2} (-1)^n \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{n+2m} \frac{\psi(n+m+1) + \psi(m+1)}{m!\,(n+m)!},$$

where $\psi(z)$ is the logarithmic derivative of the gamma function (see 14.1.2.126, Item 3°); for $n=0$, the first sum is omitted.

As $x \to +\infty$, the leading terms of the asymptotic expansion are:

$$I_\nu(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \quad K_\nu(x) \simeq \frac{\sqrt{\pi}}{\sqrt{2x}} e^{-x}.$$

The modified Bessel functions are described in Section S4.7 in more detail; see also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 2).

128. $x^2 y''_{xx} + 2xy'_x - (a^2x^2 + 2)y = 0.$

Solution: $x^2 y = C_1(ax-1)e^{ax} + C_2(ax+1)e^{-ax}$.

129. $x^2 y''_{xx} - 2axy'_x + [b^2x^2 + a(a+1)]y = 0.$

Solution: $y = \begin{cases} |x|^a (C_1 \sin bx + C_2 \cos bx) & \text{if } b \neq 0, \\ C_1|x|^a + C_2|x|^{a+1} & \text{if } b = 0. \end{cases}$

130. $x^2 y''_{xx} - 2axy'_x + [-b^2x^2 + a(a+1)]y = 0.$

Solution: $y = \begin{cases} |x|^a (C_1 e^{bx} + C_2 e^{-bx}) & \text{if } b \neq 0, \\ C_1|x|^a + C_2|x|^{a+1} & \text{if } b = 0. \end{cases}$

131. $x^2 y''_{xx} + \lambda xy'_x + (ax^2 + bx + c)y = 0.$

The substitution $y = x^k u$, where k is a root of the quadratic equation $k^2 + (\lambda-1)k + c = 0$, leads to an equation of the form 14.1.2.108: $xu''_{xx} + (\lambda + 2k)u'_x + (ax+b)u = 0$.

132. $x^2 y''_{xx} + axy'_x + (bx^n + c)y = 0, \qquad n \neq 0.$

The case $b = 0$ corresponds to the Euler equation 14.1.2.123.

For $b \neq 0$, the solution is:

$$y = x^{\frac{1-a}{2}} \left[C_1 J_\nu\left(\frac{2}{n}\sqrt{b}\, x^{\frac{n}{2}}\right) + C_2 Y_\nu\left(\frac{2}{n}\sqrt{b}\, x^{\frac{n}{2}}\right) \right],$$

where $\nu = \frac{1}{n}\sqrt{(1-a)^2 - 4c}$; $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kind.

133. $x^2 y''_{xx} + axy'_x + x^n(bx^n + c)y = 0.$

The substitution $\xi = x^n$ leads to an equation of the form 14.1.2.108:

$$n^2 \xi y''_{\xi\xi} + n(n-1+a)y'_\xi + (b\xi + c)y = 0.$$

134. $x^2 y''_{xx} + (ax+b)y'_x + cy = 0.$

The transformation $x = z^{-1}$, $y = z^k e^z w$, where k is a root of the quadratic equation $k^2 + (1-a)k + c = 0$, leads to an equation of the form 14.1.2.108:

$$zw''_{zz} + [(2-b)z + 2k + 2 - a]w'_z + [(1-b)z + 2k + 2 - a - bk]w = 0.$$

135. $x^2 y''_{xx} + ax^2 y'_x + (bx^2 + cx + d)y = 0.$

The substitution $y = w \exp\left(-\frac{1}{2}ax\right)$ leads to a linear equation of the form 14.1.2.115: $x^2 w''_{xx} + [(\frac{1}{4}a^2 + b)x^2 + cx + d]w = 0$.

136. $x^2 y''_{xx} + (ax^2 + b)y'_x + c[(a-c)x^2 + b]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

137. $x^2 y''_{xx} + (ax^2 + bx)y'_x - by = 0.$

Particular solution: $y_0 = x^{-b} e^{-ax}$.

138. $x^2 y''_{xx} + (ax^2 + bx)y'_x + [k(a-k)x^2 + (an + bk - 2kn)x + n(b-n-1)]y = 0.$

Particular solution: $y_0 = x^{-n} e^{-kx}$.

139. $a_2 x^2 y''_{xx} + (a_1 x^2 + b_1 x)y'_x + (a_0 x^2 + b_0 x + c_0)y = 0.$

The substitution $y = x^k w$, where k is a root of the quadratic equation $a_2 k^2 + (b_1 - a_2)k + c_0 = 0$, leads to an equation of the form 14.1.2.108: $a_2 x w''_{xx} + (a_1 x + 2a_2 k + b_1)w'_x + (a_0 x + a_1 k + b_0)w = 0$.

140. $x^2 y''_{xx} + [ax^2 + (ab-1)x + b]y'_x + a^2 bxy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

141. $x^2 y''_{xx} - 2x(x^2 - a)y'_x + \{2nx^2 + [(-1)^n - 1]a\}y = 0.$

For $n = 0, 1, 2, \ldots$, particular solutions are polynomials, $y_0 = P_n(x)$, where

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 2x^2 - 1 - 2a, \quad P_3(x) = 2x^3 - (3+2a)x, \quad \ldots$$

The polynomials contain only even powers of x for even n and only odd powers of x for odd n.

142. $x^2 y''_{xx} + x(ax^2 + bx + c)y'_x + (Ax^3 + Bx^2 + Cx + D)y = 0.$

1°. The substitution $y = x^k w$, where k is a root of the quadratic equation $k^2 + (c-1)k + D = 0$ leads to an equation of the form 14.1.2.84 (see also 14.1.2.80–14.1.2.83):

$$x w''_{xx} + (ax^2 + bx + c + 2k)w'_x + [Ax^2 + (B + ak)x + C + bk]y = 0.$$

2°. Let s and r be arbitrary parameters.

For $A = ar$, $B = as + br - r^2$, $C = bs + cr - 2rs$, $D = s(c - s - 1)$, a particular solution is: $y_0 = x^{-s} e^{-rx}$.

For $A = a(b-r)$, $B = a(c-s+1) + r(b-r)$, $C = bs + cr - 2rs$, $D = s(c-s-1)$, a particular solution is: $y_0 = x^{-s} \exp(-\frac{1}{2}ax^2 - rx)$.

143. $x^2 y''_{xx} + ax^n y'_x - (abx^n + acx^{n-1} + b^2 x^2 + 2bcx + c^2 - c)y = 0.$

Particular solution: $y_0 = x^c e^{bx}$.

144. $x^2 y''_{xx} + ax^n y'_x + (abx^{n+2m} - b^2 x^{4m+2} + amx^{n-1} - m^2 - m)y = 0.$

Particular solution: $y_0 = x^{-m} \exp\Bigl(-\dfrac{b}{2m+1} x^{2m+1}\Bigr)$.

145. $x^2 y''_{xx} + x(ax^n + b)y'_x + b(ax^n - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

146. $x^2y''_{xx} + x(ax^n + b)y'_x + (\alpha x^{2n} + \beta x^n + \gamma)y = 0.$

The transformation $z = x^n$, $w = yz^{-k}$, where k is a root of the quadratic equation $n^2k^2 + n(b-1)k + \gamma = 0$, leads to a linear equation of the form 14.1.2.108: $n^2zw''_{zz} + [naz + 2kn^2 + n(n-1+b)]w'_z + (\alpha z + kna + \beta)w = 0$.

147. $x^2y''_{xx} + x(2ax^n + b)y'_x + [a^2x^{2n} + a(b+n-1)x^n + \alpha x^{2m} + \beta x^m + \gamma]y = 0.$

The substitution $w = y\exp(ax^n/n)$ leads to a linear equation of the form 14.1.2.146: $x^2w''_{xx} + bxw'_x + (\alpha x^{2m} + \beta x^m + \gamma)w = 0$.

148. $x^2y''_{xx} + (ax^{n+2} + bx^2 + c)y'_x + (anx^{n+1} + acx^n + bc)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1} - bx\Bigr)$.

► **Equations of the form** $(ax^2 + bx + c)y''_{xx} + f(x)y'_x + g(x)y = 0.$

149. $(1 - x^2)y''_{xx} + n(n-1)y = 0, \qquad n = 0,\ 1,\ 2,\ \ldots$

This equation is encountered in hydrodynamics when describing axially symmetric Stokes flows.

1°. For $n \ge 2$, the solution is given by:

$$y = C_1\mathcal{J}_n(x) + C_2\mathcal{H}_n(x),$$

where $\mathcal{J}_n(x)$ and $\mathcal{H}_n(x)$ are the Gegenbauer functions which can be expressed in terms of the Legendre functions of the first and second kind (see 14.1.2.153) as follows:

$$\mathcal{J}_n(x) = \frac{P_{n-2}(x) - P_n(x)}{2n-1}, \quad \mathcal{H}_n(x) = \frac{Q_{n-2}(x) - Q_n(x)}{2n-1}.$$

2°. For $n = 0$ and $n = 1$, the solution is: $y = C_1 + C_2x$.

150. $(x^2 - a^2)y''_{xx} + by'_x - 6y = 0.$

Particular solution: $y_0 = (4x - b)|x + a|^{\frac{2a+b}{2a}}|x - a|^{\frac{2a-b}{2a}}$.

151. $(x^2 - 1)y''_{xx} + xy'_x + ay = 0.$

1°. For $a = k^2 > 0$, the solution is:

$$y = \begin{cases} C_1\cos(k\,\mathrm{arccosh}\,|x|) + C_2\sin(k\,\mathrm{arccosh}\,|x|) & \text{if } |x| > 1, \\ C_1\exp(k\arccos x) + C_2\exp(-k\arccos x) & \text{if } |x| < 1, \end{cases}$$

where $\mathrm{arccosh}\,x = \ln\bigl(x + \sqrt{x^2-1}\bigr)$.

2°. For $a = -k^2 < 0$, the solution is:

$$y = \begin{cases} C_1\exp(k\,\mathrm{arccosh}\,|x|) + C_2\exp(-k\,\mathrm{arccosh}\,|x|) & \text{if } |x| > 1, \\ C_1\cos(k\arccos x) + C_2\sin(k\arccos x) & \text{if } |x| < 1. \end{cases}$$

3°. For $a = -n^2$, where n is a nonnegative integer, particular solutions are the *Chebyshev polynomials*: $T_n(x) = \cos(n\arccos x)$.

152. $(1-x^2)y''_{xx} - xy'_x + n^2y = 0, \qquad n = 0,\ 1,\ 2,\ \dots$

This is a special case of equation 14.1.2.151 with $a = -n^2$. Particular solution:

$$y_0 = T_n(x) = \cos(n\arccos x) = \frac{(-1)^n}{2^n\left(\frac{1}{2}\right)_n}\sqrt{1-x^2}\,\frac{d^n}{dx^n}\left[(1-x^2)^{n-\frac{1}{2}}\right]$$

$$= \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m},$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind, $(a)_n = a(a+1)\dots(a+n-1)$, and $[b]$ stands for the integer part of a number b.

153. $(1-x^2)y''_{xx} - 2xy'_x + n(n+1)y = 0, \qquad n = 0,\ 1,\ 2,\ \dots$

The Legendre equation.

The solution is given by:

$$y = C_1P_n(x) + C_2Q_n(x),$$

where the Legendre polynomials $P_n(x)$ and the Legendre functions of the second kind $Q_n(x)$ are given by the formulas:

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n, \quad Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{m=1}^{n}\frac{1}{m}P_{m-1}(x)P_{n-m}(x).$$

The functions $P_n = P_n(x)$ can be conveniently calculated by the recurrence relations:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad \dots,$$

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x).$$

Three leading functions $Q_n = Q_n(x)$ are:

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1, \quad Q_2(x) = \frac{3x^2-1}{4}\ln\frac{1+x}{1-x} - \frac{3}{2}x.$$

All n zeros of the polynomial $P_n(x)$ are real and lie on the interval $-1 < x < 1$; the functions $P_n(x)$ form an orthogonal system on the closed interval $-1 \le x \le 1$, with the following relations taking place:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & \text{if } n \ne m, \\ \dfrac{2}{2n+1} & \text{if } n = m. \end{cases}$$

154. $(1-x^2)y''_{xx} - 2xy'_x + \nu(\nu+1)y = 0.$

The Legendre equation; ν is an arbitrary number. The case $\nu = n$ where n is a nonnegative integer is considered in 14.1.2.153.

The substitution $z = x^2$ leads to the hypergeometric equation. Therefore, with $|x| < 1$ the solution can be written as:

$$y = C_1F\Big(-\frac{\nu}{2},\ \frac{1+\nu}{2},\ \frac{1}{2};\ x^2\Big) + C_2xF\Big(\frac{1-\nu}{2},\ 1+\frac{\nu}{2},\ \frac{3}{2};\ x^2\Big),$$

where $F(\alpha, \beta, \gamma; x)$ is the hypergeometric series (see 14.1.2.171).

In Section S4.11, the Legendre equation is discussed in more detail. See also the books by Abramowitz & Stegun (1964), Bateman & Erdélyi (1953, Vol. 1), and Kamke (1977).

155. $(1-x^2)y''_{xx} - 3xy'_x + n(n+2)y = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Particular solution:

$$y_0 = U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}} = \frac{(-1)^n(n+1)}{2^{n+1}\left(\frac{1}{2}\right)_{n+1}}\frac{1}{\sqrt{1-x^2}}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\frac{1}{2}}\right]$$
$$= \sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)!}{m!\,(n-2m)!}(2x)^{n-2m}, \qquad (a)_n = a(a+1)\dots(a+n-1),$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind and $[b]$ stands for the integer part of a number b.

156. $(x^2-1)y''_{xx} + 2(n+1)xy'_x - (\nu+n+1)(\nu-n)y = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Solution: $y = \dfrac{d^n}{dx^n}y_\nu(x)$, where $y_\nu(x)$ is the general solution of the Legendre equation 14.1.2.154.

157. $(x^2-1)y''_{xx} - 2(n-1)xy'_x - (\nu-n+1)(\nu+n)y = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Solution: $y = |x^2-1|^n\dfrac{d^n}{dx^n}y_\nu(x)$, where $y_\nu(x)$ is the general solution of the Legendre equation 14.1.2.154.

158. $(x^2-1)y''_{xx} + (2a+1)xy'_x - b(2a+b)y = 0.$

1°. Particular solution:

$$y_0 = \frac{\Gamma(2a+b)}{\Gamma(b+1)\Gamma(2a)}F\left(2a+b, -b, a+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}x\right), \tag{1}$$

where $F(\alpha,\beta,\gamma;z)$ is the hypergeometric function (see equation 14.1.2.171 and Section S4.10).

2°. For $b = n$, where $n = 0,\ 1,\ \dots,$ the right-hand side of (1) defines the *Gegenbauer polynomials*,

$$C_n^{(a)}(x) = \frac{\Gamma(2a+n)}{\Gamma(n+1)\Gamma(2a)}F\left(2a+n, -n, a+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}x\right)$$
$$= \sum_{k=0}^{n}\frac{\Gamma(a+k)\Gamma(2a+n+k)(x-1)^k}{k!\,(n-k)!\,2^k\Gamma(a)\Gamma(2a+2k)}.$$

159. $(1-x^2)y''_{xx} + (2a-3)xy'_x + (n+1)(n+2a-1)y = 0, \qquad n = 0,\ 1,\ 2,\ \dots$

Particular solution:

$$y_0(x) = (1-x^2)^{a-1/2}C_n^{(a)}(x) = (1-x^2)^{a-1/2}\sum_{k=0}^{n}\frac{\Gamma(a+k)\Gamma(2a+n+k)(x-1)^k}{k!\,(n-k)!\,2^k\Gamma(a)\Gamma(2a+2k)},$$

where $C_n^{(a)}(x)$ are the Gegenbauer polynomials.

160. $(1-x^2)y''_{xx} + \big[\beta - \alpha - (\alpha+\beta+2)x\big]y'_x + n(n+\alpha+\beta+1)y = 0,$
$n = 0,\ 1,\ 2,\ \dots$

Particular solution:

$$y_0(x) = P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big]$$
$$= 2^{-n}\sum_{m=0}^{n} C_{n+\alpha}^{m} C_{n+\beta}^{n-m}(x-1)^{n-m}(x+1)^m,$$

where $P_n^{\alpha,\beta}(x)$ are the Jacobi polynomials and C_b^a are binomial coefficients.

161. $(1-x^2)y''_{xx} + \big[\alpha - \beta + (\alpha+\beta-2)x\big]y'_x$
$+ (n+1)(n+\alpha+\beta)y = 0, \qquad n = 0,\ 1,\ 2,\ \dots$

Particular solution: $y_0(x) = (1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x)$, where $P_n^{\alpha,\beta}(x)$ are the Jacobi polynomials (see 14.1.2.160).

162. $(ax^2+b)y''_{xx} + axy'_x + cy = 0.$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2+b}}$ leads to a constant coefficient linear equation: $y''_{zz} + cy = 0$.

163. $(x^2+a)y''_{xx} + 2bxy'_x + 2(b-1)y = 0.$

Particular solution: $y_0 = |x^2+a|^{1-b}$.

164. $(x^2-a^2)y''_{xx} + 2bxy'_x + b(b-1)y = 0.$

Solution: $y = C_1|x-a|^{1-b} + C_2|x+a|^{1-b}$.

165. $(x^2+a^2)y''_{xx} + 2bxy'_x + b(b-1)y = 0.$

Solution:

$$y = C_1\big(x^2+a^2\big)^{\frac{1-b}{2}}\sin\varphi + C_2\big(x^2+a^2\big)^{\frac{1-b}{2}}\cos\varphi, \quad \text{where} \quad \varphi = (1-b)\arctan(a/x).$$

166. $(ax^2+b)y''_{xx} + (2n+1)axy'_x + cy = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

This equation can be obtained by differentiating n times an equation of the form 14.1.2.162: $(ax^2+b)u''_{xx} + axu'_x + (c-an^2)u = 0$.

Solution: $y = u_x^{(n)}$.

167. $(1-x^2)y''_{xx} - xy'_x + (2ax^2+b)y = 0.$

This is an algebraic form of the Mathieu equation. The substitution $x = \cos z$ leads to the Mathieu equation 14.1.6.29: $y''_{zz} + (a+b+a\cos 2z)y = 0$.

168. $(1-x^2)y''_{xx} + (ax+b)y'_x + cy = 0.$

1°. The substitution $2z = 1+x$ leads to the hypergeometric equation 14.1.2.171: $z(1-z)y''_{zz} + \big[az + \frac{1}{2}(b-a)\big]y'_z + cy = 0$.

2°. For $a = -2m-3$, $b = 0$, and $c = \lambda$, the Gegenbauer functions are solutions of the equation.

3°. In the special case $a=-\alpha-\beta-2$, $b=\beta-\alpha$, and $c=n(n+\alpha+\beta+1)$, solutions of the equation are the *Jacobi polynomials*:

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\sum_{m=0}^{n}C_{n+\alpha}^{m}C_{n+\beta}^{n-m}(x-1)^{n-m}(x+1)^m,$$

where C_b^a are binomial coefficients (see Section S4.1.1).

169. $(ax^2+b)y''_{xx}+(cx^2+d)y'_x+\lambda[(c-a\lambda)x^2+d-b\lambda]y=0.$

Particular solution: $y_0=e^{-\lambda x}$.

170. $(ax^2+b)y''_{xx}+[\lambda(c+a)x^2+(c-a)x+2b\lambda]y'_x+\lambda^2(cx^2+b)y=0.$

Particular solution: $y_0=(\lambda x+1)e^{-\lambda x}$.

171. $x(x-1)y''_{xx}+[(\alpha+\beta+1)x-\gamma]y'_x+\alpha\beta y=0.$

The Gaussian hypergeometric equation. For $\gamma\neq 0,\ -1,\ -2,\ -3,\ \ldots$, a solution can be expressed in terms of the hypergeometric series:

$$F(\alpha,\beta,\gamma;x)=1+\sum_{k=1}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!},\qquad (\alpha)_k=\alpha(\alpha+1)\ldots(\alpha+k-1),$$

which, *a fortiori*, is convergent for $|x|<1$.

For $\gamma>\beta>0$, this solution can be expressed in terms of a definite integral:

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt,$$

where $\Gamma(\beta)$ is the gamma function.

If γ is not an integer, the general solution of the hypergeometric equation has the form:

$$y=C_1F(\alpha,\beta,\gamma;x)+C_2x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;\ x).$$

In the degenerate cases $\gamma=0,\ -1,\ -2,\ -3,\ \ldots$, a particular solution of the hypergeometric equation corresponds to $C_1=0$ and $C_2=1$. If γ is a positive integer, another particular solution corresponds to $C_1=1$ and $C_2=0$. In both these cases, the general solution can be constructed by means of the formula given in 14.1.1.

Table S4.3 (see Section S4.10) presents some special cases where F is expressed in terms of elementary functions.

Table 14.2 gives the general solutions of the hypergeometric equation for some values of the determining parameters.

The function F possesses the following properties:

$$\begin{aligned}
F(\alpha,\beta,\gamma;x)&=F(\beta,\alpha,\gamma;x),\\
F(\alpha,\beta,\gamma;x)&=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),\\
F(\alpha,\beta,\gamma;x)&=(1-x)^{-\alpha}F\Bigl(\alpha,\ \gamma-\beta,\ \gamma;\ \frac{x}{x-1}\Bigr),\\
\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x)&=\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\ \beta+n,\ \gamma+n;\ x).
\end{aligned}$$

The hypergeometric functions are discussed in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 1) in more detail; see also Section S4.10.

TABLE 14.2
General solutions of the hypergeometric equation for some values of the determining parameters

α	β	γ	Solution: $y=y(x)$
0	β	γ	$C_1+C_2\int \lvert x\rvert^{-\gamma}\lvert x-1\rvert^{\gamma-\beta-1}\,dx$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	$C_1\bigl(1+\sqrt{1-x}\bigr)^{-2\alpha}+C_2x^{-2\alpha}\bigl(1+\sqrt{1-x}\bigr)^{2\alpha}$
α	$\alpha-\frac{1}{2}$	$\frac{1}{2}$	$C_1\bigl(1+\sqrt{x}\bigr)^{1-2\alpha}+C_2\bigl(1-\sqrt{x}\bigr)^{1-2\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{\sqrt{x}}\Bigl[C_1\bigl(1+\sqrt{x}\bigr)^{1-2\alpha}+C_2\bigl(1-\sqrt{x}\bigr)^{1-2\alpha}\Bigr]$
1	β	γ	$\lvert x\rvert^{1-\gamma}\lvert x-1\rvert^{\gamma-\beta-1}\Bigl(C_1+C_2\int \lvert x\rvert^{\gamma-2}\lvert x-1\rvert^{\beta-\gamma}\,dx\Bigr)$
α	β	α	$\lvert x-1\rvert^{-\beta}\Bigl(C_1+C_2\int \lvert x\rvert^{-\alpha}\lvert x-1\rvert^{\beta-1}\,dx\Bigr)$
α	β	$\alpha+1$	$\lvert x\rvert^{-\alpha}\Bigl(C_1+C_2\int \lvert x\rvert^{\alpha-1}\lvert x-1\rvert^{-\beta}\,dx\Bigr)$

172. $\boldsymbol{x(x+a)y''_{xx}+(bx+c)y'_x+dy=0.}$

The substitution $x=-az$ leads to the hypergeometric equation 14.1.2.171: $z(1-z)y''_{zz}+[(c/a)-bz]y'_z-dy=0$.

173. $\boldsymbol{2x(x-1)y''_{xx}+(2x-1)y'_x+(ax+b)y=0.}$

The substitution $x=\cos^2\xi$ leads to the Mathieu equation 14.1.6.29:

$$y''_{\xi\xi}-(a+2b+a\cos 2\xi)y=0.$$

174. $\boldsymbol{(x^2+2ax+b)y''_{xx}+(x+a)y'_x-m^2y=0.}$

Solution: $y=C_1\bigl(x+a+\sqrt{x^2+2ax+b}\bigr)^m+C_2\bigl(x+a+\sqrt{x^2+2ax+b}\bigr)^{-m}$.

175. $\boldsymbol{(ax^2+bx+c)y''_{xx}+(dx+k)y'_x+(d-2a)y=0.}$

Integrating yields a first-order linear equation: $(ax^2+bx+c)y'_x+[(d-2a)x+k-b]y=C$.

176. $\boldsymbol{(ax^2+bx+c)y''_{xx}+(kx+d)y'_x-ky=0.}$

Particular solution: $y_0=kx+d$.

177. $\boldsymbol{(ax^2+2bx+c)y''_{xx}+(ax+b)y'_x+dy=0.}$

The substitution $\xi=\displaystyle\int\frac{dx}{\sqrt{ax^2+2bx+c}}$ leads to a constant coefficient linear equation of the form 14.1.2.1: $y''_{\xi\xi}+dy=0$.

178. $(ax^2 + 2bx + c)y''_{xx} + 3(ax + b)y'_x + dy = 0.$

The substitution $w = y\sqrt{|ax^2 + 2bx + c|}$ leads to a linear equation of the form 14.1.2.177: $(ax^2 + 2bx + c)w''_{xx} + (ax + b)w'_x + (d - a)w = 0$.

179. $(a_2x^2 + b_2x + c_2)y''_{xx} + (b_1x + c_1)y'_x + c_0y = 0.$

Let λ_1 and λ_2 be roots of the quadratic equation $a_2\lambda^2 + b_2\lambda + c_2 = 0$.

1°. For $\lambda_1 \neq \lambda_2$, the substitution $z = \dfrac{x - \lambda_1}{\lambda_2 - \lambda_1}$ leads to the hypergeometric equation 14.1.2.171:

$$z(1 - z)y''_{zz} - (Az + B)y'_z - Cy = 0,$$

where $A = \dfrac{b_1}{a_2}$, $B = \dfrac{b_1\lambda_1 + c_1}{a_2(\lambda_2 - \lambda_1)}$, $C = \dfrac{c_0}{a_2}$.

2°. For $\lambda_1 = \lambda_2 = \lambda$, the transformation $x = \lambda + \xi^{-1}$, $y = \xi^k u$, where k is a root of the quadratic equation $a_2k^2 + (a_2 - b_1)k + c_0 = 0$, leads to a linear equation of the form 14.1.2.108: $a_2\xi u''_{\xi\xi} - [(c_1 + \lambda b_1)\xi + b_1 - 2a_2(k + 1)]u'_\xi - k(c_1 + \lambda b_1)u = 0$.

3°. Let $c_0 = -a_2n(n - 1) - b_1n$, where n is a positive integer. Then, among solutions there exists a polynomial of degree $\le n$.

180. $(ax^2 + bx + c)y''_{xx} - (x^2 - k^2)y'_x + (x + k)y = 0.$

Particular solution: $y_0 = x - k$.

181. $(ax^2 + bx + c)y''_{xx} + (x^3 + k^3)y'_x - (x^2 - kx + k^2)y = 0.$

Particular solution: $y_0 = x + k$.

► **Equations of the form** $(a_3x^3 + a_2x^2 + a_1x + a_0)y''_{xx} + f(x)y'_x + g(x)y = 0.$

182. $x^3y''_{xx} + (ax + b)y = 0.$

This is a special case of equation 14.1.2.132 with $n = -1$.

183. $x^3y''_{xx} + (ax^2 + bx)y'_x + cxy = 0.$

The substitution $x = 1/z$ leads to an equation of the form 14.1.2.139:

$$z^2y''_{zz} + z(2 - a - bz)y'_z + cy = 0.$$

184. $x^3y''_{xx} + (ax^2 + bx)y'_x + by = 0.$

Particular solution: $y_0 = a - 2 + b/x$.

185. $x^3y''_{xx} + (ax^2 + bx)y'_x + cy = 0.$

The substitution $x = 1/z$ leads to an equation of the form 14.1.2.108:

$$zy''_{zz} + (2 - a - bz)y'_z + cy = 0.$$

186. $x^3y''_{xx} + (ax^2 + bx)y'_x + (cx + d)y = 0.$

1°. The substitution $y = x^k u$, where $k = -d/b$, leads to a linear equation of the form 14.1.2.134: $x^2u''_{xx} + [(a + 2k)x + b]u'_x + [k(a + k - 1) + c]u = 0$.

2°. If $c = 0$ and $d = b(a - 2)$, a particular solution is: $y_0 = e^{b/x}$.

187. $x^3y''_{xx} + (ax^3 - x^2 + abx + b)y'_x + a^2bxy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

188. $x^3y''_{xx} + x(ax^n + b)y'_x - (ax^n - abx^{n-1} + b)y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

189. $x(ax^2 + b)y''_{xx} + 2(ax^2 + b)y'_x - 2axy = 0.$

Particular solution: $y_0 = ax + b/x$.

190. $x(x^2 + a)y''_{xx} + (bx^2 + c)y'_x + sxy = 0.$

The substitution $az = -x^2$ leads to the hypergeometric equation 14.1.2.171: $z(1-z)y''_{zz} + \frac{1}{2}\left[1 + ca^{-1} - (1+b)z\right]y'_z - \frac{1}{4}sy = 0$.

191. $x^2(ax + b)y''_{xx} + [cx^2 + (2b + a\lambda)x + b\lambda]y'_x + \lambda(c - 2a)y = 0.$

Particular solution: $y_0 = \exp(\lambda/x)$.

192. $x^2(ax + b)y''_{xx} - 2x(ax + 2b)y'_x + 2(ax + 3b)y = 0.$

Solution: $y = \dfrac{C_1x^2 + C_2x^3}{ax+b}$.

193. $x^2(ax + b)y''_{xx} + [a(2 - n - m)x^2 - b(n + m)x]y'_x + [am(n - 1)x + bn(m + 1)]y = 0.$

$$\text{Solution:}\quad y = \begin{cases} \dfrac{C_1|x|^n + C_2|x|^{m+1}}{ax+b} & \text{if } m \neq n-1, \\ \dfrac{|x|^n(C_1 + C_2\ln|x|)}{ax+b} & \text{if } m = n-1. \end{cases}$$

194. $x^2(x + a_2)y''_{xx} + x(b_1x + a_1)y'_x + (b_0x + a_0)y = 0.$

The substitution $y = x^ku$, where k is a root of the quadratic equation $a_2k^2 + k(a_1 - a_2) + a_0 = 0$, leads to a linear equation of the form 14.1.2.172:

$$x(x + a_2)u''_{xx} + [(2k + b_1)x + 2ka_2 + a_1]u'_x + [k^2 + k(b_1 - 1) + b_0]u = 0.$$

195. $(ax^3 + bx^2 + cx)y''_{xx} + (\alpha x^2 + \beta x + 2c)y'_x + (\beta - 2b)y = 0.$

Particular solution: $y_0 = 2a - \alpha + (2b - \beta)x^{-1}$.

196. $(ax^3 + bx^2 + cx)y''_{xx} + (\alpha x^2 + \beta x + 2c)y'_x - (\alpha x + 2b - \beta)y = 0.$

Particular solution: $y_0 = \alpha x + 2(\beta - b) + \dfrac{\lambda}{x}$, where $\lambda = \dfrac{c\alpha + (b-\beta)(2b-\beta)}{\alpha - a}$.

197. $(ax^3 + bx^2 + cx)y''_{xx} + [-2ax^2 - (b + 1)x + k]y'_x + 2(ax + 1)y = 0.$

Particular solution: $y_0 = (ak + b - 1)x^2 + (c + k)(2x - k)$.

198. $(ax^3 + bx^2 + cx)y''_{xx} + (nx^2 + mx + ck)y'_x + (k - 1)[(n - ak)x + m - bk]y = 0.$

Particular solution: $y_0 = x^{1-k}$.

199. $(ax^3+bx^2+cx)y''_{xx}+[(m-a)x^2+(2cm-1)x-c]y'_x+(-2mx+1)y=0.$

Particular solution: $y_0=(a+m)x^2+(2b+4cm-1)(x+c)$.

200. $(ax^3+bx^2+cx)y''_{xx}+(nx^2+mx+k)y'_x+[-2(a+n)x+1]y=0.$

With the constraint

$$2(2a+n)(c+k)+(2b+2m+1)[m+1+2k(a+n)]=0,$$

a particular solution has the form: $y_0=(2a+n)x^2+(2b+2m+1)(x-k)$.

201. $(ax^3+x^2+b)y''_{xx}+a^2x(x^2-b)y'_x-a^3bxy=0.$

Particular solution: $y_0=(ax+2)e^{-ax}$.

202. $2x(ax^2+bx+c)y''_{xx}+(ax^2-c)y'_x+\lambda x^2y=0.$

The substitution $\xi=\int\Bigl(\frac{x}{ax^2+bx+c}\Bigr)^{1/2}dx$ leads to a constant coefficient linear equation: $2y''_{\xi\xi}+\lambda y=0$.

203. $x(ax^2+bx+1)y''_{xx}+(\alpha x^2+\beta x+\gamma)y'_x+(nx+m)y=0.$

The substitution $y=x^{1-\gamma}w$ leads to an equation of the same form:

$$x(ax^2+bx+1)w''_{xx}+[(\alpha+2a-2a\gamma)x^2+(\beta+2b-2b\gamma)x+2-\gamma]w'_x$$
$$+\bigl\{[n+(1-\gamma)(\alpha-a\gamma)]x+m+(1-\gamma)(\beta-b\gamma)\bigr\}w=0.$$

204. $x(x-1)(x-a)y''_{xx}+\bigl\{(\alpha+\beta+1)x^2$
$-[\alpha+\beta+1+a(\gamma+\delta)-\delta]x+a\gamma\bigr\}y'_x+(\alpha\beta x-q)y=0.$

Heun's equation.

1°. For $|a|\ge 1$ and $\gamma\ne 0,\ -1,\ -2,\ -3,\ \dots$, a solution can be represented as the power series:

$$F(a,q;\alpha,\beta,\gamma,\delta,x)=\sum_{n=0}^{\infty}c_nx^n,$$

where the coefficients are determined by the recurrence formulas:

$$c_0=1,\quad a\gamma c_1=q,$$
$$a(n+1)(\gamma+n)c_{n+1}=\Bigl[a(\gamma+\delta+n-1)+\alpha+\beta-\delta+n+\frac{q}{n}\Bigr]nc_n$$
$$-\bigl[(n-1)(n-2)+(n-1)(\alpha+\beta+1)+\alpha\beta\bigr]c_{n-1}.$$

A fortiori, the series is convergent for $|x|\le 1$.

2°. If γ is not an integer, the general solution of Heun's equation can be presented as follows:

$$y=C_1F(a,q;\alpha,\beta,\gamma,\delta,x)+C_2|x|^{1-\gamma}F(a,q_1;\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\delta,x),$$

where $q_1=q+(\alpha-\gamma+1)(\beta-\gamma+1)-\alpha\beta+\delta(\gamma-1)$.

Table 14.3 lists some transformations preserving the form of Heun's equation. (Whenever at least one of the indicated equations is integrable by quadrature with some values of parameters, all the other equations are also integrable for those values of the parameters.)

⊙ *Literature*: H. Bateman and A. Erdélyi (1955, Vol. 3), E. Kamke (1977), S. Yu. Slavyanov, W. Lay, and A. Seeger (1955), A. Ronveaux (1995).

TABLE 14.3
Some transformations preserving the form of Heun's equation

No	New variables	Parameters of transformed equation for $w=w(\xi)$					
1*	$\xi=x,\ w=y$	a	q	α	β	γ	δ
2	$\xi=1-x,\ w=y$	$1-a$	$\alpha\beta-q$	α	β	δ	γ
3	$\xi=x,\ w=\lvert x\rvert^{\gamma-1}y$	a	q_1	$\alpha-\gamma+1$	$\beta-\gamma+1$	$2-\gamma$	δ
4	$\xi=\dfrac{x}{a},\ w=\lvert x\rvert^{\gamma-1}y$	$\dfrac{1}{a}$	$\dfrac{q_2}{a}$	$\alpha-\gamma+1$	$\beta-\gamma+1$	$2-\gamma$	$\alpha+\beta-\gamma-\delta+1$
5	$\xi=\dfrac{1}{x},\ w=\lvert x\rvert^{\alpha}y$	$\dfrac{1}{a}$	q_3	α	$\alpha-\gamma+1$	$\alpha-\beta+1$	δ
6	$\xi=\dfrac{x}{a},\ w=y$	$\dfrac{1}{a}$	q	α	β	γ	$\alpha+\beta-\gamma-\delta+1$
7	$\xi=1-\dfrac{x}{a},\ w=y$	$1-\dfrac{1}{a}$	q	α	β	$\alpha+\beta-\gamma-\delta+1$	γ
8	$\xi=\dfrac{x}{a},\ w=\lvert x\rvert^{\alpha}y$	a	q	$\alpha-\gamma+1$	$\alpha+\gamma-1$	$\alpha-\beta+1$	$\alpha+\beta-\gamma-\delta+1$
9	$\xi=\dfrac{x-1}{x},\ w=\lvert x\rvert^{\alpha}y$	$1-\dfrac{1}{a}$	q	α	$\alpha-\gamma+1$	δ	$\alpha-\beta+1$
10	$\xi=\dfrac{a(x-1)}{x(a-1)},\ w=\lvert x\rvert^{\alpha}y$	$\dfrac{a}{a-1}$	q	α	$\alpha-\gamma+1$	δ	$\alpha+\beta-\gamma-\delta+1$
11	$\xi=\dfrac{x}{x-1},\ w=\lvert x-1\rvert^{\alpha}y$	$\dfrac{a}{a-1}$	q	α	$\alpha-\delta+1$	γ	$\alpha-\beta+1$
12	$\xi=\dfrac{x(a-1)}{a(x-1)},\ w=\lvert x-1\rvert^{\alpha}y$	$1-\dfrac{1}{a}$	q	α	$\alpha-\delta+1$	γ	$\alpha+\beta-\gamma-\delta+1$
Notation: $q_1=q+(\alpha-\gamma+1)(\beta-\gamma+1)-\alpha\beta+\delta(\gamma-1)$, $q_2=q_1+a\delta(1-\gamma)$, $q_3=qa^{-1}+\alpha(\alpha-\gamma+1)+\alpha a^{-1}(\delta-\beta)-a\delta$.							

* This row refers to the original equation, while the others refer to the transformed equation for $w=w(\xi)$.

205. $(ax^3+bx^2+cx+d)y''_{xx}-(x^2-\lambda^2)y'_x+(x+\lambda)y=0.$

Particular solution: $y_0=x-\lambda$.

206. $2(ax^3+bx^2+cx+d)y''_{xx}+(3ax^2+2bx+c)y'_x+\lambda y=0.$

The substitution $\xi=\displaystyle\int\frac{dx}{\sqrt{ax^3+bx^2+cx+d}}$ leads to a constant coefficient linear equation: $2y''_{\xi\xi}+\lambda y=0$.

207. $2(ax^3+bx^2+cx+d)y''_{xx}+3(3ax^2+2bx+c)y'_x+(6ax+2b+\lambda)y=0.$

This equation is obtained by differentiating the equation 14.1.2.206.

208. $(ax^3+bx^2+cx+d)y''_{xx}+[\alpha x^2+(\alpha\gamma+\beta)x+\beta\gamma]y'_x-(\alpha x+\beta)y=0.$

Particular solution: $y_0=x+\gamma$.

209. $(ax^3+bx^2+cx+d)y''_{xx}+(x^3+\lambda^3)y'_x-(x^2-\lambda x+\lambda^2)y=0.$

Particular solution: $y_0=x+\lambda$.

210. $2x(ax^2+bx+c)y''_{xx}+[a(2-k)x^2+b(1-k)x-ck]y'_x+\lambda x^{k+1}y=0.$

The substitution $\xi=\int x^{k/2}(ax^2+bx+c)^{-1/2}\,dx$ leads to a constant coefficient linear equation: $2y''_{\xi\xi}+\lambda y=0$.

► **Equations of the form** $(a_4x^4+a_3x^3+a_2x^2+a_1x+a_0)y''_{xx}+f(x)y'_x+g(x)y=0.$

211. $x^4y''_{xx}+ay=0.$

The transformation $z=1/x$, $u=y/x$ leads to a constant coefficient linear equation of the form 14.1.2.1: $u''_{zz}+au=0$.

212. $x^4y''_{xx}+(ax^2+bx+c)y=0.$

The transformation $z=1/x$, $u=y/x$ leads to a linear equation of the form 14.1.2.115: $z^2u''_{zz}+(cz^2+bz+a)u=0$.

213. $x^4y''_{xx}-(a+b)x^2y'_x+[(a+b)x+ab]y=0.$

Solution: $y=\begin{cases}C_1xe^{-a/x}+C_2xe^{-b/x} & \text{if } a\neq b,\\ (C_1x+C_2)e^{-a/x} & \text{if } a=b.\end{cases}$

214. $x^4y''_{xx}+2x^2(x+a)y'_x+by=0.$

The substitution $z=1/x$ leads to a constant coefficient linear equation: $y''_{zz}-2ay'_z+by=0$.

215. $x^4y''_{xx}+ax^ny'_x-(ax^{n-1}+abx^{n-2}+b^2)y=0.$

Particular solution: $y_0=xe^{-b/x}$.

216. $x^2(x-a)^2y''_{xx}+by=0.$

Solution: $y=C_1|x|^m|x-a|^{1-m}+C_2|x|^{1-m}|x-a|^m$, where m is a root of the quadratic equation $m(m-1)a^2=-b$.

217. $x^2(x-a)^2y''_{xx}+by=cx^2(x-a)^2.$

Solution:

$$y=|x|^m|x-a|^{1-m}\left(C_1+\frac{c}{a(2m-1)}\int|x|^{1-m}|x-a|^m\,dx\right)$$
$$+|x|^{1-m}|x-a|^m\left(C_2-\frac{c}{a(2m-1)}\int|x|^m|x-a|^{1-m}\,dx\right),$$

where m is a root of the quadratic equation $m(m-1)a^2=-b$.

218. $ax^2(x-1)^2y''_{xx}+(bx^2+cx+d)y=0.$

Let p and q be roots of the quadratic equations

$$ap(p-1)+d=0,\qquad aq(q-1)+b+c+d=0.$$

The substitution $y=x^p(x-1)^qw$ leads to the hypergeometric equation of the form 14.1.2.171: $ax(x-1)w''_{xx}+2a[(p+q)x-p]w'_x+(2apq-c-2d)w=0$.

219. $x^2(x^2+a)y''_{xx}+(bx^2+c)xy'_x+dy=0.$

The substitution $\xi=x^2$ leads to a linear equation of the form 14.1.2.194: $4\xi^2(\xi+a)y''_{\xi\xi}+2\xi[(b+1)\xi+a+c]y'_\xi+dy=0$.

220. $(x^2+1)^2y''_{xx}+ay=0.$

The Halm equation. Solution:

$$y=\begin{cases}\sqrt{x^2+1}\,\bigl[C_1\cos(\beta\arctan x)+C_2\sin(\beta\arctan x)\bigr] & \text{if } a+1=\beta^2>0,\\ \sqrt{x^2+1}\,\bigl[C_1\cosh(\beta\arctan x)+C_2\sinh(\beta\arctan x)\bigr] & \text{if } a+1=-\beta^2<0,\\ \sqrt{x^2+1}\,(C_1+C_2\arctan x) & \text{if } a=-1.\end{cases}$$

221. $(x^2-1)^2y''_{xx}+ay=0.$

Solution:

$$y=\begin{cases}\sqrt{|x^2-1|}\,\bigl[C_1\cos\bigl(\beta\ln|z|\bigr)+C_2\sin\bigl(\beta\ln|z|\bigr)\bigr] & \text{if } a-1=4\beta^2>0,\\ (x+1)\bigl(C_1|z|^{(2\beta-1)/2}+C_2|z|^{-(2\beta+1)/2}\bigr) & \text{if } a-1=-4\beta^2<0,\\ \sqrt{|x^2-1|}\,\bigl(C_1+C_2\ln|z|\bigr) & \text{if } a=1,\end{cases}$$

where $z=(x+1)/(x-1)$.

222. $(x^2\pm a^2)^2y''_{xx}+b^2y=0.$

This is the equation of bending of a double-walled compressed bar with a parabolic cross-section.

1°. For the upper sign (constricted bar), the solution is as follows:

$$y=\sqrt{x^2+a^2}\,(C_1\cos u+C_2\sin u),\qquad \text{where } u=\sqrt{1+(b/a)^2}\,\arctan(x/a).$$

2°. For the lower sign (bar with salients), the solution is given by:

$$y=\sqrt{a^2-x^2}\,(C_1\cos u+C_2\sin u),\qquad \text{where } u=\frac{\sqrt{b^2-a^2}}{2a}\ln\frac{a+x}{a-x};\quad |x|<a.$$

223. $4(x^2+1)^2y''_{xx}+(ax^2+a-3)y=0.$

Solution:

$$y=\begin{cases}(x^2+1)^{1/4}(C_1\cos\xi+C_2\sin\xi) & \text{if } a>1,\\ (x^2+1)^{1/4}(C_1\cosh\xi+C_2\sinh\xi) & \text{if } a<1,\end{cases}$$

where $\xi=\frac{1}{2}\sqrt{|a-1|}\,\ln\bigl(x+\sqrt{|x^2+1|}\,\bigr)$.

224. $(ax^2+b)^2y''_{xx}+2ax(ax^2+b)y'_x+cy=0.$

The substitution $\xi=\displaystyle\int\frac{dx}{ax^2+b}$ leads to a constant coefficient linear equation: $y''_{\xi\xi}+cy=0$.

225. $(x^2-1)^2y''_{xx}+2x(x^2-1)y'_x-[\nu(\nu+1)(x^2-1)+n^2]y=0.$

Here, ν is an arbitrary number and n is a nonnegative integer. This is a special case of equation 14.1.2.226.

If $n=0$, this equation coincides with the Legendre equation 14.1.2.154. Denote its general solution by $y_\nu(x)$. If $n=1,\ 2,\ 3,\ \dots$, the general solution of the original equation is given by the formula: $y=|x^2-1|^{n/2}\dfrac{d^n}{dx^n}y_\nu(x)$.

226. $(1-x^2)^2 y''_{xx} - 2x(1-x^2)y'_x + [\nu(\nu+1)(1-x^2) - \mu^2]y = 0.$

The Legendre equation, ν and μ are arbitrary parameters.

The transformation $x = 1-2\xi,\ y = |x^2-1|^{\mu/2}w$ leads to the hypergeometric equation 14.1.2.171:

$$\xi(\xi-1)w''_{\xi\xi} + (\mu+1)(1-2\xi)w'_\xi + (\nu-\mu)(\nu+\mu+1)w = 0$$

with parameters $\alpha = \mu-\nu$, $\beta = \mu+\nu+1$, $\gamma = \mu+1$.

In particular, the original equation is integrable by quadrature if $\nu = \mu$ or $\nu = -\mu-1$.

In Section S4.11, the Legendre equation is discussed in more detail. See also the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, Vol. 1).

227. $a(x^2-1)^2 y''_{xx} + bx(x^2-1)y'_x + (cx^2+dx+e)y = 0.$

The transformation $\xi = \frac{1}{2}(x+1),\ w = |x+1|^{-p}|x-1|^{-q}y$, where p and q are parameters that are determined by solving the second-order algebraic system

$$4aq(q-1) + 2bq + c + d + e = 0, \qquad (p-q)[2a(p+q-1)+b] = d,$$

leads to the hypergeometric equation 14.1.2.171 with respect to $w = w(\xi)$.

228. $(ax^2+b)^2 y''_{xx} + (2ax+c)(ax^2+b)y'_x + ky = 0.$

The substitution $\xi = \displaystyle\int \frac{dx}{ax^2+b}$ leads to a constant coefficient linear equation of the form 14.1.2.11: $y''_{\xi\xi} + cy'_\xi + ky = 0.$

229. $(ax^2+b)^2 y''_{xx} + (ax^2+b)(cx^2+d)y'_x + 2(bc-ad)xy = 0.$

Particular solution: $y_0 = \exp\Bigl(-\displaystyle\int \frac{cx^2+d}{ax^2+b}\,dx\Bigr).$

230. $(x^2+a)^2 y''_{xx} + bx^n(x^2+a)y'_x - (bx^{n+1}+a)y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}.$

231. $(x^2+a)^2 y''_{xx} + bx^n(x^2+a)y'_x - m[bx^{n+1} + (m-1)x^2 + a]y = 0.$

Particular solution: $y_0 = (x^2+a)^{m/2}.$

232. $(x-a)^2(x-b)^2 y''_{xx} - cy = 0, \qquad a \neq b.$

The transformation $\xi = \ln\left|\dfrac{x-a}{x-b}\right|,\ y = (x-b)\eta$ leads to a constant coefficient linear equation: $(a-b)^2(\eta''_{\xi\xi} - \eta'_\xi) - c\eta = 0$. Therefore, the solution is as follows:

$$y = C_1|x-a|^{(1+\lambda)/2}|x-b|^{(1-\lambda)/2} + C_2|x-a|^{(1-\lambda)/2}|x-b|^{(1+\lambda)/2},$$

where $\lambda^2 = 4c(a-b)^{-2} + 1 \neq 0$.

233. $(x-a)^2(x-b)^2 y''_{xx} + (x-a)(x-b)(2x+\lambda)y'_x + \mu y = 0.$

Let k_1 and k_2 be roots of the quadratic equation $(a-b)^2k^2 + (a-b)(a+b+\lambda)k + \mu = 0.$

Solution:

$$y = \begin{cases} C_1|z|^{k_1} + C_2|z|^{k_2} & \text{if } k_1 \neq k_2, \\ |z|^k\bigl(C_1 + C_2\ln|z|\bigr) & \text{if } k_1 = k_2 = k, \end{cases}$$

where $z = (x-a)/(x-b)$.

234. $(ax^2+bx+c)^2y''_{xx}+Ay=0.$

The transformation $\xi=\int\frac{dx}{ax^2+bx+c}$, $w=\frac{y}{\sqrt{|ax^2+bx+c|}}$ leads to a constant coefficient linear equation of the form 14.1.2.1: $w''_{\xi\xi}+(A+ac-\frac{1}{4}b^2)w=0$.

235. $(x^2-1)^2y''_{xx}+2x(x^2-1)y'_x+[(x^2-1)(a^2x^2-\lambda)-m^2]y=0.$

Equation for prolate spheroidal wave functions, $m=0,\ 1,\ \ldots$ It arises when separating variables in the wave equation written in the system of prolate spheroidal coordinates.

1°. In applications, one usually looks for eigenvalues $\lambda=\lambda_{mn}$ and eigenfunctions $y=y_{mn}(x)$ that assume finite values at $x=\pm1$. The following functions are solutions of the eigenvalue problem:

$$S^{(1)}_{mn}(a,x)=\sum_{r=0,1}^{\infty}d^{mn}_r(a)P^m_{m+r}(x)\qquad\text{(prolate angular functions of the first kind)},$$

$$S^{(2)}_{mn}(a,x)=\sum_{r=-\infty}^{\infty}d^{mn}_r(a)Q^m_{m+r}(x)\qquad\text{(prolate angular functions of the second kind)},$$

where $P^m_n(x)$ and $Q^m_n(x)$ are the associated Legendre functions of the first and second kind. For $-1\le x\le1$, we have $P^m_n(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$. The summation is performed over either even or odd values of r, depending on whether $|n-m|$ is even or odd, respectively.

2°. The following recurrence relations for the coefficients $d_k=d^{mn}_k(a)$ hold:

$$\alpha_kd_{k+2}+(\beta_k-\lambda_{mn})d_k+\gamma_kd_{k-2}=0,$$

where

$$\alpha_k=\frac{a^2(2m+k+1)(2m+k+2)}{(2m+2k+3)(2m+2k+5)},$$

$$\beta_k=(m+k)(m+k+1)+a^2\frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)},$$

$$\gamma_k=\frac{a^2k(k-1)}{(2m+2k-3)(2m+2k-1)}.$$

3°. For $a\to0$, the eigenvalues are defined by:

$$\lambda_{mn}=n(n+1)+\frac{1}{2}\Big[1-\frac{(2m-1)(2m+1)}{(2n-1)(2n+3)}\Big]a^2+O(a^4).$$

4°. For $a\to\infty$, we have:

$$\lambda_{mn}=aq+m^2-\tfrac{1}{8}(q^2+5)-\tfrac{1}{64}q(q^2+11-32m^2)a^{-1}+O(a^{-2}),\qquad q=2(n-m)+1.$$

⊙ *Literature*: H. Bateman and A. Erdélyi (1955, Vol. 3), M. Abramowitz and I. A. Stegun (1964).

236. $(x^2+1)^2y''_{xx}+2x(x^2+1)y'_x+[(x^2+1)(a^2x^2-\lambda)+m^2]y=0.$

Equation of oblate spheroidal wave functions, $m=0,\ 1,\ \ldots$ The transformations $x=\pm i\widetilde{x}$, $a=\mp i\widetilde{a}$ lead to equation 14.1.2.235.

See the books by Bateman & Erdélyi (1955, Vol. 3) and Abramowitz & Stegun (1964) for more information on this equation.

237. $(ax^2 + bx + c)^2 y''_{xx} + (2ax + k)(ax^2 + bx + c) y'_x + my = 0.$

The substitution $\xi = \int \frac{dx}{ax^2 + bx + c}$ leads to a constant coefficient linear equation of the form 14.1.2.11: $y''_{\xi\xi} + (k - b) y'_\xi + my = 0$.

► **Other equations.**

238. $x^6 y''_{xx} - x^5 y'_x + ay = 0.$

The transformation $\xi = x^{-2}$, $w = yx^{-2}$ leads to a constant coefficient linear equation of the form 14.1.2.1: $4w''_{\xi\xi} + aw = 0$.

239. $x^6 y''_{xx} + (3x^2 + a) x^3 y'_x + by = 0.$

The substitution $\xi = x^{-2}$ leads to a constant coefficient linear equation: $4y''_{\xi\xi} - 2ay'_\xi + by = 0$.

240. $$y''_{xx} + y'_x \sum_{n=1}^{3} \frac{b_n(1 - \alpha_n - \beta_n)}{b_n x - a_n} - \frac{y}{(b_1 x - a_1)(b_2 x - a_2)(b_3 x - a_3)} \sum_{n=1}^{3} \alpha_n \beta_n \frac{\Delta_n \Delta_{n-1}}{b_n x - a_n} = 0.$$

Here $\sum_{n=1}^{3} (\alpha_n + \beta_n) = 1$, $|a_n| + |b_n| > 0$, $\Delta_n = a_n b_{n+1} - a_{n+1} b_n \neq 0$, $a_{n+3} = a_n$, $b_{n+3} = b_n$.

It is the *Riemann equation*. Denote this equation by:

$$\left\{ \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \middle| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right\} = 0. \tag{1}$$

For $a_1 = b_2 = 0$, $a_3 = b_3 = 1$, $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = \alpha$, $\beta_1 = 1 - \gamma$, $\beta_2 = \beta$, and $\beta_3 = \gamma - \alpha - \beta$, equation (1) transforms into the hypergeometric equation 14.1.2.171.

The transformation

$$\xi = \frac{Ax + B}{Cx + D}, \qquad w = \frac{|b_1 x - a_1|^r |b_3 x - a_3|^s}{|b_2 x - a_2|^{r+s}} y, \tag{2}$$

where $AD - BC \neq 0$, brings the original equation into an equation of similar form:

$$\left\{ \begin{matrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{matrix} \middle| \begin{matrix} \alpha_1 + r & \alpha_2 - r - s & \alpha_3 + s \\ \beta_1 + r & \beta_2 - r - s & \beta_3 + s \end{matrix} \middle| \begin{matrix} \xi \\ w \end{matrix} \right\} = 0, \tag{3}$$

where $A_n = Aa_n + Bb_n$, $B_n = Ca_n + Db_n$.

In (2), assume $r = -\alpha_1$, $s = -\alpha_3$, $A = b_1/\Delta_3$, $B = -a_1/\Delta_3$, $C = -b_2/\Delta_2$, and $D = a_2/\Delta_2$ to obtain the hypergeometric equation (3).

241. $x^n y''_{xx} + c(ax + b)^{n-4} y = 0.$

The transformation $\xi = \frac{x}{ax + b}$, $w = \frac{y}{ax + b}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + cb^{-2} \xi^{-n} w = 0$.

242. $x^n y''_{xx} + axy'_x - (b^2x^n + 2bx^{n-1} + abx + a)y = 0.$

Particular solution: $y_0 = xe^{bx}$.

243. $x^n y''_{xx} + (ax+b)y'_x - ay = 0.$

Particular solution: $y_0 = ax + b$.

244. $x^n y''_{xx} + (ax^{n-1} + bx)y'_x + (a-1)by = 0.$

Particular solution: $y_0 = x^{1-a}$.

245. $x^n y''_{xx} + (2x^{n-1} + ax^2 + bx)y'_x + by = 0.$

Particular solution: $y_0 = a + b/x$.

246. $x^n y''_{xx} + (ax^n + b)y'_x + c[(a-c)x^n + b]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

247. $x^n y''_{xx} + (ax^n - x^{n-1} + abx + b)y'_x + a^2bxy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

248. $x^n y''_{xx} + (ax^{n+m} + 1)y'_x + ax^m(1 + mx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr)$.

249. $(ax^n + b)y''_{xx} + (cx^n + d)y'_x + \lambda[(c - a\lambda)x^n + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

250. $(ax^n + bx + c)y''_{xx} = an(n-1)x^{n-2}y.$

Particular solution: $y_0 = ax^n + bx + c$.

251. $x(x^n + 1)y''_{xx} + [(a-b)x^n + a - n]y'_x + b(1-a)x^{n-1}y = 0.$

Particular solution: $y_0 = (x^n + 1)^{b/n}$.

252. $x(x^{2n} + a)y''_{xx} + (x^{2n} + a - an)y'_x - b^2x^{2n-1}y = 0.$

Solution: $y = C_1\bigl(x^n + \sqrt{x^{2n} + a}\bigr)^{b/n} + C_2\bigl(x^n + \sqrt{x^{2n} + a}\bigr)^{-b/n}$.

253. $x^2(a^2x^{2n} - 1)y''_{xx} + x[a^2(n+1)x^{2n} + n - 1]y'_x - \nu(\nu+1)a^2n^2x^{2n}y = 0.$

Solution: $y = y_\nu(ax^n)$, where $y_\nu(x)$ is the general solution of the Legendre equation 14.1.2.154.

254. $x^2(ax^n - 1)y''_{xx} + x(apx^n + q)y'_x + (arx^n + s)y = 0.$

Find the roots A_1, A_2 and B_1, B_2 of the quadratic equations

$$A^2 - (q+1)A - s = 0, \qquad B^2 - (p-1)B + r = 0$$

and define parameters c, α, β, and γ by the relations

$$c = A_1, \quad \alpha = (A_1 + B_1)n^{-1}, \quad \beta = (A_1 + B_2)n^{-1}, \quad \gamma = 1 + (A_1 - A_2)n^{-1}.$$

Then the solution of the original equation has the form $y = x^c u(ax^n)$, where $u = u(z)$ is the general solution of the hypergeometric equation 14.1.2.171:

$$z(z-1)u''_{zz} + [(\alpha + \beta + 1)z - \gamma]u'_z + \alpha\beta u = 0.$$

255. $(x^n + a)^2 y''_{xx} - bx^{n-2}[(b-1)x^n + a(n-1)]y = 0.$

Particular solution: $y_0 = |x^n + a|^{b/n}$.

256. $(ax^n+b)^2y''_{xx}+(ax^n+b)(cx^n+d)y'_x+n(bc-ad)x^{n-1}y=0.$

Particular solution: $y_0=\exp\Bigl(-\int\frac{cx^n+d}{ax^n+b}\,dx\Bigr).$

257. $(x^n+a)^2y''_{xx}+bx^m(x^n+a)y'_x-x^{n-2}(bx^{m+1}+an-a)y=0.$

Particular solution: $y_0=(x^n+a)^{1/n}.$

258. $(ax^n+b)^2y''_{xx}+cx^m(ax^n+b)y'_x+(cx^m-anx^{n-1}-1)y=0.$

Particular solution: $y_0=\exp\Bigl(-\int\frac{dx}{ax^n+b}\Bigr).$

259. $x^2(ax^n+b)^2y''_{xx}+(n+1)x(a^2x^{2n}-b^2)y'_x+cy=0.$

The substitution $\xi=\dfrac{1}{nb}\ln\Bigl(\dfrac{ax^n}{ax^n+b}\Bigr)$ leads to a constant coefficient linear equation of the form 14.1.2.11: $y''_{\xi\xi}-b(n+2)y'_\xi+cy=0.$

260. $(ax^{n+1}+bx^n+c)y''_{xx}+(\alpha x^n+\beta x^{n-1}+\gamma)y'_x$
$+[n(\alpha-a-an)x^{n-1}+(n-1)(\beta-bn)x^{n-2}]y=0.$

Particular solution: $y_0=\exp\Bigl[\int\frac{(an+a-\alpha)x^n+(bn-\beta)x^{n-1}-\gamma}{ax^{n+1}+bx^n+c}\,dx\Bigr].$

261. $(ax^n+bx^m+c)y''_{xx}+(\lambda-x)y'_x+y=0.$

Particular solution: $y_0=x-\lambda.$

262. $(ax^n+bx^m+c)y''_{xx}+(\lambda^2-x^2)y'_x+(x+\lambda)y=0.$

Particular solution: $y_0=x-\lambda.$

263. $2(ax^n+bx^m+c)y''_{xx}+(anx^{n-1}+bmx^{m-1})y'_x+dy=0.$

The substitution $\xi=\displaystyle\int\frac{dx}{\sqrt{ax^n+bx^m+c}}$ leads to a constant coefficient linear equation: $2y''_{\xi\xi}+dy=0.$

264. $(ax^n+b)^{m+1}y''_{xx}+(ax^n+b)y'_x-anmx^{n-1}y=0.$

Particular solution: $y_0=\exp\Bigl[-\int\frac{dx}{(ax^n+b)^m}\Bigr].$

265. $xP_n(x)y''_{xx}+[2P_n(x)+(ax^2+bx)Q_{n-2}(x)]y'_x+bQ_{n-2}(x)y=0.$

Here, $P_n(x)$ and $Q_{n-2}(x)$ are arbitrary polynomials of degrees n and $n-2$, respectively.

Particular solution: $y_0=a+b/x.$

14.1.3 Equations Containing Exponential Functions

► **Equations with exponential functions.**

1. $y''_{xx}+ae^{\lambda x}y=0,\qquad \lambda\neq 0.$

Solution: $y=C_1J_0(z)+C_2Y_0(z)$, where $z=2\lambda^{-1}\sqrt{a}\,e^{\lambda x/2}$; $J_0(z)$ and $Y_0(z)$ are Bessel functions.

2. $y''_{xx} + (ae^x - b)y = 0.$

Solution: $y = C_1 J_{2\sqrt{b}}\bigl(2\sqrt{a}\, e^{x/2}\bigr) + C_2 Y_{2\sqrt{b}}\bigl(2\sqrt{a}\, e^{x/2}\bigr)$, where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions.

3. $y''_{xx} + a(\lambda e^{\lambda x} - ae^{2\lambda x})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

4. $y''_{xx} - [a^2 e^{2x} + a(2b+1)e^x + b^2]y = 0.$

Particular solution: $y_0 = \exp(ae^x + bx)$.

5. $y''_{xx} - (ae^{2\lambda x} + be^{\lambda x} + c)y = 0.$

The transformation $z = e^{\lambda x}$, $w = z^{-k}y$, where $k = \sqrt{c}/\lambda$, leads to an equation of the form 14.1.2.108: $\lambda^2 z w''_{zz} + \lambda^2(2k+1)w'_z - (az+b)w = 0$.

6. $y''_{xx} + (ae^{4\lambda x} + be^{3\lambda x} + ce^{2\lambda x} - \tfrac{1}{4}\lambda^2)y = 0.$

The transformation $\xi = e^{\lambda x}$, $w = ye^{\lambda x/2}$ leads to a linear equation of the form 14.1.2.6: $w''_{\xi\xi} + \lambda^{-2}(a\xi^2 + b\xi + c)w = 0$.

7. $y''_{xx} + [ae^{2\lambda x}(be^{\lambda x} + c)^n - \tfrac{1}{4}\lambda^2]y = 0.$

The transformation $\xi = be^{\lambda x} + c$, $w = ye^{\lambda x/2}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + a(b\lambda)^{-2}\xi^n w = 0$.

8. $y''_{xx} + ay'_x + be^{2ax}y = 0.$

The transformation $\xi = e^{ax}$, $u = ye^{ax}$ leads to a constant coefficient linear equation of the form 14.1.2.1: $u''_{\xi\xi} + ba^{-2}u = 0$.

9. $y''_{xx} - ay'_x + be^{2ax}y = 0.$

The substitution $\xi = e^{ax}$ leads to a constant coefficient linear equation of the form 14.1.2.1: $y''_{\xi\xi} + ba^{-2}y = 0$.

10. $y''_{xx} + ay'_x + (be^{\lambda x} + c)y = 0.$

Solution:

$$y = e^{-ax/2}\bigl[C_1 J_\nu\bigl(2\lambda^{-1}\sqrt{b}\, e^{\lambda x/2}\bigr) + C_2 Y_\nu\bigl(2\lambda^{-1}\sqrt{b}\, e^{\lambda x/2}\bigr)\bigr], \quad \text{where} \quad \nu = \lambda^{-1}\sqrt{a^2 - 4c};$$

$J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

11. $y''_{xx} - y'_x + \bigl(ae^{3\lambda x} + be^{2\lambda x} + \tfrac{1}{4} - \tfrac{1}{4}\lambda^2\bigr)y = 0.$

The substitution $z = e^x$ leads to a second-order linear equation of the form 14.1.2.121: $z^2 y''_{zz} + \bigl(az^{3\lambda} + bz^{2\lambda} + \tfrac{1}{4} - \tfrac{1}{4}\lambda^2\bigr)y = 0$.

12. $y''_{xx} - y'_x + \bigl[ae^{2\lambda x}(be^{\lambda x} + c)^n + \tfrac{1}{4} - \tfrac{1}{4}\lambda^2\bigr]y = 0.$

The substitution $z = e^x$ leads to a second-order linear equation of the form 14.1.2.122: $z^2 y''_{zz} + \bigl[az^{2\lambda}(bz^\lambda + c)^n + \tfrac{1}{4} - \tfrac{1}{4}\lambda^2\bigr]y = 0$.

13. $y''_{xx} + 2ae^{\lambda x}y'_x + ae^{\lambda x}(ae^{\lambda x} + \lambda)y = 0.$

Solution: $y = \exp\Bigl(-\frac{a}{\lambda}e^{\lambda x}\Bigr)(C_1 + C_2 x).$

14. $y''_{xx} + (a+b)e^{\lambda x}y'_x + ae^{\lambda x}(be^{\lambda x} + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\frac{a}{\lambda}e^{\lambda x}\Bigr).$

15. $y''_{xx} + ae^{\lambda x}y'_x - be^{\mu x}(ae^{\lambda x} + be^{\mu x} + \mu)y = 0.$

Particular solution: $y_0 = \exp\Bigl(\frac{b}{\mu}e^{\mu x}\Bigr).$

16. $y''_{xx} + 2ke^{\mu x}y'_x + (ae^{2\lambda x} + be^{\lambda x} + k^2e^{2\mu x} + k\mu e^{\mu x} + c)y = 0.$

The substitution $w = y\exp\Bigl(\frac{k}{\mu}e^{\mu x}\Bigr)$ leads to a linear equation of the form 14.1.3.5: $w''_{xx} + (ae^{2\lambda x} + be^{\lambda x} + c)w = 0.$

17. $y''_{xx} - (a + 2be^{ax})y'_x + b^2e^{2ax}y = 0.$

Particular solution: $y_0 = \exp\Bigl(\frac{b}{a}e^{ax}\Bigr).$

18. $y''_{xx} + (ae^{2\lambda x} + \lambda)y'_x - a\lambda e^{2\lambda x}y = 0.$

Particular solution: $y_0 = ae^{\lambda x} + \lambda e^{-\lambda x}.$

19. $y''_{xx} + (ae^{\lambda x} - \lambda)y'_x + be^{2\lambda x}y = 0.$

The substitution $\xi = e^{\lambda x}$ leads to a constant coefficient linear equation: $\lambda^2 y''_{\xi\xi} + a\lambda y'_\xi + by = 0.$

20. $y''_{xx} + (ae^{\lambda x} + b)y'_x + c(ae^{\lambda x} + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}.$

21. $y''_{xx} + (a + be^{2\lambda x})y'_x + \lambda(a - \lambda - be^{2\lambda x})y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}.$

22. $y''_{xx} + (abe^{\lambda x} + b - 3\lambda)y'_x + a^2\lambda(b-\lambda)e^{2\lambda x}y = 0.$

Particular solution: $y_0 = (ae^{\lambda x} + 1)\exp(-ae^{\lambda x}).$

23. $y''_{xx} + (2ae^{\lambda x} - \lambda)y'_x + (a^2e^{2\lambda x} + ce^{\mu x})y = 0.$

This is a special case of equation 14.1.3.28 with $b = k = 0$.

24. $y''_{xx} + (2ae^{\lambda x} + b)y'_x + [a^2e^{2\lambda x} + a(b+\lambda)e^{\lambda x} + c]y = 0.$

The substitution $w = y\exp\Bigl(\frac{a}{\lambda}e^{\lambda x}\Bigr)$ leads to a constant coefficient linear equation of the form 14.1.2.11: $w''_{xx} + bw'_x + cw = 0.$

25. $y''_{xx} + (ae^{\lambda x} + 2b - \lambda)y'_x + (ce^{2\lambda x} + abe^{\lambda x} + b^2 - b\lambda)y = 0.$

The transformation $\xi = e^{\lambda x}/\lambda,\ w = e^{bx}y$ leads to a constant coefficient linear equation: $w''_{\xi\xi} + aw'_\xi + cw = 0.$

26. $y''_{xx}+(ae^x+b)y'_x+[c(a-c)e^{2x}+(ak+bc+c-2ck)e^x+k(b-k)]y=0.$

Particular solution: $y_0=\exp(-ce^x-kx)$.

27. $y''_{xx}+(ae^{\lambda x}+b)y'_x+(\alpha e^{2\lambda x}+\beta e^{\lambda x}+\gamma)y=0.$

The substitution $\xi=e^x$ leads to an equation of the form 14.1.2.146:

$$\xi^2y''_{\xi\xi}+(a\xi^\lambda+b+1)\xi y'_\xi+(\alpha\xi^{2\lambda}+\beta\xi^\lambda+\gamma)y=0.$$

28. $y''_{xx}+(2ae^{\lambda x}-\lambda)y'_x+(a^2e^{2\lambda x}+be^{2\mu x}+ce^{\mu x}+k)y=0.$

The substitution $w=y\exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}-\dfrac{\lambda x}{2}\Bigr)$ leads to a linear equation of the form 14.1.3.5: $w''_{xx}+\bigl(be^{2\mu x}+ce^{\mu x}+k-\frac{1}{4}\lambda^2\bigr)w=0$.

29. $y''_{xx}+(2ae^{\lambda x}+b-\lambda)y'_x+(a^2e^{2\lambda x}+abe^{\lambda x}+ce^{2\mu x}+de^{\mu x}+k)y=0.$

The substitution $w=y\exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}+\dfrac{b-\lambda}{2}x\Bigr)$ leads to an equation of the form 14.1.3.5: $w''_{xx}+\bigl[ce^{2\mu x}+de^{\mu x}+k-\frac{1}{4}(b-\lambda)^2\bigr]w=0$.

30. $y''_{xx}+(ae^{\lambda x}+be^{\mu x})y'_x+ae^{\lambda x}(be^{\mu x}+\lambda)y=0.$

Particular solution: $y_0=\exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

31. $y''_{xx}+e^{\lambda x}(ae^{2\mu x}+b)y'_x+\mu[e^{\lambda x}(b-ae^{2\mu x})-\mu]y=0.$

Particular solution: $y_0=ae^{\mu x}+be^{-\mu x}$.

32. $y''_{xx}+(ae^{\lambda x}+be^{\mu x}+c)y'_x+(a\lambda e^{\lambda x}+b\mu e^{\mu x})y=0.$

Particular solution: $y_0=\exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}-\dfrac{b}{\mu}e^{\mu x}-cx\Bigr)$.

33. $y''_{xx}+(ae^{\lambda x}+be^{\mu x}+c)y'_x+[abe^{(\lambda+\mu)x}+ace^{\lambda x}+b\mu e^{\mu x}]y=0.$

Particular solution: $y_0=\exp\Bigl(-\dfrac{b}{\mu}e^{\mu x}-cx\Bigr)$.

34. $y''_{xx}+(ae^{\lambda x}+2be^{\mu x}-\lambda)y'_x$
$+[abe^{(\lambda+\mu)x}+ce^{2\lambda x}+b^2e^{2\mu x}+b(\mu-\lambda)e^{\mu x}]y=0.$

1°. If $\lambda=0$, the equation transforms into 2.1.3.24, and if $\mu=0$, into 2.1.3.25.

2°. For $\lambda\mu\neq0$, the transformation $\xi=\dfrac{1}{\lambda}e^{\lambda x}$, $w=y\exp\Bigl(\dfrac{b}{\mu}e^{\mu x}\Bigr)$ leads to a constant coefficient linear equation: $w''_{\xi\xi}+aw'_\xi+cw=0$.

35. $y''_{xx}+[abe^{(\lambda+\mu)x}+a\lambda e^{\lambda x}+be^{\mu x}-2\lambda]y'_x+a^2b\lambda e^{(2\lambda+\mu)x}y=0.$

Particular solution: $y_0=(ae^{\lambda x}+1)\exp(-ae^{\lambda x})$.

36. $y''_{xx}+a\exp(bx^n)y'_x+c[a\exp(bx^n)-c]y=0.$

Particular solution: $y_0=e^{-cx}$.

37. $(ae^{\lambda x}+b)y''_{xx}-a\lambda^2e^{\lambda x}y=0.$

Particular solution: $y_0=ae^{\lambda x}+b$.

38. $(a^2e^{2\lambda x}+b)y''_{xx}-b\lambda y'_x-a^2\lambda^2k^2e^{2\lambda x}y=0.$

Solution: $y=C_1\bigl(ae^{\lambda x}+\sqrt{a^2e^{2\lambda x}+b}\,\bigr)^k+C_2\bigl(ae^{\lambda x}+\sqrt{a^2e^{2\lambda x}+b}\,\bigr)^{-k}.$

39. $2(ae^{\lambda x}+b)y''_{xx}+a\lambda e^{\lambda x}y'_x+cy=0.$

The substitution $\xi=\int(ae^{\lambda x}+b)^{-1/2}\,dx$ leads to a constant coefficient linear equation of the form 14.1.2.1: $2y''_{\xi\xi}+cy=0$.

40. $(ae^{\lambda x}+b)y''_{xx}+(ce^{\lambda x}+d)y'_x+k[(c-ak)e^{\lambda x}+d-bk]y=0.$

Particular solution: $y_0=e^{-kx}$.

41. $(ae^{\lambda x}+b)y''_{xx}+(ce^{\lambda x}+d)y'_x+(ne^{\lambda x}+m)y=0.$

For the case $a=0$, see equation 14.1.3.27. For $a\neq 0$, the transformation $\xi=ae^{\lambda x}$, $w=y\xi^{-k}$, where k is a root of the quadratic equation $b\lambda^2k^2+d\lambda k+m=0$, leads to an equation of the form 14.1.2.172:

$$a\lambda^2\xi(\xi+b)w''_{\xi\xi}+\lambda[(2ak\lambda+a\lambda+c)\xi+a(2bk\lambda+b\lambda+d)]w'_\xi+(ak^2\lambda^2+ck\lambda+n)w=0.$$

42. $(e^x+k)y''_{xx}+(ae^{\lambda x}+be^{\mu x}+c)y'_x+(a\lambda e^{\lambda x}+b\mu e^{\mu x}-e^x)y=0.$

Integrating yields a first-order linear equation: $(e^x+k)y'_x+(ae^{\lambda x}+be^{\mu x}-e^x+c)y=C.$

43. $(ae^{\lambda x}+b)^2y''_{xx}+ce^{\lambda x}(\lambda b-ce^{\lambda x})y=0.$

Particular solution: $y_0=(ae^{\lambda x}+b)^k$, where $k=-\dfrac{c}{a\lambda}$.

44. $(ae^{\lambda x}+b)^2y''_{xx}+\sigma(ae^{\lambda x}+b)y'_x+ce^{\lambda x}(\sigma+\lambda b-ce^{\lambda x})y=0.$

Particular solution: $y_0=(ae^{\lambda x}+b)^k$, where $k=-\dfrac{c}{a\lambda}$.

45. $(ae^{\lambda x}+b)^2y''_{xx}+(a\lambda e^{\lambda x}+c)(ae^{\lambda x}+b)y'_x+my=0.$

The substitution $\xi=\displaystyle\int\frac{dx}{ae^{\lambda x}+b}$ leads to a constant coefficient linear equation of the form 14.1.2.11: $y''_{\xi\xi}+cy'_\xi+my=0$.

46. $(ae^{\lambda x}+b)^2y''_{xx}+ke^{\mu x}(ae^{\lambda x}+b)y'_x+ce^{\lambda x}(ke^{\mu x}-ce^{\lambda x}+\lambda b)y=0.$

Particular solution: $y_0=(ae^{\lambda x}+b)^k$, where $k=-\dfrac{c}{a\lambda}$.

47. $4(ae^{\lambda x}+b)^ny''_{xx}+[ke^{2\lambda x}(ce^{\lambda x}+d)^{n-4}-\lambda^2(ae^{\lambda x}+b)^n]y=0.$

The transformation $\xi=\dfrac{ae^{\lambda x}+b}{ce^{\lambda x}+d}$, $w=\dfrac{ye^{\lambda x/2}}{ce^{\lambda x}+d}$ leads to an equation of the form 14.1.2.7: $4w''_{\xi\xi}+k(\Delta\lambda)^{-2}\xi^{-n}w=0$, where $\Delta=ad-bc$.

► **Equations with power and exponential functions.**

48. $y''_{xx}+ae^{\lambda x}y'_x+b(ax^ne^{\lambda x}-bx^{2n}+nx^{n-1})y=0.$

Particular solution: $y_0=\exp\Bigl(-\dfrac{b}{n+1}x^{n+1}\Bigr)$.

49. $y''_{xx} + 2ae^{\lambda x}y'_x + (a^2e^{2\lambda x} + a\lambda e^{\lambda x} + bx^{2n} + cx^{n-1})y = 0.$

The substitution $w = y\exp\left(\frac{a}{\lambda}e^{\lambda x}\right)$ leads to a linear equation of the form 14.1.2.10: $w''_{xx} + (bx^{2n} + cx^{n-1})w = 0$.

50. $y''_{xx} + (ax + b)e^{\lambda x}y'_x - ae^{\lambda x}y = 0.$

Particular solution: $y_0 = ax + b$.

51. $y''_{xx} + (axe^{\lambda x} + 2b)y'_x + (abxe^{\lambda x} - ae^{\lambda x} + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

52. $y''_{xx} + x(ae^{\lambda x} + be^{\mu x})y'_x - (ae^{\lambda x} + be^{\mu x})y = 0.$

Particular solution: $y_0 = x$.

53. $y''_{xx} + (ax^n + be^{\lambda x})y'_x + (abx^ne^{\lambda x} + anx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\left(-\frac{a}{n+1}x^{n+1}\right)$.

54. $y''_{xx} + (ax + b)\exp(\lambda x^n)y'_x - a\exp(\lambda x^n)y = 0.$

Particular solution: $y_0 = ax + b$.

55. $y''_{xx} + ax^n\exp(bx^m)y'_x - ax^{n-1}\exp(bx^m)y = 0.$

Particular solution: $y_0 = x$.

56. $xy''_{xx} - (2ax^2 + 1)y'_x + 4bx^3\exp(2\lambda x^2)y = 0.$

Solution:

$$y = \exp\left(\tfrac{1}{2}ax^2\right)\left[C_1J_{\frac{a}{2\lambda}}(z) + C_2Y_{\frac{a}{2\lambda}}(z)\right], \quad \text{where} \quad z = \lambda^{-1}\sqrt{b}\,\exp\left(\lambda x^2\right);$$

$J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

57. $xy''_{xx} + axe^{\lambda x}y'_x + ae^{\lambda x}(1 + \lambda x)y = 0.$

Particular solution: $y_0 = x\exp\left(-\frac{a}{\lambda}e^{\lambda x}\right)$.

58. $xy''_{xx} + axe^{\lambda x}y'_x - [a(bx + 1)e^{\lambda x} + b(bx + 2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

59. $xy''_{xx} + (axe^{\lambda x} + b)y'_x + a(b - 1)e^{\lambda x}y = 0.$

Particular solution: $y_0 = x^{1-b}$.

60. $xy''_{xx} + [a(bx + 1)e^{\lambda x} + bx - 1]y'_x + ab^2xe^{\lambda x}y = 0.$

Particular solution: $y_0 = (bx + 1)e^{-bx}$.

61. $xy''_{xx} + [(ax^2 + bx)e^{\lambda x} + 2]y'_x + be^{\lambda x}y = 0.$

Particular solution: $y_0 = a + b/x$.

62. $xy''_{xx} + (ax^n + be^{\lambda x})y'_x + ax^{n-1}(be^{\lambda x} + n - 1)y = 0.$

Particular solution: $y_0 = \exp(-ax^n/n)$.

63. $xy''_{xx} + (axe^{\lambda x} + bx^n)y'_x + [a(bx^n - 1)e^{\lambda x} + bnx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-bx^n/n)$.

64. $xy''_{xx} + [(ax^n + 1)e^{\lambda x} + anx^n + 1 - 2n]y'_x + a^2nx^{2n-1}e^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n)$.

65. $xy''_{xx} + (ae^{\lambda x} + be^{\mu x})y'_x + (a\lambda e^{\lambda x} + b\mu e^{\mu x})y = 0.$

Integrating, we obtain a first-order linear equation: $xy'_x + (ae^{\lambda x} + be^{\mu x} - 1)y = C$.

66. $xy''_{xx} + [ax^n\exp(bx^m) + c]y'_x + a(c-1)x^{n-1}\exp(bx^m)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

67. $(x+a)y''_{xx} + (be^{\lambda x} + c)y'_x + b\lambda e^{\lambda x}y = 0.$

Particular solution: $y_0 = \exp\Bigl(\int \dfrac{1 - c - be^{\lambda x}}{x+a}\,dx\Bigr)$.

68. $4x^2y''_{xx} + [ax^{2n}\exp(bx^n) + 1 - n^2]y = 0.$

The transformation $\xi = bx^n$, $w = yx^{\frac{n-1}{2}}$ leads to a linear equation of the form 14.1.3.1: $4w''_{\xi\xi} + a(bn)^{-2}e^{\xi}w = 0$.

69. $x^2y''_{xx} + 2axy'_x + [(b^2e^{2cx} - \nu^2)c^2x^2 + a(a-1)]y = 0.$

Solution: $y = x^{-a}\bigl[C_1J_\nu(be^{cx}) + C_2Y_\nu(be^{cx})\bigr]$, where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

70. $x^2y''_{xx} + axe^{\lambda x}y'_x + b(ae^{\lambda x} - b - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

71. $x^2y''_{xx} + x(ae^{\lambda x} + 2b)y'_x + [a(cx+b)e^{\lambda x} - c^2x^2 + b(b-1)]y = 0.$

Particular solution: $y_0 = x^{-b}e^{-cx}$.

72. $x^4y''_{xx} + (e^{2/x} - \nu^2)y = 0.$

Solution: $y = x\bigl[C_1J_\nu(e^{1/x}) + C_2Y_\nu(e^{1/x})\bigr]$, where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

73. $x^4y''_{xx} + [a\exp(2\lambda/x) + b\exp(\lambda/x) + c]y = 0.$

The transformation $\xi = 1/x$, $w = y/x$ leads to a linear equation of the form 14.1.3.5: $w''_{\xi\xi} + (ae^{2\lambda\xi} + be^{\lambda\xi} + c)w = 0$.

74. $x^4y''_{xx} + ax^2e^{\lambda x}y'_x + [a(b-x)e^{\lambda x} - b^2]y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

75. $(x^2+a)^2y''_{xx} + be^{\lambda x}(x^2+a)y'_x - (bxe^{\lambda x} + a)y = 0.$

Particular solution: $y_0 = \sqrt{x^2 + a}$.

76. $(x^n+a)^2y''_{xx}+b(x^n+a)e^{\lambda x}y'_x-x^{n-2}(bxe^{\lambda x}+an-a)y=0.$

Particular solution: $y_0=(x^n+a)^{1/n}$.

77. $(ax^n+b)^2y''_{xx}+c(ax^n+b)e^{\lambda x}y'_x+(ce^{\lambda x}-anx^{n-1}-1)y=0.$

Particular solution: $y_0=\exp\Bigl(-\int\frac{dx}{ax^n+b}\Bigr)$.

78. $(ae^{\lambda x}+bx+c)y''_{xx}-a\lambda^2e^{\lambda x}y=0.$

Particular solution: $y_0=ae^{\lambda x}+bx+c$.

79. $[(ax+b)e^{\lambda x}+c]y''_{xx}-c\lambda^2y=0.$

Particular solution: $y_0=ce^{-\lambda x}+ax+b$.

14.1.4 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine.**

1. $y''_{xx}+(a\sinh^2x+b)y=0.$

Applying the formula $\sinh^2x=\frac{1}{2}\cosh 2x-\frac{1}{2}$, we obtain the modified Mathieu equation 14.1.4.9: $y''_{xx}+\bigl(b-\frac{1}{2}a+\frac{1}{2}a\cosh 2x\bigr)y=0$.

2. $y''_{xx}+a\sinh(\lambda x)y'_x+b[a\sinh(\lambda x)-b]y=0.$

Particular solution: $y_0=e^{-bx}$.

3. $y''_{xx}+[a\sinh^n(\lambda x)+c]y'_x+ab\sinh^n(\lambda x)y=0.$

Particular solution: $y_0=e^{-bx}$.

4. $y''_{xx}+(ax+b)\sinh^n(\lambda x)y'_x-a\sinh^n(\lambda x)y=0.$

Particular solution: $y_0=ax+b$.

5. $xy''_{xx}+(a\sinh^nx+bx^{m+1})y'_x+bx^m(a\sinh^nx+m)y=0.$

Particular solution: $y_0=\exp\Bigl(-\frac{b}{m+1}x^{m+1}\Bigr)$.

6. $\sinh^2(ax)y''_{xx}-by=0.$

The substitution $ax=\pm\ln\frac{z}{\sqrt{z^2+1}}$ $(z>0)$ leads to a linear equation of the form 14.1.2.190: $z(z^2+1)y''_{zz}+(3z^2+1)y'_z-4a^{-2}bzy=0$.

7. $y''_{xx}\sinh^2x-[a^2\sinh^2x+n(n-1)]y=0,\qquad a\neq 0;\quad n=1,\ 2,\ 3,\ \ldots$

Solution: $y=\sinh^nx\Bigl(\frac{1}{\sinh x}\frac{d}{dx}\Bigr)^n(C_1e^{ax}+C_2e^{-ax})$.

8. $[a\sinh(\lambda x)+bx+c]y''_{xx}-a\lambda^2\sinh(\lambda x)y=0.$

Particular solution: $y_0=a\sinh(\lambda x)+bx+c$.

► **Equations with hyperbolic cosine.**

9. $y''_{xx} - (a - 2q\cosh 2x)y = 0.$

The modified Mathieu equation. The substitution $x = i\xi$ leads to the Mathieu equation 14.1.6.29:

$$y''_{\xi\xi} + (a - 2q\cos 2\xi)y = 0.$$

For eigenvalues $a = a_n(q)$ and $a = b_n(q)$, the corresponding solutions of the modified Mathieu equation are:

$$\mathrm{Ce}_{2n+p}(x,q) = \mathrm{ce}_{2n+p}(ix,q) = \sum_{k=0}^{\infty} A_{2k+p}^{2n+p}\cosh[(2k+p)x],$$

$$\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q) = \sum_{k=0}^{\infty} B_{2k+p}^{2n+p}\sinh[(2k+p)x],$$

where p can be either 0 or 1, and the coefficients A_{2k+p}^{2n+p} and B_{2k+p}^{2n+p} are specified in 14.1.6.29.

The modified Mathieu equation is discussed in the books by Abramowitz & Stegun (1964), Bateman & Erdélyi (1955, vol. 3), and McLachlan (1947) in more detail.

10. $y''_{xx} + (a\cosh^2 x + b)y = 0.$

Applying the formula $\cosh 2x = 2\cosh^2 x - 1$, we obtain the modified Mathieu equation 14.1.4.9: $y''_{xx} + \left(\frac{1}{2}a + b + \frac{1}{2}a\cosh 2x\right)y = 0.$

11. $y''_{xx} + a\cosh(\lambda x)y'_x + b[a\cosh(\lambda x) - b]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

12. $y''_{xx} + [a\cosh^n(\lambda x) + c]y'_x + ab\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-bx}$.

13. $y''_{xx} + (ax + b)\cosh^n(\lambda x)y'_x - a\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

14. $y''_{xx} + ax^n\cosh^m(\lambda x)y'_x - ax^{n-1}\cosh^m(\lambda x)y = 0.$

Particular solution: $y_0 = x$.

15. $xy''_{xx} + ax\cosh^n(\lambda x)y'_x - [a(bx+1)\cosh^n(\lambda x) + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

16. $xy''_{xx} + [ax\cosh^n(\lambda x) + b]y'_x + a(b-1)\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = x^{1-b}$.

17. $xy''_{xx} + [(ax^2 + bx)\cosh^n(\lambda x) + 2]y'_x + b\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = a + b/x$.

18. $x^2y''_{xx} + ax\cosh^n(\lambda x)y'_x + b[a\cosh^n(\lambda x) - b - 1]y = 0.$

Particular solution: $y_0 = x^{-b}$.

19. $(a\cosh x + b)y''_{xx} + (c\cosh x + d)y'_x + \lambda[(c - a\lambda)\cosh x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

20. $\cosh^2(ax)y''_{xx} - by = 0.$

The substitution $x = \dfrac{1}{2a}\ln\dfrac{z}{1-z}$ $(0 < z < 1)$ leads to the hypergeometric equation 14.1.2.171: $z(z-1)y''_{zz} + (2z-1)y'_z + a^{-2}by = 0$.

21. $[a\cosh(\lambda x) + bx + c]y''_{xx} - a\lambda^2\cosh(\lambda x)y = 0.$

Particular solution: $y_0 = a\cosh(\lambda x) + bx + c$.

► **Equations with hyperbolic tangent.**

22. $y''_{xx} + [a\tanh(\lambda x) + b]y = 0.$

The transformation $z = \dfrac{1-\tanh(\lambda x)}{1+\tanh(\lambda x)}$, $w = yz^{-k/\lambda}$, where k is a root of the quadratic equation $4k^2 + b - a = 0$, leads to a linear equation of the form 14.1.2.172:

$$4\lambda^2 z(z+1)w''_{zz} + 4\lambda(2k+\lambda)(z+1)w'_z + (4k^2 + a + b)w = 0.$$

23. $y''_{xx} - 4a^2\tanh^2(3ax)y = 0.$

Particular solution: $y_0 = \sinh(3ax)[\cosh(3ax)]^{-1/3}$.

24. $y''_{xx} + [a\lambda - a(a+\lambda)\tanh^2(\lambda x)]y = 0.$

Particular solution: $y_0 = [\cosh(\lambda x)]^{-a/\lambda}$.

25. $y''_{xx} + [3a\lambda - \lambda^2 - a(a+\lambda)\tanh^2(\lambda x)]y = 0.$

Particular solution: $y_0 = \sinh(\lambda x)[\cosh(\lambda x)]^{-a/\lambda}$.

26. $y''_{xx} + ay'_x - \lambda[\lambda + a\tanh(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x)$.

27. $y''_{xx} + 2\tanh x\, y'_x + ay = 0.$

Solution: $y\cosh x = \begin{cases} C_1\cos(bx) + C_2\sin(bx) & \text{if } a - 1 = b^2 > 0, \\ C_1\cosh(bx) + C_2\sinh(bx) & \text{if } a - 1 = -b^2 < 0. \end{cases}$

28. $y''_{xx} + a\tanh(\lambda x)y'_x + b[a\tanh(\lambda x) - b]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

29. $y''_{xx} + 2\tanh x\, y'_x + (ax^2 + bx + c)y = 0.$

The substitution $w = y\cosh x$ leads to a second-order linear equation of the form 14.1.2.6: $w''_{xx} + (ax^2 + bx + c - 1)w = 0$.

30. $y''_{xx} + 2\tanh x\, y'_x + (ax^n + 1)y = 0.$

The substitution $w = y\cosh x$ leads to a linear equation of the form 14.1.2.7: $w''_{xx} + ax^n w = 0$.

31. $y''_{xx} + 2\tanh x\, y'_x + (ax^{2n} + bx^{n-1} + 1)y = 0.$

The substitution $w = y\cosh x$ leads to a second-order linear equation of the form 14.1.2.10: $w''_{xx} + (ax^{2n} + bx^{n-1})w = 0$.

32. $y''_{xx} + (2\tanh x + a)y'_x + (a\tanh x + b)y = 0.$

The substitution $w = y\cosh x$ leads to a constant coefficient linear equation: $w''_{xx} + aw'_x + (b-1)w = 0$.

33. $y''_{xx} + a\tanh^n(\lambda x)y'_x - \lambda[\lambda + a\tanh^{n+1}(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x)$.

34. $y''_{xx} + [a\tanh^n(\lambda x) + c]y'_x + ab\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-bx}$.

35. $y''_{xx} + (ax + b)\tanh^n(\lambda x)y'_x - a\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

36. $y''_{xx} + ax^n\tanh^m(\lambda x)y'_x - ax^{n-1}\tanh^m(\lambda x)y = 0.$

Particular solution: $y_0 = x$.

37. $xy''_{xx} + ax\tanh^n(\lambda x)y'_x - [a(bx+1)\tanh^n(\lambda x) + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

38. $xy''_{xx} + [ax\tanh^n(\lambda x) + b]y'_x + a(b-1)\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = x^{1-b}$.

39. $xy''_{xx} + [(ax^2 + bx)\tanh^n(\lambda x) + 2]y'_x + b\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = a + b/x$.

40. $xy''_{xx} + (a\tanh^n x + bx^{m+1})y'_x + bx^m(a\tanh^n x + m)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{m+1}x^{m+1}\Bigr)$.

41. $x^2y''_{xx} + ax\tanh^n(\lambda x)y'_x + b[a\tanh^n(\lambda x) - b - 1]y = 0.$

Particular solution: $y_0 = x^{-b}$.

42. $(a\tanh x + b)y''_{xx} + (c\tanh x + d)y'_x + \lambda[(c - a\lambda)\tanh x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

43. $[a\tanh(\lambda x) + b]y''_{xx} + [c\tanh(\lambda x) + d]y'_x + [n\tanh(\lambda x) + m]y = 0.$

The transformation $z = \dfrac{1 + \tanh(\lambda x)}{1 - \tanh(\lambda x)}$, $w = yz^{-k/\lambda}$, where k is a root of the quadratic equation $4(a-b)k^2 + 2(c-d)k + n - m = 0$, leads to a linear equation of the form 14.1.2.172:

$$4\lambda^2 z[(a+b)z + b - a]w''_{zz} + 2\lambda\{[2(2k+\lambda)(a+b) + c + d]z + 2(2k+\lambda)(b-a) + d - c\}w'_z + [4(a+b)k^2 + 2(c+d)k + n + m]w = 0.$$

► Equations with hyperbolic cotangent.

44. $y''_{xx} + [a \coth(\lambda x) + b]y = 0.$

The transformation $z = \dfrac{1 - \tanh(\lambda x)}{1 + \tanh(\lambda x)}$, $w = yz^{-k/\lambda}$, where k is a root of the quadratic equation $4k^2 + b - a = 0$, leads to an equation of the form 14.1.2.172:

$$4\lambda^2 z(z-1)w''_{zz} + 4\lambda(2k+\lambda)(z-1)w'_z + (4k^2 + a + b)w = 0.$$

45. $y''_{xx} - 4a^2 \coth^2(3ax)y = 0.$

Particular solution: $y_0 = \cosh(3ax)[\sinh(3ax)]^{-1/3}$.

46. $y''_{xx} + [a\lambda - a(a+\lambda)\coth^2(\lambda x)]y = 0.$

Particular solution: $y_0 = [\sinh(\lambda x)]^{-a/\lambda}$.

47. $y''_{xx} + [3a\lambda - \lambda^2 - a(a+\lambda)\coth^2(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x)[\sinh(\lambda x)]^{-a/\lambda}$.

48. $y''_{xx} + a\coth(\lambda x)y'_x + b[a\coth(\lambda x) - b]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

49. $y''_{xx} + [a\coth^n(\lambda x) + c]y'_x + ab\coth^n(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-bx}$.

50. $y''_{xx} + (ax+b)\coth^n(\lambda x)y'_x - a\coth^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

51. $y''_{xx} + 2n\coth x\, y'_x + (n^2 - a^2)y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $y = \left(\dfrac{1}{\sinh x}\dfrac{d}{dx}\right)^n (C_1 e^{ax} + C_2 e^{-ax})$.

52. $[a + b\coth(\lambda x)]y''_{xx} + [c + d\coth(\lambda x)]y'_x + [n + m\coth(\lambda x)]y = 0.$

Multiply this equation by $\tanh(\lambda x)$ to obtain equation 14.1.4.43.

► Equations containing combinations of hyperbolic functions.

53. $y''_{xx} - a[a\cosh^2(bx) + b\sinh(bx)]y = 0.$

Particular solution: $y_0 = \exp\left[\dfrac{a}{b}\sinh(bx)\right]$.

54. $y''_{xx} - a[a\sinh^2(bx) + b\cosh(bx)]y = 0.$

Particular solution: $y_0 = \exp\left[\dfrac{a}{b}\cosh(bx)\right]$.

55. $y''_{xx} + (a\cosh^2 x + b\sinh^2 x + c)y = 0.$

Apply the formulas $2\sinh^2 x = \cosh(2x) - 1$ and $2\cosh^2 x = \cosh(2x) + 1$ to obtain an equation of the form 14.1.4.9: $y''_{xx} + \left[\dfrac{a-b}{2} + c + \dfrac{a+b}{2}\cosh(2x)\right]y = 0$.

56. $y''_{xx} + a\sinh(\lambda x)y'_x - \lambda[\lambda + a\cosh(\lambda x)]y = 0.$

Particular solution: $y_0 = \sinh(\lambda x)$.

57. $y''_{xx} + a\cosh(\lambda x)y'_x - \lambda[\lambda + a\sinh(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x)$.

58. $y''_{xx} - \lambda\tanh(\lambda x)y'_x - a^2\cosh^2(\lambda x)y = 0.$

Solution: $y = C_1\exp\Big[\frac{a}{\lambda}\sinh(\lambda x)\Big] + C_2\exp\Big[-\frac{a}{\lambda}\sinh(\lambda x)\Big]$.

59. $y''_{xx} - \tanh x\, y'_x + a^2\coth^2 x\,(\sinh x)^{2m-2}y = 0.$

Solution: $y = \sqrt{\sinh x}\Big[C_1 J_{\frac{1}{2m}}\Big(\frac{a}{m}\sinh^m x\Big) + C_2 Y_{\frac{1}{2m}}\Big(\frac{a}{m}\sinh^m x\Big)\Big]$, where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

60. $\sinh^n(\lambda x)y''_{xx} + [a\cosh^{n-4}(\lambda x) - \lambda^2\sinh^n(\lambda x)]y = 0.$

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + a\lambda^{-2}\xi^{-n}w = 0$.

61. $\cosh^n(\lambda x)y''_{xx} + [a\sinh^{n-4}(\lambda x) - \lambda^2\cosh^n(\lambda x)]y = 0.$

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + a\lambda^{-2}\xi^{-n}w = 0$.

14.1.5 Equations Containing Logarithmic Functions

► **Equations of the form** $f(x)y''_{xx} + g(x)y = 0$.

1. $y''_{xx} - (a^2x^2\ln^2 x + a\ln x + a)y = 0.$

Particular solution: $y_0 = e^{-ax^2/4}x^{ax^2/2}$.

2. $y''_{xx} - (a^2x^{2n}\ln^2 x + anx^{n-1}\ln x + ax^{n-1})y = 0.$

Particular solution: $y_0 = e^{-F}x^{(n+1)F}$, where $F = \dfrac{ax^{n+1}}{(n+1)^2}$.

3. $xy''_{xx} - (a^2x\ln^2 x + a)y = 0.$

Particular solution: $y_0 = e^{-ax}x^{ax}$.

4. $xy''_{xx} - [a^2x\ln^{2n}(bx) + an\ln^{n-1}(bx)]y = 0.$

Particular solution: $y_0 = \exp\Big[a\int \ln^n(bx)\,dx\Big]$.

5. $x^2y''_{xx} + (a\ln x + b)y = 0.$

The transformation $\xi = a\ln x + b - \frac{1}{4}$, $w = yx^{-1/2}$ leads to an equation of the form 14.1.2.2: $w''_{\xi\xi} + a^{-2}\xi w = 0$.

6. $x^2 y''_{xx} + (a\ln^2 x + b\ln x + c)y = 0.$

The transformation $\xi = \ln x$, $w = yx^{-1/2}$ leads to an equation of the form 14.1.2.6: $w''_{\xi\xi} + (a\xi^2 + b\xi + c - \frac{1}{4})w = 0$.

7. $x^2 y''_{xx} + \big[a(b\ln x + c)^n + \frac{1}{4}\big]y = 0.$

The transformation $\xi = b\ln x + c$, $w = yx^{-1/2}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + ab^{-2}\xi^n w = 0$.

8. $x^2 \ln(ax) y''_{xx} + y = 0.$

Solution: $y = C_1\ln(ax) + C_2\ln(ax)\int[\ln(ax)]^{-2}\,dx.$

9. $x(ax\ln x + bx + c)y''_{xx} - ay = 0.$

Particular solution: $y_0 = ax\ln x + bx + c$.

10. $x^2(a\ln x + bx + c)y''_{xx} + ay = 0.$

Particular solution: $y_0 = a\ln x + bx + c$.

► **Equations of the form** $f(x)y''_{xx} + g(x)y'_x + h(x)y = 0.$

11. $y''_{xx} + a\ln^n(bx)y'_x + c[a\ln^n(bx) - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

12. $y''_{xx} + [a\ln^n(bx) + c]y'_x + ac\ln^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

13. $y''_{xx} + (ax + b)\ln^n(cx)y'_x - a\ln^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

14. $y''_{xx} + ax^n\ln^m(bx)y'_x - ax^{n-1}\ln^m(bx)y = 0.$

Particular solution: $y_0 = x$.

15. $xy''_{xx} + ax\ln x\,y'_x + a(\ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{1-ax}$.

16. $xy''_{xx} + (ax\ln x + b)y'_x + (ab\ln x + a)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

17. $xy''_{xx} + (2ax\ln x + 1)y'_x + (a^2x\ln^2 x + a\ln x + a)y = 0.$

Solution: $y = e^{ax}x^{-ax}(C_1 + C_2\ln x)$.

18. $xy''_{xx} + \ln x(ax + b)y'_x + a(b\ln^2 x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

19. $xy''_{xx} + ax\ln^n(bx)y'_x + an\ln^{n-1}(bx)y = 0.$

Particular solution: $y_0 = \exp\Big[-a\int\ln^n(bx)\,dx\Big].$

20. $xy''_{xx} + ax\ln^n x\,y'_x + (a\ln^n x + an\ln^{n-1} x)y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-a\int \ln^n x\,dx\Bigr)$.

21. $xy''_{xx} + (ax^n \ln x + 1)y'_x - ax^{n-1}y = 0.$

Particular solution: $y_0 = \ln x$.

22. $xy''_{xx} + (ax\ln^n x + 1)y'_x - a\ln^{n-1} x\,y = 0.$

Particular solution: $y_0 = \ln x$.

23. $xy''_{xx} + (ax\ln^n x + b)y'_x + a(b-1)\ln^n x\,y = 0.$

Particular solution: $y_0 = x^{1-b}$.

24. $xy''_{xx} + [(ax^2 + bx)\ln^n(cx) + 2]y'_x + b\ln^n(cx)y = 0.$

Particular solution: $y_0 = a + b/x$.

25. $xy''_{xx} + (ax^n + b\ln^m x)y'_x + ax^{n-1}(b\ln^m x + n - 1)y = 0.$

Particular solution: $y_0 = \exp(-ax^n/n)$.

26. $xy''_{xx} + (ax^n + bx\ln^m x)y'_x + [b(ax^n - 1)\ln^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

27. $x^2y''_{xx} + xy'_x + a\ln^n(bx)y = 0.$

Solution:

$$y = \sqrt{\ln(bx)}\left[C_1 J_{\frac{1}{2m}}\left(\frac{\sqrt{a}}{m}\ln^m(bx)\right) + C_2 Y_{\frac{1}{2m}}\left(\frac{\sqrt{a}}{m}\ln^m(bx)\right)\right], \quad m = \frac{1}{2}(n+2),$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

28. $x^2y''_{xx} + xy'_x + (a\ln^{2n} x + b\ln^{n-1} x)y = 0.$

The substitution $\xi = \ln x$ leads to an equation of the form 14.1.2.10:

$$y''_{\xi\xi} + (a\xi^{2n} + b\xi^{n-1})y = 0.$$

29. $x^2y''_{xx} + x(2a\ln x + 1)y'_x + (x^2 + a^2\ln^2 x + b)y = 0.$

The substitution $y = w\exp(-\frac{1}{2}a\ln^2 x)$ leads to the Bessel equation 14.1.2.126: $x^2w''_{\xi\xi} + xw'_\xi + (x^2 + b - a)w = 0$.

30. $x^2y''_{xx} + x(2\ln x + a + 1)y'_x + (\ln^2 x + a\ln x + b)y = 0.$

The transformation $\xi = \ln x$, $w = y\exp(\frac{1}{2}\ln^2 x)$ leads to a constant coefficient linear equation: $w''_{\xi\xi} + aw'_\xi + (b-1)w = 0$.

31. $x^2y''_{xx} + x(2\ln x + a)y'_x + [\ln^2 x + (a-1)\ln x + bx^n + c]y = 0.$

The substitution $w = y\exp(\frac{1}{2}\ln^2 x)$ leads to a linear equation of the form 14.1.2.132: $x^2w''_{xx} + axw'_x + (bx^n + c - 1)w = 0$.

32. $x^2 y''_{xx} + ax\ln^n(bx)y'_x + c[a\ln^n(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

33. $x^2 y''_{xx} + x(ax^n + b\ln x)y'_x + b(ax^n \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}b\ln^2 x\bigr)$.

34. $x(x+a)y''_{xx} + x(b\ln x + c)y'_x + by = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int \frac{b\ln x + c - 1}{x+a}\,dx\Bigr)$.

35. $x^4 y''_{xx} + ax^2\ln^n(bx)y'_x + [a(c-x)\ln^n(bx) - c^2]y = 0.$

Particular solution: $y_0 = x\exp(c/x)$.

36. $(a\ln x + b)y''_{xx} + (c\ln x + d)y'_x + \lambda[(c - a\lambda)\ln x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

37. $x\ln x\, y''_{xx} - ny'_x - a^2 x(\ln x)^{2n+1}y = 0.$

Solution: $y = C_1 e^{aF} + C_2 e^{-aF}$, where $F = \int \ln^n x\,dx$.

38. $x\ln(ax)y''_{xx} - [n\ln(ax) + m]y'_x - b^2 x^{2n+1}\ln^{2m+1}(ax)y = 0.$

Solution: $y = C_1 e^{bF} + C_2 e^{-bF}$, where $F = \int x^n \ln^m(ax)\,dx$.

39. $x\ln^2 x\, y''_{xx} + (ax+1)\ln x\, y'_x + bxy = 0.$

The substitution $\xi = \int \frac{dx}{\ln x}$ leads to a constant coefficient linear equation: $y''_{\xi\xi} + ay'_\xi + by = 0$.

40. $\ln^n(ax)y''_{xx} + (b^2 - x^2)y'_x + (x+b)y = 0.$

Particular solution: $y_0 = x - b$.

14.1.6 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y''_{xx} + a^2 y = b\sin(\lambda x).$

Equation of forced oscillations.

$$\text{Solution:}\quad y = \begin{cases} C_1\sin(ax) + C_2\cos(ax) + \dfrac{b}{a^2-\lambda^2}\sin(\lambda x) & \text{if } a \neq \lambda, \\ C_1\sin(ax) + C_2\cos(ax) - \dfrac{b}{2a}x\cos(ax) & \text{if } a = \lambda. \end{cases}$$

2. $y''_{xx} + [a\sin(\lambda x) + b]y = 0.$

Applying the substitution $\lambda x = 2\xi + \frac{1}{2}\pi$, we obtain the Mathieu equation 14.1.6.29: $y''_{\xi\xi} + (4a\lambda^{-2}\cos 2\xi + 4b\lambda^{-2})y = 0$.

3. $y''_{xx} + (a\sin^2 x + b)y = 0.$

Applying the formula $2\sin^2 x = 1 - \cos 2x$, we obtain the Mathieu equation 14.1.6.29: $y''_{xx} + (\frac{1}{2}a + b - \frac{1}{2}a\cos 2x)y = 0$.

4. $y''_{xx} + a\sin(bx)y'_x + c[ax^n\sin(bx) - cx^{2n} + nx^{n-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{n+1}x^{n+1}\Bigr)$.

5. $y''_{xx} + a\sin^n(bx)y'_x + c[a\sin^n(bx) - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

6. $y''_{xx} + [a\sin^n(bx) + c]y'_x + ac\sin^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

7. $y''_{xx} + (ax + b)\sin^n(cx)y'_x - a\sin^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

8. $y''_{xx} + ax^n\sin^m(bx)y'_x - ax^{n-1}\sin^m(bx)y = 0.$

Particular solution: $y_0 = x$.

9. $y''_{xx} + ax^n\sin^m(bx)y'_x + c[ax^{n+k}\sin^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

10. $xy''_{xx} + [(ax^2 + bx)\sin^n(cx) + 2]y'_x + b\sin^n(cx)y = 0.$

Particular solution: $y_0 = a + b/x$.

11. $xy''_{xx} + (ax^{n+1} + b\sin^m x)y'_x + ax^n(b\sin^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

12. $xy''_{xx} + (ax^n + bx\sin^m x)y'_x + [b(ax^n - 1)\sin^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

13. $xy''_{xx} + ax^n\sin^m(bx)\,y'_x - [a(cx + 1)x^{n-1}\sin^m(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

14. $xy''_{xx} + [ax^n\sin^m(bx) + c]y'_x + a(c - 1)x^{n-1}\sin^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

15. $x^2y''_{xx} + x(a\sin^n x + 1)y'_x + b(a\sin^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

16. $x^2y''_{xx} + x(a\sin^n x + b)y'_x + b(a\sin^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

17. $x^2y''_{xx} + ax^n\sin^m(bx)y'_x + c[ax^{n-1}\sin^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

18. $x^4 y''_{xx} + [a\sin(\lambda/x) + b]y = 0.$

The transformation $\xi = 1/x$, $w = y/x$ leads to a linear equation of the form 14.1.6.2: $w''_{\xi\xi} + [a\sin(\lambda\xi) + b]w = 0$.

19. $x^4 y''_{xx} + ax^2 \sin^n(bx)\, y'_x + [a(c - x)\sin^n(bx) - c^2]y = 0.$

Particular solution: $y_0 = x\exp(c/x)$.

20. $\sin(2x)\, y''_{xx} - y'_x + 2a^2 \sin^2 x\, y = 0.$

Solution: $y = C_1 \sin(au) + C_2 \cos(au)$, where $u = \int \sqrt{\tan x}\, dx$.

21. $\sin^2 x\, y''_{xx} + ay = 0.$

This is a special case of equation 14.1.6.23.

22. $\sin^2 x\, y''_{xx} - [a\sin^2 x + n(n-1)]y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $y = \sin^n x \Big(\frac{1}{\sin x}\frac{d}{dx}\Big)^n \big(C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}}\big)$.

23. $\sin^2 x\, y''_{xx} + (a\sin^2 x + b)y = 0.$

Set $x = 2\xi$. Applying the trigonometric formulas

$$\sin 2\xi = 2\sin\xi\cos\xi, \quad b = b(\sin^2\xi + \cos^2\xi)^2$$

and dividing both sides of the equation by $\sin^2 x$, we arrive at an equation of the form 14.1.6.131: $y''_{\xi\xi} + (b\tan^2\xi + b\cot^2\xi + 4a + 2b)y = 0$.

24. $\sin^2 x\, y''_{xx} - \{[(a^2 b^2 - (a+1)^2]\sin^2 x + a(a+1)b\sin 2x + a(a-1)\}y = 0.$

Particular solution: $y_0 = e^{abx}\sin^a x\,(\cos x + b\sin x)$.

25. $[a\sin(\lambda x) + bx + c]y''_{xx} + a\lambda^2 \sin(\lambda x)\, y = 0.$

Particular solution: $y_0 = a\sin(\lambda x) + bx + c$.

26. $\sin^n(ax)\, y''_{xx} + (x^2 - b^2)y'_x - (x + b)y = 0.$

Particular solution: $y_0 = x - b$.

27. $(a\sin^n x + b)y''_{xx} + (c\sin^n x + d)y'_x + \lambda[(c - a\lambda)\sin^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

► Equations with cosine.

28. $y''_{xx} + a^2 y = b\cos(\lambda x).$

Equation of forced oscillations.

$$\text{Solution: } y = \begin{cases} C_1\sin(ax) + C_2\cos(ax) + \dfrac{b}{a^2 - \lambda^2}\cos(\lambda x) & \text{if } a \neq \lambda, \\ C_1\sin(ax) + C_2\cos(ax) + \dfrac{b}{2a}x\sin(ax) & \text{if } a = \lambda. \end{cases}$$

29. $y''_{xx} + (a - 2q\cos 2x)y = 0.$

The Mathieu equation.

1°. Given numbers a and q, there exists a general solution $y(x)$ and a characteristic index μ such that

$$y(x+\pi) = e^{2\pi\mu} y(x).$$

For small values of q, an approximate value of μ can be found from the equation:

$$\cosh(\pi\mu) = 1 + 2\sin^2\left(\tfrac{1}{2}\pi\sqrt{a}\,\right) + \frac{\pi q^2}{(1-a)\sqrt{a}}\sin\left(\pi\sqrt{a}\,\right) + O(q^4).$$

If $y_1(x)$ is the solution of the Mathieu equation satisfying the initial conditions $y_1(0) = 1$ and $y_1'(0) = 0$, the characteristic index can be determined from the relation:

$$\cosh(2\pi\mu) = y_1(\pi).$$

The solution $y_1(x)$, and hence μ, can be determined with any degree of accuracy by means of numerical or approximate methods.

The general solution differs depending on the value of $y_1(\pi)$ and can be expressed in terms of two auxiliary periodical functions $\varphi_1(x)$ and $\varphi_2(x)$ (see Table 14.4).

TABLE 14.4
The general solution of the Mathieu equation 14.1.6.29 expressed in terms of auxiliary periodical functions $\varphi_1(x)$ and $\varphi_2(x)$

Constraint	General solution $y = y(x)$	Period of φ_1 and φ_2	Index
$y_1(\pi) > 1$	$C_1 e^{2\mu x}\varphi_1(x) + C_2 e^{-2\mu x}\varphi_2(x)$	π	μ is a real number
$y_1(\pi) < -1$	$C_1 e^{2\rho x}\varphi_1(x) + C_2 e^{-2\rho x}\varphi_2(x)$	2π	$\mu = \rho + \frac{1}{2}i,\ \ i^2 = -1$, ρ is the real part of μ
$\lvert y_1(\pi)\rvert < 1$	$(C_1\cos\nu x + C_2\sin\nu x)\varphi_1(x) +$ $+ (C_1\cos\nu x - C_2\sin\nu x)\varphi_2(x)$	π	$\mu = i\nu$ is a pure imaginary number, $\cos(2\pi\nu) = y_1(\pi)$
$y_1(\pi) = \pm 1$	$C_1\varphi_1(x) + C_2 x\varphi_2(x)$	π	$\mu = 0$

2°. In applications, of major interest are periodical solutions of the Mathieu equation that exist for certain values of the parameters a and q (those values of a are referred to as eigenvalues). The most important solutions (the *Mathieu functions*) are listed in Table S4.6. The main properties of the solutions are avilable in Section S4.16.

30. $y''_{xx} + (a\cos^2 x + b)y = 0.$

Applying the formula $2\cos^2 x = 1 + \cos 2x$, we obtain the Mathieu equation 14.1.6.29: $y''_{xx} + (\frac{1}{2}a + b + \frac{1}{2}a\cos 2x)y = 0$.

31. $y''_{xx} + a\cos^n(bx)y'_x + c[a\cos^n(bx) - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

32. $y''_{xx} + [a\cos^n(bx) + c]y'_x + ac\cos^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

33. $y''_{xx} + (ax + b)\cos^n(cx)y'_x - a\cos^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

34. $y''_{xx} + ax^n\cos^m(bx)y'_x - ax^{n-1}\cos^m(bx)y = 0.$

Particular solution: $y_0 = x$.

35. $y''_{xx} + ax^n\cos^m(bx)y'_x + c[ax^{n+k}\cos^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

36. $xy''_{xx} + ax\cos^n(bx)\,y'_x - [a(cx+1)\cos^n(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

37. $xy''_{xx} + [(ax^2 + bx)\cos^n(cx) + 2]y'_x + b\cos^n(cx)y = 0.$

Particular solution: $y_0 = a + b/x$.

38. $xy''_{xx} + (ax^{n+1} + b\cos^m x)y'_x + ax^n(b\cos^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

39. $xy''_{xx} + [ax^n\cos^m(bx) + c]y'_x + a(c-1)x^{n-1}\cos^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

40. $xy''_{xx} + (ax^n + bx\cos^m x)y'_x + [b(ax^n - 1)\cos^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

41. $x^2y''_{xx} + x(a\cos^n x + 1)y'_x + b(a\cos^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

42. $x^2y''_{xx} + x(a\cos^n x + b)y'_x + b(a\cos^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

43. $x^2y''_{xx} + ax^n\cos^m(bx)y'_x + c[ax^{n-1}\cos^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

44. $x^4y''_{xx} + ax^2\cos^n(bx)\,y'_x + [a(c-x)\cos^n(bx) - c^2]y = 0.$

Particular solution: $y_0 = x\exp(c/x)$.

45. $\cos^2 x\, y''_{xx} - [a\cos^2 x + n(n-1)]y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $y = \cos^n x\Bigl(\dfrac{1}{\cos x}\dfrac{d}{dx}\Bigr)^n\bigl(C_1e^{x\sqrt{a}} + C_2e^{-x\sqrt{a}}\bigr)$.

46. $\cos^2 x\, y''_{xx} + (a\cos^2 x + b)y = 0.$

The substitution $x = \xi + \frac{1}{2}\pi$ leads to a linear equation of the form 14.1.6.23: $\sin^2\xi\, y''_{\xi\xi} + (a\sin^2\xi + b)y = 0$.

47. $[a\cos(\lambda x)+bx+c]y''_{xx}+a\lambda^2\cos(\lambda x)\,y=0.$

Particular solution: $y_0=a\cos(\lambda x)+bx+c.$

48. $\cos^n(ax)\,y''_{xx}+(x^2-b^2)y'_x-(x+b)y=0.$

Particular solution: $y_0=x-b.$

49. $(a\cos^n x+b)y''_{xx}+(c\cos^n x+d)y'_x+\lambda[(c-a\lambda)\cos^n x+d-b\lambda]y=0.$

Particular solution: $y_0=e^{-\lambda x}.$

► **Equations with tangent.**

50. $y''_{xx}+a[\lambda+(\lambda-a)\tan^2(\lambda x)]y=0.$

Particular solution: $y_0=[\cos(\lambda x)]^{a/\lambda}.$

51. $y''_{xx}+(a\tan^2 x+b)y=0.$

The transformation $z=\sin^2 x$, $u=y\cos^k x$, where k is a root of the quadratic equation $k^2+k+a=0$, leads to the hypergeometric equation 14.1.2.171:

$$z(z-1)u''_{zz}+[(1-k)z-\frac{1}{2}]u'_z-\frac{1}{4}(k+b)u=0.$$

52. $y''_{xx}+(a-\lambda)\tan(\lambda x)y'_x+a\lambda y=0.$

Particular solution: $y_0=[\cos(\lambda x)]^{a/\lambda}.$

53. $y''_{xx}+a\tan x\,y'_x+by=0.$

1°. The substitution $\xi=\sin x$ leads to a linear differential equation of the form 14.1.2.168: $(\xi^2-1)y''_{\xi\xi}+(1-a)\xi y'_\xi-by=0.$

2°. Solution for $a=-2$:

$$y\cos x=\begin{cases}C_1\sin(kx)+C_2\cos(kx) & \text{if } b+1=k^2>0,\\ C_1\sinh(kx)+C_2\cosh(kx) & \text{if } b+1=-k^2<0.\end{cases}$$

3°. Solution for $a=2$ and $b=3$: $y=C_1\cos^3 x+C_2\sin x(1+2\cos^2 x).$

54. $y''_{xx}+a\tan x\,y'_x+(b\tan^2 x+c)y=0.$

This is a special case of equation 14.1.6.131.

55. $y''_{xx}-2\lambda\tan(\lambda x)\,y'_x+(ax^2+bx+c)y=0.$

The substitution $w=y\cos(\lambda x)$ leads to a second-order linear equation of the form 14.1.2.6: $w''_{xx}+(ax^2+bx+c+\lambda^2)w=0.$

56. $y''_{xx}-2\lambda\tan(\lambda x)\,y'_x+(ax^{2n}+bx^{n-1}-\lambda^2)y=0.$

Substituting $w=y\cos(\lambda x)$ yields a second-order linear equation of the form 14.1.2.10: $w''_{xx}+(ax^{2n}+bx^{n-1})w=0.$

57. $y''_{xx}+a\tan^n(bx)\,y'_x+c[a\tan^n(bx)-c]y=0.$

Particular solution: $y_0=e^{-cx}.$

58. $y''_{xx} + a\tan^n(\lambda x)\,y'_x + b[a\tan^{n+1}(\lambda x) + (\lambda - b)\tan^2(\lambda x) + \lambda]y = 0.$

Particular solution: $y_0 = [\cos(\lambda x)]^{b/\lambda}$.

59. $y''_{xx} + a\tan^n x\,y'_x + (a\tan^{n+1} x - a\tan^{n-1} x + 4)y = 0.$

Particular solution: $y_0 = \sin x\cos x$.

60. $y''_{xx} + [a\tan^n(bx) + c]y'_x + ac\tan^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

61. $y''_{xx} + \tan x\,(a\tan^n x + b - 1)y'_x + (ab\tan^{n+2} x - a\tan^n x + 2b + 2)y = 0.$

Particular solution: $y_0 = \sin x\cos^b x$.

62. $y''_{xx} + (ax + b)\tan^n(cx)\,y'_x - a\tan^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

63. $y''_{xx} + ax^n\tan^m(bx)\,y'_x - ax^{n-1}\tan^m(bx)y = 0.$

Particular solution: $y_0 = x$.

64. $y''_{xx} + ax^n\tan^m(bx)\,y'_x + c[ax^{n+k}\tan^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

65. $xy''_{xx} - 2\lambda x\tan(\lambda x)\,y'_x + (ax + b)y = 0.$

Substituting $w = y\cos(\lambda x)$ yields a second-order linear equation of the form 14.1.2.64: $xw''_{xx} + [(a + \lambda^2)x + b]w = 0$.

66. $xy''_{xx} + ax\tan^n(bx)\,y'_x - [a(cx + 1)\tan^n(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

67. $xy''_{xx} + [(ax^2 + bx)\tan^n(cx) + 2]y'_x + b\tan^n(cx)y = 0.$

Particular solution: $y_0 = a + b/x$.

68. $xy''_{xx} + (ax^{n+1} + b\tan^m x)y'_x + ax^n(b\tan^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

69. $xy''_{xx} + (ax^n + bx\tan^m x)y'_x + [b(ax^n - 1)\tan^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

70. $xy''_{xx} + [ax^n\tan^m(bx) + c]y'_x + a(c - 1)x^{n-1}\tan^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

71. $x^2y''_{xx} - 2\lambda x^2\tan(\lambda x)\,y'_x + (ax^2 + bx + c)y = 0.$

The substitution $w = y\cos(\lambda x)$ leads to an equation of the form 14.1.2.115: $x^2w''_{xx} + [(a + \lambda^2)x^2 + bx + c]w = 0$.

72. $x^2y''_{xx} + x(1 - 2x\tan x)y'_x - (x\tan x + \nu^2)y = 0.$

Solution: $y\cos x = C_1J_\nu(x) + C_2Y_\nu(x)$, where $J_\nu(x)$ and $Y_\nu(x)$ are Bessel functions.

73. $x^2 y''_{xx} - x(2x\tan x + k)y'_x + (ax^2 + bx + c + kx\tan x)y = 0.$

The substitution $w = y\cos x$ leads to a second-order linear equation of the form 14.1.2.131: $x^2 w''_{xx} - kxw'_x + [(a+1)x^2 + bx + c]w = 0$.

74. $x^2 y''_{xx} + x(a\tan^n x + 1)y'_x + b(a\tan^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

75. $x^2 y''_{xx} + x(a\tan^n x + b)y'_x + b(a\tan^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

76. $x^2 y''_{xx} + ax^n \tan^m(bx)\, y'_x + c[ax^{n-1}\tan^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

77. $x^4 y''_{xx} + ax^2 \tan^n(bx)\, y'_x + [a(c - x)\tan^n(bx) - c^2]y = 0.$

Particular solution: $y_0 = x\exp(c/x)$.

78. $(a\tan^n x + b)y''_{xx} + (cx + d)y'_x - cy = 0.$

Particular solution: $y_0 = cx + d$.

79. $(a\tan^n x + b)y''_{xx} + (c\tan^n x + d)y'_x + \lambda[(c - a\lambda)\tan^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

► **Equations with cotangent.**

80. $y''_{xx} + a[\lambda + (\lambda - a)\cot^2(\lambda x)]y = 0.$

Particular solution: $y_0 = [\sin(\lambda x)]^{a/\lambda}$.

81. $y''_{xx} + (a\cot^2 x + b)y = 0.$

The substitution $x = \xi + \frac{\pi}{2}$ leads to a linear equation of the form 14.1.6.51: $y''_{\xi\xi} + (a\tan^2\xi + b)y = 0$.

82. $y''_{xx} + \cot x\, y'_x + \nu(\nu + 1)y = 0.$

The substitution $\xi = \cos x$ leads to the Legendre equation 14.1.2.154: $(\xi^2 - 1)y''_{\xi\xi} + 2\xi y'_\xi - \nu(\nu + 1)y = 0$.

83. $y''_{xx} + 2a\cot(ax)\, y'_x + (b^2 - a^2)y = 0.$

Particular solution: $y_0 = \dfrac{\cos(bx)}{\sin(ax)}$.

84. $y''_{xx} + (\lambda - a)\cot(\lambda x)\, y'_x + a\lambda y = 0.$

Particular solution: $y_0 = [\sin(\lambda x)]^{a/\lambda}$.

85. $y''_{xx} + a\cot(\lambda x)\, y'_x + by = 0.$

The substitution $z = \lambda x + \frac{\pi}{2}$ leads to a second-order linear equation of the form 14.1.6.53: $y''_{zz} - a\lambda^{-1}\tan z\, y'_z + b\lambda^{-2}y = 0$.

86. $y''_{xx} + a\cot(\lambda x)\,y'_x + b[\lambda + (\lambda - a - b)\cot^2(\lambda x)]y = 0.$

Particular solution: $y_0 = [\sin(\lambda x)]^{b/\lambda}$.

87. $y''_{xx} + a\cot x\,y'_x + (b\cot^2 x + c)y = 0.$

This is a special case of equation 14.1.6.131.

88. $y''_{xx} + 2\lambda\cot(\lambda x)\,y'_x + (ax^2 + bx + c)y = 0.$

The substitution $w = y\sin(\lambda x)$ leads to a second-order linear equation of the form 14.1.2.6: $w''_{xx} + (ax^2 + bx + c + \lambda^2)w = 0$.

89. $y''_{xx} + 2\lambda\cot(\lambda x)\,y'_x + (ax^{2n} + bx^{n-1} - \lambda^2)y = 0.$

Substituting $w = y\sin(\lambda x)$ yields a second-order linear equation of the form 14.1.2.10: $w''_{xx} + (ax^{2n} + bx^{n-1})w = 0$.

90. $y''_{xx} + a\cot^n(bx)\,y'_x + c[a\cot^n(bx) - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

91. $y''_{xx} + [a\cot^n(bx) + c]y'_x + ac\cot^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

92. $y''_{xx} + (ax + b)\cot^n(cx)\,y'_x - a\cot^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

93. $y''_{xx} + ax^n\cot^m(bx)\,y'_x - ax^{n-1}\cot^m(bx)y = 0.$

Particular solution: $y_0 = x$.

94. $y''_{xx} + ax^n\cot^m(bx)\,y'_x + c[ax^{n+k}\cot^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

95. $xy''_{xx} + 2\lambda x\cot(\lambda x)\,y'_x + (ax + b)y = 0.$

Substituting $w = y\sin(\lambda x)$ yields a second-order linear equation of the form 14.1.2.64: $xw''_{xx} + [(a + \lambda^2)x + b]w = 0$.

96. $xy''_{xx} + ax\cot^n(bx)\,y'_x - [a(cx + 1)\cot^n(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

97. $xy''_{xx} + (ax^{n+1} + b\cot^m x)y'_x + ax^n(b\cot^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

98. $xy''_{xx} + (ax^n + bx\cot^m x)y'_x + [b(ax^n - 1)\cot^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

99. $xy''_{xx} + [ax^n\cot^m(bx) + c]y'_x + a(c - 1)x^{n-1}\cot^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

100. $x^2 y''_{xx} + 2\lambda x^2 \cot(\lambda x)\, y'_x + (ax^2 + bx + c)y = 0.$

Substituting $w = y\sin(\lambda x)$ yields a second-order linear equation of the form 14.1.2.115: $x^2 w''_{xx} + [(a+\lambda^2)x^2 + bx + c]w = 0.$

101. $x^2 y''_{xx} + x(2x\cot x + k)y'_x + (ax^2 + bx + c + kx\cot x)y = 0.$

The substitution $w = y\sin x$ leads to a second-order linear equation of the form 14.1.2.131: $x^2 w''_{xx} + kxw'_x + [(a+1)x^2 + bx + c]w = 0.$

102. $x^2 y''_{xx} + x(a\cot^n x + 1)y'_x + b(a\cot^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

103. $x^2 y''_{xx} + x(a\cot^n x + b)y'_x + b(a\cot^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

104. $x^2 y''_{xx} + ax^n \cot^m(bx)\, y'_x + c[ax^{n-1}\cot^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

105. $x^4 y''_{xx} + ax^2 \cot^n(bx)\, y'_x + [a(c-x)\cot^n(bx) - c^2]y = 0.$

Particular solution: $y_0 = x\exp(c/x)$.

106. $(a\cot^n x + b)y''_{xx} + (c\cot^n x + d)y'_x + \lambda[(c - a\lambda)\cot^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

► Equations containing combinations of trigonometric functions.

107. $y''_{xx} - a[a\sin^2(bx) + b\cos(bx)]y = 0.$

Particular solution: $y_0 = \exp\left[-\frac{a}{b}\cos(bx)\right]$.

108. $y''_{xx} - a[a\cos^2(bx) + b\sin(bx)]y = 0.$

Particular solution: $y_0 = \exp\left[-\frac{a}{b}\sin(bx)\right]$.

109. $y''_{xx} + (a\sin x + b)y'_x + a(b\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(a\cos x)$.

110. $y''_{xx} + (a\sin^n x + b\sin x)y'_x + b(a\sin^{n+1} x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

111. $y''_{xx} + (a\cos x + b)y'_x + a(b\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-a\sin x)$.

112. $y''_{xx} + (a\cos^n x + b\cos x)y'_x + b(a\cos^{n+1} x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

113. $\sin x\, y''_{xx} + \cos x\, y'_x + \nu(\nu+1)\sin x\, y = 0.$

The substitution $\xi = \cos x$ leads to the Legendre equation 14.1.2.154: $(1-\xi^2)y''_{\xi\xi} - 2\xi\, y'_\xi + \nu(\nu+1)y = 0.$

114. $\sin x\, y''_{xx} + (2n+1)\cos x\, y'_x + (\nu - n)(\nu + n + 1)\sin x\, y = 0.$

Here, ν is an arbitrary number and n is a positive integer. The substitution $\xi = \cos x$ leads to an equation of the form 14.1.2.156: $(\xi^2-1)y''_{\xi\xi}+2(n+1)\xi\, y'_\xi+(n-\nu)(\nu+n+1)y=0$.

115. $\sin^2 x\, y''_{xx} + \sin x \cos x\, y'_x + [\nu(\nu+1)\sin^2 x - n^2]y = 0.$

Here, ν is an arbitrary number and n is a nonnegative integer.

The transformation $\xi = \cos x$, $y = w\sin^n x$ leads to an equation of the form 14.1.2.156: $(\xi^2-1)w''_{\xi\xi} + 2(n+1)\xi w'_\xi + (n-\nu)(n+\nu+1)w = 0$.

116. $\sin^2 x\, y''_{xx} + \sin x(a\cos x + b)y'_x + (\alpha\cos^2 x + \beta\cos x + \gamma)y = 0.$

Set $x = 2\xi$. Applying the trigonometric formulas

$$\sin(2\xi) = 2\sin\xi\cos\xi, \quad \cos(2\xi) = \cos^2\xi - \sin^2\xi, \quad b = b(\sin^2\xi + \cos^2\xi),$$
$$\beta = \beta(\sin^2\xi + \cos^2\xi), \quad \gamma = \gamma(\sin^2\xi + \cos^2\xi)^2,$$

and dividing all the terms by $\sin^2 x$, we arrive at an equation of the form 14.1.6.131:

$$y''_{\xi\xi}+[(b-a)\tan\xi+(b+a)\cot\xi]y'_\xi+[(\alpha-\beta+\gamma)\tan^2\xi+(\alpha+\beta+\gamma)\cot^2\xi+2\gamma-2\alpha]y=0.$$

117. $\cos^2 x\, y''_{xx} + a\sin(2x)\, y'_x + [b\cos(2x) + c]y = 0.$

Dividing the equation by $\cos^2 x$ and applying the formulas

$$\sin(2x) = 2\sin x\cos x, \quad \cos(2x) = \cos^2 x - \sin^2 x, \quad c = c(\sin^2 x + \cos^2 x),$$

we obtain an equation of the form 14.1.6.131:

$$y''_{xx} + 2a\tan x\, y'_x + [(c-b)\tan^2 x + b + c]y = 0.$$

118. $\cos^2(ax)\, y''_{xx} + (n-1)a\sin(2ax)\, y'_x + na^2[(n-1)\sin^2(ax) + \cos^2(ax)]y = 0.$

Particular solution: $y_0 = \cos^n(ax)$.

119. $\cos^2 x\, y''_{xx} + \cos x\,(a\sin x + b)y'_x + (\alpha\sin^2 x + \beta\sin x + \gamma)y = 0.$

The substitution $x = \xi + \frac{\pi}{2}$ leads to a second-order linear equation of the form 14.1.6.116: $\sin^2\xi y''_{\xi\xi} - \sin\xi\,(a\cos\xi + b)y'_\xi + (\alpha\cos^2\xi + \beta\cos\xi + \gamma)y = 0$.

120. $\sin x\cos^2 x\, y''_{xx} + \cos x\,(a\sin^2 x + b)y'_x + c\sin x\, y = 0.$

1°. Dividing the equation by $\sin x\cos^2 x$ and assuming

$$b = b(\sin^2 x + \cos^2 x), \quad c = c(\sin^2 x + \cos^2 x),$$

we obtain equation 14.1.6.131: $y''_{xx} + [(a+b)\tan x + b\cot x]y'_x + c(\tan^2 x + 1)y = 0$.

2°. Particular solutions:

$$y_0 = \cos^a x \qquad \text{if } c = a(b+1),$$
$$y_0 = \tan^{1-b} x \qquad \text{if } c = (a+2)(b-1),$$
$$y_0 = \sin^{1-b} x\cos^{a+b-1} x \qquad \text{if } c = 2(a+b-1).$$

121. $\sin x\cos^2 x\, y''_{xx} + \cos x\,(a\sin^2 x - 1)y'_x + b\sin^3 x\, y = 0.$

Solution: $y = C_1(\cos x)^{k_1} + C_2(\cos x)^{k_2}$, where k_1 and k_2 are roots of the quadratic equation $k^2 - ak + b = 0$.

122. $\sin^2 x \cos^2 x\, y''_{xx} + (a\sin^2 x + b\cos^2 x + c\sin^2 x\cos^2 x)y = 0.$

Dividing the equation by $\sin^2 x\cos^2 x$ and assuming

$$a = a(\sin^2 x + \cos^2 x), \quad b = b(\sin^2 x + \cos^2 x),$$

we obtain equation 14.1.6.131: $y''_{xx} + (a\tan^2 x + b\cot^2 x + a + b + c)y = 0.$

123. $\sin^n(\lambda x)\, y''_{xx} + [\lambda^2 \sin^n(\lambda x) + a\cos^{n-4}(\lambda x)]y = 0.$

The transformation $\xi = \tan(\lambda x)$, $w = \dfrac{y}{\cos(\lambda x)}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + a\lambda^{-2}\xi^{-n}w = 0.$

124. $\cos^n(\lambda x)\, y''_{xx} + [\lambda^2 \cos^n(\lambda x) + a\sin^{n-4}(\lambda x)]y = 0.$

The substitution $\lambda x = \frac{\pi}{2} - \lambda\xi$ leads to an equation of the form 14.1.6.123.

125. $y''_{xx} + n\tan x\, y'_x + a^2(\cos x)^{2n}y = 0.$

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \displaystyle\int \cos^n x\,dx.$

126. $y''_{xx} + \tan x\, y'_x + a^2\cos^2 x(\sin x)^{2n-2}y = 0.$

Solution: $y = \sqrt{\sin x}\left[C_1 J_{\frac{1}{2n}}\left(\dfrac{a}{n}\sin^n x\right) + C_2 Y_{\frac{1}{2n}}\left(\dfrac{a}{n}\sin^n x\right)\right]$, where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions.

127. $y''_{xx} + \tan x\, y'_x - a(a-1)\cot^2 x\, y = 0.$

Solution: $y = \begin{cases} C_1|\sin x|^a + C_2|\sin x|^{1-a} & \text{if } a \neq \frac{1}{2}, \\ \sqrt{|\sin x|}\,(C_1 + C_2\ln|\sin x|) & \text{if } a = \frac{1}{2}. \end{cases}$

128. $y''_{xx} - 2a\cot(2ax)\, y'_x - b^2\sin^2(2ax)y = 0.$

Solution: $y = C_1\exp\left[\dfrac{b}{a}\sin^2(ax)\right] + C_2\exp\left[-\dfrac{b}{a}\sin^2(ax)\right].$

129. $y''_{xx} - n\cot x\, y'_x + a^2(\sin x)^{2n}y = 0.$

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \displaystyle\int \sin^n x\,dx.$

130. $y''_{xx} - 2\cot(2x)\, y'_x + a\tan^2 x\, y = 0.$

The substitution $\xi = \cos x$ leads to the Euler equation 14.1.2.123: $\xi^2 y''_{\xi\xi} - \xi y'_\xi + ay = 0.$

131. $y''_{xx} + (a\tan x + b\cot x)y'_x + (\alpha\tan^2 x + \beta\cot^2 x + \gamma)y = 0.$

The transformation $\xi = \sin^2 x$, $y = w\sin^n x\cos^m x$, where n and m are roots of the quadratic equations

$$n^2 + (b-1)n + \beta = 0, \qquad m^2 - (a+1)m + \alpha = 0,$$

leads to the hypergeometric equation 14.1.2.171:

$$4\xi(\xi-1)w''_{\xi\xi} + 2[(2n+2m+2+b-a)\xi - 2n - b - 1]w'_\xi + (2nm+n+m+bm-an-\gamma)w = 0.$$

132. $\sin(2x)\, y''_{xx} - 2ny'_x + 2a^2\sin^2 x\,(\tan x)^{2n-1}y = 0.$

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \displaystyle\int \tan^n x\,dx.$

14.1.7 Equations Containing Inverse Trigonometric Functions

► **Equations with arcsine.**

1. $y''_{xx} + (ax + b + c \arcsin x)y'_x + [c(ax + b) \arcsin x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx)$.

2. $y''_{xx} + b(\arcsin x)^n y'_x + c[b(\arcsin x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

3. $y''_{xx} + b(\arcsin x)^n y'_x + a[bx^m (\arcsin x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr)$.

4. $y''_{xx} + (ax + b)(\arcsin x)^n y'_x - a(\arcsin x)^n y = 0.$

Particular solution: $y_0 = ax + b$.

5. $y''_{xx} + ax^n (\arcsin x)^m y'_x - ax^{n-1}(\arcsin x)^m y = 0.$

Particular solution: $y_0 = x$.

6. $xy''_{xx} + ax \arcsin x\, y'_x - [a(bx + 1) \arcsin x + b(bx + 2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

7. $xy''_{xx} + [a(bx + 1) \arcsin x + bx - 1]y'_x + ab^2 x \arcsin x\, y = 0.$

Particular solution: $y_0 = (bx + 1)e^{-bx}$.

8. $xy''_{xx} + [(ax^2 + bx) \arcsin x + 2]y'_x + b \arcsin x\, y = 0.$

Particular solution: $y_0 = a + b/x$.

9. $xy''_{xx} + [ax(\arcsin x)^n + b]y'_x + a(b - 1)(\arcsin x)^n y = 0.$

Particular solution: $y_0 = x^{1-b}$.

10. $xy''_{xx} + (ax^{n+1} + b \arcsin x)y'_x + ax^n(b \arcsin x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

11. $xy''_{xx} + (ax^n + bx \arcsin x)y'_x + [b(ax^n - 1) \arcsin x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x \exp(-ax^n/n)$.

12. $x^2 y''_{xx} + bx \arcsin x\, y'_x + a(b \arcsin x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}$.

13. $x^2 y''_{xx} + x(b \arcsin x + 2)y'_x + [b(ax + 1) \arcsin x - a^2 x^2]y = 0.$

Particular solution: $y_0 = x^{-1}e^{-ax}$.

14. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\arcsin x)^n y'_x - 2a[cx(\arcsin x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b$.

15. $x^4 y''_{xx} + ax^2 \arcsin x\, y'_x + [a(b-x)\arcsin x - b^2]y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

16. $(ax^2+b)^2 y''_{xx} + (cx+d)(\arcsin x)^n y'_x - c(\arcsin x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

17. $(x^2+a)^2 y''_{xx} + b(x^2+a)(\arcsin x)^n y'_x - [bx(\arcsin x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

18. $(ax^2+b)^2 y''_{xx} + c(ax^2+b)(\arcsin x)^n y'_x + [c(\arcsin x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int \frac{dx}{ax^2+b}\Bigr)$.

► **Equations with arccosine.**

19. $y''_{xx} + (ax + b + c\arccos x)y'_x + [c(ax+b)\arccos x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx)$.

20. $y''_{xx} + b(\arccos x)^n y'_x + c[b(\arccos x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

21. $y''_{xx} + b(\arccos x)^n y'_x + a[bx^m(\arccos x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\frac{a}{m+1}x^{m+1}\Bigr)$.

22. $y''_{xx} + (ax+b)(\arccos x)^n y'_x - a(\arccos x)^n y = 0.$

Particular solution: $y_0 = ax + b$.

23. $y''_{xx} + ax^n(\arccos x)^m y'_x - ax^{n-1}(\arccos x)^m y = 0.$

Particular solution: $y_0 = x$.

24. $xy''_{xx} + ax\arccos x\, y'_x - [a(bx+1)\arccos x + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

25. $xy''_{xx} + [a(bx+1)\arccos x + bx - 1]y'_x + ab^2 x\arccos x\, y = 0.$

Particular solution: $y_0 = (bx+1)e^{-bx}$.

26. $xy''_{xx} + [(ax^2+bx)\arccos x + 2]y'_x + b\arccos x\, y = 0.$

Particular solution: $y_0 = a + b/x$.

27. $xy''_{xx} + [ax(\arccos x)^n + b]y'_x + a(b-1)(\arccos x)^n y = 0.$

Particular solution: $y_0 = x^{1-b}$.

28. $xy''_{xx} + (ax^{n+1} + b\arccos x)y'_x + ax^n(b\arccos x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\frac{a}{n+1}x^{n+1}\Bigr)$.

29. $xy''_{xx} + (ax^n + bx\arccos x)y'_x + [b(ax^n - 1)\arccos x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

30. $x^2y''_{xx} + bx\arccos x\, y'_x + a(b\arccos x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}$.

31. $x^2y''_{xx} + x(b\arccos x + 2)y'_x + [b(ax + 1)\arccos x - a^2x^2]y = 0.$

Particular solution: $y_0 = x^{-1}e^{-ax}$.

32. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\arccos x)^n y'_x - 2a[cx(\arccos x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b$.

33. $x^4y''_{xx} + ax^2\arccos x\, y'_x + [a(b - x)\arccos x - b^2]y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

34. $(ax^2 + b)^2y''_{xx} + (cx + d)(\arccos x)^n y'_x - c(\arccos x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

35. $(x^2 + a)^2y''_{xx} + b(x^2 + a)(\arccos x)^n y'_x - [bx(\arccos x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2 + a}$.

36. $(ax^2+b)^2y''_{xx}+c(ax^2+b)(\arccos x)^n y'_x+[c(\arccos x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\Big(-\int \frac{dx}{ax^2 + b}\Big)$.

► **Equations with arctangent.**

37. $y''_{xx} + (ax + b + c\arctan x)y'_x + [c(ax + b)\arctan x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx)$.

38. $y''_{xx} + b(\arctan x)^n y'_x + c[b(\arctan x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

39. $y''_{xx} + b(\arctan x)^n y'_x + a[bx^m(\arctan x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Big(-\frac{a}{m+1}x^{m+1}\Big)$.

40. $y''_{xx} + (ax + b)(\arctan x)^n y'_x - a(\arctan x)^n y = 0.$

Particular solution: $y_0 = ax + b$.

41. $y''_{xx} + ax^n(\arctan x)^m y'_x - ax^{n-1}(\arctan x)^m y = 0.$

Particular solution: $y_0 = x$.

42. $xy''_{xx} + ax\arctan x\, y'_x - [a(bx + 1)\arctan x + b(bx + 2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

43. $xy''_{xx} + [a(bx + 1)\arctan x + bx - 1]y'_x + ab^2x\arctan x\, y = 0.$

Particular solution: $y_0 = (bx + 1)e^{-bx}$.

44. $xy''_{xx} + [(ax^2 + bx)\arctan x + 2]y'_x + b\arctan x\, y = 0.$

Particular solution: $y_0 = a + b/x$.

45. $xy''_{xx} + [ax(\arctan x)^n + b]y'_x + a(b-1)(\arctan x)^n y = 0.$

Particular solution: $y_0 = x^{1-b}$.

46. $xy''_{xx} + (ax^{n+1} + b\arctan x)y'_x + ax^n(b\arctan x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

47. $xy''_{xx} + (ax^n + bx\arctan x)y'_x + [b(ax^n - 1)\arctan x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

48. $xy''_{xx} + ax(\arctan^n x + b)y'_x - a(\arctan^n x + b)y = 0.$

Particular solution: $y_0 = x$.

49. $xy''_{xx} + b\arctan^n x\, y'_x + a(b\arctan^n x - ax)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

50. $xy''_{xx} + a(\arctan^n x + bx)y'_x + ab\arctan^n x\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

51. $xy''_{xx} + b\arctan^n x\, y'_x + ax(b\arctan^n x - ax^2 + 1)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)$.

52. $x^2y''_{xx} + bx\arctan x\, y'_x + a(b\arctan x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}$.

53. $x^2y''_{xx} + x(b\arctan x + 2)y'_x + [b(ax+1)\arctan x - a^2x^2]y = 0.$

Particular solution: $y_0 = x^{-1}e^{-ax}$.

54. $x^2y''_{xx} + ax(\arctan^n x + b)y'_x - a(\arctan^n x + b)y = 0.$

Particular solution: $y_0 = x$.

55. $x^2y''_{xx} + b\arctan^n x\, y'_x + a(b\arctan^n x - ax^2)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

56. $x^2y''_{xx} + a(\arctan^n x + bx^2)y'_x + ab\arctan^n x\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

57. $x^2y''_{xx} + x[(ax+b)\arctan^n x + 2]y'_x + b\arctan^n x\, y = 0.$

Particular solution: $y_0 = a + b/x$.

58. $(x^2+1)y''_{xx} - [a^2(x^2+1)(\arctan x)^2 + a]y = 0.$

Particular solution: $y_0 = (x^2+1)^{-a/2}\exp(ax\arctan x)$.

59. $(ax^2+b)y''_{xx} + c(ax^2+b)(\arctan x)^n y'_x - 2a[cx(\arctan x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b$.

60. $x^4 y''_{xx} + ax^2 \arctan x\, y'_x + [a(b-x)\arctan x - b^2]y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

61. $(x^2+1)^2 y''_{xx} + [a(\arctan x)^2 + b\arctan x + c]y = 0.$

The transformation $\xi = \arctan x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 14.1.2.6: $w''_{\xi\xi} + (a\xi^2 + b\xi + c + 1)w = 0$.

62. $(x^2+1)^2 y''_{xx} + [b(\arctan x)^n - 1]y = 0.$

The transformation $\xi = \arctan x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + b\xi^n w = 0$.

63. $(ax^2+b)^2 y''_{xx} + (cx+d)(\arctan x)^n y'_x - c(\arctan x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

64. $(x^2+a)^2 y''_{xx} + b(x^2+a)(\arctan x)^n y'_x - [bx(\arctan x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

65. $(ax^2+b)^2 y''_{xx} + c(ax^2+b)(\arctan x)^n y'_x + [c(\arctan x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\displaystyle\int \frac{dx}{ax^2+b}\Bigr)$.

► **Equations with arccotangent.**

66. $y''_{xx} + (ax + b + c\,\operatorname{arccot} x)y'_x + [c(ax+b)\operatorname{arccot} x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx)$.

67. $y''_{xx} + b(\operatorname{arccot} x)^n y'_x + c[b(\operatorname{arccot} x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

68. $y''_{xx} + b(\operatorname{arccot} x)^n y'_x + a[bx^m(\operatorname{arccot} x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr)$.

69. $y''_{xx} + (ax+b)(\operatorname{arccot} x)^n y'_x - a(\operatorname{arccot} x)^n y = 0.$

Particular solution: $y_0 = ax + b$.

70. $y''_{xx} + ax^n(\operatorname{arccot} x)^m y'_x - ax^{n-1}(\operatorname{arccot} x)^m y = 0.$

Particular solution: $y_0 = x$.

71. $xy''_{xx} + ax\operatorname{arccot} x\, y'_x - [a(bx+1)\operatorname{arccot} x + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

72. $xy''_{xx} + [a(bx+1)\operatorname{arccot} x + bx - 1]y'_x + ab^2x\operatorname{arccot} x\, y = 0.$

Particular solution: $y_0 = (bx+1)e^{-bx}$.

73. $xy''_{xx} + [(ax^2 + bx)\operatorname{arccot} x + 2]y'_x + b\operatorname{arccot} x\, y = 0.$

Particular solution: $y_0 = a + b/x$.

74. $xy''_{xx} + [ax(\operatorname{arccot} x)^n + b]y'_x + a(b-1)(\operatorname{arccot} x)^n y = 0.$

Particular solution: $y_0 = x^{1-b}$.

75. $xy''_{xx} + (ax^{n+1} + b\operatorname{arccot} x)y'_x + ax^n(b\operatorname{arccot} x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

76. $xy''_{xx} + (ax^n + bx\operatorname{arccot} x)y'_x + [b(ax^n - 1)\operatorname{arccot} x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

77. $x^2y''_{xx} + bx\operatorname{arccot} x\, y'_x + a(b\operatorname{arccot} x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}$.

78. $x^2y''_{xx} + x(b\operatorname{arccot} x + 2)y'_x + [b(ax+1)\operatorname{arccot} x - a^2x^2]y = 0.$

Particular solution: $y_0 = x^{-1}e^{-ax}$.

79. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\operatorname{arccot} x)^n y'_x - 2a[cx(\operatorname{arccot} x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b$.

80. $x^4y''_{xx} + ax^2\operatorname{arccot} x\, y'_x + [a(b-x)\operatorname{arccot} x - b^2]y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

81. $(x^2 + 1)^2y''_{xx} + [a(\operatorname{arccot} x)^2 + b\operatorname{arccot} x + c]y = 0.$

The transformation $\xi = \operatorname{arccot} x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 14.1.2.6: $w''_{\xi\xi} + (a\xi^2 + b\xi + c + 1)w = 0$.

82. $(x^2 + 1)^2y''_{xx} + [b(\operatorname{arccot} x)^n - 1]y = 0.$

The transformation $\xi = \operatorname{arccot} x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 14.1.2.7: $w''_{\xi\xi} + b\xi^n w = 0$.

83. $(ax^2 + b)^2y''_{xx} + (cx + d)(\operatorname{arccot} x)^n y'_x - c(\operatorname{arccot} x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

84. $(x^2 + a)^2y''_{xx} + b(x^2 + a)(\operatorname{arccot} x)^n y'_x - [bx(\operatorname{arccot} x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2 + a}$.

85. $(ax^2 + b)^2y''_{xx} + c(ax^2 + b)(\operatorname{arccot} x)^n y'_x + [c(\operatorname{arccot} x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\displaystyle\int \frac{dx}{ax^2 + b}\Bigr)$.

14.1.8 Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions

1. $y''_{xx} + ae^{\lambda x}y'_x + b[b + ae^{\lambda x}\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

2. $y''_{xx} + ae^{\lambda x}y'_x + b[b - ae^{\lambda x}\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

3. $y''_{xx} + a\cosh^n(\lambda x)\, y'_x + b[b + a\cosh^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

4. $y''_{xx} + a\cosh^n(\lambda x)\, y'_x + b[b - a\cosh^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

5. $y''_{xx} + a\cosh^n(kx)\, y'_x + be^{\lambda x}[a\cosh^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

6. $y''_{xx} + a\sinh^n(\lambda x)\, y'_x + b[b + a\sinh^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

7. $y''_{xx} + a\sinh^n(\lambda x)\, y'_x + b[b - a\sinh^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

8. $y''_{xx} + a\sinh^n(kx)\, y'_x + be^{\lambda x}[a\sinh^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

9. $y''_{xx} + a\tanh^n(\lambda x)\, y'_x + b[b + a\tanh^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

10. $y''_{xx} + a\tanh^n(\lambda x)\, y'_x + b[b - a\tanh^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

11. $y''_{xx} + a\tanh^n(kx)\, y'_x + be^{\lambda x}[a\tanh^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

12. $y''_{xx} + a\coth^n(\lambda x)\, y'_x + b[b + a\coth^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

13. $y''_{xx} + a\coth^n(\lambda x)\, y'_x + b[b - a\coth^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

14. $y''_{xx} + a\coth^n(kx)\, y'_x + be^{\lambda x}[a\coth^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

15. $y''_{xx} + a\ln^n(\lambda x)\,y'_x + b[b + a\ln^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

16. $y''_{xx} + a\ln^n(\lambda x)\,y'_x + b[b - a\ln^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

17. $y''_{xx} + a\ln^n(kx)\,y'_x + be^{\lambda x}[a\ln^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big)$.

18. $y''_{xx} + a\cos^n(kx)\,y'_x + be^{\lambda x}[a\cos^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big)$.

19. $y''_{xx} + a\sin^n(kx)\,y'_x + be^{\lambda x}[a\sin^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big)$.

20. $y''_{xx} + a\tan^n(kx)\,y'_x + be^{\lambda x}[a\tan^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big)$.

21. $y''_{xx} + a\cot^n(kx)\,y'_x + be^{\lambda x}[a\cot^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big)$.

22. $y''_{xx} + (ae^{\lambda x} + b\ln^n x)y'_x + ae^{\lambda x}(b\ln^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{a}{\lambda}e^{\lambda x}\Big)$.

23. $y''_{xx} + (ae^{\lambda x} + b\cos x)y'_x + b(ae^{\lambda x}\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

24. $y''_{xx} + (ae^{\lambda x} + b\cos^n x)y'_x + ae^{\lambda x}(b\cos^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{a}{\lambda}e^{\lambda x}\Big)$.

25. $y''_{xx} + (ae^{\lambda x} + b\cos^n x)y'_x + b\cos^{n-1} x\,(ae^{\lambda x}\cos x - n\sin x)y = 0.$

Particular solution: $y_0 = \exp\Big(-b\displaystyle\int \cos^n x\,dx\Big)$.

26. $y''_{xx} + (ae^{\lambda x} + b\sin x)y'_x + b(ae^{\lambda x}\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

27. $y''_{xx} + (ae^{\lambda x} + b\sin^n x)y'_x + ae^{\lambda x}(b\sin^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{a}{\lambda}e^{\lambda x}\Big)$.

28. $y''_{xx} + (ae^{\lambda x} + b\sin^n x)y'_x + b\sin^{n-1} x\,(ae^{\lambda x}\sin x + n\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^n x\,dx\Bigr).$

29. $y''_{xx} + (ae^{\lambda x} + b\tan x)y'_x + (b+1)(ae^{\lambda x}\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x.$

30. $y''_{xx} + (ae^{\lambda x} + b\tan^n x)y'_x + ae^{\lambda x}(b\tan^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

31. $y''_{xx} + (ae^{\lambda x} + b\cot x)y'_x + (b-1)(ae^{\lambda x}\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x.$

32. $y''_{xx} + (ae^{\lambda x} + b\cot^n x)y'_x + ae^{\lambda x}(b\cot^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

33. $y''_{xx} + (a\cosh^n x + b\cos x)y'_x + b(a\cosh^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x).$

34. $y''_{xx} + (a\cosh^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\cosh^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr).$

35. $y''_{xx} + (a\cosh^n x + b\sin x)y'_x + b(a\cosh^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x).$

36. $y''_{xx} + (a\cosh^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\cosh^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr).$

37. $y''_{xx} + (a\cosh^n x + b\tan x)y'_x + (b+1)(a\cosh^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x.$

38. $y''_{xx} + (a\cosh^n x + b\cot x)y'_x + (b-1)(a\cosh^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x.$

39. $y''_{xx} + (a\sinh^n x + b\cos x)y'_x + b(a\sinh^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x).$

40. $y''_{xx} + (a\sinh^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\sinh^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr).$

41. $y''_{xx} + (a \sinh^n x + b \sin x)y'_x + b(a \sinh^n x \sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b \cos x)$.

42. $y''_{xx} + (a \sinh^n x + b \sin^m x)y'_x + b \sin^{m-1} x\,(a \sinh^n x \sin x + m \cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b \int \sin^m x\,dx\Bigr)$.

43. $y''_{xx} + (a \sinh^n x + b \tan x)y'_x + (b+1)(a \sinh^n x \tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

44. $y''_{xx} + (a \sinh^n x + b \cot x)y'_x + (b-1)(a \sinh^n x \cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

45. $y''_{xx} + (a \tanh^n x + b \cos x)y'_x + b(a \tanh^n x \cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b \sin x)$.

46. $y''_{xx} + (a \tanh^n x + b \cos^m x)y'_x + b \cos^{m-1} x\,(a \tanh^n x \cos x - m \sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b \int \cos^m x\,dx\Bigr)$.

47. $y''_{xx} + (a \tanh^n x + b \sin x)y'_x + b(a \tanh^n x \sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b \cos x)$.

48. $y''_{xx} + (a \tanh^n x + b \sin^m x)y'_x + b \sin^{m-1} x\,(a \tanh^n x \sin x + m \cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b \int \sin^m x\,dx\Bigr)$.

49. $y''_{xx} + (a \tanh^n x + b \tan x)y'_x + (b+1)(a \tanh^n x \tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

50. $y''_{xx} + (a \tanh^n x + b \cot x)y'_x + (b-1)(a \tanh^n x \cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

51. $y''_{xx} + (a \coth^n x + b \cos x)y'_x + b(a \coth^n x \cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b \sin x)$.

52. $y''_{xx} + (a \coth^n x + b \cos^m x)y'_x + b \cos^{m-1} x\,(a \coth^n x \cos x - m \sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b \int \cos^m x\,dx\Bigr)$.

53. $y''_{xx} + (a \coth^n x + b \sin x)y'_x + b(a \coth^n x \sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b \cos x)$.

54. $y''_{xx} + (a\coth^n x + b\sin^m x)y'_x$
$$+ b\sin^{m-1} x\,(a\coth^n x\sin x + m\cos x)y = 0.$$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr)$.

55. $y''_{xx} + (a\coth^n x + b\tan x)y'_x + (b+1)(a\coth^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

56. $y''_{xx} + (a\coth^n x + b\cot x)y'_x + (b-1)(a\coth^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

57. $y''_{xx} + (a\ln^n x + b\cos x)y'_x + b(a\ln^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

58. $y''_{xx} + (a\ln^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\ln^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr)$.

59. $y''_{xx} + (a\ln^n x + b\sin x)y'_x + b(a\ln^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

60. $y''_{xx} + (a\ln^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\ln^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr)$.

61. $y''_{xx} + (a\ln^n x + b\tan x)y'_x + (b+1)(a\ln^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

62. $y''_{xx} + (a\ln^n x + b\cot x)y'_x + (b-1)(a\ln^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

63. $y''_{xx} + ae^{\lambda x}\cos(bx)\,y'_x + b[b + ae^{\lambda x}\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

64. $y''_{xx} + ae^{\lambda x}\sin(bx)\,y'_x + b[b - ae^{\lambda x}\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

65. $y''_{xx} + a\cosh(bx)\ln^n(\lambda x)\,y'_x - b[b + a\sinh(bx)\ln^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx)$.

66. $y''_{xx} + a\cosh(bx)\cos^n(\lambda x)\,y'_x - b[b + a\sinh(bx)\cos^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx)$.

67. $y''_{xx} + a\cosh^n(\lambda x)\cos(bx)\,y'_x + b[b + a\cosh^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

68. $y''_{xx} + a\cosh(bx)\sin^n(\lambda x)\,y'_x - b[b + a\sinh(bx)\sin^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx)$.

69. $y''_{xx} + a\cosh^n(\lambda x)\sin(bx)\,y'_x + b[b - a\cosh^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

70. $y''_{xx} + a\cosh(bx)\tan^n(\lambda x)\,y'_x - b[b + a\sinh(bx)\tan^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx)$.

71. $y''_{xx} + a\cosh(bx)\cot^n(\lambda x)\,y'_x - b[b + a\sinh(bx)\cot^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx)$.

72. $y''_{xx} + a\sinh(bx)\ln^n(kx)\,y'_x - b[b + a\cosh(bx)\ln^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

73. $y''_{xx} + a\sinh(bx)\cos^n(kx)\,y'_x - b[b + a\cosh(bx)\cos^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

74. $y''_{xx} + a\sinh^n(\lambda x)\cos(bx)\,y'_x + b[b + a\sinh^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

75. $y''_{xx} + a\sinh(bx)\sin^n(kx)\,y'_x - b[b + a\cosh(bx)\sin^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

76. $y''_{xx} + a\sinh^n(\lambda x)\sin(bx)\,y'_x + b[b - a\sinh^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

77. $y''_{xx} + a\sinh(bx)\tan^n(kx)\,y'_x - b[b + a\cosh(bx)\tan^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

78. $y''_{xx} + a\sinh(bx)\cot^n(\lambda x)\,y'_x - b[b + a\cosh(bx)\cot^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

79. $y''_{xx} + a\tanh^n(\lambda x)\cos(bx)\,y'_x + b[b + a\tanh^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

80. $y''_{xx} + a\tanh^n(\lambda x)\sin(bx)\,y'_x + b[b - a\tanh^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

81. $y''_{xx} + a\coth^n(\lambda x)\cos(bx)\,y'_x + b[b + a\coth^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

82. $y''_{xx} + a\coth^n(\lambda x)\sin(bx)\,y'_x + b[b - a\coth^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

83. $y''_{xx} + a\ln^n(\lambda x)\cos(bx)\,y'_x + b[b + a\ln^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

84. $y''_{xx} + a\ln^n(\lambda x)\sin(bx)\,y'_x + b[b - a\ln^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

85. $y''_{xx} + (a + be^{2\lambda x}) \ln^n(kx)\, y'_x + \lambda[(a - be^{2\lambda x}) \ln^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

86. $y''_{xx} + (a + be^{2\lambda x}) \cos^n(kx)\, y'_x + \lambda[(a - be^{2\lambda x}) \cos^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

87. $y''_{xx} + (a + be^{2\lambda x}) \sin^n(kx)\, y'_x + \lambda[(a - be^{2\lambda x}) \sin^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

88. $y''_{xx} + (a + be^{2\lambda x}) \tan^n(kx)\, y'_x + \lambda[(a - be^{2\lambda x}) \tan^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

89. $y''_{xx} + (a + be^{2\lambda x}) \cot^n(kx)\, y'_x + \lambda[(a - be^{2\lambda x}) \cot^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

90. $y''_{xx} + (a\,\mathrm{sn}^2 x + b)y = 0.$

The Lamé equation in the form of Jacobi; $\mathrm{sn}\, x$ is the Jacobi elliptic function. See the books by Whittaker & Watson (1952), Bateman & Erdélyi (1955, Vol. 3), and Kamke (1977) for information on this equation.

91. $y''_{xx} + [A\wp(x) + B]y = 0.$

The Lamé equation in the form of Weierstrass; $\wp(x)$ is the Weierstrass function. See the books by Whittaker & Watson (1952), Bateman & Erdélyi (1955, Vol. 3), and Kamke (1977) for information on this equation.

92. $xy''_{xx} + (ax \ln x + be^{\lambda x})y'_x + a(be^{\lambda x} \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

93. $xy''_{xx} + (1 - axe^{\lambda x} \ln x)y'_x + ae^{\lambda x}y = 0.$

Particular solution: $y_0 = \ln x$.

94. $xy''_{xx} + (ax \ln x + b\cosh^n x)y'_x + a(b\cosh^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

95. $xy''_{xx} + (1 - ax\cosh^n x \ln x)y'_x + a\cosh^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

96. $xy''_{xx} + (ax \ln x + b\sinh^n x)y'_x + a(b\sinh^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

97. $xy''_{xx} + (1 - ax\sinh^n x \ln x)y'_x + a\sinh^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

98. $xy''_{xx} + (ax \ln x + b\tanh^n x)y'_x + a(b\tanh^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

99. $xy''_{xx} + (1 - ax \tanh^n x \ln x)y'_x + a \tanh^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

100. $xy''_{xx} + (ax \ln x + b \coth^n x)y'_x + a(b \coth^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

101. $xy''_{xx} + (1 - ax \coth^n x \ln x)y'_x + a \coth^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

102. $xy''_{xx} + (ax \ln x + b \cos^n x)y'_x + a(b \cos^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

103. $xy''_{xx} + (1 - ax \cos^n x \ln x)y'_x + a \cos^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

104. $xy''_{xx} + (ax \ln x + b \sin^n x)y'_x + a(b \sin^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

105. $xy''_{xx} + (1 - ax \sin^n x \ln x)y'_x + a \sin^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

106. $xy''_{xx} + (ax \ln x + b \tan^n x)y'_x + a(b \tan^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

107. $xy''_{xx} + (1 - ax \tan^n x \ln x)y'_x + a \tan^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

108. $xy''_{xx} + (ax \ln x + b \cot^n x)y'_x + a(b \cot^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

109. $xy''_{xx} + (1 - ax \cot^n x \ln x)y'_x + a \cot^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

110. $x^2y''_{xx} + x(a \ln x + be^{\lambda x})y'_x + a(be^{\lambda x} \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a \ln^2 x)$.

111. $x^2y''_{xx} + x(a \ln x + b \cosh^n x)y'_x + a(b \cosh^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a \ln^2 x)$.

112. $x^2y''_{xx} + x(a \ln x + b \sinh^n x)y'_x + a(b \sinh^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a \ln^2 x)$.

113. $x^2y''_{xx} + x(a \ln x + b \tanh^n x)y'_x + a(b \tanh^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a \ln^2 x)$.

114. $x^2y''_{xx} + x(a \ln x + b \coth^n x)y'_x + a(b \coth^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a \ln^2 x)$.

115. $x^2 y''_{xx} + x(a\ln x + b\cos^n x)y'_x + a(b\cos^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a\ln^2 x)$.

116. $x^2 y''_{xx} + x(a\ln x + b\sin^n x)y'_x + a(b\sin^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a\ln^2 x)$.

117. $x^2 y''_{xx} + x(a\ln x + b\tan^n x)y'_x + a(b\tan^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a\ln^2 x)$.

118. $x^2 y''_{xx} + x(a\ln x + b\cot^n x)y'_x + a(b\cot^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a\ln^2 x)$.

119. $\sin^2 x\, y''_{xx} + \sin x\,(a + be^{\lambda x})y'_x + a(be^{\lambda x} - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

120. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\cosh^n x)y'_x + a(b\cosh^n x - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

121. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\sinh^n x)y'_x + a(b\sinh^n x - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

122. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\tanh^n x)y'_x + a(b\tanh^n x - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

123. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\coth^n x)y'_x + a(b\coth^n x - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

124. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\ln^n x)y'_x + a(b\ln^n x - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

125. $\cos^2 x\, y''_{xx} + \cos x\,(a + be^{\lambda x})y'_x + a(be^{\lambda x} + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

126. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\cosh^n x)y'_x + a(b\cosh^n x + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

127. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\sinh^n x)y'_x + a(b\sinh^n x + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

128. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\tanh^n x)y'_x + a(b\tanh^n x + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

129. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\coth^n x)y'_x + a(b\coth^n x + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

130. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\ln^n x)y'_x + a(b\ln^n x + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

14.1.9 Equations with Arbitrary Functions

◆ *Notation:* $f = f(x)$ *and* $g = g(x)$ *are arbitrary functions;* a, b, c, d, n, m, k, λ, α, β, *and* γ *are arbitrary parameters.*

► **Equations containing arbitrary functions (but not containing their derivatives).**

1. $y''_{xx} + ay = f.$

Equation of forced oscillations without friction.

Solution:

$$y = \begin{cases} C_1\cos(kx) + C_2\sin(kx) + k^{-1}\displaystyle\int_{x_0}^{x} f(\xi)\sin[k(x-\xi)]\,d\xi & \text{if } a = k^2 > 0,\\ C_1\cosh(kx) + C_2\sinh(kx) + k^{-1}\displaystyle\int_{x_0}^{x} f(\xi)\sinh[k(x-\xi)]\,d\xi & \text{if } a = -k^2 < 0,\\ C_1 x + C_2 + \displaystyle\int_{x_0}^{x}(x-\xi)f(\xi)\,d\xi & \text{if } a = 0,\end{cases}$$

where x_0 is an arbitrary number.

2. $y''_{xx} + ay'_x + by = f.$

Equation of forced oscillations with friction. The substitution $y = w\exp(-\tfrac{1}{2}ax)$ leads to an equation of the form 14.1.9.1: $w''_{xx} + (b - \tfrac{1}{4}a^2)w = f\exp(\tfrac{1}{2}ax)$.

3. $y''_{xx} + fy'_x = g.$

Solution: $y = C_1 + \displaystyle\int e^{-F}\Big(C_2 + \int e^F g\,dx\Big)\,dx$, where $F = \displaystyle\int f\,dx$.

4. $y''_{xx} + fy'_x + a(f-a)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

5. $y''_{xx} + fy'_x + a(x^n f - ax^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{a}{n+1}x^{n+1}\Big)$.

6. $y''_{xx} + xfy'_x - fy = 0.$

Particular solution: $y_0 = x$.

7. $y''_{xx} + (f + ax^n + b)y'_x + [(ax^n + b)f + anx^{n-1}]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{a}{n+1}x^{n+1} - bx\Big)$.

8. $xy''_{xx} + xfy'_x - [(ax+1)f + a(ax+2)]y = 0.$

Particular solution: $y_0 = xe^{ax}$.

9. $xy''_{xx} + (xf + a)y'_x + (a-1)fy = 0.$

Particular solution: $y_0 = x^{1-a}$.

10. $xy''_{xx} + [(ax+1)f + ax - 1]y'_x + a^2xfy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

11. $xy''_{xx} + [(ax^2+bx)f + 2]y'_x + bfy = 0.$

Particular solution: $y_0 = a + b/x$.

12. $xy''_{xx} + (f + ax^{n+1})y'_x + ax^n(f+n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

13. $xy''_{xx} + (xf + ax^n)y'_x + [(ax^n - 1)f + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp(-ax^n/n)$.

14. $xy''_{xx} + [(ax^n+1)f + anx^n + 1 - 2n]y'_x + a^2nx^{2n-1}fy = 0.$

Particular solution: $y_0 = (ax^n+1)\exp(-ax^n)$.

15. $x^2y''_{xx} + \alpha xy'_x + \beta y = f.$

The nonhomogeneous Euler equation. The substitution $x = e^t$ leads to an equation of the form 14.1.9.2: $y''_{tt} + (\alpha-1)y'_t + \beta y = f(e^t)$.

16. $x^2y''_{xx} + xy'_x + (x^2 - \nu^2)y = f.$

The nonhomogeneous Bessel equation. The general solution is expressed in terms of Bessel functions:

$$y = C_1J_\nu(x) + C_2Y_\nu(x) + \tfrac{1}{2}\pi Y_\nu(x)\int xJ_\nu(x)f(x)\,dx - \tfrac{1}{2}\pi J_\nu(x)\int xY_\nu(x)f(x)\,dx.$$

17. $x^2y''_{xx} + xfy'_x + a(f - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}$.

18. $x^2y''_{xx} + x(f+2a)y'_x + [(bx+a)f - b^2x^2 + a(a-1)]y = 0.$

Particular solution: $y_0 = x^{-a}e^{-bx}$.

19. $x^2y''_{xx} + xfy'_x + [(ax^{2n+1}+n)f - a^2x^{4n+2} - n^2 - n]y = 0.$

Particular solution: $y_0 = x^{-n}\exp\Bigl(-\dfrac{a}{2n+1}x^{2n+1}\Bigr)$.

20. $(ax^2+bx+c)y''_{xx} + (x+k)fy'_x - fy = 0.$

Particular solution: $y_0 = x + k$.

21. $x^4y''_{xx} + x^2fy'_x + [(\lambda - x)f - \lambda^2]y = 0.$

Particular solution: $y_0 = x\exp(\lambda/x)$.

22. $x^2(ax^2+b)y''_{xx} + x(ax^2+b)fy'_x - [(ax^2-b)f + 2b]y = 0.$

Particular solution: $y_0 = ax + b/x$.

23. $(x^2+a)^2y''_{xx} + (x^2+a)fy'_x - (xf+a)y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

24. $(x^2+a)^2y''_{xx}+(x^2+a)fy'_x-m[xf+(m-1)x^2+a]y=0.$

Particular solution: $y_0=(x^2+a)^{m/2}$.

25. $(ax^n+b)y''_{xx}+(ax^n+b)fy'_x-anx^{n-2}(xf+n-1)y=0.$

Particular solution: $y_0=ax^n+b$.

26. $(ax^n+bx)y''_{xx}+(ax^n+bx)fy'_x-[(anx^{n-1}+b)f+an(n-1)x^{n-2}]y=0.$

Particular solution: $y_0=ax^n+bx$.

27. $(x^n+a)^2y''_{xx}+(x^n+a)fy'_x-x^{n-2}(xf+an-a)y=0.$

Particular solution: $y_0=(x^n+a)^{1/n}$.

28. $(ax^n+b)^2y''_{xx}+(ax^n+b)fy'_x+(f-anx^{n-1}-1)y=0.$

Particular solution: $y_0=\exp\Bigl(-\int\frac{dx}{ax^n+b}\Bigr)$.

29. $f(x)y''_{xx}+[ax^2+(ac+b)x+bc]y'_x-(ax+b)y=0.$

Particular solution: $y_0=x+c$.

30. $y''_{xx}+fy'_x+ae^{\lambda x}(f-ae^{\lambda x}+\lambda)y=0.$

Particular solution: $y_0=\exp\Bigl(-\frac{a}{\lambda}e^{\lambda x}\Bigr)$.

31. $y''_{xx}+(f+ae^{\lambda x})y'_x+ae^{\lambda x}(f+\lambda)y=0.$

Particular solution: $y_0=\exp\Bigl(-\frac{a}{\lambda}e^{\lambda x}\Bigr)$.

32. $y''_{xx}+(a+be^{2\lambda x})fy'_x+\lambda[(a-be^{2\lambda x})f-\lambda]y=0.$

Particular solution: $y_0=be^{\lambda x}+ae^{-\lambda x}$.

33. $(ae^{\lambda x}+b)^2y''_{xx}+(ae^{\lambda x}+b)fy'_x+ce^{\lambda x}(f-ce^{\lambda x}+\lambda b)y=0.$

Particular solution: $y_0=\bigl(ae^{\lambda x}+b\bigr)^{-\frac{c}{a\lambda}}$.

34. $y''_{xx}+f\sinh(ax)y'_x-a[a+f\cosh(ax)]y=0.$

Particular solution: $y_0=\sinh(ax)$.

35. $y''_{xx}+f\cosh(ax)y'_x-a[a+f\sinh(ax)]y=0.$

Particular solution: $y_0=\cosh(ax)$.

36. $xy''_{xx}+(1-fx\ln x)y'_x+fy=0.$

Particular solution: $y_0=\ln x$.

37. $xy''_{xx}+(f+ax\ln x)y'_x+a(f\ln x+1)y=0.$

Particular solution: $y_0=e^{ax}x^{-ax}$.

38. $x^2y''_{xx}+2x(\ln x+a)fy'_x+\bigl[\frac{1}{4}-(\ln x+a+2)f\bigr]y=0.$

Particular solution: $y_0=\sqrt{x}\,(\ln x+a)$.

39. $x^2 y''_{xx} + x(f + a\ln x)y'_x + a(f\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}a\ln^2 x)$.

40. $y''_{xx} + f\sin(ax)y'_x + a[a - f\cos(ax)]y = 0.$

Particular solution: $y_0 = \sin(ax)$.

41. $y''_{xx} + f\cos(ax)y'_x + a[a + f\sin(ax)]y = 0.$

Particular solution: $y_0 = \cos(ax)$.

42. $y''_{xx} + (f + a\sin x)y'_x + a(f\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(a\cos x)$.

43. $y''_{xx} + (f + a\cos x)y'_x + a(f\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-a\sin x)$.

44. $y''_{xx} + (f + a\cos^n x)y'_x + a\cos^{n-1} x\,(f\cos x - n\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-a\int \cos^n x\,dx\Bigr)$.

45. $y''_{xx} + (f + a\sin^n x)y'_x + a\sin^{n-1} x\,(f\sin x + n\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-a\int \sin^n x\,dx\Bigr)$.

46. $\sin^2 x\,y''_{xx} + \sin x\,(f + a)y'_x + a(f - \cos x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x\bigr)$.

47. $\cos^2 x\,y''_{xx} + \cos x\,(a + f)y'_x + a(f + \sin x)y = 0.$

Particular solution: $y_0 = \cot^a\bigl(\frac{1}{2}x + \frac{1}{4}\pi\bigr)$.

48. $y''_{xx} + fy'_x + a[\lambda + f\tan(\lambda x) + (\lambda - a)\tan^2(\lambda x)]y = 0.$

Particular solution: $y_0 = [\cos(\lambda x)]^{a/\lambda}$.

49. $y''_{xx} + (f + a\tan x)y'_x + (a + 1)(f\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{a+1} x$.

50. $y''_{xx} + \tan x\,(f + a - 1)y'_x + [(a\tan^2 x - 1)f + 2a + 2]y = 0.$

Particular solution: $y_0 = \sin x\cos^a x$.

51. $y''_{xx} + fy'_x + a[\lambda - f\cot(\lambda x) + (\lambda - a)\cot^2(\lambda x)]y = 0.$

Particular solution: $y_0 = [\sin(\lambda x)]^{a/\lambda}$.

52. $y''_{xx} + (f + a\cot x)y'_x + (a - 1)(f\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-a} x$.

► **Equations containing arbitrary functions and their derivatives.**

53. $y''_{xx} - (f^2 + f'_x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(\int f\,dx\Bigr).$

54. $y''_{xx} + fy'_x - [a(a+1)f^2 + af'_x]y = 0.$

Particular solution: $y_0 = \exp\Bigl(a\int f\,dx\Bigr).$

55. $y''_{xx} + 2fy'_x + (f^2 + f'_x)y = 0.$

Solution: $y = (C_2x + C_1)\exp\Bigl(-\int f\,dx\Bigr).$

56. $y''_{xx} + (1-a)fy'_x - a(f^2 + f'_x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(a\int f\,dx\Bigr).$

57. $y''_{xx} + fy'_x + (fg - g^2 + g'_x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int g\,dx\Bigr).$

58. $y''_{xx} + 2fy'_x + (f^2 + f'_x + a)y = 0.$

The substitution $w = y\exp\Bigl(\int f\,dx\Bigr)$ leads to a constant coefficient linear equation: $w''_{xx} + aw = 0$.

59. $y''_{xx} + 2fy'_x + (f^2 + f'_x + ax^{2n} + bx^{n-1})y = 0.$

The substitution $w = y\exp\Bigl(\int f\,dx\Bigr)$ leads to a linear equation of the form 14.1.2.10: $w''_{xx} + a(x^{2n} + bx^{n-1})w = 0$.

60. $y''_{xx} + (2f + a)y'_x + (f^2 + af + f'_x + b)y = 0.$

The substitution $w = y\exp\Bigl(\int f\,dx\Bigr)$ leads to a constant coefficient linear equation: $w''_{xx} + aw'_x + bw = 0$.

61. $y''_{xx} + (f + g)y'_x + (fg + f'_x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int f\,dx\Bigr).$

62. $xy''_{xx} + xfy'_x + (f + xf'_x)y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\int f\,dx\Bigr).$

63. $xy''_{xx} + (xf + a)y'_x + (af + xf'_x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int f\,dx\Bigr).$

64. $(x+a)y''_{xx} + (f+b)y'_x + f'_x y = 0.$

Particular solution: $y_0 = \exp\Bigl(\int \frac{1-b-f}{x+a}\,dx\Bigr).$

65. $x^2 y''_{xx} + xfy'_x + [xf'_x + af - a(a+1)]y = 0.$

Particular solution: $y_0 = x^{a+1}\exp\Bigl(-\int x^{-1} f\,dx\Bigr).$

66. $x^2 y''_{xx} + 2xfy'_x + (xf'_x + f^2 - f + ax^2 + bx + cx)y = 0.$

The transformation $w = y\exp\Bigl(\int x^{-1} f\,dx\Bigr)$ leads to an equation of the form 14.1.2.115: $x^2 w''_{xx} + (ax^2 + bx + c)w = 0.$

67. $x^2 y''_{xx} + x(2f+1)y'_x + (f^2 + xf'_x + x^2 - a)y = 0.$

The substitution $y = w\exp\Bigl(-\int x^{-1} f\,dx\Bigr)$ leads to the Bessel equation 14.1.2.126:

$$x^2 w''_{xx} + xw'_x + (x^2 - a)w = 0.$$

68. $x^2 y''_{xx} + x(2f+a)y'_x + [f^2 + (a-1)f + xf'_x + bx^n + c]y = 0.$

The substitution $w = y\exp\Bigl(\int x^{-1} f\,dx\Bigr)$ leads to a linear equation of the form 14.1.2.132: $x^2 w''_{xx} + axw'_x + (bx^n + c)w = 0.$

69. $x^2 y''_{xx} + 2x^2 fy'_x + [x^2(f'_x + f^2 + a) + b]y = 0.$

The transformation $w = y\exp\Bigl(\int f\,dx\Bigr)$ leads to a linear equation of the form 14.1.2.115: $x^2 w''_{xx} + (ax^2 + b)w = 0.$

70. $x^2 y''_{xx} + x(2f + ax^n + b)y'_x$
$+ [f^2 + (ax^n + b - 1)f + xf'_x + \alpha x^{2n} + \beta x^n + \gamma]y = 0.$

The substitution $w = y\exp\Bigl(\int x^{-1} f\,dx\Bigr)$ leads to a linear equation of the form 14.1.2.146: $x^2 w''_{xx} + (ax^n + b)xw'_x + (\alpha x^{2n} + \beta x^n + \gamma)w = 0.$

71. $2fy''_{xx} + f'_x y'_x + ay = 0.$

The substitution $\xi = \int f^{-1/2}dx$ leads to a constant coefficient linear equation: $2y''_{\xi\xi} + ay = 0.$

72. $fy''_{xx} - f'_x y'_x - af^3 y = 0.$

Solution: $y = C_1 e^u + C_2 e^{-u}$, where $u = \sqrt{a}\int f\,dx.$

73. $\boldsymbol{f y''_{xx} - a f'_x y'_x - b f^{2a+1} y = 0.}$

Solution: $y = C_1 e^u + C_2 e^{-u}$, where $u = \sqrt{b}\int f^a\,dx$.

74. $\boldsymbol{f y''_{xx} - (f'_x + a f^2) y'_x + b f^3 y = 0.}$

Solution: $y = C_1 \exp\Bigl(\lambda_1 \int f\,dx\Bigr) + C_2 \exp\Bigl(\lambda_2 \int f\,dx\Bigr)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 - a\lambda + b = 0$.

75. $\boldsymbol{f y''_{xx} - (f'_x + a f) y'_x - b f^2 (a + b f) y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-b \int f\,dx\Bigr)$.

76. $\boldsymbol{f y''_{xx} - (f'_x + 2 a f) y'_x + (a f'_x + a^2 f - b^2 f^3) y = 0.}$

Particular solution: $y_0 = e^{ax} \exp\Bigl(b \int f\,dx\Bigr)$.

77. $\boldsymbol{f^2 y''_{xx} + f (f'_x + a) y'_x + b y = 0.}$

The substitution $\xi = \int f^{-1} dx$ leads to a constant coefficient linear equation: $y''_{\xi\xi} + a y'_\xi + by = 0$.

78. $\boldsymbol{f^2 y''_{xx} + f (f'_x + 2g + a) y'_x + (f g'_x + g^2 + a g + b) y = 0.}$

The transformation $\xi = \int f^{-1} dx$, $u = y \exp\Bigl(\int f^{-1} g\,dx\Bigr)$ leads to a constant coefficient linear equation: $u''_{\xi\xi} + a u'_\xi + bu = 0$.

79. $\boldsymbol{f g y''_{xx} - (a f'_x g + b f g'_x) y'_x - \lambda f^{2a+1} g^{2b+1} y = 0.}$

Solution: $y = C_1 e^u + C_2 e^{-u}$, where $u = \sqrt{\lambda}\int f^a g^b\,dx$.

80. $\boldsymbol{y''_{xx} + 2 f y'_x + (f^2 + f'_x + a e^{2\lambda x} + b e^{\lambda x} + c) y = 0.}$

The substitution $w = y \exp\Bigl(\int f\,dx\Bigr)$ leads to a linear equation of the form 14.1.3.5: $w''_{xx} + (a e^{2\lambda x} + b e^{\lambda x} + c) w = 0$.

81. $\boldsymbol{y''_{xx} - f'_x y'_x + a^2 e^{2f} y = 0.}$

Solution: $y = C_1 \sin\Bigl(a \int e^f\,dx\Bigr) + C_2 \cos\Bigl(a \int e^f\,dx\Bigr)$.

82. $\boldsymbol{y''_{xx} - f'_x y'_x - a^2 e^{2f} y = 0.}$

Solution: $y = C_1 \exp\Bigl(a \int e^f\,dx\Bigr) + C_2 \exp\Bigl(-a \int e^f\,dx\Bigr)$.

83. $\boldsymbol{f y''_{xx} - f''_{xx} y = 0.}$

Solution: $y = C_1 f + C_2 f \int f^{-2}\,dx$.

84. $4f^2 y''_{xx} - [2ff''_{xx} - (f'_x)^2 + a]y = 0.$

Solution:

$$y = \begin{cases} C_1\sqrt{f}\,\exp\Bigl(\tfrac{1}{2}\sqrt{a}\displaystyle\int f^{-1}\,dx\Bigr) + C_2\sqrt{f}\,\exp\Bigl(-\tfrac{1}{2}\sqrt{a}\displaystyle\int f^{-1}\,dx\Bigr) & \text{if } a>0,\\ C_1\sqrt{f}\,\cos\Bigl(\tfrac{1}{2}\sqrt{|a|}\displaystyle\int f^{-1}\,dx\Bigr) + C_2\sqrt{f}\,\sin\Bigl(\tfrac{1}{2}\sqrt{|a|}\displaystyle\int f^{-1}\,dx\Bigr) & \text{if } a<0,\\ C_1\sqrt{f} + C_2\sqrt{f}\displaystyle\int f^{-1}\,dx & \text{if } a=0. \end{cases}$$

85. $y''_{xx} - \dfrac{f''_{xx}}{f'_x}y'_x + a^2(f'_x)^2 f^{2n-2}y = 0.$

Solution: $y = \sqrt{f}\left[C_1 J_{\frac{1}{2n}}\Bigl(\dfrac{a}{n}f^n\Bigr) + C_2 Y_{\frac{1}{2n}}\Bigl(\dfrac{a}{n}f^n\Bigr)\right]$, where $J_m(z)$ and $Y_m(z)$ are Bessel functions.

86. $y''_{xx} + \left(\dfrac{ff'_x}{f^2+a} - \dfrac{f''_{xx}}{f'_x}\right)y'_x - \dfrac{b^2(f'_x)^2}{f^2+a}y = 0.$

Solution: $y = C_1\bigl(f + \sqrt{f^2+a}\bigr)^b + C_2\bigl(f + \sqrt{f^2+a}\bigr)^{-b}.$

87. $y''_{xx} - \left[\dfrac{f''_{xx}}{f'_x} + (2a-1)\dfrac{f'_x}{f}\right]y'_x + \left[(a^2-b^2)\left(\dfrac{f'_x}{f}\right)^2 + (f'_x)^2\right]y = 0.$

Solution: $y = f^a[C_1 J_b(f) + C_2 Y_b(f)]$, where $J_b(f)$ and $Y_b(f)$ are Bessel functions.

88. $y''_{xx} + \left[\dfrac{1}{2}\dfrac{f'''_{xxx}}{f'_x} - \dfrac{3}{4}\left(\dfrac{f''_{xx}}{f'_x}\right)^2 + \left(\dfrac{1}{4} - a^2\right)\left(\dfrac{f'_x}{f}\right)^2 + (f'_x)^2\right]y = 0.$

Solution: $y = \sqrt{f/f'_x}\,[C_1 J_a(f) + C_2 Y_a(f)]$, where $J_a(f)$ and $Y_a(f)$ are Bessel functions.

89. $y''_{xx} + \dfrac{f'_x}{f}y'_x + \left[\dfrac{3}{4}\left(\dfrac{f'_x}{f}\right)^2 - \dfrac{1}{2}\dfrac{f''_{xx}}{f} - \dfrac{3}{4}\left(\dfrac{g''_{xx}}{g'_x}\right)^2 + \dfrac{1}{2}\dfrac{g'''_{xxx}}{g'_x} + \left(\dfrac{1}{4} - a^2\right)\left(\dfrac{g'_x}{g}\right)^2 + (g'_x)^2\right]y = 0.$

Solution: $y = \sqrt{fg/g'_x}\,[C_1 J_a(g) + C_2 Y_a(g)]$, where $J_b(g)$ and $Y_b(g)$ are Bessel functions.

90. $y''_{xx} - \left(2\dfrac{f'_x}{f} + \dfrac{g''_{xx}}{g'_x} - \dfrac{g'_x}{g}\right)y'_x + \left[\dfrac{f'_x}{f}\left(2\dfrac{f'_x}{f} + \dfrac{g''_{xx}}{g'_x} - \dfrac{g'_x}{g}\right) - \dfrac{f''_{xx}}{f} - a^2\left(\dfrac{g'_x}{g}\right)^2 + (g'_x)^2\right]y = 0.$

Solution: $y = f[C_1 J_a(g) + C_2 Y_a(g)]$, where $J_b(g)$ and $Y_b(g)$ are Bessel functions.

91. $y''_{xx} - \left(\dfrac{g''_{xx}}{g'_x} + (2a-1)\dfrac{g'_x}{g} + 2\dfrac{h'_x}{h}\right)y'_x + \left[\dfrac{h'_x}{h}\left(\dfrac{g''_{xx}}{g'_x} + (2a-1)\dfrac{g'_x}{g} + 2\dfrac{h'_x}{h}\right) - \dfrac{h''_{xx}}{h} + (g'_x)^2\right]y = 0.$

Solution: $y = hg^a[C_1 J_a(g) + C_2 Y_a(g)]$, where $J_b(g)$ and $Y_b(g)$ are Bessel functions.

14.1.10 Some Transformations

◆ *Notation: f, g, and h are arbitrary composite functions of their arguments, which are written in parentheses following the name of a function (the argument is a function of x).*

1. $y''_{xx} + x^{-4} f(1/x) y = 0.$

The transformation $\xi = 1/x$, $w = y/x$ leads to the equation $w''_{\xi\xi} + f(\xi) w = 0$.

2. $y''_{xx} + (cx+d)^{-4} f\Big(\dfrac{ax+b}{cx+d}\Big) y = 0.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to a simpler equation: $w''_{\xi\xi} + \Delta^{-2} f(\xi) w = 0$, where $\Delta = ad - bc$.

3. $x^2 y''_{xx} + \big[x^{2n} f(ax^n + b) + \frac{1}{4} - \frac{1}{4}n^2\big] y = 0.$

The transformation $\xi = ax^n + b$, $w = yx^{\frac{n-1}{2}}$ leads to a simpler equation: $w''_{\xi\xi} + (an)^{-2} f(\xi) w = 0$.

4. $x^2 y''_{xx} + x(xf + a) y'_x + (xg + b) y = 0, \qquad f = f(x),\ g = g(x).$

The substitution $y = x^k w$, where k is a root of the quadratic equation $k^2 + (a-1)k + b = 0$, leads to the equation $x w''_{xx} + (xf + a + 2k) w'_x + (g + kf) w = 0$.

5. $x P_n(x) y''_{xx} + Q_n(x) y'_x + R_{n-1}(x) y = 0,$

$$P_n(x) = \sum_{m=0}^{n} a_m x^m,\quad Q_n(x) = \sum_{m=0}^{n} b_m x^m,\quad R_{n-1}(x) = \sum_{m=0}^{n-1} c_m x^m.$$

The substitution $y = x^k w$, where $k = 1 - b_0/a_0$, leads to an equation of the same form:

$$x P_n(x) w''_{xx} + [Q_n(x) + 2k P_n(x)] w'_x + [R_{n-1}(x) + F_{n-1}(x)] w = 0,$$

where $F_{n-1}(x) = kx^{-1}[Q_n(x) + (k-1) P_n(x)]$.

6. $x(x-1) P_{n-1}(x) y''_{xx} + Q_n(x) y'_x + R_{n-1}(x) y = 0,$

$$P_{n-1}(x) = \sum_{m=0}^{n-1} a_m x^m,\quad Q_n(x) = \sum_{m=0}^{n} b_m x^m,\quad R_{n-1}(x) = \sum_{m=0}^{n-1} c_m x^m.$$

The transformation $\xi = \dfrac{x}{x-1}$, $w = |x-1|^{-k} y$, where k is a root of the quadratic equation $a_{n-1}k^2 + (b_n - a_{n-1})k + c_{n-1} = 0$, leads to an equation of the same form:

$$\xi(\xi-1)\widehat{P}_{n-1}(\xi) w''_{\xi\xi} + [2(1-k)\xi \widehat{P}_{n-1}(\xi) - \widehat{Q}_n(\xi)] w'_\xi + [k(k-1)\widehat{P}_{n-1}(\xi) + F_{n-1}(\xi)] w = 0,$$

where

$$\widehat{P}_{n-1}(\xi) = \sum_{m=0}^{n-1} a_m \xi^m (\xi-1)^{n-m-1},\quad \widehat{Q}_n(\xi) = \sum_{m=0}^{n} b_m \xi^m (\xi-1)^{n-m},$$

$$\widehat{R}_{n-1}(\xi) = \sum_{m=0}^{n-1} c_m \xi^m (\xi-1)^{n-m-1},\quad F_{n-1}(\xi) = \frac{\widehat{R}_{n-1}(\xi) + k\widehat{Q}_n(\xi) + k(k-1)\widehat{P}_{n-1}(\xi)}{\xi - 1}.$$

7. $y''_{xx} + \big[e^{2\lambda x} f(ae^{\lambda x} + b) - \frac{1}{4}\lambda^2\big] y = 0.$

The transformation $\xi = ae^{\lambda x} + b$, $w = ye^{\lambda x/2}$ leads to the equation $w''_{\xi\xi} + (a\lambda)^{-2} f(\xi) w = 0$.

8. $y''_{xx} + f(e^{\lambda x})y'_x + g(e^{\lambda x})y = 0.$

The substitution $z = e^{\lambda x}$ leads to the equation $\lambda^2 z^2 y''_{zz} + \lambda z[f(z) + \lambda]y'_z + g(z)y = 0.$

9. $y''_{xx} + \bigl[-\lambda^2 + \sinh^{-4}(\lambda x) f\bigl(\coth(\lambda x)\bigr)\bigr]y = 0.$

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to a simpler equation $w''_{\xi\xi} + \lambda^{-2} f(\xi)w = 0.$

10. $y''_{xx} + \bigl[-\lambda^2 + \cosh^{-4}(\lambda x) f\bigl(\tanh(\lambda x)\bigr)\bigr]y = 0.$

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to a simpler equation $w''_{\xi\xi} + \lambda^{-2} f(\xi)w = 0.$

11. $y''_{xx} + \left[-\frac{1}{4}\lambda^2 + \dfrac{e^{2\lambda x}}{(ce^{\lambda x} + d)^4} f\left(\dfrac{ae^{\lambda x} + b}{ce^{\lambda x} + d}\right)\right]y = 0.$

The transformation $\xi = \dfrac{ae^{\lambda x} + b}{ce^{\lambda x} + d}$, $w = \dfrac{ye^{\lambda x/2}}{ce^{\lambda x} + d}$ leads to a simpler equation: $w''_{\xi\xi} + (\Delta\lambda)^{-2} f(\xi)w = 0$, where $\Delta = ad - bc$.

12. $fy''_{xx} + (2f \tanh x + g)y'_x + (g \tanh x + h)y = 0,$

$$f = f(x),\quad g = g(x),\quad h = h(x).$$

The substitution $u = y \cosh x$ leads to a simpler equation: $fu''_{xx} + gu'_x + (h - f)u = 0.$

13. $fy''_{xx} + (2f \coth x + g)y'_x + (g \coth x + h)y = 0,$

$$f = f(x),\quad g = g(x),\quad h = h(x).$$

The substitution $u = y \sinh x$ leads to a simpler equation: $fu''_{xx} + gu'_x + (h - f)u = 0.$

14. $x^2 y''_{xx} + \bigl[f(a \ln x + b) + \frac{1}{4}\bigr]y = 0.$

The transformation $\xi = a \ln x + b$, $w = yx^{-1/2}$ leads to a simpler equation: $w''_{\xi\xi} + a^{-2} f(\xi)w = 0.$

15. $(x^2 - 1)^2 y''_{xx} + f\left(\ln \dfrac{ax - a}{x + 1}\right)y = 0.$

The transformation $\xi = \ln \dfrac{ax - a}{x + 1}$, $w = \dfrac{y}{\sqrt{|x^2 - 1|}}$ leads to a simpler equation: $4w''_{\xi\xi} + [f(\xi) - 1]w = 0.$

16. $x^2 f(\ln x)y''_{xx} + xg(\ln x)y'_x + h(\ln x)y = 0.$

The substitution $\xi = \ln x$ leads to the equation $f(\xi)y''_{\xi\xi} + [g(\xi) - f(\xi)]y'_\xi + h(\xi)y = 0.$

17. $y''_{xx} + [\lambda^2 + \sin^{-4}(\lambda x) f(\cot(\lambda x))]y = 0.$

The transformation $\xi = \cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to a simpler equation: $w''_{\xi\xi} + \lambda^{-2} f(\xi)w = 0.$

18. $y''_{xx} + \bigl[\lambda^2 + \cos^{-4}(\lambda x) f\bigl(\tan(\lambda x)\bigr)\bigr]y = 0.$

The transformation $\xi = \tan(\lambda x)$, $w = \dfrac{y}{\cos(\lambda x)}$ leads to a simpler equation: $w''_{\xi\xi} + \lambda^{-2} f(\xi)w = 0.$

19. $y''_{xx} + \left[\lambda^2 + \dfrac{1}{\sin^4(\lambda x + b)} f\left(\dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}\right)\right] y = 0.$

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{y}{\sin(\lambda x + b)}$ leads to a simpler equation: $w''_{\xi\xi} + [\lambda \sin(b - a)]^{-2} f(\xi) w = 0$.

20. $f y''_{xx} + (g - 2f \tan x) y'_x + (h - g \tan x) y = 0,$

$$f = f(x), \quad g = g(x), \quad h = h(x).$$

The substitution $u = y \cos x$ leads to a simpler equation: $f u''_{xx} + g u'_x + (f + h) u = 0$.

21. $f y''_{xx} + (g + 2f \cot x) y'_x + (h + g \cot x) y = 0,$

$$f = f(x), \quad g = g(x), \quad h = h(x).$$

The substitution $u = y \sin x$ leads to a simpler equation: $f u''_{xx} + g u'_x + (f + h) u = 0$.

22. $(x^2 + 1)^2 y''_{xx} + f(\arctan x + b) y = 0.$

The transformation $\xi = \arctan x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to a simpler equation: $w''_{\xi\xi} + [f(\xi) + 1] w = 0$.

23. $(x^2 + 1)^2 y''_{xx} + f(\operatorname{arccot} x + b) y = 0.$

The transformation $\xi = \operatorname{arccot} x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to a simpler equation: $w''_{\xi\xi} + [f(\xi) + 1] w = 0$.

24. $y''_{xx} + f(x) y = 0.$

The transformation $x = \varphi(z)$, $y = w\sqrt{|\varphi'_z|}$ leads to an equation of the same form: $w''_{zz} + \Phi(z) w = 0$, where $\Phi(z) = \dfrac{1}{2}\dfrac{\varphi'''_{zzz}}{\varphi'_z} - \dfrac{3}{4}\left(\dfrac{\varphi''_{zz}}{\varphi'_z}\right)^2 + (\varphi'_z)^2 f(\varphi)$.

14.2 Autonomous Equations $y''_{xx} = F(y, y'_x)$

Preliminary remarks. Equations of this type often arise in different areas of mechanics, applied mathematics, physics, and chemical engineering science.

1°. The substitution $w(y) = y'_x$ leads to a first-order equation:

$$w'_y = w^{-1} F(y, w). \tag{1}$$

2°. The solution of the original autonomous equation can be represented in implicit form:

$$x = \int \frac{dy}{w(y, C_1)} + C_2, \tag{2}$$

where $w = w(y, C_1)$ is the solution of the first-order equation (1).

3°. The solution of the original autonomous equation can be written in parametric form:

$$x = \int \frac{y'_\tau(\tau, C_1)}{w(\tau, C_1)}\, d\tau + C_2, \qquad y = y(\tau,\ C_1), \tag{3}$$

where $y = y(\tau, C_1)$, $w = w(\tau, C_1)$ is a parametric form of the solution of the first-order equation (1). Formula (2) is a special case of formula (3) with $y = \tau$.

4°. For the special cases $F = F(y)$ and $F = F(y'_x)$, see equations 14.9.1.1 and 14.9.4.35.

14.2.1 Equations of the Form $y''_{xx} - y'_x = f(y)$

Preliminary remarks. Equations of this type arise in the theory of combustion and the theory of chemical reactors.

1°. The substitution $w(y) = y'_x$ leads to the Abel equation $ww'_y - w = f(y)$, which is considered in Section 13.3.1 for some specific functions f.

2°. The solution of the original autonomous equation can be written in the parametric form (3), where $y = y(\tau, C_1)$, $w = w(\tau,\ C_1)$ is a parametric form of the solution to an Abel equation of the second kind $ww'_y - w = f(y)$.

1. $y''_{xx} - y'_x = -\dfrac{2(m+1)}{(m+3)^2}y \pm \dfrac{m+1}{2a^2}y^m, \qquad m \neq \pm 1, \quad m \neq -3.$

Solution in parametric form:

$$x = \frac{m+3}{m-1}\ln\biggl(aC_1^{1-m}\frac{m-1}{m+3}\int\frac{d\tau}{\sqrt{1\pm\tau^{m+1}}} + C_2\biggr),$$
$$y = C_1^2\tau\biggl(aC_1^{1-m}\frac{m-1}{m+3}\int\frac{d\tau}{\sqrt{1\pm\tau^{m+1}}} + C_2\biggr)^{\frac{2}{m-1}}.$$

2. $y''_{xx} - y'_x = \pm 2a^2y^{-1}.$

Solution in parametric form:

$$x = -\ln\Bigl[C_1\int\exp(\pm\tau^2)\,d\tau + C_2\Bigr], \qquad y = aC_1\exp(\pm\tau^2)\Bigl[C_1\int\exp(\pm\tau^2)\,d\tau + C_2\Bigr]^{-1}.$$

3. $y''_{xx} - y'_x = -\frac{2}{9}y + \frac{16}{9}a^{3/2}y^{-1/2}.$

Solution in parametric form:

$$x = -3\ln\bigl\{C_1\exp(-\tau)\bigl[\exp(3\tau) + C_2\sin\bigl(\sqrt{3}\,\tau\bigr)\bigr]\bigr\},$$
$$y = a\exp(2\tau)\frac{\bigl[2\exp(3\tau) - C_2\sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\,C_2\cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2}{\bigl[\exp(3\tau) + C_2\sin\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2}.$$

4. $y''_{xx} - y'_x = -\frac{9}{100}y \pm \frac{9}{100}a^{8/3}y^{-5/3}.$

Solution in parametric form:

$$x = -\tfrac{5}{4}\ln\bigl[\pm(\tau^4 - 6\tau^2 + 4C_1\tau - 3)\bigr] + C_2,$$
$$y = a(\tau^3 - 3\tau + C_1)^{3/2}\bigl[\pm(\tau^4 - 6\tau^2 + 4C_1\tau - 3)\bigr]^{-9/8}.$$

5. $y''_{xx} - y'_x = -\frac{3}{16}y - \frac{3}{64}a^{8/3}y^{-5/3}.$

Solution in parametric form:

$$x = C_1 - 2\ln[\sin\tau\cosh(\tau+C_2) + \cos\tau\sinh(\tau+C_2)], \qquad y = a[\tan\tau + \tanh(\tau+C_2)]^{-3/2}.$$

◆ *In the solutions of equations 6–9, the following notation is used:*

$$Z = \begin{cases} C_1J_\nu(\tau) + Y_\nu(\tau) & \textit{for the upper sign,} \\ C_1I_\nu(\tau) + K_\nu(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are Bessel functions, and $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

6. $\boldsymbol{y''_{xx} - y'_x = Ay^{-1/2}.}$

Solution in parametric form:

$$x = -2\int \tau^{-1}Z^{-1}(\tau Z'_\tau + \tfrac{1}{3}Z)\,d\tau + C_2, \qquad y = a\tau^{-4/3}Z^{-2}[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2],$$

where $\nu = \frac{1}{3}$, $A = \mp\frac{1}{3}a^{3/2}$.

7. $\boldsymbol{y''_{xx} - y'_x = Ay^{-2}.}$

Solution in parametric form:

$$x = \mp\tfrac{2}{3}\int \tau Z^2\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{-1}\,d\tau + C_2, \quad y = 2a\tau^{4/3}Z^2\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{-1},$$

where $\nu = \frac{1}{3}$, $A = -36a^3$.

8. $\boldsymbol{y''_{xx} - y'_x = 2A^2 - Ay^{1/2}.}$

Solution in parametric form:

$$x = \pm 2\int \tau^{-1}(Z'_\tau)^{-1}(\tau Z \pm 2Z'_\tau)\,d\tau + C_2, \qquad y = a(Z'_\tau)^{-2}(\tau Z \pm 2Z'_\tau)^2,$$

where $\nu = 0$, $A = a^{1/2}$.

9. $\boldsymbol{y''_{xx} - y'_x = Ay^{-1/2} + 2B^2 + By^{1/2}.}$

Solution in parametric form:

$$x = -2\int \tau^{-1}Z^{-1}(\tau Z'_\tau - Z)\,d\tau + C_2, \qquad y = B^2 Z^{-2}(\tau Z'_\tau - Z)^2,$$

where $A = (1 - \nu^2)B^3$.

◆ *In the solutions of equations 10–14, the function $\wp = \wp(\tau)$ is defined in implicit form:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_1.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_1,\, 0,\, 1)$.

10. $\boldsymbol{y''_{xx} - y'_x = Ay^2 - \frac{9}{625}A^{-1}.}$

Solution in parametric form:

$$x = 5\ln\tau + C_2, \quad y = 5a(\tau^2\wp \mp \tfrac{1}{2}), \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

11. $\boldsymbol{y''_{xx} - y'_x = Ay^2 - \frac{6}{25}y.}$

Solution in parametric form:

$$x = 5\ln\tau + C_2, \quad y = 5a\tau^2\wp, \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

12. $\boldsymbol{y''_{xx} - y'_x = Ay^2 + \frac{6}{25}y.}$

Solution in parametric form:

$$x = 5\ln\tau + C_2, \quad y = 5a(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

13. $y''_{xx} - y'_x = 12y + Ay^{-5/2}$.

Solution in parametric form:

$$x = \mp\tfrac{2}{7}\int \wp^{-1}\bigl(f \pm 2\tau\wp^2\bigr)^{-1}\,d\tau + C_2, \qquad y = a\wp^{-6/7}\bigl(f \pm 2\tau\wp^2\bigr)^{-4/7},$$

where $f = \sqrt{\pm(4\wp^3 - 1)}$, $A = \mp 147a^{7/2}$.

14. $y''_{xx} - y'_x = \frac{63}{4}y + Ay^{-5/3}$.

Solution in parametric form:

$$x = -\tfrac{3}{4}\int (f \pm 2\tau\wp^2)(\tau f + 2\wp)^{-1}\,d\tau + C_2, \qquad y = 2a\bigl(f \pm 2\tau\wp^2\bigr)^{3/2}(\tau f + 2\wp)^{-9/8},$$

where $f = \sqrt{\pm(4\wp^3 - 1)}$, $A = -\frac{128}{3}a^2(2a)^{2/3}$.

◆ *In the solutions of equations 15–18, the following notation is used:*

$$I = \int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - 1)}} + C_1 \quad (\textit{incomplete elliptic integral of the second kind}),$$
$$R = \sqrt{\pm(4\tau^3 - 1)}, \qquad I_1 = 2\tau I \mp R, \qquad I_2 = \tau^{-1}(2\tau RI \mp R^2 - 1).$$

15. $y''_{xx} - y'_x = Ay^{1/2} - \frac{12}{49}y$.

Solution in parametric form:

$$x = -7\int \tau R^{-1} I^{-1}\,d\tau + C_2, \qquad y = 7a\tau^2 I^{-4}, \qquad \text{where} \quad A = \pm\tfrac{12}{49}(7a)^{1/2}.$$

16. $y''_{xx} - y'_x = 6y + Ay^{-4}$.

Solution in parametric form:

$$x = -\tfrac{1}{5}\int \tau^{-1}R^{-1}I_1^{-1}\,d\tau + C_2, \qquad y = a\tau^{-3/5}I_1^{-2/5}, \qquad \text{where} \quad A = \mp 150a^5.$$

17. $y''_{xx} - y'_x = 20y + Ay^{-1/2}$.

Solution in parametric form:

$$x = \tfrac{1}{3}\int R^{-1}I_1^{-1}I_2\,d\tau + C_2, \qquad y = aI_1^{-4/3}I_2^2, \qquad \text{where} \quad A = \pm 108a^{3/2}.$$

18. $y''_{xx} - y'_x = \frac{15}{4}y + Ay^{-7}$.

Solution in parametric form:

$$x = \int R^{-1}I_1\bigl(4\tau I_1^2 \mp I_2^2\bigr)^{-1}d\tau + C_2, \qquad y = aI_1^{1/2}\bigl(4\tau I_1^2 \mp I_2^2\bigr)^{-3/8}, \qquad \text{where} \quad A = \pm\tfrac{3}{4}a^8.$$

19. $y''_{xx} - y'_x = Ay + By^{-1} - B^2y^{-3}$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.5:

$$ww'_y - w = Ay + By^{-1} - B^2y^{-3}.$$

20. $y''_{xx} - y'_x = -\frac{3}{16}y + Ay^{-1/3} + By^{-5/3}$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.61:

$$ww'_y - w = -\tfrac{3}{16}y + Ay^{-1/3} + By^{-5/3}.$$

21. $y''_{xx} - y'_x = -\frac{5}{36}y + Ay^{-3/5} + By^{-7/5}$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.62:

$$ww'_y - w = -\tfrac{5}{36}y + Ay^{-3/5} + By^{-7/5}.$$

22. $y''_{xx} - y'_x = \frac{4}{9}y + 2Ay^2 + 2A^2y^3$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.14:

$$ww'_y - w = \tfrac{4}{9}y + 2Ay^2 + 2A^2y^3.$$

23. $y''_{xx} - y'_x = Ay^{k-1} - kBy^k + kB^2y^{2k-1}$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.6:

$$ww'_y - w = Ay^{k-1} - kBy^k + kB^2y^{2k-1}.$$

24. $y''_{xx} - y'_x = \pm\dfrac{2a^2}{\sqrt{y^2 \pm 8a^2}}$.

Solution in parametric form:

$$x = \mp\int E^{-1}F^{-1}(F^2 \pm 2E^2)\,d\tau + C_2, \qquad y = \pm aE^{-1}F^{-1}(F^2 \mp 2E^2),$$

where $E = \displaystyle\int \exp(\mp\tau^2)\,d\tau + C_1$, $F = 2\tau E \pm \exp(\mp\tau^2)$.

25. $y''_{xx} - y'_x = A + B\exp(-2y/A)$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.8:

$$ww'_y - w = A + B\exp(-2y/A).$$

26. $y''_{xx} - y'_x = a^2\lambda e^{2\lambda y} - a(b\lambda + 1)e^{\lambda y} + b$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.73:

$$ww'_y - w = a^2\lambda e^{2\lambda y} - a(b\lambda + 1)e^{\lambda y} + b.$$

27. $y''_{xx} - y'_x = a^2\lambda e^{2\lambda y} + a\lambda ye^{\lambda y} + be^{\lambda y}$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.1.74:

$$ww'_y - w = a^2\lambda e^{2\lambda y} + a\lambda ye^{\lambda y} + be^{\lambda y}.$$

14.2.2 Equations of the Form $y''_{xx} + f(y)y'_x + y = 0$

► **Preliminary remarks.**

Equation of this form are often encountered in the theory of nonlinear oscillations, where x plays the role of time.

1°. The transformation

$$z = -\tfrac{1}{2}y^2 + a, \qquad w = y'_x$$

leads to an Abel equation:

$$ww'_z = g(z)w + 1, \qquad \text{where} \quad g(z) = f(y)/y, \quad y = \pm\sqrt{2(a-z)},$$

whose special cases are outlined in Section 13.3.2.

2°. For oscillatory systems with a weak nonlinearity

$$y''_{xx} + \varepsilon F(y)y'_x + y = 0,$$

two leading terms of the asymptotic solution, as $\varepsilon \to 0$, are described by the formula

$$y = A\cos(x + B),$$

where the functions $A = A(\xi)$ and $B = B(\xi)$ depend on the slow variable $\xi = \varepsilon x$; they are determined from the autonomous system of first-order differential equations:

$$A'_\xi = -\frac{A}{2\pi}\int_0^{2\pi} F(A\cos\varphi)\sin^2\varphi\,d\varphi, \qquad B'_\xi = -\frac{1}{2\pi}\int_0^{2\pi} F(A\cos\varphi)\sin\varphi\cos\varphi\,d\varphi.$$

The right-hand sides of these equations depend only on A. The system is solved consecutively starting from the first equation.

► **Solvable equations and their solutions.**

1. $y''_{xx} + ayy'_x + y = 0.$

Solution in parametric form:

$$x = -A\int \frac{d\tau}{\tau(C_1 + 2A^2\ln|\tau| - 2A\tau)^{1/2}} + C_2, \qquad y = (C_1 + 2A^2\ln|\tau| - 2A\tau)^{1/2},$$

where $A = 1/a$.

2. $y''_{xx} - \varepsilon(1 - y^2)y'_x + y = 0.$

Van der Pol oscillator.

1°. Solution, as $\varepsilon \to 0$:

$$y = a\cos(x - \theta) - \tfrac{1}{32}\varepsilon a^3\sin[3(x-\theta)] + O(\varepsilon^2),$$

where

$$a^2 = \frac{4}{1 + (4C_1^{-2} - 1)e^{-\varepsilon x}}, \qquad \theta = \frac{1}{8}\varepsilon\ln a - \frac{7}{64}\varepsilon a^2 + \frac{1}{16}\varepsilon^2 x + C_2.$$

In applications, x plays the role of time, C_1 is the initial oscillation amplitude, and C_2 is the initial phase with $\varepsilon = 0$.

2°. As $\varepsilon \to +\infty$, the periodic solution of the Van der Pol equation consists of intervals with fast and slow oscillations and describes damping oscillations with period $T = (3 - 2\ln 2)\varepsilon + O(\varepsilon^{-1/3})$.

3. $y''_{xx} + y(ay^2 + b)y'_x + y = 0.$

1°. The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.1: $ww'_z = (-2az + b)w + 1$.

2°. Solution in parametric form with $a < 0$:

$$x = \mp\frac{2}{3}k\int \tau^{-1/3}\left[\pm\frac{4}{3}k^2\tau^{-2/3}Z^{-1}\left(\tau Z'_\tau + \frac{1}{3}Z\right) - \frac{b}{a}\right]^{-1/2} d\tau + C_2,$$

$$y = \left[\pm\frac{4}{3}k^2\tau^{-2/3}Z^{-1}\left(\tau Z'_\tau + \frac{1}{3}Z\right) - \frac{b}{a}\right]^{1/2}, \qquad a = -\frac{9}{4}k^{-3},$$

where

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

4. $y''_{xx} + y(ay^2 + b)^{-2}y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.2: $ww'_z = (-2az + b)^{-2}w + 1$.

5. $y''_{xx} + y(ay^2 + b)^{-1/2}y'_x + y = 0.$

1°. The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.4: $ww'_z = (-2az + b)^{-1/2}w + 1$.

2°. Solution in parametric form:

$$x = -aC_1\int\left(aC_1^2E^2 - \frac{b}{a}\right)^{-1/2}\frac{E\,d\tau}{\tau^2 - \tau + a} + C_2, \qquad y = \left(aC_1^2E^2 - \frac{b}{a}\right)^{1/2},$$

where $E = \exp\left(-\int\frac{\tau\,d\tau}{\tau^2 - \tau + a}\right)$.

6. $y''_{xx} - y\left(2a + \frac{1}{ay^2 + b}\right)y'_x + y = 0.$

The transformation $z = -\dfrac{1}{2}y^2 - \dfrac{b}{2a}$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.3:

$$ww'_z = \left(A - \frac{1}{Az}\right)w + 1, \qquad \text{where} \quad A = -2a.$$

7. $y''_{xx} + ay\exp(\lambda y^2)y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.7: $ww'_z = a\exp(-2\lambda z)w + 1$.

8. $y''_{xx} + y[a\exp(\lambda y^2) + b\exp(-\lambda y^2)]y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.8: $ww'_z = [b\exp(2\lambda z) + a\exp(-2\lambda z)]w + 1$.

9. $y''_{xx} + 2ay\exp[a(b - y^2)]y'_x + y = 0.$

Solution in parametric form:

$$x = \mp 2k \int \bigl(b - 4k^2\tau^2 - \ln|kE_\mp^{-1}|\bigr)^{-1/2} d\tau + C_2, \qquad y = \bigl(b - 4k^2\tau^2 - \ln|kE_\mp^{-1}|\bigr)^{1/2},$$

where $a = \mp\frac{1}{4}k^{-2}$, $E_\mp = \int \exp(\mp\tau^2)\,d\tau + C_1$.

10. $y''_{xx} + Ay\cosh(\lambda y^2)y'_x + y = 0.$

This is a special case of equation 14.2.2.8 with $a = b = \frac{1}{2}A$.

11. $y''_{xx} + Ay\sinh(\lambda y^2)y'_x + y = 0.$

This is a special case of equation 14.2.2.8 with $a = -b = \frac{1}{2}A$.

12. $y''_{xx} + 2Ayy'_x\sqrt{\sinh^2[A(B - y^2)] + 2A^{-1}} + y = 0.$

Solution in parametric form:

$$x = 2a\int (F^2 + 2E^2)G^{-1}Q^{-1}\,d\tau + C_2, \quad y = Q; \qquad A = \tfrac{1}{4}a^{-2},$$

where

$$E = \int \exp(-\tau^2)\,d\tau + C_1, \quad F = 2\tau E + \exp(-\tau^2), \quad G = \sqrt{F^2 - 2E^2 + 8E^2F^2},$$
$$Q = \sqrt{B - 4a^2\operatorname{arcsinh}[aE^{-1}F^{-1}(F^2 - 2E^2)]}, \quad \operatorname{arcsinh} z = \ln\bigl(z + \sqrt{z^2 + 1}\,\bigr).$$

13. $y''_{xx} - 2Ayy'_x\sqrt{\cosh^2[A(y^2 - B)] - 2A^{-1}} + y = 0.$

Solution in parametric form:

$$x = 2a\int (F^2 - 2E^2)G^{-1}Q^{-1}\,d\tau + C_2, \quad y = Q; \qquad A = \tfrac{1}{4}a^{-2},$$

where

$$E = \int \exp(\tau^2)\,d\tau + C_1, \quad F = 2\tau E - \exp(\tau^2), \quad G = \sqrt{F^2 + 2E^2 - 8E^2F^2},$$
$$Q = \sqrt{B + 4a^2\operatorname{arccosh}[aE^{-1}F^{-1}(F^2 + 2E^2)]}, \quad \operatorname{arccosh} z = \pm\ln\bigl(z + \sqrt{z^2 - 1}\,\bigr).$$

14. $y''_{xx} + Ay\cos(\lambda y^2)y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.11: $ww'_z = A\cos(2\lambda z)w + 1$.

15. $y''_{xx} + Ay\sin(\lambda y^2)y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to an Abel equation of the form 13.3.2.12: $ww'_z = -A\sin(2\lambda z)w + 1$.

14.2.3 Lienard Equations $y''_{xx} + f(y)y'_x + g(y) = 0$

► **Preliminary remarks.**

Equations of this form are encountered in various fields of applied mathematics, mechanics, and physics.

1°. For $f(y) = 0$, see equation 14.9.1.1.

2°. The substitution $w(y) = y'_x$ leads to an Abel equation of the second kind:

$$ww'_y + f(y)w + g(y) = 0,$$

whose special cases are outlined in Section 13.3.3.

3°. The transformation $w(z) = y'_x$, $z = -\int f(y)\,dy$ leads to an Abel equation of the second kind:

$$ww'_z - w = \varphi(z), \qquad \text{where} \quad \varphi(z) = g(y)/f(y),$$

whose special cases are outlined in Section 13.3.1.

► **Solvable equations and their solutions.**

1. $y''_{xx} + y + ay^3 = 0.$

Duffing equation. This is a special case of equation 14.9.1.1 with $f(y) = -y - ay^3$.

1°. Solution:

$$x = \pm \int \left(C_1 - y^2 - \tfrac{1}{2}ay^4\right)^{-1/2} dy + C_2.$$

The period of oscillations with amplitude C is expressed in terms of the complete elliptic integral of the first kind:

$$T = \frac{4}{\sqrt{1 + aC^2}} K\left(\frac{aC^2}{2 + 2aC^2}\right), \qquad \text{where} \quad K(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - m\sin^2 t}}.$$

2°. The asymptotic solution, as $a \to 0$, has the form:

$$y = \widetilde{C}_1 \cos[(1 + \tfrac{3}{8}a\widetilde{C}_1^2)x + \widetilde{C}_2] + \tfrac{1}{32}a\widetilde{C}_1^3 \cos[3(1 + \tfrac{3}{8}a\widetilde{C}_1^2)x + 3\widetilde{C}_2] + O(a^2),$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. The corresponding asymptotics for the period of oscillations with amplitude C is described by the formula: $T = 2\pi(1 - \tfrac{3}{8}aC^2) + O(a^2)$.

2. $y''_{xx} + ayy'_x + by^3 + cy = 0.$

The transformation $w(z) = y'_x$, $z = -\frac{1}{2}ay^2$ leads to an Abel equation of the form 13.3.1.2: $ww'_z - w = -\dfrac{2b}{a^2}z + \dfrac{c}{a}.$

3. $y''_{xx} = (ay + 3b)y'_x + cy^3 - aby^2 - 2b^2y.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.1:

$$ww'_y = (ay + 3b)w + cy^3 - aby^2 - 2b^2y.$$

4. $y''_{xx} = (3ay + b)y'_x - a^2y^3 - aby^2 + cy.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.2:

$$ww'_y = (3ay + b)w - a^2y^3 - aby^2 + cy.$$

5. $2y''_{xx} = (7ay + 5b)y'_x - 3a^2y^3 - 2cy^2 - 3b^2y.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.3:

$$2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y.$$

6. $y''_{xx} = y^{n-1}[(1 + 2n)y + an]y'_x - ny^{2n}(y + a).$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.8:

$$ww'_y = y^{n-1}[(1 + 2n)y + an]w - ny^{2n}(y + a).$$

7. $y''_{xx} = a(y - nb)y^{n-1}y'_x + c[y^2 - (2n + 1)by + n(n + 1)b^2]y^{2n-1}.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.9:

$$ww'_y = a(y - nb)y^{n-1}w + c[y^2 - (2n + 1)by + n(n + 1)b^2]y^{2n-1}.$$

8. $y''_{xx} = [a(2n + k)y^k + b]y^{n-1}y'_x + (-a^2ny^{2k} - aby^k + c)y^{2n-1}.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.10:

$$ww'_y = [a(2n + k)y^k + b]y^{n-1}w + (-a^2ny^{2k} - aby^k + c)y^{2n-1}.$$

9. $y''_{xx} = [a(2m + k)y^{2k} + b(2m - k)]y^{m-k-1}y'_x - (a^2my^{4k} + cy^{2k} + b^2m)y^{2m-2k-1}.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.11:

$$ww'_y = [a(2m + k)y^{2k} + b(2m - k)]y^{m-k-1}w - (a^2my^{4k} + cy^{2k} + b^2m)y^{2m-2k-1}.$$

10. $y''_{xx} = ae^{\lambda y}y'_x + be^{\lambda y}.$

Solution in parametric form:

$$x = -\frac{A}{\lambda}\int \tau^{-1}\bigl(C_1 + A^2\ln|\tau| - A\tau\bigr)^{-1}\,d\tau + C_2, \qquad y = \frac{1}{\lambda}\ln\Bigl[-\frac{\lambda}{b}(C_1 + A^2\ln|\tau| - A\tau)\Bigr],$$

where $A = b/a$.

11. $y''_{xx} = (ae^y + b)y'_x + ce^{2y} - abe^y - b^2.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.67:

$$ww'_y = (ae^y + b)w + ce^{2y} - abe^y - b^2.$$

12. $y''_{xx} = [a(2\mu + \lambda)e^{\lambda y} + b]e^{\mu y}y'_x + (-a^2\mu e^{2\lambda y} - abe^{\lambda y} + c)e^{2\mu y}.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.68:

$$ww'_y = [a(2\mu + \lambda)e^{\lambda y} + b]e^{\mu y}w + (-a^2\mu e^{2\lambda y} - abe^{\lambda y} + c)e^{2\mu y}.$$

13. $y''_{xx} = (ae^{\lambda y} + b)y'_x + c[a^2e^{2\lambda y} + ab(\lambda y + 1)e^{\lambda y} + b^2\lambda y]$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.69:

$$ww'_y = (ae^{\lambda y} + b)w + c[a^2e^{2\lambda y} + ab(\lambda y + 1)e^{\lambda y} + b^2\lambda y].$$

14. $y''_{xx} = e^{\lambda y}(2a\lambda y + a + b)y'_x - e^{2\lambda y}(a^2\lambda y^2 + aby + c)$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.70:

$$ww'_y = e^{\lambda y}(2a\lambda y + a + b)w - e^{2\lambda y}(a^2\lambda y^2 + aby + c).$$

15. $y''_{xx} = e^{ay}(2ay^2 + 2y + b)y'_x + e^{2ay}(-ay^4 - by^2 + c)$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.71:

$$ww'_y = e^{ay}(2ay^2 + 2y + b)w + e^{2ay}(-ay^4 - by^2 + c).$$

16. $y''_{xx} = (a\cosh y + b)y'_x - ab\sinh y + c$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.75:

$$ww'_y = (a\cosh y + b)w - ab\sinh y + c.$$

17. $y''_{xx} = (a\sinh y + b)y'_x - ab\cosh y + c$.

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 13.3.3.76:

$$ww'_y = (a\sinh y + b)w - ab\cosh y + c.$$

18. $y''_{xx} + a\sin y = 0$.

This is the equation of oscillations of the mathematical pendulum, where the variable x plays the role of time, and y is the angle of deviation from the equilibrium state.

1°. Solution:

$$x = \pm\int(2a\cos y + C_1)^{-1/2}\,dy + C_2.$$

2°. With $a > 0$ and the initial conditions $y(0) = C > 0$ and $y'_x(0) = 0$, the oscillations of the mathematical pendulum are described by

$$\sin\frac{y}{2} = m\,\mathrm{sn}\big(\sqrt{a}\,x\big), \qquad m = \sin\frac{C}{2},$$

where $\mathrm{sn} = \mathrm{sn}(z)$ is the Jacobi elliptic function defined parametrically by the following relations:

$$\mathrm{sn}(z) = \sin\beta, \qquad z = \int_0^\beta \frac{d\beta}{\sqrt{1 - m^2\sin^2\beta}}.$$

3°. The period of oscillations of the mathematical pendulum is expressed in terms of the complete elliptic integral of the second kind:

$$T = \frac{4}{\sqrt{a}}K(m), \qquad \text{where} \quad K(m) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - m^2\sin^2\beta}}.$$

At small amplitudes, as $C \to 0$, the following asymptotic formula holds for the period:

$$T = \frac{2\pi}{\sqrt{a}}\Big(1 + \frac{1}{16}C^2\Big) + O(C^4), \qquad C \to 0.$$

19. $y''_{xx} + a\sin(\lambda y)y'_x + b\sin(\lambda y) = 0.$

Solution in parametric form:

$$x = -A\int t^{-1}\bigl(b^2 - \lambda^2 F^2\bigr)^{-1/2}\,dt + C_2, \qquad y = \frac{1}{\lambda}\arccos\Bigl(\frac{\lambda}{b}F\Bigr),$$

where $A = b/a$, $F = At - A^2\ln|t| + C_1$.

20. $y''_{xx} + a\cos(\lambda y)y'_x + b\cos(\lambda y) = 0.$

The substitution $\lambda y = \lambda u + \frac{\pi}{2}$ leads to an equation of the form 14.2.3.19:

$$u''_{xx} - a\sin(\lambda u)u'_x - b\sin(\lambda u) = 0.$$

21. $y''_{xx} + f'_y(y)y'_x = \dfrac{a}{f(y)}.$

First integral:

$$(y'_x)^2 + 2fy'_x + f^2 - 2a\int\frac{dy}{f} - 2ax = C,$$

where C is an arbitrary constant. This equation has a singular solution,

$$x + \frac{C}{2a} = -\int\frac{dy}{f(y)}, \tag{1}$$

which *is not* a one-parameter family of solutions to the original equation, as the integrating factor

$$R = 2\bigl(y' + f(y)\bigr)$$

vanishes on the integral curve (1).

⊙ *Literature*: L. V. Linchuk and V. F. Zaitsev (2015).

14.2.4 Rayleigh Equations $y''_{xx} + f(y'_x) + g(y) = 0$

► **Preliminary remarks. Some transformations.**

Equations of this form arise in the theory of nonlinear oscillations.

1°. Let us discuss the special case $g(y) = y$, which corresponds to the equation

$$y''_{xx} + f(y'_x) + y = 0. \tag{1}$$

Differentiating equation (1) with respect to x and substituting $z(x) = y'_x$, we obtain the equation of nonlinear oscillations:

$$z''_{xx} + \Phi(z)z'_x + z = 0, \qquad \text{where} \quad \Phi(z) = f'_z(z), \tag{2}$$

which is considered in Section 14.2.2.

The solution of equation (1) can be written in parametric form:

$$x = x(\tau, C_1, C_2), \qquad y = -f(z) - \frac{z'_\tau}{x'_\tau},$$

where $x = x(\tau, C_1, C_2)$, $z = z(\tau, C_1, C_2)$ is a parametric representation of the solution of equation (2).

2°. The transformation

$$\xi = -\tfrac{1}{2}(y'_x)^2 + a, \qquad w = -y - f(y'_x),$$

reduces equation (1) to an Abel equation of the second kind:

$$ww'_\xi = H(\xi)w + 1, \quad \text{where} \quad H(\xi) = z^{-1}\Phi(z), \quad z = \pm\sqrt{2(a-\xi)}, \tag{3}$$

where function $\Phi = \Phi(z)$ is defined above in equation (2). Specific equations of the form (3) are outlined in Section 13.3.2.

3°. The equation of the special form

$$y''_{xx} + a(y'_x)^2 + g(y) = 0 \tag{4}$$

is reduced, with the aid of the substitution $w(y) = (y'_x)^2$, to the first-order linear equation $w'_y + 2aw + 2g(y) = 0$. Therefore, the solution of equation (4) can be written in implicit form:

$$x = C_2 \pm \int \bigl[C_1 e^{-2ay} - G(y)\bigr]^{-1/2}\,dy, \quad \text{where} \quad G(y) = 2e^{-2ay}\int e^{2ay}g(y)\,dy.$$

4°. The equation of the special form

$$y''_{xx} + a(y'_x)^4 + b(y'_x)^2 + g(y) = 0 \tag{5}$$

is reduced, with the aid of the substitution $w(y) = (y'_x)^2$, to the Riccati equation $w'_y + 2aw^2 + 2bw + 2g(y) = 0$, which is outlined in Section 13.2.

5°. For the oscillatory systems with a weak nonlinearity

$$y''_{xx} + \varepsilon F(y'_x) + y = 0,$$

two leading terms of the asymptotic solution, as $\varepsilon \to 0$, are described by the formula

$$y = A\cos(x + B),$$

where the functions $A = A(\xi)$ and $B = B(\xi)$ depend on the slow variable $\xi = \varepsilon x$ and are determined from the autonomous system of first-order differential equations:

$$A'_\xi = \frac{1}{2\pi}\int_0^{2\pi} F(-A\sin\varphi)\sin\varphi\,d\varphi, \qquad AB'_\xi = \frac{1}{2\pi}\int_0^{2\pi} F(-A\sin\varphi)\cos\varphi\,d\varphi.$$

The right-hand sides of these equations depend only on A. The system is solved consecutively starting from the first equation.

► Solvable equations and their solutions.

1. $\boldsymbol{y''_{xx} + a(y'_x)^2 + by = 0.}$

This equation describes small oscillations in the case where the drag force is proportional to the speed squared.

Solution in implicit form: $x = C_2 \pm a\displaystyle\int \bigl[C_1 a^2 e^{-2ay} + b(\tfrac{1}{2} - ay)\bigr]^{-1/2}\,dy.$

2. $y''_{xx} + \varepsilon\left[\frac{1}{3}(y'_x)^3 - y'_x\right] + y = 0.$

Van der Pol equation.

1°. Differentiating the equation with respect to x and passing on to the new variable $w(x) = y'_x$, we arrive at an equation of the form 14.2.2.2: $w''_{xx} - \varepsilon(1 - w^2)w'_x + w = 0.$

2°. Solution, as $\varepsilon \to 0$:

$$y = \frac{2C_1}{\sqrt{1 - C_2 e^{-\varepsilon x}}} \cos x + \frac{2\sqrt{1 - C_1^2}}{\sqrt{1 - C_2 e^{-\varepsilon x}}} \sin x + O(\varepsilon^2).$$

3. $y''_{xx} + a(y'_x)^4 + b(y'_x)^2 + y = 0.$

The transformation $\xi = -\frac{1}{2}(y'_x)^2$, $w = -y - a(y'_x)^4 - b(y'_x)^2$ leads to an Abel equation of the form 13.3.2.1: $ww'_\xi = (-8a\xi + 2b)w + 1.$

4. $y''_{xx} + (y'_x)^2\left[a(y'_x)^2 + b\right]^{-1} + y = 0.$

The transformation $\xi = -\frac{1}{2}(y'_x)^2$, $w = -y - (y'_x)^2[a(y'_x)^2 + b]^{-1}$ leads to an equation of the form 13.3.2.2: $ww'_\xi = 2b(b - 2a\xi)^{-2}w + 1.$

5. $y''_{xx} + A\exp[\lambda(y'_x)^2] + B + y = 0.$

Differentiating the equation with respect to x and passing on to the new variable $w(x) = y'_x$, we arrive at an equation of the form 14.2.2.7: $w''_{xx} + 2A\lambda w \exp(\lambda w^2)w'_x + w = 0.$

6. $y''_{xx} + A\cosh[\lambda(y'_x)^2] + B + y = 0.$

Differentiating the equation with respect to x and passing on to the new variable $w(x) = y'_x$, we arrive at an equation of the form 14.2.2.11: $w''_{xx} + 2A\lambda w \sinh(\lambda w^2)w'_x + w = 0.$

7. $y''_{xx} + A\sinh[\lambda(y'_x)^2] + B + y = 0.$

Differentiating the equation with respect to x and passing on to the new variable $w(x) = y'_x$, we arrive at an equation of the form 14.2.2.10: $w''_{xx} + 2A\lambda w \cosh(\lambda w^2)w'_x + w = 0.$

8. $y''_{xx} + a(y'_x)^2 + b\sin y = 0.$

This equation describes the oscillations of the mathematical pendulum in the case where the drag force is proportional to the speed squared.

Solution in implicit form: $x = C_2 \pm \int \left[C_1 e^{-2ay} + \frac{2b}{4a^2 + 1}(\cos y - 2a\sin y)\right]^{-1/2} dy.$

9. $y''_{xx} + A\cos[\lambda(y'_x)^2] + B + y = 0.$

Differentiating the equation with respect to x and passing on to the new variable $w(x) = y'_x$, we arrive at an equation of the form 14.2.2.15: $w''_{xx} - 2A\lambda w \sin(\lambda w^2)w'_x + w = 0.$

10. $y''_{xx} + A\sin[\lambda(y'_x)^2] + B + y = 0.$

Differentiating the equation with respect to x and passing on to the new variable $w(x) = y'_x$, we arrive at an equation of the form 14.2.2.14: $w''_{xx} + 2A\lambda w \cos(\lambda w^2)w'_x + w = 0.$

14.3 Emden–Fowler Equation $y''_{xx} = Ax^n y^m$

14.3.1 Exact Solutions

▶ Preliminary remarks. Classification table.

In this subsection, the value of the insignificant parameter A is in many cases defined in the form of a function of two (one) auxiliary coefficients a and b,

$$A = \varphi(a, b), \tag{1}$$

and the corresponding solutions are represented in parametric form,

$$x = f_1(\tau, C_1, C_2, a), \qquad y = f_2(\tau, C_1, C_2, b), \tag{2}$$

where τ is the parameter, C_1 and C_2 are arbitrary constants, and f_1 and f_2 are some functions.

Having fixed the auxiliary coefficient sign $a > 0$ (or $b > 0$), one should express the coefficient b in terms of both A and a with the help of (1). As a result, one obtains:

$$b = \psi(A, a).$$

Substituting this formula into (2), we find a solution of the equation under consideration (where the specific numerical value of the coefficient a can be chosen arbitrarily). The case $a < 0$ (or $b < 0$), which may lead to another branch of the solution or to a different domain of definition of the variables x and y in (2), should be considered in a similar manner.

One can also use a different approach by setting one of the auxiliary coefficients (e.g., a) equal to ± 1 in (1) and (2); then the other coefficients will be identically expressed in terms of A by means of (1).

Table 14.5 presents all solvable Emden–Fowler equations whose solutions are outlined in Section 14.3.1. The one-parameter families (in the space of the parameters n and m) and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of m and the growth of n (for identical m). The number of the equation sought is indicated in the last column in this table.

▶ Solvable equations and their solutions.

1. $y''_{xx} = Ax^n$.

Solution: $$y = \begin{cases} \dfrac{Ax^{n+2}}{(n+1)(n+2)} + C_1 x + C_2 & \text{if } n \neq -1,\ n \neq -2; \\ -A \ln|x| + C_1 x + C_2 & \text{if } n = -2; \\ A \displaystyle\int \ln|x|\,dx + C_1 x + C_2 & \text{if } n = -1. \end{cases}$$

2. $y''_{xx} = Ay^m$.

Solution: $$x = \begin{cases} \pm \displaystyle\int \Bigl(\frac{2A}{m+1} y^{m+1} + C_1\Bigr)^{-1/2} dy + C_2 & \text{if } m \neq -1, \\ \pm \displaystyle\int (2A \ln|y| + C_1)^{-1/2}\,dy + C_2 & \text{if } m = -1. \end{cases}$$

TABLE 14.5

Solvable cases of the Emden–Fowler equation $y''_{xx} = Ax^n y^m$

No	m	n	Equation
	One-parameter families		
1	arbitrary	0	2.3.1.2
2	arbitrary	$-m-3$	2.3.1.3
3	arbitrary	$-\frac{1}{2}(m+3)$	2.3.1.4
4	0	arbitrary	2.3.1.1
5	1	arbitrary	2.3.1.5
	Isolated points		
6	-7	1	2.3.1.15
7	-7	3	2.3.1.16
8	$-5/2$	$-1/2$	2.3.1.22
9	-2	-2	2.3.1.28
10	-2	1	2.3.1.27
11	$-5/3$	$-10/3$	2.3.1.10
12	$-5/3$	$-7/3$	2.3.1.8
13	$-5/3$	$-5/6$	2.3.1.23
14	$-5/3$	$-1/2$	2.3.1.24
15	$-5/3$	1	2.3.1.7
16	$-5/3$	2	2.3.1.9
17	$-7/5$	$-13/5$	2.3.1.14
18	$-7/5$	1	2.3.1.13
19	$-1/2$	$-7/2$	2.3.1.12
20	$-1/2$	$-5/2$	2.3.1.6
21	$-1/2$	-2	2.3.1.26
22	$-1/2$	$-4/3$	2.3.1.17
23	$-1/2$	$-7/6$	2.3.1.18
24	$-1/2$	$-1/2$	2.3.1.25
25	$-1/2$	1	2.3.1.11
26	2	-5	2.3.1.19
27	2	$-20/7$	2.3.1.21
28	2	$-15/7$	2.3.1.20

Special cases.

1°. In the case $m = -1/2$, the solution can be written in the parametric form:

$$x = aC_1^3(\tau^3 - 3\tau + C_2), \quad y = bC_1^4(\tau^2 - 1)^2, \quad \text{where} \quad A = \pm\tfrac{4}{9}a^{-2}b^{3/2}.$$

2°. In the case $m = -4$, the solution can be written in the parametric form:

$$x = aC_1^5\tau^{-1}\left(2\tau\int\frac{\tau\,d\tau}{R} + C_2\tau \mp R\right), \quad y = bC_1^2\tau^{-1},$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $A = \mp 6a^{-2}b^5$.

3°. In the case $m = 2$, the solution can be written in the parametric form:

$$x = aC_1^{-1}\tau, \quad y = bC_1^2\wp; \qquad A = \pm 6a^{-2}b^{-1},$$

the function $\wp$ of the parameter τ is defined in implicit form:

$$\tau = \int\frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\, 0,\, 1)$.

4°. In the case $m = -5/2$, the solution can be written in the parametric form:

$$x = aC_1^7\wp^{-2}\left[\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2\right], \quad y = bC_1^4\wp^{-2}, \quad \text{where} \quad A = \mp 3a^{-2}b^{7/2}.$$

The function $\wp$ of the parameter τ is defined in the previous case.

3. $y''_{xx} = Ax^{-m-3}y^m$.

1°. Solution in parametric form with $m \neq -1$:

$$x = aC_1^{m-1}\Big[\int (1\pm\tau^{m+1})^{-1/2}\,d\tau + C_2\Big]^{-1},\qquad y = bC_1^{m+1}\tau\Big[\int (1\pm\tau^{m+1})^{-1/2}\,d\tau + C_2\Big]^{-1},$$

where $A = \pm\dfrac{m+1}{2}a^{m+1}b^{1-m}$.

2°. Solution in parametric form with $m = -1$:

$$x = C_1\Big[\int \exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1},\qquad y = b\exp(\mp\tau^2)\Big[\int \exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1},$$

where $A = \mp 2b^2$.

4. $y''_{xx} = Ax^{-\frac{m+3}{2}}y^m$.

1°. Solution in parametric form with $m \neq -1$:

$$x = aC_2^2\exp\Big[2\int\Big(\frac{8}{m+1}\tau^{m+1} + \tau^2 + C_1\Big)^{-1/2}d\tau\Big],$$
$$y = bC_2\tau\exp\Big[\int\Big(\frac{8}{m+1}\tau^{m+1} + \tau^2 + C_1\Big)^{-1/2}d\tau\Big],$$

where $A = \Big(\dfrac{a}{b^2}\Big)^{\frac{m-1}{2}}$.

2°. Solution in parametric form with $m = -1$:

$$x = aC_2^2\exp\Big[2\int\big(8\ln|\tau| + \tau^2 + C_1\big)^{-1/2}d\tau\Big],$$
$$y = bC_2\tau\exp\Big[\int\big(8\ln|\tau| + \tau^2 + C_1\big)^{-1/2}d\tau\Big],$$

where $A = b^2/a$.

5. $y''_{xx} = Ax^n y$.

For $n \neq -2$, see equation 14.1.2.7. For $n = -2$, see equation 14.1.2.123.

6. $y''_{xx} = Ax^{-5/2}y^{-1/2}$.

Solution in parametric form:

$$x = aC_1^{-3}(\tau^3 - 3\tau + C_2)^{-1},\quad y = bC_1(\tau^2-1)^2(\tau^3 - 3\tau + C_2)^{-1},\quad \text{where } A = \pm\tfrac{4}{9}a^{1/2}b^{3/2}.$$

7. $y''_{xx} = Axy^{-5/3}$.

Solution in parametric form:

$$x = \pm aC_1^8(\tau^4 - 6\tau^2 + 4C_2\tau - 3),\quad y = bC_1^9(\tau^3 - 3\tau + C_2)^{3/2},\quad \text{where } A = \pm\tfrac{9}{64}a^{-3}b^{8/3}.$$

8. $y''_{xx} = Ax^{-7/3}y^{-5/3}$.

Solution in parametric form:

$$x = \pm\frac{aC_1^{-8}}{\tau^4 - 6\tau^2 + 4C_2\tau - 3},\quad y = \pm\frac{bC_1(\tau^3 - 3\tau + C_2)^{3/2}}{\tau^4 - 6\tau^2 + 4C_2\tau - 3},\quad \text{where } A = \pm\tfrac{9}{64}a^{1/3}b^{8/3}.$$

9. $y''_{xx} = Ax^2 y^{-5/3}$.

1°. Solution in parametric form with $A < 0$:

$$x = aC_1^2 \cos\tau \cosh(\tau + C_2)[\tan\tau + \tanh(\tau + C_2)], \qquad y = bC_1^3[\cos\tau \cosh(\tau + C_2)]^{3/2},$$

where $A = -\frac{3}{16}a^{-4}b^{8/3}$.

2°. Solution in parametric form with $A > 0$:

$$x = aC_1^2[\sinh\tau + \cos(\tau + C_2)], \qquad y = bC_1^3[\cosh\tau - \sin(\tau + C_2)]^{3/2},$$

where $A = \frac{3}{4}a^{-4}b^{8/3}$.

10. $y''_{xx} = Ax^{-10/3} y^{-5/3}$.

1°. Solution in parametric form with $A < 0$:

$$x = aC_1^{-2}[\cos\tau \cosh(\tau + C_2)]^{-1}[\tan\tau + \tanh(\tau + C_2)]^{-1},$$
$$y = bC_1[\cos\tau \cosh(\tau + C_2)]^{1/2}[\tan\tau + \tanh(\tau + C_2)]^{-1},$$

where $A = -\frac{3}{16}a^{4/3}b^{8/3}$.

2°. Solution in parametric form with $A > 0$:

$$x = aC_1^{-2}[\sinh\tau + \cos(\tau + C_2)]^{-1},$$
$$y = bC_1[\cosh\tau - \sin(\tau + C_2)]^{3/2}[\sinh\tau + \cos(\tau + C_2)]^{-1},$$

where $A = \frac{3}{4}a^{4/3}b^{8/3}$.

11. $y''_{xx} = Axy^{-1/2}$.

Solution in parametric form:

$$x = aC_1 \exp(-\tau)\bigl[\exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr)\bigr],$$
$$y = bC_1^2 \exp(-2\tau)\bigl[2\exp(3\tau) - C_2 \sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}C_2 \cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2,$$

where $A = 16a^{-3}b^{3/2}$.

◆ *In the solutions of equations 12–14, the following notation is used:*

$$S_1 = \exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr), \quad S_2 = 2\exp(3\tau) - C_2 \sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\,C_2 \cos\bigl(\sqrt{3}\,\tau\bigr),$$
$$S_3 = 2S_1(S_2)'_\tau - (S_1)'_\tau S_2 - S_1 S_2.$$

12. $y''_{xx} = Ax^{-7/2} y^{-1/2}$.

Solution in parametric form:

$$x = aC_1^{-1}\exp(\tau)S_1^{-1}, \quad y = bC_1 \exp(-\tau)S_1^{-1}S_2^2, \qquad \text{where} \quad A = 16(ab)^{3/2}.$$

13. $y''_{xx} = Axy^{-7/5}$.

Solution in parametric form:

$$x = aC_1^4 \exp(-2\tau)S_3, \quad y = bC_1^5 \exp\bigl(-\tfrac{5}{2}\tau\bigr)S_1^{5/2}, \qquad \text{where} \quad A = \tfrac{5}{1024}a^{-3}b^{12/5}.$$

14. $y''_{xx} = Ax^{-13/5}y^{-7/5}$**.**

Solution in parametric form:

$$x = aC_1^{-4}\exp(2\tau)S_3^{-1}, \quad y = bC_1\exp\bigl(-\tfrac{1}{2}\tau\bigr)S_1^{5/2}S_3^{-1}, \qquad \text{where} \quad A = \tfrac{5}{1024}a^{3/5}b^{12/5}.$$

◆ *In the solutions of equations 15–18, the following notation is used:*

$$f = 2\tau I(\tau) + C_2\tau \mp R, \quad I(\tau) = \int \frac{\tau\,d\tau}{R}, \quad R = \sqrt{\pm(4\tau^3 - 1)},$$

where $I(\tau)$ is the incomplete elliptic integral of the second kind in the form of Weierstrass.

15. $y''_{xx} = Axy^{-7}$**.**

Solution in parametric form:

$$x = aC_1^8[4\tau f^2 \mp \tau^{-2}(fR - 1)^2], \quad y = bC_1^3 f^{1/2}, \qquad \text{where} \quad A = \pm\tfrac{3}{64}a^{-3}b^8.$$

16. $y''_{xx} = Ax^3y^{-7}$**.**

Solution in parametric form:

$$x = aC_1^8[4\tau f^2 \mp \tau^{-2}(fR - 1)^2]^{-1}, \quad y = bC_1^5 f^{1/2}[4\tau f^2 \mp \tau^{-2}(fR - 1)^2]^{-1},$$

where $A = \pm\frac{3}{64}a^{-5}b^8$.

17. $y''_{xx} = Ax^{-4/3}y^{-1/2}$**.**

Solution in parametric form:

$$x = aC_1^9 f^3, \quad y = bC_1^4\tau^{-2}(fR - 1)^2, \qquad \text{where} \quad A = \pm\tfrac{4}{3}a^{-2/3}b^{3/2}.$$

18. $y''_{xx} = Ax^{-7/6}y^{-1/2}$**.**

Solution in parametric form:

$$x = aC_1^9 f^{-3}, \quad y = bC_1^5\tau^{-2}f^{-3}(fR - 1)^2, \qquad \text{where} \quad A = \pm\tfrac{4}{3}a^{-5/6}b^{3/2}.$$

◆ *In the solutions of equations 19–24, the function $\wp = \wp(\tau)$ is defined in implicit form:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\, 0,\, 1)$.

19. $y''_{xx} = Ax^{-5}y^2$**.**

Solution in parametric form:

$$x = aC_1\tau^{-1}, \quad y = bC_1^3\tau^{-1}\wp, \qquad \text{where} \quad A = \pm 6a^3b^{-1}.$$

20. $y''_{xx} = Ax^{-15/7}y^2$.

Solution in parametric form:

$$x = aC_1^7\tau^7, \quad y = bC_1\tau(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{49}a^{1/7}b^{-1}.$$

21. $y''_{xx} = Ax^{-20/7}y^2$.

Solution in parametric form:

$$x = aC_1^7\tau^{-7}, \quad y = bC_1^6\tau^{-6}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{49}a^{6/7}b^{-1}.$$

22. $y''_{xx} = Ax^{-1/2}y^{-5/2}$.

Solution in parametric form:

$$x = aC_1^7\wp^2\big[\sqrt{\pm(4\wp^3-1)} \pm 2\tau\wp^2\big]^{-1}, \quad y = bC_1^3\big[\sqrt{\pm(4\wp^3-1)} \pm 2\tau\wp^2\big]^{-1},$$

where $A = \mp 3a^{-3/2}b^{7/2}$.

23. $y''_{xx} = Ax^{-5/6}y^{-5/3}$.

Solution in parametric form:

$$x = \frac{aC_1^{16}}{\big[\tau\sqrt{\pm(4\wp^3-1)} + 2\wp\big]^2}, \quad y = \frac{bC_1^7\big[\sqrt{\pm(4\wp^3-1)} \pm 2\tau\wp^2\big]^{3/2}}{\big[\tau\sqrt{\pm(4\wp^3-1)} + 2\wp\big]^2},$$

where $A = -\tfrac{1}{6}a^{-7/6}b^{8/3}$.

24. $y''_{xx} = Ax^{-1/2}y^{-5/3}$.

Solution in parametric form:

$$x = aC_1^{16}\big[\tau\sqrt{\pm(4\wp^3-1)} + 2\wp\big]^2, \quad y = bC_1^9\big[\sqrt{\pm(4\wp^3-1)} \pm 2\tau\wp^2\big]^{3/2},$$

where $A = -\tfrac{1}{6}a^{-3/2}b^{8/3}$.

◆ *In the solutions of equations 25–28, the following notation is used:*

$$Z = \begin{cases} C_1J_{1/3}(\tau) + C_2Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1I_{1/3}(\tau) + C_2K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

where $J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

25. $y''_{xx} = Ax^{-1/2}y^{-1/2}$.

Solution in parametric form:

$$x = a\tau^{2/3}Z^2, \quad y = b\tau^{-2/3}\big(\tau Z'_\tau + \tfrac{1}{3}Z\big)^2, \qquad \text{where} \quad A = \tfrac{1}{3}(\mp b/a)^{3/2}.$$

26. $y''_{xx} = Ax^{-2}y^{-1/2}$.

Solution in parametric form:

$$x = a\tau^{-2/3}Z^{-2}, \quad y = b\tau^{-4/3}Z^{-2}\big(\tau Z'_\tau + \tfrac{1}{3}Z\big)^2, \qquad \text{where} \quad A = \mp\tfrac{1}{3}b^{3/2}.$$

27. $y''_{xx} = Axy^{-2}$**.**

Solution in parametric form:

$$x = a\tau^{-2/3}\big[\big(\tau Z'_\tau + \tfrac{1}{3}Z\big)^2 \pm \tau^2 Z^2\big], \quad y = b\tau^{2/3}Z^2, \quad \text{where} \quad A = -\tfrac{9}{2}(b/a)^3.$$

28. $y''_{xx} = Ax^{-2}y^{-2}$**.**

Solution in parametric form:

$$x = \tau^{2/3}\big[\big(\tau Z'_\tau + \tfrac{1}{3}Z\big)^2 \pm \tau^2 Z^2\big]^{-1}, \quad y = b\tau^{4/3}Z^2\big[\big(\tau Z'_\tau + \tfrac{1}{3}Z\big)^2 \pm \tau^2 Z^2\big]^{-1},$$

where $A = -\frac{9}{2}b^3$.

14.3.2 First Integrals (Conservation Laws)

In this subsection, first integrals of the form

$$\sum_{\alpha=0}^{k} f_\alpha(x,y)(y'_x)^\alpha = C, \qquad \text{where} \quad k = 2,\ 3,\ 4,\ 5,$$

for the Emden–Fowler equation $y''_{xx} = Ax^n y^m$ are given.

► **First integrals with $k = 2$.**

1°. For $n = 0$ and arbitrary m $(m \neq -1)$,

$$(y'_x)^2 - \frac{2A}{m+1}y^{m+1} = C.$$

2°. For $n = -\frac{1}{2}(m+3)$ and arbitrary m $(m \neq -1)$,

$$x(y'_x)^2 - yy'_x - \frac{2A}{m+1}x^{-\frac{m+1}{2}}y^{m+1} = C.$$

3°. For $n = -m-3$ and arbitrary m $(m \neq -1)$,

$$x^2(y'_x)^2 - 2xyy'_x + y^2 - \frac{2A}{m+1}x^{-m-1}y^{m+1} = C.$$

4°. For $n = -\frac{20}{7}$, $m = 2$,

$$\tfrac{343}{24}Ax^{8/7}(y'_x)^2 - \big(\tfrac{49}{3}Ax^{1/7}y - x\big)y'_x - \tfrac{343}{36}A^2x^{-12/7}y^3 + \tfrac{7}{6}Ax^{-6/7}y^2 - y = C.$$

5°. For $n = -\frac{15}{7}$, $m = 2$,

$$\tfrac{343}{24}Ax^{6/7}(y'_x)^2 - \big(\tfrac{49}{4}Ax^{-1/7}y + 1\big)y'_x - \tfrac{343}{36}A^2x^{-9/7}y^3 - \tfrac{7}{8}Ax^{-8/7}y^2 = C.$$

► **First integrals with $k = 3$.**

1°. For $n = 0$, $m = -\frac{1}{2}$,

$$(y'_x)^3 - 6Ay^{1/2}y'_x + 6A^2x = C,$$
$$x(y'_x)^3 - y(y'_x)^2 - 6Axy^{1/2}y'_x + \tfrac{16}{3}Ay^{3/2} + 3A^2x^2 = C.$$

2°. For $n = 1$, $m = -\frac{1}{2}$,

$$(y'_x)^3 - 6Axy^{1/2}y'_x + 4Ay^{3/2} + 2A^2x^3 = C.$$

3°. For $n = -\frac{4}{3}$, $m = -\frac{1}{2}$,

$$x(y'_x)^3 - y(y'_x)^2 - 6Ax^{-1/3}y^{1/2}y'_x - 9A^2x^{-2/3} = C.$$

4°. For $n = -\frac{5}{2}$, $m = -\frac{1}{2}$,

$$x^2(y'_x)^3 - 2xy(y'_x)^2 + (y^2 - 6Ax^{-1/2}y^{1/2})y'_x + \tfrac{2}{3}Ax^{-3/2}y^{3/2} - 3A^2x^{-2} = C,$$
$$x^3(y'_x)^3 - 3x^2y(y'_x)^2 + 3(xy^2 - 2Ax^{1/2}y^{1/2})y'_x - y^3 + 6Ax^{-1/2}y^{3/2} - 6A^2x^{-1} = C.$$

5°. For $n = -\frac{7}{6}$, $m = -\frac{1}{2}$,

$$x^2(y'_x)^3 - 2xy(y'_x)^2 + (y^2 - 6Ax^{5/6}y^{1/2})y'_x + 6Ax^{-1/6}y^{3/2} + 9A^2x^{2/3} = C.$$

6°. For $n = -\frac{7}{2}$, $m = -\frac{1}{2}$,

$$x^3(y'_x)^3 - 3x^2y(y'_x)^2 + 3(xy^2 - 2Ax^{-1/2}y^{1/2})y'_x - y^3 + 2Ax^{-3/2}y^{3/2} - 2A^2x^{-3} = C.$$

► **First integrals with $k = 4$.**

1°. For $n = 1$, $m = -\frac{5}{3}$,

$$(y'_x)^4 + 6Axy^{-2/3}(y'_x)^2 - 18Ay^{1/3}y'_x + 9A^2x^2y^{-4/3} = C,$$
$$x(y'_x)^4 - y(y'_x)^3 + 6Ax^2y^{-2/3}(y'_x)^2 - 27Axy^{1/3}y'_x + \tfrac{81}{4}Ay^{4/3} + 9A^2x^3y^{-4/3} = C.$$

2°. For $n = 2$, $m = -\frac{5}{3}$,

$$(y'_x)^4 + 6Ax^2y^{-2/3}(y'_x)^2 - 36Axy^{1/3}y'_x + 9A^2x^4y^{-4/3} = C.$$

3°. For $n = 0$, $m = -\frac{5}{3}$,

$$x(y'_x)^4 - y(y'_x)^3 + 6Axy^{-2/3}(y'_x)^2 - 9Ay^{1/3}y'_x + 9A^2xy^{-4/3} = C,$$
$$x^2(y'_x)^4 - 2xy(y'_x)^3 + (y^2 + 6Ax^2y^{-2/3})(y'_x)^2 - 18Axy^{1/3}y'_x + 12Ay^{4/3} + 9A^2x^2y^{-4/3} = C.$$

4°. For $n = -\frac{1}{2}$, $m = -\frac{5}{3}$,

$$x(y'_x)^4 - y(y'_x)^3 + 6Ax^{1/2}y^{-2/3}(y'_x)^2 + 9A^2y^{-4/3} = C.$$

5°. For $n = -\frac{4}{3}$, $m = -\frac{5}{3}$,

$$x^2(y'_x)^4 - 2xy(y'_x)^3 + (y^2 + 6Ax^{2/3}y^{-2/3})(y'_x)^2 + 6Ax^{-1/3}y^{1/3}y'_x + 9A^2x^{-2/3}y^{-4/3} = C,$$

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + 2Ax^{2/3}y^{-2/3})(y'_x)^2 - (y^3 + 3Ax^{2/3}y^{1/3})y'_x - 3Ax^{-1/3}y^{4/3} + 9A^2x^{1/3}y^{-4/3} = C.$$

6°. For $n = -\frac{7}{3}$, $m = -\frac{5}{3}$,

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + 2Ax^{-1/3}y^{-2/3})(y'_x)^2 - (y^3 - 15Ax^{-1/3}y^{1/3})y'_x - \tfrac{3}{4}Ax^{-4/3}y^{4/3} + 9A^2x^{-5/3}y^{-4/3} = C,$$

$$x^4(y'_x)^4 - 4x^3y(y'_x)^3 + 6x^2(y^2 + Ax^{-1/3}y^{-2/3})(y'_x)^2 - 2x(2y^3 - 3Ax^{-1/3}y^{1/3})y'_x + y^4 - 12Ax^{-1/3}y^{4/3} + 9A^2x^{-2/3}y^{-4/3} = C.$$

7°. For $n = -\frac{5}{6}$, $m = -\frac{5}{3}$,

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + 2Ax^{7/6}y^{-2/3})(y'_x)^2 - (y^3 + 12Ax^{7/6}y^{1/3})y'_x + 6Ax^{1/6}y^{4/3} + 9A^2x^{4/3}y^{-4/3} = C.$$

8°. For $n = -\frac{10}{3}$, $m = -\frac{5}{3}$,

$$x^4(y'_x)^4 - 4x^3y(y'_x)^3 + 6x^2(y^2 + Ax^{-4/3}y^{-2/3})(y'_x)^2 - 4x(y^3 - 6Ax^{-4/3}y^{1/3})y'_x + y^4 - 30Ax^{-4/3}y^{4/3} + 9A^2x^{-8/3}y^{-4/3} = C.$$

9°. For $n = 1$, $m = -7$,

$$x(y'_x)^4 - y(y'_x)^3 + \tfrac{2}{3}Ax^2y^{-6}(y'_x)^2 - \tfrac{1}{3}Axy^{-5}y'_x - \tfrac{1}{12}Ay^{-4} + \tfrac{1}{9}A^2x^3y^{-12} = C.$$

10°. For $n = 3$, $m = -7$,

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + \tfrac{2}{9}Ax^5y^{-6})(y'_x)^2 - y(y^2 + Ax^5y^{-6})y'_x + \tfrac{1}{4}Ax^4y^{-4} + \tfrac{1}{9}A^2x^9y^{-12} = C.$$

Remark 14.2. In the case $k = 4$ we omitted the first integrals of the form

$$\alpha F^2 + \beta F + \gamma = C,$$

where function $F = F(x, y, y'_x)$ is the left-hand side of the above integrals for $k = 2$, and α, β, and γ are some constants.

► First integrals with $k = 5$.

1°. For $n = 0$, $m = -\frac{2}{3}$,

$$(y'_x)^5 - 15Ay^{1/3}(y'_x)^3 + \tfrac{135}{2}A^2y^{2/3}y'_x - \tfrac{135}{2}A^3x = C.$$

2°. For $n = -\frac{7}{3}$, $m = -\frac{2}{3}$,

$$x^5(y'_x)^5 - 5x^4y(y'_x)^4 + 5x^3y(2y - 3Ax^{-1/3}y^{-2/3})(y'_x)^3 - 5x^2y^2(2y - 9Ax^{-1/3}y^{-2/3})(y'_x)^2 + 5x(y^4 - 9Ax^{-1/3}y^{7/3} + \tfrac{27}{2}A^2x^{-2/3}y^{2/3})y'_x + 15A(x^{-1/3}y^{10/3} - \tfrac{9}{2}Ax^{-2/3}y^{5/3} - \tfrac{9}{2}A^2x^{-1}) = C.$$

14.3.3 Some Formulas and Transformations

1°. With $m \neq 1$, the Emden–Fowler equation has a particular solution:

$$y = \lambda x^{\frac{n+2}{1-m}}, \qquad \text{where } \lambda = \left[\frac{(n+2)(n+m+1)}{A(m-1)^2}\right]^{\frac{1}{m-1}}.$$

2°. The transformation $y = w/t$, $x = 1/t$ leads to the Emden–Fowler equation with the independent variable raised to a different power:

$$w''_{tt} = At^{-n-m-3}w^m.$$

3°. Some more complicated transformations leading to the Emden–Fowler equation are outlined in Section 14.5.3 (see also Example 10.6 with Fig. 10.2 and Fig. 10.3).

4°. With $m \neq 1$ and $m \neq -2n-3$, the transformation

$$\xi = \frac{2n+m+3}{m-1} x^{\frac{n+2}{m-1}} y, \qquad u = x^{\frac{n+2}{m-1}}\Bigl(xy'_x + \frac{n+2}{m-1}y\Bigr)$$

leads to an Abel equation:

$$uu'_\xi - u = -\frac{(n+2)(n+m+1)}{(2n+m+3)^2}\xi + A\Bigl(\frac{m-1}{2n+m+3}\Bigr)^2 \xi^m,$$

whose special cases are given in Section 13.3.1.

5°. Some more complicated transformations leading to other Abel equations are outlined in Section 14.5.3.

14.4 Equations of the Form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

See Section 14.3 for the special cases $A_1 = 0$ and $A_2 = 0$.

14.4.1 Classification Table

Table 14.6 presents all solvable equations of the form

$$y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2},$$

whose solutions are outlined in Section 14.4.2. Two-parameter families (in the space of the parameters m_1, m_2, n_1, and n_2), one-parameter families, and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of m_1, the growth of m_2 (for identical m_1), the growth of n_1 (for identical m_1 and m_2), and the growth of n_2 (for identical m_1, m_2, and n_1). The number of the equation sought is indicated in the last column in this table.

TABLE 14.6
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
1	arbitrary	arbitrary	0	0	arbitrary	arbitrary	14.4.2.1
2	arbitrary	arbitrary	$-m_1-3$	$-m_2-3$	arbitrary	arbitrary	14.4.2.2
3	arbitrary	arbitrary	$-\frac{1}{2}(m_1+3)$	$-\frac{1}{2}(m_2+3)$	arbitrary	arbitrary	14.4.2.3
4	arbitrary	0	0	0	arbitrary	arbitrary	14.4.2.19
5	arbitrary	0	$-m_1-3$	-3	arbitrary	arbitrary	14.4.2.20
6	1	arbitrary	-2	-2	$-\dfrac{2(m_2+1)}{(m_2+3)^2}$	arbitrary	14.4.2.4
7	1	arbitrary	-2	$-m_2-1$	$-\dfrac{2(m_2+1)}{(m_2+3)^2}$	arbitrary	14.4.2.5
8	1	-3	arbitrary $(n_1 \neq -2)$	0	arbitrary	arbitrary	14.4.2.83
9	-7	-7	4	3	arbitrary	arbitrary	14.4.2.39
10	-5	-5	2	0	arbitrary	arbitrary	14.4.2.16
11	-3	-7	0	1	arbitrary	arbitrary	14.4.2.42
12	-3	-7	0	3	arbitrary	arbitrary	14.4.2.43
13	-3	-4	0	0	arbitrary	arbitrary	14.4.2.17
14	-3	-4	0	1	arbitrary	arbitrary	14.4.2.18
15	-2	-3	-2	0	arbitrary	arbitrary	14.4.2.88
16	-2	-3	1	0	arbitrary	arbitrary	14.4.2.87
17	-2	-2	-1	-2	arbitrary	arbitrary	14.4.2.28
18	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	14.4.2.48
19	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	14.4.2.49
20	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	14.4.2.24
21	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	14.4.2.90
22	$-\frac{5}{3}$	$-\frac{5}{3}$	0	$-\frac{2}{3}$	arbitrary	arbitrary	14.4.2.89
23	$-\frac{5}{3}$	$-\frac{5}{3}$	2	0	arbitrary	arbitrary	14.4.2.47
24	$-\frac{5}{3}$	$-\frac{5}{3}$	2	1	arbitrary	arbitrary	14.4.2.46
25	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	-2	arbitrary	arbitrary	14.4.2.81
26	$-\frac{3}{2}$	-2	0	1	arbitrary	arbitrary	14.4.2.80
27	$-\frac{7}{5}$	$-\frac{7}{5}$	$-\frac{8}{5}$	$-\frac{13}{5}$	arbitrary	arbitrary	14.4.2.25
28	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	14.4.2.102
29	$-\frac{4}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	14.4.2.101

TABLE 14.6 (*Continued*)
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
30	$-\frac{3}{5}$	$-\frac{7}{5}$	$-\frac{12}{5}$	$-\frac{13}{5}$	arbitrary	arbitrary	2.4.2.53
31	$-\frac{3}{5}$	$-\frac{7}{5}$	0	1	arbitrary	arbitrary	14.4.2.52
32	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	arbitrary	arbitrary	14.4.2.23
33	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	14.4.2.55
34	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	14.4.2.59
35	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	14.4.2.57
36	$-\frac{1}{3}$	$-\frac{5}{3}$	0	0	arbitrary	arbitrary	14.4.2.56
37	$-\frac{1}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	14.4.2.58
38	$-\frac{1}{3}$	$-\frac{5}{3}$	0	2	arbitrary	arbitrary	14.4.2.54
39	0	-2	-3	-2	arbitrary	arbitrary	14.4.2.108
40	0	-2	0	1	arbitrary	arbitrary	14.4.2.107
41	0	-1	-3	-2	arbitrary	arbitrary	14.4.2.22
42	0	-1	0	0	arbitrary	arbitrary	14.4.2.21
43	0	$-\frac{2}{3}$	-3	$-\frac{7}{3}$	arbitrary	arbitrary	14.4.2.73
44	0	$-\frac{2}{3}$	0	0	arbitrary	arbitrary	14.4.2.72
45	0	$-\frac{1}{2}$	-4	$-\frac{5}{2}$	arbitrary	arbitrary	14.4.2.96
46	0	$-\frac{1}{2}$	-3	$-\frac{7}{2}$	arbitrary	arbitrary	14.4.2.51
47	0	$-\frac{1}{2}$	-3	$-\frac{5}{2}$	arbitrary	arbitrary	14.4.2.45
48	0	$-\frac{1}{2}$	-3	-2	arbitrary	arbitrary	14.4.2.106
49	0	$-\frac{1}{2}$	-3	$-\frac{1}{2}$	arbitrary	arbitrary	14.4.2.85
50	0	$-\frac{1}{2}$	$-\frac{5}{3}$	$-\frac{7}{6}$	arbitrary	arbitrary	14.4.2.41
51	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	arbitrary	arbitrary	14.4.2.100
52	0	$-\frac{1}{2}$	$-\frac{3}{2}$	-2	arbitrary	arbitrary	14.4.2.79
53	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	arbitrary	arbitrary	14.4.2.78
54	0	$-\frac{1}{2}$	$-\frac{3}{2}$	0	arbitrary	arbitrary	14.4.2.99
55	0	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	14.4.2.40
56	0	$-\frac{1}{2}$	0	-2	arbitrary	arbitrary	14.4.2.86
57	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	arbitrary	arbitrary	14.4.2.105
58	0	$-\frac{1}{2}$	0	0	arbitrary	arbitrary	14.4.2.44
59	0	$-\frac{1}{2}$	0	1	arbitrary	arbitrary	14.4.2.50
60	0	$-\frac{1}{2}$	1	0	arbitrary	arbitrary	14.4.2.95

TABLE 14.6 (*Continued*)
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
61	$\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{10}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	14.4.2.98
62	$\frac{1}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	14.4.2.97
63	1	-7	-2	-2	$\frac{15}{4}$	arbitrary	14.4.2.35
64	1	-7	-2	6	$\frac{15}{4}$	arbitrary	14.4.2.36
65	1	-4	-2	-2	6	arbitrary	14.4.2.31
66	1	-4	-2	3	6	arbitrary	14.4.2.32
67	1	-3	-5	0	arbitrary	arbitrary	14.4.2.84
68	1	-3	1	0	arbitrary	arbitrary	14.4.2.82
69	1	$-\frac{5}{2}$	-2	-2	12	arbitrary	14.4.2.64
70	1	$-\frac{5}{2}$	-2	$\frac{3}{2}$	12	arbitrary	14.4.2.65
71	1	-2	-2	-2	2	arbitrary	14.4.2.6
72	1	-2	-2	1	2	arbitrary	14.4.2.7
73	1	$-\frac{5}{3}$	-2	-2	$-\frac{3}{16}$	arbitrary	14.4.2.26
74	1	$-\frac{5}{3}$	-2	-2	$-\frac{9}{100}$	arbitrary	14.4.2.10
75	1	$-\frac{5}{3}$	-2	-2	$\frac{3}{4}$	arbitrary	14.4.2.12
76	1	$-\frac{5}{3}$	-2	-2	$\frac{63}{4}$	arbitrary	14.4.2.66
77	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$-\frac{3}{16}$	arbitrary	14.4.2.27
78	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$-\frac{9}{100}$	arbitrary	14.4.2.11
79	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$\frac{3}{4}$	arbitrary	14.4.2.13
80	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$\frac{63}{4}$	arbitrary	14.4.2.67
81	1	$-\frac{7}{5}$	-2	-2	$-\frac{5}{36}$	arbitrary	14.4.2.29
82	1	$-\frac{7}{5}$	-2	$\frac{2}{5}$	$-\frac{5}{36}$	arbitrary	14.4.2.30
83	1	$-\frac{1}{2}$	-2	-2	$-\frac{2}{9}$	arbitrary	14.4.2.14
84	1	$-\frac{1}{2}$	-2	-2	$-\frac{4}{25}$	arbitrary	14.4.2.8
85	1	$-\frac{1}{2}$	-2	-2	20	arbitrary	14.4.2.33
86	1	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	$-\frac{2}{9}$	arbitrary	14.4.2.15
87	1	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	$-\frac{4}{25}$	arbitrary	14.4.2.9
88	1	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	20	arbitrary	14.4.2.34
89	1	0	-5	-3	arbitrary	arbitrary	14.4.2.77
90	1	0	1	0	arbitrary	arbitrary	14.4.2.76
91	1	$\frac{1}{2}$	-2	-2	$-\frac{12}{49}$	arbitrary	14.4.2.37

TABLE 14.6 (*Continued*)
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
92	1	$\frac{1}{2}$	-2	$-\frac{3}{2}$	$-\frac{12}{49}$	arbitrary	14.4.2.38
93	2	0	-5	-4	arbitrary	arbitrary	14.4.2.92
94	2	0	-5	-3	arbitrary	arbitrary	14.4.2.69
95	2	0	$-\frac{20}{7}$	$-\frac{13}{7}$	arbitrary	arbitrary	14.4.2.94
96	2	0	$-\frac{20}{7}$	$-\frac{12}{7}$	arbitrary	arbitrary	14.4.2.71
97	2	0	$-\frac{15}{7}$	$-\frac{9}{7}$	arbitrary	arbitrary	14.4.2.70
98	2	0	$-\frac{15}{7}$	$-\frac{8}{7}$	arbitrary	arbitrary	14.4.2.93
99	2	0	0	0	arbitrary	arbitrary	14.4.2.68
100	2	0	0	1	arbitrary	arbitrary	14.4.2.91
101	2	1	-3	-2	arbitrary	$-\frac{6}{25}$	14.4.2.61
102	2	1	-3	-2	arbitrary	$\frac{6}{25}$	14.4.2.63
103	2	1	-2	-2	arbitrary	$-\frac{6}{25}$	14.4.2.60
104	2	1	-2	-2	arbitrary	$\frac{6}{25}$	14.4.2.62
105	3	1	-6	-5	arbitrary	arbitrary	14.4.2.104
106	3	1	0	1	arbitrary	arbitrary	14.4.2.103
107	3	2	$-\frac{18}{5}$	$-\frac{14}{5}$	arbitrary	arbitrary	14.4.2.74
108	3	2	$-\frac{12}{5}$	$-\frac{11}{5}$	arbitrary	arbitrary	14.4.2.75

14.4.2 Exact Solutions

1. $y''_{xx} = A_1 y^{m_1} + A_2 y^{m_2}, \qquad m_1 \neq -1, \ m_2 \neq -1.$

1°. Solution in parametric form:

$$x = a \int (C_1 + \tau^{m_1+1} \pm \tau^{m_2+1})^{-1/2}\, d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = \frac{1}{2} a^{-2} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{-2} b^{1-m_2}(m_2+1)$.

2°. Solution in parametric form:

$$x = a \int (C_1 - \tau^{m_1+1} \pm \tau^{m_2+1})^{-1/2}\, d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = -\frac{1}{2} a^{-2} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{-2} b^{1-m_2}(m_2+1)$.

2. $y''_{xx} = A_1 x^{-m_1-3} y^{m_1} + A_2 x^{-m_2-3} y^{m_2}, \qquad m_1 \neq -1, \ m_2 \neq -1.$

1°. Solution in parametric form:

$$x = a\left(\int \frac{d\tau}{\sqrt{C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}}} + C_2\right)^{-1}, \quad y = b\tau\left(\int \frac{d\tau}{\sqrt{C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}}} + C_2\right)^{-1},$$

where $A_1 = \frac{1}{2} a^{1+m_1} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{1+m_2} b^{1-m_2}(m_2+1)$.

2°. Solution in parametric form:

$$x = a\left(\int \frac{d\tau}{\sqrt{C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}}} + C_2\right)^{-1}, \quad y = b\tau\left(\int \frac{d\tau}{\sqrt{C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}}} + C_2\right)^{-1},$$

where $A_1 = -\frac{1}{2}a^{1+m_1}b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2}a^{1+m_2}b^{1-m_2}(m_2+1)$.

3. $y''_{xx} = A_1 x^{-\frac{m_1+3}{2}} y^{m_1} + A_2 x^{-\frac{m_2+3}{2}} y^{m_2}$.

1°. Solution in parametric form with $m_1 \neq -1$ and $m_2 \neq -1$:

$$x = C_1^2 \exp\left[\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + \frac{2A_2}{m_2+1}\tau^{m_2+1}\right)^{-1/2} d\tau\right],$$
$$y = C_1\tau \exp\left[\frac{1}{2}\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + \frac{2A_2}{m_2+1}\tau^{m_2+1}\right)^{-1/2} d\tau\right].$$

2°. Solution in parametric form with $m_1 \neq -1$ and $m_2 = -1$:

$$x = C_1^2 \exp\left[\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + 2A_2 \ln|\tau|\right)^{-1/2} d\tau\right],$$
$$y = C_1\tau \exp\left[\frac{1}{2}\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + 2A_2 \ln|\tau|\right)^{-1/2} d\tau\right].$$

4. $y''_{xx} = -\frac{2(m+1)}{(m+3)^2} x^{-2} y + A x^{-2} y^m, \qquad m \neq -3, \ m \neq -1.$

Solution in parametric form:

$$x = C_1\left(\int \frac{d\tau}{\sqrt{1 \pm \tau^{m+1}}} + C_2\right)^{\frac{m+3}{m-1}}, \quad y = b\tau\left(\int \frac{d\tau}{\sqrt{1 \pm \tau^{m+1}}} + C_2\right)^{\frac{2}{m-1}},$$

where $A = \pm\frac{(m+1)(m-1)^2}{2(m+3)^2} b^{1-m}$.

5. $y''_{xx} = -\frac{2(m+1)}{(m+3)^2} x^{-2} y + A x^{-m-1} y^m, \qquad m \neq -3, \ m \neq -1.$

Solution in parametric form:

$$x = C_1\left(\int \frac{d\tau}{\sqrt{1 \pm \tau^{m+1}}} + C_2\right)^{-\frac{m+3}{m-1}}, \quad y = bC_1\tau\left(\int \frac{d\tau}{\sqrt{1 \pm \tau^{m+1}}} + C_2\right)^{-\frac{m+1}{m-1}},$$

where $A = \pm\frac{(m+1)(m-1)^2}{2(m+3)^2} b^{1-m}$.

6. $y''_{xx} = 2x^{-2} y + A x^{-2} y^{-2}$.

Solution in parametric form:

$$x = C_1\left[\sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2\right]^{-1/3},$$
$$y = b\tau\left[\sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2\right]^{-2/3},$$

where $A = -\frac{9}{2}b^3$.

7. $y''_{xx} = 2x^{-2}y + Axy^{-2}.$

Solution in parametric form:

$$x = C_1\big[\sqrt{\tau(\tau+1)} - \ln\big(\sqrt{\tau} + \sqrt{\tau+1}\,\big) + C_2\big]^{1/3},$$
$$y = bC_1\tau\big[\sqrt{\tau(\tau+1)} - \ln\big(\sqrt{\tau} + \sqrt{\tau+1}\,\big) + C_2\big]^{-1/3},$$

where $A = -\frac{9}{2}b^3$.

8. $y''_{xx} = -\frac{4}{25}x^{-2}y + Ax^{-2}y^{-1/2}.$

Solution in parametric form:

$$x = C_1(\tau^3 - 3\tau + C_2)^{-5/3}, \qquad y = b(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{-4/3},$$

where $A = \pm\frac{4}{25}b^{3/2}$.

9. $y''_{xx} = -\frac{4}{25}x^{-2}y + Ax^{-1/2}y^{-1/2}.$

Solution in parametric form:

$$x = C_1(\tau^3 - 3\tau + C_2)^{5/3}, \qquad y = bC_1(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{1/3},$$

where $A = \pm\frac{4}{25}b^{3/2}$.

10. $y''_{xx} = -\frac{9}{100}x^{-2}y + Ax^{-2}y^{-5/3}.$

Solution in parametric form:

$$x = C_1\big[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\big]^{-5/4}, \quad y = b(\tau^3 - 3\tau + C_2)^{3/2}\big[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\big]^{-9/8},$$

where $A = \pm\frac{9}{100}b^{8/3}$.

11. $y''_{xx} = -\frac{9}{100}x^{-2}y + Ax^{2/3}y^{-5/3}.$

Solution in parametric form:

$$x = C_1[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{5/4}, \quad y = bC_1(\tau^3 - 3\tau + C_2)^{3/2}[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{1/8},$$

where $A = \pm\frac{9}{100}b^{8/3}$.

12. $y''_{xx} = \frac{3}{4}x^{-2}y + Ax^{-2}y^{-5/3}.$

Solution in parametric form:

$$x = C_1(\tau^3 \pm 3\tau + C_2)^{-1/2}, \quad y = b(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-3/4}, \quad \text{where } A = \pm\tfrac{4}{3}b^{8/3}.$$

13. $y''_{xx} = \frac{3}{4}x^{-2}y + Ax^{2/3}y^{-5/3}.$

Solution in parametric form:

$$x = C_1(\tau^3 \pm 3\tau + C_2)^{1/2}, \quad y = bC_1(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-1/4}, \quad \text{where } A = \pm\tfrac{4}{3}b^{8/3}.$$

14. $y''_{xx} = -\frac{2}{9}x^{-2}y + Ax^{-2}y^{-1/2}.$

Solution in parametric form:

$$x = C_1(C_1e^{2k\tau} + C_2e^{-k\tau}\sin\omega)^{-3}, \quad \omega = \sqrt{3}\,k\tau,$$
$$y = bk^2(C_1e^{2k\tau} + C_2e^{-k\tau}\sin\omega)^{-2}\big[2C_1e^{2k\tau} + C_2e^{-k\tau}\big(\sqrt{3}\cos\omega - \sin\omega\big)\big]^2,$$

where $A = \frac{16}{9}bk^3$.

15. $y''_{xx} = -\frac{2}{9}x^{-2}y + Ax^{-1/2}y^{-1/2}$**.**

Solution in parametric form:

$$x = C_1\left(C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin\omega\right)^3, \quad \omega = \sqrt{3}\,k\tau,$$
$$y = bk^2 C_1\left(C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin\omega\right)\left[2C_1 e^{2k\tau} + C_2 e^{-k\tau}\left(\sqrt{3}\cos\omega - \sin\omega\right)\right]^2,$$

where $A = \frac{16}{9}bk^3$.

16. $y''_{xx} = A_1 x^2 y^{-5} + A_2 y^{-5}$**.**

Solution in parametric form:

$$x = \left(\frac{A_2}{A_1}\right)^{1/2}\tan\left[\int\left(C_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2\right)^{-1/2} d\tau + C_2\right],$$
$$y = A_2^{1/2}\tau\left\{\cos\left[\int\left(C_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2\right)^{-1/2} d\tau + C_2\right]\right\}^{-1}.$$

17. $y''_{xx} = A_1 y^{-3} + A_2 y^{-4}$**.**

1°. Solution in parametric form:

$$x = a\left[\int\left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = \mp a^{-2}b^4$, $A_2 = -\frac{3}{2}a^{-2}b^5$.

2°. Solution in parametric form:

$$x = a\left[\int\left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = \mp a^{-2}b^4$, $A_2 = \frac{3}{2}a^{-2}b^5$.

18. $y''_{xx} = A_1 y^{-3} + A_2 x y^{-4}$**.**

1°. Solution in parametric form:

$$x = a\left[\int\left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1}, \qquad y = b\tau\left[\int\left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = \mp a^{-2}b^4$, $A_2 = -\frac{3}{2}a^{-3}b^5$.

2°. Solution in parametric form:

$$x = a\left[\int\left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1}, \qquad y = b\tau\left[\int\left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = \mp a^{-2}b^4$, $A_2 = \frac{3}{2}a^{-3}b^5$.

19. $y''_{xx} = A_1 y^m + A_2, \qquad m \neq -1$**.**

1°. Solution in parametric form:

$$x = a\left[\int\left(C_1 + \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = \frac{1}{2}a^{-2}b^{1-m}(m+1)$, $A_2 = \pm\frac{1}{2}a^{-2}b$.

2°. Solution in parametric form:

$$x = a\left[\int\left(C_1 - \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = -\frac{1}{2}a^{-2}b^{1-m}(m+1)$, $A_2 = \pm\frac{1}{2}a^{-2}b$.

3°. For the case $m = -1$, see equation 14.4.2.21.

20. $y''_{xx} = A_1 x^{-m-3} y^m + A_2 x^{-3}, \qquad m \neq -1.$

1°. Solution in parametric form:

$$x = a\Big[\int (C_1 + \tau^{m+1} \pm \tau)^{-1/2}\, d\tau + C_2\Big]^{-1}, \quad y = b\tau\Big[\int (C_1 + \tau^{m+1} \pm \tau)^{-1/2}\, d\tau + C_2\Big]^{-1},$$

where $A_1 = \frac{1}{2} a^{1+m} b^{1-m}(m+1)$, $A_2 = \pm\frac{1}{2}ab$.

2°. Solution in parametric form:

$$x = a\Big[\int (C_1 - \tau^{m+1} \pm \tau)^{-1/2}\, d\tau + C_2\Big]^{-1}, \quad y = b\tau\Big[\int (C_1 - \tau^{m+1} \pm \tau)^{-1/2}\, d\tau + C_2\Big]^{-1},$$

where $A_1 = -\frac{1}{2} a^{1+m} b^{1-m}(m+1)$, $A_2 = \pm\frac{1}{2}ab$.

21. $y''_{xx} = A_1 + A_2 y^{-1}.$

Solution: $x = \displaystyle\int (C_1 + 2A_1 y + 2A_2 \ln|y|)^{-1/2} dy + C_2.$

22. $y''_{xx} = A_1 x^{-3} + A_2 x^{-2} y^{-1}.$

Solution in parametric form:

$$x = \Big[\int (C_1 + 2A_1\tau + 2A_2 \ln|\tau|)^{-1/2}\, d\tau + C_2\Big]^{-1},$$
$$y = \tau\Big[\int (C_1 + 2A_1\tau + 2A_2 \ln|\tau|)^{-1/2}\, d\tau + C_2\Big]^{-1}.$$

23. $y''_{xx} = A_1 x^{-5/2} y^{-1/2} + A_2 x^{-7/2} y^{-1/2}.$

Solution in parametric form:

$$x = \frac{1}{F}, \quad y = \frac{k^2}{F}\big\{2C_1 e^{2k\tau} + C_2 e^{-k\tau}\big[\sqrt{3}\cos(\omega\tau) - \sin(\omega\tau)\big]\big\}^2,$$

where $F = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\omega\tau) - A_1/A_2$, $A_2 = 16k^3$, $\omega = k\sqrt{3}$.

24. $y''_{xx} = A_1 x^{-4/3} y^{-5/3} + A_2 x^{-7/3} y^{-5/3}.$

Solution in parametric form:

$$x = (\tfrac{1}{36}A_2\tau^4 + C_1\tau^3 + C_2\tau + C_3)^{-1},$$
$$y = (\tfrac{1}{9}A_2\tau^3 + 3C_1\tau^2 + C_2)^{3/2}(\tfrac{1}{36}A_2\tau^4 + C_1\tau^3 + C_2\tau + C_3)^{-1},$$

where the constants C_1, C_2, and C_3 are related by the constraint $9C_1C_2 = A_1 + A_2C_3$.

25. $y''_{xx} = A_1 x^{-8/5} y^{-7/5} + A_2 x^{-13/5} y^{-7/5}.$

Solution in parametric form:

$$x = \Big(aC_1^4 F - \frac{A_1}{A_2}\Big)^{-1}, \quad y = bC_1^5 S^{5/2}\Big(aC_1^4 F - \frac{A_1}{A_2}\Big)^{-1},$$

where $S = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\sqrt{3}\,k\tau)$, $F = (S'_\tau)^2 - 2SS''_{\tau\tau}$, $A_2 = -\dfrac{5}{1024}\dfrac{b^{12/5}}{a^3k^6}$.

26. $y''_{xx} = -\frac{3}{16}x^{-2}y + Ax^{-2}y^{-5/3}$.

1°. Solution in parametric form with $A<0$:

$$x = C_1[\cosh(\tau+C_2)\cos\tau]^{-2}[\tanh(\tau+C_2)+\tan\tau]^{-2}, \quad y = b[\tanh(\tau+C_2)+\tan\tau]^{-3/2},$$

where $A = -\frac{3}{64}b^{8/3}$.

2°. Solution in parametric form with $A>0$:

$$x = C_1[\sinh\tau+\cos(\tau+C_2)]^{-2}, \quad y = b[\cosh\tau-\sin(\tau+C_2)]^{3/2}[\sinh\tau+\cos(\tau+C_2)]^{-3/2},$$

where $A = \frac{3}{16}b^{8/3}$.

27. $y''_{xx} = -\frac{3}{16}x^{-2}y + Ax^{2/3}y^{-5/3}$.

1°. Solution in parametric form with $A<0$:

$$x = C_1[\cosh(\tau+C_2)\cos\tau]^2[\tanh(\tau+C_2)+\tan\tau]^2,$$
$$y = bC_1[\cosh(\tau+C_2)\cos\tau]^2[\tanh(\tau+C_2)+\tan\tau]^{1/2},$$

where $A = -\frac{3}{64}b^{8/3}$.

2°. Solution in parametric form with $A>0$:

$$x = C_1[\sinh\tau+\cos(\tau+C_2)]^2, \quad y = bC_1[\cosh\tau-\sin(\tau+C_2)]^{3/2}[\sinh\tau+\cos(\tau+C_2)]^{1/2},$$

where $A = \frac{3}{16}b^{8/3}$.

28. $y''_{xx} = A_1x^{-1}y^{-2} + A_2x^{-2}y^{-2}$.

Solution in parametric form:

$$x = \Bigl\{aC_1\tau^{-2/3}\bigl[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2Z^2\bigr] - \frac{A_1}{A_2}\Bigr\}^{-1},$$
$$y = bC_1\tau^{2/3}Z^2\Bigl\{aC_1\tau^{-2/3}\bigl[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2Z^2\bigr] - \frac{A_1}{A_2}\Bigr\}^{-1},$$

where

$$Z = \begin{cases} C_1J_{1/3}(\tau) + C_2Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1I_{1/3}(\tau) + C_2K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions; $A_2 = -\frac{9}{2}a^{-3}b^3$.

◆ *In the solutions of equations 29 and 30, the following notation is used:*

$$S_1 = C_1e^{2k\tau} + C_2e^{-k\tau}\sin(\sqrt{3}\,k\tau),$$
$$S_2 = 2kC_1e^{2k\tau} + kC_2e^{-k\tau}\bigl[\sqrt{3}\cos(\sqrt{3}\,k\tau) - \sin(\sqrt{3}\,k\tau)\bigr],$$
$$S_3 = S_2^2 - 2S_1(S_2)'_\tau.$$

29. $y''_{xx} = -\frac{5}{36}x^{-2}y + Ax^{-2}y^{-7/5}$.

Solution in parametric form:

$$x = C_1S_3^{-3/2}, \quad y = bS_1^{5/2}S_3^{-5/4}, \qquad \text{where} \quad A = -\tfrac{5}{2304}b^{12/5}k^{-6}.$$

30. $y''_{xx} = -\frac{5}{36}x^{-2}y + Ax^{2/5}y^{-7/5}.$

Solution in parametric form:

$$x = C_1 S_3^{3/2}, \quad y = bC_1 S_1^{5/2} S_3^{1/4}, \qquad \text{where} \quad A = -\tfrac{5}{2304} b^{12/5} k^{-6}.$$

◆ *In the solutions of equations 31–39, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad I = \int \tau R^{-1}\, d\tau, \quad F_1 = 2\tau I + C_2\tau \mp R, \quad F_2 = \tau^{-1}(RF_1 - 1),$$

where $I = I(\tau)$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

31. $y''_{xx} = 6x^{-2}y + Ax^{-2}y^{-4}.$

Solution in parametric form:

$$x = C_1 \tau^{-1/5} F_1^{1/5}, \quad y = b\tau^{-3/5} F_1^{-2/5}, \qquad \text{where} \quad A = \mp 150 b^5.$$

32. $y''_{xx} = 6x^{-2}y + Ax^3y^{-4}.$

Solution in parametric form:

$$x = C_1 \tau^{1/5} F_1^{-1/5}, \quad y = bC_1 \tau^{-2/5} F_1^{-3/5}, \qquad \text{where} \quad A = \mp 150 b^5.$$

33. $y''_{xx} = 20x^{-2}y + Ax^{-2}y^{-1/2}.$

Solution in parametric form:

$$x = C_1 F_1^{1/3}, \quad y = bF_1^{-4/3} F_2^2, \qquad \text{where} \quad A = \pm 108 b^{3/2}.$$

34. $y''_{xx} = 20x^{-2}y + Ax^{-1/2}y^{-1/2}.$

Solution in parametric form:

$$x = C_1 F_1^{-1/3}, \quad y = bC_1 F_1^{-5/3} F_2^2, \qquad \text{where} \quad A = \pm 108 b^{3/2}.$$

35. $y''_{xx} = \frac{15}{4}x^{-2}y + Ax^{-2}y^{-7}.$

Solution in parametric form:

$$x = C_1(4\tau F_1^2 \mp F_2^2)^{1/4}, \quad y = bF_1^{1/2}(4\tau F_1^2 \mp F_2^2)^{-3/8}, \qquad \text{where} \quad A = \pm\tfrac{3}{4} b^8.$$

36. $y''_{xx} = \frac{15}{4}x^{-2}y + Ax^6y^{-7}.$

Solution in parametric form:

$$x = C_1(4\tau F_1^2 \mp F_2^2)^{-1/4}, \quad y = bC_1 F_1^{1/2}(4\tau F_1^2 \mp F_2^2)^{-5/8}, \qquad \text{where} \quad A = \pm\tfrac{3}{4} b^8.$$

37. $y''_{xx} = -\frac{12}{49}x^{-2}y + Ax^{-2}y^{1/2}.$

Solution in parametric form:

$$x = C_1(I + C_2)^{-7}, \quad y = b\tau^2(I + C_2)^{-4}, \qquad \text{where} \quad A = \pm\tfrac{12}{49} b^{1/2}.$$

38. $y''_{xx} = -\frac{12}{49}x^{-2}y + Ax^{-3/2}y^{1/2}.$

Solution in parametric form:

$$x = C_1(I + C_2)^7, \quad y = bC_1\tau^2(I + C_2)^3, \qquad \text{where} \quad A = \pm\tfrac{12}{49} b^{1/2}.$$

39. $y''_{xx} = A_1x^4y^{-7} + A_2x^3y^{-7}.$

Solution in parametric form:

$$x = \left[aC_1^8(4\tau F_1^2 \mp F_2^2) - \frac{A_1}{A_2}\right]^{-1}, \quad y = bC_1^3 F_1^{1/2}\left[aC_1^8(4\tau F_1^2 \mp F_2^2) - \frac{A_1}{A_2}\right]^{-1},$$

where $A_2 = \pm\frac{3}{64} a^{-3} b^8.$

◆ *In the solutions of equations 40–43, the following notation is used:*

$$R_1 = \sqrt{C_1 + \tau^{-3} \pm \tau^{-2}}, \quad E_1 = \int \frac{d\tau}{R_1} + C_2, \quad F_1 = \tau - R_1 E_1, \quad H_1 = 3\tau^3 F_1^2 + 3(1 \pm \tau) E_1^2,$$
$$R_2 = \sqrt{C_1 - \tau^{-3} \pm \tau^{-2}}, \quad E_2 = \int \frac{d\tau}{R_2} + C_2, \quad F_2 = \tau - R_2 E_2, \quad H_2 = 3\tau^3 F_2^2 + 3(-1 \pm \tau) E_2^2.$$

40. $y''_{xx} = A_1 x^{-4/3} + A_2 x^{-4/3} y^{-1/2}$.

Solutions in parametric form:

$$x = a\tau^{-3} E_k^3, \qquad y = b F_k^2,$$

where $A_1 = \mp\frac{2}{9} a^{-2/3} b$, $A_2 = \frac{1}{3} a^{-2/3} b^{3/2} (-1)^k$; $k = 1$ and $k = 2$.

41. $y''_{xx} = A_1 x^{-5/3} + A_2 x^{-7/6} y^{-1/2}$.

Solutions in parametric form:

$$x = a\tau^3 E_k^{-3}, \qquad y = b\tau^3 E_k^{-3} F_k^2,$$

where $A_1 = \pm\frac{2}{9} a^{-1/3} b$, $A_2 = \frac{1}{3} a^{-5/6} b^{3/2} (-1)^{k+1}$; $k = 1$ and $k = 2$.

42. $y''_{xx} = A_1 y^{-3} + A_2 x y^{-7}$.

Solutions in parametric form:

$$x = a\tau^{-3} H_k, \qquad y = b\tau^{-1/2} E_k^{1/2},$$

where $A_1 = \mp\frac{1}{36} a^{-2} b^4$, $A_2 = -\frac{1}{36} a^{-3} b^8$; $k = 1$ and $k = 2$.

43. $y''_{xx} = A_1 y^{-3} + A_2 x^3 y^{-7}$.

Solutions in parametric form:

$$x = a\tau^3 H_k^{-1}, \qquad y = b\tau^{5/2} E_k^{1/2} H_k^{-1},$$

where $A_1 = \pm\frac{1}{36} a^{-2} b^4$, $A_2 = -\frac{1}{36} a^{-5} b^8$; $k = 1$ and $k = 2$.

◆ *In the solutions of equations 44 and 45, the following notation is used:*

$$f_1 = \begin{cases} C_1 e^{k\tau} + C_2 e^{-k\tau} - \dfrac{A_2}{A_1}\tau & \text{if } A_1 > 0, \\ C_1 \sin(k\tau) + C_2 \cos(k\tau) - \dfrac{A_2}{A_1}\tau & \text{if } A_1 < 0, \end{cases}$$
$$f_2 = \begin{cases} k(C_1 e^{k\tau} - C_2 e^{-k\tau}) - \dfrac{A_2}{A_1} & \text{if } A_1 > 0, \\ k[C_1 \cos(k\tau) - C_2 \sin(k\tau)] - \dfrac{A_2}{A_1} & \text{if } A_1 < 0, \end{cases}$$

where $k = \sqrt{\frac{1}{2}|A_1|}$.

44. $y''_{xx} = A_1 + A_2 y^{-1/2}$.

Solution in parametric form:

$$x = f_1, \qquad y = f_2^2.$$

45. $y''_{xx} = A_1 x^{-3} + A_2 x^{-5/2} y^{-1/2}$.

Solution in parametric form:

$$x = f_1^{-1}, \quad y = f_1^{-1} f_2^2.$$

◆ *In the solutions of equations 46 and 47, the following notation is used:*

For $A_1 > 0$,

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin(k\tau), \quad k = (\tfrac{4}{3}A_1)^{1/4},$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + kC_3 \cos(k\tau).$$

For $A_1 < 0$,

$$T_1 = e^{s\tau}[C_1 \sin(s\tau) + C_2 \cos(s\tau)] + C_3 e^{-s\tau} \sin(s\tau), \quad s = (-\tfrac{1}{3}A_1)^{1/4},$$
$$T_2 = se^{s\tau}[(C_1 - C_2)\sin(s\tau) + (C_1 + C_2)\cos(s\tau)] - sC_3 e^{-s\tau}[\sin(s\tau) - \cos(s\tau)].$$

46. $y''_{xx} = A_1 x^2 y^{-5/3} + A_2 x y^{-5/3}$.

Solution in parametric form:

$$x = T_1 - \frac{A_2}{2A_1}, \quad y = T_2^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$4C_1C_2 + C_3^2 = \tfrac{1}{4}A_1^{-2}A_2^2 \quad \text{if } A_1 > 0,$$
$$C_1C_3 = \tfrac{1}{16}A_1^{-2}A_2^2 \quad \text{if } A_1 < 0.$$

47. $y''_{xx} = A_1 x^2 y^{-5/3} + A_2 y^{-5/3}$.

Solution in parametric form:

$$x = T_1, \quad y = T_2^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$4C_1C_2 + C_3^2 = -\tfrac{1}{2}A_1^{-1}A_2 \quad \text{if } A_1 > 0,$$
$$C_1C_3 = -\tfrac{1}{4}A_1^{-1}A_2 \quad \text{if } A_1 < 0.$$

◆ *In the solutions of equations 48 and 49, the following notation is used:*

For $A_2 > 0$,

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin(k\tau), \quad k = (\tfrac{4}{3}A_2)^{1/4},$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + kC_3 \cos(k\tau).$$

For $A_2 < 0$,

$$T_1 = e^{s\tau}[C_1 \sin(s\tau) + C_2 \cos(s\tau)] + C_3 e^{-s\tau} \sin(s\tau), \quad s = (-\tfrac{1}{3}A_2)^{1/4},$$
$$T_2 = se^{s\tau}[(C_1 - C_2)\sin(s\tau) + (C_1 + C_2)\cos(s\tau)] - sC_3 e^{-s\tau}[\sin(s\tau) - \cos(s\tau)].$$

48. $y''_{xx} = A_1 x^{-7/3} y^{-5/3} + A_2 x^{-10/3} y^{-5/3}$.

Solution in parametric form:

$$x = \left(T_1 - \frac{A_1}{2A_2}\right)^{-1}, \quad y = T_2^{3/2}\left(T_1 - \frac{A_1}{2A_2}\right)^{-1},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &4C_1C_2 + C_3^2 = \tfrac{1}{4}A_1^2A_2^{-2} && \text{if } A_2 > 0, \\ &C_1C_3 = \tfrac{1}{16}A_1^2A_2^{-2} && \text{if } A_2 < 0. \end{aligned}$$

49. $y''_{xx} = A_1x^{-4/3}y^{-5/3} + A_2x^{-10/3}y^{-5/3}$.

Solution in parametric form:

$$x = T_1^{-1}, \qquad y = T_1^{-1}T_2^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &4C_1C_2 + C_3^2 = -\tfrac{1}{2}A_1A_2^{-1} && \text{if } A_2 > 0, \\ &C_1C_3 = -\tfrac{1}{4}A_1A_2^{-1} && \text{if } A_2 < 0. \end{aligned}$$

◆ *In the solutions of equations 50–53, the following notation is used:*

$$\begin{aligned} R_1 &= C_1\tau^{k_1} + C_2\tau^{k_2} + C_3\tau^{k_3}, \\ R_2 &= (C_1 + C_2\tau)e^{k\tau} + C_3e^{\omega\tau}, \\ R_3 &= C_1e^{k\tau} + e^{s\tau}(C_2\sin\omega\tau + C_3\cos\omega\tau), \\ Q_1 &= C_1k_1\tau^{k_1} + C_2k_2\tau^{k_2} + C_3k_3\tau^{k_3}, \\ Q_2 &= (kC_1 + C_2 + kC_2\tau)e^{k\tau} + \omega C_3e^{\omega\tau}, \\ Q_3 &= kC_1e^{k\tau} + e^{s\tau}[(sC_2 - \omega C_3)\sin\omega\tau + (sC_3 + \omega C_2)\cos\omega\tau], \\ S_1 &= \tau(Q_1)'_\tau, \quad S_2 = (Q_2)'_\tau, \quad S_3 = (Q_3)'_\tau, \end{aligned}$$

where k_1, k_2, and k_3 (real numbers) or k and $s \pm i\omega$ (one real and two complex numbers) are roots of the cubic equation $\lambda^3 - \frac{1}{2}B_2\lambda - \frac{1}{2}B_1 = 0$. The subscripts of the functions R_m, Q_m, and S_m ($m = 1,\ 2,\ 3$) are selected depending on the sign of the expression $\Delta = 2B_2^3 - 27B_1^2$:

$$\begin{aligned} &\Delta > 0 \quad \textit{subscript} \quad m = 1, \\ &\Delta = 0 \quad \textit{subscript} \quad m = 2, \\ &\Delta < 0 \quad \textit{subscript} \quad m = 3. \end{aligned}$$

If $2B_2^3 = 27B_1^2$ (subscript $m = 2$), then

$$\begin{aligned} &k = (\tfrac{1}{6}B_2)^{1/2}, && \omega = -2(\tfrac{1}{6}B_2)^{1/2} && \textit{if} \quad B_1 < 0, \\ &k = -(\tfrac{1}{6}B_2)^{1/2}, && \omega = 2(\tfrac{1}{6}B_2)^{1/2} && \textit{if} \quad B_1 > 0. \end{aligned}$$

Remark 14.3. *The expressions for R_m, Q_m contain three constants C_1, C_2, and C_3. One of them can be arbitrarily fixed to set it equal to any nonzero number (for example, we can set $C_3 = \pm 1$), and the other constants can be arbitrary.*

50. $y''_{xx} = A_1 + A_2xy^{-1/2}$.

Solution in parametric form:

$$x = R_m, \quad y = Q_m^2, \qquad \text{where} \quad A_1 = B_2, \quad A_2 = B_1.$$

51. $y''_{xx} = A_1 x^{-3} + A_2 x^{-7/2} y^{-1/2}$.

Solution in parametric form:

$$x = R_m^{-1}, \quad y = R_m^{-1} Q_m^2, \qquad \text{where} \quad A_1 = B_2, \quad A_2 = B_1.$$

52. $y''_{xx} = A_1 y^{-3/5} + A_2 x y^{-7/5}$.

Solution in parametric form:

$$x = a(2Q_m^2 - 4R_m S_m + B_2 R_m^2), \quad y = b R_m^{5/2},$$

where $A_1 = -ab^{-4/5} A_2 B_2$, $A_2 = -\frac{5}{32} a^{-3} b^{12/5} B_1^{-2}$.

53. $y''_{xx} = A_1 x^{-12/5} y^{-3/5} + A_2 x^{-13/5} y^{-7/5}$.

Solution in parametric form:

$$x = a(2Q_m^2 - 4R_m S_m + B_2 R_m^2)^{-1}, \quad y = b R_m^{5/2} (2Q_m^2 - 4R_m S_m + B_2 R_m^2)^{-1},$$

where $A_1 = \frac{5}{32} a^{2/5} b^{8/5} B_1^{-2} B_2$, $A_2 = -\frac{5}{32} a^{3/5} b^{12/5} B_1^{-2}$.

◆ *In the solutions of equations 54 and 55, the following notation is used:*

1°. *For* $A_2 > 0$, $A_1 \neq 0$:

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin \omega\tau, \quad T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + \omega C_3 \cos \omega\tau,$$

where $k = \{\frac{2}{3}[(A_1^2 + 3A_2)^{1/2} + A_1]\}^{1/2}$, $\omega = \{\frac{2}{3}[(A_1^2 + 3A_2)^{1/2} - A_1]\}^{1/2}$; *the constants* C_1, C_2, *and* C_3 *are related by the constraint* $4k^2 C_1 C_2 + \omega^2 C_3^2 = 0$.

2°. *For* $-A_1^2 < 3A_2 < 0$, $A_1 > 0$:

$$T_1 = C_1 \tau^{k_1} + C_2 \tau^{-k_1} + C_3 \tau^{k_2} + C_4 \tau^{-k_2}, \quad T_2 = k_1 (C_1 \tau^{k_1} - C_2 \tau^{-k_1}) + k_2 (C_3 \tau^{k_2} - C_4 \tau^{-k_2}),$$

where $k_1 = \{\frac{2}{3}[A_1 + (A_1^2 + 3A_2)^{1/2}]\}^{1/2}$, $k_2 = \{\frac{2}{3}[A_1 - (A_1^2 + 3A_2)^{1/2}]\}^{1/2}$; *the constants* C_1, C_2, *and* C_3 *are related by the constraint*

$$(C_1 C_2 + C_3 C_4)(A_1^2 + 3A_2)^{1/2} + (C_1 C_2 - C_3 C_4) A_1 = 0.$$

3°. *For* $-A_1^2 < 3A_2 < 0$, $A_1 < 0$:

$$T_1 = C_1 \sin \omega_1 \tau + C_2 \cos \omega_1 \tau + C_3 \sin \omega_2 \tau,$$
$$T_2 = \omega_1 (C_1 \cos \omega_1 \tau - C_2 \sin \omega_1 \tau) + \omega_2 C_3 \cos \omega_2 \tau,$$

where $\omega_1 = \{-\frac{2}{3}[A_1 + (A_1^2 + 3A_2)^{1/2}]\}^{1/2}$, $\omega_2 = \{-\frac{2}{3}[A_1 - (A_1^2 + 3A_2)^{1/2}]\}^{1/2}$; *the constants* C_1, C_2, *and* C_3 *are related by the constraint* $\omega_1^2 (C_1^2 + C_2^2) - \omega_2^2 C_3^2 = 0$.

4°. *For* $A_1^2 + 3A_2 = 0$, $A_1 > 0$:

$$T_1 = (C_1 + C_2 \tau) e^{k\tau} + (C_3 + C_4 \tau) e^{-k\tau},$$
$$T_2 = (kC_1 + C_2 + kC_2 \tau) e^{k\tau} - (kC_3 - C_4 + kC_4 \tau) e^{-k\tau},$$

where $k = (\frac{2}{3} A_1)^{1/2}$; *the constants* C_1, C_2, *and* C_3 *are related by the constraint* $C_2 C_4 + (C_1 C_4 - C_2 C_3)(\frac{1}{6} A_1)^{1/2} = 0$.

5°. *For* $A_1^2 + 3A_2 = 0,\ A_1 < 0$:

$$T_1 = (C_1 + C_2\tau)\sin\omega\tau + C_3\tau\cos\omega\tau,$$
$$T_2 = (\omega C_1 + C_3 + \omega C_2\tau)\cos\omega\tau + (C_2 - \omega C_3\tau)\sin\omega\tau,$$

where $\omega = (-\frac{2}{3}A_1)^{1/2}$; *the constants* C_1, C_2, *and* C_3 *are related by the constraint* $C_2^2 + C_3^2 + C_1C_3(-\frac{2}{3}A_1)^{1/2} = 0$.

6°. *For* $3A_2 < -A_1^2$:

$$T_1 = e^{k\tau}(C_1\sin\omega\tau + C_2\cos\omega\tau) + C_3 e^{-k\tau}\sin\omega\tau,$$
$$T_2 = e^{k\tau}[(kC_2 + \omega C_1)\cos\omega\tau + (kC_1 - \omega C_2)\sin\omega\tau)] + C_3 e^{-k\tau}(\omega\cos\omega\tau - k\sin\omega\tau),$$

where $k = \{\frac{1}{3}[A_1 + (-3A_2)^{1/2}]\}^{1/2}$, $\omega = \{\frac{1}{3}[-A_1 + (-3A_2)^{1/2}]\}^{1/2}$; *the constants* C_1, C_2, *and* C_3 *are related by the constraint* $C_2A_1 + C_1(-A_1^2 - 3A_2)^{1/2} = 0$.

54. $\boldsymbol{y''_{xx} = A_1 y^{-1/3} + A_2 x^2 y^{-5/3}}$.

Solution in parametric form:

$$x = T_1, \qquad y = T_2^{3/2}.$$

55. $\boldsymbol{y''_{xx} = A_1 x^{-8/3} y^{-1/3} + A_2 x^{-10/3} y^{-5/3}}$.

Solution in parametric form:

$$x = T_1^{-1}, \qquad y = T_1^{-1} T_2^{3/2}.$$

◆ *In the solutions of equations 56–59, the following notation is used:*

$$T_1 = \begin{cases} C_1 e^{\omega\tau} + C_2 e^{-\omega\tau} + C_3\tau & \text{if } A_1 > 0, \\ C_1\sin\omega\tau + C_2\cos\omega\tau + C_3\tau & \text{if } A_1 < 0, \end{cases} \qquad \text{where} \quad \omega = |\tfrac{4}{3}A_1|^{1/2},$$

$$T_2 = \begin{cases} \omega(C_1 e^{\omega\tau} - C_2 e^{-\omega\tau}) + C_3 & \text{if } A_1 > 0, \\ \omega(C_1\cos\omega\tau - C_2\sin\omega\tau) + C_3 & \text{if } A_1 < 0, \end{cases} \qquad \text{where} \quad \omega = |\tfrac{4}{3}A_1|^{1/2}.$$

56. $\boldsymbol{y''_{xx} = A_1 y^{-1/3} + A_2 y^{-5/3}}$.

Solution in parametric form:

$$x = T_1, \qquad y = T_2^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$3(A_1C_3^2 + A_2) + 16A_1^2C_1C_2 = 0 \qquad \text{if } A_1 > 0,$$
$$3(A_1C_3^2 + A_2) + 4A_1^2(C_1^2 + C_2^2) = 0 \quad \text{if } A_1 < 0.$$

57. $\boldsymbol{y''_{xx} = A_1 x^{-8/3} y^{-1/3} + A_2 x^{-4/3} y^{-5/3}}$.

Solution in parametric form:

$$x = T_1^{-1}, \qquad y = T_1^{-1} T_2^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$3(A_1C_3^2 + A_2) + 16A_1^2C_1C_2 = 0 \qquad \text{if } A_1 > 0,$$
$$3(A_1C_3^2 + A_2) + 4A_1^2(C_1^2 + C_2^2) = 0 \quad \text{if } A_1 < 0.$$

58. $y''_{xx} = A_1 y^{-1/3} + A_2 x y^{-5/3}$.

Solution in parametric form:

$$x = T_1 - \frac{A_2}{4A_1}\tau^2, \qquad y = \Big(T_2 - \frac{A_2}{2A_1}\tau\Big)^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &3A_1C_3^2 + 16A_1^2C_1C_2 + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 > 0,\\ &3A_1C_3^2 + 4A_1^2(C_1^2 + C_2^2) + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 < 0. \end{aligned}$$

59. $y''_{xx} = A_1 x^{-8/3} y^{-1/3} + A_2 x^{-7/3} y^{-5/3}$.

Solution in parametric form:

$$x = \Big(T_1 - \frac{A_2}{4A_1}\tau^2\Big)^{-1}, \qquad y = \Big(T_1 - \frac{A_2}{4A_1}\tau^2\Big)^{-1}\Big(T_2 - \frac{A_2}{2A_1}\tau\Big)^{3/2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &3A_1C_3^2 + 16A_1^2C_1C_2 + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 > 0,\\ &3A_1C_3^2 + 4A_1^2(C_1^2 + C_2^2) + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 < 0. \end{aligned}$$

◆ *In the solutions of equations 60–67, the following notation is used:*

$$f = \sqrt{\pm(4\wp^3 - 1)}, \qquad \tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

The function $\wp = \wp(\tau)$ is defined implicitly in terms of the above elliptic integral of the first kind. For the upper sign, $\wp$ coincides with the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\, 0,\, 1)$. In the solution given below, one can take $\wp$ as the parameter instead of τ and use the explicit dependence $\tau = \tau(\wp)$.

60. $y''_{xx} = Ax^{-2}y^2 - \frac{6}{25}x^{-2}y$.

Solution in parametric form:

$$x = C_1\tau^5, \quad y = b\tau^2\wp, \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

61. $y''_{xx} = Ax^{-3}y^2 - \frac{6}{25}x^{-2}y$.

Solution in parametric form:

$$x = C_1\tau^{-5}, \quad y = bC_1\tau^{-3}\wp, \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

62. $y''_{xx} = Ax^{-2}y^2 + \frac{6}{25}x^{-2}y$.

Solution in parametric form:

$$x = C_1\tau^5, \quad y = b(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

63. $y''_{xx} = Ax^{-3}y^2 + \frac{6}{25}x^{-2}y$.

Solution in parametric form:

$$x = C_1\tau^{-5}, \quad y = bC_1\tau^{-5}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

64. $\boldsymbol{y''_{xx} = 12x^{-2}y + Ax^{-2}y^{-5/2}}$**.**

Solution in parametric form:

$$x = C_1\wp^{2/7}(f \pm 2\tau\wp^2)^{-1/7}, \quad y = b\wp^{-6/7}(f \pm 2\tau\wp^2)^{-4/7}, \quad \text{where} \quad A = \mp 147\, b^{7/2}.$$

65. $\boldsymbol{y''_{xx} = 12x^{-2}y + Ax^{3/2}y^{-5/2}}$**.**

Solution in parametric form:

$$x = C_1\wp^{-2/7}(f \pm 2\tau\wp^2)^{1/7}, \quad y = bC_1\wp^{-8/7}(f \pm 2\tau\wp^2)^{-3/7}, \quad \text{where} \quad A = \mp 147\, b^{7/2}.$$

66. $\boldsymbol{y''_{xx} = \frac{63}{4}x^{-2}y + Ax^{-2}y^{-5/3}}$**.**

Solution in parametric form:

$$x = C_1(\tau f + 2\wp)^{-1/4}, \quad y = b(\tau f + 2\wp)^{-9/8}(f \pm 2\tau\wp^2)^{3/2}, \quad \text{where} \quad A = -\tfrac{32}{3}b^{8/3}.$$

67. $\boldsymbol{y''_{xx} = \frac{63}{4}x^{-2}y + Ax^{2/3}y^{-5/3}}$**.**

Solution in parametric form:

$$x = C_1(\tau f + 2\wp)^{1/4}, \quad y = bC_1(\tau f + 2\wp)^{-7/8}(f \pm 2\tau\wp^2)^{3/2}, \quad \text{where} \quad A = -\tfrac{32}{3}b^{8/3}.$$

◆ *In the solutions of equations 68–73, the following notation is used:*

$$f_1 = \sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}, \quad \tau = \int \frac{d\wp_1}{\sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}} - C_1;$$

$$f_2 = \sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}, \quad \tau = \int \frac{d\wp_2}{\sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}} - C_1.$$

The functions $\wp_1 = \wp_1(\tau)$ and $\wp_2 = \wp_2(\tau)$ are the inverses of the above elliptic integrals. For the upper signs, they are the classical Weierstrass functions $\wp_1 = \wp(\tau + C_1,\, 2,\, C_2)$ and $\wp_2 = \wp(\tau + C_1,\, -2,\, C_2)$.

68. $\boldsymbol{y''_{xx} = A_1 y^2 + A_2}$**.**

Solutions in parametric form:

$$x = a\tau, \quad y = b\wp_k,$$

where $A_1 = \pm 6a^{-2}b^{-1}$, $A_2 = a^{-2}b(-1)^k$; $k = 1$ and $k = 2$.

69. $\boldsymbol{y''_{xx} = A_1 x^{-5}y^2 + A_2 x^{-3}}$**.**

Solutions in parametric form:

$$x = a\tau^{-1}, \quad y = b\tau^{-1}\wp_k,$$

where $A_1 = \pm 6a^3b^{-1}$, $A_2 = ab(-1)^k$; $k = 1$ and $k = 2$.

70. $\boldsymbol{y''_{xx} = A_1 x^{-15/7}y^2 + A_2 x^{-9/7}}$**.**

Solutions in parametric form:

$$x = a\tau^7, \quad y = b\tau(\tau^2\wp_k \mp 1),$$

where $A_1 = \pm\frac{6}{49}a^{1/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-5/7}b(-1)^k$; $k = 1$ and $k = 2$.

71. $y''_{xx} = A_1 x^{-20/7} y^2 + A_2 x^{-12/7}$.

Solutions in parametric form:

$$x = a\tau^{-7}, \qquad y = b\tau^{-6}(\tau^2 \wp_k \mp 1),$$

where $A_1 = \pm\frac{6}{49} a^{6/7} b^{-1}$, $A_2 = \frac{1}{49} a^{-2/7} b(-1)^k$; $k = 1$ and $k = 2$.

72. $y''_{xx} = A_1 + A_2 y^{-2/3}$.

Solutions in parametric form:

$$x = a[f_k - (-1)^k \tau], \qquad y = b\wp_k^3,$$

where $A_1 = \pm\frac{1}{2} a^{-2} b$, $A_2 = \frac{1}{12} a^{-2} b^{5/3} (-1)^k$; $k = 1$ and $k = 2$.

73. $y''_{xx} = A_1 x^{-3} + A_2 x^{-7/3} y^{-2/3}$.

Solutions in parametric form:

$$x = a[f_k - (-1)^k \tau]^{-1}, \qquad y = b\wp_k^3 [f_k - (-1)^k \tau]^{-1},$$

where $A_1 = \pm\frac{1}{2} ab$, $A_2 = \frac{1}{12} a^{1/3} b^{5/3} (-1)^k$; $k = 1$ and $k = 2$.

◆ *In the solutions of equations 74 and 75, the following notation is used:*

$$E = \int (1 \pm \tau^4)^{-1/2}\, d\tau + C_2, \qquad k^2 = \pm 1.$$

The function E can be expressed in terms of elliptic integrals or lemniscate functions.

74. $y''_{xx} = A_1 x^{-18/5} y^3 + A_2 x^{-14/5} y^2$.

Solutions in parametric form:

$$x = aC_1^5 E^{-5}, \quad y = bC_1^4 E^{-4}(\tau E - k), \quad \text{where} \quad A_1 = \pm\tfrac{2}{25} a^{8/5} b^{-2}, \ A_2 = \pm\tfrac{6}{25} a^{4/5} b^{-1} k.$$

75. $y''_{xx} = A_1 x^{-12/5} y^3 + A_2 x^{-11/5} y^2$.

Solutions in parametric form:

$$x = aC_1^5 E^5, \quad y = bC_1 E(\tau E - k), \qquad \text{where} \quad A_1 = \pm\tfrac{2}{25} a^{2/5} b^{-2}, \ A_2 = \pm\tfrac{6}{25} a^{1/5} b^{-1} k.$$

◆ *In the solutions of equations 76–81, the following notation is used:*

$$f = \begin{cases} J_{1/3}(\tau) & \text{for the upper sign (Bessel function)}, \\ I_{1/3}(\tau) & \text{for the lower sign (modified Bessel function)}, \end{cases}$$

$$g = \begin{cases} Y_{1/3}(\tau) & \text{for the upper sign (Bessel function)}, \\ K_{1/3}(\tau) & \text{for the lower sign (modified Bessel function)}, \end{cases}$$

$$H = C_1 f + C_2 g + \beta\omega\Big(g \int f\, d\tau - f \int g\, d\tau\Big), \qquad \omega = \begin{cases} \frac{1}{2}\pi & \text{for the upper sign}, \\ -1 & \text{for the lower sign}. \end{cases}$$

76. $y''_{xx} = A_1 xy + A_2.$

Solutions in parametric form:

$$x = a\tau^{2/3}, \quad y = \tau^{1/3} H, \qquad \text{where} \quad A_1 = \mp\tfrac{9}{4}a^{-3}, \ \ A_2 = \tfrac{9}{4}a^{-2}\beta.$$

77. $y''_{xx} = A_1 x^{-5} y + A_2 x^{-3}.$

Solutions in parametric form:

$$x = a\tau^{-2/3}, \quad y = \tau^{-1/3} H, \qquad \text{where} \quad A_1 = \mp\tfrac{9}{4}a^{3}, \ \ A_2 = \tfrac{9}{4}a\beta.$$

78. $y''_{xx} = A_1 x^{-3/2} + A_2 x^{-1/2} y^{-1/2}.$

Solutions in parametric form:

$$x = a\tau^{2/3} H^2,$$
$$y = b\tau^{-2/3} \bigl(\tau H'_\tau + \tfrac{1}{3}H\bigr)^2,$$

where $A_1 = -\frac{1}{2}a^{-1/2}b\beta, \ A_2 = \mp\frac{1}{3}a^{-3/2}b^{3/2}.$

79. $y''_{xx} = A_1 x^{-3/2} + A_2 x^{-2} y^{-1/2}.$

Solutions in parametric form:

$$x = a\tau^{-2/3} H^{-2},$$
$$y = b\tau^{-4/3} H^{-2} \bigl(\tau H'_\tau + \tfrac{1}{3}H\bigr)^2,$$

where $A_1 = -\frac{1}{2}a^{-1/2}b\beta, \ A_2 = \mp\frac{1}{3}b^{3/2}.$

80. $y''_{xx} = A_1 y^{-3/2} + A_2 x y^{-2}.$

Solutions in parametric form:

$$x = a\tau^{-2/3} \bigl[\mp\tau^2 H^2 + 2\beta\tau H - \bigl(\tau H'_\tau + \tfrac{1}{3}H\bigr)^2\bigr],$$
$$y = b\tau^{2/3} H^2,$$

where $A_1 = -ab^{-1/2}\beta A_2, \ A_2 = \frac{9}{2}a^{-3}b^3.$

81. $y''_{xx} = A_1 x^{-3/2} y^{-3/2} + A_2 x^{-2} y^{-2}.$

Solutions in parametric form:

$$x = a\tau^{2/3} \bigl[\mp\tau^2 H^2 + 2\beta\tau H - \bigl(\tau H'_\tau + \tfrac{1}{3}H\bigr)^2\bigr]^{-1},$$
$$y = b\tau^{4/3} H^2 \bigl[\mp\tau^2 H^2 + 2\beta\tau H - \bigl(\tau H'_\tau + \tfrac{1}{3}H\bigr)^2\bigr]^{-1},$$

where $A_1 = -\frac{9}{2}a^{-1/2}b^{5/2}\beta, \ A_2 = \frac{9}{2}b^3.$

◆ *In the solutions of equations 82–88, the following notation is used:*

$$U_\nu = \begin{cases} C_1 J_\nu(\tau) & \text{for the upper sign (Bessel function)}, \\ C_1 I_\nu(\tau) & \text{for the lower sign (modified Bessel function)}, \end{cases}$$

$$V_\nu = \begin{cases} C_2 Y_\nu(\tau) & \text{for the upper sign (Bessel function)}, \\ C_2 K_\nu(\tau) & \text{for the lower sign (modified Bessel function)}, \end{cases}$$

$$Z_\nu = \alpha_1 U_\nu + \alpha_2 V_\nu,\ \ X_\nu = \beta_1 U_\nu + \beta_2 V_\nu,\ \ F_\nu = \tau Z'_\nu + \nu Z_\nu,\ \ G_\nu = \tau X'_\nu + \nu X_\nu,$$

$$N = \begin{cases} Z_\nu X_\nu & \text{if } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2, \\ \alpha U_\nu^2 + \beta U_\nu V_\nu + \gamma V_\nu^2 & \text{if } \Delta = 4\alpha\gamma - \beta^2, \end{cases}$$

$$N_1 = \begin{cases} Z_\nu G_\nu + X_\nu F_\nu & \text{if } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2, \\ \tau N' + 2\nu N & \text{if } \Delta = 4\alpha\gamma - \beta^2, \end{cases}$$

$$N_2 = N_1^2 \pm 4\tau^2 N^2 + \omega^2\Delta, \qquad \omega = \begin{cases} 2/\pi & \text{for the upper sign}, \\ -1 & \text{for the lower sign}. \end{cases}$$

The prime denotes differentiation with respect to τ.

82. $y''_{xx} = A_1 x y + A_2 y^{-3}$.

Solutions in parametric form:

$$x = a\tau^{2/3}, \qquad y = b\tau^{1/3} N^{1/2},$$

where $\nu = \frac{1}{3}$, $A_1 = \mp\frac{9}{4}a^{-3}$, $A_2 = \frac{9}{16}a^{-2}b^4\omega^2\Delta$.

83. $y''_{xx} = A_1 x^n y + A_2 y^{-3}$, $\qquad n \neq -2$.

Solutions in parametric form:

$$x = a\tau^{2\nu}, \qquad y = b\tau^{\nu} N^{1/2},$$

where $\nu = \dfrac{1}{n+2}$, $A_1 = \mp\dfrac{1}{4\nu^2}a^{-n-2}$, $A_2 = \dfrac{1}{16\nu^2}a^{-2}b^4\omega^2\Delta$.

84. $y''_{xx} = A_1 x^{-5} y + A_2 y^{-3}$.

Solutions in parametric form:

$$x = a\tau^{-2/3}, \qquad y = b\tau^{-1/3} N^{1/2},$$

where $\nu = \frac{1}{3}$, $A_1 = \mp\frac{9}{4}a^{3}$, $A_2 = \frac{9}{16}a^{-2}b^4\omega^2\Delta$.

85. $y''_{xx} = A_1 x^{-3} + A_2 x^{-1/2} y^{-1/2}$.

Solutions in parametric form:

$$x = a\tau^{2/3} N, \qquad y = b\tau^{-2/3} N^{-1} N_1^2,$$

where $\nu = \frac{1}{3}$, $A_1 = -2ab\omega^2\Delta$, $A_2 = \mp\frac{8}{3}a^{-3/2}b^{3/2}$.

86. $y''_{xx} = A_1 + A_2 x^{-2} y^{-1/2}$.

Solutions in parametric form:

$$x = a\tau^{-2/3} N^{-1}, \qquad y = b\tau^{-4/3} N^{-2} N_1^2,$$

where $\nu = \frac{1}{3}$, $A_1 = -2a^{-2}b\omega^2\Delta$, $A_2 = \mp\frac{8}{3}b^{3/2}$.

87. $y''_{xx} = A_1 x y^{-2} + A_2 y^{-3}.$

Solutions in parametric form:

$$x = a\tau^{-2/3} N^{-1} N_2, \qquad y = b\tau^{2/3} N,$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{128} a^{-3} b^3$, $A_2 = \frac{9}{64} a^{-2} b^4 \omega^2 \Delta$.

88. $y''_{xx} = A_1 x^{-2} y^{-2} + A_2 y^{-3}.$

Solutions in parametric form:

$$x = a\tau^{2/3} N N_2^{-1}, \qquad y = b\tau^{4/3} N^2 N_2^{-1},$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{128} b^3$, $A_2 = \frac{9}{64} a^{-2} b^4 \omega^2 \Delta$.

◆ *In the solutions of equations 89 and 90, the following notation is used:*

$$\Delta = C_2^2 - 2C_1, \quad R = (36\Delta + 54B\tau - 2\tau^3)^{1/2}, \quad z = 3\int \tau^{-1} R^{-1}\, d\tau,$$

$$W(z) = \begin{cases} \dfrac{\sqrt{-\Delta}}{C_1} \tan\bigl(\pm\sqrt{-\Delta}\, z\bigr) + \dfrac{C_2}{C_1} & \text{if } \Delta < 0, \\ \dfrac{\sqrt{\Delta}}{C_1} \tanh\bigl(\mp\sqrt{\Delta}\, z\bigr) + \dfrac{C_2}{C_1} & \text{if } \Delta > 0, \\ \mp\dfrac{1}{C_1 z} - \dfrac{\sqrt{2}}{\sqrt{|C_1|}} & \text{if } \Delta = 0,\ C_2 < 0, \\ \mp\dfrac{1}{C_1 z} + \dfrac{\sqrt{2}}{\sqrt{|C_1|}} & \text{if } \Delta = 0,\ C_2 > 0. \end{cases}$$

89. $y''_{xx} = A_1 y^{-5/3} + A_2 x^{-2/3} y^{-5/3}.$

Solutions in parametric form:

$$x = a\tau^{-3/2}(C_1 W^2 - 2C_2 W + 2)^{3/2},$$
$$y = b\tau^{-9/4}(C_1 W^2 - 2C_2 W + 2)^{3/4}(6C_1 W - 6C_2 \mp R)^{3/2},$$

where $A_1 = 24a^{-2} b^{8/3} C_1$, $A_2 = -36a^{-4/3} b^{8/3} B$.

90. $y''_{xx} = A_1 x^{-2/3} y^{-5/3} + A_2 x^{-4/3} y^{-5/3}.$

Solutions in parametric form:

$$x = a\tau^{3/2}(C_1 W^2 - 2C_2 W + 2)^{-3/2},$$
$$y = b\tau^{-3/4}(C_1 W^2 - 2C_2 W + 2)^{-3/4}(6C_1 W - 6C_2 \mp R)^{3/2},$$

where $A_1 = -36a^{-4/3} b^{8/3} B$, $A_2 = 24a^{-2/3} b^{8/3} C_1$.

◆ *In the solutions of equations 91–102, the following notation is used: The functions P_1 and P_2 are the general solutions of the four modifications of the first Painlevé equation:*

$$P_1'' = \pm 6P_1^2 + \tau, \qquad P_2'' = \pm 6P_2^2 - \tau$$

(in the case of the upper sign, the equation for P_1 is the canonical form of the first Painlevé equation, see Section 3.4.2). In addition,

$$Q_1 = \pm 6P_1^2 + \tau, \quad T_1 = \tau^2 P_1 \mp 1, \quad U_1 = (P_1')^2 - 2P_1 Q_1 \pm 8P_1^3, \quad V_1 = P_1' Q_1' + P_1' - Q_1^2,$$
$$Q_2 = \pm 6P_2^2 - \tau, \quad T_2 = \tau^2 P_2 \mp 1, \quad U_2 = (P_2')^2 - 2P_2 Q_2 \pm 8P_2^3, \quad V_2 = P_2' Q_2' - P_2' - Q_2^2.$$

The prime denotes differentiation with respect to τ.

91. $y''_{xx} = A_1 y^2 + A_2 x.$

Solutions in parametric form:

$$x = a\tau, \quad y = bP_k,$$

where $A_1 = \pm 6a^{-2}b^{-1}$, $A_2 = a^{-3}b(-1)^{k+1}$; $k = 1$ and $k = 2$.

92. $y''_{xx} = A_1 x^{-5} y^2 + A_2 x^{-4}.$

Solutions in parametric form:

$$x = a\tau^{-1}, \quad y = b\tau^{-1}P_k,$$

where $A_1 = \pm 6a^3 b^{-1}$, $A_2 = a^2 b(-1)^{k+1}$; $k = 1$ and $k = 2$.

93. $y''_{xx} = A_1 x^{-15/7} y^2 + A_2 x^{-8/7}.$

Solutions in parametric form:

$$x = a\tau^7, \quad y = b\tau T_k,$$

where $A_1 = \pm\frac{6}{49}a^{1/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-6/7}b(-1)^{k+1}$; $k = 1$ and $k = 2$.

94. $y''_{xx} = A_1 x^{-20/7} y^2 + A_2 x^{-13/7}.$

Solutions in parametric form:

$$x = a\tau^{-7}, \quad y = b\tau^{-6}T_k,$$

where $A_1 = \pm\frac{6}{49}a^{6/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-1/7}b(-1)^{k+1}$; $k = 1$ and $k = 2$.

95. $y''_{xx} = A_1 x + A_2 y^{-1/2}.$

Solutions in parametric form:

$$x = aP_k, \quad y = b(P'_k)^2,$$

where $A_1 = \pm 24a^{-3}b$, $A_2 = 2a^{-2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

96. $y''_{xx} = A_1 x^{-4} + A_2 x^{-5/2} y^{-1/2}.$

Solutions in parametric form:

$$x = aP_k^{-1}, \quad y = bP_k^{-1}(P'_k)^2,$$

where $A_1 = \pm 24a^2 b$, $A_2 = 2a^{1/2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

97. $y''_{xx} = A_1 y^{1/3} + A_2 x y^{-5/3}.$

Solutions in parametric form:

$$x = aU_k, \quad y = bP_k^{3/2},$$

where $A_1 = \mp 8ab^{-2}A_2$, $A_2 = -\frac{3}{16}a^{-3}b^{8/3}$; $k = 1$ and $k = 2$.

98. $y''_{xx} = A_1 x^{-10/3} y^{1/3} + A_2 x^{-7/3} y^{-5/3}.$

Solutions in parametric form:

$$x = aU_k^{-1}, \quad y = bP_k^{3/2}U_k^{-1},$$

where $A_1 = \mp 8ab^{-2}A_2$, $A_2 = -\frac{3}{16}a^{1/3}b^{8/3}$; $k = 1$ and $k = 2$.

99. $\boldsymbol{y''_{xx} = A_1x^{-3/2} + A_2y^{-1/2}}$**.**

Solutions in parametric form:

$$x = a(P'_k)^2, \quad y = bQ_k^2,$$

where $A_1 = \frac{1}{2}a^{-1/2}b(-1)^k$, $A_2 = \pm 6a^{-2}b^{3/2}$; $k = 1$ and $k = 2$.

100. $\boldsymbol{y''_{xx} = A_1x^{-3/2} + A_2x^{-5/2}y^{-1/2}}$**.**

Solutions in parametric form:

$$x = a(P'_k)^{-2}, \quad y = b(P'_k)^{-2}Q_k^2,$$

where $A_1 = \frac{1}{2}a^{-1/2}b(-1)^k$, $A_2 = \pm 6a^{1/2}b^{3/2}$; $k = 1$ and $k = 2$.

101. $\boldsymbol{y''_{xx} = A_1y^{-4/3} + A_2xy^{-5/3}}$**.**

Solutions in parametric form:

$$x = aV_k, \quad y = b(P'_k)^3,$$

where $A_1 = ab^{-1/3}A_2(-1)^k$, $A_2 = \frac{1}{36}a^{-3}b^{8/3}$; $k = 1$ and $k = 2$.

102. $\boldsymbol{y''_{xx} = A_1x^{-5/3}y^{-4/3} + A_2x^{-7/3}y^{-5/3}}$**.**

Solutions in parametric form:

$$x = aV_k^{-1}, \quad y = b(P'_k)^3V_k^{-1},$$

where $A_1 = \frac{1}{36}a^{-1/3}b^{7/3}(-1)^k$, $A_2 = \frac{1}{36}a^{1/3}b^{8/3}$; $k = 1$ and $k = 2$.

◆ *In the solutions of equations 103–108, the following notation is used:*
The functions P_1 and P_2 are the general solutions of the four modifications of the second Painlevé equation (with parameter $a = 0$):

$$P''_1 = \tau P_1 \pm 2P_1^3, \quad P''_2 = -\tau P_2 \pm 2P_2^3,$$

where the primes denote differentiation with respect to τ. In the case of the upper sign, the equation for P_1 is the canonical form of the second Painlevé equation (with parameter $a = 0$; see Section 3.4.3).

103. $\boldsymbol{y''_{xx} = A_1y^3 + A_2xy}$**.**

Solutions in parametric form:

$$x = a\tau, \quad y = bP_k,$$

where $A_1 = \pm 2a^{-2}b^{-2}$, $A_2 = a^3(-1)^{k+1}$; $k = 1$ and $k = 2$.

104. $\boldsymbol{y''_{xx} = A_1x^{-6}y^3 + A_2x^{-5}y}$**.**

Solutions in parametric form:

$$x = a\tau^{-1}, \quad y = b\tau^{-1}P_k,$$

where $A_1 = \pm 2a^4b^{-2}$, $A_2 = a^3(-1)^{k+1}$; $k = 1$ and $k = 2$.

105. $y''_{xx} = A_1 + A_2 x^{-1/2} y^{-1/2}$.

Solutions in parametric form:

$$x = aP_k^2, \quad y = b(P'_k)^2, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \pm 2a^{-2}b$, $A_2 = \frac{1}{2}a^{-3/2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

106. $y''_{xx} = A_1 x^{-3} + A_2 x^{-2} y^{-1/2}$.

Solutions in parametric form:

$$x = aP_k^{-2}, \quad y = bP_k^{-2}(P'_k)^2, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \pm 2ab$, $A_2 = \frac{1}{2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

107. $y''_{xx} = A_1 + A_2 x y^{-2}$.

Solutions in parametric form:

$$x = a[\tau P_k^2 \pm P_k^4 - (P'_k)^2], \quad y = bP_k^2, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \mp 2a^{-2}b(-1)^k$, $A_2 = 2a^{-3}b^3(-1)^{k+1}$; $k = 1$ and $k = 2$.

108. $y''_{xx} = A_1 x^{-3} + A_2 x^{-2} y^{-2}$.

Solutions in parametric form:

$$x = a\big[\tau P_k^2 \pm P_k^4 - (P'_k)^2\big]^{-1}, \quad y = bP_k^2\big[\tau P_k^2 \pm P_k^4 - (P'_k)^2\big]^{-1},$$

where $A_1 = \pm 2ab$, $A_2 = 2b^3$; $k = 1$ and $k = 2$.

14.5 Generalized Emden–Fowler Equation $y''_{xx} = Ax^n y^m (y'_x)^l$

14.5.1 Classification Table

The case $l = 0$ corresponding to the classical Emden–Fowler equation is outlined in Section 14.3. In this section, the case $l \neq 0$ is considered.

Table 14.7 presents all solvable equations of the form $y''_{xx} = Ax^n y^m (y'_x)^l$ whose solutions are outlined in Section 14.5.2. Two-parameter families (in the space of the parameters n, m, and l), one-parameter families, and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of l, the growth of m (for identical l), and the growth of n (for identical m and l). The number of the equation sought is indicated in the last column in this table.

TABLE 14.7
Solvable cases of the generalized Emden–Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
		Two-parameter families		
1	arbitrary	arbitrary	0	14.5.2.1
2	arbitrary	0	arbitrary	14.5.2.2
3	$\dfrac{2n+m+3}{n+m+2}$	arbitrary $(m \neq -1)$	arbitrary $(n \neq -1)$	14.5.2.3
		One-parameter families		
4	arbitrary $(l \neq 1, 2)$	-1	-1	14.5.2.6
5	arbitrary $(l \neq \frac{3}{2})$	$-\frac{1}{2}$	$-\frac{1}{2}$	14.5.2.97
6	$\dfrac{3m+5}{2m+3}$	arbitrary $(m \neq -\frac{3}{2})$	$-\frac{1}{2}$	14.5.2.13
7	$\dfrac{3m+5}{2m+3}$	arbitrary $(m \neq -\frac{3}{2})$	1	14.5.2.10
8	$\dfrac{3n+4}{2n+3}$	$-\frac{1}{2}$	arbitrary $(n \neq -\frac{3}{2})$	14.5.2.11
9	$\dfrac{3n+4}{2n+3}$	1	arbitrary $(n \neq -\frac{3}{2})$	14.5.2.12
10	$\dfrac{3n+4}{2n+3}$	$-n-3$	arbitrary $(n \neq -\frac{3}{2})$	14.5.2.107
11	1	arbitrary $(m \neq -1, 0)$	-1	14.5.2.5
12	2	-1	arbitrary $(n \neq -1, 0)$	14.5.2.4
13	3	arbitrary $(m \neq -2)$	1	14.5.2.96
14	3	$-n-3$	arbitrary	14.5.2.9
		Isolated points		
15	$\frac{1}{2}$	$-\frac{10}{7}$	$-\frac{11}{7}$	14.5.2.28
16	$\frac{1}{2}$	$-\frac{13}{10}$	$-\frac{17}{10}$	14.5.2.56
17	$\frac{1}{2}$	$-\frac{11}{5}$	$-\frac{4}{5}$	14.5.2.108
18	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	14.5.2.39
19	$\frac{1}{2}$	1	$-\frac{15}{8}$	14.5.2.95
20	$\frac{1}{2}$	1	$-\frac{20}{13}$	14.5.2.98
21	$\frac{1}{2}$	1	$-\frac{5}{4}$	14.5.2.92

TABLE 14.7 (*Continued*)
Solvable cases of the generalized Emden–Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
22	$\frac{1}{2}$	1	0	14.5.2.90
23	$\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{7}{6}$	14.5.2.65
24	$\frac{3}{4}$	$-\frac{11}{6}$	$-\frac{2}{3}$	14.5.2.113
25	$\frac{4}{5}$	$-\frac{5}{2}$	$-\frac{1}{2}$	14.5.2.88
26	1	-2	1	14.5.2.14
27	1	-1	-1	14.5.2.8
28	$\frac{8}{7}$	1	$-\frac{3}{4}$	14.5.2.66
29	$\frac{8}{7}$	1	$-\frac{1}{2}$	14.5.2.64
30	$\frac{6}{5}$	$-\frac{1}{2}$	$-\frac{2}{3}$	14.5.2.80
31	$\frac{16}{13}$	$-\frac{17}{7}$	2	14.5.2.58
32	$\frac{5}{4}$	1	$-\frac{1}{2}$	14.5.2.70
33	$\frac{5}{4}$	1	0	14.5.2.68
34	$\frac{9}{7}$	$-\frac{13}{8}$	1	14.5.2.45
35	$\frac{9}{7}$	$-\frac{1}{2}$	1	14.5.2.44
36	$\frac{13}{10}$	$-\frac{11}{4}$	2	14.5.2.30
37	$\frac{13}{10}$	$-\frac{1}{2}$	$-\frac{5}{2}$	14.5.2.53
38	$\frac{27}{20}$	$-\frac{1}{2}$	$-\frac{2}{3}$	14.5.2.84
39	$\frac{15}{11}$	$-\frac{10}{3}$	$-\frac{2}{3}$	14.5.2.33
40	$\frac{18}{13}$	$-\frac{1}{2}$	$-\frac{7}{2}$	14.5.2.46
41	$\frac{7}{5}$	$-\frac{7}{4}$	1	14.5.2.18
42	$\frac{7}{5}$	$-\frac{10}{7}$	1	14.5.2.52
43	$\frac{7}{5}$	$-\frac{2}{3}$	1	14.5.2.38
44	$\frac{7}{5}$	$-\frac{1}{2}$	1	14.5.2.17
45	$\frac{7}{5}$	1	0	14.5.2.101
46	$\frac{7}{5}$	1	1	14.5.2.103
47	$\frac{7}{5}$	5	1	14.5.2.87
48	$\frac{24}{17}$	$-\frac{13}{3}$	$-\frac{2}{3}$	14.5.2.61
49	$\frac{10}{7}$	$-\frac{1}{2}$	$-\frac{5}{2}$	14.5.2.19
50	$\frac{16}{11}$	2	4	14.5.2.111
51	$\frac{22}{15}$	$-\frac{1}{2}$	$-\frac{2}{3}$	14.5.2.82
52	$\frac{3}{2}$	-2	$-\frac{1}{2}$	14.5.2.124

TABLE 14.7 (*Continued*)
Solvable cases of the generalized Emden–Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
53	$\frac{3}{2}$	-2	1	14.5.2.117
54	$\frac{3}{2}$	$-\frac{1}{2}$	-2	14.5.2.118
55	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	14.5.2.35
56	$\frac{3}{2}$	$-\frac{1}{2}$	1	14.5.2.116
57	$\frac{3}{2}$	1	-2	14.5.2.123
58	$\frac{3}{2}$	1	$-\frac{1}{2}$	14.5.2.121
59	$\frac{23}{15}$	$-\frac{2}{3}$	$-\frac{1}{2}$	14.5.2.96
60	$\frac{17}{11}$	4	2	14.5.2.110
61	$\frac{11}{7}$	$-\frac{5}{2}$	$-\frac{1}{2}$	14.5.2.27
62	$\frac{27}{17}$	$-\frac{2}{3}$	$-\frac{13}{3}$	14.5.2.60
63	$\frac{8}{5}$	0	1	14.5.2.85
64	$\frac{8}{5}$	1	$-\frac{7}{4}$	14.5.2.26
65	$\frac{8}{5}$	1	$-\frac{10}{7}$	14.5.2.54
66	$\frac{8}{5}$	1	$-\frac{2}{3}$	14.5.2.41
67	$\frac{8}{5}$	1	$-\frac{1}{2}$	14.5.2.24
68	$\frac{8}{5}$	1	1	14.5.2.86
69	$\frac{8}{5}$	1	5	14.5.2.106
70	$\frac{21}{13}$	$-\frac{7}{2}$	$-\frac{1}{2}$	14.5.2.51
71	$\frac{18}{11}$	$-\frac{2}{3}$	$-\frac{10}{3}$	14.5.2.32
72	$\frac{33}{20}$	$-\frac{2}{3}$	$-\frac{1}{2}$	14.5.2.99
73	$\frac{17}{10}$	$-\frac{5}{2}$	$-\frac{1}{2}$	14.5.2.55
74	$\frac{17}{10}$	2	$-\frac{11}{4}$	14.5.2.31
75	$\frac{12}{7}$	1	$-\frac{13}{8}$	14.5.2.50
76	$\frac{12}{7}$	1	$-\frac{1}{2}$	14.5.2.48
77	$\frac{7}{4}$	$-\frac{1}{2}$	1	14.5.2.63
78	$\frac{7}{4}$	0	1	14.5.2.62
79	$\frac{23}{13}$	2	$-\frac{17}{7}$	14.5.2.59
80	$\frac{9}{5}$	$-\frac{2}{3}$	$-\frac{1}{2}$	14.5.2.93
81	$\frac{13}{7}$	$-\frac{3}{4}$	1	14.5.2.77
82	$\frac{13}{7}$	$-\frac{1}{2}$	1	14.5.2.73
83	2	-1	-1	14.5.2.7

TABLE 14.7 (*Continued*)
Solvable cases of the generalized Emden–Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
84	2	1	-2	14.5.2.16
85	$\frac{11}{5}$	$-\frac{1}{2}$	$-\frac{5}{2}$	14.5.2.107
86	$\frac{9}{4}$	$-\frac{2}{3}$	$-\frac{11}{6}$	14.5.2.112
87	$\frac{7}{3}$	$-\frac{7}{6}$	$-\frac{1}{2}$	14.5.2.76
88	$\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	14.5.2.42
89	$\frac{5}{2}$	$-\frac{15}{8}$	1	14.5.2.81
90	$\frac{5}{2}$	$-\frac{17}{10}$	$-\frac{13}{10}$	14.5.2.57
91	$\frac{5}{2}$	$-\frac{11}{7}$	$-\frac{10}{7}$	14.5.2.29
92	$\frac{5}{2}$	$-\frac{20}{13}$	1	14.5.2.83
93	$\frac{5}{2}$	$-\frac{5}{4}$	1	14.5.2.79
94	$\frac{5}{2}$	$-\frac{4}{5}$	$-\frac{11}{5}$	14.5.2.109
95	$\frac{5}{2}$	0	1	14.5.2.78
96	3	-5	2	14.5.2.91
97	3	$-\frac{7}{2}$	$-\frac{1}{2}$	14.5.2.37
98	3	$-\frac{10}{3}$	$-\frac{5}{3}$	14.5.2.43
99	3	$-\frac{20}{7}$	2	14.5.2.97
100	3	$-\frac{5}{2}$	$-\frac{1}{2}$	14.5.2.22
101	3	$-\frac{13}{5}$	$-\frac{7}{5}$	14.5.2.49
102	3	$-\frac{7}{3}$	$-\frac{5}{3}$	14.5.2.25
103	3	$-\frac{15}{7}$	2	14.5.2.94
104	3	-2	-2	14.5.2.122
105	3	-2	-1	14.5.2.15
106	3	-2	$-\frac{1}{2}$	14.5.2.119
107	3	-2	1	14.5.2.34
108	3	$-\frac{4}{3}$	$-\frac{1}{2}$	14.5.2.71
109	3	$-\frac{7}{6}$	$-\frac{1}{2}$	14.5.2.72
110	3	$-\frac{5}{6}$	$-\frac{5}{3}$	14.5.2.105
111	3	$-\frac{1}{2}$	$-\frac{5}{2}$	14.5.2.102
112	3	$-\frac{1}{2}$	$-\frac{5}{3}$	14.5.2.104
113	3	0	-4	14.5.2.67
114	3	0	$-\frac{5}{2}$	14.5.2.10

TABLE 14.7 (*Continued*)
Solvable cases of the generalized Emden–Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
115	3	0	$-\frac{1}{2}$	14.5.2.20
116	3	0	2	14.5.2.89
117	3	1	-7	14.5.2.74
118	3	1	-4	14.5.2.69
119	3	1	-2	14.5.2.120
120	3	1	$-\frac{5}{3}$	14.5.2.23
121	3	1	$-\frac{7}{5}$	14.5.2.47
122	3	1	$-\frac{1}{2}$	14.5.2.36
123	3	1	0	14.5.2.21
124	3	2	$-\frac{5}{3}$	14.5.2.40
125	3	3	-7	14.5.2.75

14.5.2 Exact Solutions

1. $y''_{xx} = Ay^m (y'_x)^l$.

1°. Solution in parametric form with $m \neq -1$, $l \neq 2$:

$$x = aC_1^{1-m-l} \int (1 \pm \tau^{m+1})^{\frac{1}{l-2}}\, d\tau + C_2, \quad y = bC_1^{2-l}\tau, \quad \text{where } A = \pm \frac{m+1}{2-l} a^{l-2} b^{1-m-l}.$$

2°. Solution in parametric form with $m = -1$, $l \neq 2$:

$$x = aC_1 \int \tau^{\frac{l}{l-2}} \exp(\mp\tau^2)\, d\tau + C_2, \quad y = bC_1 \exp(\mp\tau^2), \quad \text{where } A = \mp \frac{4b^2}{a^2(2-l)} \Big(\mp \frac{a}{2b}\Big)^l.$$

3°. Solution in parametric form with $m \neq -1$, $l = 2$:

$$x = C_1 \int \tau^{\frac{1-m}{1+m}} \exp(\mp\tau^2)\, d\tau + C_2, \quad y = b\tau^{\frac{2}{m+1}}, \quad \text{where } A = \pm(m+1)b^{-1-m}.$$

4°. Solution for $m = -1$, $l = 2$:

$$y = \begin{cases} (C_1 x + C_2)^{\frac{1}{1-A}} & \text{if } A \neq 1, \\ C_2 \exp(C_1 x) & \text{if } A = 1. \end{cases}$$

2. $y''_{xx} = Ax^n (y'_x)^l$.

1°. Solution in parametric form with $n \neq -1$, $l \neq 1$:

$$x = aC_1^{1-l}\tau, \quad y = bC_1^{2+n-l} \int (1 \pm \tau^{n+1})^{\frac{1}{1-l}}\, d\tau + C_2, \quad \text{where } A = \pm \frac{n+1}{1-l} a^{l-n-2} b^{1-l}.$$

2°. Solution in parametric form with $n=-1$, $l\neq 1$:

$$x=aC_1\exp(\mp\tau^2),\quad y=bC_1\int\tau^{\frac{3-l}{1-l}}\exp(\mp\tau^2)\,d\tau+C_2,\quad\text{with}\quad A=\mp\frac{4a^2}{b^2(1-l)}\Bigl(\mp\frac{b}{2a}\Bigr)^{3-l}.$$

3°. Solution in parametric form with $n\neq -1$, $l=1$:

$$x=a\tau^{\frac{2}{n+1}},\quad y=C_1\int\tau^{\frac{1-n}{1+n}}\exp(\mp\tau^2)\,d\tau+C_2,\quad\text{where}\quad A=\mp(n+1)a^{1-n}.$$

4°. Solution for $n=-1$, $l=1$:

$$y=\begin{cases}C_1|x|^{A+1}+C_2 & \text{if}\ \ A\neq 1,\\ C_1\ln|x|+C_2 & \text{if}\ \ A=1.\end{cases}$$

3. $\boldsymbol{y''_{xx}=Ax^ny^m(y'_x)^{\frac{2n+m+3}{n+m+2}}}$.

Solution in parametric form with $n\neq -1$, $m\neq -1$:

$$x=\exp\Bigl[\int\frac{d\tau}{f(\tau)}+C_2\Bigr],\qquad y=\tau\exp\Bigl[-\frac{n+1}{m+1}\int\frac{d\tau}{f(\tau)}-\frac{n+1}{m+1}C_2\Bigr],$$

where the function $f=f(\tau)$ is defined implicitly by the formula

$$[f+(\sigma-1)\tau](f+\sigma\tau)^{\frac{\sigma}{1-\sigma}}=C_1+\frac{A\tau^{m+2}}{n+m+2},\qquad \sigma=-\frac{n+1}{m+1}.$$

For the case $n=-1$, see equation 14.5.2.5. For $m=-1$, see equation 14.5.2.4.

4. $\boldsymbol{y''_{xx}=Ax^ny^{-1}(y'_x)^2}$.

Solution in parametric form with $n\neq -1$, $n\neq 0$:

$$x=a\tau^{\frac{1}{n}},\quad y=\pm\exp\Bigl[\int\tau^{\frac{1-n}{n}}\Bigl(\frac{n}{n+1}\tau^{\frac{n+1}{n}}+n\tau^{\frac{1}{n}}+C_1\Bigr)^{-1}d\tau+C_2\Bigr],$$

where $A=-a^{-n}$.

For the case $n=-1$, see equation 14.5.2.7. For $n=0$, see equation 14.5.2.1.

5. $\boldsymbol{y''_{xx}=Ax^{-1}y^my'_x}$.

Solution in parametric form with $m\neq -1$, $m\neq 0$:

$$x=\pm\exp\Bigl[\int\tau^{\frac{1-m}{m}}\Bigl(\frac{m}{m+1}\tau^{\frac{m+1}{m}}+m\tau^{\frac{1}{m}}+C_1\Bigr)^{-1}d\tau+C_2\Bigr],\qquad y=(A\tau)^{\frac{1}{m}}.$$

For the case $m=-1$, see equation 14.5.2.8. For $m=0$, see equation 14.5.2.2.

6. $\boldsymbol{y''_{xx}=Ax^{-1}y^{-1}(y'_x)^l}$.

Solution in parametric form with $l\neq 1$, $l\neq 2$:

$$x=\pm\Bigl(f-\frac{\tau}{\lambda}\Bigr)^{-1}\exp\Bigl[\frac{1}{\lambda}\int\frac{d\tau}{f(\tau)}+C_2\Bigr],\quad y=\pm\exp\Bigl[\int\frac{d\tau}{f(\tau)}+\lambda C_2\Bigr],\quad \lambda=\frac{l-1}{l-2},$$

where the function $f=f(\tau)$ is defined implicitly by the formula

$$\ln\Bigl(\frac{f}{\tau}-\frac{1}{\lambda}\Bigr)-\frac{\tau}{\lambda f-\tau}=\pm\frac{A}{\lambda}\tau^\lambda-\ln\tau+C_1,\qquad \lambda=\frac{l-1}{l-2}.$$

For the case $l=2$, see equation 14.5.2.7. For $l=1$, see equation 14.5.2.8.

7. $y''_{xx} = Ax^{-1}y^{-1}(y'_x)^2$.

Solution in parametric form:

$$x = \pm e^\tau, \quad y = C_2(\mp A\tau + e^\tau + C_1)\exp\Big[\pm A\int(\mp A\tau + e^\tau + C_1)^{-1}\,d\tau\Big].$$

8. $y''_{xx} = Ax^{-1}y^{-1}y'_x$.

Solution in parametric form:

$$x = C_2(\pm A\tau + e^\tau + C_1)\exp\Big[\mp A\int(\pm A\tau + e^\tau + C_1)^{-1}\,d\tau\Big], \quad y = \pm e^\tau.$$

9. $y''_{xx} = Ax^n y^{-n-3}(y'_x)^3$.

Solution in parametric form with $n \neq -1$:

$$x = aC_1^{n+1}\tau\Big(\int\frac{d\tau}{\sqrt{1\pm\tau^{n+1}}} + C_2\Big)^{-1}, \quad y = bC_1^{n-1}\Big(\int\frac{d\tau}{\sqrt{1\pm\tau^{n+1}}} + C_2\Big)^{-1},$$

where $A = \mp\dfrac{n+1}{2}a^{1-n}b^{n+1}$.

For the case $n = -1$, see equation 14.5.2.15.

10. $y''_{xx} = Axy^m(y'_x)^{\frac{3m+5}{2m+3}}$.

Solution in parametric form with $m \neq -3/2$:

$$x = aC_1^{-2}\Big\{(1\pm\tau^{\mu+1})^{1/2}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big] - \tau\Big\},$$
$$y = bC_1^{(\mu+1)(\mu-2)}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big]^{\mu+2},$$

where $\mu = -\dfrac{2m+3}{m+1}$, $A = -\dfrac{\mu b^{\frac{1}{\mu+2}}}{(\mu+2)a}\Big[\pm\dfrac{(\mu+1)a}{2(\mu+2)b}\Big]^{\frac{1}{\mu}}$.

11. $y''_{xx} = Ax^n y^{-\frac{1}{2}}(y'_x)^{\frac{3n+4}{2n+3}}$.

Solution in parametric form with $n \neq -3/2$:

$$x = aC_1^{(\mu+1)^2}\tau^{\mu+1}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big]^{-\mu-1},$$
$$y = bC_1^4\Big\{(1\pm\tau^{\mu+1})^{1/2}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big] - \tau\Big\}^2,$$

where $\mu = -\dfrac{n}{n+1}$, $A = \dfrac{\mu+3}{a(\mu+1)}a^{\frac{\mu}{\mu+1}}b^{\frac{1}{2}}\Big(\pm\dfrac{a}{b}\Big)^{\frac{1}{\mu+3}}$.

12. $y''_{xx} = Ax^n y(y'_x)^{\frac{3n+4}{2n+3}}$.

Solution in parametric form with $n \neq -3/2$:

$$x = aC_1^{(\mu-1)(\mu+2)}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big]^{\mu+2},$$
$$y = bC_1^{-2}\Big\{(1\pm\tau^{\mu+1})^{1/2}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big] - \tau\Big\},$$

where $\mu = -\dfrac{2n+3}{n+1}$, $A = \dfrac{\mu a^{\frac{1}{\mu+2}}}{(\mu+2)b}\Big[\pm\dfrac{(\mu+1)b}{2(\mu+2)a}\Big]^{\frac{1}{\mu}}$.

13. $y''_{xx} = Ax^{-\frac{1}{2}}y^m(y'_x)^{\frac{3m+5}{2m+3}}$**.**

Solution in parametric form with $m \neq -3/2$:

$$x = aC_1^4\Big\{(1\pm\tau^{\mu+1})^{1/2}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big] - \tau\Big\}^2,$$
$$y = bC_1^{(\mu+1)^2}\tau^{\mu+1}\Big[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\Big]^{-\mu-1},$$

where $\mu = -\dfrac{m}{m+1}$, $A = -\dfrac{\mu+3}{b(\mu+1)}a^{\frac{1}{2}}b^{\frac{\mu}{\mu+1}}\Big(\pm\dfrac{b}{a}\Big)^{\frac{1}{\mu+3}}$.

14. $y''_{xx} = Axy^{-2}y'_x$**.**

Solution in parametric form:

$$x = aC_1\Big\{2\tau\Big[\int\exp(\mp\tau^2)\,d\tau + C_2\Big] \pm \exp(\mp\tau^2)\Big\}, \qquad y = bC_1\Big[\int\exp(\mp\tau^2)\,d\tau + C_2\Big],$$

where $A = \mp\frac{1}{2}a^{-2}b^2$.

15. $y''_{xx} = Ax^{-1}y^{-2}(y'_x)^3$**.**

Solution in parametric form:

$$x = a\exp(\mp\tau^2)\Big[\int\exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1}, \qquad y = C_1\Big[\int\exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1},$$

where $A = \pm 2a^2$.

16. $y''_{xx} = Ax^{-2}y(y'_x)^2$**.**

Solution in parametric form:

$$x = aC_1\Big[\int\exp(\mp\tau^2)\,d\tau + C_2\Big], \qquad y = bC_1\Big\{2\tau\Big[\int\exp(\mp\tau^2)\,d\tau + C_2\Big] \pm \exp(\mp\tau^2)\Big\},$$

where $A = \pm a^2b^{-2}$.

◆ *In the solutions of equations 17–33, the following notation is used:*

$$P_2 = \pm(\tau^2-1), \quad P_3 = \tau^3 - 3\tau + C_2, \quad P_4 = \pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3),$$
$$P_6 = \pm(\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2),$$
$$P_9 = 7\tau^9 - 108\tau^7 + 84C_2\tau^6 + 378\tau^5 - 756C_2\tau^4 + 84(4C_2^2+9)\tau^3$$
$$- 756C_2\tau^2 + 567\tau + 4(4C_2^2 - 27)C_2.$$

17. $y''_{xx} = Axy^{-1/2}(y'_x)^{7/5}$**.**

Solution in parametric form:

$$x = aC_1P_2P_3^{-1/2}, \qquad y = bC_1^{16}P_4^{\,2},$$

where $A = \pm 15a^{-2}b^{1/2}\Big(\dfrac{a}{16b}\Big)^{2/5}$.

18. $y''_{xx} = Axy^{-7/4}(y'_x)^{7/5}$.

Solution in parametric form:

$$x = aC_1^{27} P_3{}^{-1/2} P_6, \quad y = bC_1^{32} P_4{}^{4/3},$$

where $A = \pm\frac{5}{12}a^{-2}b^{7/4}\Big(\frac{a}{9b}\Big)^{2/5}$.

19. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{10/7}$.

Solution in parametric form:

$$x = aC_1^{-1} P_3{}^{-1} P_4{}^{2/3}, \quad y = bC_1^{27} P_3{}^{-1} P_6{}^2,$$

where $A = 28a(ab)^{1/2}\Big(\frac{a}{27b}\Big)^{3/7}$.

20. $y''_{xx} = Ax^{-1/2}(y'_x)^3$.

Solution in parametric form:

$$x = aC_1^4 P_2{}^2, \quad y = bC_1^3 P_3, \quad \text{where} \quad A = \pm\tfrac{4}{9}a^{3/2}b^{-2}.$$

21. $y''_{xx} = Ay(y'_x)^3$.

Solution in parametric form:

$$x = aC_1^3 P_3, \quad y = bC_1\tau, \quad \text{where} \quad A = -6ab^{-3}.$$

22. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^3$.

Solution in parametric form:

$$x = aC_1 P_2{}^2 P_3{}^{-1}, \quad y = bC_1^{-3} P_3{}^{-1},$$

where $A = \mp\frac{4}{9}a^{3/2}b^{1/2}$.

23. $y''_{xx} = Ax^{-5/3}y(y'_x)^3$.

Solution in parametric form:

$$x = aC_1^9 P_3{}^{3/2}, \quad y = bC_1^8 P_4, \quad \text{where} \quad A = \mp\tfrac{9}{64}a^{8/3}b^{-3}.$$

24. $y''_{xx} = Ax^{-1/2}y(y'_x)^{8/5}$.

Solution in parametric form:

$$x = aC_1^{16} P_4{}^2, \quad y = bC_1 P_2 P_3{}^{-1/2},$$

where $A = \mp 15a^{1/2}b^{-2}\Big(\frac{b}{16a}\Big)^{2/5}$.

25. $y''_{xx} = Ax^{-5/3}y^{-7/3}(y'_x)^3$.

Solution in parametric form:

$$x = aC_1 P_3^{3/2} P_4^{-1}, \quad y = bC_1^{-8} P_4^{-1},$$

where $A = \mp\frac{9}{64}a^{8/3}b^{1/3}$.

26. $y''_{xx} = Ax^{-7/4}y(y'_x)^{8/5}$.

Solution in parametric form:

$$x = aC_1^{32}P_4{}^{4/3}, \quad y = bC_1^{27}P_3{}^{-1/2}P_6,$$

where $A = \mp\frac{5}{12}a^{7/4}b^{-2}\left(\frac{b}{9a}\right)^{2/5}$.

27. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{11/7}$.

Solution in parametric form:

$$x = aC_1^{27}P_3{}^{-1}P_6{}^2, \quad y = bC_1^{-1}P_3{}^{-1}P_4{}^{2/3},$$

where $A = -28b(ab)^{1/2}\left(\frac{b}{27a}\right)^{3/7}$.

28. $y''_{xx} = Ax^{-11/7}y^{-10/7}(y'_x)^{1/2}$.

Solution in parametric form:

$$x = -aC_1^{-27}P_9^{-1}, \quad y = bC_1P_4^{7/3}P_9^{-1},$$

where $A = \pm\frac{4\sqrt{3}}{1701}a^{1/14}b^{27/14}$.

29. $y''_{xx} = Ax^{-10/7}y^{-11/7}(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aC_1P_4^{7/3}P_9^{-1}, \quad y = -bC_1^{-27}P_9^{-1},$$

where $A = \mp\frac{4\sqrt{3}}{1701}a^{27/14}b^{1/14}$.

30. $y''_{xx} = Ax^2y^{-11/4}(y'_x)^{13/10}$.

Solution in parametric form:

$$x = -aC_1^{49}P_4^{-1/3}P_6, \quad y = bC_1^{54}P_9^{4/7},$$

where $A = \pm\frac{5\cdot 2^{7/10}3^{1/10}}{6}a^{-27/10}b^{49/20}$.

31. $y''_{xx} = Ax^{-11/4}y^2(y'_x)^{17/10}$.

Solution in parametric form:

$$x = aC_1^{54}P_9^{4/7}, \quad y = \mp bC_1^{49}P_4^{-1/3}P_6,$$

where $A = \mp\frac{5\cdot 2^{7/10}3^{1/10}}{6}a^{49/20}b^{-27/10}$.

32. $y''_{xx} = Ax^{-10/3}y^{-2/3}(y'_x)^{18/11}$.

Solution in parametric form:

$$x = aC_1^{-3}P_4^{-1}P_9^{3/7}, \quad y = bC_1^{294}P_4^{-1}P_6^3,$$

where $A = \mp\frac{2^{4/11}99}{2}a^{98/33}b^{1/33}$.

33. $\boldsymbol{y''_{xx} = Ax^{-2/3} y^{-10/3} (y'_x)^{15/11}}$**.**

Solution in parametric form:

$$x = aC_1^{294} P_4^{-1} P_6^3, \qquad y = bC_1^{-3} P_4^{-1} P_9^{3/7},$$

where $A = \pm \dfrac{2^{4/11} 99}{2} a^{1/33} b^{98/33}$.

34. $\boldsymbol{y''_{xx} = Axy^{-2} (y'_x)^3}$**.**

1°. Solution in parametric form with $A < \frac{1}{4}$:

$$x = \tau(C_1 \tau^\nu + C_2 \tau^{-\nu}), \quad y = \tau^2, \qquad \text{where} \quad \nu = \sqrt{1 - 4A}.$$

2°. Solution in parametric form with $A = \frac{1}{4}$:

$$x = \tau(C_1 \ln|\tau| + C_2), \quad y = \tau^2.$$

3°. Solution in parametric form with $A > \frac{1}{4}$:

$$x = \tau C_1 \sin(\nu \ln \tau + C_2), \quad y = \tau^2, \qquad \text{where} \quad \nu = \sqrt{4A - 1}.$$

35. $\boldsymbol{y''_{xx} = Ax^{-1/2} y^{-1/2} (y'_x)^{3/2}}$**.**

Solution in parametric form:

$$x = \pm\tau^2 (C_1 \tau^\nu + C_2 \tau^{-\nu})^2, \quad y = \tfrac{1}{4} \tau^{-2} [(1 + \nu) C_1 \tau^\nu + (1 - \nu) C_2 \tau^{-\nu}]^2,$$

where $A = \mp k^2$, $\nu = k^{-2} (k^4 + 4)^{1/2}$.

36. $\boldsymbol{y''_{xx} = Ax^{-1/2} y (y'_x)^3}$**.**

Solution in parametric form:

$$x = aC_1^2 \exp(-2\tau) \bigl[2\exp(3\tau) - C_2 \sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\, C_2 \cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2,$$
$$y = bC_1 \exp(-\tau) \bigl[\exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr)\bigr],$$

where $A = -16 a^{3/2} b^{-3}$.

37. $\boldsymbol{y''_{xx} = Ax^{-1/2} y^{-7/2} (y'_x)^3}$**.**

Solution in parametric form:

$$x = \frac{aC_1 e^{-\tau} \bigl[2\exp(3\tau) - C_2 \sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\, C_2 \cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2}{\exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr)}, \quad y = \frac{bC_1^{-1} e^\tau}{\exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr)},$$

where $A = -16 (ab)^{3/2}$.

38. $\boldsymbol{y''_{xx} = Axy^{-2/3} (y'_x)^{7/5}}$**.**

1°. Solution in parametric form with $A < 0$:

$$x = aC_1 [\cosh(\tau + C_2) \cos\tau]^{1/2} [\tanh(\tau + C_2) - \tan\tau],$$
$$y = bC_1^6 \cosh^3(\tau + C_2) \cos^3\tau\, [\tanh(\tau + C_2) + \tan\tau]^3,$$

where $A = -5 a^{-2} b^{2/3} \Bigl(\dfrac{a}{12b}\Bigr)^{2/5}$.

2°. Solution in parametric form with $A > 0$:

$$x = aC_1 [\cosh\tau - \sin(\tau + C_2)]^{-1/2} [\sinh\tau - \cos(\tau + C_2)], \quad y = bC_1^6 [\sinh\tau + \cos(\tau + C_2)]^3,$$

where $A = 5 a^{-2} b^{2/3} \Bigl(\dfrac{a}{6b}\Bigr)^{2/5}$.

39. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{1/2}.$

Solution in parametric form:

$$x = aC_1^{-1}[\cosh(\tau+C_2)\cos\tau]^{-1}, \quad y = bC_1\cosh(\tau+C_2)\cos\tau\,[\tanh(\tau+C_2)-\tan\tau]^2,$$

where $A=-4ab$.

40. $y''_{xx} = Ax^{-5/3}y^2(y'_x)^3.$

1°. Solution in parametric form with $A>0$:

$$x = aC_1^3[\cosh(\tau+C_2)\cos\tau]^{3/2}, \quad y = bC_1^2\cosh(\tau+C_2)\cos\tau\,[\tanh(\tau+C_2)+\tan\tau],$$

where $A=\frac{3}{16}a^{8/3}b^{-4}$.

2°. Solution in parametric form with $A<0$:

$$x = aC_1^3[\cosh\tau-\sin(\tau+C_2)]^{3/2}, \quad y = bC_1^2[\sinh\tau+\cos(\tau+C_2)],$$

where $A=-\frac{3}{4}a^{8/3}b^{-4}$.

41. $y''_{xx} = Ax^{-2/3}y(y'_x)^{8/5}.$

1°. Solution in parametric form with $A>0$:

$$x = aC_1^6\cosh^3(\tau+C_2)\cos^3\tau[\tanh(\tau+C_2)+\tan\tau]^3,$$
$$y = bC_1[\cosh(\tau+C_2)\cos\tau]^{1/2}[\tanh(\tau+C_2)-\tan\tau],$$

where $A=5a^{2/3}b^{-2}\left(\dfrac{b}{12a}\right)^{2/5}$.

2°. Solution in parametric form with $A<0$:

$$x=aC_1^6[\sinh\tau+\cos(\tau+C_2)]^3, \quad y=bC_1[\cosh\tau-\sin(\tau+C_2)]^{-1/2}[\sinh\tau-\cos(\tau+C_2)],$$

where $A=-5a^{2/3}b^{-2}\left(\dfrac{b}{6a}\right)^{2/5}$.

42. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{5/2}.$

Solution in parametric form:

$$x = aC_1\cosh(\tau+C_2)\cos\tau\,[\tanh(\tau+C_2)-\tan\tau]^2, \quad y = bC_1^{-1}[\cosh(\tau+C_2)\cos\tau]^{-1},$$

where $A=4ab$.

43. $y''_{xx} = Ax^{-5/3}y^{-10/3}(y'_x)^3.$

1°. Solution in parametric form with $A>0$:

$$x = aC_1[\cosh(\tau+C_2)\cos\tau]^{1/2}[\tanh(\tau+C_2)+\tan\tau]^{-1},$$
$$y = bC_1^{-2}[\cosh(\tau+C_2)\cos\tau]^{-1}[\tanh(\tau+C_2)+\tan\tau]^{-1},$$

where $A=\frac{3}{16}a^{8/3}b^{4/3}$.

2°. Solution in parametric form with $A<0$:

$$x = aC_1[\cosh\tau-\sin(\tau+C_2)]^{3/2}[\sinh\tau+\cos(\tau+C_2)]^{-1},$$
$$y = bC_1^{-2}[\sinh\tau+\cos(\tau+C_2)]^{-1},$$

where $A=-\frac{3}{4}a^{8/3}b^{4/3}$.

◆ *In the solutions of equations 44–51, the following notation is used:*

$$E = \exp(3\tau), \quad S_1 = E + C_2 \sin(\sqrt{3}\,\tau), \quad S_2 = 2E - C_2 \sin(\sqrt{3}\,\tau) + \sqrt{3}\,C_2 \cos(\sqrt{3}\,\tau),$$
$$S_3 = 2S_1(S_2)'_\tau - (S_1)'_\tau S_2 - S_1 S_2, \quad S_4 = 2S_1(S_3)'_\tau - 5(S_1)'_\tau S_3 + S_1 S_3.$$

44. $y''_{xx} = Axy^{-1/2}(y'_x)^{9/7}$.

Solution in parametric form:

$$x = aC_1 E^{-1/6} S_1^{-1/2} S_2, \quad y = bC_1^8 E^{-4/3} S_3^2, \quad \text{where } A = 7a^{-2} b^{1/2} \left(\frac{a}{64b}\right)^{2/7}.$$

45. $y''_{xx} = Axy^{-13/8}(y'_x)^{9/7}$.

Solution in parametric form:

$$x = aC_1^{25} E^{-5/6} S_1^{-1/2} S_4, \quad y = bC_1^{32} E^{-16/15} S_3^{8/5}, \quad \text{where } A = 7a^{-2} b^{13/8} \left(\frac{25a}{256b}\right)^{2/7}.$$

46. $y''_{xx} = Ax^{-7/2} y^{-1/2}(y'_x)^{18/13}$.

Solution in parametric form:

$$x = aC_1^{-1} E^{1/15} S_1^{-1} S_3^{2/5}, \quad y = bC_1^{25} E^{-5/3} S_1^{-1} S_4^2, \quad \text{where } A = -208a^{5/2} b^{1/2} \left(\frac{a}{25b}\right)^{5/13}.$$

47. $y''_{xx} = Ax^{-7/5} y (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^5 E^{-5/6} S_1^{5/2}, \quad y = bC_1^4 E^{-2/3} S_3, \quad \text{where } A = -\tfrac{5}{1024} a^{12/5} b^{-3}.$$

48. $y''_{xx} = Ax^{-1/2} y (y'_x)^{12/7}$.

Solution in parametric form:

$$x = aC_1^8 E^{-4/3} S_3^2, \quad y = bC_1 E^{-1/6} S_1^{-1/2} S_2, \quad \text{where } A = -7a^{1/2} b^{-2} \left(\frac{b}{64a}\right)^{2/7}.$$

49. $y''_{xx} = Ax^{-7/5} y^{-13/5} (y'_x)^3$.

Solution in parametric form:

$$x = aC_1 E^{-1/6} S_1^{5/2} S_3^{-1}, \quad y = bC_1^{-4} E^{2/3} S_3^{-1}, \quad \text{where } A = -\tfrac{5}{1024} a^{12/5} b^{3/5}.$$

50. $y''_{xx} = Ax^{-13/8} y (y'_x)^{12/7}$.

Solution in parametric form:

$$x = aC_1^{32} E^{-16/15} S_3^{8/5}, \quad y = bC_1^{25} E^{-5/6} S_1^{-1/2} S_4, \quad \text{where } A = -7a^{13/8} b^{-2} \left(\frac{25b}{256a}\right)^{2/7}.$$

51. $y''_{xx} = Ax^{-1/2} y^{-7/2} (y'_x)^{21/13}$.

Solution in parametric form:

$$x = aC_1^{25} E^{-5/3} S_1^{-1} S_4^2, \quad y = bC_1^{-1} E^{1/15} S_1^{-1} S_3^{2/5}, \quad \text{where } A = 208a^{1/2} b^{5/2} \left(\frac{b}{25a}\right)^{5/13}.$$

◆ *In the solutions of equations 52–61, the following notation is used:*

$$T_1 = \cosh(\tau + C_2)\cos\tau, \qquad T_2 = \tanh(\tau + C_2) + \tan\tau, \quad T_3 = \tanh(\tau + C_2) - \tan\tau,$$
$$\theta_1 = \cosh\tau - \sin(\tau + C_2), \quad \theta_2 = \sinh\tau + \cos(\tau + C_2), \quad \theta_3 = \sinh\tau - \cos(\tau + C_2),$$
$$T_4 = 3T_2T_3 - 4, \qquad \theta_4 = 3\theta_2\theta_3 - 2\theta_1^2.$$

52. $y''_{xx} = Axy^{-10/7}(y'_x)^{7/5}$.

1°. Solution in parametric form with $A < 0$:

$$x = aC_1^9 T_1^{3/2} T_4, \quad y = bC_1^{14} T_1^{7/3} T_2^{7/3}, \qquad \text{where} \quad A = -\frac{5}{9}a^{-2}b^{10/7}\Bigl(\frac{9a}{28b}\Bigr)^{2/5}.$$

2°. Solution in parametric form with $A > 0$:

$$x = aC_1^9 \theta_1^{-1/2}\theta_4, \quad y = bC_1^{14}\theta_2^{7/3}, \qquad \text{where} \quad A = \frac{5}{9}a^{-2}b^{10/7}\Bigl(\frac{9a}{14b}\Bigr)^{2/5}.$$

53. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{13/10}$.

Solution in parametric form:

$$x = aC_1^{-1} T_1^{-1/3} T_2^{2/3}, \quad y = bC_1^9 T_1^3 T_4^2, \qquad \text{where} \quad A = -20a(ab)^{1/2}\Bigl(\frac{a}{9b}\Bigr)^{3/10}.$$

54. $y''_{xx} = Ax^{-10/7}y(y'_x)^{8/5}$.

1°. Solution in parametric form with $A > 0$:

$$x = aC_1^{14} T_1^{7/3} T_2^{7/3}, \quad y = bC_1^9 T_1^{3/2} T_4, \qquad \text{where} \quad A = \frac{5}{9}a^{10/7}b^{-2}\Bigl(\frac{9b}{28a}\Bigr)^{2/5}.$$

2°. Solution in parametric form with $A < 0$:

$$x = aC_1^{14}\theta_2^{7/3}, \quad y = bC_1^9\theta_1^{-1/2}\theta_4, \qquad \text{where} \quad A = -\frac{5}{9}a^{10/7}b^{-2}\Bigl(\frac{9b}{14a}\Bigr)^{2/5}.$$

55. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{17/10}$.

Solution in parametric form:

$$x = aC_1^9 T_1^3 T_4^2, \quad y = bC_1^{-1} T_1^{-1/3} T_2^{2/3}, \qquad \text{where} \quad A = 20b(ab)^{1/2}\Bigl(\frac{b}{9a}\Bigr)^{3/10}.$$

56. $y''_{xx} = Ax^{-17/10}y^{-13/10}(y'_x)^{1/2}$.

Solution in parametric form:

$$x = -aC_1^{-9} T_1^{-3}(T_4 + 3T_2^2)^{-1}(T_4 - 3T_2^2)^{-1},$$
$$y = -bC_1 T_1^{1/3} T_2^{10/3}(T_4 + 3T_2^2)^{-1}(T_4 - 3T_2^2)^{-1},$$

where $A = \dfrac{1}{540}a^{1/5}b^{9/5}$.

57. $y''_{xx} = Ax^{-13/10}y^{-17/10}(y'_x)^{5/2}$.

Solution in parametric form:

$$x = -aC_1T_1^{1/3}T_2^{10/3}(T_4 + 3T_2^2)^{-1}(T_4 - 3T_2^2)^{-1},$$
$$y = -bC_1^{-9}T_1^{-3}(T_4 + 3T_2^2)^{-1}(T_4 - 3T_2^2)^{-1},$$

where $A = -\dfrac{1}{540}a^{9/5}b^{1/5}$.

58. $y''_{xx} = Ax^2y^{-17/7}(y'_x)^{16/13}$.

1°. Solution in parametric form with $A < 0$:

$$x = -aC_1^{50}T_1^{5/3}T_2^{-1/3}T_4,$$
$$y = -bC_1^{63}T_1^{21/10}(T_4 + 3T_2^2)^{7/10}(T_4 - 3T_2^2)^{7/10},$$

where $A = -\dfrac{13}{63}252^{10/13}a^{-36/13}b^{200/91}$.

2°. Solution in parametric form with $A > 0$:

$$x = -aC_1^{50}\theta_2^{-1/3}\theta_4,$$
$$y = bC_1^{63}\theta_1^{-7/10}(4\theta_1^4 + 9\theta_2^4 - 12\theta_1^2\theta_2\theta_3 + 9\theta_2^2\theta_3^2)^{7/10},$$

where $A = \dfrac{13}{63}126^{10/13}a^{-36/13}b^{200/91}$.

59. $y''_{xx} = Ax^{-17/7}y^2(y'_x)^{23/13}$.

1°. Solution in parametric form with $A > 0$:

$$x = -aC_1^{63}T_1^{21/10}(T_4 + 3T_2^2)^{7/10}(T_4 - 3T_2^2)^{7/10},$$
$$y = -bC_1^{50}T_1^{5/3}T_2^{-1/3}T_4,$$

where $A = \dfrac{13}{63}252^{10/13}a^{200/91}b^{-36/13}$.

2°. Solution in parametric form with $A < 0$:

$$x = aC_1^{63}\theta_1^{-7/10}(4\theta_1^4 + 9\theta_2^4 - 12\theta_1^2\theta_2\theta_3 + 9\theta_2^2\theta_3^2)^{7/10},$$
$$y = -bC_1^{50}\theta_2^{-1/3}\theta_4,$$

where $A = -\dfrac{13}{63}126^{10/13}a^{200/91}b^{-36/13}$.

60. $y''_{xx} = Ax^{-13/3}y^{-2/3}(y'_x)^{27/17}$.

1°. Solution in parametric form with $A > 0$:

$$x = -aC_1^{-3}T_1^{-1/10}T_2^{-1}(T_4 + 3T_2^2)^{3/10}(T_4 - 3T_2^2)^{3/10},$$
$$y = bC_1^{50}T_1^5T_2^{-1}T_4^3,$$

where $A = 102 \cdot 2^{3/17}a^{200/51}b^{4/51}$.

2°. Solution in parametric form with $A<0$:

$$x=-aC_1^{-3}\theta_1^{-3/10}\theta_2^{-1}(4\theta_1^4+9\theta_2^4-12\theta_1^2\theta_2\theta_3+9\theta_2^2\theta_3^2)^{3/10},$$
$$y=-bC_1^{50}\theta_2^{-1}\theta_4^3,$$

where $A=-51\cdot 2^{10/17}a^{200/51}b^{4/51}$.

61. $y''_{xx}=Ax^{-2/3}y^{-13/3}(y'_x)^{24/17}$.

1°. Solution in parametric form with $A<0$:

$$x=aC_1^{50}T_1^5T_2^{-1}T_4^3,$$
$$y=-bC_1^{-3}T_1^{-1/10}T_2^{-1}(T_4+3T_2^2)^{3/10}(T_4-3T_2^2)^{3/10},$$

where $A=-102\cdot 2^{3/17}a^{4/51}b^{200/51}$.

2°. Solution in parametric form with $A>0$:

$$x=-aC_1^{50}\theta_2^{-1}\theta_4^3,$$
$$y=-bC_1^{-3}\theta_1^{-3/10}\theta_2^{-1}(4\theta_1^4+9\theta_2^4-12\theta_1^2\theta_2\theta_3+9\theta_2^2\theta_3^2)^{3/10},$$

where $A=51\cdot 2^{10/17}a^{4/51}b^{200/51}$.

◆ *In the solutions of equations 62–113, the following notation is used:*

$$R=\sqrt{\pm(4\tau^3-1)},\quad F_1=2\tau I(\tau)+C_2\tau\mp R,\quad F_2=\tau^{-1}(RF_1-1),\quad F_3=4\tau F_1^2\mp F_2^2,$$

where $I(\tau)=\int\frac{\tau\,d\tau}{R}$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

62. $y''_{xx}=Ax(y'_x)^{7/4}$.

Solution in parametric form:

$$x=aC_1^{-3}R,\quad y=bC_1^5\tau^{-1}F_1,\qquad \text{where}\quad A=\mp\tfrac{2}{3}a^{-2}(\mp 6a/b)^{3/4}.$$

63. $y''_{xx}=Axy^{-1/2}(y'_x)^{7/4}$.

Solution in parametric form:

$$x=aC_1^{-1}F_2,\quad y=bC_1^5\tau^2F_1^{-2},\qquad \text{where}\quad A=\mp\tfrac{2}{3}a^{-2}b^{1/2}(\pm 3a/b)^{3/4}.$$

64. $y''_{xx}=Ax^{-1/2}y(y'_x)^{8/7}$.

Solution in parametric form:

$$x=aC_1^{-16}F_3^2,\quad y=bC_1^5F_1^{-3/2}F_2,\qquad \text{where}\quad A=\mp\frac{7}{16}a^{-1/2}b^{-1}\left(\frac{16a}{3b}\right)^{1/7}.$$

65. $y''_{xx}=Ax^{-7/6}y^{-1/2}(y'_x)^{2/3}$.

Solution in parametric form:

$$x=aC_1^5F_1^{-3}F_3^6,\quad y=bC_1F_1^{-3}(F_2F_3-8F_1^2)^2,\qquad \text{where}\quad A=\mp 4a^{-5/6}b^{3/2}(a/b)^{2/3}.$$

66. $y''_{xx} = Ax^{-3/4} y (y'_x)^{8/7}$.

Solution in parametric form:

$$x = aC_1^{-32} F_3^{-4}, \quad y = bC_1^3 F_1^{-3/2}(F_2 F_3 - 8F_1^2), \quad \text{where} \quad A = \mp \frac{7}{32} a^{-1/4} b^{-1} \left(\frac{32a}{3b}\right)^{1/7}.$$

67. $y''_{xx} = Ax^{-4} (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^2 \tau^{-1}, \quad y = bC_1^5 \tau^{-1} F_1, \quad \text{where} \quad A = \pm 6a^5 b^{-2}.$$

68. $y''_{xx} = Ay (y'_x)^{5/4}$.

Solution in parametric form:

$$x = aC_1^5 \tau^{-1} F_1, \quad y = bC_1^{-3} R, \quad \text{where} \quad A = \pm \tfrac{2}{3} b^{-2} (\mp 6b/a)^{3/4}.$$

69. $y''_{xx} = Ax^{-4} y (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^3 F_1^{-1}, \quad y = bC_1^5 \tau F_1^{-1}, \quad \text{where} \quad A = \pm 6a^5 b^{-3}.$$

70. $y''_{xx} = Ax^{-1/2} y (y'_x)^{5/4}$.

Solution in parametric form:

$$x = aC_1^5 \tau^2 F_1^{-2}, \quad y = bC_1^{-1} F_2, \quad \text{where} \quad A = \pm \tfrac{2}{3} a^{1/2} b^{-2} (\pm 3b/a)^{3/4}.$$

71. $y''_{xx} = Ax^{-1/2} y^{-4/3} (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^4 F_2^2, \quad y = bC_1^9 F_1^3, \quad \text{where} \quad A = \mp \tfrac{4}{3} a^{3/2} b^{-2/3}.$$

72. $y''_{xx} = Ax^{-1/2} y^{-7/6} (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^5 F_1^{-3} F_2^2, \quad y = bC_1^9 F_1^{-3}, \quad \text{where} \quad A = \mp \tfrac{4}{3} a^{3/2} b^{-5/6}.$$

73. $y''_{xx} = Axy^{-1/2} (y'_x)^{13/7}$.

Solution in parametric form:

$$x = aC_1^5 F_1^{-3/2} F_2, \quad y = bC_1^{-16} F_3^2, \quad \text{where} \quad A = \pm \frac{7}{16} a^{-1} b^{-1/2} \left(\frac{16b}{3a}\right)^{1/7}.$$

74. $y''_{xx} = Ax^{-7} y (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^3 F_1^{1/2}, \quad y = bC_1^8 F_3, \quad \text{where} \quad A = \mp \tfrac{3}{64} a^8 b^{-3}.$$

75. $y''_{xx} = Ax^{-7} y^3 (y'_x)^3$.

Solution in parametric form:

$$x = aC_1^5 F_1^{1/2} F_3^{-1}, \quad y = bC_1^8 F_3^{-1}, \quad \text{where} \quad A = \mp \tfrac{3}{64} a^8 b^{-5}.$$

76. $y''_{xx} = Ax^{-1/2}y^{-7/6}(y'_x)^{7/3}$.

Solution in parametric form:

$$x = aC_1F_1^{-3}(F_2F_3 - 8F_1^2)^2, \quad y = bC_1^5F_1^{-3}F_3^6, \quad \text{where} \quad A = \pm 4a^{3/2}b^{-5/6}(b/a)^{2/3}.$$

77. $y''_{xx} = Axy^{-3/4}(y'_x)^{13/7}$.

Solution in parametric form:

$$x = aC_1^3F_1^{-3/2}(F_2F_3 - 8F_1^2), \quad y = bC_1^{-32}F_3^{-4}, \quad \text{where} \quad A = \pm\frac{7}{32}a^{-1}b^{-1/4}\left(\frac{32b}{3a}\right)^{1/7}.$$

◆ *In the solutions of equations 78–113, the following notation is used:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2, \qquad f = \sqrt{\pm(4\wp^3 - 1)}.$$

The function $\wp = \wp(\tau)$ is defined implicitly. The upper sign in the formulas corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2, 0, 1)$. The solutions given below are written in parametric form. One can assume as the parameter either τ, hence $\wp = \wp(\tau)$, or $\wp$, hence $\tau = \tau(\wp)$.

78. $y''_{xx} = Ax(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aC_1^{-3}f, \quad y = bC_1\tau, \quad \text{where} \quad A = \mp\frac{2}{b}\left(\pm\frac{6}{ab}\right)^{1/2}.$$

79. $y''_{xx} = Axy^{-5/4}(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aC_1^{-1}(\tau f - \wp), \quad y = bC_1^2\tau^4, \quad \text{where} \quad A = -\frac{1}{2}a^{-1}b^{1/4}\left(\pm\frac{3a}{2b}\right)^{1/2}.$$

80. $y''_{xx} = Ax^{-2/3}y^{-1/2}(y'_x)^{6/5}$.

Solution in parametric form:

$$x = aC_1^9\tau^{-3}\wp^3, \quad y = bC_1^4(\tau f - \wp)^2, \quad \text{where} \quad A = \pm\frac{5}{3}a^{-1/3}b^{1/2}\left(\frac{a}{4b}\right)^{1/5}.$$

81. $y''_{xx} = Axy^{-15/8}(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aC_1^3\tau^{-6}(\tau^3 f + 3\tau^2\wp \mp 1), \quad y = bC_1^4\tau^{-8}, \quad \text{where} \quad A = \tfrac{1}{8}a^{-1}b^{7/8}(\mp 3a/b)^{1/2}.$$

82. $y''_{xx} = Ax^{-2/3}y^{-1/2}(y'_x)^{22/15}$.

Solution in parametric form:

$$x = aC_1^{-1}\tau^3(\tau^2\wp \mp 1)^3, \quad y = bC_1^4\tau^{-12}(\tau^3 f + 3\tau^2\wp \mp 1)^2,$$

where $A = -5a^{-1/3}b^{1/2}\left(\pm\frac{a}{4b}\right)^{7/15}$.

83. $\boldsymbol{y''_{xx} = Axy^{-20/13}(y'_x)^{5/2}}$.

Solution in parametric form:

$$x = aC_1\tau(\tau^3 f - 4\tau^2\wp \pm 6), \quad y = bC_1^{13}\tau^{13}, \quad \text{where} \quad A = \mp\frac{2}{13}a^{-1}b^{7/13}\Big(\pm\frac{6a}{13b}\Big)^{1/2}.$$

84. $\boldsymbol{y''_{xx} = Ax^{-2/3}y^{-1/2}(y'_x)^{27/20}}$.

Solution in parametric form:

$$x = aC_1^{-9}\tau^{-18}(\tau^2\wp \mp 1)^3, \quad y = bC_1\tau^2(\tau^3 f - 4\tau^2\wp \pm 6)^2,$$

where $A = \dfrac{20}{3}a^{-1/3}b^{1/2}\Big(\pm\dfrac{a}{4b}\Big)^{7/20}$.

85. $\boldsymbol{y''_{xx} = Ax(y'_x)^{8/5}}$.

Solution in parametric form:

$$x = aC_1^{-3}f, \quad y = bC_1^7\wp^{-2}(f \pm 2\tau\wp^2), \quad \text{where} \quad A = \mp\tfrac{5}{6}a^{-2}(3a/b)^{3/5}.$$

86. $\boldsymbol{y''_{xx} = Axy(y'_x)^{8/5}}$.

Solution in parametric form:

$$x = aC_1^{-8}(\tau f + 2\wp), \quad y = bC_1^7\wp(f \pm 2\tau\wp^2)^{-1/2}, \quad \text{where} \quad A = \tfrac{10}{3}a^{-2}b^{-1}(3a/b)^{3/5}.$$

87. $\boldsymbol{y''_{xx} = Axy^5(y'_x)^{7/5}}$.

Solution in parametric form:

$$x = aC_1^{-27}(\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{-1/2}, \quad y = bC_1^8(\tau f + 2\wp)^{-1/3},$$

where $A = -10a^{-2}b^{-5}(a/b)^{2/5}$.

88. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{4/5}}$.

Solution in parametric form:

$$x = aC_1^{27}(\tau^2\wp \mp 1)^2(f \pm 2\tau\wp^2)^{-1}, \quad y = bC_1^7(f \pm 2\tau\wp^2)^{-1}(\tau f + 2\wp)^{4/3},$$

where $A = -5a^{-3/2}b^{7/2}\Big(\dfrac{a}{2b}\Big)^{4/5}$.

89. $\boldsymbol{y''_{xx} = Ax^2(y'_x)^3}$.

Solution in parametric form:

$$x = aC_1^2\wp, \quad y = bC_1^{-1}\tau, \quad \text{where} \quad A = \mp 6a^{-1}b^{-2}.$$

90. $\boldsymbol{y''_{xx} = Ay(y'_x)^{1/2}}$.

Solution in parametric form:

$$x = aC_1\tau, \quad y = bC_1^{-3}f, \quad \text{where} \quad A = \pm\frac{2}{a}\Big(\pm\frac{6}{ab}\Big)^{1/2}.$$

91. $y''_{xx} = Ax^2 y^{-5} (y'_x)^3.$

Solution in parametric form:

$$x = aC_1^3 \tau^{-1} \wp, \quad y = bC_1 \tau^{-1}, \qquad \text{where} \quad A = \mp 6a^{-1} b^3.$$

92. $y''_{xx} = Ax^{-5/4} y (y'_x)^{1/2}.$

Solution in parametric form:

$$x = aC_1^2 \tau^4, \quad y = bC_1^{-1} (\tau f - \wp), \qquad \text{where} \quad A = \frac{1}{2} a^{1/4} b^{-1} \Big(\pm \frac{3b}{2a}\Big)^{1/2}.$$

93. $y''_{xx} = Ax^{-1/2} y^{-2/3} (y'_x)^{9/5}.$

Solution in parametric form:

$$x = aC_1^4 (\tau f - \wp)^2, \quad y = bC_1^9 \tau^{-3} \wp^3, \qquad \text{where} \quad A = \mp \frac{5}{3} a^{1/2} b^{-1/3} \Big(\frac{b}{4a}\Big)^{1/5}.$$

94. $y''_{xx} = Ax^2 y^{-15/7} (y'_x)^3.$

Solution in parametric form:

$$x = aC_1 \tau (\tau^2 \wp \mp 1), \quad y = bC_1^7 \tau^7, \qquad \text{where} \quad A = \mp \tfrac{6}{49} a^{-1} b^{1/7}.$$

95. $y''_{xx} = Ax^{-15/8} y (y'_x)^{1/2}.$

Solution in parametric form:

$$x = aC_1^4 \tau^{-8}, \quad y = bC_1^3 \tau^{-6} (\tau^3 f + 3\tau^2 \wp \mp 1), \qquad \text{where} \quad A = -\tfrac{1}{8} a^{7/8} b^{-1} (\mp 3b/a)^{1/2}.$$

96. $y''_{xx} = Ax^{-1/2} y^{-2/3} (y'_x)^{23/15}.$

Solution in parametric form:

$$x = aC_1^4 \tau^{-12} (\tau^3 f + 3\tau^2 \wp \mp 1)^2, \quad y = bC_1^{-1} \tau^3 (\tau^2 \wp \mp 1)^3,$$

where $A = 5a^{1/2} b^{-1/3} \Big(\pm \dfrac{b}{4a}\Big)^{7/15}.$

97. $y''_{xx} = Ax^2 y^{-20/7} (y'_x)^3.$

Solution in parametric form:

$$x = aC_1^6 \tau^{-6} (\tau^2 \wp \mp 1), \quad y = bC_1^7 \tau^{-7}, \qquad \text{where} \quad A = \mp \tfrac{6}{49} a^{-1} b^{6/7}.$$

98. $y''_{xx} = Ax^{-20/13} y (y'_x)^{1/2}.$

Solution in parametric form:

$$x = aC_1^{13} \tau^{13}, \quad y = bC_1 \tau (\tau^3 f - 4\tau^2 \wp \pm 6), \qquad \text{where} \quad A = \pm \frac{2}{13} a^{7/13} b^{-1} \Big(\pm \frac{6b}{13a}\Big)^{1/2}.$$

99. $y''_{xx} = Ax^{-1/2} y^{-2/3} (y'_x)^{33/20}.$

Solution in parametric form:

$$x = aC_1 \tau^2 (\tau^3 f - 4\tau^2 \wp \pm 6)^2, \quad y = bC_1^{-9} \tau^{-18} (\tau^2 \wp \mp 1)^3,$$

where $A = -\dfrac{20}{3} a^{1/2} b^{-1/3} \Big(\pm \dfrac{b}{4a}\Big)^{7/20}.$

100. $y''_{xx} = Ax^{-5/2}(y'_x)^3.$

Solution in parametric form:

$$x = aC_1^4 \wp^{-2}, \quad y = bC_1^7 \wp^{-2}(f \pm 2\tau\wp^2), \qquad \text{where} \quad A = \pm 3a^{7/2}b^{-2}.$$

101. $y''_{xx} = Ay(y'_x)^{7/5}.$

Solution in parametric form:

$$x = aC_1^7 \wp^{-2}(f \pm 2\tau\wp^2), \quad y = bC_1^{-3} f, \qquad \text{where} \quad A = \pm\tfrac{5}{6}b^{-2}(3b/a)^{3/5}.$$

102. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^3.$

Solution in parametric form:

$$x = aC_1^3 (f \pm 2\tau\wp^2)^{-1}, \quad y = bC_1^7 \wp^2 (f \pm 2\tau\wp^2)^{-1}, \qquad \text{where} \quad A = \pm 3a^{7/2}b^{-3/2}.$$

103. $y''_{xx} = Axy(y'_x)^{7/5}.$

Solution in parametric form:

$$x = aC_1^7 \wp (f \pm 2\tau\wp^2)^{-1/2}, \quad y = bC_1^{-8}(\tau f + 2\wp), \qquad \text{where} \quad A = -\tfrac{10}{3}a^{-1}b^{-2}(3b/a)^{3/5}.$$

104. $y''_{xx} = Ax^{-5/3}y^{-1/2}(y'_x)^3.$

Solution in parametric form:

$$x = aC_1^9 (f \pm 2\tau\wp^2)^{3/2}, \quad y = bC_1^{16}(\tau f + 2\wp)^2, \qquad \text{where} \quad A = \tfrac{1}{6}a^{8/3}b^{-3/2}.$$

105. $y''_{xx} = Ax^{-5/3}y^{-5/6}(y'_x)^3.$

Solution in parametric form:

$$x = aC_1^7 (f \pm 2\tau\wp^2)^{3/2}(\tau f + 2\wp)^{-2}, \quad y = bC_1^{16}(\tau f + 2\wp)^{-2}, \qquad \text{where} \quad A = \tfrac{1}{6}a^{8/3}b^{-7/6}.$$

106. $y''_{xx} = Ax^5 y(y'_x)^{8/5}.$

Solution in parametric form:

$$x = aC_1^8 (\tau f + 2\wp)^{-1/3}, \quad y = bC_1^{-27}(\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{-1/2},$$

where $A = 10a^{-5}b^{-2}(b/a)^{2/5}$.

107. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{11/5}.$

Solution in parametric form:

$$x = aC_1^7 (f \pm 2\tau\wp^2)^{-1}(\tau f + 2\wp)^{4/3}, \quad y = bC_1^{27}(\tau^2\wp \mp 1)^2 (f \pm 2\tau\wp^2)^{-1},$$

where $A = 5a^{7/2}b^{-3/2}\left(\dfrac{b}{2a}\right)^{4/5}$.

108. $y''_{xx} = Ax^{-4/5}y^{-11/5}(y'_x)^{1/2}.$

Solution in parametric form:

$$x = aC_1^{-27}\left[2(\tau^2\wp + 1)f + 8\tau\wp - \tau^3\right]^{-1},$$

$$y = bC_1^{-7}(\tau f + 2\wp)^{5/3}\left[2(\tau^2\wp + 1)f + 8\tau\wp^2 - \tau^3\right]^{-1},$$

where $A = -\dfrac{2\sqrt{2}}{5}a^{-7/10}b^{27/10}$.

109. $y''_{xx} = Ax^{-11/5}y^{-4/5}(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aC_1^{-7}(\tau f + 2\wp)^{5/3}\big[2(\tau^2\wp + 1)f + 8\tau\wp^2 - \tau^3\big]^{-1},$$
$$y = bC_1^{-27}\big[2(\tau^2\wp + 1)\wp' + 8\tau\wp - \tau^3\big]^{-1},$$

where $A = \dfrac{2\sqrt{2}}{5}a^{27/10}b^{-7/10}$.

110. $y''_{xx} = Ax^2y^4(y'_x)^{17/11}$.

Solution in parametric form:

$$x = aC_1^{50}(\tau^2\wp - 1)(\tau f + 2\wp)^{-2/3},$$
$$y = bC_1^{-27}\big[2(\tau^2\wp + 1)f + 8\tau\wp^2 - \tau^3\big]^{-1/5},$$

where $A = -22a^{-27/11}b^{-50/11}$.

111. $y''_{xx} = Ax^4y^2(y'_x)^{16/11}$.

Solution in parametric form:

$$x = aC_1^{-27}\big[2(\tau^2\wp + 1)f + 8\tau\wp^2 - \tau^3\big]^{-1/5},$$
$$y = bC_1^{50}(\tau^2\wp - 1)(\tau f + 2\wp)^{-2/3},$$

where $A = 22a^{-50/11}b^{-27/11}$.

112. $y''_{xx} = Ax^{-11/6}y^{-2/3}(y'_x)^{9/4}$.

Solution in parametric form:

$$x = aC_1^{21}(\tau f + 2\wp)^{-2}\big[2(\tau^2\wp + 1)f + 8\tau\wp^2 - \tau^3\big]^{6/5},$$
$$y = bC_1^{75}(\tau^2\wp - 1)^3(\tau f + 2\wp)^{-2},$$

where $A = \dfrac{4\sqrt{2}}{3}a^{425/132}b^{-7/12}$.

113. $y''_{xx} = Ax^{-2/3}y^{-11/6}(y'_x)^{3/4}$.

Solution in parametric form:

$$x = aC_1^{75}(\tau^2\wp - 1)^3(\tau f + 2\wp)^{-2},$$
$$y = bC_1^{21}(\tau f + 2\wp)^{-2}\big[2(\tau^2\wp + 1)f + 8\tau\wp^2 - \tau^3\big]^{6/5},$$

where $A = -\dfrac{4\sqrt{2}}{3}a^{-7/12}b^{425/132}$.

◆ *In the solutions of equations 114 and 115, the following notation is used:*

$$Z = \begin{cases} C_1J_\nu(\tau) + C_2Y_\nu(\tau) & \text{for the upper sign,} \\ C_1I_\nu(\tau) + C_2K_\nu(\tau) & \text{the lower sign,} \end{cases}$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are Bessel functions, and $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

114. $y''_{xx} = Axy^m (y'_x)^3$.

Solution in parametric form with $m \neq -2$:

$$x = \tau^\nu Z, \quad y = b\tau^{2\nu}, \qquad \text{where} \quad \nu = \frac{1}{m+2}, \quad A = \pm\Big(\frac{m+2}{2b}\Big)^2.$$

For the case $m = -2$, see equation 14.5.2.28.

115. $y''_{xx} = Ax^{-1/2} y^{-1/2} (y'_x)^l$.

Solution in parametric form with $l \neq 3/2$:

$$x = a\tau^{2\nu} Z^2, \qquad y = b\tau^{-2\nu}(\tau Z'_\tau + \nu Z)^2,$$

where $\nu = \dfrac{1-l}{3-2l}$, $A = \dfrac{1}{3-2l}\Big(\mp\dfrac{b}{a}\Big)^{\frac{3}{2}-l}$.

For the case $l = 3/2$, see equation 14.5.2.29.

◆ *In the solutions of equations 116–124, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$$U_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \quad U_2 = U_1^2 \pm \tau^2 Z^2, \quad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1 U_2,$$

where $J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

116. $y''_{xx} = Axy^{-1/2} (y'_x)^{3/2}$.

Solution in parametric form:

$$x = a\tau^{-2/3} Z^{-1} U_1, \quad y = \tau^{-4/3} U_2^2, \qquad \text{where} \quad A = -\frac{2}{a}\Big(\mp\frac{3}{a}\Big)^{1/2}.$$

117. $y''_{xx} = Axy^{-2} (y'_x)^{3/2}$.

Solution in parametric form:

$$x = a\tau^{-4/3} Z^{-1} U_3, \quad y = b\tau^{-2/3} U_2, \qquad \text{where} \quad A = -a^{-2} b(\pm 3ab)^{1/2}.$$

118. $y''_{xx} = Ax^{-2} y^{-1/2} (y'_x)^{3/2}$.

Solution in parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_2, \quad y = \tau^{-8/3} Z^{-2} U_3^2, \qquad \text{where} \quad A = \pm\tfrac{2}{3} a^{3/2}.$$

119. $y''_{xx} = Ax^{-1/2} y^{-2} (y'_x)^3$.

Solution in parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_1^2, \quad y = b\tau^{-2/3} Z^{-2}, \qquad \text{where} \quad A = \pm\tfrac{1}{3} a^{3/2}.$$

120. $y''_{xx} = Ax^{-2} y (y'_x)^3$.

Solution in parametric form:

$$x = a\tau^{2/3} Z^2, \quad y = b\tau^{-2/3} U_2, \qquad \text{where} \quad A = \tfrac{9}{2}(a/b)^3.$$

121. $y''_{xx} = Ax^{-1/2}y(y'_x)^{3/2}$.

Solution in parametric form:

$$x = \tau^{-4/3}U_2^2, \quad y = b\tau^{-2/3}Z^{-1}U_1, \qquad \text{where} \quad A = \frac{2}{b}\Big(\mp\frac{3}{b}\Big)^{1/2}.$$

122. $y''_{xx} = Ax^{-2}y^{-2}(y'_x)^3$.

Solution in parametric form:

$$x = a\tau^{4/3}Z^2U_2^{-1}, \quad y = \tau^{2/3}Z^{-1}U_2^{-1}, \qquad \text{where} \quad A = \tfrac{9}{2}a^3.$$

123. $y''_{xx} = Ax^{-2}y(y'_x)^{3/2}$.

Solution in parametric form:

$$x = a\tau^{-2/3}U_2, \quad y = b\tau^{-4/3}Z^{-1}U_3, \qquad \text{where} \quad A = ab^{-2}\big(\pm 3ab\big)^{1/2}.$$

124. $y''_{xx} = Ax^{-1/2}y^{-2}(y'_x)^{3/2}$.

Solution in parametric form:

$$x = \tau^{-8/3}Z^{-2}U_3^2, \quad y = b\tau^{-4/3}Z^{-2}U_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}b^{3/2}.$$

125. $y''_{xx} = Ax^n y^{-n-3}(y'_x)^{\frac{3n+4}{2n+3}}$.

In the books by Zaitsev & Polyanin (1993, 1994) it was shown that this equation is reducible to a Riccati equation whose solution is expressed in terms of associated Legendre functions.

14.5.3 Some Formulas and Transformations

► **Symbolic notation. A particular solution.**

For the sake of visualization, we use the symbolic notation

$$\{n,\ m,\ l\}$$

to denote the generalized Emden–Fowler equation

$$y''_{xx} = Ax^n y^m (y'_x)^l.$$

Hereinafter we omit the insignificant parameter A (which can be reduced to ± 1 by scaling the variables in accordance with the rule $x \to ax,\ y \to by,$ selecting appropriate constants a and b).

If $m + l \neq 1$, the generalized Emden–Fowler equation has a particular solution:

$$y = Bx^{\frac{n+2-l}{1-m-l}}, \qquad \text{where} \quad B = \Big(\frac{n+2-l}{1-m-l}\Big)^{\frac{1-l}{m+l-1}}\Big[\frac{n+m+1}{A(1-m-l)}\Big]^{\frac{1}{m+l-1}}.$$

► **Discrete transformations of the generalized Emden–Fowler equation.**

1°. Taking y as the independent variable and x as the dependent one, we obtain a generalized Emden–Fowler equation for $x = x(y)$ with changed parameters:

$$x''_{yy} = -Ay^m x^n (x'_y)^{3-l}.$$

Denote this transformation by $\mathcal{F}$ and represent it as follows:

$$\{n,\ m,\ l\} \quad \leftarrow - - - \rightarrow \quad \{m,\ n,\ 3-l\} \qquad \textit{transformation } \mathcal{F}.$$

The twofold transformation $\mathcal{F}$ yields the original equation.

2°. For $m \neq 0$, $n \neq -1$, $l \neq 1$, the transformation $t = (y'_x)^{1-l}$, $w = x^{n+1}$ leads to a generalized Emden–Fowler equation for $x = x(y)$ with changed parameters:

$$w''_{tt} = Bt^{\frac{1}{1-l}} w^{-\frac{n}{n+1}} (w'_t)^{\frac{2m+1}{m}},$$

where $B = -\dfrac{m}{n+1}\left[\dfrac{A(1-l)}{n+1}\right]^{\frac{1}{m}}$. Denote this transformation by $\mathcal{G}$ and represent it as follows:

$$\{n,\ m,\ l\} \quad \longmapsto \quad \left\{\frac{1}{1-l},\ -\frac{n}{n+1},\ \frac{2m+1}{m}\right\} \qquad \textit{transformation } \mathcal{G}.$$

The threefold transformation $\mathcal{G}$ yields the original equation.

Whenever the solution of the transformed equation is obtained in the form $w = w(t)$, the solution of the original equation can be written in parametric form as:

$$x = w^{\frac{1}{n+1}}, \quad y = k(w'_t)^{-\frac{1}{m}}, \qquad \text{where} \quad k = \left[\frac{n+1}{A(1-l)}\right]^{\frac{1}{m}}.$$

Different compositions of the transformations $\mathcal{F}$ and $\mathcal{G}$ generate six different generalized Emden–Fowler equations, whose parameters are shown in Figure 10.1 (see Section 10.2.2).

3°. In the special case $l = 0$, the transformation $y = w/t$, $x = 1/t$ leads to an Emden–Fowler equation with the independent variable raised to a different power:

$$w''_{tt} = At^{-n-m-3} w^m.$$

Denote this transformation by $\mathcal{H}$ and represent it as follows:

$$\{n,\ m,\ 0\} \quad \longleftrightarrow \quad \{-n-m-3,\ m,\ 0\} \qquad \textit{transformation } \mathcal{H}.$$

If $l = 0$, different compositions of the transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generate 12 different generalized Emden–Fowler equations, whose parameters are shown in Figure 10.2 (see Section 10.2.2).

If $l = 0$ and $n = 1$, different compositions of the transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generate 24 different generalized Emden–Fowler equations, whose parameters are presented in Figure 10.3 (see Section 10.2.2).

4°. In the special case $n + m + 3 = 0$, the contact transformation $t = y - xy'_x$, $w = -y'_x$ leads to a generalized Emden–Fowler equation for $w = w(t)$ with changed parameters:

$$\{-m-3,\ m,\ l\} \quad \longmapsto \quad \bigl\{-l,\ l-3,\ m+3\bigr\} \qquad \textit{transformation } \mathcal{Q}.$$

If $n+m+3=0$, different compositions of the transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{Q}$ generate 18 different generalized Emden–Fowler equations (see Section 12.3).

► **Reduction of the generalized Emden–Fowler equation to an Abel equation.**

The transformation

$$z = \frac{x}{y}y'_x, \qquad v = Ax^{n-l+2}y^{m+l-1}$$

reduces the generalized Emden–Fowler equation to the equation

$$(z^l v - z^2 + z)v'_z = [(m+l-1)z + n - l + 2]v.$$

Furthermore, using the substitution $\xi = v - z^{2-l} + z^{1-l}$, we obtain an Abel equation of the second kind:

$$\xi\xi'_z = [(m+2l-3)z+n-2l+3]z^{-l}\xi+[(m+l-1)z^2+(n-m-2l+3)z-n+l-2]z^{1-2l}.$$

14.6 Equations of the Form $y''_{xx} = A_1x^{n_1}y^{m_1}(y'_x)^{l_1} + A_2x^{n_2}y^{m_2}(y'_x)^{l_2}$

14.6.1 Modified Emden–Fowler Equation $y''_{xx} = A_1x^{-1}y'_x + A_2x^ny^m$

► **Preliminary remarks. Classification table.**

For the sake of clarity, below in this subsection we use the conventional notation

$$xy''_{xx} - ky'_x = Ax^{n+1}y^m$$

for the modified Emden–Fowler equation. For $k = 0$, see Section 14.3. For $k \neq -1$, the substitution $z = x^{k+1}$ leads to the Emden–Fowler equation:

$$y''_{zz} = \frac{A}{(k+1)^2}z^{\frac{n-2k}{k+1}}y^m,$$

which is discussed in Section 14.3.

The classification Table 14.8 represents all solvable equations whose solutions are outlined in Section 14.6.1. Equations are arranged in accordance with the growth of parameter m. The number of the equation sought is indicated in the last column in this table.

TABLE 14.8
Solvable cases of the modified Emden–Fowler equation $xy''_{xx} - ky'_x = Ax^{n+1}y^m$

No	m	n	k	Equation
1	arbitrary $(m \neq -1)$	arbitrary $(n \neq -2)$	$\frac{1}{2}n$	14.6.1.1
2	arbitrary $(m \neq -1)$	arbitrary $(n \neq -2)$	$-\dfrac{n+m+3}{m+1}$	14.6.1.2
3	arbitrary $(m \neq -1)$	arbitrary $(n \neq -2)$	$\dfrac{2n+m+3}{1-m}$	14.6.1.3
4	arbitrary $(m \neq -1)$	-2	-1	14.6.1.6

TABLE 14.8 (*Continued*)
Solvable cases of the modified Emden–Fowler equation $xy''_{xx} - ky'_x = Ax^{n+1}y^m$

No	m	n	k	Equation
5	-7	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	14.6.1.45
6	-7	arbitrary ($n \neq -2$)	$\frac{1}{5}(n-3)$	14.6.1.46
7	-4	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	14.6.1.40
8	-4	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	14.6.1.42
9	-4	-2	-1	14.6.1.41
10	-2	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	14.6.1.28
11	-2	-2	arbitrary ($k \neq -1$)	14.6.1.29
12	$-\frac{5}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	14.6.1.35
13	$-\frac{5}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(2n+1)$	14.6.1.37
14	$-\frac{5}{2}$	-2	-1	14.6.1.36
15	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$-3n-7$	14.6.1.14
16	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	14.6.1.8
17	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{2}(3n+4)$	14.6.1.9
18	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	14.6.1.13
19	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(2n+1)$	14.6.1.38
20	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{4}(n-2)$	14.6.1.18
21	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$-\frac{1}{4}(3n+10)$	14.6.1.19
22	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{7}(6n+5)$	14.6.1.39
23	$-\frac{5}{3}$	-2	-1	14.6.1.22
24	$-\frac{7}{5}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	14.6.1.24
25	$-\frac{7}{5}$	arbitrary ($n \neq -2$)	$-\frac{1}{3}(5n+13)$	14.6.1.25

TABLE 14.8 (*Continued*)
Solvable cases of the modified Emden–Fowler equation $xy''_{xx} - ky'_x = Ax^{n+1}y^m$

No	m	n	k	Equation
26	-1	arbitrary $(n \neq -2)$	$n+1$	14.6.1.5
27	-1	arbitrary $(n \neq -2)$	$\frac{1}{2}n$	14.6.1.4
28	-1	-2	arbitrary $(k \neq -1)$	14.6.1.7
29	-1	-2	-1	14.6.1.20
30	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$-2n-5$	14.6.1.12
31	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$\frac{1}{2}n$	14.6.1.11
32	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$\frac{1}{2}(3n+4)$	14.6.1.43
33	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$\frac{1}{3}(n-1)$	14.6.1.16
34	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$\frac{1}{3}(2n+1)$	14.6.1.26
35	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$-\frac{1}{3}(2n+7)$	14.6.1.17
36	$-\frac{1}{2}$	arbitrary $(n \neq -2)$	$\frac{1}{5}(6n+7)$	14.6.1.44
37	$-\frac{1}{2}$	-2	arbitrary $(k \neq -1)$	14.6.1.27
38	$-\frac{1}{2}$	-2	-1	14.6.1.21
39	$\frac{1}{2}$	arbitrary $(n \neq -2)$	$\frac{1}{2}n$	14.6.1.15
40	$\frac{1}{2}$	arbitrary $(n \neq -2)$	$-\frac{1}{3}(2n+7)$	14.6.1.10
41	$\frac{1}{2}$	-2	-1	14.6.1.23
42	2	arbitrary $(n \neq -2)$	$-7n-15$	14.6.1.33
43	2	arbitrary $(n \neq -2)$	$\frac{1}{2}n$	14.6.1.30
44	2	arbitrary $(n \neq -2)$	$-\frac{1}{3}(n+5)$	14.6.1.32
45	2	arbitrary $(n \neq -2)$	$-\frac{1}{6}(7n+20)$	14.6.1.34
46	2	-2	-1	14.6.1.31

► **Solvable equations and their solutions.**

1. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^m, \qquad m \neq -1, \ n \neq -2.$

Solution in parametric form:

$$x = aC_1^{1-m}\Big[\int (1 \pm \tau^{m+1})^{-1/2}\,d\tau + C_2\Big]^{\frac{2}{n+2}}, \qquad y = bC_1^{n+2}\tau,$$

where $A = \pm\frac{1}{8}(m+1)(n+2)^2 a^{-n-2} b^{1-m}$.

2. $xy''_{xx} + \dfrac{n+m+3}{m+1}y'_x = Ax^{n+1}y^m, \qquad m \neq -1, \ n \neq -2.$

Solution in parametric form:

$$x = aC_1^{1-m}\Big[\int (1 \pm \tau^{m+1})^{-1/2}\,d\tau + C_2\Big]^{\frac{m+1}{n+2}}, \quad y = bC_1^{n+2}\tau\Big[\int (1 \pm \tau^{m+1})^{-1/2}\,d\tau + C_2\Big]^{-1},$$

where $A = \pm\dfrac{(n+2)^2}{2(m+1)} a^{-n-2} b^{1-m}$.

3. $xy''_{xx} + \dfrac{2n+m+3}{m-1}y'_x = Ax^{n+1}y^m, \qquad m \neq -1, \ n \neq -2.$

Solution in parametric form:

$$x = \exp\Big[\frac{1-m}{n+2}C_2 \int \Big(C_1 + \frac{1}{4}\tau^2 + \frac{2B}{m+1}\tau^{m+1}\Big)^{-1/2} d\tau\Big],$$
$$y = \tau\exp\Big[C_2 \int \Big(C_1 + \frac{1}{4}\tau^2 + \frac{2B}{m+1}\tau^{m+1}\Big)^{-1/2} d\tau\Big],$$

where $A = \dfrac{4(n+2)^2}{(m-1)^2}B$.

4. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^{-1}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^2\Big[\int \exp(\mp\tau^2)\,d\tau + C_2\Big]^{\frac{2}{n+2}}, \qquad y = bC_1^{n+2}\exp(\mp\tau^2),$$

where $A = \mp\frac{1}{2}(n+2)^2 a^{-n-2} b^2$.

5. $xy''_{xx} - (n+1)y'_x = Ax^{n+1}y^{-1}, \qquad n \neq -2.$

Solution in parametric form:

$$x = \exp\Big\{\frac{2C_2}{n+2}\int \Big[C_1 + \frac{1}{4}\tau^2 + \frac{2A}{(n+2)^2}\ln|\tau|\Big]^{-1/2} d\tau\Big\},$$
$$y = \tau\exp\Big\{C_2\int \Big[C_1 + \frac{1}{4}\tau^2 + \frac{2A}{(n+2)^2}\ln|\tau|\Big]^{-1/2} d\tau\Big\}.$$

6. $xy''_{xx} + y'_x = Ax^{-1}y^m, \qquad m \neq -1.$

Solution in parametric form:

$$x = C_2\exp\Big[\int (C_1 \pm \tau^{m+1})^{-1/2}\,d\tau\Big], \quad y = b\tau, \quad \text{where } A = \pm\tfrac{1}{2}b^{1-m}(m+1).$$

7. $xy''_{xx} - ky'_x = Ax^{-1}y^{-1}, \qquad k \neq -1.$

Solution in parametric form:

$$x = \left\{\int \left[\frac{2A}{(k+1)^2}\ln\tau + C_1\right]^{-1/2} d\tau + C_2\right\}^{-\frac{1}{k+1}},$$
$$y = \tau\left\{\int \left[\frac{2A}{(k+1)^2}\ln\tau + C_1\right]^{-1/2} d\tau + C_2\right\}^{-1}.$$

8. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8(\tau^3 \pm 3\tau + C_2)^{\frac{2}{n+2}}, \quad y = bC_1^{3n+6}(\tau^2 \pm 1)^{3/2},$$

where $A = \pm\frac{1}{12}a^{-n-2}b^{8/3}(n+2)^2$.

9. $xy''_{xx} - \frac{1}{2}(3n+4)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8(\tau^3 \pm 3\tau + C_2)^{-\frac{2}{3n+6}}, \quad y = bC_1^{3n+6}(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-1},$$

where $A = \pm\frac{3}{4}a^{-n-2}b^{8/3}(n+2)^2$.

10. $xy''_{xx} + \frac{1}{3}(2n+7)y'_x = Ax^{n+1}y^{1/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1\left[\int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3-1)}} + C_2\right]^{\frac{3}{2n+4}}, \quad y = bC_1^{2n+4}\tau^2\left[\int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3-1)}} + C_2\right]^{-1},$$

where $A = \pm\frac{16}{3}a^{-n-2}b^{1/2}(n+2)^2$.

11. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^3(\tau^3 - 3\tau + C_2)^{\frac{2}{n+2}}, \quad y = bC_1^{2n+4}(\tau^2 - 1)^2,$$

where $A = \pm\frac{1}{9}(n+2)^2a^{-n-2}b^{3/2}$.

12. $xy''_{xx} + (2n+5)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^3(\tau^3 - 3\tau + C_2)^{\frac{1}{2n+4}}, \quad y = bC_1^{2n+4}(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = \pm\frac{16}{9}(n+2)^2a^{-n-2}b^{3/2}$.

13. $xy''_{xx} - \frac{1}{3}(n-1)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8\left[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\right]^{\frac{3}{n+2}}, \quad y = bC_1^{3n+6}(\tau^3 - 3\tau + C_2)^{3/2},$$

where $A = \pm\frac{1}{64}(n+2)^2a^{-n-2}b^{8/3}$.

14. $\boldsymbol{xy''_{xx} + (3n+7)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.}$

Solution in parametric form:

$$x = aC_1^8\bigl[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\bigr]^{\frac{1}{3n+6}},$$
$$y = \pm bC_1^{3n+6}(\tau^3 - 3\tau + C_2)^{3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1},$$

where $A = \pm\frac{81}{64}(n+2)^2 a^{-n-2} b^{8/3}$.

15. $\boldsymbol{xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^{1/2}, \qquad n \neq -2.}$

Solution in parametric form:

$$x = aC_1\biggl[\int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - 1)}} + C_2\biggr]^{\frac{2}{n+2}}, \qquad y = bC_1^{2n+4}\tau^2,$$

where $A = \pm 3a^{-n-2}b^{1/2}(n+2)^2$.

16. $\boldsymbol{xy''_{xx} - \frac{1}{3}(n-1)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.}$

Solution in parametric form:

$$x = \bigl[C_1e^{2s\tau} + C_2e^{-s\tau}\sin\bigl(\sqrt{3}\,s\tau\bigr)\bigr]^{\frac{3}{n+2}},$$
$$y = \bigl\{2C_1se^{2s\tau} + C_2se^{-s\tau}\bigl[\sqrt{3}\cos\bigl(\sqrt{3}\,s\tau\bigr) - \sin\bigl(\sqrt{3}\,s\tau\bigr)\bigr]\bigr\}^2,$$

where $A = \frac{16}{9}s^3(n+2)^2$.

17. $\boldsymbol{xy''_{xx} + \frac{1}{3}(2n+7)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.}$

Solution in parametric form:

$$x = \bigl[C_1e^{2s\tau} + C_2e^{-s\tau}\sin\bigl(\sqrt{3}\,s\tau\bigr)\bigr]^{\frac{3}{2n+4}},$$
$$y = \frac{\bigl\{2C_1se^{2s\tau} + C_2se^{-s\tau}\bigl[\sqrt{3}\cos\bigl(\sqrt{3}\,s\tau\bigr) - \sin\bigl(\sqrt{3}\,s\tau\bigr)\bigr]\bigr\}^2}{C_1e^{2s\tau} + C_2e^{-s\tau}\sin\bigl(\sqrt{3}\,s\tau\bigr)},$$

where $A = \frac{64}{9}s^3(n+2)^2$.

18. $\boldsymbol{xy''_{xx} - \frac{1}{4}(n-2)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.}$

1°. Solution in parametric form with $A < 0$:

$$x = aC_1^8\bigl[\cosh(\tau + C_2)\cos\tau\bigr]^{\frac{4}{n+2}}\bigl[\tanh(\tau + C_2) + \tan\tau\bigr]^{\frac{4}{n+2}},$$
$$y = bC_1^{3n+6}\bigl[\cosh(\tau + C_2)\cos\tau\bigr]^{3/2},$$

where $A = -\frac{3}{256}a^{-n-2}b^{8/3}(n+2)^2$.

2°. Solution in parametric form with $A > 0$:

$$x = aC_1^8\bigl[\sinh\tau + \cos(\tau + C_2)\bigr]^{\frac{4}{n+2}}, \qquad y = bC_1^{3n+6}\bigl[\cosh\tau - \sin(\tau + C_2)\bigr]^{3/2},$$

where $A = \frac{3}{64}a^{-n-2}b^{8/3}(n+2)^2$.

19. $xy''_{xx} + \frac{1}{4}(3n+10)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.$

1°. Solution in parametric form with $A < 0$:

$$x = aC_1^8\big[\cosh(\tau + C_2)\cos\tau\big]^{\frac{4}{3n+6}}\big[\tanh(\tau + C_2) + \tan\tau\big]^{\frac{4}{3n+6}},$$
$$y = bC_1^{3n+6}\big[\cosh(\tau + C_2)\cos\tau\big]^{1/2}\big[\tanh(\tau + C_2) + \tan\tau\big]^{-1},$$

where $A = -\frac{27}{256}a^{-n-2}b^{8/3}(n+2)^2$.

2°. Solution in parametric form with $A > 0$:

$$x = aC_1^8\big[\sinh\tau + \cos(\tau + C_2)\big]^{\frac{4}{3n+6}},$$
$$y = bC_1^{3n+6}\big[\cosh\tau - \sin(\tau + C_2)\big]^{3/2}\big[\sinh\tau + \cos(\tau + C_2)\big]^{-1},$$

where $A = \frac{27}{64}a^{-n-2}b^{8/3}(n+2)^2$.

20. $xy''_{xx} + y'_x = Ax^{-1}y^{-1}.$

Solution in parametric form:

$$x = C_2\exp\Big[\int(2A\ln|\tau| + C_1)^{-1/2}\,d\tau\Big], \qquad y = \tau.$$

21. $xy''_{xx} + y'_x = Ax^{-1}y^{-1/2}.$

Solution in parametric form:

$$x = \exp(\pm\tau^3 - 3C_1\tau + C_2), \quad y = b(\pm\tau^2 - C_1)^2, \qquad \text{where} \quad A = \pm\tfrac{4}{9}b^{3/2}.$$

22. $xy''_{xx} + y'_x = Ax^{-1}y^{-5/3}.$

Solution in parametric form:

$$x = \exp(C_1\tau^3 \pm 3\tau + C_2), \qquad y = (\pm 3A/C_1)^{3/8}(C_1\tau^2 \pm 1)^{3/2}.$$

23. $xy''_{xx} + y'_x = Ax^{-1}y^{1/2}.$

Solution in parametric form:

$$x = C_1\exp\Big[\int\frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - C_1)}}\Big], \quad y = b\tau^2, \qquad \text{where} \quad A = \pm 12\,b^{1/2}.$$

◆ *In the solutions of equations 24 and 25, the following notation is used:*

$$S_1 = C_1e^{2s\tau} + C_2e^{-s\tau}\sin\big(\sqrt{3}\,s\tau\big),$$
$$S_2 = 2C_1se^{2s\tau} + C_2se^{-s\tau}\big[\sqrt{3}\cos\big(\sqrt{3}\,s\tau\big) - \sin\big(\sqrt{3}\,s\tau\big)\big],$$
$$S_3 = S_2^2 - 2S_1(S_2)'_\tau.$$

24. $xy''_{xx} - \frac{1}{3}(n-1)y'_x = Ax^{n+1}y^{-7/5}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aS_3^{\frac{3}{n+2}}, \quad y = bS_1^{5/2}, \qquad \text{where} \quad A = -\tfrac{5}{9216}a^{-n-2}b^{12/5}s^{-6}(n+2)^2.$$

25. $xy''_{xx} + \frac{1}{3}(5n+13)y'_x = Ax^{n+1}y^{-7/5}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aS_3^{\frac{3}{5n+10}}, \quad y = bS_1^{5/2}S_3^{-1}, \qquad \text{where} \quad A = -\tfrac{125}{9216}a^{-n-2}b^{12/5}s^{-6}(n+2)^2.$$

◆ *In the solutions of equations 26–29, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \textit{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

26. $xy''_{xx} - \frac{1}{3}(2n+1)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^3\tau^{\frac{1}{n+2}}Z^{\frac{3}{n+2}}, \quad y = bC_1^{2n+4}\tau^{-2/3}(\tau Z'_\tau + \tfrac{1}{3}Z)^2,$$

where $A = \mp\frac{4}{27}a^{-n-2}b^{3/2}(n+2)^2$.

27. $xy''_{xx} - ky'_x = Ax^{-1}y^{-1/2}, \qquad k \neq -1.$

Solution in parametric form:

$$x = C_1(\tau^{1/3}Z)^{-\frac{2}{k+1}}, \quad y = b\tau^{-4/3}Z^{-2}(\tau Z'_\tau + \tfrac{1}{3}Z)^2, \qquad \text{where} \quad A = \mp\tfrac{1}{3}b^{3/2}(k+1)^2.$$

28. $xy''_{xx} - \frac{1}{3}(n-1)y'_x = Ax^{n+1}y^{-2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^3\tau^{-\frac{2}{n+2}}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}\tau^{2/3}Z^2,$$

where $A = -\frac{1}{2}a^{-n-2}b^3(n+2)^2$.

29. $xy''_{xx} - ky'_x = Ax^{-1}y^{-2}, \qquad k \neq -1.$

Solution in parametric form:

$$x = C_1\tau^{\frac{2}{3k+3}}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{-\frac{1}{k+1}}, \quad y = b\tau^{4/3}Z^2\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{-1},$$

where $A = -\frac{9}{2}b^3(k+1)^2$.

◆ *In the solutions of equations 30–39, the following notation is used:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2, \qquad f = \sqrt{\pm(4\wp^3 - 1)}.$$

The function $\wp$ is defined implicitly by a first integral; the upper sign in the formulas corresponds to the classical Weierstrass elliptic function $\wp = \wp(\tau + C_2, 0, 1)$.

30. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^2, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{\frac{2}{n+2}}, \quad y = bC_1^{n+2}\wp, \qquad \text{where} \quad A = \pm\tfrac{3}{2}a^{-n-2}b^{-1}(n+2)^2.$$

31. $xy''_{xx} + y'_x = Ax^{-1}y^2.$

Solution in parametric form:

$$x = C_2e^{\tau}, \quad y = b\wp(\tau,\, 0,\, C_1),$$

where $A = \pm 6b^{-1}$, and the elliptic Weierstrass function $\wp = \wp(\tau,\, 0,\, C_1)$ is defined implicitly by the integral $\tau = \int_{\infty}^{\wp}(4z^3 - C_1)^{-1/2}\,dz$.

32. $xy''_{xx} + \frac{1}{3}(n+5)y'_x = Ax^{n+1}y^2, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}\tau^{-1}\wp, \qquad \text{where} \quad A = \pm\tfrac{2}{3}a^{-n-2}b^{-1}(n+2)^2.$$

33. $xy''_{xx} + (7n+15)y'_x = Ax^{n+1}y^2, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{-\frac{1}{n+2}}, \quad y = bC_1^{n+2}\tau(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm 6a^{-n-2}b^{-1}(n+2)^2.$$

34. $xy''_{xx} + \frac{1}{6}(7n+20)y'_x = Ax^{n+1}y^2, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{\frac{6}{7(n+2)}}, \quad y = bC_1^{n+2}\tau^{-6}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{1}{6}a^{-n-2}b^{-1}(n+2)^2.$$

35. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^{-5/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^7\wp^{-\frac{4}{n+2}}(f \pm 2\tau\wp^2)^{\frac{2}{n+2}}, \quad y = bC_1^{2n+4}\wp^{-2}, \qquad \text{where} \quad A = \mp\tfrac{3}{4}a^{-n-2}b^{7/2}(n+2)^2.$$

36. $xy''_{xx} + y'_x = Ax^{-1}y^{-5/2}.$

Solution in parametric form:

$$x = C_2\exp[\wp^{-2}(f \pm 2\tau\wp^2)], \quad y = b\wp^{-2},$$

where $A = \mp 3b^{7/2}$, and the elliptic Weierstrass function $\wp = \wp(\tau,\, 0,\, C_1)$ is defined implicitly by the integral $\tau = \int_{\infty}^{\wp}(4z^3 - C_1)^{-1/2}\,dz$.

37. $xy''_{xx} - \frac{1}{3}(2n+1)y'_x = Ax^{n+1}y^{-5/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^7\wp^{\frac{3}{n+2}}(f \pm 2\tau\wp^2)^{-\frac{3}{2n+4}}, \quad y = bC_1^{2n+4}(f \pm 2\tau\wp^2)^{-1},$$

where $A = \mp\frac{4}{3}a^{-n-2}b^{7/2}(n+2)^2$.

38. $xy''_{xx} - \frac{1}{3}(2n+1)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8(\tau f + 2\wp)^{\frac{3}{n+2}}, \quad y = bC_1^{3n+6}(f \pm 2\tau\wp^2)^{3/2},$$

where $A = -\frac{2}{27}a^{-n-2}b^{8/3}(n+2)^2$.

39. $xy''_{xx} - \frac{1}{7}(6n+5)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8(\tau f + 2\wp)^{-\frac{7}{3n+6}}, \quad y = bC_1^{3n+6}(f \pm 2\tau\wp^2)^{3/2}(\tau f + 2\wp)^{-2},$$

where $A = -\frac{6}{49}a^{-n-2}b^{8/3}(n+2)^2$.

◆ *In the solutions of equations 40–46, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad F_1 = 2\tau I(\tau) + C_2\tau \mp R, \quad F_2 = \tau^{-1}(RF_1 - 1),$$

where $I(\tau) = \int \frac{\tau\, d\tau}{R}$ *is the incomplete elliptic integral of the second kind in the Weierstrass form.*

40. $xy''_{xx} - \frac{1}{2}ny'_x = Ax^{n+1}y^{-4}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^5(\tau^{-1}F_1)^{\frac{2}{n+2}}, \quad y = bC_1^{n+2}\tau^{-1}, \qquad \text{where} \quad A = \mp\tfrac{3}{2}a^{-n-2}b^5(n+2)^2.$$

41. $xy''_{xx} + y'_x = Ax^{-1}y^{-4}.$

Solution in parametric form:

$$x = C_2 \exp\left[2\int \frac{\tau\, d\tau}{\sqrt{\pm(4\tau^3 - C_1)}} + C_2 \mp \frac{1}{\tau}\sqrt{\pm(4\tau^3 - C_1)}\right], \qquad y = \mp(AC_1^2/6)^{1/5}\tau^{-1}.$$

42. $xy''_{xx} - \frac{1}{3}(n-1)y'_x = Ax^{n+1}y^{-4}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^5(\tau F_1^{-1})^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}F_1^{-1}, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-n-2}b^5(n+2)^2.$$

43. $xy''_{xx} - \frac{1}{2}(3n+4)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^3F_1^{\frac{2}{n+2}}, \quad y = bC_1^{2n+4}F_2^2, \qquad \text{where} \quad A = \pm 3a^{-n-2}b^{3/2}(n+2)^2.$$

44. $xy''_{xx} - \frac{1}{5}(6n+7)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^3F_1^{-\frac{5}{2n+4}}, \quad y = bC_1^{2n+4}F_1^{-3}F_2^2, \qquad \text{where} \quad A = \pm\tfrac{48}{25}a^{-n-2}b^{3/2}(n+2)^2.$$

45. $xy''_{xx} - \frac{1}{3}(n-1)y'_x = Ax^{n+1}y^{-7}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8(4\tau F_1^2 \mp F_2^2)^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}F_1^{1/2}, \qquad \text{where} \quad A = \pm\tfrac{1}{192}a^{-n-2}b^8(n+2)^2.$$

46. $xy''_{xx} - \frac{1}{5}(n-3)y'_x = Ax^{n+1}y^{-7}, \qquad n \neq -2.$

Solution in parametric form:

$$x = aC_1^8(4\tau F_1^2 \mp F_2^2)^{-\frac{5}{n+2}}, \quad y = bC_1^{n+2}F_1^{1/2}(4\tau F_1^2 \mp F_2^2)^{-1},$$

where $A = \pm\frac{3}{1600}a^{-n-2}b^8(n+2)^2$.

14.6.2 Equations of the Form $y''_{xx} = (A_1x^{n_1}y^{m_1} + A_2x^{n_2}y^{m_2})(y'_x)^l$

See Section 14.4 for the case $l = 0$; see Section 14.5 for the cases $A_1 = 0$ or $A_2 = 0$.

► **Classification table.**

Table 14.9 presents all solvable equations whose solutions are outlined in Section 14.6.2. Equations are arranged in accordance with the growth of l, the growth of m_1 (for identical l), the growth of m_2 (for identical l and m_1, $m_1 \geq m_2$), the growth of n_1 (for identical l, m_1, and m_2), and the growth of n_2 (for identical l, m_1, m_2, and n_1). The number of the equation sought is indicated in the last column in this table.

TABLE 14.9
Solvable cases of the equation $y''_{xx} = (A_1x^{n_1}y^{m_1} + A_2x^{n_2}y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
arbitrary $(l \neq 2)$	arbitrary $(m_1 \neq -1)$	arbitrary $(m_2 \neq -1)$	0	0	Any	Any	14.6.2.1
$\dfrac{m_1+2n_1+3}{m_1+n_1+2}$	arbitrary	arbitrary	arbitrary	$\dfrac{m_2(n_1+1)-m_1+n_1}{m_1+1}$	Any	Any	14.6.2.98
arbitrary $(l \neq 1)$	0	0	arbitrary $(n_1 \neq -1)$	arbitrary $(n_2 \neq -1)$	Any	Any	14.6.2.5
arbitrary $(l \neq 2)$	arbitrary $(m_1 \neq -1)$	-1	0	0	Any	Any	14.6.2.2
arbitrary $(l \neq 2)$	0	0	arbitrary $(n_1 \neq -1)$	-1	Any	Any	14.6.2.6
$\dfrac{3m_1+5}{2m_1+3}$	arbitrary	$-m_1-2$	1	0	Any	Any	14.6.2.21
$\dfrac{m_1+5}{m_1+3}$	arbitrary	$\dfrac{m_1-1}{2}$	1	0	Any	Any	14.6.2.94
$\dfrac{3n_1+4}{n_1+1}$	1	0	arbitrary	$-n_1-2$	Any	Any	14.6.2.22
$\dfrac{2(n_1+2)}{n_1+3}$	1	0	arbitrary	$\dfrac{n_1-1}{2}$	Any	Any	14.6.2.95

TABLE 14.9 (*Continued*)
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
arbitrary ($l \neq 1, 2$)	1	0	0	1	Any	Any	14.6.2.20
$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{7}{4}$	0	1	Any	Any	14.6.2.71
$\frac{1}{2}$	1	0	$-\frac{15}{8}$	$-\frac{7}{4}$	Any	Any	14.6.2.81
$\frac{1}{2}$	1	0	$-\frac{15}{8}$	$-\frac{13}{8}$	Any	Any	14.6.2.66
$\frac{1}{2}$	1	0	$-\frac{20}{13}$	$-\frac{15}{13}$	Any	Any	14.6.2.68
$\frac{1}{2}$	1	0	$-\frac{20}{13}$	$-\frac{14}{13}$	Any	Any	14.6.2.84
$\frac{1}{2}$	1	0	$-\frac{5}{4}$	$-\frac{3}{4}$	Any	Any	14.6.2.64
$\frac{1}{2}$	1	0	$-\frac{5}{4}$	$-\frac{1}{2}$	Any	Any	14.6.2.78
$\frac{1}{2}$	1	0	0	1	Any	Any	14.6.2.62
$\frac{1}{2}$	1	0	0	2	Any	Any	14.6.2.75
1	0	0	arbitrary ($n_1 \neq -1$)	arbitrary ($n_2 \neq -1$)	Any	Any	14.6.2.7
1	0	0	arbitrary ($n_1 \neq -1$)	-1	Any	Any	14.6.2.8
1	0	-2	0	1	Any	Any	14.6.2.25
1	1	0	0	1	Any	Any	14.6.2.23
$\frac{3}{2}$	arbitrary	arbitrary	m_1	m_2	Any	Any	14.6.2.97
$\frac{3}{2}$	0	-2	0	1	Any	Any	14.6.2.107
$\frac{3}{2}$	0	$-\frac{1}{2}$	0	1	Any	Any	14.6.2.105
$\frac{3}{2}$	1	0	-2	0	Any	Any	14.6.2.108
$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	Any	Any	14.6.2.106
2	arbitrary ($m_1 \neq -1$)	arbitrary ($m_2 \neq -1$)	0	0	Any	Any	14.6.2.3
2	arbitrary ($m_1 \neq -1$)	-1	0	0	Any	Any	14.6.2.4
2	1	0	-2	0	Any	Any	14.6.2.26
2	1	0	0	1	Any	Any	14.6.2.24
$\frac{5}{2}$	$-\frac{7}{4}$	$-\frac{15}{8}$	0	1	Any	Any	14.6.2.80
$\frac{5}{2}$	$-\frac{13}{8}$	$-\frac{15}{8}$	0	1	Any	Any	14.6.2.65
$\frac{5}{2}$	$-\frac{15}{13}$	$-\frac{20}{13}$	0	1	Any	Any	14.6.2.67
$\frac{5}{2}$	$-\frac{14}{13}$	$-\frac{20}{13}$	0	1	Any	Any	14.6.2.83
$\frac{5}{2}$	$-\frac{3}{4}$	$-\frac{5}{4}$	0	1	Any	Any	14.6.2.63

TABLE 14.9 (*Continued*)
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{5}{4}$	0	1	Any	Any	14.6.2.77
$\frac{5}{2}$	1	0	$-\frac{7}{4}$	$-\frac{1}{4}$	Any	Any	14.6.2.72
$\frac{5}{2}$	1	0	0	1	Any	Any	14.6.2.61
$\frac{5}{2}$	2	0	0	1	Any	Any	14.6.2.74
3	arbitrary	arbitrary	$-m_1-3$	$-m_2-3$	Any	Any	14.6.2.9
3	arbitrary	arbitrary	$-2m_1-3$	$-2m_2-3$	Any	Any	14.6.2.93
3	arbitrary ($m_1 \neq -2$)	Any	1	0	Any	Any	14.6.2.49
3	arbitrary	-3	$-m_1-3$	0	Any	Any	14.6.2.19
3	arbitrary ($m_1 \neq -2$)	0	1	-3	Any	Any	14.6.2.51
3	-5	-6	1	3	Any	Any	14.6.2.100
3	-4	-5	0	2	Any	Any	14.6.2.76
3	-3	-5	0	1	Any	Any	14.6.2.44
3	-3	-5	0	2	Any	Any	14.6.2.58
3	-3	$-\frac{7}{2}$	0	$-\frac{1}{2}$	Any	Any	14.6.2.28
3	$-\frac{14}{5}$	$-\frac{18}{5}$	2	3	Any	Any	14.6.2.112
3	$-\frac{8}{3}$	$-\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.38
3	$-\frac{5}{2}$	-4	$-\frac{1}{2}$	0	Any	Any	14.6.2.86
3	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	Any	Any	14.6.2.13
3	$-\frac{5}{2}$	-3	$-\frac{1}{2}$	0	Any	Any	14.6.2.32
3	$-\frac{12}{5}$	$-\frac{13}{5}$	$-\frac{3}{5}$	$-\frac{7}{5}$	Any	Any	14.6.2.30
3	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.34
3	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	Any	Any	14.6.2.88
3	$-\frac{7}{3}$	-3	$-\frac{2}{3}$	0	Any	Any	14.6.2.70
3	$-\frac{7}{3}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	14.6.2.42
3	$-\frac{11}{5}$	$-\frac{12}{5}$	2	3	Any	Any	14.6.2.113
3	-2	$-n_2-1$	1	arbitrary	$\dfrac{2(n_2+1)}{(n_2+3)^2}$	Any	14.6.2.132
3	-2	-3	-2	0	Any	Any	14.6.2.104
3	-2	-3	-1	0	Any	Any	14.6.2.12
3	-2	-3	1	2	$-\frac{6}{25}$	Any	14.6.2.145

TABLE 14.9 (*Continued*)
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	-2	-3	1	2	$\frac{6}{25}$	Any	14.6.2.144
3	-2	-3	$-\frac{1}{2}$	0	Any	Any	14.6.2.102
3	-2	-2	1	arbitrary	$\dfrac{2(n_2+1)}{(n_2+3)^2}$	Any	14.6.2.116
3	-2	-2	1	-7	$-\frac{15}{4}$	Any	14.6.2.117
3	-2	-2	1	-4	-6	Any	14.6.2.118
3	-2	-2	1	$-\frac{5}{2}$	-12	Any	14.6.2.119
3	-2	-2	1	-2	-2	Any	14.6.2.120
3	-2	-2	1	$-\frac{5}{3}$	$-\frac{63}{4}$	Any	14.6.2.124
3	-2	-2	1	$-\frac{5}{3}$	$-\frac{3}{4}$	Any	14.6.2.123
3	-2	-2	1	$-\frac{5}{3}$	$\frac{9}{100}$	Any	14.6.2.122
3	-2	-2	1	$-\frac{5}{3}$	$\frac{3}{16}$	Any	14.6.2.121
3	-2	-2	1	$-\frac{7}{5}$	$\frac{5}{36}$	Any	14.6.2.125
3	-2	-2	1	$-\frac{1}{2}$	-20	Any	14.6.2.128
3	-2	-2	1	$-\frac{1}{2}$	$\frac{4}{25}$	Any	14.6.2.127
3	-2	-2	1	$-\frac{1}{2}$	$\frac{2}{9}$	Any	14.6.2.126
3	-2	-2	1	$\frac{1}{2}$	$\frac{12}{49}$	Any	14.6.2.129
3	-2	-2	2	1	Any	$-\frac{6}{25}$	14.6.2.131
3	-2	-2	2	1	Any	$\frac{6}{25}$	14.6.2.130
3	$-\frac{13}{7}$	$-\frac{20}{7}$	0	2	Any	Any	14.6.2.82
3	$-\frac{12}{7}$	$-\frac{20}{7}$	0	2	Any	Any	14.6.2.60
3	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.92
3	$-\frac{8}{5}$	$-\frac{13}{5}$	$-\frac{7}{5}$	$-\frac{7}{5}$	Any	Any	14.6.2.109
3	$-\frac{3}{2}$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	Any	Any	14.6.2.90
3	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	-2	Any	Any	14.6.2.48
3	$-\frac{3}{2}$	-2	0	$-\frac{1}{2}$	Any	Any	14.6.2.46
3	$-\frac{3}{2}$	-2	$\frac{1}{2}$	1	Any	$\frac{12}{49}$	14.6.2.143
3	$-\frac{4}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.36
3	$-\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	14.6.2.40
3	$-\frac{4}{3}$	$-\frac{7}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.14
3	$-\frac{4}{3}$	$-\frac{4}{3}$	0	$-\frac{1}{2}$	Any	Any	14.6.2.15

TABLE 14.9 (*Continued*)
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	$-\frac{9}{7}$	$-\frac{15}{7}$	0	2	Any	Any	14.6.2.59
3	$-\frac{7}{6}$	$-\frac{5}{3}$	$-\frac{1}{2}$	0	Any	Any	14.6.2.16
3	$-\frac{8}{7}$	$-\frac{15}{7}$	0	2	Any	Any	14.6.2.79
3	-1	-2	-2	-2	Any	Any	14.6.2.110
3	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.115
3	$-\frac{1}{2}$	-3	$-\frac{1}{2}$	0	Any	Any	14.6.2.53
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	-20	14.6.2.141
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	$\frac{4}{25}$	14.6.2.134
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	$\frac{2}{9}$	14.6.2.137
3	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0	Any	Any	14.6.2.45
3	0	-5	-3	1	Any	Any	14.6.2.52
3	0	-2	-3	-2	Any	Any	14.6.2.56
3	0	-2	0	$-\frac{1}{2}$	Any	Any	14.6.2.54
3	0	$-\frac{3}{2}$	$-\frac{1}{2}$	0	Any	Any	14.6.2.89
3	0	$-\frac{2}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.114
3	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	Any	Any	14.6.2.101
3	0	0	$-\frac{1}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.39
3	0	0	0	-1	Any	Any	14.6.2.11
3	0	0	0	$-\frac{2}{3}$	Any	Any	14.6.2.69
3	0	0	0	$-\frac{1}{2}$	Any	Any	14.6.2.31
3	0	0	2	0	Any	Any	14.6.2.57
3	$\frac{2}{5}$	-2	$-\frac{7}{5}$	1	Any	$\frac{5}{36}$	14.6.2.139
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$-\frac{63}{4}$	14.6.2.147
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$-\frac{3}{4}$	14.6.2.136
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$\frac{9}{100}$	14.6.2.135
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$\frac{3}{16}$	14.6.2.138
3	1	-2	-2	1	Any	-2	14.6.2.133
3	1	0	-7	-3	Any	Any	14.6.2.17
3	1	0	-4	-3	Any	Any	14.6.2.10
3	1	0	-2	-3	Any	Any	14.6.2.55

TABLE 14.9 (*Continued*)
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	1	0	-2	$-\frac{3}{2}$	Any	Any	14.6.2.47
3	1	0	-2	0	Any	Any	14.6.2.103
3	1	0	$-\frac{5}{3}$	$-\frac{4}{3}$	Any	Any	14.6.2.91
3	1	0	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	14.6.2.41
3	1	0	$-\frac{5}{3}$	$\frac{1}{3}$	Any	Any	14.6.2.87
3	1	0	$-\frac{7}{5}$	$-\frac{3}{5}$	Any	Any	14.6.2.29
3	1	0	$-\frac{1}{2}$	0	Any	Any	14.6.2.27
3	1	0	0	$-\frac{1}{2}$	Any	Any	14.6.2.85
3	1	0	0	2	Any	Any	14.6.2.73
3	1	0	1	-3	Any	Any	14.6.2.50
3	1	0	1	0	Any	Any	14.6.2.43
3	1	0	1	3	Any	Any	14.6.2.99
3	$\frac{3}{2}$	-2	$-\frac{5}{2}$	1	Any	-12	14.6.2.146
3	2	0	-5	-5	Any	Any	14.6.2.96
3	2	0	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.35
3	2	0	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	14.6.2.37
3	2	1	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	14.6.2.33
3	3	-2	-4	1	Any	-6	14.6.2.140
3	3	0	-7	-3	Any	Any	14.6.2.18
3	4	3	-7	-7	Any	Any	14.6.2.111
3	6	-2	-7	1	Any	$-\frac{15}{4}$	14.6.2.142

► **Solvable equations and their solutions.**

1. $y''_{xx} = (A_1 y^{m_1} + A_2 y^{m_2})(y'_x)^l, \qquad l \neq 2, \;\; m_1 \neq -1, \;\; m_2 \neq -1.$

1°. Solution in parametric form:

$$x = a \int \left(C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{\frac{1}{l-2}} d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = \dfrac{m_1+1}{2-l} a^{l-2} b^{1-m_1-l}$, $A_2 = \pm \dfrac{m_2+1}{2-l} a^{l-2} b^{1-m_2-l}$.

2°. Solution in parametric form:

$$x = a \int \left(C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{\frac{1}{l-2}} d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = \dfrac{m_1+1}{l-2} a^{l-2} b^{1-m_1-l}$, $A_2 = \pm\dfrac{m_2+1}{2-l} a^{l-2} b^{1-m_2-l}$.

2. $\boldsymbol{y''_{xx} = (A_1 y^m + A_2 y^{-1})(y'_x)^l, \qquad l \neq 2,\ m \neq -1.}$

Solution: $x = \displaystyle\int \Bigl[C_1 + \frac{A_1(2-l)}{m+1} y^{m+1} + (2-l)A_2 \ln y\Bigr]^{\frac{1}{l-2}} dy + C_2.$

3. $\boldsymbol{y''_{xx} = (A_1 y^{m_1} + A_2 y^{m_2})(y'_x)^2, \qquad m_1 \neq -1,\ m_2 \neq -1.}$

Solution: $x = C_1 \displaystyle\int \exp\Bigl(-\frac{A_1}{m_1+1} y^{m_1+1} - \frac{A_2}{m_2+1} y^{m_2+1}\Bigr)\, dy + C_2.$

4. $\boldsymbol{y''_{xx} = (A_1 y^m + A_2 y^{-1})(y'_x)^2, \qquad m \neq -1.}$

Solution: $x = C_1 \displaystyle\int y^{-A_2} \exp\Bigl(-\frac{A_1}{m+1} y^{m+1}\Bigr)\, dy + C_2.$

5. $\boldsymbol{y''_{xx} = (A_1 x^{n_1} + A_2 x^{n_2})(y'_x)^l, \qquad l \neq 1,\ n_1 \neq -1,\ n_2 \neq -1.}$

1°. Solution in parametric form:

$$x = a\tau, \qquad y = b \int \bigl(C_1 + \tau^{n_1+1} \pm \tau^{n_2+1}\bigr)^{\frac{1}{1-l}} d\tau + C_2,$$

where $A_1 = \dfrac{n_1+1}{1-l} a^{l-n_1-2} b^{1-l}$, $A_2 = \pm\dfrac{n_2+1}{1-l} a^{l-n_2-2} b^{1-l}$.

2°. Solution in parametric form:

$$x = a\tau, \qquad y = b \int \bigl(C_1 - \tau^{n_1+1} \pm \tau^{n_2+1}\bigr)^{\frac{1}{1-l}} d\tau + C_2,$$

where $A_1 = \dfrac{n_1+1}{l-1} a^{l-n_1-2} b^{1-l}$, $A_2 = \pm\dfrac{n_2+1}{1-l} a^{l-n_2-2} b^{1-l}$.

6. $\boldsymbol{y''_{xx} = (A_1 x^n + A_2 x^{-1})(y'_x)^l, \qquad l \neq 1,\ n \neq -1.}$

Solution: $y = \displaystyle\int \Bigl[C_1 + \frac{A_1(1-l)}{n+1} x^{n+1} + (1-l)A_2 \ln x\Bigr]^{\frac{1}{1-l}} dx + C_2.$

7. $\boldsymbol{y''_{xx} = (A_1 x^{n_1} + A_2 x^{n_2}) y'_x, \qquad n_1 \neq -1,\ n_2 \neq -1.}$

Solution: $y = C_1 \displaystyle\int \exp\Bigl(\frac{A_1}{n_1+1} x^{n_1+1} + \frac{A_2}{n_2+1} x^{n_2+1}\Bigr)\, dx + C_2.$

8. $\boldsymbol{y''_{xx} = (A_1 x^n + A_2 x^{-1}) y'_x, \qquad n \neq -1.}$

Solution: $y = C_1 \displaystyle\int x^{A_2} \exp\Bigl(\frac{A_1}{n+1} x^{n+1}\Bigr)\, dx + C_2.$

9. $\boldsymbol{y''_{xx} = (A_1 x^{-m_1-3} y^{m_1} + A_2 x^{-m_2-3} y^{m_2})(y'_x)^3, \quad m_1 \neq -2,\ m_2 \neq -2.}$

1°. Solution in parametric form:

$$x = a\tau \Bigl[\int \bigl(C_1 + \tau^{-m_1-2} \pm \tau^{-m_2-2}\bigr)^{-1/2} d\tau + C_2\Bigr]^{-1},$$
$$y = b \Bigl[\int \bigl(C_1 + \tau^{-m_1-2} \pm \tau^{-m_2-2}\bigr)^{-1/2} d\tau + C_2\Bigr]^{-1},$$

where $A_1 = \frac{1}{2} a^{m_1+4} b^{-m_1-2}(m_1+2)$, $A_2 = \pm\frac{1}{2} a^{m_2+4} b^{-m_2-2}(m_2+2)$.

2°. Solution in parametric form:

$$x = a\tau \Big[\int \big(C_1 - \tau^{-m_1-2} \pm \tau^{-m_2-2}\big)^{-1/2}\,d\tau + C_2\Big]^{-1},$$
$$y = b\Big[\int \big(C_1 - \tau^{-m_1-2} \pm \tau^{-m_2-2}\big)^{-1/2}\,d\tau + C_2\Big]^{-1},$$

where $A_1 = -\frac{1}{2}a^{m_1+4}b^{-m_1-2}(m_1+2)$, $A_2 = \pm\frac{1}{2}a^{m_2+4}b^{-m_2-2}(m_2+2)$.

10. $y''_{xx} = (A_1 x^{-4} y + A_2 x^{-3})(y'_x)^3.$

1°. Solution in parametric form:

$$x = a\tau\Big(\int \frac{d\tau}{\sqrt{C_1 + \tau^{-3} \pm \tau^{-2}}} + C_2\Big)^{-1}, \quad y = b\Big(\int \frac{d\tau}{\sqrt{C_1 + \tau^{-3} \pm \tau^{-2}}} + C_2\Big)^{-1},$$

where $A_1 = \frac{3}{2}a^5 b^{-3}$, $A_2 = \pm a^4 b^{-2}$.

2°. Solution in parametric form:

$$x = a\tau\Big(\int \frac{d\tau}{\sqrt{C_1 - \tau^{-3} \pm \tau^{-2}}} + C_2\Big)^{-1}, \quad y = b\Big(\int \frac{d\tau}{\sqrt{C_1 - \tau^{-3} \pm \tau^{-2}}} + C_2\Big)^{-1},$$

where $A_1 = -\frac{3}{2}a^5 b^{-3}$, $A_2 = \pm a^4 b^{-2}$.

11. $y''_{xx} = (A_1 + A_2 x^{-1})(y'_x)^3.$

Solution: $y = \int (C_1 - 2A_1 x - 2A_2 \ln x)^{-1/2}\,dx + C_2.$

12. $y''_{xx} = (A_1 x^{-1} y^{-2} + A_2 y^{-3})(y'_x)^3.$

Solution in parametric form:

$$x = \tau\Big(\int \frac{d\tau}{\sqrt{C_1 - 2A_1 \ln\tau - 2A_2\tau}} + C_2\Big)^{-1}, \quad y = \Big(\int \frac{d\tau}{\sqrt{C_1 - 2A_1 \ln\tau - 2A_2\tau}} + C_2\Big)^{-1}.$$

13. $y''_{xx} = (A_1 x^{-1/2} y^{-5/2} + A_2 x^{-1/2} y^{-7/2})(y'_x)^3.$

Solution in parametric form:

$$x = \frac{k^2}{F}\big\{2C_1 e^{2k\tau} + C_2 e^{-k\tau}\big[\sqrt{3}\cos(\omega\tau) - \sin(\omega\tau)\big]\big\}^2, \qquad y = \frac{1}{F},$$

where $F = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\omega\tau) - \dfrac{A_1}{A_2}$, $A_2 = -16k^3$, $\omega = k\sqrt{3}$.

14. $y''_{xx} = (A_1 x^{-5/3} y^{-4/3} + A_2 x^{-5/3} y^{-7/3})(y'_x)^3.$

Solution in parametric form:

$$x = \big(\tfrac{1}{36}A_2\tau^4 + C_1\tau^3 + C_2\tau^2 + C_3\tau\big)^{-1}\big(\tfrac{1}{9}A_2\tau^3 + 3C_1\tau^2 + 2C_2\tau + C_3\big)^{3/2},$$
$$y = \big(\tfrac{1}{36}A_2\tau^4 + C_1\tau^3 + C_2\tau^2 + C_3\tau\big)^{-1},$$

where $A_1 = 9C_1C_3 - 3C_2^2$.

◆ *In the solutions of equations 15–18, the following notation is used:*

$$R_1=\sqrt{C_1+\tau^{-3}\pm\tau^{-2}},\quad E_1=\int\frac{d\tau}{R_1}+C_2,\quad F_1=\tau-R_1E_1,\quad H_1=3\tau^3F_1^2+3(1\pm\tau)E_1^2,$$
$$R_2=\sqrt{C_1-\tau^{-3}\pm\tau^{-2}},\quad E_2=\int\frac{d\tau}{R_2}+C_2,\quad F_2=\tau-R_2E_2,\quad H_2=3\tau^3F_2^2+3(-1\pm\tau)E_2^2.$$

15. $y''_{xx}=(A_1y^{-4/3}+A_2x^{-1/2}y^{-4/3})(y'_x)^3.$

Solution in parametric form:

$$x=aF_k^2,\qquad y=b\tau^{-3}E_k^3,$$

where $A_1=\pm\frac{2}{9}ab^{-2/3}$, $A_2=\frac{1}{3}a^{3/2}b^{-2/3}(-1)^{k+1}$; $k=1$ and $k=2$.

16. $y''_{xx}=(A_1x^{-1/2}y^{-7/6}+A_2y^{-5/3})(y'_x)^3.$

Solution in parametric form:

$$x=a\tau^3E_k^{-3}F_k^2,\qquad y=b\tau^3E_k^{-3},$$

where $A_1=\frac{1}{3}a^{3/2}b^{-5/6}(-1)^k$, $A_2=\mp\frac{2}{9}ab^{-1/3}$; $k=1$ and $k=2$.

17. $y''_{xx}=(A_1x^{-7}y+A_2x^{-3})(y'_x)^3.$

Solution in parametric form:

$$x=a\tau^{-1/2}E_k^{1/2},\qquad y=b\tau^{-3}H_k,$$

where $A_1=\frac{1}{36}a^8b^{-3}$, $A_2=\pm\frac{1}{36}a^4b^{-2}$; $k=1$ and $k=2$.

18. $y''_{xx}=(A_1x^{-7}y^3+A_2x^{-3})(y'_x)^3.$

Solution in parametric form:

$$x=a\tau^{5/2}E_k^{1/2}H_k^{-1},\qquad y=b\tau^3H_k^{-1},$$

where $A_1=\frac{1}{36}a^8b^{-5}$, $A_2=\mp\frac{1}{36}a^4b^{-2}$; $k=1$ and $k=2$.

◆ *In the solutions of equations 19–22, the following notation is used:*

$$R_1=\sqrt{C_1\pm\tau^{\gamma+1}+\tau},\quad E_1=\int\frac{d\tau}{R_1}+C_2,\quad F_1=2R_1-E_1,\quad H_1=4(\tau-R_1F_1)+E_1^2,$$
$$R_2=\sqrt{C_1\pm\tau^{\gamma+1}-\tau},\quad E_2=\int\frac{d\tau}{R_2}+C_2,\quad F_2=2R_2+E_2,\quad H_2=4(\tau-R_2F_2)-E_2^2.$$

19. $y''_{xx}=(A_1x^{-m-3}y^m+A_2y^{-3})(y'_x)^3,\qquad m\neq-2.$

Solution in parametric form:

$$x=a\tau E_k^{-1},\quad y=bE_k^{-1},\qquad \gamma=-m-3,$$

where $A_1=\pm\frac{1}{2}a^{m+4}b^{-m-2}(m+2)$, $A_2=\frac{1}{2}ab(-1)^k$; $k=1$ and $k=2$.

20. $y''_{xx} = (A_1 y + A_2 x)(y'_x)^l, \qquad l \neq 1, \ l \neq 2.$

Solution in parametric form:

$$x = aF_k, \quad y = bE_k, \qquad \gamma = \frac{1}{l-2},$$

where $A_1 = ab^{-1}A_2(-1)^{k+1}$, $A_2 = -\dfrac{\gamma}{2ab}\left[\pm\dfrac{(\gamma+1)a}{b}\right]^{1/\gamma}$; $k=1$ and $k=2$.

21. $y''_{xx} = (A_1 x y^m + A_2 y^{-m-2})(y'_x)^{\frac{3m+5}{2m+3}}.$

Solution in parametric form:

$$x = aH_k, \quad y = bE_k^{\gamma+2}, \qquad \gamma = -\frac{2m+3}{m+1},$$

where $A_1 = \dfrac{\gamma}{4(\gamma+2)}a^{-1}b^{\frac{1}{\gamma+2}}\left[\mp\dfrac{2(\gamma+1)a}{(\gamma+2)b}\right]^{\frac{1}{\gamma}}$, $A_2 = ab^{-\frac{2}{\gamma+2}}A_1(-1)^k$; $k=1$ and $k=2$.

22. $y''_{xx} = (A_1 x^n y + A_2 x^{-n-2})(y'_x)^{\frac{3n+4}{n+1}}.$

Solution in parametric form:

$$x = aE_k^{\gamma+2}, \quad y = bH_k, \qquad \gamma = -\frac{2n+3}{n+1},$$

where $A_1 = -\dfrac{\gamma}{4(\gamma+2)}a^{\frac{1}{\gamma+2}}b^{-1}\left[\mp\dfrac{2(\gamma+1)b}{(\gamma+2)a}\right]^{\frac{1}{\gamma}}$, $A_2 = a^{-\frac{2}{\gamma+2}}bA_1(-1)^k$; $k=1$ and $k=2$.

◆ *In the solutions of equations 23–26, the following notation is used:*

$$R_1 = \sqrt{C_1 + \tau \pm \ln \tau}, \quad E_1 = \int \frac{d\tau}{R_1} + C_2, \quad F_1 = 2R_1 - E_1, \quad H_1 = 4(\tau - R_1F_1) + E_1^2,$$
$$R_2 = \sqrt{C_1 - \tau \pm \ln \tau}, \quad E_2 = \int \frac{d\tau}{R_2} + C_2, \quad F_2 = 2R_2 + E_2, \quad H_2 = 4(\tau - R_2F_2) - E_2^2.$$

23. $y''_{xx} = (A_1 y + A_2 x)y'_x.$

Solution in parametric form:

$$x = aF_k, \quad y = bE_k,$$

where $A_1 = ab^{-1}A_2(-1)^k$, $A_2 = \pm\frac{1}{2}a^{-2}$; $k=1$ and $k=2$.

24. $y''_{xx} = (A_1 y + A_2 x)(y'_x)^2.$

Solution in parametric form:

$$x = aE_k, \quad y = bF_k,$$

where $A_1 = \mp\frac{1}{2}b^{-2}$, $A_2 = a^{-1}bA_1(-1)^k$; $k=1$ and $k=2$.

25. $y''_{xx} = (A_1 + A_2 x y^{-2})y'_x.$

Solution in parametric form:

$$x = aH_k, \quad y = bE_k,$$

where $A_1 = ab^{-2}A_2(-1)^k$, $A_2 = \pm\frac{1}{8}a^{-2}b^{-2}$; $k=1$ and $k=2$.

26. $y''_{xx} = (A_1 x^{-2} y + A_2)(y'_x)^2.$

Solution in parametric form:

$$x = aE_k, \qquad y = bH_k,$$

where $A_1 = \mp\frac{1}{8}a^2 b^{-2}$, $A_2 = a^{-2}bA_1(-1)^k$; $k = 1$ and $k = 2$.

◆ *In the solutions of equations 27–30, the following notation is used:*

$$\begin{aligned}
R_1 &= C_1\tau^{k_1} + C_2\tau^{k_2} + C_3\tau^{k_3},\\
R_2 &= (C_1 + C_2\tau)e^{k\tau} + C_3 e^{\omega\tau},\\
R_3 &= C_1 e^{k\tau} + e^{s\tau}(C_2 \sin\omega\tau + C_3\cos\omega\tau),\\
Q_1 &= C_1 k_1\tau^{k_1} + C_2 k_2\tau^{k_2} + C_3 k_3\tau^{k_3},\\
Q_2 &= (kC_1 + C_2 + kC_2\tau)e^{k\tau} + \omega C_3 e^{\omega\tau},\\
Q_3 &= kC_1 e^{k\tau} + e^{s\tau}[(sC_2 - \omega C_3)\sin\omega\tau + (sC_3 + \omega C_2)\cos\omega\tau],\\
S_1 &= \tau(Q_1)'_\tau, \quad S_2 = (Q_2)'_\tau, \quad S_3 = (Q_3)'_\tau,
\end{aligned}$$

where k_1, k_2, and k_3 (real numbers) or k and $s \pm i\omega$ (one real and two complex numbers) are roots of the cubic equation $\lambda^3 - \frac{1}{2}B_2\lambda - \frac{1}{2}B_1 = 0$. The subscripts of the functions R_m, Q_m, and S_m $(m = 1,\ 2,\ 3)$ are selected depending on the sign of the expression $\Delta = 2B_2^3 - 27B_1^2$:

$$\begin{aligned}
&\textit{if} \quad \Delta > 0 \quad \textit{subscript} \quad m = 1,\\
&\textit{if} \quad \Delta = 0 \quad \textit{subscript} \quad m = 2,\\
&\textit{if} \quad \Delta < 0 \quad \textit{subscript} \quad m = 3.
\end{aligned}$$

If $2B_2^3 = 27B_1^2$ (subscript $m = 2$), then

$$\begin{aligned}
k &= (\tfrac{1}{6}B_2)^{1/2}, \quad &\omega &= -2(\tfrac{1}{6}B_2)^{1/2} \quad \textit{if} \quad B_1 < 0,\\
k &= -(\tfrac{1}{6}B_2)^{1/2}, \quad &\omega &= 2(\tfrac{1}{6}B_2)^{1/2} \quad \textit{if} \quad B_1 > 0.
\end{aligned}$$

Remark 14.4. *The expressions for R_m, and Q_m contain three constants C_1, C_2, and C_3. One of them can be arbitrarily fixed to let it be any nonzero number (for instance, we can set $C_3 = \pm1$), while the other constants remain arbitrary.*

27. $y''_{xx} = (A_1 x^{-1/2} y + A_2)(y'_x)^3.$

Solution in parametric form:

$$x = Q_m^2, \quad y = R_m, \qquad \text{where} \quad A_1 = -B_1, \quad A_2 = -B_2.$$

28. $y''_{xx} = (A_1 y^{-3} + A_2 x^{-1/2} y^{-7/2})(y'_x)^3.$

Solution in parametric form:

$$x = R_m^{-1}Q_m^2, \quad y = R_m^{-1}, \qquad \text{where} \quad A_1 = -B_2, \quad A_2 = -B_1.$$

29. $y''_{xx} = (A_1 x^{-7/5} y + A_2 x^{-3/5})(y'_x)^3.$

Solution in parametric form:

$$x = aR_m^{5/2}, \qquad y = b(2Q_m^2 - 4R_m S_m + B_2 R_m^2),$$

where $A_1 = \frac{5}{32}a^{12/5}b^{-3}B_1^{-2}$, $A_2 = -a^{-4/5}bA_1B_2$.

30. $\boldsymbol{y''_{xx} = (A_1 x^{-3/5} y^{-12/5} + A_2 x^{-7/5} y^{-13/5})(y'_x)^3.}$

Solution in parametric form:

$$x = aR_m^{5/2}\bigl(2Q_m^2 - 4R_m S_m + B_2 R_m^2\bigr)^{-1}, \qquad y = b\bigl(2Q_m^2 - 4R_m S_m + B_2 R_m^2\bigr)^{-1},$$

where $A_1 = -\frac{5}{32} a^{8/5} b^{2/5} B_1^{-2} B_2$, $A_2 = \frac{5}{32} a^{12/5} b^{3/5} B_1^{-2}$.

◆ *In the solutions of equations 31 and 32, the following notation is used:*

$$f_1 = \begin{cases} C_1 e^{k\tau} + C_2 e^{-k\tau} - \dfrac{B_1}{B_2}\tau & \text{if } B_2 > 0, \\ C_1 \sin(k\tau) + C_2 \cos(k\tau) - \dfrac{B_1}{B_2}\tau & \text{if } B_2 < 0, \end{cases}$$

$$f_2 = \begin{cases} k(C_1 e^{k\tau} - C_2 e^{-k\tau}) - \dfrac{B_1}{B_2} & \text{if } B_2 > 0, \\ k[C_1 \cos(k\tau) - C_2 \sin(k\tau)] - \dfrac{B_1}{B_2} & \text{if } B_2 < 0, \end{cases}$$

where $k = \sqrt{\frac{1}{2}|B_2|}$.

31. $\boldsymbol{y''_{xx} = (A_1 + A_2 x^{-1/2})(y'_x)^3.}$

Solution in parametric form:

$$x = f_2^2, \quad y = f_1, \qquad \text{where} \quad A_1 = -B_2, \quad A_2 = -B_1.$$

32. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} y^{-5/2} + A_2 y^{-3})(y'_x)^3.}$

Solution in parametric form:

$$x = f_1^{-1} f_2^2, \quad y = f_1^{-1}, \qquad \text{where} \quad A_1 = -B_1, \quad A_2 = -B_2.$$

◆ *In the solutions of equations 33–36, the following notation is used:*
For $B_1 > 0$,

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin(k\tau), \quad k = \bigl(\tfrac{4}{3} B_1\bigr)^{1/4},$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + kC_3 \cos(k\tau).$$

For $B_1 < 0$,

$$T_1 = e^{s\tau}[C_1 \sin(s\tau) + C_2 \cos(s\tau)] + C_3 e^{-s\tau} \sin(s\tau), \quad s = \bigl(-\tfrac{1}{3} B_1\bigr)^{1/4},$$
$$T_2 = se^{s\tau}[(C_1 - C_2)\sin(s\tau) + (C_1 + C_2)\cos(s\tau)] - sC_3 e^{-s\tau}[\sin(s\tau) - \cos(s\tau)].$$

33. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^2 + A_2 x^{-5/3} y)(y'_x)^3.}$

Solution in parametric form:

$$x = T_2^{3/2}, \quad y = T_1 - \frac{A_2}{2A_1},$$

where $B_1 = -A_1$, $B_2 = -A_2$; the constants C_1, C_2, and C_3 are related by the constraint

$$C_1 C_3 = \tfrac{1}{16} A_1^{-2} A_2^2 \qquad \text{if } A_1 > 0,$$
$$4C_1 C_2 + C_3^2 = \tfrac{1}{4} A_1^{-2} A_2^2 \quad \text{if } A_1 < 0.$$

34. $y''_{xx} = (A_1 x^{-5/3} y^{-7/3} + A_2 x^{-5/3} y^{-10/3})(y'_x)^3.$

Solution in parametric form:

$$x = \left(T_1 - \frac{A_1}{2A_2}\right)^{-1} T_2^{3/2}, \qquad y = \left(T_1 - \frac{A_1}{2A_2}\right)^{-1},$$

where $B_1 = -A_2,\ B_2 = -A_1$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} C_1 C_3 &= \tfrac{1}{16} A_1^2 A_2^{-2} && \text{if } A_2 > 0,\\ 4C_1C_2 + C_3^2 &= \tfrac{1}{4} A_1^2 A_2^{-2} && \text{if } A_2 < 0. \end{aligned}$$

35. $y''_{xx} = (A_1 x^{-5/3} y^2 + A_2 x^{-5/3})(y'_x)^3.$

Solution in parametric form:

$$x = T_2^{3/2}, \qquad y = T_1,$$

where $B_1 = -A_1,\ B_2 = -A_2$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} C_1 C_3 &= -\tfrac{1}{4} A_1^{-1} A_2 && \text{if } A_1 > 0,\\ 4C_1C_2 + C_3^2 &= -\tfrac{1}{2} A_1^{-1} A_2 && \text{if } A_1 < 0. \end{aligned}$$

36. $y''_{xx} = (A_1 x^{-5/3} y^{-4/3} + A_2 x^{-5/3} y^{-10/3})(y'_x)^3.$

Solution in parametric form:

$$x = T_1^{-1} T_2^{3/2}, \qquad y = T_1^{-1},$$

where $B_1 = -A_2,\ B_2 = -A_1$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} C_1 C_3 &= -\tfrac{1}{4} A_1 A_2^{-1} && \text{if } A_2 > 0,\\ 4C_1C_2 + C_3^2 &= -\tfrac{1}{2} A_1 A_2^{-1} && \text{if } A_2 < 0. \end{aligned}$$

◆ *In the solutions of equations 37 and 38, the following notation is used:*

1°. *For* $B_1 > 0$, $B_2 \neq 0$:

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin \omega\tau, \qquad T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + \omega C_3 \cos \omega\tau,$$

where

$$k = \left\{\tfrac{2}{3}[(B_2^2 + 3B_1)^{1/2} + B_2]\right\}^{1/2}, \quad \omega = \left\{\tfrac{2}{3}[(B_2^2 + 3B_1)^{1/2} - B_2]\right\}^{1/2}, \quad 4k^2 C_1 C_2 + \omega^2 C_3^2 = 0.$$

2°. *For* $-B_2^2 < 3B_1 < 0$, $B_2 > 0$:

$$T_1 = C_1 \tau^{k_1} + C_2 \tau^{-k_1} + C_3 \tau^{k_2} + C_4 \tau^{-k_2}, \quad T_2 = k_1 (C_1 \tau^{k_1} - C_2 \tau^{-k_1}) + k_2 (C_3 \tau^{k_2} - C_4 \tau^{-k_2}),$$

where

$$k_1 = \left\{\tfrac{2}{3}[B_2 + (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2}, \quad k_2 = \left\{\tfrac{2}{3}[B_2 - (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2},$$
$$(C_1 C_2 + C_3 C_4)(B_2^2 + 3B_1)^{1/2} + (C_1 C_2 - C_3 C_4) B_2 = 0.$$

3°. *For* $-B_2^2 < 3B_1 < 0$, $B_2 < 0$:

$$\begin{aligned} T_1 &= C_1 \sin \omega_1\tau + C_2 \cos \omega_1\tau + C_3 \sin \omega_2\tau,\\ T_2 &= \omega_1 (C_1 \cos \omega_1\tau - C_2 \sin \omega_1\tau) + \omega_2 C_3 \cos \omega_2\tau, \end{aligned}$$

where

$$\omega_1 = \left\{-\tfrac{2}{3}[B_2 + (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2}, \quad \omega_2 = \left\{-\tfrac{2}{3}[B_2 - (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2},$$
$$\omega_1^2(C_1^2 + C_2^2) - \omega_2^2 C_3^2 = 0.$$

4°. *For* $B_2^2 + 3B_1 = 0$, $B_2 > 0$:

$$T_1 = (C_1 + C_2\tau)e^{k\tau} + (C_3 + C_4\tau)e^{-k\tau},$$
$$T_2 = (kC_1 + C_2 + kC_2\tau)e^{k\tau} - (kC_3 - C_4 + kC_4\tau)e^{-k\tau},$$

where

$$k = \left(\tfrac{2}{3}B_2\right)^{1/2}, \quad (C_1C_4 - C_2C_3)\left(\tfrac{2}{3}B_2\right)^{1/2} + 2C_2C_4 = 0.$$

5°. *For* $B_2^2 + 3B_1 = 0$, $B_2 < 0$:

$$T_1 = (C_1 + C_2\tau)\sin\omega\tau + C_3\tau\cos\omega\tau,$$
$$T_2 = (\omega C_1 + C_3 + \omega C_2\tau)\cos\omega\tau + (C_2 - \omega C_3\tau)\sin\omega\tau,$$

where

$$\omega = \left(-\tfrac{2}{3}B_2\right)^{1/2}, \quad C_1C_3\left(-\tfrac{2}{3}B_2\right)^{1/2} + C_2^2 + C_3^2 = 0.$$

6°. *For* $3B_1 < -B_2^2$:

$$T_1 = e^{k\tau}(C_1\sin\omega\tau + C_2\cos\omega\tau) + C_3e^{-k\tau}\sin\omega\tau,$$
$$T_2 = e^{k\tau}[(kC_2 + \omega C_1)\cos\omega\tau + (kC_1 - \omega C_2)\sin\omega\tau)] + C_3e^{-k\tau}(\omega\cos\omega\tau - k\sin\omega\tau),$$

where

$$k = \left\{\tfrac{1}{3}[B_2 + (-3B_1)^{1/2}]\right\}^{1/2}, \quad \omega = \left\{\tfrac{1}{3}[-B_2 + (-3B_1)^{1/2}]\right\}^{1/2},$$
$$C_2B_2 + C_1(-B_2^2 - 3B_1)^{1/2} = 0.$$

37. $\boldsymbol{y''_{xx} = (A_1x^{-5/3}y^2 + A_2x^{-1/3})(y'_x)^3.}$

Solution in parametric form:

$$x = T_2^{3/2}, \quad y = T_1, \qquad \text{where} \quad B_1 = -A_1, \quad B_2 = -A_2.$$

38. $\boldsymbol{y''_{xx} = (A_1x^{-1/3}y^{-8/3} + A_2x^{-5/3}y^{-10/3})(y'_x)^3.}$

Solution in parametric form:

$$x = T_1^{-1}T_2^{3/2}, \quad y = T_1^{-1}, \qquad \text{where} \quad B_1 = -A_2, \quad B_2 = -A_1.$$

◆ *In the solutions of equations 39–42, the following notation is used:*

$$T_1 = \begin{cases} C_1e^{\omega\tau} + C_2e^{-\omega\tau} + C_3\tau & \text{if } B > 0, \\ C_1\sin\omega\tau + C_2\cos\omega\tau + C_3\tau & \text{if } B < 0, \end{cases}$$
$$T_2 = \begin{cases} \omega(C_1e^{\omega\tau} - C_2e^{-\omega\tau}) + C_3 & \text{if } B > 0, \\ \omega(C_1\cos\omega\tau - C_2\sin\omega\tau) + C_3 & \text{if } B < 0, \end{cases}$$

where $\omega = |\tfrac{4}{3}B|^{1/2}$.

39. $y''_{xx} = (A_1 x^{-1/3} + A_2 x^{-5/3})(y'_x)^3$.

Solution in parametric form:

$$x = T_2^{3/2}, \quad y = T_1,$$

where $B = -A_1$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &3(A_1 C_3^2 + A_2) - 4A_1^2(C_1^2 + C_2^2) = 0 && \text{if } A_1 > 0,\\ &3(A_1 C_3^2 + A_2) - 16A_1^2 C_1 C_2 = 0 && \text{if } A_1 < 0. \end{aligned}$$

40. $y''_{xx} = (A_1 x^{-5/3} y^{-4/3} + A_2 x^{-1/3} y^{-8/3})(y'_x)^3$.

Solution in parametric form:

$$x = T_1^{-1} T_2^{3/2}, \quad y = T_1^{-1},$$

where $B = -A_2$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &3(A_1 + A_2 C_3^2) - 4A_2^2(C_1^2 + C_2^2) = 0 && \text{if } A_2 > 0,\\ &3(A_1 + A_2 C_3^2) - 16A_2^2 C_1 C_2 = 0 && \text{if } A_2 < 0. \end{aligned}$$

41. $y''_{xx} = (A_1 x^{-5/3} y + A_2 x^{-1/3})(y'_x)^3$.

Solution in parametric form:

$$x = \left(T_2 - \frac{A_1}{2A_2}\tau\right)^{3/2}, \quad y = T_1 - \frac{A_1}{4A_2}\tau^2,$$

where $B = -A_2$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &3A_2 C_3^2 - 4A_2^2(C_1^2 + C_2^2) - \tfrac{9}{16} A_1^2 A_2^{-2} = 0 && \text{if } A_2 > 0,\\ &3A_2 C_3^2 - 16A_2^2 C_1 C_2 - \tfrac{9}{16} A_1^2 A_2^{-2} = 0 && \text{if } A_2 < 0. \end{aligned}$$

42. $y''_{xx} = (A_1 x^{-5/3} y^{-7/3} + A_2 x^{-1/3} y^{-8/3})(y'_x)^3$.

Solution in parametric form:

$$x = \left(T_1 - \frac{A_1}{4A_2}\tau^2\right)^{-1}\left(T_2 - \frac{A_1}{2A_2}\tau\right)^{3/2}, \quad y = \left(T_1 - \frac{A_1}{4A_2}\tau^2\right)^{-1},$$

where $B = -A_2$; the constants C_1, C_2, and C_3 are related by the constraint

$$\begin{aligned} &3A_2 C_3^2 - 4A_2^2(C_1^2 + C_2^2) - \tfrac{9}{16} A_1^2 A_2^{-2} = 0 && \text{if } A_2 > 0,\\ &3A_2 C_3^2 - 16A_2^2 C_1 C_2 - \tfrac{9}{16} A_1^2 A_2^{-2} = 0 && \text{if } A_2 < 0. \end{aligned}$$

◆ *In the solutions of equations 43–48, the following notation is used:*

$$f = \begin{cases} J_{1/3}(\tau) & \textit{for the upper sign (Bessel function)},\\ I_{1/3}(\tau) & \textit{for the lower sign (modified Bessel function)}, \end{cases}$$

$$g = \begin{cases} Y_{1/3}(\tau) & \textit{for the upper sign (Bessel function)},\\ K_{1/3}(\tau) & \textit{for the lower sign (modified Bessel function)}, \end{cases}$$

$$H = C_1 f + C_2 g + \beta\omega\left(g\int f\,d\tau - f\int g\,d\tau\right), \quad \omega = \begin{cases} \frac{1}{2}\pi & \textit{for the upper sign},\\ -1 & \textit{for the lower sign}. \end{cases}$$

43. $y''_{xx} = (A_1 xy + A_2)(y'_x)^3.$

Solution in parametric form:

$$x = \tau^{1/3} H, \quad y = b\tau^{2/3}, \qquad \text{where} \quad A_1 = \pm\tfrac{9}{4} b^{-3}, \quad A_2 = -\tfrac{9}{4} b^{-2}\beta.$$

44. $y''_{xx} = (A_1 y^{-3} + A_2 xy^{-5})(y'_x)^3.$

Solution in parametric form:

$$x = \tau^{-1/3} H, \quad y = b\tau^{-2/3}, \qquad \text{where} \quad A_1 = -\tfrac{9}{4} b\beta, \quad A_2 = \pm\tfrac{9}{4} b^3.$$

45. $y''_{xx} = (A_1 x^{-1/2} y^{-1/2} + A_2 y^{-3/2})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-2/3}(\tau H'_\tau + \tfrac{1}{3}H)^2, \quad y = b\tau^{2/3} H^2,$$

where $A_1 = \pm\frac{1}{3} a^{3/2} b^{-3/2}$, $A_2 = \frac{1}{2} ab^{-1/2}\beta$.

46. $y''_{xx} = (A_1 y^{-3/2} + A_2 x^{-1/2} y^{-2})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-4/3} H^{-2}(\tau H'_\tau + \tfrac{1}{3}H)^2, \quad y = b\tau^{-2/3} H^{-2},$$

where $A_1 = \frac{1}{2} ab^{-1/2}\beta$, $A_2 = \pm\frac{1}{3} a^{3/2}$.

47. $y''_{xx} = (A_1 x^{-2} y + A_2 x^{-3/2})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{2/3} H^2, \qquad y = b\tau^{-2/3}\bigl[\mp\tau^2 H^2 + 2\beta\tau H - (\tau H'_\tau + \tfrac{1}{3}H)^2\bigr],$$

where $A_1 = -\frac{9}{2} a^3 b^{-3}$, $A_2 = -a^{-1/2} b\beta A_1$.

48. $y''_{xx} = (A_1 x^{-3/2} y^{-3/2} + A_2 x^{-2} y^{-2})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{4/3} H^2\bigl[\mp\tau^2 H^2 + 2\beta\tau H - (\tau H'_\tau + \tfrac{1}{3}H)^2\bigr]^{-1},$$
$$y = b\tau^{2/3}\bigl[\mp\tau^2 H^2 + 2\beta\tau H - (\tau H'_\tau + \tfrac{1}{3}H)^2\bigr]^{-1},$$

where $A_1 = \frac{9}{2} a^{5/2} b^{-1/2}\beta$, $A_2 = -\frac{9}{2} a^3$.

49. $y''_{xx} = (A_1 xy^{m_1} + A_2 y^{m_2})(y'_x)^3, \qquad m_1 \neq -2.$

Solution in parametric form:

$$x = \tau^\nu H, \quad y = b\tau^{2\nu}, \qquad \nu = \frac{1}{m_1 + 2},$$

where

$$H = C_1 f + C_2 g + \frac{4b^2\beta\omega}{(m_1+2)^2}\Bigl(g\int \tau^k f\,d\tau - f\int \tau^k g\,d\tau\Bigr), \quad k = \frac{2m_2 - m_1 + 1}{m_1 + 2};$$

$$f = \begin{cases} J_\nu(\tau) & \text{for the upper sign (Bessel function),} \\ I_\nu(\tau) & \text{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} Y_\nu(\tau) & \text{for the upper sign (Bessel function),} \\ K_\nu(\tau) & \text{for the lower sign (modified Bessel function),} \end{cases}$$

$$A_1 = \pm\tfrac{1}{4}(m_1+2)^2 b^{-m_1-2}, \quad A_2 = -b^{-m_2}\beta, \quad \omega = \begin{cases} \frac{1}{2}\pi & \text{for the upper sign,} \\ -1 & \text{for the lower sign.} \end{cases}$$

◆ *In the solutions of equations 50–56, the following notation is used:*

$$U_\nu = \begin{cases} C_1 J_\nu(\tau) & \textit{for the upper sign (Bessel function)},\\ C_1 I_\nu(\tau) & \textit{for the lower sign (modified Bessel function)},\end{cases}$$

$$V_\nu = \begin{cases} C_2 Y_\nu(\tau) & \textit{for the upper sign (Bessel function))},\\ C_2 K_\nu(\tau) & \textit{for the lower sign (modified Bessel function)},\end{cases}$$

$$Z_\nu = \alpha_1 U_\nu + \alpha_2 V_\nu,\quad X_\nu = \beta_1 U_\nu + \beta_2 V_\nu,\quad F_\nu = \tau Z_\nu' + \nu Z_\nu,\quad G_\nu = \tau X_\nu' + \nu X_\nu,$$

$$N = \begin{cases} Z_\nu X_\nu & \textit{if } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2,\\ \alpha U_\nu^2 + \beta U_\nu V_\nu + \gamma V_\nu^2 & \textit{if } \Delta = 4\alpha\gamma - \beta^2,\end{cases}$$

$$N_1 = \begin{cases} Z_\nu G_\nu + X_\nu F_\nu & \textit{if } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2,\\ \tau N' + 2\nu N & \textit{if } \Delta = 4\alpha\gamma - \beta^2,\end{cases}$$

$$N_2 = N_1^2 \pm 4\tau^2 N^2 + \omega^2\Delta, \qquad \omega = \begin{cases} 2/\pi & \textit{for the upper sign},\\ -1 & \textit{for the lower sign},\end{cases}$$

where the prime denotes differentiation with respect to τ.

50. $y''_{xx} = (A_1 xy + A_2 x^{-3})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{1/3} N^{1/2}, \qquad y = b\tau^{2/3},$$

where $\nu = \frac{1}{3}$, $A_1 = \pm\frac{9}{4} b^{-3}$, $A_2 = -\frac{9}{16} a^4 b^{-2}\omega^2\Delta$.

51. $y''_{xx} = (A_1 xy^m + A_2 x^{-3})(y'_x)^3, \qquad m \neq -2.$

Solution in parametric form:

$$x = a\tau^{\nu} N^{1/2}, \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{m+2}$, $A_1 = \pm\frac{1}{4} b^{-m-2}(m+2)^2$, $A_2 = -\frac{1}{16} a^4 b^{-2}\omega^2\Delta(m+2)^2$.

52. $y''_{xx} = (A_1 x^{-3} + A_2 xy^{-5})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-1/3} N^{1/2}, \qquad y = b\tau^{-2/3},$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{16} a^4 b^{-2}\omega^2\Delta$, $A_2 = \pm\frac{9}{4} b^3$.

53. $y''_{xx} = (A_1 x^{-1/2} y^{-1/2} + A_2 y^{-3})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-2/3} N^{-1} N_1^2, \qquad y = b\tau^{2/3} N,$$

where $\nu = \frac{1}{3}$, $A_1 = \pm\frac{8}{3} a^{3/2} b^{-3/2}$, $A_2 = 2ab\omega^2\Delta$.

54. $y''_{xx} = (A_1 + A_2 x^{-1/2} y^{-2})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-4/3} N^{-2} N_1^2, \qquad y = b\tau^{-2/3} N^{-1},$$

where $\nu = \frac{1}{3}$, $A_1 = 2ab^{-2}\omega^2\Delta$, $A_2 = \pm\frac{8}{3} a^{3/2}$.

55. $y''_{xx} = (A_1 x^{-2} y + A_2 x^{-3})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{2/3} N, \qquad y = b\tau^{-2/3} N^{-1} N_2,$$

where $\nu = \frac{1}{3}$, $A_1 = \frac{9}{128} a^3 b^{-3}$, $A_2 = -\frac{9}{64} a^4 b^{-2} \omega^2 \Delta$.

56. $y''_{xx} = (A_1 x^{-3} + A_2 x^{-2} y^{-2})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{4/3} N^2 N_2^{-1}, \qquad y = b\tau^{2/3} N N_2^{-1},$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{64} a^4 b^{-2} \omega^2 \Delta$, $A_2 = \frac{9}{128} a^3$.

◆ *In the solutions of equations 57–72, the following notation is used:*

$$f_1 = \sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}, \quad \tau = \int \frac{d\wp_1}{\sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}} - C_1;$$

$$f_2 = \sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}, \quad \tau = \int \frac{d\wp_2}{\sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}} - C_1.$$

The functions $\wp_1 = \wp_1(\tau)$ *and* $\wp_2 = \wp_2(\tau)$ *are defined implicitly by the above elliptic integrals. For the upper signs, they are the classical elliptic Weierstrass functions* $\wp_1 = \wp(\tau + C_1, 2, C_2)$ *and* $\wp_2 = \wp(\tau + C_1, -2, C_2)$.

57. $y''_{xx} = (A_1 x^2 + A_2)(y'_x)^3.$

Solution in parametric form:

$$x = a\wp_k, \qquad y = b\tau,$$

where $A_1 = \mp 6 a^{-1} b^{-2}$, $A_2 = a b^{-2} (-1)^{k+1}$; $k = 1$ and $k = 2$.

58. $y''_{xx} = (A_1 y^{-3} + A_2 x^2 y^{-5})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-1} \wp_k, \qquad y = b\tau^{-1},$$

where $A_1 = ab(-1)^{k+1}$, $A_2 = \mp 6 a^{-1} b^3$; $k = 1$ and $k = 2$.

59. $y''_{xx} = (A_1 y^{-9/7} + A_2 x^2 y^{-15/7})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau(\tau^2 \wp_k \mp 1), \qquad y = b\tau^7,$$

where $A_1 = \frac{1}{49} a b^{-5/7} (-1)^{k+1}$, $A_2 = \mp \frac{6}{49} a^{-1} b^{1/7}$; $k = 1$ and $k = 2$.

60. $y''_{xx} = (A_1 y^{-12/7} + A_2 x^2 y^{-20/7})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-6} (\tau^2 \wp_k \mp 1), \qquad y = b\tau^{-7},$$

where $A_1 = \frac{1}{49} a b^{-2/7} (-1)^{k+1}$, $A_2 = \mp \frac{6}{49} a^{-1} b^{6/7}$; $k = 1$ and $k = 2$.

61. $y''_{xx} = (A_1 y + A_2 x)(y'_x)^{5/2}$.

Solution in parametric form:

$$x = a[f_k - (-1)^k \tau], \qquad y = b\tau,$$

where $A_1 = ab^{-1}A_2(-1)^k$, $A_2 = -2a^{-1}b^{-1}(\pm 6a/b)^{1/2}$; $k = 1$ and $k = 2$.

62. $y''_{xx} = (A_1 y + A_2 x)(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\tau, \qquad y = b[f_k - (-1)^k \tau],$$

where $A_1 = 2a^{-1}b^{-1}(\pm 6b/a)^{1/2}$, $A_2 = a^{-1}bA_1(-1)^k$; $k = 1$ and $k = 2$.

63. $y''_{xx} = (A_1 y^{-3/4} + A_2 x y^{-5/4})(y'_x)^{5/2}$.

Solution in parametric form:

$$x = a[2\tau f_k - 2\wp_k + (-1)^{k+1}\tau^2], \qquad y = b\tau^4,$$

where $A_1 = ab^{-1/2}A_2(-1)^k$, $A_2 = -\frac{1}{4}a^{-1}b^{1/4}(\pm 3a/b)^{1/2}$; $k = 1$ and $k = 2$.

64. $y''_{xx} = (A_1 x^{-5/4} y + A_2 x^{-3/4})(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\tau^4, \qquad y = b[2\tau f_k - 2\wp_k + (-1)^{k+1}\tau^2],$$

where $A_1 = \frac{1}{4}a^{1/4}b^{-1}(\pm 3b/a)^{1/2}$, $A_2 = a^{-1/2}bA_1(-1)^k$; $k = 1$ and $k = 2$.

65. $y''_{xx} = (A_1 y^{-13/8} + A_2 x y^{-15/8})(y'_x)^{5/2}$.

Solution in parametric form:

$$x = a\tau^{-6}[2\tau^3 f_k + 6\tau^2\wp_k \mp 2 + (-1)^k\tau^4], \qquad y = b\tau^{-8},$$

where $A_1 = ab^{-1/4}A_2(-1)^k$, $A_2 = \frac{1}{16}a^{-1}b^{7/8}\bigl(\mp 6a/b\bigr)^{1/2}$; $k = 1$ and $k = 2$.

66. $y''_{xx} = (A_1 x^{-15/8} y + A_2 x^{-13/8})(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\tau^{-8}, \qquad y = b\tau^{-6}[2\tau^3 f_k + 6\tau^2\wp_k \mp 2 + (-1)^k\tau^4],$$

where $A_1 = -\frac{1}{16}a^{7/8}b^{-1}\bigl(\mp 6b/a\bigr)^{1/2}$, $A_2 = a^{-1/4}bA_1(-1)^k$; $k = 1$ and $k = 2$.

67. $y''_{xx} = (A_1 y^{-15/13} + A_2 x y^{-20/13})(y'_x)^{5/2}$.

Solution in parametric form:

$$x = a\tau[5\tau^3 f_k - 20\tau^2\wp_k \pm 30 - (-1)^k\tau^4], \qquad y = b\tau^{13},$$

where $A_1 = ab^{-5/13}A_2(-1)^k$, $A_2 = -\dfrac{2}{65}a^{-1}b^{7/13}\Bigl(\pm\dfrac{30a}{13b}\Bigr)^{1/2}$; $k = 1$ and $k = 2$.

68. $y''_{xx} = (A_1 x^{-20/13} y + A_2 x^{-15/13})(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\tau^{13}, \qquad y = b\tau[5\tau^3 f_k - 20\tau^2 \wp_k \pm 30 - (-1)^k \tau^4],$$

where $A_1 = \dfrac{2}{65} a^{7/13} b^{-1} \Big(\pm\dfrac{30b}{13a}\Big)^{1/2}$, $A_2 = a^{-5/13} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

69. $y''_{xx} = (A_1 + A_2 x^{-2/3})(y'_x)^3$.

Solution in parametric form:

$$x = a\wp_k^3, \qquad y = b[f_k - (-1)^k \tau],$$

where $A_1 = \mp\frac{1}{2} ab^{-2}$, $A_2 = \frac{1}{12} a^{5/3} b^{-2} (-1)^{k+1}$; $k = 1$ and $k = 2$.

70. $y''_{xx} = (A_1 x^{-2/3} y^{-7/3} + A_2 y^{-3})(y'_x)^3$.

Solution in parametric form:

$$x = a\wp_k^3 \big[f_k - (-1)^k \tau\big]^{-1}, \qquad y = b\big[f_k - (-1)^k \tau\big]^{-1},$$

where $A_1 = \frac{1}{12} a^{5/3} b^{1/3} (-1)^{k+1}$, $A_2 = \mp\frac{1}{2} ab$; $k = 1$ and $k = 2$.

71. $y''_{xx} = (A_1 y^{-1/4} + A_2 x y^{-7/4})(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\big(f_k^2 - \tau^2 \mp 4\wp_k^3\big), \qquad y = b\big[f_k - (-1)^k \tau\big]^{4/3},$$

where $A_1 = ab^{-3/2} A_2$, $\quad A_2 = \begin{cases} \pm a^{-3} b^{11/4} (\mp a/b)^{1/2} & \text{if } k = 1, \\ \pm a^{-3} b^{11/4} (\pm a/b)^{1/2} & \text{if } k = 2. \end{cases}$

72. $y''_{xx} = (A_1 x^{-7/4} y + A_2 x^{-1/4})(y'_x)^{5/2}$.

Solution in parametric form:

$$x = a[f_k - (-1)^k \tau]^{4/3}, \qquad y = b(f_k^2 - \tau^2 \mp 4\wp_k^3),$$

where $A_1 = \begin{cases} \mp a^{11/4} b^{-3} (\mp b/a)^{1/2} & \text{if } k = 1, \\ \mp a^{11/4} b^{-3} (\pm b/a)^{1/2} & \text{if } k = 2, \end{cases} \quad A_2 = a^{-3/2} b A_1.$

◆ *In the solutions of equations 73–92, the following notation is used:*
The functions P_1 and P_2 are the general solutions of the four modifications of the first Painlevé equation:

$$P_1'' = \pm 6P_1^2 + \tau, \qquad P_2'' = \pm 6P_2^2 - \tau$$

(in the case of the upper sign, the equation for P_1 is the canonical form of the first Painlevé

equation; see Section 3.4.2). In addition,

$$Q_1 = \pm 6P_1^2 + \tau, \qquad Q_2 = \pm 6P_2^2 - \tau,$$
$$R_1 = 2P_1' - \tau^2, \qquad R_2 = 2P_2' + \tau^2,$$
$$S_1 = 3\tau P_1' - 3P_1 - \tau^3, \qquad S_2 = 3\tau P_2' - 3P_2 + \tau^3,$$
$$T_1 = \tau^2 P_1 \mp 1, \qquad T_2 = \tau^2 P_2 \mp 1,$$
$$U_1 = (P_1')^2 - 2P_1Q_1 \pm 8P_1^3, \qquad U_2 = (P_2')^2 - 2P_2Q_2 \pm 8P_2^3,$$
$$V_1 = P_1'Q_1' + P_1' - Q_1^2, \qquad V_2 = P_2'Q_2' - P_2' - Q_2^2,$$
$$W_1 = \tau^3 P_1' + 3\tau^2 P_1 \mp 1 + \tau^5, \qquad W_2 = \tau^3 P_2' + 3\tau^2 P_2 \mp 1 - \tau^5,$$
$$Z_1 = 6(\tau^3 P_1' - 4\tau^2 P_1 \pm 6) - \tau^5, \qquad Z_2 = 6(\tau^3 P_2' - 4\tau^2 P_2 \pm 6) + \tau^5,$$

where the prime denotes differentiation with respect to τ.

73. $y''_{xx} = (A_1 y + A_2 x^2)(y'_x)^3$.

Solution in parametric form:

$$x = aP_k, \qquad y = b\tau,$$

where $A_1 = ab^{-3}(-1)^k$, $A_2 = \mp 6a^{-1}b^{-2}$; $k = 1$ and $k = 2$.

74. $y''_{xx} = (A_1 y^2 + A_2 x)(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aR_k, \qquad y = b\tau,$$

where $A_1 = ab^{-2}A_2(-1)^{k+1}$, $A_2 = -2a^{-1}b^{-1}(\pm 3a/b)^{1/2}$; $k = 1$ and $k = 2$.

75. $y''_{xx} = (A_1 y + A_2 x^2)(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\tau, \qquad y = bR_k,$$

where $A_1 = 2a^{-1}b^{-1}(\pm 3b/a)^{1/2}$, $A_2 = a^{-2}bA_1(-1)^{k+1}$; $k = 1$ and $k = 2$.

76. $y''_{xx} = (A_1 y^{-4} + A_2 x^2 y^{-5})(y'_x)^3$.

Solution in parametric form:

$$x = a\tau^{-1}P_k, \qquad y = b\tau^{-1},$$

where $A_1 = ab^2(-1)^k$, $A_2 = \mp 6a^{-1}b^3$; $k = 1$ and $k = 2$.

77. $y''_{xx} = (A_1 y^{-1/2} + A_2 x y^{-5/4})(y'_x)^{5/2}$.

Solution in parametric form:

$$x = aS_k, \qquad y = b\tau^4,$$

where $A_1 = ab^{-3/4}A_2(-1)^{k+1}$, $A_2 = -\frac{1}{4}a^{-1}b^{-1/4}(\pm 2a/b)^{1/2}$; $k = 1$ and $k = 2$.

78. $y''_{xx} = (A_1 x^{-5/4} y + A_2 x^{-1/2})(y'_x)^{1/2}$.

Solution in parametric form:

$$x = a\tau^4, \qquad y = bS_k,$$

where $A_1 = \frac{1}{4}a^{-1/4}b^{-1}(\pm 2b/a)^{1/2}$, $A_2 = a^{-3/4}bA_1(-1)^{k+1}$; $k = 1$ and $k = 2$.

79. $y''_{xx} = (A_1 y^{-8/7} + A_2 x^2 y^{-15/7})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau T_k, \quad y = b\tau^7,$$

where $A_1 = \frac{1}{49} ab^{-6/7}(-1)^k$, $A_2 = \mp\frac{6}{49} a^{-1} b^{1/7}$; $k = 1$ and $k = 2$.

80. $y''_{xx} = (A_1 y^{-7/4} + A_2 x y^{-15/8})(y'_x)^{5/2}.$

Solution in parametric form:

$$x = a\tau^{-6} W_k, \quad y = b\tau^{-8},$$

where $A_1 = ab^{-1/8} A_2 (-1)^k$, $A_2 = \frac{1}{8} a^{-1} b^{7/8} (\mp 3a/b)^{1/2}$; $k = 1$ and $k = 2$.

81. $y''_{xx} = (A_1 x^{-15/8} y + A_2 x^{-7/4})(y'_x)^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-8}, \quad y = b\tau^{-6} W_k,$$

where $A_1 = -\frac{1}{8} a^{7/8} b^{-1} (\mp 3b/a)^{1/2}$, $A_2 = a^{-1/8} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

82. $y''_{xx} = (A_1 y^{-13/7} + A_2 x^2 y^{-20/7})(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^{-6} T_k, \quad y = b\tau^{-7},$$

where $A_1 = \frac{1}{49} ab^{-1/7}(-1)^k$, $A_2 = \mp\frac{6}{49} a^{-1} b^{6/7}$; $k = 1$ and $k = 2$.

83. $y''_{xx} = (A_1 y^{-14/13} + A_2 x y^{-20/13})(y'_x)^{5/2}.$

Solution in parametric form:

$$x = a\tau Z_k, \quad y = b\tau^{13},$$

where $A_1 = ab^{-6/13} A_2 (-1)^{k+1}$, $A_2 = -\dfrac{2}{13} a^{-1} b^{7/13} \left(\pm\dfrac{a}{13b}\right)^{1/2}$; $k = 1$ and $k = 2$.

84. $y''_{xx} = (A_1 x^{-20/13} y + A_2 x^{-14/13})(y'_x)^{1/2}.$

Solution in parametric form:

$$x = a\tau^{13}, \quad y = b\tau Z_k,$$

where $A_1 = \dfrac{2}{13} a^{7/13} b^{-1} \left(\pm\dfrac{b}{13a}\right)^{1/2}$, $A_2 = a^{-6/13} b A_1 (-1)^{k+1}$; $k = 1$ and $k = 2$.

85. $y''_{xx} = (A_1 y + A_2 x^{-1/2})(y'_x)^3.$

Solution in parametric form:

$$x = a(P'_k)^2, \quad y = bP_k,$$

where $A_1 = \mp 24ab^{-3}$, $A_2 = 2a^{3/2} b^{-2} (-1)^k$; $k = 1$ and $k = 2$.

86. $y''_{xx} = (A_1 x^{-1/2} y^{-5/2} + A_2 y^{-4})(y'_x)^3.$

Solution in parametric form:

$$x = aP_k^{-1}(P_k')^2, \quad y = bP_k^{-1},$$

where $A_1 = 2a^{3/2}b^{1/2}(-1)^k,\ A_2 = \mp 24ab^2;\ k = 1$ and $k = 2$.

87. $y''_{xx} = (A_1 x^{-5/3} y + A_2 x^{1/3})(y'_x)^3.$

Solution in parametric form:

$$x = aP_k^{3/2}, \quad y = bU_k,$$

where $A_1 = \frac{3}{16}a^{8/3}b^{-3},\ A_2 = \mp 8a^{-2}bA_1;\ k = 1$ and $k = 2$.

88. $y''_{xx} = (A_1 x^{-5/3} y^{-7/3} + A_2 x^{1/3} y^{-10/3})(y'_x)^3.$

Solution in parametric form:

$$x = aP_k^{3/2}U_k^{-1}, \quad y = bU_k^{-1},$$

where $A_1 = \frac{3}{16}a^{8/3}b^{1/3},\ A_2 = \mp 8a^{-2}bA_1;\ k = 1$ and $k = 2$.

89. $y''_{xx} = (A_1 x^{-1/2} + A_2 y^{-3/2})(y'_x)^3.$

Solution in parametric form:

$$x = aQ_k^2, \quad y = b(P_k')^2,$$

where $A_1 = \mp 6a^{3/2}b^{-2},\ A_2 = \frac{1}{2}ab^{-1/2}(-1)^{k+1};\ k = 1$ and $k = 2$.

90. $y''_{xx} = (A_1 y^{-3/2} + A_2 x^{-1/2} y^{-5/2})(y'_x)^3.$

Solution in parametric form:

$$x = a(P_k')^{-2}Q_k^2, \quad y = b(P_k')^{-2},$$

where $A_1 = \frac{1}{2}ab^{-1/2}(-1)^{k+1},\ A_2 = \mp 6a^{3/2}b^{1/2};\ k = 1$ and $k = 2$.

91. $y''_{xx} = (A_1 x^{-5/3} y + A_2 x^{-4/3})(y'_x)^3.$

Solution in parametric form:

$$x = a(P_k')^3, \quad y = bV_k,$$

where $A_1 = -\frac{1}{36}a^{8/3}b^{-3},\ A_2 = a^{-1/3}bA_1(-1)^k;\ k = 1$ and $k = 2$.

92. $y''_{xx} = (A_1 x^{-4/3} y^{-5/3} + A_2 x^{-5/3} y^{-7/3})(y'_x)^3.$

Solution in parametric form:

$$x = a(P_k')^3 V_k^{-1}, \quad y = bV_k^{-1},$$

where $A_1 = \frac{1}{36}a^{7/3}b^{-1/3}(-1)^{k+1},\ A_2 = -\frac{1}{36}a^{8/3}b^{1/3};\ k = 1$ and $k = 2$.

◆ *In the solutions of equations 93–96, the following notation is used:*

$$F = C_2 \exp\Bigl(\int \frac{d\tau}{\sqrt{R}}\Bigr), \quad G = \tau + 2\sqrt{R} + 4B_2,$$

$$H = \int \Bigl(C_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2\Bigr)^{-1/2} d\tau + C_2,$$

$$R = \begin{cases} C_1 + \frac{1}{4}\tau^2 + \frac{2B_1}{k_1+1}\tau^{k_1+1} + \frac{2B_2}{k_2+1}\tau^{k_2+1} & \textit{if } k_1 \neq -1, k_2 \neq -1; \\ C_1 + \frac{1}{4}\tau^2 + \frac{2B_1}{k+1}\tau^{k+1} + 2B_2 \ln|\tau| & \textit{if } k = k_1 \neq -1, k_2 = -1. \end{cases}$$

93. $y''_{xx} = (A_1 x^{-2m_1-3} y^{m_1} + A_2 x^{-2m_2-3} y^{m_2})(y'_x)^3.$

Solution in parametric form:

$$x = \tau F^{1/2}, \quad y = F,$$

where $k_1 = -2m_1 - 3,\ k_2 = -2m_2 - 3,\ A_1 = -B_1,\ A_2 = -B_2.$

94. $y''_{xx} = \Bigl(A_1 x y^m + A_2 y^{\frac{m-1}{2}}\Bigr)(y'_x)^{\frac{m+5}{m+3}}.$

Solution in parametric form:

$$x = aF^{-1/2}G, \quad y = bF^{\frac{1}{m+1}},$$

where

$$k_1 = k = -\frac{m+3}{m+1}, \quad k_2 = 0, \quad A_1 = \frac{kb^{k+1}}{(k+1)a}\Bigl[-\frac{4aB_1}{(k+1)b}\Bigr]^{1/k}, \quad A_2 = -4ab^{-\frac{1}{k+1}}A_1B_2.$$

95. $y''_{xx} = \Bigl(A_1 x^n y + A_2 x^{\frac{n-1}{2}}\Bigr)(y'_x)^{\frac{2n+4}{n+3}}.$

Solution in parametric form:

$$x = aF^{\frac{1}{n+1}}, \quad y = F^{-1/2}G,$$

where $k_1 = k = -\dfrac{n+3}{n+1},\ k_2 = 0,\ A_1 = -\dfrac{k}{k+1}a^{\frac{2}{k+1}}b^{-1}\Bigl[-\dfrac{4bB_1}{(k+1)a}\Bigr]^{1/k},\ A_2 = -4a^{-\frac{1}{k+1}}bA_1B_2.$

96. $y''_{xx} = (A_1 x^{-5} y^2 + A_2 x^{-5})(y'_x)^3.$

Solution in parametric form:

$$x = \frac{\tau\sqrt{-A_2}}{\cos H}, \quad y = \sqrt{\frac{A_2}{A_1}}\tan H.$$

97. $y''_{xx} = (A_1 x^{m_1} y^{m_1} + A_2 x^{m_2} y^{m_2})(y'_x)^{3/2}.$

Solution in parametric form:

$$x = C_1 \tau^{1/2} \exp\Bigl(-\frac{1}{2}\int \frac{f\,d\tau}{\tau\sqrt{f^2+4}}\Bigr), \quad y = C_1^{-1}\tau^{1/2}\exp\Bigl(\frac{1}{2}\int \frac{f\,d\tau}{\tau\sqrt{f^2+4}}\Bigr),$$

where

$$f=\begin{cases}\tau^{-1/2}\Big[C_2+\dfrac{A_1}{2(m_1+1)}\tau^{m_1+1}+\dfrac{A_2}{2(m_2+1)}\tau^{m_2+1}\Big] & \text{if } m_1\neq-1,\ m_2\neq-1,\\ \tau^{-1/2}\Big[C_2+\dfrac{A_1}{2(m_1+1)}\tau^{m_1+1}+\dfrac{1}{2}A_2\ln\tau\Big] & \text{if } m_1\neq-1,\ m_2=-1.\end{cases}$$

98. $y''_{xx}=\Big(A_1x^ny^{m_1}+A_2x^{\frac{m_2(n+1)-m_1+n}{m_1+1}}y^{m_2}\Big)(y'_x)^{\frac{m_1+2n+3}{m_1+n+2}}$.

Solution in parametric form:

$$x=C_1\exp\Big(\int\frac{d\tau}{\tau z}\Big),\qquad y=C_1^{-\frac{n+1}{m_1+1}}\tau\exp\Big(-\frac{n+1}{m_1+1}\int\frac{d\tau}{\tau z}\Big),$$

where $z=z(\tau)$ is the solution of the algebraic equation

$$\Big(z-\frac{m_1+n+2}{m_1+1}\Big)\Big(z-\frac{n+1}{m_1+1}\Big)^{\frac{n+1}{m_1+n+2}}=\tau^{-\frac{m_1+1}{m_1+n+2}}\Big(C_2+\frac{A_1}{m_1+n+2}\tau^{m_1+1}+F\Big),$$

$$F=\begin{cases}\dfrac{A_2(m_1+1)}{(m_1+n+2)(m_2+1)}\tau^{m_2+1} & \text{if } m_2\neq-1,\\ \dfrac{A_2}{m_1+n+2}\ln|\tau| & \text{if } m_2=-1.\end{cases}$$

◆ *In the solutions of equations 99–108, the following notation is used:*
The functions P_1 and P_2 are the general solutions of the four modifications of the second Painlevé equation (with parameter $a=0$):

$$P_1''=\tau P_1\pm 2P_1^3,\qquad P_2''=-\tau P_2\pm 2P_2^3.$$

In the case of the upper sign, the equation for P_1 is the canonical form of the second Painlevé equation (with parameter $a=0$; see Section 3.4.3);

$$Q_1=\tau P_1^2\pm P_1^4-(P_1')^2,\quad R_1=P_1'\mp P_1Q_1,\quad S_1=2P_1'Q_1-P_1^3\mp P_1Q_1^2,$$
$$Q_2=\tau P_2^2\pm P_2^4-(P_2')^2,\quad R_2=P_2'\pm P_2Q_2,\quad S_2=2P_2'Q_2+P_2^3\pm P_2Q_2^2;$$

where the prime denotes differentiation with respect to τ.

99. $y''_{xx}=(A_1xy+A_2x^3)(y'_x)^3$.

Solution in parametric form:

$$x=aP_k,\qquad y=b\tau,$$

where $A_1=b^3(-1)^k$, $A_2=\mp 2a^{-2}b^{-2}$; $k=1$ and $k=2$.

100. $y''_{xx}=(A_1xy^{-5}+A_2x^3y^{-6})(y'_x)^3$.

Solution in parametric form:

$$x=a\tau^{-1}P_k,\qquad y=b\tau^{-1},$$

where $A_1=b^3(-1)^k$, $A_2=\mp 2a^{-2}b^4$; $k=1$ and $k=2$.

101. $y''_{xx} = (A_1 + A_2 x^{-1/2} y^{-1/2})(y'_x)^3.$

Solutions in parametric form:

$$x = a(P'_k)^2, \quad y = bP_k^2, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \mp 2ab^{-2}$, $A_2 = \frac{1}{2} a^{3/2} b^{-3/2} (-1)^k$; $k = 1$ and $k = 2$.

102. $y''_{xx} = (A_1 x^{-1/2} y^{-2} + A_2 y^{-3})(y'_x)^3.$

Solutions in parametric form:

$$x = aP_k^{-2}(P'_k)^2, \quad y = bP_k^{-2}, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \frac{1}{2} a^{3/2} (-1)^k$, $A_2 = \mp 2ab$; $k = 1$ and $k = 2$.

103. $y''_{xx} = (A_1 x^{-2} y + A_2)(y'_x)^3.$

Solutions in parametric form:

$$x = aP_k^2, \quad y = b\bigl[\tau P_k^2 \pm P_k^4 - (P'_k)^2\bigr], \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = 2a^3 b^{-3} (-1)^k$, $A_2 = \pm 2ab^{-2} (-1)^k$; $k = 1$ and $k = 2$.

104. $y''_{xx} = (A_1 x^{-2} y^{-2} + A_2 y^{-3})(y'_x)^3.$

Solutions in parametric form:

$$x = aP_k^2 \bigl[\tau P_k^2 \pm P_k^4 - (P'_k)^2\bigr]^{-1}, \quad y = b\bigl[\tau P_k^2 \pm P_k^4 - (P'_k)^2\bigr]^{-1},$$

where $A_1 = -2a^3$, $A_2 = \mp 2ab$; $k = 1$ and $k = 2$.

105. $y''_{xx} = (A_1 + A_2 x y^{-1/2})(y'_x)^{3/2}.$

Solutions in parametric form:

$$x = aP_k^{-1} R_k, \quad y = bQ_k^2,$$

where $A_1 = \mp ab^{-1/2} A_2 (-1)^k$, $A_2 = \begin{cases} 2a^{-2} b^{1/2} (2a/b)^{1/2} & \text{if } k = 1, \\ -2a^{-2} b^{1/2} (-2a/b)^{1/2} & \text{if } k = 2. \end{cases}$

106. $y''_{xx} = (A_1 x^{-1/2} y + A_2)(y'_x)^{3/2}.$

Solutions in parametric form:

$$x = aQ_k^2, \quad y = bP_k^{-1} R_k,$$

where $A_1 = \begin{cases} -2a^{1/2} b^{-2} (2b/a)^{1/2} & \text{if } k = 1, \\ 2a^{1/2} b^{-2} (-2b/a)^{1/2} & \text{if } k = 2, \end{cases}$ $A_2 = \mp a^{-1/2} b A_1 (-1)^k.$

107. $y''_{xx} = (A_1 + A_2 x y^{-2})(y'_x)^{3/2}.$

Solutions in parametric form:

$$x = aS_k, \quad y = bQ_k,$$

where $A_1 = \mp ab^{-2} A_2 (-1)^k$, $A_2 = \begin{cases} a^{-2} b^2 (2a/b)^{1/2} & \text{if } k = 1, \\ a^{-2} b^2 (-2a/b)^{1/2} & \text{if } k = 2. \end{cases}$

108. $y''_{xx} = (A_1 x^{-2} y + A_2)(y'_x)^{3/2}$.

Solutions in parametric form:

$$x = aQ_k, \qquad y = bS_k,$$

where $A_1 = \begin{cases} -a^2 b^{-2}(2b/a)^{1/2} & \text{if } k = 1, \\ -a^2 b^{-2}(-2b/a)^{1/2} & \text{if } k = 2, \end{cases}$ $A_2 = \mp a^{-2} b A_1 (-1)^k$.

109. $y''_{xx} = (A_1 x^{-7/5} y^{-8/5} + A_2 x^{-7/5} y^{-13/5})(y'_x)^3$.

Solution in parametric form:

$$x = aC_1^5 S^{5/2}\Bigl(bC_1^4 F - \frac{A_1}{A_2}\Bigr)^{-1}, \qquad y = \Bigl(bC_1^4 F - \frac{A_1}{A_2}\Bigr)^{-1},$$

where $S = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin\bigl(\sqrt{3}\,k\tau\bigr)$, $F = (S'_\tau)^2 - 2SS''_{\tau\tau}$, $A_2 = \frac{5}{1024} a^{12/5} b^{-3} k^{-6}$.

110. $y''_{xx} = (A_1 x^{-2} y^{-1} + A_2 x^{-2} y^{-2})(y'_x)^3$.

Solution in parametric form:

$$x = aC_1 \tau^{2/3} Z^2 \Bigl\{ bC_1 \tau^{-2/3}\bigl[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\bigr] - \frac{A_1}{A_2} \Bigr\}^{-1},$$
$$y = \Bigl\{ bC_1 \tau^{-2/3}\bigl[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\bigr] - \frac{A_1}{A_2} \Bigr\}^{-1},$$

where

$$A_2 = \frac{9}{2} a^3 b^{-3}, \qquad Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

111. $y''_{xx} = (A_1 x^{-7} y^4 + A_2 x^{-7} y^3)(y'_x)^3$.

Solution in parametric form:

$$x = aC_1^3 F^{1/2}\Bigl(bC_1^8 G - \frac{A_1}{A_2}\Bigr)^{-1}, \qquad y = \Bigl(bC_1^8 G - \frac{A_1}{A_2}\Bigr)^{-1},$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $F = 2\tau \int \tau R^{-1}\,d\tau + C_2\tau \mp R$, $G = 4\tau F^2 \mp \tau^{-2}(RF - 1)^2$, $A_2 = \mp\frac{3}{64} a^8 b^{-3}$.

◆ *In the solutions of equations 112 and 113, the following notation is used:*

$$E = \int (1 \pm \tau^4)^{-1/2}\,d\tau + C_2, \qquad k^2 = \pm 1;$$

the function E can be expressed in terms of elliptic integrals or lemniscate functions.

112. $y''_{xx} = (A_1 x^2 y^{-14/5} + A_2 x^3 y^{-18/5})(y'_x)^3.$

Solutions in parametric form:

$$x = aC_1^4 E^{-4}(\tau E - k), \quad y = bC_1^5 E^{-5}, \quad \text{where } A_1 = \mp\tfrac{6}{25}a^{-1}b^{4/5}k, \quad A_2 = \mp\tfrac{2}{25}a^{-2}b^{8/5}.$$

113. $y''_{xx} = (A_1 x^2 y^{-11/5} + A_2 x^3 y^{-12/5})(y'_x)^3.$

Solutions in parametric form:

$$x = aC_1 E(\tau E - k), \quad y = bC_1^5 E^5, \quad \text{where } A_1 = \mp\tfrac{6}{25}a^{-1}b^{1/5}k, \quad A_2 = \mp\tfrac{2}{25}a^{-2}b^{2/5}.$$

◆ *In the solutions of equations 114 and 115, the following notation is used:*

$$\Delta = C_2^2 - 2C_1, \quad R = (36\Delta + 54B\tau - 2\tau^3)^{1/2}, \quad z = 3\int \frac{d\tau}{\tau R},$$

$$W(z) = \begin{cases} \dfrac{\sqrt{-\Delta}}{C_1}\tan\bigl(\pm\sqrt{-\Delta}\,z\bigr) + \dfrac{C_2}{C_1} & \text{if } \Delta < 0; \\ \dfrac{\sqrt{\Delta}}{C_1}\tanh\bigl(\mp\sqrt{\Delta}\,z\bigr) + \dfrac{C_2}{C_1} & \text{if } \Delta > 0; \\ \mp\dfrac{1}{C_1 z} - \dfrac{\sqrt{2}}{\sqrt{|C_1|}} & \text{if } \Delta = 0,\ C_2 < 0; \\ \mp\dfrac{1}{C_1 z} + \dfrac{\sqrt{2}}{\sqrt{|C_1|}} & \text{if } \Delta = 0,\ C_2 > 0. \end{cases}$$

114. $y''_{xx} = (A_1 x^{-5/3} + A_2 x^{-5/3} y^{-2/3})(y'_x)^3.$

Solutions in parametric form:

$$x = a\tau^{-9/4}(C_1W^2 - 2C_2W + 2)^{3/4}(6C_1W - 6C_2 \mp R)^{3/2},$$
$$y = b\tau^{-3/2}(C_1W^2 - 2C_2W + 2)^{3/2},$$

where $A_1 = -24a^{8/3}b^{-2}C_1$, $A_2 = 36a^{8/3}b^{-4/3}B$.

115. $y''_{xx} = (A_1 x^{-5/3} y^{-2/3} + A_2 x^{-5/3} y^{-4/3})(y'_x)^3.$

Solutions in parametric form:

$$x = a\tau^{-3/4}(C_1W^2 - 2C_2W + 2)^{-3/4}(6C_1W - 6C_2 \mp R)^{3/2},$$
$$y = b\tau^{3/2}(C_1W^2 - 2C_2W + 2)^{-3/2},$$

where $A_1 = 36a^{8/3}b^{-4/3}B$, $A_2 = -24a^{8/3}b^{-2/3}C_1$.

116. $y''_{xx} = \left[\dfrac{2(n+1)}{(n+3)^2}x + Ax^n\right]y^{-2}(y'_x)^3, \qquad n \neq -3, \ n \neq -1.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.4 with respect to $x = x(y)$: $x''_{yy} = y^{-2}\left[-\dfrac{2(n+1)}{(n+3)^2}x - Ax^n\right]$.

117. $y''_{xx} = (-\tfrac{15}{4}x + Ax^{-7})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.35 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(\tfrac{15}{4}x - Ax^{-7})$.

118. $y''_{xx} = (-6x + Ax^{-4})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.31 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(6x - Ax^{-4})$.

119. $y''_{xx} = (-12x + Ax^{-5/2})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.64 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(12x - Ax^{-5/2})$.

120. $y''_{xx} = (-2x + Ax^{-2})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.6 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(2x - Ax^{-2})$.

121. $y''_{xx} = (\frac{3}{16}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.26 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-\frac{3}{16}x - Ax^{-5/3})$.

122. $y''_{xx} = (\frac{9}{100}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.10 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-\frac{9}{100}x - Ax^{-5/3})$.

123. $y''_{xx} = (-\frac{3}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.12 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(\frac{3}{4}x - Ax^{-5/3})$.

124. $y''_{xx} = (-\frac{63}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.66 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(\frac{63}{4}x - Ax^{-5/3})$.

125. $y''_{xx} = (\frac{5}{36}x + Ax^{-7/5})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.29 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-\frac{5}{36}x - Ax^{-7/5})$.

126. $y''_{xx} = (\frac{2}{9}x + Ax^{-1/2})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.14 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-\frac{2}{9}x - Ax^{-1/2})$.

127. $y''_{xx} = (\frac{4}{25}x + Ax^{-1/2})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.8 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-\frac{4}{25}x - Ax^{-1/2})$.

128. $y''_{xx} = (-20x + Ax^{-1/2})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.33 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(20x - Ax^{-1/2})$.

129. $y''_{xx} = (\frac{12}{49}x + Ax^{1/2})y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.37 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-\frac{12}{49}x - Ax^{1/2})$.

130. $y''_{xx} = (Ax^2 + \frac{6}{25}x)y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.60 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-Ax^2 - \frac{6}{25}x)$.

131. $y''_{xx} = (Ax^2 - \frac{6}{25}x)y^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.62 with respect to $x = x(y)$: $x''_{yy} = y^{-2}(-Ax^2 + \frac{6}{25}x)$.

132. $y''_{xx} = \left[\frac{2(n+1)}{(n+3)^2}xy^{-2} + Ax^n y^{-n-1}\right](y'_x)^3, \qquad n \neq -3,\ -1.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.5 with respect to $x = x(y)$: $x''_{yy} = -\dfrac{2(n+1)}{(n+3)^2}y^{-2}x - Ay^{-n-1}x^n$.

133. $y''_{xx} = (Ax^{-2}y - 2xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.7 with respect to $x = x(y)$: $x''_{yy} = 2y^{-2}x - Ayx^{-2}$.

134. $y''_{xx} = (Ax^{-1/2}y^{-1/2} + \frac{4}{25}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.9 with respect to $x = x(y)$: $x''_{yy} = -\frac{4}{25}y^{-2}x - Ay^{-1/2}x^{-1/2}$.

135. $y''_{xx} = (Ax^{-5/3}y^{2/3} + \frac{9}{100}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.11 with respect to $x = x(y)$: $x''_{yy} = -\frac{9}{100}y^{-2}x - Ay^{2/3}x^{-5/3}$.

136. $y''_{xx} = (Ax^{-5/3}y^{2/3} - \frac{3}{4}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.13 with respect to $x = x(y)$: $x''_{yy} = \frac{3}{4}y^{-2}x - Ay^{2/3}x^{-5/3}$.

137. $y''_{xx} = (Ax^{-1/2}y^{-1/2} + \frac{2}{9}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.15 with respect to $x = x(y)$: $x''_{yy} = -\frac{2}{9}y^{-2}x - Ay^{-1/2}x^{-1/2}$.

138. $y''_{xx} = (Ax^{-5/3}y^{2/3} + \frac{3}{16}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.27 with respect to $x = x(y)$: $x''_{yy} = -\frac{3}{16}y^{-2}x - Ay^{2/3}x^{-5/3}$.

139. $y''_{xx} = (Ax^{-7/5}y^{2/5} + \frac{5}{36}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.30 with respect to $x = x(y)$: $x''_{yy} = -\frac{5}{36}y^{-2}x - Ay^{2/5}x^{-7/5}$.

140. $y''_{xx} = (Ax^{-4}y^3 - 6xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.32 with respect to $x = x(y)$: $x''_{yy} = 6y^{-2}x - Ay^3x^{-4}$.

141. $y''_{xx} = (Ax^{-1/2}y^{-1/2} - 20xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.34 with respect to $x = x(y)$: $x''_{yy} = 20y^{-2}x - Ay^{-1/2}x^{-1/2}$.

142. $y''_{xx} = (Ax^{-7}y^6 - \frac{15}{4}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.36 with respect to $x = x(y)$: $x''_{yy} = \frac{15}{4}y^{-2}x - Ay^6x^{-7}$.

143. $y''_{xx} = (Ax^{1/2}y^{-3/2} + \frac{12}{49}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.38 with respect to $x = x(y)$: $x''_{yy} = -\frac{12}{49}y^{-2}x - Ay^{-3/2}x^{1/2}$.

144. $y''_{xx} = (\frac{6}{25}xy^{-2} + Ax^2y^{-3})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.61 with respect to $x = x(y)$: $x''_{yy} = -Ay^{-3}x^2 - \frac{6}{25}y^{-2}x$.

145. $y''_{xx} = (-\frac{6}{25}xy^{-2} + Ax^2y^{-3})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.63 with respect to $x = x(y)$: $x''_{yy} = -Ay^{-3}x^2 + \frac{6}{25}y^{-2}x$.

146. $y''_{xx} = (Ax^{-5/2}y^{3/2} - 12xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.65 with respect to $x = x(y)$: $x''_{yy} = 12y^{-2}x - Ay^{3/2}x^{-5/2}$.

147. $y''_{xx} = (Ax^{-5/3}y^{2/3} - \frac{63}{4}xy^{-2})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.4.2.67 with respect to $x = x(y)$: $x''_{yy} = \frac{63}{4}y^{-2}x - Ay^{2/3}x^{-5/3}$.

14.6.3 Equations of the Form $y''_{xx} = \sigma Ax^n y^m (y'_x)^l + Ax^{n-1}y^{m+1}(y'_x)^{l-1}$

► **Classification table.**

Table 14.10 presents all solvable equations whose solutions are outlined in Section 14.6.3. Two-parameter families (in the space of the parameters n, m, and l), one-parameter families, and isolated points are presented in a consecutive fashion. Equations are arranged in accordance with the growth of l. The number of the equation sought is indicated in the last column in this table.

TABLE 14.10
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
arbitrary ($l \neq 2$)	arbitrary ($m \neq -1$)	$-m-1$	-1	14.6.3.75
arbitrary ($l \neq 2$)	$1-l$	$l-2$	-1	14.6.3.76
arbitrary ($l \neq 3$)	-2	1	-1	14.6.3.79
arbitrary ($l \neq 1$)	0	-1	-1	14.6.3.80
$\dfrac{m+3}{m+2}$	arbitrary ($m \neq -1, -2$)	1	$m+1$	14.6.3.74
$\dfrac{3n+2}{n+1}$	0	arbitrary ($n \neq 0, -1$)	$\dfrac{1}{n}$	14.6.3.73
1	arbitrary ($m \neq -1$)	$-m-2$	-1	14.6.3.1
1	0	arbitrary ($n \neq -1$)	$\dfrac{1}{n}$	14.6.3.23
1	1	arbitrary ($n \neq 0, -2$)	$\dfrac{2}{n}$	14.6.3.37
$\dfrac{3}{2}$	0	arbitrary ($n \neq 0, -1$)	$\dfrac{1}{n}$	14.6.3.41
2	arbitrary ($m \neq -1$)	arbitrary ($n \neq 0$)	-1	14.6.3.85
2	arbitrary ($m \neq -1$)	$m+1$	-1	14.6.3.82
2	arbitrary ($m \neq -1$)	$-2m-2$	-1	14.6.3.83
2	arbitrary ($m \neq -1$)	$-\dfrac{m+1}{2}$	-1	14.6.3.84
2	arbitrary ($m \neq -1$)	0	arbitrary	14.6.3.87
2	-1	arbitrary ($n \neq 0$)	arbitrary	14.6.3.86
$\dfrac{5}{2}$	arbitrary ($m \neq -1, -2$)	1	$m+1$	14.6.3.42
3	arbitrary ($m \neq -2$)	$-m-2$	-1	14.6.3.3
3	arbitrary ($m \neq -2$)	1	$m+1$	14.6.3.24
3	arbitrary ($m \neq -1, -3$)	2	$\dfrac{m+1}{2}$	14.6.3.38

TABLE 14.10 (*Continued*)
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
0	-3	-1	2	14.6.3.65
0	-3	$\frac{1}{2}$	-4	14.6.3.61
0	-3	2	-1	14.6.3.35
0	0	-2	-1	14.6.3.48
0	0	-1	-2	14.6.3.50
0	0	-1	-1	14.6.3.33
0	0	$-\frac{2}{5}$	$-\frac{5}{2}$	14.6.3.59
0	0	2	$\frac{1}{2}$	14.6.3.63
1	-3	1	-1	14.6.3.15
1	-2	-2	$\frac{1}{2}$	14.6.3.51
1	-2	-1	arbitrary	14.6.3.71
1	-2	-1	-1	14.6.3.7
1	-2	-1	1	14.6.3.5
1	-2	$-\frac{1}{2}$	2	14.6.3.55
1	-2	$\frac{1}{2}$	-1	14.6.3.13
1	-2	1	arbitrary	14.6.3.69
1	-2	1	-2	14.6.3.11
1	-2	1	-1	14.6.3.29
1	-1	-1	-1	14.6.3.2
1	$-\frac{1}{2}$	-2	$-\frac{1}{4}$	14.6.3.53
1	$-\frac{1}{2}$	-1	-1	14.6.3.45
1	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	14.6.3.31
1	$-\frac{1}{2}$	1	$\frac{1}{2}$	14.6.3.57
1	0	-1	-1	14.6.3.77
1	0	1	arbitrary	14.6.3.67
1	0	1	-1	14.6.3.9
1	1	-4	$-\frac{1}{2}$	14.6.3.21
1	1	-1	-2	14.6.3.39
1	1	-2	-1	14.6.3.25
1	1	1	1	14.6.3.17
$\frac{3}{2}$	0	-1	-1	14.6.3.27

TABLE 14.10 (*Continued*)
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
$\frac{3}{2}$	0	$-\frac{1}{2}$	-2	14.6.3.43
$\frac{3}{2}$	0	1	1	14.6.3.19
2	-1	0	arbitrary	14.6.3.81
$\frac{5}{2}$	-2	1	-1	14.6.3.28
$\frac{5}{2}$	$-\frac{3}{2}$	1	$-\frac{1}{2}$	14.6.3.44
$\frac{5}{2}$	0	1	1	14.6.3.20
3	-5	2	-2	14.6.3.22
3	-3	-1	2	14.6.3.52
3	-3	$\frac{1}{2}$	-4	14.6.3.54
3	-3	2	-1	14.6.3.26
3	-2	-1	arbitrary	14.6.3.72
3	-2	-1	-1	14.6.3.8
3	-2	-1	1	14.6.3.6
3	-2	0	-1	14.6.3.4
3	-2	$\frac{1}{2}$	-1	14.6.3.46
3	-2	1	-1	14.6.3.78
3	-2	2	$-\frac{1}{2}$	14.6.3.40
3	$-\frac{3}{2}$	-1	$\frac{1}{2}$	14.6.3.56
3	$-\frac{3}{2}$	$\frac{1}{2}$	-1	14.6.3.32
3	$-\frac{1}{2}$	-1	-1	14.6.3.14
3	0	-2	-1	14.6.3.16
3	0	-1	arbitrary	14.6.3.70
3	0	-1	-1	14.6.3.30
3	0	-1	$-\frac{1}{2}$	14.6.3.12
3	0	$\frac{1}{2}$	2	14.6.3.58
3	0	1	arbitrary	14.6.3.68
3	0	1	-1	14.6.3.10
3	0	2	1	14.6.3.18
4	-3	1	-1	14.6.3.47
4	-2	-2	$\frac{1}{2}$	14.6.3.66
4	-2	1	-1	14.6.3.34

TABLE 14.10 (*Continued*)
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
4	-2	1	$-\frac{1}{2}$	14.6.3.49
4	$-\frac{1}{2}$	-2	$-\frac{1}{4}$	14.6.3.62
4	$-\frac{2}{5}$	1	$-\frac{2}{5}$	14.6.3.60
4	1	-2	-1	14.6.3.36
4	1	1	2	14.6.3.64

► **Solvable equations and their solutions.**

1. $\boldsymbol{y''_{xx} = Ax^{-m-2}y^m y'_x - Ax^{-m-3}y^{m+1}, \qquad m \neq -1.}$

Solution in parametric form:

$$x = aC_1^m\Big(\int \frac{d\tau}{1 \pm \tau^{m+1}} + C_2\Big)^{-1}, \qquad y = bC_1^{m+1}\tau\Big(\int \frac{d\tau}{1 \pm \tau^{m+1}} + C_2\Big)^{-1},$$

where $A = \mp(m+1)a^{m+1}b^{-m}$.

2. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-1}y'_x - Ax^{-2}.}$

Solution in parametric form:

$$x = C_1\Big[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1}, \quad y = -\frac{A}{2}\exp(\mp\tau^2)\Big[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1}.$$

3. $\boldsymbol{y''_{xx} = Ax^{-m-2}y^m (y'_x)^3 - Ax^{-m-3}y^{m+1}(y'_x)^2, \qquad m \neq -2.}$

Solution in parametric form:

$$x = aC_1^{m+2}\tau\Big(\int \frac{d\tau}{1 \pm \tau^{-m-2}} + C_2\Big)^{-1}, \qquad y = bC_1^{m+3}\Big(\int \frac{d\tau}{1 \pm \tau^{-m-2}} + C_2\Big)^{-1},$$

where $A = \pm(m+2)a^{m+3}b^{-m-2}$.

4. $\boldsymbol{y''_{xx} = Ay^{-2}(y'_x)^3 - Ax^{-1}y^{-2}(y'_x)^2.}$

Solution in parametric form:

$$x = -\frac{A}{2}\exp(\mp\tau^2)\Big[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1}, \quad y = C_1\Big[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1}.$$

◆ *In the solutions of equations 5–12, the following notation is used:*

$$f = \int \exp(\mp\tau^2)\,d\tau + C_2.$$

5. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-2}y'_x + Ax^{-2}y^{-1}.}$

Solution in parametric form:

$$x = C_1\exp(\mp\tau^2)f^{-1}, \quad y = b[2\tau \pm \exp(\mp\tau^2)f^{-1}], \qquad \text{where} \quad A = \pm 2b^2.$$

6. $y''_{xx} = Ax^{-1}y^{-2}(y'_x)^3 + Ax^{-2}y^{-1}(y'_x)^2.$

Solution in parametric form:

$$x = a[2\tau \pm \exp(\mp\tau^2) f^{-1}], \quad y = C_1 \exp(\mp\tau^2) f^{-1}, \qquad \text{where} \quad A = \mp 2a^2.$$

7. $y''_{xx} = Ax^{-1}y^{-2}y'_x - Ax^{-2}y^{-1}.$

Solution in parametric form:

$$x = C_1 \bigl[2\tau f \pm \exp(\mp\tau^2)\bigr]^{-1}, \quad y = bf \bigl[2\tau f \pm \exp(\mp\tau^2)\bigr]^{-1}, \qquad \text{where} \quad A = \pm\tfrac{1}{2} b^2.$$

8. $y''_{xx} = Ax^{-1}y^{-2}(y'_x)^3 - Ax^{-2}y^{-1}(y'_x)^2.$

Solution in parametric form:

$$x = af \bigl[2\tau f \pm \exp(\mp\tau^2)\bigr]^{-1}, \quad y = C_1 \bigl[2\tau f \pm \exp(\mp\tau^2)\bigr]^{-1}, \qquad \text{where} \quad A = \pm\tfrac{1}{2} a^2.$$

9. $y''_{xx} = Axy'_x - Ay.$

Solution in parametric form:

$$x = a\tau, \quad y = C_1[2\tau f \pm \exp(\mp\tau^2)], \qquad \text{where} \quad A = \mp 2a^{-2}.$$

10. $y''_{xx} = Ax(y'_x)^3 - Ay(y'_x)^2.$

Solution in parametric form:

$$x = C_1[2\tau f \pm \exp(\mp\tau^2)], \quad y = b\tau, \qquad \text{where} \quad A = \mp 2b^{-2}.$$

11. $y''_{xx} = 2Axy^{-2}y'_x - Ay^{-1}.$

Solution in parametric form:

$$x = aC_1[2\tau^2 f \pm \tau \exp(\mp\tau^2) \pm f], \quad y = bC_1[2\tau f \pm \exp(\mp\tau^2)], \qquad \text{where} \quad A = \mp\tfrac{1}{2} a^{-2} b^2.$$

12. $y''_{xx} = Ax^{-1}(y'_x)^3 - 2Ax^{-2}y(y'_x)^2.$

Solution in parametric form:

$$x = aC_1[2\tau f \pm \exp(\mp\tau^2)], \quad y = bC_1[2\tau^2 f \pm \tau \exp(\mp\tau^2) \pm f], \qquad \text{where} \quad A = \mp\tfrac{1}{2} a^2 b^{-2}.$$

◆ *In the solutions of equations 13–22, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\bigl(\sqrt{\tau} + \sqrt{\tau+1}\,\bigr) + C_2, \qquad F = E\sqrt{\frac{\tau+1}{\tau}} - \tau.$$

13. $y''_{xx} = Ax^{1/2}y^{-2}y'_x - Ax^{-1/2}y^{-1}.$

Solution in parametric form:

$$x = aC_1^4 F^{-2}, \quad y = bC_1^3 \tau^{-1} E F^{-2}, \qquad \text{where} \quad A = -a^{-3/2} b^2.$$

14. $y''_{xx} = Ax^{-1}y^{-1/2}(y'_x)^3 - Ax^{-2}y^{1/2}(y'_x)^2.$

Solution in parametric form:

$$x = aC_1^3 \tau^{-1} E F^{-2}, \quad y = bC_1^4 F^{-2}, \qquad \text{where} \quad A = -a^2 b^{-3/2}.$$

15. $y''_{xx} = Axy^{-3}y'_x - Ay^{-2}$.

Solution in parametric form:

$$x = aC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}}, \quad y = bC_1^2F^{-1}, \qquad \text{where} \quad A = -2a^{-2}b^3.$$

16. $y''_{xx} = Ax^{-2}(y'_x)^3 - Ax^{-3}y(y'_x)^2$.

Solution in parametric form:

$$x = aC_1^2F^{-1}, \quad y = bC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad \text{where} \quad A = -2a^3b^{-2}.$$

17. $y''_{xx} = Axyy'_x + Ay^2$.

Solution in parametric form:

$$x = aC_1^{-1}\tau^{-1}E^{-1}(\tau F^2 + \tau^2F - E^2), \quad y = bC_1^2F^{-1}, \qquad \text{where} \quad A = a^{-2}b^{-1}.$$

18. $y''_{xx} = Ax^2(y'_x)^3 + Axy(y'_x)^2$.

Solution in parametric form:

$$x = aC_1^2F^{-1}, \quad y = bC_1^{-1}\tau^{-1}E^{-1}(\tau F^2 + \tau^2F - E^2), \qquad \text{where} \quad A = -a^{-1}b^{-2}.$$

19. $y''_{xx} = Ax(y'_x)^{3/2} + Ay(y'_x)^{1/2}$.

Solution in parametric form:

$$x = aC_1^{-1}\left(F\sqrt{\frac{\tau+1}{\tau}} - E\tau^{-1}\right), \quad y = bC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}}, \quad \text{where} \quad A = 2a^{-2}\left(-\frac{a}{b}\right)^{1/2}.$$

20. $y''_{xx} = Ax(y'_x)^{5/2} + Ay(y'_x)^{3/2}$.

Solution in parametric form:

$$x = aC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}}, \quad y = bC_1^{-1}\left(F\sqrt{\frac{\tau+1}{\tau}} - E\tau^{-1}\right), \quad \text{where} \quad A = -2b^{-2}\left(-\frac{b}{a}\right)^{1/2}.$$

21. $y''_{xx} = Ax^{-4}yy'_x - 2Ax^{-5}y^2$.

Solution in parametric form:

$$x = aC_1\tau E\bigl(\tau F^2 + \tau^2F - E^2\bigr)^{-1}, \quad y = bC_1^3\tau EF^{-1}\bigl(\tau F^2 + \tau^2F - E^2\bigr)^{-1},$$

where $A = -2a^3b^{-1}$.

22. $y''_{xx} = 2Ax^2y^{-5}(y'_x)^3 - Axy^{-4}(y'_x)^2$.

Solution in parametric form:

$$x = aC_1^3\tau EF^{-1}\bigl(\tau F^2 + \tau^2F - E^2\bigr)^{-1}, \quad y = bC_1\tau E\bigl(\tau F^2 + \tau^2F - E^2\bigr)^{-1},$$

where $A = -2a^{-1}b^3$.

23. $y''_{xx} = Ax^n y'_x + nAx^{n-1} y, \qquad n \neq -1.$

Solution in parametric form:

$$x = a\tau^{\frac{1}{n+1}}, \quad y = C_1 e^{\beta\tau}\Big(\int \tau^{-\frac{n}{n+1}} e^{-\beta\tau}\, d\tau + C_2\Big), \qquad \text{where} \quad A = (n+1)a^{-n-1}\beta.$$

24. $y''_{xx} = Axy^m (y'_x)^3 + \dfrac{A}{m+1} y^{m+1} (y'_x)^2, \qquad m \neq -2.$

Solution in parametric form:

$$x = C_1 e^{\beta\tau}\Big(\int \tau^{-\frac{m+1}{m+2}} e^{-\beta\tau}\, d\tau + C_2\Big), \quad y = b\tau^{\frac{1}{m+2}}, \qquad \text{where} \quad A = -\frac{m+2}{m+1} b^{-m-2}\beta.$$

◆ *In the solutions of equations 25–36, the following notation is used:*

$$R = \begin{cases} \tau^\nu + C_2 \tau^{-\nu} & \textit{for the upper sign,} \\ \sin(\nu \ln \tau) + C_2 \cos(\nu \ln \tau) & \textit{for the lower sign,} \\ \ln \tau + C_2 & \textit{for } \nu = 0, \end{cases}$$

$$Q = \begin{cases} (1+\nu)\tau^\nu + (1-\nu)C_2 \tau^{-\nu} & \textit{for the upper sign,} \\ (1-\nu C_2)\sin(\nu \ln \tau) + (C_2+\nu)\cos(\nu \ln \tau) & \textit{for the lower sign,} \\ \ln \tau + 1 + C_2 & \textit{for } \nu = 0. \end{cases}$$

25. $y''_{xx} = Ax^{-2} yy'_x - Ax^{-3} y^2.$

Solution in parametric form:

$$x = a\tau^{-2}, \quad y = b\tau^{-2} R^{-1} Q, \qquad \text{where} \quad \nu = C_1, \ \ A = ab^{-1}.$$

The solution is valid for all three cases of the functions R and Q given above.

26. $y''_{xx} = Ax^2 y^{-3} (y'_x)^3 - Axy^{-2} (y'_x)^2.$

Solution in parametric form:

$$x = a\tau^{-2} R^{-1} Q, \quad y = b\tau^{-2}, \qquad \text{where} \quad \nu = C_1, \ \ A = a^{-1} b.$$

The solution is valid for all three cases of the functions R and Q given above.

27. $y''_{xx} = Ax^{-1} (y'_x)^{3/2} - Ax^{-2} y (y'_x)^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-2}, \quad y = \tfrac{1}{2} b\tau^{-2}\big(2QR^{-1} - 1 \pm \nu^2\big), \qquad \text{where} \quad \nu = C_1, \ \ A = (2a/b)^{1/2}.$$

28. $y''_{xx} = Axy^{-2} (y'_x)^{5/2} - Ay^{-1} (y'_x)^{3/2}.$

Solution in parametric form:

$$x = \tfrac{1}{2} a\tau^{-2}(2QR^{-1} - 1 \pm \nu^2), \quad y = b\tau^{-2}, \qquad \text{where} \quad \nu = C_1, \ \ A = (2b/a)^{1/2}.$$

29. $y''_{xx} = Axy^{-2} y'_x - Ay^{-1}.$

Solution in parametric form:

$$x = aC_1 \tau R, \quad y = bC_1 \tau Q, \qquad \text{where} \quad A = a^{-2} b^2 (1 \mp \nu^2).$$

30. $y''_{xx} = Ax^{-1}(y'_x)^3 - Ax^{-2}y(y'_x)^2.$

Solution in parametric form:

$$x = aC_1\tau Q, \quad y = bC_1\tau R, \qquad \text{where} \quad A = a^2b^{-2}(1 \mp \nu^2).$$

31. $y''_{xx} = Ax^{-1/2}y^{-1/2}y'_x - Ax^{-3/2}y^{1/2}.$

Solution in parametric form:

$$x = a\tau^2R^2, \quad y = b\tau^2Q^2, \qquad \text{where} \quad \nu = C_1, \ \ A = a^{-1/2}b^{1/2}.$$

32. $y''_{xx} = Ax^{1/2}y^{-3/2}(y'_x)^3 - Ax^{-1/2}y^{-1/2}(y'_x)^2.$

Solution in parametric form:

$$x = a\tau^2Q^2, \quad y = b\tau^2R^2, \qquad \text{where} \quad \nu = C_1, \ \ A = a^{1/2}b^{-1/2}.$$

33. $y''_{xx} = Ax^{-1} - Ax^{-2}y(y'_x)^{-1}.$

Solution in parametric form:

$$x = a\tau^2R^2, \quad y = b\tau^2[Q^2 + (1 \mp \nu^2)R^2], \qquad \text{where} \quad \nu = C_1, \ \ A = 2a^{-1}b.$$

34. $y''_{xx} = Axy^{-2}(y'_x)^4 - Ay^{-1}(y'_x)^3.$

Solution in parametric form:

$$x = a\tau^2[Q^2 + (1 \mp \nu^2)R^2], \quad y = b\tau^2R^2, \qquad \text{where} \quad \nu = C_1, \ \ A = 2ab^{-1}.$$

35. $y''_{xx} = Ax^2y^{-3} - Axy^{-2}(y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1\tau R, \quad y = bC_1\tau\bigl[Q^2 + (1 \mp \nu^2)R^2\bigr]^{1/2}, \qquad \text{where} \quad A = 4(1 \mp \nu^2)a^{-4}b^4.$$

36. $y''_{xx} = Ax^{-2}y(y'_x)^4 - Ax^{-3}y^2(y'_x)^3.$

Solution in parametric form:

$$x = aC_1\tau\bigl[Q^2 + (1 \mp \nu^2)R^2\bigr]^{1/2}, \quad y = bC_1\tau R, \qquad \text{where} \quad A = 4(1 \mp \nu^2)a^4b^{-4}.$$

◆ *In the solutions of equations 37–50, the following notation is used:*

$$Z = \begin{cases} J_\nu(\tau) + C_2Y_\nu(\tau) & \textit{for the upper sign,} \\ I_\nu(\tau) + C_2K_\nu(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are Bessel functions, and $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

37. $y''_{xx} = 2Ax^nyy'_x + nAx^{n-1}y^2, \qquad n \neq 0, \ \ n \neq -2.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{2-2\nu}, \quad y = bC_1^{n+1}\tau^{-2\nu}Z^{-1}(\tau Z'_\tau + \nu Z),$$

where $\nu = \dfrac{n+1}{n+2}$, $A = -\dfrac{n+2}{2}a^{-n-1}b^{-1}.$

38. $y''_{xx} = (m+1)Ax^2 y^m (y'_x)^3 + 2Axy^{m+1}(y'_x)^2, \quad m \neq -1, \ m \neq -3.$

Solution in parametric form:

$$x = aC_1^{-m-2}\tau^{-2\nu} Z^{-1}(\tau Z'_\tau + \nu Z), \qquad y = bC_1 \tau^{2-2\nu},$$

where $\nu = \dfrac{m+2}{m+3}$, $A = \dfrac{m+3}{2} a^{-1} b^{-m-2}$.

39. $y''_{xx} = 2Ax^{-1}yy'_x - Ax^{-2}y^2.$

Solution in parametric form:

$$x = C_1\tau^2, \quad y = b\tau Z^{-1} Z'_\tau, \qquad \text{where} \quad \nu = 0, \ A = -\tfrac{1}{2} b^{-1}.$$

40. $y''_{xx} = Ax^2 y^{-2} (y'_x)^3 - 2Axy^{-1}(y'_x)^2.$

Solution in parametric form:

$$x = a\tau Z^{-1} Z'_\tau, \quad y = C_1\tau^2, \qquad \text{where} \quad \nu = 0, \ A = -\tfrac{1}{2} a^{-1}.$$

41. $y''_{xx} = Ax^n (y'_x)^{3/2} + nAx^{n-1} y (y'_x)^{1/2}, \qquad n \neq 0, \ n \neq -1.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{4-2\nu}, \qquad y = bC_1^{2n+1}\tau^{-2\nu}\Big[Z^{-1}(\tau Z'_\tau + \nu Z) \pm \frac{1}{2(1-\nu)}\tau^2\Big],$$

where $\nu = \dfrac{2n+1}{n+1}$, $A = -(n+1)a^{-n-1}\Big[-\dfrac{2a}{(n+1)b}\Big]^{1/2}$.

42. $y''_{xx} = (m+1)Axy^m (y'_x)^{5/2} + Ay^{m+1}(y'_x)^{3/2}, \quad m \neq -1, \ m \neq -2.$

Solution in parametric form:

$$x = aC_1^{2m-3}\tau^{-2\nu}\Big[Z^{-1}(\tau Z'_\tau + \nu Z) \pm \frac{1}{2(1-\nu)}\tau^2\Big], \qquad y = bC_1\tau^{4-2\nu},$$

where $\nu = \dfrac{2m+3}{m+2}$, $A = (m+2)b^{-m-2}\Big[-\dfrac{2b}{(m+2)a}\Big]^{1/2}$.

43. $y''_{xx} = 2Ax^{-1/2}(y'_x)^{3/2} - Ax^{-3/2} y (y'_x)^{1/2}.$

Solution in parametric form:

$$x = C_1\tau^4, \quad y = b(\tau Z^{-1} Z'_\tau \pm \tfrac{1}{2}\tau^2), \qquad \text{where} \quad \nu = 0, \ A = -\tfrac{1}{2}(-b)^{-1/2}.$$

44. $y''_{xx} = Axy^{-3/2}(y'_x)^{5/2} - 2Ay^{-1/2}(y'_x)^{3/2}.$

Solution in parametric form:

$$x = a(\tau Z^{-1} Z'_\tau \pm \tfrac{1}{2}\tau^2), \quad y = C_1\tau^4, \qquad \text{where} \quad \nu = 0, \ A = -\tfrac{1}{2}(-a)^{-1/2}.$$

45. $y''_{xx} = Ax^{-1} y^{-1/2} y'_x - Ax^{-2} y^{1/2}.$

Solution in parametric form:

$$x = C_1 Z^{-2}, \quad y = b\tau^2 Z^{-2}(Z'_\tau)^2, \qquad \text{where} \quad \nu = 0, \ A = -b^{1/2}.$$

46. $y''_{xx} = Ax^{1/2}y^{-2}(y'_x)^3 - Ax^{-1/2}y^{-1}(y'_x)^2.$

Solution in parametric form:

$$x = a\tau^2 Z^{-2}(Z'_\tau)^2, \quad y = C_1 Z^{-2}, \qquad \text{where} \quad \nu = 0, \;\; A = -a^{1/2}.$$

47. $y''_{xx} = Axy^{-3}(y'_x)^4 - Ay^{-2}(y'_x)^3.$

Solution in parametric form:

$$x = aZ^{-1}(2\tau Z'_\tau \pm \tau^2 Z), \quad y = C_1 Z^{-1}, \qquad \text{where} \quad \nu = 0, \;\; A = 4a.$$

48. $y''_{xx} = Ax^{-2} - Ax^{-3}y(y'_x)^{-1}.$

Solution in parametric form:

$$x = C_1 Z^{-1}, \quad y = bZ^{-1}(2\tau Z'_\tau \pm \tau^2 Z), \qquad \text{where} \quad \nu = 0, \;\; A = 4b.$$

49. $y''_{xx} = Axy^{-2}(y'_x)^4 - 2Ay^{-1}(y'_x)^3.$

Solution in parametric form:

$$x = aC_1[\tau^2(Z'_\tau)^2 + 2\tau Z Z'_\tau \pm \tau^2 Z^2], \quad y = bC_1 Z^2, \qquad \text{where} \quad \nu = 0, \;\; A = \tfrac{1}{2}ab^{-1}.$$

50. $y''_{xx} = 2Ax^{-1} - Ax^{-2}y(y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1 Z^2, \quad y = bC_1[\tau^2(Z'_\tau)^2 + 2\tau Z Z'_\tau \pm \tau^2 Z^2], \qquad \text{where} \quad \nu = 0, \;\; A = \tfrac{1}{2}a^{-1}b.$$

◆ *In the solutions of equations 51–66, the following notation is used:*

$$Z = \begin{cases} J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$
$$U_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \;\; U_2 = U_1^2 \pm \tau^2 Z^2, \;\; U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1U_2,$$

where $J_{1/3}(\tau)$ *and* $Y_{1/3}(\tau)$ *are Bessel functions, and* $I_{1/3}(\tau)$ *and* $K_{1/3}(\tau)$ *are modified Bessel functions.*

51. $y''_{xx} = Ax^{-2}y^{-2}y'_x + 2Ax^{-3}y^{-1}.$

Solution in parametric form:

$$x = aC_1^{-2}\tau^{4/3}Z^2U_2^{-1}, \quad y = bC_1\tau^{-2/3}Z^{-1}U_2^{-1}U_3, \qquad \text{where} \quad A = 2ab^2.$$

52. $y''_{xx} = 2Ax^{-1}y^{-3}(y'_x)^3 + Ax^{-2}y^{-2}(y'_x)^2.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{-2/3}Z^{-1}U_2^{-1}U_3, \quad y = bC_1^2\tau^{4/3}Z^2U_2^{-1}, \qquad \text{where} \quad A = -2a^2b.$$

53. $y''_{xx} = Ax^{-2}y^{-1/2}y'_x - 4Ax^{-3}y^{1/2}.$

Solution in parametric form:

$$x = aC_1^{-1}\tau^{-4/3}Z^{-2}U_2, \quad y = bC_1^2\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}ab^{1/2}.$$

54. $y''_{xx} = 4Ax^{1/2} y^{-3} (y'_x)^3 - Ax^{-1/2} y^{-2} (y'_x)^2.$

Solution in parametric form:

$$x = aC_1^{-2} \tau^{-4/3} Z^{-2} U_2^{-2} U_3^2, \quad y = bC_1 \tau^{-4/3} Z^{-2} U_2, \qquad \text{where} \quad A = \mp \tfrac{2}{3} a^{1/2} b.$$

55. $y''_{xx} = 2Ax^{-1/2} y^{-2} y'_x + Ax^{-3/2} y^{-1}.$

Solution in parametric form:

$$x = aC_1^4 \tau^{-4/3} Z^{-2} U_1^2, \quad y = bC_1 \tau^{-4/3} Z^{-2} U_2, \qquad \text{where} \quad A = \pm \tfrac{1}{6} a^{-1/2} b^2.$$

56. $y''_{xx} = Ax^{-1} y^{-3/2} (y'_x)^3 + 2Ax^{-2} y^{-1/2} (y'_x)^2.$

Solution in parametric form:

$$x = aC_1 \tau^{-4/3} Z^{-2} U_2, \quad y = bC_1^4 \tau^{-4/3} Z^{-2} U_1^2, \qquad \text{where} \quad A = \mp \tfrac{1}{6} a^2 b^{-1/2}.$$

57. $y''_{xx} = Axy^{-1/2} y'_x + 2Ay^{1/2}.$

Solution in parametric form:

$$x = aC_1 \tau^{-2/3} Z^{-1} U_1, \quad y = bC_1^4 \tau^{-8/3} Z^{-4} U_2^2, \qquad \text{where} \quad A = 2a^{-2} b^{1/2}.$$

58. $y''_{xx} = 2Ax^{1/2} (y'_x)^3 + Ax^{-1/2} y (y'_x)^2.$

Solution in parametric form:

$$x = aC_1^4 \tau^{-8/3} Z^{-4} U_2^2, \quad y = bC_1 \tau^{-2/3} Z^{-1} U_1, \qquad \text{where} \quad A = -2a^{1/2} b^{-2}.$$

59. $y''_{xx} = 5Ax^{-2/5} - 2Ax^{-7/5} y (y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1^5 \tau^{-5/3} Z^{-5/2} U_1^{5/2}, \quad y = bC_1^8 \tau^{-8/3} Z^{-4} (U_2^2 \pm \tfrac{4}{3} \tau^2 Z^3 U_1),$$

where $A = \frac{32}{125} a^{-8/5} b.$

60. $y''_{xx} = 2Axy^{-2/5} (y'_x)^4 - 5Ay^{-7/5} (y'_x)^3.$

Solution in parametric form:

$$x = aC_1^8 \tau^{-8/3} Z^{-4} (U_2^2 \pm \tfrac{4}{3} \tau^2 Z^3 U_1), \quad y = bC_1^5 \tau^{-5/3} Z^{-5/2} U_1^{5/2}, \quad \text{where} \quad A = \tfrac{32}{125} ab^{-8/5}.$$

61. $y''_{xx} = 4Ax^{1/2} y^{-3} - Ax^{-1/2} y^{-2} (y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1^8 \tau^{-4/3} Z^{-2} U_1^2, \quad y = bC_1^5 \tau^{-4/3} Z^{-2} (U_2^2 \pm \tfrac{4}{3} \tau^2 Z^3 U_1)^{1/2}, \quad \text{where} \quad A = \pm \tfrac{1}{3} a^{-5/2} b^4.$$

62. $y''_{xx} = Ax^{-2} y^{-1/2} (y'_x)^4 - 4Ax^{-3} y^{1/2} (y'_x)^3.$

Solution in parametric form:

$$x = aC_1^5 \tau^{-4/3} Z^{-2} \bigl(U_2^2 \pm \tfrac{4}{3} \tau^2 Z^3 U_1\bigr)^{1/2}, \quad y = bC_1^8 \tau^{-4/3} Z^{-2} U_1^2, \quad \text{where} \quad A = \pm \tfrac{1}{3} a^4 b^{-5/2}.$$

63. $y''_{xx} = Ax^2 + 2Axy (y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1 \tau^{2/3} Z U_2^{-1/2}, \quad y = bC_1^4 \tau^{-4/3} Z^{-2} U_2^{-2} (U_3^2 - 4U_2^3), \qquad \text{where} \quad A = \tfrac{32}{9} a^{-4} b.$$

64. $y''_{xx} = 2Axy(y'_x)^4 + Ay^2(y'_x)^3.$

Solution in parametric form:

$$x = aC_1^4\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3), \quad y = bC_1\tau^{2/3}ZU_2^{-1/2}, \quad \text{where} \quad A = -\tfrac{32}{9}ab^{-4}.$$

65. $y''_{xx} = 2Ax^{-1}y^{-3} + Ax^{-2}y^{-2}(y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1^4\tau^{4/3}Z^2U_2^{-1}, \quad y = bC_1\tau^{-2/3}Z^{-1}U_2^{-1}\bigl(U_3^2 - 4U_2^3\bigr)^{1/2}, \quad \text{where} \quad A = -\tfrac{8}{9}a^{-1}b^4.$$

66. $y''_{xx} = Ax^{-2}y^{-2}(y'_x)^4 + 2Ax^{-3}y^{-1}(y'_x)^3.$

Solution in parametric form:

$$x = aC_1\tau^{-2/3}Z^{-1}U_2^{-1}\bigl(U_3^2 - 4U_2^3\bigr)^{1/2}, \quad y = bC_1^4\tau^{4/3}Z^2U_2^{-1}, \quad \text{where} \quad A = \tfrac{8}{9}a^4b^{-1}.$$

◆ *In the solutions of equations 67–72, the following notation is used:*

$$M = C_1\Phi(\lambda, \tfrac{1}{2}; \pm\tau) + C_2\Psi(\lambda, \tfrac{1}{2}; \pm\tau),$$

where Φ *and* Ψ *are linearly independent solutions of the degenerate hypergeometric equation:*

$$\tau M''_{\tau\tau} + (\tfrac{1}{2} \pm \tau)M'_\tau - \lambda M = 0.$$

The function $\Phi = \Phi(\lambda, \frac{1}{2}, \pm\tau)$ *can be expressed in terms of a degenerate hypergeometric series (see equation 14.1.2.70).*

67. $y''_{xx} = A_1xy'_x + A_2y.$

Solution in parametric form:

$$x = a\tau^{1/2}, \quad y = M, \quad \text{where} \quad A_1 = \pm 2a^{-2}, \ A_2 = \pm 4a^{-2}\lambda.$$

68. $y''_{xx} = A_1x(y'_x)^3 + A_2y(y'_x)^2.$

Solution in parametric form:

$$x = M, \quad y = b\tau^{1/2}, \quad \text{where} \quad A_1 = \mp 4b^{-2}\lambda, \ A_2 = \mp 2b^{-2}.$$

69. $y''_{xx} = A_1xy^{-2}y'_x + A_2y^{-1}.$

Solution in parametric form:

$$x = M, \quad y = \pm b\tau^{1/2}M'_\tau, \quad \text{where} \quad A_1 = \mp b^2\lambda, \ A_2 = \pm b^2\bigl(\lambda + \tfrac{1}{2}\bigr).$$

70. $y''_{xx} = A_1x^{-1}(y'_x)^3 + A_2x^{-2}y(y'_x)^2.$

Solution in parametric form:

$$x = \pm a\tau^{1/2}M'_\tau, \quad y = M, \quad \text{where} \quad A_1 = \mp a^2\bigl(\lambda + \tfrac{1}{2}\bigr), \ A_2 = \pm a^2\lambda.$$

71. $y''_{xx} = A_1x^{-1}y^{-2}y'_x + A_2x^{-2}y^{-1}.$

Solution in parametric form:

$$x = M^{-1}, \quad y = \pm b\tau^{1/2}M^{-1}M'_\tau, \quad \text{where} \quad A_1 = \pm b^2\lambda, \ A_2 = \pm\tfrac{1}{2}b^2.$$

72. $y''_{xx} = A_1 x^{-1} y^{-2} (y'_x)^3 + A_2 x^{-2} y^{-1} (y'_x)^2.$

Solution in parametric form:

$$x = \pm a\tau^{1/2} M^{-1} M'_\tau, \quad y = M^{-1}, \qquad \text{where} \quad A_1 = \mp\tfrac{1}{2}a^2, \ \ A_2 = \mp a^2\lambda.$$

73. $y''_{xx} = Ax^n (y'_x)^{\frac{3n+2}{n+1}} + nAx^{n-1} y (y'_x)^{\frac{2n+1}{n+1}}, \qquad n \neq 0, \ \ n \neq -1.$

Solution in parametric form:

$$x = aC_1^{-2n-1}\Big(\int \frac{d\tau}{\beta\tau^k + 1} + C_2\Big)^{-1/n}, \quad y = bC_1^{n^2}\Big(\tau - \int \frac{d\tau}{\beta\tau^k + 1} - C_2\Big),$$

where $k = -\dfrac{n+1}{n}$, $A = \dfrac{n+1}{n^2\beta} a^{1-n} b^{-2}\Big(-\dfrac{nb\beta}{a}\Big)^{\frac{1}{n+1}}$.

74. $y''_{xx} = A(m+1)xy^m (y'_x)^{\frac{m+3}{m+2}} + Ay^{m+1} (y'_x)^{\frac{1}{m+2}}, \qquad m \neq -1, -2.$

Solution in parametric form:

$$x = aC_1^{(m+1)^2}\Big(\tau - \int \frac{d\tau}{\beta\tau^k + 1} - C_2\Big), \quad y = bC_1^{-2m-3}\Big(\int \frac{d\tau}{\beta\tau^k + 1} + C_2\Big)^{-\frac{1}{m+1}},$$

where $k = -\dfrac{m+2}{m+1}$, $A = -\dfrac{m+2}{(m+1)^2\beta} a^{-2} b^{-m}\Big[-\dfrac{a(m+1)\beta}{b}\Big]^{\frac{1}{m+2}}$.

75. $y''_{xx} = Ax^{-m-1} y^m (y'_x)^l - Ax^{-m-2} y^{m+1} (y'_x)^{l-1},$

$m \neq -1, \ \ l \neq 2, \ \ m + l - 1 \neq 0.$

Solution in parametric form:

$$x = aC_1 \exp\Big(\frac{l-2}{m+l-1}\int \frac{d\tau}{F}\Big), \quad y = bC_1 \tau^{\frac{l-2}{m+l-1}} \exp\Big(\frac{l-2}{m+l-1}\int \frac{d\tau}{F}\Big),$$

where

$$F = \frac{m+l-1}{l-2}\big(\beta + C_2\tau^k\big)^{\frac{1}{2-l}} - \tau, \quad k = \frac{(m+1)(l-2)}{m+l-1},$$
$$A = -\frac{(m+1)(m+l-1)}{(l-2)^3} a^{m-1} b^{1-m} \beta \Big[\frac{(l-2)a}{(m+l-1)b}\Big]^l.$$

76. $y''_{xx} = Ax^{l-2} y^{1-l} (y'_x)^l - Ax^{l-3} y^{2-l} (y'_x)^{l-1}, \qquad l \neq 2.$

Solution in parametric form:

$$x = C_1 \exp\Big(\int \frac{d\tau}{F}\Big), \quad y = C_2 \exp\Big(\tau + \int \frac{d\tau}{F}\Big),$$

where $F = (2-l)\big[\beta + e^{(l-2)\tau}\big]^{\frac{1}{2-l}} - 1$, $A = (2-l)^{2-l}\beta$.

77. $y''_{xx} = Ax^{-1} y'_x - Ax^{-2} y.$

Solution: $y = \begin{cases} C_1 x + C_2 |x|^A & \text{if } A \neq 1, \\ x\big(C_1 + C_2 \ln|x|\big) & \text{if } A = 1. \end{cases}$

78. $y''_{xx} = Axy^{-2}(y'_x)^3 - Ay^{-1}(y'_x)^2.$

Solution in implicit form: $x = \begin{cases} C_1 y + C_2 |y|^A & \text{if } A \neq 1, \\ y\bigl(C_1 + C_2 \ln|y|\bigr) & \text{if } A = 1. \end{cases}$

◆ *In the solutions of equations 79 and 80, the following notation is used:*

$$f = \begin{cases} \dfrac{1}{\beta+1}\tau^{\beta+1} + \dfrac{1}{\beta}\tau^{\beta} + C_2 & \text{if } \beta \neq 0, \\ \tau + \ln|\tau| + C_2 & \text{if } \beta = 0. \end{cases}$$

79. $y''_{xx} = Axy^{-2}(y'_x)^l - Ay^{-1}(y'_x)^{l-1}, \qquad l \neq 3.$

Solution in parametric form:

$$x = aC_1[\tau^{\beta+1} - (\beta+1)f]\exp\Bigl(-\int \tau^{\beta-1} f^{-1}\,d\tau\Bigr), \quad y = bC_1 \exp\Bigl(-\int \tau^{\beta-1} f^{-1}\,d\tau\Bigr),$$

where $\beta = \dfrac{2-l}{l-3}$, $A = -a^{l-3} b^{3-l}$.

80. $y''_{xx} = Ax^{-1}(y'_x)^l - Ax^{-2}y(y'_x)^{l-1}, \qquad l \neq 1.$

Solution in parametric form:

$$x = aC_1 \exp\Bigl(-\int \tau^{\beta-1} f^{-1}\,d\tau\Bigr), \quad y = bC_1[\tau^{\beta+1} - (\beta+1)f]\exp\Bigl(-\int \tau^{\beta-1} f^{-1}\,d\tau\Bigr),$$

where $\beta = \dfrac{l-2}{1-l}$, $A = -a^{l-1} b^{1-l}$.

81. $y''_{xx} = A_1 y^{-1}(y'_x)^2 + A_2 x^{-1} y'_x.$

Solution: $y = \begin{cases} \pm(C_1|x|^{A_2+1} + C_2)^{A_1-1} & \text{if } A_1 \neq 1,\ A_2 \neq -1; \\ \pm(C_1 \ln|x| + C_2)^{A_1-1} & \text{if } A_1 \neq 1,\ A_2 = -1; \\ C_2 \exp(C_1|x|^{A_2+1}) & \text{if } A_1 = 1,\ A_2 \neq -1; \\ C_2|x|^{C_1} & \text{if } A_1 = 1,\ A_2 = -1. \end{cases}$

◆ *In the solutions of equations 82–84, the following notation is used:*

$$U = \exp\Bigl(\int \frac{W\,d\tau}{\tau\sqrt{W^2+4}}\Bigr), \quad W = C_2 \tau^{-1/2} \exp\Bigl[\frac{A}{2(k+1)}\tau^{k+1}\Bigr].$$

82. $y''_{xx} = Ax^{m+1}y^m(y'_x)^2 - Ax^m y^{m+1} y'_x, \qquad m \neq -1.$

Solution in parametric form:

$$x = C_1 \tau^{1/2} U^{-1/2}, \quad y = C_1^{-1} \tau^{1/2} U^{1/2}, \qquad k = m.$$

83. $y''_{xx} = Ax^{-2m-2}y^m(y'_x)^2 - Ax^{-2m-3} y^{m+1} y'_x, \qquad m \neq -1.$

Solution in parametric form:

$$x = C_1 \tau^{-1/2} U^{1/2}, \quad y = C_1^2 U, \qquad k = m.$$

84. $y''_{xx} = Ax^{-\frac{m+1}{2}} y^m (y'_x)^2 - Ax^{-\frac{m+3}{2}} y^{m+1} y'_x, \qquad m \neq -1.$

Solution in parametric form:

$$x = C_1^2 U, \quad y = C_1 \tau^{-1/2} U^{1/2}, \qquad k = -\frac{m+3}{2}.$$

85. $y''_{xx} = Ax^n y^m (y'_x)^2 - Ax^{n-1} y^{m+1} y'_x, \qquad m \neq -1, \ n \neq 0.$

Solution in parametric form:

$$x = C_1 \exp\Big(\int \frac{d\tau}{\tau F}\Big), \quad y = C_1^k \tau \exp\Big(k \int \frac{d\tau}{\tau F}\Big), \qquad k = -\frac{n}{m+1},$$

where $F = F(\tau)$ is the solution of the transcendental equation

$$\frac{(F+k)^k}{(F+k-1)^{k-1}} = C_2 \tau^{-1} \exp\Big(\frac{A}{m+1} \tau^{m+1}\Big).$$

86. $y''_{xx} = A_1 x^n y^{-1} (y'_x)^2 + A_2 x^{n-1} y'_x, \qquad n \neq 0.$

Solution:

$$y = C_1 \exp\Big[\int \exp\Big(\frac{A_2}{n} x^n\Big) (F + C_2)^{-1} dx\Big], \quad \text{where} \quad F = \int (1 - A_1 x^n) \exp\Big(\frac{A_2}{n} x^n\Big)\, dx.$$

87. $y''_{xx} = A_1 y^m (y'_x)^2 + A_2 x^{-1} y^{m+1} y'_x, \qquad m \neq -1.$

Solution:

$$x = C_1 \exp\Big[\int \exp\Big(-\frac{A_1 y^{m+1}}{m+1}\Big) \frac{dy}{F + C_2}\Big] \ \text{with} \ F = \int (1 + A_2 y^{m+1}) \exp\Big(-\frac{A_1 y^{m+1}}{m+1}\Big)\, dy.$$

14.6.4 Other Equations $(l_1 \neq l_2)$

► **Classification table.**

Table 14.11 presents all solvable equations whose solutions are outlined in Section 14.6.4. Equations are arranged in accordance with the growth of l_1 $(l_1 > l_2)$. The number of the equation sought is indicated in the last column in this table.

► **Solvable equations and their solutions.**

1. $y''_{xx} = A_1 y^{-1/2} y'_x + A_2 y^{-1/2}.$

Solution in parametric form:

$$x = C_1 \exp(A_1 \tau) - \frac{A_2}{4A_1} \tau^2 + C_2, \quad y = \Big[A_1 C_1 \exp(A_1 \tau) - \frac{A_2}{2A_1} \tau\Big]^2.$$

TABLE 14.11
Solvable cases of the equation $y''_{xx} = A_1 x^{n_1} y^{m_1} (y'_x)^{l_1} + A_2 x^{n_2} y^{m_2} (y'_x)^{l_2}$, $l_1 \neq l_2$

l_1	l_2	m_1	m_2	n_1	n_2	Equation
arbitrary	arbitrary	arbitrary	m_1	0	0	14.6.4.3
arbitrary	arbitrary	0	0	arbitrary	n_1	14.6.4.4
arbitrary	arbitrary	$1-l_1$	$1-l_2$	l_1-2	l_2-2	14.6.4.9
arbitrary	$3-l_1$	$1-l_1$	l_1-2	l_1-2	$1-l_1$	14.6.4.8
1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	14.6.4.1
1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	14.6.4.14
1	0	$-\frac{1}{2}$	0	0	0	14.6.4.16
2	0	-1	1	arbitrary	arbitrary	14.6.4.18
2	1	arbitrary	arbitrary	0	-1	14.6.4.13
2	1	arbitrary ($m_1 \neq -1$)	0	0	arbitrary ($n_2 \neq -1$)	14.6.4.5
2	1	arbitrary ($m_1 \neq -1$)	0	0	-1	14.6.4.7
2	1	-1	0	arbitrary	arbitrary	14.6.4.12
2	1	-1	0	0	arbitrary ($n_2 \neq -1$)	14.6.4.6
$\frac{5}{2}$	$\frac{1}{2}$	arbitrary	m_1+2	m_1+2	m_1	14.6.4.11
3	0	arbitrary* ($m_1 \neq -2$)	m_1+3	m_1+3	m_1	14.6.4.10
3	1	arbitrary	arbitrary	1	-1	14.6.4.19
3	2	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	14.6.4.2
3	2	0	0	0	$-\frac{1}{2}$	14.6.4.17
3	2	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	14.6.4.15

* For $m_1 = -2$, see Equation 14.6.4.8 with $l = 3$.

2. $y''_{xx} = A_1 x^{-1/2} (y'_x)^3 + A_2 x^{-1/2} (y'_x)^2$.

Solution in parametric form:

$$x = \left[A_2 C_1 \exp(-A_2\tau) + \frac{A_1}{2A_2}\tau\right]^2, \qquad y = C_1 \exp(-A_2\tau) - \frac{A_1}{4A_2}\tau^2 + C_2.$$

3. $y''_{xx} = A_1 y^m (y'_x)^{l_1} + A_2 y^m (y'_x)^{l_2}$.

1°. Solution in parametric form with $m \neq -1$:

$$x = C_2 + \int \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} f^{-\frac{m}{m+1}}\, d\tau, \qquad y = f^{\frac{1}{m+1}},$$

where $f = C_1 + (m+1) \displaystyle\int \tau \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} d\tau$.

2°. Solution in parametric form with $m = -1$:

$$x = C_2 + \int \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} e^f\, d\tau, \qquad y = e^f,$$

where $f = C_1 \displaystyle\int \tau \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} d\tau$.

4. $y''_{xx} = A_1 x^n (y'_x)^{l_1} + A_2 x^n (y'_x)^{l_2}$.

1°. Solution in parametric form with $n \neq -1$:

$$x = f^{\frac{1}{n+1}}, \qquad y = C_2 + \int \tau \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} f^{-\frac{n}{n+1}}\, d\tau,$$

where $f = C_1 + (n+1) \displaystyle\int \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} d\tau$.

2°. Solution in parametric form with $n = -1$:

$$x = e^f, \qquad y = C_2 + \int \tau \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} e^f\, d\tau,$$

where $f = C_1 \displaystyle\int \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2}\bigr)^{-1} d\tau$.

5. $y''_{xx} = A_1 y^m (y'_x)^2 + A_2 x^n y'_x, \qquad n \neq -1, \; m \neq -1.$

Solution: $\displaystyle\int \exp\Bigl(-\frac{A_1}{m+1} y^{m+1}\Bigr) dy = C_1 \int \exp\Bigl(\frac{A_2}{n+1} x^{n+1}\Bigr) dx + C_2.$

6. $y''_{xx} = A_1 y^{-1} (y'_x)^2 + A_2 x^n y'_x, \qquad n \neq -1.$

1°. Solution for $A_1 \neq 1$:

$$y = \Bigl[C_1 \int \exp\Bigl(\frac{A_2}{n+1} x^{n+1}\Bigr) dx + C_2\Bigr]^{A_1 - 1}.$$

2°. Solution for $A_1 = 1$:

$$y = C_2 \exp\Bigl[C_1 \int \exp\Bigl(\frac{A_2}{n+1} x^{n+1}\Bigr) dx\Bigr].$$

7. $y''_{xx} = A_1 y^m (y'_x)^2 + A_2 x^{-1} y'_x, \qquad m \neq -1.$

1°. Solution for $A_2 \neq -1$:

$$x = \Bigl[C_1 \int \exp\Bigl(-\frac{A_1}{m+1} y^{m+1}\Bigr) dy + C_2\Bigr]^{-A_2 - 1}.$$

2°. Solution for $A_2=-1$:

$$x=C_2\exp\Bigl[C_1\int\exp\Bigl(-\frac{A_1}{m+1}y^{m+1}\Bigr)\,dy\Bigr].$$

8. $\boldsymbol{y''_{xx}=A_1x^{l-2}y^{1-l}(y'_x)^l+A_2x^{1-l}y^{l-2}(y'_x)^{3-l}.}$

Solution in parametric form:

$$x=C_2\exp\Bigl(\int\frac{d\tau}{A_1\tau^l+A_2\tau^{3-l}-\tau^2+\tau}\Bigr),\quad y=C_1\exp\Bigl(\int\frac{\tau\,d\tau}{A_1\tau^l+A_2\tau^{3-l}-\tau^2+\tau}\Bigr).$$

9. $\boldsymbol{y''_{xx}=A_1x^{l_1-2}y^{1-l_1}(y'_x)^{l_1}+A_2x^{l_2-2}y^{1-l_2}(y'_x)^{l_2}.}$

Solution in parametric form:

$$x=C_2\exp\Bigl(\int\frac{d\tau}{A_1\tau^{l_1}+A_2\tau^{l_2}-\tau^2+\tau}\Bigr),\quad y=C_1\exp\Bigl(\int\frac{\tau\,d\tau}{A_1\tau^{l_1}+A_2\tau^{l_2}-\tau^2+\tau}\Bigr).$$

10. $\boldsymbol{y''_{xx}=Ax^{m+3}y^m(y'_x)^3-Ax^my^{m+3},\qquad m\neq-2.}$

Solution in parametric form:

$$x=C_1\tau^{1/2}\exp\Bigl(-\frac12\int\frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr),\quad y=C_1^{-1}\tau^{1/2}\exp\Bigl(\frac12\int\frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr),$$

where $V=\tau^{-1/2}\exp\Bigl(\frac{3A}{2m+4}\tau^{m+2}\Bigr)\Bigl[C_2-A\int\tau^m\exp\Bigl(\frac{3A}{m+2}\tau^{m+2}\Bigr)\,d\tau\Bigr]^{-1/2}.$

11. $\boldsymbol{y''_{xx}=Ax^{m+2}y^m(y'_x)^{5/2}+Ax^my^{m+2}(y'_x)^{1/2}.}$

Solution in parametric form:

$$x=C_1\tau^{1/2}\exp\Bigl(-\frac12\int\frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr),\quad y=C_1^{-1}\tau^{1/2}\exp\Bigl(\frac12\int\frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr).$$

Here, the function $V=V(\tau)$ is defined in parametric form, $\tau=\tau(u)$, $V=V(u)$, as follows:

1°. For $m\neq-1$, $m\neq-3/2$:

$$\tau=au^{\frac{2}{2m+3}},\qquad V=\frac{b(2m+3)}{2(m+1)}u^{-1}Z^{-1}(\tau Z'_u+\nu Z),$$

where $Z=\begin{cases}C_2J_\nu(u)+Y_\nu(u) & \text{for the upper sign,}\\ C_2I_\nu(u)+K_\nu(u) & \text{for the lower sign,}\end{cases}$ $J_\nu(u)$ and $Y_\nu(u)$ are Bessel functions, and $I_\nu(u)$ and $K_\nu(u)$ are modified Bessel functions,

$$\nu=\frac{m+1}{2m+3},\quad a=\Bigl(-\frac{2m+2}{Ab}\Bigr)^{-\frac{1}{m+1}},\quad \pm\frac{(2m+3)^2}{8(m+1)^2}b^{\frac{2m+3}{m+1}}=\Bigl(-\frac{2m+2}{A}\Bigr)^{\frac{1}{m+1}}.$$

2°. For $m=-1$:

$$\tau=\frac{u^2}{2A^2},\quad V=\frac{A}{\sqrt2}Z^{-1}Z'_u,\qquad\text{where}\quad Z=C_2J_0(u)+Y_0(u).$$

3°. For $m = -3/2$:

$$\tau = A^2 u^{-4}, \qquad V = \begin{cases} \dfrac{1}{2A} \dfrac{(1+k)C_2 u^k + (1-k)u^{-k}}{C_2 u^k + u^{-k}} & \text{if } A^2 < \frac{1}{8}, \\ \dfrac{1}{2A} \dfrac{C_2 \ln u + C_2 + 1}{C_2 \ln u + 1} & \text{if } A^2 = \frac{1}{8}, \\ \dfrac{1}{2A} \dfrac{(C_2 - k)\sin(k \ln u) + (1 + kC_2)\cos(k \ln u)}{C_2 \sin(k \ln u) + \cos(k \ln u)} & \text{if } A^2 > \frac{1}{8}, \end{cases}$$

where $k = \sqrt{|1 - 8A^2|}$.

12. $\boldsymbol{y''_{xx} = A_1 x^{n_1} y^{-1} (y'_x)^2 + A_2 x^{n_2} y'_x}$.

1°. Solution for $n_2 \neq -1$:

$$y = C_1 \exp\Bigl(\int F\,dx\Bigr),$$

where $F = \exp\Bigl(\dfrac{A_2 x^{n_2+1}}{n_2+1}\Bigr)\Bigl[C_2 + \displaystyle\int (1 - A_1 x^{n_1}) \exp\Bigl(\dfrac{A_2 x^{n_2+1}}{n_2+1}\Bigr)\,dx\Bigr]^{-1}$.

2°. Solution for $n_2 = -1$, $A_2 \neq -1$, $A_2 \neq -n_1 - 1$:

$$y = C_1 \exp\Bigl[\int x^{A_2}\Bigl(C_2 + \frac{1}{A_2+1} x^{A_2+1} - \frac{A_1}{n_1 + A_2 + 1} x^{n_1+A_2+1}\Bigr)^{-1} dx\Bigr].$$

3°. Solution for $n_2 = -1$, $A_2 = -1$:

$$y = C_1 \exp\Bigl[\int x^{-1}\Bigl(C_2 + \ln x - \frac{A_1}{n_1} x^{n_1}\Bigr)^{-1} dx\Bigr].$$

4°. Solution for $n_2 = -1$, $A_2 = -n_1 - 1$:

$$y = C_1 \exp\Bigl[\int x^{-n_1-1}\Bigl(C_2 - \frac{1}{n_1} x^{-n_1} - A_1 \ln x\Bigr)^{-1} dx\Bigr].$$

13. $\boldsymbol{y''_{xx} = A_1 y^{m_1} (y'_x)^2 + A_2 x^{-1} y^{m_2} y'_x}$.

1°. Solution for $m_1 \neq -1$:

$$x = C_1 \exp\Bigl(\int F\,dy\Bigr),$$

where $F = \exp\Bigl(-\dfrac{A_1 y^{m_1+1}}{m_1+1}\Bigr)\Bigl[C_2 + \displaystyle\int (1 + A_2 y^{m_2}) \exp\Bigl(-\dfrac{A_1 y^{m_1+1}}{m_1+1}\Bigr)\,dy\Bigr]^{-1}$.

2°. Solution for $m_1 = -1$, $A_1 \neq 1$, $A_1 \neq m_2 + 1$:

$$x = C_1 \exp\Bigl[\int y^{-A_1}\Bigl(C_2 + \frac{1}{1-A_1} y^{1-A_1} + \frac{A_2}{m_2 - A_1 + 1} y^{m_2-A_1+1}\Bigr)^{-1} dy\Bigr].$$

3°. Solution for $m_1 = -1$, $A_1 = 1$:

$$x = C_1 \exp\Bigl[\int y^{-1}\Bigl(C_2 + \ln y + \frac{A_2}{m_2} y^{m_2}\Bigr)^{-1} dy\Bigr].$$

4°. Solution for $m_1 = -1$, $A_1 = m_2 + 1$:

$$x = C_1 \exp\Bigl[\int y^{-m_2-1}\Bigl(C_2 - \frac{1}{m_2} y^{-m_2} + A_2 \ln y\Bigr)^{-1} dy\Bigr].$$

◆ *In the solutions of equations 14 and 15, the following notations are used:*

$$R=\begin{cases} C_1\tau^{k_1}+C_2\tau^{k_2}+C_3\tau^{k_3} & \text{if } B_2(8B_1^3+27B_2)<0,\\ C_1\tau e^{k\tau}+C_2e^{\sigma\tau} & \text{if } 8B_1^3+27B_2=0,\\ C_1e^{k\tau}+C_2e^{\rho\tau}\cos\omega\tau & \text{if } B_2(8B_1^3+27B_2)>0,\end{cases}$$

$$Q=\begin{cases} C_1k_1\tau^{k_1}+C_2k_2\tau^{k_2}+C_3k_3\tau^{k_3} & \text{if } B_2(8B_1^3+27B_2)<0,\\ C_1(1+k\tau)e^{k\tau}+C_2\sigma e^{\sigma\tau} & \text{if } 8B_1^3+27B_2=0,\\ C_1ke^{k\tau}+C_2e^{\rho\tau}(\rho\cos\omega\tau-\omega\sin\omega\tau) & \text{if } B_2(8B_1^3+27B_2)>0,\end{cases}$$

where k_1, k_2, and k_3 (real numbers) or k and $\rho\pm i\omega$ (one real and two complex numbers) are roots of the cubic equation

$$\lambda^3-B_1\lambda^2-\tfrac{1}{2}B_2=0.$$

In the special case $8B_1^3=-27B_2$, we have $k=\frac{2}{3}B_1$ (multiple root) and $\sigma=-\frac{1}{3}B_1$ (simple root).

Remark 14.5. *In the expressions for R and Q, the constant C_3 can be set to any nonzero number (for example, one can set $C_3=\pm1$).*

14. $y''_{xx}=A_1y^{-1/2}y'_x+A_2xy^{-1/2}$.

Solution in parametric form:

$$x=R,\quad y=Q^2,\qquad \text{where}\quad B_1=A_1,\ \ B_2=A_2.$$

15. $y''_{xx}=A_1x^{-1/2}y(y'_x)^3+A_2x^{-1/2}(y'_x)^2$.

Solution in parametric form:

$$x=Q^2,\quad y=R,\qquad \text{where}\quad B_1=-A_2,\ \ B_2=-A_1.$$

◆ *In the solutions of equations 16 and 17, the following notations are used:*

$$R=\begin{cases} \tau^{B_1/2}(C_1\tau^k+C_2\tau^{-k})+C_3 & \text{if } B_1^2+2B_2>0,\\ C_1\tau\exp(\frac{1}{2}B_1\tau)+C_2 & \text{if } B_1^2+2B_2=0,\\ C_1\exp(\frac{1}{2}B_1\tau)\cos(\omega\tau)+C_2 & \text{if } B_1^2+2B_2<0,\end{cases}$$

$$Q=\begin{cases} \tau^{B_1/2}[(C_1(B_1+2k)\tau^k+C_2(B_1-2k)\tau^{-k}] & \text{if } B_1^2+2B_2>0,\\ C_1(B_1\tau+2)\exp(\frac{1}{2}B_1\tau) & \text{if } B_1^2+2B_2=0,\\ C_1\exp(\frac{1}{2}B_1\tau)[B_1\cos(\omega\tau)-2\omega\sin(\omega\tau)] & \text{if } B_1^2+2B_2<0,\end{cases}$$

where $k=\frac{1}{2}\sqrt{B_1^2+2B_2}$ and $\omega=\frac{1}{2}\sqrt{-(B_1^2+2B_2)}$.

16. $y''_{xx}=A_1y^{-1/2}y'_x+A_2$.

Solution in parametric form:

$$x=R,\quad y=\tfrac{1}{4}Q^2,\qquad \text{where}\quad B_1=A_1,\ \ B_2=A_2.$$

17. $y''_{xx}=A_1(y'_x)^3+A_2x^{-1/2}(y'_x)^2$.

Solution in parametric form:

$$x=\tfrac{1}{4}Q^2,\quad y=R,\qquad \text{where}\quad B_1=-A_2,\ \ B_2=-A_1.$$

18. $y''_{xx} = A_1 x^{n_1} y^{-1}(y'_x)^2 + A_2 x^{n_2} y.$

Solution: $y = C_1 \exp\Bigl[-\int \frac{w'_x\,dx}{(A_1 x^{n_1} - 1)w}\Bigr]$, where $w = w(x)$ is the general solution of the second-order linear equation

$$(A_1 x^{n_1} - 1)w''_{xx} - A_1 n_1 x^{n_1 - 1} w'_x + A_1 x^{n_2}(A_1 x^{n_1} - 1)^2 w = 0.$$

19. $y''_{xx} = A_1 x y^{m_1}(y'_x)^3 + A_2 x^{-1} y^{m_2} y'_x.$

Solution: $x = C_1 \exp\Bigl[\int \frac{w'_y\,dy}{(A_2 y^{m_2} + 1)w}\Bigr]$, where $w = w(y)$ is the general solution of the second-order linear equation

$$(A_2 y^{m_2} + 1)w''_{yy} - A_2 m_2 y^{m_2 - 1} w'_y - A_1 y^{m_1}(A_2 y^{m_2} + 1)^2 w = 0.$$

14.7 Equations of the Form $y''_{xx} = f(x)g(y)h(y'_x)$

See Section 14.3 for the case $f(x) = \mathrm{const}\,x^n$, $g(y) = \mathrm{const}\,y^m$, $h(w) = \mathrm{const}$.

See Section 14.5 for the case $f(x) = \mathrm{const}\,x^n$, $g(y) = \mathrm{const}\,y^m$, $h(w) = \mathrm{const}\,w^l$.

14.7.1 Equations of the Form $y''_{xx} = f(x)g(y)$

1. $y''_{xx} = x^{-2}\Bigl[-\frac{2(m+1)}{(m+3)^2}y + Ay^m\Bigr], \qquad m \neq -3, \ m \neq -1.$

See equation 14.4.2.4.

2. $y''_{xx} = x^{-2}(\frac{15}{4}y + Ay^{-7}).$

See equation 14.4.2.35.

3. $y''_{xx} = x^{-2}(6y + Ay^{-4}).$

See equation 14.4.2.31.

4. $y''_{xx} = x^{-2}(12y + Ay^{-5/2}).$

See equation 14.4.2.64.

5. $y''_{xx} = x^{-2}(2y + Ay^{-2}).$

See equation 14.4.2.6.

6. $y''_{xx} = x^{-2}(-\frac{3}{16}y + Ay^{-5/3}).$

See equation 14.4.2.26.

7. $y''_{xx} = x^{-2}(-\frac{9}{100}y + Ay^{-5/3}).$

See equation 14.4.2.10.

8. $y''_{xx} = x^{-2}(\frac{3}{4}y + Ay^{-5/3}).$

See equation 14.4.2.12.

9. $y''_{xx} = x^{-2}(\frac{63}{4}y + Ay^{-5/3})$.

See equation 14.4.2.66.

10. $y''_{xx} = x^{-2}(-\frac{5}{36}y + Ay^{-7/5})$.

See equation 14.4.2.29.

11. $y''_{xx} = x^{-2}(-\frac{2}{9}y + Ay^{-1/2})$.

See equation 14.4.2.14.

12. $y''_{xx} = x^{-2}(-\frac{4}{25}y + Ay^{-1/2})$.

See equation 14.4.2.8.

13. $y''_{xx} = x^{-2}(20y + Ay^{-1/2})$.

See equation 14.4.2.33.

14. $y''_{xx} = x^{-2}(-\frac{12}{49}y + Ay^{1/2})$.

See equation 14.4.2.37.

15. $y''_{xx} = x^{-2}(Ay^2 - \frac{6}{25}y)$.

See equation 14.4.2.60.

16. $y''_{xx} = x^{-2}(Ay^2 + \frac{6}{25}y)$.

See equation 14.4.2.62.

17. $y''_{xx} = x^{-4/3}(A + By^{-1/2})$.

See equation 14.4.2.40.

18. $y''_{xx} = (Ax^4 + Bx^3)y^{-7}$.

See equation 14.4.2.39.

19. $y''_{xx} = (Ax^2 + B)y^{-5}$.

See equation 14.4.2.16.

20. $y''_{xx} = (Ax^{-1} + Bx^{-2})y^{-2}$.

See equation 14.4.2.28.

21. $y''_{xx} = (Ax^{-7/3} + Bx^{-10/3})y^{-5/3}$.

See equation 14.4.2.48.

22. $y''_{xx} = (Ax^{-4/3} + Bx^{-10/3})y^{-5/3}$.

See equation 14.4.2.49.

23. $y''_{xx} = (Ax^{-4/3} + Bx^{-7/3})y^{-5/3}$.

See equation 14.4.2.24.

24. $y''_{xx} = (Ax^{-2/3} + Bx^{-4/3})y^{-5/3}$.

See equation 14.4.2.90.

25. $y''_{xx} = (A + Bx^{-2/3})y^{-5/3}$.

See equation 14.4.2.89.

26. $y''_{xx} = (Ax^2 + B)y^{-5/3}$.

See equation 14.4.2.47.

27. $y''_{xx} = (Ax^2 + Bx)y^{-5/3}$.

See equation 14.4.2.46.

28. $y''_{xx} = A(ax^{-2/3} + bx^{-5/3})^2 y^{-5/3}$.

This is a special case of equation 14.7.1.37 with $c = 1$ and $d = 0$.

29. $y''_{xx} = (Ax^{-8/5} + Bx^{-13/5})y^{-7/5}$.

See equation 14.4.2.25.

30. $y''_{xx} = (Ax^{-5/2} + Bx^{-7/2})y^{-1/2}$.

See equation 14.4.2.23.

31. $y''_{xx} = A(ax^5 + bx^4)^{-1/2} y^{-1/2}$.

This is a special case of equation 14.7.1.38 with $c = 1$ and $d = 0$.

32. $y''_{xx} = A(ax^{15/8} + bx^{7/8})^{-4/3} y^{-1/2}$.

This is a special case of equation 14.7.1.39 with $c = 1$ and $d = 0$.

33. $y''_{xx} = A(ax^{7/3} + bx^{4/3})^{-15/7} y^2$.

This is a special case of equation 14.7.1.40 with $c = 1$ and $d = 0$.

34. $y''_{xx} = (ax^2 + bx + c)y^{-5/3}$.

The transformation $x = x(t)$, $y = (x'_t)^{3/2}$ leads to a third-order equation: $2x'_t x'''_{ttt} - (x''_{tt})^2 = \frac{4}{3}(ax^2 + bx + c)$. Differentiating the latter equation with respect to t and dividing it by x'_t, we obtain a fourth-order constant coefficient linear equation: $3x''''_{tttt} = 4ax + 2b$.

35. $y''_{xx} = (ax^{-10/3} + bx^{-7/3} + cx^{-4/3})y^{-5/3}$.

The transformation $x = 1/t$, $y = w/t$ leads to an equation of the form 14.7.1.34: $w''_{tt} = (at^2 + bt + c)w^{-5/3}$.

36. $y''_{xx} = k(ax^2 + bx + c)^n y^{-2n-3}$.

This is a special case of equation 14.9.1.21 with $f(u) = ku^{-2n}$. Setting $u(x) = y(ax^2+bx+c)^{-1/2}$ and integrating the equation, we obtain a first-order separable equation: $(ax^2 + bx + c)^2 (u'_x)^2 = (\frac{1}{4}b^2 - ac)u^2 - \frac{k}{n+1}u^{-2n-2} + C_1$.

37. $y''_{xx} = A(ax + b)^2 (cx + d)^{-10/3} y^{-5/3}$.

The transformation $\xi = \dfrac{ax + b}{cx + d}$, $w = \dfrac{y}{cx + d}$ leads to an Emden–Fowler equation of the form 14.3.1.9: $w''_{\xi\xi} = A\Delta^{-2}\xi^2 w^{-5/3}$, where $\Delta = ad - bc$.

38. $y''_{xx} = A(ax+b)^{-1/2}(cx+d)^{-2}y^{-1/2}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to an Emden–Fowler equation of the form 14.3.1.25: $w''_{\xi\xi} = A\Delta^{-2}\xi^{-1/2}w^{-1/2}$, where $\Delta = ad - bc$.

39. $y''_{xx} = A(ax+b)^{-4/3}(cx+d)^{-7/6}y^{-1/2}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to an Emden–Fowler equation of the form 14.3.1.17: $w''_{\xi\xi} = A\Delta^{-2}\xi^{-4/3}w^{-1/2}$, where $\Delta = ad - bc$.

40. $y''_{xx} = A(ax+b)^{-15/7}(cx+d)^{-20/7}y^{2}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to an Emden–Fowler equation of the form 14.3.1.20: $w''_{\xi\xi} = A\Delta^{-2}\xi^{-15/7}w^{2}$, where $\Delta = ad - bc$.

41. $y''_{xx} = A\exp(ax^2+bx)\exp(ky).$

The substitution $kw = ky + ax^2 + bx$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = Ae^{kw} + 2ak^{-1}$.

14.7.2 Equations Containing Power Functions ($h \not\equiv$ const)

1. $y''_{xx} = \left[\dfrac{2(n+1)}{(n+3)^2}x + Ax^n\right]y^{-2}(y'_x)^3, \qquad n \neq -3,\ n \neq -1.$

See equation 14.6.2.116.

2. $y''_{xx} = (-\frac{15}{4}x + Ax^{-7})y^{-2}(y'_x)^3.$

See equation 14.6.2.117.

3. $y''_{xx} = (-6x + Ax^{-4})y^{-2}(y'_x)^3.$

See equation 14.6.2.118.

4. $y''_{xx} = (-12x + Ax^{-5/2})y^{-2}(y'_x)^3.$

See equation 14.6.2.119.

5. $y''_{xx} = (-2x + Ax^{-2})y^{-2}(y'_x)^3.$

See equation 14.6.2.120.

6. $y''_{xx} = (\frac{3}{16}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

See equation 14.6.2.121.

7. $y''_{xx} = (\frac{9}{100}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

See equation 14.6.2.122.

8. $y''_{xx} = (-\frac{3}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

See equation 14.6.2.123.

9. $y''_{xx} = (-\frac{63}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

See equation 14.6.2.124.

10. $y''_{xx} = (\frac{5}{36}x + Ax^{-7/5})y^{-2}(y'_x)^3.$

See equation 14.6.2.125.

11. $y''_{xx} = (\frac{2}{9}x + Ax^{-1/2})y^{-2}(y'_x)^3.$

See equation 14.6.2.126.

12. $y''_{xx} = (\frac{4}{25}x + Ax^{-1/2})y^{-2}(y'_x)^3.$

See equation 14.6.2.127.

13. $y''_{xx} = (-20x + Ax^{-1/2})y^{-2}(y'_x)^3.$

See equation 14.6.2.128.

14. $y''_{xx} = (\frac{12}{49}x + Ax^{1/2})y^{-2}(y'_x)^3.$

See equation 14.6.2.129.

15. $y''_{xx} = (Ax^2 + \frac{6}{25}x)y^{-2}(y'_x)^3.$

See equation 14.6.2.130.

16. $y''_{xx} = (Ax^2 - \frac{6}{25}x)y^{-2}(y'_x)^3.$

See equation 14.6.2.131.

17. $y''_{xx} = (A + Bx^{-1/2})y^{-4/3}(y'_x)^3.$

See equation 14.6.2.15.

18. $y''_{xx} = x^{-7}(Ay^4 + By^3)(y'_x)^3.$

See equation 14.6.2.111.

19. $y''_{xx} = x^{-5}(Ay^2 + B)(y'_x)^3.$

See equation 14.6.2.96.

20. $y''_{xx} = x^{-2}(Ay^{-1} + By^{-2})(y'_x)^3.$

See equation 14.6.2.110.

21. $y''_{xx} = x^{-5/3}(Ay^{-7/3} + By^{-10/3})(y'_x)^3.$

See equation 14.6.2.34.

22. $y''_{xx} = x^{-5/3}(Ay^{-4/3} + By^{-10/3})(y'_x)^3.$

See equation 14.6.2.36.

23. $y''_{xx} = x^{-5/3}(Ay^{-4/3} + By^{-7/3})(y'_x)^3.$

See equation 14.6.2.14.

24. $y''_{xx} = x^{-5/3}(Ay^{-2/3} + By^{-4/3})(y'_x)^3.$

See equation 14.6.2.115.

25. $y''_{xx} = x^{-5/3}(A + By^{-2/3})(y'_x)^3.$

See equation 14.6.2.114.

26. $y''_{xx} = x^{-5/3}(Ay^2 + B)(y'_x)^3.$

See equation 14.6.2.35.

27. $y''_{xx} = x^{-5/3}(Ay^2 + By)(y'_x)^3.$

See equation 14.6.2.33.

28. $y''_{xx} = Ax^{-5/3}(ay^{-2/3} + by^{-5/3})^2(y'_x)^3.$

This is a special case of equation 14.7.2.37 with $c = 1$ and $d = 0$.

29. $y''_{xx} = x^{-7/5}(Ay^{-8/5} + By^{-13/5})(y'_x)^3.$

See equation 14.6.2.109.

30. $y''_{xx} = x^{-1/2}(Ay^{-5/2} + By^{-7/2})(y'_x)^3.$

See equation 14.6.2.13.

31. $y''_{xx} = Ax^{-1/2}(ay^5 + by^4)^{-1/2}(y'_x)^3.$

This is a special case of equation 14.7.2.38 with $c = 1$ and $d = 0$.

32. $y''_{xx} = Ax^{-1/2}(ay^{15/8} + by^{7/8})^{-4/3}(y'_x)^3.$

This is a special case of equation 14.7.2.39 with $c = 1$ and $d = 0$.

33. $y''_{xx} = Ax^2(ay^{7/3} + by^{4/3})^{-15/7}(y'_x)^3.$

This is a special case of equation 14.7.2.40 with $c = 1$ and $d = 0$.

34. $y''_{xx} = x^{-5/3}(ay^2 + by + c)(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.34 with respect to $x = x(y)$: $x''_{yy} = -(ay^2 + by + c)x^{-5/3}$.

35. $y''_{xx} = x^{-5/3}(ay^{-10/3} + by^{-7/3} + cy^{-4/3})(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.35 with respect to $x = x(y)$: $x''_{yy} = -(ay^{-10/3} + by^{-4/3} + cy^{-4/3})x^{-5/3}$.

36. $y''_{xx} = x^{-2n-3}(ay^2 + by + c)^n(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.9.1.21 (for $f(\xi) = -\xi^{-2n}$) with respect to $x = x(y)$: $x''_{yy} = -(ay^2 + by + c)^n x^{-2n-3}$.

37. $y''_{xx} = Ax^{-5/3}(ay + b)^2(cy + d)^{-10/3}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.37 with respect to $x = x(y)$: $x''_{yy} = -A(ay + b)^2(cy + d)^{-10/3}x^{-5/3}$.

38. $y''_{xx} = Ax^{-1/2}(ay + b)^{-1/2}(cy + d)^{-2}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.38 with respect to $x = x(y)$: $x''_{yy} = -A(ay + b)^{-1/2}(cy + d)^{-2}x^{-1/2}$.

39. $y''_{xx} = Ax^{-1/2}(ay+b)^{-4/3}(cy+d)^{-7/6}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.39 with respect to $x = x(y)$: $x''_{yy} = -A(ay+b)^{-4/3}(cy+d)^{-7/6}x^{-1/2}$.

40. $y''_{xx} = Ax^2(ay+b)^{-15/7}(cy+d)^{-20/7}(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.40 with respect to $x = x(y)$: $x''_{yy} = -A(ay+b)^{-15/7}(cy+d)^{-20/7}x^2$.

41. $y''_{xx} = Ax^{-1/2}y^{-2}[(y'_x)^2 + B^2]^{1/2}.$

Solution in parametric form:

$$x = a(u^2-1)^{-1}(\tau u \pm R)^2, \qquad y = b\tau^{-1}(u^2-1)^{-1/2},$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \mp\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{2}a^{-1/2}b^2$, $B = \frac{1}{2}a^{-1}b$.

42. $y''_{xx} = Ax^{-1/2}y^{-2}[(y'_x)^2 - B^2]^{1/2}.$

Solution in parametric form:

$$x = a(u^2+1)^{-1}(\tau u \pm R)^2, \qquad y = b\tau^{-1}(u^2+1)^{-1/2},$$

where $R = \sqrt{C_1 - \tau^2 - 2\tau^{-1}}$, $u = \pm\tan\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{2}a^{-1/2}b^2$, $B = \frac{1}{2}a^{-1}b$.

43. $y''_{xx} = Ax^{-1/2}y^{-2}[B^2 - (y'_x)^2]^{1/2}.$

Solution in parametric form:

$$x = a(1-u^2)^{-1}(\tau u \mp R)^2, \qquad y = b\tau^{-1}(1-u^2)^{-1/2},$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{2}a^{-1/2}b^2$, $B = \frac{1}{2}a^{-1}b$.

44. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^2[(y'_x)^2 + B^2]^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-1}(u^2-1)^{-1/2}, \qquad y = b(u^2-1)^{-1}(\tau u \pm R)^2,$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \mp\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{4}a^3b^{-3/2}$, $B = 2a^{-1}b$.

45. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^2[(y'_x)^2 - B^2]^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-1}(1-u^2)^{-1/2}, \qquad y = b(1-u^2)^{-1}(\tau u \mp R)^2,$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{4}a^3b^{-3/2}$, $B = 2a^{-1}b$.

46. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^2[B^2 - (y'_x)^2]^{1/2}.$

Solution in parametric form:

$$x = a\tau^{-1}(1+u^2)^{-1/2}, \quad y = b(1+u^2)^{-1}(\tau u \pm R)^2,$$

where $R = \sqrt{-\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tan\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{4}a^3b^{-3/2}$, $B = 2a^{-1}b$.

14.7.3 Equations Containing Exponential Functions $(h \not\equiv \text{const})$

▶ **Preliminary remarks.**

1°. If $l \neq 1-m$, the equation

$$y''_{xx} = Ae^x y^m (y'_x)^l \tag{1}$$

has a particular solution:

$$y = Be^{\lambda x}, \quad \text{where} \quad \lambda = \frac{1}{1-m-l}, \quad B = (A\lambda^{l-2})^{\lambda}.$$

2°. If $m \neq 0$ and $l \neq 1$, equation (1) can be reduced with the aid of the transformation

$$t = (y'_x)^{1-l}, \quad w = e^x$$

to a generalized Emden–Fowler equation with respect to $w = w(t)$:

$$w''_{tt} = Bt^{\frac{1}{1-l}} w^{-1} (w'_t)^{\frac{2m+1}{m}}, \tag{2}$$

where $B = -m\bigl[A(1-l)\bigr]^{\frac{1}{m}}$. Equations of the form (2) are outlined in Section 14.5.

Whenever the general solution $w = w(t)$ of the Emden–Fowler equation (2) is obtained, the solution of the original equation (1) can be written out in parametric form as:

$$x = \ln w, \quad y = k(w'_t)^{-\frac{1}{m}}, \quad \text{where} \quad k = \bigl[A(1-l)\bigr]^{-\frac{1}{m}}.$$

3°. If $l \neq n+2$, the equation

$$y''_{xx} = Ax^n e^y (y'_x)^l \tag{3}$$

has a particular solution:

$$y = \lambda \ln(Bx), \quad \text{where} \quad \lambda = l-n-2, \quad B = \Bigl(-\frac{\lambda^{1-l}}{A}\Bigr)^{\frac{1}{\lambda}}.$$

4°. Taking y to be the independent variable and x to be the dependent one, we obtain from equation (3) an equation of the form (1) for $x = x(y)$:

$$x''_{yy} = -Ae^y x^n (x'_y)^{3-l}.$$

5°. If $n \neq -1$ and $l \neq 1$, equation (3) can be reduced with the aid of the transformation

$$t = (y'_x)^{1-l}, \quad u = x^{n+1}$$

to a generalized Emden–Fowler equation for $u = u(t)$:

$$u''_{tt} = -\frac{1}{n+1} t^{\frac{1}{1-l}} u^{-\frac{n}{n+1}} (u'_t)^2. \tag{4}$$

Equations of this form are outlined in Section 14.5.

Whenever the general solution $u = u(t)$ of the Emden–Fowler equation (4) is obtained, the solution of the original equation (3) can be written out in parametric form as:

$$x = u^{\frac{1}{n+1}}, \qquad y = -\ln(u'_t) + \ln\frac{n+1}{A(1-l)}.$$

► **Solvable equations and their solutions.**

1. $y''_{xx} = ae^{\lambda y}$**.**

Solution: $$y = \begin{cases} -\dfrac{1}{\lambda}\ln\left[\dfrac{a\lambda}{2C_1^2}\sin^2(C_1x + C_2)\right] & \text{if } a < 0,\ \lambda < 0, \\ -\dfrac{1}{\lambda}\ln\left[\dfrac{a\lambda}{2C_1^2}\sinh^2(C_1x + C_2)\right] & \text{if } a > 0,\ \lambda > 0, \\ -\dfrac{1}{\lambda}\ln\left[-\dfrac{a\lambda}{2C_1^2}\cosh^2(C_1x + C_2)\right] & \text{if } a\lambda < 0. \end{cases}$$

2. $y''_{xx} = Ae^x(y'_x)^l$**.**

1°. Solution in parametric form with $l \neq 1$:

$$x = \ln\left[\pm\frac{1}{A(1-l)}C_1^{1-l}\tau\right], \qquad y = C_1\int\frac{1}{\tau}(1 \pm \tau)^{\frac{1}{1-l}}\,d\tau + C_2.$$

2°. Solution in parametric form with $l = 1$:

$$x = \ln\left(\pm\frac{\tau}{A}\right), \qquad y = C_1\int\frac{1}{\tau}\exp(\pm\tau)\,d\tau + C_2.$$

3. $y''_{xx} = Ae^x y^m (y'_x)^2$**.**

1°. Solution in parametric form with $m \neq -1$:

$$x = \int\frac{d\tau}{f} + C_2, \qquad y = \tau\exp\left[-\frac{1}{m+1}\left(\int\frac{d\tau}{f} + C_2\right)\right],$$

where the function $f = f(\tau)$ is defined implicitly by the relation

$$\ln\left(\frac{f}{\tau} - \frac{1}{m+1}\right) - \frac{\tau}{(m+1)f - \tau} = \frac{A}{m+1}\tau^{m+1} - \ln\tau + C_1.$$

2°. Solution for $m = -1$:

$$y = C_2\exp\left(\int\frac{dx}{x + Ae^x + C_1}\right).$$

4. $y''_{xx} = Ae^x y$**.**

1°. Solution for $A > 0$:

$$y = C_1 I_0\big(2\sqrt{A}\,e^{x/2}\big) + C_2 K_0\big(2\sqrt{A}\,e^{x/2}\big),$$

where $I_0(z)$ and $K_0(z)$ are modified Bessel functions.

2°. Solution for $A < 0$:

$$y = C_1 J_0\big(2\sqrt{-A}\,e^{x/2}\big) + C_2 Y_0\big(2\sqrt{-A}\,e^{x/2}\big),$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions.

5. $y''_{xx} = Ae^x y^{-1/2}(y'_x)^{3/2}$.

Solution in parametric form:

$$x = \tau^2 - \ln(Af), \quad y = C_1\bigl[2\tau f - \exp(\tau^2)\bigr]^2, \qquad \text{where} \quad f = \int \exp(\tau^2)\,d\tau + C_2.$$

6. $y''_{xx} = Ae^x y(y'_x)^{3/2}$.

Solution in parametric form:

$$x = -\ln\bigl[AC_1^3\bigl(\sqrt{\tau^2+\tau} - f\bigr)\bigr], \quad y = 2C_1^2\biggl(1 - f\sqrt{\frac{\tau+1}{\tau}}\biggr),$$

where $f = \ln\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr) + C_2$.

7. $y''_{xx} = Ae^y (y'_x)^l$.

1°. Solution in parametric form with $l \neq 2$:

$$x = C_1 \int \frac{1}{\tau}(1 \pm \tau)^{\frac{1}{l-2}}\,d\tau + C_2, \qquad y = \ln\Bigl[\pm\frac{1}{A(2-l)}C_1^{l-2}\tau\Bigr].$$

2°. Solution in parametric form with $l = 2$:

$$x = C_1 \int \frac{1}{\tau}\exp(\mp\tau)\,d\tau + C_2, \qquad y = \ln\Bigl(\pm\frac{\tau}{A}\Bigr).$$

8. $y''_{xx} = Ax^n e^y y'_x$.

1°. Solution in parametric form with $n \neq -1$:

$$x = \tau \exp\Bigl[-\frac{1}{n+1}\Bigl(\int\frac{d\tau}{f} + C_2\Bigr)\Bigr], \quad y = \int \frac{d\tau}{f} + C_2,$$

where the function $f = f(\tau)$ is defined implicitly by the relation

$$\ln\Bigl(\frac{f}{\tau} - \frac{1}{n+1}\Bigr) - \frac{\tau}{(n+1)f - \tau} = -\frac{A}{n+1}\tau^{n+1} - \ln\tau + C_1.$$

2°. Solution for $n = -1$:

$$x = C_2 \exp\Bigl(\int \frac{dy}{y - Ae^y + C_1}\Bigr).$$

9. $y''_{xx} = Ax^{-1/2} e^y (y'_x)^{3/2}$.

Solution in parametric form:

$$x = C_1\bigl[2\tau f - \exp(\tau^2)\bigr]^2, \quad y = \tau^2 - \ln(-Af), \qquad \text{where} \quad f = \int \exp(\tau^2)\,d\tau + C_2.$$

10. $y''_{xx} = Axe^y (y'_x)^{3/2}$.

Solution in parametric form:

$$x = 2C_1^2\biggl(1 - f\sqrt{\frac{\tau+1}{\tau}}\biggr), \quad y = -\ln\bigl[AC_1^3\bigl(f - \sqrt{\tau^2+\tau}\bigr)\bigr],$$

where $f = \ln\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr) + C_2$.

11. $y''_{xx} = Axe^y(y'_x)^3.$

1°. Solution in parametric form with $A > 0$:

$$x = C_1 J_0(2\tau) + C_2 Y_0(2\tau), \qquad y = \ln\bigl(\tau/\sqrt{A}\,\bigr),$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions.

2°. Solution in parametric form with $A < 0$:

$$x = C_1 I_0(2\tau) + C_2 K_0(2\tau), \qquad y = \ln\bigl(\tau/\sqrt{-A}\,\bigr),$$

where $I_0(z)$ and $K_0(z)$ are modified Bessel functions.

12. $y''_{xx} = Ae^x e^y (y'_x)^l.$

Solution in parametric form:

$$x = \int \frac{d\tau}{f\tau^l} + C_2, \qquad y = \ln\Bigl(\frac{f}{A}\Bigr) - \int \frac{d\tau}{f\tau^l} - C_2,$$

where $f = \begin{cases} \dfrac{1}{2-l}\tau^{2-l} + \dfrac{1}{1-l}\tau^{1-l} + C_1 & \text{if } l \neq 1,\, 2; \\ \tau + \ln|\tau| + C_1 & \text{if } l = 1; \\ \ln|\tau| - \dfrac{1}{\tau} + C_1 & \text{if } l = 2. \end{cases}$

13. $y''_{xx} = A\exp(kx)\exp(ay^2 + by)(y'_x)^3.$

Taking y to be the independent variable, we obtain an equation of the form 14.7.1.41 with respect to $x = x(y)$: $x''_{yy} = -A\exp(ay^2 + by)\exp(kx)$.

14. $y''_{xx} = Ae^x y^{-1/2}(y'_x)^{3/2}\sqrt{y'_x - 2B}.$

Solution in parametric form:

$$x = \ln[a\tau(\cosh u)^{-1}], \qquad y = B\cosh^2 u\,(\tau\tanh u \pm R)^2,$$

where $a = -A^{-1}B^{-1/2}$, $R = \sqrt{2\ln\tau + \tau^2 + C_1}$, $u = C_2 \mp \int R^{-1}\,d\tau$.

15. $y''_{xx} = Ae^x y^{-1/2}(y'_x)^{3/2}\sqrt{2B - y'_x}.$

Solution in parametric form:

$$x = \ln[a\tau(\cos u)^{-1}], \qquad y = B\cos^2 u\,(\tau\tan u \pm R)^2,$$

where $a = -A^{-1}B^{-1/2}$, $R = \sqrt{2\ln\tau - \tau^2 + C_1}$, $u = C_2 \pm \int R^{-1}\,d\tau$.

16. $y''_{xx} = Ax^{-1/2}e^y y'_x\sqrt{y'_x - B}.$

Solution in parametric form:

$$x = \frac{1}{2B}\cos^2 u\,(\tau\tan u \pm R)^2, \qquad y = \ln[b\tau(\cos u)^{-1}],$$

where $b = A^{-1}\sqrt{2}$, $R = \sqrt{2\ln\tau - \tau^2 + C_1}$, $u = C_2 \pm \int R^{-1}\,d\tau$.

17. $y''_{xx} = Ax^{-1/2}e^{y}y'_x\sqrt{B - y'_x}$.

Solution in parametric form:

$$x = \frac{1}{2B}\cosh^2 u\,(\tau\tanh u \pm R)^2, \qquad y = \ln[b\tau(\cosh u)^{-1}],$$

where $b = A^{-1}\sqrt{2},\ R = \sqrt{2\ln\tau + \tau^2 + C_1},\ u = C_2 \mp \int R^{-1}\,d\tau.$

14.7.4 Equations Containing Hyperbolic Functions $(h \not\equiv \text{const})$

1. $y''_{xx} = Ax[\cosh(\lambda y)]^{-2}y'_x$.

Solution in parametric form:

$$x = a\cosh u\,(\tau\tanh u \pm R), \qquad y = u/\lambda,$$

where $A = a^{-2},\ R = \sqrt{2\ln\tau + \tau^2 + C_1},\ u = C_2 \mp \int R^{-1}\,d\tau.$

2. $y''_{xx} = Ax[\sinh(\lambda y)]^{-2}y'_x$.

Solution in parametric form:

$$x = a\sinh u\,(\tau\coth u \pm R), \qquad y = u/\lambda,$$

where $A = a^{-2},\ R = \sqrt{2\ln\tau + \tau^2 + C_1},\ u = C_2 \mp \int R^{-1}\,d\tau.$

3. $y''_{xx} = Ax\cosh(\lambda y)(y'_x)^{3/2}$.

Solution in parametric form:

$$x = a(u^2+1)^{-1/2}(\tau u \pm R), \qquad y = \lambda^{-1}\ln\bigl(u + \sqrt{u^2+1}\,\bigr),$$

where $A = 2a^{-2}\sqrt{a\lambda},\ R = \sqrt{C_1 - \tau^2 - 2\tau^{-1}},\ u = \pm\tan\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr).$

4. $y''_{xx} = Ax\sinh(\lambda y)(y'_x)^{3/2}$.

Solution in parametric form:

$$x = a(u^2-1)^{-1/2}(\tau u \pm R), \qquad y = \pm\lambda^{-1}\ln\bigl(u + \sqrt{u^2-1}\,\bigr),$$

where $A = \pm 2a^{-2}\sqrt{a\lambda},\ R = \sqrt{C_1 + \tau^2 - 2\tau^{-1}},\ u = \mp\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr).$

5. $y''_{xx} = A\cosh(\lambda x)y(y'_x)^{3/2}$.

Solution in parametric form:

$$x = \lambda^{-1}\ln(u + \sqrt{u^2+1}), \qquad y = b(u^2+1)^{-1/2}(\tau u \pm R),$$

where $A = -2b^{-2}\sqrt{b\lambda},\ R = \sqrt{C_1 - \tau^2 - 2\tau^{-1}},\ u = \pm\tan\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr).$

6. $y''_{xx} = A\sinh(\lambda x) y (y'_x)^{3/2}$.

Solution in parametric form:

$$x = \pm\lambda^{-1}\ln(u + \sqrt{u^2 - 1}), \qquad y = b(u^2 - 1)^{-1/2}(\tau u \pm R),$$

where $A = \mp 2b^{-2}\sqrt{b\lambda}$, $R = \sqrt{C_1 + \tau^2 - 2\tau^{-1}}$, $u = \mp\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$.

7. $y''_{xx} = A[\cosh(\lambda x)]^{-2} y (y'_x)^2$.

Solution in parametric form:

$$x = u/\lambda, \qquad y = b\cosh u\,(\tau\tanh u \pm R),$$

where $A = -b^{-2}$, $R = \sqrt{2\ln\tau + \tau^2 + C_1}$, $u = C_2 \mp \int R^{-1}\,d\tau$.

8. $y''_{xx} = A[\sinh(\lambda x)]^{-2} y (y'_x)^2$.

Solution in parametric form:

$$x = u/\lambda, \qquad y = b\sinh u\,(\tau\coth u \pm R),$$

where $A = -b^{-2}$, $R = \sqrt{2\ln\tau + \tau^2 + C_1}$, $u = C_2 \mp \int R^{-1}\,d\tau$.

14.7.5 Equations Containing Trigonometric Functions ($h \not\equiv \text{const}$)

◆ *In the solutions of equations 1–4, the following notation is used:*

$$R = \sqrt{2\ln\tau - \tau^2 + C_1}, \qquad u = C_2 \pm \int R^{-1}\,d\tau.$$

1. $y''_{xx} = Ax[\cos(\lambda y)]^{-2} y'_x$.

Solution in parametric form:

$$x = a\cos u\,(\tau\tan u \pm R), \quad y = u/\lambda, \qquad \text{where} \quad A = a^{-2}.$$

2. $y''_{xx} = Ax[\sin(\lambda y)]^{-2} y'_x$.

Solution in parametric form:

$$x = a\sin u\,(\tau\cot u \mp R), \quad y = u/\lambda, \qquad \text{where} \quad A = a^{-2}.$$

3. $y''_{xx} = A[\cos(\lambda x)]^{-2} y (y'_x)^2$.

Solution in parametric form:

$$x = \lambda^{-1}u, \quad y = b\cos u\,(\tau\tan u \pm R), \qquad \text{where} \quad A = -b^2.$$

4. $y''_{xx} = A[\sin(\lambda x)]^{-2} y (y'_x)^2$.

Solution in parametric form:

$$x = \lambda^{-1}u, \quad y = b\sin u\,(\tau\cot u \mp R), \qquad \text{where} \quad A = -b^2.$$

◆ *In the solutions of equations 5–8, the following notation is used:*

$$R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}, \qquad u = \pm\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr).$$

5. $y''_{xx} = Ax\cos(\lambda y)\,(y'_x)^{3/2}$.

Solution in parametric form:

$$x = a\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \quad y = \lambda^{-1}\arccos u, \qquad \text{where} \quad A = 2a^{-2}(-a\lambda)^{1/2}.$$

6. $y''_{xx} = Ax\sin(\lambda y)\,(y'_x)^{3/2}$.

Solution in parametric form:

$$x = a\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \quad y = \lambda^{-1}\arccos u, \qquad \text{where} \quad A = 2a^{-2}(a\lambda)^{1/2}.$$

7. $y''_{xx} = A\cos(\lambda x)\,y(y'_x)^{3/2}$.

Solution in parametric form:

$$x = \lambda^{-1}\arccos u, \quad y = b\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \qquad \text{where} \quad A = -2b^{-2}(-b\lambda)^{1/2}.$$

8. $y''_{xx} = A\sin(\lambda x)\,y(y'_x)^{3/2}$.

Solution in parametric form:

$$x = \lambda^{-1}\arccos u, \quad y = b\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \qquad \text{where} \quad A = -2b^{-2}(b\lambda)^{1/2}.$$

14.7.6 Some Transformations

For the sake of visualization, we also use the symbolic notation $\{f,\, g,\, h\}$ to denote the equation

$$y''_{xx} = f_1(x)g_1(y)h_1(y'_x). \tag{1}$$

1°. Taking y to be the independent variable and x to be the dependent one, we obtain an equation of similar form for $x = x(y)$:

$$x''_{yy} = g_1(y)f_1(x)h_1^*(x'_y), \qquad \text{where} \quad h_1^*(w) = -w^3 h_1(1/w).$$

Denote this transformation by $\mathcal{F}$.

2°. The Bäcklund transformation

$$\bar{x} = \int\frac{dw}{h_1(w)}, \quad \bar{y} = \int f_1(x)\,dx, \qquad \text{where} \quad w = y'_x, \tag{2}$$

leads to an equation of similar form for the function $\bar{y} = \bar{y}(\bar{x})$:

$$\bar{y}''_{\bar{x}\bar{x}} = f_2(\bar{x})g_2(\bar{y})h_2(\bar{y}'_{\bar{x}}),$$

where the functions f_2, g_2, and h_2 are defined in terms of the original functions f_1, g_1, and h_1 parametrically by the relations

$$f_2(\bar{x}) = w, \qquad \bar{x} = \int\frac{dw}{h_1(w)};$$

$$g_2(\bar{y}) = \frac{1}{f_1(x)}, \qquad \bar{y} = \int f_1(x)\,dx;$$

$$h_2(\bar{w}) = -\frac{1}{[g_1(y)]^3}\frac{dg_1}{dy}, \qquad \bar{w} = \frac{1}{g_1(y)}.$$

Denote transformation (2) by $\mathcal{G}$.

For equations of the form (1) in which f_1, g_1, and h_1 are power functions of their arguments, the transformation $\mathcal{G}$ (up to a constant factor) is considered in Section 14.5.3. For equations (1) with exponential functions f_1 and g_1, the transformation $\mathcal{G}$ is discussed in Section 14.7.3.

Whenever the solution $\bar{y} = \bar{y}(\bar{x})$ of the transformed equation is found, the formulas

$$\bar{y} = \int f_1(x)\,dx, \qquad \bar{y}'_{\bar{x}} = \frac{1}{g_1(y)},$$

can be used to obtain the solution of the original equation (1) in parametric form, $x = x(\bar{x})$, $y = y(\bar{x})$.

3°. The twofold application of the transformation $\mathcal{G}$ to the original equation yields an equation of similar form:

$$\bar{\bar{y}}''_{\bar{\bar{x}}\bar{\bar{x}}} = f_3(\bar{\bar{x}})g_3(\bar{\bar{y}})h_3(\bar{\bar{y}}'_{\bar{\bar{x}}}),$$

where the functions f_3, g_3, and h_3 are defined in terms of the original functions f_1, g_1, and h_1 perimetrically by

$$\begin{aligned} f_3(\bar{\bar{x}}) &= \frac{1}{g_1(y)}, & \bar{\bar{x}} &= \int g_1(y)\,dy; \\ g_3(\bar{\bar{y}}) &= \frac{1}{w}, & \bar{\bar{y}} &= \int \frac{w\,dw}{h_1(w)}; \\ h_3(\bar{\bar{w}}) &= \frac{df_1}{dx}, & \bar{\bar{w}} &= f_1(x). \end{aligned}$$

The threefold transformation $\mathcal{G}$ yields the original equation.

Different compositions of the transformations $\mathcal{F}$ and $\mathcal{G}$ generate six different equations of the analogous form, which are shown in Figure 10.4 (see Section 10.2.2).

4°. In the special case $g(y) = y^m$, $h = 1$, the transformation $x = \dfrac{1}{\tau}$, $y = \dfrac{u}{\tau}$ leads to an equation of similar form:

$$u''_{\tau\tau} = \tau^{-m-3} f\Big(\frac{1}{\tau}\Big) u^m.$$

Denote this transformation by $\mathcal{H}$.

For $g(y) = y^m$ and $h = 1$, different compositions of the transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generate twelve different equations of the form (1).

14.8 Some Nonlinear Equations with Arbitrary Parameters

14.8.1 Equations Containing Power Functions

► **Equations of the form $f(x,y)y''_{xx} + g(x,y) = 0$.**

1. $y''_{xx} = ay^2 + bx$.

The transformation $y = bk^3 w$, $x = kz$, where $k = 6^{1/5}(ab)^{-1/5}$, leads to the first Painlevé transcendent: $w''_{zz} = 6w^2 + z$ (see Section 3.4.2).

2. $y''_{xx} = ay^3 + bxy + c.$

The transformation $y = (2/a)^{1/2} b^{1/3} w$, $x = b^{-1/3} z$ leads to the second Painlevé transcendent: $w''_{zz} = 2w^3 + zw + (a/2)^{1/2} b^{-1} c$ (see Section 3.4.3).

3. $y''_{xx} = bx^n y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -bx^n$.

4. $y''_{xx} = ax^n y + bkx^{k-1} y^{-1} - b^2 x^{2k} y^{-3}.$

This is a special case of equation 14.9.1.3 with $f(x) = -ax^n$ and $g(x) = bx^k$.

5. $y''_{xx} = (ax^2 + bx + c)y^{-5}.$

This is a special case of equation 14.7.1.36 with $n = 1$.

6. $y''_{xx} = abx^{-1} y^{-1/2} + ax^{-2} + 2b^2.$

The solution is determined by the first-order equation $ax(y'_x + 2by^{1/2}) = w(x, y, C)$, where the function w is defined implicitly by $w - \ln|w + a^{1/2}| = a^{-1/2}(y^{1/2} + bx)^2 + C$.

7. $y''_{xx} = (ay^2 + bxy + cx^2 + \alpha y + \beta x + \gamma)^{-3/2}, \qquad a \neq 0.$

The substitution $2aw = 2ay + bx + \alpha$ leads to an equation of the form 14.9.1.21:

$$w''_{xx} = w^{-3} f\left(\frac{w}{\sqrt{Ax^2 + Bx + C}}\right),$$

where $f(\xi) = \xi^3(a\xi^2 + 1)^{-3/2}$, $A = \dfrac{4ac - b^2}{4a}$, $B = \dfrac{2a\beta - b\alpha}{2a}$, $C = \dfrac{4a\gamma - \alpha^2}{4a}$.

8. $y''_{xx} = \lambda y^{-1/3} + (ax^2 + bx + c)y^{-5/3}.$

The transformation $x = x(t)$, $y = (x'_t)^{3/2}$ leads to a third-order equation: $2x'_t x'''_{ttt} - (x''_{tt})^2 = \frac{4}{3}\lambda(x'_t)^2 + \frac{4}{3}(ax^2 + bx + c)$. Differentiating the latter equation with respect to t and dividing it by x'_t, we arrive at a fourth-order constant coefficient linear equation: $3x''''_{tttt} = 2\lambda x''_{tt} + 4ax + 2b$.

9. $y''_{xx} = \lambda x^{-8/3} y^{-1/3} + (ax^{-10/3} + bx^{-7/3} + cx^{-4/3})y^{-5/3}.$

The transformation $x = 1/t$, $y = w/t$ leads to an equation of the form 148.1.8: $w''_{tt} = \lambda w^{-1/3} + (at^2 + bt + c)w^{-5/3}$.

10. $y''_{xx} = (ay + bx + c)^n.$

This is a special case of equation 14.9.1.4 with $f(\xi) = \xi^n$.

11. $y''_{xx} = (ay + bx^2)^n + c.$

The substitution $aw = ay + bx^2$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = a^n w^n + c + 2a^{-1} b$.

12. $y''_{xx} = \lambda x^{-2n-3}(xy + a)^n.$

This is a special case of equation 14.9.1.15 with $f(\xi) = \lambda \xi^n$ and $b = c = 0$.

13. $y''_{xx} = A(ax+b)^n(cx+d)^{-n-m-3}y^m.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to the Emden–Fowler equation $w''_{\xi\xi} = A(ad-bc)^{-2}\xi^n w^m$, whose solvable cases are outlined in Section 14.3.

14. $y''_{xx} = cx^m y^{-nk-m-3}(ay^n + bx^n)^k.$

This is a special case of equation 14.9.1.8 with $f(\xi) = c\xi^{-nk-m-3}(a\xi^n + b)^k$.

15. $y''_{xx} = cx^m y^{-2nk-2m-3}(ay^{2n} + bx^n)^k.$

This is a special case of equation 14.9.1.9 with $f(\xi) = c\xi^{-2nk-2m-3}(a\xi^{2n} + b)^k$.

16. $x^2y''_{xx} = ax^n y^{m+1} + by.$

This is a special case of equation 14.9.1.11 with $f(z) = az + b$.

17. $x^2y''_{xx} = n(n+1)y + ax^{3n+2} + bx^{nm+3n+2}y^m.$

This is a special case of equation 14.9.1.12 with $f(\xi) = a + b\xi^m$.

18. $x^2y''_{xx} = k(k+1)y + ax^{km+3k+2}(bx^{2k+1}+c)^n y^m.$

The transformation $\xi = bx^{2k+1} + c$, $w = yx^k$ leads to the Emden–Fowler equation $w''_{\xi\xi} = ab^{-2}(2k+1)^{-2}\xi^n w^m$, whose solvable cases are outlined in Section 14.3.

19. $(ay + bx^2)y''_{xx} = 1.$

This is a special case of equation 14.8.1.11 with $n = -1$ and $c = 0$.

20. $(x+a)^2y^2y''_{xx} = bx.$

The transformation $\xi = \ln\left|\dfrac{x+a}{x}\right|$, $w = \dfrac{y}{x}$ leads to an autonomous equation of the form 14.2.1.7: $w''_{\xi\xi} - w'_\xi = a^{-2}w^{-2}$.

21. $(y^2 + ax^2 + 2bx + c)^2 y''_{xx} + sy = 0.$

Dividing by the coefficient of y''_{xx} and multiplying by $ax(xy'_x - y) + b(2xy'_x - y) + cy'_x$, we arrive at an exact differential equation. Integrating the latter, we obtain a first-order equation: $(ax^2+2bx+c)(y'_x)^2 - 2(ax+b)yy'_x + ay^2 + \dfrac{sy^2}{y^2+ax^2+2bx+c} = C.$

22. $(ax+b)^2(cx+d)^2y''_{xx} = sy + A(ax+b)^k(cx+d)^{1-m-k}y^m.$

The transformation $\xi = \ln\left(\dfrac{ax+b}{cx+d}\right)$, $w = \left(\dfrac{ax+b}{cx+d}\right)^{\frac{k}{m-1}}\dfrac{y}{cx+d}$ leads to an autonomous equation: $w''_{\xi\xi} - (2n+1)w'_\xi + (n^2+n-s\Delta^{-2})w = A\Delta^{-2}w^m$, where $n = \frac{k}{m-1}$, $\Delta = ad - bc$.

23. $(ax + by + c)^n y''_{xx} = k(\alpha x + \beta y + \gamma)^{n-1}.$

This is a special case of equation 14.9.1.16 with $f(w) = kw^{1-n}$.

24. $(ax + by + c)^n y''_{xx} = k(\alpha x + \beta y + \gamma)^{n-3}.$

This is a special case of equation 14.9.1.17 with $f(w) = kw^{3-n}$.

25. $(ay^n + bx^n)y''_{xx} + cx^{n-3} = 0.$

This is a special case of equation 14.9.1.8 with $f(\xi) = -c(a\xi^n + b)^{-1}$.

26. $(ay^n + bx^n)y''_{xx} + cy^{n-3} = 0.$

This is a special case of equation 14.9.1.8 with $f(\xi) = -c\xi^{n-3}(a\xi^n + b)^{-1}$.

27. $(ay^{2n} + bx^n)y''_{xx} + cy^{2n-3} = 0.$

This is a special case of equation 14.9.1.9 with $f(\xi) = -c\xi^{2n-3}(a\xi^{2n} + b)^{-1}$.

28. $(ay^n + bx^n)y''_{xx} + cx^m y^{n-m-3} = 0.$

This is a special case of equation 14.9.1.8 with $f(\xi) = -c\xi^{n-m-3}(a\xi^n + b)^{-1}$.

29. $(ay^{2n} + bx^n)y''_{xx} + cx^m y^{2n-2m-3} = 0.$

This is a special case of equation 14.9.1.9 with $f(\xi) = -c\xi^{2n-2m-3}(a\xi^{2n} + b)^{-1}$.

◆ *See also equations 14.7.1.1–14.7.1.40.*

► **Equations of the form** $f(x,y)y''_{xx} + g(x,y)y'_x + h(x,y) = 0.$

30. $y''_{xx} + 3yy'_x + y^3 + ax^n y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = ax^n$.

31. $y''_{xx} + (ay + bx^n)y'_x + bnx^{n-1}y = 0.$

This is a special case of equation 14.9.2.4 with $f(x) = bx^n$.

32. $y''_{xx} + (2ay + bx^n)y'_x + abx^n y^2 = cx^m.$

This is a special case of equation 14.9.2.5 with $f(x) = bx^n$ and $g(x) = cx^m$.

33. $xy''_{xx} = ny'_x + bx^m y + ax^{2n+1}y^{-3}.$

This is a special case of equation 14.9.2.9 with $f(x) = -bx^m$.

34. $xy''_{xx} = ny'_x + ax^{2n+1} + bx^{2n+1}y^m.$

This is a special case of equation 14.9.2.20 with $f(y) = a + by^m$.

35. $xy''_{xx} = -(n+1)y'_x + ax^{n-1} + bx^{nm+n-1}y^m.$

This is a special case of equation 14.9.2.30 with $f(\xi) = a + b\xi^m$.

36. $xy''_{xx} = (ax^k y^n + k - 1)y'_x.$

Solution:

$$\int \frac{dy}{F(y) + C_1} = C_2 + \frac{1}{k}x^k, \qquad \text{where} \quad F(y) = a\frac{1}{n+1}y^{n+1}.$$

37. $x^2 y''_{xx} + xy'_x = ay^n + b.$

This is a special case of equation 14.9.2.23 with $f(y) = ay^n + b$.

38. $x^2 y''_{xx} = -(n+m+1)xy'_x - nmy + ax^{nk+n-2m}y^k.$

This is a special case of equation 14.9.2.31 with $f(\xi) = a\xi^k$.

39. $x^2y''_{xx} + axy'_x + by = cx^ny^m.$

The transformation $x = \xi^\alpha$, $y = \xi^\beta w$, where $\alpha = \pm\dfrac{1}{\sqrt{D}}$, $\beta = \pm\dfrac{1-a}{2\sqrt{D}} - \dfrac{1}{2}$, $D = (1-a)^2 - 4b$, leads to the Emden–Fowler equation $w''_{\xi\xi} = c\alpha^2\xi^{n\alpha+m\beta-\beta-2}w^m$, whose solvable cases are outlined in Section 14.3.

40. $(ax^2 + b)y''_{xx} + axy'_x + cy^n = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = cy^n$.

► **Equations of the form** $f(x,y)y''_{xx} + g(x,y)(y'_x)^2 + h(x,y)y'_x + r(x,y) = 0.$

41. $y''_{xx} = (y'_x)^2 - 2axy'_x + 2ay + b.$

The substitution $y = w + \frac{1}{2}ax^2$ leads to an autonomous equation of the form 14.9.3.25: $w''_{xx} = (w'_x)^2 + 2aw - a + b$.

42. $y''_{xx} = a(y'_x + by + cx)^2 + b^2y + kx + s.$

The substitution $w = y'_x + by + cx$ leads to a Riccati equation:

$$w'_x = aw^2 + bw + (k - bc)x + c + s.$$

43. $y''_{xx} = ax^n(xy'_x - y)^2 + bx^m.$

This is a special case of equation 14.9.3.2 with $f(x) = bx^m$, $g(x) = 0$, and $h(x) = ax^n$.

44. $y''_{xx} = ax^n(y'_x + by)^2 + b^2y + cx^m.$

The substitution $w = y'_x + by$ leads to a Riccati equation: $w'_x = ax^nw^2 + bw + cx^m$.

45. $y''_{xx} = (a^2x^2 + a)y + bx^n(y'_x - axy)^2 + cx^m.$

The substitution $w = y'_x - axy$ leads to a Riccati equation: $w'_x = bx^nw^2 - axw + cx^m$.

46. $y''_{xx} = (ax + by + c)^n[\alpha(y'_x)^2 + \beta]^k.$

This is a special case of equation 14.9.4.37 with $f(u) = u^n$ and $g(v) = (\alpha v^2 + \beta)^k$.

47. $xy''_{xx} + a(y'_x)^2 + \frac{1}{2}y'_x + by^2 + cy + k = 0.$

The substitution $w(y) = x(y'_x)^2$ leads to a first-order linear equation: $w'_y + 2aw + 2by^2 + 2cy + 2k = 0$.

48. $xy''_{xx} + a(y'_x)^2 - by^ky'_x = 0.$

Solution:

$$\int \frac{e^{ay}\,dy}{F(y) + C_1} = C_2 + \ln|x|, \qquad \text{where} \quad F(y) = b\int e^{ay}y^k\,dy + \frac{1}{a}e^{ay}.$$

49. $xy''_{xx} + a(y'_x)^2 = (bx^ky^n + k - 1)y'_x.$

Solution:

$$\int \frac{e^{ay}\,dy}{F(y) + C_1} = C_2 + \frac{1}{k}x^k, \qquad \text{where} \quad F(y) = b\int e^{ay}y^n\,dy.$$

50. $x^2y''_{xx} = 2y + ax^n(xy'_x + y)^2 + bx^m.$

The substitution $w = xy'_x + y$ leads to a Riccati equation: $xw'_x = ax^nw^2 + 2w + bx^m$.

51. $x^2 y''_{xx} = a(a+1)y + bx^n(xy'_x + ay)^2 + cx^m.$

The substitution $w = xy'_x + ay$ leads to a Riccati equation: $xw'_x = bx^n w^2 + (a+1)w + cx^m$.

52. $yy''_{xx} = (y'_x)^2 - a.$

1°. Solution (a is any):

$$y = C_1 \exp(C_2 x) - \frac{a}{4C_1 C_2^2} \exp(-C_2 x).$$

2°. Solution for $a < 0$:

$$y = C_1 \sin\left(\frac{x\sqrt{a}}{\sqrt{C_1^2 + C_2^2}}\right) + C_2 \cos\left(\frac{x\sqrt{a}}{\sqrt{C_1^2 + C_2^2}}\right).$$

There are also singular solutions: $y = \pm x\sqrt{a} + C$.

53. $yy''_{xx} - \frac{1}{4}(y'_x)^2 = ax^2 + bx + c.$

The substitution $y = w^{4/3}$ leads to a special case of the equation 14.8.1.8 with $\lambda = 0$: $4w''_{xx} = 3(ax^2 + bx + c)w^{-5/3}$.

54. $3yy''_{xx} - 2(y'_x)^2 = ax^2 + bx + c.$

The substitution $y = w^3$ leads to an equation of the form 14.8.1.5:

$$9w''_{xx} = (ax^2 + bx + c)w^{-5}.$$

55. $2yy''_{xx} = (y'_x)^2 + bx^n y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -bx^n$.

56. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + bx^m y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -bx^m$.

57. $yy''_{xx} = n(y'_x)^2 + ax^k y^2 + bx^m y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^k$ and $g(x) = -bx^m$.

58. $(n+2)yy''_{xx} - (n+1)(y'_x)^2 = (ax^2 + bx + c)^n.$

The substitution $y = w^{n+2}$ leads to an equation of the form 147.1.36:

$$w''_{xx} = \frac{1}{(n+2)^2}(ax^2 + bx + c)^n w^{-2n-3}.$$

59. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + bx^m y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -bx^m$.

60. $ayy''_{xx} + b(y'_x)^2 + (x^n + \lambda)^m yy'_x = 0.$

Solution: $y^{\frac{a+b}{a}} = C_1 \int \exp\left[-\frac{1}{a}\int (x^n + \lambda)^m\, dx\right] dx + C_2.$

61. $yy''_{xx} - (y'_x)^2 = ay''_{xx} + by + c.$

1°. Solution:

$$y = C_1 \sinh(C_3 x) + C_2 \cosh(C_3 x) + a + bC_3^{-2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$(C_1^2 - C_2^2)C_3^2 + ab + c + b^2 C_3^{-2} = 0.$$

2°. Solution:

$$y = C_1 \sin(C_3 x) + C_2 \cos(C_3 x) + a - bC_3^{-2},$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$(C_1^2 + C_2^2)C_3^2 + ab + c - b^2 C_3^{-2} = 0.$$

There is also a singular solution: $y = -c/b$.

62. $yy''_{xx} - (y'_x)^2 = a_2 y''_{xx} + a_1 y'_x + a_0 y + b.$

Particular solutions: $y = Ce^{\lambda x} - ba_0^{-1}$, where C is an arbitrary constant and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $(a_2 a_0 + b)\lambda^2 + a_1 a_0 \lambda + a_0^2 = 0$.

63. $(y + ax)y''_{xx} = bx^n(xy'_x - y)^2.$

The substitution $y = -ax + xz$ leads to the equation $xzz''_{xx} + 2zz'_x - bx^{n+3}(z'_x)^2 = 0$. Having set $w = z'_x/z$, we obtain a Bernoulli equation: $xw'_x + 2w + x(1 - bx^{n+2})w^2 = 0$.

64. $(2y + ax + b)y''_{xx} - (y'_x)^2 - ay'_x + c = 0.$

Solution:

$$y = C_1 x^2 + C_2 x + C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$4C_1 C_3 - C_2^2 + 2bC_1 - aC_2 + c = 0.$$

65. $xyy''_{xx} = x(y'_x)^2 - yy'_x + ax^k y^s.$

This is a special case of equation 14.9.4.64 with $f(\xi) = a\xi$, $g(\xi) = 1$, $k = n - 1$, and $s = m + 2$.

66. $y^2 y''_{xx} + y(y'_x)^2 = ax + b.$

Having set $1/y = u'_x(x)$, we obtain a third-order equation: $-u'_x u'''_{xxx} + 3(u''_{xx})^2 = (ax + b)(u'_x)^5$. Taking u to be the independent variable, we obtain a constant coefficient linear equation for $x = x(u)$: $x'''_{uuu} = ax + b$.

67. $(a^2 - x^2)(b^2 - y^2)y''_{xx} + (a^2 - x^2)y(y'_x)^2 = x(b^2 - y^2)y'_x.$

Solution: $\arcsin \dfrac{y}{b} = C_1 + C_2 \arcsin \dfrac{x}{b}$.

68. $(xy'_x - y + a)y''_{xx} = b_2(y'_x)^2 + b_1 y'_x + b_0.$

The contact transformation

$$X = y'_x, \quad Y = xy'_x - y + a, \quad Y'_X = x, \quad Y''_{XX} = 1/y''_{xx},$$

where $Y = Y(X)$, leads to a linear equation: $(b_2 X^2 + b_1 X + b_0)Y''_{XX} - Y = 0$.

Inverse transformation:

$$x = Y'_X, \quad y = XY'_X - Y + a, \quad y'_x = X, \quad y''_{xx} = 1/Y''_{XX}.$$

► **Other equations.**

69. $y''_{xx} = ax^n(xy'_x - y)^m.$

This is a special case of equation 14.9.4.58 with $f(x) = ax^n$ and $g(\xi) = \xi^m$.

70. $y''_{xx} = a^2y + bx^n(y'_x + ay)^m.$

The substitution $w = y'_x + ay$ leads to a Bernoulli equation: $w'_x = aw + bx^n w^m$.

71. $y''_{xx} = (a^2x^2 + a)y + bx^n(y'_x - axy)^m.$

The substitution $w = y'_x - axy$ leads to a Bernoulli equation: $w'_x = -axw + bx^n w^m$.

72. $y''_{xx} = ax^{-n-3}y^n(xy'_x - y)^m.$

This is a special case of equation 14.9.4.59 with $f(\xi) = a\xi^m$.

73. $y''_{xx} = ax^{-1}y^n y'_x(xy'_x - y)^m.$

This is a special case of equation 14.9.4.60 with $f(y) = ay^n$ and $g(\xi) = \xi^m$.

74. $y''_{xx} = ax^{nk-1}y^{mk-1}(xy'_x - y)^{\frac{2n+m}{n}}.$

This is a special case of equation 14.9.4.24 with $f(\xi) = a\xi^k$.

75. $y''_{xx} = kx^\alpha(y'_x)^\beta(xy'_x - y)^\gamma.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$, where $w = w(t)$, leads to the generalized Emden–Fowler equation: $w''_{tt} = \dfrac{1}{k}t^{-\beta}w^{-\gamma}(w'_t)^{-\alpha}$. Solvable equations of this type are outlined in Section 14.3 and Section 14.5.

76. $y''_{xx} = ax^{n-1}y^{m-1}(y'_x)^{\frac{2n+m-nk}{n+m}}(xy'_x - y)^k.$

This is a special case of equation 14.9.4.25 with $f(\xi) = a\xi$.

77. $y''_{xx} = ax^n(xy'_x - y) + bx^m(xy'_x - y)^k.$

This is a special case of equation 14.9.4.4 with $f(x) = ax^n$ and $g(x) = bx^m$.

78. $x^2y''_{xx} = 2y + ax^n(xy'_x + y)^m.$

The substitution $w = xy'_x + y$ leads to a Bernoulli equation: $xw'_x = 2w + ax^n w^m$.

79. $x^2y''_{xx} = n(n-1)y + ax^n(xy'_x - ny)^k.$

This is a special case of equation 14.9.4.3 with $f(x) = ax^{n-2}$.

80. $x^2y''_{xx} = a(a+1)y + bx^n(xy'_x + ay)^m.$

The substitution $w = xy'_x + ay$ leads to a Bernoulli equation: $xw'_x = (a+1)w + bx^n w^m$.

81. $(y''_{xx})^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma.$

Differentiating the equation with respect to x yields:

$$y''_{xx}(2y'''_{xxx} - \alpha x - \beta) = 0. \qquad (1)$$

Equating the second factor to zero and integrating, one obtains:

$$y = \tfrac{1}{48}\alpha x^4 + \tfrac{1}{12}\beta x^3 + C_2x^2 + C_1x + C_0. \qquad (2)$$

The integration constants C_i and the parameters α, β, and γ are related by the constraint $4C_2^2 = \beta C_1 - \alpha C_0 + \gamma$, which is obtained by substituting the above solution (2) into the original equation.

In addition, there is a singular solution, which corresponds to setting the first factor in (1) equal to zero:

$$y = \widetilde{C}_1 x + \widetilde{C}_0, \qquad \text{where} \quad \beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0.$$

82. $(y''_{xx})^2 + 2ayy''_{xx} + bxy''_{xx} + cy''_{xx} - a(y'_x)^2 - by'_x + k = 0.$

Solution:

$$y = C_1 x^2 + C_2 x + C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint $4C_1^2 + a(4C_1C_3 - C_2^2) - bC_2 + 2cC_1 + k = 0$.

14.8.2 Equations Containing Exponential Functions

► **Equations of the form** $f(x,y)y''_{xx} + g(x,y) = 0$.

1. $y''_{xx} = ae^{\lambda x + \beta y} + b.$

The substitution $w = y + (\lambda/\beta)x$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = ae^{\beta w} + b$.

2. $y''_{xx} = ae^{\lambda x}y^n.$

The transformation $z = e^{\lambda x}y^{n-1}$, $w = y'_x/y$ leads to a first-order equation:

$$z[(n-1)w + \lambda]w'_z = az - w^2.$$

3. $y''_{xx} = ax^n e^{\lambda y}.$

The transformation $z = x^{n+2}e^{\lambda y}$, $w = xy'_x$ leads to a first-order equation:

$$z(\lambda w + n + 2)w'_z = az + w.$$

4. $y''_{xx} = be^{\lambda x}y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -be^{\lambda x}$.

5. $y''_{xx} = \lambda^2 y + a\exp[\lambda(n+3)x]y^n.$

This is a special case of equation 14.9.1.29 with $f(\xi) = a\xi^n$.

6. $y''_{xx} = \lambda^2 y + ae^{\mu x}y^m, \qquad \lambda \neq 0.$

The transformation $\xi = e^{2\lambda x}$, $u = ye^{\lambda x}$ leads to the Emden–Fowler equation $u''_{\xi\xi} = \dfrac{a}{4\lambda^2}\xi^n u^m$, where $n = \dfrac{\mu - 3\lambda - m\lambda}{2\lambda}$, whose special cases are given in Section 14.3.

7. $y''_{xx} = \lambda^2 y + ae^{\lambda(m+3)x}(be^{2\lambda x} + c)^n y^m, \qquad \lambda \neq 0.$

The transformation $\xi = be^{2\lambda x} + c$, $w = ye^{\lambda x}$ leads to the Emden–Fowler equation $w''_{\xi\xi} = \dfrac{a}{4b^2\lambda^2}\xi^n w^m$, whose special cases are given in Section 14.3.

8. $y''_{xx} = \lambda^2 y + Ae^{\lambda(m+3)x}(ae^{2\lambda x} + b)^n (ce^{2\lambda x} + d)^{-n-m-3} y^m.$

The transformation $\xi = \dfrac{ae^{2\lambda x} + b}{ce^{2\lambda x} + d}$, $w = \dfrac{ye^{\lambda x}}{ce^{2\lambda x} + d}$ leads to the Emden–Fowler equation $w''_{\xi\xi} = A(2\Delta\lambda)^{-2}\xi^n w^m$, where $\Delta = ad - bc$ (see Section 14.3).

9. $y''_{xx} = a\exp(\alpha x^2 + \beta x)\exp(\gamma y) + b.$

The substitution $w = y + (\alpha x^2 + \beta x)/\gamma$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = ae^{\gamma w} + b + 2\alpha\gamma^{-1}$.

10. $x^2 y''_{xx} = ax^{n+2}e^y + n.$

This is a special case of equation 14.9.1.31 with $f(\xi) = a\xi$.

▶ Equations of the form $f(x,y)y''_{xx} + g(x,y)y'_x + h(x,y) = 0$.

11. $y''_{xx} = ay'_x + be^{2ax}y^n.$

This is a special case of equation 14.9.2.17 with $f(y) = by^n$.

12. $y''_{xx} = -ay'_x + be^{anx}y^{n-1}.$

This is a special case of equation 14.9.2.36 with $f(\xi) = b\xi^{n-1}$.

13. $y''_{xx} + ay'_x + by = ce^{\lambda x}y^m.$

The substitution $\xi = e^x$ leads to an equation of the form 148.1.39:

$$\xi^2 y''_{\xi\xi} + (a+1)\xi y'_\xi + by = c\xi^\lambda y^m.$$

14. $y''_{xx} = -(\mu + \nu)y'_x - \nu\mu y + ae^{(n\mu - 2\nu)x}y^{n-1}.$

This is a special case of equation 14.9.2.37 with $f(\xi) = a\xi^{n-1}$.

15. $y''_{xx} = \lambda y'_x + bxy + ae^{2\lambda x}y^{-3}.$

This is a special case of equation 14.9.2.14 with $f(x) = -bx$.

16. $y''_{xx} = \lambda y'_x + be^{\mu x}y + ae^{2\lambda x}y^{-3}.$

This is a special case of equation 14.9.2.14 with $f(x) = -be^{\mu x}$.

17. $y''_{xx} = ay'_x + b\exp(2ax + cy^n).$

This is a special case of equation 14.9.2.17 with $f(y) = b\exp(cy^n)$.

18. $y''_{xx} + 3yy'_x + y^3 + ae^{\lambda x}y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = ae^{\lambda x}$.

19. $y''_{xx} = axe^y y'_x + ae^y.$

Solution: $y = C_1 x - \ln\Bigl(-a\displaystyle\int xe^{C_1 x}\,dx + C_2\Bigr).$

20. $y''_{xx} = 2ae^x yy'_x + ae^x y^2.$

Solution in parametric form:

$$x = \ln\Big(\frac{\tau^2}{2C_1}\Big), \qquad y = -a^{-1}C_1\tau^{-2}Z^{-1}(\tau Z'_\tau + Z).$$

Here, $Z = C_1 J_1(\tau) + C_2 Y_1(\tau)$ or $Z = C_1 I_1(\tau) + C_2 K_1(\tau)$, where $J_1(\tau)$ and $Y_1(\tau)$ are Bessel functions, and $I_1(\tau)$ and $K_1(\tau)$ are modified Bessel functions.

21. $y''_{xx} = ax^n e^y y'_x + anx^{n-1}e^y.$

Solution: $y = C_1 x - \ln\Big[C_2 - a\displaystyle\int x^n \exp(C_1 x)\,dx\Big].$

22. $y''_{xx} = ae^x y^{-1/2} y'_x + 2ae^x y^{1/2}.$

Solution in parametric form:

$$x = \ln\Big(\pm\frac{C_1}{a}f\Big) \mp \tau^2, \quad y = C_1^2\big[2\tau \pm \exp(\mp\tau^2)f\big]^2, \quad \text{where } f = \Big[\int \exp(\mp\tau^2)\,d\tau + C_2\Big]^{-1}.$$

23. $y''_{xx} + (2ay + be^{\lambda x})y'_x + \lambda be^{\lambda x}y = 0.$

Integrating yields a Riccati equation: $y'_x + ay^2 + be^{\lambda x}y = C$.

24. $y''_{xx} = (ae^{\beta x}y + \beta)y'_x.$

1°. Solution: $\displaystyle\frac{2}{a}\int \frac{dy}{y^2 + C_1} = \frac{1}{\beta}e^{\beta x} + C_2.$

2°. Solution in explicit form:

$$y = \begin{cases} \sqrt{C_1}\tan\left[\dfrac{a\sqrt{C_1}}{2}\left(\dfrac{e^{\beta x}}{\beta} + C_2\right)\right] & \text{if } C_1 > 0, \\ -\sqrt{|C_1|}\tanh\left[\dfrac{a\sqrt{|C_1|}}{2}\left(\dfrac{e^{\beta x}}{\beta} + C_2\right)\right] & \text{if } C_1 < 0, \\ -\dfrac{2}{a}\left(\dfrac{e^{\beta x}}{\beta} + C_2\right)^{-1} & \text{if } C_1 = 0. \end{cases}$$

25. $y''_{xx} = ae^{x+y}(y'_x + 1).$

Solution: $y = -\ln\Big(C_1 e^{-C_2 x} - \dfrac{a}{1 + C_2}e^x\Big)$. To the limiting case $C_2 \to -1$ there corresponds $y = -x - \ln(C_1 - ax)$.

26. $xy''_{xx} + y'_x = ax^n e^{\lambda y}.$

This equation is encountered in combustion theory and hydrodynamics. The transformation $\xi = \ln x$, $w = \lambda y + (n+1)\ln x$ leads to an autonomous equation of the form 14.7.3.1: $w''_{\xi\xi} = a\lambda e^w$.

Solution in parametric form:

$$x = \exp[C_1 \pm f(t)], \qquad y = \frac{t}{\lambda} - \frac{n+1}{\lambda}[C_1 \pm f(t)],$$

where

$$f(t) = \begin{cases} \dfrac{1}{\sqrt{C_2}} \ln \dfrac{\sqrt{C_2 + 2a\lambda e^t} - \sqrt{C_2}}{\sqrt{C_2 + 2a\lambda e^t} + \sqrt{C_2}} & \text{if } C_2 > 0, \\ -\dfrac{2}{\sqrt{2a\lambda e^t}} & \text{if } C_2 = 0, \\ \dfrac{2}{\sqrt{-C_2}} \arctan \dfrac{\sqrt{C_2 + 2a\lambda e^t}}{\sqrt{-C_2}} & \text{if } C_2 < 0. \end{cases}$$

27. $xy''_{xx} = ny'_x + ax^{2n+1}e^{\lambda y}$.

This is a special case of equation 14.9.2.20 with $f(y) = ae^{\lambda y}$.

28. $xy''_{xx} = ny'_x + ax^{2n+1}\exp(\lambda y^m)$.

This is a special case of equation 14.9.2.20 with $f(y) = a\exp(\lambda y^m)$.

29. $x^2y''_{xx} + xy'_x = ae^{\lambda y}$.

The substitution $t = \ln|x|$ leads to an equation of the form 14.7.3.1: $y''_{tt} = ae^{\lambda y}$.

Solution:

$$y = -\frac{1}{\lambda}\ln\left[\frac{a\lambda}{2C_1^2}\sin^2(C_1\ln|x| + C_2)\right] \quad \text{if} \quad a\lambda > 0,$$
$$y = -\frac{1}{\lambda}\ln\left[\frac{a\lambda}{2C_1^2}\sinh^2(C_1\ln|x| + C_2)\right] \quad \text{if} \quad a\lambda > 0,$$
$$y = -\frac{1}{\lambda}\ln\left[-\frac{a\lambda}{2C_1^2}\cosh^2(C_1\ln|x| + C_2)\right] \quad \text{if} \quad a\lambda < 0.$$

30. $x^2y''_{xx} + xy'_x = ae^{\lambda y} + b$.

This is a special case of equation 14.9.2.23 with $f(y) = ae^{\lambda y} + b$.

31. $x^2y''_{xx} + xy'_x = kx^n e^{ay} + b$.

This is a special case of equation 14.9.2.40 with $f(\xi) = k\xi + b$.

32. $(ax^2 + b)y''_{xx} + axy'_x + ce^{\lambda y} = 0$.

This is a special case of equation 14.9.2.24 with $f(y) = ce^{\lambda y}$.

33. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x + cy^n = 0$.

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = -cy^n$.

34. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x + ce^{\lambda y} = 0$.

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = -ce^{\lambda y}$.

► **Equations of the form** $f(x,y)y''_{xx} + g(x,y)(y'_x)^2 + h(x,y)y'_x + r(x,y) = 0$.

35. $y''_{xx} = a(y'_x)^2 - be^{4ay} + cx^n$.

This is a special case of equation 14.9.3.18 with $f(x) = -cx^n$.

36. $y''_{xx} = a(y'_x)^2 - be^{4ay} + ce^{\lambda x}$.

This is a special case of equation 14.9.3.18 with $f(x) = -ce^{\lambda x}$.

37. $y''_{xx} = a(y'_x)^2 + bx^n e^{ay} + cx^m.$

This is a special case of equation 14.9.3.17 with $f(x) = -bx^n$ and $g(x) = -cx^m$.

38. $y''_{xx} = a(y'_x)^2 + be^{ay+cx} + kx^m.$

This is a special case of equation 14.9.3.17 with $f(x) = -be^{cx}$ and $g(x) = -kx^m$.

39. $y''_{xx} = a(y'_x)^2 + be^{ay+\lambda x} + ce^{\mu x}.$

This is a special case of equation 14.9.3.17 with $f(x) = -be^{\lambda x}$ and $g(x) = -ce^{\mu x}$.

40. $y''_{xx} + ay^n(y'_x)^2 + be^{\lambda y} + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = be^{\lambda y} + c$.

41. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + by^n + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ae^{\lambda y}$ and $g(y) = by^n + c$.

42. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + be^{\mu y} + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ae^{\lambda y}$ and $g(y) = be^{\mu y} + c$.

43. $y''_{xx} = ay^n(y'_x)^2 + be^{\lambda x}y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ay^n$ and $g(x) = be^{\lambda x}$.

44. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + bx^n y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ae^{\lambda y}$ and $g(x) = bx^n$.

45. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + be^{\mu x}y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ae^{\lambda y}$ and $g(x) = be^{\mu x}$.

46. $y''_{xx} = ae^{\lambda x}(xy'_x - y)^2 + be^{\mu x}.$

This is a special case of equation 14.9.3.2 with $f(x) = be^{\mu x}$, $g(x) = 0$, and $h(x) = ae^{\lambda x}$.

47. $yy''_{xx} - (y'_x)^2 = ae^{\lambda x}.$

The substitutions $y = \pm\exp\left(\frac{1}{2}w + \frac{1}{2}\lambda x\right)$ lead to an autonomous equation of the form 14.7.3.1: $w''_{xx} = 2ae^{-w}$.

48. $2yy''_{xx} = (y'_x)^2 + be^{\lambda x}y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -be^{\lambda x}$.

49. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + be^{\lambda x}y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -be^{\lambda x}$.

50. $yy''_{xx} - (y'_x)^2 = ae^{\lambda x}y^k.$

1°. For $k \neq 2$, the substitution $y = \exp\left(w + \dfrac{\lambda}{2-k}x\right)$ leads to an autonomous equation of the form 14.7.3.1: $w''_{xx} = ae^{(k-2)w}$.

2°. Solution for $k = 2$: $\ln|y| = C_1 x + C_2 + a\lambda^{-2}e^{\lambda x}$.

51. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + be^{\lambda x} y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -be^{\lambda x}$.

52. $yy''_{xx} = n(y'_x)^2 + ae^{\lambda x} y^2 + bx^m y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ae^{\lambda x}$ and $g(x) = -bx^m$.

53. $yy''_{xx} = n(y'_x)^2 + ae^{\lambda x} y^2 + be^{\mu x} y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ae^{\lambda x}$ and $g(x) = -be^{\mu x}$.

54. $yy''_{xx} - (y'_x)^2 = a\exp\left(\beta x^2 + \lambda x\right).$

The substitutions $y = \pm\exp\left(\frac{1}{2}w + \frac{1}{2}\beta x^2 + \frac{1}{2}\lambda x\right)$ lead to an autonomous equation of the form 14.9.1.1: $w''_{xx} = 2ae^{-w} - 2\beta$.

55. $yy''_{xx} - (y'_x)^2 + ay^2 = b\exp\left(\beta x^2 + \lambda x\right).$

The substitutions $y = \pm\exp\left(\frac{1}{2}w + \frac{1}{2}\beta x^2 + \frac{1}{2}\lambda x\right)$ lead to an autonomous equation of the form 14.9.1.1: $w''_{xx} = 2be^{-w} - 2(a+\beta)$.

56. $yy''_{xx} - (y'_x)^2 = a\exp\left(\beta x^2 + \lambda x\right) y^k.$

1°. For $k \neq 2$, the substitution $y = \exp\left[w + \dfrac{1}{2-k}\left(\beta x^2 + \lambda x\right)\right]$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = ae^{(k-2)w} - \dfrac{2\beta}{2-k}$.

2°. Solution for $k = 2$:

$$\ln|y| = C_1 x + C_2 + a\int_{x_0}^{x} (x-t)\exp\left(\beta t^2 + \lambda t\right) dt.$$

57. $yy''_{xx} - (y'_x)^2 + ay^2 = b\exp\left(\beta x^2 + \lambda x\right) y^k.$

1°. For $k \neq 2$, the substitution $y = \exp\left[w + \dfrac{1}{2-k}\left(\beta x^2 + \lambda x\right)\right]$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = be^{(k-2)w} - a - \dfrac{2\beta}{2-k}$.

2°. For $k = 2$, the substitutions $y = \pm e^w$ lead to a second-order linear equation: $w''_{xx} = b\exp\left(\beta x^2 + \lambda x\right) - a$.

Solution:

$$\ln|y| = -\frac{a}{2}x^2 + C_1 x + C_2 + b\int_{x_0}^{x} (x-t)\exp\left(\beta t^2 + \lambda t\right) dt.$$

58. $y''_{xx} + a(y'_x)^2 - \frac{1}{2}y'_x = e^x(b_2 y^2 + b_1 y + b_0).$

The substitution $w(y) = e^{-x}(y'_x)^2$ leads to a first-order linear equation: $w'_y + 2aw = 2b_2 y^2 + 2b_1 y + 2b_0$.

59. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + be^{\lambda x} y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -be^{\lambda x}$.

60. $yy''_{xx} = (y'_x)^2 + ae^{\lambda x} yy'_x + bx^n y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ae^{\lambda x}$ and $g(x) = -bx^n$.

61. $yy''_{xx} = (y'_x)^2 + ae^{\lambda x}yy'_x + be^{\mu x}y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ae^{\lambda x}$ and $g(x) = -be^{\mu x}$.

62. $y''_{xx} + \alpha(y'_x)^2 = \bigl(be^{\beta x+\gamma y} + \beta\bigr)y'_x.$

1°. Solution with $\gamma \neq -\alpha$:

$$\int \frac{e^{\alpha y}\,dy}{F(y)+C_1} = C_2 + \frac{1}{\beta}e^{\beta x}, \qquad \text{where} \quad F(y) = \frac{b}{\alpha+\gamma}e^{(\alpha+\gamma)y}.$$

2°. Solution with $\gamma = -\alpha$:

$$\int \frac{e^{\alpha y}\,dy}{by+C_1} = C_2 + \frac{1}{\beta}e^{\beta x}.$$

63. $y''_{xx} = ae^{\beta x}(y'_x + by)^2 + b^2y + ce^{\lambda x}.$

The substitution $w = y'_x + by$ leads to a Riccati equation: $w'_x = ae^{\beta x}w^2 + bw + ce^{\lambda x}$.

► **Other equations.**

64. $y''_{xx} + be^{ax}y^m(y'_x)^3 + ay'_x = 0.$

This is a special case of equation 14.9.3.35 with $f(y) = by^m$.

65. $y''_{xx} + be^{ax+\lambda y}(y'_x)^3 + ay'_x = 0.$

This is a special case of equation 14.9.3.35 with $f(y) = be^{\lambda y}$.

66. $y''_{xx} = ae^x(y'_x)^3 + ae^xy(y'_x)^2.$

Solution: $x = C_1y - \ln\Bigl(a\displaystyle\int ye^{C_1y}\,dy + C_2\Bigr)$.

67. $y''_{xx} = axe^y(y'_x)^3 + ae^y(y'_x)^2.$

Solution in parametric form:

$$x = C_1e^{-a\tau}\Bigl(\int \tau^{-1}e^{a\tau}\,d\tau + C_2\Bigr), \qquad y = \ln\tau.$$

68. $y''_{xx} = ax^2e^y(y'_x)^3 + 2axe^y(y'_x)^2.$

Solution in parametric form:

$$x = a^{-1}C_1\tau^{-2}Z^{-1}(\tau Z'_\tau + Z), \qquad y = \ln\Bigl(\frac{\tau^2}{2C_1}\Bigr).$$

Here, $Z = C_1J_1(\tau) + C_2Y_1(\tau)$ or $Z = C_1I_1(\tau) + C_2K_1(\tau)$, where $J_1(\tau)$ and $Y_1(\tau)$ are Bessel functions, and $I_1(\tau)$ and $K_1(\tau)$ are modified Bessel functions.

69. $y''_{xx} = 2ax^{1/2}e^y(y'_x)^3 + ax^{-1/2}e^y(y'_x)^2.$

Solution in parametric form:

$$x = C_1^2\bigl[2\tau \pm \exp(\mp\tau^2)f\bigr]^2, \quad y = \ln\Bigl(\mp\frac{C_1}{a}f\Bigr) \mp \tau^2, \quad \text{where} \quad f = \Bigl[\int \exp(\mp\tau^2)\,d\tau + C_2\Bigr]^{-1}.$$

70. $y''_{xx} = ane^xy^{n-1}(y'_x)^3 + ae^xy^n(y'_x)^2.$

Solution: $x = C_1y - \ln\Bigl(a\displaystyle\int y^ne^{C_1y}\,dy + C_2\Bigr)$.

71. $y''_{xx} = ae^{x+y}[(y'_x)^3 + (y'_x)^2]$.

Solution: $x = -\ln\Big(C_1 e^{C_2 y} + \dfrac{a}{1-C_2}e^y\Big)$. To the limiting case $C_2 \to 1$ there corresponds $x = -y - \ln(C_1 + ay)$.

72. $y''_{xx} = ae^x (y'_x)^{3/2} + ae^x y (y'_x)^{1/2}$.

Solution in parametric form:

$$x = \ln\tau^2, \qquad y = -2a^{-2}\tau^{-4}\big[Z^{-1}(\tau Z'_\tau + 2Z) \mp \tfrac{1}{2}\tau^2\big],$$

where

$$Z = \begin{cases} C_1 J_2(\tau) + C_2 Y_2(\tau) & \text{for the upper sign,} \\ C_1 I_2(\tau) + C_2 K_2(\tau) & \text{for the lower sign,} \end{cases}$$

$J_2(\tau)$ and $Y_2(\tau)$ are Bessel functions, and $I_2(\tau)$ and $K_2(\tau)$ are modified Bessel functions.

73. $y''_{xx} = axe^y (y'_x)^{5/2} + ae^y (y'_x)^{3/2}$.

Solution in parametric form:

$$x = -2a^{-2}\tau^{-4}\big[Z^{-1}(\tau Z'_\tau + 2Z) \mp \tfrac{1}{2}\tau^2\big], \qquad y = \ln\tau^2,$$

where

$$Z = \begin{cases} C_1 J_2(\tau) + C_2 Y_2(\tau) & \text{for the upper sign,} \\ C_1 I_2(\tau) + C_2 K_2(\tau) & \text{for the lower sign,} \end{cases}$$

$J_2(\tau)$ and $Y_2(\tau)$ are Bessel functions, and $I_2(\tau)$ and $K_2(\tau)$ are modified Bessel functions.

74. $y''_{xx} = -ay'_x + be^{amx} y^k (y'_x)^{m+2}$.

This is a special case of equation 14.9.4.17 with $f(y) = -by^k$ and $n = m + 2$.

75. $y''_{xx} = -\dfrac{a}{m}\dfrac{2-k}{1-k}y'_x + be^{ax} y^{m-k+1}(y'_x)^k$.

This is a special case of equation 14.9.4.31 with $f(\xi) = b\xi$.

76. $y''_{xx} = y'_x + A\exp[(n+2-l)x] y^m (y'_x)^l$.

The substitution $\xi = e^x$ leads to the generalized Emden–Fowler equation $y''_{\xi\xi} = A\xi^n y^m (y'_\xi)^l$, which is discussed in Section 14.5.

77. $y''_{xx} = -(y'_x)^2 + Ax^n \exp[(m+l-1)y]\,(y'_x)^l$.

The substitution $u = e^y$ leads to the generalized Emden–Fowler equation $u''_{xx} = Ax^n u^m (u'_x)^l$, which is discussed in Section 14.5.

78. $y''_{xx} = \dfrac{a}{n}\dfrac{1-k}{2-k}(y'_x)^2 + bx^{n+k-2}e^{ay}(y'_x)^k$.

This is a special case of equation 14.9.4.30 with $f(\xi) = b\xi$.

79. $y''_{xx} = ay^n (y'_x)^2 + be^{\lambda y}(y'_x)^k$.

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = be^{\lambda y}$.

80. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + by^n (y'_x)^k$.

This is a special case of equation 14.9.4.13 with $f(y) = ae^{\lambda y}$ and $g(y) = by^n$.

81. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + be^{\mu y}(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ae^{\lambda y}$ and $g(y) = be^{\mu y}$.

82. $y''_{xx} = a^2 y + be^{\beta x}(y'_x + ay)^k.$

The substitution $w = y'_x + ay$ leads to a Bernoulli equation: $w'_x = aw + be^{\beta x}w^k$.

83. $y''_{xx} = ae^{x+y}[(y'_x)^k + (y'_x)^{k-1}], \qquad k \neq 2.$

Solution in parametric form:

$$x = \tau - \int \frac{d\tau}{F} - C_2, \quad y = \int \frac{d\tau}{F} + C_2, \qquad \text{where} \quad F = \bigl[a(2-k)e^{\tau} + C_1\bigr]^{\frac{1}{k-2}} + 1.$$

84. $y''_{xx} = ax^n(xy'_x - y) + be^{\lambda x}(xy'_x - y)^k.$

This is a special case of equation 14.9.4.4 with $f(x) = ax^n$ and $g(x) = be^{\lambda x}$.

85. $y''_{xx} = ae^{\lambda x}(xy'_x - y) + bx^n(xy'_x - y)^k.$

This is a special case of equation 14.9.4.4 with $f(x) = ae^{\lambda x}$ and $g(x) = bx^n$.

86. $y''_{xx} = ae^{\lambda x}(xy'_x - y) + be^{\mu x}(xy'_x - y)^k.$

This is a special case of equation 14.9.4.4 with $f(x) = ae^{\lambda x}$ and $g(x) = be^{\mu x}$.

87. $xy''_{xx} + y'_x = ax^n e^{\lambda y}(y'_x)^m.$

The transformation $\zeta = xy'_x$, $w = x^{n-m+1}e^{\lambda y}$ leads to a first-order linear equation: $a\zeta^m w'_\zeta = \lambda\zeta + n - m + 1$.

88. $xy''_{xx} + my'_x + ax^{nm-2m+1}e^{\lambda y}(y'_x)^n = 0.$

This is a special case of equation 14.9.4.14 with $f(y) = ae^{\lambda y}$.

89. $xy''_{xx} + y'_x = (ax^n e^{\lambda y} + bx^{m-1})(y'_x)^m.$

The transformation $\zeta = xy'_x$, $w = x^{n-m+1}e^{\lambda y}$ leads to a first-order separable equation: $\zeta^m(aw + b)w'_\zeta = (\lambda\zeta + n - m + 1)w$.

90. $yy''_{xx} = (y'_x)^2 + be^{ax}y^n(y'_x)^k.$

This is a special case of equation 14.9.4.68 with $f(\xi) = b\xi$, $g(\zeta) = \zeta^k$, and $n = m - k + 2$.

91. $yy''_{xx} = (y'_x)^2 + (ae^{\lambda x}y^n + by^{2-m})(y'_x)^m.$

The transformation $\xi = y'_x/y$, $w = e^{\lambda x}y^{n+m-2}$ leads to a first-order separable equation: $\xi^m(aw + b)w'_\xi = [(n + m - 2)\xi + \lambda]w$.

14.8.3 Equations Containing Hyperbolic Functions

► Equations with hyperbolic sine.

1. $y''_{xx} = \lambda^2 y + a(\sinh \lambda x)^{-n-3}y^n.$

This is a special case of equation 14.9.1.34 with $f(\xi) = a\xi^n$.

2. $y''_{xx} = b\sinh(\lambda x)\, y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\sinh(\lambda x)$.

3. $y''_{xx} = \alpha\sinh^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\sinh^n w + \beta$ and $c = 0$.

4. $y''_{xx} = a(y + b\sinh x)^n - b\sinh x + c.$

The substitution $w = y + b\sinh x$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = aw^n + c$.

5. $y''_{xx} + 3yy'_x + y^3 + a\sinh(\lambda x)\, y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = a\sinh(\lambda x)$.

6. $y''_{xx} + ay^n(y'_x)^2 + b\sinh^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\sinh^m y + c$.

7. $y''_{xx} + a\sinh^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\sinh^n y$ and $g(y) = by^m + c$.

8. $y''_{xx} + a\sinh^n y\,(y'_x)^2 + b\sinh^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\sinh^n y$ and $g(y) = b\sinh^m(\lambda y) + c$.

9. $y''_{xx} = ay^n(y'_x)^2 + b\sinh^m y\,(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\sinh^m y$.

10. $y''_{xx} = a\sinh^n y\,(y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\sinh^n y$ and $g(y) = by^m$.

11. $xy''_{xx} = ny'_x + ax^{2n+1}\sinh^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\sinh^m(\lambda y)$.

12. $2yy''_{xx} = (y'_x)^2 + b\sinh^m(\lambda x)\, y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -b\sinh^m(\lambda x)$.

13. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + b\sinh^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -b\sinh^m(\lambda x)$.

14. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\sinh^k(\lambda x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\sinh^k(\lambda x)$.

15. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\sinh^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\sinh^m(\lambda x)$.

16. $yy''_{xx} = (y'_x)^2 + a\sinh^n(\lambda x)\, yy'_x + bx^m y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -a\sinh^n(\lambda x)$ and $g(x) = -bx^m$.

17. $x^2 y''_{xx} + xy'_x = a\sinh^n(\lambda y) + b.$

This is a special case of equation 14.9.2.23 with $f(y) = a\sinh^n(\lambda y) + b$.

18. $(ax^2 + b)y''_{xx} + axy'_x + \sinh^n(\lambda y) + c = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = \sinh^n(\lambda y) + c$.

► **Equations with hyperbolic cosine.**

19. $y''_{xx} = \lambda^2 y + a(\cosh\lambda x)^{-n-3} y^n.$

This is a special case of equation 14.9.1.35 with $f(\xi) = a\xi^n$.

20. $y''_{xx} = b\cosh(\lambda x)\, y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\cosh(\lambda x)$.

21. $y''_{xx} = \alpha\cosh^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\cosh^n w + \beta$ and $c = 0$.

22. $y''_{xx} = a(y + b\cosh x)^n - b\cosh x + c.$

The substitution $w = y + b\cosh x$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = aw^n + c$.

23. $y''_{xx} + 3yy'_x + y^3 + a\cosh(\lambda x)\, y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = a\cosh(\lambda x)$.

24. $y''_{xx} + ay^n (y'_x)^2 + b\cosh^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\cosh^m y + c$.

25. $y''_{xx} + a\cosh^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\cosh^n y$ and $g(y) = by^m + c$.

26. $y''_{xx} + a\cosh^n y\,(y'_x)^2 + b\cosh^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\cosh^n y$ and $g(y) = b\cosh^m(\lambda y) + c$.

27. $y''_{xx} = ay^n (y'_x)^2 + b\cosh^m y\,(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\cosh^m y$.

28. $y''_{xx} = a\cosh^n y\,(y'_x)^2 + by^m (y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\cosh^n y$ and $g(y) = by^m$.

29. $xy''_{xx} = ny'_x + ax^{2n+1}\cosh^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\cosh^m(\lambda y)$.

30. $2yy''_{xx} = (y'_x)^2 + b\cosh^m(\lambda x)\, y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -b\cosh^m(\lambda x)$.

31. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + b\cosh^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -b\cosh^m(\lambda x)$.

32. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\cosh^k(\lambda x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\cosh^k(\lambda x)$.

33. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\cosh^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\cosh^m(\lambda x)$.

34. $yy''_{xx} = (y'_x)^2 + a\cosh^n(\lambda x)\,yy'_x + bx^m y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -a\cosh^n(\lambda x)$ and $g(x) = -bx^m$.

35. $x^2 y''_{xx} + xy'_x = a\cosh^n(\lambda y) + b.$

This is a special case of equation 14.9.2.23 with $f(y) = a\cosh^n(\lambda y) + b$.

36. $(ax^2 + b)y''_{xx} + axy'_x + \cosh^n(\lambda y) + c = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = \cosh^n(\lambda y) + c$.

► **Equations with hyperbolic tangent.**

37. $y''_{xx} = b\tanh(\lambda x)\,y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\tanh(\lambda x)$.

38. $y''_{xx} = \alpha\tanh^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\tanh^n w + \beta$ and $c = 0$.

39. $y''_{xx} + 3yy'_x + y^3 + a\tanh(\lambda x)\,y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = a\tanh(\lambda x)$.

40. $y''_{xx} + ay^n(y'_x)^2 + b\tanh^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\tanh^m y + c$.

41. $y''_{xx} + a\tanh^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\tanh^n y$ and $g(y) = by^m + c$.

42. $y''_{xx} + a\tanh^n y\,(y'_x)^2 + b\tanh^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\tanh^n y$ and $g(y) = b\tanh^m(\lambda y) + c$.

43. $y''_{xx} = ay^n(y'_x)^2 + b\tanh^m y\,(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\tanh^m y$.

44. $y''_{xx} = a\tanh^n y\,(y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\tanh^n y$ and $g(y) = by^m$.

45. $xy''_{xx} = ny'_x + ax^{2n+1}\tanh^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\tanh^m(\lambda y)$.

46. $2yy''_{xx} = (y'_x)^2 + b\tanh^m(\lambda x)\,y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -b\tanh^m(\lambda x)$.

47. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + b\tanh^m(\lambda x)\,y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -b\tanh^m(\lambda x)$.

48. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\tanh^k(\lambda x)\,y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\tanh^k(\lambda x)$.

49. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\tanh^m(\lambda x)\,y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\tanh^m(\lambda x)$.

50. $yy''_{xx} = (y'_x)^2 + a\tanh^n(\lambda x)\,yy'_x + bx^m y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -a\tanh^n(\lambda x)$ and $g(x) = -bx^m$.

51. $x^2 y''_{xx} + xy'_x = a\tanh^n(\lambda y) + b.$

This is a special case of equation 14.9.2.23 with $f(y) = a\tanh^n(\lambda y) + b$.

52. $(ax^2 + b)y''_{xx} + axy'_x + \tanh^n(\lambda y) + c = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = \tanh^n(\lambda y) + c$.

► **Equations with hyperbolic cotangent.**

53. $y''_{xx} = b\coth(\lambda x)\,y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\coth(\lambda x)$.

54. $y''_{xx} = \alpha\coth^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\coth^n w + \beta$ and $c = 0$.

55. $y''_{xx} + 3yy'_x + y^3 + a\coth(\lambda x)\,y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = a\coth(\lambda x)$.

56. $y''_{xx} + ay^n(y'_x)^2 + b\coth^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\coth^m y + c$.

57. $y''_{xx} + a\coth^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\coth^n y$ and $g(y) = by^m + c$.

58. $y''_{xx} + a\coth^n y\,(y'_x)^2 + b\coth^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\coth^n y$ and $g(y) = b\coth^m(\lambda y) + c$.

59. $y''_{xx} = ay^n(y'_x)^2 + b\coth^m y\,(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\coth^m y$.

60. $y''_{xx} = a\coth^n y\,(y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\coth^n y$ and $g(y) = by^m$.

61. $xy''_{xx} = ny'_x + ax^{2n+1}\coth^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\coth^m(\lambda y)$.

62. $2yy''_{xx} = (y'_x)^2 + b\coth^m(\lambda x)\,y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -b\coth^m(\lambda x)$.

63. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + b\coth^m(\lambda x)\,y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -b\coth^m(\lambda x)$.

64. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\coth^k(\lambda x)\, y^{n+1}$.

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\coth^k(\lambda x)$.

65. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\coth^m(\lambda x)\, y^2$.

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\coth^m(\lambda x)$.

66. $yy''_{xx} = (y'_x)^2 + a\coth^n(\lambda x)\, yy'_x + bx^m y^2$.

This is a special case of equation 14.9.3.7 with $f(x) = -a\coth^n(\lambda x)$ and $g(x) = -bx^m$.

67. $x^2 y''_{xx} + xy'_x = a\coth^n(\lambda y) + b$.

This is a special case of equation 14.9.2.23 with $f(y) = a\coth^n(\lambda y) + b$.

68. $(ax^2 + b)y''_{xx} + axy'_x + \coth^n(\lambda y) + c = 0$.

This is a special case of equation 14.9.2.24 with $f(y) = \coth^n(\lambda y) + c$.

► **Equations containing combinations of hyperbolic functions.**

69. $y''_{xx} = \lambda^2 y + a\sinh^n(\lambda x)\cosh^{-n-m-3}(\lambda x)\, y^m$.

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to the Emden–Fowler equation $w''_{\xi\xi} = a\lambda^{-2}\xi^n w^m$, which is discussed in Section 14.3.

70. $y''_{xx} = \lambda^2 y + a\cosh^n(\lambda x)\sinh^{-n-m-3}(\lambda x)\, y^m$.

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to the Emden–Fowler equation $w''_{\xi\xi} = a\lambda^{-2}\xi^n w^m$, which is discussed in Section 14.3.

71. $y''_{xx} = a(y'_x \sinh x - y\cosh x)^k + y$.

The substitution $w = y'_x \sinh x - y\cosh x$ leads to a first-order separable equation: $w'_x = a\sinh x\, w^k$.

72. $y''_{xx} = a(y'_x \cosh x - y\sinh x)^k + y$.

The substitution $w = y'_x \cosh x - y\sinh x$ leads to a first-order separable equation: $w'_x = a\cosh x\, w^k$.

73. $\sinh x\, y''_{xx} + \frac{1}{2}\cosh x\, y'_x = a\sinh(\lambda y) + b$.

This is a special case of equation 14.9.2.34 with $g(x) = \sinh x$ and $f(y) = a\sinh(\lambda y) + b$.

74. $\cosh x\, y''_{xx} + \frac{1}{2}\sinh x\, y'_x = a\cosh(\lambda y) + b$.

This is a special case of equation 14.9.2.34 with $g(x) = \cosh x$ and $f(y) = a\cosh(\lambda y) + b$.

75. $yy''_{xx} + (y'_x)^2 + a\sinh(\beta x)yy'_x + b\cosh(\lambda x) + c = 0$.

This is a special case of equation 14.9.3.6 with $f(x) = a\sinh(\beta x)$ and $g(x) = b\cosh(\lambda x) + c$.

76. $yy''_{xx} - (y'_x)^2 + a\sinh(\beta x)yy'_x + b\cosh(\lambda x)y^2 = 0$.

This is a special case of equation 14.9.3.7 with $f(x) = a\sinh(\beta x)$ and $g(x) = b\cosh(\lambda x)$.

14.8.4 Equations Containing Logarithmic Functions

► **Equations of the form** $f(x,y)y''_{xx} + g(x,y)y'_x + h(x,y) = 0$.

1. $y''_{xx} = by(ax + m\ln y)$.

This is a special case of equation 14.9.1.27 with $f(z) = b\ln z$.

2. $y''_{xx} = bx^{-2}(ay + n\ln x)$.

This is a special case of equation 14.9.1.28 with $f(z) = b\ln z$.

3. $y''_{xx} = ax^{-3}(\ln y - \ln x)$.

This is a special case of equation 14.9.1.8 with $f(\xi) = a\ln\xi$.

4. $y''_{xx} = ax^{-3/2}(2\ln y - \ln x)$.

This is a special case of equation 14.9.1.9 with $f(\xi) = 2a\ln\xi$.

5. $y''_{xx} = k\ln^n(ay + bx) + s$.

This is a special case of equation 14.9.1.4 with $f(w) = k\ln^n w + s$ and $c = 0$.

6. $y''_{xx} = y^{-3}\big[2\ln y - \ln(ax^2 + c)\big]$.

This is a special case of equation 14.9.1.21 with $f(w) = 2\ln w$ and $b = 0$.

7. $x^2 y''_{xx} = x^2(y + a\ln x + b)^n + a$.

This is a special case of equation 14.9.1.36 with $f(\xi) = \xi^n$.

8. $x^2 y''_{xx} = n(n+1)y + ax^{3n+2}(\ln y + n\ln x)$.

This is a special case of equation 14.9.1.12 with $f(\xi) = a\ln\xi$.

9. $x^2 y''_{xx} + \frac{1}{4}y + Ax^{\frac{1-m}{2}}(a\ln x + b)^n y^m = 0$.

The transformation $\xi = a\ln x + b$, $w = yx^{-1/2}$ leads to the Emden–Fowler equation: $w''_{\xi\xi} + Aa^{-2}\xi^n w^m = 0$ (see Section 14.3).

10. $xy''_{xx} = ny'_x + ax^{2n+1}\ln^m(\lambda y)$.

This is a special case of equation 14.9.2.20 with $f(y) = a\ln^m(\lambda y)$.

11. $xy''_{xx} = -(n+1)y'_x + ax^{n-1}(\ln y + n\ln x)$.

This is a special case of equation 14.9.2.30 with $f(\xi) = a\ln\xi$.

12. $xy''_{xx} = (ay + n\ln x)y'_x$.

This is a special case of equation 14.9.2.39 with $f(\xi) = \ln\xi$.

13. $xy''_{xx} + x(2ay + \ln x + b)y'_x + y = 0$.

Integrating yields a Riccati equation: $y'_x + ay^2 + (\ln x + b)y = C$.

14. $xy''_{xx} = a\ln^k(by)y'_x$.

This is a special case of equation 14.9.2.21 with $f(y) = a\ln^k(by)$.

15. $x^2 y''_{xx} + xy'_x = a\ln^n(\lambda y) + b$.

This is a special case of equation 14.9.2.23 with $f(y) = a\ln^n(\lambda y) + b$.

16. $(ax^2 + b)y''_{xx} + axy'_x + c\ln^n(\lambda y) = 0$.

This is a special case of equation 14.9.2.24 with $f(y) = c\ln^n(\lambda y)$.

► **Other equations.**

17. $y''_{xx} + ay^n(y'_x)^2 + b\ln^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\ln^m y + c$.

18. $y''_{xx} = ay^n(y'_x)^2 + b\ln^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ay^n$ and $g(x) = b\ln^m(\lambda x)$.

19. $y''_{xx} + a\ln^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\ln^n y$ and $g(y) = by^m + c$.

20. $y''_{xx} = a\ln^n y\,(y'_x)^2 + bx^m y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\ln^n y$ and $g(x) = bx^m$.

21. $y''_{xx} = a\ln^n y\,(y'_x)^2 + b\ln^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\ln^n y$ and $g(x) = b\ln^m(\lambda x)$.

22. $y''_{xx} = ax^{-2}y^{-1}(y'_x)^4 - 2ax^{-3}\ln y\,(y'_x)^3.$

Solution in parametric form:

$$x = \lambda\bigl[(F \pm 2\tau)^2 \pm 4\ln(C_1F)\bigr]^{1/2}, \qquad y = C_1F,$$

where $F = \exp(\mp\tau^2)\Bigl[\int \exp(\mp\tau^2)\,d\tau + C_2\Bigr]^{-1}$, $\lambda = \bigl(\pm\frac{1}{2}aC_1^2\bigr)^{1/4}$.

23. $y''_{xx} = 2a\ln x\, y^{-3} - ax^{-1}y^{-2}(y'_x)^{-1}.$

Solution in parametric form:

$$x = C_1F, \qquad y = \lambda\bigl[(F \pm 2\tau)^2 \pm 4\ln(C_1F)\bigr]^{1/2},$$

where $F = \exp(\mp\tau^2)\Bigl[\int \exp(\mp\tau^2)\,d\tau + C_2\Bigr]^{-1}$, $\lambda = \bigl(\pm\frac{1}{2}aC_1^2\bigr)^{1/4}$.

24. $y''_{xx} = ay^n(y'_x)^2 + b\ln^m y\,(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\ln^m y$.

25. $y''_{xx} = a\ln^n y\,(y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\ln^n y$ and $g(y) = by^m$.

26. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\ln^k(\lambda x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\ln^k(\lambda x)$.

27. $yy''_{xx} = (y'_x)^2 + a\ln^n(\lambda x)\, yy'_x + bx^m y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -a\ln^n(\lambda x)$ and $g(x) = -bx^m$.

28. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\ln^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\ln^m(\lambda x)$.

29. $yy''_{xx} = (ax + n\ln y)(y'_x)^2.$

This is a special case of equation 14.9.3.40 with $f(\xi) = \ln\xi$.

14.8.5 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y''_{xx} = -\lambda^2 y + a(\sin\lambda x)^n y^{-n-3}.$

This is a special case of equation 14.9.1.40 with $f(\xi) = a\xi^{-n-3}$.

2. $y''_{xx} = -\lambda^2 y + A\sin^n(\lambda x + a)\sin^m(\lambda x + b)\, y^{-n-m-3}.$

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{y}{\sin(\lambda x + b)}$ leads to the Emden–Fowler equation: $w''_{\xi\xi} = A[\lambda\sin(b-a)]^{-2}\xi^n w^{-n-m-3}$ (see Section 14.3).

3. $y''_{xx} = b\sin(\lambda x)\, y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\sin(\lambda x)$.

4. $y''_{xx} = \alpha\sin^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\sin^n w + \beta$ and $c = 0$.

5. $y''_{xx} = a(y + b\sin x)^n + b\sin x + c.$

The substitution $w = y + b\sin x$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = aw^n + c$.

6. $y''_{xx} + 3yy'_x + y^3 + a\sin(\lambda x)\, y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = a\sin(\lambda x)$.

7. $y''_{xx} + ay^n(y'_x)^2 + b\sin^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\sin^m y + c$.

8. $y''_{xx} = ay^n(y'_x)^2 + b\sin^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ay^n$ and $g(x) = b\sin^m(\lambda x)$.

9. $y''_{xx} = ay^n(y'_x)^2 + b\sin^m y\,(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\sin^m y$.

10. $y''_{xx} + a\sin^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\sin^n y$ and $g(y) = by^m + c$.

11. $y''_{xx} + a\sin^n y\,(y'_x)^2 + b\sin^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\sin^n y$ and $g(y) = b\sin^m(\lambda y) + c$.

12. $y''_{xx} = a\sin^n y\,(y'_x)^2 + bx^m y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\sin^n y$ and $g(x) = bx^m$.

13. $y''_{xx} = a\sin^n y\,(y'_x)^2 + b\sin^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\sin^n y$ and $g(x) = b\sin^m(\lambda x)$.

14. $y''_{xx} = a\sin^n y\,(y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\sin^n y$ and $g(y) = by^m$.

15. $xy''_{xx} = ny'_x + ax^{2n+1}\sin^m(\lambda y)$.

This is a special case of equation 14.9.2.20 with $f(y) = a\sin^m(\lambda y)$.

16. $2yy''_{xx} = (y'_x)^2 + b\sin(\lambda x)\,y^2 - a$.

This is a special case of equation 14.9.3.5 with $f(x) = -b\sin(\lambda x)$.

17. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + b\sin(\lambda x)\,y^2$.

This is a special case of equation 14.9.3.8 with $f(x) = -b\sin(\lambda x)$.

18. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\sin^k(\lambda x)\,y^{n+1}$.

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\sin^k(\lambda x)$.

19. $yy''_{xx} = n(y'_x)^2 + a\sin(\lambda x)\,y^2 + b\sin(\mu x)\,y^{n+1}$.

This is a special case of equation 14.9.3.9 with $f(x) = -a\sin(\lambda x)$ and $g(x) = -b\sin(\mu x)$.

20. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\sin^m(\lambda x)\,y^2$.

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\sin^m(\lambda x)$.

21. $yy''_{xx} = (y'_x)^2 + a\sin^n(\lambda x)\,yy'_x + bx^m y^2$.

This is a special case of equation 14.9.3.7 with $f(x) = -a\sin^n(\lambda x)$ and $g(x) = -bx^m$.

22. $x^2y''_{xx} + xy'_x = a\sin^n(\lambda y) + b$.

This is a special case of equation 14.9.2.23 with $f(y) = a\sin^n(\lambda y) + b$.

23. $(ax^2 + b)y''_{xx} + axy'_x + \sin^n(\lambda y) + c = 0$.

This is a special case of equation 14.9.2.24 with $f(y) = \sin^n(\lambda y) + c$.

24. $\sin^2 x\, y''_{xx} = n(n + 1 - n\sin^2 x)y + a(\sin x)^{nm+3n+2}y^m$.

This is a special case of equation 14.9.1.44 with $f(\xi) = a\xi^m$.

► Equations with cosine.

25. $y''_{xx} = -\lambda^2 y + a(\cos\lambda x)^n y^{-n-3}$.

This is a special case of equation 14.9.1.41 with $f(\xi) = a\xi^{-n-3}$.

26. $y''_{xx} = -\lambda^2 y + A\cos^n(\lambda x + a)\cos^m(\lambda x + b)\,y^{-n-m-3}$.

The transformation $\xi = \dfrac{\cos(\lambda x + a)}{\cos(\lambda x + b)}$, $w = \dfrac{y}{\cos(\lambda x + b)}$ leads to the Emden–Fowler equation: $w''_{\xi\xi} = A[\lambda\sin(b - a)]^{-2}\xi^n w^{-n-m-3}$ (see Section 14.3).

27. $y''_{xx} = b\cos(\lambda x)\,y + ay^{-3}$.

This is a special case of equation 14.9.1.2 with $f(x) = -b\cos(\lambda x)$.

28. $y''_{xx} = \alpha\cos^n(ay + bx) + \beta$.

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\cos^n w + \beta$ and $c = 0$.

29. $y''_{xx} = a(y + b\cos x)^n + b\cos x + c.$

The substitution $w = y + b\cos x$ leads to an autonomous equation of the form 14.9.1.1: $w''_{xx} = aw^n + c$.

30. $y''_{xx} + 3yy'_x + y^3 + a\cos(\lambda x)\, y = 0.$

This is a special case of equation 14.9.2.1 with $f(x) = a\cos(\lambda x)$.

31. $y''_{xx} + ay^n(y'_x)^2 + b\cos^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\cos^m y + c$.

32. $y''_{xx} = ay^n(y'_x)^2 + b\cos^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ay^n$ and $g(x) = b\cos^m(\lambda x)$.

33. $y''_{xx} = ay^n(y'_x)^2 + b\cos^m y\, (y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\cos^m y$.

34. $y''_{xx} + a\cos^n y\, (y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\cos^n y$ and $g(y) = by^m + c$.

35. $y''_{xx} + a\cos^n y\, (y'_x)^2 + b\cos^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\cos^n y$ and $g(y) = b\cos^m(\lambda y) + c$.

36. $y''_{xx} = a\cos^n y\, (y'_x)^2 + bx^m y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\cos^n y$ and $g(x) = bx^m$.

37. $y''_{xx} = a\cos^n y\, (y'_x)^2 + b\cos^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\cos^n y$ and $g(x) = b\cos^m(\lambda x)$.

38. $y''_{xx} = a\cos^n y\, (y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\cos^n y$ and $g(y) = by^m$.

39. $xy''_{xx} = ny'_x + ax^{2n+1}\cos^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\cos^m(\lambda y)$.

40. $2yy''_{xx} = (y'_x)^2 + b\cos(\lambda x)\, y^2 - a.$

This is a special case of equation 14.9.3.5 with $f(x) = -b\cos(\lambda x)$.

41. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + b\cos(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.8 with $f(x) = -b\cos(\lambda x)$.

42. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\cos^k(\lambda x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\cos^k(\lambda x)$.

43. $yy''_{xx} = n(y'_x)^2 + a\cos(\lambda x)\, y^2 + b\cos(\mu x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -a\cos(\lambda x)$ and $g(x) = -b\cos(\mu x)$.

44. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\cos^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\cos^m(\lambda x)$.

45. $yy''_{xx} = (y'_x)^2 + a\cos^n(\lambda x)\, yy'_x + bx^m y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -a\cos^n(\lambda x)$ and $g(x) = -bx^m$.

46. $x^2 y''_{xx} + xy'_x = a\cos^n(\lambda y) + b.$

This is a special case of equation 14.9.2.23 with $f(y) = a\cos^n(\lambda y) + b$.

47. $(ax^2 + b)y''_{xx} + axy'_x + \cos^n(\lambda y) + c = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = \cos^n(\lambda y) + c$.

48. $\cos^2 x\, y''_{xx} = n(n + 1 - n\cos^2 x)y + a(\cos x)^{nm+3n+2} y^m.$

This is a special case of equation 14.9.1.45 with $f(\xi) = a\xi^m$.

► **Equations with tangent.**

49. $y''_{xx} = b\tan(\lambda x)\, y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\tan(\lambda x)$.

50. $y''_{xx} = \alpha\tan^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\tan^n w + \beta$ and $c = 0$.

51. $y''_{xx} + ay^n(y'_x)^2 + b\tan^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\tan^m y + c$.

52. $y''_{xx} = ay^n(y'_x)^2 + b\tan^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ay^n$ and $g(x) = b\tan^m(\lambda x)$.

53. $y''_{xx} = ay^n(y'_x)^2 + b\tan^m y\, (y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\tan^m y$.

54. $y''_{xx} + a\tan^n y\, (y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\tan^n y$ and $g(y) = by^m + c$.

55. $y''_{xx} + a\tan^n y\, (y'_x)^2 + b\tan^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\tan^n y$ and $g(y) = b\tan^m(\lambda y) + c$.

56. $y''_{xx} = a\tan^n y\, (y'_x)^2 + bx^m y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\tan^n y$ and $g(x) = bx^m$.

57. $y''_{xx} = a\tan^n y\, (y'_x)^2 + b\tan^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\tan^n y$ and $g(x) = b\tan^m(\lambda x)$.

58. $y''_{xx} = a\tan^n y\, (y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\tan^n y$ and $g(y) = by^m$.

59. $xy''_{xx} = ny'_x + ax^{2n+1}\tan^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\tan^m(\lambda y)$.

60. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\tan^k(\lambda x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\tan^k(\lambda x)$.

61. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\tan^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\tan^m(\lambda x)$.

62. $x^2 y''_{xx} + xy'_x = a\tan^n(\lambda y) + b.$

This is a special case of equation 14.9.2.23 with $f(y) = a\tan^n(\lambda y) + b$.

63. $(ax^2 + b)y''_{xx} + axy'_x + \tan^n(\lambda y) + c = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = \tan^n(\lambda y) + c$.

► **Equations with cotangent.**

64. $y''_{xx} = b\cot(\lambda x)\, y + ay^{-3}.$

This is a special case of equation 14.9.1.2 with $f(x) = -b\cot(\lambda x)$.

65. $y''_{xx} = \alpha\cot^n(ay + bx) + \beta.$

This is a special case of equation 14.9.1.4 with $f(w) = \alpha\cot^n w + \beta$ and $c = 0$.

66. $y''_{xx} + ay^n(y'_x)^2 + b\cot^m y + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = ay^n$ and $g(y) = b\cot^m y + c$.

67. $y''_{xx} = ay^n(y'_x)^2 + b\cot^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = ay^n$ and $g(x) = b\cot^m(\lambda x)$.

68. $y''_{xx} = ay^n(y'_x)^2 + b\cot^m y\, (y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = ay^n$ and $g(y) = b\cot^m y$.

69. $y''_{xx} + a\cot^n y\, (y'_x)^2 + by^m + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\cot^n y$ and $g(y) = by^m + c$.

70. $y''_{xx} + a\cot^n y\, (y'_x)^2 + b\cot^m(\lambda y) + c = 0.$

This is a special case of equation 14.9.3.25 with $f(y) = a\cot^n y$ and $g(y) = b\cot^m(\lambda y) + c$.

71. $y''_{xx} = a\cot^n y\, (y'_x)^2 + bx^m y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\cot^n y$ and $g(x) = bx^m$.

72. $y''_{xx} = a\cot^n y\, (y'_x)^2 + b\cot^m(\lambda x)\, y'_x.$

This is a special case of equation 14.9.3.38 with $f(y) = a\cot^n y$ and $g(x) = b\cot^m(\lambda x)$.

73. $y''_{xx} = a\cot^n y\, (y'_x)^2 + by^m(y'_x)^k.$

This is a special case of equation 14.9.4.13 with $f(y) = a\cot^n y$ and $g(y) = by^m$.

74. $xy''_{xx} = ny'_x + ax^{2n+1}\cot^m(\lambda y).$

This is a special case of equation 14.9.2.20 with $f(y) = a\cot^m(\lambda y)$.

75. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\cot^k(\lambda x)\, y^{n+1}.$

This is a special case of equation 14.9.3.9 with $f(x) = -ax^m$ and $g(x) = -b\cot^k(\lambda x)$.

76. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\cot^m(\lambda x)\, y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ax^n$ and $g(x) = -b\cot^m(\lambda x)$.

77. $x^2 y''_{xx} + xy'_x = a\cot^n(\lambda y) + b.$

This is a special case of equation 14.9.2.23 with $f(y) = a\cot^n(\lambda y) + b$.

78. $(ax^2 + b)y''_{xx} + axy'_x + \cot^n(\lambda y) + c = 0.$

This is a special case of equation 14.9.2.24 with $f(y) = \cot^n(\lambda y) + c$.

► **Equations containing combinations of trigonometric functions.**

79. $y''_{xx} = -\lambda^2 y + a\cos^n(\lambda x)\sin^m(\lambda x)\, y^{-n-m-3}.$

The transformation $\xi = \cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to the Emden–Fowler equation: $w''_{\xi\xi} = a\lambda^{-2}\xi^n w^{-n-m-3}$ (see Section 14.3).

80. $y''_{xx} = -\lambda^2 y + a\sin^n(\lambda x)[\sin(\lambda x) + b\cos(\lambda x)]^m y^{-n-m-3}.$

The transformation $\xi = 1 + b\cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to the Emden–Fowler equation: $w''_{\xi\xi} = a(b\lambda)^{-2}\xi^m w^{-n-m-3}$ (see Section 14.3).

81. $y''_{xx} = a(y'_x \sin x - y\cos x)^k - y.$

The substitution $w = y'_x \sin x - y\cos x$ leads to a first-order separable equation: $w'_x = a\sin x\, w^k$.

82. $y''_{xx} = a(y'_x \cos x + y\sin x)^k - y.$

The substitution $w = y'_x \cos x + y\sin x$ leads to a first-order separable equation: $w'_x = a\cos x\, w^k$.

83. $y''_{xx} = 2(\cos x)^{-2} y + a(\cot x)^{n+3} y^n.$

This is a special case of equation 14.9.1.43 with $f(\xi) = a\xi^n$.

84. $y''_{xx} = 2(\sin x)^{-2} y + a(\tan x)^{n+3} y^n.$

This is a special case of equation 14.9.1.42 with $f(\xi) = a\xi^n$.

85. $y''_{xx} = (n+1)(\tan x)y'_x + ny + a(\cos x)^{nm-2} y^{m-1}.$

This is a special case of equation 14.9.2.44 with $f(\xi) = a\xi^{m-1}$.

86. $y''_{xx} + (2ay + b\sin x)y'_x + b(\cos x)y = 0.$

Integrating yields a Riccati equation: $y'_x + ay^2 + b(\sin x)y = C$.

87. $x^2 y''_{xx} + ax^2 \tan x\, y'_x + n(ax\tan x - n - 1)y = bx^{nm+2}(\cos x)^{2a} y^{m-3}.$

This is a special case of equation 14.9.2.47 with $f(\xi) = b\xi^{m-3}$.

88. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = ay^n + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = ay^n + b$.

89. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\sin^n(\lambda y) + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\sin^n(\lambda y) + b$.

90. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\cos^n(\lambda y) + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\cos^n(\lambda y) + b$.

91. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\tan^n(\lambda y) + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\tan^n(\lambda y) + b$.

92. $yy''_{xx} + (y'_x)^2 + a\sin(\beta x)yy'_x + b\cos(\lambda x) + c = 0.$

This is a special case of equation 14.9.3.6 with $f(x) = a\sin(\beta x)$ and $g(x) = b\cos(\lambda x) + c$.

93. $yy''_{xx} - (y'_x)^2 + a\sin(\beta x)yy'_x + b\cos(\lambda x)y^2 = 0.$

This is a special case of equation 14.9.3.7 with $f(x) = a\sin(\beta x)$ and $g(x) = b\cos(\lambda x)$.

14.8.6 Equations Containing the Combinations of Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. $y''_{xx} = \lambda^2 y + ae^{3\lambda x}(\ln y + \lambda x).$

This is a special case of equation 14.9.1.29 with $f(\xi) = a\ln\xi$.

2. $y''_{xx} = -ay'_x + be^{ax}(\ln y + ax).$

This is a special case of equation 14.9.2.36 with $f(\xi) = b\ln\xi$.

3. $y''_{xx} = ay'_x + be^{2ax}\ln^n(\lambda y).$

This is a special case of equation 14.9.2.17 with $f(y) = b\ln^n(\lambda y)$.

4. $y''_{xx} = ay'_x + be^{2ax}\sin^n(\lambda y).$

This is a special case of equation 14.9.2.17 with $f(y) = b\sin^n(\lambda y)$.

5. $y''_{xx} = ay'_x + be^{2ax}\tan^n(\lambda y).$

This is a special case of equation 14.9.2.17 with $f(y) = b\tan^n(\lambda y)$.

6. $y''_{xx} + a\tan x\, y'_x + b(a\tan x - b)y = ce^{bmx}(\cos x)^{2a}y^{m-3}.$

This is a special case of equation 14.9.2.46 with $f(\xi) = c\xi^{m-3}$.

7. $y''_{xx} = a(y'_x)^2 - be^{4ay} + c\sinh(\lambda x).$

This is a special case of equation 14.9.3.18 with $f(x) = -c\sinh(\lambda x)$.

8. $y''_{xx} = a(y'_x)^2 + b\cosh^n(\lambda x)e^{ay} + cx^m.$

This is a special case of equation 14.9.3.17 with $f(x) = -b\cosh^n(\lambda x)$ and $g(x) = -cx^m$.

9. $y''_{xx} = a(y'_x)^2 + b\ln^n(\lambda x)e^{ay} + cx^m.$

This is a special case of equation 14.9.3.17 with $f(x) = -b\ln^n(\lambda x)$ and $g(x) = -cx^m$.

10. $y''_{xx} = a(y'_x)^2 + b\ln^n(\lambda x)e^{ay} + ce^{\nu x}$.

This is a special case of equation 14.9.3.17 with $f(x) = -b\ln^n(\lambda x)$ and $g(x) = -ce^{\nu x}$.

11. $y''_{xx} = a(y'_x)^2 - be^{4ay} + c\sin(\lambda x)$.

This is a special case of equation 14.9.3.18 with $f(x) = -c\sin(\lambda x)$.

12. $y''_{xx} = a(y'_x)^2 + b\sin^n(\lambda x)e^{ay} + cx^m$.

This is a special case of equation 14.9.3.17 with $f(x) = -b\sin^n(\lambda x)$ and $g(x) = -cx^m$.

13. $y''_{xx} = a(y'_x)^2 + b\sin^n(\lambda x)e^{ay} + ce^{\nu x}$.

This is a special case of equation 14.9.3.17 with $f(x) = -b\sin^n(\lambda x)$ and $g(x) = -ce^{\nu x}$.

14. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + b\ln^n y + c = 0$.

This is a special case of equation 14.9.3.25 with $f(y) = ae^{\lambda y}$ and $g(y) = b\ln^n y + c$.

15. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + b\sin^n y + c = 0$.

This is a special case of equation 14.9.3.25 with $f(y) = ae^{\lambda y}$ and $g(y) = b\sin^n y + c$.

16. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + b\ln^n(\mu x)y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = ae^{\lambda y}$ and $g(x) = b\ln^n(\mu x)$.

17. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + b\sin^n(\mu x)y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = ae^{\lambda y}$ and $g(x) = b\sin^n(\mu x)$.

18. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + b\tan^n(\mu x)y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = ae^{\lambda y}$ and $g(x) = b\tan^n(\mu x)$.

19. $y''_{xx} + a\ln^n y\,(y'_x)^2 + be^{\lambda y} + c = 0$.

This is a special case of equation 14.9.3.25 with $f(y) = a\ln^n y$ and $g(y) = be^{\lambda y} + c$.

20. $y''_{xx} = a\ln^n y\,(y'_x)^2 + be^{\lambda x}y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = a\ln^n y$ and $g(x) = be^{\lambda x}$.

21. $y''_{xx} = a\ln^n y\,(y'_x)^2 + b\sin^m(\lambda x)y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = a\ln^n y$ and $g(x) = b\sin^m(\lambda x)$.

22. $y''_{xx} + a\sin^n y\,(y'_x)^2 + be^{\lambda y} + c = 0$.

This is a special case of equation 14.9.3.25 with $f(y) = a\sin^n y$ and $g(y) = be^{\lambda y} + c$.

23. $y''_{xx} = a\sin^n y\,(y'_x)^2 + be^{\lambda x}y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = a\sin^n y$ and $g(x) = be^{\lambda x}$.

24. $y''_{xx} = a\sin^n y\,(y'_x)^2 + b\ln^m(\lambda x)y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = a\sin^n y$ and $g(x) = b\ln^m(\lambda x)$.

25. $y''_{xx} = a\tan^n y\,(y'_x)^2 + be^{\lambda x}y'_x$.

This is a special case of equation 14.9.3.38 with $f(y) = a\tan^n y$ and $g(x) = be^{\lambda x}$.

26. $y''_{xx} + be^{ax}\cosh(\lambda y)(y'_x)^3 + ay'_x = 0.$

This is a special case of equation 14.9.3.35 with $f(y) = b\cosh(\lambda y)$.

27. $y''_{xx} + be^{ax}\sin(\lambda y)(y'_x)^3 + ay'_x = 0.$

This is a special case of equation 14.9.3.35 with $f(y) = b\sin(\lambda y)$.

28. $xy''_{xx} = ax\ln x\, e^y y'_x + ae^y.$

Solution: $y = -\ln\Big[e^{C_1 x}\Big(C_2 - \dfrac{a}{C_1}\displaystyle\int x^{-1}e^{-C_1 x}\,dx\Big) + \dfrac{a}{C_1}\ln x\Big].$

29. $yy''_{xx} = ae^x(y'_x)^3 + ae^x y\ln y\,(y'_x)^2.$

Solution: $x = -\ln\Big[e^{C_1 y}\Big(C_2 + \dfrac{a}{C_1}\displaystyle\int y^{-1}e^{-C_1 y}\,dy\Big) - \dfrac{a}{C_1}\ln y\Big].$

30. $yy''_{xx} = (y'_x)^2 + ae^{\lambda x}yy'_x + b\sin^n(\nu x)y^2.$

This is a special case of equation 14.9.3.7 with $f(x) = -ae^{\lambda x}$ and $g(x) = -b\sin^n(\nu x)$.

31. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x = \cosh^n(\lambda y) + c.$

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = \cosh^n(\lambda y) + c$.

32. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x = \tanh^n(\lambda y) + c.$

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = \tanh^n(\lambda y) + c$.

33. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x = \ln^n(\lambda y) + c.$

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = \ln^n(\lambda y) + c$.

34. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x = \sin^n(\lambda y) + c.$

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = \sin^n(\lambda y) + c$.

35. $(ae^{2x} + b)y''_{xx} + ae^{2x}y'_x = \tan^n(\lambda y) + c.$

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2x} + b$ and $f(y) = \tan^n(\lambda y) + c$.

36. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = ae^{\lambda y} + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = ae^{\lambda y} + b$.

37. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\cosh^n(\lambda y) + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\cosh^n(\lambda y) + b$.

38. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\sinh^n(\lambda y) + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\sinh^n(\lambda y) + b$.

39. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\tanh^n(\lambda y) + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\tanh^n(\lambda y) + b$.

40. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\ln^n y + b.$

This is a special case of equation 14.9.2.34 with $g(x) = \sin x$ and $f(y) = a\ln^n y + b$.

14.9 Equations Containing Arbitrary Functions

◆ *Notation:* f, g, h, φ, *and* ψ *are arbitrary composite functions of their arguments indicated in parentheses just after the function name (the arguments can depend on* x, y, y'_x*).*

14.9.1 Equations of the Form $F(x,y)y''_{xx} + G(x,y) = 0$

► **Arguments of arbitrary functions are algebraic and power functions of x and y.**

1. $y''_{xx} = f(y)$.

This autonomous equation arises in mechanics, combustion theory, and the theory of mass transfer with chemical reactions. The substitution $u(y) = y'_x$ leads to a first-order separated equation: $uu'_y = f(y)$.

Solution: $\displaystyle\int\Big[C_1 + 2\int f(y)\,dy\Big]^{-1/2} dy = C_2 \pm x.$

Particular solutions: $y = A_k$, where A_k are roots of the algebraic (transcendental) equation $f(A_k) = 0$.

2. $y''_{xx} + f(x)y = ay^{-3}$.

Yermakov's equation. Let $w = w(x)$ be a nontrivial solution of the second-order linear equation $w''_{xx} + f(x)w = 0$. The transformation $\xi = \displaystyle\int\frac{dx}{w^2}$, $z = \dfrac{y}{w}$ leads to an autonomous equation of the form 14.9.1.1: $z''_{\xi\xi} = az^{-3}$.

Solution: $\displaystyle C_1y^2 = aw^2 + w^2\Big(C_2 + C_1\int\frac{dx}{w^2}\Big)^2.$

⊙ *Literature*: V. P. Yermakov (1880).

3. $y''_{xx} + f(x)y = g'_x(x)y^{-1} - g^2(x)y^{-3}$.

Generalized Yermakov's equation.

Solution: $\displaystyle y = w\Big[C + 2\int\frac{g(x)\,dx}{w^2}\Big]^{1/2}$, where $w = w(x)$ is the general solution of the linear equation $w''_{xx} + f(x)w = 0$.

4. $y''_{xx} = f(ay + bx + c)$.

The substitution $w = ay + bx + c$ leads to an equation of the form 14.9.1.1: $w''_{xx} = af(w)$.

5. $y''_{xx} = f(y + ax^2 + bx + c)$.

The substitution $w = y + ax^2 + bx + c$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w) + 2a$.

6. $y''_{xx} = f(y + ax^n + bx^2 + cx) - an(n-1)x^{n-2}$.

The substitution $w = y + ax^n + bx^2 + cx$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w) + 2b$.

7. $y''_{xx} = x^{-1}f(yx^{-1})$.

Homogeneous equation. The transformation $t = -\ln|x|$, $z = y/x$ leads to an autonomous equation: $z''_{tt} - z'_t = f(z)$. Reducing its order with the substitution $w(z) = w'_t$, we arrive at the Abel equation $ww'_z - w = f(w)$, which is discussed in Section 13.3.1.

8. $y''_{xx} = x^{-3} f(yx^{-1})$**.**

The transformation $\xi = 1/x,\ w = y/x$ leads to an equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

9. $y''_{xx} = x^{-3/2} f(yx^{-1/2})$**.**

Having set $w = yx^{-1/2}$, we obtain $\frac{d}{dx}(xw'_x)^2 = \frac{1}{2}ww'_x + 2f(w)w'_x$. Integrating the latter equation, we arrive at a separable equation.

Solution: $\displaystyle\int \Big[C_1 + \tfrac{1}{4}w^2 + 2\int f(w)\,dw\Big]^{-1/2} dw = C_2 \pm \ln x.$

10. $y''_{xx} = x^{k-2} f(x^{-k}y)$**.**

Generalized homogeneous equation.

1°. The transformation $t = \ln x,\ z = x^{-k}y$ leads to an autonomous equation of the form 14.9.6.2: $z''_{tt} + (2k-1)z'_t + k(k-1)z = f(z)$.

2°. The transformation $z = x^{-k}y,\ w = xy'_x/y$ leads to a first-order equation of the form: $z(w-k)w'_z = z^{-1}f(z) + w - w^2$.

11. $y''_{xx} = yx^{-2} f(x^n y^m)$**.**

Generalized homogeneous equation. The transformation $z = x^n y^m,\ w = xy'_x/y$ leads to a first-order equation: $z(mw+n)w'_z = f(z) + w - w^2$.

12. $y''_{xx} = n(n+1)x^{-2}y + x^{3n} f(yx^n)$**.**

This is a special case of equation 14.9.1.46 with $\psi = x^{-n}$.

13. $y''_{xx} = x^{-3/2} f(ayx^{-1/2} + bx^{1/2})$**.**

The substitution $w = ay + bx$ leads to an equation of the form 14.9.1.9:

$$w''_{xx} = ax^{-3/2} f(wx^{-1/2}).$$

14. $x(x+a)^2 y''_{xx} = f(y/x)$**.**

The transformation $\xi = \ln\left|\dfrac{x+a}{x}\right|,\ z = \dfrac{y}{x}$ leads to an autonomous equation of the form 14.9.6.2: $z''_{\xi\xi} - z'_\xi = a^{-2}f(z)$. Reducing its order with the substitution $w(z) = z'_\xi$, we arrive at an Abel equation of the second kind: $ww'_z - w = a^{-2}f(z)$. See Section 13.3.1 for information about an Abel equation of the second kind.

15. $y''_{xx} = \dfrac{1}{x^3} f\Big(\dfrac{y}{x} + \dfrac{a}{x^2} + \dfrac{b}{x} + c\Big)$**.**

The transformation $\xi = 1/x,\ w = y/x$ leads to an equation of the form 14.9.1.5: $w''_{\xi\xi} = f(w + a\xi^2 + b\xi + c)$.

16. $y''_{xx} = \dfrac{1}{ax+by+c} f\Big(\dfrac{ax+by+c}{\alpha x+\beta y+\gamma}\Big)$**.**

This is a special case of equation 14.9.1.18.

1°. For $a\beta - b\alpha = 0$, we have an equation of the form 14.9.1.4.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system of equations

$$ax_0 + by_0 + c = 0, \qquad \alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to a homogeneous equation of the form 14.9.1.7:

$$w''_{zz} = \frac{1}{z} F\Big(\frac{w}{z}\Big), \qquad \text{where} \quad F(\xi) = \frac{1}{a + b\xi} f\Big(\frac{a + b\xi}{\alpha + \beta\xi}\Big).$$

17. $\boldsymbol{y''_{xx} = \dfrac{1}{(ax + by + c)^3} f\Big(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma}\Big).}$

This is a special case of equation 14.9.1.20.

1°. For $a\beta - b\alpha = 0$, we have an equation of the form 14.9.1.4.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system of equations

$$ax_0 + by_0 + c = 0, \qquad \alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to a solvable equation of the form 14.9.1.8:

$$w''_{zz} = \frac{1}{z^3} F\Big(\frac{w}{z}\Big), \qquad \text{where} \quad F(\xi) = \frac{1}{(a + b\xi)^3} f\Big(\frac{a + b\xi}{\alpha + \beta\xi}\Big).$$

18. $\boldsymbol{y''_{xx} = \dfrac{1}{a_1 x + b_1 y + c_1} f\Big(\dfrac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}\Big).}$

Let the following condition be satisfied: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

For $a_2 b_3 - a_3 b_2 \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system of equations

$$a_2 x_0 + b_2 y_0 + c_2 = 0, \qquad a_3 x_0 + b_3 y_0 + c_3 = 0,$$

leads to a homogeneous equation of the form 14.9.1.7:

$$w''_{zz} = \frac{1}{z} F\Big(\frac{w}{z}\Big), \qquad \text{where} \quad F(\xi) = \frac{1}{a_1 + b_1\xi} f\Big(\frac{a_2 + b_2\xi}{a_3 + b_3\xi}\Big).$$

19. $\boldsymbol{y''_{xx} = \dfrac{1}{x^2(ax + by + c)} f\Big(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma}\Big).}$

For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system of equations

$$ax_0 + by_0 + c = 0, \qquad \alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to an equation of the form 14.9.1.14:

$$z(z + x_0)^2 w''_{zz} = F\left(\frac{w}{z}\right), \qquad \text{where} \quad F(\xi) = \frac{1}{a + b\xi} f\left(\frac{a + b\xi}{\alpha + \beta\xi}\right).$$

20. $\boldsymbol{y''_{xx} = \dfrac{1}{(a_1 x + b_1 y + c_1)^3} f\left(\dfrac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}\right).}$

Let the following condition be satisfied: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

For $a_2 b_3 - a_3 b_2 \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system of equations

$$a_2 x_0 + b_2 y_0 + c_2 = 0, \qquad a_3 x_0 + b_3 y_0 + c_3 = 0,$$

leads to a solvable equation of the form 14.9.1.8:

$$w''_{zz} = \frac{1}{z^3} F\left(\frac{w}{z}\right), \qquad \text{where} \quad F(\xi) = \frac{1}{(a_1 + b_1\xi)^3} f\left(\frac{a_2 + b_2\xi}{a_3 + b_3\xi}\right).$$

21. $\boldsymbol{y''_{xx} = y^{-3} f\left(\dfrac{y}{\sqrt{ax^2 + bx + c}}\right).}$

Setting $u(x) = y(ax^2 + bx + c)^{-1/2}$ and integrating the equation, we obtain a first-order separable equation:

$$(ax^2 + bx + c)^2 (u'_x)^2 = (\tfrac{1}{4}b^2 - ac)u^2 + 2\int u^{-3} f(u)\,du + C_1.$$

22. $\boldsymbol{(ax^2 + bx + c)^{3/2} y''_{xx} = f\left(\dfrac{\alpha y + \beta x + \gamma}{\sqrt{ax^2 + bx + c}}\right).}$

Setting $w = \alpha y + \beta x + \gamma$ and denoting $f(z) = \dfrac{1}{\alpha z^3}\varphi(z)$, we obtain an equation of the form 14.9.1.21: $w''_{xx} = w^{-3}\varphi\left(\dfrac{w}{\sqrt{ax^2 + bx + c}}\right)$.

23. $\boldsymbol{(ax^n + b) y''_{xx} = an(n-1)x^{n-2} y + y^{-2} f\left(\dfrac{y}{ax^n + b}\right).}$

The transformation $\xi = \displaystyle\int \frac{dx}{(ax^n + b)^2}$, $w = \dfrac{y}{ax^n + b}$ leads to an autonomous equation of the form 14.9.1.1: $w''_{\xi\xi} = w^{-2} f(w)$.

24. $y''_{xx} = (ax+b)^{-2} y^{-1/2} f(y^{1/2}+cx) + 2c^2.$

The solution is determined by the first-order equation

$$(ax+b)(y'_x + 2c\sqrt{y}\,) = \varphi(cx+\sqrt{y}\,, C), \tag{1}$$

where the function $\varphi = (u, C)$ is the general solution of the Abel equation of the second kind:

$$\varphi\varphi'_u - 2(au+bc)\varphi = 2f(u).$$

Abel equations of this type are discussed in Section 13.3.3. By the change of variable $y^{1/2} = u - cx$, equation (1) is reduced to the form $2(ax+b)(u-cx)u'_x = \varphi(u, C)$.

25. $\sqrt{x^2+2ay}\, y''_{xx} + f\bigl(\sqrt{x^2+2ay} - x\bigr) = 0.$

First integral in implicit form:

$$y'_x = \varphi(u, C), \qquad \sqrt{x^2+2ay} - x = u,$$

where the function $\varphi = (u, C)$ is the general solution of the Abel equation of the second kind

$$(a\varphi - u)\varphi'_u = f(u).$$

With the transformation $a\varphi - u = -w(u)$, this equation is reduced to the canonical form

$$ww'_u - w = af(u).$$

Abel equations of this type are discussed in Section 13.3.1.

⊙ *Literature*: V. F. Zaitsev and L. V. Linchuk (2016).

26. $(x+b)^2\sqrt{x^2+2ay}\, y''_{xx} + f\bigl(\sqrt{x^2+2ay} - x\bigr) = 0.$

First integral in implicit form:

$$(x+b)y'_x - \frac{x}{a}\Bigl(\sqrt{x^2+2ay} - x\Bigr) = \varphi(u, C), \qquad \sqrt{x^2+2ay} - x = u,$$

where the function $\varphi = (u, C)$ is the general solution of the Abel equation of the second kind

$$a(a\varphi - bu)\varphi'_u = (a\varphi - bu)u - af(u).$$

With the transformation $a\varphi(u) - bu = w(u)$, $\frac{1}{1}2u^2 - bu + c = \tau$, this equation is reduced to the canonical form

$$ww'_\tau - w = \mp\frac{f\bigl(b \pm \sqrt{b^2-2c+2\tau}\,\bigr)}{\sqrt{b^2-2c+2\tau}}.$$

Abel equations of this type are discussed in Section 13.3.1.

⊙ *Literature*: V. F. Zaitsev and L. V. Linchuk (2016).

27. $\sqrt{y^2-2ax}\, y''_{xx} + f\bigl(y - \sqrt{y^2-2ax}\,\bigr) = 0.$

First integral in implicit form:

$$y'_x = \varphi(u, C), \qquad y - \sqrt{y^2-2ax} = u.$$

The function $\varphi = (u, C)$ is the general solution of the Abel equation of the second kind

$$(a - u\varphi)\varphi'_u = f(u).$$

Abel equations of this type are discussed in Section 13.3.3.

⊙ *Literature*: V. F. Zaitsev and L. V. Linchuk (2016).

28. $(x+b)^2\sqrt{y^2-2ax}\,y''_{xx}+f\bigl(y-\sqrt{y^2-2ax}\,\bigr)=0.$

First integral in implicit form:

$$a(x+b)y'_x-\frac{a}{2}\left(y+\sqrt{y^2-2ax}\right)=\varphi(u,C),\quad y-\sqrt{y^2-2ax}=u.$$

The function $\varphi=(u,C)$ is the general solution of the Abel equation of the second kind

$$\varphi'_u=\frac{a^2f(u)}{a^2b-u\varphi}+\frac{a}{2}.$$

Abel equations of this type are discussed in Section 13.3.3.

⊙ *Literature*: V. F. Zaitsev and L. V. Linchuk (2016).

29. $y''_{xx}=\dfrac{1}{(x+b)^2\sqrt{a^2+4xy}}\Phi\left(\dfrac{y}{\sqrt{a^2+4xy}-a}\right).$

First integral in implicit form:

$$(x+b)y'_x-\frac{y\sqrt{a^2+4xy}}{\sqrt{a^2+4xy}-a}=\varphi(u,C),\qquad \frac{y}{\sqrt{a^2+4xy}-a}=u.$$

The function $\varphi=(u,C)$ is the general solution of the Abel equation of the second kind

$$\varphi'_u+\frac{2\Phi(u)}{\varphi-4bu^2-au}+a=0.$$

Abel equations of this type are discussed in Section 13.3.3.

⊙ *Literature*: V. F. Zaitsev and L. V. Linchuk (2016).

30. $y''_{xx}=\dfrac{1}{\sqrt{a^2+4xy}}\Phi\left(\dfrac{y}{\sqrt{a^2+4xy}-a}\right).$

First integral in implicit form:

$$y'_x=\varphi(u,C),\qquad \frac{y}{\sqrt{a^2+4xy}-a}=u.$$

The function $\varphi=(u,C)$ is the general solution of the Abel equation of the second kind

$$\varphi'_u+\frac{2\Phi(u)}{\varphi-4u^2}=0.$$

Abel equations of this type are discussed in Section 13.3.3.

⊙ *Literature*: V. F. Zaitsev and L. V. Linchuk (2016).

31. $y''_{xx}=\dfrac{(cx+d)^{n-1}}{(ax+b)^{n+2}}f\left(\dfrac{(ax+b)^ny}{(cx+d)^{n+1}}\right).$

The transformation $\xi=\ln\Bigl(\dfrac{ax+b}{cx+d}\Bigr)$, $w=\dfrac{(ax+b)^ny}{(cx+d)^{n+1}}$ leads to an autonomous equation of the form 14.9.6.2:

$$w''_{\xi\xi}-(2n+1)w'_\xi+n(n+1)w=\Delta^{-2}f(w),\qquad \text{where}\quad \Delta=ad-bc.$$

▶ **Arguments of the arbitrary functions are other functions.**

32. $y''_{xx} = e^{-ax} f(e^{ax} y)$.

1°. The substitution $z = e^{ax} y$ leads to an autonomous equation: $z''_{xx} - 2az'_x + a^2 z = f(z)$.

2°. The transformation $z = e^{ax} y$, $w = y'_x / y$ leads to a first-order equation: $z(w + a)w'_z = z^{-1} f(z) - w^2$.

33. $y''_{xx} = y f(e^{ax} y^m)$.

Equation invariant under "translation–dilatation" transformation. The transformation $z = e^{ax} y^m$, $w = y'_x / y$ leads to a first-order equation: $z(mw + a)w'_z = f(z) - w^2$.

34. $y''_{xx} = x^{-2} f(x^n e^{ay})$.

Equation invariant under "dilatation–translation" transformation. The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to a first-order equation: $z(aw + n)w'_z = f(z) + w$.

35. $y''_{xx} = \lambda^2 y + e^{3\lambda x} f(y e^{\lambda x})$.

This is a special case of equation 14.9.1.46 with $\psi = e^{-\lambda x}$.

36. $y''_{xx} = f(y + ae^{\lambda x} + b) - a\lambda^2 e^{\lambda x}$.

The substitution $w = y + ae^{\lambda x} + b$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w)$.

37. $x^2 y''_{xx} = x^2 f(x^n e^y) + n$.

The substitution $y = w - n \ln x$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(e^w)$.

38. $y''_{xx} = f(y + a \sinh x + b) - a \sinh x$.

The substitution $w = y + a \sinh x + b$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w)$.

39. $y''_{xx} = f(y + a \cosh x + b) - a \cosh x$.

The substitution $w = y + a \cosh x + b$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w)$.

40. $y''_{xx} = \lambda^2 y + (\sinh \lambda x)^{-3} f\left(\dfrac{y}{\sinh \lambda x}\right)$.

This is a special case of equation 14.9.1.46 with $\psi = \sinh \lambda x$.

41. $y''_{xx} = \lambda^2 y + (\cosh \lambda x)^{-3} f\left(\dfrac{y}{\cosh \lambda x}\right)$.

This is a special case of equation 14.9.1.46 with $\psi = \cosh \lambda x$.

42. $x^2 y''_{xx} = x^2 f(y + a \ln x + b) + a$.

The substitution $w = y + a \ln x + b$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w)$.

43. $y''_{xx} = -\dfrac{y}{x^2 \ln x} + \dfrac{1}{(\ln x)^3} f\left(\dfrac{y}{\ln x}\right)$.

This is a special case of equation 14.9.1.46 with $\psi = \ln x$.

44. $y''_{xx} = f(y + a \sin x + b) + a \sin x$.

The substitution $w = y + a \sin x + b$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w)$.

45. $y''_{xx} = f(y + a\cos x + b) + a\cos x.$

The substitution $w = y + a\cos x + b$ leads to an equation of the form 14.9.1.1: $w''_{xx} = f(w)$.

46. $y''_{xx} = -\lambda^2 y + (\sin\lambda x)^{-3} f\Big(\dfrac{y}{\sin\lambda x}\Big).$

This is a special case of equation 14.9.1.46 with $\psi = \sin\lambda x$.

47. $y''_{xx} = -\lambda^2 y + (\cos\lambda x)^{-3} f\Big(\dfrac{y}{\cos\lambda x}\Big).$

This is a special case of equation 14.9.1.46 with $\psi = \cos\lambda x$.

48. $y''_{xx} = 2(\sin x)^{-2} y + (\tan x)^3 f(y\tan x).$

This is a special case of equation 14.9.1.46 with $\psi = \cot x$.

49. $y''_{xx} = 2(\cos x)^{-2} y + (\cot x)^3 f(y\cot x).$

This is a special case of equation 14.9.1.46 with $\psi = \tan x$.

50. $\sin^2 x\, y''_{xx} = n(n+1 - n\sin^2 x)y + \sin^{3n+2} x\, f(y\sin^n x).$

The substitution $x = \xi + \frac{\pi}{2}$ leads to an equation of the form 14.9.1.45:

$$\cos^2\xi\, y''_{\xi\xi} = n(n+1 - n\cos^2\xi)y + \cos^{3n+2}\xi\, f(y\cos^n\xi).$$

51. $\cos^2 x\, y''_{xx} = n(n+1 - n\cos^2 x)y + \cos^{3n+2} x\, f(y\cos^n x).$

The transformation $\xi = \displaystyle\int \cos^{2n} x\, dx$, $w = y\cos^n x$ leads to an autonomous equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

52. $y''_{xx} = \dfrac{\psi''_{xx}}{\psi} y + \psi^{-3} f\Big(\dfrac{y}{\psi}\Big), \qquad \psi = \psi(x).$

The transformation $\xi = \displaystyle\int \frac{dx}{\psi^2}$, $w = \dfrac{y}{\psi}$ leads to an equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

Solution: $\displaystyle\int \Big[C_1 + 2\int f(w)\,dw\Big]^{-1/2} dw = C_2 \pm \int \frac{dx}{\psi^2(x)}.$

53. $y''_{xx} = \varphi^{-3} f\Big(\dfrac{y}{\varphi} + \psi\Big) + \dfrac{\varphi''_{xx}}{\varphi} y - \varphi\psi''_{xx} - 2\varphi'_x\psi'_x, \qquad \varphi = \varphi(x),\ \ \psi = \psi(x).$

The transformation $t = \displaystyle\int \frac{dx}{\varphi^2}$, $w = \dfrac{y}{\varphi} + \psi$ leads to an autonomous equation of the form 14.9.1.1: $w''_{tt} = f(w)$.

Solution: $\displaystyle\int \frac{dw}{\sqrt{2F(w) + C_1}} = \pm\int \frac{dx}{\varphi^2} + C_2$, where $F(w) = \displaystyle\int f(w)\,dw$.

14.9.2 Equations of the Form $F(x,y)y''_{xx} + G(x,y)y'_x + H(x,y) = 0$

► **Argument of the arbitrary functions is x.**

1. $y''_{xx} + 3yy'_x + y^3 + f(x)y = 0.$

The substitution $y = w\Big(\displaystyle\int w\,dx\Big)^{-1}$ leads to a second-order linear equation: $w''_{xx} + f(x)w = 0$.

2. $y''_{xx} + 6yy'_x + 4y^3 + 4f(x)y + f'_x(x) = 0.$

The substitution $y = w'_x/w$ leads to a third-order equation of the form 3.5.3.11: $ww'''_{xxx} + 3w'_x w''_{xx} + 4f(x)ww'_x + f'_x(x)w^2 = 0$.

3. $y''_{xx} + 3fyy'_x + f^2y^3 + f'_x y^2 + g(x)y = 0, \qquad f = f(x).$

The substitution $y = w\left(\int fw\,dx\right)^{-1}$ leads to a second-order linear equation: $w''_{xx} + g(x)w = 0$.

4. $y''_{xx} + [ay + f(x)]y'_x + f'_x(x)y = 0.$

Integrating yields a Riccati equation: $y'_x + f(x)y + \frac{1}{2}ay^2 = C$.

5. $y''_{xx} + [2ay + f(x)]y'_x + af(x)y^2 = g(x).$

On setting $u = y'_x + ay^2$, we obtain $u'_x + f(x)u = g(x)$. Thus, the original equation is reduced to a first-order linear equation and a Riccati equation.

6. $y''_{xx} + [3y + f(x)]y'_x + y^3 + f(x)y^2 + g(x)y + h(x) = 0.$

The substitution $y = u'_x/u$ leads to a third-order linear equation: $u'''_{xxx} + f(x)u''_{xx} + g(x)u'_x + h(x)u = 0$.

7. $y''_{xx} + [y + 3f(x)]y'_x - y^3 + f(x)y^2 + [f'_x(x) + 2f^2(x)]y = 0.$

The transformation

$$y = F(x)w(z), \quad z = \int F(x)\,dx, \quad \text{where} \quad F(x) = \exp\left[-\int f(x)\,dx\right],$$

leads to an autonomous equation of the form 14.2.3.2: $w''_{zz} + ww'_z - w^3 = 0$.

8. $y''_{xx} - (n+1)g(x)y^{n-1}y'_x = f(x)y + g'_x(x)y^n - g^2(x)y^{2n-1}.$

Solution: $y = w\left[C + (1-n)\int g(x)w^{n-1}\,dx\right]^{\frac{1}{1-n}}$, where $w = w(x)$ is the general solution of the second-order linear equation $w''_{xx} = f(x)w$.

9. $xy''_{xx} - ny'_x + f(x)y = ax^{2n+1}y^{-3}.$

The substitution $w = yx^{-n/2}$ leads to Yermakov's equation 14.9.1.2:

$$w''_{xx} + x^{-2}[xf(x) - \tfrac{1}{4}n(n+2)]w = aw^{-3}.$$

10. $y''_{xx} + (2fy + g)y'_x + f'_x y^2 + g'_x y = 0, \qquad f = f(x), \;\; g = g(x).$

Integrating yields a Riccati equation: $y'_x + fy^2 + gy = C$.

11. $y''_{xx} + \left(3fy + \frac{g}{y}\right)y'_x + f^2y^3 + f'_x y^2 + (2fg + h)y + g'_x + \frac{g^2}{y} = 0.$

Here, $f = f(x)$, $g = g(x)$, $h = h(x)$.

Solution: $y = \dfrac{u'_x}{f(x)u}$, where $u = u(x)$ is the general solution of the linear equation:

$$u''_{xx} - \left(\frac{f'_x}{f} + \frac{w'_x}{w}\right)u'_x + fgu = 0,$$

and $w = w(x)$ is the general solution of another linear equation: $w''_{xx} + h(x)w = 0$.

12. $y''_{xx} + [3f(x)y + 2g(x) + h(x)y^{-1}]y'_x + f^2(x)y^3$
$+ [f'_x(x) + 2f(x)g(x)]y^2 + [g'_x(x) + g^2(x) + 2f(x)g(x) - p(x)]y$
$+ h'_x(x) + 2g(x)h(x) + h^2(x)y^{-1} = 0.$

The solution satisfies the Riccati equation $y'_x + f(x)y^2 + [g(x) - z(x,C)]y + h(x) = 0$, where $z = z(x,C)$ is the general solution of another Riccati equation: $z'_x + z^2 = p(x)$.

13. $y''_{xx} + \left(2fy + 2g - \dfrac{f}{y} - \dfrac{2g}{y^2}\right)y'_x + hy + f'_x + \dfrac{f^2 + g'_x}{y} + \dfrac{2fg}{y^2} + \dfrac{g^2}{y^3} = 0.$

Here, $f = f(x)$, $g = g(x)$, and $h = h(x)$.

The solution is determined by the Abel equation of the second kind $yy'_x = (\ln w)'_x\, y^2 - f(x)y - g(x)$, where $w = w(x)$ is the general solution of the linear equation: $w''_{xx} + h(x)w = 0$. Abel equations of the second kind are discussed in Section 13.3.

14. $y''_{xx} - \lambda y'_x + f(x)y = ae^{2\lambda x}y^{-3}.$

The substitution $w = ye^{-\lambda x/2}$ leads to Yermakov's equation 14.9.1.2:

$$w''_{xx} + \left[f(x) - \tfrac{1}{4}\lambda^2\right]w = aw^{-3}.$$

15. $y''_{xx} + g'_x y'_x + fy = ae^{-2g}y^{-3}, \qquad f = f(x),\ g = g(x).$

The substitution $w = ye^{g/2}$ leads to Yermakov's equation 14.9.1.2:

$$w''_{xx} + [f - \tfrac{1}{4}(g'_x)^2 - \tfrac{1}{2}g''_{xx}]w = aw^{-3}.$$

16. $y''_{xx} + \lambda m \tan(\lambda x)y'_x + f(x)y = a[\cos(\lambda x)]^{2m}y^{-3}.$

This is a special case of equation 14.9.2.15 with $g = -m\ln\cos(\lambda x)$.

► **Argument of the arbitrary functions is y.**

17. $y''_{xx} = ay'_x + e^{2ax}f(y).$

Multiplying both sides by e^{-2ax}, we obtain an equation of the form 14.9.2.34.

Solution: $\displaystyle\int\left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = C_2 \pm \frac{1}{a}e^{ax}.$

18. $y''_{xx} = f(y)y'_x.$

Solution: $\displaystyle\int \frac{dy}{F(y) + C_1} = C_2 + x$, where $F(y) = \displaystyle\int f(y)\,dy.$

19. $y''_{xx} = [e^{\alpha x}f(y) + \alpha]y'_x.$

The substitution $w(y) = e^{-\alpha x}y'_x$ leads to a first-order separable equation: $w'_y = f(y)$.

Solution: $\displaystyle\int \frac{dy}{F(y) + C_1} = C_2 + \frac{1}{\alpha}e^{\alpha x}$, where $F(y) = \displaystyle\int f(y)\,dy.$

20. $xy''_{xx} = ny'_x + x^{2n+1}f(y).$

Multiplying both sides by x^{-2n-1}, we obtain an equation of the form 14.9.2.34.

1°. Solution for $n \neq -1$:

$$\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = \pm\frac{x^{n+1}}{n+1} + C_2.$$

2°. Solution for $n = -1$:

$$\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = \pm \ln|x| + C_2.$$

21. $\boldsymbol{xy''_{xx} = f(y)y'_x.}$

The substitution $w(y) = xy'_x/y$ leads to a first-order linear equation: $yw'_y = -w + 1 + f(y)$.

22. $\boldsymbol{xy''_{xx} = [x^k f(y) + k - 1]y'_x.}$

Solution: $\displaystyle\int \frac{dy}{F(y) + C_1} = C_2 + \frac{1}{k}x^k$, where $F(y) = \displaystyle\int f(y)\,dy$.

23. $\boldsymbol{x^2 y''_{xx} + xy'_x = f(y).}$

The substitution $x = \pm e^t$ leads to an equation of the form 14.9.1.1: $y''_{tt} = f(y)$.

24. $\boldsymbol{(ax^2 + b)y''_{xx} + axy'_x + f(y) = 0.}$

The substitution $\xi = \displaystyle\int \frac{dx}{\sqrt{ax^2 + b}}$ leads to an autonomous equation of the form 14.9.1.1: $y''_{\xi\xi} + f(y) = 0$.

25. $\boldsymbol{(ae^{2\lambda x} + b)y''_{xx} + a\lambda e^{2\lambda x} y'_x + f(y) = 0.}$

This is a special case of equation 14.9.2.34 with $g(x) = ae^{2\lambda x} + b$.

26. $\boldsymbol{\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = f(y).}$

This is a special case of equation 14.9.2.34 with $g = \sin x$.

27. $\boldsymbol{\cos x\, y''_{xx} - \frac{1}{2}\sin x\, y'_x = f(y).}$

This is a special case of equation 14.9.2.34 with $g = \cos x$.

► **Other arguments of the arbitrary functions.**

28. $\boldsymbol{y''_{xx} = f(y)y'_x + g(x).}$

Integrating yields a first-order equation: $y'_x = \displaystyle\int f(y)\,dy + \int g(x)\,dx + C$.

29. $\boldsymbol{y''_{xx} + [f(x) + g(y)]y'_x + f'_x(x)y = 0.}$

Integrating yields a first-order equation: $y'_x + f(x)y + \displaystyle\int g(y)\,dy = C$.

30. $\boldsymbol{xy''_{xx} + (n+1)y'_x = x^{n-1} f(yx^n).}$

The transformation $\xi = x^n$, $w = yx^n$ leads to an autonomous equation of the form 14.9.1.1: $n^2 w''_{\xi\xi} = f(w)$.

31. $x^2y''_{xx} + (n+m+1)xy'_x + nmy = x^{n-2m}f(yx^n).$

1°. For $n \neq m$, the transformation $\xi = x^{n-m}$, $w = yx^n$ leads to an autonomous equation of the form 14.9.1.1: $(n-m)^2 w''_{\xi\xi} = f(w)$.

2°. For $n = m$, the transformation $\xi = \ln x$, $w = yx^n$ leads to an autonomous equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

32. $x^2y''_{xx} = f(y/x)(xy'_x - y).$

This is a special case of equation 14.9.3.42 with $g(x) = h(x) = 0$.

33. $x^3y''_{xx} = f(y/x)(xy'_x - y).$

This is a special case of equation 14.9.4.61 with $g(z) = z$.

34. $gy''_{xx} + \frac{1}{2}g'_x y'_x = f(y), \qquad g = g(x).$

Integrating yields a first-order separable equation: $g(x)(y'_x)^2 = 2\int f(y)\,dy + C_1$.

Solution for $g(x) \geq 0$:

$$\int \Bigl[C_1 + 2\int f(y)\,dy\Bigr]^{-1/2} dy = C_2 \pm \int \frac{dx}{\sqrt{g(x)}}.$$

35. $y''_{xx} - \dfrac{\varphi'_x}{\varphi}y'_x + \Bigl(\dfrac{\varphi'_x}{\varphi}\dfrac{\psi'_x}{\psi} - \dfrac{\psi''_{xx}}{\psi}\Bigr)y = \dfrac{\varphi^2}{\psi^3}f\Bigl(\dfrac{y}{\psi}\Bigr), \qquad \varphi = \varphi(x),\ \ \psi = \psi(x).$

The transformation $\xi = \displaystyle\int \frac{\varphi}{\psi^2}\,dx$, $w = \dfrac{y}{\psi}$ leads to an autonomous equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

36. $y''_{xx} = -ay'_x + e^{ax}f(ye^{ax}).$

The transformation $\xi = e^{ax}$, $w = ye^{ax}$ leads to an equation of the form 14.9.1.1: $w''_{\xi\xi} = a^{-2}f(w)$.

37. $y''_{xx} + (\mu+\nu)y'_x + \nu\mu y = e^{(\mu-2\nu)x}f(ye^{\mu x}).$

1°. For $\mu \neq \nu$, the transformation $\xi = e^{(\mu-\nu)x}$, $w = ye^{\mu x}$ leads to an autonomous equation of the form 14.9.1.1: $(\mu-\nu)^2 w''_{\xi\xi} = f(w)$.

2°. For $\mu = \nu$, the substitution $w = ye^{\mu x}$ leads to an autonomous equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

38. $xy''_{xx} - ny'_x - a(ax+n)y = x^{2n+1}e^{3ax}f(ye^{ax}).$

This is a special case of equation 14.9.2.35 with $\varphi = x^n$ and $\psi = e^{-ax}$.

39. $xy''_{xx} = f(x^n e^{ay})y'_x.$

The transformation $z = x^n e^{ay}$, $w = xy'_x$ reduces this equation to a first-order separable equation: $z(aw+n)w'_z = [f(z)+1]w$.

40. $x^2y''_{xx} + xy'_x = f(x^n e^{ay}).$

The transformation $z = x^n e^{ay}$, $w = xy'_x$ reduces this equation to a first-order separable equation: $z(aw+n)w'_z = f(z)$.

41. $x^2y''_{xx} - ax^2y'_x - n(ax+n+1)y = x^{3n+2}e^{2ax}f(yx^n).$

This is a special case of equation 14.9.2.35 with $\varphi = e^{ax}$ and $\psi = x^{-n}$.

42. $y''_{xx} - \frac{\varphi'_x}{\varphi}y'_x - a\left(\frac{\varphi'_x}{\varphi} + a\right)y = e^{3ax}\varphi^2 f\left(ye^{ax}\right), \qquad \varphi = \varphi(x).$

The transformation $\xi = \int \varphi e^{2ax}\,dx$, $w = ye^{ax}$ leads to an equation of the form 14.9.1.1: $w''_{\xi\xi} = f(w)$.

43. $x^2y''_{xx} + xy'_x = f(y + a\ln x + b\ln^2 x).$

The substitution $x = e^t$ leads to an equation of the form 14.9.1.5: $y''_{tt} = f(y + at + bt^2)$.

44. $y''_{xx} - (n+1)\tan x\, y'_x - ny = \cos^{n-2}x\, f(y\cos^n x).$

This is a special case of equation 14.9.2.35 with $\varphi = \cos^{-n-1}x$ and $\psi = \cos^{-n}x$.

45. $y''_{xx} + (m-n)\tan x\, y'_x - n[(m+1)\tan^2 x + 1]y = \cos^{2m+n}x\, f(y\cos^n x).$

This is a special case of equation 14.9.2.35 with $\varphi = \cos^{m-n}x$ and $\psi = \cos^{-n}x$.

46. $y''_{xx} + a\tan x\, y'_x + b(a\tan x - b)y = \cos^{2a}x\, e^{3bx}f(ye^{bx}).$

This is a special case of equation 14.9.2.35 with $\varphi = \cos^a x$ and $\psi = e^{-bx}$.

47. $x^2y''_{xx} + ax^2\tan x\, y'_x + n(ax\tan x - n - 1)y = x^{3n+2}\cos^{2a}x\, f(yx^n).$

This is a special case of equation 14.9.2.35 with $\varphi = \cos^a x$ and $\psi = x^{-n}$.

48. $x^2y''_{xx} - ax^2\cot x\, y'_x - n(ax\cot x + n + 1)y = x^{3n+2}\sin^{2a}x\, f(yx^n).$

This is a special case of equation 14.9.2.35 with $\varphi = \sin^a x$ and $\psi = x^{-n}$.

14.9.3 Equations of the Form $F(x,y)y''_{xx} + \sum\limits_{m=0}^{M} G_m(x,y)(y'_x)^m = 0 \;\; (M = 2,\,3,\,4)$

► **Argument of the arbitrary functions is x.**

1. $y''_{xx} = f(x)(y'_x + ay)^2 + a^2y.$

The substitution $w = y'_x + ay$ leads to a Bernoulli equation: $w'_x = aw + f(x)w^2$.

2. $y''_{xx} = f(x) + g(x)(xy'_x - y) + h(x)(xy'_x - y)^2.$

The substitution $w(x) = xy'_x - y$ leads to a Riccati equation: $w'_x = xf(x) + xg(x)w + xh(x)w^2$.

3. $xy''_{xx} + (a+1)y'_x = f(x)(xy'_x + ay)^2.$

The substitution $w = xy'_x + ay$ leads to a first-order separable equation: $w'_x = f(x)w^2$.

4. $yy''_{xx} - (y'_x)^2 = f(x).$

1°. The substitution $y = ae^{\lambda x}w$ leads to a similar equation: $ww''_{xx} - (w'_x)^2 = a^{-2}e^{-2\lambda x}f(x)$.

2°. The substitutions $y = \pm e^{u/2}$ lead to the equation $u''_{xx} = 2f(x)e^{-u}$. For $f(x) = ke^{\alpha x^2+\beta x}$, see 14.7.1.41.

5. $2yy''_{xx} - (y'_x)^2 + f(x)y^2 + a = 0.$

1°. Differentiating with respect to x, we obtain a third-order linear equation:

$$2y'''_{xxx} + 2f(x)y'_x + f'_x(x)y = 0.$$

2°. The substitution $y = z^2$ leads to Yermakov's equation 14.9.1.2: $z''_{xx} + \frac{1}{4}f(x)z = -\frac{1}{4}az^{-3}$.

3°. If u and v are two solutions of the second-order linear equation $4y''_{xx} + f(x)y = 0$, which satisfy the condition $(uv'_x - u'_x v)^2 = a$, then $y = uv$ is a solution of the original equation.

6. $yy''_{xx} + (y'_x)^2 + f(x)yy'_x + g(x) = 0.$

The substitution $u = y^2$ leads to a linear equation: $u''_{xx} + f(x)u'_x + 2g(x) = 0$.

7. $yy''_{xx} - (y'_x)^2 + f(x)yy'_x + g(x)y^2 = 0.$

The substitution $u = y'_x/y$ leads to a first-order linear equation: $u'_x + f(x)u + g(x) = 0$.

8. $yy''_{xx} - n(y'_x)^2 + f(x)y^2 + ay^{4n-2} = 0.$

1°. For $n = 1$, this is an equation of the form 14.9.3.7.

2°. For $n \neq 1$, the substitution $w = y^{1-n}$ leads to Yermakov's equation 14.9.1.2: $w''_{xx} + (1-n)f(x)w + a(1-n)w^{-3} = 0$.

9. $yy''_{xx} - n(y'_x)^2 + f(x)y^2 + g(x)y^{n+1} = 0.$

The substitution $w = y^{1-n}$ leads to a nonhomogeneous second-order linear equation: $w''_{xx} + (1-n)f(x)w + (1-n)g(x) = 0$.

10. $yy''_{xx} + a(y'_x)^2 + f(x)yy'_x + g(x)y^2 = 0.$

The substitution $w = y^{a+1}$ leads to a linear equation: $w''_{xx} + f(x)w'_x + (a+1)g(x)w = 0$.

11. $yy''_{xx} - 2(y'_x)^2 - (fy + 2g)y'_x + f'_x y^2 + g'_x y = 0, \qquad f = f(x),\ g = g(x).$

Integrating yields a Riccati equation: $y'_x + Cy^2 + fy + g = 0$.

12. $yy''_{xx} - (y'_x)^2 + (fy^2 + g)y'_x + f'_x y^3 - g'_x y = 0, \qquad f = f(x),\ g = g(x).$

Integrating yields a Riccati equation: $y'_x + fy^2 + Cy - g = 0$.

13. $y''_{xx} + (y'_x)^2 + [2fy^2 + 2(f+g)y + g + 2h]y'_x + f^2y^4 + 2fgy^3 + (2f'_x + g^2 + 2fh)y^2 + (g'_x + 2gh)y + h'_x + h^2 - p = 0.$

Here, $f = f(x)$, $g = g(x)$, $h = h(x)$, $p = p(x)$.

The solution satisfies the Riccati equation $y'_x + f(x)y^2 + g(x)y + h(x) - z(x, C) = 0$, where $z = z(x, C)$ is the general solution of another Riccati equation: $z'_x + z^2 = p(x)$.

14. $yy''_{xx} = f(x)(y'_x)^2.$

The substitution $w(x) = xy'_x/y$ leads to a Bernoulli equation 1.1.5:

$$xw'_x = w + [f(x) - 1]w^2.$$

15. $yy''_{xx} + f(x)(y'_x)^2 + g(x)yy'_x + h(x)y^2 = 0.$

The substitution $u = y'_x/y$ leads to a Riccati equation: $u'_x + (1+f)u^2 + gu + h = 0$.

16. $(y+ax)y''_{xx}=f(x)(xy'_x-y)^2.$

The substitution $y=-ax+xz$ leads to the equation $xzz''_{xx}+2zz'_x-x^3f(x)(z'_x)^2=0$. On setting $w=z'_x/z$, we obtain a Bernoulli equation: $xw'_x+2w+[x-x^3f(x)]w^2=0$.

17. $y''_{xx}-a(y'_x)^2+f(x)e^{ay}+g(x)=0.$

The substitution $w=e^{-ay}$ leads to a nonhomogeneous linear equation: $w''_{xx}-ag(x)w=af(x)$.

18. $y''_{xx}-a(y'_x)^2+be^{4ay}+f(x)=0.$

The substitution $w=e^{-ay}$ leads to Yermakov's equation 14.9.1.2: $w''_{xx}-af(x)w=abw^{-3}$.

19. $y''_{xx}=f(x)(y'_x\sinh x-y\cosh x)^2+y.$

The substitution $w=y'_x\sinh x-y\cosh x$ leads to a first-order separable equation: $w'_x=\sinh x\,f(x)w^2$.

20. $y''_{xx}=f(x)(y'_x\cosh x-y\sinh x)^2+y.$

The substitution $w=y'_x\cosh x-y\sinh x$ leads to a first-order separable equation: $w'_x=\cosh x\,f(x)w^2$.

21. $y''_{xx}=f(x)(y'_x\sin x-y\cos x)^2-y.$

The substitution $w=y'_x\sin x-y\cos x$ leads to a first-order separable equation: $w'_x=\sin x\,f(x)w^2$.

22. $y''_{xx}=f(x)(y'_x\cos x+y\sin x)^2-y.$

The substitution $w=y'_x\cos x+y\sin x$ leads to a first-order separable equation: $w'_x=\cos x\,f(x)w^2$.

► **Argument of the arbitrary functions is y.**

23. $y''_{xx}+a(y'_x)^2-\frac{1}{2}y'_x=e^xf(y).$

The substitution $w(y)=e^{-x}(y'_x)^2$ leads to a first-order linear equation: $w'_y+2aw=2f(y)$.

24. $y''_{xx}+\alpha(y'_x)^2=\left[e^{\beta x}f(y)+\beta\right]y'_x.$

Solution:

$$\int\frac{e^{\alpha y}\,dy}{F(y)+C_1}=C_2+\frac{1}{\beta}e^{\beta x},\qquad\text{where}\quad F(y)=\int e^{\alpha y}f(y)\,dy.$$

25. $y''_{xx}+f(y)(y'_x)^2+g(y)=0.$

The substitution $w(y)=(y'_x)^2$ leads to a first-order linear equation: $w'_y+2f(y)w+2g(y)=0$.

26. $y''_{xx}+f(y)(y'_x)^2-\frac{1}{2}y'_x=e^xg(y).$

The substitution $w(y)=e^{-x}(y'_x)^2$ leads to a first-order linear equation: $w'_y+2f(y)w=2g(y)$.

27. $y''_{xx} = xf(y)(y'_x)^3.$

Taking y to be the independent variable, we obtain a linear equation with respect to $x = x(y)$: $x''_{yy} = -f(y)x$.

28. $y''_{xx} + [xf(y) + g(y)](y'_x)^3 + h(y)(y'_x)^2 = 0.$

Taking y to be the independent variable, we obtain a linear equation with respect to $x = x(y)$: $x''_{yy} - h(y)x'_y - f(y)x - g(y) = 0$.

29. $y''_{xx} + f(y)(y'_x)^4 + g(y)(y'_x)^2 + h(y) = 0.$

The substitution $w(y) = (y'_x)^2$ leads to a Riccati equation: $w'_y + 2f(y)w^2 + 2g(y)w + 2h(y) = 0$ (see Section 13.2).

30. $xy''_{xx} + ax(y'_x)^2 - f(y)y'_x = 0.$

Solution:

$$\int \frac{e^{ay}\,dy}{F(y) + C_1} = C_2 + \ln|x|, \qquad \text{where} \quad F(y) = \int e^{ay} f(y)\,dy + \frac{1}{a}e^{ay}.$$

31. $xy''_{xx} + \frac{1}{2}y'_x = xf(y)(y'_x)^2 + g(y).$

The substitution $w(y) = x(y'_x)^2$ leads to a first-order linear equation: $w'_y = 2f(y)w + 2g(y)$.

32. $xy''_{xx} + ax(y'_x)^2 = \left[x^k f(y) + k - 1\right]y'_x.$

Solution:

$$\int \frac{e^{ay}\,dy}{F(y) + C_1} = C_2 + \frac{1}{k}x^k, \qquad \text{where} \quad F(y) = \int e^{ay} f(y)\,dy.$$

33. $x^3 y''_{xx} + [x^4 f(y) + a](y'_x)^3 = 0.$

Taking y to be the independent variable, we obtain an equation of the form 14.9.1.2 for $x = x(y)$: $x''_{yy} - f(y)x - ax^{-3} = 0$.

34. $y''_{xx} = x^{-1}[f(y) + g(y)(xy'_x - y) + h(y)(xy'_x - y)^2]y'_x.$

The substitution $w(y) = xy'_x - y$ leads to a Riccati equation: $w'_y = f(y) + g(y)w + h(y)w^2$.

35. $y''_{xx} + e^{ax} f(y)(y'_x)^3 + ay'_x = 0.$

The substitution $\xi = e^{-ax}$ leads to an equation of the form 14.9.4.36 with $g(z) = az^3$: $y''_{\xi\xi} - af(y)(y'_\xi)^3 = 0$.

36. $y''_{xx} + f(y)(y'_x)^4 + g(y)(y'_x)^2 + h(y) = 0.$

The substitution $w(y) = (y'_x)^2$ leads to a Riccati equation: $w'_y + 2f(y)w^2 + 2g(y)w + 2h(y) = 0$.

37. $xy''_{xx} + x^{2m+1} f(y)(y'_x)^4 + my'_x = 0.$

This is a special case of equation 14.9.4.18 with $n = 4$ and $\varphi = x^{-m}$.

► **Other arguments of arbitrary functions.**

38. $y''_{xx} = f(y)(y'_x)^2 + g(x)y'_x.$

Dividing by y'_x, we obtain an exact differential equation. Its solution follows from the equation:

$$\ln|y'_x| = \int f(y)\,dy + \int g(x)\,dx + C.$$

Solving the latter for y'_x, we arrive at a separable equation. In addition, $y = C_1$ is a singular solution, with C_1 being an arbitrary constant.

39. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(xy'_x - y)y'_x.$

The transformation $z = x^n y^m$, $w = xy'_x/y$ reduces this equation to a first-order separable equation: $z(mw + n)w'_z = [f(z) - 1](w^2 - w)$.

40. $yy''_{xx} = f(e^{ax}y^n)(y'_x)^2.$

The transformation $z = e^{ax}y^n$, $w = y'_x/y$ reduces this equation to a first-order separable equation: $z(nw + a)w'_z = [f(z) - 1]w^2$.

41. $x^2y''_{xx} = f(y/x)(xy'_x - y)(y'_x)^2.$

This is a special case of equation 14.9.4.20 with $k = 2$.

42. $y''_{xx} = x^{-2}(xy'_x - y)\left[f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)y'_x + h\left(\frac{y}{x}\right)(y'_x)^2\right].$

The transformation $z = y/x$, $w = xy'_x/y$ leads to a Riccati equation: $zw'_z = z^2h(z)w^2 + [zg(z) - 1]w + f(z)$.

43. $y''_{xx} = x^{-5/2}\varphi(yx^{-3/2})(y'_x)^4.$

This is a special case of equation 14.9.4.21 with $n = -3/2$, $m = 1$, and $f(z) = z\varphi(z)$.

14.9.4 Equations of the Form $F(x, y, y'_x)y''_{xx} + G(x, y, y'_x) = 0$

► **Arguments of the arbitrary functions depend on x or y.**

1. $y''_{xx} = f(x)(y'_x + ay)^k + a^2y.$

The substitution $w = y'_x + ay$ leads to a Bernoulli equation: $w'_x = aw + f(x)w^k$.

2. $y''_{xx} + f(x)y'_x + g(x)(y'_x)^k = 0.$

The substitution $u(x) = y'_x$ leads to a Bernoulli equation: $u'_x + f(x)u + g(x)u^k = 0$.

3. $y''_{xx} = n(n-1)x^{-2}y + f(x)(xy'_x - ny)^k.$

The substitution $w = x^n y'_x - nx^{n-1}y$ leads to a first-order separable equation: $w'_x = x^{n+k-nk}f(x)w^k$.

4. $y''_{xx} = f(x)(xy'_x - y) + g(x)(xy'_x - y)^k.$

The substitution $w(x) = xy'_x - y$ leads to a Bernoulli equation: $w'_x = xf(x)w + xg(x)w^k$.

5. $xy''_{xx} + (a+1)y'_x = f(x)(xy'_x + ay)^k.$

The substitution $w = xy'_x + ay$ leads to a first-order separable equation: $w'_x = f(x)w^k$.

6. $y''_{xx} = a + f(x)\sqrt{(y'_x)^2 - 2ay}.$

Setting $u = y'_x$, we rewrite the equation as follows: $(u'_x - a)^2[f(x)]^{-2} = u^2 - 2ay$. Differentiating both sides with respect to x and dividing by $(u'_x - a)$, we obtain a second-order linear equation: $fu''_{xx} = f'_x u'_x + f^3 u - af'_x$.

There is also the solution: $y = \frac{1}{2}a(x+C)^2$.

7. $y''_{xx} = f(x)\sqrt{y'_x(xy'_x - y)}.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 14.9.4.40: $w''_{tt} = \dfrac{1}{f(w'_t)\sqrt{tw}}$.

8. $f_1(x)y'_x y''_{xx} + f_2(x)yy''_{xx} + f_3(x)(y'_x)^2 + f_4(x)yy'_x + f_5(x)y^2 = 0.$

The substitution $w(x) = y'_x/y$ leads to the Abel equation

$$(f_1 w + f_2)w'_x + f_1 w^3 + (f_2 + f_3)w^2 + f_4 w + f_5 = 0, \quad \text{where} \quad f_k = f_k(x).$$

9. $y''_{xx} = f(x)(y'_x \sinh x - y\cosh x)^k + y.$

The substitution $w = y'_x \sinh x - y\cosh x$ leads to a first-order separable equation: $w'_x = \sinh x\, f(x)w^k$.

10. $y''_{xx} = f(x)(y'_x \cosh x - y\sinh x)^k + y.$

The substitution $w = y'_x \cosh x - y\sinh x$ leads to a first-order separable equation: $w'_x = \cosh x\, f(x)w^k$.

11. $y''_{xx} = f(x)(y'_x \sin x - y\cos x)^k - y.$

The substitution $w = y'_x \sin x - y\cos x$ leads to a first-order separable equation: $w'_x = \sin x\, f(x)w^k$.

12. $y''_{xx} = f(x)(y'_x \cos x + y\sin x)^k - y.$

The substitution $w = y'_x \cos x + y\sin x$ leads to a first-order separable equation: $w'_x = \cos x\, f(x)w^k$.

13. $y''_{xx} = f(y)(y'_x)^2 + g(y)(y'_x)^k.$

The substitution $w(y) = y'_x$ leads to a Bernoulli equation: $w'_y = f(y)w + g(y)w^{k-1}$.

14. $xy''_{xx} + x^{nm-2m+1}f(y)(y'_x)^n + my'_x = 0.$

This is a special case of equation 14.9.4.18 with $\varphi = x^{-m}$.

15. $y''_{xx} = x^{-1}\big[f(y)(xy'_x - y) + g(y)(xy'_x - y)^k\big]y'_x.$

The substitution $w(y) = xy'_x - y$ leads to a Bernoulli equation: $w'_y = f(y)w + g(y)w^k$.

16. $y''_{xx} = f(y)(xy'_x - y)^{1/2}(y'_x)^2.$

The transformation $x = tw'_t - w$, $y = -w'_t$, where $w = w(t)$, leads to an equation of the form 14.9.4.40: $w''_{tt} = \dfrac{1}{f(-w'_t)\sqrt{tw}}$.

17. $y''_{xx} + e^{a(n-2)x} f(y)(y'_x)^n + a y'_x = 0.$

This is a special case of equation 14.9.4.18 with $\varphi = e^{-ax}$.

18. $y''_{xx} + \varphi^{2-n} f(y)(y'_x)^n - \dfrac{\varphi'_x}{\varphi} y'_x = 0, \qquad \varphi = \varphi(x).$

The substitution $\xi = \displaystyle\int \varphi(x)\,dx$ leads to an equation of the form 14.9.4.36:

$$y''_{\xi\xi} + f(y)(y'_\xi)^n = 0.$$

19. $f y''_{xx} + \frac{1}{2} f'_x y'_x = f g(y)(y'_x)^2 + f^n h(y)(y'_x)^{2n}, \qquad f = f(x).$

The substitution $\xi = \displaystyle\int \frac{dx}{\sqrt{f(x)}}$ leads to an autonomous equation of the form 14.9.4.13: $y''_{\xi\xi} = g(y)(y'_\xi)^2 + h(y)(y'_\xi)^{2n}$.

► **Arguments of the arbitrary functions depend on x and y.**

20. $y''_{xx} = x^{-2} f(y/x)(xy'_x - y)(y'_x)^k.$

The transformation $z = y/x$, $w = xy'_x/y$ leads to a Bernoulli equation: $zw'_z = -w + z^k f(z) w^k$.

There are particular solutions: $y = Cx$ and $y = C_1$ (for $k > 0$).

21. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m}{n+m}}.$

The transformation $z = x^n y^m$, $w = xy'_x/y$ yields: $z(mw+n)w'_z = z^{-\frac{1}{m+1}} f(z) w^{\frac{2n+m}{n+m}} + w - w^2$. We divide both sides of this equation by $w^{\frac{2n+m}{n+m}}$ and introduce the new dependent variable $\zeta = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}$. As a result, we obtain a first-order linear equation:

$$(n+m)z\zeta'_z = -\zeta + z^{-\frac{1}{n+m}} f(z).$$

22. $y''_{xx} = \dfrac{n - kn - km}{km} x^{-1} y'_x + x^{k-1} y^{-k} f(x^n y^m)(y'_x)^{k+1}.$

Passing on to the new variables $z = x^n y^m$ and $w = xy'_x/y$, we arrive at a first-order equation:

$$z(mw+n)w'_z = \frac{n(1-k)}{km} w - w^2 + f(z) w^{k+1}.$$

The substitution $\zeta = \dfrac{m}{1-k} w^{1-k} - \dfrac{n}{k} w^{-k}$ leads to a linear equation: $z\zeta'_z = \dfrac{k-1}{m}\zeta + f(z)$.

23. $y''_{xx} = \dfrac{m + km + kn}{kn} y^{-1}(y'_x)^2 + x^k y^{-k-1} f(x^n y^m)(y'_x)^{k+2}.$

Passing on to the new variables $z = x^n y^m$, $w = xy'_x/y$, we arrive at a first-order equation:

$$z(mw+n)w'_z = w + \frac{m(1+k)}{kn} w^2 + f(z) w^{k+2}.$$

The substitution $\zeta = \dfrac{m}{k} w^{-k} + \dfrac{n}{k+1} w^{-k-1}$ leads to a linear equation: $z\zeta'_z = -\dfrac{k+1}{n}\zeta - f(z)$.

24. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(xy'_x - y)^{\frac{2n+m}{n}}.$

This is a special case of equation 14.9.4.25 with $k = \dfrac{2n+m}{n}$.

25. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m-nk}{n+m}}(xy'_x - y)^k.$

The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to a first-order equation:

$$z(mw+n)w'_z = z^{\frac{k-1}{n+m}} f(z) w^{\frac{2n+m-nk}{n+m}}(w-1)^k + w - w^2.$$

Multiplying both sides by $w^{-\frac{2n+m}{n+m}}$ and passing on to the new variable $\zeta = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}$, we arrive at a Bernoulli equation: $(n+m)z\zeta'_z = -\zeta + z^{\frac{k-1}{n+m}} f(z)\zeta^k$.

26. $y''_{xx} = x^{-2}(xy'_x - y)\Big[f\Big(\dfrac{y}{x}\Big)y'_x + g\Big(\dfrac{y}{x}\Big)(y'_x)^k\Big].$

The transformation $z = y/x$, $u = xy'_x/y$ leads to a Bernoulli equation:

$$zu'_z = [zf(z) - 1]u + z^k g(z) u^k.$$

27. $y''_{xx} = x^{-3}(xy'_x - y)^2 f\Big(\dfrac{y}{x}\Big) + x^{-3}(xy'_x - y)^k g\Big(\dfrac{y}{x}\Big).$

The transformation $x = -1/t$, $y = -w/t$ leads to an autonomous equation of the form 14.9.4.13: $w''_{tt} = f(w)(w'_t)^2 + g(w)(w'_t)^k$.

28. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m-nk}{n+m}}(xy'_x - y)^k + \dfrac{g(x^n y^m)}{xy}y'_x(xy'_x - y).$

The transformation $z = x^n y^m$, $w = xy'_x/y$, followed by the substitution $\zeta = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}$, leads to a Bernoulli equation: $(n+m)z\zeta'_z = [g(z) - 1]\zeta + z^{\frac{k-1}{n+m}} f(z)\zeta^k$.

29. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m}{n+m}} + \dfrac{g(x^n y^m)}{xy}y'_x(xy'_x - y)$

$$+ \frac{h(x^n y^m)}{xy}(y'_x)^{\frac{m}{n+m}}(xy'_x - y)^2.$$

The transformation $z = x^n y^m$, $w = xy'_x/y$, followed by the substitution $\zeta(z) = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}$, leads to a Riccati equation: $(n+m)z\zeta'_z = z^{\frac{1}{n+m}} h(z)\zeta^2 + [g(z) - 1]\zeta + z^{-\frac{1}{n+m}} f(z)$.

30. $y''_{xx} = \dfrac{a}{n}\dfrac{1-k}{2-k}(y'_x)^2 + x^{k-2} f(x^n e^{ay})(y'_x)^k.$

Passing on to the new variables $z = x^n e^{ay}$ and $w = xy'_x$, we have

$$z(aw+n)w'_z = \frac{a}{n}\frac{1-k}{2-k}w^2 + w + f(z)w^k.$$

Multiplying both sides by w^{-k} and introducing the new variable $v = \dfrac{a}{2-k}w^{2-k} + \dfrac{n}{1-k}w^{1-k}$, we obtain a first-order linear equation: $zv'_z = \dfrac{1-k}{n}v + f(z)$.

31. $y''_{xx} = -\dfrac{a}{m}\dfrac{2-k}{1-k}y'_x + y^{1-k}f(e^{ax}y^m)(y'_x)^k.$

Passing on to the new variables $z = e^{ax}y^m$ and $w = y'_x/y$, we have

$$z(mw+a)w'_z = -w^2 - \frac{a}{m}\frac{2-k}{1-k}w + f(z)w^k.$$

Multiplying both sides by w^{-k} and introducing the new variable $v = \dfrac{m}{2-k}w^{2-k} + \dfrac{a}{1-k}w^{1-k}$, we obtain a first-order linear equation: $mzv'_z = (k-2)v + mf(z)$.

32. $y''_{xx} = -\dfrac{a}{m}y'_x \ln\left(\dfrac{y'_x}{y}\right) + f(e^{ax}y^m)y'_x.$

The transformation $z = e^{ax}y^m$, $w = y'_x/y$ leads to a first-order equation:

$$z(mw+a)w'_z = -\frac{a}{m}w\ln w - w^2 + f(z)w.$$

Dividing both sides by w and passing on to the new variable $v = mw + a\ln w$, we obtain a first-order linear equation: $mzv'_z = -v + mf(z)$.

33. $y''_{xx} = x^{-2}f(x^n e^{ay})\exp\left(-\dfrac{a}{n}xy'_x\right).$

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to the first-order equation

$$z(aw+n)w'_z = w + f(z)\exp\Bigl(-\frac{a}{n}w\Bigr),$$

which can be reduced, with the aid of the substitution $\zeta = w\exp\Bigl(\dfrac{a}{n}w\Bigr)$, to a linear equation: $nz\zeta'_z = \zeta + f(z)$.

34. $y''_{xx} = -\dfrac{a}{n}(y'_x)^2\ln(xy'_x) + f(x^n e^{ay})(y'_x)^2.$

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to the equation

$$z(aw+n)w'_z = w - \frac{a}{n}w^2\ln w + w^2 f(z).$$

Dividing both sides by w^2 and passing on to the new variable $v = a\ln w - nw^{-1}$, we obtain a first-order linear equation: $nzv'_z = -v + nf(z)$.

► Arguments of the arbitrary functions depend on x, y, and y'_x.

35. $y''_{xx} = f(x)g(y'_x).$

The substitution $u(x) = y'_x$ leads to a first-order separable equation: $u'_x = f(x)g(u)$.

36. $y''_{xx} = f(y)g(y'_x).$

The substitution $u(y) = y'_x$ leads to a first-order separable equation: $uu'_y = f(y)g(u)$.

In addition, there may exist solutions of the form $y = Ax + C$, where A are roots of the equation $g(A) = 0$, C is an arbitrary number, or $y = B$, where B are roots of the equation $f(B) = 0$.

37. $y''_{xx} = f(ax + by + c)g(y'_x).$

For $b = 0$, we have an equation of the form 14.9.4.35. For $b \neq 0$, the substitution $u(x) = y + (ax + c)/b$ leads to an equation of the form 14.9.4.36: $u''_{xx} = f(bu)g\Bigl(u'_x - \dfrac{a}{b}\Bigr)$.

38. $y''_{xx} = x^{-n-1} f(x^n y'_x).$

The substitution $w(x) = x^n y'_x$ leads to a first-order separable equation: $xw'_x = f(w) + nw$.

39. $y''_{xx} = y^{-2n-1} f(y^n y'_x).$

The substitution $w(y) = y^n y'_x$ leads to a first-order separable equation: $yww'_y = f(w) + nw^2$.

40. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{xy}}.$

Setting $u = y'_x$ and passing on to the new variables $t = \displaystyle\int \frac{du}{f(u)}$ and $w = 2\sqrt{x}$, we have $y = (w'_t)^2$. Differentiating the latter with respect to x, we obtain a second-order linear equation: $w''_{tt} = g(t)w$. Here, the function $g(t)$ is defined parametrically: $g = \frac{1}{4}u$, $t = \displaystyle\int \frac{du}{f(u)}$.

41. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy + b}}.$

Setting $u = y'_x$, we rewrite the equation as follows: $\bigl[\sqrt{x}\, u'_x / f(u)\bigr]^{-2} = ay + bx^{-1}$. Differentiating both sides with respect to x and passing on to the new variables $t = \dfrac{1}{2}\displaystyle\int \frac{du}{f(u)}$, $z = \sqrt{x}$, we obtain an equation of the form 14.9.1.2: $z''_{tt} = au(t)z - bz^{-3}$.

42. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{ay + bx^2}}.$

Setting $u = y'_x$, we rewrite the equation as follows: $\bigl[u'_x / f(u)\bigr]^{-2} = ay + bx^2$. Differentiating both sides with respect to x and passing on to the new variable $t = \displaystyle\int \frac{du}{f(u)}$, we obtain a second-order linear equation for $x = x(t)$ integrable by quadrature: $2x''_{tt} = 2bx + au(t)$. Here, the function $u = u(t)$ is defined implicitly: $t = \displaystyle\int \frac{du}{f(u)}$.

43. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{ax + by^2}}.$

Taking y to be the independent variable, we obtain an equation of the form 14.9.4.42 for $x = x(y)$: $x''_{yy} = -(ax + by^2)^{-1/2} f(1/x'_y)(x'_y)^3$.

44. $y''_{xx} = (ax^2 + bxy + cy^2 + \alpha x + \beta y + \gamma)^{-1/2} f(y'_x).$

The transformation $x = At + Bu + C$, $y = Dt + Pu + Q$, where $u = u(t)$, reduces this equation by selecting appropriate constants A, B, C, D, P, and Q, to an equation of the form 14.9.4.41, 2.9.4.42, or 2.9.4.43.

45. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy + bx^{3/2} + cx}}.$

Setting $u = y'_x$, we rewrite the equation as follows: $\big[\sqrt{x}\,u'_x/f(u)\big]^{-2} = ay + b\sqrt{x} + c$. Differentiating both sides with respect to x and passing to the new variables $t = \dfrac{1}{2}\displaystyle\int \frac{du}{f(u)}$ and $z = \sqrt{x}$, we obtain a second-order linear equation: $2z''_{tt} = 2au(t)z + b$. Here, the function $u = u(t)$ is defined implicitly: $t = \dfrac{1}{2}\displaystyle\int \frac{du}{f(u)}$.

46. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy + by^{3/2} + cy}}.$

Taking y to be the independent variable, we obtain an equation of the form 14.9.4.45 for $x = x(y)$: $x''_{yy} = -(axy + by^{3/2} + cy)^{-1/2} f(1/x'_y)(x'_y)^3$.

47. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy + bx^2 + cx^{3/2} + dx}}.$

The substitution $aw = ay + bx$ leads to an equation of the form 14.9.4.45 for $w = w(x)$: $w''_{xx} = \dfrac{f(w'_x - b/a)}{\sqrt{axw + cx^{3/2} + dx}}$.

48. $y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy + by^2 + cy^{3/2} + dy}}.$

Taking y to be the independent variable, we obtain an equation of the form 14.9.4.47 for $x = x(y)$: $x''_{yy} = -(axy + by^2 + cy^{3/2} + dy)^{-1/2} f(1/x'_y)(x'_y)^3$.

49. $y''_{xx} = x^{-2}(xy'_x - y)f(y'_x).$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ ($y'_x = t$, $y''_{xx} = 1/w''_{tt}$) leads to an equation of the form 14.9.3.14: $ww''_{tt} = [f(t)]^{-1}(w'_t)^2$.

50. $y''_{xx} = \big[xf(y'_x) + yg(y'_x) + h(y'_x)\big]^{-1}.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ ($y'_x = t$, $y''_{xx} = 1/w''_{tt}$) leads to a second-order linear equation: $w''_{tt} = [f(t) + tg(t)]w'_t - g(t)w + h(t)$.

51. $(xy'_x + ay'_x - y + b)y''_{xx} = f(y'_x).$

The contact transformation

$$X = y'_x, \quad Y = xy'_x + ay'_x - y + b, \quad Y'_X = x + a, \quad Y''_{XX} = 1/y''_{xx},$$

where $Y = Y(X)$, leads to a linear equation: $f(X)Y''_{XX} - Y = 0$.

Inverse transformation:

$$x = Y'_X - a, \quad y = XY'_X - Y + b, \quad y'_x = X, \quad y''_{xx} = 1/Y''_{XX}.$$

52. $f(y'_x)y''_{xx} + g(y)y'_x + h(x) = 0.$

Integrating yields a first-order equation:

$$\int f(u)\,du + \int g(y)\,dy + \int h(x)\,dx = C, \qquad \text{where} \quad u = y'_x.$$

53. $y''_{xx} = a^2 y + f(x)g(y'_x + ay).$

The substitution $w = y'_x + ay$ leads to a first-order equation: $w'_x = aw + f(x)g(w)$.

54. $f(y'_x + ax)y''_{xx} + (y'^2_x + 2ay)(y''_{xx} + a) = 0.$

The contact transformation

$$X = y'_x + ax, \quad Y = \tfrac{1}{2}(y'_x)^2 + ay, \quad Y'_X = y'_x, \quad Y''_{XX} = \frac{y''_{xx}}{y''_{xx} + a},$$

where $Y = Y(X)$, leads to a linear equation: $f(X)Y''_{XX} + 2Y = 0$.

Inverse transformation:

$$x = \frac{1}{a}(X - Y'_X), \quad y = \frac{1}{2a}\bigl[2Y - (Y'_X)^2\bigr], \quad y'_x = Y'_X.$$

55. $y''_{xx} = x^{-1} f\Bigl(y'_x - \dfrac{y}{x}\Bigr).$

The substitution $w = y'_x - \dfrac{y}{x}$ leads to a first-order separable equation: $xw'_x = -w + f(w)$.

56. $(xy'_x - y)(x^2 y''_{xx} + axy'_x - ay) = x^2 f\Bigl(y'_x + a\dfrac{y}{x}\Bigr).$

The contact transformation $(a \neq -1)$

$$X = y'_x + a\frac{y}{x}, \quad Y = x^{a+1}y'_x - x^a y, \quad Y'_X = x^{a+1}, \quad Y''_{XX} = \frac{(a+1)x^{a+2}}{x^2 y''_{xx} + axy'_x - ay}$$

leads to a linear equation: $f(X)Y''_{XX} = (a+1)Y$.

Inverse transformation:

$$x = (Y'_X)^{\frac{1}{a+1}}, \quad y = \frac{1}{a+1}(XY'_X - Y)(Y'_X)^{-\frac{a}{a+1}}, \quad y'_x = \frac{XY'_X + aY}{(a+1)Y'_X}.$$

57. $y''_{xx} = x^{-3} f(xy'_x - y)(y'_x)^{-1}.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 14.9.3.27: $w''_{tt} = t[f(w)]^{-1}(w'_t)^3$.

58. $y''_{xx} = f(x)g(xy'_x - y).$

The substitution $w = xy'_x - y$ leads to a first-order separable equation: $w'_x = xf(x)g(w)$.

59. $y''_{xx} = x^{-n-3} y^n f(xy'_x - y).$

The transformation $x = 1/t$, $y = w/t$ leads to an autonomous equation of the form 14.9.4.36: $w''_{tt} = w^n f(-w'_t)$.

60. $y''_{xx} = x^{-1} f(y)g(xy'_x - y)y'_x.$

The substitution $w(y) = xy'_x - y$ leads to a first-order separable equation: $w'_y = f(y)g(w)$.

61. $y''_{xx} = x^{-3} f(y/x) g(xy'_x - y).$

The transformation $x = 1/t$, $y = w/t$ leads to an autonomous equation of the form 14.9.4.36: $w''_{tt} = f(w)g(-w'_t)$.

62. $x^3 y'_x y''_{xx} + 2x^2 (y'_x)^2 = f(xy'_x + y).$

The contact transformation

$$X = xy'_x + y, \quad Y = x^2 y'_x, \quad Y'_X = x, \quad Y''_{XX} = \frac{1}{xy''_{xx} + 2y'_x},$$

where $Y = Y(X)$, leads to a linear equation: $f(X)Y''_{XX} = Y$.

Inverse transformation:

$$x = Y'_X, \quad y = X - \frac{Y}{Y'_X}, \quad y'_x = \frac{Y}{(Y'_X)^2}, \quad y''_{xx} = \frac{1}{Y'_X Y''_{XX}} - \frac{2Y}{(Y'_X)^3}.$$

63. $y''_{xx} = \frac{y}{x^2} f\left(\frac{xy'_x}{y}\right).$

The substitution $w(x) = xy'_x/y$ leads to a first-order separable equation: $xw'_x = f(w) + w - w^2$.

64. $y''_{xx} = y^{-1}(y'_x)^2 - x^{-1}y'_x + x^{-2} y f(x^n y^m) g\left(\frac{xy'_x}{y}\right).$

The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to a first-order separable equation: $z(mw + n)w'_z = f(z)g(w)$.

65. $y''_{xx} = n(n-1)x^{-2}y + f(x)g(x^n y'_x - nx^{n-1}y).$

The substitution $w = x^n y'_x - nx^{n-1}y$ leads to a first-order separable equation: $w'_x = x^n f(x)g(w)$.

66. $x^2 y''_{xx} + axy'_x - ay = x^{a+2} f\big(x^{a+1}y'_x - x^a y\big).$

The contact transformation $(a \neq -1)$

$$X = y'_x + a\frac{y}{x}, \quad Y = x^{a+1}y'_x - x^a y, \quad Y'_X = x^{a+1}, \quad Y''_{XX} = \frac{(a+1)x^{a+2}}{x^2 y''_{xx} + axy'_x - ay}$$

leads to an autonomous equation of the form 14.9.1.1: $f(Y)Y''_{XX} = a + 1$.

Inverse transformation:

$$x = (Y'_X)^{\frac{1}{a+1}}, \quad y = \frac{1}{a+1}(XY'_X - Y)(Y'_X)^{-\frac{a}{a+1}}, \quad y'_x = \frac{XY'_X + aY}{(a+1)Y'_X}.$$

67. $y''_{xx} = -x^{-1}y'_x + x^{-2} f(x^n e^{ay}) g(xy'_x).$

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to a first-order separable equation:

$$z(aw + n)w'_z = f(z)g(w).$$

68. $y''_{xx} = y^{-1}(y'_x)^2 + y f(e^{ax} y^m) g(y'_x/y).$

The transformation $z = e^{ax}y^m$, $w = y'_x/y$ leads to a first-order separable equation: $z(mw + a)w'_z = f(z)g(w)$.

69. $(y'_x + y)(y''_{xx} + y'_x) = e^{-2x} f(e^x y'_x)$.

The contact transformation

$$X = e^x y'_x, \quad Y = y'_x + y, \quad Y'_X = e^{-x}, \quad Y''_{XX} = -\frac{e^{-2x}}{y''_{xx} + y'_x},$$

where $Y = Y(X)$, leads to a linear equation: $f(X)Y''_{XX} = -Y$.

Inverse transformation:

$$x = -\ln Y'_X, \quad y = Y - XY'_X, \quad y'_x = XY'_X.$$

70. $y''_{xx} - y'_x = e^{2x} f\big(e^x y'_x - e^x y\big)\big(y''_{xx} - y\big)$.

1°. The substitution $w = y'_x - y$ leads to a first-order equation: $w'_x = e^{2x} f(e^x w)(w'_x + w)$.

2°. The contact transformation

$$X = e^x(y'_x - y), \quad Y = (y'_x)^2 - y^2, \quad Y'_X = 2e^{-x} y'_x, \quad Y''_{XX} = 2e^{-2x}\frac{y''_{xx} - y'_x}{y''_{xx} - y}$$

leads to a linear equation: $Y''_{XX} = 2f(X)$.

71. $y''_{xx} = f(x)g(y'_x \sinh x - y\cosh x) + y$.

The substitution $w = y'_x \sinh x - y\cosh x$ leads to a first-order separable equation: $w'_x = \sinh x\, f(x)g(w)$.

72. $y''_{xx} = f(x)g(y'_x \cosh x - y\sinh x) + y$.

The substitution $w = y'_x \cosh x - y\sinh x$ leads to a first-order separable equation: $w'_x = \cosh x\, f(x)g(w)$.

73. $y''_{xx} = f(x)g(y'_x \sin x - y\cos x) - y$.

The substitution $w = y'_x \sin x - y\cos x$ leads to a first-order separable equation: $w'_x = \sin x\, f(x)g(w)$.

74. $y''_{xx} = f(x)g(y'_x \cos x + y\sin x) - y$.

The substitution $w = y'_x \cos x + y\sin x$ leads to a first-order separable equation: $w'_x = \cos x\, f(x)g(w)$.

75. $gy''_{xx} + \frac{1}{2}g'_x y'_x = f(y)h\big(y'_x\sqrt{g}\,\big), \qquad g = g(x)$.

The substitution $w(y) = y'_x\sqrt{g}$ leads to a first-order separable equation: $ww'_y = f(y)h(w)$.

76. $f\big(ay'^2_x - bx\big)y''_{xx} + \big(2ay'^3_x - 3by\big)\big(2ay'_x y''_{xx} - b\big) = 0, \qquad b \neq 0$.

The contact transformation

$$X = a(y'_x)^2 - bx, \quad Y = 2a(y'_x)^3 - 3by, \quad Y'_X = 3y'_x, \quad Y''_{XX} = \frac{3y''_{xx}}{2ay'_x y''_{xx} - b}$$

leads to a linear equation: $f(X)Y''_{XX} + 3Y = 0$.

Inverse transformation:

$$x = \frac{a}{9b}(Y'_X)^2 - \frac{1}{b}X, \quad y = \frac{2a}{81b}(Y'_X)^3 - \frac{1}{3b}Y, \quad y'_x = \frac{1}{3}Y'_X, \quad y''_{xx} = \frac{3bY''_{XX}}{2aY'_X Y''_{XX} - 9}.$$

77. $y''_{xx} = f\big(y'^2_x + ay\big)$.

The change of variable $w(y) = (y'_x)^2 + ay$ results in a first-order separable equation: $w'_y = 2f(w) + a$.

78. $y''_{xx} = f(y)g(y'^2_x + 2ay) - a.$

The change of variable $w(y) = (y'_x)^2 + 2ay$ results in a first-order separable equation: $w'_y = 2f(y)g(w)$.

79. $y''_{xx} = f(y'^2_x - 2axy'_x + 2ay).$

The substitution $y = w + \frac{1}{2}ax^2$ leads to an autonomous equation of the form 14.9.4.78: $w''_{xx} = f(w'^2_x + 2aw) - a$.

80. $y''_{xx} - y'_x = e^{2x}f(y'^2_x - y^2)(y''_{xx} - y).$

The contact transformation

$$X = e^x(y'_x - y), \quad Y = (y'_x)^2 - y^2, \quad Y'_X = 2e^{-x}y'_x, \quad Y''_{XX} = 2e^{-2x}\frac{y''_{xx} - y'_x}{y''_{xx} - y}$$

leads to an autonomous equation of the form 14.9.1.1: $Y''_{XX} = 2f(Y)$.

81. $xy''_{xx} + \frac{1}{2}y'_x = f(xy'^2_x + ay).$

The change of variable $w(y) = x(y'_x)^2 + ay$ leads to a first-order separable equation: $w'_y = 2f(w) + a$.

82. $y''_{xx} + \frac{1}{2}y'_x = e^{-x}f(e^x y'^2_x + ay).$

The substitution $w(y) = e^x(y'_x)^2 + ay$ leads to a first-order separable equation: $w'_y = 2f(w) + a$.

83. $f(a(y'_x)^k - bx)y''_{xx} = [ak(y'_x)^{k+1} - b(k+1)y][ak(y'_x)^{k-1}y''_{xx} - b].$

The contact transformation ($ab \neq 0$, $k \neq -1$)

$$X = a(y'_x)^k - bx, \quad Y = ak(y'_x)^{k+1} - b(k+1)y, \quad Y'_X = (k+1)y'_x, \quad Y''_{XX} = \frac{(k+1)y''_{xx}}{ak(y'_x)^{k-1}y''_{xx} - b}$$

leads to a linear equation: $f(X)Y''_{XX} = (k+1)Y$.

Inverse transformation:

$$x = \frac{a(Y'_X)^k}{b(k+1)^k} - \frac{X}{b}, \quad y = \frac{ak(Y'_X)^{k+1}}{b(k+1)^{k+2}} - \frac{Y}{b(k+1)}, \quad y'_x = \frac{Y'_X}{k+1}.$$

84. $\varphi y''_{xx} + \frac{1}{2}\varphi'_x y'_x = f(\varphi y'^2_x + ay), \qquad \varphi = \varphi(x).$

The substitution $w(y) = \varphi(x)(y'_x)^2 + ay$ leads to a first-order separated equation: $w'_y = 2f(w) + a$.

85. $\varphi y''_{xx} + \frac{1}{2}\varphi'_x y'_x = f(y)g(\varphi y'^2_x + 2ay) - a, \qquad \varphi = \varphi(x).$

The substitution $w(y) = \varphi(x)(y'_x)^2 + 2ay$ leads to a first-order separable equation: $w'_y = 2f(y)g(w)$.

86. $y''_{xx} = x^n y^m (y'_x)^{\frac{2n+m+3}{n+m+2}} F(\zeta), \qquad \zeta = (xy'_x - y)(y'_x)^{-\frac{n+1}{n+m+2}}.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 14.9.4.25:

$$w''_{tt} = \frac{1}{tw}\left[\frac{t^a w}{F(t^a w)}\right](w'_t)^{\frac{2a+1-ab}{a+1}}(tw'_t - w)^b, \quad \text{where} \quad a = -\frac{n+1}{n+m+2}, \quad b = -m.$$

14.9.5 Equations Not Solved for Second Derivative

1. $f(x)(y''_{xx} - a)^2 = (y'_x)^2 - 2ay + b.$

Differentiating with respect to x, we obtain

$$(y''_{xx} - a)(2fy'''_{xxx} + f'_x y''_{xx} - 2y'_x - af'_x) = 0. \tag{1}$$

Equating the second factor to zero and making the transformation $\xi = \int \frac{dx}{\sqrt{f}}$, $w = y'_x$, we arrive at a second-order constant coefficient linear equation of the form 14.1.9.1:

$$w''_{\xi\xi} - w = \tfrac{1}{2}af'_x, \tag{2}$$

whose right-hand side is to be expressed in terms of ξ. Substituting the solution of equation (2) into the original one, we obtain a relation connecting integration constants.

Equating the first factor in (1) to zero, we find the singular solution:

$$y = \frac{1}{2}a(x+C)^2 + \frac{b}{2a}.$$

2. $f(x)(y''_{xx} - ay)^2 = (y'_x)^2 - ay^2 + b.$

Differentiating with respect to x, we obtain

$$(y''_{xx} - ay)[2f(y'''_{xxx} - ay'_x) + f'_x(y''_{xx} - ay) - 2y'_x] = 0.$$

Equating the second factor to zero, we arrive at a third-order linear equation:

$$2f(y'''_{xxx} - ay'_x) + f'_x(y''_{xx} - ay) - 2y'_x = 0.$$

Equating the first factor to zero, one can find the singular solution.

3. $x = f(y''_{xx}).$

The substitution $w(x) = y'_x$ leads to an equation of the form 13.8.1.7: $x = f(w'_x)$.

4. $y = f(y''_{xx}).$

The substitution $w(y) = \frac{1}{2}(y'_x)^2$ leads to an equation of the form 13.8.1.8: $y = f(w'_y)$.

5. $y = ax^2 + bx + c + f(y''_{xx}).$

The substitution $w = y - ax^2 - bx - c$ leads to an equation of the form 14.9.5.4: $w = f(w''_{xx} + 2a)$.

6. $xy'_x = y + a(y'_x)^2 + by'_x + c + f(y''_{xx}).$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ ($y'_x = t$, $y''_{xx} = 1/w''_{tt}$) leads to an equation of the form 14.9.5.5: $w = at^2 + bt + c + f(1/w''_{tt})$.

7. $f(y''_{xx}) + xy''_{xx} = y'_x.$

Solution: $y = \frac{1}{2}C_1x^2 + xf(C_1) + C_2$.

8. $f(y''_{xx}+y)=(y''_{xx})^2+(y'_x)^2.$

Differentiating with respect to x, we obtain

$$[f'(y''_{xx}+y)-2y''_{xx}](y'''_{xxx}+y'_x)=0.$$

From the equation $y'''_{xxx}+y'_x=0$, it follows that:

$$y=A\sin(x+C_1)+C_2,\qquad \text{where}\quad A^2=f(C_2).$$

Equating the expression in square brackets to zero, we arrive at the singular solution in parametric form:

$$x=\int\frac{[2-f''_{uu}(u)]\,du}{\sqrt{4f(u)-\bigl[f'_u(u)\bigr]^2}},\qquad y=u-\tfrac{1}{2}f'_u(u).$$

9. $y'_x=yf(y''_{xx}/y).$

The transformation $t=y^2$, $w=(y'_x)^2$ leads to an equation of the form 13.8.1.11: $w=tf^2(w'_t)$.

10. $yy''_{xx}=(y'_x)^2+f(y''_{xx}/y).$

1°. Solution:

$$y=C_1\exp(C_2x)+\frac{f(C_2^2)}{4C_1C_2^2}\exp(-C_2x).$$

2°. Solution:

$$y=C_1\sin(C_3x)+C_2\cos(C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$(C_1^2+C_2^2)C_3^2+f(-C_3^2)=0.$$

3°. Solutions: $y=\pm x\sqrt{-f(0)}+C$.

11. $y=xf(x^3y''_{xx}).$

The transformation $x=1/t$, $y=w/t$ leads to an equation of the form 14.9.5.4: $w=f(w''_{tt})$.

12. $xy'_x-y=f(x^ny''_{xx}).$

This is a special case of equation 14.9.5.16 with $\varphi=x^n$.

13. $xy'_x-y=f(e^{\lambda x}y''_{xx}).$

This is a special case of equation 14.9.5.16 with $\varphi=e^{\lambda x}$.

14. $xy'_x-y=f(\ln x\,y''_{xx}).$

This is a special case of equation 14.9.5.16 with $\varphi=\ln x$.

15. $xy'_x-y=f(\sin x\,y''_{xx}).$

This is a special case of equation 14.9.5.16 with $\varphi=\sin x$.

16. $xy'_x-y=f(\varphi y''_{xx}),\qquad \varphi=\varphi(x).$

The transformation $\xi=\displaystyle\int\frac{x}{\varphi}\,dx$, $w=xy'_x-y$ leads to an equation of the form 13.8.1.8: $w=f(w'_\xi)$.

14.9.6 Equations of General Form

► **Equations solved for the y''_{xx} containing arbitrary functions of two variables.**

1. $y''_{xx} = F(x,\, y'_x)$.

The substitution $w(x) = y'_x$ leads to a first-order equation: $w'_x = F(x, w)$.

2. $y''_{xx} = F(y,\, y'_x)$.

Autonomous equation. The substitution $w(y) = y'_x$ leads to a first-order equation: $ww'_y = F(y,\, w)$.

3. $y''_{xx} = F(ax + by,\, y'_x)$.

The substitution $bw = ax + by$ leads to an equation of the form 14.9.6.2: $w''_{xx} = F\Bigl(bw,\, w'_x - \dfrac{a}{b}\Bigr)$.

4. $y''_{xx} = \frac{1}{x} F\Bigl(\frac{y}{x},\, y'_x\Bigr)$.

Homogeneous equation. This is a special case of equation 14.9.6.6 with $k = 1$.

5. $y''_{xx} = \dfrac{1}{ax + by + c} F\Bigl(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma},\, y'_x\Bigr)$.

1°. For $a\beta - b\alpha = 0$, the substitution $bw = ax + by + c$ leads to an autonomous equation of the form 14.9.6.2.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system of equations

$$ax_0 + by_0 + c = 0, \qquad \alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to a homogeneous equation of the form 14.9.6.4:

$$w''_{zz} = \frac{1}{z} \Phi\Bigl(\frac{w}{z},\, w'_z\Bigr), \qquad \text{where} \quad \Phi(\xi, u) = \frac{1}{a + b\xi} F\Bigl(\frac{a + b\xi}{\alpha + \beta\xi},\, u\Bigr).$$

6. $y''_{xx} = x^{k-2} F(x^{-k} y,\, x^{1-k} y'_x)$.

Generalized homogeneous equation. The transformation $t = \ln x$, $w = x^{-k} y$ leads to an equation of the form 14.9.6.2: $w''_{tt} + (2k-1)w'_t + k(k-1)w = F(w,\, w'_t + kw)$.

7. $y''_{xx} = \frac{y}{x^2} F\Bigl(x^n y^m,\, \frac{x}{y} y'_x\Bigr)$.

Generalized homogeneous equation. The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to a first-order equation: $z(mw + n)w'_z = F(z, w) + w - w^2$.

8. $y''_{xx} = a^2 y + F(x, y'_x + ay)$.

The substitution $w = y'_x + ay$ leads to a first-order equation: $w'_x = aw + F(x, w)$.

9. $y''_{xx} = (a^2 x^2 + a)y + F(x, y'_x - axy)$.

The substitution $w = y'_x - axy$ leads to a first-order equation: $w'_x = -axw + F(x, w)$.

10. $y''_{xx} = F\left(x,\, y'_x - \frac{y}{x}\right)$.

The substitution $w(x) = y'_x - \dfrac{y}{x}$ leads to a first-order equation: $xw'_x = -w + xF(x, w)$.

11. $y''_{xx} = F(x,\, xy'_x - y)$.

The substitution $w(x) = xy'_x - y$ leads to a first-order equation: $w'_x = xF(x, w)$.

12. $y''_{xx} = x^{-2}F(y,\, xy'_x - y)$.

The substitution $w(y) = xy'_x - y$ leads to a first-order equation: $(y + w)w'_y = F(y, w)$.

13. $xy''_{xx} + (a + 1)y'_x = F(x, xy'_x + ay)$.

The substitution $w = xy'_x + ay$ leads to a first-order equation: $w'_x = F(x, w)$.

14. $x^2y''_{xx} = 2y + F(x, xy'_x + y)$.

The substitution $w = xy'_x + y$ leads to a first-order equation: $xw'_x = 2w + F(x, w)$.

15. $x^2y''_{xx} = a(a + 1)y + F(x, xy'_x + ay)$.

The substitution $w = xy'_x + ay$ leads to a first-order equation: $xw'_x = (a+1)w + F(x, w)$.

16. $y''_{xx} = 2ayy'_x + F(x, y'_x - ay^2)$.

The substitution $w = y'_x - ay^2$ leads to a first-order equation: $w'_x = F(x, w)$.

17. $y''_{xx} = e^{-ax}F(e^{ax}y,\, e^{ax}y'_x)$.

The substitution $w = e^{ax}y$ leads to a second-order autonomous equation of the form 14.9.6.2: $w''_{xx} - 2aw'_x + a^2w = F(w,\, w'_x - aw)$.

18. $y''_{xx} = yF(e^{ax}y^m,\, y'_x/y)$.

Equation invariant under "translation–dilatation" transformation. The transformation $z = e^{ax}y^m$, $w = y'_x/y$ leads to a first-order equation: $z(mw + a)w'_z = F(z, w) - w^2$. See also Section 8.3.4.

19. $y''_{xx} = x^{-2}F(x^ne^{ay},\, xy'_x)$.

Equation invariant under "dilatation–translation" transformation. The transformation $z = x^ne^{ay}$, $w = xy'_x$ leads to a first-order equation: $z(aw + n)w'_z = F(z, w) + w$. See also Section 8.3.4.

20. $y''_{xx} = e^{2ay}F(xe^{ay},\, e^{-ay}y'_x)$.

The transformation $z = xe^{ay}$, $w = e^{-ay}y'_x$ leads to a first-order equation: $(azw + 1)w'_z = F(z, w) - aw^2$.

21. $y''_{xx} = ae^yy'_x + F(x,\, y'_x - ae^y)$.

The substitution $w = y'_x - ae^y$ leads to a first-order equation: $w'_x = F(x, w)$.

22. $y''_{xx} = (e^{2x} + e^x)y + F\big(x, y'_x - e^xy\big)$.

The substitution $w = y'_x - e^xy$ leads to a first-order equation: $w'_x = -e^xw + F(x, w)$.

23. $y''_{xx} = F(x, y'_x \sinh x - y \cosh x) + y.$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a first-order equation of the form: $w'_x = F(x, w) \sinh x$.

24. $y''_{xx} = F(x, y'_x \cosh x - y \sinh x) + y.$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a first-order equation of the form: $w'_x = F(x, w) \cosh x$.

25. $y''_{xx} = x^{-2} F(ay + b \ln x,\ xy'_x).$

The transformation $z = ay + b \ln x$, $w = xy'_x$ leads to a first-order equation: $(aw + b)w'_z = F(z, w) + w$.

26. $y''_{xx} = yF(ax + b \ln y,\ y'_x/y).$

The transformation $z = ax + b \ln y$, $w = y'_x/y$ leads to a first-order equation: $(bw + a)w'_z = F(z, w) - w^2$.

27. $y''_{xx} = F(x, y'_x \sin x - y \cos x) - y.$

The substitution $w = y'_x \sin x - y \cos x$ leads to a first-order equation: $w'_x = F(x, w) \sin x$.

28. $y''_{xx} = F(x, y'_x \cos x + y \sin x) - y.$

The substitution $w = y'_x \cos x + y \sin x$ leads to a first-order equation: $w'_x = F(x, w) \cos x$.

29. $y''_{xx} = (\varphi^2 + \varphi'_x)y + F(x, y'_x - \varphi y), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \varphi y$ leads to a first-order equation: $w'_x = -\varphi w + F(x, w)$.

30. $y''_{xx} = \dfrac{\varphi''_{xx}}{\varphi} y + F\Big(x, y'_x - \dfrac{\varphi'_x}{\varphi} y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi} y$ leads to a first-order equation: $w'_x = -\dfrac{\varphi'_x}{\varphi} w + F(x, w)$.

31. $y''_{xx} = \varphi_y y'_x + \varphi_x + F(x,\ y'_x - \varphi), \qquad \varphi = \varphi(x, y).$

The substitution $w = y'_x - \varphi(x, y)$ leads to a first-order equation: $w'_x = F(x, w)$.

32. $f^2 y''_{xx} + f f'_x y'_x = \Phi(y,\ f y'_x), \qquad f = f(x).$

The substitution $w(y) = f y'_x$ leads to a first-order equation: $w w'_y = \Phi(y, w)$.

33. $y''_{xx} = F(x, y).$

Let $F \neq \varphi(x)y + \psi(x)$, i.e., the equation is nonlinear. Then its order can be reduced by one if the right-hand side of the equation has the following form:

$$F(x,\ y) = f^{-3/2} E\Big\{\Phi(u) + \int \big[\tfrac{1}{2} f f'''_{xxx}(u + V) + f^{1/2} g''_{xx} E^{-1}\big]\, dx\Big\}, \tag{1}$$

where

$$E = \exp\Big(k \int f^{-1}\, dx\Big), \quad V = \int f^{-3/2} g E^{-1}\, dx, \quad u = f^{-1/2} E^{-1} y - V;$$

$\Phi = \Phi(u)$, $f = f(x)$, and $g = g(x)$ are arbitrary functions, and k is an arbitrary constant.

The integral in (1) can always be expressed in terms of E and V. The following cases are possible:

1°. For $f'''_{xxx} \neq 0$,

$$F(x,y)=f^{-3/2}E\Phi(u)+\tfrac{1}{4}f^{-2}\left[2ff''_{xx}-(f'_x)^2\right]y+\tfrac{1}{2}f^{-2}\left(2fg'_x-f'_xg+2kg\right)+k^2f^{-3/2}EV.$$

2°. For $f = ax^2+bx+c$, $f'_x \neq -2k$, $f'_x \neq \frac{2}{3}k$,

$$F(x,y) = f^{-3/2}E\Phi(u) + \tfrac{1}{2}f^{-2}\left(2fg'_x - f'_xg + 2kg\right) + \left(k^2 + \tfrac{1}{4}\Delta\right)f^{-3/2}EV,$$

where $\Delta = 4ac - b^2$.

3°. For $f = \beta - 2kx$,

$$F(x,y) = f^{-2}[\Phi(y+W) + fg'_x + 2kg], \qquad \text{where} \quad W = -\int f^{-1}g\,dx.$$

4°. For $f = \frac{2}{3}kx + \beta$,

$$F(x,y) = \Phi\left(f^{-2}y - U\right) + f^{-2}\left(fg'_x + \tfrac{2}{3}kg\right) + \tfrac{8}{9}k^2U, \qquad \text{where} \quad U = \int f^{-3}g\,dx.$$

In all these cases, the transformation

$$t = \int f^{-1}dx, \qquad u = f^{-1/2}E^{-1}y - V$$

leads to the autonomous equation $u''_{tt} + 2ku'_t + k^2u = \Phi(u)$, which is reducible, with the aid of the substitution $z(u) = u'_t$, to an Abel equation: $zz'_u + 2kz + k^2u = \Phi(u)$ (see Section 13.3.1).

If $k=0$, the solution of the original equation for case 1° is as follows:

$$\int \frac{du}{\sqrt{2\Psi(u)+C_1}} = \pm\int \frac{dx}{f} + C_2, \qquad \text{where} \quad \Psi(u) = \int \Phi(u)\,du.$$

If $k=0$, the solution of the original equation for case 2° is given by:

$$2\int \frac{du}{\sqrt{8\Psi(u)-\Delta u^2+C_1}} = \pm\int \frac{dx}{ax^2+bx+c} + C_2, \qquad \text{where} \quad \Psi(u) = \int \Phi(u)\,du.$$

► **Equations not solved for the y''_{xx} containing arbitrary functions of two variables.**

34. $y''_{xx} + ay'_x + by = e^{\lambda x}F\left(y'_x/y,\ y''_{xx}/y\right).$

1°. The substitution $y = e^{\lambda x}w$ leads to an autonomous equation:

$$w''_{xx} + (2\lambda + a)w'_x + (\lambda^2 + a\lambda + b)w = F\left(w'_x + \lambda w,\ w''_{xx} + 2\lambda w'_x + \lambda^2 w\right).$$

2°. Particular solution: $y = ke^{\lambda x}$, where k is a root of the algebraic (transcendental) equation $k(\lambda^2 + a\lambda + b) = F(k\lambda, k\lambda^2)$.

35. $a_1(y''_{xx})^2 + a_2y'_xy''_{xx} + a_3yy''_{xx} + a_4(y'_x)^2 + a_5yy'_x + a_6y^2 = e^{\lambda x}F\left(y'_x/y,\ y''_{xx}/y\right).$

The substitution $y = e^{\lambda x/2}w$ leads to an autonomous equation.

36. $y''_{xx} = y'_x F_1(y'_x/y,\ y''_{xx}/y) + yF_2(y'_x/y,\ y''_{xx}/y) + e^{\lambda x} F_3(y'_x/y,\ y''_{xx}/y)$.

The substitution $y = e^{\lambda x} w$ leads to an autonomous equation.

37. $F(xy''_{xx},\ x^2 y''_{xx} - xy'_x + y) = 0$.

Solution:

$$y = C_1 x \ln x + C_2 x + C_3,$$

where C_2 is an arbitrary constant and the constants C_1 and C_3 are related by the constraint $F(C_1, C_3) = 0$.

38. $F(y''_{xx}/y,\ yy''_{xx} - y'^2_x) = 0$.

1°. Solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(C_3^2,\ 4C_1C_2C_3^2) = 0$.

2°. Solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F\bigl(-C_3^2, -(C_1^2 + C_2^2)C_3^2\bigr) = 0.$$

39. $F(y^3 y''_{xx},\ yy''_{xx} + y'^2_x) = 0$.

Solutions:

$$y = \pm\sqrt{C_1 x^2 + 2C_2 x + C_3},$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(C_1C_3 - C_2^2, C_1) = 0$.

40. $F\left(y + \dfrac{y'^2_x}{y''_{xx}},\ x + \dfrac{y'_x - y'^3_x}{2y''_{xx}}\right) = 0$.

Solutions can be found from the relation

$$(y - C_1)^2 = 2C_2(x - A) + C_2^2,$$

where $F(C_1,\ A) = 0$. The question of whether there are other solutions calls for further investigation.

41. $F\left(y\dfrac{y''_{xx}}{y'_x} + ay'_x,\ y^{a+1}\dfrac{y''_{xx}}{y'_x}\right) = 0$.

A solution of this equation is any function that solves the first-order separable equation:

$$y'_x = C_1 y^{-a} + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(aC_2, -aC_1) = 0$.

42. $F\left(\dfrac{y''_{xx}}{y'_x} + y'_x,\ e^y\dfrac{y''_{xx}}{y'_x}\right) = 0$.

A solution of this equation is any function that solves the first-order separable equation:

$$y'_x = C_1 e^{-y} + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(C_2, -C_1) = 0$.

► **Equations containing arbitrary functions of three variables.**

43. $F(y''_{xx},\ xy''_{xx}-y'_x,\ x^2y''_{xx}-2xy'_x+2y)=0.$

Solution:

$$y=C_1x^2+C_2x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(2C_1,-C_2,2C_3)=0$.

⊙ *Literature*: E. L. Ince (1964).

44. $F\big(y''_{xx},\ xy''_{xx}-y'_x,\ 2yy''_{xx}-(y'_x)^2\big)=0.$

Solution:

$$y=C_1x^2+C_2x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F(2C_1,-C_2,4C_1C_3-C_2^2)=0.$$

45. $F\big(x^3y''_{xx},\ xy''_{xx}+2y'_x,\ x^2y''_{xx}+xy'_x-y\big)=0.$

Solution:

$$y=C_1x+C_2+\frac{C_3}{x},$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(2C_3,2C_1,-C_2)=0$.

46. $F\big(x^{a+2}y''_{xx},\ xy''_{xx}+(a+1)y'_x,\ x^2y''_{xx}+axy'_x-ay\big)=0.$

Solution:

$$y=C_1x^{-a}+C_2x+C_3.$$

The constants C_1, C_2, and C_3 are related by the constraint

$$F\big(a(a+1)C_1,\ (a+1)C_2,\ -aC_3\big)=0.$$

47. $F\big(y^3y''_{xx},\ yy''_{xx}+y'^2_x,\ xyy''_{xx}+xy'^2_x-yy'_x\big)=0.$

Solution:

$$y^2=C_1x^2+2C_2x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F(C_1C_3-C_2^2,C_1,-C_2)=0.$$

48. $F\big(x,y'_x-y,y''_{xx}-y,y''_{xx}-y'_x\big)=0.$

The substitution $w=y'_x-y$ leads to a first-order equation: $F(x,w,w'_x+w,w'_x)=0$.

49. $F\big(x,y'_x+ay,y''_{xx}-a^2y,y''_{xx}+ay'_x\big)=0.$

The substitution $w=y'_x+ay$ leads to a first-order equation: $F(x,w,w'_x-aw,w'_x)=0$.

50. $F\left(x-\dfrac{y'^2_x+1}{y''_{xx}}y'_x,\ y+\dfrac{y'^2_x+1}{y''_{xx}},\ \dfrac{(y'^2_x+1)^{3/2}}{y''_{xx}}\right)=0.$

It is known that all functions of the form $(x-C_1)^2+(y-C_2)^2=A^2$, where $A=A(C_1,C_2)$ is determined from the algebraic (transcendental) equation $F(C_1,C_2,A)=0$, are solutions of the original equation.

51. $F\big(e^x y''_{xx},\, y''_{xx}+y'_x,\, y-xy'_x-(x+1)y''_{xx}\big)=0.$

Solution:

$$y=C_1e^{-x}+C_2x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(C_1,C_2,C_3)=0$.

52. $F\Big(y''_{xx}/y,\, yy''_{xx}-(y'_x)^2,\, \big(y+y'_x\sqrt{y/y''_{xx}}\,\big)\exp\big(-x\sqrt{y''_{xx}/y}\,\big)\Big)=0.$

Solution:

$$y=C_1\exp(C_3x)+C_2\exp(-C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F(C_3^2,\, 4C_1C_2C_3^2,\, 2C_1)=0.$$

53. $F\big(y''_{xx}-y,\, y''_{xx}\sinh x-y'_x\cosh x,\, (y''_{xx})^2-(y'_x)^2\big)=0.$

Solution:

$$y=C_1\sinh x+C_2\cosh x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F(-C_3,-C_1,C_1^2-C_2^2)=0.$$

54. $F\big(xy''_{xx},\, x^2y''_{xx}-xy'_x+y,\, y'_x-y''_{xx}x\ln x\big)=0.$

Solution:

$$y=C_1x\ln x+C_2x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(C_1,C_3,C_1+C_2)=0$.

55. $F\left(\dfrac{y''_{xx}}{y'_x},\, y'_x-y\dfrac{y''_{xx}}{y'_x},\, x\dfrac{y''_{xx}}{y'_x}-\ln\dfrac{(y'_x)^2}{y''_{xx}}\right)=0.$

Solution:

$$y=C_1\exp(C_2x)+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F(C_2,-C_2C_3,-\ln C_1)=0.$$

56. $F\big(y''_{xx}+y,\, y''_{xx}\sin x-y'_x\cos x,\, (y''_{xx})^2+(y'_x)^2\big)=0.$

Solution:

$$y=C_1\sin x+C_2\cos x+C_3,$$

where the constants C_1, C_2, and C_3 are related by the constraint $F(C_3,-C_1,C_1^2+C_2^2)=0$.

57. $F\big(y,\, xy'^2_x,\, xy''_{xx}+\tfrac{1}{2}y'_x\big)=0.$

The substitution $w(y)=x(y'_x)^2$ leads to a first-order equation: $F\big(y,w,\tfrac{1}{2}w'_y\big)=0$.

58. $F\big(y,\, \varphi y'^2_x,\, \varphi y''_{xx}+\tfrac{1}{2}\varphi'_x y'_x\big)=0, \qquad \varphi=\varphi(x).$

The substitution $w(y)=\varphi(x)(y'_x)^2$ leads to a first-order equation: $F\big(y,w,\tfrac{1}{2}w'_y\big)=0$.

14.9.7 Equations Defined Parametrically

1. $x = \varphi(t), \quad y''_{xx} = \psi(t).$

General solution in parametric form:

$$x = \varphi(t), \qquad y = C_1\varphi(t) + C_2 + \int f(t)\varphi'_t(t)\,dt,$$
$$f(t) = \int \psi(t)\varphi'_t(t)\,dt,$$

where C_1 and C_2 are arbitrary constants.

2. $y = \varphi(t), \quad y''_{xx} = \psi(t).$

General solution in parametric form:

$$x = \int f(t)\,dt + C_1, \qquad y = \varphi(t),$$
$$f(t) = \pm\varphi'_t(t)\left[2\int \psi(t)\varphi'_t(t)\,dt + C_2\right]^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

3. $y'_x = \varphi(t), \quad y''_{xx} = \psi(t).$

General solution in parametric form:

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \quad y = \int \frac{\varphi(t)\varphi'_t(t)}{\psi(t)}\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

4. $y'_x = a(t)x + b(t), \quad y''_{xx} = c(t).$

The equation is reduced to the following system of equations for $x = x(t)$ and $y = y(t)$ (see Section 3.2.8):

$$(c-a)x'_t = a'_t x + b'_t,$$
$$(c-a)y'_t = (ax+b)(a'_t x + b'_t).$$

1°. Let $c - a \not\equiv 0$. Then the general solution of the first (linear) equation of the system is

$$x = C_1 E + E\int \frac{b'_t\,dt}{E(c-a)}, \qquad E = \exp\left(\int \frac{a'_t\,dt}{c-a}\right),$$

where C_1 is an arbitrary constant. Substituting this expression of $x = x(t)$ into the second equation yields a separable equation for $y = y(t)$ (its solution is omitted).

2°. Let $c - a \equiv 0$. Then the general solution of the original equation is

$$y = \frac{1}{2}a(C_1)x^2 + b(C_1)x + C_2,$$

where C_1 and C_2 are arbitrary constants. In addition, there is a one-parameter singular solution:

$$x = -\frac{b'_t}{a'_t}, \qquad y = \int\left(b - \frac{ab'_t}{a'_t}\right)dt + C.$$

5. $y'_x = a(t)x + ky + b(t), \quad y''_{xx} = ka(t)x + k^2 y + c(t).$

The equation is reduced to the following system of equations for $x = x(t)$ and $y = y(t)$ (see Section 3.2.8):

$$(c - a - kb)x'_t = a'_t x + b'_t,$$
$$(c - a - kb)y'_t = (a'_t x + b'_t)(ax + ky + b).$$

1°. Let $c - a - kb \not\equiv 0$. Then the general solution of the first (linear) equation of the system is

$$x = C_1 E + E\int \frac{b'_t\,dt}{E(c-a-kb)}, \qquad E = \exp\biggl(\int \frac{a'_t\,dt}{c-a-kb}\biggr),$$

where C_1 is an arbitrary constant. Substituting this expression of $x = x(t)$ into the second equation yields a separable equation for $y = y(t)$ (its solution is omitted).

2°. Let $c - a - kb \equiv 0$. Then the general solution of the original equation is

$$y = C_2 e^{kx} - \frac{a(C_1)}{k}x - \frac{b(C_1)}{k} - \frac{a(C_1)}{k^2},$$

where C_1 and C_2 are arbitrary constants. In addition, there is a one-parameter singular solution:

$$x = -\frac{b'_t}{a'_t}, \qquad y = Ce^{kt} + e^{kt}\int e^{-kt}\biggl(b - \frac{ab'_t}{a'_t}\biggr)\,dt.$$

6. $y'_x = a(t)x, \quad y''_{xx} = a(t) + b(t)x^2 y.$

General solution in parametric form:

$$x = \pm\biggl[2\int \frac{a'_t(t)}{b(t)\varphi(t)}\,dt + C_1\biggr]^{1/2},$$
$$y = \pm\varphi(t), \qquad \varphi(t) = \biggl[2\int \frac{a(t)a'_t(t}{b(t)})\,dt + C_2\biggr]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

7. $y'_x = a(t)x, \quad y''_{xx} = a(t)x + b(t)x^2 y^k.$

General solution in parametric form:

$$x = \pm\biggl[2\int \frac{a'_t(t)}{b(t)\varphi^k(t)}\,dt + C_1\biggr]^{1/2},$$
$$y = \pm\varphi(t), \qquad \varphi(t) = \biggl[(k+1)\int \frac{a(t)a'_t(t}{b(t)})\,dt + C_2\biggr]^{\frac{1}{k+1}},$$

where C_1 and C_2 are arbitrary constants.

8. $y'_x = a(t)y, \quad y''_{xx} = b(t)y.$

General solution in parametric form:

$$x = \int \frac{a'_t(t)\,dt}{b(t) - a^2(t)} + C_1, \qquad y = C_2 \exp\biggl[\int \frac{a(t)a'_t(t)\,dt}{b(t) - a^2(t)}\biggr],$$

where C_1 and C_2 are arbitrary constants.

9. $xy'_x = y + a(t), \quad y''_{xx} = b(t)x^k.$

General solution in parametric form:

$$x = \varphi(t), \quad \varphi(t) = \left[(k+2)\int \frac{a'_t(t)}{b(t)}\,dt + C_1\right]^{\frac{1}{k+2}},$$
$$y = C_2 E(t) + E(t)\int \frac{a(t)a'_t(t)\,dt}{E(t)b(t)\varphi^{k+2}(t)}, \quad E(t) = \exp\left[\int \frac{a'_t(t)\,dt}{b(t)\varphi^{k+2}(t)}\right],$$

where C_1 and C_2 are arbitrary constants.

10. $y'_x = a(t)y^k, \quad y''_{xx} = b(t)y^{2k-1}.$

General solution in parametric form:

$$x = \int \frac{a'_t(t)\varphi^{1-k}(t)\,dt}{b(t) - ka^2(t)} + C_1,$$
$$y = \varphi(t), \quad \varphi(t) = C_2 \exp\left[\int \frac{a(t)a'_t(t)\,dt}{b(t) - ka^2(t)}\right],$$

where C_1 and C_2 are arbitrary constants.

11. $y'_x = 2x\sqrt{y}\,h(t), \quad y''_{xx} = 2x^2h^2(t) - 2\sqrt{y}\,\lambda(t).$

General solution in parametric form:

$$x = C_1 E(t), \quad E(t) = \exp\left[-\int \frac{h'_t(t)\,dt}{h(t) + \lambda(t)}\right],$$
$$y = \left[C_2 - C_1^2 \int \frac{h(t)h'_t(t)}{h(t) + \lambda(t)} E^2(t)\,dt\right]^2,$$

where C_1 and C_2 are arbitrary constants.

12. $y'_x = e^x g(y)h(t), \quad y''_{xx} = e^{2x}g(y)g'_y(y)h^2(t) - e^x g(y)\lambda(t).$

General solution in parametric form:

$$e^x = C_1 E(t), \quad E(t) = \exp\left[-\int \frac{h'_t(t)\,dt}{h(t) + \lambda(t)}\right],$$
$$\int \frac{dy}{g(y)} = -C_1 \int \frac{h(t)h'_t(t)}{h(t) + \lambda(t)} E(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

13. $y'_x = f(x)g(x, y, t), \quad y''_{xx} = h(x)g(x, y, t).$

General solution:

$$y = C_1 \int E(x)\,dx + C_2, \quad E(x) = \exp\left[\int \frac{h(x)}{f(x)}\,dx\right],$$

where C_1 and C_2 are arbitrary constants.

The dependence $t = t(x)$ is defined implicitly by the equation $f(x)g(x, y, t) = C_1 E(x)$.

14. $y'_x = f(x)g(x,y,t), \quad y''_{xx} = \dfrac{h(x)}{g(x,y,t)}.$

General solution:

$$y = \int E(x)\,dx + C_2, \quad E(x) = \pm\left[2\int f(x)h(x)\,dx\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

The dependence $t = t(x)$ is defined implicitly by the equation $E(x) = f(x)g(x,y,t)$.

14.9.8 Some Transformations

1. $y''_{xx} + x^{-3}F\left(\frac{1}{x},\, \frac{y}{x}\right) = 0.$

The transformation $\xi = 1/x$, $w = y/x$ leads to the equation $w''_{\xi\xi} + F(\xi, w) = 0$.

2. $y''_{xx} = n(n+1)x^{-2}y + x^{3n}F(x^{2n+1},\, x^n y).$

The transformation $\xi = x^{2n+1}$, $w = x^n y$ leads to the equation $(2n+1)^2 w''_{\xi\xi} = F(\xi, w)$.

3. $y''_{xx} + (ax+b)^{-3}F\left(\dfrac{cx+d}{ax+b},\, \dfrac{y}{ax+b}\right) = 0.$

The transformation $\xi = \dfrac{cx+d}{ax+b}$, $w = \dfrac{y}{ax+b}$ leads to the equation $w''_{\xi\xi} + \Delta^{-2}F(\xi, w) = 0$, where $\Delta = ad - bc$.

4. $x^2 y''_{xx} + axy'_x + by + F(x,\, y) = 0.$

The transformation $x = \xi^\nu$, $y = \xi^\mu w$, where the parameters ν and μ are found from the simultaneous algebraic equations

$$2\mu + 1 + (a-1)\nu = 0, \quad \mu^2 + (a-1)\mu\nu + b\nu^2 = 0,$$

leads to an equation of the form

$$w''_{\xi\xi} + \nu^2 \xi^{-\mu-2} F(\xi^\nu,\, \xi^\mu w) = 0.$$

5. $y''_{xx} = n(n+1)x^{-2}y + x^{3n}F(ax^{2n+1} + b,\, x^n y).$

The transformation $\xi = ax^{2n+1} + b$, $w = x^n y$ leads to the equation $a^2(2n+1)^2 w''_{\xi\xi} = F(\xi, w)$.

6. $y''_{xx} = \lambda^2 y + e^{3\lambda x}F(ae^{2\lambda x} + b,\, e^{\lambda x}y).$

The transformation $\xi = ae^{2\lambda x} + b$, $w = e^{\lambda x}y$ leads to the equation $w''_{\xi\xi} = (2a\lambda)^{-2}F(\xi, w)$.

7. $y''_{xx} = \lambda^2 y + \dfrac{e^{3\lambda x}}{(ce^{2\lambda x}+d)^3}F\left(\dfrac{ae^{2\lambda x}+b}{ce^{2\lambda x}+d},\, \dfrac{e^{\lambda x}y}{ce^{2\lambda x}+d}\right).$

The transformation $\xi = \dfrac{ae^{2\lambda x}+b}{ce^{2\lambda x}+d}$, $w = \dfrac{e^{\lambda x}y}{ce^{2\lambda x}+d}$ leads to the equation

$$w''_{\xi\xi} = (2\Delta\lambda)^{-2}F(\xi,\, w), \qquad \text{where} \quad \Delta = ad - bc.$$

8. $y''_{xx} = \lambda^2 y + \sinh^{-3}(\lambda x)\, F\left(\coth(\lambda x),\, \dfrac{y}{\sinh(\lambda x)}\right).$

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to the equation $w''_{\xi\xi} = \lambda^{-2}F(\xi,\, w)$.

9. $y''_{xx} = \lambda^2 y + \cosh^{-3}(\lambda x)\, F\left(\tanh(\lambda x),\ \frac{y}{\cosh(\lambda x)}\right).$

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to the equation $w''_{\xi\xi} = \lambda^{-2} F(\xi,\ w)$.

10. $x^2 y''_{xx} + \frac{1}{4} y + \sqrt{x}\, F\left(a \ln x + b,\ \frac{y}{\sqrt{x}}\right) = 0.$

The transformation $\xi = a \ln x + b$, $w = \dfrac{y}{\sqrt{x}}$ leads to the equation $w''_{\xi\xi} + a^{-2} F(\xi,\ w) = 0$.

11. $|x^2 - 1|^{3/2} y''_{xx} = F\left(\ln \frac{ax - a}{x + 1},\ \frac{y}{\sqrt{|x^2 - 1|}}\right).$

The transformation $\xi = \ln \dfrac{ax - a}{x + 1}$, $w = \dfrac{y}{\sqrt{|x^2 - 1|}}$ leads to the equation $4w''_{\xi\xi} = F(\xi, w) + w$.

12. $y''_{xx} + \lambda^2 y + \sin^{-3}(\lambda x)\, F\left(\cot(\lambda x),\ \frac{y}{\sin(\lambda x)}\right) = 0.$

The transformation $\xi = \cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to an equation of the form $w''_{\xi\xi} + \lambda^{-2} F(\xi,\ w) = 0$.

13. $y''_{xx} + \lambda^2 y + \cos^{-3}(\lambda x)\, F\left(\tan(\lambda x),\ \frac{y}{\cos(\lambda x)}\right) = 0.$

The transformation $\xi = \tan(\lambda x)$, $w = \dfrac{y}{\cos(\lambda x)}$ leads to an equation of the form $w''_{\xi\xi} + \lambda^{-2} F(\xi,\ w) = 0$.

14. $y''_{xx} + \lambda^2 y + \sin^{-3}(\lambda x + b)\, F\left(\frac{\sin(\lambda x + a)}{\sin(\lambda x + b)},\ \frac{y}{\sin(\lambda x + b)}\right) = 0.$

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{y}{\sin(\lambda x + b)}$ leads to the equation

$$w''_{\xi\xi} + [\lambda \sin(b - a)]^{-2} F(\xi,\ w) = 0.$$

15. $(x^2 + 1)^{3/2} y''_{xx} + F\left(\arctan x + b,\ \frac{y}{\sqrt{x^2 + 1}}\right) = 0.$

The transformation $\xi = \arctan x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to the equation $w''_{\xi\xi} + w + F(\xi,\ w) = 0$.

16. $(x^2 + 1)^{3/2} y''_{xx} + F\left(\operatorname{arccot} x + b,\ \frac{y}{\sqrt{x^2 + 1}}\right) = 0.$

The transformation $\xi = \operatorname{arccot} x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to the equation $w''_{\xi\xi} + w + F(\xi, w) = 0$.

17. $y''_{xx} + F(x, y) = 0.$

The transformation $x = \varphi(z),\ y = w\sqrt{a\varphi'_z}$ leads to the equation

$$w''_{zz} + \left[\frac{1}{2}\frac{\varphi'''_{zzz}}{\varphi'_z} - \frac{3}{4}\left(\frac{\varphi''_{zz}}{\varphi'_z}\right)^2\right]w + a^{-2}(a\varphi'_z)^{3/2}F\bigl(\varphi,\, w\sqrt{a\varphi'_z}\,\bigr) = 0.$$

The sign of the parameter a must coincide with that of the derivative φ'_z.

18. $y''_{xx} + f(x, y)(y'_x)^3 + g(x, y)(y'_x)^2 = 0.$

Taking y to be the independent variable, we obtain the following equation with respect to $x = x(y)$: $x''_{yy} - g(x, y)x'_y - f(x, y) = 0$.

19. $F(x, y, y'_x, y''_{xx}) = 0.$

Applying the Legendre transformation $x = w'_t,\ y = tw'_t - w,$ where $w = w(t),$ and using the relations $y'_x = t$ and $y''_{xx} = 1/w''_{tt},$ we arrive at the equation

$$F\left(w'_t,\, tw'_t - w,\, t,\, \frac{1}{w''_{tt}}\right) = 0.$$

Given a solution of the original equation, the corresponding solution of the transformed equation is written in parametric form as:

$$t = y'_x, \quad w = xy'_x - y, \qquad \text{where} \quad y = y(x).$$

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Chapter 15

Third-Order Ordinary Differential Equations

15.1 Linear Equations

15.1.1 Preliminary Remarks

1°. A homogeneous linear equation of the third order has the general form

$$f_3(x)y'''_{xxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution of this equation. The substitution

$$y = y_0(x) \int z(x)\,dx$$

leads to a second-order linear equation:

$$f_3 y_0 z'' + (3f_3 y_0' + f_2 y_0)z' + (3f_3 y_0'' + 2f_2 y_0' + f_1 y_0)z = 0, \tag{2}$$

where the prime denotes differentiation with respect to x.

2°. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent particular solutions of equation (1). Then the general solution of this equation can be written in the form:

$$y = C_1 y_1 + C_2 y_2 + C_3\Big(y_2 \int y_1 \psi\,dx - y_1 \int y_2 \psi\,dx\Big), \tag{3}$$

where

$$\psi = \exp\Big(-\int \frac{f_2}{f_3}\,dx\Big)\big(y_1 y_2' - y_1' y_2\big)^{-2}.$$

For specific equations described below in Sections 15.1.2–15.1.9, often only particular solutions will be given, and the general solution can be obtained by formula (3).

3°. A nonhomogeneous linear equation of the third-order has the form

$$f_3(x)y'''_{xxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x). \tag{4}$$

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two linearly independent particular solutions of the corresponding homogeneous equation (1). Then the general solution of equation (4) is defined by formula (3) with:

$$\psi = \Delta^{-2} e^{-F}\Bigl(1 + \frac{1}{C_3}\int \frac{g}{f_3}\Delta e^F\,dx\Bigr), \qquad \text{where} \quad F = \int \frac{f_2}{f_3}\,dx, \;\; \Delta = y_1 y_2' - y_1' y_2.$$

4°. The substitution $y = z\exp\Bigl(-\dfrac{1}{3}\displaystyle\int \frac{f_2}{f_3}\,dx\Bigr)$ reduces equation (1) to a form from which the second derivative is absent:

$$z''' + \bigl(-\varphi_2' - \tfrac{1}{3}\varphi_2^2 + \varphi_1\bigr)z' + \bigl(-\tfrac{1}{3}\varphi_2'' - \tfrac{1}{3}\varphi_1\varphi_2 + \tfrac{2}{27}\varphi_2^3 + \varphi_0\bigr)z = 0,$$

where $\varphi_k = f_k/f_3$ $(k = 0,\,1,\,2)$.

15.1.2 Equations Containing Power Functions

► **Equations of the form $f_3(x)y'''_{xxx} + f_0(x)y = g(x)$.**

1. $y'''_{xxx} + \lambda y = 0.$

Solution:

$$y = \begin{cases} C_1 + C_2 x + C_3 x^2 & \text{if } \lambda = 0, \\ C_1\exp(-kx) + C_2\exp\bigl(\tfrac{1}{2}kx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}kx\bigr) + C_3\exp\bigl(\tfrac{1}{2}kx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}kx\bigr) & \text{if } \lambda \neq 0, \end{cases}$$

where $k = \lambda^{1/3}$.

2. $y'''_{xxx} + \lambda y = ax^2 + bx + c.$

Solution: $y = w + \dfrac{1}{\lambda}(ax^2 + bx + c)$, where w is the general solution of the equation 15.1.2.1: $w'''_{xxx} + \lambda w = 0$.

3. $y'''_{xxx} = axy + b.$

This is a special case of equation 17.1.2.3 with $n = 3$.

4. $y'''_{xxx} + (ax + b)y = 0.$

For $a = 0$, this is an equation of the form 15.1.2.1. For $a \neq 0$, the substitution $a\xi = ax + b$ leads to an equation of the form 15.1.2.3: $y'''_{\xi\xi\xi} + a\xi y = 0$.

5. $y'''_{xxx} + ax^3 y = bx.$

The substitution $\xi = x^2$ leads to an equation of the form 15.1.2.126: $2\xi y'''_{\xi\xi\xi} + 3y''_{\xi\xi} + \frac{1}{4}a\xi y = \frac{1}{4}b$.

6. $y'''_{xxx} + (3a^2 x - a^3 x^3)y = 0.$

Integrating, we obtain a second-order nonhomogeneous linear equation: $y''_{xx} + axy'_x + (a^2x^2 - a)y = C\exp\bigl(\frac{1}{2}ax^2\bigr)$ (see 14.1.2.31 for the solution of the corresponding homogeneous equation).

7. $y'''_{xxx} = ax^\beta y.$

This is a special case of equation 17.1.2.4 with $n = 3$.

1°. For $\beta = -9,\ -7,\ -6,\ -9/2,\ -3,\ -3/2,\ 1$, and 3, see equations 15.1.2.17, 15.1.2.14, 15.1.2.11, 15.1.2.19, 15.1.2.10, 15.1.2.18, 15.1.2.3, and 15.1.2.5, respectively.

2°. The transformation $x = t^{-1}$, $y = ut^{-2}$ leads to an equation of similar form: $u'''_{xxx} = -at^{-\beta-6}u$.

3°. For $\beta \neq -3$, the transformation $\xi = x^{(\beta+3)/3}$, $w = x^{\beta/3}y$ leads to an equation of the form 15.1.2.69: $\xi^3 w'''_{\xi\xi\xi} + (1-\nu^2)\xi w'_\xi + (\nu^2 - 1 - a\nu^3\xi^3)w = 0$, where $\nu = \frac{3}{\beta+3}$.

8. $y'''_{xxx} + [a^3x^{3n} - 3a^2nx^{2n-1} + an(n-1)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{ax^{n+1}}{n+1}\Bigr)$. The substitution $y = \exp\Bigl(-\dfrac{ax^{n+1}}{n+1}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.2.47:

$$z''_{xx} - 3ax^n z'_x + (3a^2x^{2n} - 3anx^{n-1})z = 0.$$

9. $xy'''_{xxx} - a^2(ax+3)y = 0.$

Particular solution: $y_0 = xe^{ax}$. The substitution $y = xe^{ax}\displaystyle\int z(x)\,dx$ leads to a second-order equation of the form 14.1.2.108: $xz''_{xx} + 3(ax+1)z'_x + 3a(ax+2)z = 0$.

10. $x^3y'''_{xxx} = a(a^2-1)y.$

This is a special case of equation 15.1.2.175. Solution: $y = x(C_1x^{n_1} + C_2x^{n_2} + C_3x^a)$, where n_1 and n_2 are roots of the quadratic equation $n^2 + an + a^2 - 1 = 0$.

11. $x^6y'''_{xxx} = ay + bx^2.$

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to a constant coefficient linear equation of the form 15.1.2.2: $w'''_{ttt} + aw + b = 0$.

12. $(x-a)^3(x-b)^3y'''_{xxx} - cy = 0, \qquad a \neq b.$

The transformation $t = \ln\Bigl|\dfrac{x-a}{x-b}\Bigr|$, $w = \dfrac{y}{(x-b)^2}$ leads to a constant coefficient linear equation: $(a-b)^3(w'''_{ttt} - 3w''_{tt} + 2w'_t) - cw = 0$.

13. $(ax^2+bx+c)^3y'''_{xxx} = ky.$

The transformation $\xi = \displaystyle\int\frac{dx}{ax^2+bx+c}$, $w = \dfrac{y}{ax^2+bx+c}$ leads to a constant coefficient linear equation: $w'''_{\xi\xi\xi} + (4ac-b^2)w'_\xi = kw$.

14. $x^7y'''_{xxx} = ay + bx^3.$

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to a linear equation of the form 15.1.2.3: $w'''_{ttt} + atw + b = 0$.

15. $x^7y'''_{xxx} + (ax+b)y = 0.$

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to a linear equation of the form 15.1.2.4: $w'''_{ttt} - (bt+a)w = 0$.

16. $x^9 y'''_{xxx} + (a^3 - 3a^2x^2)y = 0.$

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to a linear equation of the form 15.1.2.6: $w'''_{ttt} + (3a^2t - a^3t^3)w = 0$.

17. $x^9 y'''_{xxx} = ay.$

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 15.1.2.5: $w'''_{ttt} + at^3w = 0$.

18. $x^{3/2} y'''_{xxx} = ay.$

This is a special case of equation 17.1.2.8 with $n = 1$.

19. $x^{9/2} y'''_{xxx} = ay.$

This is a special case of equation 17.1.2.9 with $n = 1$.

► **Equations of the form** $f_3(x)y'''_{xxx} + f_1(x)y'_x + f_0(x)y = g(x)$.

20. $y'''_{xxx} + aby'_x + a^2x(3 - b - ax^2)y = 0.$

Integrating, we obtain a second-order nonhomogeneous linear equation: $y''_{xx} + axy'_x + (a^2x^2 + ab - a)y = C\exp\bigl(\frac{1}{2}ax^2\bigr)$ (see 14.1.2.31 for the solution of the corresponding homogeneous equation).

21. $y'''_{xxx} + axy'_x + any = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $y = u_x^{(n-1)}$, where u is the solution of the second-order linear equation $u''_{xx} + axu = C$.

22. $y'''_{xxx} + axy'_x - 2ay = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.2: $w''_{xx} + axw = 0$.

23. $y'''_{xxx} + axy'_x + b(ax + b^2)y = 0.$

Particular solution: $y_0 = e^{-bx}$. The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.2.12: $w''_{xx} - bw'_x + (ax + b^2)w = 0$.

24. $y'''_{xxx} + axy'_x + (abx + a + b^3)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} - by'_x + (ax + b^2)y = Ce^{-bx}$ (see 14.1.2.108 for the solution of the corresponding homogeneous equation with $C = 0$).

25. $y'''_{xxx} + (ax + b)y'_x + ay = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $y''_{xx} + (ax + b)y = C$ (see 14.1.2.2 for the solution of the corresponding homogeneous equation).

26. $y'''_{xxx} + (ax + b)y'_x - ay = 0.$

Particular solution: $y_0 = ax + b$. The transformation $\xi = ax + b$, $z = \dfrac{y'_x}{ax + b} - \dfrac{ay}{(ax + b)^2}$ leads to a second-order linear equation of the form 14.1.2.67: $\xi z''_{\xi\xi} + 3z'_\xi + a^{-2}\xi^2 z = 0$.

27. $y'''_{xxx} + (ax + b)y'_x + 3ay = 0.$

The substitution $a\xi = ax + b$ leads to a linear equation of the form 15.1.2.21 with $n = 3$: $y'''_{\xi\xi\xi} + a\xi y'_\xi + 3ay = 0$.

28. $y'''_{xxx} + (2ax + b)y'_x + ay = 0.$

The substitution $a\xi = ax + \frac{1}{2}b$ leads to a linear equation of the form 15.1.2.48 with $n = 1$: $y'''_{\xi\xi\xi} + 2a\xi y'_\xi + ay = 0$.

29. $y'''_{xxx} + (ax - b^2)y'_x + abxy = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.2.108: $w''_{xx} - bw'_x + axw = 0$.

30. $y'''_{xxx} + (ax - b^2)y'_x + a(bx + 1)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} - by'_x + axy = Ce^{-bx}$ (see 14.1.2.108 for the solution of the corresponding homogeneous equation with $C = 0$).

31. $y'''_{xxx} + (ax + b)y'_x + c(ax + b + c^2)y = 0.$

The substitution $w = y'_x + cy$ leads to a second-order linear equation of the form 14.1.2.12: $w''_{xx} - cw'_x + (ax + b + c^2)w = 0$.

32. $y'''_{xxx} + (ax + b)y'_x + cx(c^2x^2 + ax + b - 3c)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}cx^2\bigr)$. The substitution $y = \exp\bigl(-\frac{1}{2}cx^2\bigr)\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.2.31:

$$z''_{xx} - 3cxz'_x + (3c^2x^2 + ax + b - 3c)z = 0.$$

33. $y'''_{xxx} + ax^2y'_x + axy = 0.$

This is a special case of equation 15.1.2.47 with $n = 1$.

34. $y'''_{xxx} + ax^2y'_x - 2axy = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + ax^2w = 0$.

35. $y'''_{xxx} - a^2x^2y'_x + a^2xy = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + axy'_x - ay = C\exp\bigl(\frac{1}{2}ax^2\bigr)$ (see 14.1.2.108 for the solution of the corresponding homogeneous equation with $C = 0$).

36. $y'''_{xxx} + ax^2y'_x + b(ax^2 + b^2)y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.2.31: $w''_{xx} - bw'_x + (ax^2 + b^2)w = 0$.

37. $y'''_{xxx} + (a - 1)b^2x^2y'_x + b^2x(abx^2 + 2a + 1)y = 0.$

Integrating, we obtain a second-order nonhomogeneous linear equation: $y''_{xx} - bxy'_x + (ab^2x^2 + b)y = C\exp\bigl(-\frac{1}{2}bx^2\bigr)$ (see 14.1.2.31 for the solution of the corresponding homogeneous equation).

38. $y'''_{xxx} + (ax^2 + b)y'_x + 2axy = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $y''_{xx} + (ax^2 + b)y = C$ (see 14.1.2.4 for the solution of the corresponding homogeneous equation).

39. $y'''_{xxx} + (ax^2 - b^2)y'_x + ax(2 - bx)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + by'_x + ax^2 y = Ce^{bx}$ (see 14.1.2.13 for the solution of the corresponding homogeneous equation with $C = 0$).

40. $y'''_{xxx} + (ax^2 + b)y'_x + c(ax^2 + b + c^2)y = 0.$

The substitution $w = y'_x + cy$ leads to a second-order linear equation of the form 14.1.2.13: $w''_{xx} - cw'_x + (ax^2 + b + c^2)w = 0.$

41. $y'''_{xxx} - (3b^2x^2 + a + 3b)y'_x + 2bx(b^2x^2 - a)y = 0.$

1°. Particular solutions with $a > 0$:

$$y_1 = \exp\bigl(\tfrac{1}{2}bx^2 + x\sqrt{a}\,\bigr), \quad y_2 = \exp\bigl(\tfrac{1}{2}bx^2 - x\sqrt{a}\,\bigr).$$

2°. Particular solutions with $a < 0$:

$$y_1 = \exp\bigl(\tfrac{1}{2}bx^2\bigr)\cos\bigl(x\sqrt{|a|}\,\bigr), \quad y_2 = \exp\bigl(\tfrac{1}{2}bx^2\bigr)\sin\bigl(x\sqrt{|a|}\,\bigr).$$

3°. Particular solutions with $a = 0$:

$$y_1 = \exp\bigl(\tfrac{1}{2}bx^2\bigr), \quad y_2 = x\exp\bigl(\tfrac{1}{2}bx^2\bigr).$$

42. $y'''_{xxx} + (ax^2 + bx + c)y'_x + kx[(a + k^2)x^2 + bx + c - 3k]y = 0.$

Particular solution: $y_0 = \exp\bigl(-\tfrac{1}{2}kx^2\bigr)$. The substitution $y = \exp\bigl(-\tfrac{1}{2}kx^2\bigr)\displaystyle\int z(x)\,dx$ leads to a second-order equation of the form 14.1.2.31:

$$z''_{xx} - 3kxz'_x + [(a + 3k^2)x^2 + bx + c - 3k]z = 0.$$

43. $y'''_{xxx} + (ax^4 + bx)y'_x - 2(ax^3 + b)y = 0.$

This is a special case of equation 15.1.2.49 with $n = 2$.

44. $y'''_{xxx} + ax^n y'_x - 2ax^{n-1}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + ax^n w = 0.$

45. $y'''_{xxx} + ax^n y'_x + anx^{n-1}y = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $y''_{xx} + ax^n y = C$ (see 14.1.2.7 for the solution of the corresponding homogeneous equation).

46. $y'''_{xxx} + ax^{m+1}y'_x + a(m + 3)x^m y = 0.$

The substitution $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 15.1.2.45 with $n = -m - 5$: $w'''_{ttt} + at^{-m-5}w'_t - a(m + 5)t^{-m-6}w = 0.$

47. $y'''_{xxx} + ax^{2n}y'_x + anx^{2n-1}y = 0.$

Solution:

$$y = C_1 x J_\nu^2(u) + C_2 x J_\nu(u) Y_\nu(u) + C_3 x Y_\nu^2(u),$$

where $\nu = \dfrac{1}{2(n+1)}$, $u = \dfrac{\sqrt{a}\,x^{n+1}}{2(n+1)}$; $J_\nu(u)$ and $Y_\nu(u)$ are Bessel functions.

48. $y'''_{xxx} + 2ax^n y'_x + anx^{n-1}y = 0.$

Solution: $y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2$. Here, w_1 and w_2 are a fundamental set of solutions of a second-order linear equation of the form 14.1.2.7: $2w''_{xx} + ax^n w = 0$.

49. $y'''_{xxx} + (ax^{2n} + bx^{n-1})y'_x - 2(ax^{2n-1} + bx^{n-2})y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.10: $w''_{xx} + (ax^{2n} + bx^{n-1})w = 0$.

50. $y'''_{xxx} + (ax^{2n} + bx^{n-1})y'_x$
$\qquad - c[(a + c^2)x^{3n} + (b + 3cn)x^{2n-1} + n(n-1)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(\dfrac{cx^{n+1}}{n+1}\Bigr)$. The substitution $y = \exp\Bigl(\dfrac{cx^{n+1}}{n+1}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second-order equation of the form 14.1.2.47:

$$z''_{xx} + 3cx^n z'_x + [(a + 3c^2)x^{2n} + (b + 3cn)x^{n-1}]z = 0.$$

51. $xy'''_{xxx} + ay'_x + b(b^2x + a)y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} - bxw'_x + (b^2x + a)w = 0$.

52. $xy'''_{xxx} + axy'_x - [b(a + b^2)x + a + 3b^2]y = 0.$

Particular solution: $y_0 = xe^{bx}$. The substitution $y = xe^{bx}\displaystyle\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.2.108: $xz''_{xx} + 3(bx + 1)z'_x + [(a + 3b^2)x + 6b]z = 0$.

53. $xy'''_{xxx} + (b - a^2x)y'_x + aby = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} - axw'_x + bw = 0$.

54. $xy'''_{xxx} + (ax^2 + bx)y'_x - 2(ax + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.2: $w''_{xx} + (ax + b)w = 0$.

55. $xy'''_{xxx} + (ax^3 + bx)y'_x - 2(ax^2 + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.4: $w''_{xx} + (ax^2 + b)w = 0$.

56. $(ax + b)y'''_{xxx} + cy'_x + k(ak^2x + bk^2 + c)y = 0.$

The substitution $w = y'_x + ky$ leads to a second-order linear equation of the form 14.1.2.108: $(ax + b)w''_{xx} - k(ax + b)w'_x + (ak^2x + bk^2 + c)w = 0$.

57. $(ax+2)y'''_{xxx} - a^3xy'_x + 2a^3y = 0.$

Particular solutions: $y_1 = x^2$, $y_2 = e^{-ax}$.

58. $(acx+bc-a)y'''_{xxx} - c^3(ax+b)y'_x + ac^3y = 0.$

Particular solutions: $y_1 = ax+b$, $y_2 = e^{cx}$.

59. $(ax+b)y'''_{xxx} + (cx+d)y'_x + s[(as^2+c)x+bs^2+d]y = 0.$

The substitution $w = y'_x + sy$ leads to a second-order linear equation of the form 14.1.2.108: $(ax+b)w''_{xx} - s(ax+b)w'_x + [(as^2+c)x+bs^2+d]w = 0$.

60. $(ax+b)y'''_{xxx} + [(c-ak^2)x+d-bk^2]y'_x + k(cx+d)y = 0.$

The substitution $w = y'_x + ky$ leads to a second-order linear equation of the form 14.1.2.108: $(ax+b)w''_{xx} - k(ax+b)w'_x + (cx+d)w = 0$.

61. $(ax+b)y'''_{xxx} - (a^3x^3 - 3a^2x + b^3)y'_x + abx(a^2x^2 - 3a - b^2)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$.

62. $x^2y'''_{xxx} - 6y'_x + ax^2y + 2bx = 0.$

The substitution $y = x^2w$ leads to an equation of the form 15.1.2.173: $x^3w'''_{xxx} + 6x^2w''_{xx} + (ax^3-12)w + 2b = 0$.

63. $x^2y'''_{xxx} + (ax^2+bx-m^2-m)y'_x + (m-1)(ax+b)y = 0.$

The substitution $w = xy'_x + (m-1)y$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} - (m+1)w'_x + (ax+b)w = 0$.

64. $x^2y'''_{xxx} + (ax^2+bx+c)y'_x - k[(a+k^2)x^2+bx+c]y = 0.$

The substitution $w = y'_x - ky$ leads to a second-order linear equation of the form 14.1.2.135: $x^2w''_{xx} + kx^2w'_x + [(a+k^2)x^2+bx+c]w = 0$.

65. $x^2y'''_{xxx} + (ax^n - b^2 - b)y'_x + a(b-1)x^{n-1}y = 0.$

The substitution $w = xy'_x + (b-1)y$ leads to a second-order equation of the form 14.1.2.67: $xw''_{xx} - (b+1)w'_x + ax^{n-1}w = 0$.

66. $x^2y'''_{xxx} + (ax^{n+1} - b^2 - b)y'_x + a(b-1)x^ny = 0.$

The substitution $w = xy'_x + (b-1)y$ leads to a second-order linear equation of the form 14.1.2.67: $xw''_{xx} - (b+1)w'_x + ax^nw = 0$.

67. $x^2y'''_{xxx} - 3ax^{n+1}(n + ax^{n+1})y'_x + ax^n(n - n^2 + 2a^2x^{2n+2})y = 0.$

1°. Particular solutions with $n \neq -1$: $y_1 = \exp\Bigl(\dfrac{ax^{n+1}}{n+1}\Bigr)$, $y_2 = x\exp\Bigl(\dfrac{ax^{n+1}}{n+1}\Bigr)$.

2°. Particular solutions with $n = -1$: $y_1 = x^a$, $y_2 = x^{a+1}$.

68. $x(ax+b)y'''_{xxx} + x(cx+d)y'_x - 2(cx+d)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.108: $(ax+b)w''_{xx} + (cx+d)w = 0$.

69. $\boldsymbol{x^3 y'''_{xxx} + (1-a^2)xy'_x + (bx^3 + a^2 - 1)y = 0.}$

For $a = \pm 1$, we have a constant coefficient equation of the form 15.1.2.1. For $b = 0$, we obtain the Euler equation 15.1.2.175.

1°. If $b \neq 0$ and a is a positive integer greater than 1, then the solution is:

$$y = x^{1-a} \sum_{k=1}^{3} C_k \exp(-\lambda_k x) P_k(x),$$

where λ_1, λ_2, and λ_3 are roots of the cubic equation $\lambda^3 = b$ and $P_k(x)$ are polynomials of degree $\leq 3(a-1)$.

2°. Denote the solution of the original equation for arbitrary (including complex) a by y_a. Then the following recurrence relation holds:

$$y_{a+3} = by_a + (2a+3)x^{-1}y''_a - (a+1)(2a+3)(x^{-2}y'_a - x^{-3}y_a), \qquad (1)$$

where the prime denotes differentiation with respect to x.

Since the functions $y_{\pm 1} = e^{-\lambda x}$, corresponding to three values of λ determined by the equation $\lambda^3 = b$, form a fundamental set of solutions, formula (1) makes it possible to find all y_n for any integer values of n not divisible by 3. In particular, $y_2 = (x^{-1} + \lambda)e^{-\lambda x}$, where $\lambda^3 = b$.

70. $\boldsymbol{x^3 y'''_{xxx} + (4x^3 + ax)y'_x - ay = 0.}$

Solution: $y = C_1 x J_\nu^2(x) + C_2 x J_\nu(x) Y_\nu(x) + C_3 x Y_\nu^2(x)$, where $J_\nu(x)$ and $Y_\nu(x)$ are Bessel functions; $4\nu^2 = 1 - a$.

71. $\boldsymbol{x^3 y'''_{xxx} + x[ax^2 + 3b(1-b)]y'_x + 2b(ax^2 + b^2 - 1)y = 0.}$

1°. Particular solutions with $a > 0$: $y_1 = x^b \sin\bigl(x\sqrt{a}\bigr)$, $y_2 = x^b \cos\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = x^b \exp\bigl(-x\sqrt{-a}\bigr)$, $y_2 = x^b \exp\bigl(x\sqrt{-a}\bigr)$.

3°. Particular solutions with $a = 0$: $y_1 = x^b$, $y_2 = x^{b+1}$.

72. $\boldsymbol{x^3 y'''_{xxx} + x(ax^2 + bx + c)y'_x + (k-1)(ax^2 + bx + c + k^2 + k)y = 0.}$

The substitution $w = xy'_x + (k-1)y$ leads to a second-order linear equation of the form 14.1.2.131: $x^2 w''_{xx} - (k+1)xw'_x + (ax^2 + bx + c + k^2 + k)w = 0$.

73. $\boldsymbol{x^3 y'''_{xxx} + ax^n y'_x + (b-1)(ax^{n-1} + b^2 + b)y = 0.}$

The substitution $w = xy'_x + (b-1)y$ leads to a second-order linear equation of the form 14.1.2.132: $x^2 w''_{xx} - (b+1)xw'_x + (ax^{n-1} + b^2 + b)w = 0$.

74. $\boldsymbol{x^3 y'''_{xxx} + x(ax^n + b)y'_x - 2(ax^n + b)y = 0.}$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.118: $x^2 w''_{xx} + (ax^n + b)w = 0$.

75. $\boldsymbol{x^3 y'''_{xxx} + x(ax^n + b - c)y'_x + (c-1)(ax^n + b + c^2)y = 0.}$

The substitution $w = xy'_x + (c-1)y$ leads to a second-order linear equation of the form 14.1.2.132: $x^2 w''_{xx} - (c+1)xw'_x + (ax^n + b + c^2)w = 0$.

76. $x^3y'''_{xxx} + (ax^{2n} + 1 - n^2)xy'_x + [bx^{3n} + a(n-1)x^{2n} + n^2 - 1]y = 0.$

The transformation $\xi = x^n/n,\ z = x^{n-1}y$ leads to a constant coefficient linear equation: $z'''_{\xi\xi\xi} + az'_\xi + bz = 0$.

77. $x^2(ax+b)y'''_{xxx} + (cx - bm^2 - bm)y'_x + (m-1)(c + am^2 + am)y = 0.$

The substitution $w = xy'_x + (m-1)y$ leads to a second-order linear equation of the form 14.1.2.172: $x(ax+b)w''_{xx} - (m+1)(ax+b)w'_x + (c + am^2 + am)w = 0$.

78. $x(ax^2 + bx + c)y'''_{xxx} + xy'_x - 2y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.179: $(ax^2 + bx + c)w''_{xx} + w = 0$.

79. $x^5y'''_{xxx} = a(xy'_x - 2y).$

Solution: $y = \begin{cases} x^2\bigl[C_1 + C_2\exp\bigl(\sqrt{a}/x\bigr) + C_3\exp\bigl(-\sqrt{a}/x\bigr)\bigr] & \text{if } a > 0, \\ x^2\bigl[C_1 + C_2\cos\bigl(\sqrt{-a}/x\bigr) + C_3\sin\bigl(\sqrt{-a}/x\bigr)\bigr] & \text{if } a < 0. \end{cases}$

80. $x^6y'''_{xxx} + ax^2y'_x + (b - 2ax)y = 0.$

The transformation $x = t^{-1},\ y = wt^{-2}$ leads to a constant coefficient linear equation of the form 15.1.2.82 with $a_2 = 0$: $w'''_{ttt} + aw'_t - bw = 0$.

► Equations of the form $f_3(x)y'''_{xxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x)$.

81. $y'''_{xxx} + 3ay''_{xx} + 3a^2y'_x + a^3y = 0.$

Solution: $y = e^{-ax}(C_1 + C_2x + C_3x^2)$.

82. $y'''_{xxx} + a_2y''_{xx} + a_1y'_x + a_0y = 0.$

A third-order constant coefficient linear equation.

Denote $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.

1°. Let the characteristic polynomial $P(\lambda)$ be factorizable:

$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, where λ_1, λ_2, and λ_3 are real numbers.

Solution: $y = \begin{cases} C_1e^{\lambda_1x} + C_2e^{\lambda_2x} + C_3e^{\lambda_3x} & \text{if all the roots } \lambda_k \text{ are different}, \\ (C_1 + C_2x)e^{\lambda_1x} + C_3e^{\lambda_3x} & \text{if } \lambda_1 = \lambda_2 \neq \lambda_3, \\ (C_1 + C_2x + C_3x^2)e^{\lambda_1x} & \text{if } \lambda_1 = \lambda_2 = \lambda_3. \end{cases}$

2°. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda^2 + 2b_1\lambda + b_0)$, where $b_1^2 < b_0$.

Solution: $y = C_1e^{\lambda_1x} + e^{-b_1x}(C_2\cos\mu x + C_3\sin\mu x)$, where $\mu = \sqrt{b_0 - b_1^2}$.

83. $y'''_{xxx} + ay''_{xx} + (bx + c)y'_x + (abx + ac + b)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + (bx + c)y = Ce^{-ax}$ (see 14.1.2.2 for the solution of the corresponding homogeneous equation with $C = 0$).

84. $y'''_{xxx} + 3ay''_{xx} + 2(bx + a^2)y'_x + b(2ax + 1)y = 0.$

This is a special case of equation 15.1.2.113 with $n = 0$ and $m = 1$.

85. $y'''_{xxx} + ay''_{xx} + (bx^2 + cx + d)y'_x + a(bx^2 + cx + d)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.2.6: $w''_{xx} + (bx^2 + cx + d)w = 0$.

86. $y'''_{xxx} + ay''_{xx} + bx^n y'_x + abx^n y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + bx^n w = 0$.

87. $y'''_{xxx} + 3axy''_{xx} + 3a^2x^2y'_x + (a^3x^3 + b)y = 0.$

The substitution $y = w\exp\bigl(-\frac{1}{2}ax^2\bigr)$ leads to a constant coefficient linear equation of the form 15.1.2.82 with $a_2 = 0$: $w'''_{xxx} - 3aw'_x + bw = 0$.

88. $y'''_{xxx} + axy''_{xx} + (abx + a - b^2)y'_x + aby = 0.$

Particular solutions: $y_1 = e^{-bx}$, $y_2 = e^{-bx}\int \exp\bigl(2bx - \frac{1}{2}ax^2\bigr)\,dx$.

89. $y'''_{xxx} + 3axy''_{xx} + (2a^2x^2 + a + b)y'_x + abxy = 0.$

Solution: $y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2$. Here, w_1 and w_2 are linearly independent solutions of a second-order equation of the form 14.1.2.28: $w''_{xx} + axw'_x + \frac{1}{4}bw = 0$.

90. $y'''_{xxx} + 3axy''_{xx} + (2a^2x^2 + 2bx + a)y'_x + b(2ax^2 + 1)y = 0.$

This is a special case of equation 15.1.2.113 with $n = 1$ and $m = 1$.

91. $y'''_{xxx} + 3axy''_{xx} + 3(a^2x^2 + a)y'_x + (a^3x^3 + bx + c)y = 0.$

The substitution $y = \exp\bigl(-\frac{1}{2}ax^2\bigr)w$ leads to a linear equation of the form 15.1.2.4: $w'''_{xxx} + [(b - 3a^2)x + c]w = 0$.

92. $y'''_{xxx} + 3axy''_{xx} + [2(a^2 + b)x^2 + a]y'_x + 2bx(ax^2 + 1)y = 0.$

This is a special case of equation 15.1.2.113 with $n = 1$ and $m = 2$.

93. $y'''_{xxx} + (ax + b)y''_{xx} + (abx + a + c)y'_x + bcy = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + axy'_x + cy = Ce^{-bx}$ (see 14.1.2.28 for the solution of the corresponding homogeneous equation with $C = 0$).

94. $y'''_{xxx} + (abx + a + b)y''_{xx} + ab^2xy'_x - ab^2y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

95. $y'''_{xxx} + (ax + b)y''_{xx} + [(ab + c)x + a]y'_x + c(bx + 1)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + axy'_x + cxy = Ce^{-bx}$ (see 14.1.2.28 for the solution of the corresponding homogeneous equation with $C = 0$).

96. $y'''_{xxx} + (ax + b + c)y''_{xx} + (acx + bc + s)y'_x + s(ax + b)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + c\lambda + s = 0$.

97. $y'''_{xxx} + (ax+b)y''_{xx} + (cx+2a)y'_x + a[(c-ab)x^2+b]y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$. The substitution $y = \exp\bigl(-\frac{1}{2}ax^2\bigr)\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.2.31:

$$z''_{xx} + (b-2ax)z'_x + [a^2x^2 + (c-2ab)x - a]z = 0.$$

98. $y'''_{xxx} + (ax+b)y''_{xx} + (cx+d)y'_x + [acx^2 + (ad+bc)x + c + bd]y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + (cx+d)y = C\exp\bigl(-\frac{1}{2}ax^2 - bx\bigr)$ (see 14.1.2.2 for the solution of the corresponding homogeneous equation with $C=0$).

99. $y'''_{xxx} + (ax+b)y''_{xx} + (\alpha x^2 + \beta x + \gamma)y'_x - k[\alpha x^2 + (ak+\beta)x + k^2 + bk + \gamma]y = 0.$

The substitution $w = y'_x - ky$ leads to a second-order linear equation of the form 14.1.2.31: $w''_{xx} + (ax+b+k)w'_x + [\alpha x^2 + (ak+\beta)x + k^2 + bk + \gamma]w = 0.$

100. $y'''_{xxx} - x^2y''_{xx} + (a+b-1)xy'_x - aby = 0.$

The following three series, converging for any x, make up a fundamental set of solutions:

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{ab(a-3)(b-3)\dots(a-3n+3)(b-3n+3)}{(3n)!}x^{3n},$$
$$y_2 = x + \sum_{n=1}^{\infty} \frac{(a-1)(b-1)(a-4)(b-4)\dots(a-3n+2)(b-3n+2)}{(3n+1)!}x^{3n+1},$$
$$y_3 = \frac{x^2}{2} + \sum_{n=1}^{\infty} \frac{(a-2)(b-2)(a-5)(b-5)\dots(a-3n+1)(b-3n+1)}{(3n+2)!}x^{3n+2}.$$

101. $y'''_{xxx} + ax^ny''_{xx} - 2ax^{n-2}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.45: $w''_{xx} + ax^nw'_x + ax^{n-1}w = 0.$

102. $y'''_{xxx} + ax^ny''_{xx} - by'_x - abx^ny = 0.$

1°. Particular solutions with $b>0$: $y_1 = \exp(-x\sqrt{b})$, $y_2 = \exp(x\sqrt{b})$.

2°. Particular solutions with $b<0$: $y_1 = \cos\bigl(x\sqrt{-b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\bigr)$.

103. $y'''_{xxx} + ax^ny''_{xx} - 2ax^{n-1}y'_x + 2ax^{n-2}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

104. $y'''_{xxx} + ax^ny''_{xx} + bx^{n-1}y'_x - 2(a+b)x^{n-2}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.2.45: $w''_{xx} + ax^nw'_x + (a+b)x^{n-1}w = 0.$

105. $y'''_{xxx} + ax^ny''_{xx} + abx^ny'_x + b^2(ax^n - b)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

106. $y'''_{xxx} + ax^n y''_{xx} - (ax^{n-1} - bx^2)y'_x + bx(ax^{n+1} + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr),\ y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr).$

107. $y'''_{xxx} + ax^n y''_{xx} + (abx^n + anx^{n-1} - b^2)y'_x + abnx^{n-1}y = 0.$

Particular solutions: $y_1 = e^{-bx},\ y_2 = e^{-bx}\int \exp\Bigl(2bx - \frac{a}{n+1}x^{n+1}\Bigr)\,dx.$

108. $y'''_{xxx} + ax^n y''_{xx} + bx^m y'_x - bx^{m-1}y = 0.$

Particular solution: $y_0 = x.$

109. $y'''_{xxx} + ax^n y''_{xx} + bx^m y'_x + bx^{m-1}(ax^{n+1} + m)y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation: $w'_x + ax^n w = 0.$

110. $y'''_{xxx} + ax^n y''_{xx} - b(2ax^n + 3b)y'_x + b^2(ax^n + 2b)y = 0.$

Particular solutions: $y_1 = e^{bx},\ y_2 = xe^{bx}.$

111. $y'''_{xxx} + ax^n y''_{xx} + (abx^n - b^2 + c)y'_x + c(ax^n - b)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0.$

112. $y'''_{xxx} + ax^n y''_{xx} + (bx^m - c^2)y'_x - c(acx^n + bx^m)y = 0.$

Particular solution: $y_0 = e^{cx}.$

113. $y'''_{xxx} + 3ax^n y''_{xx} + (2a^2x^{2n} + 2bx^m + anx^{n-1})y'_x + b(2ax^{n+m} + mx^{m-1})y = 0.$

Solution: $y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2$. Here, w_1 and w_2 form a fundamental set of solutions of the second-order linear equation: $w''_{xx} + ax^n w'_x + \tfrac{1}{2}bx^m w = 0.$

114. $y'''_{xxx} = (x^n - a)y''_{xx} + (ax^n - b)y'_x + bx^n y.$

Particular solutions: $y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

115. $y'''_{xxx} + (ax^n + b)y''_{xx} + (acx^n + bc + m)y'_x + (m + c^2)(ax^n + b - c)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + c\lambda + m + c^2 = 0.$

116. $y'''_{xxx} + (ax^n - b)y''_{xx} + cx^m y'_x - b(abx^n + cx^m)y = 0.$

Particular solution: $y_0 = e^{bx}.$

117. $y'''_{xxx} + (x^n + a)y''_{xx} + (ax^n + bx^m)y'_x + abx^m y = 0.$

Particular solution: $y_0 = e^{-ax}.$

118. $y'''_{xxx} + (ax^n + c)y''_{xx} + [acx^n + (an + b)x^{n-1}]y'_x + b[cx^{n-1} + (n - 1)x^{n-2}]y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + ax^n y'_x + bx^{n-1}y = Ce^{-cx}$ (see 14.1.2.45 for the solution of the corresponding homogeneous equation with $C = 0$).

119. $y'''_{xxx} + (ax^n + bx)y''_{xx} + b(ax^{n+1} + 2)y'_x + abx^n y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\int \exp\bigl(\frac{1}{2}bx^2\bigr)\,dx$.

120. $y'''_{xxx} + (ax^n + bx)y''_{xx} + (abx^{n+1} + bcx + b - c^2)y'_x + c(abx^{n+1} - acx^n + b)y = 0.$

Particular solutions: $y_1 = e^{-cx}$, $y_2 = e^{-cx}\int \exp\bigl(2cx - \frac{1}{2}bx^2\bigr)\,dx$.

121. $y'''_{xxx} + (abx^n + ax^{n-1} + b)y''_{xx} + ab^2x^n y'_x - ab^2x^{n-1}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

122. $y'''_{xxx} + (ax^n + bx^m)y''_{xx} + cy'_x + c(ax^n + bx^m)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

123. $y'''_{xxx} + (ax^n + bx^m)y''_{xx} + (abx^{n+m} + bcx^m + bmx^{m-1} - c^2)y'_x + c(abx^{n+m} - acx^n + bmx^{m-1})y = 0.$

Particular solutions: $y_1 = e^{-cx}$, $y_2 = e^{-cx}\int \exp\Bigl(2cx - \frac{bx^{m+1}}{m+1}\Bigr)\,dx$.

124. $xy'''_{xxx} + 3y''_{xx} + axy = 0.$

The substitution $w = xy$ leads to a constant coefficient linear equation of the form 15.1.2.1: $w'''_{xxx} + aw = 0$.

125. $xy'''_{xxx} - 3ny''_{xx} + axy = 0, \qquad n = 0,\ 1,\ 2,\ \ldots$

Solution: $y = x^{3n+2}\Bigl(\dfrac{1}{x^2}\dfrac{d}{dx}\Bigr)^n\Bigl(\dfrac{w}{x^2}\Bigr)$, where w is the general solution of equation 15.1.2.1: $w'''_{xxx} + aw = 0$.

126. $2xy'''_{xxx} + 3y''_{xx} + axy = b, \qquad a \neq 0.$

Solution: $y = \displaystyle\sum_{\nu=1}^{4} C_\nu \int_0^{\lambda_\nu} \frac{e^{xz}\,dz}{\sqrt{2z^3 + a}}$, where λ_1, λ_2, and λ_3 are roots of the cubic equation $2\lambda^3 + a = 0$; $\lambda_4 = -\infty$ for $x > 0$ and $\lambda_4 = +\infty$ for $x < 0$. In addition, the constants C_ν are related by the constraint $\sqrt{a}\,(C_1 + C_2 + C_3 + C_4) + b = 0$, and the integrals are taken along straight lines.

127. $xy'''_{xxx} + 3y''_{xx} + ax^2y = b.$

The substitution $w = xy$ leads to an equation of the form 15.1.2.3: $w'''_{xxx} + axw = b$.

128. $xy'''_{xxx} + 3y''_{xx} + ax^4y = bx.$

The substitution $w = xy$ leads to an equation of the form 15.1.2.5: $w'''_{xxx} + ax^3w = bx$.

129. $xy'''_{xxx} + ay''_{xx} + aby'_x + b^3xy = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} + (a - bx)w'_x + b^2xw = 0$.

130. $xy'''_{xxx} + (a+b)y''_{xx} - xy'_x - ay = 0, \qquad a > 0, \quad b > 0.$

Solution:

$$y = \sum_{\nu=1}^{3} C_\nu \int_{\gamma_\nu}^{\beta_\nu} |t|^{a-1}|t^2-1|^{(b-2)/2} e^{-tx}\, dt,$$

where $\gamma_1 = -1$, $\beta_1 = \gamma_2 = 0$, $\beta_2 = 1$; for $x > 0$, $\gamma_3 = 1$ and $\beta_3 = +\infty$; for $x < 0$, $\gamma_3 = -\infty$ and $\beta_3 = -1$.

131. $xy'''_{xxx} + ay''_{xx} + (b-c^2)xy'_x - c(ac+bx)y = 0.$

The substitution $w = y'_x - cy$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} + (cx+a)w'_x + (bx+ac)w = 0.$

132. $xy'''_{xxx} + ay''_{xx} + [(c-b^2)x + ab]y'_x + c(a-bx)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0.$

133. $xy'''_{xxx} + ay''_{xx} + bx^n y'_x + b(a+n-1)x^{n-1}y = 0.$

The substitution $w = y''_{xx} + bx^{n-1}y$ leads to a first-order linear equation: $xw'_x + aw = 0.$

134. $xy'''_{xxx} + (ax+b)y''_{xx} - a^2by = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} + bw'_x - abw = 0.$

135. $xy'''_{xxx} + (ax+b)y''_{xx} + cxy'_x - cy = 0.$

The substitution $w = xy'_x - y$ leads to a second-order linear equation of the form 14.1.2.108: $xw''_{xx} + (ax+b-1)w'_x + cxw = 0.$

136. $xy'''_{xxx} + (ax+3)y''_{xx} + (bx+2a)y'_x + (cx+b)y = 0.$

The substitution $w = xy$ leads to a constant coefficient linear equation of the form 15.1.2.82: $w'''_{xxx} + aw''_{xx} + bw'_x + cw = 0.$

137. $xy'''_{xxx} + (ax+3)y''_{xx} + a(bx+2)y'_x + b[b(a-b)x+a]y = 0.$

Solution: $y = x^{-1}\bigl[C_1 e^{(b-a)x} + C_2 e^{-bx/2}\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr) + C_3 e^{-bx/2}\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)\bigr].$

138. $xy'''_{xxx} + [a(b+1)x+b]y''_{xx} + a^2bxy'_x - a^2by = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}.$

139. $xy'''_{xxx} - (x+2a)y''_{xx} - (x-2a-1)y'_x + (x-1)y = 0.$

Solution: $y = C_1 e^x + x^{a+1}[C_2 I_{a+1}(x) + C_3 K_{a+1}(x)]$, where $I_a(x)$ and $K_a(x)$ are modified Bessel functions.

140. $2xy'''_{xxx} - 4(x+a-1)y''_{xx} + (2x+6a-5)y'_x + (1-2a)y = 0.$

Solution: $y = C_1 e^x + x^a e^{x/2}[C_2 I_a(x/2) + C_3 K_a(x/2)]$, where $I_a(z)$ and $K_a(z)$ are modified Bessel functions.

141. $2xy'''_{xxx} + 3(2ax+k)y''_{xx} + 6(bx+ak)y'_x + (2cx+3bk)y = 0, \qquad k > 0.$

Solution:

$$y = \sum_{\nu=1}^{4} C_\nu \int_0^{\lambda_\nu} e^{xz}[P(z)]^{(k-2)/2}\,dz, \qquad C_4 = -C_1 - C_2 - C_3,$$

where $P(z) = z^3 + 3az^2 + 3bz + c$; λ_1, λ_2, and λ_3 are roots of this polynomial, which are assumed to be different; $\lambda_4 = -\infty$ for $x > 0$ and $\lambda_4 = +\infty$ for $x < 0$.

142. $xy'''_{xxx} + (ax+b)y''_{xx} + [(ac+s-c^2)x + bc]y'_x + s[(a-c)x+b]y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + c\lambda + s = 0$.

143. $xy'''_{xxx} + (ax^2+b+2)y''_{xx} - ab(b+1)y = 0.$

This is a special case of equation 15.1.2.145 with $n = 2$.

144. $xy'''_{xxx} + (ax^2+b)y''_{xx} + 4axy'_x + 2ay = 0.$

Integrating the equation twice, we arrive at a first-order linear equation:

$$xy'_x + (ax^2+b-2)y = C_1 + C_2 x.$$

145. $xy'''_{xxx} + (ax^n+b+2)y''_{xx} - ab(b+1)x^{n-2}y = 0.$

The substitution $w = xy'_x + by$ leads to a second-order linear equation of the form 14.1.2.45: $w''_{xx} + ax^{n-1}w'_x - a(b+1)x^{n-2}w = 0$.

146. $xy'''_{xxx} + (ax^n+3)y''_{xx} + (2ax^{n-1}+bx)y'_x + b(ax^n+1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{b}\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{b}\bigr)$.

147. $xy'''_{xxx} + (ax^{n+1}+3)y''_{xx} + a(bx+2)x^n y'_x + b(abx^{n+1}+ax^n-b^2x)y = 0.$

Particular solutions: $y_1 = x^{-1}\exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = x^{-1}\exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

148. $xy'''_{xxx} + (ax^n+3)y''_{xx} + (abx^n+2ax^{n-1}-b^2x)y'_x + b(ax^{n-1}-b)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^{-1}e^{-bx}$.

149. $(ax+b)y'''_{xxx} + [b(a+1)x+b^2+1]y''_{xx} + b^2xy'_x - b^2y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

150. $(ax+b)y'''_{xxx} + k(ax+b)y''_{xx} + (cx+d)y'_x + k(cx+d)y = 0.$

The substitution $w = y'_x + ky$ leads to a second-order linear equation of the form 14.1.2.108: $(ax+b)w''_{xx} + (cx+d)w = 0$.

151. $(ax+b)y'''_{xxx} + (cx+d)y''_{xx} + [(a\lambda+c\mu)x + b\lambda + d\mu]y'_x$
$\qquad + (\lambda+\mu^2)[(c-a\mu)x + d - b\mu]y = 0.$

Particular solutions: $y_1 = \exp(s_1 x)$, $y_2 = \exp(s_2 x)$, where s_1 and s_2 are roots of the quadratic equation $s^2 + \mu s + \lambda + \mu^2 = 0$.

152. $(ax+b)y'''_{xxx}+(cx+d)y''_{xx}-k[(3ak+2c)x+3bk+2d]y'_x$
$+k^2[(2ak+c)x+2bk+d]y=0.$

Particular solutions: $y_1=e^{kx}$, $y_2=xe^{kx}$.

153. $(ax+b)y'''_{xxx}+(cx+d)y''_{xx}+sx(ax+b)y'_x+s[cx^2+(a+d)x+b]y=0.$

The substitution $w=y''_{xx}+sxy$ leads to a first-order linear equation: $(ax+b)w'_x+(cx+d)w=0$.

154. $(1-x)y'''_{xxx}+x(ax-2a+1)y''_{xx}+(-ax^2+2a-1)y'_x+2a(x-1)y=0.$

Particular solutions: $y_1=x^2$, $y_2=e^x$.

155. $(ax+b)y'''_{xxx}+(cx+d)y''_{xx}+sx^n(ax+b)y'_x$
$+sx^{n-1}[cx^2+(an+d)x+bn]y=0.$

The substitution $w=y''_{xx}+sx^ny$ leads to a first-order linear equation: $(ax+b)w'_x+(cx+d)w=0$.

156. $(ax-1)y'''_{xxx}+x(abx^{n+1}-2bx^n-a^2)y''_{xx}+(2bx^n-a^2bx^{n+2}+a^2)y'_x$
$+2ab(ax-1)x^ny=0.$

Particular solutions: $y_1=x^2$, $y_2=e^{ax}$.

157. $x^2y'''_{xxx}+3xy''_{xx}-3y'_x+ax^2y+b=0.$

Solution: $y=(w/x)'_x$, where the function $w=w(x)$ satisfies a constant coefficient linear equation of the form 15.1.2.2: $w'''_{xxx}+aw=b$.

158. $x^2y'''_{xxx}+6xy''_{xx}+6y'_x+ax^2y=b.$

The substitution $w=x^2y$ leads to a constant coefficient linear equation of the form 15.1.2.2: $w'''_{xxx}+aw=b$.

159. $x^2y'''_{xxx}-3(n+m)xy''_{xx}+3n(3m+1)y'_x-x^2y=0,\quad m,n=1,\ 2,\ 3,\ \ldots$

Solution:

$$y=\prod_{\mu=0}^{n-1}(\delta-3\mu-1)\prod_{\nu=0}^{m-1}(\delta-3\nu-2)\sum_{k=1}^{3}C_ke^{\omega_kx},\qquad \delta=x\frac{d}{dx},$$

where the ω_k are three roots of the cubic equation $\omega^3=1$.

160. $x^2y'''_{xxx}+6xy''_{xx}+6y'_x+ax^3y=b.$

The substitution $w=x^2y$ leads to a linear equation of the form 15.1.2.3: $w'''_{xxx}+axw=b$.

161. $x^2y'''_{xxx}-2(n+1)xy''_{xx}-(ax^2-6n)y'_x+2axy=0,\quad n=1,\ 2,\ 3,\ \ldots$

Solution:

$$y=\begin{cases}C_1+C_2x^4+C_3x^{2n+1} & \text{if } a=0,\\ C_1(ax^2-4n+2)+C_2e^{x\sqrt{a}}P(x)+C_3e^{-x\sqrt{a}}Q(x) & \text{if } a\neq 0,\end{cases}$$

where $P(x)$ and $Q(x)$ are some polynomials of degree $\le 2n+2$.

162. $x^2y'''_{xxx} + 3xy''_{xx} + (4a^2x^{2a} + 1 - 4a^2b^2)y'_x + 4a^3x^{2a-1}y = 0.$

Solution: $y = C_1J_b^2(x^a) + C_2J_b(x^a)Y_b(x^a) + C_3Y_b^2(x^a)$, where $J_b(z)$ and $Y_b(z)$ are Bessel functions.

163. $x^2y'''_{xxx} + ax^2y''_{xx} + (bx + c)y'_x + a(bx + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.2.111: $x^2w''_{xx} + (bx + c)w = 0$.

164. $x^2y'''_{xxx} + ax^2y''_{xx} + (bx^n + c)y'_x + a(bx^n + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.2.118: $x^2w''_{xx} + (bx^n + c)w = 0$.

165. $x^2y'''_{xxx} - (x + a)xy''_{xx} + a(2x + 1)y'_x - a(x + 1)y = 0.$

Solution: $y = C_1e^x + x^{(a+1)/2}\left[C_2J_{a+1}\left(2\sqrt{ax}\right) + C_3Y_{a+1}\left(2\sqrt{ax}\right)\right]$, where $J_a(z)$ and $Y_a(z)$ are Bessel functions.

166. $x^2y'''_{xxx} - (x^2 - 2x)y''_{xx} - (x^2 + a^2 - \frac{1}{4})y'_x + (x^2 - 2x + a^2 - \frac{1}{4})y = 0.$

Solution: $y = C_1e^x + \sqrt{x}\left[C_2I_a(x) + C_3K_a(x)\right]$, where $I_a(x)$ and $K_a(x)$ are modified Bessel functions.

167. $x^2y'''_{xxx} - 2x(x - 1)y''_{xx} + (x^2 - 2x + \frac{1}{4} - a^2)y'_x + (a^2 - \frac{1}{4})y = 0.$

Solution: $y = C_1e^x + \sqrt{x}\,e^{x/2}\left[C_2I_a(x/2) + C_3K_a(x/2)\right]$, where $I_a(z)$ and $K_a(z)$ are modified Bessel functions.

168. $x^2y'''_{xxx} - 3(x - a)xy''_{xx} + [2x^2 + 4(b - a)x + a(2a - 1)]y'_x - 2b(2x - 2a + 1)y = 0.$

Solution: $y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2$. Here, w_1 and w_2 are a fundamental set of solutions of a second-order linear equation of the form 14.1.2.108: $xw''_{xx} + (a - x)w'_x + bw = 0$.

169. $x^2y'''_{xxx} + x[(a + c)x + b]y''_{xx} + [(ac + \alpha)x^2 + (bc + \beta)x + \gamma]y'_x + c(\alpha x^2 + \beta x + \gamma)y = 0.$

The substitution $w = y'_x + cy$ leads to a second-order linear equation of the form 14.1.2.146 with $n = 1$: $x^2w''_{xx} + x(ax + b)w'_x + (\alpha x^2 + \beta x + \gamma)w = 0$.

170. $x^2y'''_{xxx} + (ax^{n+1} + bx)y''_{xx} + [a(b - 2)x^n + c]y'_x + a(c - b + 2)x^{n-1}y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b - 3)m + c - b + 2 = 0$.

171. $2x(x - 1)y'''_{xxx} + 3(2x - 1)y''_{xx} + (2ax + b)y'_x + ay = 0.$

Solution: $y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2$. Here, w_1 and w_2 form a fundamental set of solutions of the equation $2x(x - 1)w''_{xx} + (2x - 1)w'_x + \left(\frac{1}{2}ax + \frac{1}{4}b - \frac{1}{2}\right)w = 0$, which is reduced, by means of the substitution $x = \cos^2\xi$, to the Mathieu equation 2.1.6.29: $2w''_{\xi\xi} = (a + b - 2 + a\cos 2\xi)w$.

172. $(a_2x^2 + a_1x + a_0)y'''_{xxx} + (b_1x + b_0)y''_{xx} + (c_1x + c_0)y'_x - mc_1y = 0.$

Here, $c_1 \neq 0$ and m is a positive integer. A solution of this equation is a polynomial of degree m that can represented as follows:

$$y = \sum_{k=0}^{m} \Big(-\frac{1}{c_1}\Big)^k \big\{x^m I x^{-m-1}[(ax^2 + a_1x + a_0)D^3 + (b_1x + b_0)D^2 + c_0D]\big\}^k x^m,$$

where $D = \dfrac{d}{dx}$, $Ix^\nu = \dfrac{x^{\nu+1}}{\nu+1}$ with $\nu \neq -1$.

173. $x^3y'''_{xxx} + 6x^2y''_{xx} + (ax^3 - 12)y + 2b = 0.$

Solution: $y = (w/x^2)'_x$, where $w = w(x)$ satisfies a constant coefficient linear equation of the form 15.1.2.2: $w'''_{xxx} + aw = b$.

174. $x^3y'''_{xxx} + ax^2y''_{xx} + bxy'_x + (a - 2)by = 0.$

This is a special case of equation 15.1.2.175. Solution: $y = C_1x^{2-a} + C_2x^{n_1} + C_3x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 - n + b = 0$.

175. $x^3y'''_{xxx} + ax^2y''_{xx} + bxy'_x + cy = 0.$

The Euler equation. The substitution $t = \ln|x|$ leads to a constant coefficient linear equation of the form 15.1.2.82: $y'''_{ttt} + (a - 3)y''_{tt} + (2 - a + b)y'_t + cy = 0$.

176. $x^3y'''_{xxx} + 3ax^2y''_{xx} + 3a(a - 1)xy'_x + [bx^3 + a(a - 1)(a - 2)]y = 0.$

The substitution $w = x^a y$ leads to a constant coefficient linear equation of the form 15.1.2.1: $w'''_{xxx} + bw = 0$.

177. $x^3y'''_{xxx} + 3ax^2y''_{xx} + 3a(a - 1)xy'_x + [bx^n + a(a - 1)(a - 2)]y = 0.$

The substitution $w = x^a y$ leads to an equation of the form 3.1.2.7: $w'''_{xxx} + bx^{n-3}w = 0$.

178. $x^3y'''_{xxx} + 3(1 - a)x^2y''_{xx} + x[4b^2c^2x^{2c} + 1 - 4\nu^2c^2 + 3a(a - 1)]y'_x + [4b^2c^2(c - a)x^{2c} + a(4\nu^2c^2 - a^2)]y = 0.$

Solution: $y = C_1x^aJ^2_\nu(bx^c) + C_2x^aJ_\nu(bx^c)Y_\nu(bx^c) + C_3x^aY^2_\nu(bx^c)$, where $J_\nu(u)$ and $Y_\nu(u)$ are Bessel functions.

179. $x^3y'''_{xxx} + (ax^2 + b)y''_{xx} + 2(2a - 9)xy'_x + 2(a - 6)y = 0.$

Integrating the equation twice, we arrive at a first-order linear equation:

$$x^3y'_x + [(a - 6)x^2 + b]y = C_1 + C_2x.$$

180. $x^3y'''_{xxx} + x^2(ax + b)y''_{xx} + cxy'_x + c(ax + b - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 - n + c = 0$.

181. $x^3y'''_{xxx} + x^2(2ax + b)y''_{xx} + x(a^2x^2 + 2abx + c)y'_x + (a^2bx^2 + bc - 2c)y = 0.$

Particular solutions: $y_1 = e^{-ax}x^{n_1}$, $y_2 = e^{-ax}x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 - n + c = 0$.

182. $x^3y'''_{xxx} + ax^ny''_{xx} + bxy'_x + b(ax^{n-2} - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 - m + b = 0$.

183. $x^3y'''_{xxx} + x^2(ax^n + b)y''_{xx} + x(ax^n + b - 1)y'_x + (ax^n + b - 3)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

184. $x^3y'''_{xxx} + x^2(ax^n + b + c + 1)y''_{xx}$
$+ x[\alpha x^{2n} + (ac + \beta)x^n + \gamma + bc]y'_x + (c - 1)(\alpha x^{2n} + \beta x^n + \gamma)y = 0.$

The substitution $w = xy'_x + (c - 1)y$ leads to a second-order linear equation of the form 14.1.2.146: $x^2w''_{xx} + x(ax^n + b)w'_x + (\alpha x^{2n} + \beta x^n + \gamma)w = 0$.

185. $(ax+b)x^3y'''_{xxx}+(cx+d)x^2y''_{xx}+s(ax+b)xy'_x+s[(c-2a)x+d-2b]y=0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 - n + s = 0$.

186. $x^6y'''_{xxx} + 6x^5y''_{xx} - ay + 2bx = 0.$

The substitution $x = t^{-1}$ leads to an equation of the form 15.1.2.62:

$$t^2y'''_{ttt} - 6y'_t + at^2y - 2bt = 0.$$

187. $x^2(x^n + a)y'''_{xxx} + x(bx^{k+1} + 2nx^n + cx)y''_{xx}$
$+ [2bkx^{k+1} + n(n - 1)x^n]y'_x + bk(k - 1)x^ky = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $(x^n + a)y'_x + (bx^k + c)y = C_1 + C_2x$.

15.1.3 Equations Containing Exponential Functions

► Equations with exponential functions.

1. $y'''_{xxx} - ae^{\lambda x}(a^2e^{2\lambda x} + 3a\lambda e^{\lambda x} + \lambda^2)y = 0.$

Particular solution: $y_0 = \exp\left(\frac{a}{\lambda}e^{\lambda x}\right)$. The substitution $y = \exp\left(\frac{a}{\lambda}e^{\lambda x}\right)\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.3.27:

$$z''_{xx} + 3ae^{\lambda x}z'_x + (3a^2e^{2\lambda x} + 3a\lambda e^{\lambda x})z = 0.$$

2. $y'''_{xxx} + ae^{\lambda x}y'_x + a\lambda e^{\lambda x}y = be^{\mu x}.$

Integrating yields a second-order linear equation: $y''_{xx}+ae^{\lambda x}y=b\mu^{-1}e^{\mu x}+C$ (see 14.1.3.1 for the solution of the corresponding homogeneous equation with $b = C = 0$).

3. $y'''_{xxx} + ae^{\lambda x}y'_x + b(ae^{\lambda x} + b^2)y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.3.10: $w''_{xx} - bw'_x + (ae^{\lambda x} + b^2)w = 0$.

4. $y'''_{xxx} + ae^{\lambda x}y'_x + [a(\lambda - b)e^{\lambda x} - b^3]y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + by'_x + (ae^{\lambda x} + b^2)y = Ce^{bx}$ (see 14.1.3.10 for the solution of the corresponding homogeneous equation with $C = 0$).

5. $y'''_{xxx} + (ae^{\lambda x} - b^2)y'_x + abe^{\lambda x}y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.3.10: $w''_{xx} - bw'_x + ae^{\lambda x}w = 0$.

6. $y'''_{xxx} + (ae^{\lambda x} - b^2)y'_x + a(\lambda - b)e^{\lambda x}y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + by'_x + ae^{\lambda x}y = Ce^{bx}$ (see 14.1.3.10 for the solution of the corresponding homogeneous equation with $C = 0$).

7. $y'''_{xxx} + (ae^{\lambda x} + b)y'_x + c(ae^{\lambda x} + b + c^2)y = 0.$

The substitution $w = y'_x + cy$ leads to a second-order linear equation of the form 14.1.3.10: $w''_{xx} - cw'_x + (ae^{\lambda x} + b + c^2)w = 0$.

8. $y'''_{xxx} + (ae^{2\lambda x} + be^{\lambda x})y'_x - c(ae^{2\lambda x} + be^{\lambda x} + c^2)y = 0.$

The substitution $w = y'_x - cy$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + cw'_x + (ae^{2\lambda x} + be^{\lambda x} + c^2)w = 0$.

9. $y'''_{xxx} - 3ae^{\lambda x}(ae^{\lambda x} + \lambda)y'_x + ae^{\lambda x}(2a^2e^{2\lambda x} - \lambda^2)y = 0.$

Particular solutions: $y_1 = \exp\left(\frac{a}{\lambda}e^{\lambda x}\right)$, $y_2 = x\exp\left(\frac{a}{\lambda}e^{\lambda x}\right)$.

10. $y'''_{xxx} - (3a^2e^{2\lambda x} + 3a\lambda e^{\lambda x} + b)y'_x + ae^{\lambda x}(2a^2e^{2\lambda x} - 2b - \lambda^2)y = 0.$

1°. Particular solutions with $b > 0$:

$$y_1 = \exp\left(\frac{a}{\lambda}e^{\lambda x} - x\sqrt{b}\right), \quad y_2 = \exp\left(\frac{a}{\lambda}e^{\lambda x} + x\sqrt{b}\right).$$

2°. Particular solutions with $b < 0$:

$$y_1 = \exp\left(\frac{a}{\lambda}e^{\lambda x}\right)\cos\left(x\sqrt{-b}\right), \quad y_2 = \exp\left(\frac{a}{\lambda}e^{\lambda x}\right)\sin\left(x\sqrt{-b}\right).$$

11. $y'''_{xxx} + ay''_{xx} + be^{\lambda x}y'_x + abe^{\lambda x}y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.3.1: $w''_{xx} + be^{\lambda x}w = 0$.

12. $y'''_{xxx} + ay''_{xx} + (be^{\lambda x} + c)y'_x + [b(a + \lambda)e^{\lambda x} + ac]y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + (be^{\lambda x} + c)y = Ce^{-ax}$ (see 14.1.3.2 for the solution of the corresponding homogeneous equation with $C = 0$).

13. $y'''_{xxx} + ay''_{xx} + (be^{2\lambda x} + ce^{\lambda x})y'_x + a(be^{2\lambda x} + ce^{\lambda x})y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (be^{2\lambda x} + ce^{\lambda x})w = 0$.

14. $y'''_{xxx} + ae^{\lambda x}y''_{xx} - b^2(ae^{\lambda x} + b)y = 0.$

The substitution $w = y'_x - by$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (ae^{\lambda x} + b)w'_x + b(ae^{\lambda x} + b)w = 0$.

15. $y'''_{xxx} + ae^{\lambda x}y''_{xx} - by'_x - abe^{\lambda x}y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\left(-x\sqrt{b}\right)$, $y_2 = \exp\left(x\sqrt{b}\right)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\left(x\sqrt{-b}\right)$, $y_2 = \sin\left(x\sqrt{-b}\right)$.

16. $y'''_{xxx} + ae^{\lambda x}y''_{xx} + abe^{\lambda x}y'_x + b^3y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (ae^{\lambda x} - b)w'_x + b^2w = 0.$

17. $y'''_{xxx} + ae^{\lambda x}y''_{xx} + abe^{\lambda x}y'_x + b^2(ae^{\lambda x} - b)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

18. $y'''_{xxx} + ae^{\lambda x}y''_{xx} - b(2ae^{\lambda x} + 3b)y'_x + b^2(ae^{\lambda x} + 2b)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

19. $y'''_{xxx} + ae^{\lambda x}y''_{xx} + (be^{\mu x} - c^2)y'_x - c(ace^{\lambda x} + be^{\mu x})y = 0.$

Particular solution: $y_0 = e^{cx}$.

20. $y'''_{xxx} + ae^{\lambda x}y''_{xx} + (abe^{\lambda x} - b^2 + c)y'_x + c(ae^{\lambda x} - b)y = 0.$

Particular solutions: $y_1 = e^{\beta_1 x}$, $y_2 = e^{\beta_2 x}$, where β_1 and β_2 are roots of the quadratic equation $\beta^2 + b\beta + c = 0$.

21. $y'''_{xxx} + ae^{\lambda x}y''_{xx} + [a(b + \lambda)e^{\lambda x} - b^2]y'_x + ab\lambda e^{\lambda x}y = 0.$

Particular solutions: $y_1 = e^{-bx}$, $y_2 = e^{-bx}\int \exp\Bigl(2bx - \frac{a}{\lambda}e^{\lambda x}\Bigr)\,dx$.

22. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} - ab^2e^{\lambda x}y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + ae^{\lambda x}w'_x - abe^{\lambda x}w = 0.$

23. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} - c^2(ae^{\lambda x} + b + c)y = 0.$

The substitution $w = y'_x - cy$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (ae^{\lambda x} + b + c)w'_x + c(ae^{\lambda x} + b + c)w = 0.$

24. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + c(ae^{\lambda x} + b)y'_x + c^3y = 0.$

The substitution $w = y'_x + cy$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (ae^{\lambda x} + b - c)w'_x + c^2w = 0.$

25. $y'''_{xxx} + (be^{ax} + 2a)y''_{xx} - a(be^{ax} + a)y'_x - 2a^3y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = e^{-ax} + b/a$.

26. $y'''_{xxx} = (e^{\lambda x} - a)y''_{xx} + (ae^{\lambda x} - b)y'_x + be^{\lambda x}y.$

Particular solutions: $y_1 = e^{\beta_1 x}$, $y_2 = e^{\beta_2 x}$, where β_1 and β_2 are roots of the quadratic equation $\beta^2 + a\beta + b = 0$.

27. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + (ce^{\lambda x} + d)y'_x - s[(as + c)e^{\lambda x} + bs + d + s^2]y = 0.$

The substitution $w = y'_x - sy$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (ae^{\lambda x} + b + s)w'_x + [(as + c)e^{\lambda x} + bs + d + s^2]w = 0.$

28. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + (ce^{2\lambda x} + d)y'_x - s(ce^{2\lambda x} + ase^{\lambda x} + bs + d + s^2)y = 0.$

The substitution $w = y'_x - sy$ leads to a second-order linear equation of the form 14.1.3.27: $w''_{xx} + (ae^{\lambda x} + b + s)w'_x + (ce^{2\lambda x} + ase^{\lambda x} + bs + d + s^2)w = 0.$

29. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + (ce^{\lambda x} + d)y'_x - ke^{\lambda x}[k(a+k)e^{2\lambda x}$
$+ (a\lambda + 3k\lambda + bk + c)e^{\lambda x} + \lambda^2 + b\lambda + d]y = 0.$

Particular solution: $y_0 = \exp\Bigl(\frac{k}{\lambda}e^{\lambda x}\Bigr)$. The substitution $y = \exp\Bigl(\frac{k}{\lambda}e^{\lambda x}\Bigr)\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.3.27.

30. $y'''_{xxx} + (2ae^{\lambda x} + b)y''_{xx} + ae^{\lambda x}(ae^{\lambda x} + 2b + 3\lambda)y'_x$
$+ ae^{\lambda x}[a(b + 2\lambda)e^{\lambda x} + b\lambda + \lambda^2]y = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\frac{a}{\lambda}e^{\lambda x}\Bigr)$, $y_2 = x\exp\Bigl(-\frac{a}{\lambda}e^{\lambda x}\Bigr)$.

31. $y'''_{xxx} + (ae^{\lambda x} - b)y''_{xx} + ce^{\mu x}y'_x - b(abe^{\lambda x} + ce^{\mu x})y = 0.$

Particular solution: $y_0 = e^{bx}$.

32. $y'''_{xxx} + (e^{\lambda x} + a)y''_{xx} + (ae^{\lambda x} + be^{\mu x})y'_x + abe^{\mu x}y = 0.$

Particular solution: $y_0 = e^{-ax}$.

33. $y'''_{xxx} + (ae^{\lambda x} + be^{\mu x})y''_{xx} + cy'_x + c(ae^{\lambda x} + be^{\mu x})y = 0.$

The substitution $w = y''_{xx} + cy$ leads to a first-order linear equation:

$$w'_x + (ae^{\lambda x} + be^{\mu x})w = 0.$$

34. $y'''_{xxx} + (ae^{\lambda x} + be^{\nu x})y''_{xx} + [abe^{(\lambda+\nu)x} + b(c+\nu)e^{\nu x} - c^2]y'_x$
$+ c[abe^{(\lambda+\nu)x} - ace^{\lambda x} + b\nu e^{\nu x}]y = 0.$

Particular solutions: $y_1 = e^{-cx}$, $y_2 = e^{-cx}\int \exp\Bigl(2cx - \frac{b}{\nu}e^{\nu x}\Bigr)\,dx.$

35. $y'''_{xxx} + ae^{\lambda x}(be^{\mu x} + 2\mu)y''_{xx} - \mu[abe^{(\lambda+\mu)x} + \mu]y'_x - 2a\mu^3e^{\lambda x}y = 0.$

Particular solutions: $y_1 = e^{\mu x}$, $y_2 = e^{-\mu x} + b/\mu$.

36. $(ae^x + b)y'''_{xxx} - ae^x y = 0.$

Particular solution: $y_0 = ae^x + b$.

37. $(bce^{ax} + a + c)y'''_{xxx} - (bc^3e^{ax} + a^3 + c^3)y'_x + ac(a^2 - c^2)y = 0.$

Particular solutions: $y_1 = e^{cx}$, $y_2 = e^{-ax} + b$.

38. $(ae^{\lambda x} + b)y'''_{xxx} + (ce^{\lambda x} + d)y''_{xx} + k(ae^{\lambda x} + b)y'_x + k(ce^{\lambda x} + d)y = 0.$

1°. Particular solutions with $k > 0$: $y_1 = \cos\bigl(x\sqrt{k}\bigr)$, $y_2 = \sin\bigl(x\sqrt{k}\bigr)$.

2°. Particular solutions with $k < 0$: $y_1 = \exp\bigl(-x\sqrt{-k}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-k}\bigr)$.

► **Equations with power and exponential functions.**

39. $y'''_{xxx} + ae^{\lambda x}y'_x + bx(ae^{\lambda x} + b^2x^2 - 3b)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$.

40. $y'''_{xxx} + (ax + b)e^{\lambda x}y'_x - ae^{\lambda x}y = 0.$

Particular solution: $y_0 = ax + b$.

41. $y'''_{xxx} + (ax+b)e^{\lambda x}y'_x - 2ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

42. $y'''_{xxx} + ae^{\lambda x}y''_{xx} + bx^n y'_x + bx^{n-1}(axe^{\lambda x} + n)y = 0.$

The substitution $w = y''_{xx} + bx^n y$ leads to a first-order linear equation: $w'_x + ae^{\lambda x}w = 0$.

43. $y'''_{xxx} + axe^{\lambda x}y''_{xx} + (bx^2 - ae^{\lambda x})y'_x + bx(ax^2e^{\lambda x} + 3)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-\frac{1}{2}x^2\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}x^2\sqrt{-b}\bigr)$.

44. $y'''_{xxx} + ax^2e^{\lambda x}y''_{xx} - 2axe^{\lambda x}y'_x + 2ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

45. $y'''_{xxx} + (ax + be^{\lambda x})y''_{xx} + a(bxe^{\lambda x} + 2)y'_x + abe^{\lambda x}y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\int\exp\bigl(\frac{1}{2}ax^2\bigr)\,dx$.

46. $y'''_{xxx} + (abxe^{\lambda x} + be^{\lambda x} + a)y''_{xx} + a^2bxe^{\lambda x}y'_x - a^2be^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

47. $y'''_{xxx} + ax^n(be^{\lambda x} + 2\lambda)y''_{xx} - \lambda(abx^ne^{\lambda x} + \lambda)y'_x - 2a\lambda^3x^ny = 0.$

Particular solutions: $y_1 = e^{\lambda x}$, $y_2 = e^{-\lambda x} + b/\lambda$.

48. $y'''_{xxx} + (ax^n - 2be^{\lambda x})y''_{xx} - be^{\lambda x}(2ax^n - be^{\lambda x} + 3\lambda)y'_x$
$+ be^{\lambda x}[ax^n(be^{\lambda x} - \lambda) + 2b\lambda e^{\lambda x} - \lambda^2]y = 0.$

Particular solutions: $y_1 = \exp\Bigl(\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$, $y_2 = x\exp\Bigl(\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

49. $xy'''_{xxx} + ay''_{xx} + x(be^{\lambda x} + c)y'_x + [b(\lambda x + a)e^{\lambda x} + ac]y = 0.$

The substitution $w = y''_{xx} + (be^{\lambda x} + c)y$ leads to a first-order linear equation: $xw'_x + aw = 0$.

50. $xy'''_{xxx} + axe^{\lambda x}y'_x - 2ae^{\lambda x}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.3.1: $w''_{xx} + ae^{\lambda x}w = 0$.

51. $xy'''_{xxx} = (e^{\lambda x} - ax)y''_{xx} + (ae^{\lambda x} - bx)y'_x + be^{\lambda x}y.$

Particular solutions: $y_1 = e^{\beta_1 x}$, $y_2 = e^{\beta_2 x}$, where β_1 and β_2 are roots of the quadratic equation $\beta^2 + a\beta + b = 0$.

52. $xy'''_{xxx} + (axe^{\lambda x} + 3)y''_{xx} + a(bx + 2)e^{\lambda x}y'_x + b(abxe^{\lambda x} + ae^{\lambda x} - b^2x)y = 0.$

Particular solutions:

$$y_1 = x^{-1}\exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr), \quad y_2 = x^{-1}\exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr).$$

53. $x^2y'''_{xxx} + x(axe^{\lambda x} + b)y''_{xx} + [a(b-2)xe^{\lambda x} + c]y'_x + a(c - b + 2)e^{\lambda x}y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 + (b-3)n + c - b + 2 = 0$.

54. $x^3 y'''_{xxx} + bx^2 e^{\lambda x} y''_{xx} + axy'_x + a(be^{\lambda x} - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 - n + a = 0$.

55. $x^3 y'''_{xxx} + x^2(ae^{\lambda x} + b)y''_{xx} + x(abe^{\lambda x} + c - b)y'_x + c(ae^{\lambda x} - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 + (b-1)n + c = 0$.

56. $(ae^x + bx)y'''_{xxx} - ae^x y = 0.$

Particular solution: $y_0 = ae^x + bx$.

57. $(ae^x + bx^2)y'''_{xxx} - ae^x y = 0.$

Particular solution: $y_0 = ae^x + bx^2$.

58. $(axe^x + b)y'''_{xxx} + by = 0.$

Particular solution: $y_0 = ax + be^{-x}$.

59. $(ax^2 e^x + b)y'''_{xxx} + by = 0.$

Particular solution: $y_0 = ax^2 + be^{-x}$.

15.1.4 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine.**

1. $y'''_{xxx} + ay''_{xx} + b\sinh^2 x\, y'_x + ab\sinh^2 x\, y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.4.1: $w''_{xx} + b\sinh^2 x\, w = 0$.

2. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} + by'_x + ab\sinh^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

3. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\sinh^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\sinh^n(\lambda x)w = 0.$$

4. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} + ab\sinh^n(\lambda x)y'_x + b^2[a\sinh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

5. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} - b[2a\sinh^n(\lambda x) + 3b]y'_x + b^2[a\sinh^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

6. $y'''_{xxx} + a\sinh^n x\, y''_{xx} + (ab\sinh^n x + c - b^2)y'_x + c(a\sinh^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

7. $y'''_{xxx} + ax\sinh^n x\, y''_{xx} + (bx^2 - a\sinh^n x)y'_x + bx(ax^2\sinh^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\,\bigr)$.

8. $y'''_{xxx} + ax^2\sinh^n(\lambda x)y''_{xx} - 2ax\sinh^n(\lambda x)y'_x + 2a\sinh^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

9. $y'''_{xxx} = (\sinh^n x - a)y''_{xx} + (a\sinh^n x - b)y'_x + b\sinh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

10. $y'''_{xxx} + (a\sinh^n x + bx)y''_{xx} + b(ax\sinh^n x + 2)y'_x + ab\sinh^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\,dx$.

11. $xy'''_{xxx} + x(a\sinh^2 x + b)y'_x - 2(a\sinh^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.4.1: $w''_{xx} + (a\sinh^2 x + b)w = 0$.

12. $x^2y'''_{xxx} + (ax^2\sinh^n x + bx)y''_{xx} + [a(b-2)x\sinh^n x + c]y'_x + a(c - b + 2)\sinh^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

13. $x^3y'''_{xxx} + x^2(a\sinh^n x + b)y''_{xx} + x(ab\sinh^n x + c - b)y'_x + c(a\sinh^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

14. $\sinh^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\sinh^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

15. $\sinh^n x\, y'''_{xxx} + ay''_{xx} + b\sinh^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

16. $\sinh^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\sinh^n x)y'_x + b^2(a + 2b\sinh^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

17. $\sinh^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\sinh^n x + a]y'_x + b(1 - a\sinh^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

18. $\sinh^n(\lambda x)y'''_{xxx} + ax^2y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

19. $\sinh^n x\, y'''_{xxx} + (a\sinh^n x + ax + 1)y''_{xx} + a^2xy'_x - a^2y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

20. $\sinh^n x\, y'''_{xxx} + (ax\sinh^n x + 1)y''_{xx} + a(x + 2\sinh^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\int \exp\bigl(\frac{1}{2}ax^2\bigr)\,dx.$

21. $x\sinh^n x\, y'''_{xxx} + (3\sinh^n x + x)y''_{xx} + (ax\sinh^n x + 2)y'_x + a(\sinh^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr).$

22. $x^3\sinh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\sinh^n x\, y'_x + 2(2\sinh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2.$

23. $x^3\sinh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\sinh^n x\, y'_x + 6(2\sinh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3.$

24. $x^3\sinh^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \sinh^n x)y'_x + (a - 3\sinh^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x).$

25. $x^3\sinh^n x\, y'''_{xxx} + x^2(\sinh^n x + a)y''_{xx} + x[a - (b+1)\sinh^n x]y'_x + b(2\sinh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}.$

► **Equations with hyperbolic cosine.**

26. $y'''_{xxx} + ay''_{xx} + b\cosh(2x)y'_x + ab\cosh(2x)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.4.9: $w''_{xx} + b\cosh(2x)w = 0.$

27. $y'''_{xxx} + ay''_{xx} + b\cosh^2 x\, y'_x + ab\cosh^2 x\, y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.4.10: $w''_{xx} + b\cosh^2 x\, w = 0.$

28. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} + by'_x + ab\cosh^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr).$

29. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\cosh^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\cosh^n(\lambda x)w = 0.$$

30. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} + ab\cosh^n(\lambda x)y'_x + b^2[a\cosh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr).$

31. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} - b[2a\cosh^n(\lambda x) + 3b]y'_x + b^2[a\cosh^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}.$

32. $y'''_{xxx} + a\cosh^n x\, y''_{xx} + (ab\cosh^n x + c - b^2)y'_x + c(a\cosh^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

33. $y'''_{xxx} + ax\cosh^n x\, y''_{xx} + (bx^2 - a\cosh^n x)y'_x + bx(ax^2\cosh^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$.

34. $y'''_{xxx} + ax^2\cosh^n(\lambda x)y''_{xx} - 2ax\cosh^n(\lambda x)y'_x + 2a\cosh^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

35. $y'''_{xxx} = (\cosh^n x - a)y''_{xx} + (a\cosh^n x - b)y'_x + b\cosh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

36. $y'''_{xxx} + (a\cosh^n x + bx)y''_{xx} + b(ax\cosh^n x + 2)y'_x + ab\cosh^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\int \exp\bigl(\frac{1}{2}bx^2\bigr)\,dx$.

37. $xy'''_{xxx} + x[a\cosh(2x) + b]y'_x - 2[a\cosh(2x) + b]y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.4.9: $w''_{xx} + [a\cosh(2x) + b]w = 0$.

38. $xy'''_{xxx} + x(a\cosh^2 x + b)y'_x - 2(a\cosh^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.4.10: $w''_{xx} + (a\cosh^2 x + b)w = 0$.

39. $xy'''_{xxx} = (\cosh^n x - ax)y''_{xx} + (a\cosh^n x - bx)y'_x + b\cosh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

40. $x^2 y'''_{xxx} + (ax^2\cosh^n x + bx)y''_{xx} + [a(b-2)x\cosh^n x + c]y'_x + a(c - b + 2)\cosh^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

41. $x^3 y'''_{xxx} + x^2(a\cosh^n x + b)y''_{xx} + x(ab\cosh^n x + c - b)y'_x + c(a\cosh^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

42. $\cosh^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\cosh^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

43. $\cosh^n x\, y'''_{xxx} + ay''_{xx} + b\cosh^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

44. $\cosh^n x\, y'''_{xxx} + a y''_{xx} - b(2a + 3b\cosh^n x)y'_x + b^2(a + 2b\cosh^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

45. $\cosh^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\cosh^n x + a]y'_x + b(1 - a\cosh^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

46. $\cosh^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

47. $\cosh^n x\, y'''_{xxx} + (a\cosh^n x + ax + 1)y''_{xx} + a^2 x y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

48. $\cosh^n x\, y'''_{xxx} + (ax\cosh^n x + 1)y''_{xx} + a(x + 2\cosh^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

49. $x\cosh^n x\, y'''_{xxx} + (3\cosh^n x + x)y''_{xx} + (ax\cosh^n x + 2)y'_x + a(\cosh^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

50. $x^3\cosh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\cosh^n x\, y'_x + 2(2\cosh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

51. $x^3\cosh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\cosh^n x\, y'_x + 6(2\cosh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

52. $x^3\cosh^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \cosh^n x)y'_x + (a - 3\cosh^n x)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\ln|x|\bigr)$, $y_2 = \sin\bigl(\ln|x|\bigr)$.

53. $x^3\cosh^n x\, y'''_{xxx} + x^2(\cosh^n x + a)y''_{xx} + x[a - (b+1)\cosh^n x]y'_x + b(2\cosh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

► **Equations with hyperbolic sine and cosine.**

54. $y'''_{xxx} + [a\sinh(2x) + b]y'_x + a\cosh(2x)y = 0.$

This is a special case of equation 15.1.9.26 with $f(x) = \frac{1}{2}[a\sinh(2x) + b]$.

55. $y'''_{xxx} + [a\sinh(2x) + b]y'_x + 2a\cosh(2x)\,y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + [a\sinh(2x) + b]y = C$.

56. $y'''_{xxx} + [a\cosh(2x) + b]y'_x + a\sinh(2x)y = 0.$

Solution: $y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2$. Here, w_1 and w_2 form a fundamental set of solutions of the modified Mathieu equation 14.1.4.9: $4w''_{xx} + [a\cosh(2x) + b]w = 0$.

57. $y'''_{xxx} + [a\cosh(2x) + b]y'_x + 2a\sinh(2x)\,y = 0.$

Integrating yields a second-order linear equation: $y''_{xx}+[a\cosh(2x)+b]y=C$ (see 14.1.4.9 for the solution of the corresponding modified homogeneous Mathieu equation with $C=0$).

58. $y'''_{xxx} + (b\cosh x - a^2)y'_x + b(\sinh x - a\cosh x)y = 0.$

This is a special case of equation 15.1.9.29 with $f(x) = b\cosh x$.

► **Equations with hyperbolic tangent.**

59. $y'''_{xxx} - a^3\tanh(ax)y = 0.$

Particular solution: $y_0 = \cosh(ax)$. The substitution $y = \cosh(ax)\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.4.43: $z''_{xx} + 3a\tanh(ax)z'_x + 3a^2 z = 0$.

60. $y'''_{xxx} = ay'_x + (1-a)\tanh x\,y.$

This is a special case of equation 15.1.9.30 with $f(x) = a$ and $g(x) = \cosh x$.

61. $y'''_{xxx} - 3a^2y'_x + 2a^3\tanh(ax)y = 0.$

Particular solutions: $y_1 = \cosh(ax)$, $y_2 = x\cosh(ax)$.

62. $y'''_{xxx} = a\tanh^n x y'_x + \tanh x(1 - a\tanh^n x)y.$

This is a special case of equation 15.1.9.30 with $f(x) = a\tanh^n x$ and $g(x) = \cosh x$.

63. $y'''_{xxx} + ay''_{xx} + [b\tanh(\lambda x) + c]y'_x + a[b\tanh(\lambda x) + c]y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.4.22: $w''_{xx} + [b\tanh(\lambda x) + c]w = 0$.

64. $$y'''_{xxx} + ay''_{xx} - \lambda[2a\tanh(\lambda x) + 3\lambda]y'_x + \lambda^2[2a\tanh^2(\lambda x) + 2\lambda\tanh(\lambda x) - a]y = 0.$$

Particular solutions: $y_1 = \cosh(\lambda x)$, $y_2 = x\cosh(\lambda x)$.

65. $y'''_{xxx} - \tanh x\,y''_{xx} - ay'_x + a\tanh x\,y = 0.$

1°. Solution for $a > 0$: $y = C_1\exp(-x\sqrt{a}) + C_2\exp(x\sqrt{a}) + C_3\cosh x$.

2°. Solution for $a < 0$: $y = C_1\cos(x\sqrt{-a}) + C_2\sin(x\sqrt{-a}) + C_3\cosh x$.

66. $y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\tanh^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\tanh^n(\lambda x)w = 0.$$

67. $y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} + ab\tanh^n(\lambda x)y'_x + b^2[a\tanh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

68. $$y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} - b[2a\tanh^n(\lambda x) + 3b]y'_x + b^2[a\tanh^n(\lambda x) + 2b]y = 0.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

69. $y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} + [ab\tanh^n(\lambda x) + c - b^2]y'_x + c[a\tanh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp(\beta_1 x)$, $y_2 = \exp(\beta_2 x)$, where β_1 and β_2 are roots of the quadratic equation $\beta^2 + b\beta + c = 0$.

70. $y'''_{xxx} + ax^n y''_{xx} - (2ax^n\tanh x + 3)y'_x + [ax^n(2\tanh^2 x - 1) + 2\tanh x]y = 0.$

Particular solutions: $y_1 = \cosh x$, $y_2 = x\cosh x$.

71. $y'''_{xxx} + a\tanh^n x\, y''_{xx} - (2a\tanh^{n+1} x + 3)y'_x + (2a\tanh^{n+2} x - a\tanh^n x + 2\tanh x)y = 0.$

Particular solutions: $y_1 = \cosh x$, $y_2 = x\cosh x$.

72. $y'''_{xxx} + ax\tanh^n x\, y''_{xx} + (bx^2 - a\tanh^n x)y'_x + bx(ax^2\tanh^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$.

73. $y'''_{xxx} + ax^2\tanh^n(\lambda x)y''_{xx} - 2ax\tanh^n(\lambda x)y'_x + 2a\tanh^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

74. $y'''_{xxx} = (\tanh^n x - a)y''_{xx} + (a\tanh^n x - b)y'_x + b\tanh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

75. $y'''_{xxx} + (a\tanh^n x + bx)y''_{xx} + b(ax\tanh^n x + 2)y'_x + ab\tanh^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\frac{1}{2}bx^2\bigr)\,dx$.

76. $y'''_{xxx} + [ax^n(\tanh x - b) - b]y''_{xx} + [a(b^2 - 1)x^n - 1]y'_x + b[ax^n(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

77. $y'''_{xxx} + [\lambda\tanh(\lambda x)(ax^n - 1) - ax^{n-1}]y''_{xx} - a\lambda^2 x^n y'_x + a\lambda^2 x^{n-1}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cosh(\lambda x)$.

78. $y'''_{xxx} + (a\tanh^{n+1} x - ab\tanh^n x - b)y''_{xx} + [a(b^2 - 1)\tanh^n x - 1]y'_x + b(-ab\tanh^{n+1} x + a\tanh^n x + 1)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

79. $x^2 y'''_{xxx} + (ax^2\tanh^n x + bx)y''_{xx} + [a(b - 2)x\tanh^n x + c]y'_x + a(c - b + 2)\tanh^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b - 3)m + c - b + 2 = 0$.

80. $x^3 y'''_{xxx} + x^2(a\tanh^n x + b)y''_{xx} + x(ab\tanh^n x + c - b)y'_x + c(a\tanh^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b - 1)m + c = 0$.

81. $\tanh^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\tanh^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

82. $\tanh^n x\, y'''_{xxx} + ay''_{xx} + b\tanh^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

83. $\tanh^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\tanh^n x)y'_x + b^2(a + 2b\tanh^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

84. $\tanh^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\tanh^n x + a]y'_x + b(1 - a\tanh^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

85. $\tanh^n(\lambda x)y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

86. $\tanh^n x\, y'''_{xxx} + (a\tanh^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

87. $\tanh^n x\, y'''_{xxx} + (ax\tanh^n x + 1)y''_{xx} + a(x + 2\tanh^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

88. $x\tanh^n x\, y'''_{xxx} + (3\tanh^n x + x)y''_{xx} + (ax\tanh^n x + 2)y'_x + a(\tanh^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

89. $x^3\tanh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\tanh^n x\, y'_x + 2(2\tanh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

90. $x^3\tanh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\tanh^n x\, y'_x + 6(2\tanh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

91. $x^3\tanh^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \tanh^n x)y'_x + (a - 3\tanh^n x)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\ln|x|\bigr)$, $y_2 = \sin\bigl(\ln|x|\bigr)$.

92. $x^3\tanh^n x\, y'''_{xxx} + x^2(\tanh^n x + a)y''_{xx} + x[a - (b + 1)\tanh^n x]y'_x + b(2\tanh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

► **Equations with hyperbolic cotangent.**

93. $y'''_{xxx} - a^3\coth(ax)y = 0.$

Particular solution: $y_0 = \sinh(ax)$. The substitution $y = \sinh(ax)\displaystyle\int z(x)\,dx$ leads to a second-order linear equation of the form 14.1.4.52: $z''_{xx} + 3a\coth(ax)z'_x + 3a^2 z = 0$.

94. $y'''_{xxx} = ay'_x + (1-a)\coth x\, y.$

This is a special case of equation 15.1.9.30 with $f(x) = a$ and $g(x) = \sinh x$.

95. $y'''_{xxx} - 3a^2 y'_x + 2a^3 \coth(ax) y = 0.$

Particular solutions: $y_1 = \sinh(ax)$, $y_2 = x\sinh(ax)$.

96. $y'''_{xxx} = a\coth^n x\, y'_x + \coth x(1 - a\coth^n x)y.$

This is a special case of equation 15.1.9.30 with $f(x) = a\coth^n x$ and $g(x) = \sinh x$.

97. $y'''_{xxx} + ay''_{xx} + [b\coth(\lambda x) + c]y'_x + a[b\coth(\lambda x) + c]y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.4.44: $w''_{xx} + [b\coth(\lambda x) + c]w = 0$.

98. $y'''_{xxx} + ay''_{xx} - \lambda[2a\coth(\lambda x) + 3\lambda]y'_x + \lambda^2[2a\coth^2(\lambda x) + 2\lambda\coth(\lambda x) - a]y = 0.$

Particular solutions: $y_1 = \sinh(\lambda x)$, $y_2 = x\sinh(\lambda x)$.

99. $y'''_{xxx} - \coth x\, y''_{xx} - ay'_x + a\coth x\, y = 0.$

1°. Solution for $a > 0$: $y = C_1\exp(-x\sqrt{a}\,) + C_2\exp(x\sqrt{a}\,) + C_3\sinh x$.

2°. Solution for $a < 0$: $y = C_1\cos(x\sqrt{-a}) + C_2\sin(x\sqrt{-a}) + C_3\sinh x$.

100. $y'''_{xxx} + (a\coth x - ab - b)y''_{xx} + (ab^2 - a - 1)y'_x + b(-ab\coth x + a + 1)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

101. $y'''_{xxx} + a\coth^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\coth^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\coth^n(\lambda x)w = 0.$$

102. $y'''_{xxx} + a\coth^n(\lambda x)y''_{xx} + ab\coth^n(\lambda x)y'_x + b^2[a\coth^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

103. $y'''_{xxx} + a\coth^n(\lambda x)y''_{xx} - b[2a\coth^n(\lambda x) + 3b]y'_x + b^2[a\coth^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

104. $y'''_{xxx} + ax\coth^n x\, y''_{xx} + (bx^2 - a\coth^n x)y'_x + bx(ax^2\coth^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$.

105. $y'''_{xxx} + ax^2\coth^n(\lambda x)y''_{xx} - 2ax\coth^n(\lambda x)y'_x + 2a\coth^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

106. $y'''_{xxx} + ax^n y''_{xx} - (2ax^n\coth x + 3)y'_x + [ax^n(2\coth^2 x - 1) + 2\coth x]y = 0.$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

107. $y'''_{xxx} + a\coth^n x\, y''_{xx} - (2a\coth^{n+1}x + 3)y'_x + (2a\coth^{n+2}x - a\coth^n x + 2\coth x)y = 0.$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

108. $y'''_{xxx} + (a\coth^n x + bx)y''_{xx} + b(ax\coth^n x + 2)y'_x + ab\coth^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\,dx$.

109. $y'''_{xxx} + [\lambda\coth(\lambda x)(ax^n - 1) - ax^{n-1}]y''_{xx} - a\lambda^2 x^n y'_x + a\lambda^2 x^{n-1} y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sinh(\lambda x)$.

110. $x^2 y'''_{xxx} + (ax^2\coth^n x + bx)y''_{xx} + [a(b-2)x\coth^n x + c]y'_x + a(c - b + 2)\coth^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

111. $x^3 y'''_{xxx} + x^2(a\coth^n x + b)y''_{xx} + x(ab\coth^n x + c - b)y'_x + c(a\coth^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

112. $\coth^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\coth^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

113. $\coth^n x\, y'''_{xxx} + ay''_{xx} + b\coth^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

114. $\coth^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\coth^n x)y'_x + b^2(a + 2b\coth^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

115. $\coth^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\coth^n x + a]y'_x + b(1 - a\coth^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

116. $\coth^n(\lambda x)y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

117. $\coth^n x\, y'''_{xxx} + (a\coth^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

118. $\coth^n x\, y'''_{xxx} + (ax\coth^n x + 1)y''_{xx} + a(x + 2\coth^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

119. $x \coth^n x \, y'''_{xxx} + (3\coth^n x + x)y''_{xx} + (ax\coth^n x + 2)y'_x + a(\coth^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

120. $x^3\coth^n x \, y'''_{xxx} + ax^2 y''_{xx} - 2x\coth^n x \, y'_x + 2(2\coth^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

121. $x^3\coth^n x \, y'''_{xxx} + ax^2 y''_{xx} - 6x\coth^n x \, y'_x + 6(2\coth^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

122. $x^3\coth^n x \, y'''_{xxx} + ax^2 y''_{xx} + x(a - \coth^n x)y'_x + (a - 3\coth^n x)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\ln x\bigr)$, $y_2 = \sin\bigl(\ln x\bigr)$.

123. $x^3\coth^n x \, y'''_{xxx} + x^2(\coth^n x + a)y''_{xx} + x[a - (b+1)\coth^n x]y'_x + b(2\coth^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

15.1.5 Equations Containing Logarithmic Functions

► **Equations with logarithmic functions.**

1. $y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + by'_x + ab\ln^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

2. $y'''_{xxx} + a\ln^n x \, y''_{xx} + ab\ln^n x \, y'_x + b^2(a\ln^n x - b)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

3. $y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} - b[2a\ln^n(\lambda x) + 3b]y'_x + b^2[a\ln^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

4. $y'''_{xxx} + a\ln^n x \, y''_{xx} + (ab\ln^n x + c - b^2)y'_x + c(a\ln^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

5. $y'''_{xxx} + (a\ln^n x + b)y''_{xx} + c\,y'_x + c(a\ln^n x + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

6. $y'''_{xxx} = (\ln^n x - a)y''_{xx} + (a\ln^n x - b)y'_x + b\ln^n x \, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

7. $\ln^n x \, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\ln^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

8. $\ln^n x\, y'''_{xxx} + y''_{xx} + [(a - b^2)\ln^n x + b]y'_x + a(1 - b\ln^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + a = 0$.

9. $\ln^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\ln^n x)y'_x + b^2(a + 2b\ln^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

10. $\ln^n(\lambda x)y'''_{xxx} + ay''_{xx} + b\ln^n(\lambda x)y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

► **Equations with power and logarithmic functions.**

11. $y'''_{xxx} + (ax + b)\ln^n(\lambda x)y'_x - a\ln^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

12. $y'''_{xxx} + (ax + b)\ln^n(\lambda x)y'_x - 2a\ln^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

13. $y'''_{xxx} + a\ln^n x\, y''_{xx} + (bx + c)\, y'_x + (abx\ln^n x + ac\ln^n x + b)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + (bx + c)y = C\exp\bigl(-a\int \ln^n x\, dx\bigr)$ (see 14.1.2.2 for the solution of the corresponding homogeneous equation with $C = 0$).

14. $y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\ln^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\ln^n(\lambda x)w = 0.$$

15. $y'''_{xxx} + ax\ln^n x\, y''_{xx} + (bx^2 - a\ln^n x)y'_x + bx(ax^2\ln^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$.

16. $y'''_{xxx} + (a\ln^n x + bx)y''_{xx} + b(ax\ln^n x + 2)y'_x + ab\ln^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\, dx$.

17. $y'''_{xxx} + (ax\ln^n x + b)y''_{xx} + a(bx - 1)\ln^n x\, y'_x - ab\ln^n x\, y = 0.$

The substitution $w = y'_x + by$ leads to a second-order linear equation of the form 14.1.5.13: $w''_{xx} + ax\ln^n x\, w'_x - a\ln^n x\, w = 0$.

18. $y'''_{xxx} + (abx\ln^n x + a\ln^n x + b)y''_{xx} + ab^2x\ln^n x\, y'_x - ab^2\ln^n x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

19. $y'''_{xxx} + ax^2\ln^n(\lambda x)y''_{xx} - 2ax\ln^n(\lambda x)y'_x + 2a\ln^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

20. $xy'''_{xxx} + axy''_{xx} - b(bx\ln^2 x + 1)y'_x - ab(bx\ln^2 x + 1)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.5.3: $xw''_{xx} - (b^2x\ln^2 x + b)w = 0$.

21. $xy'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bxy'_x + ab\ln^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

22. $xy'''_{xxx} + ax\ln x\, y''_{xx} + (abx\ln x - b^2x + a)y'_x + aby = 0.$

Particular solutions: $y_1 = e^{-bx}$, $y_2 = e^{-bx}\displaystyle\int x^{-ax}e^{(a+2b)x}\,dx$.

23. $xy'''_{xxx} = (\ln^n x - ax)y''_{xx} + (a\ln^n x - bx)y'_x + b\ln^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

24. $xy'''_{xxx} + (ax\ln^n x + 3)y''_{xx} + (2a\ln^n x + bx)y'_x + b(ax\ln^n x + 1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{b}\,\bigr)$.

25. $xy'''_{xxx} + (ax\ln^n x + 3)y''_{xx} + (abx\ln^n x + 2a\ln^n x - b^2x)y'_x + b(a\ln^n x - b)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^{-1}e^{-bx}$.

26. $xy'''_{xxx} + [a(b - \ln x)x^n + 2]y''_{xx} + ax^{n-1}y'_x - ax^{n-2}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - b + 1$.

27. $x^2y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bx^2y'_x + ab\ln^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

28. $x^2y'''_{xxx} + x^2(a\ln x + b)y''_{xx} + 2axy'_x - ay = 0.$

Integrating the equation twice, we arrive at a first-order linear equation: $y'_x + (a\ln x + b)y = C_1 + C_2x$.

29. $x^2y'''_{xxx} - 3ax[ax\ln^2(\lambda x) + 1]y'_x + a[2a^2x^2\ln^3(\lambda x) + 1]y = 0.$

Particular solutions: $y_1 = \exp\Bigl[a\displaystyle\int \ln(\lambda x)\,dx\Bigr]$, $y_2 = x\exp\Bigl[a\displaystyle\int \ln(\lambda x)\,dx\Bigr]$.

30. $x^2y'''_{xxx} + x^2(a\ln x + bx)y''_{xx} + 2x(bx + a)y'_x - ay = 0.$

Integrating the equation twice, we arrive at a first-order linear equation:

$$y'_x + (a\ln x + bx)y = C_1 + C_2x.$$

31. $x^2y'''_{xxx} + (ax^2\ln^n x + bx)y''_{xx} + [a(b - 2)x\ln^n x + c]y'_x + a(c - b + 2)\ln^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b - 3)m + c - b + 2 = 0$.

32. $x^3y'''_{xxx} + x^2(a\ln x + b)y''_{xx} + 2axy'_x - ay = 0.$

Integrating the equation twice, we obtain a first-order linear equation:

$$xy'_x + (a\ln x + b - 2)y = C_1 + C_2x.$$

33. $x^3y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bx^3y'_x + ab\ln^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

34. $x^3y'''_{xxx} + ax^2\ln^n x\, y''_{xx} - 2xy'_x + 2(2 - a\ln^n x)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

35. $x^3y'''_{xxx} + ax^2\ln^n x\, y''_{xx} - 6xy'_x + 6(2 - a\ln^n x)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

36. $x^3y'''_{xxx} + x^2(a\ln x + bx)y''_{xx} + 2x(bx + a)y'_x - ay = 0.$

Integrating the equation twice, we obtain a first-order linear equation:

$$xy'_x + (a\ln x + bx - 2)y = C_1 + C_2x.$$

37. $x^3y'''_{xxx} + x^2(a\ln^n x + b)y''_{xx} + x(ab\ln^n x + c - b)y'_x + c(a\ln^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 + (b - 1)m + c = 0$.

38. $\ln^n x\, y'''_{xxx} + (ax\ln^n x + 1)y''_{xx} + a(x + 2\ln^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

39. $\ln^n x\, y'''_{xxx} + (a\ln^n x + ax + 1)y''_{xx} + a^2x\, y'_x - a^2y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

40. $\ln^n(\lambda x)y'''_{xxx} + ax^2y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

15.1.6 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y'''_{xxx} + a\sin x\, y'_x - b(a\sin x + b^2)y = 0.$

The substitution $w = e^{bx/2}(y'_x - by)$ leads to a second-order linear equation of the form 14.1.6.2: $w''_{xx} + \bigl(a\sin x + \tfrac{3}{4}b^2\bigr)w = 0$.

2. $y'''_{xxx} + a\sin^2 x\, y'_x - b(a\sin^2 x + b^2)y = 0.$

The substitution $w = e^{bx/2}(y'_x - by)$ leads to a second-order linear equation of the form 14.1.6.3: $w''_{xx} + \bigl(a\sin^2 x + \tfrac{3}{4}b^2\bigr)w = 0$.

3. $y'''_{xxx} + [a\sin(\lambda x) + b]y'_x - c[a\sin(\lambda x) + b + c^2]y = 0.$

The substitution $w = e^{cx/2}(y'_x - cy)$ leads to a second-order linear equation of the form 14.1.6.2: $w''_{xx} + \left[a\sin(\lambda x) + b + \frac{3}{4}c^2\right]w = 0$.

4. $y'''_{xxx} + ay''_{xx} + [b\sin(\lambda x) + c]y'_x + a[b\sin(\lambda x) + c]y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.6.2: $w''_{xx} + [b\sin(\lambda x) + c]w = 0$.

5. $y'''_{xxx} + ay''_{xx} + b\sin^2(\lambda x)y'_x + ab\sin^2(\lambda x)\,y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.6.3: $w''_{xx} + b\sin^2(\lambda x)\,w = 0$.

6. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} - by'_x - ab\sin^n(\lambda x)\,y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\bigr)$.

7. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} + ab\sin^n(\lambda x)y'_x + b^2[a\sin^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

8. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} - b[2a\sin^n(\lambda x) + 3b]y'_x + b^2[a\sin^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

9. $y'''_{xxx} + a\sin^n x\,y''_{xx} + (ab\sin^n x + c - b^2)y'_x + c(a\sin^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

10. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\sin^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\sin^n(\lambda x)\,w = 0.$$

11. $y'''_{xxx} + (a\sin^n x + b)y''_{xx} + cy'_x + c(a\sin^n x + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

12. $y'''_{xxx} = (\sin^n x - a)y''_{xx} + (a\sin^n x - b)y'_x + b\sin^n x\,y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

13. $y'''_{xxx} + (a\sin^n x + bx)y''_{xx} + b(ax\sin^n x + 2)y'_x + ab\sin^n x\,y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\,dx$.

14. $y'''_{xxx} + ax\sin^n x\,y''_{xx} + (bx^2 - a\sin^n x)y'_x + bx(ax^2\sin^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$.

15. $y'''_{xxx} + (abx\sin^n x + a\sin^n x + b)y''_{xx} + ab^2 x\sin^n x\, y'_x - ab^2\sin^n x\, y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = e^{-bx}$.

16. $y'''_{xxx} + ax^2\sin^n(\lambda x)y''_{xx} - 2ax\sin^n(\lambda x)y'_x + 2a\sin^n(\lambda x)\, y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = x^2$.

17. $xy'''_{xxx} + x[a\sin(\lambda x) + b]y'_x - 2[a\sin(\lambda x) + b]y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.6.2: $w''_{xx} + [a\sin(\lambda x) + b]w = 0$.

18. $xy'''_{xxx} = (\sin^n x - ax)y''_{xx} + (a\sin^n x - bx)y'_x + b\sin^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x),\ y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

19. $xy'''_{xxx} + (ax\sin^n x + 3)y''_{xx} + (2a\sin^n x + bx)y'_x + b(ax\sin^n x + 1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{b}\,\bigr),\ \ y_2 = x^{-1}\sin\bigl(x\sqrt{b}\,\bigr)$.

20. $x^2y'''_{xxx} = (\sin^n x - ax^2)y''_{xx} + (a\sin^n x - bx^2)y'_x + b\sin^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x),\ y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

21. $x^3y'''_{xxx} + ax^2\sin^n(\lambda x)y''_{xx} + bxy'_x + b[a\sin^n(\lambda x) - 2]y = 0.$

Particular solutions: $y_1 = x^{m_1},\ y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 - m + b = 0$.

22. $\sin^2 x\, y'''_{xxx} + a\sin^2 x\, y''_{xx} + by'_x + aby = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.6.21: $\sin^2 x\, w''_{xx} + bw = 0$.

23. $\sin^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\sin^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr),\ \ y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

24. $\sin^n x\, y'''_{xxx} + ay''_{xx} + b\sin^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr),\ \ y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr),\ \ y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

25. $\sin^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\sin^n x)y'_x + b^2(a + 2b\sin^n x)y = 0.$

Particular solutions: $y_1 = e^{bx},\ \ y_2 = xe^{bx}$.

26. $\sin^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\sin^n x + a]y'_x + b(1 - a\sin^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

27. $\sin^n(\lambda x)y'''_{xxx} + ax^2y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = x^2$.

28. $\sin^n x\, y'''_{xxx} + (a\sin^n x + ax + 1)y''_{xx} + a^2 x y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = e^{-ax}.$

29. $\sin^n x\, y'''_{xxx} + (ax\sin^n x + 1)y''_{xx} + a(x + 2\sin^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr),\ \ y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\int \exp\bigl(\frac{1}{2}ax^2\bigr)\,dx.$

30. $x\sin^n x\, y'''_{xxx} + (3\sin^n x + x)y''_{xx} + (ax\sin^n x + 2)y'_x + a(\sin^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr),\ \ y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr).$

31. $x^3\sin^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\sin^n x\, y'_x + 2(2\sin^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1},\ \ y_2 = x^2.$

32. $x^3\sin^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\sin^n x\, y'_x + 6(2\sin^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2},\ \ y_2 = x^3.$

33. $x^3\sin^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \sin^n x)y'_x + (a - 3\sin^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x),\ \ y_2 = \sin(\ln x).$

34. $x^3\sin^n x\, y'''_{xxx} + x^2(\sin^n x + a)y''_{xx}$
$\qquad + x[a - (b+1)\sin^n x]y'_x + b(2\sin^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}},\ \ y_2 = x^{\sqrt{b}}.$

► Equations with cosine.

35. $y'''_{xxx} + a\cos(2x)y'_x - b[a\cos(2x) + b^2]y = 0.$

The substitution $w = e^{bx/2}(y'_x - by)$ leads to a Mathieu equation of the form 14.1.6.29: $w''_{xx} + \bigl[a\cos(2x) + \frac{3}{4}b^2\bigr]w = 0.$

36. $y'''_{xxx} + [a\cos(\lambda x) + b]y'_x - c[a\cos(\lambda x) + b + c^2]y = 0.$

The transformation $\xi = \frac{1}{2}\lambda x$, $w = e^{cx/2}(y'_x - cy)$ leads to the Mathieu equation 2.1.6.29: $w''_{\xi\xi} + 4\lambda^{-2}\bigl[a\cos(2\xi) + b + \frac{3}{4}c^2\bigr]w = 0.$

37. $y'''_{xxx} + ay''_{xx} + (b\cos 2x + c)y'_x + a(b\cos 2x + c)y = 0.$

The substitution $w = y'_x + ay$ leads to the Mathieu equation 2.1.6.29:

$$w''_{xx} + (b\cos 2x + c)w = 0.$$

38. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} + by'_x + ab\cos^n(\lambda x)\,y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr),\ \ y_2 = \sin\bigl(x\sqrt{b}\,\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr),\ \ y_2 = \exp\bigl(x\sqrt{-b}\,\bigr).$

39. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} + ab\cos^n(\lambda x)y'_x + b^2[a\cos^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr),\ \ y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr).$

40. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} - b[2a\cos^n(\lambda x) + 3b]y'_x + b^2[a\cos^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx},\ \ y_2 = xe^{bx}.$

41. $y'''_{xxx} + a\cos^n x\, y''_{xx} + (ab\cos^n x + c - b^2)y'_x + c(a\cos^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

42. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\cos^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\cos^n(\lambda x)\,w = 0.$$

43. $y'''_{xxx} + (a\cos^n x + b)y''_{xx} + cy'_x + c(a\cos^n x + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

44. $y'''_{xxx} = (\cos^n x - a)y''_{xx} + (a\cos^n x - b)y'_x + b\cos^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

45. $y'''_{xxx} + (a\cos^n x + bx)y''_{xx} + b(ax\cos^n x + 2)y'_x + ab\cos^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx^2\bigr)\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\,dx.$

46. $y'''_{xxx} + ax\cos^n x\, y''_{xx} + (bx^2 - a\cos^n x)y'_x + bx(ax^2\cos^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$.

47. $y'''_{xxx} + (abx\cos^n x + a\cos^n x + b)y''_{xx} + ab^2 x\cos^n x\, y'_x - ab^2\cos^n x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

48. $y'''_{xxx} + ax^2\cos^n(\lambda x)y''_{xx} - 2ax\cos^n(\lambda x)y'_x + 2a\cos^n(\lambda x)\,y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

49. $xy'''_{xxx} + x(a\cos 2x + b)y'_x - 2(a\cos 2x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to the Mathieu equation 14.1.6.29:

$$w''_{xx} + (a\cos 2x + b)w = 0.$$

50. $xy'''_{xxx} + x(a\cos^2 x + b)y'_x - 2(a\cos^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.6.30: $w''_{xx} + (a\cos^2 x + b)w = 0$.

51. $xy'''_{xxx} = (\cos^n x - ax)y''_{xx} + (a\cos^n x - bx)y'_x + b\cos^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

52. $xy'''_{xxx} + (ax\cos^n x + 3)y''_{xx} + (2a\cos^n x + bx)y'_x + b(ax\cos^n x + 1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{b}\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{b}\bigr)$.

53. $x^2 y'''_{xxx} = (\cos^n x - ax^2) y''_{xx} + (a\cos^n x - bx^2) y'_x + b\cos^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

54. $x^3 y'''_{xxx} + ax^2 \cos^n(\lambda x) y''_{xx} + bxy'_x + b[a\cos^n(\lambda x) - 2]y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are roots of the quadratic equation $m^2 - m + b = 0$.

55. $\cos^2 x\, y'''_{xxx} + a\cos^2 x\, y''_{xx} + by'_x + aby = 0.$

The substitution $x = \xi + \frac{\pi}{2}$ leads to an equation of the form 3.1.6.22: $\sin^2\xi\, y'''_{\xi\xi\xi} + a\sin^2\xi\, y''_{\xi\xi} + by'_\xi + aby = 0$.

56. $\cos^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\cos^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx\bigr)\cos\bigl(\frac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx\bigr)\sin\bigl(\frac{\sqrt{3}}{2}bx\bigr)$.

57. $\cos^n x\, y'''_{xxx} + ay''_{xx} + b\cos^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

58. $\cos^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\cos^n x)y'_x + b^2(a + 2b\cos^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

59. $\cos^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\cos^n x + a]y'_x + b(1 - a\cos^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

60. $\cos^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

61. $\cos^n x\, y'''_{xxx} + (a\cos^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

62. $\cos^n x\, y'''_{xxx} + (ax\cos^n x + 1)y''_{xx} + a(x + 2\cos^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

63. $x\cos^n x\, y'''_{xxx} + (3\cos^n x + x)y''_{xx} + (ax\cos^n x + 2)y'_x + a(\cos^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

64. $x^3\cos^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\cos^n x\, y'_x + 2(2\cos^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

65. $x^3\cos^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\cos^n x\, y'_x + 6(2\cos^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

66. $x^3\cos^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \cos^n x)y'_x + (a - 3\cos^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln|x|)$, $y_2 = \sin(\ln|x|)$.

67. $x^3 \cos^n x\, y'''_{xxx} + x^2(\cos^n x + a)y''_{xx} + x[a - (b+1)\cos^n x]y'_x + b(2\cos^n x - a)y = 0.$

Particular solutions: $y_1 = |x|^{-\sqrt{b}}, \quad y_2 = |x|^{\sqrt{b}}.$

► **Equations with sine and cosine.**

68. $y'''_{xxx} + [a\sin(\lambda x) + b]y'_x + a\lambda\cos(\lambda x)\, y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + [a\sin(\lambda x) + b]y = C$ (see 14.1.6.2 for the solution of the corresponding homogeneous equation with $C = 0$).

69. $y'''_{xxx} + [a\sin(\lambda x) - b^2]y'_x + a[\lambda\cos(\lambda x) - b\sin(\lambda x)]y = 0.$

By integrating and substituting $w = ye^{bx/2}$, we obtain a second-order nonhomogeneous linear equation: $w''_{xx} + \left[a\sin(\lambda x) - \frac{1}{4}b^2\right]w = Ce^{3bx/2}$ (see 14.1.6.2 for the solution of the corresponding homogeneous equation).

70. $y'''_{xxx} + [a\cos(2x) + b]y'_x - a\sin(2x)\, y = 0.$

Solution: $y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2$. Here, w_1, w_2 is a fundamental set of solutions of the Mathieu equation 2.1.6.29: $4w''_{xx} + [a\cos(2x) + b]w = 0$.

71. $y'''_{xxx} + [a\cos(2x) + b]y'_x - 2a\sin(2x)\, y = 0.$

Integrating yields a nonhomogeneous Mathieu equation: $y''_{xx} + [a\cos(2x) + b]y = C$.

72. $y'''_{xxx} + [a\cos(2x) - b^2]y'_x - a[b\cos(2x) + 2\sin(2x)]y = 0.$

By integrating and substituting $w = ye^{bx/2}$, we obtain a nonhomogeneous Mathieu equation: $w''_{xx} + [a\cos(2x) - \frac{1}{4}b^2]w = Ce^{3bx/2}$.

73. $y'''_{xxx} - 3a[a\sin^2(bx) + b\cos(bx)]y'_x + a\sin(bx)[b^2 + 2a^2\sin^2(bx)]y = 0.$

Particular solutions: $y_1 = \exp\left[-\frac{a}{b}\cos(bx)\right], \quad y_2 = x\exp\left[-\frac{a}{b}\cos(bx)\right].$

74. $a\sin(\lambda x)y'''_{xxx} + by''_{xx} + 3a\lambda^2\sin(\lambda x)y'_x + 2a\lambda^3\cos(\lambda x)y = 0.$

This is a special case of equation 15.1.9.105 with $f(x) = 0$.

75. $\sin(\lambda x)y'''_{xxx} + [a + (2\lambda + 1)\cos(\lambda x)]y''_{xx} - (\lambda^2 + 2\lambda)\sin(\lambda x)y'_x - \lambda^2\cos(\lambda x)y = 0.$

This is a special case of equation 15.1.9.106 with $f(x) = 0$.

76. $\sin^2 x\, y'''_{xxx} + 3\sin x\cos x\, y''_{xx} + [\cos 2x + 4\nu(\nu+1)\sin^2 x]y'_x + 2\nu(\nu+1)\sin 2x\, y = 0.$

Solution: $y = C_1 y_1^2 + C_2 y_1 y_2 + C_3 y_2^2$. Here, y_1, y_2 form a fundamental set of solutions of the Legendre equation 2.1.2.154, with the argument x of the functions y_1 and y_2 substituted by $\cos x$.

► **Equations with tangent.**

77. $y'''_{xxx} + a^3 \tan(ax)\, y = 0.$

Integrating yields a second-order nonhomogeneous linear equation: $y''_{\xi\xi} + \tan\xi\, y'_{\xi} - y = C/\cos\xi$, where $\xi = ax$ (see 14.1.6.53 for the solution of the corresponding homogeneous equation).

78. $y'''_{xxx} - a^3 \tan(ax)\, y = 0.$

Particular solution: $y_0 = \cos(ax)$. The substitution $y = \cos(ax) \int z(x)\, dx$ leads to a second-order linear equation of the form 14.1.6.53: $z''_{\xi\xi} - 3\tan\xi\, z'_{\xi} - 3z = 0$, where $\xi = ax$.

79. $y'''_{xxx} + 3a^2 y'_x + 2a^3 \tan(ax)\, y = 0.$

Particular solutions: $y_1 = \cos(ax)$, $y_2 = x\cos(ax)$.

80. $y'''_{xxx} + ay'_x + (a-1)\tan x\, y = 0.$

Particular solution: $y_0 = \cos x$. The substitution $y = \cos x \int z(x)\, dx$ leads to a second-order linear equation: $z''_{xx} - 3\tan x\, z'_x + (a-3)z = 0$.

81. $y'''_{xxx} + ay'_x + \lambda(a - \lambda^2)\tan(\lambda x)\, y = 0.$

Particular solution: $y_0 = \cos(\lambda x)$. The substitution $y = \cos(\lambda x) \int z(x)\, dx$ leads to a second-order linear equation of the form 14.1.6.53: $z''_{\xi\xi} - 3\tan\xi\, z'_{\xi} + (a\lambda^{-2} - 3)z = 0$, where $\xi = \lambda x$.

82. $y'''_{xxx} + a\tan^2 x\, y'_x - b(a\tan^2 x + b^2)y = 0.$

The substitution $w = e^{bx/2}(y'_x - by)$ leads to a second-order linear equation of the form 14.1.6.51: $w''_{xx} + \big(a\tan^2 x + \frac{3}{4}b^2\big)w = 0$.

83. $y'''_{xxx} + [a\tan^2(\lambda x) + b]y'_x - c[a\tan^2(\lambda x) + b + c^2]y = 0.$

The transformation $\xi = \lambda x$, $w = e^{cx/2}(y'_x - cy)$ leads to an equation of the form 2.1.6.51: $w''_{\xi\xi} + \lambda^{-2}(a\tan^2\xi + b + \frac{3}{4}c^2)w = 0$.

84. $y'''_{xxx} + a\tan^n x\, y'_x + \tan x(a\tan^n x - 1)y = 0.$

Particular solution: $y_0 = \cos x$. The substitution $y = \cos x \int z(x)\, dx$ leads to a second-order linear equation: $z''_{xx} - 3\tan x\, z'_x + (a\tan^n x - 3)z = 0$.

85. $y'''_{xxx} + ay''_{xx} + (b\tan^2 x + c)y'_x + a(b\tan^2 x + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.6.51: $w''_{xx} + (b\tan^2 x + c)w = 0$.

86. $y'''_{xxx} + ay''_{xx} + \lambda[3\lambda + 2a\tan(\lambda x)]y'_x + \lambda^2\{a[1 + 2\tan^2(\lambda x)] + 2\lambda\tan(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \cos(\lambda x)$, $y_2 = x\cos(\lambda x)$.

87. $y'''_{xxx} + \lambda \tan(\lambda x) y''_{xx} - a y'_x - a\lambda \tan(\lambda x)\, y = 0.$

1°. Solution for $a > 0$: $y = C_1 \exp\bigl(-x\sqrt{a}\,\bigr) + C_2 \exp\bigl(x\sqrt{a}\,\bigr) + C_3 \cos(\lambda x)$.

2°. Solution for $a < 0$: $y = C_1 \cos\bigl(x\sqrt{-a}\,\bigr) + C_2 \sin\bigl(x\sqrt{-a}\,\bigr) + C_3 \cos(\lambda x)$.

88. $y'''_{xxx} + a \tan(\lambda x) y''_{xx} + b y'_x + \lambda(a\lambda + b - \lambda^2) \tan(\lambda x)\, y = 0.$

Particular solution: $y_0 = \cos(\lambda x)$. The transformation $x = \dfrac{z}{\lambda}$, $y = \cos(\lambda x) \displaystyle\int w\, dx$ leads to a second-order linear equation of the form 14.1.6.131: $w''_{zz} + \bigl(a\lambda^{-1} - 3\bigr) \tan z\, w'_z + \bigl(b\lambda^{-2} - 3 - 2a\lambda^{-1} \tan^2 z\bigr) w = 0$.

89. $y'''_{xxx} + a \tan^n(\lambda x) y''_{xx} + b y'_x + ab \tan^n(\lambda x)\, y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

90. $y'''_{xxx} + a \tan^n(\lambda x) y''_{xx} + ab \tan^n(\lambda x) y'_x + b^2 [a \tan^n(\lambda x) - b] y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2} bx\bigr) \cos\bigl(\tfrac{\sqrt{3}}{2} bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2} bx\bigr) \sin\bigl(\tfrac{\sqrt{3}}{2} bx\bigr)$.

91. $y'''_{xxx} + a \tan^n(\lambda x) y''_{xx} - b[2a \tan^n(\lambda x) + 3b] y'_x + b^2 [a \tan^n(\lambda x) + 2b] y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = x e^{bx}$.

92. $y'''_{xxx} + a \tan^n x\, y''_{xx} + (ab \tan^n x + c - b^2) y'_x + c(a \tan^n x - b) y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

93. $y'''_{xxx} + a \tan^n(\lambda x) y''_{xx} + b x^m y'_x + b x^{m-1} [a x \tan^n(\lambda x) + m] y = 0.$

The substitution $w = y''_{xx} + b x^m y$ leads to a first-order linear equation:

$$w'_x + a \tan^n(\lambda x)\, w = 0.$$

94. $y'''_{xxx} + (a \tan^n x + b) y''_{xx} + c y'_x + c(a \tan^n x + b) y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

95. $y'''_{xxx} = (\tan^n x - a) y''_{xx} + (a \tan^n x - b) y'_x + b \tan^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

96. $y'''_{xxx} + (a \tan^n x + bx) y''_{xx} + b(ax \tan^n x + 2) y'_x + ab \tan^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2} bx^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2} bx^2\bigr) \displaystyle\int \exp\bigl(\tfrac{1}{2} bx^2\bigr)\, dx$.

97. $y'''_{xxx} + ax \tan^n x\, y''_{xx} + (bx^2 - a \tan^n x) y'_x + bx(ax^2 \tan^n x + 3) y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2} x^2 \sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2} x^2 \sqrt{b}\,\bigr)$.

98. $y'''_{xxx} + (abx \tan^n x + a \tan^n x + b) y''_{xx} + ab^2 x \tan^n x\, y'_x - ab^2 \tan^n x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

99. $y'''_{xxx} + ax^2 \tan^n(\lambda x) y''_{xx} - 2ax \tan^n(\lambda x) y'_x + 2a \tan^n(\lambda x)\, y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = x^2.$

100. $y'''_{xxx} - [b(a + \tan x)x^n + a] y''_{xx} + [b(a^2 + 1)x^n + 1] y'_x + a[b(a \tan x - 1)x^n - 1] y = 0.$

Particular solutions: $y_1 = e^{ax}, \;\; y_2 = \cos x.$

101. $y'''_{xxx} - (ab \tan^n x + b \tan^{n+1} x + a) y''_{xx} + [b(a^2 + 1) \tan^n x + 1] y'_x + a(ab \tan^{n+1} x - b \tan^n x - 1) y = 0.$

Particular solutions: $y_1 = e^{ax}, \;\; y_2 = \cos x.$

102. $y'''_{xxx} + [\lambda \tan(\lambda x)(ax^n + 1) + ax^{n-1}] y''_{xx} - a\lambda^2 x^n y'_x + a\lambda^2 x^{n-1} y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cos(\lambda x).$

103. $xy'''_{xxx} + x(a \tan^2 x + b) y'_x - 2(a \tan^2 x + b) y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.6.51: $w''_{xx} + (a \tan^2 x + b) w = 0.$

104. $xy'''_{xxx} = (\tan^n x - ax) y''_{xx} + (a \tan^n x - bx) y'_x + b \tan^n x \, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x), \;\; y_2 = \exp(\lambda_2 x),$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

105. $xy'''_{xxx} + (ax \tan^n x + 3) y''_{xx} + (2a \tan^n x + bx) y'_x + b(ax \tan^n x + 1) y = 0.$

Particular solutions: $y_1 = x^{-1} \cos\bigl(x\sqrt{b}\bigr), \;\; y_2 = x^{-1} \sin\bigl(x\sqrt{b}\bigr).$

106. $x^2 y'''_{xxx} = (\tan^n x - ax^2) y''_{xx} + (a \tan^n x - bx^2) y'_x + b \tan^n x \, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x), \;\; y_2 = \exp(\lambda_2 x),$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

107. $x^3 y'''_{xxx} + ax^2 \tan^n(\lambda x) y''_{xx} + bxy'_x + b[a \tan^n(\lambda x) - 2] y = 0.$

Particular solutions: $y_1 = x^{m_1}, \;\; y_2 = x^{m_2},$ where m_1 and m_2 are roots of the quadratic equation $m^2 - m + b = 0.$

108. $\tan^n x \, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b \tan^n x) y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr) \cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr), \;\; y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr) \sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr).$

109. $\tan^n x \, y'''_{xxx} + ay''_{xx} + b \tan^n x \, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr), \;\; y_2 = \sin\bigl(x\sqrt{b}\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr), \;\; y_2 = \exp\bigl(x\sqrt{-b}\bigr).$

110. $\tan^n x \, y'''_{xxx} + ay''_{xx} - b(2a + 3b \tan^n x) y'_x + b^2(a + 2b \tan^n x) y = 0.$

Particular solutions: $y_1 = e^{bx}, \;\; y_2 = xe^{bx}.$

111. $\tan^n x \, y'''_{xxx} + y''_{xx} + [(b - a^2) \tan^n x + a] y'_x + b(1 - a \tan^n x) y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}, \;\; y_2 = e^{\lambda_2 x},$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

112. $\tan^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x, \quad y_2 = x^2.$

113. $\tan^n x\, y'''_{xxx} + (a\tan^n x + ax + 1)y''_{xx} + a^2 x y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = e^{-ax}.$

114. $\tan^n x\, y'''_{xxx} + (ax\tan^n x + 1)y''_{xx} + a(x + 2\tan^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr), \quad y_2 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx.$

115. $x\tan^n x\, y'''_{xxx} + (3\tan^n x + x)y''_{xx} + (ax\tan^n x + 2)y'_x + a(\tan^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr), \quad y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr).$

116. $x^3\tan^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\tan^n x\, y'_x + 2(2\tan^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}, \quad y_2 = x^2.$

117. $x^3\tan^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\tan^n x\, y'_x + 6(2\tan^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}, \quad y_2 = x^3.$

118. $x^3\tan^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \tan^n x)y'_x + (a - 3\tan^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x).$

119. $x^3\tan^n x\, y'''_{xxx} + x^2(\tan^n x + a)y''_{xx}$
$+ x[a - (b+1)\tan^n x]y'_x + b(2\tan^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}, \quad y_2 = x^{\sqrt{b}}.$

► **Equations with cotangent.**

120. $y'''_{xxx} + a^3\cot(ax)\,y = 0.$

The substitution $x = t + \dfrac{\pi}{2a}$ leads to a linear equation of the form 15.1.6.78: $y'''_{ttt} - a^3\tan(at)\,y = 0.$

121. $y'''_{xxx} - a^3\cot(ax)\,y = 0.$

The substitution $x = t + \dfrac{\pi}{2a}$ leads to a linear equation of the form 15.1.6.77: $y'''_{ttt} + a^3\tan(at)\,y = 0.$

122. $y'''_{xxx} + 3a^2 y'_x - 2a^3\cot(ax)\,y = 0.$

Particular solutions: $y_1 = \sin(ax), \quad y_2 = x\sin(ax).$

123. $y'''_{xxx} + ay'_x + (1-a)\cot x\, y = 0.$

Particular solution: $y_0 = \sin x.$

124. $y'''_{xxx} + a\cot^2 x\, y'_x - b(a\cot^2 x + b^2)y = 0.$

The substitution $w = e^{bx/2}(y'_x - by)$ leads to a second-order linear equation of the form 14.1.6.81: $w''_{xx} + \bigl(a\cot^2 x + \tfrac{3}{4}b^2\bigr)w = 0.$

125. $y'''_{xxx} + ay''_{xx} + (b\cot^2 x + c)y'_x + a(b\cot^2 x + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second-order linear equation of the form 14.1.6.81: $w''_{xx} + (b\cot^2 x + c)w = 0$.

126. $y'''_{xxx} + a\cot^n x\, y'_x + \cot x(1 - a\cot^n x)y = 0.$

Particular solution: $y_0 = \sin x$.

127. $y'''_{xxx} + ay''_{xx} + \lambda[3\lambda - 2a\cot(\lambda x)]y'_x$
$+ \lambda^2\{a[1 + 2\cot^2(\lambda x)] - 2\lambda\cot(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \sin(\lambda x)$, $y_2 = x\sin(\lambda x)$.

128. $y'''_{xxx} - \lambda\cot(\lambda x)y''_{xx} - ay'_x + a\lambda\cot(\lambda x)\, y = 0.$

1°. Solution for $a > 0$: $y = C_1\exp\bigl(-x\sqrt{a}\,\bigr) + C_2\exp\bigl(x\sqrt{a}\,\bigr) + C_3\sin(\lambda x)$.

2°. Solution for $a < 0$: $y = C_1\cos\bigl(x\sqrt{-a}\,\bigr) + C_2\sin\bigl(x\sqrt{-a}\,\bigr) + C_3\sin(\lambda x)$.

129. $y'''_{xxx} + a\cot^n(\lambda x)y''_{xx} + ab\cot^n(\lambda x)y'_x + b^2[a\cot^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr)$.

130. $y'''_{xxx} + a\cot^n(\lambda x)y''_{xx} - b[2a\cot^n(\lambda x) + 3b]y'_x + b^2[a\cot^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

131. $y'''_{xxx} + a\cot^n x\, y''_{xx} + (ab\cot^n x + c - b^2)y'_x + c(a\cot^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

132. $y'''_{xxx} + a\cot^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\cot^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first-order linear equation:

$$w'_x + a\cot^n(\lambda x)\, w = 0.$$

133. $y'''_{xxx} + (a\cot^n x + b)y''_{xx} + cy'_x + c(a\cot^n x + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

134. $y'''_{xxx} = (\cot^n x - a)y''_{xx} + (a\cot^n x - b)y'_x + b\cot^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

135. $y'''_{xxx} + ax\cot^n x\, y''_{xx} + (bx^2 - a\cot^n x)y'_x + bx(ax^2\cot^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{b}\,\bigr)$.

136. $y'''_{xxx} + (abx\cot^n x + a\cot^n x + b)y''_{xx} + ab^2 x\cot^n x\, y'_x - ab^2\cot^n x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

137. $y'''_{xxx} + ax^2\cot^n(\lambda x)y''_{xx} - 2ax\cot^n(\lambda x)y'_x + 2a\cot^n(\lambda x)\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

138. $axy'''_{xxx} + [1 - \lambda(a+1)x\cot(\lambda x)]y''_{xx} - \lambda^2 xy'_x + \lambda^2 y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sin(\lambda x).$

139. $xy'''_{xxx} + x(a\cot^2 x + b)y'_x - 2(a\cot^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second-order linear equation of the form 14.1.6.81: $w''_{xx} + (a\cot^2 x + b)w = 0.$

140. $xy'''_{xxx} + (ax\cot^n x + 3)y''_{xx} + (2a\cot^n x + bx)y'_x + b(ax\cot^n x + 1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{b}\,\bigr), \;\; y_2 = x^{-1}\sin\bigl(x\sqrt{b}\,\bigr).$

141. $x^2 y'''_{xxx} = (\cot^n x - ax^2)y''_{xx} + (a\cot^n x - bx^2)y'_x + b\cot^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x), \;\; y_2 = \exp(\lambda_2 x),$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

142. $x^3 y'''_{xxx} + ax^2\cot^n(\lambda x)y''_{xx} + bxy'_x + b[a\cot^n(\lambda x) - 2]y = 0.$

Particular solutions: $y_1 = x^{m_1}, \;\; y_2 = x^{m_2},$ where m_1 and m_2 are roots of the quadratic equation $m^2 - m + b = 0.$

143. $\cot^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\cot^n x)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}bx\bigr), \;\; y_2 = \exp\bigl(-\tfrac{1}{2}bx\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}bx\bigr).$

144. $\cot^n x\, y'''_{xxx} + ay''_{xx} + b\cot^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr), \;\; y_2 = \sin\bigl(x\sqrt{b}\,\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr), \;\; y_2 = \exp\bigl(x\sqrt{-b}\,\bigr).$

145. $\cot^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\cot^n x)y'_x + b^2(a + 2b\cot^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}, \;\; y_2 = xe^{bx}.$

146. $\cot^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\cot^n x + a]y'_x + b(1 - a\cot^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}, \;\; y_2 = e^{\lambda_2 x},$ where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

147. $\cot^n(\lambda x)y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = x^2.$

148. $\cot^n x\, y'''_{xxx} + (a\cot^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = e^{-ax}.$

149. $\cot^n x\, y'''_{xxx} + (ax\cot^n x + 1)y''_{xx} + a(x + 2\cot^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr), \;\; y_2 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx.$

150. $x\cot^n x\, y'''_{xxx} + (3\cot^n x + x)y''_{xx} + (ax\cot^n x + 2)y'_x + a(\cot^n x + x)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr), \;\; y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr).$

151. $x^3\cot^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\cot^n x\, y'_x + 2(2\cot^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}, \;\; y_2 = x^2.$

152. $x^3 \cot^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x \cot^n x\, y'_x + 6(2\cot^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

153. $x^3 \cot^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \cot^n x)y'_x + (a - 3\cot^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

154. $x^3 \cot^n x\, y'''_{xxx} + x^2(\cot^n x + a)y''_{xx} + x[a - (b+1)\cot^n x]y'_x + b(2\cot^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

15.1.7 Equations Containing Inverse Trigonometric Functions

1. $y'''_{xxx} + ay''_{xx} + by'_x + cy = \arcsin^k x.$

This is a special case of equation 17.1.6.26.

2. $y'''_{xxx} + \arcsin^k x\, y''_{xx} + ay'_x + a\arcsin^k x\, y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a first-order linear equation: $w'_x + \arcsin^k x\, w = 0$.

3. $y'''_{xxx} + \arcsin^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x\arcsin^k x + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first-order linear equation: $w'_x + \arcsin^k x\, w = 0$.

4. $y'''_{xxx} + \arcsin^k x\, y''_{xx} + a\arcsin^k x\, y'_x + a^2(\arcsin^k x - a)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}ax\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}ax\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}ax\bigr)$.

5. $y'''_{xxx} + \arcsin^k x\, y''_{xx} - a(2\arcsin^k x + 3a)y'_x + a^2(\arcsin^k x + 2a)y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

6. $y'''_{xxx} + \arcsin^k x\, y''_{xx} + (a\arcsin^k x + b - a^2)y'_x + b(\arcsin^k x - a)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

7. $y'''_{xxx} = (\arcsin^k x - a)y''_{xx} + (a\arcsin^k x - b)y'_x + b\arcsin^k x\, y.$

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first-order linear equation: $w'_x = \arcsin^k x\, w$.

8. $y'''_{xxx} + x\arcsin^k x\, y''_{xx} + (ax^2 - \arcsin^k x)y'_x + ax(x^2\arcsin^k x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{a}\,\bigr)$.

9. $y'''_{xxx} + (\arcsin^k x + ax)y''_{xx} + a(x\arcsin^k x + 2)y'_x + a\arcsin^k x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

10. $\boldsymbol{y'''_{xxx} + x^2 \arcsin^k x \, y''_{xx} - 2x \arcsin^k x \, y'_x + 2 \arcsin^k x \, y = 0.}$

Solution:

$$y = C_1 x + C_2 x^2 + C_3 \Big(x^2 \int x^{-3} \psi \, dx - x \int x^{-2} \psi \, dx \Big),$$

where $\psi = \exp\Big(- \int x^2 \arcsin^k x \, dx \Big)$.

11. $\boldsymbol{y'''_{xxx} + (ax \arcsin^k x + \arcsin^k x + a) y''_{xx} + a^2 x \arcsin^k x \, y'_x - a^2 \arcsin^k x \, y = 0.}$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

12. $\boldsymbol{y'''_{xxx} + \arccos^k x \, y''_{xx} + a y'_x + a \arccos^k x \, y = 0.}$

1°. Particular solutions with $a > 0$: $y_1 = \cos\big(x\sqrt{a}\,\big)$, $y_2 = \sin\big(x\sqrt{a}\,\big)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\big(-x\sqrt{-a}\,\big)$, $y_2 = \exp\big(x\sqrt{-a}\,\big)$.

The substitution $w = y''_{xx} + ay$ leads to a first-order linear equation: $w'_x + \arccos^k x \, w = 0$.

13. $\boldsymbol{y'''_{xxx} + \arccos^k x \, y''_{xx} + a x^n y'_x + a x^{n-1} (x \arccos^k x + n) y = 0.}$

The substitution $w = y''_{xx} + a x^n y$ leads to a first-order linear equation: $w'_x + \arccos^k x \, w = 0$.

14. $\boldsymbol{y'''_{xxx} + \arccos^k x \, y''_{xx} + a \arccos^k x \, y'_x + a^2 (\arccos^k x - a) y = 0.}$

Particular solutions: $y_1 = \exp\big(-\tfrac{1}{2} ax\big) \cos\big(\tfrac{\sqrt{3}}{2} ax\big)$, $y_2 = \exp\big(-\tfrac{1}{2} ax\big) \sin\big(\tfrac{\sqrt{3}}{2} ax\big)$.

15. $\boldsymbol{y'''_{xxx} + \arccos^k x \, y''_{xx} - a(2 \arccos^k x + 3a) y'_x + a^2 (\arccos^k x + 2a) y = 0.}$

Particular solutions: $y_1 = e^{ax}$, $y_2 = x e^{ax}$.

16. $\boldsymbol{y'''_{xxx} + \arccos^k x \, y''_{xx} + (a \arccos^k x + b - a^2) y'_x + b(\arccos^k x - a) y = 0.}$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

17. $\boldsymbol{y'''_{xxx} = (\arccos^k x - a) y''_{xx} + (a \arccos^k x - b) y'_x + b \arccos^k x \, y.}$

The substitution $w = y''_{xx} + a y'_x + b y$ leads to a first-order linear equation: $w'_x = \arccos^k x \, w$.

18. $\boldsymbol{y'''_{xxx} + x \arccos^k x \, y''_{xx} + (a x^2 - \arccos^k x) y'_x + a x (x^2 \arccos^k x + 3) y = 0.}$

Particular solutions: $y_1 = \cos\big(\tfrac{1}{2} x^2 \sqrt{a}\,\big)$, $y_2 = \sin\big(\tfrac{1}{2} x^2 \sqrt{a}\,\big)$.

19. $\boldsymbol{y'''_{xxx} + (\arccos^k x + ax) y''_{xx} + a(x \arccos^k x + 2) y'_x + a \arccos^k x \, y = 0.}$

Particular solutions: $y_1 = \exp\big(-\tfrac{1}{2} a x^2\big)$, $y_2 = \exp\big(-\tfrac{1}{2} a x^2\big) \displaystyle\int \exp\big(\tfrac{1}{2} a x^2\big)\, dx$.

20. $\boldsymbol{y'''_{xxx} + x^2 \arccos^k x \, y''_{xx} - 2x \arccos^k x \, y'_x + 2 \arccos^k x \, y = 0.}$

Solution:

$$y = C_1 x + C_2 x^2 + C_3 \Big(x^2 \int x^{-3} \psi \, dx - x \int x^{-2} \psi \, dx \Big),$$

where $\psi = \exp\Big(- \int x^2 \arccos^k x \, dx \Big)$.

21. $y'''_{xxx} + (ax \arccos^k x + \arccos^k x + a)y''_{xx}$
$$+ a^2 x \arccos^k x\, y'_x - a^2 \arccos^k x\, y = 0.$$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

22. $y'''_{xxx} + \arctan^k x\, y''_{xx} + ay'_x + a \arctan^k x\, y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos(x\sqrt{a})$, $y_2 = \sin(x\sqrt{a})$.

2°. Particular solutions with $a < 0$: $y_1 = \exp(-x\sqrt{-a})$, $y_2 = \exp(x\sqrt{-a})$.

The substitution $w = y''_{xx} + ay$ leads to a first-order linear equation: $w'_x + \arctan^k x\, w = 0$.

23. $y'''_{xxx} + \arctan^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x \arctan^k x + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first-order linear equation: $w'_x + \arctan^k x\, w = 0$.

24. $y'''_{xxx} + \arctan^k x\, y''_{xx} + a \arctan^k x\, y'_x + a^2(\arctan^k x - a)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax\bigr)\cos\bigl(\frac{\sqrt{3}}{2}ax\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax\bigr)\sin\bigl(\frac{\sqrt{3}}{2}ax\bigr)$.

25. $y'''_{xxx} + \arctan^k x\, y''_{xx} - a(2\arctan^k x + 3a)y'_x + a^2(\arctan^k x + 2a)y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

26. $y'''_{xxx} + \arctan^k x\, y''_{xx} + (a \arctan^k x + b - a^2)y'_x + b(\arctan^k x - a)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

27. $y'''_{xxx} = (\arctan^k x - a)y''_{xx} + (a \arctan^k x - b)y'_x + b \arctan^k x\, y.$

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first-order linear equation: $w'_x = \arctan^k x\, w$.

28. $y'''_{xxx} + x \arctan^k x\, y''_{xx} + (ax^2 - \arctan^k x)y'_x + ax(x^2 \arctan^k x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{a}\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{a}\bigr)$.

29. $y'''_{xxx} + (\arctan^k x + ax)y''_{xx} + a(x \arctan^k x + 2)y'_x + a \arctan^k x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

30. $y'''_{xxx} + x^2 \arctan^k x\, y''_{xx} - 2x \arctan^k x\, y'_x + 2 \arctan^k x\, y = 0.$

Solution:
$$y = C_1 x + C_2 x^2 + C_3\Bigl(x^2 \int x^{-3}\psi\,dx - x\int x^{-2}\psi\,dx\Bigr),$$
where $\psi = \exp\Bigl(-\displaystyle\int x^2 \arctan^k x\,dx\Bigr)$.

31. $y'''_{xxx} + (ax \arctan^k x + \arctan^k x + a)y''_{xx}$
$$+ a^2 x \arctan^k x\, y'_x - a^2 \arctan^k x\, y = 0.$$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

32. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + ay'_x + a \operatorname{arccot}^k x\, y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos(x\sqrt{a})$, $y_2 = \sin(x\sqrt{a})$.

2°. Particular solutions with $a < 0$: $y_1 = \exp(-x\sqrt{-a})$, $y_2 = \exp(x\sqrt{-a})$.

The substitution $w = y''_{xx} + ay$ leads to a first-order linear equation: $w'_x + \operatorname{arccot}^k x\, w = 0$.

33. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x \operatorname{arccot}^k x + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first-order linear equation: $w'_x + \operatorname{arccot}^k x\, w = 0$.

34. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + a \operatorname{arccot}^k x\, y'_x + a^2(\operatorname{arccot}^k x - a)y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax\bigr)\cos\bigl(\frac{\sqrt{3}}{2}ax\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax\bigr)\sin\bigl(\frac{\sqrt{3}}{2}ax\bigr)$.

35. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} - a(2\operatorname{arccot}^k x + 3a)y'_x + a^2(\operatorname{arccot}^k x + 2a)y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

36. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + (a \operatorname{arccot}^k x + b - a^2)y'_x + b(\operatorname{arccot}^k x - a)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

37. $y'''_{xxx} = (\operatorname{arccot}^k x - a)y''_{xx} + (a \operatorname{arccot}^k x - b)y'_x + b \operatorname{arccot}^k x\, y.$

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first-order linear equation: $w'_x = \operatorname{arccot}^k x\, w$.

38. $y'''_{xxx} + x \operatorname{arccot}^k x\, y''_{xx} + (ax^2 - \operatorname{arccot}^k x)y'_x + ax(x^2 \operatorname{arccot}^k x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{a}\,\bigr)$.

39. $y'''_{xxx} + (\operatorname{arccot}^k x + ax)y''_{xx} + a(x \operatorname{arccot}^k x + 2)y'_x + a \operatorname{arccot}^k x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

40. $y'''_{xxx} + x^2 \operatorname{arccot}^k x\, y''_{xx} - 2x \operatorname{arccot}^k x\, y'_x + 2 \operatorname{arccot}^k x\, y = 0.$

Solution:

$$y = C_1 x + C_2 x^2 + C_3\Bigl(x^2 \int x^{-3}\psi\,dx - x\int x^{-2}\psi\,dx\Bigr),$$

where $\psi = \exp\Bigl(-\displaystyle\int x^2 \operatorname{arccot}^k x\,dx\Bigr)$.

41. $y'''_{xxx} + (ax \operatorname{arccot}^k x + \operatorname{arccot}^k x + a)y''_{xx} + a^2 x \operatorname{arccot}^k x\, y'_x - a^2 \operatorname{arccot}^k x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

42. $xy'''_{xxx} + (ax^2 + b)y''_{xx} + 4axy'_x + 2ay = \arcsin^k x.$

Integrating the equation twice, we arrive at a first-order linear equation:

$$xy'_x + (ax^2 + b - 2)y = C_1 + C_2 x + \int_{x_0}^{x}(x - t)\arcsin^k t\,dt, \qquad x_0 \text{ is any number.}$$

43. $xy'''_{xxx} + (x \arcsin^k x + 3)y''_{xx} + (2\arcsin^k x + ax)y'_x + a(x \arcsin^k x + 1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

44. $xy'''_{xxx} + (x \arccos^k x + 3)y''_{xx} + (2\arccos^k x + ax)y'_x + a(x \arccos^k x + 1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

45. $xy'''_{xxx}+(x\arctan^k x+3)y''_{xx}+(2\arctan^k x+ax)y'_x+a(x\arctan^k x+1)y=0.$

Particular solutions: $y_1=x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2=x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

46. $xy'''_{xxx}+(x\,\mathrm{arccot}^k x+3)y''_{xx}+(2\,\mathrm{arccot}^k x+ax)y'_x+a(x\,\mathrm{arccot}^k x+1)y=0.$

Particular solutions: $y_1=x^{-1}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2=x^{-1}\sin\bigl(x\sqrt{a}\,\bigr)$.

47. $x^3y'''_{xxx}+[(a+6)x^2+b]y''_{xx}+2(2a+3)xy'_x+2ay=\arcsin^k x.$

Integrating the equation twice, we arrive at a first-order linear equation:

$$x^3y'_x+(ax^2+b)y=C_1+C_2x+\int_{x_0}^{x}(x-t)\arcsin^k t\,dt,\qquad x_0 \text{ is any number.}$$

48. $x^3y'''_{xxx}+x^2\arcsin^k x y''_{xx}-2xy'_x+2(2-\arcsin^k x)y=0.$

Particular solutions: $y_1=x^{-1}$, $y_2=x^2$.

49. $x^3y'''_{xxx}+x^2\arcsin^k x\,y''_{xx}-6xy'_x+6(2-\arcsin^k x)y=0.$

Particular solutions: $y_1=x^{-2}$, $y_2=x^3$.

50. $x^3y'''_{xxx}+x^2\arcsin^k x\,y''_{xx}+x(\arcsin^k x-1)y'_x+(\arcsin^k x-3)y=0.$

Particular solutions: $y_1=\cos(\ln x)$, $y_2=\sin(\ln x)$.

51. $x^3y'''_{xxx}+x^2(\arcsin^k x+1)y''_{xx}+x(\arcsin^k x-a-1)y'_x$
$-a(\arcsin^k x-2)y=0.$

Particular solutions: $y_1=x^{-\sqrt{a}}$, $y_2=x^{\sqrt{a}}$.

52. $x^3y'''_{xxx}+x^2(\arcsin^k x+a)y''_{xx}+x(a\arcsin^k x+b-a)y'_x$
$+b(\arcsin^k x-2)y=0.$

Particular solutions: $y_1=x^{n_1}$, $y_2=x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2+(a-1)n+b=0$.

53. $x^3y'''_{xxx}+x^2\arccos^k x\,y''_{xx}-2xy'_x+2(2-\arccos^k x)y=0.$

Particular solutions: $y_1=x^{-1}$, $y_2=x^2$.

54. $x^3y'''_{xxx}+x^2\arccos^k x\,y''_{xx}-6xy'_x+6(2-\arccos^k x)y=0.$

Particular solutions: $y_1=x^{-2}$, $y_2=x^3$.

55. $x^3y'''_{xxx}+x^2\arccos^k x\,y''_{xx}+x(\arccos^k x-1)y'_x+(\arccos^k x-3)y=0.$

Particular solutions: $y_1=\cos(\ln x)$, $y_2=\sin(\ln x)$.

56. $x^3y'''_{xxx}+x^2(\arccos^k x+1)y''_{xx}+x(\arccos^k x-a-1)y'_x$
$-a(\arccos^k x-2)y=0.$

Particular solutions: $y_1=x^{-\sqrt{a}}$, $y_2=x^{\sqrt{a}}$.

57. $x^3y'''_{xxx}+x^2(\arccos^k x+a)y''_{xx}+x(a\arccos^k x+b-a)y'_x$
$+b(\arccos^k x-2)y=0.$

Particular solutions: $y_1=x^{n_1}$, $y_2=x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2+(a-1)n+b=0$.

58. $x^3 y'''_{xxx} + x^2 \arctan^k x \, y''_{xx} - 2xy'_x + 2(2 - \arctan^k x)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

59. $x^3 y'''_{xxx} + x^2 \arctan^k x \, y''_{xx} - 6xy'_x + 6(2 - \arctan^k x)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

60. $x^3 y'''_{xxx} + x^2 \arctan^k x \, y''_{xx} + x(\arctan^k x - 1)y'_x + (\arctan^k x - 3)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

61. $x^3 y'''_{xxx} + x^2(\arctan^k x + 1)y''_{xx} + x(\arctan^k x - a - 1)y'_x - a(\arctan^k x - 2)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{a}}$, $y_2 = x^{\sqrt{a}}$.

62. $x^3 y'''_{xxx} + x^2(\arctan^k x + a)y''_{xx} + x(a \arctan^k x + b - a)y'_x + b(\arctan^k x - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

63. $x^3 y'''_{xxx} + x^2 \operatorname{arccot}^k x \, y''_{xx} - 2xy'_x + 2(2 - \operatorname{arccot}^k x)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

64. $x^3 y'''_{xxx} + x^2 \operatorname{arccot}^k x \, y''_{xx} - 6xy'_x + 6(2 - \operatorname{arccot}^k x)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

65. $x^3 y'''_{xxx} + x^2 \operatorname{arccot}^k x \, y''_{xx} + x(\operatorname{arccot}^k x - 1)y'_x + (\operatorname{arccot}^k x - 3)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

66. $x^3 y'''_{xxx} + x^2(\operatorname{arccot}^k x + 1)y''_{xx} + x(\operatorname{arccot}^k x - a - 1)y'_x - a(\operatorname{arccot}^k x - 2)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{a}}$, $y_2 = x^{\sqrt{a}}$.

67. $x^3 y'''_{xxx} + x^2(\operatorname{arccot}^k x + a)y''_{xx} + x(a \operatorname{arccot}^k x + b - a)y'_x + b(\operatorname{arccot}^k x - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

15.1.8 Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions

1. $y'''_{xxx} = \tan x \, y + ae^{\lambda x}(y'_x + \tan x \, y).$

Particular solution: $y_0 = \cos x$.

2. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + (2ae^{\lambda x} \tan x + 3)y'_x + [ae^{\lambda x}(2\tan^2 x + 1) + 2\tan x]y = 0.$

Particular solutions: $y_1 = \cos x$, $y_2 = x \cos x$.

3. $y'''_{xxx}+ae^{\lambda x}y''_{xx}+(3-2ae^{\lambda x}\cot x)y'_x+[ae^{\lambda x}(2\cot^2 x+1)-2\cot x]y=0.$

Particular solutions: $y_1=\sin x,\ \ y_2=x\sin x.$

4. $$y'''_{xxx}+a\cosh^n x\, y''_{xx}+(2a\cosh^n x\tan x+3)y'_x + [a\cosh^n x\,(2\tan^2 x+1)+2\tan x]y=0.$$

Particular solutions: $y_1=\cos x,\ \ y_2=x\cos x.$

5. $$y'''_{xxx}+a\cosh^n x\, y''_{xx}+(3-2a\cosh^n x\cot x)y'_x + [a\cosh^n x\,(2\cot^2 x+1)-2\cot x]y=0.$$

Particular solutions: $y_1=\sin x,\ \ y_2=x\sin x.$

6. $$y'''_{xxx}+a\sinh^n x\, y''_{xx}+(2a\sinh^n x\tan x+3)y'_x + [a\sinh^n x\,(2\tan^2 x+1)+2\tan x]y=0.$$

Particular solutions: $y_1=\cos x,\ \ y_2=x\cos x.$

7. $$y'''_{xxx}+a\sinh^n x\, y''_{xx}+(3-2a\sinh^n x\cot x)y'_x + [a\sinh^n x\,(2\cot^2 x+1)-2\cot x]y=0.$$

Particular solutions: $y_1=\sin x,\ \ y_2=x\sin x.$

8. $$y'''_{xxx}+a\tanh^n x\, y''_{xx}+(2a\tanh^n x\tan x+3)y'_x + [a\tanh^n x\,(2\tan^2 x+1)+2\tan x]y=0.$$

Particular solutions: $y_1=\cos x,\ \ y_2=x\cos x.$

9. $$y'''_{xxx}+a\tanh^n x\, y''_{xx}+(3-2a\tanh^n x\cot x)y'_x + [a\tanh^n x\,(2\cot^2 x+1)-2\cot x]y=0.$$

Particular solutions: $y_1=\sin x,\ \ y_2=x\sin x.$

10. $$y'''_{xxx}+a\coth^n x\, y''_{xx}+(2a\coth^n x\tan x+3)y'_x + [a\coth^n x\,(2\tan^2 x+1)+2\tan x]y=0.$$

Particular solutions: $y_1=\cos x,\ \ y_2=x\cos x.$

11. $$y'''_{xxx}+a\coth^n x\, y''_{xx}+(3-2a\coth^n x\cot x)y'_x + [a\coth^n x\,(2\cot^2 x+1)-2\cot x]y=0.$$

Particular solutions: $y_1=\sin x,\ \ y_2=x\sin x.$

12. $$y'''_{xxx}+a\ln^n x\, y''_{xx}-(2a\ln^n x\tanh x+3)y'_x + [a\ln^n x\,(2\tanh^2 x-1)+2\tanh x]y=0.$$

Particular solutions: $y_1=\cosh x,\ \ y_2=x\cosh x.$

13. $$y'''_{xxx}+a\ln^n x\, y''_{xx}-(2a\ln^n x\coth x+3)y'_x + [a\ln^n x\,(2\coth^2 x-1)+2\coth x]y=0.$$

Particular solutions: $y_1=\sinh x,\ \ y_2=x\sinh x.$

14. $$y'''_{xxx}+a\ln^n x\, y''_{xx}+(2a\ln^n x\tan x+3)y'_x + [a\ln^n x\,(2\tan^2 x+1)+2\tan x]y=0.$$

Particular solutions: $y_1=\cos x,\ \ y_2=x\cos x.$

15. $y'''_{xxx} + a\ln^n x\, y''_{xx} + (3 - 2a\ln^n x\cot x)y'_x + [a\ln^n x\,(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x,\ \ y_2 = x\sin x.$

16. $y'''_{xxx} + a\cos^n x\, y''_{xx} - (2a\cos^n x\tanh x + 3)y'_x + [a\cos^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$

Particular solutions: $y_1 = \cosh x,\ \ y_2 = x\cosh x.$

17. $y'''_{xxx} + a\cos^n x\, y''_{xx} - (2a\cos^n x\coth x + 3)y'_x + [a\cos^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$

Particular solutions: $y_1 = \sinh x,\ \ y_2 = x\sinh x.$

18. $y'''_{xxx} + a\sin^n x\, y''_{xx} - (2a\sin^n x\tanh x + 3)y'_x + [a\sin^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$

Particular solutions: $y_1 = \cosh x,\ \ y_2 = x\cosh x.$

19. $y'''_{xxx} + a\sin^n x\, y''_{xx} - (2a\sin^n x\coth x + 3)y'_x + [a\sin^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$

Particular solutions: $y_1 = \sinh x,\ \ y_2 = x\sinh x.$

20. $y'''_{xxx} + a\tan^n x\, y''_{xx} - (2a\tan^n x\tanh x + 3)y'_x + [a\tan^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$

Particular solutions: $y_1 = \cosh x,\ \ y_2 = x\cosh x.$

21. $y'''_{xxx} + a\tan^n x\, y''_{xx} - (2a\tan^n x\coth x + 3)y'_x + [a\tan^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$

Particular solutions: $y_1 = \sinh x,\ \ y_2 = x\sinh x.$

22. $y'''_{xxx} + a\cot^n x\, y''_{xx} - (2a\cot^n x\tanh x + 3)y'_x + [a\cot^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$

Particular solutions: $y_1 = \cosh x,\ \ y_2 = x\cosh x.$

23. $y'''_{xxx} + a\cot^n x\, y''_{xx} - (2a\cot^n x\coth x + 3)y'_x + [a\cot^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$

Particular solutions: $y_1 = \sinh x,\ \ y_2 = x\sinh x.$

24. $y'''_{xxx} + (be^{ax} + 2a)\cosh^n x\, y''_{xx} - a(be^{ax}\cosh^n x + a)y'_x - 2a^3\cosh^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

25. $y'''_{xxx} + (be^{ax} + 2a)\sinh^n x\, y''_{xx} - a(be^{ax}\sinh^n x + a)y'_x - 2a^3\sinh^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

26. $y'''_{xxx} + (be^{ax} + 2a)\tanh^n x\, y''_{xx} - a(be^{ax}\tanh^n x + a)y'_x - 2a^3\tanh^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

27. $y'''_{xxx} + (be^{ax} + 2a)\coth^n x\, y''_{xx} - a(be^{ax}\coth^n x + a)y'_x - 2a^3\coth^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

28. $y'''_{xxx} + (be^{ax} + 2a)\ln^n x\, y''_{xx} - a(be^{ax}\ln^n x + a)y'_x - 2a^3\ln^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

29. $y'''_{xxx} + (a\ln^n x - 2be^x)y''_{xx} - be^x(2a\ln^n x - be^x + 3)y'_x$
$+ be^x[a\ln^n x\,(be^x - 1) + 2be^x - 1]y = 0.$

Particular solutions: $y_1 = \exp(be^x),\ \ y_2 = x\exp(be^x).$

30. $y'''_{xxx} + (a\cos^n x - 2be^x)y''_{xx} - be^x(2a\cos^n x - be^x + 3)y'_x$
$+ be^x[a\cos^n x\,(be^x - 1) + 2be^x - 1]y = 0.$

Particular solutions: $y_1 = \exp(be^x),\ \ y_2 = x\exp(be^x).$

31. $y'''_{xxx} + (be^{ax} + 2a)\cos^n x\, y''_{xx} - a(be^{ax}\cos^n x + a)y'_x - 2a^3\cos^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

32. $y'''_{xxx} + (a\sin^n x - 2be^x)y''_{xx} - be^x(2a\sin^n x - be^x + 3)y'_x$
$+ be^x[a\sin^n x\,(be^x - 1) + 2be^x - 1]y = 0.$

Particular solutions: $y_1 = \exp(be^x),\ \ y_2 = x\exp(be^x).$

33. $y'''_{xxx} + (be^{ax} + 2a)\sin^n x\, y''_{xx} - a(be^{ax}\sin^n x + a)y'_x - 2a^3\sin^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

34. $y'''_{xxx} - [e^{\lambda x}(\tan x + a) + a]y''_{xx} + [(a^2 + 1)e^{\lambda x} + 1]y'_x$
$+ a[e^{\lambda x}(a\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = \cos x.$

35. $y'''_{xxx} + [\tan x\,(axe^{\lambda x} + 1) + ae^{\lambda x}]y''_{xx} - axe^{\lambda x}y'_x + ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \cos x.$

36. $y'''_{xxx} + (a\tan^n x - 2be^x)y''_{xx} - be^x(2a\tan^n x - be^x + 3)y'_x$
$+ be^x[a\tan^n x\,(be^x - 1) + 2be^x - 1]y = 0.$

Particular solutions: $y_1 = \exp(be^x),\ \ y_2 = x\exp(be^x).$

37. $y'''_{xxx} + (be^{ax} + 2a)\tan^n x\, y''_{xx} - a(be^{ax}\tan^n x + a)y'_x - 2a^3\tan^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax},\ \ y_2 = e^{-ax} + b/a.$

38. $y'''_{xxx} + [e^{\lambda x}(\cot x + a) + a]y''_{xx} + [(a^2 + 1)e^{\lambda x} + 1]y'_x$
$+ a[e^{\lambda x}(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax},\ \ y_2 = \sin x.$

39. $y'''_{xxx} + [ae^{\lambda x} - \cot x\,(axe^{\lambda x} + 1)]y''_{xx} - axe^{\lambda x}y'_x + ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \sin x.$

40. $y'''_{xxx} + (a\cot^n x - 2be^x)y''_{xx} - be^x(2a\cot^n x - be^x + 3)y'_x$
$+ be^x[a\cot^n x\,(be^x - 1) + 2be^x - 1]y = 0.$

Particular solutions: $y_1 = \exp(be^x),\ \ y_2 = x\exp(be^x).$

41. $y'''_{xxx} + (be^{ax} + 2a)\cot^n x\, y''_{xx} - a(be^{ax}\cot^n x + a)y'_x - 2a^3\cot^n x\, y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = e^{-ax} + b/a$.

42. $y'''_{xxx} - [\cosh^n x\,(\tan x + a) + a]y''_{xx} + [(a^2+1)\cosh^n x + 1]y'_x$
$+ a[\cosh^n x\,(a\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

43. $y'''_{xxx} + [\cosh^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\cosh^n x + 1]y'_x$
$+ a[\cosh^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

44. $y'''_{xxx} - [\sinh^n x\,(\tan x + a) + a]y''_{xx} + [(a^2+1)\sinh^n x + 1]y'_x$
$+ a[\sinh^n x\,(a\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

45. $y'''_{xxx} + [\sinh^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\sinh^n x + 1]y'_x$
$+ a[\sinh^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

46. $y'''_{xxx} - [\tanh^n x\,(\tan x + a) + a]y''_{xx} + [(a^2+1)\tanh^n x + 1]y'_x$
$+ a[\tanh^n x\,(a\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

47. $y'''_{xxx} + [\tanh^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\tanh^n x + 1]y'_x$
$+ a[\tanh^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

48. $y'''_{xxx} - [\coth^n x\,(\tan x + a) + a]y''_{xx} + [(a^2+1)\coth^n x + 1]y'_x$
$+ a[\coth^n x\,(\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

49. $y'''_{xxx} + [\coth^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\coth^n x + 1]y'_x$
$+ a[\coth^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

50. $y'''_{xxx} + [a\tan^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\tan^n x - 1]y'_x$
$+ b[a\tan^n x(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

51. $y'''_{xxx} + (a\tan^n x + b\tanh^m x)y''_{xx} + cy'_x + c(a\tan^n x + b\tanh^m x)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos(x\sqrt{c})$, $y_2 = \sin(x\sqrt{c})$.

2°. Particular solutions with $c < 0$: $y_1 = \exp(-x\sqrt{-c})$, $y_2 = \exp(x\sqrt{-c})$.

52. $y'''_{xxx} + [a\tan^n x(\coth x - b) - b]y''_{xx} + [a(b^2-1)\tan^n x - 1]y'_x$
$+ b[a\tan^n x(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

53. $y'''_{xxx} + (a\tan^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\tan^n x + b\coth^m x)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

54. $y'''_{xxx} + [a\cot^n x(\tanh x - b) - b]y''_{xx} + [a(b^2 - 1)\cot^n x - 1]y'_x$
$+ b[a\cot^n x(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

55. $y'''_{xxx} + a\cot^n x\tanh^m x\, y''_{xx} - by'_x - ab\cot^n x\tanh^m x\, y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\,\bigr)$.

56. $y'''_{xxx} + [a\cot^n x(\coth x - b) - b]y''_{xx} + [a(b^2 - 1)\cot^n x - 1]y'_x$
$+ b[a\cot^n x(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

57. $y'''_{xxx} + (a\cot^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\cot^n x + b\coth^m x)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

58. $y'''_{xxx} + [a\ln^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2 - 1)\ln^n x - 1]y'_x$
$+ b[a\ln^n x\,(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

59. $y'''_{xxx} + a\ln^n x\tanh^m x\, y''_{xx} - by'_x - ab\ln^n x\tanh^m x\, y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\,\bigr)$.

60. $y'''_{xxx} + [a\ln^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2 - 1)\ln^n x - 1]y'_x$
$+ b[a\ln^n x\,(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

61. $y'''_{xxx} + (a\ln^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\ln^n x + b\coth^m x)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

62. $y'''_{xxx} - [\ln^n x\,(\tan x + a) + a]y''_{xx} + [(a^2 + 1)\ln^n x + 1]y'_x$
$+ a[\ln^n x\,(a\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

63. $y'''_{xxx} + [\ln^n x\,(\cot x + a) + a]y''_{xx} + [(a^2 + 1)\ln^n x + 1]y'_x$
$+ a[\ln^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

64. $y'''_{xxx} + [a\cos^n x(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\cos^n x - 1]y'_x + b[a\cos^n x(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

65. $y'''_{xxx} + a\cos^n x\tanh^m x\, y''_{xx} + by'_x + ab\cos^n x\tanh^m x\, y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

66. $y'''_{xxx} + [a\cos^n x(\coth x - b) - b]y''_{xx} + [a(b^2-1)\cos^n x - 1]y'_x + b[a\cos^n x(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

67. $y'''_{xxx} + (a\cos^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\cos^n x + b\coth^m x)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

68. $y'''_{xxx} + [a\sin^n x(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\sin^n x - 1]y'_x + b[a\sin^n x(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

69. $y'''_{xxx} + a\sin^n x\tanh^m x\, y''_{xx} + by'_x + ab\sin^n x\tanh^m x\, y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

70. $y'''_{xxx} + [a\sin^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2-1)\sin^n x - 1]y'_x + b[a\sin^n x\,(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

71. $y'''_{xxx} + (a\sin^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\sin^n x + b\coth^m x)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

72. $xy'''_{xxx} + [ax^2e^{\lambda x}(b - \ln x) + 2]y''_{xx} + axe^{\lambda x}y'_x - ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - b + 1$.

73. $(e^{\lambda x} - 1)y'''_{xxx} - (ae^{\lambda x} + \tan x)y''_{xx} + (e^{\lambda x} + a^2)y'_x + a(a\tan x - e^{\lambda x})y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

74. $a\cosh^n x\, y'''_{xxx} + [\tan x\,(a\cosh^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos x$.

75. $a\cosh^n x\, y'''_{xxx} + [1 - \cot x\,(a\cosh^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sin x$.

76. $a\sinh^n x\, y'''_{xxx} + [\tan x\,(a\sinh^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cos x.$

77. $a\sinh^n x\, y'''_{xxx} + [1 - \cot x\,(a\sinh^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sin x.$

78. $a\tanh^n x\, y'''_{xxx} + [\tan x\,(a\tanh^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cos x.$

79. $a\tanh^n x\, y'''_{xxx} + [1 - \cot x\,(a\tanh^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sin x.$

80. $a\coth^n x\, y'''_{xxx} + [\tan x\,(a\coth^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cos x.$

81. $a\coth^n x\, y'''_{xxx} + [1 - \cot x\,(a\coth^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sin x.$

82. $a\ln^n x\, y'''_{xxx} + [\tanh x\,(x - a\ln^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cosh x.$

83. $a\ln^n x\, y'''_{xxx} + [\coth x\,(x - a\ln^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sinh x.$

84. $a\ln^n x\, y'''_{xxx} + [\tan x\,(a\ln^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cos x.$

85. $a\ln^n x\, y'''_{xxx} + [1 - \cot x\,(a\ln^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sin x.$

86. $a\cos^n x\, y'''_{xxx} + [\tanh x\,(x - a\cos^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cosh x.$

87. $a\cos^n x\, y'''_{xxx} + [\coth x\,(x - a\cos^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sinh x.$

88. $ax\cos^n x\, y'''_{xxx} + (2a\cos^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \ln x - b + 1.$

89. $a\sin^n x\, y'''_{xxx} + [\tanh x\,(x - a\sin^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \cosh x.$

90. $a\sin^n x\, y'''_{xxx} + [\coth x\,(x - a\sin^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \sinh x.$

91. $ax\sin^n x\, y'''_{xxx} + (2a\sin^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x, \;\; y_2 = \ln x - b + 1.$

92. $a\tan^n x\, y'''_{xxx} + [\tanh x\,(x - a\tan^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \cosh x.$

93. $a\tan^n x\, y'''_{xxx} + [\coth x\,(x - a\tan^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \sinh x.$

94. $ax\tan^n x\, y'''_{xxx} + (2a\tan^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \ln x - b + 1.$

95. $a\cot^n x\, y'''_{xxx} + [\tanh x\,(x - a\cot^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \cosh x.$

96. $a\cot^n x\, y'''_{xxx} + [\coth x\,(x - a\cot^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \sinh x.$

97. $ax\cot^n x\, y'''_{xxx} + (2a\cot^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x,\ \ y_2 = \ln x - b + 1.$

15.1.9 Equations Containing Arbitrary Functions

◆ *Notation:* $f = f(x)$, $g = g(x)$, *and* $h = h(x)$ *are arbitrary functions of* x*;* a, b, c, n, *and* λ *are arbitrary parameters.*

► **Equations of the form** $f_3(x)y'''_{xxx} + f_1(x)y'_x + f_0(x)y = g(x)$**.**

1. $y'''_{xxx} = f(x)y.$

The transformation $x = t^{-1}$, $y = ut^{-2}$ leads to an equation of the same form: $u'''_{ttt} = -t^{-6}f(1/t)u$.

2. $y'''_{xxx} = f\Big(\dfrac{ax+b}{cx+d}\Big)\dfrac{y}{(cx+d)^6}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^2}$ leads to a simpler equation: $w'''_{\xi\xi\xi} = \Delta^{-3}f(\xi)w$, where $\Delta = ad - bc$.

3. $fy'''_{xxx} - f'''_{xxx}y = 0.$

Particular solution: $y_0 = f$. The substitution $y = f\displaystyle\int z\,dx$ leads to a second-order linear equation: $fz''_{xx} + 3f'_x z'_x + 3f''_{xx}z = 0$.

4. $fy'''_{xxx} + f'''_{xxx}y = g.$

Integrating yields a second-order linear equation: $fy''_{xx} - f'_x y'_x + f''_{xx}y = \displaystyle\int g\,dx + C.$

5. $y'''_{xxx} + fy'_x - (af + a^3)y = 0.$

Particular solution: $y_0 = e^{ax}$. The substitution $w = y'_x - ay$ leads to a second-order linear equation: $w''_{xx} + aw'_x + (f + a^2)w = 0$.

6. $y'''_{xxx} + fy'_x + ax(f + a^2x^2 - 3a)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$. The substitution $y = \exp\bigl(-\frac{1}{2}ax^2\bigr)\int z(x)\,dx$ leads to a second-order linear equation: $z''_{xx} - 3axz'_x + (f + 3a^2x^2 - 3a)z = 0$.

7. $y'''_{xxx} + (f - a^2)y'_x + afy = 0.$

Particular solution: $y_0 = e^{-ax}$. The substitution $w = y'_x + ay$ leads to a second-order linear equation: $w''_{xx} - aw'_x + fw = 0$.

8. $y'''_{xxx} + xfy'_x - 2fy = 0.$

Particular solution: $y_0 = x^2$. The substitution $w = xy'_x - 2y$ leads to a second-order linear equation: $w''_{xx} + xfw = 0$.

9. $y'''_{xxx} + (ax + b)fy'_x - afy = 0.$

Particular solution: $y_0 = ax + b$.

10. $y'''_{xxx} + (ax + b)fy'_x - 2afy = 0.$

Particular solution: $y_0 = (ax + b)^2$. The substitution $w = (ax + b)y'_x - 2ay$ leads to a second-order linear equation: $w''_{xx} + (ax + b)fw = 0$.

11. $y'''_{xxx} + (f - a^2x^2)y'_x + ax(f - 3a)y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$. The substitution $y = \exp\bigl(-\frac{1}{2}ax^2\bigr)\int z(x)\,dx$ leads to a second-order linear equation: $z''_{xx} - 3axz'_x + (2a^2x^2 - 3a + f)z = 0$.

12. $y'''_{xxx} + (f - a^2x^{2n})y'_x - a[x^nf + 3anx^{2n-1} + n(n-1)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(\frac{a}{n+1}x^{n+1}\Bigr)$. Substituting $y = \exp\Bigl(\frac{a}{n+1}x^{n+1}\Bigr)\int z(x)\,dx$ yields a second-order linear equation:

$$z''_{xx} + 3ax^nz'_x + (2a^2x^{2n} + 3anx^{n-1} + f)z = 0.$$

13. $xy'''_{xxx} + xfy'_x - [(ax + 1)f + a^3x + 3a^2]y = 0.$

Particular solution: $y_0 = xe^{ax}$.

14. $x^2y'''_{xxx} + (xf - a^2 - a)y'_x + (a - 1)fy = 0.$

Particular solution: $y_0 = x^{1-a}$. The substitution $w = xy'_x + (a-1)y$ leads to a second-order linear equation: $xw''_{xx} - (a + 1)w'_x + fw = 0$.

15. $x^2y'''_{xxx} + [x(ax + 1)f - 6]y'_x + fy = 0.$

Particular solution: $y_0 = a + x^{-1}$.

16. $x(x + 1)y'''_{xxx} + x(f - x - 3)y'_x - (x + 1)fy = 0.$

Particular solution: $y_0 = xe^x$.

17. $x^3y'''_{xxx} + xfy'_x + (a - 1)(f + a^2 + a)y = 0.$

Particular solution: $y_0 = x^{1-a}$. The substitution $w = xy'_x + (a-1)y$ leads to a second-order linear equation: $x^2w''_{xx} - (a + 1)xw'_x + (f + a^2 + a)w = 0$.

18. $x^6 y'''_{xxx} + x^2 f y'_x + (a^3 + af - 2xf)y = 0.$

Particular solution: $y_0 = x^2 e^{a/x}$.

19. $y'''_{xxx} + (f - a^2 e^{2\lambda x})y'_x - ae^{\lambda x}(f + 3a\lambda e^{\lambda x} + \lambda^2)y = 0.$

Particular solution: $y_0 = \exp\Bigl(\frac{a}{\lambda}e^{\lambda x}\Bigr)$. The substitution $y = \exp\Bigl(\frac{a}{\lambda}e^{\lambda x}\Bigr)\int z(x)\,dx$ leads to a second-order linear equation: $z''_{xx} + 3ae^{\lambda x}z'_x + (f + 2a^2 e^{2\lambda x} + 3a\lambda e^{\lambda x})z = 0$.

20. $y'''_{xxx} + [(1 + be^{ax})f - a^2]y'_x + afy = 0.$

Particular solution: $y_0 = e^{-ax} + b$.

21. $y'''_{xxx} = fy'_x + \tanh x\,(1 - f)y.$

This is a special case of equation 15.1.9.30 with $g(x) = \cosh x$.

22. $y'''_{xxx} = fy'_x + \coth x\,(1 - f)y.$

This is a special case of equation 15.1.9.30 with $g(x) = \sinh x$.

23. $y'''_{xxx} + fy'_x + \tan x\,(f - 1)y = 0.$

Particular solution: $y_0 = \cos x$. The substitution $y = \cos x \int z(x)\,dx$ leads to a second-order linear equation: $z''_{xx} - 3\tan x\, z'_x + (f - 3)z = 0$.

24. $y'''_{xxx} + fy'_x + \cot x\,(1 - f)y = 0.$

Particular solution: $y_0 = \sin x$.

25. $y'''_{xxx} + fy'_x + f'_x y = g.$

Integrating yields a second-order linear equation: $y''_{xx} + fy = \int g\,dx + C$.

26. $y'''_{xxx} + 2fy'_x + f'_x y = 0.$

Solution: $y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2$. Here, w_1 and w_2 are linearly independent solutions of the second-order linear equation: $2w''_{xx} + fw = 0$.

27. $y'''_{xxx} + f'_x y'_x + f(2f'_x - f^2)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + fy'_x + f^2 y = C\exp\Bigl(\int f\,dx\Bigr)$.

28. $y'''_{xxx} + (a - 1)f^2 y'_x - [f''_{xx} - (2a + 1)ff'_x + af^3]y = 0.$

Integrating yields a second-order equation: $y''_{xx} + fy'_x + (af^2 - f'_x)y = C\exp\Bigl(\int f\,dx\Bigr)$.

29. $y'''_{xxx} + (f - a^2)y'_x + (f'_x - af)y = 0.$

The substitution $w = y''_{xx} + ay'_x + fy$ leads to a first-order linear equation: $w'_x - aw = 0$.

30. $y'''_{xxx} = \dfrac{g'''_{xxx}}{g}y + f(x)\Bigl(y'_x - \dfrac{g'_x}{g}y\Bigr).$

The substitution $w = y'_x - \dfrac{g'_x}{g}y$ leads to a second-order linear equation.

► **Equations of the form** $f_3(x)y'''_{xxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x)$.

31. $y'''_{xxx} + ay''_{xx} + by'_x + cy = f(x)$.

This is a special case of equation 17.1.6.26 with $n = 3$.

32. $y'''_{xxx} + ay''_{xx} + fy'_x + afy = 0$.

The substitution $w = y'_x + ay$ leads to a second-order linear equation: $w''_{xx} + fw = 0$.

33. $y'''_{xxx} + fy''_{xx} - a^2(f + a)y = 0$.

Particular solution: $y_0 = e^{ax}$. The substitution $w = y'_x - ay$ leads to a second-order linear equation: $w''_{xx} + (f + a)w'_x + a(f + a)w = 0$.

34. $y'''_{xxx} + fy''_{xx} + ay'_x + afy = 0$.

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a first-order linear equation: $w'_x + fw = 0$.

35. $y'''_{xxx} + fy''_{xx} + ax^n y'_x + ax^{n-1}(xf + n)y = 0$.

The substitution $w = y''_{xx} + ax^n y$ leads to a first-order linear equation: $w'_x + fw = 0$.

36. $y'''_{xxx} + fy''_{xx} + afy'_x + a^3 y = 0$.

The substitution $w = y'_x + ay$ leads to a second-order linear equation: $w''_{xx} + (f - a)w'_x + a^2 w = 0$.

37. $y'''_{xxx} + fy''_{xx} + afy'_x + a^2(f - a)y = 0$.

Particular solutions: $y_1 = \exp\bigl(-\tfrac{1}{2}ax\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}ax\bigr)$, $y_2 = \exp\bigl(-\tfrac{1}{2}ax\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}ax\bigr)$.

38. $y'''_{xxx} + fy''_{xx} + gy'_x + h = 0$.

The substitution $w = y'_x$ leads to a second-order linear equation: $w''_{xx} + fw'_x + gw + h = 0$.

39. $y'''_{xxx} + fy''_{xx} - a(2f + 3a)y'_x + a^2(f + 2a)y = 0$.

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

40. $y'''_{xxx} + fy''_{xx} + xgy'_x - gy = 0$.

The substitution $w = xy'_x - y$ leads to a second-order equation: $xw''_{xx} + (xf - 1)w'_x + x^2 gw = 0$.

41. $y'''_{xxx} + fy''_{xx} + (g - a^2)y'_x - a(af + g)y = 0$.

Particular solution: $y_0 = e^{ax}$. The substitution $w = y'_x - ay$ leads to a second-order linear equation: $w''_{xx} + (f + a)w'_x + (af + g)w = 0$.

42. $y'''_{xxx} + fy''_{xx} + (af + b - a^2)y'_x + b(f - a)y = 0$.

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

43. $y'''_{xxx} + (f - a)y''_{xx} - a^2 f y = 0.$

Particular solution: $y_0 = e^{ax}$. The substitution $w = y'_x - ay$ leads to a second-order linear equation: $w''_{xx} + f w'_x + a f w = 0$.

44. $y'''_{xxx} = (f - a)y''_{xx} + (af - b)y'_x + bfy.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$. The substitution $w = y''_{xx} + ay'_x + by$ leads to a first-order linear equation: $w'_x = fw$.

45. $y'''_{xxx} + (f - a)y''_{xx} + gy'_x - a(af + g)y = 0.$

Particular solution: $y_0 = e^{ax}$.

46. $y'''_{xxx} + (f + a)y''_{xx} + (af + g)y'_x + agy = 0.$

Particular solution: $y_0 = e^{-ax}$.

47. $y'''_{xxx} + xfy''_{xx} + (ax^2 - f)y'_x + ax(x^2 f + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{a}\,\bigr)$.

48. $y'''_{xxx} + (ax + b)fy''_{xx} + xfy'_x - 2fy = 0.$

Particular solution: $y_0 = x^2 + 2ax + b$.

49. $y'''_{xxx} + (f + ax)y''_{xx} + a(xf + 2)y'_x + afy = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}ax^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}ax^2\bigr)\,dx$.

50. $y'''_{xxx} + x^2 fy''_{xx} - 2xfy'_x + 2fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.
Solution:

$$y = C_1 x + C_2 x^2 + C_3\Bigl(x^2\int x^{-3}\psi\,dx - x\int x^{-2}\psi\,dx\Bigr),$$

where $\psi = \exp\Bigl(-\displaystyle\int x^2 f\,dx\Bigr)$.

51. $y'''_{xxx} + (f + ax)y''_{xx} + (g + 2a)y'_x + a[xg + (1 - ax^2)f]y = 0.$

Particular solution: $y_0 = \exp\bigl(-\frac{1}{2}ax^2\bigr)$. The substitution $w = y'_x + axy$ leads to a second-order linear equation: $w''_{xx} + fw'_x + (g - axf)w = 0$.

52. $y'''_{xxx} + (axf + f + a)y''_{xx} + a^2 xfy'_x - a^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

53. $y'''_{xxx} + (ax^2 + bx + c)fy''_{xx} - 2afy = 0.$

Particular solution: $y_0 = ax^2 + bx + c$.

54. $y'''_{xxx} + x(xf + g)y''_{xx} - gy'_x - 2fy = 0.$

Particular solution: $y_0 = x^2$. The substitution $w = xy'_x - 2y$ leads to a second-order linear equation: $w''_{xx} + x(xf + g)w'_x + xfw = 0$.

55. $y'''_{xxx} - x(ax+b)fy''_{xx} + (b-a^2)fy'_x + 2afy = 0.$

Particular solution: $y_0 = x^2 + ax + \frac{1}{2}(a^2 - b)$.

56. $y'''_{xxx} - [(2x+a)f + (x^2+ax+b)g]y''_{xx} + 2fy'_x + 2gy = 0.$

Particular solution: $y_0 = x^2 + ax + b$.

57. $xy'''_{xxx} + 3y''_{xx} + x(ax^2+1)fy'_x - (ax^2-1)fy = 0.$

Particular solution: $y_0 = ax + x^{-1}$.

58. $xy'''_{xxx} + (ax^2+b)y''_{xx} + 4axy'_x + 2ay = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$xy'_x + (ax^2+b-2)y = C_1 + C_2x + \int_{x_0}^{x}(x-t)f(t)\,dt, \qquad x_0 \text{ is any number.}$$

59. $xy'''_{xxx} + x(f-2a)y''_{xx} + x(g+a^2)y'_x - [a(ax+2)f + (ax+1)g]y = 0.$

Particular solution: $y_0 = xe^{ax}$.

60. $xy'''_{xxx} + (xf+3)y''_{xx} + (2f+ax)y'_x + a(xf+1)y = 0.$

Particular solutions: $y_1 = x^{-1}\cos(x\sqrt{a})$, $y_2 = x^{-1}\sin(x\sqrt{a})$.

61. $xy'''_{xxx} + (xf+3)y''_{xx} + (ax+2)fy'_x + a(axf + f - a^2x)y = 0.$

Particular solutions:

$$y_1 = x^{-1}\exp\bigl(-\tfrac{1}{2}ax\bigr)\cos\bigl(\tfrac{\sqrt{3}}{2}ax\bigr), \quad y_2 = x^{-1}\exp\bigl(-\tfrac{1}{2}ax\bigr)\sin\bigl(\tfrac{\sqrt{3}}{2}ax\bigr).$$

62. $xy'''_{xxx} + (xf+3)y''_{xx} + (axf + 2f - a^2x)y'_x + a(f-a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^{-1}e^{-ax}$.

63. $xy'''_{xxx} + (xf+3)y''_{xx} + (2f + ax^{n+1})y'_x + ax^n(xf+n+1)y = 0.$

The substitution $w = xy$ leads to an equation of the form 3.1.9.35: $w'''_{xxx} + fw''_{xx} + ax^n w'_x + ax^{n-1}(xf+n)w = 0$.

64. $xy'''_{xxx} + (x^2f + a + 2)y''_{xx} - a(a+1)fy = 0.$

Particular solution: $y_0 = x^{-a}$. The substitution $w = xy'_x + ay$ leads to a second-order linear equation: $w''_{xx} + xfw'_x - (a+1)fw = 0$.

65. $xy'''_{xxx} + [x^2(ax^2+1)f + 3]y''_{xx} - 2fy = 0.$

Particular solution: $y_0 = ax + x^{-1}$.

66. $xy'''_{xxx} + [x(ax^2-1)f + x^2(ax^2+1)g + 3]y''_{xx} - 2fy'_x - 2gy = 0.$

Particular solution: $y_0 = ax + x^{-1}$.

67. $(ax-1)y'''_{xxx} + x[(ax-2)f - a^2]y''_{xx} + [(2-a^2x^2)f + a^2]y'_x + 2a(ax-1)fy = 0.$

Particular solutions: $y_1 = x^2$, $y_2 = e^{ax}$.

68. $x^2 y'''_{xxx} + xfy''_{xx} + [x(ax+1)g + 2f - 6]y'_x + gy = 0.$

Particular solution: $y_0 = a + x^{-1}$.

69. $x^2 y'''_{xxx} + x[x(ax+1)f + 3]y''_{xx} - 2fy = 0.$

Particular solution: $y_0 = a + x^{-1}$.

70. $x^2 y'''_{xxx} + x(xf + a)y''_{xx} + [(a-2)xf + b]y'_x + (b - a + 2)fy = 0.$

By integrating, we obtain the second-order nonhomogeneous Euler equation 14.1.9.15: $x^2 y''_{xx} + (a-2)xy'_x + (b-a+2)y = C\exp\Bigl(-\int f\,dx\Bigr)$.

71. $(ax+b)xy'''_{xxx} + (\alpha x + \beta)y''_{xx} + xy'_x + y = f.$

Integrating yields a second-order linear equation: $(ax+b)xy''_{xx} + [(\alpha - 2a)x + \beta - b]y'_x + (x + 2a - \alpha)y = \int f\,dx + C$.

72. $x^3 y'''_{xxx} + ax^2 y''_{xx} + bxy'_x + cy = f(x).$

Nonhomogeneous Euler equation. The substitution $t = \ln|x|$ leads to an equation of the form 15.1.9.31: $y'''_{ttt} + (a-3)y''_{tt} + (b-a+2)y'_t + cy = f(\pm e^t)$.

73. $x^3 y'''_{xxx} + (a+2)x^2 y''_{xx} + xfy'_x + afy = 0.$

Particular solution: $y_0 = x^{-a}$.

74. $x^3 y'''_{xxx} + [(a+6)x^2 + b]y''_{xx} + 2(2a+3)xy'_x + 2ay = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$x^3 y'_x + (ax^2 + b)y = C_1 + C_2 x + \int_{x_0}^{x} (x-t)f(t)\,dt, \qquad x_0 \text{ is any number}.$$

75. $x^3 y'''_{xxx} + x^2(bx^{2a+1} - 3a)y''_{xx} + 2a(a+1)(2a+1)y = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$x^{-2a} y'_x + (ax^{-2a-1} + b)y = C_1 + C_2 x + \int_{x_0}^{x} (x-t)t^{-2a-3} f(t)\,dt, \qquad x_0 \text{ is any number}.$$

76. $x^3 y'''_{xxx} + x^2 fy''_{xx} - 2xy'_x + 2(2-f)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

77. $x^3 y'''_{xxx} + x^2 fy''_{xx} - 6xy'_x + 6(2-f)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

78. $x^3 y'''_{xxx} + x^2 fy''_{xx} + x(f-1)y'_x + (f-3)y = 0.$

Particular solutions: $y_1 = \cos(\ln|x|)$, $y_2 = \sin(\ln|x|)$.

79. $x^3 y'''_{xxx} + x^2(f+1)y''_{xx} + x(f-a-1)y'_x - a(f-2)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{a}}$, $y_2 = x^{\sqrt{a}}$.

80. $x^3y'''_{xxx} + x^2(f+a)y''_{xx} + x(af+b-a)y'_x + b(f-2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

81. $x^3y'''_{xxx} + x^2(f+a)y''_{xx} + x[g+(a-1)f]y'_x + (a-2)gy = 0.$

Particular solution: $y_0 = x^{2-a}$. The substitution $w = xy'_x + (a-2)y$ leads to a second-order linear equation: $x^2w''_{xx} + xfw'_x + gw = 0$.

82. $x^3y'''_{xxx} + x^2(f+2ax)y''_{xx} + x(2axf+a^2x^2+b)y'_x + (a^2x^2f+bf-2b)y = 0.$

Particular solutions: $y_1 = e^{-ax}x^{n_1}$, $y_2 = e^{-ax}x^{n_2}$, where n_1 and n_2 are roots of the quadratic equation $n^2 - n + b = 0$.

83. $y'''_{xxx} + ae^{\lambda x}y''_{xx} - 3\lambda^2 y'_x + 2\lambda^3 y = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$e^{-\lambda x}y'_x + (a + 2\lambda e^{-\lambda x})y = C_1 + C_2x + \int_{x_0}^{x}(x-t)e^{-\lambda t}f(t)\,dt, \qquad x_0 \text{ is any number.}$$

84. $y'''_{xxx} + (f+a)y''_{xx} + [af + (1+be^{ax})g]y'_x + agy = 0.$

Particular solution: $y_0 = e^{-ax} + b$.

85. $y'''_{xxx} + (be^{ax} + 2a)fy''_{xx} - a(be^{ax}f + a)y'_x - 2a^3fy = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = e^{-ax} + b/a$.

86. $y'''_{xxx} + (f - 2ae^{\lambda x})y''_{xx} - ae^{\lambda x}(2f - ae^{\lambda x} + 3\lambda)y'_x$
$\qquad + ae^{\lambda x}[(ae^{\lambda x} - \lambda)f + 2a\lambda e^{\lambda x} - \lambda^2]y = 0.$

Particular solutions: $y_1 = \exp\Big(\dfrac{a}{\lambda}e^{\lambda x}\Big)$, $y_2 = x\exp\Big(\dfrac{a}{\lambda}e^{\lambda x}\Big)$.

87. $y'''_{xxx} + (f - ae^{\lambda x})y''_{xx} + (g - 2a\lambda e^{\lambda x})y'_x - ae^{\lambda x}[(ae^{\lambda x} + \lambda)f + g + \lambda^2]y = 0.$

Particular solution: $y_0 = \exp\Big(\dfrac{a}{\lambda}e^{\lambda x}\Big)$. The substitution $y = \exp\Big(\dfrac{a}{\lambda}e^{\lambda x}\Big)\displaystyle\int z(x)\,dx$ leads to a second-order linear equation:

$$z''_{xx} + (f + 2ae^{\lambda x})z'_x + (2ae^{\lambda x}f + g + a^2e^{2\lambda x} + a\lambda e^{\lambda x})z = 0.$$

88. $y'''_{xxx} - [(a + c + be^{ax})f - a + c]y''_{xx}$
$\qquad + [(c^2 - a^2 + bce^{ax})f - ac]y'_x + ac(a+c)fy = 0.$

Particular solutions: $y_1 = e^{cx}$, $y_2 = e^{-ax} + b/c$.

89. $e^{\lambda x}y'''_{xxx} + (2\lambda e^{\lambda x} + \beta e^{\mu x} + \gamma)y''_{xx} + (\lambda^2 e^{\lambda x} + 2\beta\mu e^{\mu x})y'_x + \beta\mu^2 e^{\mu x}y = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$e^{\lambda x}y'_x + (\beta e^{\mu x} + \gamma)y = C_1 + C_2x + \int_{x_0}^{x}(x-t)f(t)\,dt, \qquad x_0 \text{ is any number.}$$

90. $y'''_{xxx} + fy''_{xx} + gy'_x - \lambda[\lambda f + \tanh(\lambda x)(g + \lambda^2)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x)$. The substitution $y = \cosh(\lambda x)\displaystyle\int z(x)\,dx$ leads to a second-order linear equation: $z''_{xx} + [f + 3\lambda\tanh(\lambda x)]z'_x + [g + 3\lambda^2 + 2\lambda f\tanh(\lambda x)]z = 0$.

91. $y'''_{xxx} + fy''_{xx} - \lambda[2f\tanh(\lambda x) + 3\lambda]y'_x + \lambda^2\{[2\tanh^2(\lambda x) - 1]f + 2\lambda\tanh(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \cosh(\lambda x)$, $y_2 = x\cosh(\lambda x)$.

92. $y'''_{xxx} + fy''_{xx} - \lambda[2f\coth(\lambda x) + 3\lambda]y'_x + \lambda^2\{[2\coth^2(\lambda x) - 1]f + 2\lambda\coth(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \sinh(\lambda x)$, $y_2 = x\sinh(\lambda x)$.

93. $y'''_{xxx} + [(\tanh x - a)f - a]y''_{xx} + [(a^2 - 1)f - 1]y'_x + a[(1 - a\tanh x)f + 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cosh x$.

94. $y'''_{xxx} + [(\coth x - a)f - a]y''_{xx} + [(a^2 - 1)f - 1]y'_x + a[(1 - a\coth x)f + 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \sinh x$.

95. $y'''_{xxx} + [\lambda\tanh(\lambda x)(xf - 1) - f]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cosh(\lambda x)$.

96. $y'''_{xxx} + [\lambda\coth(\lambda x)(xf - 1) - f]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sinh(\lambda x)$.

97. $xy'''_{xxx} + [x^2(a - \ln x)f + 2]y''_{xx} + xfy'_x - fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - a + 1$.

98. $y'''_{xxx} + fy''_{xx} + gy'_x + \lambda[\lambda f + \tan(\lambda x)(g - \lambda^2)]y = 0.$

Particular solution: $y_0 = \cos(\lambda x)$. The substitution $y = \cos(\lambda x)\int z(x)\,dx$ leads to a second-order linear equation: $z''_{xx} + [f - 3\lambda\tan(\lambda x)]z'_x + [g - 3\lambda^2 - 2\lambda f\tan(\lambda x)]z = 0$.

99. $y'''_{xxx} + fy''_{xx} + \lambda[2f\tan(\lambda x) + 3\lambda]y'_x + \lambda^2\{[1 + 2\tan^2(\lambda x)]f + 2\lambda\tan(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \cos(\lambda x)$, $y_2 = x\cos(\lambda x)$.

100. $y'''_{xxx} + fy''_{xx} + \lambda[3\lambda - 2f\cot(\lambda x)]y'_x + \lambda^2\{[1 + 2\cot^2(\lambda x)]f - 2\lambda\cot(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \sin(\lambda x)$, $y_2 = x\sin(\lambda x)$.

101. $y'''_{xxx} - [(a + \tan x)f + a]y''_{xx} + [(a^2 + 1)f + 1]y'_x + a[(a\tan x - 1)f - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

102. $y'''_{xxx} + [(\cot x + a)f + a]y''_{xx} + [(a^2 + 1)f + 1]y'_x + a[(1 - a\cot x)f + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

103. $y'''_{xxx} + [f + \lambda\tan(\lambda x)(xf + 1)]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos(\lambda x)$.

104. $y'''_{xxx} + [f - \lambda \cot(\lambda x)(xf + 1)]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sin(\lambda x)$.

105. $a \sin(\lambda x)y'''_{xxx} + by''_{xx} + 3a\lambda^2 \sin(\lambda x)y'_x + 2a\lambda^3 \cos(\lambda x)y = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$a \sin(\lambda x)y'_x + [b - 2a\lambda \cos(\lambda x)]y = C_1 + C_2 x + \int_{x_0}^{x} (x - t)f(t)\,dt, \qquad x_0 \text{ is any number.}$$

106. $\sin(\lambda x)y'''_{xxx} + [a + (2\lambda + 1)\cos(\lambda x)]y''_{xx} - (\lambda^2 + 2\lambda)\sin(\lambda x)y'_x - \lambda^2 \cos(\lambda x)y = f(x).$

Integrating the equation twice, we arrive at a first-order linear equation:

$$\sin(\lambda x)y'_x + [a + \cos(\lambda x)]y = C_1 + C_2 x + \int_{x_0}^{x} (x - t)f(t)\,dt, \qquad x_0 \text{ is any number.}$$

107. $(f - 1)y'''_{xxx} - [af + \lambda \tan(\lambda x)]y''_{xx} + (\lambda^2 f + a^2)y'_x + a\lambda[a \tan(\lambda x) - \lambda f]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos(\lambda x)$.

108. $y'''_{xxx} + fy''_{xx} + gy'_x + (fg + g'_x)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + gy = C \exp\Bigl(-\int f\,dx\Bigr)$.

109. $y'''_{xxx} + 3fy''_{xx} + (f'_x + 2f^2 + 2g)y'_x + (2fg + g'_x)y = 0.$

Solution: $y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2$. Here, w_1, w_2 is a fundamental set of solutions of the second-order linear equation: $w''_{xx} + fw'_x + \frac{1}{2}gw = 0$.

110. $y'''_{xxx} + (f + g)y''_{xx} + (f'_x + fg + h)y'_x + (h'_x + gh)y = 0.$

Integrating yields a second-order linear equation: $y''_{xx} + fy'_x + hy = C \exp\Bigl(-\int g\,dx\Bigr)$.

111. $y'''_{xxx} + (f + g)y''_{xx} + (2g'_x + fg + h)y'_x + (g''_{xx} + fg'_x + gh)y = 0.$

The substitution $w = y'_x + gy$ leads to a second-order linear equation: $w''_{xx} + fw'_x + hw = 0$.

15.2 Equations of the Form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

15.2.1 Classification Table

Table 15.1 presents below all solvable equations whose solutions are outlined in Sections 15.2.2–15.2.4. Two-parameter families (in the space of the parameters α, β, γ, and δ), one-parameter families, and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of δ, the growth of γ (for identical δ), the growth of β (for identical δ and γ), and the growth of α (for identical δ, γ, and β). The number of the equation sought is indicated in the last column in this table.

TABLE 15.1
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
arbitrary	arbitrary	0	arbitrary	15.2.4.15*
arbitrary ($\delta \neq 2$)	arbitrary ($\gamma \neq -1$)	0	0	15.2.4.1
$\dfrac{\gamma+4\beta+5}{\gamma+2\beta+3}$	arbitrary ($\gamma \neq -1$)	arbitrary ($\beta \neq -1$)	0	15.2.4.192
$\dfrac{3\gamma+7}{2(\gamma+2)}$	arbitrary ($\gamma \neq -2$)	$-\frac{1}{2}$	0	15.2.4.10
$\dfrac{3\gamma+7}{2(\gamma+2)}$	arbitrary ($\gamma \neq -2$)	1	0	15.2.4.7
arbitrary ($\delta \neq 1,\ 2$)	-1	-1	0	15.2.4.193
arbitrary ($\delta \neq 2$)	-1	0	0	15.2.4.11
$\dfrac{3\beta+4}{2\beta+3}$	0	arbitrary ($\beta \neq -\frac{3}{2}$)	0	15.2.4.8
arbitrary ($\delta \neq \frac{3}{2}$)	0	$-\frac{1}{2}$	0	15.2.4.99
arbitrary ($\delta \neq 1$)	1	arbitrary ($\beta \neq -1$)	0	15.2.4.2
arbitrary ($\delta \neq 1$)	1	-1	0	15.2.4.13
arbitrary ($\delta \neq 2$)	1	1	0	15.2.4.4
$\dfrac{3\beta+4}{2\beta+3}$	3	arbitrary ($\beta \neq -\frac{3}{2}$)	0	15.2.4.9
-1	3	$-\frac{7}{5}$	0	15.2.4.186
-1	3	0	0	15.2.4.182
0	arbitrary ($\gamma \neq -1$)	0	0	15.2.4.3
0	arbitrary	$-\frac{1}{4}(\gamma+5)$	0	15.2.4.189
0	$-2\beta-5$	arbitrary ($\beta \neq -2$)	0	15.2.4.5
0	-13	1	0	15.2.4.171
0	-13	3	0	15.2.4.173
0	-7	0	0	15.2.4.159
0	-7	1	0	15.2.4.163

* Given are formulas of reduction to the generalized Emden–Fowler equation.

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

δ	γ	β	α	Equation
0	-4	$-\frac{1}{2}$	0	15.2.4.139
0	-4	0	0	15.2.4.135
0	-3	-2	0	15.2.4.107
0	-3	-1	0	15.2.4.30
0	-3	0	0	15.2.4.26
0	-3	1	0	15.2.4.103
0	$-\frac{7}{3}$	$-\frac{10}{3}$	0	15.2.4.82
0	$-\frac{7}{3}$	$-\frac{7}{3}$	0	15.2.4.42
0	$-\frac{7}{3}$	$-\frac{4}{3}$	0	15.2.4.52
0	$-\frac{7}{3}$	$-\frac{5}{6}$	0	15.2.4.145
0	$-\frac{7}{3}$	$-\frac{1}{2}$	0	15.2.4.143
0	$-\frac{7}{3}$	0	0	15.2.4.48
0	$-\frac{7}{3}$	1	0	15.2.4.38
0	$-\frac{7}{3}$	2	0	15.2.4.76
0	$-\frac{9}{5}$	$-\frac{13}{5}$	0	15.2.4.70
0	$-\frac{9}{5}$	1	0	15.2.4.66
0	-1	-2	0	15.2.4.22
0	-1	0	0	15.2.4.18
0	0	$-\frac{7}{2}$	0	15.2.2.2
0	0	$-\frac{7}{2}$	3	15.2.3.3
0	0	$-\frac{5}{2}$	0	15.2.2.3
0	0	$-\frac{5}{2}$	1	15.2.3.4
0	0	-2	0	15.2.2.6
0	0	$-\frac{4}{3}$	$-\frac{4}{3}$	15.2.3.5
0	0	$-\frac{4}{3}$	0	15.2.2.4
0	0	$-\frac{5}{4}$	$-\frac{3}{2}$	15.2.3.7
0	0	$-\frac{5}{4}$	0	15.2.2.8
0	0	$-\frac{7}{6}$	$-\frac{5}{3}$	15.2.3.6
0	0	$-\frac{7}{6}$	0	15.2.2.5
0	0	$-\frac{1}{2}$	-3	15.2.3.8
0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	15.2.3.9

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
0	0	$-\frac{1}{2}$	0	15.2.2.7
0	0	0	arbitrary	15.2.3.1
0	0	0	0	15.2.2.1
0	0	1	arbitrary	15.2.3.2
0	1	1	0	15.2.4.35
0	2	$-\frac{7}{2}$	0	15.2.4.183
0	2	0	0	15.2.4.179
0	3	arbitrary ($\beta \neq -2$)	0	15.2.4.97
0	3	-2	0	15.2.4.94
0	5	-5	0	15.2.4.117
0	5	$-\frac{20}{7}$	0	15.2.4.129
0	5	$-\frac{15}{7}$	0	15.2.4.123
0	5	0	0	15.2.4.113
$\frac{1}{2}$	$-\frac{17}{5}$	$-\frac{4}{5}$	0	15.2.4.151
$\frac{1}{2}$	$-\frac{13}{7}$	$-\frac{11}{7}$	0	15.2.4.58
$\frac{1}{2}$	$-\frac{8}{5}$	$-\frac{17}{10}$	0	15.2.4.88
$\frac{1}{2}$	0	$-\frac{5}{2}$	0	15.2.4.80
$\frac{1}{2}$	3	$-\frac{15}{8}$	0	15.2.4.126
$\frac{1}{2}$	3	$-\frac{20}{13}$	0	15.2.4.132
$\frac{1}{2}$	3	$-\frac{5}{4}$	0	15.2.4.120
$\frac{1}{2}$	3	0	0	15.2.4.116
$\frac{2}{3}$	0	$-\frac{7}{6}$	0	15.2.4.175
$\frac{3}{4}$	$-\frac{8}{3}$	$-\frac{2}{3}$	0	15.2.4.156
$\frac{4}{5}$	-4	$-\frac{1}{2}$	0	15.2.4.149
1	arbitrary ($\gamma \neq 1$)	-1	0	15.2.4.158
1	-3	$-\frac{1}{2}$	0	15.2.4.32
1	-3	1	0	15.2.4.24
1	-1	-1	0	15.2.4.195
1	1	arbitrary ($\beta \neq -1$)	0	15.2.4.14

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

δ	γ	β	α	Equation
1	1	-1	0	15.2.4.17
1	1	1	0	15.2.4.21
$\frac{8}{7}$	3	$-\frac{3}{4}$	0	15.2.4.177
$\frac{8}{7}$	3	$-\frac{1}{2}$	0	15.2.4.169
$\frac{6}{5}$	0	$-\frac{2}{3}$	0	15.2.4.121
$\frac{16}{13}$	$-\frac{27}{7}$	2	0	15.2.4.90
$\frac{5}{4}$	-4	$-\frac{1}{2}$	0	15.2.4.57
$\frac{5}{4}$	3	$-\frac{1}{2}$	0	15.2.4.166
$\frac{5}{4}$	3	0	0	15.2.4.162
$\frac{9}{7}$	$-\frac{9}{4}$	1	0	15.2.4.72
$\frac{9}{7}$	0	$-\frac{1}{3}$	0	15.2.4.187
$\frac{9}{7}$	0	1	0	15.2.4.68
$\frac{13}{10}$	$-\frac{9}{2}$	2	0	15.2.4.60
$\frac{13}{10}$	0	$-\frac{5}{2}$	0	15.2.4.86
$\frac{27}{20}$	0	$-\frac{2}{3}$	0	15.2.4.133
$\frac{15}{11}$	$-\frac{17}{3}$	$-\frac{2}{3}$	0	15.2.4.63
$\frac{18}{13}$	0	$-\frac{7}{2}$	0	15.2.4.74
$\frac{7}{5}$	-7	1	0	15.2.4.54
$\frac{7}{5}$	$-\frac{5}{2}$	1	0	15.2.4.45
$\frac{7}{5}$	$-\frac{13}{7}$	1	0	15.2.4.84
$\frac{7}{5}$	$-\frac{1}{3}$	1	0	15.2.4.78
$\frac{7}{5}$	0	1	0	15.2.4.40
$\frac{7}{5}$	1	1	0	15.2.4.50
$\frac{7}{5}$	3	0	0	15.2.4.138
$\frac{7}{5}$	3	1	0	15.2.4.142
$\frac{7}{5}$	11	1	0	15.2.4.147
$\frac{24}{17}$	$-\frac{23}{3}$	$-\frac{2}{3}$	0	15.2.4.93
$\frac{10}{7}$	0	$-\frac{5}{2}$	0	15.2.4.43
$\frac{16}{11}$	5	4	0	15.2.4.154
$\frac{22}{15}$	0	$-\frac{2}{3}$	0	15.2.4.127
$\frac{3}{2}$	arbitrary	$\frac{1}{2}(\gamma-1)$	0	15.2.4.191

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
$\frac{3}{2}$	-3	$-\frac{1}{2}$	0	15.2.4.112
$\frac{3}{2}$	-3	1	0	15.2.4.109
$\frac{3}{2}$	0	-2	0	15.2.4.111
$\frac{3}{2}$	0	$-\frac{1}{2}$	0	15.2.4.96
$\frac{3}{2}$	0	1	0	15.2.4.105
$\frac{3}{2}$	1	1	0	15.2.4.28
$\frac{3}{2}$	3	-2	0	15.2.4.110
$\frac{3}{2}$	3	$-\frac{1}{2}$	0	15.2.4.106
$\frac{3}{2}$	3	0	0	15.2.4.29
$\frac{23}{15}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	15.2.4.128
$\frac{17}{11}$	9	2	0	15.2.4.153
$\frac{11}{7}$	-4	$-\frac{1}{2}$	0	15.2.4.47
$\frac{27}{17}$	$-\frac{1}{3}$	$-\frac{13}{3}$	0	15.2.4.92
$\frac{8}{5}$	1	1	0	15.2.4.137
$\frac{8}{5}$	3	-4	0	15.2.4.55
$\frac{8}{5}$	3	$-\frac{7}{4}$	0	15.2.4.46
$\frac{8}{5}$	3	$-\frac{10}{7}$	0	15.2.4.85
$\frac{8}{5}$	3	$-\frac{2}{3}$	0	15.2.4.79
$\frac{8}{5}$	3	$-\frac{1}{2}$	0	15.2.4.41
$\frac{8}{5}$	3	0	0	15.2.4.51
$\frac{8}{5}$	3	1	0	15.2.4.141
$\frac{8}{5}$	3	5	0	15.2.4.148
$\frac{21}{13}$	-6	$-\frac{1}{2}$	0	15.2.4.75
$\frac{18}{11}$	$-\frac{1}{3}$	$-\frac{10}{3}$	0	15.2.4.62
$\frac{33}{20}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	15.2.4.134
$\frac{17}{10}$	-4	$-\frac{1}{2}$	0	15.2.4.87
$\frac{17}{10}$	5	$-\frac{11}{4}$	0	15.2.4.61
$\frac{12}{7}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	15.2.4.188
$\frac{12}{7}$	3	$-\frac{13}{8}$	0	15.2.4.73
$\frac{12}{7}$	3	$-\frac{1}{2}$	0	15.2.4.69
$\frac{7}{4}$	0	$-\frac{5}{2}$	0	15.2.4.56

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

δ	γ	β	α	Equation
$\frac{7}{4}$	0	1	0	15.2.4.165
$\frac{7}{4}$	1	1	0	15.2.4.161
$\frac{23}{13}$	5	$-\frac{17}{7}$	0	15.2.4.91
$\frac{9}{5}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	15.2.4.122
$\frac{13}{7}$	$-\frac{1}{2}$	1	0	15.2.4.178
$\frac{13}{7}$	0	1	0	15.2.4.170
2	arbitrary ($\gamma \neq -1$)	0	0	15.2.4.12
2	-1	arbitrary ($\beta \neq 0$)	0	15.2.4.157
2	-1	-1	0	15.2.4.194
2	-1	0	0	15.2.4.16
2	0	-2	0	15.2.4.33
2	3	-2	0	15.2.4.25
2	3	0	0	15.2.4.19
$\frac{11}{5}$	0	$-\frac{5}{2}$	0	15.2.4.150
$\frac{9}{4}$	$-\frac{1}{3}$	$-\frac{11}{6}$	0	15.2.4.155
$\frac{7}{3}$	$-\frac{4}{3}$	$-\frac{1}{2}$	0	15.2.4.176
$\frac{5}{2}$	-4	$-\frac{1}{2}$	0	15.2.4.81
$\frac{5}{2}$	$-\frac{11}{4}$	1	0	15.2.4.125
$\frac{5}{2}$	$-\frac{12}{5}$	$-\frac{13}{10}$	0	15.2.4.89
$\frac{5}{2}$	$-\frac{15}{7}$	$-\frac{10}{7}$	0	15.2.4.59
$\frac{5}{2}$	$-\frac{27}{13}$	1	0	15.2.4.131
$\frac{5}{2}$	$-\frac{3}{2}$	1	0	15.2.4.119
$\frac{5}{2}$	$-\frac{3}{5}$	$-\frac{11}{5}$	0	15.2.4.152
$\frac{5}{2}$	1	1	0	15.2.4.115
3	arbitrary ($\gamma \neq -3$)	1	0	15.2.4.98
3	$-2\beta - 5$	arbitrary ($\beta \neq -2$)	0	15.2.4.6
3	arbitrary	$-\gamma - 2$	0	15.2.4.190
3	-9	2	0	15.2.4.118
3	-6	$-\frac{1}{2}$	0	15.2.4.65

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
3	-6	$\frac{1}{2}$	0	15.2.4.184
3	$-\frac{17}{3}$	$-\frac{5}{3}$	0	15.2.4.83
3	$-\frac{33}{7}$	2	0	15.2.4.130
3	$-\frac{21}{5}$	$-\frac{7}{5}$	0	15.2.4.71
3	-4	$-\frac{1}{2}$	0	15.2.4.37
3	$-\frac{11}{3}$	$-\frac{5}{3}$	0	15.2.4.44
3	$-\frac{23}{7}$	2	0	15.2.4.124
3	-3	-2	0	15.2.4.108
3	-3	-1	0	15.2.4.23
3	-3	$-\frac{1}{2}$	0	15.2.4.102
3	-3	1	0	15.2.4.95
3	$-\frac{5}{3}$	$-\frac{5}{3}$	0	15.2.4.53
3	$-\frac{5}{3}$	$-\frac{1}{2}$	0	15.2.4.167
3	$-\frac{4}{3}$	$-\frac{1}{2}$	0	15.2.4.168
3	-1	-2	0	15.2.4.31
3	$-\frac{2}{3}$	$-\frac{5}{3}$	0	15.2.4.146
3	0	$-\frac{5}{2}$	0	15.2.4.140
3	0	$-\frac{5}{3}$	0	15.2.4.144
3	0	$-\frac{1}{2}$	0	15.2.4.101
3	0	1	-3	15.2.4.100
3	1	-4	0	15.2.4.160
3	1	$-\frac{5}{2}$	0	15.2.4.136
3	1	-2	0	15.2.4.27
3	1	$-\frac{5}{3}$	0	15.2.4.49
3	1	-1	0	15.2.4.20
3	1	$-\frac{1}{2}$	0	15.2.4.34
3	1	$\frac{1}{2}$	0	15.2.4.180
3	1	2	0	15.2.4.114
3	3	-7	0	15.2.4.172
3	3	-4	0	15.2.4.164
3	3	-2	0	15.2.4.104

TABLE 15.1 (*Continued*)
Solvable equations of the form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

δ	γ	β	α	Equation
3	3	$-\frac{5}{3}$	0	15.2.4.39
3	3	$-\frac{7}{5}$	0	15.2.4.67
3	3	$-\frac{1}{2}$	0	15.2.4.64
3	3	0	0	15.2.4.36
3	5	$-\frac{5}{3}$	0	15.2.4.77
3	7	-7	0	15.2.4.174
4	$-\frac{9}{5}$	1	0	15.2.4.185
4	1	1	0	15.2.4.181

In Sections 15.2.2–15.2.4, the value of the insignificant parameter A is in many cases defined in the form of a function of two (one) auxiliary coefficients a and b,

$$A = \varphi(a, b) \tag{1}$$

and the corresponding solutions are represented in parametric form,

$$x = f_1(\tau, C_1, C_2, C_3, a), \qquad y = f_2(\tau, C_1, C_2, C_3, b), \tag{2}$$

where τ is the parameter, C_1, C_2, and C_3 are arbitrary constants, and f_1 and f_2 are some functions.

Having fixed the auxiliary coefficient sign $a > 0$ (or $b > 0$), one should express the coefficient b in terms of both A and a with the help of

$$b = \psi(A, a).$$

Substituting this formula into (2) yields a solution of the equation under consideration (where the specific numerical value of the coefficient a can be chosen arbitrarily). The case $a < 0$ (or $b < 0$), which may lead to a branch of the solution or to a different domain of definition of the variables x and y in (2), should be considered in a similar manner.

15.2.2 Equations of the Form $y'''_{xxx} = Ay^\beta$

1. $y'''_{xxx} = A.$

Solution: $y = \frac{1}{6}Ax^3 + C_2x^2 + C_1x + C_0.$

2. $y'''_{xxx} = Ay^{-7/2}.$

Solution in parametric form:

$$x = aC_1^3 \int \big[C_1 e^{2\sigma\tau} + C_2 e^{-\sigma\tau} \sin\big(\sqrt{3}\,\sigma\tau\big)\big]^{-3/2} d\tau + C_3,$$
$$y = bC_1^2 \big[C_1 e^{2\sigma\tau} + C_2 e^{-\sigma\tau} \sin\big(\sqrt{3}\,\sigma\tau\big)\big]^{-1},$$

where $A = -8a^{-3}b^{9/2}\sigma^3.$

3. $y'''_{xxx} = Ay^{-5/2}$**.**

Solution in parametric form:

$$x = aC_1^7 \int (\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3, \qquad y = bC_1^6(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = -6a^{-3}b^{7/2}$.

4. $y'''_{xxx} = Ay^{-4/3}$**.**

Solution in parametric form:

$$x = aC_1^7 \int R^{-1}(2\tau I \mp R)^2\,d\tau + C_3, \qquad y = bC_1^9(2\tau I \mp R)^3,$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $I = \int \tau R^{-1}\,d\tau + C_2$, $A = \pm 18a^{-3}b^{7/3}$.

5. $y'''_{xxx} = Ay^{-7/6}$**.**

Solution in parametric form:

$$x = aC_1^{13} \int R^{-1}(2\tau I \mp R)^{-5/2}\,d\tau + C_3, \qquad y = bC_1^{18}(2\tau I \mp R)^{-3},$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $I = \int \tau R^{-1}\,d\tau + C_2$, $A = \mp 18a^{-3}b^{13/6}$.

◆ *In the solutions of equations 6 and 7, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

where $J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

6. $y'''_{xxx} = Ay^{-2}$**.**

Solution in parametric form:

$$x = aC_1 \int \tau^{-1} Z^{-2}\,d\tau + C_3, \quad y = bC_1\tau^{-2/3}Z^{-2}, \qquad \text{where} \quad A = \pm\tfrac{4}{3}a^{-3}b^3.$$

7. $y'''_{xxx} = Ay^{-1/2}$**.**

Solution in parametric form:

$$x = aC_1 \int Z\,d\tau + C_3, \quad y = bC_1^2\tau^{2/3}Z^2, \qquad \text{where} \quad A = \mp\tfrac{4}{3}a^{-3}b^{3/2}.$$

8. $y'''_{xxx} = Ay^{-5/4}$**.**

This is a special case of equation 15.2.4.189 with $\gamma = 0$.

15.2.3 Equations of the Form $y'''_{xxx} = Ax^\alpha y^\beta$

For $\alpha = 0$, see Section 3.2.2.

1. $y'''_{xxx} = Ax^\alpha$.

Solution: $y = Af(x) + C_2x^2 + C_1x + C_0$, where

$$f(x) = \begin{cases} \dfrac{x^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} & \text{if } \alpha \neq -1,\ -2,\ -3; \\ \frac{1}{2}x^2 \ln|x| - \frac{3}{4}x^2 & \text{if } \alpha = -1; \\ -x\ln|x| + x & \text{if } \alpha = -2; \\ \frac{1}{2}\ln|x| & \text{if } \alpha = -3. \end{cases}$$

2. $y'''_{xxx} = Ax^\alpha y$.

See equation 15.1.2.7.

3. $y'''_{xxx} = Ax^3 y^{-7/2}$.

Solution in parametric form:

$$x = aC_1^3\Bigl(\int f^{-3/2}\,d\tau + C_3\Bigr)^{-1}, \qquad y = bC_1^4 f^{-1}\Bigl(\int f^{-3/2}\,d\tau + C_3\Bigr)^{-2},$$

where $f = C_1e^{2\sigma\tau} + C_2e^{-\sigma\tau}\sin\bigl(\sqrt{3}\,\sigma\tau\bigr)$, $A = 8a^{-6}b^{9/2}\sigma^3$.

4. $y'''_{xxx} = Axy^{-5/2}$.

Solution in parametric form:

$$x = aC_1^7\Bigl[\int(\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3\Bigr]^{-1},$$
$$y = bC_1^8(\tau^3 - 3\tau + C_2)^{-1}\Bigl[\int(\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3\Bigr]^{-2},$$

where $A = 6a^{-4}b^{7/2}$.

◆ *In the solutions of equations 5 and 6, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \qquad I = \int \tau R^{-1}\,d\tau + C_2.$$

5. $y'''_{xxx} = Ax^{-4/3}y^{-4/3}$.

Solution in parametric form:

$$x = aC_1^7\Bigl[\int R^{-1}(2\tau I \mp R)^2\,d\tau + C_3\Bigr]^{-1},$$
$$y = bC_1^5(2\tau I \mp R)^3\Bigl[\int R^{-1}(2\tau I \mp R)^2\,d\tau + C_3\Bigr]^{-2},$$

where $A = \mp 18a^{-5/3}b^{7/3}$.

6. $y'''_{xxx} = Ax^{-5/3}y^{-7/6}$.

Solution in parametric form:

$$x = aC_1^{13}\Big[\int R^{-1}(2\tau I \mp R)^{-5/2}\,d\tau + C_3\Big]^{-1},$$
$$y = bC_1^{8}(2\tau I \mp R)^{-3}\Big[\int R^{-1}(2\tau I \mp R)^{-5/2}\,d\tau + C_3\Big]^{-2},$$

where $A = \mp 18a^{-4/3}b^{13/6}$.

7. $y'''_{xxx} = Ax^{-3/2}y^{-5/4}$.

Solution in parametric form:

$$x = aC_1^3\Big(\int \tau^{-1/2}z^{-1/2}f^{3/4}\,d\tau + C_3\Big)^{-1}, \quad y = bC_1^2 f\Big(\int \tau^{-1/2}z^{-1/2}f^{3/4}\,d\tau + C_3\Big)^{-2},$$

where $z = C_2 + \frac{1}{4}\tau^2 + 4B\tau^{1/2}$, $f = \exp\Big(\int z^{-1/2}\,d\tau\Big)$, $A = \frac{1}{2}Ba^{-3/2}b^{9/4}$.

8. $y'''_{xxx} = Ax^{-3}y^{-1/2}$.

Solution in parametric form:

$$x = C_1\Big(\int Z\,d\tau + C_3\Big)^{-1}, \quad y = b\tau^{2/3}Z^2\Big(\int Z\,d\tau + C_3\Big)^{-2},$$

where

$$A = \pm\tfrac{4}{3}b^{3/2}, \qquad Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}(\tau)$ and $Y_{1/3}(\tau)$ are Bessel functions, and $I_{1/3}(\tau)$ and $K_{1/3}(\tau)$ are modified Bessel functions.

9. $y'''_{xxx} = Ax^{-3/2}y^{-1/2}$.

Solution in parametric form:

$$x = aC_3\exp\Big(2\int P\,d\tau\Big), \quad x = bC_3P^2\exp\Big(2\int P\,d\tau\Big).$$

Here, $P = P(\tau, C_1, C_2)$ is the general solution of the second Painlevé transcendent: $P''_{\tau\tau} = \pm\tau P + 2P^3$, and $A = \pm\frac{1}{4}a^{-3/2}b^{3/2}$.

15.2.4 Equations with $|\gamma| + |\delta| \neq 0$

1. $y'''_{xxx} = A(y'_x)^\gamma(y''_{xx})^\delta$, $\quad \gamma \neq -1,\ \delta \neq 2$.

Solution in parametric form:

$$x = aC_1^{\gamma+\delta-1}\int \tau^{-1/2}\Big(1 \pm \tau^{\frac{\gamma+1}{2}}\Big)^{\frac{1}{\delta-2}}\,d\tau + C_3, \quad y = bC_1^{\gamma+2\delta-3}\int\Big(1 \pm \tau^{\frac{\gamma+1}{2}}\Big)^{\frac{1}{\delta-2}}\,d\tau + C_2,$$

where $A = \pm\dfrac{\gamma+1}{2-\delta}2^{\delta-2}a^{\gamma+2\delta-3}b^{1-\gamma-\delta}$.

2. $y'''_{xxx} = Ay^\beta y'_x (y''_{xx})^\delta, \qquad \beta \neq -1, \;\; \delta \neq 1.$

Solution in parametric form:

$$x = aC_1^{\beta+\delta} \int \Bigl[\int \bigl(1 \pm \tau^{\beta+1}\bigr)^{\frac{1}{1-\delta}} \, d\tau + C_2 \Bigr]^{-1/2} d\tau + C_3, \qquad y = bC_1^{2\delta-2}\tau,$$

where $A = \pm \dfrac{\beta+1}{1-\delta} 2^{\delta-1} a^{2\delta-2} b^{-\beta-\delta}$.

◆ *In the solutions of equations 3–10, the following notation is used:*

$$R = \sqrt{1 \pm \tau^{m+1}}, \qquad E = \int \bigl(1 \pm \tau^{m+1}\bigr)^{-1/2} d\tau + C_2, \qquad F = RE - \tau.$$

3. $y'''_{xxx} = A(y'_x)^\gamma, \qquad \gamma \neq -1.$

Solution in parametric form:

$$x = aC_1^m \int \tau^{-1/2} R^{-1} \, d\tau + C_3, \qquad y = bC_1^{m-1} E,$$

where $m = \dfrac{\gamma - 1}{2}$, $A = \pm \dfrac{m+1}{4} a^{2m-2} b^{-2m}$.

4. $y'''_{xxx} = Ayy'_x (y''_{xx})^\delta, \qquad \delta \neq 2.$

Solution in parametric form:

$$x = aC_1^{3m+1} \int E^{-1/2} R^{-1} \, d\tau + C_3, \qquad y = bC_1^{2m+2} R,$$

where $m = \dfrac{1}{\delta - 2}$, $A = -8ma^2 b^{-3} \Bigl[\pm \dfrac{2a^2}{(m+1)b} \Bigr]^{1/m}$.

5. $y'''_{xxx} = Ay^\beta (y'_x)^{-2\beta-5}, \qquad \beta \neq -2.$

Solution in parametric form:

$$x = aC_1^{m-3} \int \tau^{-1/2} E^{-3/2} R^{-1} \, d\tau + C_3, \qquad y = bC_1^{2m-2} E^{-1},$$

where $m = -\beta - 3$, $A = \pm \frac{1}{4} (-1)^{-2m} (m+1) a^{2m-2} b^{3-m}$.

6. $y'''_{xxx} = Ay^\beta (y'_x)^{-2\beta-5} (y''_{xx})^3, \qquad \beta \neq -2.$

Solution in parametric form:

$$x = aC_1^{m+3} \int E^{-3/2} R^{-1} F \, d\tau + C_3, \qquad y = bC_1^{2m+2} \tau E^{-1},$$

where $m = \beta$, $A = \mp 2(m+1) a^{-2m-2} b^{m+3}$.

7. $y'''_{xxx} = Ay(y'_x)^\gamma (y''_{xx})^{\frac{3\gamma+7}{2\gamma+4}}, \qquad \gamma \neq -2.$

Solution in parametric form:

$$x = aC_1^{m^2+m+2} \int E^{-m/2} R^{-1} \, d\tau + C_3, \qquad y = bC_1^4 F,$$

where $m = -\dfrac{2(\gamma+2)}{\gamma+1}$, $A = -\dfrac{2mb^{-1}}{m+2} \Bigl[\pm \dfrac{(m+1)b}{2a} \Bigr]^{\frac{2}{m+2}} \Bigl[\pm \dfrac{4a^2}{(m+1)(m+2)b} \Bigr]^{\frac{1}{m}}$.

8. $y'''_{xxx} = Ay^{\beta}(y''_{xx})^{\frac{3\beta+4}{2\beta+3}}, \qquad \beta \neq -3/2.$

Solution in parametric form:

$$x = aC_1^{m^2+2m-1} \int \tau^m E^{-m-2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{(m+1)^2} \tau^{m+1} E^{-m-1},$$

where $m = -\dfrac{\beta}{\beta+1}$, $A = (m+3)a^{-1} b^{\frac{m}{m+1}} \Big[\pm \dfrac{2a^2}{(m+1)^2 b}\Big]^{\frac{1}{m+3}}$.

9. $y'''_{xxx} = Ay^{\beta}(y'_x)^3 (y''_{xx})^{\frac{3\beta+4}{2\beta+3}}, \qquad \beta \neq -3/2.$

Solution in parametric form:

$$x = aC_1^{m^2+m-1} \int E^{m+1} F^{-1/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{(m-1)(m+2)} E^{m+2},$$

where $m = -\dfrac{2\beta+3}{\beta+1}$, $A = \dfrac{m}{(m+2)^3} a^2 b^{-\frac{2m+1}{m+1}} \Big[\pm \dfrac{(m+1)(m+2)b}{4a^2}\Big]^{\frac{1}{m}}$.

10. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{\gamma} (y''_{xx})^{\frac{3\gamma+7}{2\gamma+4}}, \qquad \gamma \neq -2.$

Solution in parametric form:

$$x = aC_1^{m^2+2m-7} \int \tau^{\frac{m-1}{2}} R^{-1} E^{\frac{m+3}{2}} F\, d\tau + C_3, \qquad y = bC_1^{-8} F^2,$$

where $m = \dfrac{1-\gamma}{1+\gamma}$, $A = -\dfrac{2(m+3)}{m+1} b^{1/2} \Big[\pm \dfrac{a}{(m+1)b}\Big]^{\frac{2}{m+1}} \Big[\pm \dfrac{(m+1)^2 b}{2a^2}\Big]^{\frac{1}{m+3}}$.

11. $y'''_{xxx} = A(y'_x)^{-1} (y''_{xx})^{\delta}, \qquad \delta \neq 2.$

Solution in parametric form:

$$x = aC_1 \int \tau^{\frac{\delta}{\delta-2}} \exp(\mp\tau^2)\, d\tau + C_2, \qquad y = bC_1^2 \int \tau^{\frac{\delta}{\delta-2}} \exp(\mp 2\tau^2)\, d\tau + C_3,$$

where $A = \mp \dfrac{4b^2}{(2-\delta)a^4} \Big(\mp \dfrac{a^2}{2b}\Big)^{\delta}$.

12. $y'''_{xxx} = A(y'_x)^{\gamma} (y''_{xx})^2, \qquad \gamma \neq -1.$

Solution in parametric form:

$$x = aC_1 \int \tau^{\frac{1-\gamma}{1+\gamma}} \exp(\mp\tau^2)\, d\tau + C_2, \qquad y = bC_1 \int \tau^{\frac{3-\gamma}{1+\gamma}} \exp(\mp\tau^2)\, d\tau + C_3,$$

where $A = \pm(\gamma+1)a^{\gamma+1} b^{-\gamma-1}$.

13. $y'''_{xxx} = Ay^{-1} y'_x (y''_{xx})^{\delta}, \qquad \delta \neq 1.$

Solution in parametric form:

$$x = aC_1 \int \tau \exp(\mp\tau^2) \Big[\int \tau^{\frac{3-\delta}{1-\delta}} \exp(\mp\tau^2)\, d\tau + C_2\Big]^{-1/2} d\tau + C_3, \qquad y = bC_1^2 \exp(\mp\tau^2),$$

where $A = \dfrac{1}{1-\delta} (\mp 1)^{-\delta} a^{2\delta-2} b^{1-\delta}$.

14. $y'''_{xxx} = Ay^\beta y'_x y''_{xx}, \qquad \beta \neq -1.$

Solution in parametric form:

$$x = C_1 \int \tau^{\frac{1-\beta}{1+\beta}} \left[\int \tau^{\frac{1-\beta}{1+\beta}} \exp(\mp\tau^2)\, d\tau + C_2 \right]^{-1/2} d\tau + C_3, \qquad y = b\tau^{\frac{2}{1+\beta}},$$

where $A = \mp(\beta+1)b^{-1-\beta}$.

15. $y'''_{xxx} = Ax^\alpha (y'_x)^\gamma (y''_{xx})^\delta.$

Solution in parametric form:

$$x = aC_1^{\gamma+\delta-1} X(\tau), \qquad y = bC_1^{\gamma+2\delta-\alpha-3} \int Y(\tau) \frac{dX(\tau)}{d\tau}\, d\tau + C_3.$$

Here, $X = X(\tau)$, $Y = Y(\tau)$ is the general solution of the generalized Emden–Fowler equation:

$$Y''_{XX} = BX^\alpha Y^\gamma (Y'_X)^\delta, \qquad \text{where} \quad A = Ba^{\gamma+2\delta-\alpha-3} b^{1-\gamma-\delta}.$$

16. $y'''_{xxx} = A(y'_x)^{-1}(y''_{xx})^2.$

Solution: $y = \begin{cases} \dfrac{1-A}{(2-A)C_1} (C_1x + C_2)^{\frac{2-A}{1-A}} + C_3 & \text{if } A \neq 1,\ A \neq 2; \\ \dfrac{C_2}{C_1} \exp(C_1 x) + C_3 & \text{if } A = 1; \\ \dfrac{1}{C_1} \ln(C_1 x + C_2) + C_3 & \text{if } A = 2. \end{cases}$

17. $y'''_{xxx} = Ay^{-1} y'_x y''_{xx}.$

Solution: $x = \begin{cases} \displaystyle\int (C_1 y^{A+1} + C_2)^{-1/2}\, dy + C_3 & \text{if } A \neq -1; \\ \displaystyle\int (C_1 \ln y + C_2)^{-1/2}\, dy + C_3 & \text{if } A = -1. \end{cases}$

18. $y'''_{xxx} = A(y'_x)^{-1}.$

Solution in parametric form:

$$x = aC_1 \int \exp(\mp\tfrac{1}{2}\tau^2)\, d\tau + C_2, \quad y = bC_1^2 \int \exp(\mp\tau^2)\, d\tau + C_3, \quad \text{where} \quad A = \mp a^{-4} b^2.$$

19. $y'''_{xxx} = A(y'_x)^3 (y''_{xx})^2.$

Solution in parametric form:

$$x = aC_1 \int \tau^{-1/2} \exp(\mp\tau^2)\, d\tau + C_2, \quad y = bC_1 \int \exp(\mp\tau^2)\, d\tau + C_3,$$

where $A = \pm 4a^4 b^{-4}$.

◆ *In the solutions of equations 20–25, the following notation is used:*

$$E = \int \exp(\mp\tau^2)\,d\tau + C_2, \qquad F = 2\tau E \pm \exp(\mp\tau^2).$$

20. $y'''_{xxx} = Ay^{-1}y'_x(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1 \int \tau\exp(\mp\tau^2)E^{-1/2}\,d\tau + C_3, \quad y = bC_1^2\exp(\mp\tau^2), \qquad \text{where} \quad A = \pm\tfrac{1}{2}a^4b^{-2}.$$

21. $y'''_{xxx} = Ayy'_xy''_{xx}$.

Solution in parametric form:

$$x = C_1 \int E^{-1/2}\,d\tau + C_3, \quad y = b\tau, \qquad \text{where} \quad A = \mp 2b^{-2}.$$

22. $y'''_{xxx} = Ay^{-2}(y'_x)^{-1}$.

Solution in parametric form:

$$x = aC_1 \int E^{-3/2}\exp\bigl(\mp\tfrac{1}{2}\tau^2\bigr)\,d\tau + C_3, \quad y = bC_1E^{-1}, \qquad \text{where} \quad A = \mp a^{-4}b^4.$$

23. $y'''_{xxx} = Ay^{-1}(y'_x)^{-3}(y''_{xx})^3$.

Solution in parametric form:

$$x = C_1 \int E^{-3/2}F\exp(\mp\tau^2)\,d\tau + C_3, \quad y = bE^{-1}\exp(\mp\tau^2), \qquad \text{where} \quad A = \mp 8b^2.$$

24. $y'''_{xxx} = Ay(y'_x)^{-3}y''_{xx}$.

Solution in parametric form:

$$x = aC_1 \int E^{1/2}\,d\tau + C_3, \quad y = bC_1^2F, \qquad \text{where} \quad A = \mp 8a^{-4}b^2.$$

25. $y'''_{xxx} = Ay^{-2}(y'_x)^3(y''_{xx})^2$.

Solution in parametric form:

$$x = aC_1 \int F^{-1/2}\exp(\mp\tau^2)\,d\tau + C_3, \quad y = bC_1^2E, \qquad \text{where} \quad A = \pm a^4b^{-2}.$$

◆ *In the solutions of equations 26–33, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr) + C_2, \qquad R = \sqrt{\frac{\tau+1}{\tau}}, \qquad F = RE - \tau.$$

26. $y'''_{xxx} = A(y'_x)^{-3}$.

Solution in parametric form:

$$x = 2aC_1^2\sqrt{\tau+1} + C_3, \quad y = bC_1^3E, \qquad \text{where} \quad A = -\tfrac{1}{4}a^{-6}b^4.$$

27. $y'''_{xxx} = Ay^{-2}y'_x(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1 \int E^{-1/2}\,d\tau + C_3, \quad y = bC_1^4\tau, \qquad \text{where} \quad A = 2a^4b^{-1}.$$

28. $y'''_{xxx} = Ayy'_x(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = aC_1^5 \int \tau^{-2}R^{-1}E^{-1/2}\,d\tau + C_3, \quad y = bC_1^2R, \qquad \text{where} \quad A = -8a(-b)^{-5/2}.$$

29. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = aC_1^7 \int R^{-3/2}\,d\tau + C_3, \quad y = bC_1^6E, \qquad \text{where} \quad A = 4a^3(-b)^{-7/2}.$$

30. $y'''_{xxx} = Ay^{-1}(y'_x)^{-3}$.

Solution in parametric form:

$$x = aC_1^5 \int \tau^{-1/2}R^{-1}E^{-3/2}\,d\tau + C_3, \quad y = bC_1^6E^{-1}, \qquad \text{where} \quad A = -\tfrac{1}{4}a^{-6}b^5.$$

31. $y'''_{xxx} = Ay^{-2}(y'_x)^{-1}(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^{-1} \int R^{-1}E^{-3/2}F\,d\tau + C_3, \quad y = bC_1^2\tau E^{-1}, \qquad \text{where} \quad A = 2a^2b.$$

32. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-3}y''_{xx}$.

Solution in parametric form:

$$x = aC_1^7 \int \tau^{-3/2}R^{-1}E^{1/2}F\,d\tau + C_3, \quad y = bC_1^8F^2, \qquad \text{where} \quad A = a^{-4}b^{7/2}.$$

33. $y'''_{xxx} = Ay^{-2}(y''_{xx})^2$.

Solution in parametric form:

$$x = aC_1^{-1} \int \tau^{-2}R^{-1}\,d\tau + C_3, \quad y = bC_1\tau^{-1}E, \qquad \text{where} \quad A = 2ab.$$

34. $y'''_{xxx} = Ay^{-1/2}y'_x(y''_{xx})^3$.

Solution in parametric form:

$$x = \pm aC_1^5 \int \tau(\tau^2-1)(\tau^3-3\tau+C_2)^{-1/2}\,d\tau + C_3, \qquad y = bC_1^8(\tau^2-1)^2,$$

where $A = \mp\frac{1}{144}a^4b^{-5/2}$.

35. $y'''_{xxx} = Ayy'_x$.

Solution in parametric form:

$$x = aC_1^{-1}\int(\tau^3 - 3\tau + C_2)^{-1/2}\,d\tau + C_3, \quad y = bC_1^2\tau, \qquad \text{where} \quad A = 3a^{-2}b^{-1}.$$

36. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^3$.

Solution in parametric form:

$$x = \pm\tfrac{2}{5}aC_1^5\tau^{1/2}(\tau^2 - 5) + C_3, \quad y = bC_1^6(\tau^3 - 3\tau + C_2), \qquad \text{where} \quad A = -\tfrac{8}{243}a^6b^{-5}.$$

37. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^5\int(\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\,d\tau + C_3,$$

$$y = bC_1^2(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = \mp\frac{16}{9}a^{-1}b^{5/2}$.

38. $y'''_{xxx} = Ay(y'_x)^{-7/3}$.

Solution in parametric form:

$$x = aC_1^7\int(\tau^3 - 3\tau + C_2)^{1/4}\,d\tau + C_3, \quad y = \pm bC_1^{16}(\tau^4 - 6\tau^2 + 4C_2\tau - 3),$$

where $A = \pm 72a^{-5}b^2(4b/a)^{1/3}$.

39. $y'''_{xxx} = Ay^{-5/3}(y'_x)^3(y''_{xx})^3$.

Solution in parametric form:

$$x = \pm aC_1^5\int(\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{1/2}[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{-1/2}\,d\tau + C_3,$$

$$y = bC_1^9(\tau^3 - 3\tau + C_2)^{3/2},$$

where $A = \mp 8\times 9^{-5}a^6b^{-10/3}$.

40. $y'''_{xxx} = Ay(y''_{xx})^{7/5}$.

Solution in parametric form:

$$x = aC_1^{-7}\int(\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3, \quad y = \pm bC_1(\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-1/2},$$

where $A = \pm\dfrac{15}{2}a^{-1}b^{-1}\left(\dfrac{a^2}{2b}\right)^{2/5}$.

41. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{8/5}$.

Solution in parametric form:

$$x = \pm aC_1^{31}\int[\pm(\tau^2 - 1)]^{-1/2}(\tau^3 - 3\tau + C_2)^{5/4}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\,d\tau + C_3,$$

$$y = bC_1^{32}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^2,$$

where $A = -15\times 2^{-10}a^2b^{-5/2}\left(\dfrac{a^2}{2b}\right)^{3/5}$.

42. $y'''_{xxx} = Ay^{-7/3}(y'_x)^{-7/3}$.

Solution in parametric form:

$$x = aC_1^{17}\int (\tau^3 - 3\tau + C_2)^{1/4}[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{-3/2}\,d\tau + C_3,$$
$$y = \pm bC_1^{16}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1},$$

where $A = \pm 72a^{-5}b^{17/3}(a/4)^{-1/3}$.

43. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{10/7}$.

Solution in parametric form:

$$x = \pm aC_1^{29}\int (\tau^3 - 3\tau + C_2)^{-3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1/3}\,d\tau + C_3,$$
$$y = bC_1^2(\tau^3 - 3\tau + C_2)^{-1}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{2/3},$$

where $A = -\dfrac{28}{3}a^{-1}b^{5/2}\Big(\dfrac{2a^2}{3b}\Big)^{3/7}$.

◆ *In the solutions of equations 44–47, the following notation is used:*

$$P_6(\tau) = \pm(\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2).$$

44. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-11/3}(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^5\int (\tau^3 - 3\tau + C_2)^{1/2}[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{-3/2}P_6(\tau)\,d\tau + C_3,$$
$$y = \pm bC_1(\tau^3 - 3\tau + C_2)^{3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1},$$

where $A = \mp\frac{9}{16}b^{10/3}(2a)^{-2/3}$.

45. $y'''_{xxx} = Ay(y'_x)^{-5/2}(y''_{xx})^{7/5}$.

Solution in parametric form:

$$x = aC_1^{11}\int (\tau^3 - 3\tau + C_2)^{-3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{4/3}\,d\tau + C_3,$$
$$y = bC_1^{27}(\tau^3 - 3\tau + C_2)^{-1/2}P_6(\tau),$$

where $A = \dfrac{405}{8}a^{-3}b\Big(\pm\dfrac{b}{2a}\Big)^{1/2}\Big(\dfrac{a^2}{12b}\Big)^{2/5}$.

46. $y'''_{xxx} = Ay^{-7/4}(y'_x)^3(y''_{xx})^{8/5}$.

Solution in parametric form:

$$x = \pm aC_1^{37}\int (\tau^3 - 3\tau + C_2)^{5/4}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{1/3}[P_6(\tau)]^{-1/2}\,d\tau + C_3,$$
$$y = bC_1^{64}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{4/3},$$

where $A = -45\times 2^{-13}a^2b^{-5/4}\Big(\dfrac{a^2}{12b}\Big)^{3/5}$.

47. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{11/7}$.

Solution in parametric form:

$$x = \pm aC_1^{55} \int (\tau^3 - 3\tau + C_2)^{-3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{5/3} P_6(\tau)\, d\tau + C_3,$$
$$y = bC_1^{54}(\tau^3 - 3\tau + C_2)^{-1}[P_6(\tau)]^2,$$

where $A = \mp 28 \times 3^7 a^{-5} b^{9/2}\Big(\dfrac{2a^2}{3b}\Big)^{4/7}$.

48. $y'''_{xxx} = A(y'_x)^{-7/3}$.

Solution in parametric form:

$$x = aC_1^5 \int (\tau^2 \pm 1)^{1/4}\, d\tau + C_3, \qquad y = bC_1^8(\tau^3 \pm 3\tau + C_2),$$

where $A = \pm\frac{81}{2}a^{-5}b^3(3b/a)^{1/3}$.

49. $y'''_{xxx} = Ay^{-5/3}y'_x(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1 \int \tau(\tau^2 \pm 1)^{1/2}(\tau^3 \pm 3\tau + C_2)^{-1/2}\, d\tau + C_3, \qquad y = bC_1^3(\tau^2 \pm 1)^{3/2},$$

where $A = \mp\frac{4}{243}a^4 b^{-4/3}$.

50. $y'''_{xxx} = Ayy'_x(y''_{xx})^{7/5}$.

Solution in parametric form:

$$x = aC_1^3 \int (\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{-1/2}\, d\tau + C_3, \qquad y = bC_1\tau(\tau^2 \pm 1)^{-1/2},$$

where $A = \pm 5b^{-2}\Big(\dfrac{2a^2}{3b}\Big)^{2/5}$.

51. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{8/5}$.

Solution in parametric form:

$$x = aC_1^9 \int \tau^{-1/2}(\tau^2 \pm 1)^{5/4}\, d\tau + C_3, \qquad y = bC_1^8(\tau^3 \pm 3\tau + C_2),$$

where $A = \mp\dfrac{4}{27}a^2 b^{-3}\Big(\dfrac{2a^2}{3b}\Big)^{3/5}$.

52. $y'''_{xxx} = Ay^{-4/3}(y'_x)^{-7/3}$.

Solution in parametric form:

$$x = aC_1^7 \int (\tau^2 \pm 1)^{1/4}(\tau^3 \pm 3\tau + C_2)^{-3/2}\, d\tau + C_3, \qquad y = bC_1^8(\tau^3 \pm 3\tau + C_2)^{-1},$$

where $A = \pm\frac{81}{2}a^{-5}b^{13/3}(3b/a)^{1/3}$.

53. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-5/3}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^{-1}\int(\pm\tau^2 + C_2\tau - 1)(\tau^2 \pm 1)^{1/2}(\tau^3 \pm 3\tau + C_2)^{-3/2}\,d\tau + C_3,$$
$$y = bC_1(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-1},$$

where $A = \mp\frac{4}{27}a^2b^{2/3}(3b/a)^{2/3}$.

54. $y'''_{xxx} = Ay(y'_x)^{-7}(y''_{xx})^{7/5}.$

Solution in parametric form:

$$x = aC_1^7\int(\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{5/6}\,d\tau + C_3,\quad y = bC_1^9(\pm\tau^2 + C_2\tau - 1)(\tau^2 \pm 1)^{-1/2},$$

where $A = \pm 5a^{-8}b^6(2a^2/b)^{2/5}$.

55. $y'''_{xxx} = Ay^{-4}(y'_x)^3(y''_{xx})^{8/5}.$

Solution in parametric form:

$$x = aC_1^{-1}\int(\tau^2 \pm 1)^{5/4}(\pm\tau^2 + C_2\tau - 1)^{-1/2}(\tau^3 \pm 3\tau + C_2)^{-2/3}\,d\tau + C_3,$$
$$y = bC_1^8(\tau^3 \pm 3\tau + C_2)^{1/3},$$

where $A = \mp 5a^2b(2a^2/b)^{3/5}$.

56. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{7/4}.$

Solution in parametric form:

$$x = aC_1^{-7}\int(\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{-1/3}\,d\tau + C_3,\quad y = bC_1^2(\tau^2 \pm 1)^{-1}(\tau^3 \pm 3\tau + C_2)^{2/3},$$

where $A = 4a^{-1}b^{5/2}\left(\mp\dfrac{a^2}{2b}\right)^{3/4}$.

57. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{5/4}.$

Solution in parametric form:

$$x = aC_1^{17}\int(\pm\tau^2 + C_2\tau - 1)(\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{2/3}\,d\tau + C_3,$$
$$y = bC_1^{18}(\pm\tau^2 + C_2\tau - 1)^2(\tau^2 \pm 1)^{-1},$$

where $A = -64a^{-5}b^{9/2}\left(\mp\dfrac{a^2}{2b}\right)^{1/4}$.

◆ *In the solutions of equations 58–63, the following notation is used:*

$$P_4 = \tau^4 - 6\tau^2 + 4C_2\tau - 3,$$
$$P_6 = \tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau - 8C_2^2 + 27,$$
$$P_9 = 7\tau^9 - 108\tau^7 + 84C_2\tau^6 + 378\tau^5 - 756C_2\tau^4 +$$
$$+ 84(4C_2^2 + 9)\tau^3 - 756C_2\tau^2 + 567\tau + 4(4C_2^2 - 27)C_2.$$

58. $y'''_{xxx} = Ay^{-11/7}(y'_x)^{-13/7}(y''_{xx})^{1/2}$.

Solution in parametric form:

$$x = C_3 + aC_1^{55}\int P_4^{5/6}P_9^{-3/2}\,d\tau, \quad y = -bC_1^{54}P_9^{-1},$$

where $A = -294\cdot 63^{6/7}\sqrt{6}a^{-27/7}b^{55/14}$.

59. $y'''_{xxx} = Ay^{-10/7}(y'_x)^{-15/7}(y''_{xx})^{5/2}$.

Solution in parametric form:

$$x = C_3 - aC_1^{29}\int P_4^{4/3}P_6^2P_9^{-3/2}\,d\tau, \quad y = -bC_1^2P_4^{7/3}P_9^{-1},$$

where $A = \dfrac{8\cdot 7^{1/7}\cdot 3^{5/14}\sqrt{2}}{1701}a^{-1/7}b^{29/14}$.

60. $y'''_{xxx} = Ay^2(y'_x)^{-9/2}(y''_{xx})^{13/10}$.

Solution in parametric form:

$$x = C_3 - aC_1^{22}\int P_4^{-4/3}P_9^{5/7}\,d\tau, \quad y = bC_1^{49}P_4^{-1/3}P_6,$$

where $A = \dfrac{80\cdot 2^{9/10}\cdot 3^{1/5}}{729}a^{-49/10}b^{11/5}$.

61. $y'''_{xxx} = Ay^{-11/4}(y'_x)^5(y''_{xx})^{17/10}$.

Solution in parametric form:

$$x = C_3 - aC_1^{59}\int P_4^{-13/6}P_6^{-1/2}P_9^{-3/7}\,d\tau, \quad y = bC_1^{108}P_9^{4/7},$$

where $A = -\dfrac{5\cdot 2^{3/5}\cdot 3^{3/10}}{1088391168}a^{27/5}b^{-59/20}$.

62. $y'''_{xxx} = Ay^{-10/3}(y'_x)^{-1/3}(y''_{xx})^{18/11}$.

Solution in parametric form:

$$x = C_3 - aC_1^{50}\int P_4^{-3/2}P_6^{1/2}P_9^{-4/7}\,d\tau, \quad y = bC_1P_4^{-1}P_9^{3/7},$$

where $A = 99a^{-2/33}b^{100/33}$.

63. $y'''_{xxx} = Ay^{-2/3}(y'_x)^{-17/3}(y''_{xx})^{15/11}$.

Solution in parametric form:

$$x = C_3 - aC_1^{-591}\int P_4^{-3/2}P_6^2P_9^{-11/14}\,d\tau, \quad y = bC_1^{588}P_4^{-1}P_6^3,$$

where $A = -3168\cdot 2^{2/3}a^{-196/33}b^{197/33}$.

◆ *In the solutions of equations 64–75, the following notation is used:*

$$S_1 = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\omega\tau), \qquad \omega = k\sqrt{3},$$
$$S_2 = 2kC_1 e^{2k\tau} + kC_2 e^{-k\tau}\big[\sqrt{3}\cos(\omega\tau) - \sin(\omega\tau)\big],$$
$$S_3 = 4k^2 C_1 e^{2k\tau} - 2k^2 C_2 e^{-k\tau}\big[\sqrt{3}\cos(\omega\tau) + \sin(\omega\tau)\big],$$
$$S_4 = S_2^2 - 2S_1S_3, \qquad S_5 = 5S_2S_4 + 32k^3 S_1^3.$$

64. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^3 \int S_1^{-1/2} S_2 S_3\, d\tau + C_3, \quad y = bC_1^4 S_2^2, \qquad \text{where} \quad A = -a^6 b^{-9/2} k^3.$$

65. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-6}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^3 \int S_1^{-3/2} S_2 S_4\, d\tau + C_3, \quad y = bC_1^2 S_1^{-1} S_2^2, \qquad \text{where} \quad A = 16a^{-3} b^{9/2} k^3.$$

66. $y'''_{xxx} = Ay(y'_x)^{-9/5}.$

Solution in parametric form:

$$x = aC_1^3 \int S_1^{3/4}\, d\tau + C_3, \quad y = bC_1^8 S_4, \qquad \text{where} \quad A = -160a^{-4} b k^6 (16bk^3/a)^{4/5}.$$

67. $y'''_{xxx} = Ay^{-7/5}(y'_x)^3(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^3 \int S_1^{3/2} S_2 S_4^{-1/2}\, d\tau + C_3, \quad y = bC_1^5 S_1^{5/2}, \quad \text{where } A = \tfrac{1}{4} \times 5^{-5} a^6 b^{-18/5} k^{-6}.$$

68. $y'''_{xxx} = Ay(y''_{xx})^{9/7}.$

Solution in parametric form:

$$x = aC_1^{-3} \int S_1^{-3/2}\, d\tau + C_3, \quad y = bC_1 S_1^{-1/2} S_2, \quad \text{where } A = \frac{7}{2} a^{-1} b^{-1} \Big(\frac{a^2}{8bk^3}\Big)^{2/7}.$$

69. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{12/7}.$

Solution in parametric form:

$$x = aC_1^{15} \int S_1^{9/4} S_2^{-1/2} S_4\, d\tau + C_3, \qquad y = bC_1^{16} S_4^2,$$

where $A = 7 \times 2^{-16} a^2 b^{-5/2} k^{-9} \Big(\dfrac{a^2}{8bk^3}\Big)^{5/7}.$

70. $y'''_{xxx} = Ay^{-13/5}(y'_x)^{-9/5}.$

Solution in parametric form:

$$x = aC_1^9 \int S_1^{3/4} S_4^{-3/2}\, d\tau + C_3, \qquad y = bC_1^8 S_4^{-1},$$

where $A = -160a^{-4} b^{23/5} k^6 (16bk^3/a)^{4/5}.$

71. $y'''_{xxx} = Ay^{-7/5}(y'_x)^{-21/5}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^3 \int S_1^{3/2} S_4^{-3/2} S_5 \, d\tau + C_3, \qquad y = bC_1 S_1^{5/2} S_4^{-1},$$

where $A = \dfrac{5}{512} a^{-1} b^{17/5} k^{-6} \Big(\dfrac{b}{2a}\Big)^{1/5}.$

72. $y'''_{xxx} = Ay(y'_x)^{-9/4}(y''_{xx})^{9/7}.$

Solution in parametric form:

$$x = aC_1^9 \int S_1^{-3/2} S_4^{6/5} \, d\tau + C_3, \qquad y = bC_1^{25} S_1^{-1/2} S_5,$$

where $A = \dfrac{35}{8} a^{-3} b \Big(-\dfrac{5b}{2a}\Big)^{1/4} \Big(\dfrac{a^2}{32bk^3}\Big)^{2/7}.$

73. $y'''_{xxx} = Ay^{-13/8}(y'_x)^3(y''_{xx})^{12/7}.$

Solution in parametric form:

$$x = aC_1^{39} \int S_1^{9/4} S_4^{3/5} S_5^{-1/2} \, d\tau + C_3, \qquad y = bC_1^{64} S_4^{8/5},$$

where $A = 175 \times 2^{-22} a^2 b^{-11/8} k^{-9} \Big(\dfrac{a^2}{32bk^3}\Big)^{5/7}.$

74. $y'''_{xxx} = Ay^{-7/2}(y''_{xx})^{18/13}.$

Solution in parametric form:

$$x = aC_1^{27} \int S_1^{-3/2} S_4^{-3/5} \, d\tau + C_3, \qquad y = bC_1^2 S_1^{-1} S_4^{2/5},$$

where $A = -\tfrac{208}{5} a^{-1} b^{7/2} k^3 (2a^2/b)^{5/13}.$

75. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-6}(y''_{xx})^{21/13}.$

Solution in parametric form:

$$x = aC_1^{51} \int S_1^{-3/2} S_4^{9/5} S_5 \, d\tau + C_3, \qquad y = bC_1^{50} S_1^{-1} S_5^2,$$

where $A = 208 \times 5^5 a^{-7} b^{13/2} k^3 (2a^2/b)^{8/13}.$

◆ *In the solutions of equations 76–93, the following notation is used:*

$$T_1 = \cosh(\tau + C_2)\cos\tau, \qquad T_2 = \tanh(\tau + C_2) + \tan\tau, \quad T_3 = \tanh(\tau + C_2) - \tan\tau,$$
$$\theta_1 = \cosh\tau - \sin(\tau + C_2), \quad \theta_2 = \sinh\tau + \cos(\tau + C_2), \quad \theta_3 = \sinh\tau - \cos(\tau + C_2),$$
$$T_4 = 3T_2T_3 - 4, \quad \theta_4 = 3\theta_2\theta_3 - 2\theta_1^2.$$

76. $y'''_{xxx} = Ay^2(y'_x)^{-7/3}.$

1°. Solution in parametric form:

$$x = aC_1 \int T_1^{1/4} \, d\tau + C_3, \quad y = bC_1^4 T_1 T_2, \qquad \text{where} \quad A = -3a^{-5} b (2b/a)^{1/3}.$$

2°. Solution in parametric form:

$$x = aC_1 \int \theta_1^{1/4}\, d\tau + C_3, \quad y = bC_1^4\theta_2, \qquad \text{where} \quad A = \tfrac{3}{8}a^{-5}b(b/a)^{1/3}.$$

77. $y'''_{xxx} = Ay^{-5/3}(y'_x)^5(y''_{xx})^3.$

1°. Solution in parametric form:

$$x = aC_1^2 \int T_1 T_2^{-1/2} T_3\, d\tau + C_3, \quad y = bC_1^3 T_1^{3/2}, \qquad \text{where} \quad A = 64 \times 3^{-7} a^8 b^{-16/3}.$$

2°. Solution in parametric form:

$$x = aC_1^2 \int \theta_1^{1/2}\theta_2^{-1/2}\theta_3\, d\tau + C_3, \quad y = bC_1^3\theta_1^{3/2}, \qquad \text{where} \quad A = -256 \times 3^{-7} a^8 b^{-16/3}.$$

78. $y'''_{xxx} = Ay(y'_x)^{-1/3}(y''_{xx})^{7/5}.$

1°. Solution in parametric form:

$$x = aC_1^{-2} \int T_1^{-1} T_2^{1/2}\, d\tau + C_3, \quad y = bC_1 T_1^{1/2} T_3, \quad \text{where } A = -\frac{5}{2ab}\left(\frac{b}{2a}\right)^{1/3}\left(\frac{2a^2}{3b}\right)^{2/5}.$$

2°. Solution in parametric form:

$$x = aC_1^{-2} \int \theta_1^{-3/2}\theta_2^{1/2}\, d\tau + C_3, \quad y = bC_1\theta_1^{-1/2}\theta_3, \quad \text{where } A = \frac{5}{2ab}\left(\frac{b}{2a}\right)^{1/3}\left(\frac{4a^2}{3b}\right)^{2/5}.$$

79. $y'''_{xxx} = Ay^{-2/3}(y'_x)^3(y''_{xx})^{8/5}.$

1°. Solution in parametric form:

$$x = aC_1^{11} \int T_1^{11/4} T_2^2 T_3^{-1/2}\, d\tau + C_3, \quad y = bC_1^{12} T_1^3 T_2^3, \quad \text{where } A = \frac{5}{432}a^2 b^{-7/3}\left(\frac{2a^2}{3b}\right)^{3/5}.$$

2°. Solution in parametric form:

$$x = aC_1^{11} \int \theta_1^{5/4}\theta_2^2\theta_3^{-1/2}\, d\tau + C_3, \quad y = bC_1^{12}\theta_2^3, \quad \text{where } A = -\frac{5}{54}a^2 b^{-7/3}\left(\frac{4a^2}{3b}\right)^{3/5}.$$

80. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{1/2}.$

1°. Solution in parametric form:

$$x = aC_1^3 \int T_1^{-3/2}\, d\tau + C_3, \quad y = bC_1^2 T_1^{-1}, \qquad \text{where} \quad A = 2a^{-2}b^{7/2}(2/b)^{1/2}.$$

2°. Solution in parametric form:

$$x = aC_1^3 \int \theta_1^{-3/2}\, d\tau + C_3, \quad y = bC_1^2\theta_1^{-1}, \qquad \text{where} \quad A = -a^{-2}b^{7/2}(-2/b)^{1/2}.$$

81. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{5/2}.$

1°. Solution in parametric form:

$$x = aC_1^3 \int T_1^{3/2} T_2^2 T_3\, d\tau + C_3, \quad y = bC_1^2 T_1 T_3^2, \qquad \text{where} \quad A = -32a^{-2}b^{7/2}(b/2)^{-1/2}.$$

2°. Solution in parametric form:

$$x = aC_1^3 \int \theta_1^{-3/2}\theta_2^2\theta_3\,d\tau + C_3, \quad y = bC_1^2\theta_1^{-1}\theta_3^2, \qquad \text{where} \quad A = 16a^{-2}b^{7/2}(-b/2)^{-1/2}.$$

82. $y'''_{xxx} = Ay^{-10/3}(y'_x)^{-7/3}$.

1°. Solution in parametric form:

$$x = aC_1^5 \int T_1^{-5/4}T_2^{-3/2}\,d\tau + C_3, \quad y = bC_1^4T_1^{-1}T_2^{-1}, \quad \text{where} \quad A = -3a^{-5}b^{20/3}(2/a)^{1/3}.$$

2°. Solution in parametric form:

$$x = aC_1^5 \int \theta_1^{1/4}\theta_2^{-3/2}\,d\tau + C_3, \quad y = bC_1^4\theta_2^{-1}, \qquad \text{where} \quad A = \tfrac{3}{8}a^{-16/3}b^{20/3}.$$

83. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-17/3}(y''_{xx})^3$.

1°. Solution in parametric form:

$$x = aC_1^2 \int T_1T_2^{-3/2}T_4\,d\tau + C_3, \quad y = bC_1T_1^{1/2}T_2^{-1}, \quad \text{where} \ A = \frac{3}{16}a^{-2}b^{14/3}\Big(\frac{b}{2a}\Big)^{2/3}.$$

2°. Solution in parametric form:

$$x = aC_1^2 \int \theta_1^{1/2}\theta_2^{-3/2}\theta_4\,d\tau + C_3, \quad y = bC_1\theta_1^{3/2}\theta_2^{-1}, \quad \text{where} \ A = -\frac{3}{4}a^{-2}b^{14/3}\Big(\frac{b}{2a}\Big)^{2/3}.$$

84. $y'''_{xxx} = Ay(y'_x)^{-13/7}(y''_{xx})^{7/5}$.

1°. Solution in parametric form:

$$x = aC_1^2 \int T_1^{1/3}T_2^{11/6}\,d\tau + C_3, \quad y = bC_1^9T_1^{3/2}T_4, \quad \text{where} \ A = -\frac{5}{4}a^{-2}\Big(\frac{3b}{2a}\Big)^{6/7}\Big(\frac{2a^2}{7b}\Big)^{2/5}.$$

2°. Solution in parametric form:

$$x = aC_1^2 \int \theta_1^{-3/2}\theta_2^{11/6}\,d\tau + C_3, \quad y = bC_1^9\theta_1^{-1/2}\theta_4, \quad \text{where} \ A = \frac{5}{4}a^{-2}\Big(\frac{3b}{2a}\Big)^{6/7}\Big(\frac{4a^2}{7b}\Big)^{2/5}.$$

85. $y'''_{xxx} = Ay^{-10/7}(y'_x)^3(y''_{xx})^{8/5}$.

1°. Solution in parametric form:

$$x = aC_1^{19} \int T_1^{19/12}T_2^{4/3}T_4^{-1/2}\,d\tau + C_3, \qquad y = bC_1^{28}T_1^{7/3}T_2^{7/3},$$

where $A = \dfrac{45}{16} \times 7^{-3}a^2b^{-11/7}\Big(\dfrac{2a^2}{7b}\Big)^{3/5}$.

2°. Solution in parametric form:

$$x = aC_1^{19} \int \theta_1^{5/4}\theta_2^{4/3}\theta_4^{-1/2}\,d\tau + C_3, \qquad y = bC_1^{28}\theta_2^{7/3},$$

where $A = -\dfrac{45}{2} \times 7^{-3}a^2b^{-11/7}\Big(\dfrac{4a^2}{7b}\Big)^{3/5}$.

86. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{13/10}$.

1°. Solution in parametric form:

$$x = aC_1^{-11}\int T_1^{-11/6}T_2^{-1/3}\,d\tau + C_3, \qquad y = bC_1^4 T_1^{-1/3}T_2^{2/3},$$

where $A = \frac{20}{3}a^{-1}b^{5/2}(2a^2/b)^{3/10}$.

2°. Solution in parametric form:

$$x = aC_1^{-11}\int \theta_1^{-3/2}\theta_2^{-1/3}\,d\tau + C_3, \qquad y = bC_1^4\theta_1^{-1}\theta_2^{2/3},$$

where $A = \frac{10}{3}a^{-1}b^{5/2}(-2a^2/b)^{3/10}$.

87. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{17/10}$.

1°. Solution in parametric form:

$$x = aC_1^{19}\int T_1^{19/6}T_2^{8/3}T_4\,d\tau + C_3, \qquad y = bC_1^{18}T_1^3T_4^2,$$

where $A = -540a^{-5}b^{9/2}(2a^2/b)^{7/10}$.

2°. Solution in parametric form:

$$x = aC_1^{19}\int \theta_1^{-3/2}\theta_2^{8/3}\theta_4\,d\tau + C_3, \qquad y = bC_1^{18}\theta_1^{-1}\theta_4^2,$$

where $A = -270a^{-5}b^{9/2}(-2a^2/b)^{7/10}$.

88. $y'''_{xxx} = Ay^{-17/10}(y'_x)^{-8/5}(y''_{xx})^{1/2}$.

Solution in parametric form:

$$x = C_3 - aC_1^{19}\int T_1^{-19/6}T_2^{4/3}(3T_2^2 - T_4)^{-3/2}(3T_2^2 + T_4)^{-3/2}\,d\tau,$$
$$y = -bC_1^{18}T_1^{-3}(3T_2^2 - T_4)^{-1}(3T_2^2 + T_4)^{-1},$$

where $A = 200\sqrt{2}\cdot 60^{3/5}a^{-18/5}b^{19/5}$.

89. $y'''_{xxx} = Ay^{-13/10}(y'_x)^{-12/5}(y''_{xx})^{5/2}$.

Solution in parametric form:

$$x = C_3 - aC_1^{11}\int T_1^{11/6}T_2^{7/3}T_4^2(T_4 - 3T_2^2)^{-3/2}(3T_2^2 + T_4)^{-3/2}\,d\tau,$$
$$y = bC_1^2T_1^{1/3}T_2^{10/3}(3T_2^2 - T_4)^{-1}(3T_2^2 + T_4)^{-1},$$

where $A = \dfrac{\sqrt{2}\cdot 20^{2/5}\cdot 3^{3/5}}{810}a^{-2/5}b^{11/5}$.

90. $y'''_{xxx} = Ay^2(y'_x)^{-27/7}(y''_{xx})^{16/13}$.

1°. Solution in parametric form:

$$x = C_3 + aC_1^{37} \int T_1^{37/60} T_2^{-4/3} (T_4^2 - 9T_2^4)^{13/20}\, d\tau,$$
$$y = bC_1^{100} T_1^{5/3} T_2^{-1/3} T_4,$$

where $A = -\dfrac{13}{2^{239/91} \cdot 3^{34/7} \cdot 7^{3/13}} a^{-400/91} b^{148/91}$.

2°. Solution in parametric form:

$$x = C_3 + aC_1^{37} \int \theta_1^{-13/20} \theta_2^{-4/3} \theta_4^{-13/20}\, d\tau,$$
$$y = bC_1^{100} \theta_2^{-1/3} (2\theta_1^2 - 3\theta_2\theta_3),$$

where $A = \dfrac{26 \cdot 3^{358/91}}{7^{3/13}} a^{-400/91} b^{148/91}$.

91. $y'''_{xxx} = Ay^{-17/7}(y'_x)^5(y''_{xx})^{23/13}$.

1°. Solution in parametric form:

$$x = C_3 - aC_1^{38} \int T_1^{13/30} T_2^{19/6} T_4^{-1/2} (T_4^2 - 9T_2^4)^{-3/10}\, d\tau,$$
$$y = bC_1^{63} T_1^{21/10} (T_4^2 - 9T_2^4)^{7/10},$$

where $A = \dfrac{13}{2^{42/13} \cdot 3^6 \cdot 7^{75/13}} a^{72/13} b^{-304/91}$.

2°. Solution in parametric form:

$$x = C_3 + aC_1^{38} \int \theta_1^{3/10} \theta_2^{19/6} (2\theta_1^2 - 3\theta_2\theta_3)^{-1/2} \theta_4^{-3/10}\, d\tau,$$
$$y = bC_1^{63} \theta_1^{-7/10} \theta_4^{7/10},$$

where $A = \dfrac{13 \cdot 2^{20/13}}{3^6 \cdot 7^{75/13}} a^{72/13} b^{-304/91}$.

92. $y'''_{xxx} = Ay^{-13/3}(y'_x)^{-1/3}(y''_{xx})^{27/17}$.

1°. Solution in parametric form:

$$x = C_3 - aC_1^{78} \int T_1^{-13/5} T_2^{-3/2} T_4^{1/2} (T_4^2 - 9T_2^4)^{-7/10}\, d\tau,$$
$$y = bC_1^{3} T_1^{-1/10} T_2^{-1} (T_4^2 - 9T_2^4)^{3/10},$$

where $A = 51 \cdot 2^{3/17} a^{-8/51} b^{208/51}$.

2°. Solution in parametric form:

$$x = C_3 + aC_1^{78} \int \theta_1^{7/10} \theta_2^{-3/2} (2\theta_1^2 - 3\theta_2\theta_3)^{1/2} \theta_4^{-7/10}\, d\tau,$$
$$y = bC_1^{3} \theta_1^{-3/10} \theta_2^{-1} \theta_4^{3/10},$$

where $A = -102 \cdot 2^{98/51} a^{-8/51} b^{208/51}$.

93. $y'''_{xxx} = Ay^{-2/3}(y'_x)^{-23/3}(y''_{xx})^{24/17}$.

1°. Solution in parametric form:

$$x = C_3 - aC_1^{101} \int T_1^{51/10} T_2^{-3/2} T_4^2 (T_4^2 - 9T_2^4)^{17/20}\, d\tau,$$
$$y = bC_1^{100} T_1^5 T_2^{-1} T_4^3,$$

where $A = 102a^{-400/51} b^{404/51}$.

2°. Solution in parametric form:

$$x = C_3 + \frac{a}{2} C_1^{101} \int \theta_1^{-17/20} \theta_2^{-3/2} (2\theta_1^2 - 3\theta_2\theta_3)^2 \theta_4^{17/20}\, d\tau,$$
$$y = bC_1^{100} \theta_2^{-1} (2\theta_1^2 - 3\theta_2\theta_3)^3,$$

where $A = -102 \cdot 2^{98/51} a^{-8/51} b^{208/51}$.

◆ *In the solutions of equations 94–96, the following notation is used:*

$$L_1 = C_1\tau^k + C_2\tau^{-k}, \qquad N_1 = (1+k)C_1\tau^k + (1-k)C_2\tau^{-k},$$
$$L_2 = C_1 \ln\tau + C_2, \qquad N_2 = C_1\ln\tau + C_1 + C_2,$$
$$L_3 = C_1 \sin(k\ln\tau) + C_2\cos(k\ln\tau), \qquad N_3 = (C_1 - kC_2)\sin(k\ln\tau) + (C_2 + kC_1)\cos(k\ln\tau).$$

94. $y'''_{xxx} = Ay^{-2}(y'_x)^3$.

Solution in parametric form:

$$x = \int \tau^{1/2} L_m^{-1/2}\, d\tau + C_3, \quad y = \tau^2, \quad \text{where} \quad k = \sqrt{|1+8A|}, \quad m = \begin{cases} 1 & \text{if } A > -1/8, \\ 2 & \text{if } A = -1/8, \\ 3 & \text{if } A < -1/8. \end{cases}$$

95. $y'''_{xxx} = Ay(y'_x)^{-3}(y''_{xx})^3$.

Solution in parametric form:

$$x = \int \tau^{-1} N_m\, d\tau + C_3, \quad y = \tau L_m, \quad \text{where} \quad k = \sqrt{|A-1|}, \quad m = \begin{cases} 1 & \text{if } A < 1, \\ 2 & \text{if } A = 1, \\ 3 & \text{if } A > 1. \end{cases}$$

96. $y'''_{xxx} = Ay^{-1/2}(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = \mp 4 \int \tau^2 L_1\, d\tau + C_3, \quad y = \tau^2 L_1^2, \quad \text{where} \quad k = \sqrt{1 + 8A^{-2}}.$$

◆ *In the solutions of equations 97–112, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \textit{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \textit{for the lower sign,} \end{cases}$$

$$U_1 = \tau Z'_\tau + \nu Z, \quad U_2 = U_1^2 \pm \tau^2 Z^2, \quad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1U_2,$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are Bessel functions, and $I_\nu(\tau)$ and $K_\nu(\tau)$ are modified Bessel functions.

97. $y'''_{xxx} = Ay^{\beta}(y'_x)^3, \qquad \beta \neq -2.$

Solution in parametric form:

$$x = C_1 \int \tau^{\frac{3\nu-2}{2}} Z^{-1/2}\,d\tau + C_3, \quad y = b\tau^{2\nu}, \qquad \text{where} \quad \nu = \frac{1}{\beta+2},\ A = \mp\frac{1}{8\nu^2}b^{-\beta-2}.$$

98. $y'''_{xxx} = Ay(y'_x)^{\gamma}(y''_{xx})^3, \qquad \gamma \neq -3.$

Solution in parametric form:

$$x = aC_1 \int \tau^{-1}U_1\,d\tau + C_3, \quad y = bC_1\tau^{\nu}Z, \qquad \text{where} \quad \nu = \frac{2}{\gamma+3},\ A = \pm\frac{1}{\nu^2}a^{\gamma+3}b^{-\gamma-3}.$$

99. $y'''_{xxx} = Ay^{-1/2}(y''_{xx})^{\delta}, \qquad \delta \neq 3/2.$

Solution in parametric form:

$$x = aC_1 \int \tau^{3\nu-1} Z\,d\tau + C_3, \quad y = bC_1^2\tau^{2\nu}Z^2,$$

where $\nu = \dfrac{1-\delta}{3-2\delta},\ A = \mp\dfrac{4a^{-3}b^{3/2}}{3-2\delta}\left(\mp\dfrac{a^2}{2b}\right)^{\delta}.$

100. $y'''_{xxx} = Ax^{-3}y(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1 \int Z\,d\tau + C_3, \quad y = bC_1^2\tau^{-1/3}\Big(\tau Z^2 - U_1 \int Z\,d\tau - C_3U_1\Big),$$

where $\nu = \frac{1}{3},\ A = \frac{9}{4}a^6b^{-3}.$

101. $y'''_{xxx} = Ay^{-1/2}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1 \int \tau Z\,d\tau + C_3, \quad y = bC_1^2\tau^{4/3}Z^2, \qquad \text{where} \quad \nu = \tfrac{2}{3},\ A = -\tfrac{1}{6}a^3b^{-3/2}.$$

102. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-3}(y''_{xx})^3.$

Solution in parametric form:

$$x = C_1 \int \tau^{-2}Z^{-2}U_1U_2\,d\tau + C_3, \quad y = b\tau^{-4/3}Z^{-2}U_1^2, \qquad \text{where} \quad \nu = \tfrac{1}{3},\ A = \pm\tfrac{4}{3}b^{3/2}.$$

103. $y'''_{xxx} = Ay(y'_x)^{-3}.$

Solution in parametric form:

$$x = aC_1 \int Z\,d\tau + C_3, \quad y = bC_1^2\tau^{-2/3}U_2, \qquad \text{where} \quad \nu = \tfrac{1}{3},\ A = -\tfrac{16}{81}a^{-6}b^3.$$

104. $y'''_{xxx} = Ay^{-2}(y'_x)^3(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1 \int ZU_1U_2^{-1/2}\,d\tau + C_3, \quad y = bC_1^2\tau^{2/3}Z^2, \qquad \text{where} \quad \nu = \tfrac{1}{3},\ A = \tfrac{9}{32}a^6b^{-3}.$$

105. $y'''_{xxx} = Ay(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = C_1 \int \tau^{-1} Z^{-2}\, d\tau + C_3, \quad y = b\tau^{-2/3} Z^{-1} U_1, \quad \text{where} \quad \nu = \tfrac{1}{3}, \; A = 2b^{-1}(\mp 6/b)^{1/2}.$$

106. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = aC_1 \int Z^{5/2} U_1^{-1/2} U_2\, d\tau + C_3, \quad y = bC_1 \tau^{-4/3} U_2^2,$$

where $\nu = \frac{1}{3}$, $A = \mp\frac{27}{32} a^3 b^{-5/2} (\mp 6/b)^{1/2}$.

107. $y'''_{xxx} = Ay^{-2}(y'_x)^{-3}$.

Solution in parametric form:

$$x = aC_1 \int \tau Z U_2^{-3/2}\, d\tau + C_3, \quad y = bC_1 \tau^{2/3} U_2^{-1}, \quad \text{where} \quad \nu = \tfrac{1}{3}, \; A = -\tfrac{16}{81} a^{-6} b^6.$$

108. $y'''_{xxx} = Ay^{-2}(y'_x)^{-3}(y''_{xx})^3$.

Solution in parametric form:

$$x = C_1 \int Z U_2^{-3/2} U_3\, d\tau + C_3, \quad y = b\tau^{4/3} Z^2 U_2^{-1}, \quad \text{where} \quad \nu = \tfrac{1}{3}, \; A = 18b^{-3}.$$

109. $y'''_{xxx} = Ay(y'_x)^{-3}(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = aC_1 \int \tau^{-2} Z^{-2} U_2^{3/2}\, d\tau + C_3, \quad y = bC_1^2 \tau^{-4/3} Z^{-1} U_3,$$

where $\nu = \frac{1}{3}$, $A = -8a^{-3} b^2 (\pm 6/b)^{1/2}$.

110. $y'''_{xxx} = Ay^{-2}(y'_x)^3(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = aC_1 \int \tau Z^{5/2} U_3^{-1/2}\, d\tau + C_3, \quad y = bC_1^2 \tau^{-2/3} U_2,$$

where $\nu = \frac{1}{3}$, $A = \pm\frac{27}{8} a^3 b^{-1} (\pm 6/b)^{1/2}$.

111. $y'''_{xxx} = Ay^{-2}(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = C_1 \int \tau^{-1} Z^{-2}\, d\tau + C_3, \quad y = b\tau^{-4/3} Z^{-2} U_2, \quad \text{where} \quad \nu = \tfrac{1}{3}, \; A = \pm\tfrac{4}{3} b^2 (2b)^{-1/2}.$$

112. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-3}(y''_{xx})^{3/2}$.

Solution in parametric form:

$$x = aC_1 \int \tau^{-3} Z^{-2} U_2^{3/2} U_3\, d\tau + C_3, \quad y = bC_1 \tau^{-8/3} Z^{-2} U_3^2,$$

where $\nu = \frac{1}{3}$, $A = \mp\frac{256}{3} a^{-3} b^{7/2} (2b)^{-1/2}$.

◆ *In the solutions of equations 113–156, the following notation is used:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2, \qquad f = \sqrt{\pm(4\wp^3 - 1)}.$$

The function $\wp = \wp(\tau)$ is defined implicitly by the above elliptic integral of the first kind. For the upper sign, the function $\wp$ coincides with the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\, 0,\, 1)$. In the solution given below, we can take $\wp$ to be the parameter instead of τ and use the explicit dependence $\tau = \tau(\wp)$.

113. $y'''_{xxx} = A(y'_x)^5.$

Solution in parametric form:

$$x = aC_1^2 \int \wp^{-1/2}\, d\tau + C_3, \quad y = bC_1\tau, \qquad \text{where} \quad A = \pm 3a^2 b^{-4}.$$

114. $y'''_{xxx} = Ay^2 y'_x (y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^5 \int \tau^{-1/2} f\, d\tau + C_3, \quad y = bC_1^4 \wp, \qquad \text{where} \quad A = \mp 24a^4 b^{-5}.$$

115. $y'''_{xxx} = Ayy'_x (y''_{xx})^{5/2}.$

Solution in parametric form:

$$x = aC_1^7 \int \tau^{-1/2} \wp^2\, d\tau + C_3, \quad y = bC_1^6 f, \qquad \text{where} \quad A = -\tfrac{1}{9} a^3 b^{-3} (\pm 3b)^{-1/2}.$$

116. $y'''_{xxx} = A(y'_x)^3 (y''_{xx})^{1/2}.$

Solution in parametric form:

$$x = aC_1^5 \int f^{-1/2}\, d\tau + C_3, \quad y = bC_1^2 \tau, \qquad \text{where} \quad A = \pm 6ab^{-2} (\pm 3b)^{-1/2}.$$

117. $y'''_{xxx} = Ay^{-5} (y'_x)^5.$

Solution in parametric form:

$$x = aC_1^{-1} \int \tau^{-3/2} \wp^{-1/2}\, d\tau + C_3, \quad y = bC_1^2 \tau^{-1}, \qquad \text{where} \quad A = \pm 3a^2 b.$$

118. $y'''_{xxx} = Ay^2 (y'_x)^{-9} (y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^5 \int \tau^{-3/2} (\tau f - \wp)\, d\tau + C_3, \quad y = bC_1^6 \tau^{-1} \wp, \qquad \text{where} \quad A = \mp 24a^{-6} b^5.$$

119. $y'''_{xxx} = Ay(y'_x)^{-3/2} (y''_{xx})^{5/2}.$

Solution in parametric form:

$$x = aC_1^2 \int \tau^{-1} \wp^2\, d\tau + C_3, \quad y = bC_1 (\tau f - \wp), \qquad \text{where} \quad A = \mp \tfrac{1}{2} ab (\pm 2/a)^{1/2}.$$

120. $\boldsymbol{y'''_{xxx} = Ay^{-5/4}(y'_x)^3(y''_{xx})^{1/2}.}$

Solution in parametric form:

$$x = aC_1^5 \int \tau^3(\tau f - \wp)^{-1/2}\,d\tau + C_3, \quad y = bC_1^4\tau^4, \quad \text{where } A = \pm\tfrac{3}{16}ab^{-3/4}(\pm 3b)^{-1/2}.$$

121. $\boldsymbol{y'''_{xxx} = Ay^{-2/3}(y''_{xx})^{6/5}.}$

Solution in parametric form:

$$x = aC_1^7 \int \tau^{-4}\wp^2\,d\tau + C_3, \quad y = bC_1^9\tau^{-3}\wp^3, \quad \text{where } A = \pm 5a^{-1}b^{-2/3}\left(\frac{a^2}{18b}\right)^{1/5}.$$

122. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-1/3}(y''_{xx})^{9/5}.}$

Solution in parametric form:

$$x = aC_1^{-1} \int \tau^{5/2}\wp^{1/2}(\tau f - \wp)\,d\tau + C_3, \quad y = bC_1^8(\tau f - \wp)^2,$$

where $A = \mp 5a^{-1}b^{1/2}\left(\frac{12b}{a}\right)^{1/3}\left(\frac{a^2}{18b}\right)^{4/5}$.

123. $\boldsymbol{y'''_{xxx} = Ay^{-15/7}(y'_x)^5.}$

Solution in parametric form:

$$x = aC_1^{13} \int \tau^{11/2}(\tau^2\wp \mp 1)^{-1/2}\,d\tau + C_3, \quad y = bC_1^{14}\tau^7, \quad \text{where } A = \pm 3\times 7^{-4}a^2b^{-13/7}.$$

124. $\boldsymbol{y'''_{xxx} = Ay^2(y'_x)^{-23/7}(y''_{xx})^3.}$

Solution in parametric form:

$$x = aC_1^{-5} \int \tau^{-7/2}(\tau^3 f + 3\tau^2\wp \mp 1)\,d\tau + C_3, \quad y = bC_1^2\tau(\tau^2\wp \mp 1),$$

where $A = \mp\frac{24}{49}a^{-2/7}b^{-5/7}$.

125. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-11/4}(y''_{xx})^{5/2}.}$

Solution in parametric form:

$$x = aC_1 \int \tau^{-3}(\tau^2\wp \mp 1)^2\,d\tau + C_3, \quad y = bC_1^3\tau^{-6}(\tau^3 f + 3\tau^2\wp \mp 1),$$

where $A = \pm\frac{3}{2}(\pm 6b/a)^{3/4}(\mp 6b)^{-1/2}$.

126. $\boldsymbol{y'''_{xxx} = Ay^{-15/8}(y'_x)^3(y''_{xx})^{1/2}.}$

Solution in parametric form:

$$x = aC_1^5 \int \tau^{-6}(\tau^3 f + 3\tau^2\wp \mp 1)^{-1/2}\,d\tau + C_3, \quad y = bC_1^8\tau^{-8},$$

where $A = \pm\frac{3}{64}ab^{-1/8}(\mp 6b)^{-1/2}$.

127. $y'''_{xxx} = Ay^{-2/3}(y''_{xx})^{22/15}$.

Solution in parametric form:

$$x = aC_1^3 \int \tau^8 (\tau^2 \wp \mp 1)^2 \, d\tau + C_3, \quad y = bC_1 \tau^3 (\tau^2 \wp \mp 1)^3,$$

where $A = \mp 15a^{-1/15} b^{2/3} (18b)^{-7/15}$.

128. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-1/3}(y''_{xx})^{23/15}$.

Solution in parametric form:

$$x = aC_1^9 \int \tau^{-29/2} (\tau^3 f + 3\tau^2 \wp \mp 1)(\tau^2 \wp \mp 1)^{1/2} \, d\tau + C_3,$$
$$y = bC_1^8 \tau^{-12} (\tau^3 f + 3\tau^2 \wp \mp 1)^2,$$

where $A = \pm 15a^{-1} b^{1/2} \Big(\dfrac{12b}{a}\Big)^{1/3} \Big(\dfrac{a^2}{18b}\Big)^{8/15}$.

129. $y'''_{xxx} = Ay^{-20/7}(y'_x)^5$.

Solution in parametric form:

$$x = aC_1^4 \int \tau^{-5} (\tau^2 \wp \mp 1)^{-1/2} \, d\tau + C_3, \quad y = bC_1^7 \tau^{-7}, \qquad \text{where} \quad A = \pm 3 \times 7^{-4} a^2 b^{-8/7}.$$

130. $y'''_{xxx} = Ay^2(y'_x)^{-33/7}(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^5 \int \tau^{-7/2} (\tau^3 f - 4\tau^2 \wp \pm 6) \, d\tau + C_3, \quad y = bC_1^{12} \tau^{-6} (\tau^2 \wp \mp 1),$$

where $A = \mp \frac{24}{49} a^{-12/7} b^{5/7}$.

131. $y'''_{xxx} = Ay(y'_x)^{-27/13}(y''_{xx})^{5/2}$.

Solution in parametric form:

$$x = aC_1^{-11} \int \tau^{-13/2} (\tau^2 \wp \mp 1)^2 \, d\tau + C_3, \quad y = bC_1^2 \tau (\tau^3 f - 4\tau^2 \wp \pm 6),$$

where $A = -\frac{24}{13} (6b/a)^{1/13} (\pm 39b)^{-1/2}$.

132. $y'''_{xxx} = Ay^{-20/13}(y'_x)^3(y''_{xx})^{1/2}$.

Solution in parametric form:

$$x = aC_1^{25} \int \tau^{23/2} (\tau^3 f - 4\tau^2 \wp \pm 6)^{-1/2} \, d\tau + C_3, \quad y = bC_1^{26} \tau^{13},$$

where $A = \pm \frac{6}{169} ab^{-6/13} (\pm 39b)^{-1/2}$.

133. $y'''_{xxx} = Ay^{-2/3}(y''_{xx})^{27/20}$.

Solution in parametric form:

$$x = aC_1^{19} \int \tau^{-20} (\tau^2 \wp \mp 1)^2 \, d\tau + C_3, \quad y = bC_1^{18} \tau^{-18} (\tau^2 \wp \mp 1)^3,$$

where $A = 20a^{-1} b^{2/3} \Big(\pm \dfrac{a^2}{18b}\Big)^{7/20}$.

134. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-1/3}(y''_{xx})^{33/20}$**.**

Solution in parametric form:

$$x = aC_1^{11}\int \tau^{10}(\tau^3 f - 4\tau^2\wp \pm 6)(\tau^2\wp \mp 1)^{1/2}\,d\tau + C_3, \quad y = bC_1^2\tau^2(\tau^3 f - 4\tau^2\wp \pm 6)^2,$$

where $A = \mp 20a^{-1}b^{1/2}\Bigl(\frac{12b}{a}\Bigr)^{1/3}\Bigl(\pm\frac{a^2}{18b}\Bigr)^{13/20}$.

135. $y'''_{xxx} = A(y'_x)^{-4}$**.**

Solution in parametric form:

$$x = aC_1^5\int \wp^{-2}\,d\tau + C_3, \quad y = bC_1^7\wp^{-2}(f \pm 2\tau\wp^2), \quad \text{where} \quad A = -192a^{-7}b^5.$$

136. $y'''_{xxx} = Ay^{-5/2}y'_x(y''_{xx})^3$**.**

Solution in parametric form:

$$x = aC_1\int f\wp^{-2}(f \pm 2\tau\wp^2)^{-1/2}\,d\tau + C_3, \quad y = bC_1^8\wp^{-2}, \quad \text{where} \quad A = \pm\tfrac{3}{4}a^4b^{-1/2}.$$

137. $y'''_{xxx} = Ayy'_x(y''_{xx})^{8/5}$**.**

Solution in parametric form:

$$x = aC_1^{13}\int \wp^3(f \pm 2\tau\wp^2)^{-1/2}\,d\tau + C_3, \quad y = bC_1^6 f, \quad \text{where} \quad A = \mp\frac{5}{6}b^{-2}\Bigl(\frac{a^2}{6b}\Bigr)^{3/5}.$$

138. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{7/5}$**.**

Solution in parametric form:

$$x = aC_1^{17}\int \wp^{-3}f^{-1/2}\,d\tau + C_3, \quad y = bC_1^{14}\wp^{-2}(f \pm 2\tau\wp^2),$$

where $A = \mp\frac{5}{8}a^2b^{-3}\Bigl(\frac{a^2}{6b}\Bigr)^{2/5}$.

139. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}$**.**

Solution in parametric form:

$$x = aC_1^{11}\int \frac{\wp\,d\tau}{(f \pm 2\tau\wp^2)^{3/2}} + C_3, \quad y = bC_1^{14}\frac{\wp^2}{f \pm 2\tau\wp^2}, \quad \text{where} \quad A = 192a^{-7}b^{11/2}.$$

140. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^3$**.**

Solution in parametric form:

$$x = aC_1^{-1}\int (f \pm 2\tau\wp^2)^{-3/2}(\tau f + 2\wp)\,d\tau + C_3, \quad y = bC_1^6(f \pm 2\tau\wp^2)^{-1},$$

where $A = -\frac{3}{16}a^3b^{1/2}$.

141. $y'''_{xxx} = Ay(y'_x)^3(y''_{xx})^{8/5}.$

Solution in parametric form:

$$x = aC_1^{23}\int \wp^{-1/2}(f \pm 2\tau\wp^2)^{5/4}\,d\tau + C_3, \quad y = bC_1^{16}(\tau f + 2\wp),$$

where $A = \dfrac{10}{27}a^2b^{-4}\Big(\dfrac{2a^2}{3b}\Big)^{3/5}.$

142. $y'''_{xxx} = Ay(y'_x)^3(y''_{xx})^{7/5}.$

Solution in parametric form:

$$x = aC_1^{11}\int (f \pm 2\tau\wp^2)^{-3/2}(\tau f + 2\wp)^{-1/2}\,d\tau + C_3, \quad y = bC_1^{7}\wp(f \pm 2\tau\wp^2)^{-1/2},$$

where $A = -10a^2b^{-4}\Big(\dfrac{2a^2}{3b}\Big)^{2/5}.$

143. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-7/3}.$

Solution in parametric form:

$$x = aC_1^{23}\int (f \pm 2\tau\wp^2)^{1/4}(\tau f + 2\wp)\,d\tau + C_3, \quad y = bC_1^{32}(\tau f + 2\wp)^2,$$

where $A = -648a^{-5}b^{7/2}(6b/a)^{1/3}.$

144. $y'''_{xxx} = Ay^{-5/3}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1\int \wp(f \pm 2\tau\wp^2)^{1/2}\,d\tau + C_3, \quad y = bC_1^{9}(f \pm 2\tau\wp^2)^{3/2},$$

where $A = \pm\frac{1}{324}a^3b^{-1/3}.$

145. $y'''_{xxx} = Ay^{-5/6}(y'_x)^{-7/3}.$

Solution in parametric form:

$$x = aC_1^{25}\int (f \pm 2\tau\wp^2)^{1/4}(\tau f + 2\wp)^{-2}\,d\tau + C_3, \quad y = bC_1^{32}(\tau f + 2\wp)^{-2},$$

where $A = -648a^{-5}b^{23/6}(6b/a)^{1/3}.$

146. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-2/3}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^{-1}\int (\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{1/2}(\tau f + 2\wp)^{-2}\,d\tau + C_3,$$
$$y = bC_1^{7}(f \pm 2\tau\wp^2)^{3/2}(\tau f + 2\wp)^{-2},$$

where $A = -\frac{1}{324}a^3b^{-1/3}(6b/a)^{2/3}.$

147. $y'''_{xxx} = Ay(y'_x)^{11}(y''_{xx})^{7/5}$.

Solution in parametric form:

$$x = aC_1^{31}\int (f \pm 2\tau\wp^2)^{-3/2}(\tau f + 2\wp)^{13/6}\,d\tau + C_3, \quad y = bC_1^{27}(\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{-1/2},$$

where $A = -20a^{10}b^{-12}(2a^2/b)^{2/5}$.

148. $y'''_{xxx} = Ay^5(y'_x)^3(y''_{xx})^{8/5}$.

Solution in parametric form:

$$x = aC_1^{43}\int (\tau^2\wp \mp 1)^{-1/2}(f \pm 2\tau\wp^2)^{5/4}(\tau f + 2\wp)^{-4/3}\,d\tau + C_3, \quad y = bC_1^{16}(\tau f + 2\wp)^{-1/3},$$

where $A = 20a^2b^{-8}(2a^2/b)^{3/5}$.

149. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{4/5}$.

Solution in parametric form:

$$x = aC_1^{47}\int (\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{-3/2}(\tau f + 2\wp)^{4/3}\,d\tau + C_3,$$
$$y = bC_1^{54}(\tau^2\wp \mp 1)^2(f \pm 2\tau\wp^2)^{-1},$$

where $A = -320a^{-7}b^{11/2}\left(\frac{a^2}{4b}\right)^{4/5}$.

150. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{11/5}$.

Solution in parametric form:

$$x = aC_1^{-13}\int (f \pm 2\tau\wp^2)^{-3/2}(\tau f + 2\wp)^{1/3}\,d\tau + C_3, \quad y = bC_1^{14}(f \pm 2\tau\wp^2)^{-1}(\tau f + 2\wp)^{4/3},$$

where $A = \frac{5}{4}ab^{3/2}\left(\frac{a^2}{4b}\right)^{1/5}$.

151. $y'''_{xxx} = Ay^{-4/5}(y'_x)^{-17/5}(y''_{xx})^{1/2}$.

Solution in parametric form:

$$x = C_3 - aC_1^{47}\int (\tau\wp' + 2\wp)^{7/6}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{-3/2}\,d\tau,$$
$$y = bC_1^{54}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{-1},$$

where $A = -1250\cdot 5^{2/5}a^{-27/5}b^{47/10}$.

152. $y'''_{xxx} = Ay^{-11/5}(y'_x)^{-3/5}(y''_{xx})^{5/2}$.

Solution in parametric form:

$$x = C_3 - aC_1^{13}\int \frac{(\tau^2\wp - 1)^2(\tau\wp' + 2\wp)^{2/3}}{(\tau^3 - 2\tau^2\wp\wp' - \wp' - 8\tau\wp^2)^{3/2}}\,d\tau,$$
$$y = -bC_1^{-14}(\tau\wp' + 2\wp)^{5/3}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{-1},$$

where $A = \frac{2\cdot 10^{3/5}}{125}a^{7/5}b^{13/10}$.

153. $y'''_{xxx} = Ay^2(y'_x)^9(y''_{xx})^{17/11}$.

Solution in parametric form:

$$x = C_3 - aC_1^{127} \int \frac{(\tau^3 - 2\tau^2\wp\wp' - \wp' - 8\tau\wp^2)^{11/10}}{(\tau\wp' + 2\wp)^{5/3}}\, d\tau,$$

$$y = bC_1^{100}(\tau^2\wp - 1)(\tau\wp' + 2\wp)^{-2/3},$$

where $A = -11 \cdot 2^{17/11} a^{100/11} b^{-127/11}$.

154. $y'''_{xxx} = Ay^4(y'_x)^5(y''_{xx})^{16/11}$.

Solution in parametric form:

$$x = C_3 - aC_1^{52} \int \frac{(\tau\wp' + 2\wp)^{7/3}}{(\tau^2\wp - 1)^{1/2}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{6/5}}\, d\tau,$$

$$y = bC_1^{27}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{-1/5},$$

where $A = 11 \cdot 2^{-16/11} a^{54/11} b^{-104/11}$.

155. $y'''_{xxx} = Ay^{-11/6}(y'_x)^{-1/3}(y''_{xx})^{9/4}$.

Solution in parametric form:

$$x = C_3 + aC_1^{-33} \int \frac{(\tau^2\wp - 1)^{1/2}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{1/5}}{(\tau\wp' + 2\wp)^2}\, d\tau,$$

$$y = bC_1^{42}(\tau\wp' + 2\wp)^{-2}(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{6/5},$$

where $A = 2^{17/12} \cdot 3^{-13/6} \cdot a^{7/6} b^{11/12}$.

156. $y'''_{xxx} = Ay^{-2/3}(y'_x)^{-8/3}(y''_{xx})^{3/4}$.

Solution in parametric form:

$$x = C_3 + aC_1^{129} \int \frac{(\tau^2\wp - 1)^2(2\tau^2\wp\wp' + \wp' + 8\tau\wp^2 - \tau^3)^{2/5}}{(\tau\wp' + 2\wp)^2}\, d\tau,$$

$$y = bC_1^{150}(\tau^2\wp - 1)^3(\tau\wp' + 2\wp)^{-2},$$

where $A = -108 \cdot 2^{1/4} \cdot 3^{1/6} \cdot a^{-25/6} b^{43/12}$.

◆ *In the solutions of equations 157 and 158, the following notation is used:*

$$U = \int \frac{\tau^{k-1}\, d\tau}{z(\tau)}, \qquad z = \begin{cases} \dfrac{1}{k+1}\tau^{k+1} + \dfrac{1}{k}\tau^k + C_2 & \textit{if } k \neq 0,\ k \neq -1; \\ \tau + \ln|\tau| + C_2 & \textit{if } k = 0; \\ \ln|\tau| - \dfrac{1}{\tau} + C_2 & \textit{if } k = -1. \end{cases}$$

157. $y'''_{xxx} = Ay^{\beta}(y'_x)^{-1}(y''_{xx})^2, \qquad \beta \neq 0.$

Solution in parametric form:

$$x = C_1 \int \tau^{\frac{1-\beta}{\beta}} \exp(-\tfrac{1}{2}U)\, d\tau + C_3, \quad y = b\tau^{1/\beta}, \quad \text{where} \quad k = 1/\beta,\ \ A = -2b^{-\beta}.$$

158. $y'''_{xxx} = Ay^{-1}(y'_x)^\gamma y''_{xx}, \qquad \gamma \neq 1.$

Solution in parametric form:

$$x = aC_1 \int \tau^{\frac{2-\gamma}{\gamma-1}} z^{-1} e^U \, d\tau + C_3, \quad y = e^U, \qquad \text{where} \quad k = \frac{2}{\gamma - 1}, \quad A = a^{\gamma-1} b^{1-\gamma}.$$

◆ *In the solutions of equations 159–188, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad I_1 = 2\tau I \mp R, \quad I_2 = \tau^{-1}(RI_1 - 1),$$
$$I_3 = 4\tau I_1^2 \mp I_2^2, \quad I_4 = I_2 I_3 - 8I_1^2, \quad I_5 = 2RI - \tau^2,$$

where $I = \int \frac{\tau \, d\tau}{R} + C_2$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

159. $y'''_{xxx} = A(y'_x)^{-7}.$

Solution in parametric form:

$$x = aC_1^4 \int \tau^{-3/2} R^{-1} \, d\tau + C_3, \quad y = bC_1^5 \tau^{-1} I_1, \qquad \text{where} \quad A = \mp 3a^{-10} b^8.$$

160. $y'''_{xxx} = Ay^{-4} y'_x (y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^{-1} \int \tau^{-3/2} I_1^{-1/2} \, d\tau + C_3, \quad y = bC_1^4 \tau^{-1}, \qquad \text{where} \quad A = \pm 24a^4 b.$$

161. $y'''_{xxx} = Ayy'_x (y''_{xx})^{7/4}.$

Solution in parametric form:

$$x = aC_1^{11} \int \tau^{5/2} I_1^{-1/2} R^{-1} \, d\tau + C_3, \quad y = bC_1^6 R, \qquad \text{where} \quad A = \mp \frac{2}{3} b^{-2} \Big(\mp \frac{a^2}{3b}\Big)^{3/4}.$$

162. $y'''_{xxx} = A(y'_x)^3 (y''_{xx})^{5/4}.$

Solution in parametric form:

$$x = aC_1^{13} \int \tau^{-2} R^{-3/2} \, d\tau + C_3, \quad y = bC_1^{10} \tau^{-1} I_1, \qquad \text{where} \quad A = -4a^2 b^{-3} \Big(\mp \frac{a^2}{3b}\Big)^{1/4}.$$

163. $y'''_{xxx} = Ay(y'_x)^{-7}.$

Solution in parametric form:

$$x = aC_1^7 \int I_1^{-3/2} R^{-1} \, d\tau + C_3, \quad y = bC_1^{10} \tau I_1^{-1}, \qquad \text{where} \quad A = \mp 3a^{-10} b^7.$$

164. $y'''_{xxx} = Ay^{-4} (y'_x)^3 (y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1 \int \tau^{-1/2} I_1^{-3/2} I_2 R^{-1} \, d\tau + C_3, \quad y = bC_1^6 I_1^{-1}, \qquad \text{where} \quad A = \pm 24a^6 b^{-1}.$$

165. $y'''_{xxx} = Ay(y''_{xx})^{7/4}$.

Solution in parametric form:

$$x = aC_1^7 \int I_1^2 R^{-1}\,d\tau + C_3, \quad y = bC_1^2 I_2, \qquad \text{where} \quad A = -4a^{-1}b^{-1}\Big(\pm\frac{a^2}{6b}\Big)^{3/4}.$$

166. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{5/4}$.

Solution in parametric form:

$$x = aC_1^{11} \int \tau I_1^{-3} I_2^{-1/2} R^{-1}\,d\tau + C_3, \quad y = bC_1^{10}\tau^2 I_1^{-2}, \quad \text{where } A = \frac{1}{2}a^2 b^{-5/2}\Big(\pm\frac{a^2}{6b}\Big)^{5/4}.$$

167. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-5/3}(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^{-1} \int \tau I_1^{-1/2} I_2 R^{-1}\,d\tau + C_3, \quad y = bC_1^8 I_2^2, \qquad \text{where } A = \mp\tfrac{1}{27}a^2 b^{-1/2}(12b/a)^{2/3}.$$

168. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4/3}(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1 \int I_1^{-5/2} I_2 I_3 R^{-1}\,d\tau + C_3, \quad y = bC_1^{10} I_1^{-3} I_2^2, \quad \text{with } A = \mp\tfrac{16}{27}a^2 b^{-1/2}(\pm 3b/a)^{1/3}.$$

169. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{8/7}$.

Solution in parametric form:

$$x = aC_1^{37} \int I_1^{7/4} I_2^{-1/2} I_3 R^{-1}\,d\tau + C_3, \quad y = bC_1^{32} I_3^2,$$

where $A = \mp 7 \times 2^{-10} a^2 b^{-5/2}\Big(\frac{a^2}{6b}\Big)^{1/7}$.

170. $y'''_{xxx} = Ay(y''_{xx})^{13/7}$.

Solution in parametric form:

$$x = aC_1^{13} \int I_1^{-5/2} R^{-1}\,d\tau + C_3, \quad y = bC_1^5 I_1^{-3/2} I_2, \qquad \text{where} \quad A = \frac{7}{2}a^{-1}b^{-1}\Big(\frac{a^2}{6b}\Big)^{6/7}.$$

171. $y'''_{xxx} = Ay(y'_x)^{-13}$.

Solution in parametric form:

$$x = aC_1^{13} \int I_1^{3/4} R^{-1}\,d\tau + C_3, \quad y = bC_1^{16} I_3, \qquad \text{where} \quad A = \pm 3 \times 2^{25} a^{-16} b^{13}.$$

172. $y'''_{xxx} = Ay^{-7}(y'_x)^3(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^{-1} \int I_1^{-1/2} I_2 I_3^{-1/2} R^{-1}\,d\tau + C_3, \quad y = bC_1^3 I_1^{1/2}, \qquad \text{where} \quad A = \mp 12 a^6 b^2.$$

173. $y'''_{xxx} = Ay^3(y'_x)^{-13}$.

Solution in parametric form:

$$x = aC_1^{11}\int I_1^{3/4} I_3^{-3/2} R^{-1}\,d\tau + C_3, \quad y = bC_1^{16} I_3^{-1}, \qquad \text{where} \quad A = \pm 3\times 2^{25} a^{-16} b^{11}.$$

174. $y'''_{xxx} = Ay^{-7}(y'_x)^7(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1\int I_1^{-1/2} I_3^{-3/2} I_4 R^{-1}\,d\tau + C_3, \quad y = bC_1^5 I_1^{1/2} I_3^{-1}, \qquad \text{where} \quad A = \mp 192 a^{10} b^{-2}.$$

175. $y'''_{xxx} = Ay^{-7/6}(y''_{xx})^{2/3}$.

Solution in parametric form:

$$x = aC_1^9\int I_1^{-5/2} I_3^5 R^{-1}\,d\tau + C_3, \quad y = bC_1^{10} I_1^{-3} I_3^6, \quad \text{where} \quad A = \pm 54 a^{-3} b^{13/6}\Bigl(\frac{2a^2}{9b}\Bigr)^{2/3}.$$

176. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4/3}(y''_{xx})^{7/3}$.

Solution in parametric form:

$$x = aC_1^3\int I_1^{-5/2} I_3^{-1} I_4 R^{-1}\,d\tau + C_3, \quad y = bC_1^2 I_1^{-3} I_4^2, \qquad \text{where} \quad A = 8b^{1/2}(2a/3)^{1/3}.$$

177. $y'''_{xxx} = Ay^{-3/4}(y'_x)^3(y''_{xx})^{8/7}$.

Solution in parametric form:

$$x = aC_1^{67}\int I_1^{7/4} I_3^{-5} I_4^{-1/2} R^{-1}\,d\tau + C_3, \quad y = bC_1^{64} I_3^{-4},$$

where $A = \mp 7\times 2^{-13} a^2 b^{-9/4}\Bigl(\dfrac{a^2}{12b}\Bigr)^{1/7}$.

178. $y'''_{xxx} = Ay(y'_x)^{-1/2}(y''_{xx})^{13/7}$.

Solution in parametric form:

$$x = aC_1^{19}\int I_1^{-5/2} I_3^4 R^{-1}\,d\tau + C_3, \quad y = bC_1^3 I_1^{-3/2} I_4,$$

where $A = \dfrac{7}{2} a^{-1} b^{-1}\Bigl(\pm\dfrac{3b}{a}\Bigr)^{1/2}\Bigl(\dfrac{a^2}{12b}\Bigr)^{6/7}$.

179. $y'''_{xxx} = A(y'_x)^2$.

Solution in parametric form:

$$x = aC_1^{-1}\int R^{-1}\,d\tau + C_3, \quad y = bC_1\int \tau R^{-1}\,d\tau + C_2, \qquad \text{where} \quad A = \pm 6a^{-1}b^{-1}.$$

180. $y'''_{xxx} = Ay^{1/2}y'_x(y''_{xx})^3$.

Solution in parametric form:

$$x = aC_1^7\int \tau I^{-1/2}\,d\tau + C_3, \quad y = bC_1^8\tau^2, \qquad \text{where} \quad A = \mp 24 a^4 b^{-7/2}.$$

181. $y'''_{xxx} = Ayy'_x(y''_{xx})^4.$

Solution in parametric form:

$$x = aC_1^5 \int \tau^2 I^{-1/2} R^{-1}\, d\tau + C_3, \quad y = bC_1^6 R, \qquad \text{where} \quad A = -\tfrac{1}{162} a^6 b^{-5}.$$

182. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{-1}.$

Solution in parametric form:

$$x = aC_1^{-1} \int \tau R^{-3/2}\, d\tau + C_3, \quad y = bC_1^2 \int \tau R^{-1}\, d\tau + C_2, \qquad \text{where} \quad A = 9a^{-2}b^{-1}.$$

183. $y'''_{xxx} = Ay^{-7/2}(y'_x)^2.$

Solution in parametric form:

$$x = aC_1^5 \int I^{-3/2} R^{-1}\, d\tau + C_3, \quad y = bC_1^2 I^{-1}, \qquad \text{where} \quad A = \mp 6a^{-1}b^{5/2}.$$

184. $y'''_{xxx} = Ay^{1/2}(y'_x)^{-6}(y''_{xx})^3.$

Solution in parametric form:

$$x = aC_1^7 \int \tau I^{-3/2} I_5 R^{-1}\, d\tau + C_3, \quad y = bC_1^6 \tau^2 I^{-1}, \qquad \text{where} \quad A = \mp 48a^{-3}b^{7/2}.$$

185. $y'''_{xxx} = Ay(y'_x)^{-9/5}(y''_{xx})^4.$

Solution in parametric form:

$$x = aC_1^{11} \int \tau^2 I^{-1/4} R^{-1}\, d\tau + C_3, \quad y = bC_1^{16} I_5, \qquad \text{where} \quad A = -\tfrac{2}{1125} a^4 b^{-3} (12b/a)^{4/5}.$$

186. $y'''_{xxx} = Ay^{-7/5}(y'_x)^3(y''_{xx})^{-1}.$

Solution in parametric form:

$$x = aC_1 \int \tau I^{3/2} I_5^{-1/2} R^{-1}\, d\tau + C_3, \quad y = bC_1 I^{5/2}, \qquad \text{where} \quad A = \tfrac{36}{5} a^{-2} b^{2/5}.$$

187. $y'''_{xxx} = Ay^{-1/3}(y''_{xx})^{9/7}.$

Solution in parametric form:

$$x = aC_1 \int \tau^2 I^{-5/2} R^{-1}\, d\tau + C_3, \quad y = bC_1^9 \tau^3 I^{-3/2}, \qquad \text{where} \quad A = \frac{7}{2} a^{-1} b^{1/3} \left(\frac{a^2}{18b}\right)^{2/7}.$$

188. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{1/3}(y''_{xx})^{12/7}.$

Solution in parametric form:

$$x = aC_1^{23} \int \tau^{1/2} I^{7/4} I_5 R^{-1}\, d\tau + C_3, \qquad y = bC_1^{32} I_5^2,$$

where $A = -\dfrac{7}{4} a^{-1} b^{1/2} \left(\dfrac{a}{3b}\right)^{1/3} \left(\dfrac{a^2}{18b}\right)^{5/7}.$

◆ *In the solutions of equations 189 and 190, the following notation is used:*

$$f = \exp\left(\int \frac{d\tau}{\sqrt{z}}\right), \qquad z = \begin{cases} C_2 + \dfrac{1}{4}\tau^2 + \dfrac{2B}{k+1}\tau^{k+1} & \textit{if } k \neq -1, \\ C_2 + \dfrac{1}{4}\tau^2 + 2B\ln|\tau| & \textit{if } k = -1. \end{cases}$$

189. $y'''_{xxx} = Ay^{-\frac{\gamma+5}{4}}(y'_x)^\gamma.$

Solution in parametric form:

$$x = aC_1^3 \int \tau^{-1/2} f^{3/4} z^{-1/2}\, d\tau + C_3, \quad y = bC_1^4 f,$$

where $k = \frac{1}{2}(\gamma - 1)$, $A = \frac{1}{2}Ba^{2(k-1)}b^{\frac{3}{2}(1-k)}$.

190. $y'''_{xxx} = Ay^{-\gamma-2}(y'_x)^\gamma (y''_{xx})^3.$

Solution in parametric form:

$$x = a\int (\tau z^{-1/2} + 2)\, d\tau + C_3, \quad y = C_1 \tau f^{1/2}, \quad \text{where} \quad k = -\gamma - 2, \ A = -2^{3-k}a^{1-k}B.$$

191. $y'''_{xxx} = Ay^{\frac{\gamma-1}{2}}(y'_x)^\gamma (y''_{xx})^{3/2}.$

Solution in parametric form:

$$x = aC_1^3 \int \frac{V - \sqrt{V^2+4}}{(\tau U)^{3/4}\sqrt{V^2+4}}\, d\tau + C_3, \quad y = bC_1^2 \tau^{1/2} U^{1/2},$$

where

$$U = \exp\left(\int \frac{V\, d\tau}{\tau\sqrt{V^2+4}}\right), \quad V = \begin{cases} \tau^{-1/2}\left(C_2 + \dfrac{B}{\gamma+1}\tau^{\frac{\gamma+1}{2}}\right) & \text{if } \gamma \neq -1, \\ \tau^{-1/2}(C_2 + \frac{1}{2}B\ln|\tau|) & \text{if } \gamma = -1, \end{cases}$$

$$A = 2^{3/2}ab^{-\frac{\gamma}{2}-1}B(-2a/b)^{\gamma-1}.$$

192. $y'''_{xxx} = Ay^\beta (y'_x)^\gamma (y''_{xx})^{\frac{\gamma+4\beta+5}{\gamma+2\beta+3}}, \qquad \beta \neq -1, \ \gamma \neq -1.$

Solution in parametric form:

$$x = aC_1^{\gamma+\beta+2} \int \tau^{-3/2} U^{-\frac{\gamma+4\beta+5}{2(\gamma+1)}} z^{-1}\, d\tau + C_3, \quad y = bC_1^{\gamma+1} U,$$

where $A = a^{\gamma-1}b^{-\beta-\gamma}\left(\frac{2a^2}{b}\right)^{\frac{2(\beta+1)}{\gamma+2\beta+3}}B$, $U = \exp\left(\int \frac{d\tau}{\tau z}\right)$; $z = z(\tau)$ is the solution of the transcendental equation

$$(z + k - 1)(z + k)^{\frac{k}{1-k}} = \left(C_2 + \frac{2B}{\gamma + 2\beta + 3}\tau^{\frac{\gamma+1}{2}}\right)\tau^{\frac{1}{k-1}}, \qquad k = -\frac{2(\beta+1)}{\gamma+1}.$$

193. $y'''_{xxx} = Ay^{-1}(y'_x)^{-1}(y''_{xx})^\delta, \qquad \delta \neq 1, \ \delta \neq 2.$

Solution in parametric form:

$$x = aC_1^{\delta-3} \int \tau^{k-1} U^{\frac{2-k}{2k}} z^{-1}\, d\tau + C_3, \quad y = bC_1^{2\delta-4} k(kz - \tau)^{-1} U^{1/k},$$

where $k=\dfrac{\delta-1}{\delta-2}$, $A=\dfrac{1-k}{2}a^{-4}b^3\Big(-\dfrac{2a^2}{b}\Big)^{\delta}B$, $U=\exp\Big(\displaystyle\int\frac{d\tau}{z}\Big)$; $z=z(\tau)$ is the solution of the transcendental equation

$$\ln|kz-\tau|-\frac{\tau}{kz-\tau}=\frac{1}{k}\tau^k+C_2.$$

194. $\boldsymbol{y'''_{xxx}=Ay^{-1}(y'_x)^{-1}(y''_{xx})^2}$.

Solution in parametric form:

$$x=\pm C_1\int e^{\tau}z^{-1/2}U^{-1/2}\,d\tau+C_3,\qquad y=\pm\tfrac{1}{2}e^{\tau},$$

where $z=\mp A\tau+e^{\tau}+C_2$, $U=\exp\Big(\pm A\displaystyle\int\frac{d\tau}{z}\Big)$.

195. $\boldsymbol{y'''_{xxx}=Ay^{-1}(y'_x)^{-1}y''_{xx}}$.

Solution in parametric form:

$$x=C_1\int e^{\tau/2}U\,d\tau+C_3,\quad y=\pm C_1zU,\quad \text{where}\ z=\pm A\tau+e^{\tau}+C_2,\ U=\exp\Big(\mp\int\frac{d\tau}{z}\Big).$$

15.2.5 Some Transformations

Let us consider some transformations of the equation

$$y'''_{xxx}=Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}.$$

1°. In the special case $\gamma=\delta=0$, the transformation $x=1/t$, $y=w/t^2$ reduces the equation

$$y'''_{xxx}=Ax^{\alpha}y^{\beta}$$

to an equation of similar form (with other parameters):

$$w'''_{ttt}=-At^{-\alpha-2\beta-4}w^{\beta}.$$

2°. In the special case $\alpha=\delta=0$, the transformation $x=-\displaystyle\int\frac{d\tau}{[z(\tau)]^{3/2}}$, $y=\dfrac{1}{z(\tau)}$ reduces the equation

$$y'''_{xxx}=Ay^{\beta}(y'_x)^{\gamma}$$

to an equation of similar form (with other parameters):

$$z'''_{\tau\tau\tau}=Az^{-\frac{2\beta+\gamma+5}{2}}(w'_{\tau})^{\gamma}.$$

3°. In the special case $\beta=0$, the substitution $u(x)=y'_x$ brings the equation

$$y'''_{xxx}=Ax^{\alpha}(y'_x)^{\gamma}(y''_{xx})^{\delta}$$

to the generalized Emden–Fowler equation:

$$u''_{xx}=Ax^{\alpha}u^{\gamma}(u'_x)^{\delta},$$

which is discussed in Sections 14.3 and 14.5.

4°. In the special case $\alpha = 0$, the substitution $v(y) = (y'_x)^2$ reduces the equation

$$y'''_{xxx} = Ay^\beta (y'_x)^\gamma (y''_{xx})^\delta$$

to the generalized Emden–Fowler equation:

$$v''_{yy} = A \times 2^{1-\delta} y^\beta v^{\frac{\gamma-1}{2}} (v'_y)^\delta,$$

which is discussed in Sections 14.3 and 14.5.

15.3 Equations of the Form $y'''_{xxx} = f(y)g(y'_x)h(y''_{xx})$

15.3.1 Equations Containing Power Functions

1. $y'''_{xxx} = (ay+b)^{-5/2}$.

This is a special case of equation 15.3.1.2 with $b^2 - 4ac = 0$. Three autonomous first integrals two of which are functionally independent:

$$(ay+b)^2(y''_{xx})^2 - a(ay+b)(y'_x)^2 y''_{xx} + \frac{1}{4}a^2(y'_x)^4 - 2(ay+b)^{-1/2} y'_x = C_1,$$

$$\begin{aligned}\frac{(ay+b)^2 y}{b^2}(y''_{xx})^3 - \frac{(3ay+b)(ay+b)}{2b^2}(y'_x)^2(y''_{xx})^2 + \\ + \left[\frac{(3ay+2b)a}{4b^2}(y'_x)^3 - \frac{3y}{(ay+b)^{1/2}b^2}\right] y'_x y''_{xx} - \frac{a^2}{8b^2}(y'_x)^6 + \\ + \frac{9a^3y^3 + 26a^2by^2 + 25ab^2y + 8b^3}{6(ay+b)^{7/2}b^2}(y'_x)^3 - \frac{3}{2}\frac{2aby + b^2}{a^2b^3(ay+b)^2} = C_2,\end{aligned}$$

$$\begin{aligned}\frac{(2ay-b)(ay+b)^2}{b^3}(y''_{xx})^3 - \frac{3a^2y(ay+b)}{b^3}(y'_x)^2(y''_{xx})^2 + \\ + \frac{3}{b^3}\left[\frac{a^2(2ay+b)}{4}(y'_x)^3 + \frac{2ay-b}{(ay+b)^{1/2}}\right] y'_x y''_{xx} + \frac{a^3}{4b^3}(y'_x)^6 + \\ + \frac{a(6a^3y^3 + 17a^2by^2 + 16ab^2y + 5b^3)}{2b^3(ay+b)^{7/2}}(y'_x)^3 - \frac{3}{2}\frac{4ay+b}{ab^3(ay+b)^2} = C_3.\end{aligned}$$

2. $y'''_{xxx} = (ay^2+by+c)^{-5/4}$.

This is a special case of equation 15.5.2.29 with $f(w) = 1$. Autonomous first integral:

$$(ay^2+by+c)\big(y''_{xx}\big)^2 - \frac{1}{2}(2ay+b)\big(y'_x\big)^2 y''_{xx} + \frac{a}{4}\big(y'_x\big)^4 - 2(ay^2+by+c)^{-1/4} y'_x = C.$$

3. $y'''_{xxx} = (Ay^n + By^m)y'_x$.

This is a special case of equation 15.5.2.1 with $f(y) = Ay^n + By^m$.

4. $y'''_{xxx} = (Ay^n + By^m)\big[a(y'_x)^3 + by'_x\big]$.

This is a special case of equation 15.5.2.3 in which $f(y) = b(Ay^n + By^m)$ and $g(y) = a(Ay^n + By^m)$.

5. $y'''_{xxx} = y^{-2}\Big[-\frac{(m+1)}{(m+3)^2}(y'_x)^3 + A(y'_x)^{2m+1}\Big], \qquad m \neq -3, \ m \neq -1.$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.4: $w''_{yy} = y^{-2}\Big[-\frac{2(m+1)}{(m+3)^2}w + 2Aw^m\Big]$.

6. $y'''_{xxx} = y^{-2}\big[\frac{15}{8}(y'_x)^3 + A(y'_x)^{-13}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.35: $w''_{yy} = y^{-2}(\frac{15}{4}w + 2Aw^{-7})$.

7. $y'''_{xxx} = y^{-2}\big[3(y'_x)^3 + A(y'_x)^{-7}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.31: $w''_{yy} = y^{-2}(6w + 2Aw^{-4})$.

8. $y'''_{xxx} = y^{-2}\big[6(y'_x)^3 + A(y'_x)^{-4}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.64: $w''_{yy} = y^{-2}(12w + 2Aw^{-5/2})$.

9. $y'''_{xxx} = y^{-2}\big[(y'_x)^3 + A(y'_x)^{-3}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.6: $w''_{yy} = y^{-2}(2w + 2Aw^{-2})$.

10. $y'''_{xxx} = y^{-2}\big[-\frac{3}{32}(y'_x)^3 + A(y'_x)^{-7/3}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.26: $w''_{yy} = y^{-2}(-\frac{3}{16}w + 2Aw^{-5/3})$.

11. $y'''_{xxx} = y^{-2}\big[-\frac{9}{200}(y'_x)^3 + A(y'_x)^{-7/3}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.10: $w''_{yy} = y^{-2}(-\frac{9}{100}w + 2Aw^{-5/3})$.

12. $y'''_{xxx} = y^{-2}\big[\frac{3}{8}(y'_x)^3 + A(y'_x)^{-7/3}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.12: $w''_{yy} = y^{-2}(\frac{3}{4}w + 2Aw^{-5/3})$.

13. $y'''_{xxx} = y^{-2}\big[\frac{63}{8}(y'_x)^3 + A(y'_x)^{-7/3}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.66: $w''_{yy} = y^{-2}(\frac{63}{4}w + 2Aw^{-5/3})$.

14. $y'''_{xxx} = y^{-2}\big[-\frac{5}{72}(y'_x)^3 + A(y'_x)^{-9/5}\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.29: $w''_{yy} = y^{-2}(-\frac{5}{36}w + 2Aw^{-7/5})$.

15. $y'''_{xxx} = y^{-2}\big[-\frac{1}{9}(y'_x)^3 + A\big].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.14: $w''_{yy} = y^{-2}(-\frac{2}{9}w + 2Aw^{-1/2})$.

16. $y'''_{xxx} = y^{-2}\bigl[-\frac{2}{25}(y'_x)^3 + A\bigr].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.8: $w''_{yy} = y^{-2}(-\frac{4}{25}w + 2Aw^{-1/2})$.

17. $y'''_{xxx} = y^{-2}\bigl[10(y'_x)^3 + A\bigr].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.33: $w''_{yy} = y^{-2}(20w + 2Aw^{-1/2})$.

18. $y'''_{xxx} = y^{-2}\bigl[-\frac{6}{49}(y'_x)^3 + A(y'_x)^2\bigr].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.37: $w''_{yy} = y^{-2}(-\frac{12}{49}w + 2Aw^{1/2})$.

19. $y'''_{xxx} = y^{-2}\bigl[A(y'_x)^5 - \frac{3}{25}(y'_x)^3\bigr].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.60: $w''_{yy} = y^{-2}(2Aw^2 - \frac{6}{25}w)$.

20. $y'''_{xxx} = y^{-2}\bigl[A(y'_x)^5 + \frac{3}{25}(y'_x)^3\bigr].$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.62: $w''_{yy} = y^{-2}(2Aw^2 + \frac{6}{25}w)$.

21. $y'''_{xxx} = y^{-4/3}(Ay'_x + B).$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.40: $w''_{yy} = y^{-4/3}(2A + 2Bw^{-1/2})$.

22. $y'''_{xxx} = (Ay^4 + By^3)(y'_x)^{-13}.$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.39: $w''_{yy} = (2Ay^4 + 2By^3)w^{-7}$.

23. $y'''_{xxx} = (Ay^2 + B)(y'_x)^{-9}.$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.16: $w''_{yy} = (2Ay^2 + 2B)w^{-5}$.

24. $y'''_{xxx} = (Ay^{-1} + By^{-2})(y'_x)^{-3}.$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.28: $w''_{yy} = (2Ay^{-1} + 2By^{-2})w^{-2}$.

25. $y'''_{xxx} = (Ay^{-7/3} + By^{-10/3})(y'_x)^{-7/3}.$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.48: $w''_{yy} = (2Ay^{-7/3} + 2By^{-10/3})w^{-5/3}$.

26. $y'''_{xxx} = (Ay^{-4/3} + By^{-10/3})(y'_x)^{-7/3}.$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.49: $w''_{yy} = (2Ay^{-4/3} + 2By^{-10/3})w^{-5/3}$.

27. $y'''_{xxx} = (Ay^{-4/3} + By^{-7/3})(y'_x)^{-7/3}$.

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.24: $w''_{yy} = (2Ay^{-4/3} + 2By^{-7/3})w^{-5/3}$.

28. $y'''_{xxx} = (Ay^{-2/3} + By^{-4/3})(y'_x)^{-7/3}$.

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.90: $w''_{yy} = (2Ay^{-2/3} + 2By^{-4/3})w^{-5/3}$.

29. $y'''_{xxx} = (A + By^{-2/3})(y'_x)^{-7/3}$.

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.89: $w''_{yy} = (2A + 2By^{-2/3})w^{-5/3}$.

30. $y'''_{xxx} = (Ay^2 + B)(y'_x)^{-7/3}$.

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.47: $w''_{yy} = (2Ay^2 + 2B)w^{-5/3}$.

31. $y'''_{xxx} = (Ay^2 + By)(y'_x)^{-7/3}$.

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation of the form 14.4.2.46: $w''_{yy} = (2Ay^2 + 2By)w^{-5/3}$.

32. $y'''_{xxx} = (Ay^n + By^k)y'_x(y''_{xx})^m$.

This is a special case of equation 15.5.4.12 with $f(y) = Ay^n + By^k$.

15.3.2 Equations Containing Exponential Functions

Tables 15.2–15.4 present the equations whose solutions are given in this subsection.

◆ *In the solutions of equations 1–6, the following notation is used:*

$$E = \int \exp(\tau^2)\,d\tau + C_2, \quad F = 2\tau E - \exp(\tau^2),$$
$$G = \int (1 \pm \tau)^k \frac{d\tau}{\tau} + C_2, \quad H = \int \exp(\mp\tau)\frac{d\tau}{\tau} + C_2.$$

1. $y'''_{xxx} = Ae^y y'_x (y''_{xx})^\delta, \qquad \delta \neq 1.$

Solution in parametric form:

$$x = aC_1 \int \tau^{-1} G^{-1/2}\,d\tau + C_3, \qquad y = \ln(bC_1^{2\delta-2}\tau),$$

where $k = \dfrac{1}{1-\delta}$, $A = \pm\dfrac{1}{2(1-\delta)}a^{-2}b^{-1}(2a^2)^\delta$.

2. $y'''_{xxx} = Ae^y y'_x y''_{xx}$.

Solution in parametric form:

$$x = C_1 \int \tau^{-1} H^{-1/2}\,d\tau + C_3, \qquad y = \ln\Bigl(\mp\frac{\tau}{A}\Bigr).$$

TABLE 15.2
Solvable equations of the form
$y'''_{xxx} = Ae^y(y'_x)^\gamma(y''_{xx})^\delta$

δ	γ	Equation
arbitrary ($\delta \neq 1$)	1	15.3.2.1
0	3	15.3.2.9
1	1	15.3.2.2
$\frac{3}{2}$	0	15.3.2.3
$\frac{3}{2}$	3	15.3.2.7
2	arbitrary ($\gamma \neq -1$)	15.3.2.11
2	-1	15.3.2.13

TABLE 15.3
Solvable equations of the form
$y'''_{xxx} = Ay^\beta y'_x \exp[(y'_x)^2](y''_{xx})^\delta$

δ	β	Equation
arbitrary ($\delta \neq 2$)	0	15.3.2.4
1	arbitrary ($\beta \neq -1$)	15.3.2.12
1	-1	15.3.2.14
$\frac{3}{2}$	$-\frac{1}{2}$	15.3.2.5
$\frac{3}{2}$	1	15.3.2.8
2	0	15.3.2.6
3	1	15.3.2.10

TABLE 15.4
Other solvable equations of the type considered

Form of equation	Equation
$y'''_{xxx} = Ae^y y'_x \exp[(y'_x)^2](y''_{xx})^\delta$	15.3.2.20
$y'''_{xxx} = A(y'_x)^\gamma \exp(y''_{xx}), \quad \gamma \neq -1$	15.3.2.15
$y'''_{xxx} = A(y'_x)^{-1} \exp(y''_{xx})$	15.3.2.16
$y'''_{xxx} = Ay^\beta y'_x \exp(y''_{xx}), \quad \beta \neq -1$	15.3.2.17
$y'''_{xxx} = Ay^{-1} y'_x \exp(y''_{xx})$	15.3.2.18
$y'''_{xxx} = Ae^y y'_x \exp[(y'_x)^2 + y''_{xx}]$	15.3.2.19

3. $\boldsymbol{y'''_{xxx} = Ae^y(y''_{xx})^{3/2}}$.

Solution in parametric form:

$$x = C_1 \int E^{-1}\,d\tau + C_3, \qquad y = \tau^2 + \ln\bigl(\sqrt{2}\,A^{-1}E^{-1}\bigr).$$

4. $\boldsymbol{y'''_{xxx} = Ay'_x \exp\left[(y'_x)^2\right](y''_{xx})^\delta, \qquad \delta \neq 2.}$

Solution in parametric form:

$$x = aC_1 \int \tau^{-1}(1 \pm \tau)^{\frac{1}{\delta-2}}\left[\ln(bC_1^{\delta-2}\tau)\right]^{-1/2} d\tau + C_3, \qquad y = aC_1 G,$$

where $k = \dfrac{1}{\delta - 2}$, $A = \pm\dfrac{1}{2-\delta}a^{\delta-3}b^{-1}$.

5. $y'''_{xxx} = Ay^{-1/2} y'_x \exp\left[(y'_x)^2\right](y''_{xx})^{3/2}.$

Solution in parametric form:

$$x = 4C_1 \int EF[\tau^2 + \ln(aE^{-1})]^{-1/2}\,d\tau + C_3, \quad y = C_1 F^2, \quad \text{where} \quad A = -\sqrt{2}\,a^{-1}.$$

6. $y'''_{xxx} = Ay'_x \exp\left[(y'_x)^2\right](y''_{xx})^2.$

Solution in parametric form:

$$x = C_1 \int \tau^{-1} \exp(\mp\tau)\Big[\ln\Big(\pm\frac{\tau}{A}\Big)\Big]^{-1/2} d\tau + C_3, \quad y = C_1 H.$$

◆ *In the solutions of equations 7 and 8, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\big[C_2\big(\sqrt{\tau} + \sqrt{\tau+1}\,\big)\big], \quad R = \sqrt{\frac{\tau+1}{\tau}},$$
$$F = 1 - \sqrt{\frac{\tau+1}{\tau}}\ln\big[C_2\big(\sqrt{\tau} + \sqrt{\tau+1}\,\big)\big].$$

7. $y'''_{xxx} = Ae^y (y'_x)^3 (y''_{xx})^{3/2}.$

Solution in parametric form:

$$x = aC_1 \int R^{-1}E^{-1}F^{-1/2}\,d\tau + C_3, \quad y = -\ln(bC_1^{-3}E), \quad \text{where } A = 2^{3/2}a^3 b.$$

8. $y'''_{xxx} = Ayy'_x \exp\left[(y'_x)^2\right](y''_{xx})^{3/2}.$

Solution in parametric form:

$$x = -\tfrac{1}{2}bC_1 \int \tau^{-2}R^{-1}E\,[\ln(aC_1^{-3/2}E^{-1})]^{-1/2} d\tau + C_3, \quad y = bC_1 F,$$

where $A = -4a^{-1}b^{-3/2}$.

◆ *In the solutions of equations 9 and 10, the following notation is used:*

$$Z = \begin{cases} C_1 J_0(\tau) + C_2 Y_0(\tau) & \textit{for the upper sign,} \\ C_1 I_0(\tau) + C_2 K_0(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_0(\tau)$ and $Y_0(\tau)$ are Bessel functions, and $I_0(\tau)$ and $K_0(\tau)$ are modified Bessel functions.

9. $y'''_{xxx} = Ae^y (y'_x)^3.$

Solution in parametric form:

$$x = 2C_1 \int \tau^{-1} Z^{-1/2}\,d\tau + C_3, \quad y = \ln(\mp\tfrac{1}{8}A^{-1}\tau^2).$$

10. $y'''_{xxx} = Ayy'_x \exp\left[(y'_x)^2\right](y''_{xx})^3.$

Solution in parametric form:

$$x = C_1 \int Z'_\tau \Big[\ln\Big(\pm\frac{\tau^2}{A}\Big)\Big]^{-1/2} d\tau + C_3, \quad y = C_1 Z.$$

11. $y'''_{xxx} = Ae^y(y'_x)^\gamma(y''_{xx})^2, \qquad \gamma \neq -1.$

Solution in parametric form:

$$x = a\int \tau^{-1/2} f^{-1} \exp\Big(\frac{U}{\gamma+1}\Big)\,d\tau + C_3, \qquad y = U,$$

where $A = \frac{1}{2}a^{\gamma+1}k$, $U = \displaystyle\int \frac{d\tau}{f} + C_1$; $f = f(\tau)$ is the solution of the transcendental equation

$$\ln(\lambda f - \tau) - \frac{\tau}{\lambda f - \tau} = \frac{k}{\lambda}\tau^\lambda + C_2, \qquad \lambda = \frac{\gamma+1}{2}.$$

12. $y'''_{xxx} = Ay^\beta y'_x \exp\big[(y'_x)^2\big]y''_{xx}, \qquad \beta \neq -1.$

Solution in parametric form:

$$x = a\int f^{-1}\Big(f - \frac{\tau}{\beta+1}\Big)U^{-1/2}\exp\Big(-\frac{U}{\beta+1}\Big)\,d\tau + C_3, \qquad y = a\tau\exp\Big(-\frac{U}{\beta+1}\Big),$$

where $A = a^{-\beta-1}k$, $U = \displaystyle\int \frac{d\tau}{f} + C_1$; $f = f(\tau)$ is the solution of the transcendental equation

$$\ln(\lambda f - \tau) - \frac{\tau}{\lambda f - \tau} = -\frac{k}{\lambda}\tau^\lambda + C_2, \qquad \lambda = \beta + 1.$$

13. $y'''_{xxx} = Ae^y(y'_x)^{-1}(y''_{xx})^2.$

Solution in parametric form:

$$x = C_1\int W^{-1/2}\,d\tau + C_3, \quad y = \tau, \qquad \text{where} \quad W = \exp\Big(\int \frac{d\tau}{\tau - 2Ae^\tau + C_2}\Big).$$

14. $y'''_{xxx} = Ay^{-1}y'_x \exp\big[(y'_x)^2\big]y''_{xx}.$

Solution in parametric form:

$$x = C_1\int \tau^{-1/2}(\tau + Ae^\tau + C_2)^{-1}W\,d\tau + C_3, \qquad y = C_1 W,$$

where $W = \exp\Big(\displaystyle\int \frac{d\tau}{\tau + Ae^\tau + C_2}\Big)$.

◆ *In the solutions of equations 15–19, the following notation is used:*

$$V = C_1 - \frac{1}{\lambda}(m+1)(\tau+1)e^{-\tau}, \qquad M = C_1 - \frac{1}{\lambda}(\tau+2)e^{-\tau/2},$$

$$W = \begin{cases} C_2 - \displaystyle\int \ln\Big(C_1 - \frac{\lambda}{n+1}\tau^{n+1}\Big)\,d\tau & \text{if } n \neq -1, \\ C_2 - \displaystyle\int \ln(C_1 - \lambda\ln|\tau|)\,d\tau & \text{if } n = -1, \end{cases}$$

$$N = \ln M - \frac{1}{2\lambda}\int e^{-\tau/2}M^{-1}\,d\tau - \frac{1}{2} + C_2.$$

15. $y'''_{xxx} = A(y'_x)^\gamma \exp(y''_{xx}), \qquad \gamma \neq -1.$

Solution in parametric form:

$$x = \frac{1}{\lambda}\int e^{-\tau}V^{-\frac{2m+1}{2m+2}}\,d\tau + C_3, \quad y = \frac{2}{\lambda}\int e^{-\tau}V^{-\frac{m}{m+1}}\,d\tau + C_2,$$

where $m = \frac{1}{2}(\gamma - 1)$, $A = 2^{-\gamma}\lambda$.

16. $y'''_{xxx} = A(y'_x)^{-1} \exp(y''_{xx})$.

Solution in parametric form:

$$x = \frac{1}{\lambda}\int \exp\bigl(-\tau + \tfrac{1}{2}V\bigr)\,d\tau + C_3, \quad y = \frac{2}{\lambda}\int \exp\bigl(-\tau + V\bigr)\,d\tau + C_2,$$

where $m = 0$, $A = 2\lambda$.

17. $y'''_{xxx} = Ay^{\beta} y'_x \exp(y''_{xx}), \qquad \beta \neq -1$.

Solution in parametric form:

$$x = \int W^{-1/2}\,d\tau + C_3, \quad y = 2\tau, \qquad \text{where} \quad n = \beta, \ \ A = 2^{-\beta-1}\lambda.$$

18. $y'''_{xxx} = Ay^{-1} y'_x \exp(y''_{xx})$.

Solution in parametric form:

$$x = \int W^{-1/2}\,d\tau + C_3, \quad y = 2\tau, \qquad \text{where} \quad n = -1, \ \ A = \lambda.$$

19. $y'''_{xxx} = Ae^{y} y'_x \exp\bigl[(y'_x)^2 + y''_{xx}\bigr]$.

Solution in parametric form:

$$x = \frac{1}{2\lambda}\int e^{-\tau/2} M^{-1} N^{-1/2}\,d\tau + C_3, \quad y = \frac{1}{2\lambda}\int e^{-\tau/2} M^{-1}\,d\tau + C_2, \quad \text{where} \quad A = \lambda.$$

20. $y'''_{xxx} = Ae^{y} y'_x \exp[(y'_x)^2](y''_{xx})^{\delta}$.

Solution in parametric form:

$$x = \int \tau^{-\delta} z^{-1}\Bigl(\ln\frac{z}{A} - U - C_1\Bigr)^{-1/2}\,d\tau + C_3, \qquad y = U,$$

where

$$U = \int \frac{d\tau}{z\tau^{\delta}}, \qquad z = \begin{cases} \dfrac{1}{2-\delta}\tau^{2-\delta} + \dfrac{1}{1-\delta}\tau^{1-\delta} + C_2 & \text{if } \ \delta \neq 2, \ \delta \neq 1; \\ \tau + \ln|\tau| + C_2 & \text{if } \ \delta = 1; \\ \ln|\tau| - \dfrac{1}{\tau} + C_2 & \text{if } \ \delta = 2. \end{cases}$$

15.3.3 Other Equations

1. $y'''_{xxx} = Ayy'_x\{\cosh[\lambda(y'_x)^2]\}^{-2} y''_{xx}$.

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.1: $w''_{yy} = Ay[\cosh(\lambda w)]^{-2} w'_y$.

2. $y'''_{xxx} = Ayy'_x\{\sinh[\lambda(y'_x)^2]\}^{-2} y''_{xx}$.

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.2: $w''_{yy} = Ay[\sinh(\lambda w)]^{-2} w'_y$.

3. $y'''_{xxx} = Ayy'_x \cosh[\lambda(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.3: $w''_{yy} = \frac{\sqrt{2}}{2}Ay\cosh(\lambda w)(w'_y)^{3/2}$.

4. $y'''_{xxx} = Ayy'_x \sinh[\lambda(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.4: $w''_{yy} = \frac{\sqrt{2}}{2}Ay\sinh(\lambda w)(w'_y)^{3/2}$.

5. $y'''_{xxx} = A\cosh(\lambda y)\,(y'_x)^3(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.5: $w''_{yy} = \frac{\sqrt{2}}{2}A\cosh(\lambda y)w(w'_y)^{3/2}$.

6. $y'''_{xxx} = A\sinh(\lambda y)\,(y'_x)^3(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.6: $w''_{yy} = \frac{\sqrt{2}}{2}A\sinh(\lambda y)w(w'_y)^{3/2}$.

7. $y'''_{xxx} = A[\cosh(\lambda y)]^{-2}(y'_x)^3(y''_{xx})^2.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.7: $w''_{yy} = \frac{1}{2}A[\cosh(\lambda y)]^{-2}w(w'_y)^2$.

8. $y'''_{xxx} = A[\sinh(\lambda y)]^{-2}(y'_x)^3(y''_{xx})^2.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.4.8: $w''_{yy} = \frac{1}{2}A[\sinh(\lambda y)]^{-2}w(w'_y)^2$.

9. $y'''_{xxx} = A\cosh^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\cosh^n(\lambda y)$.

10. $y'''_{xxx} = A\sinh^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\sinh^n(\lambda y)$.

11. $y'''_{xxx} = A\tanh^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\tanh^n(\lambda y)$.

12. $y'''_{xxx} = A\coth^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\coth^n(\lambda y)$.

13. $y'''_{xxx} = A\ln^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\ln^n(\lambda y)$.

14. $y'''_{xxx} = Ayy'_x\{\cos[\lambda(y'_x)^2]\}^{-2}y''_{xx}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.1: $w''_{yy} = Ay[\cos(\lambda w)]^{-2}w'_y$.

15. $y'''_{xxx} = Ayy'_x\{\sin[\lambda(y'_x)^2]\}^{-2}y''_{xx}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.2: $w''_{yy} = Ay[\sin(\lambda w)]^{-2}w'_y$.

16. $y'''_{xxx} = A[\cos(\lambda y)]^{-2}(y'_x)^3(y''_{xx})^2.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.3: $w''_{yy} = \frac{1}{2}A[\cos(\lambda y)]^{-2}w(w'_y)^2$.

17. $y'''_{xxx} = A[\sin(\lambda y)]^{-2}(y'_x)^3(y''_{xx})^2.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.4: $w''_{yy} = \frac{1}{2}A[\sin(\lambda y)]^{-2}w(w'_y)^2$.

18. $y'''_{xxx} = Ayy'_x \cos[\lambda(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.5: $w''_{yy} = \frac{\sqrt{2}}{2}Ay\cos(\lambda w)(w'_y)^{3/2}$.

19. $y'''_{xxx} = Ayy'_x \sin[\lambda(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.6: $w''_{yy} = \frac{\sqrt{2}}{2}Ay\sin(\lambda w)(w'_y)^{3/2}$.

20. $y'''_{xxx} = A\cos(\lambda y)\,(y'_x)^3(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.7: $w''_{yy} = \frac{\sqrt{2}}{2}A\cos(\lambda y)w(w'_y)^{3/2}$.

21. $y'''_{xxx} = A\sin(\lambda y)\,(y'_x)^3(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{w(y)}$ leads to a second-order equation of the form 14.7.5.8: $w''_{yy} = \frac{\sqrt{2}}{2}A\sin(\lambda y)w(w'_y)^{3/2}$.

22. $y'''_{xxx} = A\cos^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\cos^n(\lambda y)$.

23. $y'''_{xxx} = A\sin^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\sin^n(\lambda y)$.

24. $y'''_{xxx} = A\tan^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\tan^n(\lambda y)$.

25. $y'''_{xxx} = A\cot^n(\lambda y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A\cot^n(\lambda y)$.

26. $y'''_{xxx} = A(\arcsin y)^n y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A(\arcsin y)^n$.

27. $y'''_{xxx} = A(\arctan y)^n y'_x(y''_{xx})^m.$

This is a special case of equation 15.5.4.12 with $f(y) = A(\arctan y)^n$.

15.4 Nonlinear Equations with Arbitrary Parameters

15.4.1 Equations Containing Power Functions

► **Equations of the form** $f(x,y)y'''_{xxx} = g(x,y)$.

1. $y'''_{xxx} = ay^n$.

See Section 3.2.2. The substitution $w(y) = (y'_x)^2$ leads to the Emden–Fowler equation $w''_{yy} = \pm 2ay^n w^{-1/2}$, which is discussed in Section 2.3.

2. $y'''_{xxx} = ax^n y^{-1}$.

This is a special case of equation 15.5.1.2 with $f(x) = ax^n$. On integrating the equation, we have $yy''_{xx} - \dfrac{1}{2}(y'_x)^2 = \dfrac{a}{n+1}x^{n+1} + C$.

3. $y'''_{xxx} = ax^n y^m$.

See Sections 15.2.3 and 15.2.5 (Item 1°). The transformation $z = x^{n+3}y^{m-1}$, $w = xy'_x/y$ leads to a second-order equation.

4. $y'''_{xxx} = ay^{-5/2} + by^{-7/2}$.

Using the transformation given in 15.5.2.15 (Item 2°), we reduce this equation to a constant coefficient nonhomogeneous linear equation.

Solution in parametric form ($b \neq 0$):

$$x = \int \frac{d\tau}{[\varphi(\tau)]^{3/2}} + C_3, \qquad y = \frac{1}{\varphi(\tau)},$$

where $\varphi(\tau) = -\dfrac{a}{b} + C_1 e^{-k\tau} + C_2 e^{k\tau/2} \sin \dfrac{k\tau\sqrt{3}}{2}$, $k = b^{1/3}$.

5. $y'''_{xxx} = axy^{-5/2} + bx^3 y^{-7/2}$.

The transformation $x = 1/t$, $y = w/t^2$ leads to an autonomous equation of the form 15.4.1.4: $w'''_{ttt} = -aw^{-5/2} - bw^{-7/2}$.

6. $y'''_{xxx} = k(ay^2 + by + c)^{-5/4}$.

This is a special case of equation 15.5.2.29 with $f(u) = k$.

7. $y'''_{xxx} = x(ay^2 + bx^2 y + cx^4)^{-5/4}$.

This is a special case of equation 15.5.2.30 with $f(\xi) \equiv 1$.

8. $y'''_{xxx} = k(y + ax^2 + bx + c)^n$.

The substitution $z = y + ax^2 + bx + c$ leads to an equation $z'''_{xxx} = kz^n$, whose solvable cases are outlined in Section 15.2.2.

9. $y'''_{xxx} = ky^n(ax^2 + bx + c)^{-n-2}$.

This is a special case of equation 15.5.1.13 with $f(w) = kw^n$.

10. $y'''_{xxx} = (ax+b)^n (cx+d)^{-n-2m-4} y^m.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^2}$ leads to a simpler equation: $w'''_{\xi\xi\xi} = \Delta^{-3}\xi^n w^m$, where $\Delta = ad - bc$ (see Sections 15.2.2 and 15.2.3).

11. $y'''_{xxx} = bx^{2n-1}(x-a)^{-3} y^{-n}.$

This is a special case of equation 15.5.1.11 with $f(\xi) = b\xi^{-n}$.

12. $(y + ax^2 + bx + c)y'''_{xxx} = kx^n.$

The substitution $w = y + ax^2 + bx + c$ leads to an equation of the form 15.4.1.2: $ww'''_{xxx} = kx^n$.

► Equations of the form $y'''_{xxx} = f(x, y, y'_x)$.

13. $y'''_{xxx} = ay^n y'_x + bx^m.$

Integrating yields a second-order equation: $y''_{xx} = \dfrac{a}{n+1} y^{n+1} + \dfrac{b}{m+1} x^{m+1} + C$.

14. $y'''_{xxx} = ax^{-n-2} y^n y'_x - ax^{-n-3} y^{n+1}.$

This is a special case of equation 15.5.2.5 with $f(\xi) = a\xi^n$.

15. $y'''_{xxx} = ax^{-2n-4} y^n y'_x - 2ax^{-2n-5} y^{n+1}.$

This is a special case of equation 15.5.2.8 with $f(\xi) = a\xi^n$.

16. $y'''_{xxx} = \lambda y^{-3} y'_x + ay^{-5/2} + by^{-7/2}.$

The transformation $x = \int [\varphi(\tau)]^{-3/2}\, d\tau$, $y = [\varphi(\tau)]^{-1}$ leads to a constant coefficient linear equation: $\varphi'''_{\tau\tau\tau} - \lambda\varphi'_\tau + b\varphi + a = 0$.

17. $y'''_{xxx} = -x^{-2} y'_x + x^{-3} y + ax^{1/2} y^{-5/2}.$

This is a special case of equation 15.5.2.31 with $f(\xi) = a$.

18. $y'''_{xxx} = -x^{-2} y'_x + x^{-3} y + ax^{-3/4} y^{-5/4}.$

This is a special case of equation 15.5.2.32 with $f(\xi) = a$.

19. $y'''_{xxx} = ay^n y'_x + by^m (y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = ay^n$ and $g(y) = by^m$.

20. $y'''_{xxx} = by^n (y'_x)^3 + a(y'_x)^{-5}.$

This is a special case of equation 15.5.2.4 with $f(y) = by^n$.

21. $y'''_{xxx} = (ay^2 + by + c)^{-\frac{m+5}{4}} (y'_x)^m.$

This is a special case of equation 15.5.2.29 with $f(\xi) = \xi^m$.

22. $y'''_{xxx} = -a^3 y + b(y'_x + ay)^n.$

The substitution $w = y'_x + ay$ leads to a second-order autonomous equation: $w''_{xx} - aw'_x + a^2 w = bw^n$.

23. $y'''_{xxx} = ax(xy'_x - y)^n.$

This is a special case of equation 15.5.2.18 with $f(\xi) = a\xi^n$.

24. $y'''_{xxx} = ax^{-n-2}(xy'_x - y)^n.$

This is a special case of equation 15.5.2.23 with $F(\xi) = a\xi^n$.

25. $y'''_{xxx} = ax^n(xy'_x - y)^m.$

The substitution $w(x) = xy'_x - y$ leads to a second-order generalized homogeneous equation: $(w'_x/x)'_x = ax^n w^m$.

26. $y'''_{xxx} = ax^n(xy'_x - y)^m + bx^k.$

The substitution $w(x) = xy'_x - y$ leads to a second-order equation: $(w'_x/x)'_x = ax^n w^m + bx^k$.

27. $y'''_{xxx} = ax^{n-5}y^{-n}(xy'_x - y)^3.$

This is a special case of equation 15.5.2.6 with $f(\xi) = a\xi^{-n}$.

28. $y'''_{xxx} = ax^{-n-4}(xy'_x - 2y)^n.$

This is a special case of equation 15.5.2.24 with $f(\xi) = a\xi^n$.

29. $y'''_{xxx} = ax^n(xy'_x - 2y)^m + bx^k.$

The substitution $w(x) = xy'_x - 2y$ leads to a second-order equation: $w''_{xx} = ax^{n+1}w^m + bx^{k+1}$.

30. $y'''_{xxx} = ax^{2n-7}y^{-n}(xy'_x - 2y)^3.$

This is a special case of equation 15.5.2.9 with $f(\xi) = a\xi^{-n}$.

31. $y'''_{xxx} = ax^n y^m(xy'_x - 2y)^l.$

The transformation $t = 1/x$, $z = y/x^2$ leads to an equation discussed in Section 15.2: $z'''_{ttt} = -a(-1)^l t^{-n-2m-l-4} z^m (z'_t)^l$.

◆ *See also equations* 15.3.1.2–15.3.1.30.

► **Equations of the form** $f(x,y,y'_x)y'''_{xxx} + g(x,y,y'_x)y''_{xx} + h(x,y,y'_x) = 0.$

32. $y'''_{xxx} + ayy''_{xx} - a(y'_x)^2 = 0.$

1°. The substitution $w(y) = (y'_x)^2$ leads to a second-order generalized homogeneous equation: $\pm\sqrt{w}\,w''_{yy} + ayw'_y - 2aw = 0$.

2°. Particular solutions:

$$y = C_1 \exp(C_2 x) - a^{-1}C_2,$$
$$y = 6(ax + C_1)^{-1}.$$

33. $y'''_{xxx} + ayy''_{xx} + by''_{xx} - a(y'_x)^2 + cy'_x = 0.$

Particular solution: $y = C_1 \exp(C_2 x) - \dfrac{C_2^2 + bC_2 + c}{aC_2}$.

34. $y'''_{xxx} + a_2 y''_{xx} + a_1 y'_x + a_0 y = byy''_{xx} - b(y'_x)^2 + k.$

Particular solutions: $y = Ce^{\lambda x} + \dfrac{k}{a_0}$, where C is an arbitrary constant and $\lambda = \lambda_n$ are roots of the cubic equation $\lambda^3 + \Big(a_2 - \dfrac{bk}{a_0}\Big)\lambda^2 + a_1\lambda + a_0 = 0.$

35. $y'''_{xxx} = \dfrac{ay''_{xx}}{by+c} + \dfrac{d}{(by+c)^{5/2}}.$

Autonomous first integral:

$$\Big[(by+c)y''_{xx} - \frac{1}{2}b(y'_x)^2 - ay'_x\Big]^2 - \frac{2bdy'_x + 4ad}{b\sqrt{by+c}} = C.$$

The equation in question is the only equation of the form $y'''_{xxx} = f(y)y''_{xx} + g(y)$ that has an autonomous first integral quadratic in the second derivative y''_{xx}.

36. $y'''_{xxx} = ax^n y''_{xx} + bx^m(xy'_x - y)^k + cx^s.$

The substitution $w(x) = xy'_x - y$ leads to a second-order equation: $(w'_x/x)'_x = ax^{n-1}w'_x + bx^m w^k + cx^s$.

37. $y'''_{xxx} = ayy'_x y''_{xx} - a(y'_x)^3 + by'_x.$

1°. Particular solution:

$$y = C_1\exp(C_3x) + C_2\exp(-C_3x),$$

where the constants C_1, C_2, and C_2 are related by the constraint $C_3^2 - 4aC_1C_2C_3^2 - b = 0$.

2°. Particular solution:

$$y = C_1\cos(C_3x) + C_2\sin(C_3x),$$

where the constants C_1, C_2, and C_2 are related by the constraint $C_3^2 - a(C_1^2 + C_2^2)C_3^2 + b = 0$.

38. $xy'''_{xxx} + 3y''_{xx} = ax^n y^n.$

The substitution $w(x) = xy$ leads to the equation $w'''_{xxx} = aw^n$, which is discussed in Section 15.2.2.

39. $xy'''_{xxx} - y''_{xx} - 2axyy'_x + ay^2 + b = 0.$

Integrating yields a second-order equation: $y''_{xx} = ay^2 + Cx + b$. The transformation $y = Ck^3w$, $x = kz - \dfrac{b}{C}$, where $k = \Big(\dfrac{6}{aC}\Big)^{1/3}$, leads to the first Painlevé transcendent: $w''_{zz} = w^2 + 6z$ (see Section 3.4.2).

40. $xy'''_{xxx} + (a+2)y''_{xx} = b(xy'_x + ay)^n.$

The substitution $w = xy'_x + ay$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{xx} = bw^n$.

41. $xy'''_{xxx} = -\frac{3}{2}y''_{xx} + ax^{-n-2}y^{2n}(2xy'_x - y).$

This is a special case of equation 15.5.3.37 with $f(\xi) = a\xi^{2n}$.

42. $xy'''_{xxx} = -\frac{3}{2}y''_{xx} + ax^{-n-3}y^{2n}(2xy'_x - y)^3.$

This is a special case of equation 15.5.3.39 with $f(\xi) = a\xi^{2n}$.

43. $xy'''_{xxx} + y''_{xx} = ax^{-n-3}(xy'_x - y)^n.$

This is a special case of equation 15.5.3.47 with $f(\xi) = a\xi^n$.

44. $xy'''_{xxx} + (1-a)y''_{xx} = bx^{2a}(xy'_x - y)^n.$

This is a special case of equation 15.5.3.42 with $f(\xi) = b\xi^n$.

45. $x^2y'''_{xxx} + 6xy''_{xx} + 6y'_x = ax^{2n}y^n.$

The substitution $w(x) = x^2y$ leads to the equation $w'''_{xxx} = aw^n$, which is discussed in Section 15.2.2.

46. $yy'''_{xxx} + \frac{1}{2}y'_xy''_{xx} = ax + b.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a fourth-order constant coefficient nonhomogeneous linear equation of the form 16.1.2.2: $2x''''_{tttt} = ax + b$.

47. $yy'''_{xxx} - \frac{1}{3}y'_xy''_{xx} = ax + b.$

1°. On integrating the equation, we obtain $yy''_{xx} - \frac{2}{3}(y'_x)^2 = \frac{1}{2}ax^2 + bx + C$. The substitution $y = w^3$ leads to a solvable equation of the form 14.8.1.5:

$$w''_{xx} = \tfrac{1}{3}\big(\tfrac{1}{2}ax^2 + bx + C\big)w^{-5}.$$

2°. Particular solution:

$$y = C_1x^3 + C_2x^2 + C_3x + C_4,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$4C_1C_3 - \tfrac{4}{3}C_2^2 = a,$$
$$6C_1C_4 - \tfrac{2}{3}C_2C_3 = b.$$

48. $yy'''_{xxx} + \frac{1}{2}y'_xy''_{xx} = Ax^{-5/3}.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to an equation of the form 16.2.1.1: $2x''''_{tttt} = Ax^{-5/3}$.

49. $yy'''_{xxx} = y'_xy''_{xx} + ay'_x.$

Integrating yields a second-order constant coefficient linear equation: $y''_{xx} = Cy - a$.

Solutions:

$$y = C_1\sinh(C_3x) + C_2\cosh(C_3x) + aC_3^{-2},$$
$$y = C_1\sin(C_3x) + C_2\cos(C_3x) - aC_3^{-2},$$
$$y = -\tfrac{1}{2}ax^2 + C_1x + C_2.$$

50. $yy'''_{xxx} - 2y'_xy''_{xx} + 2axy'_x - ay = 0.$

Integrating yields a second-order equation: $y''_{xx} = Cy^2 + ax$. The transformation

$$y = \frac{1}{Ck^2}w, \quad x = kz, \quad \text{where} \quad k = \left(\frac{6}{aC}\right)^{1/5},$$

leads to the first Painlevé transcendent: $w''_{zz} = w^2 + 6z$ (see Section 3.4.2).

There is also the trivial solution $y = 0$.

51. $yy'''_{xxx} + 3y'_x y''_{xx} + 4axyy'_x + ay^2 = 0.$

Multiplying by y^2, we arrive at an exact differential equation. Integrating it yields Yermakov's equation 14.9.1.2: $y''_{xx} + axy = Cy^{-3}$.

There is also the trivial solution $y = 0$.

52. $yy'''_{xxx} + \frac{1}{2}y'_x y''_{xx} = k\sqrt{y}\, y''_{xx} + my'_x + a\sqrt{y} + bx + c.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a fourth-order constant coefficient nonhomogeneous linear equation:

$$2x''''_{tttt} = \pm 2kx'''_{ttt} + 2mx''_{tt} \pm ax'_t + bx + c.$$

Here, the plus sign corresponds to $x'_t > 0$ and the minus sign to $x'_t < 0$.

53. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = bx^m.$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = bx^m$.

54. $yy'''_{xxx} + 3y'_x y''_{xx} + a[yy''_{xx} + (y'_x)^2] = bx^n.$

This is a special case of equation 15.5.3.13 with $f(x) = bx^n$.

55. $yy'''_{xxx} - y'_x y''_{xx} = a[yy''_{xx} - (y'_x)^2] + bx + c.$

This is a special case of equation 15.5.3.16 with $f(x) = a$ and $g(x) = bx + c$. The substitution $w = yy''_{xx} - (y'_x)^2$ leads to a first-order linear equation: $w'_x = aw + bx + c$.

56. $yy'''_{xxx} + (3y'_x + 2ay)y''_{xx} + 2a(y'_x)^2 + a^2 yy'_x = bx^n.$

This is a special case of equation 15.5.3.29 with $f(x) = e^{ax}$ and $g(x) = bx^n e^{ax}$.

57. $yy'''_{xxx} + (3y'_x + ax^n y)y''_{xx} + ax^n (y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = ax^n$.

58. $(y + a)y'''_{xxx} + by'_x y''_{xx} + cy^n y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = cy^n$.

59. $(y + ax)y'''_{xxx} - y'_x y''_{xx} + bxy''_{xx} = 0.$

A solution of this equation is any function that solves the first-order linear equation $y'_x = C_1 y + (aC_1 + b)x$.

Particular solution: $y = C_2 \exp(C_1 x) - \dfrac{aC_1 + b}{C_1^2}(C_1 x + 1).$

60. $(y + ax)y'''_{xxx} = (y'_x + a)y''_{xx} + bxy'_x - by.$

1°. Integrating yields a second-order constant coefficient linear equation: $y''_{xx} + Cy = -(aC + b)x$.

2°. Solutions:

$$\begin{aligned}
&y = C_1 \sin(C_3 x) + C_2 \cos(C_3 x) - (a + bC_3^{-2})x && \text{if } C = C_3^2 > 0,\\
&y = C_1 \sinh(C_3 x) + C_2 \cosh(C_3 x) - (a - bC_3^{-2})x && \text{if } C = -C_3^2 < 0,\\
&y = -\tfrac{1}{6}bx^3 + C_1 x + C_2 && \text{if } C = 0.
\end{aligned}$$

61. $(y+ax+b)y'''_{xxx}-\frac{1}{3}y'_xy''_{xx}+cy''_{xx}=kx+s.$

Particular solution:

$$y=C_1x^3+C_2x^2+C_3x+C_4,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$4C_1C_3-\tfrac{4}{3}C_2^2+6(a+c)C_1=k,$$
$$6C_1C_4-\tfrac{2}{3}C_2C_3+6bC_1+2cC_2=s.$$

62. $(y+ax+b)y'''_{xxx}+3(y'_x+a)y''_{xx}=cx^n.$

This is a special case of equation 15.5.3.23 with $f(x)=cx^n$.

63. $xyy'''_{xxx}=xy'_xy''_{xx}+ayy''_{xx}.$

Integrating yields a second-order linear equation of the form 14.1.2.7: $y''_{xx}=Cx^ay$.

64. $xyy'''_{xxx}=(ay+bxy'_x)y''_{xx}.$

Integrating yields the Emden–Fowler equation: $y''_{xx}=Cx^ay^b$ (see Section 15.3).

65. $x(yy'''_{xxx}+3y'_xy''_{xx})+a\bigl[yy''_{xx}+(y'_x)^2\bigr]=bx^n.$

This is a special case of equation 15.5.3.25 with $f(x)=bx^n$.

66. $x^2yy'''_{xxx}+x(3xy'_x+2ay)y''_{xx}+2ax(y'_x)^2+a(a-1)yy'_x=bx^n.$

This is a special case of equation 15.5.3.29 with $f(x)=x^a$ and $g(x)=bx^{n+a-2}$.

67. $y^2y'''_{xxx}-3yy'_xy''_{xx}+2(y'_x)^3=ax^ny^3.$

This is a special case of equation 15.5.3.26 with $f(x)=ax^n$.

68. $y^2y'''_{xxx}+3myy'_xy''_{xx}+m(m-1)(y'_x)^3=ax^ky^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x)=ax^k$ and $n=m+1$.

69. $y'_xy'''_{xxx}-(y''_{xx})^2=ayy''_{xx}-a(y'_x)^2+b.$

1°. Particular solution:

$$y=C_1\exp(C_3x)+C_2\exp(-C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint $4C_1C_2(C_3^4+aC_3^2)+b=0$.

2°. Particular solution:

$$y=C_1\cos(C_3x)+C_2\sin(C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint $(C_1^2+C_2^2)(C_3^4-aC_3^2)+b=0$.

3°. Particular solutions: $y=\pm x\sqrt{b/a}+C$.

► Other equations.

70. $y'''_{xxx} = \alpha(y''_{xx})^2 - \dfrac{a y''_{xx}}{\alpha(ay+b)} + \dfrac{c}{(ay+b)^4}$.

Autonomous first integral:

$$\left\{\left[\alpha(ay+b)y''_{xx} + a y'_x\right]^2 + \frac{\alpha c}{(ay+b)^2}\right\} e^{-2\alpha y'_x} = C.$$

71. $2y'_x y'''_{xxx} - (y''_{xx})^2 = \lambda(y'_x)^2 + ay^2 + by + c$.

Differentiating with respect to x and dividing by y'_x, we arrive at a fourth-order constant coefficient linear equation: $2y''''_{xxxx} = 2\lambda y''_{xx} + 2ay + b$.

72. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^{4-2n}(y'_x)^n$.

This is a special case of equation 15.5.4.14 with $f(\xi) = a\xi^n$.

73. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^{2n-8}y^{4-2n}(y'_x)^n$.

This is a special case of equation 15.5.4.16 with $f(\xi) = a\xi^n$.

74. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^{n-4}y^{4-2n}(y'_x)^n$.

This is a special case of equation 15.5.4.15 with $f(\xi) = a\xi^n$.

75. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + by^m(y'_x)^4$.

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = by^m$.

76. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = by^m$.

77. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = bx^m$.

78. $2y'_x y'''_{xxx} - (y''_{xx})^2 = \lambda x^n(y'_x)^2 + ay^2 + 2by + c$.

This is a special case of equation 15.5.4.5 with $f(x) = -\lambda x^n$.

79. $xy'_x y'''_{xxx} - 3x(y''_{xx})^2 + 3y'_x y''_{xx} = axy^n(y'_x)^4 + by^m(y'_x)^5$.

This is a special case of equation 15.5.4.11 with $f(y) = ay^n$ and $g(y) = by^m$.

80. $y'''_{xxx} = ax^{-2n-5}(xy'_x - y)^n(y''_{xx})^3$.

This is a special case of equation 15.5.4.18 with $f(\xi) = a\xi^n$.

81. $y'''_{xxx} = ax^{-4n-5}(xy'_x - y)^n(y''_{xx})^3$.

This is a special case of equation 15.5.4.19 with $f(\xi) = a\xi^n$.

82. $y'''_{xxx} = \left[ax^{-5} + bx^3(xy'_x - y)^n\right](y''_{xx})^3$.

This is a special case of equation 15.5.4.17 with $f(\xi) = b\xi^n$.

83. $y'''_{xxx} = \left[ax(y'_x)^n + by(y'_x)^m + c(y'_x)^k\right](y''_{xx})^3 + s(y'_x)^l(y''_{xx})^2$.

This is a special case of equation 15.5.4.20 with $f(\xi) = a\xi^n$, $g(\xi) = b\xi^m$, $h(\xi) = c\xi^k$, and $\varphi(\xi) = s\xi^l$.

84. $y'''_{xxx} = ax^n(y''_{xx})^m + bx^k(xy'_x - y)^l + cx^s$.

The substitution $w(x) = xy'_x - y$ leads to a second-order equation.

85. $xy'''_{xxx} + y''_{xx} = ax^n(xy'_x - y)^m(y''_{xx})^n$.

This is a special case of equation 15.5.5.5 with $f(z) = az^m$ and $g(\xi) = \xi^n$.

86. $y'''_{xxx} = Ax^n(y'_x)^m(xy'_x - y)^l(y''_{xx})^k$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ $(y'_x = t)$ leads to the equation $w'''_{ttt} = -At^m w^l(w'_t)^n(w''_{tt})^{3-k}$, which is discussed in Section 15.2.

87. $y'''_{xxx} = ax(xy'_x - y)^n(y''_{xx})^2 + bx(xy'_x - y)^m(y''_{xx})^k$.

This is a special case of equation 15.5.4.21 with $f(\xi) = a\xi^n$ and $g(\xi) = b\xi^m$.

88. $yy'''_{xxx} = y'_x y''_{xx} + ay^{-n-m}(y'_x)^n(y''_{xx})^m\big[yy''_{xx} - (y'_x)^2\big]$.

This is a special case of equation 15.5.5.7 with $f(\xi) = a\xi^n$ and $g(\xi) = \xi^m$.

89. $(y'''_{xxx})^2 = a(x^2y''_{xx} - 2xy'_x + 2y) + by''_{xx} + c$.

Differentiating with respect to x, we obtain $y'''_{xxx}(2y''''_{xxxx} - ax^2 - b) = 0$. Equating the second factor to zero and integrating, we find the solution:

$$y = \tfrac{1}{720}ax^6 + \tfrac{1}{48}bx^4 + C_3x^3 + C_2x^2 + C_1x + C_0.$$

The integration constants C_i and the parameters a, b, and c are related by:

$$36C_3^2 = 2aC_0 + 2bC_2 + c.$$

This constraint is obtained by substituting the above solution into the original equation. In addition, to the first factor there corresponds the solution $y = \widetilde{C}_2x^2 + \widetilde{C}_1x + \widetilde{C}_0$, where the constants $\widetilde{C}_i$ are related by the constraint $2a\widetilde{C}_0 + 2b\widetilde{C}_2 + c = 0$.

15.4.2 Equations Containing Exponential Functions

► **Equations of the form** $y'''_{xxx} = f(x, y, y'_x)$.

1. $y'''_{xxx} = ae^{\lambda y}$.

Autonomous equation. This is a special case of equation 15.5.1.1 with $f(y) = ae^{\lambda y}$. The substitution $u(y) = (y'_x)^2$ leads to a second-order equation: $u''_{yy} = \pm 2ae^{\lambda y}u^{-1/2}$. The transformation $z = e^{\lambda y}u^{-3/2}$, $w = u'_y/u$ leads to a first-order equation: $z\big(\lambda - \frac{3}{2}w\big)w'_z = \pm 2az - w^2$.

2. $y'''_{xxx} = ae^{\lambda y + \beta x}$.

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 15.4.2.1: $w'''_{xxx} = ae^{\lambda w}$.

3. $y'''_{xxx} = ae^{\lambda x}y^n$.

The transformation $z = e^{\lambda x}y^{n-1}$, $w = y'_x/y$ leads to a second-order equation.

4. $y'''_{xxx} = ax^n e^{\lambda y}$.

The transformation $z = x^{n+3}e^{\lambda y}$, $w = xy'_x$ leads to a second-order equation.

5. $y'''_{xxx} = a(y + be^x + c)^n - be^x$.

The substitution $w = y + be^x + c$ leads to the equation $w'''_{xxx} = aw^n$, whose solvable cases are outlined in Section 15.2.2.

6. $y'''_{xxx} = ae^{\lambda y} y'_x$.

Solution: $C_3 \pm x = \int \left(C_2 y + C_1 + 2a\lambda^{-2}e^{\lambda y}\right)^{-1/2} dy.$

7. $y'''_{xxx} = ae^{\lambda y} y'_x + be^{\mu x}$.

This is a special case of equation 15.5.2.2 with $f(y) = ae^{\lambda y}$ and $g(x) = be^{\mu x}$. Integrating yields a second-order equation: $y''_{xx} = \dfrac{a}{\lambda}e^{\lambda y} + \dfrac{b}{\mu}e^{\mu x} + C$.

8. $y'''_{xxx} = ae^{\lambda y} y'_x + be^{\mu y}(y'_x)^3$.

This is a special case of equation 15.5.2.3 with $f(y) = ae^{\lambda y}$ and $g(y) = be^{\mu y}$.

9. $y'''_{xxx} = ae^{\lambda y} y'_x + by^n(y'_x)^3$.

This is a special case of equation 15.5.2.3 with $f(y) = ae^{\lambda y}$ and $g(y) = by^n$.

10. $y'''_{xxx} = ay^n y'_x + be^{\lambda y}(y'_x)^3$.

This is a special case of equation 15.5.2.3 with $f(y) = ay^n$ and $g(y) = be^{\lambda y}$.

11. $y'''_{xxx} = be^{\lambda y}(y'_x)^3 + a(y'_x)^{-5}$.

This is a special case of equation 15.5.2.4 with $f(y) = be^{\lambda y}$.

12. $y'''_{xxx} = 2\lambda^2(y'_x)^3 + ae^{\lambda m y}(y'_x)^{m-5}$.

This is a special case of equation 15.5.2.34 with $f(\xi) = a\xi^{m-6}$.

13. $y'''_{xxx} = a^3 y + be^{\lambda x}(y'_x - ay)^n$.

The substitution $w = y'_x - ay$ leads to a second-order equation: $w''_{xx} + aw'_x + a^2 w = be^{\lambda x}w^n$.

14. $y'''_{xxx} = ae^{\lambda x}(xy'_x - y)^n$.

The substitution $w = xy'_x - y$ leads to a second-order equation: $(w'_x/x)'_x = ae^{\lambda x}w^n$.

15. $y'''_{xxx} = ae^{\lambda x}(xy'_x - 2y)^n$.

The substitution $w = xy'_x - 2y$ leads to a second-order equation: $w''_{xx} = axe^{\lambda x}w^n$.

► **Other equations.**

16. $y'''_{xxx} = -3y''_{xx} + ae^{mx}y^m y'_x + ae^{mx}y^{m+1} + 2y$.

This is a special case of equation 15.5.3.33 with $f(\xi) = a\xi^m$.

17. $y'''_{xxx} + 3\lambda y'_x y''_{xx} + \lambda^2(y'_x)^3 = ae^{\beta x - \lambda y}$.

This is a special case of equation 15.5.3.1 with $f(x) = ae^{\beta x}$.

18. $y'''_{xxx} + 3\lambda y'_x y''_{xx} + \lambda^2 (y'_x)^3 = ax^n e^{-\lambda y}.$

This is a special case of equation 15.5.3.1 with $f(x) = ax^n$.

19. $xy'''_{xxx} + (1 - ax) y''_{xx} = be^{2ax} (xy'_x - y)^n.$

This is a special case of equation 15.5.3.44 with $f(\xi) = b\xi^n$.

20. $yy'''_{xxx} + 3y'_x y''_{xx} = ae^{\lambda x}.$

Solution: $y^2 = C_2 x^2 + C_1 x + C_0 + 2a\lambda^{-3} e^{\lambda x}$.

21. $yy'''_{xxx} + 3y'_x y''_{xx} = ae^{\lambda y} + b.$

This is a special case of equation 15.5.3.8 with $f(y) = ae^{\lambda y} + b$.

22. $yy'''_{xxx} + 3y'_x y''_{xx} + ae^{\lambda x} yy'_x = be^{\mu x}.$

This is a special case of equation 15.5.3.10 with $f(x) = ae^{\lambda x}$ and $g(x) = be^{\mu x}$.

23. $yy'''_{xxx} + 3y'_x y''_{xx} + ae^{\lambda x} yy'_x = bx^n.$

This is a special case of equation 15.5.3.10 with $f(x) = ae^{\lambda x}$ and $g(x) = bx^n$.

24. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = be^{\lambda x}.$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = be^{\lambda x}$.

25. $yy'''_{xxx} + 3y'_x y''_{xx} + a\bigl[yy''_{xx} + (y'_x)^2\bigr] = be^{\lambda x}.$

This is a special case of equation 15.5.3.13 with $f(x) = be^{\lambda x}$.

26. $yy'''_{xxx} - y'_x y''_{xx} = a\bigl[yy''_{xx} - (y'_x)^2\bigr] + be^{\lambda x} + c.$

This is a special case of equation 15.4.3.16 with $f(x) = a$ and $g(x) = be^{\lambda x} + c$. The substitution $w = yy''_{xx} - (y'_x)^2$ leads to a first-order linear equation: $w'_x = aw + be^{\lambda x} + c$.

27. $yy'''_{xxx} + (3y'_x + 2ay) y''_{xx} + 2a(y'_x)^2 + a^2 yy'_x = be^{\lambda x}.$

This is a special case of equation 15.5.3.29 with $f(x) = e^{ax}$ and $g(x) = be^{(\lambda + a)x}$.

28. $yy'''_{xxx} + (3y'_x + ae^{\lambda x} y) y''_{xx} + ae^{\lambda x} (y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = ae^{\lambda x}$.

29. $(y + a) y'''_{xxx} + by'_x y''_{xx} + ce^{\lambda y} y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = ce^{\lambda y}$.

30. $(y + ax + b) y'''_{xxx} + 3(y'_x + a) y''_{xx} = ke^{\lambda x}.$

Solution: $(y + ax + b)^2 = C_2 x^2 + C_1 x + C_0 + 2k\lambda^{-3} e^{\lambda x}$.

31. $y^2 y'''_{xxx} - (y'_x)^3 + ay^2 y'_x = be^{\lambda x}.$

Integrating yields a second-order equation: $y^2 y''_{xx} - y(y'_x)^2 + \frac{1}{3} ay^3 = b\lambda^{-1} e^{\lambda x} + C$. For $C = 0$, we have an equation of the form 14.8.3.57 with $k = -1$: $yy''_{xx} - (y'_x)^2 + \frac{1}{3} ay^2 = b\lambda^{-1} e^{\lambda x} y^{-1}$.

32. $y^2 y'''_{xxx} - (y'_x)^3 + ay^2 y'_x = bx\exp(\lambda x^2)$.

Integrating yields a second-order equation: $y^2 y''_{xx} - y(y'_x)^2 + \frac{1}{3}ay^3 = \frac{1}{2}b\lambda^{-1}\exp(\lambda x^2) + C$. For $C=0$, we have an equation of the form 14.8.3.5.7 with $k=-1$: $yy''_{xx} - (y'_x)^2 + \frac{1}{3}ay^2 = \frac{1}{2}b\lambda^{-1}\exp(\lambda x^2)y^{-1}$.

33. $y^2 y'''_{xxx} - 3yy'_x y''_{xx} + 2(y'_x)^3 = ae^{\lambda x}y^3$.

Solution: $\ln|y| = C_2 x^2 + C_1 x + C_0 + a\lambda^{-3}e^{\lambda x}$.

34. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = ae^{\lambda x}y^{2-m}$.

This is a special case of equation 15.5.3.27 with $f(x) = ae^{\lambda x}$ and $n = m+1$.

35. $x^2 yy'''_{xxx} + x(3xy'_x + 2ay)y''_{xx} + 2ax(y'_x)^2 + a(a-1)yy'_x = be^{\lambda x}$.

This is a special case of equation 15.5.3.29 with $f(x) = x^a$ and $g(x) = bx^{a-2}e^{\lambda x}$.

36. $2y'_x y'''_{xxx} - (y''_{xx})^2 = ke^{\lambda x}(y'_x)^2 + ay^2 + 2by + c$.

This is a special case of equation 15.5.4.5 with $f(x) = -ke^{\lambda x}$.

37. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + be^{\mu y}(y'_x)^4$.

This is a special case of equation 15.5.4.7 with $f(x) = ae^{\lambda x}$ and $g(y) = be^{\mu y}$.

38. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + by^m(y'_x)^4$.

This is a special case of equation 15.5.4.7 with $f(x) = ae^{\lambda x}$ and $g(y) = by^m$.

39. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + be^{\mu y}(y'_x)^4$.

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = be^{\mu y}$.

40. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + be^{\mu x}y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = ae^{\lambda x}$ and $g(x) = be^{\mu x}$.

41. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + bx^n y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = ae^{\lambda x}$ and $g(x) = bx^n$.

42. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + be^{\lambda x}y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = be^{\lambda x}$.

43. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda y}(y'_x)^4 + bx^{-1}e^{\mu y}(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = ae^{\lambda y}$ and $g(y) = be^{\mu y}$.

44. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda y}(y'_x)^4 + bx^{-1}y^n(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = ae^{\lambda y}$ and $g(y) = by^n$.

45. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}e^{\lambda y}(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = be^{\lambda y}$.

15.4.3 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine.**

1. $y'''_{xxx} = a\sinh(\lambda y)y'_x$.

Solution: $C_3 \pm x = \int \bigl[C_2 y + C_1 + 2a\lambda^{-2}\sinh(\lambda y)\bigr]^{-1/2}\,dy$.

2. $y'''_{xxx} = a\sinh(\lambda y)y'_x + b\sinh(\mu x)$.

This is a special case of equation 15.5.2.2 with $f(y) = a\sinh(\lambda y)$ and $g(x) = b\sinh(\mu x)$. Integrating yields a second-order equation: $y''_{xx} = \dfrac{a}{\lambda}\cosh(\lambda y) + \dfrac{b}{\mu}\cosh(\mu x) + C$.

3. $y'''_{xxx} = a\sinh^n(\lambda y)y'_x + b\sinh(\mu y)(y'_x)^3$.

This is a special case of equation 15.5.2.3 with $f(y) = a\sinh^n(\lambda y)$ and $g(y) = b\sinh(\mu y)$.

4. $y'''_{xxx} = \frac{1}{2}\lambda^2(y'_x)^3 + a(\sinh\lambda y)^{-m-3}(y'_x)^{2m+1}$.

This is a special case of equation 15.5.2.36 with $f(\xi) = a\xi^{2m}$.

5. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sinh(\lambda x)$.

Solution: $y^2 = C_2x^2 + C_1x + C_0 + 2a\lambda^{-3}\cosh(\lambda x)$.

6. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sinh^n(\lambda x) + b$.

This is a special case of equation 15.5.3.6 with $f(x) = a\sinh^n(\lambda x) + b$.

7. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sinh^n(\lambda y) + b$.

This is a special case of equation 15.5.3.8 with $f(y) = a\sinh^n(\lambda y) + b$.

8. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\sinh^m(\lambda x)$.

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\sinh^m(\lambda x)$.

9. $yy'''_{xxx} + 3y'_x y''_{xx} + a\bigl[yy''_{xx} + (y'_x)^2\bigr] = b\sinh^n(\lambda x)$.

This is a special case of equation 15.5.3.13 with $f(x) = b\sinh^n(\lambda x)$.

10. $yy'''_{xxx} + (3y'_x + ay\sinh^n x)y''_{xx} + a\sinh^n x\,(y'_x)^2 = 0$.

This is a special case of equation 15.5.3.17 with $f(x) = a\sinh^n x$.

11. $(y + a)y'''_{xxx} + by'_x y''_{xx} + c\sinh^n(\lambda y)y'_x = 0$.

This is a special case of equation 15.5.3.21 with $f(y) = c\sinh^n(\lambda y)$.

12. $y^2y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\sinh^k(\lambda x)y^{2-m}$.

This is a special case of equation 15.5.3.27 with $f(x) = a\sinh^k(\lambda x)$ and $n = m + 1$.

► **Equations with hyperbolic cosine.**

13. $y'''_{xxx} = a\cosh(\lambda y)y'_x$.

Solution: $C_3 \pm x = \int \bigl[C_2 y + C_1 + 2a\lambda^{-2}\cosh(\lambda y)\bigr]^{-1/2}\,dy$.

14. $y'''_{xxx} = a\cosh(\lambda y)y'_x + b\cosh(\mu x).$

This is a special case of equation 15.5.2.2 with $f(y) = a\cosh(\lambda y)$ and $g(x) = b\cosh(\mu x)$. Integrating yields a second-order equation: $y''_{xx} = \dfrac{a}{\lambda}\sinh(\lambda y) + \dfrac{b}{\mu}\sinh(\mu x) + C$.

15. $y'''_{xxx} = a\cosh^n(\lambda y)y'_x + b\cosh(\mu y)(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = a\cosh^n(\lambda y)$ and $g(y) = b\cosh(\mu y)$.

16. $y'''_{xxx} = ay^n y'_x + b\cosh(\mu y)(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = ay^n$ and $g(y) = b\cosh(\mu y)$.

17. $y'''_{xxx} = b\cosh(\lambda y)(y'_x)^3 + a(y'_x)^{-5}.$

This is a special case of equation 15.5.2.4 with $f(y) = b\cosh(\lambda y)$.

18. $y'''_{xxx} = \frac{1}{2}\lambda^2(y'_x)^3 + a(\cosh\lambda y)^{-m-3}(y'_x)^{2m+1}.$

This is a special case of equation 15.5.2.35 with $f(\xi) = a\xi^{2m}$.

19. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cosh(\lambda x).$

Solution: $y^2 = C_2x^2 + C_1x + C_0 + 2a\lambda^{-3}\sinh(\lambda x)$.

20. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cosh^n(\lambda x).$

This is a special case of equation 15.5.3.6 with $f(x) = a\cosh^n(\lambda x)$.

21. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cosh^n(\lambda y) + b.$

This is a special case of equation 15.5.3.8 with $f(y) = a\cosh^n(\lambda y) + b$.

22. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\cosh^m(\lambda x).$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\cosh^m(\lambda x)$.

23. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\cosh^n(\lambda x).$

This is a special case of equation 15.5.3.13 with $f(x) = b\cosh^n(\lambda x)$.

24. $yy'''_{xxx} + (3y'_x + ay\cosh^n x)y''_{xx} + a\cosh^n x\,(y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = a\cosh^n x$.

25. $(y + a)y'''_{xxx} + by'_x y''_{xx} + c\cosh^n(\lambda y)y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = c\cosh^n(\lambda y)$.

26. $y^2y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\cosh^k(\lambda x)y^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x) = a\cosh^k(\lambda x)$ and $n = m + 1$.

27. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\cosh^n(\lambda x)$ and $g(y) = by^m$.

28. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cosh^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\cosh^m(\lambda y)$.

29. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda x)(y'_x)^2 + b\cosh^m(\mu x)y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = a\cosh^n(\lambda x)$ and $g(x) = b\cosh^m(\mu x)$.

30. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = a\cosh^n(\lambda x)$ and $g(x) = bx^m$.

31. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cosh^m(\lambda x)y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\cosh^m(\lambda x)$.

32. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda y)(y'_x)^4 + bx^{-1}\cosh^m(\mu y)(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = a\cosh^n(\lambda y)$ and $g(y) = b\cosh^m(\mu y)$.

33. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = a\cosh^n(\lambda y)$ and $g(y) = by^m$.

34. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\cosh^m(\lambda y)(y'_x)^{7/2}$.

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\cosh^m(\lambda y)$.

► **Equations with hyperbolic tangent.**

35. $y'''_{xxx} = a\tanh(\lambda y)y'_x$.

This is a special case of equation 15.5.2.1 with $f(y) = a\tanh(\lambda y)$.

36. $y'''_{xxx} = a\tanh(\lambda y)y'_x + b\tanh(\mu x)$.

This is a special case of equation 15.5.2.2 with $f(y) = a\tanh(\lambda y)$ and $g(x) = b\tanh(\mu x)$.

37. $yy'''_{xxx} + 3y'_x y''_{xx} = a\tanh^n(\lambda x) + b$.

This is a special case of equation 15.5.3.6 with $f(x) = a\tanh^n(\lambda x) + b$.

38. $yy'''_{xxx} + 3y'_x y''_{xx} = a\tanh^n(\lambda y) + b$.

This is a special case of equation 15.5.3.8 with $f(y) = a\tanh^n(\lambda y) + b$.

39. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\tanh^m(\lambda x)$.

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\tanh^m(\lambda x)$.

40. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\tanh^n(\lambda x)$.

This is a special case of equation 15.5.3.13 with $f(x) = b\tanh^n(\lambda x)$.

41. $yy'''_{xxx} + (3y'_x + ay\tanh^n x)y''_{xx} + a\tanh^n x\,(y'_x)^2 = 0$.

This is a special case of equation 15.5.3.17 with $f(x) = a\tanh^n x$.

42. $(y + a)y'''_{xxx} + by'_x y''_{xx} + c\tanh^n(\lambda y)y'_x = 0$.

This is a special case of equation 15.5.3.21 with $f(y) = c\tanh^n(\lambda y)$.

43. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\tanh^k(\lambda x)y^{2-m}$.

This is a special case of equation 15.5.3.27 with $f(x) = a\tanh^k(\lambda x)$ and $n = m + 1$.

44. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + b\tanh^m(\mu y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\tanh^n(\lambda x)$ and $g(y) = b\tanh^m(\mu y)$.

45. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\tanh^n(\lambda x)$ and $g(y) = by^m$.

46. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tanh^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\tanh^m(\lambda y)$.

47. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + b\tanh^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\tanh^n(\lambda x)$ and $g(x) = b\tanh^m(\mu x)$.

48. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\tanh^n(\lambda x)$ and $g(x) = bx^m$.

49. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tanh^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\tanh^m(\lambda x)$.

50. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda y)(y'_x)^4 + bx^{-1}\tanh^m(\mu y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\tanh^n(\lambda y)$ and $g(y) = b\tanh^m(\mu y)$.

51. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\tanh^n(\lambda y)$ and $g(y) = by^m$.

52. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\tanh^m(\lambda y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\tanh^m(\lambda y)$.

► **Equations with hyperbolic cotangent.**

53. $y'''_{xxx} = a\coth(\lambda y)y'_x.$

This is a special case of equation 15.5.2.1 with $f(y) = a\coth(\lambda y)$.

54. $y'''_{xxx} = a\coth(\lambda y)y'_x + b\coth(\mu x).$

This is a special case of equation 15.5.2.2 with $f(y) = a\coth(\lambda y)$ and $g(x) = b\coth(\mu x)$.

55. $yy'''_{xxx} + 3y'_x y''_{xx} = a\coth^n(\lambda x) + b.$

This is a special case of equation 15.5.3.6 with $f(x) = a\coth^n(\lambda x) + b$.

56. $yy'''_{xxx} + 3y'_x y''_{xx} = a\coth^n(\lambda y) + b.$

This is a special case of equation 15.5.3.8 with $f(y) = a\coth^n(\lambda y) + b$.

57. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\coth^m(\lambda x).$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\coth^m(\lambda x)$.

58. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\coth^n(\lambda x).$

This is a special case of equation 15.5.3.13 with $f(x) = b\coth^n(\lambda x)$.

59. $yy'''_{xxx} + (3y'_x + ay\coth^n x)y''_{xx} + a\coth^n x\,(y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = a\coth^n x$.

60. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\coth^n(\lambda y)y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = c\coth^n(\lambda y)$.

61. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\coth^k(\lambda x)y^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x) = a\coth^k(\lambda x)$ and $n = m+1$.

62. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\coth^n(\lambda x)(y'_x)^2 + b\coth^m(\mu y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\coth^n(\lambda x)$ and $g(y) = b\coth^m(\mu y)$.

63. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\coth^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\coth^n(\lambda x)$ and $g(y) = by^m$.

64. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\coth^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\coth^m(\lambda y)$.

65. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\coth^n(\lambda x)(y'_x)^2 + b\coth^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\coth^n(\lambda x)$ and $g(x) = b\coth^m(\mu x)$.

66. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\coth^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\coth^n(\lambda x)$ and $g(x) = bx^m$.

67. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\coth^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\coth^m(\lambda x)$.

68. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\coth^n(\lambda y)(y'_x)^4 + bx^{-1}\coth^m(\mu y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\coth^n(\lambda y)$ and $g(y) = b\coth^m(\mu y)$.

69. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\coth^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\coth^n(\lambda y)$ and $g(y) = by^m$.

70. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\coth^m(\lambda y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\coth^m(\lambda y)$.

15.4.4 Equations Containing Logarithmic Functions

► **Equations of the form $y'''_{xxx} = f(x, y, y'_x)$.**

1. $y'''_{xxx} = ay(\lambda x + m\ln y).$

This is a special case of equation 15.5.1.16 with $f(z) = a\ln z$.

2. $y'''_{xxx} = ax^{-3}(\lambda y + m\ln x).$

This is a special case of equation 15.5.1.17 with $f(z) = a\ln z$.

3. $y'''_{xxx} = ax^{-4}(\ln y - 2\ln x).$

This is a special case of equation 15.5.1.7 with $f(z) = a\ln z$.

4. $y'''_{xxx} = a\ln(\lambda y)y'_x.$

This is a special case of equation 15.5.2.1 with $f(y) = a\ln(\lambda y)$.

5. $y'''_{xxx} = a\ln(\lambda y)y'_x + b\ln(\mu x).$

This is a special case of equation 15.5.2.2 with $f(y) = a\ln(\lambda y)$ and $g(x) = b\ln(\mu x)$.

6. $y'''_{xxx} = a\ln^n(\lambda y)y'_x + b\ln^m(\mu y)(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = a\ln^n(\lambda y)$ and $g(y) = b\ln^m(\mu y)$.

7. $y'''_{xxx} = a\ln^n(\lambda y)y'_x + by^m(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = a\ln^n(\lambda y)$ and $g(y) = by^m$.

8. $y'''_{xxx} = ay^n y'_x + b\ln^m(\lambda y)(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = ay^n$ and $g(y) = b\ln^m(\lambda y)$.

9. $y'''_{xxx} = ax^{-3}(\ln y - \ln x)(xy'_x - y).$

This is a special case of equation 15.5.2.5 with $f(\xi) = a\ln\xi$.

10. $y'''_{xxx} = ax^{-5}(\ln y - 2\ln x)(xy'_x - 2y).$

This is a special case of equation 15.5.2.8 with $f(\xi) = a\ln\xi$.

11. $y'''_{xxx} = ay^{-5/2}(2\ln y'_x - \ln y).$

This is a special case of equation 15.5.2.27 with $f(\xi) = 2a\ln\xi$.

12. $y'''_{xxx} = ay^{-5/4}(4\ln y'_x - \ln y).$

This is a special case of equation 15.5.2.28 with $f(\xi) = 4a\ln\xi$.

13. $y'''_{xxx} = a^3 y + b\ln x\,(y'_x - ay)^n.$

This is a special case of equation 15.5.2.16 with $f(x,w) = bw^n\ln x$.

14. $y'''_{xxx} = a\ln x\,(xy'_x - y)^n.$

This is a special case of equation 15.5.2.20 with $f(x,w) = aw^n\ln x$.

15. $y'''_{xxx} = a\ln x\,(xy'_x - 2y)^n.$

This is a special case of equation 15.5.2.21 with $f(x,w) = aw^n\ln x$.

▶ Other equations.

16. $y'''_{xxx} = -3y''_{xx} + a(x + \ln y)^n(y'_x + y) + 2y.$

This is a special case of equation 15.5.3.33 with $f(\xi) = a\ln^n\xi$.

17. $xy'''_{xxx} = b(xy'_x - y + a\ln x)y''_{xx}.$

This is a special case of equation 15.5.3.45 with $f(\xi) = b\xi$.

18. $yy'''_{xxx} + 3y'_x y''_{xx} = a\ln^n(bx).$

This is a special case of equation 15.5.3.6 with $f(x) = a\ln^n(bx)$.

19. $yy'''_{xxx} + 3y'_x y''_{xx} = a\ln^n(by).$

This is a special case of equation 15.5.3.8 with $f(y) = a\ln^n(by)$.

20. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\ln^m(\lambda x).$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\ln^m(\lambda x)$.

21. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\ln^n(\lambda x).$

This is a special case of equation 15.5.3.13 with $f(x) = b\ln^n(\lambda x)$.

22. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\ln^n(\lambda y)y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = c\ln^n(\lambda y)$.

23. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\ln^k(bx)y^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x) = a\ln^k(bx)$ and $n = m+1$.

24. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^4(\ln y'_x - 2\ln y).$

This is a special case of equation 15.5.4.14 with $f(\xi) = a\ln\xi$.

25. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda x)(y'_x)^2 + b\ln^m(\mu y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\ln^n(\lambda x)$ and $g(y) = b\ln^m(\mu y)$.

26. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\ln^n(\lambda x)$ and $g(y) = by^m$.

27. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\ln^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\ln^m(\lambda y)$.

28. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda x)(y'_x)^2 + b\ln^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\ln^n(\lambda x)$ and $g(x) = b\ln^m(\mu x)$.

29. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n x\,(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\ln^n x$ and $g(x) = bx^m$.

30. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\ln^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\ln^m(\lambda x)$.

31. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda y)(y'_x)^4 + bx^{-1}\ln^m(\mu y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\ln^n(\lambda y)$ and $g(y) = b\ln^m(\mu y)$.

32. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\ln^m(\lambda y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\ln^m(\lambda y)$.

33. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\ln^n(\lambda y)$ and $g(y) = by^m$.

34. $yy'''_{xxx}\ln y - y'_x y''_{xx}\ln y''_{xx} + ay'_x y''_{xx} = 0.$

Integrating yields a second-order autonomous equation of the form 14.9.1.1: $y''_{xx} = e^a y^C$.

15.4.5 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y'''_{xxx} = a\sin(\lambda y)y'_x$.

Solution: $C_3 \pm x = \int \bigl[C_2 y + C_1 - 2a\lambda^{-2}\sin(\lambda y)\bigr]^{-1/2}\,dy$.

2. $y'''_{xxx} = a\sin(\lambda y)y'_x + b\sin(\mu x)$.

This is a special case of equation 15.5.2.2 with $f(y) = a\sin(\lambda y)$ and $g(x) = b\sin(\mu x)$. Integrating yields a second-order equation: $y''_{xx} = -\dfrac{a}{\lambda}\cos(\lambda y) - \dfrac{b}{\mu}\cos(\mu x) + C$.

3. $y'''_{xxx} = a\sin^n(\lambda y)y'_x + b\sin(\mu y)(y'_x)^3$.

This is a special case of equation 15.5.2.3 with $f(y) = a\sin^n(\lambda y)$ and $g(y) = b\sin(\mu y)$.

4. $y'''_{xxx} = \frac{1}{2}\lambda^2(y'_x)^3 + a(\sin\lambda y)^{-m-3}(y'_x)^{2m+1}$.

This is a special case of equation 15.5.2.36 with $f(\xi) = a\xi^{2m}$.

5. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sin(\lambda x)$.

Solution: $y^2 = C_2 x^2 + C_1 x + C_0 + 2a\lambda^{-3}\cos(\lambda x)$.

6. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sin^n(\lambda x) + b$.

This is a special case of equation 15.5.3.6 with $f(x) = a\sin^n(\lambda x) + b$.

7. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sin^n(\lambda y) + b$.

This is a special case of equation 15.5.3.8 with $f(y) = a\sin^n(\lambda y) + b$.

8. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\sin^m(\lambda x)$.

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\sin^m(\lambda x)$.

9. $yy'''_{xxx} + 3y'_x y''_{xx} + a\bigl[yy''_{xx} + (y'_x)^2\bigr] = b\sin^n(\lambda x)$.

This is a special case of equation 15.5.3.13 with $f(x) = b\sin^n(\lambda x)$.

10. $yy'''_{xxx} + (3y'_x + ay\sin^n x)y''_{xx} + a\sin^n x\,(y'_x)^2 = 0$.

This is a special case of equation 15.5.3.17 with $f(x) = a\sin^n x$.

11. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\sin^n(\lambda y)y'_x = 0$.

This is a special case of equation 15.5.3.21 with $f(y) = c\sin^n(\lambda y)$.

12. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\sin^k(\lambda x)y^{2-m}$.

This is a special case of equation 15.5.3.27 with $f(x) = a\sin^k(\lambda x)$ and $n = m+1$.

► **Equations with cosine.**

13. $y'''_{xxx} = a\cos(\lambda y)y'_x$.

Solution: $C_3 \pm x = \int \bigl[C_2 y + C_1 - 2a\lambda^{-2}\cos(\lambda y)\bigr]^{-1/2}\,dy$.

14. $y'''_{xxx} = a\cos(\lambda y)y'_x + b\cos(\mu x).$

This is a special case of equation 15.5.2.2 with $f(y) = a\cos(\lambda y)$ and $g(x) = b\cos(\mu x)$. Integrating, we obtain a second-order equation: $y''_{xx} = \dfrac{a}{\lambda}\sin(\lambda y) + \dfrac{b}{\mu}\sin(\mu x) + C.$

15. $y'''_{xxx} = a\cos^n(\lambda y)y'_x + b\cos(\mu y)(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = a\cos^n(\lambda y)$ and $g(y) = b\cos(\mu y)$.

16. $y'''_{xxx} = ay^n y'_x + b\cos(\mu y)(y'_x)^3.$

This is a special case of equation 15.5.2.3 with $f(y) = ay^n$ and $g(y) = b\cos(\mu y)$.

17. $y'''_{xxx} = b\cos(\lambda y)(y'_x)^3 + a(y'_x)^{-5}.$

This is a special case of equation 15.5.2.4 with $f(y) = b\cos(\lambda y)$.

18. $y'''_{xxx} = \frac{1}{2}\lambda^2(y'_x)^3 + a(\cos\lambda y)^{-m-3}(y'_x)^{2m+1}.$

This is a special case of equation 15.5.2.35 with $f(\xi) = a\xi^{2m}$.

19. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cos(\lambda x).$

Solution: $y^2 = C_2x^2 + C_1x + C_0 - 2a\lambda^{-3}\sin(\lambda x).$

20. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cos^n(\lambda x).$

This is a special case of equation 15.5.3.6 with $f(x) = a\cos^n(\lambda x)$.

21. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cos^n(\lambda y) + b.$

This is a special case of equation 15.5.3.8 with $f(y) = a\cos^n(\lambda y) + b$.

22. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\cos^m(\lambda x).$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\cos^m(\lambda x)$.

23. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\cos^n(\lambda x).$

This is a special case of equation 15.5.3.13 with $f(x) = b\cos^n(\lambda x)$.

24. $yy'''_{xxx} + (3y'_x + ay\cos^n x)y''_{xx} + a\cos^n x\,(y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = a\cos^n x$.

25. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\cos^n(\lambda y)y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = c\cos^n(\lambda y)$.

26. $y^2y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\cos^k(\lambda x)y^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x) = a\cos^k(\lambda x)$ and $n = m+1$.

27. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\cos^n(\lambda x)$ and $g(y) = by^m$.

28. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cos^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\cos^m(\lambda y)$.

29. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda x)(y'_x)^2 + b\cos^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\cos^n(\lambda x)$ and $g(x) = b\cos^m(\mu x)$.

30. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\cos^n(\lambda x)$ and $g(x) = bx^m$.

31. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cos^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\cos^m(\lambda x)$.

32. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda y)(y'_x)^4 + bx^{-1}\cos^m(\mu y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\cos^n(\lambda y)$ and $g(y) = b\cos^m(\mu y)$.

33. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\cos^n(\lambda y)$ and $g(y) = by^m$.

34. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\cos^m(\lambda y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\cos^m(\lambda y)$.

► Equations with tangent.

35. $y'''_{xxx} = a\tan(\lambda y)y'_x.$

This is a special case of equation 15.5.2.1 with $f(y) = a\tan(\lambda y)$.

36. $y'''_{xxx} = a\tan(\lambda y)y'_x + b\tan(\mu x).$

This is a special case of equation 15.5.2.2 with $f(y) = a\tan(\lambda y)$ and $g(x) = b\tan(\mu x)$.

37. $yy'''_{xxx} + 3y'_x y''_{xx} = a\tan^n(\lambda x) + b.$

This is a special case of equation 15.5.3.6 with $f(x) = a\tan^n(\lambda x) + b$.

38. $yy'''_{xxx} + 3y'_x y''_{xx} = a\tan^n(\lambda y) + b.$

This is a special case of equation 15.5.3.8 with $f(y) = a\tan^n(\lambda y) + b$.

39. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\tan^m(\lambda x).$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\tan^m(\lambda x)$.

40. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\tan^n(\lambda x).$

This is a special case of equation 15.5.3.13 with $f(x) = b\tan^n(\lambda x)$.

41. $yy'''_{xxx} + (3y'_x + ay\tan^n x)y''_{xx} + a\tan^n x\,(y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = a\tan^n x$.

42. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\tan^n(\lambda y)y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = c\tan^n(\lambda y)$.

43. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\tan^k(\lambda x)y^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x) = a\tan^k(\lambda x)$ and $n = m + 1$.

44. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda x)(y'_x)^2 + b\tan^m(\mu y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\tan^n(\lambda x)$ and $g(y) = b\tan^m(\mu y)$.

45. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\tan^n(\lambda x)$ and $g(y) = by^m$.

46. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tan^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\tan^m(\lambda y)$.

47. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda x)(y'_x)^2 + b\tan^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\tan^n(\lambda x)$ and $g(x) = b\tan^m(\mu x)$.

48. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\tan^n(\lambda x)$ and $g(x) = bx^m$.

49. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tan^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\tan^m(\lambda x)$.

50. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda y)(y'_x)^4 + bx^{-1}\tan^m(\mu y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\tan^n(\lambda y)$ and $g(y) = b\tan^m(\mu y)$.

51. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\tan^n(\lambda y)$ and $g(y) = by^m$.

52. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\tan^m(\lambda y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\tan^m(\lambda y)$.

► **Equations with cotangent.**

53. $y'''_{xxx} = a\cot(\lambda y)y'_x.$

This is a special case of equation 15.5.2.1 with $f(y) = a\cot(\lambda y)$.

54. $y'''_{xxx} = a\cot(\lambda y)y'_x + b\cot(\mu x).$

This is a special case of equation 15.5.2.2 with $f(y) = a\cot(\lambda y)$ and $g(x) = b\cot(\mu x)$.

55. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cot^n(\lambda x) + b.$

This is a special case of equation 15.5.3.6 with $f(x) = a\cot^n(\lambda x) + b$.

56. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cot^n(\lambda y) + b.$

This is a special case of equation 15.5.3.8 with $f(y) = a\cot^n(\lambda y) + b$.

57. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\cot^m(\lambda x).$

This is a special case of equation 15.5.3.10 with $f(x) = ax^n$ and $g(x) = b\cot^m(\lambda x)$.

58. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\cot^n(\lambda x).$

This is a special case of equation 15.5.3.13 with $f(x) = b\cot^n(\lambda x)$.

59. $yy'''_{xxx} + (3y'_x + ay\cot^n x)y''_{xx} + a\cot^n x\,(y'_x)^2 = 0.$

This is a special case of equation 15.5.3.17 with $f(x) = a\cot^n x$.

60. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\cot^n(\lambda y)y'_x = 0.$

This is a special case of equation 15.5.3.21 with $f(y) = c\cot^n(\lambda y)$.

61. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\cot^k(\lambda x)y^{2-m}.$

This is a special case of equation 15.5.3.27 with $f(x) = a\cot^k(\lambda x)$ and $n = m+1$.

62. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cot^n(\lambda x)(y'_x)^2 + b\cot^m(\mu y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\cot^n(\lambda x)$ and $g(y) = b\cot^m(\mu y)$.

63. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cot^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = a\cot^n(\lambda x)$ and $g(y) = by^m$.

64. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cot^m(\lambda y)(y'_x)^4.$

This is a special case of equation 15.5.4.7 with $f(x) = ax^n$ and $g(y) = b\cot^m(\lambda y)$.

65. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cot^n(\lambda x)(y'_x)^2 + b\cot^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\cot^n(\lambda x)$ and $g(x) = b\cot^m(\mu x)$.

66. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cot^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = a\cot^n(\lambda x)$ and $g(x) = bx^m$.

67. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cot^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 15.5.4.8 with $f(x) = ax^n$ and $g(x) = b\cot^m(\lambda x)$.

68. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cot^n(\lambda y)(y'_x)^4 + bx^{-1}\cot^m(\mu y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\cot^n(\lambda y)$ and $g(y) = b\cot^m(\mu y)$.

69. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cot^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = a\cot^n(\lambda y)$ and $g(y) = by^m$.

70. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\cot^m(\lambda y)(y'_x)^{7/2}.$

This is a special case of equation 15.5.4.9 with $f(y) = ay^n$ and $g(y) = b\cot^m(\lambda y)$.

15.5 Nonlinear Equations Containing Arbitrary Functions

15.5.1 Equations of the Form $F(x,y)y'''_{xxx} + G(x,y) = 0$

▶ **Arguments of the arbitrary functions are x or y.**

1. $y'''_{xxx} = f(y).$

The substitution $w(y) = (y'_x)^2$ leads to a second-order equation: $w''_{yy} = \pm 2f(y)w^{-1/2}$. In particular, with $f(y) = ay^n$ the obtained equation is an Emden–Fowler equation; see Section 2.3.

2. $y'''_{xxx} = f(x)y^{-1}$.

1°. On integrating the equation, we have $yy''_{xx} - \frac{1}{2}(y'_x)^2 = \int f(x)\,dx + C$. The substitution $y = w^2$ reduces the latter equation to the form $w''_{xx} = \frac{1}{2}\Bigl[\int f(x)\,dx + C\Bigr]w^{-3}$.

2°. The transformation $x = 1/t$, $y = u/t^2$ leads to an equation of the same form: $u'''_{ttt} = -t^{-2}f(1/t)u^{-1}$.

3. $y'''_{xxx} = x^{-3}f(y)$.

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 15.5.5.9: $y'''_{ttt} - 3y''_{tt} + 2y'_t = f(y)$.

4. $(y + ax^2 + bx + c)y'''_{xxx} = f(x)$.

The substitution $w = y + ax^2 + bx + c$ leads to an equation of the form 15.5.1.2: $ww'''_{xxx} = f(x)$.

5. $(ay + be^x)y'''_{xxx} + be^x y = f(x)$.

Integrating yields a second-order equation:

$$(ay + be^x)y''_{xx} - \tfrac{1}{2}a(y'_x)^2 - be^x y'_x + be^x y = \int f(x)\,dx + C.$$

► **Arguments of the arbitrary functions depend on x and y.**

6. $y'''_{xxx} = x^{-2}f(yx^{-1})$.

The transformation $t = \ln|x|$, $w = yx^{-1}$ leads to an autonomous equation of the form 15.5.5.9: $w'''_{ttt} - w'_t = f(w)$.

7. $y'''_{xxx} = x^{-4}f(yx^{-2})$.

The transformation $t = x^{-1}$, $w = yx^{-2}$ leads to an autonomous equation of the form 15.5.1.1: $w'''_{ttt} = -f(w)$.

8. $y'''_{xxx} = x^{-k-3}f(x^k y)$.

Generalized homogeneous equation.

1°. The transformation $t = \ln x$, $z = x^k y$ leads to an autonomous equation.

2°. The transformation $z = x^k y$, $w = xy'_x/y$ leads to a second-order equation.

9. $y'''_{xxx} = yx^{-3}f(x^n y^m)$.

Generalized homogeneous equation. The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to a second-order equation.

10. $y'''_{xxx} = f(y + ax^3 + bx^2 + cx + k)$.

The substitution $w = y + ax^3 + bx^2 + cx + k$ leads to an autonomous equation of the form 15.5.1.1: $w'''_{xxx} = f(w) + 6a$.

11. $x(x-a)^3 y'''_{xxx} = f(yx^{-2}), \qquad a \neq 0.$

The transformation $\xi = \ln\left|\dfrac{x-a}{x}\right|$, $w = \dfrac{y}{x^2}$ leads to an autonomous equation of the form 15.5.5.9: $w'''_{\xi\xi\xi} - 3w''_{\xi\xi} + 2w'_{\xi} = a^{-3} f(w)$.

12. $y'''_{xxx} = (ax+by+c)^{-2} f\left(\dfrac{ax+by+c}{\alpha x+\beta y+\gamma}\right).$

This is a special case of equation 17.2.6.19 with $n = 3$.

13. $(ax^2+bx+c)^2 y'''_{xxx} = f\left(\dfrac{y}{ax^2+bx+c}\right).$

The transformation $\xi = \displaystyle\int \frac{dx}{ax^2+bx+c}$, $w = \dfrac{y}{ax^2+bx+c}$ leads to an autonomous equation of the form 15.5.5.9: $w'''_{\xi\xi\xi} + (4ac-b^2)w'_{\xi} = f(w)$.

14. $y'''_{xxx} = y^{-2} f\left(\dfrac{y}{ax^2+bx+c}\right).$

Setting $f(u) = u^2 f_1(u)$, we have equation 15.5.1.13 with the function f_1 (instead of f).

15. $y'''_{xxx} = e^{\lambda x} f(ye^{-\lambda x}).$

This is a special case of equation 15.5.3.32 with $a = b = c = 0$. The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation.

16. $y'''_{xxx} = yf(e^{\lambda x} y^m).$

This is a special case of equation 15.5.5.21. The transformation $z = e^{\lambda x} y^m$, $w(z) = y'_x/y$ leads to a second-order equation.

17. $y'''_{xxx} = x^{-3} f(x^m e^{\lambda y}).$

The transformation $z = x^m e^{\lambda y}$, $w(z) = xy'_x$ leads to a second-order equation.

18. $y'''_{xxx} = f(y+ae^{\lambda x}) - a\lambda^3 e^{\lambda x}.$

The substitution $w = y + ae^{\lambda x}$ leads to an autonomous equation of the form 15.5.1.1: $w'''_{xxx} = f(w)$.

19. $y'''_{xxx} = F(x, y).$

The transformation $x = 1/t$, $y = w/t^2$ leads to an equation of the same form: $w'''_{ttt} = -t^{-4} F(1/t, w/t^2)$.

15.5.2 Equations of the Form $F(x,y,y'_x)y'''_{xxx} + G(x,y,y'_x) = 0$

► **Arguments of the arbitrary functions depend on x and y.**

1. $y'''_{xxx} = f(y)y'_x.$

Solution: $C_3 \pm x = \displaystyle\int \left[C_2 y + C_1 + 2\int F(y)\,dy\right]^{-1/2} dy$, where $F(y) = \displaystyle\int f(y)\,dy$.

2. $y'''_{xxx} = f(y)y'_x + g(x).$

Integrating yields a second-order equation: $y''_{xx} = \displaystyle\int f(y)\,dy + \int g(x)\,dx + C$.

3. $y'''_{xxx} = f(y)y'_x + g(y)(y'_x)^3.$

The substitution $z(y) = (y'_x)^2$ leads to a second-order linear equation: $z''_{yy} = 2g(y)z + 2f(y)$.

4. $y'''_{xxx} = f(y)(y'_x)^3 + a(y'_x)^{-5}.$

The substitution $z(y) = \pm(y'_x)^2$ leads to Yermakov's equation 2.9.1.2: $z''_{yy} = 2f(y)z + 2az^{-3}$.

5. $y'''_{xxx} = x^{-3}f\Big(\dfrac{y}{x}\Big)(xy'_x - y).$

The transformation $z = y/x$, $w = x^{-2}(xy'_x - y)^2$ leads to a second-order linear equation: $w''_{zz} = 2f(z) + 2$. Integrating the latter equation twice, we arrive at a first-order homogeneous equation for $y(x)$:

$$y'_x = z \pm \Big[z^2 + C_2 z + C_1 + 2\int_{z_0}^{z}(z-t)f(t)\,dt\Big]^{1/2}, \quad \text{where} \quad z = \frac{y}{x},$$

and z_0 is an arbitrary number.

6. $y'''_{xxx} = x^{-5}f\Big(\dfrac{y}{x}\Big)(xy'_x - y)^3.$

The transformation $z = y/x$, $w = x^{-2}(xy'_x - y)^2$ leads to a second-order linear equation: $w''_{zz} = 2f(z)w + 2$.

7. $y'''_{xxx} = x^{-3}f\Big(\dfrac{y}{x}\Big)(xy'_x - y) + x^{-5}g\Big(\dfrac{y}{x}\Big)(xy'_x - y)^3.$

This is a special case of equation 15.5.3.38 with $k = -1$. The transformation $t = \ln x$, $z = y/x$, followed by the substitution $w(z) = (z'_t)^2$, leads to a second-order linear equation: $w''_{zz} = 2g(z)w + 2f(z) + 2$.

8. $y'''_{xxx} = x^{-5}f\Big(\dfrac{y}{x^2}\Big)(xy'_x - 2y).$

The transformation $t = 1/x$, $z = y/x^2$ leads to an autonomous equation: $z'''_{ttt} = f(z)z'_t$. The substitution $w(z) = (z'_t)^2$ then yields the second-order linear equation $w''_{zz} = 2f(z)$, whose solution is given by:

$$w = C_2 z + C_1 + 2\int_{z_0}^{z}(z-\xi)f(\xi)\,d\xi, \qquad z_0 \text{ is an arbitrary number.}$$

9. $y'''_{xxx} = x^{-7}f\Big(\dfrac{y}{x^2}\Big)(xy'_x - 2y)^3.$

The transformation $t = 1/x$, $z = y/x^2$, followed by the substitution $w(z) = (z'_t)^2$, leads to a second-order linear equation: $w''_{zz} = 2f(z)w$.

10. $y'''_{xxx} = x^{-5}f\Big(\dfrac{y}{x^2}\Big)(xy'_x - 2y) + x^{-7}g\Big(\dfrac{y}{x^2}\Big)(xy'_x - 2y)^3.$

The transformation $t = 1/x$, $z = y/x^2$, followed by the substitution $w(z) = (z'_t)^2$, leads to a second-order linear equation: $w''_{zz} = 2g(z)w + 2f(z)$.

11. $y'''_{xxx} + fy'_x + gy = -3hy^{-4}(y'_x)^2 - 3(h'_x y^3 + 2h^2)y^{-6}y'_x - h^3y^{-8} - 3hh'_x y^{-5} - (fh + h''_{xx})y^{-2}.$

Here, $f = f(x)$, $g = g(x)$, and $h = h(x)$ are arbitrary functions.

Solution: $y = w\Bigl[C + 3\int \frac{h(x)\,dx}{w^3}\Bigr]^{1/3}$, where $w = w(x)$ is the general solution of the linear equation: $w'''_{xxx} + f(x)w'_x + g(x)w = 0$.

12. $[ay + f(x)]y'''_{xxx} + g(y)y'_x + f'''_{xxx}(x)y + h(x) = 0.$

The equation admits a first integral:

$$[ay + f(x)]y''_{xx} - \tfrac{1}{2}a(y'_x)^2 - f'_x(x)y'_x + f''_{xx}(x)y + \int g(y)\,dy + \int h(x)\,dx = C.$$

► **Arguments of the arbitrary functions depend on x, y, and y'_x.**

13. $y'''_{xxx} = f(y'_x).$

Solution in parametric form:

$$x = \int_{C_2}^{\tau} \frac{d\tau}{\varphi(\tau)}, \qquad y = \int_{C_3}^{\tau} \frac{\tau\,d\tau}{\varphi(\tau)}, \qquad \text{where} \quad \varphi = \pm\Bigl[C_1 + 2\int f(\tau)\,d\tau\Bigr]^{1/2}.$$

14. $y'''_{xxx} = f(x, y'_x).$

The substitution $w(x) = y'_x$ leads to a second-order equation: $w''_{xx} = f(x, w)$.

15. $y'''_{xxx} = f(y, y'_x).$

Autonomous equation (this is a special case of equation 15.5.5.9).

1°. The substitution $u(y) = (y'_x)^2$ leads to a second-order equation:

$$u''_{yy} = \pm 2u^{-1/2} f(y, \pm u^{1/2}).$$

2°. The transformation

$$x = \int [\varphi(\tau)]^{-3/2}\,d\tau, \qquad y = [\varphi(\tau)]^{-1} \tag{1}$$

leads to an analogous equation with respect to $\varphi = \varphi(\tau)$:

$$\varphi'''_{\tau\tau\tau} = -\varphi^{-5/2} f(\varphi^{-1},\ -\varphi^{-1/2}\varphi'_\tau).$$

Note two important cases of transforming equations of special form:

$$y'''_{xxx} = f(y) \quad \xrightarrow{\text{transformation (1)}} \quad \varphi'''_{\tau\tau\tau} = -\varphi^{-5/2} f(\varphi^{-1}),$$
$$y'''_{xxx} = Ay^n \quad \xrightarrow{\text{transformation (1)}} \quad \varphi'''_{\tau\tau\tau} = -A\varphi^{-n-5/2}.$$

16. $y'''_{xxx} = a^3 y + f(x, y'_x - ay).$

The substitution $w = y'_x - ay$ leads to a second-order equation: $w''_{xx} + aw'_x + a^2 w = f(x, w)$.

17. $y'''_{xxx} = (3a^2x - a^3x^3)y + f(x, y'_x + axy)$.

The substitution $w = y'_x + axy$ leads to a second-order equation:

$$w''_{xx} - axw'_x + (a^2x^2 - 2a)w = f(x, w).$$

18. $y'''_{xxx} = xf(xy'_x - y)$.

The substitution $z = xy'_x - y$ leads to a second-order equation of the form 14.9.2.20 with $n = 1$: $xz''_{xx} = z'_x + x^3 f(z)$.

19. $y'''_{xxx} = x^{-3} f(xy'_x - y)$.

The transformation $t = \ln|x|$, $z = xy'_x - y$ leads to the second-order autonomous equation $z''_{tt} - 2z'_t = f(z)$, which is reduced, with the aid of the substitution $w(z) = \frac{1}{2}z'_t$, to the Abel equation $ww'_z - w = \frac{1}{4}f(z)$ (for some functions f, solutions of this Abel equation are given in Section 13.3.1).

20. $y'''_{xxx} = f(x, xy'_x - y)$.

The substitution $w = xy'_x - y$ leads to a second-order equation: $(w'_x/x)'_x = f(x, w)$.

21. $y'''_{xxx} = f(x,\, xy'_x - 2y)$.

The substitution $w = xy'_x - 2y$ leads to a second-order equation: $w''_{xx} = xf(x, w)$.

22. $x^3 y'''_{xxx} = f(x, xy'_x + ay) - a(a+1)(a+2)y$.

The substitution $w = xy'_x + ay$ leads to a second-order equation:

$$x^2 w''_{xx} - (a+2)xw'_x + (a+1)(a+2)w = f(x, w).$$

23. $y'''_{xxx} = x^{-2} F\Big(y'_x - \dfrac{y}{x}\Big)$.

The substitution $z = xy'_x - y$ leads to the second-order equation $xz''_{xx} = z'_x + F(z/x)$, which is a special case of the equation 14.9.4.22 with $n = -1$, $m = 1$, $k = -1$, $F(\xi) = \xi f(\xi)$.

24. $y'''_{xxx} = x^{-4} f\Big(y'_x - 2\dfrac{y}{x}\Big)$.

The transformation $t = 1/x$, $z = y/x^2$ yields $z'''_{ttt} = -f(-z'_t)$. The substitution $w = -z'_t$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{tt} = f(w)$.

25. $y'''_{xxx} = \dfrac{1}{x^4} f\Big(\dfrac{y}{x^2},\, y'_x - 2\dfrac{y}{x}\Big)$.

The transformation $t = 1/x$, $z = y/x^2$ leads to an equation of the form 15.5.2.15: $z'''_{ttt} = -f(z, -z'_t)$, which admits, with the aid of the substitution $w(z) = (z'_t)^2$, reduction of its order: $w''_{zz} = \pm 2w^{-1/2} f(z, \pm w^{1/2})$.

26. $y'''_{xxx} = yx^{-3} f(xy'_x/y)$.

The transformation $z = xy'_x/y$, $w = x^2 y''_{xx}/y$ leads to a first-order equation:

$$(w + z - z^2)w'_z = 2w - zw + f(z).$$

27. $y'''_{xxx} = y^{-5/2} f\left(\dfrac{y'_x}{\sqrt{y}}\right)$.

With the substitution $w(y) = (y'_x)^2$, one can reduce this equation to an equation of the form 14.9.1.8: $w''_{yy} = y^{-3} F(w/y)$, where $F(\xi) = \pm 2\xi^{-1/2} f(\pm \xi^{1/2})$.

28. $y'''_{xxx} = y^{-5/4} f\left(\dfrac{y'_x}{y^{1/4}}\right).$

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 14.9.1.9:

$$w''_{yy} = y^{-3/2} F\bigl(wy^{-1/2}\bigr), \quad \text{where} \quad F(\xi) = \pm\xi^{-1/2} f(\pm\xi^{1/2}).$$

29. $y'''_{xxx} = (ay^2 + by + c)^{-5/4} f\left(\dfrac{y'_x}{(ay^2 + by + c)^{1/4}}\right).$

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 14.9.1.21:

$$w''_{yy} = w^{-3} F\left(\frac{w}{\sqrt{ay^2 + by + c}}\right), \quad \text{where} \quad F(\xi) = \pm 2\xi^{5/2} f(\pm\xi^{1/2}).$$

30. $y'''_{xxx} = x(ay^2 + bx^2 y + cx^4)^{-5/4} f\left(\dfrac{xy'_x - 2y}{(ay^2 + bx^2 y + cx^4)^{1/4}}\right).$

The transformation $t = 1/x$, $z = y/x^2$ leads to an equation of the form 15.5.2.29:

$$z'''_{ttt} = -(az^2 + bz + c)^{-5/4} f\left(\frac{-z'_t}{(az^2 + bz + c)^{1/4}}\right).$$

31. $y'''_{xxx} = -x^{-2} y'_x + x^{-3} y + x^{1/2} y^{-5/2} f\left(\dfrac{xy'_x - y}{\sqrt{xy}}\right).$

The transformation $t = \ln x$, $z = y/x$, followed by the substitution $w(z) = (z'_t)^2$, leads to a second-order equation of the form 14.9.1.8: $w''_{zz} = z^{-3} F(w/z)$, where $F(\xi) = \pm 2\xi^{-1/2} f\bigl(\pm\sqrt{\xi}\,\bigr)$.

32. $y'''_{xxx} = -x^{-2} y'_x + x^{-3} y + x^{-3/4} y^{-5/4} f\left(\dfrac{xy'_x - y}{x^{3/4} y^{1/4}}\right).$

The transformation $t = \ln x$, $z = y/x$, followed by the substitution $w(z) = (z'_t)^2$, leads to a second-order equation of the form 14.9.1.9: $w''_{zz} = z^{-3/2} F(wz^{-1/2})$, where $F(\xi) = \pm 2\xi^{-1/2} f\bigl(\pm\sqrt{\xi}\,\bigr)$.

33. $y'''_{xxx} = y'_x f\bigl(y'^2_x + ay\bigr).$

The substitution $w(y) = (y'_x)^2 + ay$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{yy} = 2f(w)$.

34. $y'''_{xxx} = 2\lambda^2 (y'_x)^3 + e^{6\lambda y} f(e^{\lambda y} y'_x) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = e^{-2\lambda y}$.

35. $y'''_{xxx} = \frac{1}{2}\lambda^2 (y'_x)^3 + (\cosh \lambda y)^{-3} f\left(\dfrac{y'_x}{\sqrt{\cosh \lambda y}}\right) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \cosh \lambda y$.

36. $y'''_{xxx} = \frac{1}{2}\lambda^2 (y'_x)^3 + (\sinh \lambda y)^{-3} f\left(\dfrac{y'_x}{\sqrt{\sinh \lambda y}}\right) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \sinh \lambda y$.

37. $y'''_{xxx} = y \tanh x + f(x, y'_x - y \tanh x).$

This is a special case of equation 15.5.2.47 with $\varphi(x) = \cosh x$.

38. $y'''_{xxx} = (\sinh y)^{-2}(y'_x)^3 + (\tanh y)^3 f\big(y'_x \sqrt{\tanh y}\,\big) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \coth y$.

39. $y'''_{xxx} = y \coth x + f(x, y'_x - y \coth x).$

This is a special case of equation 15.5.2.47 with $\varphi(x) = \sinh x$.

40. $y'''_{xxx} = -(\cosh y)^{-2}(y'_x)^3 + (\coth y)^3 f\big(y'_x \sqrt{\coth y}\,\big) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \tanh y$.

41. $y'''_{xxx} = -\frac{1}{2}\lambda^2 (y'_x)^3 + (\cos \lambda y)^{-3} f\left(\dfrac{y'_x}{\sqrt{\cos \lambda y}}\right) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \cos \lambda y$.

42. $y'''_{xxx} = -\frac{1}{2}\lambda^2 (y'_x)^3 + (\sin \lambda y)^{-3} f\left(\dfrac{y'_x}{\sqrt{\sin \lambda y}}\right) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \sin \lambda y$.

43. $y'''_{xxx} = y \tan x + f(x, y'_x + y \tan x).$

This is a special case of equation 15.5.2.47 with $\varphi(x) = \cos x$.

44. $y'''_{xxx} = (\sin y)^{-2}(y'_x)^3 + (\tan y)^3 f\big(y'_x \sqrt{\tan y}\,\big) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \cot y$.

45. $y'''_{xxx} = -y \cot x + f(x, y'_x - y \cot x).$

This is a special case of equation 15.5.2.47 with $\varphi(x) = \sin x$.

46. $y'''_{xxx} = (\cos y)^{-2}(y'_x)^3 + (\cot y)^3 f\big(y'_x \sqrt{\cot y}\,\big) y'_x.$

This is a special case of equation 15.5.2.48 with $\psi(y) = \tan y$.

47. $y'''_{xxx} = \dfrac{\varphi'''_{xxx}}{\varphi} y + f\left(x, y'_x - \dfrac{\varphi'_x}{\varphi} y\right), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi} y$ leads to a second-order equation:

$$w''_{xx} + \frac{\varphi'_x}{\varphi} w'_x + \left[2\frac{\varphi''_{xx}}{\varphi} - \frac{(\varphi'_x)^2}{\varphi^2}\right] w = f(x, w).$$

48. $y'''_{xxx} = \dfrac{1}{2}\dfrac{\psi''_{yy}}{\psi}(y'_x)^3 + \psi^{-3} f\left(\dfrac{y'_x}{\sqrt{\psi}}\right) y'_x, \qquad \psi = \psi(y).$

The substitution $z = (y'_x)^2$ leads to a second-order equation of the form 14.9.1.46:

$$z''_{yy} = \frac{\psi''_{yy}}{\psi} z + 2\psi^{-3} f\left(\pm\sqrt{\frac{z}{\psi}}\right).$$

49. $y'''_{xxx} = F(x, y, y'_x)$.

Let $F \neq \varphi(x)y'_x + \psi(x)y + \chi(x)$, i.e., the equation is nonlinear. Then its order can be reduced by one if the right-hand side of the equation has the following form:

$$F(x, y, y'_x) = f^{-2}E\Big\{\Phi(u, w) + \int \big[2ff'''_{xxx}w + (f^2 f''''_{xxxx} + 2ff'_x f'''_{xxx})u - (2f'''_{xxx}g + fg'''_{xxx})E^{-1} - (2ff'_x f'''_{xxx} + f^2 f''''_{xxxx} - 2kff'''_{xxx})V\big]\,dx\Big\},$$

where

$$E = \exp\Big(-k\int \frac{dx}{f}\Big), \quad V = \int \frac{g\,dx}{f^2 E}, \quad u = \frac{y}{fE} + V, \quad w = \frac{fy'_x - f'_x y + g}{fE} - kV;$$

$\Phi = \Phi(u, w)$, $f = f(x)$, and $g = g(x)$ are arbitrary functions; k is an arbitrary constant.

In this case, the transformation $t = \int f^{-1}\,dx$, $u = f^{-1}E^{-1}y + V$, followed by the substitution $z(u) = u'_t$, leads to a second-order equation:

$$z^2 z''_{uu} + z(z'_u)^2 - 3kzz'_u + 3k^2 z - k^3 u = \Phi(u,\, z - ku).$$

50. $y'''_{xxx} = f(y)^{-3/2}y'_x\Phi(u) + \dfrac{2f(y)f''_{yy}(y) - [f'_y(y)]^2}{8f(y)^2}(y'_x)^3 + \dfrac{f'_y(y)g(y) - 2f(y)g'_y(y)}{4f(y)^2}y'_x$.

The functions $f(y)$, $g(y)$, and $\Phi(u)$ are arbitrary and $u = f^{-1/2}(y'_x)^2 + \int gf^{-3/2}\,dy$. Autonomous first integral:

$$f(y)(y''_{xx})^2 + \Big[-\frac{1}{2}f'_y(y)(y'_x)^2 + g(y)\Big]y''_{xx} + \frac{1}{16}\frac{[f'_y(y)]^2}{f(y)}(y'_x)^4 - \int \Phi(u)\,du - \frac{f'_y(y)g(y)}{4f(y)}(y'_x)^2 + \frac{1}{4}\frac{g(y)^2}{f(y)} = C.$$

51. $y'''_{xxx} = y'_x\big[a\big(y'_x\big)^2 + f(y)\big]g(y) - \frac{1}{2a}f''_{yy}(y)y'_x$.

The autonomous first integral $P = P(y, y'_x, y''_{xx})$ satisfies the linear first-order partial differential equation

$$\frac{\partial P}{\partial y} + \big[g(y) - 2a\omega^2\big]\frac{\partial P}{\partial \omega} = 0, \quad \omega = \frac{2ay'_x + f'_y(y)}{a\big(y'_x\big)^2 + f(y)},$$

which is reduced to a Riccati equation. Therefore, in a large number of cases, the first integral can be expressed in terms of elementary or standard special functions. Its representation is essentially dependent on the function $g(y)$. For example, if $g(y) = y^k$ or $g(y) = e^y$, the second differential invariant is expressible in terms of Bessel functions; furthermore, for the power-law function, we obtain a special Riccati equation and if $\frac{k+3}{k+2}$ is half-integer, the first integral is an elementary function. For example, if $k = 0$, we get

$$P = \sqrt{2a}y - \operatorname{arctanh}\frac{2ay''_{xx} + f'_y(y)}{\sqrt{2a}\,\big[C(y'_x)^2 + f(y)\big]}.$$

15.5.3 Equations of the Form $F(x,y,y'_x)y'''_{xxx} + G(x,y,y'_x)y''_{xx} + H(x,y,y'_x) = 0$

► **The arbitrary functions depend on x or y.**

1. $y'''_{xxx} + 3\lambda y'_x y''_{xx} + \lambda^2(y'_x)^3 = f(x)e^{-\lambda y}$.

Solution: $e^{\lambda y} = C_2x^2 + C_1x + C_0 + \dfrac{\lambda}{2}\displaystyle\int_{x_0}^{x}(x-t)^2 f(t)\,dt$, where x_0 is an arbitrary number.

2. $y'''_{xxx} = f(y)y'_x y''_{xx}$.

Integrating yields a second-order autonomous equation of the form 14.9.1.1: $y''_{xx} = F(y)$, where $F(y) = C\exp\Bigl[\displaystyle\int f(y)\,dy\Bigr]$.

3. $y'''_{xxx} = [f(y)y'_x + g(x)]y''_{xx}$.

Integrating yields a second-order equation: $y''_{xx} = C\exp\Bigl[\displaystyle\int f(y)\,dy + \int g(x)\,dx\Bigr]$.

4. $x^3y'''_{xxx} + ax^2y''_{xx} + bxy'_x = f(y)$.

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 15.5.5.9:

$$y'''_{ttt} + (a-3)y''_{tt} + (b-a+2)y'_t = f(y).$$

5. $y(y'''_{xxx} + 3ay''_{xx} + 2a^2y'_x) = f(x)$.

Integrating yields a second-order equation:

$$2yy''_{xx} + 2ayy'_x - (y'_x)^2 = e^{-2ax}\Bigl[2\int e^{2ax}f(x)\,dx + C\Bigr].$$

6. $yy'''_{xxx} + 3y'_x y''_{xx} = f(x)$.

Solution: $y^2 = C_2x^2 + C_1x + C_0 + \displaystyle\int_{x_0}^{x}(x-t)^2 f(t)\,dt$, where x_0 is an arbitrary number.

7. $yy'''_{xxx} + ay'_x y''_{xx} = f(x)$.

Integrating yields a second-order equation: $yy''_{xx} + \tfrac{1}{2}(a-1)(y'_x)^2 = \displaystyle\int f(x)\,dx + C$.

8. $yy'''_{xxx} + 3y'_x y''_{xx} = f(y)$.

The substitution $w = y^2$ leads to an autonomous equation of the form 15.5.1.1: $w'''_{xxx} = 2f\bigl(\pm\sqrt{w}\bigr)$.

9. $yy'''_{xxx} - y'_x y''_{xx} = f(x)y^2$.

Integrating yields a second-order linear equation: $y''_{xx} = \Bigl[\displaystyle\int f(x)\,dx + C\Bigr]y$.

10. $yy'''_{xxx} + 3y'_x y''_{xx} + f(x)yy'_x = g(x)$.

The substitution $w = yy'_x$ leads to a second-order linear nonhomogeneous equation: $w''_{xx} + f(x)w = g(x)$.

11. $yy'''_{xxx} + 3y'_x y''_{xx} + 4f(x)yy'_x + f'_x(x)y^2 = 0.$

Multiplying by y^2, we arrive at an exact differential equation. Integrating it yields Yermakov's equation 14.9.1.2: $y''_{xx} + f(x)y = Cy^{-3}$.

There is also the trivial solution $y = 0$.

12. $yy'''_{xxx} + ay'_x y''_{xx} = f(y)y'_x + g(x).$

Integrating yields a second-order equation:

$$yy''_{xx} + \tfrac{1}{2}(a-1)(y'_x)^2 = \int f(y)\,dy + \int g(x)\,dx + C.$$

13. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = f(x).$

Solution:

$$y^2 = C_3 e^{-ax} + C_2 x + C_1 + 2\int_{x_0}^{x} (x-t)e^{-at}F(t)\,dt, \quad \text{where} \quad F(t) = \int e^{at} f(t)\,dt,$$

x_0 is an arbitrary number.

14. $yy'''_{xxx} + (3y'_x + 2ay)y''_{xx} + 2a(y'_x)^2 + a^2 yy'_x = f(x).$

Integrating the equation twice, we arrive at a first-order separable equation:

$$e^{ax} yy'_x = C_2 x + C_1 + \int_{x_0}^{x} (x-t)e^{at} f(t)\,dt.$$

15. $yy'''_{xxx} = y'_x y''_{xx} + f(x)yy''_{xx}.$

Integrating yields a second-order linear equation: $y''_{xx} = C\exp\Big[\int f(x)\,dx\Big]y.$

16. $yy'''_{xxx} - y'_x y''_{xx} = f(x)\big[yy''_{xx} - (y'_x)^2\big] + g(x).$

The substitution $w = yy''_{xx} - (y'_x)^2$ leads to a first-order linear equation: $w'_x = f(x)w + g(x)$.

17. $yy'''_{xxx} + [3y'_x + f(x)y]y''_{xx} + f(x)(y'_x)^2 = 0.$

Solution: $y^2 = C_3 x + C_2 + C_1 \displaystyle\int_{x_0}^{x} (x-t)e^{-F(t)}\,dt$, where $F(t) = \displaystyle\int f(t)\,dt.$

18. $yy'''_{xxx} + [3y'_x + f(x)y]y''_{xx} + f(x)(y'_x)^2 + g(x)yy'_x + h(x) = 0.$

The substitution $w = yy'_x$ leads to a second-order linear nonhomogeneous equation: $w''_{xx} + f(x)w'_x + g(x)w + h(x) = 0$.

19. $yy'''_{xxx} + (f-1)y'_x y''_{xx} + fgyy'_x + g'_x y^2 = 0, \qquad f = f(x), \quad g = g(x).$

A solution of this equation is any function that solves the second-order linear equation $y''_{xx} + g(x)y = 0$.

20. $(y+a)y'''_{xxx} + by'_x y''_{xx} = f(x).$

Having integrated the equation, we obtain $(y+a)y''_{xx} + \tfrac{1}{2}(b-1)(y'_x)^2 = \displaystyle\int f(x)\,dx + C.$

For $b \neq -1$, the substitution $y = w^{\frac{2}{b+1}} - a$ leads to the equation:

$$w''_{xx} = \frac{b+1}{2}\Big[\int f(x)\,dx + C\Big]w^{\frac{b-3}{b+1}}$$

(with $C = 0$ and $f(x) = \lambda x^n$, see Section 14.3).

21. $(y+a)y'''_{xxx}+by'_x y''_{xx}+f(y)y'_x=0.$

Having integrated the equation, we obtain a second-order autonomous equation:

$$(y+a)y''_{xx}+\tfrac{1}{2}(b-1)(y'_x)^2+\int f(y)\,dy=C,$$

which is reduced with the aid of the substitution $w(y)=(y'_x)^2$ to a first-order linear equation:

$$(y+a)w'_y+(b-1)w+2\int f(y)\,dy=2C.$$

22. $(y+a)y'''_{xxx}+by'_x y''_{xx}+f(y)y'_x=g(x).$

Having integrated the equation, we obtain a second-order equation:

$$(y+a)y''_{xx}+\tfrac{1}{2}(b-1)(y'_x)^2+\int f(y)\,dy=\int g(x)\,dx+C.$$

23. $(y+ax+b)y'''_{xxx}+3(y'_x+a)y''_{xx}=f(x).$

Solution: $(y+ax+b)^2=C_2x^2+C_1x+C_0+\int_{x_0}^{x}(x-t)^2f(t)\,dt,$ where x_0 is an arbitrary number.

24. $[y+f(x)]y'''_{xxx}=[y'_x+f'_x(x)]y''_{xx}+af(x)y'_x-af'_x(x)y.$

Integrating yields a second-order constant coefficient linear equation of the form 14.1.9.1: $y''_{xx}+Cy=-(a+C)f(x)$. There is also the trivial solution $y=0$.

25. $x(yy'''_{xxx}+3y'_x y''_{xx})+a\big[yy''_{xx}+(y'_x)^2\big]=f(x).$

Solution:

$$y^2=C_3x^{2-a}+C_2x+C_1+2\int_{x_0}^{x}(x-t)t^{-a}F(t)\,dt,\qquad\text{where}\quad F(t)=\int t^{a-1}f(t)\,dt;$$

x_0 is an arbitrary number.

26. $y^2y'''_{xxx}-3yy'_x y''_{xx}+2(y'_x)^3=f(x)y^3.$

Solution: $\ln|y|=C_2x^2+C_1x+C_0+\frac{1}{2}\int_{x_0}^{x}(x-t)^2f(t)\,dt,$ where x_0 is an arbitrary number.

27. $y^2y'''_{xxx}+3(n-1)yy'_x y''_{xx}+(n-1)(n-2)(y'_x)^3=f(x)y^{3-n}.$

Solution for $n\neq 0$:

$$y^n=C_2x^2+C_1x+C_0+\frac{n}{2}\int_{x_0}^{x}(x-t)^2f(t)\,dt,\qquad\text{where } x_0 \text{ is an arbitrary number.}$$

For the case $n=0$, see equation 15.5.3.26.

28. $y\big(fy'''_{xxx} + \frac{3}{2}f'_x y''_{xx} + \frac{1}{2}f''_{xx}y'_x\big) = g(x), \qquad f = f(x).$

Having integrated the equation, we obtain a second-order equation:

$$2fyy''_{xx} + f'_x yy'_x - f(y'_x)^2 = 2\int g(x)\,dx + C.$$

29. $fyy'''_{xxx} + (3fy'_x + 2f'_x y)y''_{xx} + 2f'_x(y'_x)^2 + f''_{xx}yy'_x = g(x), \qquad f = f(x).$

Integrating the equation twice, we arrive at a first-order separable equation: $f(x)yy'_x = C_2x + C_1 + \int_{x_0}^{x}(x-t)g(t)\,dt.$

30. $y'''_{xxx} + fy'_x + gy = -(n+2)hy^{n-1}y''_{xx}$
$- (n-1)(n+1)hy^{n-2}(y'_x)^2 - [(2n+1)h'_x + 3nh^2y^{n-1}]y^{n-1}y'_x$
$- h^3y^{3n-2} - 3hh'_x y^{2n-1} - (fh + h''_{xx})y^n.$

Here, $f = f(x)$, $g = g(x)$, and $h = h(x)$ are arbitrary functions.

Solution: $y = w\Big[C + (1-n)\int h(x)w^{n-1}\,dx\Big]^{\frac{1}{1-n}}$, where $w = w(x)$ is the general solution of the linear equation: $w'''_{xxx} + f(x)w'_x + g(x)w = 0.$

31. $y'_x y'''_{xxx} + f(x)y'_x y''_{xx} + g(x)yy''_{xx} + h(x)(y'_x)^2$
$+ \big[g'_x(x) + f(x)g(x)\big]yy'_x + g^2(x)y^2 = 0.$

The solution satisfies the second-order linear equation $y''_{xx} - z(x,C)y'_x + g(x)y = 0$, where $z = z(x,C)$ is the general solution of the Riccati equation $z'_x + z^2 + f(x)z - g(x) + h(x) = 0.$

▶ Arguments of arbitrary functions depend on x and y.

32. $y'''_{xxx} + ay''_{xx} + by'_x + cy = e^{\lambda x}f(ye^{-\lambda x}).$

The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation of the form 15.5.5.9:

$$w'''_{xxx} + (3\lambda + a)w''_{xx} + (3\lambda^2 + 2a\lambda + b)w'_x + (\lambda^3 + a\lambda^2 + b\lambda + c)w = f(w).$$

33. $y'''_{xxx} = -3y''_{xx} + 2y + f(e^x y)(y'_x + y).$

The transformation $z = e^x y$, $w = e^{2x}(y'_x + y)^2$ leads to a second-order linear equation: $w''_{zz} = 2f(z) + 6$. Integrating the latter, we find the solution:

$$\int\frac{dz}{\sqrt{3z^2 + C_2z + C_1 + 2\Phi(z)}} = \pm x + C_3, \quad \text{where } z = e^x y,\ \Phi(z) = \int\Big[\int f(z)\,dz\Big]dz.$$

34. $xy'''_{xxx} + 3y''_{xx} = f(xy).$

The substitution $w(x) = xy$ leads to an autonomous equation of the form 15.5.1.1: $w'''_{xxx} = f(w).$

35. $x^2y'''_{xxx} + 6xy''_{xx} + 6y'_x = f(x^2y).$

The substitution $w(x) = x^2y$ leads to an autonomous equation of the form 15.5.1.1: $w'''_{xxx} = f(w).$

36. $x^3 y'''_{xxx} + ax^2 y''_{xx} + bxy'_x = f(x^m e^{\lambda y})$.

The transformation $t = \ln x$, $\lambda w = \lambda y + mt$ leads to an autonomous equation of the form 15.5.5.9: $w'''_{ttt} + (a-3)w''_{tt} + (b-a+2)w'_t = f(e^{\lambda w}) + \dfrac{m}{\lambda}(b-a+2)$.

37. $x^3 y'''_{xxx} = -\frac{3}{2} x^2 y''_{xx} + f\Big(\frac{y}{\sqrt{x}}\Big)(2xy'_x - y)$.

The transformation $t = \dfrac{y}{\sqrt{x}}$, $z = \dfrac{1}{x}\big(xy'_x - \frac{1}{2}y\big)^2$ leads to the second-order linear equation $2z''_{tt} = 8f(t) + 1$, whose solution is given by:

$$z = \tfrac{1}{4}t^2 + C_2 t + C_1 + 4\int_{t_0}^{t} (t-\xi) f(\xi)\,d\xi, \qquad t_0 \text{ is an arbitrary number.}$$

Passing on to the variables x, $t = yx^{-1/2}$, we obtain a separable equation.

38. $x^3 y'''_{xxx} = -3(k+1)x^2 y''_{xx} + k(k+1)(2k+1)y + f(x^k y)(xy'_x + ky) + x^{2k} g(x^k y)(xy'_x + ky)^3$.

The transformation $t = \ln x$, $z = x^k y$, followed by the substitution $w(z) = (z'_t)^2$, leads to a second-order linear equation: $w''_{zz} = 2g(z)w + 2f(z) + 6k^2 + 6k + 2$.

39. $x^4 y'''_{xxx} = -\frac{3}{2} x^3 y''_{xx} + f\Big(\frac{y}{\sqrt{x}}\Big)(2xy'_x - y)^3$.

The transformation $t = \dfrac{y}{\sqrt{x}}$, $z = \dfrac{1}{x}(xy'_x - \frac{1}{2}y)^2$ leads to a second-order linear equation: $2z''_{tt} = 16f(t)z + \frac{1}{2}$.

40. $y^2 y'''_{xxx} = -3y^2 y''_{xx} + 2y^3 + y^2 f(e^x y)(y'_x + y) + g(e^x y)(y'_x + y)^3$.

The substitution $z(x) = e^x y$, followed by reduction of the equation order and the substitution $w(z) = (z'_x)^2$, leads to a second-order linear equation: $w''_{zz} = 2z^{-2} g(z)w + 2f(z) + 6$.

► **Arguments of arbitrary functions depend on x, y, and y'_x.**

41. $xy'''_{xxx} = f(xy'_x - y)y''_{xx}$.

The substitution $z = xy'_x - y$ leads to a second-order equation of the form 14.9.2.21: $xz''_{xx} = [f(z) + 1]z'_x$.

42. $xy'''_{xxx} + (1-a)y''_{xx} = x^{2a} f(xy'_x - y)$.

The substitution $z = xy'_x - y$ leads to a second-order equation of the form 14.9.2.20: $xz''_{xx} = az'_x + x^{2a+1} f(z)$.

43. $xy'''_{xxx} + (a+2)y''_{xx} = f(x, xy'_x + ay)$.

The substitution $w = xy'_x + ay$ leads to a second-order equation: $w''_{xx} = f(x, w)$.

44. $xy'''_{xxx} + (1-ax)y''_{xx} = e^{2ax} f(xy'_x - y)$.

The substitution $z = xy'_x - y$ leads to a second-order equation of the form 14.9.2.17: $z''_{xx} - az'_x = e^{2ax} f(z)$.

45. $xy'''_{xxx} = f(xy'_x - y + a\ln x)y''_{xx}$.

The substitution $z = xy'_x - y$ leads to a second-order equation of the form 14.9.2.39: $xz''_{xx} = \left[f\left(\ln(x^a e^z)\right) + 1\right]z'_x$.

46. $x^3y'''_{xxx} + x^2y''_{xx} = f(xy'_x - y)$.

The transformation $t = \ln|x|$, $z = xy'_x - y$ leads to an autonomous equation of the form 14.9.6.2: $z''_{tt} - z'_t = f(z)$, which is reduced, with the aid of the substitution $w = z'_t$, to the Abel equation $ww'_z - w = f(w)$ (see Section 13.3.1).

47. $x^4y'''_{xxx} + x^3y''_{xx} = f\Big(y'_x - \dfrac{y}{x}\Big)$.

The substitution $w(x) = xy'_x - y$ leads to an equation of the form 14.9.1.8: $w''_{xx} = x^{-3}f(w/x)$.

15.5.4 Equations of the Form $F(x,y,y'_x)y'''_{xxx} + \sum_\alpha G_\alpha(x,y,y'_x)(y''_{xx})^\alpha = 0$

▶ **Arbitrary functions depend on x or y.**

1. $yy'''_{xxx} + (y''_{xx})^2 - \frac{1}{4}(y'_x)^2 - \frac{1}{4}yy'_x - f(x)y^2 = 0$.

This is a special case of equation 15.5.4.4. The solution satisfies the second-order linear equation $y''_{xx} + \frac{1}{2}y'_x - z(x,C)y = 0$, where $z = z(x,C)$ is the general solution of the Riccati equation $z'_x + z^2 - \frac{1}{2}z = f(x)$.

2. $yy'''_{xxx} + (y''_{xx})^2 - y'_x y''_{xx} - f(x)y^2 = 0$.

The solution satisfies the second-order linear equation $y''_{xx} - z(x,C)y = 0$, where $z = z(x,C)$ is the general solution of the Riccati equation $z'_x + z^2 = f(x)$.

3. $yy'''_{xxx} + (y''_{xx})^2 + [2h(x) - 1]y'_x y''_{xx} + [f(x) + h(x)]yy''_{xx}$
$+ h(x)[h(x) - 1](y'_x)^2 + [h'_x(x) + f(x)h(x)]yy'_x = 0$.

This is a special case of equation 15.5.4.4 with $g(x) = 0$. The solution satisfies the second-order linear equation

$$y''_{xx} + h(x)y'_x - z(x,C)y = 0,$$

where $z(x,C) = F(x)\Big[C + \int F(x)\,dx\Big]^{-1}$, $F(x) = \exp\Big[-\int f(x)\,dx\Big]$.

4. $yy'''_{xxx} + (y''_{xx})^2 + [2h(x) - 1]y'_x y''_{xx} + [f(x) + h(x)]yy''_{xx}$
$+ h(x)[h(x) - 1](y'_x)^2 + [h'_x(x) + f(x)h(x)]yy'_x = g(x)y^2$.

The solution satisfies the second-order linear equation $y''_{xx} + h(x)y'_x - z(x,C)y = 0$, where $z = z(x,C)$ is the general solution of the Riccati equation $z'_x + z^2 + f(x)z = g(x)$.

5. $2y'_x y'''_{xxx} - (y''_{xx})^2 + f(x)(y'_x)^2 = ay^2 + 2by + c$.

Differentiating both sides of the equation with respect to x and dividing by y'_x, we arrive at a fourth-order linear equation: $y''''_{xxxx} + fy''_{xx} + \frac{1}{2}f'_x y'_x = ay + b$.

6. $2y'_x y'''_{xxx} - (y''_{xx})^2 - \lambda y'_x y''_{xx} + F(x)(y'_x)^2 = e^{\lambda x}(ay^2 + 2by + c).$

Multiplying both sides by $e^{-\lambda x}$, we arrive at an equation of the form 15.5.4.13 with $f(x) = e^{-\lambda x}$ and $g(x) = e^{-\lambda x}F(x)$.

7. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = f(x)(y'_x)^2 + g(y)(y'_x)^4.$

Solution:

$$\int \frac{dy}{u^2(y)} = \int \frac{dx}{w^2(x)} + C,$$

where $u = u(y)$ and $w = w(x)$ are the general solutions of the second-order linear equations:

$$4u''_{yy} - g(y)u = 0 \quad \text{and} \quad 4w''_{xx} + f(x)w = 0.$$

8. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = f(x)(y'_x)^2 + g(x)y^{-1}(y'_x)^{5/2}.$

The substitution $w(x) = \dfrac{y}{\sqrt{y'_x}}$ leads to a second-order nonhomogeneous linear equation: $4w''_{xx} + f(x)w + g(x) = 0.$

9. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = f(y)(y'_x)^4 + x^{-1}g(y)(y'_x)^{7/2}.$

Taking y to be the independent variable, we obtain an equation of the form 15.5.4.8 for $x = x(y)$: $2x'_y x'''_{yyy} - 3(x''_{yy})^2 = -f(y)(x'_y)^2 - g(y)x^{-1}(x'_y)^{5/2}.$

10. $2xy'_x y'''_{xxx} - x(y''_{xx})^2 + ny'_x y''_{xx} + F(x)(y'_x)^2 = x^{1-n}(ay^2 + 2by + c).$

Multiplying both sides by x^{n-1}, we arrive at an equation of the form 15.5.4.13 with $f(x) = x^n$ and $g(x) = x^{n-1}F(x)$.

11. $xy'_x y'''_{xxx} - 3x(y''_{xx})^2 + 3y'_x y''_{xx} = xf(y)(y'_x)^4 + g(y)(y'_x)^5.$

Taking y to be the independent variable, we obtain an equation of the form 15.5.3.10 for $x = x(y)$: $xx'''_{yyy} + 3x'_y x''_{yy} = -g(y) - f(y)xx'_y.$

12. $y'''_{xxx} = f(y)y'_x(y''_{xx})^m.$

Solution for $m \neq 1$:

$$C_3 \pm x = \int \left\{2\int \Big[(1-m)F(y) + C_2\Big]^{\frac{1}{1-m}} dy + C_1\right\}^{-\frac{1}{2}} dy, \quad \text{where} \quad F(y) = \int f(y)\,dy.$$

Solution for $m = 1$:

$$C_3 \pm x = \int \left[C_2 \int e^{F(y)}\,dy + C_1\right]^{-\frac{1}{2}} dy, \quad \text{where} \quad F(y) = \int f(y)\,dy.$$

13. $2fy'_x y'''_{xxx} - f(y''_{xx})^2 + f'_x y'_x y''_{xx} + g(x)(y'_x)^2 = ay^2 + 2by + c, \quad f = f(x).$

Differentiating both sides of the equation with respect to x and dividing by y'_x, we arrive at a fourth-order linear equation: $fy''''_{xxxx} + \frac{3}{2}f'_x y'''_{xxx} + (g + \frac{1}{2}f''_{xx})y''_{xx} + \frac{1}{2}g'_x y'_x = ay + b.$

► **Arguments of arbitrary functions depend on x, y, and y'_x.**

14. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = y^4 f\left(\frac{y'_x}{y^2}\right)$.

The substitution $w(x) = y(y'_x)^{-1/2}$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{xx} = F(w)$, where $F(w) = -\frac{1}{4}w^5 f(w^{-2})$.

15. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = x^{-4} y^4 f\left(\frac{xy'_x}{y^2}\right)$.

The substitution $w(x) = y(y'_x)^{-1/2}$ leads to a second-order equation of the form 14.9.1.9: $w''_{xx} = x^{-3/2}F(wx^{-1/2})$, where $F(\xi) = -\frac{1}{4}\xi^5 f(\xi^{-2})$.

16. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = x^{-8} y^4 f\left(\frac{x^2 y'_x}{y^2}\right)$.

The substitution $w(x) = y(y'_x)^{-1/2}$ leads to a second-order equation of the form 14.9.1.8: $w''_{xx} = x^{-3}F(wx^{-1})$, where $F(\xi) = -\frac{1}{4}\xi^5 f(\xi^{-2})$.

17. $y'''_{xxx} = \left[x^3 f(xy'_x - y) + ax^{-5}\right](y''_{xx})^3$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 15.5.2.4: $w'''_{ttt} = -f(w)(w'_t)^3 - a(w'_t)^{-5}$.

18. $y'''_{xxx} = x^{-5} f\left(\frac{xy'_x - y}{x^2}\right)(y''_{xx})^3$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 15.5.2.27: $w'''_{ttt} = w^{-5/2}F(w'_t w^{-1/2})$, where $F(\xi) = -\xi^{-5} f(\xi^{-2})$.

19. $y'''_{xxx} = x^{-5} f\left(\frac{xy'_x - y}{x^4}\right)(y''_{xx})^3$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 15.5.2.28: $w'''_{ttt} = w^{-5/4}F(w'_t w^{-1/4})$, where $F(\xi) = -\xi^{-5} f(\xi^{-4})$.

20. $y'''_{xxx} = [xf(y'_x) + yg(y'_x) + h(y'_x)](y''_{xx})^3 + \varphi(y'_x)(y''_{xx})^2$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to a linear equation:

$$w'''_{ttt} = -\varphi(t)w''_{tt} - [f(t) + tg(t)]w'_t + g(t)w - h(t).$$

21. $y'''_{xxx} = xf(xy'_x - y)(y''_{xx})^2 + xg(xy'_x - y)(y''_{xx})^k$.

With the Legendre transformation $x = w'_t$, $y = tw'_t - w$, one can reduce this equation to $w'''_{ttt} = -f(w)w'_t w''_{tt} - g(w)w'_t(w''_{tt})^{3-k}$. Further lowering the order with the substitution $z(w) = w''_{tt}$ $(z'_w = w'''_{ttt}/w'_t)$, one obtains a Bernoulli equation: $z'_w = -f(w)z - g(w)z^{3-k}$.

15.5.5 Other Equations

► **Equations of the form $F(x, y, y'_x, y''_{xx})y'''_{xxx} + G(x, y, y'_x, y''_{xx}) = 0$.**

1. $y'''_{xxx} = f(y''_{xx})$.

Solution in parametric form:

$$x = \int_{C_1}^{t} \frac{dt_1}{f(t_1)}, \qquad y = \int_{C_2}^{t} \frac{dt_1}{f(t_1)} \int_{C_3}^{t_1} \frac{t_2\, dt_2}{f(t_2)}.$$

2. $y'''_{xxx} = f(y)y'_x g(y''_{xx})$.

Integrating the equation and substituting $w(y) = \frac{1}{2}(y'_x)^2$, we arrive at a first-order equation:

$$\int \frac{d\xi}{g(\xi)} = \int f(y)\,dy + C, \qquad \text{where} \quad \xi = w'_y.$$

Solving this equation for w'_y, we obtain a separable equation.

3. $y'''_{xxx} = f(y)g(y'_x)h(y''_{xx})$.

The substitution $w(y) = \frac{1}{2}(y'_x)^2$ leads to a second-order equation:

$$w''_{yy} = f(y)\varphi(w)h(w'_y), \qquad \text{where} \quad \varphi(w) = \pm\frac{g\bigl(\pm\sqrt{2w}\,\bigr)}{\sqrt{2w}},$$

whose solvable cases for some functions f, g, and h are outlined in Section 14.7.

4. $y'''_{xxx} = xf(xy'_x - y)g(y''_{xx})$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$, where $w = w(t)$, leads to an equation of the form 15.5.5.2: $w'''_{ttt} = -f(w)w'_t g(1/w''_{tt})(w''_{tt})^3$.

5. $xy'''_{xxx} + y''_{xx} = f(xy'_x - y)g(xy''_{xx})$.

The substitution $w(x) = xy'_x - y$ leads to an equation of the form 14.9.4.36: $w''_{xx} = f(w)g(w'_x)$.

6. $y'''_{xxx} = f(x)g(x^2y''_{xx} - 2xy'_x + 2y)$.

The substitution $w(x) = x^2y''_{xx} - 2xy'_x + 2y$ leads to a first-order separable equation: $w'_x = x^2f(x)g(w)$.

7. $y'''_{xxx} = \dfrac{y'_x y''_{xx}}{y} + \left[y''_{xx} - \dfrac{(y'_x)^2}{y}\right] f\left(\dfrac{y'_x}{y}\right) g\left(\dfrac{y''_{xx}}{y}\right)$.

The transformation $t = y'_x/y$, $w = y''_{xx}/y$ leads to a first-order separable equation: $w'_t = f(t)g(w)$.

8. $y'''_{xxx} = F(x, y'_x, y''_{xx})$.

The substitution $u(x) = y'_x$ leads to a second-order equation: $u''_{xx} = F(x, u, u'_x)$.

9. $y'''_{xxx} = F(y, y'_x, y''_{xx})$.

Autonomous equation. The substitution $w(y) = (y'_x)^2$ leads to a second-order equation: $w''_{yy} = \pm\dfrac{2}{\sqrt{w}} F\bigl(y, \pm\sqrt{w}, \frac{1}{2}w'_y\bigr)$.

10. $y'''_{xxx} = yF(y'_x/y,\, y''_{xx}/y)$.

This is a special case of equation 15.5.5.9. The transformation $t = y'_x/y$, $w = y''_{xx}/y$ leads to a first-order equation: $(w - t^2)w'_t = -tw + F(t, w)$.

11. $y'''_{xxx} = x^{-k-3}F(x^k y,\ x^{k+1}y'_x,\ x^{k+2}y''_{xx}).$

Generalized homogeneous equation. The transformation $t = \ln x,\ z = x^k y,$ followed by the substitution $w(z) = (z'_t)^2,$ leads to a second-order equation:

$$w''_{zz} = \pm 3(k+1)w^{-1/2}w'_z - 6k^2 - 12k - 4 \pm 2k(k+1)(k+2)zw^{-1/2} \pm 2w^{-1/2}F\bigl(z,\ \pm w^{1/2} - kz,\ \tfrac{1}{2}w'_z \mp (2k+1)w^{1/2} + k(k+1)z\bigr).$$

12. $y'''_{xxx} = yx^{-3}F(x^k y^m,\ xy'_x/y,\ x^2y''_{xx}/y).$

Generalized homogeneous equation. The transformation $t = x^k y^m,\ z = xy'_x/y$ leads to a second-order equation.

13. $y'''_{xxx} = yx^{-3}F(xy'_x/y,\ x^2y''_{xx}/y).$

This is a special case of equation 15.5.5.12. The transformation $z = xy'_x/y,\ w = x^2y''_{xx}/y$ leads to a first-order equation: $(w + z - z^2)w'_z = 2w - zw + F(z, w).$

14. $y'''_{xxx} = F(x,\ xy'_x - y,\ y''_{xx}).$

The substitution $z = xy'_x - y$ leads to a second-order equation: $xz''_{xx} = z'_x + x^2F(x, z, z'_x/x).$

15. $y'''_{xxx} = f(x, y''_{xx} + y'_x + y) + y.$

The substitution $w = y''_{xx} + y'_x + y$ leads to a first-order equation: $w'_x = f(x, w) + w.$

16. $y'''_{xxx} = f(x, y''_{xx} - y'_x + y) - y.$

The substitution $w = y''_{xx} - y'_x + y$ leads to a first-order equation: $w'_x = f(x, w) - w.$

17. $y'''_{xxx} = F(x, y, y'_x, y''_{xx}).$

The Legendre transformation $x = w'_t,\ y = tw'_t - w$ leads to the equation

$$w'''_{ttt} = -F\bigl(w'_t,\ tw'_t - w,\ t,\ 1/w''_{tt}\bigr)(w''_{tt})^3.$$

18. $y'''_{xxx} = y'_x f(yy''_{xx} - y'^2_x).$

1°. Particular solution:

$$y = C_1\exp(C_3x) + C_2\exp(-C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint $C_3^2 = f\bigl(4C_1C_2C_3^2\bigr).$

2°. Particular solution:

$$y = C_1\cos(C_3x) + C_2\sin(C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint $C_3^2 + f\bigl(-(C_1^2 + C_2^2)C_3^2\bigr) = 0.$

19. $y'_x y'''_{xxx} - (y''_{xx})^2 = f(yy''_{xx} - y'^2_x).$

1°. Particular solution:

$$y = C_1\exp(C_3x) + C_2\exp(-C_3x),$$

where the three constants C_1, C_2, and C_3 are related by the constraint $4C_1C_2C_3^4 + f(4C_1C_2C_3^2) = 0.$

2°. Particular solution:

$$y = C_1\cos(C_3x) + C_2\sin(C_3x),$$

with C_1, C_2, and C_3 related by the constraint $(C_1^2 + C_2^2)C_3^4 + f\bigl(-(C_1^2 + C_2^2)C_3^2\bigr) = 0.$

20. $y'''_{xxx} = e^{-\lambda x} F(e^{\lambda x} y,\ e^{\lambda x} y'_x,\ e^{\lambda x} y''_{xx})$.

The substitution $z = e^{\lambda x} y$ leads to an autonomous equation of the form 15.5.5.9:

$$z'''_{xxx} - 3\lambda z''_{xx} + 3\lambda^2 z'_x - \lambda^3 z = F(z,\ z'_x - \lambda z,\ z''_{xx} - 2\lambda z'_x + \lambda^2 z).$$

21. $y'''_{xxx} = yF(e^{\lambda x} y,\ y'_x/y,\ y''_{xx}/y)$.

Equation invariant under "translation–dilatation" transformation. The transformation $z = e^{\lambda x} y$, $w = y'_x/y$ leads to a second-order equation:

$$z^2(w+\lambda)^2 w''_{zz} + z^2(w+\lambda)(w'_z)^2 + z(w+\lambda)(4w+\lambda)w'_z + w^3 = F\bigl(z,\ w,\ z(w+\lambda)w'_z + w^2\bigr).$$

22. $y'''_{xxx} = x^{-3} F(x^m e^y,\ xy'_x,\ x^2 y''_{xx})$.

Equation invariant under "dilatation–translation" transformation. The transformation $z = x^m e^y$, $w = xy'_x + m$ leads to a second-order equation:

$$z^2 w^2 w''_{zz} + z^2 w(w'_z)^2 + zw^2 w'_z - 3zww'_z + 2w - 2m = F(z,\ w-m,\ zww'_z - w + m).$$

► **Equations of the form** $F(x, y, y'_x, y''_{xx}, y'''_{xxx}) = 0$.

23. $yy'''_{xxx} - \frac{1}{3} y'_x y''_{xx} = xf(y'''_{xxx}) + g(y'''_{xxx})$.

Particular solution:

$$y = \tfrac{1}{6} C_1 x^3 + \tfrac{1}{2} C_2 x^2 + C_3 x + C_4,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$2C_1 C_3 - C_2^2 = 3f(C_1),$$
$$3C_1 C_4 - C_2 C_3 = 3g(C_1).$$

Here, C_3 and C_4 are defined in terms of two arbitrary constants C_1 and C_2.

24. $yy'''_{xxx} - \frac{1}{3} y'_x y''_{xx} = y''_{xx} f(y'''_{xxx}) + xg(y'''_{xxx}) + h(y'''_{xxx})$.

Particular solution:

$$y = \tfrac{1}{6} C_1 x^3 + \tfrac{1}{2} C_2 x^2 + C_3 x + C_4,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$\tfrac{2}{3} C_1 C_3 - \tfrac{1}{3} C_2^2 = C_1 f(C_1) + g(C_1),$$
$$C_1 C_4 - \tfrac{1}{3} C_2 C_3 = C_2 f(C_1) + h(C_1).$$

Here, C_3 and C_4 are defined in terms of two arbitrary constants C_1 and C_2.

25. $F(x,\ yy''_{xx} - y'^2_x,\ yy'''_{xxx} - y'_x y''_{xx}) = 0$.

The substitution $w = yy''_{xx} - (y'_x)^2$ leads to a first-order equation: $F(x, w, w'_x) = 0$.

26. $F\bigl(y''_{xx}/y,\ yy''_{xx} - (y'_x)^2,\ y'''_{xxx}/y'_x,\ y'_x y'''_{xxx} - (y''_{xx})^2\bigr) = 0$.

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F\left(C_3^2,\, 4C_1C_2C_3^2,\, C_3^2,\, -4C_1C_2C_3^4\right) = 0.$$

2°. Particular solution:

$$y = C_1\cos(C_3x) + C_2\sin(C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$F\left(-C_3^2, -(C_1^2 + C_2^2)C_3^2, -C_3^2, -(C_1^2 + C_2^2)C_3^4\right) = 0.$$

27. $\boldsymbol{F\left(\dfrac{y''_{xx}}{y'_x},\, y'_x - y\dfrac{y''_{xx}}{y'_x},\, \dfrac{y'''_{xxx}}{y'_x}\right) = 0.}$

Particular solution:

$$y = C_1\exp(C_2x) + C_3,$$

where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $F(C_2, -C_2C_3, C_2^2) = 0$.

28. $\boldsymbol{F\left(y\dfrac{y'''_{xxx}}{y'_x} + ay''_{xx},\, y^{a+1}\dfrac{y'''_{xxx}}{y'_x}\right) = 0.}$

A solution of this equation is any function that solves the following second-order autonomous equation of the form 14.9.1.1:

$$y''_{xx} = C_1y^{-a} + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(aC_2, -aC_1) = 0$.

29. $\boldsymbol{F\left(\dfrac{y'''_{xxx}}{y'_x} + y''_{xx},\, e^y\dfrac{y'''_{xxx}}{y'_x}\right) = 0.}$

A solution of this equation is any function that solves the following second-order autonomous equation of the form 14.9.1.1:

$$y''_{xx} = C_1e^{-y} + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(C_2, -C_1) = 0$.

30. $\boldsymbol{F\left(\dfrac{1}{\varphi'_y}\dfrac{y'''_{xxx}}{y'_x},\, y''_{xx} - \dfrac{\varphi}{\varphi'_y}\dfrac{y'''_{xxx}}{y'_x}\right) = 0, \qquad \varphi = \varphi(y).}$

A solution of this equation is any function that solves the following second-order autonomous equation of the form 14.9.1.1:

$$y''_{xx} = C_1\varphi(y) + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(C_1, C_2) = 0$.

31. $\boldsymbol{F(y''_{xx},\, xy''_{xx} - y'_x,\, 2yy''_{xx} - y'^2_x,\, y'''_{xxx}) = 0.}$

Particular solution:

$$y = C_1x^2 + C_2x + C_3,$$

where the constants C_1, C_2, and C_3 are related by $F(2C_1, -C_2, 4C_1C_3 - C_2^2, 0) = 0$.

32. $\boldsymbol{F(y'''_{xxx},\, xy'''_{xxx} - y''_{xx},\, x^2y'''_{xxx} - 2xy''_{xx} + 2y'_x,}$
$\boldsymbol{x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y) = 0.}$

Solution:

$$y = C_1x^3 + C_2x^2 + C_3x + C_4,$$

where the constants C_1, C_2, C_3, and C_4 are related by $F(6C_1, -2C_2, 2C_3, -6C_4) = 0$.

Chapter 16

Fourth-Order Ordinary Differential Equations

16.1 Linear Equations

16.1.1 Preliminary Remarks

1°. A nonhomogeneous linear equation of the fourth order has the form

$$f_4 y''''_{xxxx} + f_3 y'''_{xxx} + f_2 y''_{xx} + f_1 y'_x + f_0 y = g(x), \qquad f_k = f_k(x). \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution of the corresponding homogeneous equation (with $g \equiv 0$). Then the substitution

$$y = y_0(x) \int z(x)\,dx \tag{2}$$

leads to a third-order linear equation:

$$f_4 y_0 z''' + (4f_4 y_0' + f_3 y_0) z'' + (6f_4 y_0'' + 3f_3 y_0' + f_2 y_0) z' + (4f_4 y_0''' + 3f_3 y_0'' + 2f_2 y_0' + f_1 y_0) z = g,$$

where the prime denotes differentiation with respect to x.

2°. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent particular solutions of equation (1) with $g \equiv 0$. Then the substitution

$$y = y_1 \int y_2 w\,dx - y_2 \int y_1 w\,dx$$

leads to a second-order linear equation:

$$f_4 \Delta_1 w'' + (3f_4 \Delta_2 + f_3 \Delta_1) w' + [f_4(3\Delta_3 + 2\varepsilon) + 2f_3 \Delta_2 + f_2 \Delta_1] w = g,$$

where

$$\Delta_1 = y_1' y_2 - y_1 y_2', \quad \Delta_2 = y_1'' y_2 - y_1 y_2'', \quad \Delta_3 = y_1''' y_2 - y_1 y_2''', \quad \varepsilon = y_1'' y_2' - y_1' y_2''.$$

See also Sections 4.1 and 4.2.

16.1.2 Equations Containing Power Functions

► **Equations of the form** $f_4(x)y''''_{xxxx} + f_0(x)y = g(x)$.

1. $y''''_{xxxx} + ay = 0$.

1°. Solution for $a = 0$:

$$y = C_1 + C_2x + C_3x^2 + C_4x^3.$$

2°. Solution for $a = 4k^4 > 0$:

$$y = C_1 \cosh kx \cos kx + C_2 \cosh kx \sin kx + C_3 \sinh kx \cos kx + C_4 \sinh kx \sin kx.$$

3°. Solution for $a = -k^4 < 0$:

$$y = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx.$$

2. $y''''_{xxxx} + \lambda y = ax^3 + bx^2 + cx + s, \qquad \lambda \neq 0$.

Solution: $y = \dfrac{1}{\lambda}(ax^3 + bx^2 + cx + s) + w(x)$, where $w(x)$ is the general solution of equation 16.1.2.1: $w''''_{xxxx} + \lambda w = 0$.

3. $y''''_{xxxx} = axy + b$.

This is a special case of equation 17.1.2.3 with $n = 4$.

4. $y''''_{xxxx} = ax^\beta y$.

This is a special case of equation 17.1.2.4 with $n = 4$. For $\beta = -2, -4, -6, -8$, and -9, see equations 16.1.2.5, 16.1.2.6, 16.1.2.7, 16.1.2.8, and 16.1.2.12, respectively.

The transformation $x = t^{-1}$, $y = ut^{-3}$ leads to an equation of the same form: $u''''_{tttt} = at^{-\beta-8}u$.

5. $x^2y''''_{xxxx} = ay$.

This is a special case of equation 17.1.2.6 with $n = 2$.

6. $x^4y''''_{xxxx} = ay$.

Solution:

$$y = C_1x^{k_1} + C_2x^{k_2} + C_3x^{k_3} + C_4x^{k_4},$$

$$k_{1,2} = \tfrac{3}{2} \pm \left(\tfrac{5}{4} + \sqrt{a+1}\right)^{1/2}, \quad k_{3,4} = \tfrac{3}{2} \pm \left(\tfrac{5}{4} - \sqrt{a+1}\right)^{1/2}.$$

7. $x^6y''''_{xxxx} = ay$.

This is a special case of equation 17.1.2.7 with $n = 2$.

8. $x^8y''''_{xxxx} = ay$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to a constant coefficient linear equation of the form 16.1.2.1: $w''''_{tttt} = aw$.

9. $(ax + b)^4(cx + d)^4y''''_{xxxx} = ky$.

The transformation $\xi = \ln\left|\dfrac{ax+b}{cx+d}\right|$, $w = \dfrac{y}{(cx+d)^3}$ leads to a constant coefficient linear equation.

10. $(ax^2 + bx + c)^4 y''''_{xxxx} = ky.$

The transformation $\xi = \int \frac{dx}{ax^2 + bx + c}$, $w = \frac{y}{(ax^2 + bx + c)^{3/2}}$ leads to a constant coefficient linear equation: $w''''_{\xi\xi\xi\xi} - \frac{5}{2} D w''_{\xi\xi} + \left(\frac{9}{16} D^2 - k\right) w = 0$, where $D = b^2 - 4ac$.

11. $(ax + b)^2 (cx + d)^6 y''''_{xxxx} = ky.$

The transformation $\xi = \frac{ax + b}{cx + d}$, $w = \frac{y}{(cx + d)^3}$ leads to an equation of the form 16.1.2.5: $\xi^2 w''''_{\xi\xi\xi\xi} = k\Delta^{-4} w$, where $\Delta = ad - bc$.

12. $x^9 y''''_{xxxx} = ay + bx^4.$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 16.1.2.3: $4wt = atw + b$.

13. $(ax + b)^9 y''''_{xxxx} = (cx + d)y.$

The transformation $\xi = \frac{cx + d}{ax + b}$, $w = \frac{y}{(ax + b)^3}$ leads to an equation of the form 16.1.2.3: $w''''_{\xi\xi\xi\xi} = \Delta^{-4} \xi w$, where $\Delta = ad - bc$.

► **Equations of the form** $f_4(x) y''''_{xxxx} + f_1(x) y'_x + f_0(x) y = g(x).$

14. $y''''_{xxxx} + ay'_x + by = 0.$

This is a special case of equation 16.1.2.41 with $a_2 = a_3 = 0$.

15. $y''''_{xxxx} + 2ay'_x - a^2 x^2 y = 0.$

This is a special case of equation 16.1.2.25 with $n = 1$.

16. $y''''_{xxxx} + 4axy'_x + (2a - a^2 x^4) y = 0.$

This is a special case of equation 16.1.2.25 with $n = 2$.

17. $y''''_{xxxx} + (a_1 x + b_1) y'_x + (a_2 x + b_2) y = 0.$

This is a special case of equation 17.1.2.35 with $n = 4$.

18. $y''''_{xxxx} + ax(2b - 3a - a^2 x^2) y'_x + b(2a - b + a^2 x^2) y = 0.$

The substitution $w = y''_{xx} - axy'_x + by$ leads to a second-order equation of the form 14.1.2.31: $w''_{xx} + axw'_x + (2a - b + a^2 x^2) w = 0$.

19. $y''''_{xxxx} + ax^k y'_x - ax^{k-1} y = bx^n.$

For $b = 0$, a particular solution is: $y_0 = x$. The substitution $z = xy'_x - y$ leads to a third-order linear equation.

20. $y''''_{xxxx} + ax^k y'_x - 2ax^{k-1} y = bx^n.$

For $b = 0$, a particular solution is: $y_0 = x^2$. The substitution $z = xy'_x - 2y$ leads to a third-order linear equation.

21. $y''''_{xxxx} + ax^k y'_x - 3ax^{k-1}y = bx^n.$

For $b = 0$, a particular solution is: $y_0 = x^3$. The substitution $z = xy'_x - 3y$ leads to a third-order linear equation: $z'''_{xxx} + ax^k z = bx^{n+1}$ (for $b = 0$, see 3.1.2.7).

22. $y''''_{xxxx} + ax^k y'_x + akx^{k-1}y = bx^n.$

Integrating yields a third-order linear equation: $y'''_{xxx} + ax^k y = \dfrac{b}{n+1}x^{n+1} + C.$

23. $y''''_{xxxx} + ax^k y'_x + a(k+3)x^{k-1}y = 0.$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 16.1.2.22 with $b = 0$: $w''''_{tttt} + ct^m w'_t + cmt^{m-1}w = 0$, where $c = -a$, $m = -k-6$.

24. $y''''_{xxxx} + bx^k y'_x - a(a^3 + bx^k)y = 0.$

This is a special case of equation 16.1.6.4 with $f = bx^k$.

25. $y''''_{xxxx} + 2anx^{n-1}y'_x + a[n(n-1)x^{n-2} - ax^{2n}]y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a second-order equation of the form 14.1.2.7: $w''_{xx} - ax^n w = 0$.

26. $y''''_{xxxx} + (ax+b)x^k y'_x - ax^k y = 0.$

Particular solution: $y_0 = ax + b$.

27. $y''''_{xxxx} + (ax+b)x^k y'_x - 2ax^k y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

28. $y''''_{xxxx} + (ax+b)x^k y'_x - 3ax^k y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

29. $y''''_{xxxx} + (ax^k + b^3)y'_x + abx^k y = 0.$

Particular solution: $y_0 = e^{-bx}$.

30. $xy''''_{xxxx} + ax^{k+1}y'_x - [a(x+1)x^k + x + 4]y = 0.$

Particular solution: $y_0 = xe^x$.

► Equations of the form $f_4(x)y''''_{xxxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x)$.

31. $y''''_{xxxx} + 2ay''_{xx} + a^2 y = 0.$

Solution: $y = \begin{cases} (C_1 + C_2x)\cos(kx) + (C_3 + C_4x)\sin(kx) & \text{if } a = k^2 > 0, \\ (C_1 + C_2x)\exp(kx) + (C_3 + C_4x)\exp(-kx) & \text{if } a = -k^2 < 0. \end{cases}$

32. $y''''_{xxxx} + (a+b)y''_{xx} + aby = 0.$

The case $a = b$ is given in 16.1.2.31. Let $a \neq b$.

1°. Solution for $a = \alpha^2 > 0,\ b = \beta^2 > 0$:

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

2°. Solution for $a = \alpha^2 > 0,\ b = -\beta^2 < 0$:

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + C_3 \exp(\beta x) + C_4 \exp(-\beta x).$$

3°. Solution for $a = -\alpha^2 < 0,\ b = \beta^2 > 0$:

$$y = C_1 \exp(\alpha x) + C_2 \exp(-\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

4°. Solution for $a = -\alpha^2 < 0,\ b = -\beta^2 < 0$:

$$y = C_1 \exp(\alpha x) + C_2 \exp(-\alpha x) + C_3 \exp(\beta x) + C_4 \exp(-\beta x).$$

33. $\boldsymbol{y''''_{xxxx} + ay''_{xx} + bx^n y'_x + bnx^{n-1}y = sx^m.}$

Integrating yields a third-order linear equation: $y'''_{xxx} + ay'_x + bx^n y = \dfrac{s}{m+1}x^{m+1} + C.$

34. $\boldsymbol{y''''_{xxxx} - 2a^2 y''_{xx} + a^4 y - \lambda(ax - b)(y''_{xx} - a^2 y) = 0.}$

This equation arises in the turbulence theory. Setting $z(x) = y''_{xx} - a^2 y$, one obtains a second-order linear equation of the form 14.1.2.12:

$$z''_{xx} - a^2 z - \lambda(ax - b)z = 0. \tag{1}$$

Let the following boundary conditions be given:

$$y(0) = y'_x(0) = 0, \quad y(1) = y'_x(1) = 0, \tag{2}$$

The solution of the original equation satisfying the first two conditions in (2) can represented as:

$$2ay = e^{ax} \int_0^x e^{-ax} z\, dx - e^{-ax} \int_0^x e^{ax} z\, dx.$$

To meet the last two conditions in (2), one should take the solution of (1) that satisfies the integral relations $\displaystyle\int_0^1 e^{-ax} z\, dx = \int_0^1 e^{ax} z\, dx = 0.$

35. $\boldsymbol{y''''_{xxxx} + (ax^2 + b)y''_{xx} - 2ay = 0.}$

Particular solution: $y_0 = ax^2 + b.$

36. $\boldsymbol{y''''_{xxxx} + ax^n y''_{xx} + b(ax^n - b)y = 0.}$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr),\ y_2 = \sin\bigl(x\sqrt{b}\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr),\ y_2 = \exp\bigl(x\sqrt{-b}\bigr).$

The substitution $w = y''_{xx} + by$ leads to a second-order linear equation: $w''_{xx} + (ax^n - b)w = 0.$

37. $\boldsymbol{y''''_{xxxx} + ax^{n+1} y''_{xx} - 4ax^n y'_x + 6ax^{n-1} y = 0.}$

Particular solutions: $y_1 = x^2,\ \ y_2 = x^3$. The substitution $w = x^2 y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + ax^{n+1} w = 0.$

38. $y''''_{xxxx} + 10ax^n y''_{xx} + 10anx^{n-1}y'_x + [3an(n-1)x^{n-2} + 9a^2x^{2n}]y = 0.$

This is a special case of equation 16.1.6.25 with $f = ax^n$.

39. $y''''_{xxxx} + (ax^n + b)y''_{xx} + abx^n y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

The substitution $w = y''_{xx} + by$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + ax^n w = 0$.

40. $x^2 y''''_{xxxx} - 2(ax^2 + 6)y''_{xx} + a(ax^2 + 4)y = 0.$

Particular solutions: $y_1 = x^{-1/2} I_{1/2}\bigl(x\sqrt{a}\bigr)$, $y_2 = x^{-1/2} K_{1/2}\bigl(x\sqrt{a}\bigr)$, where $I_{1/2}(z)$ and $K_{1/2}(z)$ are modified Bessel functions.

► Other equations.

41. $y''''_{xxxx} + a_3 y'''_{xxx} + a_2 y''_{xx} + a_1 y'_x + a_0 y = 0.$

A fourth-order constant coefficient linear equation. For $a_0 = 0$, the substitution $w(x) = y'_x$ leads to a third-order equation. Let $a_0 \neq 0$ and $P(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ be the characteristic polynomial.

1°. Let P be factorizable, so that $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$, where λ_1, λ_2, λ_3, and λ_4 are real numbers. The following cases are possible:

a) λ_i are all different, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x};$$

b) $\lambda_1 = \lambda_2$; λ_3 and λ_4 are different and not equal to λ_1, then

$$y = (C_1 + C_2 x)e^{\lambda_1 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x};$$

c) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, then

$$y = (C_1 + C_2 x + C_3 x^2)e^{\lambda_1 x} + C_4 e^{\lambda_4 x};$$

d) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, then

$$y = (C_1 + C_2 x + C_3 x^2 + C_4 x^3)e^{\lambda_1 x}.$$

2°. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda^2 + 2b_1\lambda + b_0)$, where λ_1 and λ_2 are real numbers, and $b_1^2 - b_0 < 0$. The following cases are possible:

a) $\lambda_1 \neq \lambda_2$, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + e^{-b_1 x}[C_3 \cos(\mu x) + C_4 \sin(\mu x)], \quad \mu = \sqrt{b_0 - b_1^2};$$

b) $\lambda_1 = \lambda_2$, then

$$y = (C_1 + C_2 x)e^{\lambda_1 x} + e^{-b_1 x}[C_3 \cos(\mu x) + C_4 \sin(\mu x)], \quad \mu = \sqrt{b_0 - b_1^2}.$$

3°. Let us assume that $P(\lambda) = (\lambda^2 + 2b_1\lambda + b_0)(\lambda^2 + 2\beta_1\lambda + \beta_0)$, where $b_1^2 - b_0 < 0$ and $\beta_1^2 - \beta_0 < 0$. The following cases are possible:

a) $(b_1-\beta_1)^2+(b_0-\beta_0)^2\neq 0$, then

$$y=e^{-b_1x}[C_1\cos(\mu x)+C_2\sin(\mu x)]+e^{-\beta_1 x}[C_3\cos(\nu x)+C_4\sin(\nu x)],$$

where $\mu=\sqrt{b_0-b_1^2}$, $\nu=\sqrt{\beta_0-\beta_1^2}$;

b) $b_1=\beta_1$, $b_0=\beta_0$, then

$$y=e^{-b_1x}[(C_1+C_2x)\cos(\mu x)+(C_3+C_4x)\sin(\mu x)],\qquad \mu=\sqrt{b_0-b_1^2}.$$

42. $\boldsymbol{y''''_{xxxx}+4axy'''_{xxx}+6a^2x^2y''_{xx}+4a^3x^3y'_x+a^4x^4y=0.}$

Solution: $y=\sum\limits_{i=1}^{4}C_i\exp(\lambda_i x-\frac{1}{2}ax^2)$, where the λ_i are roots of the biquadratic equation $\lambda^4-6a\lambda^2+3a^2=0$.

43. $\boldsymbol{y''''_{xxxx}+(ax+b)y'''_{xxx}+[b(a+c)x+c]y''_{xx}+b^2cxy'_x-b^2cy=0.}$

Particular solutions: $y_1=x$, $y_2=e^{-bx}$.

44. $\boldsymbol{y''''_{xxxx}=ax^ny'''_{xxx}+by'_x-abx^ny.}$

Particular solutions: $y_k=\exp(\lambda_k x)$ $(k=1,\,2,\,3)$, where the λ_k are roots of the cubic equation $\lambda^3-b=0$.

45. $\boldsymbol{y''''_{xxxx}+ax^{n+3}y'''_{xxx}-3ax^{n+2}y''_{xx}+6ax^{n+1}y'_x-6ax^ny=0.}$

Particular solutions: $y_1=x$, $y_2=x^2$, $y_3=x^3$. The substitution $w=x^3y'''_{xxx}-3x^2y''_{xx}+6xy'_x-6y$ leads to a first-order linear equation: $w'_x+ax^{n+3}w=0$.

46. $\boldsymbol{y''''_{xxxx}+ax^ny'''_{xxx}+bx^{m+1}y''_{xx}-2bx^my'_x+2bx^{m-1}y=0.}$

Particular solutions: $y_1=x$, $y_2=x^2$. The substitution $w=x^2y''_{xx}-2xy'_x+2y$ leads to a second-order linear equation: $xw''_{xx}+(ax^{n+1}-2)w'_x+bx^{m+2}w=0$.

47. $\boldsymbol{y''''_{xxxx}+ax^ny'''_{xxx}+bx^my''_{xx}+acx^ny'_x+c(bx^m-c)y=0.}$

1°. Particular solutions with $c>0$: $y_1=\cos\bigl(x\sqrt{c}\bigr)$, $y_2=\sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c<0$: $y_1=\exp\bigl(-x\sqrt{-c}\bigr)$, $y_2=\exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w=y''_{xx}+cy$ leads to a second-order linear equation: $w''_{xx}+ax^nw'_x+(bx^m-c)w=0$.

48. $\boldsymbol{y''''_{xxxx}+ax^ny'''_{xxx}+(bx^m+c)y''_{xx}+acx^ny'_x+bcx^my=0.}$

1°. Particular solutions with $c>0$: $y_1=\cos\bigl(x\sqrt{c}\bigr)$, $y_2=\sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c<0$: $y_1=\exp\bigl(-x\sqrt{-c}\bigr)$, $y_2=\exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w=y''_{xx}+cy$ leads to a second-order linear equation: $w''_{xx}+ax^nw'_x+bx^mw=0$.

49. $\boldsymbol{xy''''_{xxxx}+4y'''_{xxx}+axy=0.}$

The substitution $w(x)=xy$ leads to a constant coefficient linear equation of the form 16.1.2.1: $w''''_{xxxx}+aw=0$.

50. $xy''''_{xxxx} - 4ny'''_{xxx} + axy = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Solution: $y = x^{4n+3}(x^{-3}D)^n(x^{-3}w)$, where $D = \frac{d}{dx}$ and $w = w(x)$ is the general solution of a linear constant coefficient equation of the form 16.1.2.1: $w''''_{xxxx} + aw = 0$.

51. $x^2y''''_{xxxx} + 6xy'''_{xxx} + 6y''_{xx} - a^2y = 0.$

Equation of transverse vibrations of a pointed bar.

Solution: $y = \dfrac{1}{\sqrt{x}}\big[C_1J_1\big(2\sqrt{ax}\,\big) + C_2Y_1\big(2\sqrt{ax}\,\big) + C_3I_1\big(2\sqrt{ax}\,\big) + C_4K_1\big(2\sqrt{ax}\,\big)\big]$, where $J_1(z)$ and $Y_1(z)$ are Bessel functions, and $I_1(z)$ and $K_1(z)$ are modified Bessel functions.

52. $x^2y''''_{xxxx} + 2(a+2)xy'''_{xxx} + (a+1)(a+2)y''_{xx} - b^4y = 0.$

Solution: $y = x^{-a/2}\big[C_1J_a\big(2b\sqrt{x}\,\big) + C_2Y_a\big(2b\sqrt{x}\,\big) + C_3I_a\big(2b\sqrt{x}\,\big) + C_4K_a\big(2b\sqrt{x}\,\big)\big]$, where $J_a(z)$ and $Y_a(z)$ are Bessel functions, and $I_a(z)$ and $K_a(z)$ are modified Bessel functions.

53. $x^2y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} + ax^2y = 0.$

The substitution $w(x) = x^2y$ leads to a constant coefficient linear equation of the form 16.1.2.1: $w''''_{xxxx} + aw = 0$.

54. $x^2y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = ax^3y + b.$

The substitution $w(x) = x^2y$ leads to an equation of the form 16.1.2.3: $w''''_{xxxx} = axw + b$.

55. $x^2y''''_{xxxx} + axy'''_{xxx} + (bx^{n+1} + c)y''_{xx}$
$\qquad + (a-4)bx^ny'_x + b(c - 2a + 6)x^{n-1}y = 0.$

The substitution $w(x) = x^2y''_{xx} + (a-4)xy'_x + (c - 2a + 6)y$ leads to a second-order equation of the form 14.1.2.7: $w''_{xx} + bx^{n-1}w = 0$.

56. $x^3y''''_{xxxx} + 2x^2y'''_{xxx} - xy''_{xx} + y'_x - a^4x^3y = 0.$

Solution: $y = C_1J_0(ax) + C_2Y_0(ax) + C_3I_0(ax) + C_4K_0(ax)$, where $J_0(z)$ and $Y_0(z)$ are Bessel functions, and $I_0(z)$ and $K_0(z)$ are modified Bessel functions.

57. $x^4y''''_{xxxx} + A_3x^3y'''_{xxx} + A_2x^2y''_{xx} + A_1xy'_x + A_0y = 0.$

The Euler equation. The substitution $t = \ln|x|$ leads to a constant coefficient linear equation of the form 16.1.2.41:

$$y''''_{tttt} + (A_3 - 6)y'''_{ttt} + (11 - 3A_3 + A_2)y''_{tt} + (2A_3 - A_2 + A_1 - 6)y'_t + A_0y = 0.$$

58. $x^4y''''_{xxxx} + 2x^3y'''_{xxx} - (2a^2+1)x^2y''_{xx}$
$\qquad + (2a^2+1)xy'_x - \big[b^4x^4 - a^2(a^2-4)\big]y = 0.$

This equation governs free transverse vibration modes of a thin round elastic plate. The equation arises from separation of variables in the two-dimensional equation

$$\Delta\Delta w - b^4w = 0,$$

where Δ is the Laplace operator written in the polar coordinate system, with x being the polar radius.

Solution: $y = C_1J_a(bx) + C_2Y_a(bx) + C_3I_a(bx) + C_4K_a(bx)$, where $J_a(z)$ and $Y_a(z)$ are Bessel functions, and $I_a(z)$ and $K_a(z)$ are modified Bessel functions. In applications, one usually sets $a = n$, where $n = 0,\ 1,\ 2,\ \dots$

⊙ The solution is specified by Popov (1998).

59. $x^4 y''''_{xxxx} - 2n(n+1)x^2 y''_{xx}$
$$+ 4n(n+1)xy'_x + [ax^4 + n(n+1)(n+3)(n-2)]y = 0.$$

Here, n is a positive integer and $a \neq 0$ (for $a = 0$, we have the Euler equation 16.1.2.57).

Solution: $y = x^{-n} \sum\limits_{\nu=1}^{4} C_\nu \exp(\lambda_\nu x) P_\nu(x)$, where the λ_ν are four different roots of the equation $\lambda^4 + a = 0$, and $P_\nu(x)$ is some definite polynomial of degree $\leq 4n$.

60. $x^4 y''''_{xxxx} + 2(2-n)x^3 y'''_{xxx} + (1-n)(2-n)x^2 y''_{xx} - a^4 x^{2n} y = 0.$

Solution: $y = \sqrt{x}\,[C_1 J_{1/n}(\xi) + C_2 Y_{1/n}(\xi) + C_3 I_{1/n}(\xi) + C_4 K_{1/n}(\xi)]$, where $\xi = 2(a/n)x^{n/2}$; $J_\nu(\xi)$ and $Y_\nu(\xi)$ are Bessel functions, and $I_\nu(\xi)$ and $K_\nu(\xi)$ are modified Bessel functions.

61. $x^4 y''''_{xxxx} + 6x^3 y'''_{xxx} + [4x^4 + (7 - a^2 - b^2)x^2]y''_{xx}$
$$+ x(16x^2 + 1 - a^2 - b^2)y'_x + (8x^2 + a^2 b^2)y = 0.$$

Solution for $ab \neq 0$: $y = C_1 J_\mu(x) J_\nu(x) + C_2 J_\mu(x) Y_\nu(x) + C_3 Y_\mu(x) J_\nu(x) + C_4 Y_\mu(x) Y_\nu(x)$, where $J_\mu(x)$ and $Y_\mu(x)$ are Bessel functions; $\mu = \frac{1}{2}(a+b)$ and $\nu = \frac{1}{2}(a-b)$.

62. $x^8 y''''_{xxxx} + 4x^7 y'''_{xxx} = ay.$

The substitution $w(x) = xy$ leads to an equation of the form 16.1.2.8: $x^8 w''''_{xxxx} = aw$.

16.1.3 Equations Containing Exponential and Hyperbolic Functions

► **Equations with exponential functions.**

1. $y''''_{xxxx} + a^3 y'_x + be^{ax}(a^2 - be^{ax})y = 0.$

The substitution $w = y''_{xx} + ay'_x + be^{ax}y$ leads to a second-order linear equation of the form 14.1.3.10: $w''_{xx} - aw'_x + (a^2 - be^{ax})w = 0$.

2. $y''''_{xxxx} + ae^{\lambda x} y'_x + a\lambda e^{\lambda x} y = be^{\mu x}.$

Integrating yields a third-order linear equation: $y'''_{xxx} + ae^{\lambda x}y = b\mu^{-1}e^{\mu x} + C$.

3. $y''''_{xxxx} + ae^{\lambda x} y'_x - (abe^{\lambda x} + b^4)y = 0.$

Particular solution: $y_0 = e^{bx}$.

4. $y''''_{xxxx} + 2a\lambda e^{\lambda x} y'_x + a(\lambda^2 e^{\lambda x} - ae^{2\lambda x})y = 0.$

The substitution $w = y''_{xx} + ae^{\lambda x}y$ leads to a second-order linear equation of the form 14.1.3.1: $w''_{xx} - ae^{\lambda x}w = 0$.

5. $y''''_{xxxx} + (ae^{\lambda x} + b^3)y'_x + abe^{\lambda x} y = 0.$

Particular solution: $y_0 = e^{-bx}$.

6. $y''''_{xxxx} + (ax + b)e^{\lambda x} y'_x - ae^{\lambda x} y = 0.$

Particular solution: $y_0 = ax + b$.

7. $y''''_{xxxx} + (ax+b)e^{\lambda x}y'_x - 2ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

8. $y''''_{xxxx} + (ax+b)e^{\lambda x}y'_x - 3ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

9. $y''''_{xxxx} + ae^{\lambda x}y''_{xx} - b(ae^{\lambda x}+b)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\bigr)$.

The substitution $w = y''_{xx} - by$ leads to a second-order linear equation of the form 14.1.3.2: $w''_{xx} + (ae^{\lambda x}+b)w = 0$.

10. $y''''_{xxxx} + (a+be^{\lambda x})y''_{xx} + abe^{\lambda x}y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation of the form 14.1.3.1: $w''_{xx} + be^{\lambda x}w = 0$.

11. $y''''_{xxxx} + 10ae^{\lambda x}y''_{xx} + 10a\lambda e^{\lambda x}y'_x + (3a\lambda^2e^{\lambda x} + 9a^2e^{2\lambda x})y = 0.$

This is a special case of equation 16.1.6.25 with $f(x) = ae^{\lambda x}$.

12. $y''''_{xxxx} + ay'''_{xxx} + be^{\lambda x}y'_x + abe^{\lambda x}y = 0.$

Particular solution: $y_0 = e^{-ax}$.

13. $y''''_{xxxx} = ae^{\lambda x}y'''_{xxx} + by'_x - abe^{\lambda x}y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\,2,\,3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

14. $y''''_{xxxx} + ae^{\lambda x}y'''_{xxx} + be^{\mu x}y''_{xx} + ace^{\lambda x}y'_x + c(be^{\mu x} - c)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second-order linear equation: $w''_{xx} + ae^{\lambda x}w'_x + (be^{\mu x} - c)w = 0$.

15. $y''''_{xxxx} + ae^{\lambda x}y'''_{xxx} + (be^{\mu x} + c)y''_{xx} + ace^{\lambda x}y'_x + bce^{\mu x}y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second-order equation: $w''_{xx} + ae^{\lambda x}w'_x + be^{\mu x}w = 0$.

16. $y''''_{xxxx} + ax^3e^{\lambda x}y'''_{xxx} - 3ax^2e^{\lambda x}y''_{xx} + 6axe^{\lambda x}y'_x - 6ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$. The substitution $w = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to a first-order linear equation: $w'_x + ax^3e^{\lambda x}w = 0$.

17. $xy''''_{xxxx} + axe^{\lambda x}y'_x - [a(x+1)e^{\lambda x} + x + 4]y = 0.$

Particular solution: $y_0 = xe^x$.

18. $(ae^x + b)y''''_{xxxx} = ae^x y.$

Particular solution: $y_0 = ae^x + b$.

19. $(ax^m + be^x + c)y''''_{xxxx} = be^x y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + be^x + c$.

20. $(ax^m e^x + b)y''''_{xxxx} = by, \qquad m = 0,\ 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + be^{-x}$.

21. $y''''_{xxxx} + b\exp(\lambda x^n)\, y''_{xx} + a[b\exp(\lambda x^n) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\exp(\lambda x^n)$.

22. $y''''_{xxxx} + [a + b\exp(\lambda x^n)]y''_{xx} + ab\exp(\lambda x^n)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\exp(\lambda x^n)$.

► Equations with hyperbolic functions.

23. $y''''_{xxxx} + a\sinh^n(\lambda x)\, y'_x + b[a\sinh^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

24. $y''''_{xxxx} + [a\sinh^n(\lambda x) + b^3]y'_x + ab\sinh^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

25. $y''''_{xxxx} + (ax + b)\sinh^n(\lambda x)y'_x - a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

26. $y''''_{xxxx} + (ax + b)\sinh^n(\lambda x)y'_x - 2a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

27. $y''''_{xxxx} + (ax + b)\sinh^n(\lambda x)y'_x - 3a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

28. $y''''_{xxxx} + b\sinh^n(\lambda x)\, y''_{xx} + a[b\sinh^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\sinh^n(\lambda x)$.

29. $y''''_{xxxx} + [a + b\sinh^n(\lambda x)]y''_{xx} + ab\sinh^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\sinh^n(\lambda x)$.

30. $(ax^m + b\sinh x)y''''_{xxxx} = b\sinh x\, y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + b\sinh x$.

31. $y''''_{xxxx} + a\cosh^n(\lambda x)\, y'_x + b[a\cosh^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

32. $y''''_{xxxx} + [a\cosh^n(\lambda x) + b^3]y'_x + ab\cosh^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

33. $y''''_{xxxx} + (ax+b)\cosh^n(\lambda x) y'_x - a\cosh^n(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

34. $y''''_{xxxx} + (ax+b)\cosh^n(\lambda x) y'_x - 2a\cosh^n(\lambda x) y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

35. $y''''_{xxxx} + (ax+b)\cosh^n(\lambda x) y'_x - 3a\cosh^n(\lambda x) y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

36. $y''''_{xxxx} + b\cosh^n(\lambda x)\, y''_{xx} + a[b\cosh^n(\lambda x) - a] y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\cosh^n(\lambda x)$.

37. $y''''_{xxxx} + [a + b\cosh^n(\lambda x)] y''_{xx} + ab\cosh^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\cosh^n(\lambda x)$.

38. $(ax^m + b\cosh x) y''''_{xxxx} = b\cosh x\, y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + b\cosh x$.

39. $y''''_{xxxx} = y + a(y'_x \cosh x - y\sinh x).$

The substitution $w = y'_x \cosh x - y\sinh x$ leads to a third-order linear equation.

40. $y''''_{xxxx} = y + a(y'_x \sinh x - y\cosh x).$

The substitution $w = y'_x \sinh x - y\cosh x$ leads to a third-order linear equation.

41. $y''''_{xxxx} + a\tanh^n(\lambda x)\, y'_x + b[a\tanh^n(\lambda x) - b^3] y = 0.$

Particular solution: $y_0 = e^{-bx}$.

42. $y''''_{xxxx} + [a\tanh^n(\lambda x) + b^3] y'_x + ab\tanh^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

43. $y''''_{xxxx} + (ax+b)\tanh^n(\lambda x) y'_x - a\tanh^n(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

44. $y''''_{xxxx} + (ax+b)\tanh^n(\lambda x) y'_x - 2a\tanh^n(\lambda x) y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

45. $y''''_{xxxx} + (ax+b)\tanh^n(\lambda x) y'_x - 3a\tanh^n(\lambda x) y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

46. $y''''_{xxxx} + b\tanh^n(\lambda x)\, y''_{xx} + a[b\tanh^n(\lambda x) - a] y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\tanh^n(\lambda x)$.

47. $y''''_{xxxx} + [a + b\tanh^n(\lambda x)] y''_{xx} + ab\tanh^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\tanh^n(\lambda x)$.

48. $y''''_{xxxx} + a\coth^n(\lambda x)\, y'_x + b[a\coth^n(\lambda x) - b^3] y = 0.$

Particular solution: $y_0 = e^{-bx}$.

49. $y''''_{xxxx} + [a\coth^n(\lambda x) + b^3]y'_x + ab\coth^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

50. $y''''_{xxxx} + b\coth^n(\lambda x)\,y''_{xx} + a[b\coth^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\coth^n(\lambda x)$.

51. $y''''_{xxxx} + [a + b\coth^n(\lambda x)]y''_{xx} + ab\coth^n(\lambda x)\,y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\coth^n(\lambda x)$.

16.1.4 Equations Containing Logarithmic Functions

1. $y''''_{xxxx} + a\ln^k x\,y'_x - (ab\ln^k x + b^4)y = 0.$

Particular solution: $y_0 = e^{bx}$.

2. $y''''_{xxxx} + (ax + b)\ln^k(\lambda x)y'_x - a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

3. $y''''_{xxxx} + (ax + b)\ln^k(\lambda x)y'_x - 2a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

4. $y''''_{xxxx} + (ax + b)\ln^k(\lambda x)y'_x - 3a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

5. $x^2y''''_{xxxx} + 2axy'_x - a[1 + ax^2\ln^2(bx)]y = 0.$

The substitution $w = y''_{xx} + a\ln(bx)\,y$ leads to a second-order linear equation: $w''_{xx} - a\ln(bx)w = 0$.

6. $y''''_{xxxx} + b\ln^k(\lambda x)\,y''_{xx} + a[b\ln^k(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\ln^k(\lambda x)$.

7. $y''''_{xxxx} + [a + b\ln^k(\lambda x)]y''_{xx} + ab\ln^k(\lambda x)\,y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\ln^k(\lambda x)$.

8. $y''''_{xxxx} + \ln^k(\lambda x)(x^2y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

9. $y''''_{xxxx} + \ln^k(\lambda x)(x^2y''_{xx} - 4xy'_x + 6y) = 0.$

Particular solutions: $y_1 = x^2$, $y_2 = x^3$.

10. $y''''_{xxxx} + ax^2\ln^k(\lambda x)y''_{xx} - 2a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = x^2$.

11. $y''''_{xxxx} + ay'''_{xxx} + b\ln^k(\lambda x)y'_x + ab\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

16.1.5 Equations Containing Trigonometric Functions

► **Equations with sine and cosine.**

1. $y''''_{xxxx} + a\sin^n(\lambda x)\,y'_x + b[a\sin^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

2. $y''''_{xxxx} + [a\sin^n(\lambda x) + b^3]y'_x + ab\sin^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

3. $y''''_{xxxx} + (ax+b)\sin^n(\lambda x)y'_x - a\sin^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

4. $y''''_{xxxx} + (ax+b)\sin^n(\lambda x)y'_x - 2a\sin^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

5. $y''''_{xxxx} + (ax+b)\sin^n(\lambda x)y'_x - 3a\sin^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

6. $y''''_{xxxx} + a\sin^n(\lambda x)\,y''_{xx} + b[a\sin^n(\lambda x) - b]y = 0.$

The substitution $u = y''_{xx} + by$ leads to a second-order equation: $u''_{xx} + [a\sin^n(\lambda x) - b]u = 0$.

7. $y''''_{xxxx} + [a + b\sin^n(\lambda x)]y''_{xx} + ab\sin^n(\lambda x)\,y = 0.$

The substitution $w = y''_{xx} + ay$ leads to a second-order equation: $w''_{xx} + b\sin^n(\lambda x)\,w = 0$.

8. $y''''_{xxxx} = a\sin^n(\lambda x)\,y'''_{xxx} + by'_x - ab\sin^n(\lambda x)\,y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

9. $y''''_{xxxx} + a\sin^n(\lambda x)\,(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y) = 0.$

This is a special case of equation 16.1.6.33 with $f(x) = a\sin^n(\lambda x)$.

10. $x^2y''''_{xxxx} + a\sin^n(\lambda x)\,(x^2y''_{xx} - 4xy'_x + 6y) = 0.$

The substitution $u = x^2y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation: $u''_{xx} + a\sin^n(\lambda x)u = 0$.

11. $(a\sin x + b)y''''_{xxxx} = a\sin x\,y.$

Particular solution: $y_0 = a\sin x + b$.

12. $(ax + b\sin x)y''''_{xxxx} = b\sin x\,y.$

Particular solution: $y_0 = ax + b\sin x$.

13. $(ax^m + b\sin x)y''''_{xxxx} = b\sin x\,y, \qquad m = 2,\ 3.$

Particular solution: $y_0 = ax^m + b\sin x$.

14. $y''''_{xxxx} + a\cos^n(\lambda x)\,y'_x + b[a\cos^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

15. $y''''_{xxxx} + [a\cos^n(\lambda x) + b^3]y'_x + ab\cos^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

16. $y''''_{xxxx} + (ax+b)\cos^n(\lambda x)y'_x - a\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

17. $y''''_{xxxx} + (ax+b)\cos^n(\lambda x)y'_x - 2a\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

18. $y''''_{xxxx} + (ax+b)\cos^n(\lambda x)y'_x - 3a\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

19. $y''''_{xxxx} + a\cos^n(\lambda x)\,y''_{xx} + b[a\cos^n(\lambda x) - b]y = 0.$

The substitution $u = y''_{xx} + by$ leads to a second-order equation: $u''_{xx} + [a\cos^n(\lambda x) - b]u = 0$.

20. $y''''_{xxxx} + [a + b\cos^n(\lambda x)]y''_{xx} + ab\cos^n(\lambda x)\,y = 0.$

The substitution $u = y''_{xx} + ay$ leads to a second-order equation: $u''_{xx} + b\cos^n(\lambda x)\,u = 0$.

21. $y''''_{xxxx} = a\cos^n(\lambda x)\,y'''_{xxx} + by'_x - ab\cos^n(\lambda x)\,y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\,2,\,3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

22. $y''''_{xxxx} + a\cos^n(\lambda x)\,(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y) = 0.$

This is a special case of equation 16.1.6.33 with $f(x) = a\cos^n(\lambda x)$.

23. $x^2y''''_{xxxx} + a\cos^n(\lambda x)\,(x^2y''_{xx} - 4xy'_x + 6y) = 0.$

The substitution $u = x^2y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation: $u''_{xx} + a\cos^n(\lambda x)u = 0$.

24. $(a\cos x + b)y''''_{xxxx} = a\cos x\,y.$

Particular solution: $y_0 = a\cos x + b$.

25. $(ax + b\cos x)y''''_{xxxx} = b\cos x\,y.$

Particular solution: $y_0 = ax + b\cos x$.

26. $(ax^m + b\cos x)y''''_{xxxx} = b\cos x\,y, \qquad m = 2,\;3.$

Particular solution: $y_0 = ax^m + b\cos x$.

27. $y''''_{xxxx} + 2ab\cos(bx)\,y'_x - a[b^2\sin(bx) + a\sin^2(bx)]y = 0.$

The substitution $w = y''_{xx} + a\sin(bx)\,y$ leads to a second-order linear equation of the form 14.1.6.2: $w''_{xx} - a\sin(bx)w = 0$.

28. $\sin^4 x\, y''''_{xxxx} + 2\sin^3 x\cos x\, y'''_{xxx} + \sin^2 x\,(\sin^2 x - 3)\, y''_{xx} + \sin x\cos x\,(2\sin^2 x + 3)\, y'_x + (a^4\sin^4 x - 3)y = 0.$

Equation of a loaded rigid spherical shell. If $a^4 = 1 - \lambda^2$, the equation can be rewritten as

$$\mathbf{LL}(y) - \lambda^2 y = 0, \qquad \text{where} \quad \mathbf{L} \equiv \frac{d^2}{dx^2} + \cot x\frac{d}{dx} - \cot^2 x.$$

This equation falls into two second-order equations:

$$\mathbf{L}(y) + \lambda y = 0, \qquad \mathbf{L}(y) - \lambda y = 0,$$

which differ only in the sign of the parameter λ. The transformation $\xi = \sin^2 x$, $w = y/\sin x$ reduces the latter equations to the hypergeometric equations 2.1.2.171:

$$\xi(\xi - 1)w''_{\xi\xi} + (\tfrac{5}{2}\xi - 2)w'_\xi + \tfrac{1}{4}(1 \mp \lambda)w = 0.$$

29. $y''''_{xxxx} = y + a(y'_x \sin x - y\cos x).$

The substitution $w = y'_x \sin x - y\cos x$ leads to a third-order linear equation.

30. $y''''_{xxxx} = y + a(y'_x \cos x + y\sin x).$

The substitution $w = y'_x \cos x + y\sin x$ leads to a third-order linear equation.

► **Equations with tangent and cotangent.**

31. $y''''_{xxxx} + ay'_x + (a\tan x - 1)y = 0.$

Particular solution: $y_0 = \cos x$.

32. $y''''_{xxxx} + (ax + b)\tan^n(\lambda x)y'_x - a\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

33. $y''''_{xxxx} + (ax + b)\tan^n(\lambda x)y'_x - 2a\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

34. $y''''_{xxxx} + (ax + b)\tan^n(\lambda x)y'_x - 3a\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

35. $y''''_{xxxx} + a\tan^n(\lambda x)\, y'_x + b[a\tan^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

36. $y''''_{xxxx} + [a\tan^n(\lambda x) + b^3]y'_x + ab\tan^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

37. $y''''_{xxxx} + b\tan^n(\lambda x)\, y''_{xx} + a[b\tan^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\tan^n(\lambda x)$.

38. $y''''_{xxxx} + [a + b\tan^n(\lambda x)]y''_{xx} + ab\tan^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\tan^n(\lambda x)$.

39. $y''''_{xxxx} = a\tan^n(\lambda x)\, y'''_{xxx} + by'_x - ab\tan^n(\lambda x)\, y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\, 2,\, 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

40. $y''''_{xxxx} + ay'_x - (1 + a\cot x)y = 0.$

Particular solution: $y_0 = \sin x$.

41. $y''''_{xxxx} + (ax + b)\cot^n(\lambda x)y'_x - a\cot^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

42. $y''''_{xxxx} + (ax + b)\cot^n(\lambda x)y'_x - 2a\cot^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

43. $y''''_{xxxx} + (ax + b)\cot^n(\lambda x)y'_x - 3a\cot^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

44. $y''''_{xxxx} + a\cot^n(\lambda x)\, y'_x + b[a\cot^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

45. $y''''_{xxxx} + [a\cot^n(\lambda x) + b^3]y'_x + ab\cot^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

46. $y''''_{xxxx} + b\cot^n(\lambda x)\, y''_{xx} + a[b\cot^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\cot^n(\lambda x)$.

47. $y''''_{xxxx} + [a + b\cot^n(\lambda x)]y''_{xx} + ab\cot^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\cot^n(\lambda x)$.

48. $y''''_{xxxx} = a\cot^n(\lambda x)\, y'''_{xxx} + by'_x - ab\cot^n(\lambda x)\, y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\, 2,\, 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

16.1.6 Equations Containing Arbitrary Functions

► **Equations of the form** $f_4(x)y''''_{xxxx} + f_1(x)y'_x + f_0(x)y = g(x)$.

1. $y''''_{xxxx} = f(x)y.$

The transformation $x = t^{-1}$, $y = ut^{-3}$ leads to an equation of the same form: $u''''_{tttt} = t^{-8}f(1/t)u$.

2. $y''''_{xxxx} = f\left(\dfrac{ax+b}{cx+d}\right)\dfrac{y}{(cx+d)^8}.$

The transformation $z = \dfrac{ax+b}{cx+d}$, $u = \dfrac{y}{(cx+d)^3}$ leads to a simpler equation: $u''''_{zzzz} = \Delta^{-4}f(z)u$, where $\Delta = ad - bc$.

3. $fy''''_{xxxx} - f''''_{xxxx}y = 0, \qquad f = f(x).$

Particular solution: $y_0 = f(x)$.

4. $y''''_{xxxx} + f y'_x - a(f + a^3)y = 0, \qquad f = f(x).$

Particular solution: $y_0 = e^{ax}$.

5. $y''''_{xxxx} + (f + a^3)y'_x + afy = 0, \qquad f = f(x).$

Particular solution: $y_0 = e^{-ax}$.

6. $y''''_{xxxx} + (ax + b)f(x)y'_x - af(x)y = 0.$

Particular solution: $y_0 = ax + b$.

7. $y''''_{xxxx} + (ax + b)f(x)y'_x - 2af(x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

8. $y''''_{xxxx} + (ax + b)f(x)y'_x - 3af(x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$. The substitution $z = (ax + b)y'_x - 3ay$ leads to a third-order linear equation: $z'''_{xxx} + (ax + b)f(x)z = 0$.

9. $y''''_{xxxx} + f(x)y'_x + f'_x(x)y = g(x).$

Integrating yields a third-order linear equation: $y'''_{xxx} + f(x)y = \int g(x)\,dx + C$.

10. $y''''_{xxxx} + 2f'_x y'_x + (f''_{xx} - f^2)y = 0, \qquad f = f(x).$

The substitution $w = y''_{xx} + f(x)y$ leads to a second-order equation: $w''_{xx} - f(x)w = 0$.

11. $xy''''_{xxxx} + xf(x)y'_x - [(x + 1)f(x) + x + 4]y = 0.$

Particular solution: $y_0 = xe^x$.

12. $y''''_{xxxx} + f(x)y'_x + g(x)y + h(x) = 0.$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the same form:

$$w''''_{tttt} - t^{-6}f(1/t)w'_t + \left[3t^{-7}f(1/t) + t^{-8}g(1/t)\right]w + t^{-5}h(1/t) = 0.$$

13. $y''''_{xxxx} = y + f(x)(y'_x \cosh x - y \sinh x).$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a third-order linear equation.

14. $y''''_{xxxx} = y + f(x)(y'_x \sinh x - y \cosh x).$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a third-order linear equation.

15. $y''''_{xxxx} = y + f(x)(y'_x \sin x - y \cos x).$

The substitution $w = y'_x \sin x - y \cos x$ leads to a third-order linear equation.

16. $y''''_{xxxx} = y + f(x)(y'_x \cos x + y \sin x).$

The substitution $w = y'_x \cos x + y \sin x$ leads to a third-order linear equation.

17. $y''''_{xxxx} + fy'_x + (f \tan x - 1)y = 0, \qquad f = f(x).$

Particular solution: $y_0 = \cos x$.

18. $y''''_{xxxx} + fy'_x - (1 + f \cot x)y = 0, \qquad f = f(x).$

Particular solution: $y_0 = \sin x$.

19. $y''''_{xxxx} = \dfrac{\varphi''''_{xxxx}}{\varphi}y + f(x)\Big(y'_x - \dfrac{\varphi'_x}{\varphi}y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi}y$ leads to a third-order linear equation.

► **Equations of the form $f_4(x)y''''_{xxxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x)$.**

20. $y''''_{xxxx} + fy''_{xx} + a(f-a)y = 0, \qquad f = f(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation: $w''_{xx} + (f-a)w = 0$.

21. $y''''_{xxxx} + (f+a)y''_{xx} + afy = 0, \qquad f = f(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation: $w''_{xx} + f(x)w = 0$.

22. $y''''_{xxxx} + f(x)(x^2y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$. The substitution $z = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second-order linear equation: $xz''_{xx} - 2z'_x + x^3f(x)z = 0$.

23. $y''''_{xxxx} + f(x)(x^2y''_{xx} - 4xy'_x + 6y) = 0.$

Particular solutions: $y_1 = x^2$, $y_2 = x^3$. The substitution $w = x^2y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation: $w''_{xx} + x^2f(x)w = 0$.

24. $y''''_{xxxx} + (ax^2 + bx + c)f(x)y''_{xx} - 2af(x)y = 0.$

Particular solution: $y_0 = ax^2 + bx + c$.

25. $y''''_{xxxx} + 10fy''_{xx} + 10f'_xy'_x + (3f''_{xx} + 9f^2)y = 0, \qquad f = f(x).$

Solution:

$$y = C_1w_1^3 + C_2w_1^2w_2 + C_3w_1w_2^2 + C_4w_2^3,$$

where w_1 and w_2 are nontrivial linearly independent solutions of the second-order linear equation: $w''_{xx} + fw = 0$.

26. $y''''_{xxxx} + (f+g)y''_{xx} + 2f'_xy'_x + (f''_{xx} + fg)y = 0, \qquad f = f(x),\ g = g(x).$

The substitution $w = y''_{xx} + fy$ leads to a second-order linear equation: $w''_{xx} + gw = 0$.

► **Other equations.**

27. $y''''_{xxxx} + f(x)y'''_{xxx} + xg(x)y'_x - 2g(x)y = 0.$

Particular solution: $y_0 = x^2$.

28. $y''''_{xxxx} + f(x)y'''_{xxx} - 2a^2y''_{xx} - a^2f(x)y'_x + a^4y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = e^{ax}$.

29. $y''''_{xxxx} + fy'''_{xxx} + gy''_{xx} + afy'_x + a(g-a)y = 0, \qquad f = f(x),\ g = g(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order equation: $w''_{xx} + fw'_x + (g-a)w = 0$.

30. $y''''_{xxxx} + fy'''_{xxx} + (g+a)y''_{xx} + afy'_x + agy = 0, \qquad f = f(x), \ g = g(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\bigr), \ y_2 = \sin\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\bigr), \ y_2 = \exp\bigl(x\sqrt{-a}\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation: $w''_{xx} + fw'_x + gw = 0$.

31. $y''''_{xxxx} + f(x)y'''_{xxx} + g(x)y''_{xx} + xh(x)y'_x - h(x)y = 0.$

Particular solution: $y_0 = x$.

32. $y''''_{xxxx} + f(x)y'''_{xxx} + g(x)(x^2y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x, \ y_2 = x^2$. The substitution $z = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second-order linear equation: $xz''_{xx} + [xf(x) - 2]z'_x + x^3g(x)z = 0$.

33. $y''''_{xxxx} + f(x)(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y) = 0.$

Particular solutions: $y_1 = x, \ y_2 = x^2, \ y_3 = x^3$. The substitution $w = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to a first-order linear equation: $w'_x + x^3f(x)w = 0$.

34. $y''''_{xxxx} = f(x)y'''_{xxx} + ay'_x - af(x)y.$

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1, 2, 3)$, where the λ_k are roots of the cubic equation $\lambda^3 - a = 0$.

35. $y''''_{xxxx} = (f-a)y'''_{xxx} + (af-b)y''_{xx} + (bf-c)y'_x + cfy, \qquad f = f(x).$

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1, 2, 3)$, where the λ_k are roots of the cubic equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$.

36. $y''''_{xxxx} + (f+a)y'''_{xxx} + (af + g + axg)y''_{xx} + a^2xgy'_x - a^2gy = 0,$
$f = f(x), \quad g = g(x).$

Particular solutions: $y_1 = x, \ y_2 = e^{-ax}$.

37. $y''''_{xxxx} + (f_3+a)y'''_{xxx} + (f_2 + af_3)y''_{xx} + (f_1 + af_2)y'_x + af_1y = 0,$
$f_n = f_n(x) \ \ (n = 1, 2, 3).$

Particular solution: $y_0 = e^{-ax}$.

38. $xy''''_{xxxx} + 4y'''_{xxx} + axy = f(x).$

The substitution $w(x) = xy$ leads to a constant coefficient nonhomogeneous linear equation: $w''''_{xxxx} + aw = f(x)$.

39. $x^2y''''_{xxxx} + axy'''_{xxx} + (x^2f + b)y''_{xx}$
$+ (a-4)xfy'_x + (b - 2a + 6)fy = 0, \qquad f = f(x).$

The substitution $w = x^2y''_{xx} + (a-4)xy'_x + (b-2a+6)y$ leads to a second-order linear equation: $w''_{xx} + fw = 0$.

40. $x^4y''''_{xxxx} + ax^3y'''_{xxx} + xf(x)y'_x + (a-3)f(x)y = 0.$

Particular solution: $y_0 = x^{3-a}$.

41. $y''''_{xxxx} + 6fy'''_{xxx} + (4f'_x + 11f^2 + 10g)y''_{xx} + (f''_{xx} + 7ff'_x + 6f^3 + 30fg + 10g'_x)y'_x + 3(2f'_x g + 5fg'_x + 6f^2 g + g''_{xx} + 3g^2)y = 0.$

Here $f = f(x)$ and $g = g(x)$. Solution:

$$y = C_1 w_1^3 + C_2 w_1^2 w_2 + C_3 w_1 w_2^2 + C_4 w_2^3,$$

where w_1 and w_2 are nontrivial linearly independent solutions of the second-order linear equation: $w''_{xx} + fw'_x + gw = 0$.

42. $(fy''_{xx})''_{xx} = 0, \qquad f = f(x).$

Equation of transverse vibrations of a bar. Solution:

$$y = C_1 + C_2 x + \int_{x_0}^{x} \frac{x-t}{f(t)}(C_3 + C_4 t)\,dt.$$

16.2 Nonlinear Equations

16.2.1 Equations Containing Power Functions

▶ **Equations of the form** $y''''_{xxxx} = f(x, y)$.

1. $y''''_{xxxx} = Ay^{-5/3}$.

Multiply both sides of the equation by $y^{5/3}$ and differentiate the resulting expression with respect to x to obtain

$$3yy^{(5)}_x + 5y'_x y''''_{xxxx} = 0.$$

Integrating this equation three times, we arrive at the chain of equalities:

$$3yy''''_{xxxx} + 2y'_x y'''_{xxx} - (y''_{xx})^2 = 2C_2, \qquad (1)$$
$$3yy'''_{xxx} - y'_x y''_{xx} = 2C_2 x + C_1, \qquad (2)$$
$$3yy''_{xx} - 2(y'_x)^2 = C_2 x^2 + C_1 x + C_0, \qquad (3)$$

where C_0, C_1, and C_2 are arbitrary constants. By eliminating the highest derivatives from (1)–(3) with the help of the original equation, we obtain a first-order equation:

$$(2Py'_x - 3P'_x y)^2 = 9(C_1^2 - 4C_0 C_2)y^2 - 2P^3 + 54APy^{4/3},$$

where $P = C_2 x^2 + C_1 x + C_0$. The substitution $y = (P/w)^{3/2}$ leads to a separable equation, the integration of which finally yields:

$$\int \bigl[9(C_1^2 - 4C_0 C_2) + 54Aw - 2w^3\bigr]^{-1/2} \frac{dw}{w} \pm \int \frac{dx}{3P} = C_3.$$

2. $y''''_{xxxx} = Ay^m$.

This is a special case of equation 16.2.6.1 with $f(w) = Ay^m$.

1°. By integrating, we obtain

$$2y'_x y'''_{xxx} - (y''_{xx})^2 = \frac{2A}{m+1} y^{m+1} + \frac{4}{3} C,$$

where C is an arbitrary constant ($m \neq -1$). The substitution $w(y) = (y'_x)^{3/2}$ leads to a second-order equation:

$$w''_{yy} = \Big(\frac{3A}{2m+2} y^{m+1} + C\Big) w^{-5/3}.$$

The value $C = 0$ corresponds to the Emden–Fowler equation, whose integrable cases are specified in Section 2.3 for some values of m (to those cases there correspond three-parameter families of particular solutions of the original equation).

2°. Particular solution: $y = \left[\frac{8(m+1)(m+3)(3m+1)}{A(m-1)^4}\right]^{\frac{1}{m-1}} (x+C)^{\frac{4}{1-m}}$.

3. $y''''_{xxxx} = ax^{-3m-5} y^m$.

The transformation $x = t^{-1}$, $y = t^{-3} w(t)$ leads to an equation of the form 16.2.1.2: $w''''_{tttt} = aw^m$.

4. $y''''_{xxxx} = ax^{-\frac{3m+5}{2}} y^m$.

This is a special case of equation 16.2.6.5 with $f(w) = aw^m$.

5. $y''''_{xxxx} = ax^n y^m$.

Generalized homogeneous equation.

1°. The transformation $t = x^{n+4} y^{m-1}$, $u = xy'_x/y$ leads to a third-order equation.

2°. The transformation $x = z^{-1}$, $y = z^{-3} w(z)$ leads to an equation of the same form: $w''''_{zzzz} = z^{-n-3m-5} w^m$.

6. $y''''_{xxxx} = (ay + bx^k)^m$, $\quad k = 0,\ 1,\ 2,\ 3$.

The substitution $aw = ay + bx^k$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = a^m w^m$.

7. $x^{3m+1}(ax+b)^4 y''''_{xxxx} = cy^m$.

This is a special case of equation 16.2.6.10 with $f(w) = cw^m$.

8. $y''''_{xxxx} = (ax^2 + bx + c)^{-\frac{3m+5}{2}} y^m$.

This is a special case of equation 16.2.6.12 with $f(w) = w^m$.

► **Equations of the form $y''''_{xxxx} = f(x, y, y'_x)$.**

9. $y''''_{xxxx} = ay^n y'_x + bx^k$.

By integrating, we find $y'''_{xxx} = \frac{a}{n+1} y^{n+1} + \frac{b}{k+1} x^{k+1} + C$. For $b = 0$, the order of this equation can be reduced by one with the help of the substitution $w(y) = y'_x$.

10. $y''''_{xxxx} = ax^{-4}(xy'_x - y)^n.$

The transformation $t = \ln|x|$, $w = xy'_x - y$ leads to a third-order autonomous equation: $w'''_{ttt} - 5w''_{tt} + 6w'_t = aw^n$.

11. $y''''_{xxxx} = ax^n(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = ax^n w^k$. The substitution $w = xy'_x - y$ leads to a third-order generalized homogeneous equation: $(w'_x/x)''_{xx} = ax^n w^k$.

12. $y''''_{xxxx} = ax^n(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = ax^n w^k$. The substitution $w = xy'_x - 2y$ leads to a third-order generalized homogeneous equation: $xw'''_{xxx} - w''_{xx} = ax^{n+2}w^k$.

13. $y''''_{xxxx} = ax^n(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = ax^n w^k$. The substitution $w = xy'_x - 3y$ leads to a third-order generalized homogeneous equation: $w'''_{xxx} = ax^{n+1}w^k$.

14. $y''''_{xxxx} = a^4 y + bx^n(y'_x - ay)^k.$

The substitution $u = y'_x - ay$ leads to a third-order equation: $u'''_{xxx} + au''_{xx} + a^2u'_x + a^3u = bx^n u^k$.

▶ **Equations of the form** $y''''_{xxxx} = f(x, y, y'_x, y''_{xx})$.

15. $y''''_{xxxx} + ay''_{xx} = by^n + c.$

This is a special case of equation 16.2.6.33 with $f(y) = by^n + c$.

16. $y''''_{xxxx} - \frac{5}{2}ay''_{xx} + \frac{9}{16}a^2y = by^{-5/3}.$

The transformation $\xi = e^{x\sqrt{a}}$, $w(\xi) = \xi^{3/2}y$ leads to an autonomous equation of the form 16.2.1.1: $w''''_{\xi\xi\xi\xi} = a^{-2}bw^{-5/3}$.

17. $y''''_{xxxx} + ay''_{xx} + by = cyy''_{xx} - c(y'_x)^2 + k.$

1°. Particular solution:

$$y = C_1 \sinh(C_4x) + C_2 \cosh(C_4x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^4 + (a - cC_3)C_4^2 + b = 0,$$
$$c(C_2^2 - C_1^2)C_4^2 - bC_3 + k = 0.$$

2°. Particular solution:

$$y = C_1 \sin(C_4x) + C_2 \cos(C_4x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^4 - (a - cC_3)C_4^2 + b = 0,$$
$$c(C_1^2 + C_2^2)C_4^2 + bC_3 - k = 0.$$

18. $y''''_{xxxx} + ayy''_{xx} + by''_{xx} - a(y'_x)^2 + cy'_x = 0.$

Particular solution: $y = C_1 \exp(C_2x) - \dfrac{C_2^3 + bC_2 + c}{aC_2}$.

19. $y''''_{xxxx} = ay^2 y''_{xx} - ay(y'_x)^2 + by.$

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_2 are related by the constraint $C_3^4 - 4aC_1C_2C_3^2 - b = 0$.

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

where the constants C_1, C_2, and C_2 are related by the constraint $C_3^4 + a(C_1^2 + C_2^2)C_3^2 - b = 0$.

3°. There are also solutions $y = \pm x\sqrt{b/a} + C$ and $y = 0$.

20. $y''''_{xxxx} + a(y'_x)^n y''_{xx} = by^k + c.$

This is a special case of equation 16.2.6.34 with $f(u) = au^n$ and $g(y) = by^k + c$.

21. $y''''_{xxxx} = ax^{-2}(xy'_x - y)^n y''_{xx}.$

This is a special case of equation 16.2.6.35 with $f(w) = aw^n$.

22. $yy''''_{xxxx} - (y''_{xx})^2 = 0.$

1°. Particular solutions:

$$\begin{aligned}
y &= C_1 x + C_2,\\
y &= C_1 (x + C_2)^{-3/2},\\
y &= C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),\\
y &= C_1 \cos(C_3 x) + C_2 \sin(C_3 x).
\end{aligned}$$

2°. Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = C_1 x + C_2.$$

The substitution $z = C_1 x + C_2$ leads to a generalized homogeneous equation.

23. $yy''''_{xxxx} - (y''_{xx})^2 + ay + b = 0.$

Particular solutions:

$$\begin{aligned}
y &= C_1 \exp(\lambda x) + C_2 \exp(-\lambda x) - b/a, \quad && \lambda = (a^2/b)^{1/4};\\
y &= C_1 \sin(\lambda x) + C_2 \cos(\lambda x) - b/a, && \lambda = (a^2/b)^{1/4}.
\end{aligned}$$

24. $yy''''_{xxxx} - (y''_{xx})^2 + ay''_{xx} = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 + ay = C_1 x + C_2$.

2°. Particular solutions:

$$\begin{aligned}
y &= C_1 \exp(C_3 x) + C_2 \exp(-C_3 x) - aC_3^{-2},\\
y &= C_1 \sin(C_3 x) + C_2 \cos(C_3 x) + aC_3^{-2},\\
y &= C_1 x + C_2.
\end{aligned}$$

25. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b.$

1°. The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient linear equation of the form 14.1.9.1: $w''_{xx} = aw + b$.

2°. Particular solutions:

$$
\begin{aligned}
&y = C_1\exp\bigl(C_2x\bigr) - \frac{b}{4aC_1C_2^2}\exp\bigl(-C_2x\bigr) && \text{if } a\neq 0,\\
&y = C_1\exp\bigl(x\sqrt{a}\,\bigr) - \frac{b}{4a^2C_1}\bigl(-x\sqrt{a}\,\bigr) + C_2 && \text{if } a>0,\\
&y = C_1\sin\bigl(\lambda x\bigr) + C_2\cos\bigl(\lambda x\bigr), \quad \lambda^2 = \frac{b}{a(C_1^2+C_2^2)} && \text{if } ab>0,\\
&y = \frac{\sqrt{-b}}{a}\sin\bigl(x\sqrt{-a}+C_1\bigr) + C_2 && \text{if } a<0,\ b<0,\\
&y = \pm x\sqrt{b/a} + C_1 && \text{if } ab>0.
\end{aligned}
$$

26. $yy''''_{xxxx} - (y''_{xx})^2 = a\bigl[yy''_{xx} - (y'_x)^2\bigr] + bx^k + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + bx^k + c$.

27. $yy''''_{xxxx} - \frac{1}{6}(y''_{xx})^2 = ax^2 + bx + c.$

Particular solution:

$$
y = \tfrac{1}{24}C_1x^4 + \tfrac{1}{6}C_2x^3 + \tfrac{1}{2}C_3x^2 + C_4x + C_5,
$$

where the constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints

$$
\begin{aligned}
\tfrac{1}{3}C_1C_3 - \tfrac{1}{6}C_2^2 &= a,\\
C_1C_4 - \tfrac{1}{3}C_2C_3 &= b,\\
C_1C_5 - \tfrac{1}{6}C_3^2 &= c.
\end{aligned}
$$

28. $y''''_{xxxx} = a^2y + b(y''_{xx} + ay)^k.$

The substitution $w = y''_{xx} + ay$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{xx} = aw + bw^k$.

29. $y''''_{xxxx} = ay\bigl[yy''_{xx} - (y'_x)^2\bigr]^n.$

This is a special case of equation 16.2.6.42 with $f(w) = 0$ and $g(w) = aw^n$.

► Equations of the form $y''''_{xxxx} = f(x, y, y'_x, y''_{xx}, y'''_{xxx})$.

30. $y''''_{xxxx} + a_3y'''_{xxx} + a_2y''_{xx} + a_1y'_x + a_0y = byy''_{xx} - b(y'_x)^2 + k.$

Particular solutions: $y = C\exp(\lambda_nx) + ka_0^{-1}$, where C is an arbitrary constant and $\lambda = \lambda_n$ are roots of the algebraic equation $\lambda^4 + a_3\lambda^3 + \Bigl(a_2 - \dfrac{bk}{a_0}\Bigr)\lambda^2 + a_1\lambda + a_0 = 0.$

31. $y''''_{xxxx} + ayy'''_{xxx} = bx^n.$

Integrating, we arrive at a third-order equation: $y'''_{xxx} + ayy''_{xx} - \dfrac{1}{2}a(y'_x)^2 = \dfrac{b}{n+1}x^{n+1} + C.$

32. $y''''_{xxxx} + ayy'''_{xxx} - ay'_x y''_{xx} = 0.$

This equation arises in hydrodynamics.

1°. Particular solutions:

$$y = C_1 x + C_2,$$
$$y = C_1 \exp(C_2 x) - a^{-1} C_2,$$
$$y = 6(ax + C_1)^{-1}.$$

2°. Integrating, we arrive at a third-order autonomous equation:

$$y'''_{xxx} + ayy''_{xx} - a(y'_x)^2 = C.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (2002).

33. $y''''_{xxxx} + ayy'''_{xxx} - ay'_x y''_{xx} = by.$

Particular solutions:

$$y = C_1 \exp(\lambda x) + C_2 \exp(-\lambda x), \quad \lambda = b^{1/4},$$
$$y = C_1 \sin(\lambda x) + C_2 \cos(\lambda x), \quad \lambda = b^{1/4}.$$

34. $y''''_{xxxx} + ayy'''_{xxx} + b(y'_x)^n y''_{xx} = cx^k + d.$

This is a special case of equation 16.2.6.50 with $f(u) = bu^n$ and $g(x) = cx^k + d$.

35. $y''''_{xxxx} = ay^n y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = ay^n$.

36. $xy''''_{xxxx} + 4y'''_{xxx} = ax^{-5/3} y^{-5/3}.$

The substitution $w(x) = xy$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

37. $xy''''_{xxxx} + 4y'''_{xxx} = a(xy)^k.$

The substitution $w(x) = xy$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = aw^k$.

38. $xy''''_{xxxx} + 2y'''_{xxx} = a(xy'_x - y)^k.$

The substitution $w(x) = xy'_x - y$ leads to a third-order equation: $w'''_{xxx} = aw^k$ (Section 15.2 presents its solutions for $k = -\frac{7}{2}, -\frac{5}{2}, -2, -\frac{4}{3}, -\frac{7}{6}, -\frac{1}{2}, 0$, and 1).

39. $xy''''_{xxxx} + (a+3)y'''_{xxx} = bx^n(xy'_x + ay)^k.$

The substitution $w = xy'_x + ay$ leads to a third-order generalized homogeneous equation: $w'''_{xxx} = bx^n w^k$.

40. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = ax^{-10/3} y^{-5/3}.$

The substitution $w(x) = x^2 y$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

41. $x^4 y''''_{xxxx} + 6x^3 y'''_{xxx} + 7x^2 y''_{xx} + xy'_x = ay^{-5/3}.$

The substitution $t = \ln|x|$ leads to an equation of the form 16.2.1.1: $y''''_{xxxx} = ay^{-5/3}$.

42. $yy''''_{xxxx} = ay'_x y'''_{xxx}.$

Having integrated this equation, we obtain the third-order equation $y'''_{xxx} = Cy^a$, whose solvable cases are specified in Section 15.2.2.

43. $yy''''_{xxxx} - y'_x y'''_{xxx} = ax^n y^2.$

Integrating yields a third-order linear equation: $y'''_{xxx} = \Big(\dfrac{a}{n+1}x^{n+1} + C\Big)y.$

44. $yy''''_{xxxx} - y'_x y'''_{xxx} = ay'_x.$

Integrating yields a third-order nonhomogeneous linear equation with constant coefficients: $y'''_{xxx} = Cy - a.$

45. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ax^n.$

This is a special case of equation 16.2.6.58 with $f(x) = ax^n$.

46. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ay^{-10/3}.$

The substitution $w = y^2$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = 2aw^{-5/3}$.

47. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ay^n.$

The substitution $w = y^2$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = 2aw^{n/2}$.

48. $yy''''_{xxxx} + \frac{2}{3}y'_x y'''_{xxx} - \frac{1}{3}(y''_{xx})^2 = a.$

This is a special case of equation 16.2.6.59 with $a = \frac{2}{3}$ and $f(x) = a$. Integrating the equation twice, we arrive at a second-order equation of the form 14.8.1.54:

$$3yy''_{xx} - 2(y'_x)^2 = \tfrac{3}{2}ax^2 + C_1x + C_2.$$

49. $yy''''_{xxxx} + \frac{3}{2}y'_x y'''_{xxx} + \frac{1}{2}(y''_{xx})^2 = a.$

Integrating the equation twice, we arrive at a second-order equation of the form 14.8.1.53:

$$yy''_{xx} - \tfrac{1}{4}(y'_x)^2 = \tfrac{1}{2}ax^2 + C_1x + C_2.$$

50. $yy''''_{xxxx} + \frac{3}{2}y'_x y'''_{xxx} + \frac{1}{2}(y''_{xx})^2 = (ax+b)y^{-1/2}.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a constant coefficient linear equation: $2x^{(5)}_t = ax + b$.

51. $yy''''_{xxxx} - y'_x y'''_{xxx} = ax^n yy'''_{xxx}.$

Integrating yields a third-order linear equation: $y'''_{xxx} = C\exp\Big(\dfrac{a}{n+1}x^{n+1}\Big)y.$

52. $xyy''''_{xxxx} - x(y''_{xx})^2 = a.$

Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = ax\ln|x| + C_1x + C_2.$$

53. $xyy''''_{xxxx} = xy'_x y'''_{xxx} + ayy'''_{xxx}.$

Integrating yields a third-order linear equation of the form 15.1.2.7: $y'''_{xxx} = Cx^a y.$

54. $y^3 y''''_{xxxx} = 4y^2 y'_x y'''_{xxx} + 3y^2 (y''_{xx})^2 - 6(y'_x)^4.$

This is a special case of equation 16.2.6.67 with $f \equiv 0$.

Solution in parametric form:

$$x = \pm \int \frac{dx}{\sqrt{2\xi^4 + C_2\xi + C_1}} + C_3, \qquad y = C_4 \exp\biggl(\pm \int \frac{\xi\, d\xi}{\sqrt{2\xi^4 + C_2\xi + C_1}}\biggr).$$

55. $y''_{xx} y''''_{xxxx} = a(y'''_{xxx})^2.$

$$\text{Solution: } y = \begin{cases} C_0 + C_1 x + (C_2 + C_3 x)^{\frac{3-2a}{1-a}} & \text{if } a \neq 1, \\ C_0 + C_1 x + C_2 \exp(C_3 x) & \text{if } a = 1. \end{cases}$$

56. $y''_{xx} y''''_{xxxx} - \frac{1}{2}(y'''_{xxx})^2 = \alpha(x y'_x - y) + \beta y'_x + \gamma.$

Differentiating with respect to x yields

$$y''_{xx}\bigl[y^{(5)}_x - \alpha x - \beta\bigr] = 0. \tag{1}$$

Equating the second factor in (1) with zero and integrating it, we find the solution:

$$y = \alpha \frac{x^6}{6!} + \beta \frac{x^5}{5!} + C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0.$$

The constants C_k and parameters α, β, and γ are related by the constraint

$$48\, C_2 C_4 - 18\, C_3^2 = -\alpha C_0 + \beta C_1 + \gamma,$$

obtained by means of substituting the solution into the original equation.

The other solution, which corresponds to setting the first factor in (1) to zero, is given by:

$$y = \widetilde{C}_1 x + \widetilde{C}_0, \qquad \text{where} \quad \alpha \widetilde{C}_0 - \beta \widetilde{C}_1 - \gamma = 0.$$

57. $y''''_{xxxx} = a y^k y'_x (y'''_{xxx})^s.$

This is a special case of equation 16.2.6.72 with $f(y) = a y^k$ and $g(w) = w^s$. For $k = -1$ and $s = 1$, see equation 16.2.1.42.

The first integral has the form:

$$\frac{1}{1-s}(y'''_{xxx})^{1-s} - \frac{a}{k+1} y^{k+1} = C \quad \text{if } k \neq -1,\ s \neq 1; \tag{1}$$

$$\ln|y'''_{xxx}| - \frac{a}{k+1} y^{k+1} = C \quad \text{if } k \neq -1,\ s = 1; \tag{2}$$

$$\frac{1}{1-s}(y'''_{xxx})^{1-s} - a \ln|y| = C \quad \text{if } k = -1,\ s \neq 1. \tag{3}$$

For $C = 0$, equality (1) is changing to the equation

$$y'''_{xxx} = \left[\frac{a(1-s)}{k+1}\right]^{\frac{1}{1-s}} y^{\frac{k+1}{1-s}},$$

which is discussed in Section 15.2.2 (the solutions given there generate 3-parametric families of particular solutions of the original equation for $k = (1-s)\beta - 1$, where $\beta = -\frac{7}{2}$, $-\frac{5}{2}$, -2, $-\frac{4}{3}$, $-\frac{7}{6}$, $-\frac{1}{2}$, 0, and 1).

58. $a_1(y''''_{xxxx})^2 + (a_2 y'''_{xxx} + a_3 y''_{xx} + 6a_4 y + a_5 x + a_6) y''''_{xxxx} + b_1 (y'''_{xxx})^2$
$+ (b_2 x + b_3) y'''_{xxx} - a_4 (y''_{xx})^2 + b_4 y''_{xx} + b_5 x^2 + b_6 x + b_7 = 0.$

There are particular solutions of the form $y = C_1 x^4 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$, where the five constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints.

16.2.2 Equations Containing Exponential Functions

► **Equations of the form $y''''_{xxxx} = f(x, y)$.**

1. $y''''_{xxxx} = ae^{\lambda y} + b$.

This is a special case of equation 16.2.6.1 with $f(y) = ae^{\lambda y} + b$.

2. $y''''_{xxxx} = ae^{\lambda y + \beta x} + b$.

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = ae^{\lambda w} + b$.

3. $y''''_{xxxx} = ax^{-4}e^{\lambda y}$.

This is a special case of equation 16.2.6.2 with $f(y) = ae^{\lambda y}$. The substitution $t = \ln|x|$ leads to an autonomous equation.

4. $y''''_{xxxx} = ax^k e^{\lambda y}$.

This is a special case of equation 16.2.6.15 with $f(w) = aw$ and $m = k + 4$.

5. $y''''_{xxxx} = ae^{\lambda x}y^n$.

This is a special case of equation 16.2.6.14 with $f(w) = aw$ and $m = n - 1$.

6. $y''''_{xxxx} = a\exp(\lambda y + \beta x^2) + b$.

The substitution $w = y + (\beta/\lambda)x^2$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = ae^{\lambda w} + b$.

7. $y''''_{xxxx} = a(y + be^x)^{-5/3} - be^x$.

The substitution $w = y + be^x$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

8. $y''''_{xxxx} = a(y + be^x)^m - be^x$.

The substitution $w = y + be^x$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = aw^m$.

► **Other equations.**

9. $y''''_{xxxx} = ae^{\lambda y}y'_x + be^{\beta x}$.

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}e^{\lambda y} + \dfrac{b}{\beta}e^{\beta x} + C$.

10. $y''''_{xxxx} = a^4 y + be^{\lambda x}(y'_x - ay)^k$.

This is a special case of equation 16.2.6.27 with $f(x, w) = be^{\lambda x}w^k$.

11. $y''''_{xxxx} = be^{\lambda x}(xy'_x - y)^k$.

This is a special case of equation 16.2.6.23 with $f(x, w) = be^{\lambda x}w^k$.

12. $y''''_{xxxx} = be^{\lambda x}(xy'_x - 2y)^k$.

This is a special case of equation 16.2.6.24 with $f(x, w) = be^{\lambda x}w^k$.

13. $y''''_{xxxx} = be^{\lambda x}(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x, w) = be^{\lambda x}w^k$.

14. $yy''''_{xxxx} - (y''_{xx})^2 = ae^{\lambda x}.$

1°. Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = a\lambda^{-2}e^{\lambda x} + C_1x + C_2.$$

For $C_1 = C_2 = 0$, it is an equation of the form 14.8.3.47.

2°. Particular solution: $y = C\exp(\lambda x) + \dfrac{a}{C\lambda^4}$.

15. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + be^{\lambda x} = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_xy''_{xx} - ay + b\lambda^{-1}e^{\lambda x} = C$.

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{aC\lambda - b}{C\lambda^4},$$
$$y = \frac{b}{2a\lambda}\exp(\lambda x) + C\exp(-\lambda x) - \frac{a}{\lambda^3}.$$

16. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + be^{\lambda x} = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1x + C_2 + b\lambda^{-2}e^{\lambda x} = 0$.

2°. Particular solution: $y = C\exp(\lambda x) + \dfrac{aC\lambda^2 - b}{C\lambda^4}$.

17. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + be^{\lambda x}.$

1°. The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient linear equation of the form 14.1.9.1: $w''_{xx} = aw + be^{\lambda x}$.

2°. Particular solution: $y = C\exp(\lambda x) + \dfrac{b}{C\lambda^2(\lambda^2 - a)}$.

18. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + be^{\lambda x} = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1x + C_2 + b\lambda^{-2}e^{\lambda x} = 0$.

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{aC\lambda^3 - b}{C\lambda^4},$$
$$y = \frac{b}{2a\lambda^3}\exp(\lambda x) + C\exp(-\lambda x) - \frac{a}{\lambda}.$$

19. $y''''_{xxxx} = ae^{\lambda y}y'_xy'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = ae^{\lambda y}$.

20. $y''''_{xxxx} = (ae^{\lambda y}y'_x + be^{\beta x})y'''_{xxx}.$

This is a special case of equation 17.2.6.58 with $n = 4$, $f(y) = ae^{\lambda y}$, and $g(x) = be^{\beta x}$.

21. $y''''_{xxxx} - 4\lambda y'''_{xxx} + 6\lambda^2y''_{xx} - 4\lambda^3y'_x + \lambda^4y = a\exp\left(\tfrac{8}{3}\lambda x\right)y^{-5/3}.$

The substitution $w(x) = ye^{-\lambda x}$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

22. $y''''_{xxxx} - 4\lambda y'''_{xxx} + 6\lambda^2 y''_{xx} - 4\lambda^3 y'_x + \lambda^4 y = ae^{\lambda(1-m)x} y^m.$

The substitution $w(x) = ye^{-\lambda x}$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = aw^m$.

23. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ae^{\lambda x}.$

Solution: $y^2 = C_3 x^3 + C_2 x^2 + C_1 x + C_0 + 2a\lambda^{-4} e^{\lambda x}$.

24. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ae^{\lambda y} + b.$

This is a special case of equation 16.2.6.60 with $f(y) = ae^{\lambda y} + b$.

16.2.3 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine.**

1. $y''''_{xxxx} = a \sinh^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a \sinh^m(\lambda y) + b$.

2. $y''''_{xxxx} = a \sinh(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a \sinh(\lambda w) + b$.

3. $y''''_{xxxx} = a(y + b \sinh x)^2 - b \sinh x.$

The substitution $w = y + b \sinh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

4. $y''''_{xxxx} = ax^{-4} \sinh^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a \sinh^m(\lambda y)$.

5. $y''''_{xxxx} = a \sinh(\lambda y) y'_x + b \sinh(\beta x).$

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda} \cosh(\lambda y) + \dfrac{b}{\beta} \cosh(\beta x) + C$.

6. $y''''_{xxxx} = a^4 y + b \sinh(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b \sinh(\lambda x) w^k$.

7. $y''''_{xxxx} = b \sinh(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b \sinh(\lambda x) w^k$.

8. $y''''_{xxxx} = b \sinh(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b \sinh(\lambda x) w^k$.

9. $y''''_{xxxx} = b \sinh(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = b \sinh(\lambda x) w^k$.

10. $yy''''_{xxxx} - (y''_{xx})^2 = a \sinh(\lambda x).$

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = a\lambda^{-2} \sinh(\lambda x) + C_1 x + C_2$.

2°. Particular solution: $y = C \sinh(\lambda x) + \dfrac{a}{C\lambda^4}$.

11. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\sinh(\lambda x) = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}\cosh(\lambda x) = C$.

2°. Particular solution: $y = \dfrac{b}{\lambda(a^2 - C^2\lambda^6)}\big[C\lambda^3\sinh(\lambda x) + a\cosh(\lambda x)\big] + C$.

12. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\sinh(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1x + C_2 + b\lambda^{-2}\sinh(\lambda x) = 0$.

2°. Particular solution: $y = C\sinh(\lambda x) + \dfrac{aC\lambda^2 - b}{C\lambda^4}$.

13. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\sinh(\lambda x) + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\sinh(\lambda x) + c$.

14. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\sinh(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1x + C_2 + b\lambda^{-2}\sinh(\lambda x) = 0$.

2°. Particular solution: $y = \dfrac{b}{\lambda^3(a^2 - C^2\lambda^2)}\big[C\lambda\sinh(\lambda x) + a\cosh(\lambda x)\big] + C$.

15. $y''''_{xxxx} = a\sinh^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\sinh^k(\lambda y)$.

16. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\sinh(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\sinh(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \left[\dfrac{a}{\lambda}\cosh(\lambda x) + C\right]y$.

17. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\sinh(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\sinh(\lambda x)$.

18. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\sinh^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\sinh^k(\lambda y) + b$.

► **Equations with hyperbolic cosine.**

19. $y''''_{xxxx} = a\cosh^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\cosh^m(\lambda y) + b$.

20. $y''''_{xxxx} = a\cosh(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\cosh(\lambda w) + b$.

21. $y''''_{xxxx} = a(y + b\cosh x)^2 - b\cosh x.$

The substitution $w = y + b\cosh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

22. $y''''_{xxxx} = ax^{-4}\cosh^m(\lambda y)$.

This is a special case of equation 16.2.6.2 with $f(y) = a\cosh^m(\lambda y)$.

23. $y''''_{xxxx} = a\cosh(\lambda y)y'_x + b\cosh(\beta x)$.

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}\sinh(\lambda y) + \dfrac{b}{\beta}\sinh(\beta x) + C$.

24. $y''''_{xxxx} = a^4 y + b\cosh(\lambda x)(y'_x - ay)^k$.

This is a special case of equation 16.2.6.27 with $f(x,w) = b\cosh(\lambda x)w^k$.

25. $y''''_{xxxx} = b\cosh(\lambda x)(xy'_x - y)^k$.

This is a special case of equation 16.2.6.23 with $f(x,w) = b\cosh(\lambda x)w^k$.

26. $y''''_{xxxx} = b\cosh(\lambda x)(xy'_x - 2y)^k$.

This is a special case of equation 16.2.6.24 with $f(x,w) = b\cosh(\lambda x)w^k$.

27. $y''''_{xxxx} = b\cosh(\lambda x)(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x,w) = b\cosh(\lambda x)w^k$.

28. $yy''''_{xxxx} - (y''_{xx})^2 = a\cosh(\lambda x)$.

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = a\lambda^{-2}\cosh(\lambda x) + C_1 x + C_2$.

2°. Particular solution: $y = C\cosh(\lambda x) + \dfrac{a}{C\lambda^4}$.

29. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\cosh(\lambda x) = 0$.

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}\sinh(\lambda x) = C$.

2°. Particular solution: $y = \dfrac{b}{\lambda(a^2 - C^2\lambda^6)}\big[C\lambda^3\cosh(\lambda x) + a\sinh(\lambda x)\big] + C$.

30. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\cosh(\lambda x) = 0$.

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1 x + C_2 + b\lambda^{-2}\cosh(\lambda x) = 0$.

2°. Particular solution: $y = C\cosh(\lambda x) + \dfrac{aC\lambda^2 - b}{C\lambda^4}$.

31. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\cosh(\lambda x) + c$.

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\cosh(\lambda x) + c$.

32. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\cosh(\lambda x) = 0$.

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1 x + C_2 + b\lambda^{-2}\cosh(\lambda x) = 0$.

2°. Particular solution: $y = \dfrac{b}{\lambda^3(a^2 - C^2\lambda^2)}\big[C\lambda\cosh(\lambda x) + a\sinh(\lambda x)\big] + C$.

33. $y''''_{xxxx} = a\cosh^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\cosh^k(\lambda y)$.

34. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\cosh(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\cosh(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \left[\frac{a}{\lambda}\sinh(\lambda x) + C\right]y$.

35. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cosh(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\cosh(\lambda x)$.

36. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cosh^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\cosh^k(\lambda y) + b$.

► **Equations with hyperbolic tangent.**

37. $y''''_{xxxx} = a\tanh^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\tanh^m(\lambda y) + b$.

38. $y''''_{xxxx} = a\tanh(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\tanh(\lambda w) + b$.

39. $y''''_{xxxx} = ax^{-4}\tanh^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\tanh^m(\lambda y)$.

40. $y''''_{xxxx} = a\tanh(\lambda y)y'_x + b\tanh(\beta x).$

This is a special case of equation 16.2.6.21 with $f(y) = a\tanh(\lambda y)$ and $g(x) = b\tanh(\beta x)$.

41. $y''''_{xxxx} = a^4y + b\tanh(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b\tanh(\lambda x)w^k$.

42. $y''''_{xxxx} = b\tanh(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b\tanh(\lambda x)w^k$.

43. $y''''_{xxxx} = b\tanh(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b\tanh(\lambda x)w^k$.

44. $y''''_{xxxx} = b\tanh(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = b\tanh(\lambda x)w^k$.

45. $y''''_{xxxx} = y + a(y'_x - y\tanh x)^k.$

This is a special case of equation 17.2.6.32 with $f(x, u) = au^k$ and $\varphi(x) = \cosh x$.

46. $y''''_{xxxx} = a\tanh^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\tanh^k(\lambda y)$.

47. $yy''''_{xxxx} - y'_x y'''_{xxx} = a \tanh(\lambda x) y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\tanh(\lambda x)$.

48. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a \tanh^k(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\tanh^k(\lambda x) + b$.

49. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a \tanh^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\tanh^k(\lambda y) + b$.

► **Equations with hyperbolic cotangent.**

50. $y''''_{xxxx} = a \coth^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\coth^m(\lambda y) + b$.

51. $y''''_{xxxx} = a \coth(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\coth(\lambda w) + b$.

52. $y''''_{xxxx} = a x^{-4} \coth^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\coth^m(\lambda y)$.

53. $y''''_{xxxx} = a \coth(\lambda y) y'_x + b \coth(\beta x).$

This is a special case of equation 16.2.6.21 with $f(y) = a\coth(\lambda y)$ and $g(x) = b\coth(\beta x)$.

54. $y''''_{xxxx} = a^4 y + b \coth(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b\coth(\lambda x) w^k$.

55. $y''''_{xxxx} = b \coth(\lambda x)(x y'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b\coth(\lambda x) w^k$.

56. $y''''_{xxxx} = b \coth(\lambda x)(x y'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b\coth(\lambda x) w^k$.

57. $y''''_{xxxx} = b \coth(\lambda x)(x y'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = b\coth(\lambda x) w^k$.

58. $y''''_{xxxx} = y + a(y'_x - y \coth x)^k.$

This is a special case of equation 17.2.6.32 with $f(x, u) = au^k$ and $\varphi(x) = \sinh x$.

59. $y''''_{xxxx} = a \coth^k(\lambda y) y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\coth^k(\lambda y)$.

60. $yy''''_{xxxx} - y'_x y'''_{xxx} = a \coth(\lambda x) y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\coth(\lambda x)$.

61. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a \coth^k(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\coth^k(\lambda x) + b$.

62. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a \coth^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\coth^k(\lambda y) + b$.

16.2.4 Equations Containing Logarithmic Functions

► **Equations of the form $y''''_{xxxx} = f(x, y)$.**

1. $y''''_{xxxx} = a \ln^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a \ln^m(\lambda y) + b$.

2. $y''''_{xxxx} = a \ln(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a \ln(\lambda w) + b$.

3. $y''''_{xxxx} = a \ln(\lambda y + \beta x^2) + b.$

The substitution $w = y + (\beta/\lambda)x^2$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a \ln(\lambda w) + b$.

4. $y''''_{xxxx} = ax^{-4} \ln^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a \ln^m(\lambda y)$.

5. $y''''_{xxxx} = ay(\lambda x + m \ln y).$

This is a special case of equation 16.2.6.14 with $f(w) = a \ln w$.

6. $y''''_{xxxx} = ax^{-4}(\lambda y + m \ln x).$

This is a special case of equation 16.2.6.15 with $f(w) = a \ln w$.

7. $y''''_{xxxx} = ax^{-3}(\ln y - \ln x).$

This is a special case of equation 16.2.6.3 with $f(w) = a \ln w$.

8. $y''''_{xxxx} = ax^{-5}(\ln y - 3 \ln x).$

This is a special case of equation 16.2.6.4 with $f(w) = a \ln w$.

9. $y''''_{xxxx} = ax^{-5/2}(2 \ln y - 3 \ln x).$

This is a special case of equation 16.2.6.5 with $f(w) = 2a \ln w$.

10. $y''''_{xxxx} = ax^{n-4}(\ln y - n \ln x).$

This is a special case of equation 16.2.6.6 with $f(w) = a \ln w$ and $k = -n$.

► **Other equations.**

11. $y''''_{xxxx} = a^4 y + b \ln(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b \ln(\lambda x) w^k$.

12. $y''''_{xxxx} = b \ln(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b \ln(\lambda x) w^k$.

13. $y''''_{xxxx} = b \ln(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b \ln(\lambda x) w^k$.

14. $y''''_{xxxx} = b\ln(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = b\ln(\lambda x)w^k$.

15. $yy''''_{xxxx} - (y''_{xx})^2 = a\ln(\lambda x).$

This is a special case of equation 16.2.6.36 with $f(x) = a\ln(\lambda x)$.

16. $xy''''_{xxxx} + 4y'''_{xxx} = a(\ln x + \ln y).$

The substitution $w(x) = xy$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = a\ln w$.

17. $y''''_{xxxx} = a\ln(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\ln(\lambda y)$.

18. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\ln(\lambda x)y^2.$

Integrating yields a third-order linear equation: $y'''_{xxx} = \big[ax\ln(\lambda x) - ax + C\big]y$.

19. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\ln^m(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\ln^m(\lambda x) + b$.

20. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\ln^m(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\ln^m(\lambda y) + b$.

16.2.5 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y''''_{xxxx} = a\sin^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\sin^m(\lambda y) + b$.

2. $y''''_{xxxx} = a\sin(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\sin(\lambda w) + b$.

3. $y''''_{xxxx} = a(y + b\sin x)^2 - b\sin x.$

The substitution $w = y + b\sin x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

4. $y''''_{xxxx} = ax^{-4}\sin^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\sin^m(\lambda y)$.

5. $y''''_{xxxx} = a\sin(\lambda y)y'_x + b\sin(\beta x).$

Integrating yields a third-order equation: $y'''_{xxx} = -\dfrac{a}{\lambda}\cos(\lambda y) - \dfrac{b}{\beta}\cos(\beta x) + C$.

6. $y''''_{xxxx} = a^4 y + b\sin(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = b\sin(\lambda x)w^k$.

7. $y''''_{xxxx} = b\sin(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = b\sin(\lambda x)w^k$.

8. $y''''_{xxxx} = b\sin(\lambda x)(xy'_x - 2y)^k$.

This is a special case of equation 16.2.6.24 with $f(x,w) = b\sin(\lambda x)w^k$.

9. $y''''_{xxxx} = b\sin(\lambda x)(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x,w) = b\sin(\lambda x)w^k$.

10. $yy''''_{xxxx} - (y''_{xx})^2 = a\sin(\lambda x)$.

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = -a\lambda^{-2}\sin(\lambda x) + C_1 x + C_2$.

2°. Particular solution: $y = C\sin(\lambda x) + \dfrac{a}{C\lambda^4}$.

11. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\sin(\lambda x) = 0$.

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay - b\lambda^{-1}\cos(\lambda x) = C$.

2°. Particular solution: $y = -\dfrac{b}{\lambda(a^2 + C^2\lambda^6)}\big[a\cos(\lambda x) + C\lambda^3\sin(\lambda x)\big] + C$.

12. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\sin(\lambda x) = 0$.

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1 x + C_2 - b\lambda^{-2}\sin(\lambda x) = 0$.

2°. Particular solution: $y = C\sin(\lambda x) - \dfrac{b + aC\lambda^2}{C\lambda^4}$.

13. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\sin(\lambda x) + c$.

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\sin(\lambda x) + c$.

14. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\sin(\lambda x) = 0$.

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1 x + C_2 - b\lambda^{-2}\sin(\lambda x) = 0$.

2°. Particular solution: $y = \dfrac{b}{\lambda^3(a^2 + C^2\lambda^2)}\big[a\cos(\lambda x) - C\lambda\sin(\lambda x)\big] + C$.

15. $y''''_{xxxx} = a\sin^k(\lambda y)y'_x y'''_{xxx}$.

This is a special case of equation 16.2.6.51 with $f(y) = a\sin^k(\lambda y)$.

16. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\sin(\lambda x)y^2$.

This is a special case of equation 16.2.6.57 with $f(x) = a\sin(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \Big[C - \dfrac{a}{\lambda}\cos(\lambda x)\Big]y$.

17. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\sin(\lambda x)$.

Solution: $y^2 = C_3 x^3 + C_2 x^2 + C_1 x + C_0 + 2a\lambda^{-4}\sin(\lambda x)$.

18. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\sin^k(\lambda y) + b$.

This is a special case of equation 16.2.6.60 with $f(y) = a\sin^k(\lambda y) + b$.

► **Equations with cosine.**

19. $y''''_{xxxx} = a\cos^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\cos^m(\lambda y) + b$.

20. $y''''_{xxxx} = a\cos(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\cos(\lambda w) + b$.

21. $y''''_{xxxx} = a(y + b\cos x)^2 - b\cos x.$

The substitution $w = y + b\cos x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

22. $y''''_{xxxx} = ax^{-4}\cos^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\cos^m(\lambda y)$.

23. $y''''_{xxxx} = a\cos(\lambda y)y'_x + b\cos(\beta x).$

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}\sin(\lambda y) + \dfrac{b}{\beta}\sin(\beta x) + C$.

24. $y''''_{xxxx} = a^4 y + b\cos(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b\cos(\lambda x)w^k$.

25. $y''''_{xxxx} = b\cos(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b\cos(\lambda x)w^k$.

26. $y''''_{xxxx} = b\cos(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b\cos(\lambda x)w^k$.

27. $y''''_{xxxx} = b\cos(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = b\cos(\lambda x)w^k$.

28. $yy''''_{xxxx} - (y''_{xx})^2 = a\cos(\lambda x).$

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = -a\lambda^{-2}\cos(\lambda x) + C_1 x + C_2$.

2°. Particular solution: $y = C\cos(\lambda x) + \dfrac{a}{C\lambda^4}$.

29. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\cos(\lambda x) = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}\sin(\lambda x) = C$.

2°. Particular solution: $y = \dfrac{b}{\lambda(a^2 + C^2\lambda^6)}\bigl[a\sin(\lambda x) - C\lambda^3\cos(\lambda x)\bigr] + C$.

30. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\cos(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1 x + C_2 - b\lambda^{-2}\cos(\lambda x) = 0$.

2°. Particular solution: $y = C\cos(\lambda x) - \dfrac{aC\lambda^2 + b}{C\lambda^4}$.

31. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\cos(\lambda x) + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\cos(\lambda x) + c$.

32. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\cos(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1x + C_2 - b\lambda^{-2}\cos(\lambda x) = 0$.

2°. Particular solution: $y = -\dfrac{b}{\lambda^3(a^2 + C^2\lambda^2)}\big[C\lambda\cos(\lambda x) + a\sin(\lambda x)\big] + C$.

33. $y''''_{xxxx} = a\cos^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\cos^k(\lambda y)$.

34. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\cos(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\cos(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \Big[\dfrac{a}{\lambda}\sin(\lambda x) + C\Big]y$.

35. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cos(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\cos(\lambda x)$.

36. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cos^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\cos^k(\lambda y) + b$.

► Equations with tangent.

37. $y''''_{xxxx} = a\tan^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\tan^m(\lambda y) + b$.

38. $y''''_{xxxx} = a\tan(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\tan(\lambda w) + b$.

39. $y''''_{xxxx} = ax^{-4}\tan^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\tan^m(\lambda y)$.

40. $y''''_{xxxx} = a\tan(\lambda y)y'_x + b\tan(\beta x).$

This is a special case of equation 16.2.6.21 with $f(y) = a\tan(\lambda y)$ and $g(x) = b\tan(\beta x)$.

41. $y''''_{xxxx} = a^4y + b\tan(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = b\tan(\lambda x)w^k$.

42. $y''''_{xxxx} = b\tan(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = b\tan(\lambda x)w^k$.

43. $y''''_{xxxx} = b\tan(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = b\tan(\lambda x)w^k$.

44. $y''''_{xxxx} = b\tan(\lambda x)(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x,w) = b\tan(\lambda x)w^k$.

45. $y''''_{xxxx} = y + a(y'_x + y\tan x)^k$.

This is a special case of equation 17.2.6.32 with $f(x,u) = au^k$ and $\varphi(x) = \cos x$.

46. $y''''_{xxxx} = a\tan^k(\lambda y)y'_x y'''_{xxx}$.

This is a special case of equation 16.2.6.51 with $f(y) = a\tan^k(\lambda y)$.

47. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\tan(\lambda x)y^2$.

This is a special case of equation 16.2.6.57 with $f(x) = a\tan(\lambda x)$.

48. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tan^k(\lambda x) + b$.

This is a special case of equation 16.2.6.58 with $f(x) = a\tan^k(\lambda x) + b$.

49. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tan^k(\lambda y) + b$.

This is a special case of equation 16.2.6.60 with $f(y) = a\tan^k(\lambda y) + b$.

► **Equations with cotangent.**

50. $y''''_{xxxx} = a\cot^m(\lambda y) + b$.

This is a special case of equation 16.2.6.1 with $f(y) = a\cot^m(\lambda y) + b$.

51. $y''''_{xxxx} = a\cot(\lambda y + \beta x) + b$.

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\cot(\lambda w) + b$.

52. $y''''_{xxxx} = ax^{-4}\cot^m(\lambda y)$.

This is a special case of equation 16.2.6.2 with $f(y) = a\cot^m(\lambda y)$.

53. $y''''_{xxxx} = a\cot(\lambda y)y'_x + b\cot(\beta x)$.

This is a special case of equation 16.2.6.21 with $f(y) = a\cot(\lambda y)$ and $g(x) = b\cot(\beta x)$.

54. $y''''_{xxxx} = a^4 y + b\cot(\lambda x)(y'_x - ay)^k$.

This is a special case of equation 16.2.6.27 with $f(x,w) = b\cot(\lambda x)w^k$.

55. $y''''_{xxxx} = b\cot(\lambda x)(xy'_x - y)^k$.

This is a special case of equation 16.2.6.23 with $f(x,w) = b\cot(\lambda x)w^k$.

56. $y''''_{xxxx} = b\cot(\lambda x)(xy'_x - 2y)^k$.

This is a special case of equation 16.2.6.24 with $f(x,w) = b\cot(\lambda x)w^k$.

57. $y''''_{xxxx} = b\cot(\lambda x)(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x,w) = b\cot(\lambda x)w^k$.

58. $y''''_{xxxx} = y + a(y'_x - y\cot x)^k$.

This is a special case of equation 17.2.6.32 with $f(x,u) = au^k$ and $\varphi(x) = \sin x$.

59. $y''''_{xxxx} = a\cot^k(\lambda y) y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\cot^k(\lambda y)$.

60. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\cot(\lambda x) y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\cot(\lambda x)$.

61. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cot^k(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\cot^k(\lambda x) + b$.

62. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cot^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\cot^k(\lambda y) + b$.

16.2.6 Equations Containing Arbitrary Functions

▶ Equations of the form $y''''_{xxxx} = f(x, y)$.

1. $y''''_{xxxx} = f(y).$

Autonomous equation. By integrating, we obtain $2y'_x y'''_{xxx} - (y''_{xx})^2 = 2\int f(y)\,dy + 2C$. The substitution $w(y) = |y'_x|^{3/2}$ leads to a second-order equation:

$$2wy = \tfrac{3}{2}\Big[\int f(y)\,dy + C\Big]w^{-5/3}.$$

2. $y''''_{xxxx} = x^{-4} f(y).$

This is a special case of equation 16.2.6.55 with $a_1 = a_2 = a_3 = 0$. The substitution $t = \ln|x|$ leads to an autonomous equation.

3. $y''''_{xxxx} = x^{-3} f(y/x).$

Homogeneous equation. The transformation $t = \ln x$, $w = y/x$ leads to an autonomous equation of the form 16.2.6.79.

4. $y''''_{xxxx} = x^{-5} f(yx^{-3}).$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{tttt} = f(w)$.

5. $y''''_{xxxx} = x^{-5/2} f(yx^{-3/2}).$

The transformation $x = e^t$, $y = x^{3/2} w$ leads to an autonomous equation of the form 16.2.6.33: $w''''_{tttt} - \frac{5}{2} w''_{tt} = -\frac{9}{16} w + f(w)$.

6. $y''''_{xxxx} = x^{-k-4} f(x^k y).$

Generalized homogeneous equation.

1°. The transformation $t = \ln x$, $z = x^k y$ leads to an autonomous equation.

2°. The transformation $z = x^k y$, $w = xy'_x/y$ leads to a third-order equation.

7. $y''''_{xxxx} = yx^{-4} f(x^k y^m)$.

Generalized homogeneous equation. The transformation $z = x^k y^m$, $w = xy'_x/y$ leads to a third-order equation.

8. $y''''_{xxxx} = f(y + a_3x^3 + a_2x^2 + a_1x + a_0)$.

The substitution $w = y + a_3x^3 + a_2x^2 + a_1x + a_0$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

9. $y''''_{xxxx} = f(y + ax^4)$.

The substitution $w = y + ax^4$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w) + 24a$.

10. $x(ax + b)^4 y''''_{xxxx} = f(yx^{-3})$.

The transformation $\xi = \ln\left|\dfrac{ax+b}{x}\right|$, $w = \dfrac{y}{x^3}$ leads to an autonomous equation of the form 16.2.6.79.

11. $y''''_{xxxx} = (ax + by + c)^{-3} f\left(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma}\right)$.

This is a special case of equation 17.2.6.19 with $n = 4$.

12. $y''''_{xxxx} = (ax^2 + bx + c)^{-5/2} f\left(\dfrac{y}{(ax^2 + bx + c)^{3/2}}\right)$.

1°. The transformation $\xi = \displaystyle\int \frac{dx}{ax^2 + bx + c}$, $w = \dfrac{y}{(ax^2 + bx + c)^{3/2}}$ leads to an autonomous equation of the form 16.2.6.33 with respect to $w = w(\xi)$:

$$w''''_{\xi\xi\xi\xi} - \tfrac{5}{2}\Delta w''_{\xi\xi} + \tfrac{9}{16}\Delta^2 w = f(w), \qquad \text{where} \quad \Delta = b^2 - 4ac.$$

Therefore, having integrated the latter equation, we obtain

$$w'_\xi w'''_{\xi\xi\xi} - \tfrac{1}{2}(w''_{\xi\xi})^2 - \tfrac{5}{4}\Delta(w'_\xi)^2 = -\tfrac{9}{32}\Delta^2 w^2 + \int f(w)\,dw + C.$$

The substitution $z(w) = |w'_\xi|^{3/2}$ leads to a second-order equation:

$$z''_{ww} = \tfrac{15}{8}\Delta z^{-1/3} + \tfrac{3}{2}\left[-\tfrac{9}{32}\Delta^2 w^2 + \int f(w)\,dw + C\right] z^{-5/3}.$$

2°. The first integral of the original equation has the form:

$$(Py'_x - \tfrac{3}{2}P'_x y)y'''_{xxx} - \tfrac{1}{2}P(y''_{xx})^2 + \tfrac{1}{2}P'_x y'_x y''_{xx} + 3ayy''_{xx} - 2a(y'_x)^2 = \int f(w)\,dw + C,$$

where $P = ax^2 + bx + c$, $w = yP^{-3/2}$.

13. $y''''_{xxxx} = e^{\lambda x} f(ye^{-\lambda x})$.

This is a special case of equation 16.2.6.47 with $a = b = c = 0$. The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation.

14. $y''''_{xxxx} = yf(e^{\lambda x} y^m)$.

The transformation $z = e^{\lambda x} y^m$, $w(z) = y'_x/y$ leads to a third-order equation.

15. $y''''_{xxxx} = x^{-4} f(x^m e^{\lambda y})$.

The transformation $z = x^m e^{\lambda y}$, $w(z) = xy'_x$ leads to a third-order equation.

16. $y''''_{xxxx} = f(y + ae^x) - ae^x$.

The substitution $w = y + ae^x$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

17. $y''''_{xxxx} = f(y + a \cosh x) - a \cosh x$.

The substitution $w = y + a \cosh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

18. $y''''_{xxxx} = f(y + a \sinh x) - a \sinh x$.

The substitution $w = y + a \sinh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

19. $y''''_{xxxx} = f(y + a \cos x) - a \cos x$.

The substitution $w = y + a \cos x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

20. $y''''_{xxxx} = f(y + a \sin x) - a \sin x$.

The substitution $w = y + a \sin x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

► Equations of the form $y''''_{xxxx} = f(x, y, y'_x)$.

21. $y''''_{xxxx} = f(y)y'_x + g(x)$.

By integrating, we find $y'''_{xxx} = \int f(y)\,dy + \int g(x)\,dx + C$. For $g(x) \equiv 0$, the order of this equation can reduced by one with the help of the substitution $w(y) = y'_x$.

22. $y''''_{xxxx} = x^{-4} f(xy'_x - y)$.

The transformation $t = \ln|x|$, $w = xy'_x - y$ leads to a third-order autonomous equation: $w'''_{ttt} - 5w''_{tt} + 6w'_t = f(w)$.

23. $y''''_{xxxx} = f(x, xy'_x - y)$.

The substitution $w = xy'_x - y$ leads to a third-order equation: $(w'_x/x)''_{xx} = f(x, w)$.

24. $y''''_{xxxx} = f(x, xy'_x - 2y)$.

The substitution $w = xy'_x - 2y$ leads to a third-order equation: $xw'''_{xxx} - w''_{xx} = x^2 f(x, w)$.

25. $y''''_{xxxx} = f(x, xy'_x - 3y)$.

The substitution $w = xy'_x - 3y$ leads to a third-order equation: $w'''_{xxx} = xf(x, w)$.

26. $y''''_{xxxx} = yx^{-4} f(xy'_x/y)$.

The transformation $z = xy'_x/y$, $w = x^2 y''_{xx}/y$ leads to a second-order equation.

27. $y''''_{xxxx} = a^4 y + f(x, y'_x - ay)$.

The substitution $w = y'_x - ay$ leads to a third-order equation: $w'''_{xxx} + aw''_{xx} + a^2 w'_x + a^3 w = f(x, w)$.

28. $y''''_{xxxx} = f(x, y'_x \sinh x - y \cosh x) + y.$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a third-order equation.

29. $y''''_{xxxx} = f(x, y'_x \cosh x - y \sinh x) + y.$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a third-order equation.

30. $y''''_{xxxx} = f(x, y'_x \sin x - y \cos x) + y.$

The substitution $w = y'_x \sin x - y \cos x$ leads to a third-order equation.

31. $y''''_{xxxx} = f(x, y'_x \cos x + y \sin x) + y.$

The substitution $w = y'_x \cos x + y \sin x$ leads to a third-order equation.

32. $y''''_{xxxx} = \dfrac{\varphi''''_{xxxx}}{\varphi} y + f\Big(x, y'_x - \dfrac{\varphi'_x}{\varphi} y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi} y$ leads to a third-order equation.

► **Equations of the form** $y''''_{xxxx} = f(x, y, y'_x, y''_{xx})$**.**

33. $y''''_{xxxx} + a y''_{xx} = f(y).$

Having integrated this equation, we obtain $2y'_x y'''_{xxx} - (y''_{xx})^2 + a(y'_x)^2 = 2\int f(y)\,dy + 2C$, where C is an arbitrary constant. The substitution $w(y) = |y'_x|^{3/2}$ leads to a second-order equation:

$$w''_{yy} = -\tfrac{3}{4} a w^{-1/3} + \tfrac{3}{2}\Big[\int f(y)\,dy + C\Big] w^{-5/3}.$$

34. $y''''_{xxxx} + f(y'_x) y''_{xx} = g(y).$

Having integrated this equation, we obtain a third-order autonomous equation:

$$2y'_x y'''_{xxx} - (y''_{xx})^2 + 2F(y'_x) = 2\int g(y)\,dy + 2C, \qquad \text{where} \quad F(u) = \int u f(u)\,du.$$

The substitution $w(y) = y'_x$ leads to a second-order equation.

35. $y''''_{xxxx} = x^{-2} f(xy'_x - y) y''_{xx}.$

The transformation $t = \ln|x|$, $w = xy'_x - y$ leads to a third-order equation:

$$w'''_{ttt} - 5w''_{tt} + 6w'_t = f(w) w'_t.$$

Integrating it, we obtain a second-order autonomous equation:

$$w''_{tt} - 5w'_t + 6w = \int f(w)\,dw + C.$$

The substitution $z(w) = \frac{1}{5} w'_t$ leads to an Abel equation of the second kind:

$$z z'_w - z = \tfrac{1}{25}\Big[-6w + \int f(w)\,dw + C\Big]$$

(see Section 13.3.1).

36. $yy''''_{xxxx} - (y''_{xx})^2 = f(x).$

Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = \int_0^x (x-t)f(t)\,dt + C_1 x + C_2.$$

37. $yy''''_{xxxx} - (y''_{xx})^2 = f(x)[yy''_{xx} - (y'_x)^2] + g(x).$

This is a special case of equation 16.2.6.93. The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order linear equation: $w''_{xx} = f(x)w + g(x)$.

38. $y''''_{xxxx} = a^2 y + f(y''_{xx} + ay).$

The substitution $w = y''_{xx} + ay$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{xx} = aw + f(w)$.

39. $y''''_{xxxx} = f(y, y''_{xx}).$

The substitution $w(y) = \pm(y'_x)^2$ leads to a third-order equation: $ww'''_{yyy} + \frac{1}{2}w'_y w''_{yy} = 2f\bigl(y, \pm\frac{1}{2}w'_y\bigr)$.

40. $y''''_{xxxx} = a^2 y + f(x, y''_{xx} + ay).$

The substitution $w = y''_{xx} + ay$ leads to a second-order equation: $w''_{xx} = aw + f(x, w)$.

41. $y''''_{xxxx} = yf\bigl(yy''_{xx} - y'^2_x\bigr).$

This is a special case of equation 16.2.6.42.

42. $y''''_{xxxx} = y''_{xx} f\bigl(yy''_{xx} - y'^2_x\bigr) + yg\bigl(yy''_{xx} - y'^2_x\bigr).$

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$C_3^4 - C_3^2 f(4C_1C_2C_3^2) - g(4C_1C_2C_3^2) = 0.$$

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$C_3^4 + C_3^2 f(-C_1^2C_3^2 - C_2^2C_3^2) - g(-C_1^2C_3^2 - C_2^2C_3^2) = 0.$$

43. $y''''_{xxxx} = x^m f(x^2 y''_{xx} - 2xy'_x + 2y).$

The substitution $w = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation: $xw''_{xx} - 2w'_x = x^{m+3} f(w)$. For $m = -4$, the substitution $z(w) = \frac{1}{3}xw'_x$ leads to an Abel equation of the second kind: $zz'_w - z = \frac{1}{9}f(w)$ (see Section 13.3.1).

44. $y''''_{xxxx} = f(x, xy'_x - y, y''_{xx}).$

The substitution $w(x) = xy'_x - y$ leads to a third-order equation.

45. $y''''_{xxxx} = f(x, x^2 y''_{xx} - 2xy'_x + 2y)$.

The substitution $w = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation: $xw''_{xx} - 2w'_x = x^3 f(x, w)$.

46. $y''''_{xxxx} = y'_x f\Big(\dfrac{y''_{xx}}{y'_x},\ y'_x - y\dfrac{y''_{xx}}{y'_x}\Big)$.

Particular solution: $y = C_1 \exp(C_2 x) + C_3$, where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $C_2^3 = f(C_2, -C_2 C_3)$.

► **Equations of the form** $y''''_{xxxx} = f(x, y, y'_x, y''_{xx}, y'''_{xxx})$.

47. $y''''_{xxxx} + ay'''_{xxx} + by''_{xx} + cy'_x = e^{\lambda x} f(ye^{-\lambda x})$.

The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation:

$$\begin{aligned} &w''''_{xxxx} + (4\lambda + a)w'''_{xxx} + (6\lambda^2 + 3a\lambda + b)w''_{xx} \\ &\qquad + (4\lambda^3 + 3a\lambda^2 + 2b\lambda + c)w'_x + (\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda)w = f(w), \end{aligned}$$

which can be reduced to a third-order equation by means of the substitution $z(w) = w'_x$. For $a = -4\lambda$ and $c = 8\lambda^3 - 2b\lambda$, the above equation coincides, up to notation, with equation 16.2.6.33 and can be reduced to a second-order equation.

48. $y''''_{xxxx} + ayy'''_{xxx} = f(x)$.

Integrating, we arrive at a third-order equation: $y'''_{xxx} + ayy''_{xx} - \frac{1}{2}a(y'_x)^2 = \int f(x)\,dx + C$.

49. $y''''_{xxxx} + ayy'''_{xxx} - ay'_x y''_{xx} = f(x)$.

Integrating, we arrive at a third-order equation: $y'''_{xxx} + ayy''_{xx} - a(y'_x)^2 = \int f(x)\,dx + C$.

50. $y''''_{xxxx} + ayy'''_{xxx} + f(y'_x)y''_{xx} = g(x)$.

Integrating, we arrive at a third-order equation:

$$y'''_{xxx} + ayy''_{xx} - \tfrac{1}{2}a(y'_x)^2 + F(y'_x) = \int g(x)\,dx + C, \qquad \text{where} \quad F(u) = \int f(u)\,du.$$

51. $y''''_{xxxx} = f(y)y'_x y'''_{xxx}$.

Integrating, we arrive at a third-order autonomous equation of the form 15.5.1.1: $y'''_{xxx} = C\exp\Big[\int f(y)\,dy\Big]$.

52. $xy''''_{xxxx} + 4y'''_{xxx} = f(xy)$.

The substitution $w(x) = xy$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

53. $xy''''_{xxxx} + (a + 3)y'''_{xxx} = f(x, xy'_x + ay)$.

The substitution $w = xy'_x + ay$ leads to a third-order equation: $w'''_{xxx} = f(x, w)$.

54. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = f(x^2 y)$.

The substitution $w(x) = x^2 y$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

55. $x^4 y''''_{xxxx} + a_3 x^3 y'''_{xxx} + a_2 x^2 y''_{xx} + a_1 x y'_x = f(y).$

The substitution $t = \ln|x|$ leads to an autonomous equation:

$$y''''_{tttt} + (a_3 - 6)y'''_{ttt} + (11 - 3a_3 + a_2)y''_{tt} + (2a_3 - a_2 + a_1 - 6)y'_t = f(y), \qquad (1)$$

the order of which can be lowered with the help of the substitution $w(y) = y'_t$. For $a_3 = 6$ and $a_1 = a_2 - 6$, equation (1) coincides, up to notation, with equation 16.2.6.33 and can be reduced to a second-order equation.

56. $x^4 y''''_{xxxx} + a x^3 y'''_{xxx} + b x^2 y''_{xx} + c x y'_x + s y = x^{-k} f(y x^k).$

The transformation $t = \ln x$, $w = y x^k$ leads to an autonomous equation of the form 16.2.6.79.

57. $y y''''_{xxxx} - y'_x y'''_{xxx} = f(x) y^2.$

Integrating yields a third-order linear equation: $y'''_{xxx} = \Big[\int f(x)\,dx + C\Big] y.$

58. $y y''''_{xxxx} + 4 y'_x y'''_{xxx} + 3 (y''_{xx})^2 = f(x).$

Solution: $y^2 = C_3 x^3 + C_2 x^2 + C_1 x + C_0 + \frac{1}{3}\int_{x_0}^{x} (x-t)^3 f(t)\,dt.$

59. $y y''''_{xxxx} + a y'_x y'''_{xxx} + (a-1)(y''_{xx})^2 = f(x).$

Integrating the equation two times, we obtain a second-order equation:

$$y y''_{xx} + \frac{a-2}{2}(y'_x)^2 = C_1 x + C_0 + \int_{x_0}^{x} (x-t) f(t)\,dt.$$

60. $y y''''_{xxxx} + 4 y'_x y'''_{xxx} + 3 (y''_{xx})^2 = f(y).$

The substitution $w = y^2$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = 2f\big(\pm\sqrt{w}\,\big)$.

61. $y y''''_{xxxx} - y'_x y'''_{xxx} = f(x) y y'''_{xxx}.$

Integrating yields a third-order linear equation: $y'''_{xxx} = C \exp\Big[\int f(x)\,dx\Big] y.$

62. $y y''''_{xxxx} + y'_x y'''_{xxx} = f(x) y y'''_{xxx}.$

Integrating yields a third-order equation of the form 15.5.1.2: $y y'''_{xxx} = C \exp\Big[\int f(x)\,dx\Big].$

63. $y y''''_{xxxx} + (f-1) y'_x y'''_{xxx} + f g y y'_x + g'_x y^2 = 0, \qquad f = f(x), \quad g = g(x).$

The functions that solve the third-order linear equation $y'''_{xxx} + g(x) y = 0$ are solutions of the given equation.

64. $y y''''_{xxxx} + (4 y'_x + f y) y'''_{xxx} + 3 (y''_{xx})^2 + 3 f y'_x y''_{xx} + g(x) = 0, \qquad f = f(x).$

The substitution $w = (y y'_x)''_{xx}$ leads to a first-order linear equation: $w'_x + f w + g = 0$.

Solution:

$$y^2 = C_2 x^2 + C_1 x + C_0 + \int_{x_0}^{x} (x-t)^2 w(t)\,dt,$$

where $w(x) = e^{-F(x)}\Big[C_3 - \int e^{F(x)} g(x)\,dx\Big]$, $F(x) = \int f(x)\,dx$; x_0 is an arbitrary number.

65. $yy''''_{xxxx} + (4y'_x + fy)y'''_{xxx} + 3(y''_{xx})^2 + (3fy'_x + gy)y''_{xx} + g(y'_x)^2 + hyy'_x + s = 0.$

Here, $f = f(x)$, $g = g(x)$, $h = h(x)$, $s = s(x)$. The substitution $w = yy'_x$ leads to a third-order nonhomogeneous linear equation: $w'''_{xxx} + fw''_{xx} + gw'_x + hw + s = 0$.

66. $(y + ax + b)y''''_{xxxx} + 4(y'_x + a)y'''_{xxx} + 3(y''_{xx})^2 = f(x).$

Solution: $(y + ax + b)^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + \frac{1}{3}\int_{x_0}^{x}(x - t)^3 f(t)\,dt.$

67. $yy''''_{xxxx} = 4y'_x y'''_{xxx} + 3(y''_{xx})^2 - 6\dfrac{(y'_x)^4}{y^2} + \left[yy''_{xx} - (y'_x)^2\right] f\left(\dfrac{y'_x}{y}\right).$

The transformation $\xi = \dfrac{y'_x}{y}$, $w = \dfrac{y''_{xx}}{y} - \left(\dfrac{y'_x}{y}\right)^2$ leads to a second-order linear equation with respect to w^2: $(w^2)''_{\xi\xi} = 24\xi^2 + 2f(\xi)$. Integrating it, we obtain

$$w^2 = C_2\xi + C_1 + 2\xi^4 + 2\int_{\xi_0}^{\xi}(\xi - t)f(t)\,dt.$$

Taking into account that $\xi'_x = w$, $y'_x = \xi y$, $y'_\xi = \xi y/w$, we find the solution in parametric form:

$$x = \int\frac{d\xi}{w} + C_3, \qquad y = C_4\exp\left(\int\frac{\xi\,d\xi}{w}\right),$$

where $w = \pm\left[C_2\xi + C_1 + 2\xi^4 + 2\int_{\xi_0}^{\xi}(\xi - t)f(t)\,dt\right]^{1/2}$.

68. $y^2y''''_{xxxx} - 2yy'_x y'''_{xxx} + f(x)y^2y'''_{xxx} + 2(y'_x)^2y''_{xx} - f(x)yy'_x y''_{xx} + 2f'_x(x)y^2y''_{xx} + 2f(x)(y'_x)^3 + [f^2(x) - 2f'_x(x)]y(y'_x)^2 + f''_{xx}(x)y^2y'_x = 0.$

The solution satisfies the second-order linear equation $y''_{xx} + f(x)y'_x - z(x, C_1, C_2)y = 0$, where $z = z(x, C_1, C_2)$ is the Weierstrass elliptic function determined by the second-order autonomous equation $z''_{xx} + z^2 = 0$.

69. $y^2y''''_{xxxx} - 2yy'_x y'''_{xxx} + f(x)y^2y'''_{xxx} + 2(y'_x)^2y''_{xx} - f(x)yy'_x y''_{xx} + 2f'_x(x)y^2y''_{xx} + 2f(x)(y'_x)^3 + [f^2(x) - 2f'_x(x)]y(y'_x)^2 + f''_{xx}(x)y^2y'_x = Axy^3.$

The solution satisfies the second-order linear equation $y''_{xx} + f(x)y'_x - z(x, C_1, C_2)y = 0$, where $z = z(x, C_1, C_2)$ is the solution of the first Painlevé transcendent $z''_{xx} + z^2 = Ax$.

70. $y^2y''''_{xxxx} - 2yy'_x y'''_{xxx} + [a + f(x)]y^2y'''_{xxx} - y(y''_{xx})^2 - [3a + f(x)]yy'_x y''_{xx} + 2(y'_x)^2y''_{xx} + [af(x) + g(x)]y^2y''_{xx} + 2a(y'_x)^3 - af(x)y(y'_x)^2 + ag(x)y^2y'_x = h(x)y^3.$

The solution satisfies the second-order linear equation $y''_{xx} + ay'_x - z(x, C_1, C_2)y = 0$, where $z = z(x, C_1, C_2)$ is the solution of the second-order linear equation $z''_{xx} + f(x)z'_x + g(x)z = h(x)$.

71. $y''_{xx}y''''_{xxxx} - 3(y'''_{xxx})^2 = f(xy'_x - y)(y''_{xx})^5.$

The Legendre transformation $x = u'_t$, $y = tu'_t - u$ leads to an equation of the form 16.2.6.1: $u''''_{tttt} = -f(u)$.

72. $y''''_{xxxx} = f(y)y'_x g(y'''_{xxx})$.

Integrating yields a third-order autonomous equation:

$$\int \frac{dw}{g(w)} = \int f(y)\,dy + C, \qquad \text{where} \quad w = y'''_{xxx},$$

the order of which can be lowered by means of the substitution $z(y) = y'_x$.

73. $xy''''_{xxxx} + 2y'''_{xxx} = (xy''_{xx})^{-5} f\Big(\frac{xy''_{xx}}{\sqrt{xy'_x - y}}\Big)$.

The substitution $w(x) = xy'_x - y$ leads to a third-order equation of the form 15.5.2.27:

$$w'''_{xxx} = w^{-5/2} F\Big(\frac{w'_x}{\sqrt{w}}\Big), \qquad \text{where} \quad F(\xi) = \xi^{-5} f(\xi).$$

74. $x^2 y''''_{xxxx} + 2xy'''_{xxx} = f(x^2 y''_{xx} - 2xy'_x + 2y) g(x^2 y'''_{xxx})$.

The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation of the form 14.9.4.36: $w''_{xx} = f(w) g(w'_x)$.

75. $y''''_{xxxx} = f(x) g(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y)$.

The substitution $w(x) = x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y$ leads to a first-order separable equation: $w'_x = x^3 f(x) g(w)$.

► Other equations.

76. $yy''''_{xxxx} - \frac{1}{6}(y''_{xx})^2 = x^2 f_1(y''''_{xxxx}) + x f_2(y''''_{xxxx}) + f_3(y''''_{xxxx})$.

Particular solution:

$$y = \tfrac{1}{24} C_1 x^4 + \tfrac{1}{6} C_2 x^3 + \tfrac{1}{2} C_3 x^2 + C_4 x + C_5,$$

where the constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints

$$\begin{aligned} \tfrac{1}{3} C_1 C_3 - \tfrac{1}{6} C_2^2 &= f_1(C_1), \\ C_1 C_4 - \tfrac{1}{3} C_2 C_3 &= f_2(C_1), \\ C_1 C_5 - \tfrac{1}{6} C_3^2 &= f_3(C_1). \end{aligned}$$

77. $yy''''_{xxxx} - \frac{1}{6}(y''_{xx})^2 = y'''_{xxx} f_1(y''''_{xxxx}) + y''_{xx} f_2(y''''_{xxxx})$
$\qquad + x^2 f_3(y''''_{xxxx}) + x f_4(y''''_{xxxx}) + f_5(y''''_{xxxx})$.

Particular solution:

$$y = \tfrac{1}{24} C_1 x^4 + \tfrac{1}{6} C_2 x^3 + \tfrac{1}{2} C_3 x^2 + C_4 x + C_5,$$

where the constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints

$$\begin{aligned} \tfrac{1}{3} C_1 C_3 - \tfrac{1}{6} C_2^2 &= \tfrac{1}{2} C_1 f_2(C_1) + f_3(C_1), \\ C_1 C_4 - \tfrac{1}{3} C_2 C_3 &= C_1 f_1(C_1) + C_2 f_2(C_1) + f_4(C_1), \\ C_1 C_5 - \tfrac{1}{6} C_3^2 &= C_2 f_1(C_1) + C_3 f_2(C_1) + f_5(C_1). \end{aligned}$$

78. $y''''_{xxxx} = F(x, y'_x, y''_{xx}, y'''_{xxx})$.

The substitution $w(x) = y'_x$ leads to a third-order equation: $w'''_{xxx} = F(x, w, w'_x, w''_{xx})$.

79. $y''''_{xxxx} = F(y,\, y'_x,\, y''_{xx},\, y'''_{xxx})$.

Autonomous equation. The substitution $w(y) = (y'_x)^2$ leads to a third-order equation:

$$ww'''_{yyy} + \tfrac{1}{2}w'_y w''_{yy} = 2F(y,\, \pm\sqrt{w},\, \tfrac{1}{2}w'_y,\, \pm\tfrac{1}{2}\sqrt{w}\, w''_{yy}).$$

80. $y''''_{xxxx} = x^{-3}F(y/x,\, y'_x,\, xy''_{xx},\, x^2 y'''_{xxx})$.

Homogeneous equation. The transformation $t = \ln x,\ w = y/x$ leads to an autonomous equation of the form 16.2.6.79.

81. $y''''_{xxxx} = x^{-k-4}F(x^k y,\, x^{k+1}y'_x,\, x^{k+2}y''_{xx},\, x^{k+3}y'''_{xxx})$.

Generalized homogeneous equation. The transformation $t = \ln x,\ w = x^k y$ leads to an autonomous equation of the form 16.2.6.79.

82. $y''''_{xxxx} = F(x,\, xy'_x - y,\, y''_{xx},\, y'''_{xxx})$.

This is a special case of equation 17.2.6.78 with $n = 4$. The substitution $w = xy'_x - y$ leads to a third-order equation.

83. $y''''_{xxxx} = F(x,\, xy'_x - 2y,\, y'''_{xxx})$.

The substitution $w = xy'_x - 2y$ leads to a third-order equation: $\zeta'_x = F(x,\, w,\, \zeta)$, where $\zeta = w''_{xx}/x$.

84. $y''''_{xxxx} = F(x,\, x^2 y''_{xx} - 2xy'_x + 2y,\, y'''_{xxx})$.

The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation: $(x^{-2}w'_x)'_x = F(x, w, x^{-2}w'_x)$.

85. $y''''_{xxxx} = yF\left(\dfrac{y'_x}{y},\, \dfrac{y''_{xx}}{y},\, \dfrac{y'''_{xxx}}{y}\right)$.

The transformation $\xi = \dfrac{y'_x}{y},\ w = \dfrac{y''_{xx}}{y} - \left(\dfrac{y'_x}{y}\right)^2$ leads to a second-order equation:

$$w^2 w''_{\xi\xi} + w(w'_\xi)^2 + 4\xi w w'_\xi + 3w^2 + 6\xi^2 w + \xi^4 = F(\xi,\, w + \xi^2,\, ww'_\xi + 3\xi w + \xi^3).$$

86. $y''''_{xxxx} = yx^{-4}F\left(x^k y^m,\, \dfrac{xy'_x}{y},\, \dfrac{x^2 y''_{xx}}{y},\, \dfrac{x^3 y'''_{xxx}}{y}\right)$.

Generalized homogeneous equation. The transformation $t = x^k y^m,\ z = \dfrac{xy'_x}{y}$ leads to a third-order equation.

87. $y''''_{xxxx} = yx^{-4}F\left(\dfrac{xy'_x}{y},\, \dfrac{x^2 y''_{xx}}{y},\, \dfrac{x^3 y'''_{xxx}}{y}\right)$.

The transformation $z = \dfrac{xy'_x}{y},\ w = \dfrac{x^2 y''_{xx}}{y}$ leads to a second-order equation.

88. $y''''_{xxxx} = y'_x F\left(\dfrac{y''_{xx}}{y'_x},\, y'_x - y\dfrac{y''_{xx}}{y'_x},\, \dfrac{y'''_{xxx}}{y'_x}\right)$.

Autonomous equation. This is a special case of equation 16.2.6.79.

Particular solution:

$$y = C_1 \exp(C_2 x) + C_3,$$

where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $C_2^3 = F(C_2, -C_2 C_3, C_2^2)$.

89. $y''''_{xxxx} = e^{-\alpha x} F\big(e^{\alpha x} y,\ e^{\alpha x} y'_x,\ e^{\alpha x} y''_{xx},\ e^{\alpha x} y'''_{xxx}\big)$.

This equation is *invariant under "translation–dilatation" transformation.* The substitution $u = e^{\alpha x} y$ leads to an autonomous equation of the form 16.2.6.79.

90. $y''''_{xxxx} = x^{-4} F(x^m e^{\alpha y},\ x y'_x,\ x^2 y''_{xx},\ x^3 y'''_{xxx})$.

The transformation $z = x^m e^{\alpha y}$, $w = x y'_x$ leads to a third-order equation.

91. $y''''_{xxxx} = y F\Big(e^{\alpha x} y^m,\ \dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y},\ \dfrac{y'''_{xxx}}{y}\Big)$.

The transformation $z = e^{\alpha x} y^m$, $w = y'_x/y$ leads to a third-order equation.

92. $F(y''_{xx}/y,\ y y''_{xx} - y'^2_x,\ y'''_{xxx}/y'_x,\ y''''_{xxxx}/y) = 0$.

Autonomous equation. This is a special case of equation 16.2.6.79.

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where C_1, C_2, and C_2 are related by the constraint $F\big(C_3^2,\ 4C_1C_2C_3^2,\ C_3^2,\ C_3^4\big) = 0$.

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

with C_1, C_2, and C_2 related by the constraint $F\big(-C_3^2, -(C_1^2 + C_2^2)C_3^2, -C_3^2, C_3^4\big) = 0$.

93. $F\big(x,\ y y''_{xx} - (y'_x)^2,\ y y'''_{xxx} - y'_x y''_{xx},\ y y''''_{xxxx} - (y''_{xx})^2\big) = 0$.

The substitution $w(x) = y y''_{xx} - (y'_x)^2$ leads to a second-order equation of the form $F(x, w, w'_x, w''_{xx}) = 0$.

94. $F\Big(\dfrac{y''''_{xxxx}}{y'_x},\ y\dfrac{y''''_{xxxx}}{y'_x} - y'''_{xxx}\Big) = 0$.

A solution of this equation is any function that solves the third-order linear equation:

$$y'''_{xxx} = C_1 y + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(C_1, -C_2) = 0$.

95. $F(x,\ y''_{xx} + ay,\ y''''_{xxxx} - a^2 y,\ y''''_{xxxx} + a y''_{xx}) = 0$.

The substitution $u = y''_{xx} + ay$ leads to a second-order equation: $F(x, u, u''_{xx} - au, u''_{xx}) = 0$.

Chapter 17

Higher-Order Ordinary Differential Equations

17.1 Linear Equations

17.1.1 Preliminary Remarks

In this chapter, we denote higher derivatives by $y_x^{(n)}$ to mean $d^n y/dx^n$.

1°. The general solution of a homogeneous linear equation of the nth-order

$$f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = 0 \tag{1}$$

has the form:

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x). \tag{2}$$

Here, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ make up a fundamental set of solutions (the y_k are linearly independent solutions; $y_k \not\equiv 0$); C_1, C_2, ..., C_n are arbitrary constants.

2°. Let $y_0 = y_0(x)$ be a nontrivial particular solution of equation (1). Then the substitution

$$y = y_0(x) \int z(x)\,dx$$

leads to a linear $(n-1)$st-order equation for $z(x)$.

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent particular solutions of equation (1) with $g \equiv 0$. Then the substitution

$$y = y_1 \int y_2 w\,dx - y_2 \int y_1 w\,dx$$

leads to a linear $(n-2)$nd-order equation for $w = w(x)$.

3°. Further information about higher-order linear equations can be found in Chapter 4.

17.1.2 Equations Containing Power Functions

► **Equations of the form** $f_n(x)y_x^{(n)} + f_0(x)y = g(x)$.

1. $y_x^{(6)} + ay = 0$.

1°. Solution for $a = 0$:

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 x^5.$$

2°. Solution for $a = k^6 > 0$:

$$y = C_1 \cos kx + C_2 \sin kx + \cos\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_3 \cosh\xi + C_4 \sinh\xi\bigr) + \sin\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_5 \cosh\xi + C_6 \sinh\xi\bigr), \quad \text{where} \quad \xi = \tfrac{\sqrt{3}}{2}kx.$$

3°. Solution for $a = -k^6 < 0$:

$$y = C_1 \cosh kx + C_2 \sinh kx + \cosh\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_3 \cos\xi + C_4 \sin\xi\bigr) + \sinh\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_5 \cos\xi + C_6 \sin\xi\bigr), \quad \text{where} \quad \xi = \tfrac{\sqrt{3}}{2}kx.$$

2. $\boldsymbol{y_x^{(2n)} = a^{2n} y.}$

Solution:

$$y = C_1 e^{ax} + C_2 e^{-ax} + \sum_{k=1}^{n-1} e^{\varphi_k}(A_k \cos\theta_k + B_k \sin\theta_k),$$

where $\varphi_k = ax\cos\dfrac{k\pi}{n}$, $\theta_k = ax\sin\dfrac{k\pi}{n}$; C_1, C_2, A_k, B_k $(k = 1,\,2,\,\dots,\,n-1)$ are arbitrary constants.

3. $\boldsymbol{y_x^{(n)} = axy + b, \qquad a > 0.}$

Solution:

$$y = \sum_{\nu=0}^{n} C_\nu \varepsilon_\nu \int_0^\infty \exp\left[\varepsilon_\nu xt - \frac{t^{n+1}}{a(n+1)}\right] dt, \qquad \varepsilon_\nu = \exp\left(\frac{2\pi\nu i}{n+1}\right),$$

where $\displaystyle\sum_{\nu=0}^{n} C_\nu = \frac{b}{a}$ and $i^2 = -1$.

4. $\boldsymbol{y_x^{(n)} = ax^\beta y.}$

For specific β, see equations 17.1.2.2, 17.1.2.3 (with $b = 0$), 17.1.2.5 to 17.1.2.9, and 17.1.2.10 (with $b = 0$).

1°. Let $n \ge 2$, $\beta > -n$, and $(n+\beta)(s+1) \ne 1,\,2,\,\dots,\,n-1$, where $s = 0,\,1,\,\dots$ Then the equation has n solutions that can be represented as:

$$y_j(x) = x^{j-1} E_{n,1+\beta/n,(\beta+j-1)/n}\bigl(ax^{\beta+n}\bigr), \qquad j = 1,\,2,\,\dots,\,n. \tag{1}$$

Here, $E_{n,m,l}(z)$ is a Mittag-Leffler type special function defined by:

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$
$$b_k = \prod_{s=0}^{k-1} \frac{\Gamma\bigl(n(ms+l)+1\bigr)}{\Gamma\bigl(n(ms+l+1)+1\bigr)} = \prod_{s=0}^{k-1} \frac{1}{[n(ms+l)+1]\dots[n(ms+l)+n]}, \tag{2}$$

where $\Gamma(\xi)$ is the gamma function, l is an arbitrary number, and $m > 0$.

If $\beta \ge 0$, solutions (1) are linearly independent. Series expansions of (1) are convenient for small x.

2°. Let $n \geq 2$, $\beta < -n$, and $(n+\beta)(s+1) \neq -1,\ -2,\ \dots,\ -(n-1)$, where $s = 0,\ 1,\ \dots$ Then the equation in question has n solutions that can be represented as:

$$y_j(x) = x^{j-1} E_{n,-1-\beta/n,-1-(\beta+j)/n}\bigl(a(-1)^n x^{\beta+n}\bigr), \qquad j = 1,\ 2,\ \dots,\ n, \tag{3}$$

where $E_{n,m,l}(z)$ is the Mittag-Leffler type special function defined by (2). If $\beta \leq -2n$, solutions (3) are linearly independent. Series expansions of (3) are convenient for large x.

3°. The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of similar form:

$$w_t^{(n)} = a(-1)^{n+1} t^{-2n-\beta} w.$$

⊙ *Literature*: M. Saigo and A. A. Kilbas (2000).

5. $\boldsymbol{x^{2n} y_x^{(n)} = ay.}$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to a constant coefficient linear equation: $w_t^{(n)} = (-1)^n aw$.

6. $\boldsymbol{x^{n} y_x^{(2n)} = ay.}$

Solution:

$$y = x^{n/2} \sum_{k=1}^{n} \bigl[C_{k1} I_n\bigl(2\beta_k\sqrt{x}\,\bigr) + C_{k2} K_n\bigl(2\beta_k\sqrt{x}\,\bigr)\bigr],$$

where $I_n(z)$ and $K_n(z)$ are modified Bessel functions; $\beta_1,\ \beta_2,\ \dots,\ \beta_n$ are roots of the equation $\beta^n = \sqrt{a}$.

7. $\boldsymbol{x^{3n} y_x^{(2n)} = ay.}$

The transformation $x = t^{-1}$, $y = wt^{1-2n}$ leads to an equation of the form 17.1.2.6: $t^n w_t^{(2n)} = aw$.

8. $\boldsymbol{x^{n+1/2} y_x^{(2n+1)} = ay.}$

Solution:

$$y = x^{(2n+1)/4} \sum_{k=0}^{2n} C_k \bigl[J_{-n-1/2}\bigl(2\beta_k\sqrt{x}\,\bigr) + iJ_{n+1/2}\bigl(2\beta_k\sqrt{x}\,\bigr)\bigr],$$

where $J_m(z)$ are Bessel functions; $\beta_0, \beta_1, \dots, \beta_{2n}$ are roots of the equation $\beta^{2n+1} = -ai$; $i^2 = -1$.

9. $\boldsymbol{x^{3n+3/2} y_x^{(2n+1)} = ay.}$

The transformation $x = t^{-1}$, $y = wt^{-2n}$ leads to a linear equation of the form 17.1.2.8: $t^{n+1/2} w_t^{(2n+1)} = -aw$.

10. $\boldsymbol{x^{2n+1} y_x^{(n)} = ay + bx^n.}$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to a linear equation of the form 17.1.2.3: $w_t^{(n)} = (-1)^n (atw + b)$.

11. $(ax+b)^{2n+1}y_x^{(n)}=(cx+d)y.$

The transformation $\xi=\dfrac{cx+d}{ax+b},\ w=\dfrac{y}{(ax+b)^{n-1}}$ leads to an equation of the form 17.1.2.3: $w_\xi^{(n)}=\Delta^{-n}\xi w$, where $\Delta=bc-ad$.

12. $(ax+b)^n(cx+d)^ny_x^{(n)}=ky.$

1°. The transformation $\xi=\ln\dfrac{ax+b}{cx+d},\ w=\dfrac{y}{(cx+d)^{n-1}}$ leads to a constant coefficient linear equation.

2°. The transformation $\zeta=\dfrac{ax+b}{cx+d},\ w=\dfrac{y}{(cx+d)^{n-1}}$ leads to the Euler equation 17.1.2.39: $\zeta^n w_\zeta^{(n)}=k\Delta^{-n}w$, where $\Delta=ad-bc$.

13. $(ax^2+bx+c)^ny_x^{(n)}=ky.$

The transformation $\xi=\displaystyle\int\frac{dx}{ax^2+bx+c},\ w=y(ax^2+bx+c)^{\frac{1-n}{2}}$ leads to a constant coefficient linear equation.

14. $(ax+b)^n(cx+d)^{3n}y_x^{(2n)}=ky.$

The transformation $\xi=\dfrac{ax+b}{cx+d},\ w=\dfrac{y}{(cx+d)^{2n-1}}$ leads to an equation of the form 17.1.2.6: $\xi^n w_\xi^{(2n)}=k\Delta^{-2n}w$, where $\Delta=ad-bc$.

15. $(ax+b)^{n+1/2}(cx+d)^{3n+3/2}y_x^{(2n+1)}=ky.$

The transformation $\xi=\dfrac{ax+b}{cx+d},\ w=\dfrac{y}{(cx+d)^{2n}}$ leads to an equation of the form 17.1.2.8: $\xi^{n+1/2}w_\xi^{(2n+1)}=k\Delta^{-2n-1}w$, where $\Delta=ad-bc$.

► Equations of the form $f_n(x)y_x^{(n)}+f_1(x)y_x'+f_0(x)y=g(x)$.

16. $y_x^{(n)}+ax^ky_x'+akx^{k-1}y=0.$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)}+ax^ky=C$.

17. $y_x^{(n)}+ax^{k+1}y_x'-a(n-1)x^ky=0.$

The substitution $z=xy_x'-(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)}+ax^{k+1}z=0$.

18. $y_x^{(n)}+ax^{k+1}y_x'+a(k+n)x^ky=0.$

The transformation $x=t^{-1},\ y=wt^{1-n}$ leads to an equation of the form 17.1.2.16: $w_t^{(n)}+bt^\nu w_t'+b\nu t^{\nu-1}w=0$, where $b=a(-1)^{n+1}$, $\nu=1-k-2n$.

19. $y_x^{(n)}+(ax+b)x^ky_x'-ax^ky=0.$

Particular solution: $y_0=ax+b$.

20. $y_x^{(n)} + (ax+b)x^k y'_x - 2ax^k y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

21. $y_x^{(n)} + (ax+b)x^k y'_x - 3ax^k y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

22. $y_x^{(n)} + (ax+b)x^k y'_x - a(n-1)x^k y = 0.$

Particular solution: $y_0 = (ax+b)^{n-1}$. The substitution $z = (ax+b)y'_x - a(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + (ax+b)x^k z = 0$.

23. $y_x^{(n)} + ax^{k+1} y'_x - amx^k y = 0, \qquad m = 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $y_0 = x^m$. The substitution $z = xy'_x - my$ leads to an $(n-1)$st-order linear equation:

$$D^{n-m-1}\left(\frac{z_x^{(m)}}{x}\right) + ax^k z = 0, \qquad \text{where} \quad D = \frac{d}{dx}.$$

► **Other equations.**

24. $y_x^{(2n)} = a^n y + bx^k (y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = bx^k$. The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order linear equation: $w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = bx^k w$.

25. $y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y'_x + a_0 y = 0.$

Constant coefficient homogeneous linear equation. To solve this equation, determine the n roots of the characteristic equation:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

If the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all real and different, then the general solution of the original equation is:

$$y = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x) + \cdots + C_n \exp(\lambda_n x).$$

The general case, which involves the cases of multiple and/or complex roots of the characteristic equation, is discussed in Section 4.1.1.

26. $y_x^{(n)} + ax^k y_x^{(m)} - (ab^m x^k + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

27. $y_x^{(n)} + (ax^k - b^{n-m}) y_x^{(m)} - ab^m x^k y = 0.$

Particular solution: $y_0 = e^{bx}$.

28. $y_x^{(n)} + ay_x^{(n-1)} + bx^m y'_x + abx^m y = 0.$

Particular solution: $y_0 = e^{-ax}$.

29. $xy_x^{(n)} - nmy_x^{(n-1)} + axy = 0, \qquad n = 2,\ 3,\ 4,\ \ldots, \qquad m = 1,\ 2,\ 3,\ \ldots$

Solution:

$$y = x^{(m+1)n-1}\left(x^{1-n}\frac{d}{dx}\right)^m (x^{1-n}w),$$

where w is the general solution of the constant coefficient linear equation: $w_x^{(n)} + aw = 0$.

30. $xy_x^{(n)} + ny_x^{(n-1)} = axy + b.$

The substitution $w = xy$ leads to a constant coefficient linear equation: $w_x^{(n)} = aw + b$.

31. $xy_x^{(n)} + ny_x^{(n-1)} = ax^2y + b.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.3: $w_x^{(n)} = axw + b$.

32. $xy_x^{(n)} + (n - m - 1)y_x^{(n-1)} + ax^k y'_x - amx^{k-1}y = 0.$

Particular solution: $y_0 = x^m$.

33. $xy_x^{(n)} + ax^k y_x^{(m)} - (ax^k + amx^{k-1} + x + n)y = 0.$

Particular solution: $y_0 = xe^x$.

34. $xy_x^{(n)} = \sum\limits_{\nu=0}^{n-1} [(aA_{\nu+1} - A_\nu)x + A_{\nu+1}]y_x^{(\nu)}.$

Here, $A_n = 1$, $A_0 = 0$; a and A_ν are arbitrary numbers ($\nu = 1, 2, \ldots, n-1$).

Denote $f(\lambda) = \sum\limits_{\nu=0}^{n-1} A_{\nu+1}\lambda^\nu$. Let the roots $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of the algebraic equation $f(\lambda) = 0$ be all different, and $f(a) \neq 0$. Then the solution is as follows:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_{n-1} e^{\lambda_{n-1} x} + C_n e^{ax}\left[x - \frac{f'_a(a)}{f(a)}\right].$$

35. $\sum\limits_{\nu=0}^{n} (a_\nu x + b_\nu)y_x^{(\nu)} = 0.$

The Laplace equation. Particular solutions:

$$y_k = \int_{\alpha_k}^{\beta_k} \frac{1}{P(t)} \exp\left[xt + \int \frac{Q(t)}{P(t)}\,dt\right] dt,$$

where $P(t) = \sum\limits_{\nu=0}^{n} a_\nu t^\nu$, $Q(t) = \sum\limits_{\nu=0}^{n} b_\nu t^\nu$; α_k and β_k are found from the condition

$$\exp\left(xt + \int \frac{Q(t)}{P(t)}\,dt\right)\bigg|_{\alpha_k}^{\beta_k} = 0.$$

In many cases, the path of integration has to be chosen on the complex plane.

36. $x^2y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^2y + b.$

The substitution $w = x^2y$ leads to a constant coefficient linear equation: $w_x^{(n)} = aw + b$.

37. $x^2y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^3y + b.$

The substitution $w = x^2y$ leads to an equation of the form 17.1.2.3: $w_x^{(n)} = axw + b$.

38. $x(x+m)y_x^{(n)} + x(ax^k - x - n)y_x^{(m)} - a(x+m)x^k y = 0.$

Particular solution: $y_0 = xe^x$.

39. $a_n x^n y_x^{(n)} + a_{n-1} x^{n-1} y_x^{(n-1)} + \dots + a_1 x y'_x + a_0 y = 0.$

Euler equation. If all roots λ_k $(k = 1, 2, \dots, n)$ of the algebraic equation

$$\sum_{\nu=1}^{n} a_\nu \lambda(\lambda-1)\dots(\lambda-\nu+1) = -a_0$$

are different, the general solution of the original differential equation is given by:

$$y = C_1|x|^{\lambda_1} + C_2|x|^{\lambda_2} + \dots + C_n|x|^{\lambda_n}.$$

In the general case, the substitution $t = \ln|x|$ leads to a constant coefficient linear equation of the form 17.1.2.25:

$$\sum_{\nu=1}^{n} a_\nu D(D-1)\dots(D-\nu+1)y = -a_0 y, \qquad \text{where} \quad D = \frac{d}{dx}.$$

40. $x^{2n+1} y_x^{(n)} + n x^{2n} y_x^{(n-1)} = axy.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.5: $x^{2n} w_x^{(n)} = aw$.

41. $x^{2n+1} y_x^{(n)} + n x^{2n} y_x^{(n-1)} = ay.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.10: $x^{2n+1} w_x^{(n)} = aw$.

42. $x^n y_x^{(2n)} + 2n x^{n-1} y_x^{(2n-1)} = ay.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.6: $x^n w_x^{(2n)} = aw$.

43. $x^{3n} y_x^{(2n)} + 2n x^{3n-1} y_x^{(2n-1)} = ay.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.7: $x^{3n} w_x^{(2n)} = aw$.

44. $x^{n+1} y_x^{(2n+1)} + (2n+1) x^n y_x^{(2n)} = a\sqrt{x}\, y.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.8: $x^{n+1/2} w_x^{(2n+1)} = aw$.

45. $x^{3n+3/2} y_x^{(2n+1)} + (2n+1) x^{3n+1/2} y_x^{(2n)} = ay.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.9: $x^{3n+3/2} w_x^{(2n+1)} = aw$.

46. $P_{n-1}(x) y_x^{(n)} + P_{n-2}(x) y_x^{(n-1)} + \dots + P_1(x) y''_{xx} + (a_1 x + b_1) y'_x - m a_1 y = 0.$

Here, the $P_\nu(x)$ are polynomials of degree $\le \nu$, m is a positive integer, $a_1 \neq 0$.

A particular solution of this equation is the polynomial of degree m that can be written as:

$$y_0 = \sum_{k=0}^{m} \left(-\frac{1}{a_1}\right)^k [x^m I x^{-m-1}(P_{n-1}D^n + \dots + P_1 D^2 + b_1 D)]^k x^m,$$

where $D = \dfrac{d}{dx}$, $Ix^\nu = \dfrac{x^{\nu+1}}{\nu+1}$ with $\nu \neq -1$.

⊙ *Literature*: E. Kamke (1977).

47. $[a_n x^n + P_{n-1}(x)]y_x^{(n)} + \cdots + [a_1 x + P_0(x)]y'_x + a_0 y = 0.$

Here, the $P_\nu(x)$ are polynomials of degree $\le \nu$.

Assume that for some integer $m \ge 0$:

$$\sum_{\nu=0}^{n} C_m^\nu \nu!\, a_\nu = 0, \qquad \text{where} \quad C_m^\nu = \frac{m!}{\nu!\,(m-\nu)!},$$

and m is the least of the numbers satisfying this condition. Then there exists a solution in the form of a polynomial of degree m such that no polynomial of a smaller degree satisfies the original equation.

⊙ *Literature*: E. Kamke (1977).

48. $\big[xP(\delta) - Q(\delta)\big]y = 0, \qquad \delta \equiv x\dfrac{d}{dx}.$

Here, $P = P(z)$ and $Q = Q(z)$ are arbitrary polynomials of degree p and q, respectively.

Suppose $Q(z+1) = (z+1)Q_1(z+1)$, where the polynomial $Q_1(z+1)$ is such that $P(z)$ and $Q_1(z+1)$ do not have common factors. Then the original equation admits a formal solution in the power series form:

$$y_0 = \sum_{n=0}^{\infty} A_n x^n, \qquad \text{where} \quad \frac{A_{n+1}}{A_n} = \frac{P(n)}{Q(n+1)}.$$

⊙ *Literature*: H. Bateman and A. Erdélyi (1953, Vol. 1).

17.1.3 Equations Containing Exponential and Hyperbolic Functions

► **Equations with exponential functions.**

1. $y_x^{(n)} + (ax+b)e^{\lambda x}y'_x - ae^{\lambda x}y = 0.$

Particular solution: $y_0 = ax + b$.

2. $y_x^{(n)} + (ax+b)e^{\lambda x}y'_x - 2ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

3. $y_x^{(n)} + (ax+b)e^{\lambda x}y'_x - 3ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

4. $y_x^{(n)} + (ax+b)e^{\lambda x}y'_x - (n-1)ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^{n-1}$. The substitution $z = (ax+b)y'_x - a(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + (ax+b)e^{\lambda x}z = 0$.

5. $y_x^{(n)} + axe^{\lambda x}y'_x - ame^{\lambda x}y = 0, \qquad m = 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $y_0 = x^m$.

6. $y_x^{(2n)} = a^n y + be^{\lambda x}(y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = be^{\lambda x}$. The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order linear equation: $w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = be^{\lambda x}w$.

7. $y_x^{(n)} + (ae^{\lambda x} - b^{n-m})y_x^{(m)} - ab^m e^{\lambda x} y = 0.$

Particular solution: $y_0 = e^{bx}$.

8. $y_x^{(n)} + ay_x^{(n-1)} + be^{\lambda x} y_x' + abe^{\lambda x} y = 0.$

Particular solution: $y_0 = e^{-ax}$.

9. $y_x^{(n)} + ae^{\lambda x} y_x^{(m)} - (ab^m e^{\lambda x} + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

10. $y_x^{(n)} = \sum\limits_{k=0}^{n-1} (A_{k+1}e^{\lambda x} + bA_{k+1} - A_k) y_x^{(k)}.$

Here, $A_n = 1$, $A_0 = 0$; b and A_k are arbitrary numbers ($k = 1, 2, \ldots, n-1$).

Particular solutions: $y_m = e^{\mu_m x}$, where the μ_m are roots of the polynomial equation $\sum\limits_{k=0}^{n-1} A_{k+1}\mu^k = 0$.

11. $xy_x^{(n)} + axe^{\lambda x} y_x^{(m)} - [a(x+m)e^{\lambda x} + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

12. $xy_x^{(n)} + (n-m-1)y_x^{(n-1)} + axe^{\lambda x} y_x' - ame^{\lambda x} y = 0.$

Particular solution: $y_0 = x^m$.

13. $x(x+m)y_x^{(n)} + x(ae^{\lambda x} - x - n)y_x^{(m)} - a(x+m)e^{\lambda x} y = 0.$

Particular solution: $y_0 = xe^x$.

14. $(ax^m + be^x + c)y_x^{(n)} = be^x y, \qquad m = 0,\ 1,\ \ldots,\ n-1.$

Particular solution: $y_0 = ax^m + be^x + c$.

15. $(ax^m e^x + b)y_x^{(n)} = (-1)^n by, \qquad m = 0,\ 1,\ \ldots,\ n-1.$

Particular solution: $y_0 = ax^m + be^{-x}$.

16. $\Big(ae^x + \sum\limits_{k=0}^{n-1} b_k x^k\Big) y_x^{(n)} = ae^x y.$

Particular solution: $y_0 = ae^x + \sum\limits_{k=0}^{n-1} b_k x^k$.

► **Equations with hyperbolic functions.**

17. $y_x^{(2n)} = a^n y + b\sinh^k(\lambda x)(y_{xx}'' - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\sinh^k(\lambda x)$.

18. $y_x^{(n)} + a\sinh^k x\, y_x^{(m)} - (ab^m \sinh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

19. $y_x^{(n)} + (a\sinh^k x - b^{n-m})y_x^{(m)} - ab^m \sinh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

20. $y_x^{(n)} + (ax+b)\sinh^m(\lambda x)y_x' - a\sinh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

21. $xy_x^{(n)} + ax\sinh^k x\, y_x^{(m)} - [a(x+m)\sinh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

22. $y_x^{(2n)} = a^n y + b\cosh^k(\lambda x)(y_{xx}'' - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\cosh^k(\lambda x)$.

23. $y_x^{(n)} + a\cosh^k x\, y_x^{(m)} - (ab^m\cosh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

24. $y_x^{(n)} + (a\cosh^k x - b^{n-m})y_x^{(m)} - ab^m\cosh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

25. $y_x^{(n)} + (ax+b)\cosh^m(\lambda x)y_x' - a\cosh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

26. $xy_x^{(n)} + ax\cosh^k x\, y_x^{(m)} - [a(x+m)\cosh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

27. $y_x^{(2n)} = y + a(y_x'\cosh x - y\sinh x).$

The substitution $w = y_x'\cosh x - y\sinh x$ leads to a $(2n-1)$st-order linear equation.

28. $y_x^{(2n)} = y + a(y_x'\sinh x - y\cosh x).$

The substitution $w = y_x'\sinh x - y\cosh x$ leads to a $(2n-1)$st-order linear equation.

29. $y_x^{(n)} + a\tanh^k x\, y_x^{(m)} - (ab^m\tanh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

30. $y_x^{(n)} + (a\tanh^k x - b^{n-m})y_x^{(m)} - ab^m\tanh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

31. $y_x^{(n)} + (ax+b)\tanh^m(\lambda x)y_x' - a\tanh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

32. $xy_x^{(n)} + ax\tanh^k x\, y_x^{(m)} - [a(x+m)\tanh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

33. $y_x^{(n)} + a\coth^k x\, y_x^{(m)} - (ab^m\coth^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

34. $y_x^{(n)} + (a\coth^k x - b^{n-m})y_x^{(m)} - ab^m\coth^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

35. $y_x^{(n)} + (ax+b)\coth^m(\lambda x)\, y_x' - a\coth^m(\lambda x)\, y = 0.$

Particular solution: $y_0 = ax + b$.

36. $xy_x^{(n)} + ax\coth^k x\, y_x^{(m)} - [a(x+m)\coth^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

17.1.4 Equations Containing Logarithmic Functions

1. $y_x^{(2n)} = a^n y + b \ln x (y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b \ln x$.

2. $y_x^{(n)} + a \ln^k x\, y_x^{(m)} - (ab^m \ln^k x + b^n) y = 0.$

Particular solution: $y_0 = e^{bx}$.

3. $y_x^{(n)} + (a \ln^k x - b^{n-m}) y_x^{(m)} - ab^m \ln^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

4. $y_x^{(n)} + a y_x^{(n-1)} + b \ln^k(\lambda x) y'_x + ab \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

5. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - a \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

6. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - 2a \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

7. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - 3a \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

8. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - a(n - 1) \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = (ax + b)^{n-1}$.

9. $y_x^{(n)} + ax \ln^k(\lambda x) y'_x - am \ln^k(\lambda x) y = 0, \qquad m = 1, 2, \ldots, n - 1.$

Particular solution: $y_0 = x^m$.

10. $x y_x^{(n)} + ax \ln^k(\lambda x) y_x^{(m)} - [a(x + m) \ln^k(\lambda x) + x + n] y = 0.$

Particular solution: $y_0 = xe^x$.

17.1.5 Equations Containing Trigonometric Functions

► **Equations with sine and cosine.**

1. $y_x^{(n)} + a \sin^k x\, y_x^{(m)} - (ab^m \sin^k x + b^n) y = 0.$

Particular solution: $y_0 = e^{bx}$.

2. $y_x^{(n)} + (a \sin^k x - b^{n-m}) y_x^{(m)} - ab^m \sin^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

3. $y_x^{(n)} + a y_x^{(n-1)} + b \sin^m(\lambda x) y'_x + ab \sin^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

4. $y_x^{(n)} + (ax + b) \sin^m(\lambda x) y'_x - a \sin^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

5. $y_x^{(n)} + (ax+b)\sin^m(\lambda x)y_x' - 2a\sin^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

6. $y_x^{(n)} + (ax+b)\sin^m(\lambda x)y_x' - 3a\sin^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

7. $y_x^{(2n)} = a^n y + b\sin^k(\lambda x)(y_{xx}'' - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\sin^k(\lambda x)$.

8. $xy_x^{(n)} + ax\sin^k(\lambda x)y_x^{(m)} - [a(x+m)\sin^k(\lambda x) + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

9. $(ax^m + b\sin x)y_x^{(n)} = b\sin\left(x + \frac{1}{2}\pi n\right)y, \qquad m = 0,\ 1,\ \ldots,\ n-1.$

Particular solution: $y_0 = ax^m + b\sin x$.

10. $\left(a\sin x + \sum\limits_{k=0}^{n-1} b_k x^k\right)y_x^{(n)} = a\sin\left(x + \frac{1}{2}\pi n\right)y.$

Particular solution: $y_0 = a\sin x + \sum\limits_{k=0}^{n-1} b_k x^k$.

11. $y_x^{(n)} + a\cos^k x\, y_x^{(m)} - (ab^m\cos^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

12. $y_x^{(n)} + (a\cos^k x - b^{n-m})y_x^{(m)} - ab^m\cos^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

13. $y_x^{(n)} + ay_x^{(n-1)} + b\cos^m(\lambda x)y_x' + ab\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

14. $y_x^{(n)} + (ax+b)\cos^m(\lambda x)y_x' - a\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

15. $y_x^{(n)} + (ax+b)\cos^m(\lambda x)y_x' - 2a\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

16. $y_x^{(n)} + (ax+b)\cos^m(\lambda x)y_x' - 3a\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

17. $y_x^{(2n)} = a^n y + b\cos^k(\lambda x)(y_{xx}'' - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\cos^k(\lambda x)$.

18. $xy_x^{(n)} + ax\cos^k(\lambda x)y_x^{(m)} - [a(x+m)\cos^k(\lambda x) + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

19. $(ax^m + b\cos x)y_x^{(n)} = b\cos\left(x + \frac{1}{2}\pi n\right)y, \qquad m = 0,\ 1,\ \ldots,\ n-1.$

Particular solution: $y_0 = ax^m + b\cos x$.

20. $\left(a\cos x + \sum\limits_{k=0}^{n-1} b_k x^k\right) y_x^{(n)} = a\cos\left(x + \frac{1}{2}\pi n\right) y.$

Particular solution: $y_0 = a\cos x + \sum\limits_{k=0}^{n-1} b_k x^k.$

21. $y_x^{(2n)} = (-1)^n y + a(y'_x \sin x - y\cos x).$

The substitution $w = y'_x \sin x - y\cos x$ leads to a $(2n-1)$st-order linear equation.

22. $y_x^{(2n)} = (-1)^n y + a(y'_x \cos x + y\sin x).$

The substitution $w = y'_x \cos x + y\sin x$ leads to a $(2n-1)$st-order linear equation.

► **Equations with tangent and cotangent.**

23. $y_x^{(n)} + a\tan^k x\, y_x^{(m)} - (ab^m \tan^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}.$

24. $y_x^{(n)} + (a\tan^k x - b^{n-m}) y_x^{(m)} - ab^m \tan^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}.$

25. $y_x^{(n)} + a y_x^{(n-1)} + b\tan^m(\lambda x) y'_x + ab\tan^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}.$

26. $y_x^{(n)} + (ax+b)\tan^m(\lambda x) y'_x - a\tan^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b.$

27. $y_x^{(n)} + (ax+b)\tan^m(\lambda x) y'_x - 2a\tan^m(\lambda x) y = 0.$

Particular solution: $y_0 = (ax+b)^2.$

28. $y_x^{(n)} + (ax+b)\tan^m(\lambda x) y'_x - 3a\tan^m(\lambda x) y = 0.$

Particular solution: $y_0 = (ax+b)^3.$

29. $x y_x^{(n)} + ax\tan^k(\lambda x) y_x^{(m)} - [a(x+m)\tan^k(\lambda x) + x + n] y = 0.$

Particular solution: $y_0 = xe^x.$

30. $y_x^{(n)} + a\cot^k x\, y_x^{(m)} - (ab^m \cot^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}.$

31. $y_x^{(n)} + (a\cot^k x - b^{n-m}) y_x^{(m)} - ab^m \cot^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}.$

32. $y_x^{(n)} + a y_x^{(n-1)} + b\cot^m(\lambda x) y'_x + ab\cot^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}.$

33. $y_x^{(n)} + (ax+b)\cot^m(\lambda x) y'_x - a\cot^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b.$

34. $y_x^{(n)} + (ax+b)\cot^m(\lambda x)y_x' - 2a\cot^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

35. $y_x^{(n)} + (ax+b)\cot^m(\lambda x)y_x' - 3a\cot^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

36. $xy_x^{(n)} + ax\cot^k(\lambda x)y_x^{(m)} - [a(x+m)\cot^k(\lambda x) + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

17.1.6 Equations Containing Arbitrary Functions

► **Equations of the form** $f_n(x)y_x^{(n)} + f_1(x)y_x' + f_0(x)y = g(x)$.

1. $y_x^{(n)} = f(x).$

Solution: $y = \sum_{\nu=0}^{n-1} C_\nu x^\nu + \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,dt$, where x_0 is an arbitrary number.

2. $y_x^{(n)} = f(x)y.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of similar form: $w_t^{(n)} = (-1)^n t^{-2n} f(1/t)w$.

3. $y_x^{(n)} = (cx+d)^{-2n} f\left(\frac{ax+b}{cx+d}\right)y.$

The transformation $\xi = \frac{ax+b}{cx+d}$, $w = \frac{y}{(cx+d)^{n-1}}$ leads to a simpler equation: $w_\xi^{(n)} = \Delta^{-n} f(\xi)w$, where $\Delta = ad - bc$.

4. $fy_x^{(n)} - f_x^{(n)}y = 0, \qquad f = f(x).$

Particular solution: $y_0 = f(x)$.

5. $fy_x^{(2n+1)} + f_x^{(2n+1)}y = g(x), \qquad f = f(x).$

First integral: $\sum_{k=0}^{2n} (-1)^k f_x^{(2n-k)} y_x^{(k)} = \int g(x)\,dx + C$.

6. $y_x^{(n)} + (ax+b)f(x)y_x' - af(x)y = 0.$

Particular solution: $y_0 = ax+b$.

7. $y_x^{(n)} + (ax+b)f(x)y_x' - 2af(x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

8. $y_x^{(n)} + (ax+b)f(x)y_x' - 3af(x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

9. $y_x^{(n)} + (ax+b)f(x)y_x' - (n-1)af(x)y = 0.$

Particular solution: $y_0 = (ax+b)^{n-1}$. The substitution $z = (ax+b)y_x' - a(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + (ax+b)f(x)z = 0$.

10. $y_x^{(n)} + xf(x)y_x' - mf(x)y = 0, \qquad m = 1,\, 2,\, \dots,\, n-1.$

Particular solution: $y_0 = x^m$. The substitution $z = xy_x' - my$ leads to an $(n-1)$st-order equation:

$$D^{n-m-1}\left(\frac{z_x^{(m)}}{x}\right) + f(x)z = 0, \quad \text{where} \quad D = \frac{d}{dx}.$$

11. $y_x^{(n)} + f(x)y_x' + g(x)y + h(x) = 0.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of similar form:

$$w_t^{(n)} + (-1)^n t^{-2n}\Big\{-t^2 f(1/t)w_t' + \big[(n-1)tf(1/t) + g(1/t)\big]w + t^{n-1}h(1/t)\Big\} = 0.$$

12. $y_x^{(n)} + f(x)y_x' + f_x'(x)y = g(x).$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} + f(x)y = \int g(x)\,dx + C.$

13. $y_x^{(2n)} = y + f(x)(y_x' \cosh x - y \sinh x).$

The substitution $w = y_x' \cosh x - y \sinh x$ leads to a $(2n-1)$st-order linear equation.

14. $y_x^{(2n)} = y + f(x)(y_x' \sinh x - y \cosh x).$

The substitution $w = y_x' \sinh x - y \cosh x$ leads to a $(2n-1)$st-order linear equation.

15. $y_x^{(2n)} = (-1)^n y + f(x)(y_x' \sin x - y \cos x).$

The substitution $w = y_x' \sin x - y \cos x$ leads to a $(2n-1)$st-order linear equation.

16. $y_x^{(2n)} = (-1)^n y + f(x)(y_x' \cos x + y \sin x).$

The substitution $w = y_x' \cos x + y \sin x$ leads to a $(2n-1)$st-order linear equation.

17. $y_x^{(n)} = \dfrac{\varphi_x^{(n)}}{\varphi} y + f(x)\Big(y_x' - \dfrac{\varphi_x'}{\varphi} y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y_x' - \dfrac{\varphi_x'}{\varphi} y$ leads to an $(n-1)$st-order linear equation.

▶ **Other equations.**

18. $y_x^{(2n)} = a^n y + f(x)(y_{xx}'' - ay).$

The substitution $w = y_{xx}'' - ay$ leads to a $(2n-2)$nd-order linear equation:

$$w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = f(x)w.$$

19. $y_x^{(n)} + f(x)(x^2 y_{xx}'' - 2xy_x' + 2y) = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$. The substitution $z = x^2 y_{xx}'' - 2xy_x' + 2y$ leads to a linear equation of the $(n-2)$nd-order.

20. $y_x^{(n+2)} + f(x)[x^2 y_{xx}'' - 2nxy_x' + n(n+1)y] = 0.$

The substitution $w(x) = x^2 y_{xx}'' - 2nxy_x' + n(n+1)y$ leads to an nth-order linear equation: $w_x^{(n)} + x^2 f(x)w = 0.$

21. $y_x^{(2n)} = a^2 y + f(x)[y_x^{(n)} + ay]$.

The substitution $w = y_x^{(n)} + ay$ leads to an nth-order linear equation: $w_x^{(n)} = [f(x) + a]w$.

22. $y_x^{(n)} + f(x)y_x^{(m)} - [a^n + a^m f(x)]y = 0$.

Particular solution: $y_0 = e^{ax}$.

23. $y_x^{(n)} + (f - a^{n-m})y_x^{(m)} - a^m f y = 0, \qquad f = f(x)$.

Particular solution: $y_0 = e^{ax}$.

24. $y_x^{(n)} + ay_x^{(n-1)} + fy_x' + afy = 0, \qquad f = f(x)$.

Particular solution: $y_0 = e^{-ax}$.

25. $y_x^{(n)} + f(x)y_x^{(n-1)} + g(x)y_x^{(n-2)} + h(x) = 0$.

The substitution $w(x) = y_x^{(n-2)}$ leads to a second-order linear equation: $w''_{xx} + f(x)w'_x + g(x)w + h(x) = 0$.

26. $y_x^{(n)} + a_{n-1}y_x^{(n-1)} + \cdots + a_1 y_x' + a_0 y = f(x)$.

Constant coefficient nonhomogeneous linear equation. The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation (see 5.1.2.25) and any particular solution of the nonhomogeneous equation.

If the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

are all different, the original equation has the general solution:

$$y = \sum_{\nu=1}^{n} C_\nu e^{\lambda_\nu x} + \sum_{\nu=1}^{n} \frac{e^{\lambda_\nu x}}{P'_\lambda(\lambda_\nu)} \int f(x)e^{-\lambda_\nu x}\,dx$$

(with complex roots, the real part should be taken).

Section 4.1.2 lists the forms of particular solutions corresponding to some special forms of the right-hand side function of the nonhomogeneous linear equation.

27. $y_x^{(n)} + f(x)\sum_{k=0}^{n-1}(-1)^k k!\, C_{n-1}^k x^{n-k-1} y_x^{(n-k-1)} = 0$.

Here, $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

Particular solutions: $y_m = x^m$, where $m = 1, 2, \ldots, n-1$.

The substitution $z = \sum_{k=0}^{n-1}(-1)^k k!\, C_{n-1}^k x^{n-k-1} y_x^{(n-k-1)}$ leads to a first-order linear equation: $z'_x + x^{n-1}f(x)z = 0$. Having solved this equation, we obtain an $(n-1)$st-order linear equation of the form 17.1.6.34 for $y(x)$.

28. $y_x^{(n)} = \sum_{k=0}^{n-1}(a_{k+1}f - a_k)y_x^{(k)}$.

Here, $f = f(x)$; $a_n = 1$, $a_0 = 0$; a_k are arbitrary numbers ($k = 1, 2, \ldots, n-1$).

Particular solutions: $y_k = e^{\lambda_k x}$ ($k = 1, 2, \ldots, n-1$), where the λ_k are roots of the polynomial equation $\sum_{k=0}^{n-1} a_{k+1}\lambda^k = 0$.

29. $xy_x^{(n)} + (a+n-1)y_x^{(n-1)} = f(x)(xy'_x + ay).$

The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order linear equation: $w_x^{(n-1)} = f(x)w$.

30. $xy_x^{(n)} + xfy_x^{(m)} - [(x+m)f + x + n]y = 0, \qquad f = f(x).$

Particular solution: $y_0 = xe^x$.

31. $xy_x^{(n)} + ny_x^{(n-1)} = x^{1-2n}f(1/x)y + x^{-n-1}g(1/x).$

The transformation $t = x^{-1}$, $w = yx^{2-n}$ leads to an nth-order linear equation: $w_t^{(n)} = (-1)^n[f(t)w + g(t)]$.

32. $x^2y_x^{(n+2)} + \alpha xy_x^{(n+1)} + \beta y_x^{(n)} + f(x)[x^2y''_{xx} + (\alpha-2n)xy'_x + (\beta - \alpha n + n^2 + n)y] = 0.$

The substitution $w(x) = x^2y''_{xx} + (\alpha-2n)xy'_x + (\beta-\alpha n+n^2+n)y$ leads to an nth-order linear equation: $w_x^{(n)} + f(x)w = 0$.

33. $x(x+m)y_x^{(n)} + x(f-x-n)y_x^{(m)} - (x+m)fy = 0, \qquad f = f(x).$

Particular solution: $y_0 = xe^x$.

34. $x^ny_x^{(n)} + b_{n-1}x^{n-1}y_x^{(n-1)} + \cdots + b_1xy'_x + b_0y = f(x).$

Nonhomogeneous Euler equation. The substitution $x = ae^t$ $(a \neq 0)$ leads to a constant coefficient nonhomogeneous linear equation of the form 17.1.6.26.

35. $x^ny_x^{(n)} + (n-m-1)x^{n-1}y_x^{(n-1)} + xfy'_x - mfy = 0, \qquad f = f(x).$

Particular solution: $y_0 = x^m$.

36. $x^ny_x^{(n)} + x^mfy_x^{(m)} - (n!\,C_a^n + m!\,C_a^m f)y = 0.$

Here, $f = f(x)$, $C_a^n = \dfrac{\Gamma(a+1)}{n!\,\Gamma(a-n+1)}$ are binomial coefficients, and $\Gamma(a)$ is the gamma function.

Particular solution: $y_0 = x^a$.

37. $x^my_x^{(n)} = \sum\limits_{k=0}^{n-1}[x^m(a_{k+1}f - a_k) + a_{k+1}]y_x^{(k)}.$

Here, $f = f(x)$; $a_n = 1$, $a_0 = 0$; m and a_k are arbitrary numbers ($k = 1, 2, \ldots, n-1$).

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1, 2, \ldots, n-1)$, where the λ_k are roots of the polynomial equation $\sum\limits_{k=0}^{n-1} a_{k+1}\lambda^k = 0$.

38. $\sin x\, y_x^{(n)} + \sin x\, f(x)y_x^{(m)} - \bigl[\sin\bigl(x+\tfrac{1}{2}\pi n\bigr) + f(x)\sin\bigl(x+\tfrac{1}{2}\pi m\bigr)\bigr]y = 0.$

Particular solution: $y_0 = \sin x$.

39. $\cos x\, y_x^{(n)} + \cos x\, f(x)y_x^{(m)} - \bigl[\cos\bigl(x+\tfrac{1}{2}\pi n\bigr) + f(x)\cos\bigl(x+\tfrac{1}{2}\pi m\bigr)\bigr]y = 0.$

Particular solution: $y_0 = \cos x$.

40. $\sum\limits_{k=2}^{n} f_k(x)y_x^{(k)} = g(x)(xy'_x - y).$

Particular solution: $y_0 = x$. The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$st-order linear equation.

41. $\sum\limits_{k=m+1}^{n} f_k(x) y_x^{(k)} = g(x)(xy'_x - my), \qquad m = 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $y_0 = x^m$. The substitution $w(x) = xy'_x - my$ leads to an $(n-1)$st-order linear equation.

42. $\sum\limits_{k=3}^{n} f_k(x) y_x^{(k)} = g(x)(x^2 y''_{xx} - 2xy'_x + 2y).$

Particular solutions: $y_1 = x$, $y_2 = x^2$. The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to an $(n-2)$nd-order linear equation.

43. $\sum\limits_{k=4}^{n} f_k(x) y_x^{(k)} = g(x)(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y).$

Particular solutions: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$. The substitution $w(x) = x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y$ leads to an $(n-3)$rd-order linear equation.

44. $\sum\limits_{k=m+1}^{n} f_k(x) y_x^{(k)} + g(x) \sum\limits_{k=0}^{m} (-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)} = 0.$

Here, $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

Particular solutions: $y_s = x^s$, where $s = 1,\ 2,\ \ldots,\ m$.

The substitution $z = \sum\limits_{k=0}^{m} (-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)}$ leads to an $(n-m)$th-order linear equation: $\sum\limits_{k=m+1}^{n} f_k(x) D^{k-m-1}\bigl(x^{-m} z'_x\bigr) + g(x) z = 0$, where $D = d/dx$.

45. $\sum\limits_{k=0}^{n} (f_k - a f_{k+1}) y_x^{(k)} = 0.$

Here, $f_k = f_k(x)$ $(k = 1,\ 2,\ \ldots,\ n)$; $f_{n+1} \equiv f_0 \equiv 0$.

Particular solution: $y_0 = e^{ax}$.

46. $\sum\limits_{k=0}^{n} x^k [f_k + (k-m) f_{k+1}] y_x^{(k)} = 0.$

Here, $f_k = f_k(x)$ $(k = 1,\ 2,\ \ldots,\ n)$; $f_{n+1} \equiv f_0 \equiv 0$.

Particular solution: $y_0 = x^m$.

17.2 Nonlinear Equations

17.2.1 Equations Containing Power Functions

► Fifth- and sixth-order equations.

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + bx + c.$

This is a special case of equation 17.2.6.1 with $f(x) = ax + b$.

2. $yy_x^{(5)} = ax + b.$

This is a special case of equation 17.2.6.17 with $n = 2$ and $f(x) = ax + b$.

3. $yy_x^{(5)} = ay_x'y_{xxxx}''''$.

1°. For $a \neq -1$, integrating the equation two times, we arrive at a third-order autonomous equation: $y_x'y_{xxx}''' - \frac{1}{2}(y_{xx}'')^2 = C_1 y^{a+1} + C_2$. The substitution $w(y) = \frac{1}{2}(y_x')^2$ leads to a second-order equation:

$$ww_{yy}'' - \tfrac{1}{4}(w_y')^2 = \tfrac{1}{2}C_1 y^{a+1} + \tfrac{1}{2}C_2.$$

For $a = 1$, this is a solvable equation of the form 2.8.1.53.

2°. For $a = -1$, integrating the equation two times, we arrive at a third-order autonomous equation: $y_x'y_{xxx}''' - \frac{1}{2}(y_{xx}'')^2 = C_1 \ln|y| + C_2$.

3°. Particular solution: $y = C_1x^3 + C_2x^2 + C_3x + C_4$.

4. $3yy_x^{(5)} + 5y_x'y_{xxxx}'''' = 0$.

This is a special case of equation 17.2.1.3 with $a = -\frac{5}{3}$. Integrating the equation three times, we arrive at a second-order equation: $3yy_{xx}'' - 2(y_x')^2 = C_1x^2 + C_2x + C_3$. The substitution $y = w^3$ leads to a solvable equation of the form 14.8.1.5: $w_{xx}'' = \frac{1}{9}(C_1x^2 + C_2x + C_3)w^{-5}$.

5. $2yy_x^{(5)} + 5y_x'y_{xxxx}'''' + 5y_{xx}''y_{xxx}''' = 0$.

This is a special case of equation 17.2.6.4 with $a = \frac{5}{2}$ and $f(x) = 0$. Integrating the equation three times, we arrive at a second-order equation of the form 14.8.1.53: $yy_{xx}'' - \frac{1}{4}(y_x')^2 = C_1x^2 + C_2x + C_3$.

6. $yy_x^{(5)} + ay_x'y_{xxxx}'''' + (3a-5)y_{xx}''y_{xxx}''' = 0$.

Integrating the equation three times, we obtain a second-order nonlinear equation: $yy_{xx}'' + \frac{1}{2}(a-3)(y_x')^2 = C_1x^2 + C_2x + C_3$.

7. $yy_x^{(5)} + ay_x'y_{xxxx}'''' + by_{xx}''y_{xxx}''' = 0$.

Integrating yields a fourth-order equation: $yy_{xxxx}'''' + (a-1)y_x'y_{xxx}''' + \frac{1}{2}(1-a+b)(y_{xx}'')^2 = C$.

8. $yy_x^{(5)} + 5y_x'y_{xxxx}'''' + 10y_{xx}''y_{xxx}''' = ax^n$.

This is a special case of equation 17.2.6.2 with $f(x) = ax^n$.

9. $yy_x^{(5)} + 5y_x'y_{xxxx}'''' + 10y_{xx}''y_{xxx}''' = ay^n$.

This is a special case of equation 17.2.6.3 with $f(y) = ay^n$.

10. $y_x^{(6)} = Ay^{-7/5}$.

This is a special case of equation 17.2.1.12 with $n = 3$.

Multiplying by $y^{7/5}$ and differentiating with respect to x, we obtain $5yy_x^{(7)} + 7y_x'y_x^{(6)} = 0$. Having integrated this equation three times, we arrive at the chain of equations:

$$5yy_x^{(6)} + 2y_x'y_x^{(5)} - 2y_{xx}''y_{xxxx}'''' + (y_{xxx}''')^2 = 2C_2, \qquad (1)$$

$$5yy_x^{(5)} - 3y_x'y_{xxxx}'''' + y_{xx}''y_{xxx}''' = 2C_2x + C_1, \qquad (2)$$

$$5yy_{xxxx}'''' - 8y_x'y_{xxx}''' + \tfrac{9}{2}(y_{xx}'')^2 = C_2x^2 + C_1x + C_0, \qquad (3)$$

where C_0, C_1, and C_2 are arbitrary constants. Eliminating the highest derivatives from (1)–(3), with the aid of the original equation, we obtain a third-order equation that can be reduced to a second-order equation.

11. $yy_x^{(6)} + 6y_x'y_x^{(5)} + 15y_{xx}''y_{xxxx}'''' + 10(y_{xxx}''')^2 = ax^n$.

This is a special case of equation 17.2.6.6 with $f(x) = ax^n$.

► **Equations of the form** $y_x^{(n)} = f(x, y)$.

12. $y_x^{(2n)} = Ay^{\frac{1+2n}{1-2n}}$.

Multiply both sides by $y^{\frac{2n+1}{2n-1}}$ and differentiate with respect to x. As a result, we obtain

$$(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = 0.$$

Three integrals containing arbitrary constants C_0, C_1, and C_2 are presented in 5.2.6.62, where one should let $f \equiv 0$. Eliminating the highest derivatives from those integrals and the original equation, one can always obtain a $(2n-3)$rd-order equation. With the aid of the transformation

$$t = \int \frac{dx}{P}, \quad w = yP^{\frac{1-2n}{2}}, \qquad \text{where} \quad P = C_2x^2 + C_1x + C_0,$$

this equation can be reduced to the autonomous form 17.2.6.77. Therefore, the substitution $z(w) = w'_t$ finally leads to a $(2n-4)$th-order equation with respect to $z = z(w)$.

13. $y_x^{(2n)} = ay^k + b, \qquad k \neq -1.$

This is a special case of equation 17.2.6.8. Integrating yields a $(2n-1)$st-order equation:

$$\sum_{m=1}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + \frac{1}{2}(-1)^n \left[y_x^{(n)}\right]^2 = -\frac{a}{k+1}y^{k+1} - by + C,$$

where C is an arbitrary constant. Furthermore, the order of the obtained autonomous equation can be reduced by one by the substitution $w(y) = y'_x$.

14. $y_x^{(n)} = ax^{-n}y^m$.

This is a special case of equation 17.2.6.11 with $f(y) = ay^m$.

15. $y_x^{(n)} = ax^k y^m$.

$1°$. The transformation $x = t^{-1}$, $y = t^{1-n}w(t)$ leads to an equation of the same form: $w_t^{(n)} = (-1)^n At^{-k-(n-1)m-n-1}w^m$.

$2°$. The transformation $\xi = x^{n+k}y^{m-1}$, $z = xy'_x/y$ leads to an $(n-1)$st-order equation.

16. $yy_x^{(2n+1)} = ax^k + b$.

This is a special case of equation 17.2.6.17 with $f(x) = ax^k + b$.

17. $y_x^{(n)} = x^{m-nm-n-1}(ay + bx^{n-1})^m$.

This is a special case of equation 17.2.6.13 with $f(w) = (aw + b)^m$.

18. $y_x^{(2n)} = x^{\frac{m-2nm-2n-1}{2}}\left(ay + bx^{\frac{2n-1}{2}}\right)^m$.

This is a special case of equation 17.2.6.14 with $f(w) = (aw + b)^m$.

19. $y_x^{(n)} = (ay + bx^k)^m; \qquad k = 1,\ 2,\ \ldots,\ n-1.$

The substitution $aw = ay + bx^k$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = a^m w^m$ (see also 5.2.1.12 and 5.2.1.13 with $b = 0$).

20. $y_x^{(n)} = (ax^2 + bx + c)^{\frac{m-nm-n-1}{2}} y^m.$

This is a special case of equation 17.2.6.22 with $f(w) = w^m$.

21. $y_x^{(n)} = (ax + b)^{-n}(cx + d)^{m-nm-1} y^m.$

This is a special case of equation 17.2.6.21 with $f(w) = w^m$.

► **Equations of the form** $y_x^{(n)} = f(x, y, y'_x, y''_{xx})$.

22. $y_x^{(n)} = ay^k y'_x + bx^m.$

This is a special case of equation 17.2.6.34 with $f(y) = ay^k$ and $g(x) = bx^m$. Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = \dfrac{a}{k+1} y^{k+1} + \dfrac{b}{m+1} x^{m+1} + C$.

23. $y_x^{(n)} = a^n y + b(y'_x - ay)^k.$

This is a special case of equation 17.2.6.38 with $f(x, w) = bw^k$. The substitution $w = y'_x - ay$ leads to an $(n-1)$st-order autonomous equation:

$$w_x^{(n-1)} + aw_x^{(n-2)} + \cdots + a^{n-1} w = bw^k.$$

24. $y_x^{(n)} = ax^{m-n} y^{1-m} (y'_x)^m.$

This is a special case of equation 17.2.6.37 with $f(w) = aw^m$.

25. $y_x^{(n)} = ax^m (xy'_x - y)^k.$

This is a special case of equation 17.2.6.39 with $f(x, w) = ax^m w^k$. The substitution $w = xy'_x - y$ leads to an $(n-1)$st-order equation.

26. $y_x^{(n)} = ax^k (xy'_x - my)^l.$

Here, m is a positive integer and $n \geq m + 1$. The substitution $w = xy'_x - my$ leads to an $(n-1)$st-order equation: $\zeta_x^{(n-m-1)} = ax^k w^l$, where $\zeta = w_x^{(m)}/x$.

27. $y_x^{(2n)} + ay''_{xx} + by = cyy''_{xx} - c(y'_x)^2 + k.$

1°. Particular solution:

$$y = C_1 \sinh(C_4 x) + C_2 \cosh(C_4 x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^{2n} + (a - cC_3)C_4^2 + b = 0,$$
$$c(C_2^2 - C_1^2)C_4^2 - bC_3 + k = 0.$$

2°. Particular solution:

$$y = C_1 \sin(C_4 x) + C_2 \cos(C_4 x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^{2n} - (a - cC_3)C_4^2 + b = 0,$$
$$c(C_1^2 + C_2^2)C_4^2 + bC_3 - k = 0.$$

28. $y_x^{(n)} + ayy''_{xx} - a(y'_x)^2 + by''_{xx} + cy'_x = 0.$

Particular solution: $y = C_1 \exp(C_2 x) - \dfrac{C_2^{n-1} + bC_2 + c}{aC_2}$.

29. $y_x^{(2n)} = a^n y + b(y''_{xx} - ay)^k.$

This is a special case of equation 17.2.6.50 with $f(x, w) = bw^k$. The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order autonomous equation: $w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = bw^k$.

30. $y_x^{(n)} = ax^m(xy'_x - y)^k(y''_{xx})^l.$

The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$st-order equation:

$$\frac{d^{n-2}}{dx^{n-2}}\left(\frac{w'_x}{x}\right) = ax^{m-l}w^k(w'_x)^l.$$

31. $y_x^{(2n)} = ay\bigl(yy''_{xx} - y'^2_x\bigr)^k.$

This is a special case of equation 17.2.6.52 with $f(w) = 0$ and $g(w) = aw^k$.

► Other equations.

32. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2.$

1°. Integrating the equation two times, we obtain an $(n-2)$nd-order equation:

$$y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2.$$

2°. Particular solutions:

$$\begin{aligned}
&y = C_1\exp(C_3x) + C_2\exp(-C_3x) + a^{-1}C_3^{n-4} && \text{if } n \text{ is an even number},\\
&y = C_1\sin(C_3x) + C_2\cos(C_3x) + (-1)^{n/2}a^{-1}C_3^{n-4} && \text{if } n \text{ is an even number},\\
&y = C_1\exp(C_2x) + a^{-1}C_2^{n-4} && \text{if } n \text{ is an odd number},\\
&y = C_1x + C_2 && \text{if } n \ge 2 \text{ is any number}.
\end{aligned}$$

33. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + bx + k.$

This is a special case of equation 17.2.6.54 with $f(x) = bx + k$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + \frac{1}{6}bx^3 + \frac{1}{2}kx^2 + C_1x + C_2$.

34. $y_x^{(2n)} = a^2y + b[y_x^{(n)} + ay]^k.$

The substitution $w = y_x^{(n)} + ay$ leads to an nth-order autonomous equation: $w_x^{(n)} = aw + bw^k$.

35. $y_x^{(n)} + axy_x^{(n-1)} + 2byy'_x + abxy^2 + cx = 0.$

The functions that solve the $(n-1)$st-order autonomous equation $y_x^{(n-1)} = -by^2 - c/a$ are solutions of the original equation.

36. $y_x^{(n)} + ayy_x^{(n-1)} + by'_x + aby^2 + cy = 0.$

The functions that solve the $(n-1)$st-order constant coefficient nonhomogeneous linear equation $y_x^{(n-1)} + by = -c/a$ are solutions of the original equation.

37. $xy_x^{(n)} + ny_x^{(n-1)} = ax^m y^m.$

This is a special case of equation 17.2.6.59 with $f(w) = aw^m$.

38. $xy_x^{(n)} + (a+n-1)y_x^{(n-1)} = b(xy'_x + ay)^k.$

This is a special case of equation 17.2.6.60 with $f(x,w) = bw^k$. The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order autonomous equation: $w_x^{(n-1)} = bw^k$.

39. $x^2y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^{2m}y^m.$

This is a special case of equation 17.2.6.61 with $f(w) = aw^m$.

40. $yy_x^{(2n+1)} = ay'_x y_x^{(2n)}.$

The equation admits two different (with $a \neq -1$) first integrals:

$$y_x^{(2n)} = \widetilde{C}_1 y^a,$$

$$yy_x^{(2n)} + (a+1)\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n(a+1)\big[y_x^{(n)}\big]^2 = \widetilde{C}_2,$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. Eliminating the highest derivative from the first integrals, we arrive at a $(2n-1)$st-order autonomous equation:

$$\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n\big[y_x^{(n)}\big]^2 = C_1 y^{a+1} + C_2,$$

where $C_1 = -\dfrac{\widetilde{C}_1}{a+1}$, $C_2 = \dfrac{\widetilde{C}_2}{a+1}$. The order of the obtained equation next can be lowered by the standard substitution $w(y) = y'_x$.

41. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = ax^m.$

This is a special case of equation 17.2.6.62 with $f(x) = ax^m$.

42. $yy_x^{(n)} - y'_x y_x^{(n-1)} = ay^2.$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = (ax+C)y$. The transformation $z = x + C/a$ brings it to an equation of the form 17.1.2.3 with $b = 0$.

43. $yy_x^{(n)} = y'_x y_x^{(n-1)} + ay'_x.$

Integrating yields an $(n-1)$st-order constant coefficient nonhomogeneous linear equation: $y_x^{(n-1)} = Cy - a$.

44. $\displaystyle\sum_{k=0}^{n} a_k y_x^{(k)} = byy''_{xx} - b(y'_x)^2 + k.$

Particular solutions: $y = Ce^{\lambda x} + ka_0^{-1}$, where C is an arbitrary constant and λ are roots of the algebraic equation $a_0 \sum\limits_{k=0}^{n} a_k\lambda^n = bk\lambda^2$.

45. $xyy_x^{(n)} = (xy'_x + ay)y_x^{(n-1)}.$

Integrating yields an $(n-1)$st-order linear equation of the form 17.1.2.4: $y_x^{(n-1)} = Cx^a y$.

46. $(y+ax^{m-1})y_x^{(n)} - y_x^{(m)}y_x^{(n-m)} + bx^{m-1}y_x^{(m)} = 0, \qquad n > m.$

The functions that solve the $(n-m)$th-order linear equation

$$y_x^{(n-m)} = Cy + (aC+b)x^{m-1}$$

are solutions of the original equation.

47. $y_x^{(n-2)}y_x^{(n)} = a\big[y_x^{(n-1)}\big]^2.$

Solution: $y = \begin{cases} C_0 + C_1x + \cdots + C_{n-3}x^{n-3} + (C_{n-2}+C_{n-1}x)^{n-2+\frac{1}{1-a}} & \text{if } a \neq 1, \\ C_0 + C_1x + \cdots + C_{n-3}x^{n-3} + C_{n-2}\exp(C_{n-1}x) & \text{if } a = 1. \end{cases}$

48. $y_x^{(n)} = ay^k y'_x\big[y_x^{(n-1)}\big]^m.$

This is a special case of equation 17.2.6.73 with $f(y) = y^k$, $g(w) = aw^m$.

49. $y_x^{(n)} = ax^{m_1}y^{m_2}(y'_x)^{m_3}\cdots(y_x^{(n-1)})^{m_{n+1}}.$

Generalized homogeneous equation. The transformation $\xi = x^\lambda y^\mu$, $w = xy'_x/y$, where

$$\lambda = n + m_1 - m_3 - 2m_4 - \cdots - (n-1)m_{n+1}, \quad \mu = m_2 + m_3 + \cdots + m_{n+1} - 1,$$

leads to an $(n-1)$st-order equation.

50. $\left(\sqrt{y}\,\frac{d}{dx}\right)^{n-1}(y'_x) = ax + b.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a constant coefficient linear equation: $2x_t^{(n+1)} = ax + b$.

51. $2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)}y_x^{(2n-m)} + (-1)^n\big[y_x^{(n)}\big]^2 + \lambda(y'_x)^2 = ay^2 + by + c.$

Differentiating both sides with respect to x and dividing by y'_x, we arrive at a constant coefficient linear equation: $2y_x^{(2n)} - 2\lambda y''_{xx} + 2ay + b = 0$.

52. $2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)}y_x^{(2n-m)} + (-1)^n\big[y_x^{(n)}\big]^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma.$

Differentiating both sides of the equation with respect to x, we have

$$y''_{xx}\big[2y_x^{(2n-1)} - \alpha x - \beta\big] = 0. \tag{1}$$

Equate the second factor to zero to obtain:

$$y = \frac{\alpha x^{2n}}{2(2n)!} + \frac{\beta x^{2n-1}}{2(2n-1)!} + \sum_{k=0}^{2n-2} C_k x^k.$$

The integration constants C_k and parameters α, β, and γ are related by

$$2\sum_{m=2}^{n-1}(-1)^m m!\,(2n-m)!\,C_m C_{2n-m} + (-1)^n(n!)^2C_n^2 = \beta C_1 - \alpha C_0 + \gamma,$$

which is obtained as a result of substituting the above solution into the original equation.

In addition, there is a solution corresponding to equating the first factor in (1) to zero: $y = \widetilde{C}_1 x + \widetilde{C}_0$, where $\beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0$.

53. $\displaystyle 2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n\bigl[y_x^{(n)}\bigr]^2 + s(y''_{xx})^2$
$$= \alpha(xy'_x - y) + \beta y'_x + \gamma, \qquad n \ge 3.$$

For the case $s=0$, see equation 17.2.1.52. Let now $s \neq 0$. Differentiating the equation with respect to x, we have

$$y''_{xx}\bigl[2y_x^{(2n-1)} + 2sy'''_{xxx} - \alpha x - \beta\bigr] = 0.$$

Equate the second factor to zero and integrate to obtain:

$$y = \frac{\alpha x^4}{48s} + \frac{\beta x^3}{12s} + C_2x^2 + C_1x + C_0 + \iiint w\,dx\,dx\,dx,$$

where $w=w(x)$ is the general solution of a linear constant coefficient linear equation of the form 17.1.2.2: $w_x^{(2n-4)} + sw = 0$. The constants of integration are related by the constraint that results from substituting the obtained solution into the original equation.

In addition, there is the solution $y = \widetilde{C}_1 x + \widetilde{C}_0$, where the constants of integration are related by $\beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0$.

54. $\displaystyle \sum_{m=1}^{n} a_m\biggl\{2\sum_{\nu=1}^{m-1}(-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + (-1)^m\bigl[y_x^{(m)}\bigr]^2\biggr\} = \alpha y^2 + 2\beta y + \gamma.$

Differentiating with respect to x, we arrive at a constant coefficient linear equation:

$$\sum_{m=1}^{n} a_m y_x^{(2m)} + \alpha y + \beta = 0.$$

17.2.2 Equations Containing Exponential Functions

▶ Fifth- and sixth-order equations.

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + be^{\lambda x}.$

1°. This is a special case of equation 17.2.6.1 with $f(x) = be^{\lambda x}$. Integrating the equation two times, we obtain a third-order equation: $y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}e^{\lambda x}$.

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{C\lambda^5 - b}{aC\lambda^4},$$
$$y = \frac{b}{2\lambda^5}\exp(\lambda x) + C\exp(-\lambda x) - \frac{\lambda}{a}.$$

2. $y_x^{(5)} = ae^{\lambda y} y'_x y''''_{xxxx}.$

Integrating yields a fourth-order autonomous equation of the form 17.2.6.8 with $n=4$: $y''''_{xxxx} = C\exp\Bigl(\dfrac{a}{\lambda}e^{\lambda y}\Bigr).$

3. $yy_x^{(5)} = ae^{\lambda x} + b.$

This is a special case of equation 17.2.6.17 with $n=2$ and $f(x) = ae^{\lambda x} + b$.

4. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = ae^{\lambda x}.$

Solution: $y^2 = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0 + 2a\lambda^{-5} e^{\lambda x}$.

5. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = ae^{\lambda y}.$

This is a special case of equation 17.2.6.3 with $f(y) = ae^{\lambda y}$.

6. $y_x^{(6)} = ae^{\lambda y} + b.$

This is a special case of equation 17.2.6.8 with $n = 6$ and $f(y) = ae^{\lambda y} + b$.

7. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = ae^{\lambda x}.$

This is a special case of equation 17.2.6.6 with $f(x) = ae^{\lambda x}$.

► **Equations of the form** $y_x^{(n)} = f(x, y)$.

8. $y_x^{(n)} = ae^{\lambda y} + b.$

This is a special case of equation 17.2.6.8 with $f(y) = ae^{\lambda y} + b$.

9. $y_x^{(n)} = ae^{\lambda y + \beta x} + b.$

This is a special case of equation 17.2.6.9 with $m = 1$. The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = ae^{\lambda w} + b$.

10. $y_x^{(n)} = ax^{-n} e^{\lambda y}.$

This is a special case of equation 17.2.6.11 with $f(y) = ae^{\lambda y}$.

11. $y_x^{(n)} = ax^k e^{\alpha y}.$

This is a special case of equation 17.2.6.26 with $f(w) = aw$ and $m = k + n$.

12. $y_x^{(n)} = Ae^{\alpha x} y^m.$

This is a special case of equation 17.2.2.23 with $m = m_1$ and $m_2 = m_3 = \cdots = m_n = 0$.

13. $yy_x^{(2n+1)} = ae^{\lambda x} + b.$

This is a special case of equation 17.2.6.17 with $f(x) = ae^{\lambda x} + b$.

14. $y_x^{(n)} = a\exp(\lambda y + \beta x^m) + b, \qquad m = 1,\ 2,\ \ldots,\ n-1.$

The substitution $w = y + (\beta/\lambda)x^m$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = ae^{\lambda w} + b$.

► **Other equations.**

15. $y_x^{(n)} = ae^{\lambda y} y_x' + be^{\beta x}.$

Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = \dfrac{a}{\lambda} e^{\lambda y} + \dfrac{b}{\beta} e^{\beta x} + C$.

16. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + be^{\lambda x}$.

1°. This is a special case of equation 17.2.6.54 with $f(x) = be^{\lambda x}$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1 x + C_2 + b\lambda^{-2}e^{\lambda x}$.

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{C\lambda^n - b}{aC\lambda^4} \qquad (n \text{ is any number}),$$

$$y = \frac{b}{2\lambda^n}\exp(\lambda x) + C\exp(-\lambda x) - \frac{\lambda^{n-4}}{a} \qquad (n \text{ is an odd number}).$$

17. $y_x^{(n)} = a^n y + be^{\lambda x}(y'_x - ay)^k$.

This is a special case of equation 17.2.6.38 with $f(x,w) = be^{\lambda x}w^k$.

18. $y_x^{(n)} = be^{\lambda x}(xy'_x - y)^k$.

This is a special case of equation 17.2.6.39 with $f(x,w) = be^{\lambda x}w^k$.

19. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = ae^{\lambda x}$.

This is a special case of equation 17.2.6.62 with $f(x) = ae^{\lambda x}$.

20. $y_x^{(n)} = ae^{\lambda y}y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = ae^{\lambda y}$.

21. $y_x^{(n)} = (ae^{\lambda y}y'_x + be^{\beta x})y_x^{(n-1)}$.

This is a special case of equation 17.2.6.58 with $f(y) = ae^{\lambda y}$ and $g(x) = be^{\beta x}$.

22. $y_x^{(n)} = ae^{\lambda y}y'_x\big[y_x^{(n-1)}\big]^m$.

This is a special case of equation 17.2.6.73 with $f(y) = ae^{\lambda y}$ and $g(w) = w^m$.

23. $y_x^{(n)} = Ae^{\alpha x}y^{m_1}(y'_x)^{m_2}\dots\big(y_x^{(n-1)}\big)^{m_n}$.

The substitution $w(x) = ye^{\beta x}$, where $\beta = \dfrac{\alpha}{m_1 + m_2 + \dots + m_n - 1}$, leads to an autonomous equation of the form 17.2.6.77.

24. $y_x^{(n)} = Ae^{\alpha y}x^{m_1}(y'_x)^{m_2}(y''_{xx})^{m_3}\dots\big(y_x^{(n-1)}\big)^{m_n}$.

The transformation $z = x^\sigma e^{\alpha y}$, $w = xy'_x$, where $\sigma = n + m_1 - m_2 - 2m_3 - 3m_4 - \dots - (n-1)m_n$, leads to an $(n-1)$st-order equation.

17.2.3 Equations Containing Hyperbolic Functions

▶ Equations with hyperbolic sine.

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sinh(\lambda x)$.

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\sinh(\lambda x)$. Integrating the equation two times, we obtain a third-order equation: $y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1 x + C_2 + b\lambda^{-2}\sinh(\lambda x)$.

2°. Particular solution: $y = \dfrac{b}{\lambda^4(\lambda^2 - a^2C^2)}\big[aC\sinh(\lambda x) + \lambda\cosh(\lambda x)\big] + C$.

2. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\sinh(\lambda x).$

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}\cosh(\lambda x).$

3. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\sinh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\sinh^m(\lambda x) + b$.

4. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\sinh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\sinh^m(\lambda y) + b$.

5. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = a\sinh^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\sinh^m(\lambda x)$.

6. $y_x^{(n)} = a\sinh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\sinh^m(\lambda y) + b$.

7. $y_x^{(n)} = ax^{-n}\sinh^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\sinh^m(\lambda y)$.

8. $yy_x^{(2n+1)} = a\sinh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\sinh^m(\lambda x) + b$.

9. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sinh(\lambda x).$

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\sinh(\lambda x)$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}\sinh(\lambda x)$.

2°. Particular solutions:

$$y = C\sinh(\lambda x) + \frac{C\lambda^n - b}{aC\lambda^4} \qquad \text{if } n \text{ is an even number,}$$

$$y = \frac{b}{\lambda^{2n-4} - a^2C^2\lambda^4}\big[aC\sinh(\lambda x) + \lambda^{n-4}\cosh(\lambda x)\big] + C \qquad \text{if } n \text{ is an odd number.}$$

10. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\sinh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\sinh^m(\lambda x) + b$.

11. $y_x^{(n)} = a\sinh^k(\lambda y) y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\sinh^k(\lambda y)$.

12. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\sinh(\lambda x) y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\sinh(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Big[\dfrac{a}{\lambda}\cosh(\lambda x) + C\Big]y$.

► **Equations with hyperbolic cosine.**

13. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cosh(\lambda x)$.

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\cosh(\lambda x)$. Integrating the equation twice yields the third-order equation

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}\cosh(\lambda x).$$

2°. Particular solution: $y = \dfrac{b}{\lambda^4(\lambda^2 - a^2C^2)}\bigl[aC\cosh(\lambda x) + \lambda\sinh(\lambda x)\bigr] + C.$

14. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cosh(\lambda x)$.

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}\sinh(\lambda x)$.

15. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cosh^m(\lambda x) + b$.

This is a special case of equation 17.2.6.2 with $f(x) = a\cosh^m(\lambda x) + b$.

16. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cosh^m(\lambda y) + b$.

This is a special case of equation 17.2.6.3 with $f(y) = a\cosh^m(\lambda y) + b$.

17. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\cosh^m(\lambda x)$.

This is a special case of equation 17.2.6.6 with $f(x) = a\cosh^m(\lambda x)$.

18. $y_x^{(n)} = a\cosh^m(\lambda y) + b$.

This is a special case of equation 17.2.6.8 with $f(y) = a\cosh^m(\lambda y) + b$.

19. $y_x^{(n)} = ax^{-n}\cosh^m(\lambda y)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\cosh^m(\lambda y)$.

20. $yy_x^{(2n+1)} = a\cosh^m(\lambda x) + b$.

This is a special case of equation 17.2.6.17 with $f(x) = a\cosh^m(\lambda x) + b$.

21. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cosh(\lambda x)$.

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\cosh(\lambda x)$. Integrating the equation twice yields the $(n-2)$nd-order equation

$$y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}\cosh(\lambda x).$$

2°. Particular solutions:

$$y = C\cosh(\lambda x) + \frac{C\lambda^n - b}{aC\lambda^4} \qquad \text{if } n \text{ is an even number,}$$

$$y = \frac{b}{\lambda^{2n-4} - a^2C^2\lambda^4}\bigl[aC\cosh(\lambda x) + \lambda^{n-4}\sinh(\lambda x)\bigr] + C \qquad \text{if } n \text{ is an odd number.}$$

22. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\cosh^m(\lambda x) + b$.

This is a special case of equation 17.2.6.62 with $f(x) = a\cosh^m(\lambda x) + b$.

23. $y_x^{(n)} = a\cosh^k(\lambda y)y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\cosh^k(\lambda y)$.

24. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\cosh(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\cosh(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Bigl[\dfrac{a}{\lambda}\sinh(\lambda x) + C\Bigr]y$.

▶ **Equations with hyperbolic tangent.**

25. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\tanh(\lambda x) + c.$

This is a special case of equation 17.2.6.1 with $f(x) = b\tanh(\lambda x) + c$.

26. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\tanh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\tanh^m(\lambda x) + b$.

27. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\tanh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\tanh^m(\lambda y) + b$.

28. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = a\tanh^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\tanh^m(\lambda x)$.

29. $y_x^{(n)} = a\tanh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\tanh^m(\lambda y) + b$.

30. $y_x^{(n)} = ax^{-n}\tanh^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\tanh^m(\lambda y)$.

31. $yy_x^{(2n+1)} = a\tanh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\tanh^m(\lambda x) + b$.

32. $y_x^{(2n)} = y + a(y'_x - y\tanh x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cosh x$.

33. $y_x^{(2n+1)} = y\tanh x + a(y'_x - y\tanh x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cosh x$.

34. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\tanh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\tanh^m(\lambda x) + b$.

35. $y_x^{(n)} = a\tanh^k(\lambda y) y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\tanh^k(\lambda y)$.

36. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\tanh(\lambda x) y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\tanh(\lambda x)$.

▶ **Equations with hyperbolic cotangent.**

37. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\coth(\lambda x) + c.$

This is a special case of equation 17.2.6.1 with $f(x) = b\coth(\lambda x) + c$.

38. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\coth^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\coth^m(\lambda x) + b$.

39. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\coth^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\coth^m(\lambda y) + b$.

40. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = a \coth^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a \coth^m(\lambda x)$.

41. $y_x^{(n)} = a \coth^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a \coth^m(\lambda y) + b$.

42. $y_x^{(n)} = ax^{-n} \coth^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a \coth^m(\lambda y)$.

43. $yy_x^{(2n+1)} = a \coth^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a \coth^m(\lambda x) + b$.

44. $y_x^{(2n)} = y + a(y_x' - y \coth x)^k.$

This is a special case of equation 17.2.6.47 with $f(x, u) = au^k$ and $\varphi(x) = \sinh x$.

45. $y_x^{(2n+1)} = y \coth x + a(y_x' - y \coth x)^k.$

This is a special case of equation 17.2.6.47 with $f(x, u) = au^k$ and $\varphi(x) = \sinh x$.

46. $(2n-1)yy_x^{(2n+1)} + (2n+1)y_x' y_x^{(2n)} = a \coth^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a \coth^m(\lambda x) + b$.

47. $y_x^{(n)} = a \coth^k(\lambda y) y_x' y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a \coth^k(\lambda y)$.

48. $yy_x^{(n)} - y_x' y_x^{(n-1)} = a \coth(\lambda x) y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a \coth(\lambda x)$.

17.2.4 Equations Containing Logarithmic Functions

► **Equations of the form $y_x^{(n)} = f(x, y)$.**

1. $y_x^{(n)} = a \ln^m(by) + c.$

This is a special case of equation 17.2.6.8 with $f(y) = a \ln^m(by) + c$.

2. $yy_x^{(2n+1)} = a \ln^m(bx).$

This is a special case of equation 17.2.6.17 with $f(x) = a \ln^m(bx)$.

3. $y_x^{(n)} = y(\alpha x + m \ln y + b).$

This is a special case of equation 17.2.6.25 with $f(w) = \ln w + b$.

4. $y_x^{(n)} = x^{-n}(\alpha y + m \ln x + b).$

This is a special case of equation 17.2.6.26 with $f(w) = \ln w + b$.

5. $y_x^{(n)} = ax^{-n} \ln^m(by).$

This is a special case of equation 17.2.6.11 with $f(y) = a \ln^m(by)$.

6. $y_x^{(n)} = ax^{-n-1}[\ln y + (1-n) \ln x].$

This is a special case of equation 17.2.6.13 with $f(w) = a \ln w$.

7. $y_x^{(n)} = ax^{-n-k}(\ln y + k\ln x).$

This is a special case of equation 17.2.6.15 with $f(w) = a\ln w$.

8. $y_x^{(n)} = ayx^{-n}(m\ln y + k\ln x).$

This is a special case of equation 17.2.6.16 with $f(w) = a\ln w$.

9. $y_x^{(2n)} = ax^{-\frac{2n+1}{2}}[2\ln y + (1-2n)\ln x].$

This is a special case of equation 17.2.6.14 with $f(w) = 2a\ln w$.

10. $y_x^{(n)} = (ax^2+c)^{-\frac{n+1}{2}}[2\ln y + (1-n)\ln(ax^2+c)].$

This is a special case of equation 17.2.6.22 with $b=0$ and $f(w) = 2\ln w$.

11. $y_x^{(n)} = be^{\alpha x}(\ln y - \alpha x).$

This is a special case of equation 17.2.6.24 with $f(w) = b\ln w$.

▶ **Other equations.**

12. $y_x^{(n)} = ay'_x\ln y + b\ln x.$

This is a special case of equation 17.2.6.34 with $f(y) = a\ln y$ and $g(x) = b\ln x$.

13. $y_x^{(n)} = a^n y + b\ln x\,(y'_x - ay)^k.$

This is a special case of equation 17.2.6.38 with $f(x,w) = bw^k\ln x$.

14. $y_x^{(n)} = a\ln x\,(xy'_x - y)^k.$

This is a special case of equation 17.2.6.39 with $f(x,w) = aw^k\ln x$.

15. $y_x^{(n)} = a\ln x\,(xy'_x - 2y)^k.$

This is a special case of equation 17.2.6.40 with $m=2$ and $f(x,w) = aw^k\ln x$.

16. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\ln x + c.$

This is a special case of equation 17.2.6.54 with $f(x) = b\ln x + c$.

17. $xy_x^{(n)} + ny_x^{(n-1)} = a\ln x + a\ln y.$

This is a special case of equation 17.2.6.59 with $f(w) = a\ln w$.

18. $x^2y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = 2a\ln x + a\ln y.$

This is a special case of equation 17.2.6.61 with $f(w) = a\ln w$.

19. $yy_x^{(n)} - y'_x y_x^{(n-1)} = ay^2\ln x.$

This is a special case of equation 17.2.6.64 with $f(w) = a\ln x$.

20. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\ln^m(bx) + c.$

This is a special case of equation 17.2.6.62 with $f(x) = a\ln^m(bx) + c$.

21. $y_x^{(n)} = a\ln^k(by)\,y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\ln^k(by)$.

22. $y_x^{(n)} = ay^m y'_x \ln y_x^{(n-1)}.$

This is a special case of equation 17.2.6.73 with $f(y) = ay^m$ and $g(w) = \ln w$.

17.2.5 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sin(\lambda x).$

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\sin(\lambda x)$. Integrating the equation twice, we obtain a third-order equation:

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1 x + C_2 - b\lambda^{-2}\sin(\lambda x).$$

2°. Particular solution: $y = -\dfrac{b}{\lambda^4(a^2C^2+\lambda^2)}\big[aC\sin(\lambda x) + \lambda\cos(\lambda x)\big] + C.$

2. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\sin(\lambda x).$

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 - 2a\lambda^{-5}\cos(\lambda x).$

3. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\sin^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\sin^m(\lambda x) + b$.

4. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\sin^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\sin^m(\lambda y) + b$.

5. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\sin^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\sin^m(\lambda x)$.

6. $y_x^{(n)} = a\sin^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\sin^m(\lambda y) + b$.

7. $y_x^{(n)} = ax^{-n}\sin^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\sin^m(\lambda y)$.

8. $yy_x^{(2n+1)} = a\sin^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\sin^m(\lambda x) + b$.

9. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sin(\lambda x).$

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\sin(\lambda x)$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\sin(\lambda x)$.

2°. Particular solutions:

$$y = C\sin(\lambda x) + \frac{(-1)^{n/2}C\lambda^n - b}{aC\lambda^4} \qquad \text{if } n \text{ is even,}$$

$$y = -\frac{b}{\lambda^{2n-4} + a^2C^2\lambda^4}\big[aC\sin(\lambda x) + (-1)^{\frac{n-1}{2}}\lambda^{n-4}\cos(\lambda x)\big] + C \qquad \text{if } n \text{ is odd.}$$

10. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\sin^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\sin^m(\lambda x) + b$.

11. $y_x^{(n)} = a\sin^k(\lambda y)y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\sin^k(\lambda y)$.

12. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\sin(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\sin(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Big[C - \dfrac{a}{\lambda}\cos(\lambda x)\Big]y$.

► **Equations with cosine.**

13. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cos(\lambda x)$.

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\cos(\lambda x)$. Integrating the equation twice, we obtain a third-order equation:

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\cos(\lambda x).$$

2°. Particular solution: $y = -\dfrac{b}{\lambda^4(a^2C^2+\lambda^2)}\big[aC\cos(\lambda x) - \lambda\sin(\lambda x)\big] + C$.

14. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cos(\lambda x)$.

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}\sin(\lambda x)$.

15. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cos^m(\lambda x) + b$.

This is a special case of equation 17.2.6.2 with $f(x) = a\cos^m(\lambda x) + b$.

16. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cos^m(\lambda y) + b$.

This is a special case of equation 17.2.6.3 with $f(y) = a\cos^m(\lambda y) + b$.

17. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\cos^m(\lambda x)$.

This is a special case of equation 17.2.6.6 with $f(x) = a\cos^m(\lambda x)$.

18. $y_x^{(n)} = a\cos^m(\lambda y) + b$.

This is a special case of equation 17.2.6.8 with $f(y) = a\cos^m(\lambda y) + b$.

19. $y_x^{(n)} = ax^{-n}\cos^m(\lambda y)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\cos^m(\lambda y)$.

20. $yy_x^{(2n+1)} = a\cos^m(\lambda x) + b$.

This is a special case of equation 17.2.6.17 with $f(x) = a\cos^m(\lambda x) + b$.

21. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cos(\lambda x)$.

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\cos(\lambda x)$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\cos(\lambda x)$.

2°. Particular solutions:

$$y = C\cos(\lambda x) + \frac{(-1)^{n/2}C\lambda^n - b}{aC\lambda^4} \qquad \text{if } n \text{ is even,}$$
$$y = -\frac{b}{\lambda^{2n-4} + a^2C^2\lambda^4}\Big[aC\cos(\lambda x) + (-1)^{\frac{n+1}{2}}\lambda^{n-4}\sin(\lambda x)\Big] + C \qquad \text{if } n \text{ is odd.}$$

22. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\cos^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\cos^m(\lambda x) + b$.

23. $y_x^{(n)} = a\cos^k(\lambda y)y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\cos^k(\lambda y)$.

24. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\cos(\lambda x)y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\cos(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Big[\frac{a}{\lambda}\sin(\lambda x) + C\Big]y$.

► **Equations with tangent.**

25. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\tan(\lambda x) + c.$

This is a special case of equation 17.2.6.1 with $f(x) = b\tan(\lambda x) + c$.

26. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\tan^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\tan^m(\lambda x) + b$.

27. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\tan^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\tan^m(\lambda y) + b$.

28. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\tan^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\tan^m(\lambda x)$.

29. $y_x^{(n)} = a\tan^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\tan^m(\lambda y) + b$.

30. $y_x^{(n)} = ax^{-n}\tan^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\tan^m(\lambda y)$.

31. $yy_x^{(2n+1)} = a\tan^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\tan^m(\lambda x) + b$.

32. $y_x^{(2n)} = (-1)^n y + a(y'_x + y\tan x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cos x$.

33. $y_x^{(2n+1)} = (-1)^{n+1}y\tan x + a(y'_x + y\tan x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cos x$.

34. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\tan^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\tan^m(\lambda x) + b$.

35. $y_x^{(n)} = a\tan^k(\lambda y)y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\tan^k(\lambda y)$.

36. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\tan(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\tan(\lambda x)$.

► **Equations with cotangent.**

37. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cot(\lambda x) + c$.

This is a special case of equation 17.2.6.1 with $f(x) = b\cot(\lambda x) + c$.

38. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cot^m(\lambda x) + b$.

This is a special case of equation 17.2.6.2 with $f(x) = a\cot^m(\lambda x) + b$.

39. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cot^m(\lambda y) + b$.

This is a special case of equation 17.2.6.3 with $f(y) = a\cot^m(\lambda y) + b$.

40. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\cot^m(\lambda x)$.

This is a special case of equation 17.2.6.6 with $f(x) = a\cot^m(\lambda x)$.

41. $y_x^{(n)} = a\cot^m(\lambda y) + b$.

This is a special case of equation 17.2.6.8 with $f(y) = a\cot^m(\lambda y) + b$.

42. $y_x^{(n)} = ax^{-n}\cot^m(\lambda y)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\cot^m(\lambda y)$.

43. $yy_x^{(2n+1)} = a\cot^m(\lambda x) + b$.

This is a special case of equation 17.2.6.17 with $f(x) = a\cot^m(\lambda x) + b$.

44. $y_x^{(2n)} = (-1)^n y + a(y'_x - y\cot x)^k$.

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \sin x$.

45. $y_x^{(2n+1)} = (-1)^n y\cot x + a(y'_x - y\cot x)^k$.

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \sin x$.

46. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\cot^m(\lambda x) + b$.

This is a special case of equation 17.2.6.62 with $f(x) = a\cot^m(\lambda x) + b$.

47. $y_x^{(n)} = a\cot^k(\lambda y)y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\cot^k(\lambda y)$.

48. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\cot(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\cot(\lambda x)$.

17.2.6 Equations Containing Arbitrary Functions

► **Fifth- and sixth-order equations.**

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + f(x).$

Integrating the equation two times, we obtain a third-order equation:

$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + \int_{x_0}^{x}(x-t)f(t)\,dt,$ where x_0 is an arbitrary number.

2. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = f(x).$

Solution:

$$y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + \frac{1}{12}\int_{x_0}^{x}(x-t)^4 f(t)\,dt,$$

where x_0 is an arbitrary number.

3. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = f(y).$

The substitution $w = y^2$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(5)} = 2f\big(\pm\sqrt{w}\,\big).$

4. $yy_x^{(5)} + ay'_x y''''_{xxxx} + (3a-5)y''_{xx}y'''_{xxx} = f(x).$

Integrating the equation three times, we obtain a second-order equation:

$$yy''_{xx} + \frac{a-3}{2}(y'_x)^2 = C_2x^2 + C_1x + C_0 + \frac{1}{2}\int_{x_0}^{x}(x-t)^2 f(t)\,dt,$$

where x_0 is an arbitrary number.

5. $(a+y)y_x^{(5)} + by'_x y''''_{xxxx} + cy''_{xx}y'''_{xxx} = f(x).$

Integrating yields a fourth-order equation:

$$(a+y)y''''_{xxxx} + (b-1)y'_x y'''_{xxx} + \frac{1}{2}(1-b+c)(y''_{xx})^2 = \int f(x)\,dx + C.$$

6. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = f(x).$

Solution: $y^2 = C_5x^5 + C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + \dfrac{1}{60}\displaystyle\int_{x_0}^{x}(x-t)^5 f(t)\,dt.$

7. $y_x^{(6)} = (ax^2+bx+c)^{-7/2} f\big(y(ax^2+bx+c)^{-5/2}\big).$

This is a special case of equation 17.2.6.22 with $n = 6$.

► **Equations of the form** $y_x^{(n)} = f(x,y)$.

8. $y_x^{(n)} = f(y).$

Autonomous equation. This is a special case of equation 17.2.6.77.

1°. The substitution $w(y) = y'_x$ leads to an $(n-1)$st-order equation.

2°. For even $n = 2m$, the first integral of the equation is:

$$\sum_{k=1}^{m-1}(-1)^k y_x^{(k)} y_x^{(2m-k)} + \tfrac{1}{2}(-1)^m\big[y_x^{(m)}\big]^2 + \int f(y)\,dy = C.$$

Furthermore, the order of the obtained equation can be reduced by one by the substitution $w(y) = y'_x$.

9. $y_x^{(n)} = f(y + ax^m), \qquad m = 0,\ 1,\ \ldots,\ n-1.$

The substitution $w = y + ax^m$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

10. $y_x^{(n)} = f\big(y + a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0\big).$

The substitution $w = y + a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = a_n n! + f(w)$.

11. $y_x^{(n)} = x^{-n} f(y).$

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 17.2.6.77.

12. $y_x^{(n)} = x^{1-n} f(y/x).$

Homogeneous equation. This is a special case of equation 17.2.6.83. The transformation $t = \ln x,\ w = y/x$ leads to an autonomous equation of the form 17.2.6.77.

13. $y_x^{(n)} = x^{-n-1} f(x^{1-n} y).$

The transformation $x = t^{-1},\ y = t^{1-n} w$ leads to an autonomous equation of the form 17.2.6.8: $w_t^{(n)} = (-1)^n f(w)$.

14. $y_x^{(2n)} = x^{-\frac{2n+1}{2}} f\Big(x^{\frac{1-2n}{2}} y\Big).$

The transformation $x = e^t,\ y = x^{\frac{2n-1}{2}} w(t)$ leads to an autonomous equation of the form 17.2.6.68, whose order can be reduced by two.

15. $y_x^{(n)} = x^{-n-k} f(yx^k).$

This is a special case of equation 17.2.6.86.

1°. The transformation $t = \ln x,\ z = yx^k$ leads to an autonomous equation of the form 17.2.6.77.

2°. The transformation $z = yx^k,\ w = xy'_x/y$ leads to an $(n-1)$st-order equation.

16. $y_x^{(n)} = yx^{-n} f(x^k y^m).$

This is a special case of equation 17.2.6.89. The transformation $t = x^k y^m,\ w = xy'_x/y$ leads to an $(n-1)$st-order equation.

17. $yy_x^{(2n+1)} = f(x).$

Integrating yields a $2n$th-order equation:

$$2\sum_{m=0}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n \big[y_x^{(n)}\big]^2 = 2\int f(x)\,dx + C,$$

where the notation $y_x^{(0)} \equiv y$ is used.

18. $y_x^{(n)} = f(x, y).$

The transformation $x = z^{-1},\ y = z^{1-n} w(z)$ leads to an equation of the same form: $w_z^{(n)} = (-1)^n z^{-n-1} f(z^{-1},\ z^{1-n} w)$.

19. $y_x^{(n)} = (ax + by + c)^{1-n} f\left(\frac{ax + by + c}{\alpha x + \beta y + \gamma}\right).$

1°. For $a\beta - b\alpha = 0$, the substitution $bw = ax + by + c$ leads to an autonomous equation of the form 17.2.6.8.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants which are determined by solving the linear algebraic system

$$\begin{aligned} ax_0 + by_0 + c &= 0, \\ \alpha x_0 + \beta y_0 + \gamma &= 0, \end{aligned}$$

leads to a homogeneous equation of the form 17.2.6.12:

$$w_z^{(n)} = z^{1-n} F\left(\frac{w}{z}\right), \qquad \text{where} \quad F(\xi) = (a + b\xi)^{1-n} f\left(\frac{a + b\xi}{\alpha + \beta\xi}\right).$$

20. $y_x^{(n)} = (a_1 x + b_1 y + c_1)^{1-n} f\left(\frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}\right).$

Suppose the following condition holds: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

For $a_2 b_3 - a_3 b_2 \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system

$$\begin{aligned} a_2 x_0 + b_2 y_0 + c_2 &= 0, \\ a_3 x_0 + b_3 y_0 + c_3 &= 0, \end{aligned}$$

leads to a homogeneous equation of the form 17.2.6.12:

$$w_z^{(n)} = z^{1-n} F\left(\frac{w}{z}\right), \qquad \text{where} \quad F(\xi) = (a_1 + b_1\xi)^{1-n} f\left(\frac{a_2 + b_2\xi}{a_3 + b_3\xi}\right).$$

21. $(ax + b)^n (cx + d) y_x^{(n)} = f\left(\frac{y}{(cx + d)^{n-1}}\right).$

The transformation $\xi = \ln\left|\frac{ax + b}{cx + d}\right|$, $w = \frac{y}{(cx + d)^{n-1}}$ leads to an autonomous equation of the form 17.2.6.77.

22. $y_x^{(n)} = (ax^2 + bx + c)^{-\frac{1+n}{2}} f\left(y(ax^2 + bx + c)^{\frac{1-n}{2}}\right).$

1°. The transformation

$$t = \int \frac{dx}{ax^2 + bx + c}, \qquad w = y(ax^2 + bx + c)^{\frac{1-n}{2}} \tag{1}$$

leads to an autonomous equation with respect to $w = w(t)$, which admits reduction of order by the substitution $z(w) = w_t'$.

2°. Let $n=2m$ be an even integer ($m=1,\,2,\,3,\,\dots$). In this case, transformation (1) yields an equation of the form 17.2.6.68, whose order can be reduced by two.

Setting $P=ax^2+bx+c$, $y=wP^{\frac{2m-1}{2}}$ and multiplying both sides of the original equation by $w'_x=P^{-\frac{1+2m}{2}}\Bigl(Py'_x+\dfrac{1-2m}{2}P'_xy\Bigr)$, we obtain

$$\Bigl(Py'_x+\frac{1-2m}{2}P'_xy\Bigr)y^{(2m)}_x=f(w)w'_x.$$

Integrating both sides of this equality with respect to x (the left-hand side is integrated by parts), we have

$$\sum_{k=0}^{m-2}(-1)^k\psi^{(k)}_x y^{(2m-1-k)}_x+(-1)^{m-1}\int\psi^{(m-1)}_x y^{(m+1)}_x\,dx=\int f(w)\,dw+C,\qquad (2)$$

where

$$\psi^{(k)}_x=\frac{d^k}{dx^k}\Bigl(Py'_x+\frac{1-2m}{2}P'_xy\Bigr)=Py^{(k+1)}_x+\Bigl(k-m+\frac12\Bigr)P'_xy^{(k)}_x+ak(k-2m)y^{(k-1)}_x$$

(remember that $n=2m$). It can be shown that the integrand on the left-hand side of (2) is a total differential. Finally, we arrive at the first integral

$$\begin{aligned}&\sum_{k=0}^{m-2}(-1)^k\bigl[Py^{(k+1)}_x+\bigl(k-m+\tfrac12\bigr)P'_xy^{(k)}_x+ak(k-2m)y^{(k-1)}_x\bigr]y^{(2m-1-k)}_x\\&\quad+(-1)^{m-1}\Bigl\{\tfrac12P\bigl[y^{(m)}_x\bigr]^2-\tfrac12P'_xy^{(m-1)}_xy^{(m)}_x+a(1-m^2)y^{(m-2)}_xy^{(m)}_x+\tfrac12am^2\bigl[y^{(m-1)}_x\bigr]^2\Bigr\}\\&\quad=\int f(w)\,dw+C.\end{aligned}$$

23. $\boldsymbol{y^{(n)}_x=y^{\frac{1+n}{1-n}}f\Bigl(y(ax^2+bx+c)^{\frac{1-n}{2}}\Bigr).}$

1°. Setting $f(u)=u^{\frac{n+1}{n-1}}f_1(u)$, we have equation 17.2.6.22 with the function f_1 (instead of f).

2°. The transformation $x=z^{-1}$, $y=z^{1-n}w(z)$ leads to an equation of similar form: $w^{(n)}_z=(-1)^nw^{\frac{1+n}{1-n}}f\Bigl(w(cz^2+bz+a)^{\frac{1-n}{2}}\Bigr)$.

24. $\boldsymbol{y^{(n)}_x=e^{\alpha x}f(ye^{-\alpha x}).}$

The substitution $w(x)=ye^{-\alpha x}$ leads to an autonomous equation of the form 17.2.6.77.

25. $\boldsymbol{y^{(n)}_x=yf(e^{\alpha x}y^m).}$

The transformation $z=e^{\alpha x}y^m$, $w(z)=y'_x/y$ leads to an $(n-1)$st-order equation.

26. $\boldsymbol{y^{(n)}_x=x^{-n}f(x^me^{\alpha y}).}$

The transformation $z=x^me^{\alpha y}$, $w(z)=xy'_x$ leads to an $(n-1)$st-order equation.

27. $\boldsymbol{y^{(n)}_x=f(y+ae^{\lambda x})-a\lambda^ne^{\lambda x}.}$

The substitution $w(x)=y+ae^{\lambda x}$ leads to an autonomous equation of the form 17.2.6.8: $w^{(n)}_x=f(w)$.

28. $y_x^{(2n)} = f(y + a\cosh x) - a\cosh x.$

The substitution $w(x) = y + a\cosh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n)} = f(w)$.

29. $y_x^{(2n)} = f(y + a\sinh x) - a\sinh x.$

The substitution $w(x) = y + a\sinh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n)} = f(w)$.

30. $y_x^{(2n+1)} = f(y + a\cosh x) - a\sinh x.$

The substitution $w(x) = y + a\cosh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n+1)} = f(w)$.

31. $y_x^{(2n+1)} = f(y + a\sinh x) - a\cosh x.$

The substitution $w(x) = y + a\sinh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n+1)} = f(w)$.

32. $y_x^{(n)} = f(y + a\cos x) - a\cos\left(x + \frac{1}{2}\pi n\right).$

The substitution $w(x) = y + a\cos x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

33. $y_x^{(n)} = f(y + a\sin x) - a\sin\left(x + \frac{1}{2}\pi n\right).$

The substitution $w(x) = y + a\sin x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

► **Equations of the form** $y_x^{(n)} = f(x, y, y'_x)$.

34. $y_x^{(n)} = f(y)y'_x + g(x).$

Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = \int f(y)\,dy + \int g(x)\,dx + C.$

35. $y_x^{(n)} = f(x, y'_x).$

The substitution $w(x) = y'_x$ leads to an $(n-1)$st-order equation: $w_x^{(n-1)} = f(x, w)$.

36. $y_x^{(n)} = f(y, y'_x).$

Autonomous equation. This is a special case of equation 17.2.6.77.

The substitution $w(y) = y'_x$ leads to an $(n-1)$st-order equation.

37. $y_x^{(n)} = yx^{-n} f(xy'_x/y).$

The transformation $z = xy'_x/y$, $w = x^2 y''_{xx}/y$ leads to an $(n-2)$nd-order equation.

38. $y_x^{(n)} = a^n y + f(x, y'_x - ay).$

The substitution $w = y'_x - ay$ leads to an $(n-1)$st-order equation:

$$w_x^{(n-1)} + aw_x^{(n-2)} + \cdots + a^{n-1}w = f(x, w).$$

39. $y_x^{(n)} = f(x, xy'_x - y).$

The substitution $w = xy'_x - y$ leads to an $(n-1)$st-order equation: $\dfrac{d^{n-2}}{dx^{n-2}}\left(\dfrac{w'_x}{x}\right) = f(x, w)$.

40. $y_x^{(n)} = f(x, xy'_x - my)$.

Here, m is a positive integer and $n \geq m+1$. The substitution $w = xy'_x - my$ leads to an $(n-1)$st-order equation: $\zeta_x^{(n-m-1)} = f(x, w)$, where $\zeta = w_x^{(m)}/x$.

41. $x^n y_x^{(n)} = f(x, xy'_x + ay) - (a)_n y$.

Here, $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol. The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order equation.

42. $y_x^{(n)} = f(x, P_m y'_x - P'_m y), \quad P_m = \sum_{k=0}^{m} a_k x^k, \ P'_m = \sum_{k=0}^{m} a_k k x^{k-1}, \ n > m.$

The substitution $w = P_m y'_x - P'_m y$ leads to an $(n-1)$st-order equation.

43. $y_x^{(2n)} = y + f(x, y'_x \cosh x - y \sinh x)$.

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a $(2n-1)$st-order equation.

44. $y_x^{(2n)} = y + f(x, y'_x \sinh x - y \cosh x)$.

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a $(2n-1)$st-order equation.

45. $y_x^{(2n)} = (-1)^n y + f(x, y'_x \sin x - y \cos x)$.

The substitution $w = y'_x \sin x - y \cos x$ leads to a $(2n-1)$st-order equation.

46. $y_x^{(2n)} = (-1)^n y + f(x, y'_x \cos x + y \sin x)$.

The substitution $w = y'_x \cos x + y \sin x$ leads to a $(2n-1)$st-order equation.

47. $y_x^{(n)} = \dfrac{\varphi_x^{(n)}}{\varphi} y + f\Big(x, y'_x - \dfrac{\varphi'_x}{\varphi} y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi} y$ leads to an $(n-1)$st-order equation.

► **Equations of the form** $y_x^{(n)} = f(x, y, y'_x, y''_{xx})$.

48. $y_x^{(n)} = f(x, xy'_x - y, y''_{xx})$.

This is a special case of equation 17.2.6.78. The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$st-order equation.

49. $y_x^{(n)} = f(x, x^2 y''_{xx} - 2xy'_x + 2y)$.

This is a special case of equation 17.2.6.81. The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to an $(n-2)$nd-order equation.

50. $y_x^{(2n)} = a^n y + f(x, y''_{xx} - ay)$.

The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order equation:

$$w_x^{(2n-2)} + a w_x^{(2n-4)} + \dots + a^{n-1} w = f(x, w).$$

51. $y_x^{(2n)} = y f(y y''_{xx} - y'^2_x)$.

This is a special case of equation 17.2.6.52.

52. $y_x^{(2n)} = y''_{xx} f\bigl(yy''_{xx} - y'^2_x\bigr) + yg\bigl(yy''_{xx} - y'^2_x\bigr).$

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$C_3^{2n} - C_3^2 f(4C_1C_2C_3^2) - g(4C_1C_2C_3^2) = 0.$$

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$(-1)^n C_3^{2n} + C_3^2 f(-C_1^2C_3^2 - C_2^2C_3^2) - g(-C_1^2C_3^2 - C_2^2C_3^2) = 0.$$

53. $y_x^{(n)} = y'_x f\Bigl(\dfrac{y''_{xx}}{y'_x},\ y'_x - y\dfrac{y''_{xx}}{y'_x}\Bigr).$

Particular solution: $y = C_1 \exp(C_2 x) + C_3$, where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $C_2^{n-1} = f(C_2, -C_2C_3)$.

► Equations of the form $f(x,y)y_x^{(n)}+g(x,y,y'_x)y_x^{(n-1)}=h\bigl(x,y,y'_x,\dots,y_x^{(n-2)}\bigr)$.

54. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + f(x).$

Integrating the equation two times, we obtain an $(n-2)$nd-order equation:

$$y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + \int_{x_0}^{x} (x-t)f(t)\,dt, \quad \text{where } x_0 \text{ is an arbitrary number.}$$

55. $y_x^{(2n)} = a^2 y + f\bigl(x, y_x^{(n)} + ay\bigr).$

The substitution $w = y_x^{(n)} + ay$ leads to an nth-order equation: $w_x^{(n)} = aw + f(x,w)$.

56. $y_x^{(n)} = f\bigl(y_x^{(n-2)}\bigr).$

Having set $u(x) = y_x^{(n-2)}$, we obtain a second-order equation $u''_{xx} = f(u)$, whose solution has the form:

$$x = \int \frac{du}{\varphi(u)} + C_2, \quad \text{where} \quad \varphi(u) = \pm\Bigl[C_1 + 2\int f(u)\,du\Bigr]^{1/2}.$$

Expressing u in terms of x and integrating the resulting relation $n-2$ times, we find y.

Solution in parametric form:

$$x = \int_{C_2}^{u} \frac{du}{\varphi(u)}, \quad y = \int_{C_3}^{u} \frac{du_1}{\varphi(u_1)} \int_{C_4}^{u_1} \frac{du_2}{\varphi(u_2)} \dots \int_{C_{n-1}}^{u_{n-4}} \frac{du_{n-3}}{\varphi(u_{n-3})} \int_{C_n}^{u_{n-3}} \frac{u_{n-2}\,du_{n-2}}{\varphi(u_{n-2})}.$$

57. $y_x^{(n)} = f(y)y'_x y_x^{(n-1)}.$

Integrating yields an $(n-1)$st-order autonomous equation of the form 17.2.6.8:

$$y_x^{(n-1)} = F(y), \quad \text{where} \quad F(y) = C\exp\Bigl[\int f(y)\,dy\Bigr].$$

58. $y_x^{(n)} = \bigl[f(y)y'_x + g(x)\bigr]y_x^{(n-1)}.$

Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = C\exp\Bigl[\int f(y)\,dy + \int g(x)\,dx\Bigr]$.

59. $xy_x^{(n)} + ny_x^{(n-1)} = f(xy)$.

The substitution $w(x) = xy$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

60. $xy_x^{(n)} + (a+n-1)y_x^{(n-1)} = f(x, xy'_x + ay)$.

The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order equation: $w_x^{(n-1)} = f(x,w)$.

61. $x^2y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = f(x^2y)$.

The substitution $w(x) = x^2y$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

62. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = f(x)$.

Having integrated the equation, we obtain

$$(2n-1)yy_x^{(2n)} + 2\sum_{k=1}^{n-1}(-1)^{k+1}y_x^{(k)}y_x^{(2n-k)} + (-1)^{n+1}\big[y_x^{(n)}\big]^2 = \int f(x)\,dx + 2C_2.$$

The second integration leads to a $(2n-1)$st-order equation:

$$\sum_{k=0}^{n-1}(2n-1-2k)(-1)^k y_x^{(k)}y_x^{(2n-1-k)} = 2C_2x + C_1 + \int_{x_0}^{x}(x-t)f(t)\,dt.$$

The third integration leads to a $(2n-2)$nd-order equation:

$$\sum_{k=0}^{n-2}(k+1)(2n-k-1)(-1)^k y_x^{(k)}y_x^{(2n-2-k)} + \frac{1}{2}(-1)^{n-1}n^2\big[y_x^{(n-1)}\big]^2 = C_2x^2 + C_1x + C_0 + \frac{1}{2}\int_{x_0}^{x}(x-t)^2 f(t)\,dt.$$

63. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = f(y)y'_x + g(x)$.

Integrating yields an $(n-1)$st-order equation:

$$(2n-1)yy_x^{(2n)} + 2\sum_{k=1}^{n-1}(-1)^{k+1}y_x^{(k)}y_x^{(2n-k)} + (-1)^{n+1}\big[y_x^{(n)}\big]^2 = \int f(y)\,dy + \int g(x)\,dx + C.$$

64. $yy_x^{(n)} - y'_x y_x^{(n-1)} = f(x)y^2$.

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Big[\int f(x)\,dx + C\Big]y$.

65. $yy_x^{(n)} = y'_x y_x^{(n-1)} + f(x)yy_x^{(n-1)}$.

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = C\exp\Big[\int f(x)\,dx\Big]y$.

66. $yy_x^{(n)} + (f-1)y'_x y_x^{(n-1)} + fgyy'_x + g'_x y^2 = 0, \qquad f = f(x), \quad g = g(x)$.

This equation is solved by the functions that are solutions of the $(n-1)$st-order linear equation $y_x^{(n-1)} + g(x)y = 0$.

67. $[y + f(x)]y_x^{(n)} = [y'_x + f'_x(x)]y_x^{(n-1)} + af(x)y'_x - af'_x(x)y$.

Integrating yields an $(n-1)$st-order constant coefficient nonhomogeneous linear equation: $y_x^{(n-1)} - Cy = (C-a)f(x)$. There is also the trivial solution $y = 0$.

68. $\displaystyle\sum_{m=1}^{n} a_m y_x^{(2m)} = f(y).$

The first integral has the form:

$$\sum_{m=1}^{n} a_m \left\{ \sum_{\nu=1}^{m-1} (-1)^{\nu} y_x^{(\nu)} y_x^{(2m-\nu)} + \frac{1}{2}(-1)^m \left[y_x^{(m)}\right]^2 \right\} + \int f(y)\,dy = C,$$

where C is an arbitrary constant. Furthermore, the order of the obtained equation can be reduced by one by the substitution $w(y) = y'_x$.

69. $\displaystyle\sum_{m=1}^{n} a_m x^m y_x^{(m)} = f(y).$

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 17.2.6.77.

70. $\displaystyle y \sum_{m=0}^{n} a_m y_x^{(2m+1)} = f(x).$

Integrating yields a $2n$th-order equation:

$$\sum_{m=0}^{n} a_m \left\{ 2 \sum_{\nu=0}^{m-1} (-1)^{\nu} y_x^{(\nu)} y_x^{(2m-\nu)} + (-1)^m \left[y_x^{(m)}\right]^2 \right\} = 2 \int f(x)\,dx + C,$$

where $y_x^{(0)}$ stands for y.

71. $\displaystyle\sum_{m=0}^{n} a_m y_x^{(m)} y_x^{(2n+1-m)} = f(x).$

The first integral has the form:

$$2 \sum_{m=0}^{n-1} A_m y_x^{(m)} y_x^{(2n-m)} + A_n \left[y_x^{(n)}\right]^2 = 2 \int f(x)\,dx + C,$$

where

$$A_m = \sum_{k=0}^{m} (-1)^{m+k} a_k = a_m - a_{m-1} + a_{m-2} - \cdots .$$

If the condition

$$A_n = 2 \sum_{m=0}^{n-1} (-1)^{n-1+m} A_m$$

is satisfied, the obtained equation can be integrated two times more (in particular, see equation 17.2.6.62).

► Equations of the form $y_x^{(n)} = f\big(x, y, y'_x, \ldots, y_x^{(n-1)}\big)$.

72. $y_x^{(n)} = f\big(y_x^{(n-1)}\big).$

Having set $u(x) = y_x^{(n-1)}$, we obtain a first-order equation $u'_x = f(u)$. Further, find u from the relation $x = \displaystyle\int \frac{du}{f(u)} + C_1$. Then the $(n-1)$-fold integration yields y.

Solution in parametric form:

$$x = \int_{C_1}^{u} \frac{du}{f(u)}, \qquad y = \int_{C_2}^{u} \frac{du_1}{f(u_1)} \int_{C_3}^{u_1} \frac{du_2}{f(u_2)} \cdots \int_{C_{n-1}}^{u_{n-3}} \frac{du_{n-2}}{f(u_{n-2})} \int_{C_n}^{u_{n-2}} \frac{u_{n-1}\,du_{n-1}}{f(u_{n-1})}.$$

73. $\boldsymbol{y_x^{(n)} = f(y)y'_x g\big(y_x^{(n-1)}\big)}$.

Integrating yields an $(n-1)$st-order equation:

$$\int \frac{dw}{g(w)} = \int f(y)\,dy + C, \qquad \text{where} \quad w = y_x^{(n-1)}.$$

Furthermore, the order of this equation can be reduced by one by the substitution $z(y) = y'_x$.

74. $\boldsymbol{y_x^{(n)} = \big[f(y)y'_x + g(x)\big]h(y_x^{(n-1)})}$.

Integrating yields an $(n-1)$st-order equation:

$$\int \frac{dw}{h(w)} = \int f(y)\,dy + \int g(x)\,dx + C, \qquad w = y_x^{(n-1)}.$$

75. $\boldsymbol{y_x^{(n)} = f\big(x,\, y_x^{(n-2)},\, y_x^{(n-1)}\big)}$.

The substitution $w(x) = y_x^{(n-2)}$ leads to a second-order equation: $w''_{xx} = f(x, w, w'_x)$.

► **Equations of the general form** $\boldsymbol{F\big(x, y, y'_x, \ldots, y_x^{(n)}\big) = 0}$.

76. $\boldsymbol{F\big(x,\, y'_x,\, y''_{xx},\, \ldots,\, y_x^{(n)}\big) = 0}$.

The equation does not depend on y explicitly. Hence, the substitution $w(x) = y'_x$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, w'_x,\, \ldots,\, w_x^{(n-1)}\big) = 0.$$

77. $\boldsymbol{F\big(y,\, y'_x,\, y''_{xx},\, \ldots,\, y_x^{(n)}\big) = 0}$.

Autonomous equation. It does not depend on x explicitly. The substitution $w(y) = y'_x$ leads to an $(n-1)$st-order equation. The derivatives of the original equation and the transformed one are related by

$$y''_{xx} = ww'_y, \quad y'''_{xxx} = w^2 w''_{yy} + w(w'_y)^2, \quad \ldots, \quad y_x^{(n)} = w\big(y_x^{(n-1)}\big)'_y.$$

78. $\boldsymbol{F\big(x,\, xy'_x - y,\, y''_{xx},\, y'''_{xxx},\, \ldots,\, y_x^{(n)}\big) = 0}$.

The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta'_x,\, \ldots,\, \zeta_x^{(n-2)}\big) = 0, \quad \text{where} \quad \zeta = w'_x/x.$$

79. $\boldsymbol{F\big(x,\, xy'_x - 2y,\, y'''_{xxx},\, y''''_{xxxx},\, \ldots,\, y_x^{(n)}\big) = 0}$.

The substitution $w = xy'_x - 2y$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta'_x,\, \ldots,\, \zeta_x^{(n-3)}\big) = 0, \quad \text{where} \quad \zeta = w''_{xx}/x.$$

80. $\boldsymbol{F\big(x,\, xy'_x - my,\, y_x^{(m+1)},\, y_x^{(m+2)},\, \ldots,\, y_x^{(n)}\big) = 0,}$
$\boldsymbol{n \ge m+1, \quad m = 1,\, \ldots,\, n-1.}$

The substitution $w = xy'_x - my$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta'_x,\, \ldots,\, \zeta_x^{(n-m-1)}\big) = 0, \quad \text{where} \quad \zeta = w_x^{(m)}/x.$$

81. $\boldsymbol{F\big(x,\, x^2 y''_{xx} - 2xy'_x + 2y,\, y'''_{xxx},\, \ldots,\, y_x^{(n)}\big) = 0}$.

The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to an $(n-2)$nd-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta'_x,\, \ldots,\, \zeta_x^{(n-3)}\big) = 0, \quad \text{where} \quad \zeta = x^{-2} w'_x.$$

82. $\displaystyle\sum_{k=0}^{m}(-1)^k k!\,C_m^k x^{m-k} y_x^{(m-k)} = F\bigl(x,\, y_x^{(m+1)},\, \dots,\, y_x^{(n)}\bigr).$

Here, $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

The substitution $w(x) = \sum\limits_{k=0}^{m}(-1)^k k!\,C_m^k x^{m-k} y_x^{(m-k)}$ leads to an $(n-m)$th-order equation; the derivatives on the right-hand side are calculated in consecutive manner using the formula $y_x^{(m+1)} = x^{-m} w'_x$.

83. $F\Bigl(\dfrac{y}{x},\, y'_x,\, xy''_{xx},\, \dots,\, x^{n-1} y_x^{(n)}\Bigr) = 0.$

Homogeneous equation. The transformation $t = \ln x$, $w = y/x$ leads to an autonomous equation of the form 17.2.6.77.

84. $F\Bigl(\dfrac{ax+by+c}{\alpha x+\beta y+\gamma},\, y'_x,\, \dots,\, (ax+by+c)^{n-1} y_x^{(n)}\Bigr) = 0.$

1°. For $a\beta - b\alpha = 0$, the substitution $bw = ax+by+c$ leads to an autonomous equation of the form 17.2.6.77.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system

$$ax_0 + by_0 + c = 0, \qquad \alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to a homogeneous equation of the form 17.2.6.83:

$$F\Bigl(\frac{a+bw/z}{\alpha+\beta w/z},\, w'_z,\, \dots,\, (a+bw/z)^{n-1} z^{n-1} w_z^{(n)}\Bigr) = 0.$$

85. $F\Bigl(\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2},\, y'_x,\, \dots,\, (a_3x+b_3y+c_3)^{n-1} y_x^{(n)}\Bigr) = 0.$

Let the following condition hold: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

For $a_1b_2 - a_2b_1 \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system

$$a_1x_0 + b_1y_0 + c_1 = 0, \qquad a_2x_0 + b_2y_0 + c_2 = 0,$$

leads to a homogeneous equation of the form 17.2.6.83:

$$F\Bigl(\frac{a_1+b_1w/z}{a_2+b_2w/z},\, w'_z,\, \dots,\, (a_3+b_3w/z)^{n-1} z^{n-1} w_z^{(n)}\Bigr) = 0.$$

86. $F\bigl(x^k y,\, x^{k+1} y'_x,\, \dots,\, x^{k+n} y_x^{(n)}\bigr) = 0.$

Generalized homogeneous equation. The transformation $t = \ln x$, $w = x^k y$ leads to an autonomous equation of the form 17.2.6.77.

87. $F\left(\dfrac{xy'_x}{y}, \dfrac{x^2y''_{xx}}{y}, \dots, \dfrac{x^ny^{(n)}_x}{y}\right) = 0.$

Generalized homogeneous equation. The transformation $z = xy'_x/y$, $w = x^2y''_{xx}/y$ leads to an $(n-2)$nd-order equation.

88. $F\left(y'_x - y\dfrac{y''_{xx}}{y'_x}, \dfrac{y''_{xx}}{y'_x}, \dfrac{y'''_{xxx}}{y'_x}, \dots, \dfrac{y^{(n)}_x}{y'_x}\right) = 0.$

Autonomous equation. Particular solution: $y = C_1 \exp(C_2x) + C_3$, where C_1 is an arbitrary constant and the constants C_2 and C_3 are related by $F(-C_2C_3, C_2, C_2^2, \dots, C_2^{n-1}) = 0$.

89. $F\left(x^ky^m, \dfrac{xy'_x}{y}, \dfrac{x^2y''_{xx}}{y}, \dots, \dfrac{x^ny^{(n)}_x}{y}\right) = 0.$

Generalized homogeneous equation. The transformation $t = x^ky^m$, $z = xy'_x/y$ leads to an $(n-1)$st-order equation.

90. $F\left(\dfrac{y^{(n)}_x}{y'_x}, y\dfrac{y^{(n)}_x}{y'_x} - y^{(n-1)}_x\right) = 0.$

A solution of this equation is any function that satisfies the $(n-1)$st-order constant coefficient linear equation $y^{(n-1)}_x = C_1y + C_2$, where the constants C_1 and C_2 are related by the constraint $F(C_1, -C_2) = 0$.

91. $F\left(\dfrac{y^{(n)}_x}{y^{(k)}_x}, x^{1-k}y\dfrac{y^{(n)}_x}{y^{(k)}_x} - x^{1-k}y^{(n-k)}_x\right) = 0, \qquad n > k.$

A solution of this equation is any function that satisfies the $(n-k)$th-order linear equation $y^{(n-k)}_x = C_1y + C_2x^{k-1}$, where the constants C_1 and C_2 are related by $F(C_1, -C_2) = 0$.

92. $F\big(x, y^{(n)}_x - y, y^{(m)}_x - y\big) = 0.$

The substitution $w = y'_x - y$ reduces the order of the equation by one.

93. $F\big(y^{(n)}_x - y, y^{(2n)}_x - y, y^{(2n)}_x - y^{(n)}_x\big) = 0.$

The substitution $u = y^{(n)}_x - y$ leads to an nth-order autonomous equation of the form $F\big(u, u^{(n)}_x + u, u^{(n)}_x\big) = 0$.

94. $F\big(x, y^{(n)}_x + ay, y^{(2n)}_x - a^2y, y^{(2n)}_x + ay^{(n)}_x\big) = 0.$

The substitution $u = y^{(n)}_x + ay$ leads to an nth-order equation $F\big(x, u, u^{(n)}_x - au, u^{(n)}_x\big) = 0$.

95. $F\big(e^{\alpha x}y, e^{\alpha x}y'_x, e^{\alpha x}y''_{xx}, \dots, e^{\alpha x}y^{(n)}_x\big) = 0.$

Equation invariant under "translation–dilatation" transformation. The substitution $u = e^{\alpha x}y$ leads to an autonomous equation of the form 17.2.6.77.

96. $F\left(e^{\alpha x}y^m, \dfrac{y'_x}{y}, \dfrac{y''_{xx}}{y}, \dots, \dfrac{y^{(n)}_x}{y}\right) = 0.$

Equation invariant under "translation–dilatation" transformation. The transformation $z = e^{\alpha x}y^m$, $w = y'_x/y$ leads to an $(n-1)$st-order equation. See also Section 5.2.4 (the first paragraph).

97. $F\big(x^me^{\alpha y}, xy'_x, x^2y''_{xx}, \dots, x^ny^{(n)}_x\big) = 0.$

Equation invariant under "dilatation–translation" transformation. The transformation $z = x^me^{\alpha y}$, $w = xy'_x$ leads to an $(n-1)$st-order equation. See also Section 5.2.4 (the second paragraph).

Chapter 18

Some Systems of Ordinary Differential Equations

18.1 Linear Systems of Two Equations

18.1.1 Systems of First-Order Equations

1. $\boldsymbol{x'_t = ax + by, \quad y'_t = cx + dy.}$

System of two constant-coefficient first-order linear homogeneous differential equations.

Let us write out the characteristic equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0 \tag{1}$$

and find its discriminant

$$D = (a-d)^2 + 4bc. \tag{2}$$

1°. Case $ad - bc \neq 0$. The origin of coordinates $x = y = 0$ is the only one stationary point; it is

$$\begin{aligned} &\text{a node if } \ D = 0;\\ &\text{a node if } \ D > 0 \ \text{ and } \ ad - bc > 0;\\ &\text{a saddle if } \ D > 0 \ \text{ and } \ ad - bc < 0;\\ &\text{a focus if } \ D < 0 \ \text{ and } \ a + d \neq 0;\\ &\text{a center if } \ D < 0 \ \text{ and } \ a + d = 0. \end{aligned}$$

1.1. Suppose $D > 0$. The characteristic equation (1) has two distinct real roots, λ_1 and λ_2. The general solution of the original system of differential equations is expressed as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t},$$
$$y = C_1(\lambda_1 - a)e^{\lambda_1 t} + C_2(\lambda_2 - a)e^{\lambda_2 t},$$

where C_1 and C_2 are arbitrary constants.

1.2. Suppose $D < 0$. The characteristic equation (1) has two complex conjugate roots, $\lambda_{1,2} = \sigma \pm i\beta$. The general solution of the original system of differential equations is given by

$$x = be^{\sigma t}\big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\big],$$
$$y = e^{\sigma t}\big\{[(\sigma - a)C_1 - \beta C_2]\sin(\beta t) + [\beta C_1 + (\sigma - a)C_2]\cos(\beta t)\big\},$$

where C_1 and C_2 are arbitrary constants.

1.3. Suppose $D = 0$ and $a \neq d$. The characteristic equation (1) has two equal real roots, $\lambda_1 = \lambda_2$. The general solution of the original system of differential equations is

$$x = 2b\left(C_1 + \frac{C_2}{a-d} + C_2 t\right)\exp\left(\frac{a+d}{2}t\right),$$
$$y = [(d-a)C_1 + C_2 + (d-a)C_2 t]\exp\left(\frac{a+d}{2}t\right),$$

where C_1 and C_2 are arbitrary constants.

1.4. Suppose $a = d \neq 0$ and $b = 0$. Solution:

$$x = C_1 e^{at}, \qquad y = (cC_1 t + C_2)e^{at}.$$

1.5. Suppose $a = d \neq 0$ and $c = 0$. Solution:

$$x = (bC_1 t + C_2)e^{at}, \qquad y = C_1 e^{at}.$$

2°. Case $ad - bc = 0$ and $a^2 + b^2 > 0$. The whole of the line $ax + by = 0$ consists of singular points. The system in question may be rewritten in the form

$$x'_t = ax + by, \qquad y'_t = k(ax + by).$$

2.1. Suppose $a + bk \neq 0$. Solution:

$$x = bC_1 + C_2 e^{(a+bk)t}, \qquad y = -aC_1 + kC_2 e^{(a+bk)t}.$$

2.2. Suppose $a + bk = 0$. Solution:

$$x = C_1(bkt - 1) + bC_2 t, \qquad y = k^2 bC_1 t + (bk^2 t + 1)C_2.$$

2. $\boldsymbol{x'_t = a_1x + b_1y + c_1, \qquad y'_t = a_2x + b_2y + c_2.}$

The general solution of this system is given by the sum of any one of its particular solutions and the general solution of the corresponding homogeneous system (see system 18.1.1.1).

1°. Suppose $a_1b_2 - a_2b_1 \neq 0$. A particular solution:

$$x = x_0, \qquad y = y_0,$$

where the constants x_0 and y_0 are determined by solving the linear algebraic system of equations

$$a_1x_0 + b_1y_0 + c_1 = 0, \qquad a_2x_0 + b_2y_0 + c_2 = 0.$$

2°. Suppose $a_1b_2 - a_2b_1 = 0$ and $a_1^2 + b_1^2 > 0$. Then the original system can be rewritten as

$$x'_t = ax + by + c_1, \qquad y'_t = k(ax + by) + c_2.$$

2.1. If $\sigma = a + bk \neq 0$, the original system has a particular solution of the form

$$x = b\sigma^{-1}(c_1k - c_2)t - \sigma^{-2}(ac_1 + bc_2), \qquad y = kx + (c_2 - c_1k)t.$$

2.2. If $\sigma = a + bk = 0$, the original system has a particular solution of the form

$$x = \tfrac{1}{2}b(c_2 - c_1k)t^2 + c_1t, \qquad y = kx + (c_2 - c_1k)t.$$

3. $\boldsymbol{x'_t = f(t)x + g(t)y, \qquad y'_t = g(t)x + f(t)y.}$

Solution:

$$x = e^F(C_1e^G + C_2e^{-G}), \qquad y = e^F(C_1e^G - C_2e^{-G}),$$

where C_1 and C_2 are arbitrary constants, and

$$F = \int f(t)\,dt, \qquad G = \int g(t)\,dt.$$

4. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = -g(t)x + f(t)y.}$

Solution:

$$x = F(C_1 \cos G + C_2 \sin G), \qquad y = F(-C_1 \sin G + C_2 \cos G),$$

where C_1 and C_2 are arbitrary constants, and

$$F = \exp\left[\int f(t)\,dt\right], \qquad G = \int g(t)\,dt.$$

5. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = ag(t)x + [f(t) + bg(t)]y.}$

The transformation

$$x = \exp\left[\int f(t)\,dt\right]u, \qquad y = \exp\left[\int f(t)\,dt\right]v, \qquad \tau = \int g(t)\,dt$$

leads to a system of constant coefficient linear differential equations of the form 18.1.1.1:

$$u'_\tau = v, \qquad v'_\tau = au + bv.$$

6. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = a[f(t) + ah(t)]x + a[g(t) - h(t)]y.}$

Let us multiply the first equation by $-a$ and add it to the second equation to obtain

$$y'_t - ax'_t = -ah(t)(y - ax).$$

By setting $U = y - ax$ and then integrating, one obtains

$$y - ax = C_1 \exp\left[-a\int h(t)\,dt\right], \qquad (*)$$

where C_1 is an arbitrary constant. On solving $(*)$ for y and on substituting the resulting expression into the first equation of the system, one arrives at a first-order linear differential equation for x.

7. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = h(t)x + p(t)y.}$

1°. Let us express y from the first equation and substitute into the second one to obtain a second-order linear equation:

$$gx''_{tt} - (fg + gp + g'_t)x'_t + (fgp - g^2h + fg'_t - f'_tg)x = 0. \qquad (1)$$

This equation is easy to integrate if, for example, the following conditions are met:

1) $fgp - g^2h + fg'_t - f'_tg = 0;$
2) $fgp - g^2h + fg'_t - f'_tg = ag, \quad fg + gp + g'_t = bg.$

In the first case, equation (1) has a particular solution $u = C = \text{const}$. In the second case, it is a constant-coefficient equation.

A considerable number of other solvable cases of equation (1) can be found in Section 14.1.

2°. Suppose a particular solution of the system in question is known,

$$x = x_0(t), \quad y = y_0(t).$$

Then the general solution can be written out in the form

$$x(t) = C_1 x_0(t) + C_2 x_0(t) \int \frac{g(t)F(t)P(t)}{x_0^2(t)}\,dt,$$
$$y(t) = C_1 y_0(t) + C_2 \left[\frac{F(t)P(t)}{x_0(t)} + y_0(t) \int \frac{g(t)F(t)P(t)}{x_0^2(t)}\,dt \right],$$

where C_1 and C_2 are arbitrary constants, and

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad P(t) = \exp\left[\int p(t)\,dt\right].$$

18.1.2 Systems of Second-Order Equations

1. $x''_{tt} = ax + by, \quad y''_{tt} = cx + dy.$

System of two constant-coefficient second-order linear homogeneous differential equations.

The characteristic equation has the form

$$\lambda^4 - (a+d)\lambda^2 + ad - bc = 0.$$

1°. Case $ad - bc \neq 0$.

1.1. Suppose $(a-d)^2 + 4bc \neq 0$. The characteristic equation has four distinct roots $\lambda_1, \dots, \lambda_4$. The general solution of the system in question is written as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t} + C_3 b e^{\lambda_3 t} + C_4 b e^{\lambda_4 t},$$
$$y = C_1(\lambda_1^2 - a)e^{\lambda_1 t} + C_2(\lambda_2^2 - a)e^{\lambda_2 t} + C_3(\lambda_3^2 - a)e^{\lambda_3 t} + C_4(\lambda_4^2 - a)e^{\lambda_4 t},$$

where $C_1, \dots, C_4$ are arbitrary constants.

1.2. Solution with $(a-d)^2 + 4bc = 0$ and $a \neq d$:

$$x = 2C_1\left(bt + \frac{2bk}{a-d}\right)e^{kt/2} + 2C_2\left(bt - \frac{2bk}{a-d}\right)e^{-kt/2} + 2bC_3 t e^{kt/2} + 2bC_4 t e^{-kt/2},$$
$$y = C_1(d-a)te^{kt/2} + C_2(d-a)te^{-kt/2} + C_3[(d-a)t + 2k]e^{kt/2}$$
$$+ C_4[(d-a)t - 2k]e^{-kt/2},$$

where $C_1, \dots, C_4$ are arbitrary constants and $k = \sqrt{2(a+d)}$.

1.3. Solution with $a = d \neq 0$ and $b = 0$:

$$x = 2\sqrt{a}\,C_1 e^{\sqrt{a}\,t} + 2\sqrt{a}\,C_2 e^{-\sqrt{a}\,t},$$
$$y = cC_1 t e^{\sqrt{a}\,t} - cC_2 t e^{-\sqrt{a}\,t} + C_3 e^{\sqrt{a}\,t} + C_4 e^{-\sqrt{a}\,t}.$$

1.4. Solution with $a = d \neq 0$ and $c = 0$:

$$x = bC_1 t e^{\sqrt{a}\,t} - bC_2 t e^{-\sqrt{a}\,t} + C_3 e^{\sqrt{a}\,t} + C_4 e^{-\sqrt{a}\,t},$$
$$y = 2\sqrt{a}\,C_1 e^{\sqrt{a}\,t} + 2\sqrt{a}\,C_2 e^{-\sqrt{a}\,t}.$$

2°. Case $ad - bc = 0$ and $a^2 + b^2 > 0$. The original system can be rewritten in the form

$$x''_{tt} = ax + by, \quad y''_{tt} = k(ax + by).$$

2.1. Solution with $a + bk \neq 0$:

$$x = C_1 \exp\bigl(t\sqrt{a+bk}\,\bigr) + C_2 \exp\bigl(-t\sqrt{a+bk}\,\bigr) + C_3 bt + C_4 b,$$
$$y = C_1 k \exp\bigl(t\sqrt{a+bk}\,\bigr) + C_2 k \exp\bigl(-t\sqrt{a+bk}\,\bigr) - C_3 at - C_4 a.$$

2.2. Solution with $a + bk = 0$:

$$x = C_1 bt^3 + C_2 bt^2 + C_3 t + C_4,$$
$$y = kx + 6C_1 t + 2C_2.$$

2. $x''_{tt} = a_1 x + b_1 y + c_1, \quad y''_{tt} = a_2 x + b_2 y + c_2.$

The general solution of this system is expressed as the sum of any one of its particular solutions and the general solution of the corresponding homogeneous system (see system 18.1.2.1).

1°. Suppose $a_1 b_2 - a_2 b_1 \neq 0$. A particular solution:

$$x = x_0, \qquad y = y_0,$$

where the constants x_0 and y_0 are determined by solving the linear algebraic system of equations

$$a_1 x_0 + b_1 y_0 + c_1 = 0, \qquad a_2 x_0 + b_2 y_0 + c_2 = 0.$$

2°. Suppose $a_1 b_2 - a_2 b_1 = 0$ and $a_1^2 + b_1^2 > 0$. Then the system can be rewritten as

$$x''_{tt} = ax + by + c_1, \qquad y''_{tt} = k(ax + by) + c_2.$$

2.1. If $\sigma = a + bk \neq 0$, the original system has a particular solution

$$x = \tfrac{1}{2} b\sigma^{-1}(c_1 k - c_2)t^2 - \sigma^{-2}(ac_1 + bc_2), \qquad y = kx + \tfrac{1}{2}(c_2 - c_1 k)t^2.$$

2.2. If $\sigma = a + bk = 0$, the system has a particular solution

$$x = \tfrac{1}{24} b(c_2 - c_1 k)t^4 + \tfrac{1}{2} c_1 t^2, \qquad y = kx + \tfrac{1}{2}(c_2 - c_1 k)t^2.$$

3. $x''_{tt} - ay'_t + bx = 0, \quad y''_{tt} + ax'_t + by = 0.$

This system is used to describe the horizontal motion of a pendulum taking into account the rotation of the earth.

Solution with $a^2 + 4b > 0$:

$$x = C_1 \cos(\alpha t) + C_2 \sin(\alpha t) + C_3 \cos(\beta t) + C_4 \sin(\beta t),$$
$$y = -C_1 \sin(\alpha t) + C_2 \cos(\alpha t) - C_3 \sin(\beta t) + C_4 \cos(\beta t),$$

where $C_1, \dots, C_4$ are arbitrary constants and

$$\alpha = \tfrac{1}{2} a + \tfrac{1}{2}\sqrt{a^2 + 4b}, \qquad \beta = \tfrac{1}{2} a - \tfrac{1}{2}\sqrt{a^2 + 4b}.$$

4. $x''_{tt} + a_1 x'_t + b_1 y'_t + c_1 x + d_1 y = k_1 e^{i\omega t},$
$y''_{tt} + a_2 x'_t + b_2 y'_t + c_2 x + d_2 y = k_2 e^{i\omega t}.$

Systems of this type often arise in oscillation theory (e.g., oscillations of a ship and a ship gyroscope). The general solution of this constant-coefficient linear nonhomogeneous system of differential equations is expressed as the sum of any one of its particular solutions and the general solution of the corresponding homogeneous system (with $k_1 = k_2 = 0$).

1°. A particular solution is sought by the method of undetermined coefficients in the form

$$x = A_* e^{i\omega t}, \quad y = B_* e^{i\omega t}.$$

On substituting these expressions into the system of differential equations in question, one arrives at a linear nonhomogeneous system of algebraic equations for the coefficients A_* and B_*.

2°. The general solution of a homogeneous system of differential equations is determined by a linear combination of its linearly independent particular solutions, which are sought using the method of undetermined coefficients in the form of exponential functions,

$$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}.$$

On substituting these expressions into the system and on collecting the coefficients of the unknowns A and B, one obtains

$$(\lambda^2 + a_1\lambda + c_1)A + (b_1\lambda + d_1)B = 0,$$
$$(a_2\lambda + c_2)A + (\lambda^2 + b_2\lambda + d_2)B = 0.$$

For a nontrivial solution to exist, the determinant of this system must vanish. This requirement results in the characteristic equation

$$(\lambda^2 + a_1\lambda + c_1)(\lambda^2 + b_2\lambda + d_2) - (b_1\lambda + d_1)(a_2\lambda + c_2) = 0,$$

which is used to determine λ. If the roots of this equation, $k_1, \dots, k_4$, are all distinct, then the general solution of the original system of differential equations has the form

$$x = -C_1(b_1\lambda_1 + d_1)e^{\lambda_1 t} - C_2(b_1\lambda_2 + d_1)e^{\lambda_2 t} - C_3(b_1\lambda_1 + d_1)e^{\lambda_3 t} - C_4(b_1\lambda_4 + d_1)e^{\lambda_4 t},$$
$$y = C_1(\lambda_1^2 + a_1\lambda_1 + c_1)e^{\lambda_1 t} + C_2(\lambda_2^2 + a_1\lambda_2 + c_1)e^{\lambda_2 t}$$
$$+ C_3(\lambda_3^2 + a_1\lambda_3 + c_1)e^{\lambda_3 t} + C_4(\lambda_4^2 + a_1\lambda_4 + c_1)e^{\lambda_4 t},$$

where $C_1, \dots, C_4$ are arbitrary constants.

5. $x''_{tt} = a(ty'_t - y), \quad y''_{tt} = b(tx'_t - x).$

The transformation

$$u = tx_t - x, \qquad v = ty'_t - y \tag{1}$$

leads to a first-order system:

$$u'_t = atv, \qquad v'_t = btu.$$

The general solution of this system is expressed as

$$\text{with } ab > 0: \quad \begin{cases} u(t) = C_1 a \exp\bigl(\tfrac{1}{2}\sqrt{ab}\, t^2\bigr) + C_2 a \exp\bigl(-\tfrac{1}{2}\sqrt{ab}\, t^2\bigr), \\ v(t) = C_1\sqrt{ab}\, \exp\bigl(\tfrac{1}{2}\sqrt{ab}\, t^2\bigr) - C_2\sqrt{ab}\, \exp\bigl(-\tfrac{1}{2}\sqrt{ab}\, t^2\bigr); \end{cases}$$
$$\text{with } ab < 0: \quad \begin{cases} u(t) = C_1 a \cos\bigl(\tfrac{1}{2}\sqrt{|ab|}\, t^2\bigr) + C_2 a \sin\bigl(\tfrac{1}{2}\sqrt{|ab|}\, t^2\bigr), \\ v(t) = -C_1\sqrt{|ab|}\, \sin\bigl(\tfrac{1}{2}\sqrt{|ab|}\, t^2\bigr) + C_2\sqrt{|ab|}\, \cos\bigl(\tfrac{1}{2}\sqrt{|ab|}\, t^2\bigr), \end{cases} \tag{2}$$

where C_1 and C_2 are arbitrary constants. On substituting (2) into (1) and integrating, one arrives at the general solution of the original system in the form

$$x = C_3 t + t\int \frac{u(t)}{t^2}\, dt, \qquad y = C_4 t + t\int \frac{v(t)}{t^2}\, dt,$$

where C_3 and C_4 are arbitrary constants.

6. $x''_{tt} = f(t)(a_1x + b_1y), \quad y''_{tt} = f(t)(a_2x + b_2y).$

Let k_1 and k_2 be roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0.$$

Then, on multiplying the equations of the system by appropriate constants and on adding them together, one can rewrite the system in the form of two independent equations:

$$z''_1 = k_1 f(t) z_1, \quad z_1 = a_2x + (k_1 - a_1)y;$$
$$z''_2 = k_2 f(t) z_2, \quad z_2 = a_2x + (k_2 - a_1)y.$$

Here, a prime stands for a derivative with respect to t.

7. $x''_{tt} = f(t)(a_1x'_t + b_1y'_t), \quad y''_{tt} = f(t)(a_2x'_t + b_2y'_t).$

Let k_1 and k_2 be roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0.$$

Then, on multiplying the equations of the system by appropriate constants and on adding them together, one can reduce the system to two independent equations:

$$z''_1 = k_1 f(t) z'_1, \quad z_1 = a_2x + (k_1 - a_1)y;$$
$$z''_2 = k_2 f(t) z'_2, \quad z_2 = a_2x + (k_2 - a_1)y.$$

Integrating these equations and returning to the original variables, one arrives at a linear algebraic system for the unknowns x and y:

$$a_2x + (k_1 - a_1)y = C_1 \int \exp\bigl[k_1F(t)\bigr]\,dt + C_2,$$
$$a_2x + (k_2 - a_1)y = C_3 \int \exp\bigl[k_2F(t)\bigr]\,dt + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants and $F(t) = \int f(t)\,dt$.

8. $x''_{tt} = af(t)(ty'_t - y), \quad y''_{tt} = bf(t)(tx'_t - x).$

The transformation

$$u = tx_t - x, \qquad v = ty'_t - y \tag{1}$$

leads to a system of first-order equations:

$$u'_t = atf(t)v, \qquad v'_t = btf(t)u.$$

The general solution of this system is expressed as

$$\begin{aligned}
&\text{if } ab>0, \quad \begin{cases} u(t) = C_1 a \exp\Bigl(\sqrt{ab}\int tf(t)\,dt\Bigr) + C_2 a \exp\Bigl(-\sqrt{ab}\int tf(t)\,dt\Bigr),\\ v(t) = C_1\sqrt{ab}\,\exp\Bigl(\sqrt{ab}\int tf(t)\,dt\Bigr) - C_2\sqrt{ab}\,\exp\Bigl(-\sqrt{ab}\int tf(t)\,dt\Bigr); \end{cases}\\
&\text{if } ab<0, \quad \begin{cases} u(t) = C_1 a \cos\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr) + C_2 a \sin\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr),\\ v(t) = -C_1\sqrt{|ab|}\,\sin\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr) + C_2\sqrt{|ab|}\,\cos\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr), \end{cases}
\end{aligned} \tag{2}$$

where C_1 and C_2 are arbitrary constants. On substituting (2) into (1) and integrating, one obtains the general solution of the original system

$$x = C_3t + t\int \frac{u(t)}{t^2}\,dt, \qquad y = C_4t + t\int \frac{v(t)}{t^2}\,dt,$$

where C_3 and C_4 are arbitrary constants.

9. $\boldsymbol{t^2x''_{tt}+a_1tx'_t+b_1ty'_t+c_1x+d_1y=0,\quad t^2y''_{tt}+a_2tx'_t+b_2ty'_t+c_2x+d_2y=0.}$

Linear system homogeneous in the independent variable (an Euler-type system).

1°. The general solution is determined by a linear combination of linearly independent particular solutions that are sought by the method of undetermined coefficients in the form of power-law functions

$$x = A|t|^k, \quad y = B|t|^k.$$

On substituting these expressions into the system and on collecting the coefficients of the unknowns A and B, one obtains

$$A + (b_1k + d_1)B = 0,$$
$$(a_2k + c_2)A + [k^2 + (b_2 - 1)k + d_2]B = 0.$$

For a nontrivial solution to exist, the determinant of this system must vanish. This requirement results in the characteristic equation

$$[k^2 + (a_1 - 1)k + c_1][k^2 + (b_2 - 1)k + d_2] - (b_1k + d_1)(a_2k + c_2) = 0,$$

which is used to determine k. If the roots of this equation, $k_1, \ldots, k_4$, are all distinct, then the general solution of the system of differential equations in question has the form

$$x = -C_1(b_1k_1 + d_1)|t|^{k_1} - C_2(b_1k_2 + d_1)|t|^{k_2} - C_3(b_1k_1 + d_1)|t|^{k_3} - C_4(b_1k_4 + d_1)|t|^{k_4},$$
$$y = C_1[k_1^2 + (a_1 - 1)k_1 + c_1]|t|^{k_1} + C_2[k_2^2 + (a_1 - 1)k_2 + c_1]|t|^{k_2}$$
$$+ C_3[k_3^2 + (a_1 - 1)k_3 + c_1]|t|^{k_3} + C_4[k_4^2 + (a_1 - 1)k_4 + c_1]|t|^{k_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

2°. The substitution $t = \sigma e^\tau$ $(\sigma \neq 0)$ leads to a system of constant-coefficient linear differential equations:

$$x''_{\tau\tau} + (a_1 - 1)x'_\tau + b_1y'_\tau + c_1x + d_1y = 0,$$
$$y''_{\tau\tau} + a_2x'_\tau + (b_2 - 1)y'_\tau + c_2x + d_2y = 0.$$

10. $\boldsymbol{(\alpha t^2 + \beta t + \gamma)^2 x''_{tt} = ax + by, \quad (\alpha t^2 + \beta t + \gamma)^2 y''_{tt} = cx + dy.}$

The transformation

$$\tau = \int \frac{dt}{\alpha t^2 + \beta t + \gamma}, \quad u = \frac{x}{\sqrt{|\alpha t^2 + \beta t + \gamma|}}, \quad v = \frac{y}{\sqrt{|\alpha t^2 + \beta t + \gamma|}}$$

leads to a constant-coefficient linear system of equations of the form 18.1.2.1:

$$u''_{\tau\tau} = (a - \alpha\gamma + \tfrac{1}{4}\beta^2)u + bv,$$
$$v''_{\tau\tau} = cu + (d - \alpha\gamma + \tfrac{1}{4}\beta^2)v.$$

11. $\boldsymbol{x''_{tt} = f(t)(tx'_t - x) + g(t)(ty'_t - y), \quad y''_{tt} = h(t)(tx'_t - x) + p(t)(ty'_t - y).}$

The transformation

$$u = tx_t - x, \quad v = ty'_t - y \tag{1}$$

leads to a linear system of first-order equations

$$u'_t = tf(t)u + tg(t)v, \quad v'_t = th(t)u + tp(t)v. \tag{2}$$

In order to find the general solution of this system, it suffices to know any one of its particular solutions (see system 18.1.1.7).

For solutions of some systems of the form (2), see systems 18.1.1.3–18.1.1.6.

If all functions in (2) are proportional, that is,

$$f(t) = a\varphi(t), \quad g(t) = b\varphi(t), \quad h(t) = c\varphi(t), \quad p(t) = d\varphi(t),$$

then the introduction of the new independent variable $\tau = \int t\varphi(t)\,dt$ leads to a constant-coefficient system of the form 18.1.1.1.

2°. Suppose a solution of system (2) has been found in the form

$$u = u(t, C_1, C_2), \qquad v = v(t, C_1, C_2), \tag{3}$$

where C_1 and C_2 are arbitrary constants. Then, on substituting (3) into (1) and integrating, one obtains a solution of the original system:

$$x = C_3 t + t\int \frac{u(t, C_1, C_2)}{t^2}\,dt, \qquad y = C_4 t + t\int \frac{v(t, C_1, C_2)}{t^2}\,dt,$$

where C_3 and C_4 are arbitrary constants.

18.2 Linear Systems of Three and More Equations

1. $x'_t = ax, \quad y'_t = bx + cy, \quad z'_t = dx + ky + pz.$

Solution:

$$\begin{aligned} x &= C_1 e^{at}, \\ y &= \frac{bC_1}{a-c} e^{at} + C_2 e^{ct}, \\ z &= \frac{C_1}{a-p}\Bigl(d + \frac{bk}{a-c}\Bigr)e^{at} + \frac{kC_2}{c-p} e^{ct} + C_3 e^{pt}, \end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants.

2. $x'_t = cy - bz, \quad y'_t = az - cx, \quad z'_t = bx - ay.$

1°. First integrals:

$$ax + by + cz = A, \tag{1}$$

$$x^2 + y^2 + z^2 = B^2, \tag{2}$$

where A and B are arbitrary constants. It follows that the integral curves are circles formed by the intersection of planes (1) and spheres (2).

2°. Solution:

$$\begin{aligned} x &= aC_0 + kC_1\cos(kt) + (cC_2 - bC_3)\sin(kt), \\ y &= bC_0 + kC_2\cos(kt) + (aC_3 - cC_1)\sin(kt), \\ z &= cC_0 + kC_3\cos(kt) + (bC_1 - aC_2)\sin(kt), \end{aligned}$$

where $k = \sqrt{a^2 + b^2 + c^2}$ and the three of four constants of integration $C_0, \dots, C_3$ are related by the constraint

$$aC_1 + bC_2 + cC_3 = 0.$$

3. $ax'_t = bc(y - z), \quad by'_t = ac(z - x), \quad cz'_t = ab(x - y).$

1°. First integral:

$$a^2x + b^2y + c^2z = A,$$

where A is an arbitrary constant. It follows that the integral curves are plane curves.

2°. Solution:

$$x = C_0 + kC_1\cos(kt) + a^{-1}bc(C_2 - C_3)\sin(kt),$$
$$y = C_0 + kC_2\cos(kt) + ab^{-1}c(C_3 - C_1)\sin(kt),$$
$$z = C_0 + kC_3\cos(kt) + abc^{-1}(C_1 - C_2)\sin(kt),$$

where $k = \sqrt{a^2+b^2+c^2}$ and three of the four constants of integration $C_0, \dots, C_3$ are related by the constraint

$$a^2C_1 + b^2C_2 + c^2C_3 = 0.$$

4. $x'_t = (a_1 f + g)x + a_2 f y + a_3 f z,$
$\quad y'_t = b_1 f x + (b_2 f + g)y + b_3 f z,\ \ z'_t = c_1 f x + c_2 f y + (c_3 f + g)z.$

Here, $f = f(t)$ and $g = g(t)$.

The transformation

$$x = \exp\left[\int g(t)\,dt\right]u, \quad y = \exp\left[\int g(t)\,dt\right]v, \quad z = \exp\left[\int g(t)\,dt\right]w, \quad \tau = \int f(t)\,dt$$

leads to the system of constant coefficient linear differential equations

$$u'_\tau = a_1u + a_2v + a_3w, \quad v'_\tau = b_1u + b_2v + b_3w, \quad w'_\tau = c_1u + c_2v + c_3w.$$

5. $x'_t = h(t)y - g(t)z,\ \ y'_t = f(t)z - h(t)x,\ \ z'_t = g(t)x - f(t)y.$

1°. First integral:

$$x^2 + y^2 + z^2 = C^2,$$

where C is an arbitrary constant.

2°. The system concerned can be reduced to a Riccati equation (see Kamke, 1977).

6. $x'_k = a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n; \qquad k = 1, 2, \dots, n.$

System of n constant-coefficient first-order linear homogeneous differential equations.

The general solution of a homogeneous system of differential equations is determined by a linear combination of linearly independent particular solutions, which are sought by the method of undetermined coefficients in the form of exponential functions,

$$x_k = A_k e^{\lambda t}; \qquad k = 1, 2, \dots, n.$$

On substituting these expressions into the system and on collecting the coefficients of the unknowns A_k, one obtains a linear homogeneous system of algebraic equations:

$$a_{k1}A_1 + a_{k2}A_2 + \dots + (a_{kk} - \lambda)A_k + \dots + a_{kn}A_n = 0; \qquad k = 1, 2, \dots, n.$$

For a nontrivial solution to exist, the determinant of this system must vanish. This requirement results in a characteristic equation that serves to determine λ.

18.3 Nonlinear Systems of Two Equations

18.3.1 Systems of First-Order Equations

1. $x'_t = x^n F(x, y), \quad y'_t = g(y)F(x, y).$

Solution:

$$x = \varphi(y), \qquad \int \frac{dy}{g(y)F(\varphi(y), y)} = t + C_2,$$

where

$$\varphi(y) = \begin{cases} \left[C_1 + (1-n)\int \dfrac{dy}{g(y)}\right]^{\frac{1}{1-n}} & \text{if } n \neq 1, \\ C_1 \exp\left[\int \dfrac{dy}{g(y)}\right] & \text{if } n = 1, \end{cases}$$

C_1 and C_2 are arbitrary constants.

2. $\boldsymbol{x'_t = e^{\lambda x} F(x,y), \quad y'_t = g(y)F(x,y).}$

Solution:

$$x = \varphi(y), \qquad \int \frac{dy}{g(y)F(\varphi(y),y)} = t + C_2,$$

where

$$\varphi(y) = \begin{cases} -\dfrac{1}{\lambda}\ln\left[C_1 - \lambda\int \dfrac{dy}{g(y)}\right] & \text{if } \lambda \neq 0, \\ C_1 + \displaystyle\int \dfrac{dy}{g(y)} & \text{if } \lambda = 0, \end{cases}$$

C_1 and C_2 are arbitrary constants.

3. $\boldsymbol{x'_t = F(x,y), \quad y'_t = G(x,y).}$

Autonomous system of general form.

Suppose

$$y = y(x, C_1),$$

where C_1 is an arbitrary constant, is the general solution of the first-order equation

$$F(x,y)y'_x = G(x,y).$$

Then the general solution of the system in question results in the following dependence for the variable x:

$$\int \frac{dx}{F(x, y(x, C_1))} = t + C_2.$$

4. $\boldsymbol{x'_t = f_1(x)g_1(y)\Phi(x,y,t), \quad y'_t = f_2(x)g_2(y)\Phi(x,y,t).}$

First integral:

$$\int \frac{f_2(x)}{f_1(x)}\,dx - \int \frac{g_1(y)}{g_2(y)}\,dy = C, \tag{$*$}$$

where C is an arbitrary constant.

On solving ($*$) for x (or y) and on substituting the resulting expression into one of the equations of the system concerned, one arrives at a first-order equation for y (or x).

5. $\boldsymbol{x = tx'_t + F(x'_t, y'_t), \quad y = ty'_t + G(x'_t, y'_t).}$

Clairaut system.

The following are solutions of the system:

(i) straight lines

$$x = C_1 t + F(C_1, C_2), \qquad y = C_2 t + G(C_1, C_2),$$

where C_1 and C_2 are arbitrary constants;

(ii) envelopes of these lines;

(iii) continuously differentiable curves that are formed by segments of curves (i) and (ii).

18.3.2 Systems of Second-Order Equations

1. $x''_{tt} = xf(ax - by) + g(ax - by), \quad y''_{tt} = yf(ax - by) + h(ax - by).$

Let us multiply the first equation by a and the second one by $-b$ and add them together to obtain the autonomous equation

$$z''_{tt} = zf(z) + ag(z) - bh(z), \qquad z = ax - by. \tag{1}$$

We will consider this equation in conjunction with the first equation of the system,

$$x''_{tt} = xf(z) + g(z). \tag{2}$$

Autonomous equation (1) can be treated separately; its general solution can be written out in implicit form (see Eq. 14.9.1.1). The function $x = x(t)$ can be determined by solving the linear equation (2), and the function $y = y(t)$ is found as $y = (ax - z)/b$.

2. $x''_{tt} = xf(y/x), \quad y''_{tt} = yg(y/x).$

A periodic particular solution:

$$x = C_1 \sin(kt) + C_2 \cos(kt), \qquad k = \sqrt{-f(\lambda)},$$
$$y = \lambda[C_1 \sin(kt) + C_2 \cos(kt)],$$

where C_1 and C_2 are arbitrary constants and λ is a root of the transcendental (algebraic) equation

$$f(\lambda) = g(\lambda). \tag{1}$$

2°. Particular solution:

$$x = C_1 \exp(kt) + C_2 \exp(-kt), \qquad k = \sqrt{f(\lambda)},$$
$$y = \lambda[C_1 \exp(kt) + C_2 \exp(-kt)],$$

where C_1 and C_2 are arbitrary constants and λ is a root of the transcendental (algebraic) equation (1).

3. $x''_{tt} = kxr^{-3}, \quad y''_{tt} = kyr^{-3}, \quad$ **where** $\quad r = \sqrt{x^2 + y^2}.$

Equation of motion of a point mass in the xy-plane under gravity.

Passing to polar coordinates by the formulas

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad r = r(t), \qquad \varphi = \varphi(t),$$

one may obtain the first integrals

$$r^2\varphi'_t = C_1, \qquad (r'_t)^2 + r^2(\varphi'_t)^2 = -2kr^{-1} + C_2, \tag{1}$$

where C_1 and C_2 are arbitrary constants. Assuming that $C_1 \neq 0$ and integrating further, one finds that

$$r[C\cos(\varphi - \varphi_0) - k] = C_1^2, \qquad C^2 = C_1^2 C_2 + k^2.$$

This is an equation of a conic section. The dependence $\varphi(t)$ may be found from the first equation in (1).

4. $x''_{tt} = xf(r), \quad y''_{tt} = yf(r), \quad$ **where** $\quad r = \sqrt{x^2 + y^2}.$

Equation of motion of a point mass in the xy-plane under a central force.

Passing to polar coordinates by the formulas

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad r = r(t), \qquad \varphi = \varphi(t),$$

one may obtain the first integrals

$$r^2\varphi'_t = C_1, \qquad (r'_t)^2 + r^2(\varphi'_t)^2 = 2\int r f(r)\,dr + C_2,$$

where C_1 and C_2 are arbitrary constants. Integrating further, one finds that

$$t + C_3 = \pm\int \frac{r\,dr}{\sqrt{2r^2F(r) + r^2C_2 - C_1^2}}, \qquad \varphi = C_1\int\frac{dt}{r} + C_4, \tag{$*$}$$

where C_3 and C_4 are arbitrary constants and

$$F(r) = \int r f(r)\,dr.$$

It is assumed in the second relation in ($*$) that the dependence $r = r(t)$ is obtained by solving the first equation in ($*$) for $r(t)$.

5. $\boldsymbol{x''_{tt} + a(t)x = x^{-3}f(y/x), \quad y''_{tt} + a(t)y = y^{-3}g(y/x).}$

Generalized Ermakov system.

1°. First integral:

$$\frac{1}{2}(xy'_t - yx'_t)^2 + \int^{y/x}\big[uf(u) - u^{-3}g(u)\big]\,du = C,$$

where C is an arbitrary constant.

2°. Suppose $\varphi = \varphi(t)$ is a nontrivial solution of the second-order linear differential equation

$$\varphi''_{tt} + a(t)\varphi = 0. \tag{1}$$

Then the transformation

$$\tau = \int\frac{dt}{\varphi^2(t)}, \qquad u = \frac{x}{\varphi(t)}, \qquad v = \frac{y}{\varphi(t)} \tag{2}$$

leads to the autonomous system of equations

$$u''_{\tau\tau} = u^{-3}f(v/u), \qquad v''_{\tau\tau} = v^{-3}g(v/u). \tag{3}$$

3°. Particular solution of system (3) is

$$u = A\sqrt{C_2\tau^2 + C_1\tau + C_0}, \qquad v = Ak\sqrt{C_2\tau^2 + C_1\tau + C_0}, \qquad A = \left[\frac{f(k)}{C_0C_2 - \frac{1}{4}C_1^2}\right]^{1/4},$$

where C_0, C_1, and C_2 are arbitrary constants, and k is a root of the algebraic (transcendental) equation

$$k^4 f(k) = g(k).$$

6. $\boldsymbol{x''_{tt} = f(y'_t/x'_t), \qquad y''_{tt} = g(y'_t/x'_t).}$

1°. The transformation

$$u = x'_t, \qquad w = y'_t \tag{1}$$

leads to a system of the first-order equations

$$u'_t = f(w/u), \qquad w'_t = g(w/u). \tag{2}$$

Eliminating t yields a homogeneous first-order equation, whose solution is given by

$$\int\frac{f(\xi)\,d\xi}{g(\xi) - \xi f(\xi)} = \ln|u| + C, \qquad \xi = \frac{w}{u}, \tag{3}$$

where C is an arbitrary constant. On solving (3) for w, one obtains $w = w(u, C)$. On substituting this expression into the first equation of (2), one can find $u = u(t)$ and then $w = w(t)$. Finally, one can determine $x = x(t)$ and $y = y(t)$ from (1) by simple integration.

2°. *The Suslov problem.* The problem of a point particle sliding down an inclined rough plane is described by the equations

$$x''_{tt} = 1 - \frac{kx'_t}{\sqrt{(x'_t)^2 + (y'_t)^2}}, \qquad y''_{tt} = -\frac{ky'_t}{\sqrt{(x'_t)^2 + (y'_t)^2}},$$

which correspond to a special case of the system in question with

$$f(z) = 1 - \frac{k}{\sqrt{1+z^2}}, \qquad g(z) = -\frac{kz}{\sqrt{1+z^2}}.$$

The solution of the corresponding Cauchy problem under the initial conditions

$$x(0) = y(0) = x'_t(0) = 0, \qquad y'_t(0) = 1$$

leads, for the case $k = 1$, to the following dependences $x(t)$ and $y(t)$ written in parametric form:

$$x = -\tfrac{1}{16} + \tfrac{1}{16}\xi^4 - \tfrac{1}{4}\ln\xi, \quad y = \tfrac{2}{3} - \tfrac{1}{2}\xi - \tfrac{1}{6}\xi^3, \quad t = \tfrac{1}{4} - \tfrac{1}{4}\xi^2 - \tfrac{1}{2}\ln\xi \qquad (0 \le \xi \le 1).$$

7. $\boldsymbol{x''_{tt} = x\Phi(x, y, t, x'_t, y'_t), \quad y''_{tt} = y\Phi(x, y, t, x'_t, y'_t).}$

1°. First integral:

$$xy'_t - yx'_t = C,$$

where C is an arbitrary constant.

Remark 18.1. The function Φ can also be dependent on the second and higher derivatives with respect to t.

2°. Particular solution: $y = C_1 x$, where C_1 is an arbitrary constant and the function $x = x(t)$ is determined by the ordinary differential equation

$$x''_{tt} = x\Phi(x, C_1 x, t, x'_t, C_1 x'_t).$$

8. $\boldsymbol{x''_{tt} + x^{-3} f(y/x) = x\Phi(x, y, t, x'_t, y'_t), \; y''_{tt} + y^{-3} g(y/x) = y\Phi(x, y, t, x'_t, y'_t).}$

First integral:

$$\frac{1}{2}(xy'_t - yx'_t)^2 + \int^{y/x} \bigl[u^{-3} g(u) - u f(u)\bigr]\, du = C,$$

where C is an arbitrary constant.

Remark 18.2. The function Φ can also be dependent on the second and higher derivatives with respect to t.

9. $\boldsymbol{x''_{tt} = F(t, tx'_t - x, ty'_t - y), \quad y''_{tt} = G(t, tx'_t - x, ty'_t - y).}$

1°. The transformation

$$u = tx_t - x, \qquad v = ty'_t - y \tag{1}$$

leads to a system of first-order equations

$$u'_t = tF(t, u, v), \qquad v'_t = tG(t, u, v). \tag{2}$$

2°. Suppose a solution of system (2) has been found in the form

$$u = u(t, C_1, C_2), \qquad v = v(t, C_1, C_2), \tag{3}$$

where C_1 and C_2 are arbitrary constants. Then, substituting (3) into (1) and integrating, one obtains a solution of the original system,

$$x = C_3 t + t\int \frac{u(t, C_1, C_2)}{t^2}\, dt, \qquad y = C_4 t + t\int \frac{v(t, C_1, C_2)}{t^2}\, dt.$$

3°. If the functions F and G are independent of t, then, on eliminating t from system (2), one arrives at a first-order equation

$$g(u, v) u'_v = F(u, v).$$

18.4 Nonlinear Systems of Three or More Equations

18.4.1 Systems of Three Equations

1. $ax'_t = (b-c)yz, \quad by'_t = (c-a)zx, \quad cz'_t = (a-b)xy.$

First integrals:

$$ax^2 + by^2 + cz^2 = C_1,$$
$$a^2x^2 + b^2y^2 + c^2z^2 = C_2,$$

where C_1 and C_2 are arbitrary constants. On solving the first integrals for y and z and on substituting the resulting expressions into the first equation of the system, one arrives at a separable first-order equation.

2. $ax'_t = (b-c)yzF(x,y,z,t),$
$by'_t = (c-a)zxF(x,y,z,t), \quad cz'_t = (a-b)xyF(x,y,z,t).$

First integrals:

$$ax^2 + by^2 + cz^2 = C_1,$$
$$a^2x^2 + b^2y^2 + c^2z^2 = C_2,$$

where C_1 and C_2 are arbitrary constants. On solving the first integrals for y and z and on substituting the resulting expressions into the first equation of the system, one arrives at a separable first-order equation; if F is independent of t, this equation will be separable.

3. $x'_t = cF_2 - bF_3, \quad y'_t = aF_3 - cF_1, \quad z'_t = bF_1 - aF_2,$
where $F_n = F_n(x,y,z).$

First integral:

$$ax + by + cz = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one arrives at the first-order equation

$$\frac{dy}{dx} = \frac{aF_3(x,y,z) - cF_1(x,y,z)}{cF_2(x,y,z) - bF_3(x,y,z)}, \quad \text{where} \quad z = \frac{1}{c}(C_1 - ax - by).$$

4. $x'_t = czF_2 - byF_3, \quad y'_t = axF_3 - czF_1, \quad z'_t = byF_1 - axF_2.$

Here, $F_n = F_n(x,y,z)$ are arbitrary functions ($n = 1,\ 2,\ 3$).

First integral:

$$ax^2 + by^2 + cz^2 = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one arrives at the first-order equation

$$\frac{dy}{dx} = \frac{axF_3(x,y,z) - czF_1(x,y,z)}{czF_2(x,y,z) - byF_3(x,y,z)}, \quad \text{where} \quad z = \pm\sqrt{\frac{1}{c}(C_1 - ax^2 - by^2)}.$$

5. $x'_t = x(cF_2 - bF_3), \quad y'_t = y(aF_3 - cF_1), \quad z'_t = z(bF_1 - aF_2).$

Here, $F_n = F_n(x,y,z)$ are arbitrary functions ($n = 1,\ 2,\ 3$).

First integral:

$$|x|^a|y|^b|z|^c = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one may obtain a first-order equation.

6. $\boldsymbol{x'_t = h(z)F_2 - g(y)F_3, \quad y'_t = f(x)F_3 - h(z)F_1, \quad z'_t = g(y)F_1 - f(x)F_2.}$

Here, $F_n = F_n(x, y, z)$ are arbitrary functions ($n = 1,\ 2,\ 3$).

First integral:

$$\int f(x)\,dx + \int g(y)\,dy + \int h(z)\,dz = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one may obtain a first-order equation.

7. $\boldsymbol{x''_{tt} = \frac{\partial F}{\partial x}, \quad y''_{tt} = \frac{\partial F}{\partial y}, \quad z''_{tt} = \frac{\partial F}{\partial z},}$ **where** $\boldsymbol{F = F(r),\ r = \sqrt{x^2 + y^2 + z^2}.}$

Equations of motion of a point particle under gravity.

The system can be rewritten as a single vector equation:

$$\mathbf{r}''_{tt} = \operatorname{grad} F \qquad \text{or} \qquad \mathbf{r}''_{tt} = \frac{F'(r)}{r}\mathbf{r},$$

where $\mathbf{r} = (x, y, z)$.

1°. First integrals:

$$(\mathbf{r}'_t)^2 = 2F(r) + C_1 \quad \text{(law of conservation of energy)},$$
$$[\mathbf{r} \times \mathbf{r}'_t] = \mathbf{C} \quad \text{(law of conservation of areas)},$$
$$(\mathbf{r} \cdot \mathbf{C}) = 0 \quad \text{(all trajectories are plane curves)}.$$

2°. Solution:

$$\mathbf{r} = \mathbf{a}\,r\cos\varphi + \mathbf{b}\,r\sin\varphi.$$

Here, the constant vectors **a** and **b** must satisfy the conditions

$$|\mathbf{a}| = |\mathbf{b}| = 1, \qquad (\mathbf{a} \cdot \mathbf{b}) = 0,$$

and the functions $r = r(t)$ and $\varphi = \varphi(t)$ are given by

$$t = \int \frac{r\,dr}{\sqrt{2r^2F(r) + C_1r^2 - C_3^2}} + C_2, \quad \varphi = C_3 \int \frac{dr}{r\sqrt{2r^2F(r) + C_1r^2 - C_3^2}}, \quad C_3 = |\mathbf{C}|.$$

8. $\boldsymbol{x''_{tt} = xF, \ y''_{tt} = yF, \ z''_{tt} = zF,}$ **where** $\boldsymbol{F = F(x, y, z, t, x'_t, y'_t, z'_t).}$

First integrals (laws of conservation of areas):

$$zy'_t - yz'_t = C_1,$$
$$xz'_t - zx'_t = C_2,$$
$$yx'_t - xy'_t = C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

Corollary of the conservation laws:

$$C_1x + C_2y + C_3z = 0.$$

This implies that all integral curves are plane curves.

Remark 18.3. The function F can also be dependent on the second and higher derivatives with respect to t.

9. $\boldsymbol{x''_{tt} = F_1, \quad y''_{tt} = F_2, \quad z''_{tt} = F_3,}$ **where** $\boldsymbol{F_n = F_n(t, tx'_t - x, ty'_t - y, tz'_t - z).}$

1°. The transformation

$$u = tx_t - x, \qquad v = ty'_t - y, \qquad w = tz'_t - z \tag{1}$$

leads to the system of first-order equations

$$u'_t = tF_1(t,u,v,w), \quad v'_t = tF_2(t,u,v,w), \quad w'_t = tF_3(t,u,v,w). \tag{2}$$

2°. Suppose a solution of system (2) has been found in the form

$$u(t) = u(t, C_1, C_2, C_3), \quad v(t) = v(t, C_1, C_2, C_3), \quad w(t) = w(t, C_1, C_2, C_3), \tag{3}$$

where C_1, C_2, and C_3 are arbitrary constants. Then, substituting (3) into (1) and integrating, one obtains a solution of the original system:

$$x = C_4 t + t\int \frac{u(t)}{t^2}\,dt, \quad y = C_5 t + t\int \frac{v(t)}{t^2}\,dt, \quad z = C_6 t + t\int \frac{w(t)}{t^2}\,dt,$$

where C_4, C_5, and C_6 are arbitrary constants.

18.4.2 Dynamics of a Rigid Body with a Fixed Point*

► **Kinematic and dynamic Euler equations.**

The motion (rotation) of a rigid about a fixed point under the action of external forces is governed by a system of six first-order coupled ODEs:

$$Ap'_t + (C - B)qr = M_1, \tag{1}$$
$$Bq'_t + (A - C)pr = M_2, \tag{2}$$
$$Cr'_t + (B - A)pq = M_3, \tag{3}$$
$$p = \psi'_t \sin\theta \sin\varphi + \theta'_t \cos\varphi, \tag{4}$$
$$q = \psi'_t \sin\theta \cos\varphi - \theta'_t \sin\varphi, \tag{5}$$
$$r = \psi'_t \cos\theta + \varphi'_t, \tag{6}$$

where p, q, and r are the components of the body's angular velocity in a moving orthonormal reference frame, $\xi\eta\zeta$, rigidly connected with the body and formed by the principal axes of inertia (the origin placed at the fixed point); xyz is a fixed orthonormal reference frame with origin at the same point; A, B, and C are the moments of inertia about the principal axes; and M_1, M_2, and M_3 are the components of the moment of external forces in the frame $\xi\eta\zeta$, which usually depend of the Euler angles ψ, θ, and φ defining the position of the moving frame relative to the fixed one. The entries of the rotation matrix, $[a_{ij}]$, are expressed in terms of the Euler angles as follows:

$$a_{11} = \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi, \tag{7}$$
$$a_{11} = -\sin\varphi\cos\psi - \cos\varphi\cos\theta\sin\psi, \tag{8}$$
$$a_{13} = -\sin\theta\sin\varphi, \tag{9}$$
$$a_{21} = \cos\varphi\sin\psi + \sin\varphi\cos\theta\cos\psi, \tag{10}$$
$$a_{22} = -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi, \tag{11}$$
$$a_{23} = -\sin\theta\cos\varphi, \tag{12}$$
$$a_{31} = \sin\varphi\sin\theta, \tag{13}$$
$$a_{32} = \cos\varphi\sin\theta, \tag{14}$$
$$a_{33} = \cos\theta. \tag{15}$$

*This section was written by Alexander Fomichev.

It is required to determine p, q, and r as functions of ψ, θ, and φ and time t from system (1)–(6).

From now on, the following quantities will be used in this section: m is the mass of the body, $\mathbf{r}$ is the position vector of the center of mass, $\mathbf{K} = (K_1, K_2, K_3)^{\mathrm{T}} = (Ap, Bq, Cr)^{\mathrm{T}}$ is the angular momentum of the body (in the frame $\xi\eta\zeta$), $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is a vertical unit vector ($\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$), which is introduced when the body is in a homogeneous gravitational field so that the direction of $\boldsymbol{\gamma}$ is opposite to the gravitational acceleration $\mathbf{g}$, with $g = |\mathbf{g}|$.

Equations (1)–(3) are known as *Euler's dynamic equations* and (4)–(6) as *Euler's kinematic equations*. In general, system (1)–(6) cannot be solved by quadrature. However, there are three special cases where the system is reduced to quadratures for any initial conditions; this is due to the availability of first integrals, which do not exist in the general case. The three solvable cases are discussed below.

► Euler's case.

Euler's case takes place when the body has an arbitrary shape and the external moments are all zero:

$$M_1 = M_2 = M_3 = 0. \tag{16}$$

With formulas (16), the dynamic equations (1)–(3) can be solved independently of the kinematic equations.

To be specific, we assume that $A \geq B \geq C$ and $A > C$ (the case $A = B = C$ is trivial). System (1)–(3) with (16) has the following first integrals:

$$\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= 2T \qquad \text{(conservation of energy)}, \\ A^2p^2 + B^2q^2 + C^2r^2 &= K^2 \qquad \text{(conservation of angular momentum)}, \end{aligned}$$

where $T > 0$ and K are arbitrary constants. In Euler's case, the angular momentum $\mathbf{K}$ is constant in the fixed frame xyz.

For $A > C$, p and r can always be expressed via q:

$$p = \pm\sqrt{a - bq^2}, \qquad r = \pm\sqrt{c - dq^2}, \tag{17}$$

with the constants a, b, c, and d expressible in terms of the initial parameters of the problem. Substituting (17) into the equation for q yields

$$bq'_t \pm (A - C)\sqrt{(a - bq^2)(c - dq^2)} = 0.$$

Integrating gives the solution in implicit form

$$t - t_0 = \pm\frac{B}{A - C}\int_0^q \frac{dq}{\sqrt{(a - bq^2)(c - dq^2)}}.$$

Effectively, the problem is reduced to the inversion of an elliptic integral, resulting in expressions of $p(t)$, $q(t)$, and $r(t)$ in terms of elliptic functions of time.

To solve the kinematic equations, it is convenient to direct the z-axis of the fixed frame along the constant angular momentum **K**, in which case we obtain

$$K_1 = K\sin\theta\sin\varphi,$$
$$K_2 = K\sin\theta\cos\varphi, \quad \Longrightarrow \quad \cos\theta(t) = \frac{Cr(t)}{K}, \quad \Longrightarrow \quad \cos\varphi(t) = \frac{Bq(t)}{K\sin\theta(t)},$$
$$K_3 = K\cos\theta,$$
$$\psi(t) = \psi_0 + \int_0^t \frac{p(t)\sin\varphi(t) + q(t)\cos\varphi(t)}{\sin\theta(t)}\,dt.$$

This solution is known to have geometric interpretations suggested by Poinsot and MacCullagh (e.g., see Zhuravlev (1996), Borisov and Mamaev (2001), and Teodorescu (2009)).

► Lagrange's case.

The body, which is in a homogeneous gravitational field, is dynamically symmetric and its center of mass lies on the dynamic symmetry axis (the ζ-azis). Then, in equations (1)–(3), one should set

$$A = B, \qquad \mathbf{M} = (M_1, M_2, M_3)^T = mg(\mathbf{r}\times\boldsymbol{\gamma}). \tag{18}$$

The easiest way to integrate the equations is to use the Euler angles. System (1)–(6) with (18) admits the following three first integrals:

$K_3 = \text{const}$ (conservation of the angular momentum projection onto the ζ-axis);

$(\mathbf{K}\cdot\boldsymbol{\gamma}) = K_1\gamma_1 + K_2\gamma_2 + K_3\gamma_3 = C_1$ (conservation of the angular momentum projection onto the direction of $\boldsymbol{\gamma}$);

$$\frac{h}{2}(\theta'_t)^2 + \frac{K_3^2}{2C} + \frac{(C_1 - K_3\cos\theta)^2}{2A\sin^\theta} + mgl\cos\theta = h = \text{const} \qquad \text{(energy integral)}.$$

The availability of these integrals reduces the problem to the equation

$$(\theta'_t)^2 = 2h - \frac{K_3^2}{C} - \frac{(C_1 - K_3\cos\theta)^2}{\sin^\theta} - 2\cos\theta,$$

which is obtained if one sets $A = mgl = 1$ (without loss of generality). With the change of variable $u = \cos\theta$, this equation can be reduced to the elliptic quadrature

$$u'_t = \sqrt{R(u)},$$
$$R(u) = 2(h_1 - u)(1-u^2) - (C_1 - K_3 u)^2, \quad h_1 = h - \frac{K_3^2}{2C}.$$

To determine the full motion of the system, one has to integrate the following two equations:

$$\psi'_t = \frac{C_1 - K_3 u}{1-u^2}, \qquad \varphi'_t = \Bigl(\frac{1}{C} - 1\Bigr)K_3 + \frac{C_1 - K_3 u}{1-u^2}.$$

Depending on the initial data and specific parameters of the problem, the solution defines four types of motion, in one of which the axis of the top asymptotically tends to a vertical positions.

► **Sofia Kovalevskaya's case.**

The body is dynamically symmetric with $A = B$ and, in addition, the condition $A = 2C$ holds. The center of mass lies in the equatorial plane of the inertia ellipsoid (its center at the fixed point) and its position in the frame $\xi\eta\zeta$ is $\mathbf{r} = (L, 0, 0)^T$. The system is in a homogeneous gravitational field, so that $\mathbf{M} = mg(\mathbf{r} \times \boldsymbol{\gamma})$. For simplicity, we assume that $A = 1$, $mg = 1$, and $L = 1$.

This case is much more complex than the previous two, both in the way how the equations are integrated and from the viewpoint of the qualitative analysis of the motion. The Euler equations (1)–(6) admit the following three first integrals:

$$(\mathbf{K}\cdot\boldsymbol{\gamma}) = K_1\gamma_1 + K_2\gamma_2 + K_3\gamma_3 = c = \text{const} \quad \text{(conservation of the angular momentum projection onto the vertical)};$$

$$\tfrac{1}{2}(K_1^2 + K_2^2 + K_3^2) - L\gamma_1 = h = \text{const} \quad \text{(energy integral)};$$

$$\left(\frac{K_1^2 + K_2^2}{2} + \gamma_1 x\right)^2 + (K_1K_2 + \gamma_2 x)^2 = k = \text{const} \quad \text{(integral having no clear physical meaning)}.$$

The equations of motion are integrated using Kovalevskaya's variables (s_1, s_2), which are defined as follows:

$$s_1 = \frac{R - \sqrt{R_1R_2}}{2(z_1 - z_2)^2}, \quad s_2 = \frac{R + \sqrt{R_1R_2}}{2(z_1 - z_2)^2},$$
$$z_1 = K_\xi + iK_\eta, \quad z_2 = K_\xi - iK_\eta, \quad i^2 = -1,$$
$$R = R(z_1, z_2) = \tfrac{1}{4}z_1^2z_2^2 - \tfrac{1}{2}h(z_1^2 + z_2^2) + c(z_1 + z_2) + \tfrac{1}{4}k^2 - 1,$$
$$R_1 = R(z_1, z_1), \quad R_2 = R(z_2, z_2).$$

In these variables, the equations of motion become

$$\frac{ds_1}{dt} = \frac{\sqrt{P(s_1)}}{s_1 - s_2}, \quad \frac{ds_2}{dt} = \frac{\sqrt{P(s_2)}}{s_2 - s_1}, \tag{19}$$

where

$$P(s) = \left[(2s + \tfrac{1}{2}h)^2 - \tfrac{1}{16}k^2\right]\left[4s^3 + 2hs^2 + \tfrac{1}{16}(4h^2 - k^2 + 4)s + \tfrac{1}{16}c^2\right].$$

By eliminating t, system (19) can be reduced to a separable equation, which is easy to integrate. As a results, equations (19) also convert into separable equations.

⊙ *Literature for Section 18*: C. G. J. Jacobi (1884), S. Kowalewsky (1889, 1890), J. L. Lagrange (1889), F. Klein and A. Sommerfeld (1965), E. Kamke (1977), J. R. Ray and J. L. Reid (1979), V. F. Zhuravlev (1996), A. V. Borisov and I. S. Mamaev (2001), A. P. Markeev (2001), V. Ph. Zhuravlev (2001), F. R. Gantmakher (2002), D. M. Klimov and V. Ph. Zhuravlev (2002), A. D. Polyanin (2006), A. D. Polyanin and A. V. Manzhirov (2007), P. P. Teodorescu (2009).

Part III

Symbolic and Numerical Solutions of ODEs with Maple, Mathematica, and MATLAB

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Chapter 19

Symbolic and Numerical Solutions of ODEs with Maple

19.1 Introduction

19.1.1 Preliminary Remarks

In recent years, with the development of computers, supercomputers, computer algebra systems (such as Maple™ and Mathematica®), and interactive programming environments for scientific computing (such as MATLAB®), there has been an increasing trend in mathematical research towards modern and powerful computational methods for analytical, symbolic, numerical, and graphical solution of ODEs. Moreover, the use of mathematical computer packages is now a standard part of the modern undergraduate and graduate curriculum and an important tool in the core curriculum in mathematics, science, and engineering.

Maple is a general-purpose computer algebra system in which symbolic computation can readily be combined with exact and approximate (floating-point) numerical computation as well as with arbitrary-precision numerical computation. Maple provides powerful scientific graphics capabilities [for details, see Kreyszig (1994), Corless (1995), Heck (2003), Richards (2002), Abel (2005), Meade *et al.* (2009), Shingareva and Lizárraga-Celaya (2011), etc.].

In general, Maple offers the most comprehensive software support available for differential equations. For example, in Maple (Ver. ≥ 15) we can (in one step) obtain symbolic solutions of 97.5% of the 1345 solvable linear and nonlinear ODEs in the classical handbook by Kamke (1977) [see Maplesoft (2012)] with the aid of the general ODE solver `dsolve`. Moreover, one can obtain solutions (of various types) to ODEs: in *one step* (automatically), i.e., without all details of the mathematical methods applied, or *step by step*, i.e., with control of the choice of the solution strategy at each step, or *by hand*, i.e., by developing appropriate procedures and functions for solving ODEs.

In this chapter, following the most important ideas and methods, we propose and develop new computer algebra ideas and methods to obtain analytical, symbolic, numerical, and graphical solutions for studying ordinary differential equations. We compute analytical and numerical solutions in terms of predefined functions (which are an implementation of known methods for solving ODEs) and develop new procedures for constructing new solutions using Maple. We show a very helpful role that computer algebra systems play in the analytical derivation of numerical methods, computing numerical solutions, and comparing numerical and analytical solutions.

Remark 19.1. The first concept of Maple and its initial versions were developed by the Symbolic Computation Group at the University of Waterloo in the early 1980s. The Maplesoft company was created in 1988. Maple was mainly developed in research labs at Waterloo University and at the University of Western Ontario [see Char *et al.* (1992) and Geddes, Czapor, and Labahn (1992)], with important contributions from research groups at other universities worldwide.

19.1.2 Brief Introduction to Maple

▶ **Maple's conventions and terminology.**

In this chapter, we use the following conventions introduced in Maple:

- `_Cn` ($n = 1, 2, \dots$), for arbitrary constants
- `_Fn`, for arbitrary functions
- `_c[n]`, for arbitrary constants arising in separation of variables
- `_s`, for the parameter in the characteristic system
- `&where`, for the solution structure
- $\_\varepsilon$, for a Lie group parameter

Also we introduce the following notation for the Maple solutions:

- `Eqn`, for equations ($n = 1, 2, \dots$)
- `ODEn`, for ODEs
- `IVPn`, for initial value problems
- `BVPn`, for boundary value problems
- `Soln`, for solutions
- `Trn`, for transformations
- `Sysn`, for systems
- `ICn`, `BCn`, for initial and boundary conditions
- `Ln`, for lists of expressions
- `Gn`, for graphs of solutions
- `ops`, for options (various optional arguments) in predefined functions
- `vars`, for independent variables
- `funcs`, for dependent variables (indeterminate functions)

▶ **Most important features.**

The most important features of Maple are as follows: fast symbolic and numerical computation and interactive visualization; simplicity of use; simplicity of incorporating new user-defined capabilities; understandability, open-source software development path; availability for almost all operating systems; powerful programming language, intuitive syntax, and easy debugging; extensive library of mathematical functions and specialized packages; free resources and the collaborative character of development (for example, see Maple Web Site `www.maplesoft.com` (`MWS`), Maple Application Center `MWS/applications`, Maple Community `MWS/community`, Student Help Center `MWS/studentcenter`, and Teacher Resource Center `MWS/TeacherResource`).

► Basic parts.

Maple consists of three parts: the interface, the kernel (basic computational engine), and the library.

The interface and the kernel (written in C programming language) form a smaller part of the system (they are loaded when a Maple session is started).

The interface handles the input of mathematical expressions, output display, plotting of functions, and support of other user communication with the system. The user interface is the Maple worksheet.

The kernel interprets the user input, carries out basic algebraic operations, and deals with storage management.

The library consists of two parts, the main library and additional packages. The main library (written in the Maple programming language) includes many functions in which most of the common mathematical knowledge of Maple resides.

► Basic concepts.

The *prompt symbol* (>) indicates where to type a Maple expression or function; pressing `Enter` after a semicolon (;) or colon (:) symbol* at the end of the expression tells Maple to evaluate the expression, display the result (no result is displayed after a colon), and insert a new prompt.

Maple contains a complete *online help system* and a command line help system, which can be used, e.g., by typing `?NameOfFunction` or `help(NameOfFunction);` or `?help`; reference information can also be accessed by using the `Help` menu, by highlighting a function and then pressing `Ctrl-F1` or `F1` or `F2` (for Ver. ≥ 9), and by pressing `Ctrl-F2`.

Maple worksheets are files that keep track of the working process and organize it as a collection of expandable groups (see `?worksheet`, `?shortcut`). It is best to begin a new worksheet (or a new problem) with the statement `restart` to clean Maple's memory. All examples and problems in the book are assumed to begin with `restart`.

Previous results (during a session) can be referenced with symbols `%` (the last result), `%%` (the next-to-last result), and `%%...%` (k times) (the kth next-to-last result).

Comments can be included with the sharp symbol # and all characters following it up to the right end of a line. Also text can be inserted with `Insert` → `Text`.

Incorrect response. If you get no response or an incorrect response, you may have entered or executed a function incorrectly. Correct the function or interrupt the computation (click the stop button in the Tool Bar menu).

Maple source code can be viewed for most of the functions, general and specialized (package functions); e.g., `interface(verboseproc=2);print(factor);`

Palettes can be used for building or editing mathematical expressions without needing to remember the Maple syntax.

The Maplet User Interface (for Ver. ≥ 8) consists of *Maplet applications* that are collections of windows, dialogs, and actions (see `?Maplets`).

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*In earlier versions of Maple and in *Classic Worksheet Maple*, we have to end an expression with a colon or semicolon. In these chapters, we follow this tradition in every example and problem.

A number of specialized functions are available in various specialized *packages (subpackages)* (see `?index[package],with`).

Numerical approximations: numerical approximation of `expr` to 10 significant digits, `evalf(expr);` global change of precision `Digits:=n` (see `?environment`); local change of precision, `evalf(expr,n);` numerical approximation to `expr` using a binary hardware floating-point system, `evalhf(expr);` performing numerical approximations using hardware or software floating-point systems, `UseHardwareFloats:= value` (for details, see `?UseHardwareFloats`, `?environment`).

19.1.3 Maple Language

► **Basic elements of the Maple language.**

Maple language is a high-level programming language, which is well-structured and comprehensible. It supports a large collection of data structures, or Maple objects (functions, sequences, sets, lists, arrays, tables, matrices, vectors, etc.), and operations on these objects (type-testing, selection, composition, etc.). Maple procedures in the library are available in readable form. The library can be supplemented with locally developed user programs and packages.

Arithmetic operators: `+ - * / ^`, *logic operators*: `and`, `or`, `xor`, `implies`, `not`, *relation operators*: <, <=, >, >=, =, <>.

A variable name is a combination of letters, digits, and the underline symbol (_), starting from a letter; e.g., `a12_new`.

Abbreviations for longer Maple functions or any expressions: `alias`, for example, `alias(H=Heaviside); diff(H(t),t);` to remove this abbreviation, type the following: `alias(H=H);`

Maple is case sensitive; i.e., there is a difference between lowercase and uppercase letters; e.g., `evalf(Pi)` and `evalf(pi)` are different commands.

Various reserved keywords, symbols, names, and functions: these words cannot be used as variable names; e.g., operator keywords, additional language keywords, and global names that start with (_) (see `?reserved`, `?ininames`, `?inifncs`, `?names`).

The assignment/unassignment operators: a variable can be "free" (with no assigned value) or can be assigned any value (symbolic or numeric) by the assignment operators `a:=b` or `assign(a=b)`. To unassign (clear) an assigned variable (see `?:=` and `?'`), type, e.g., `x:='x'`, `evaln(x)`, or `unassign('x')`.

The difference between the *operators (:=) and (=)* is as follows: the operator `A:=B` is used to assign `B` to the variable `A`, and the operator `A=B` is used to indicate equality (not assignment) between the left- and right-hand sides (see `?rhs`), e.g., `Equation:=A=B; Equation; rhs(Equation); lhs(Equation);`

The range operator `(..)`, an expression of type range `expr1..expr2`; for example, `a[i]$ i=1..9; plot(sin(x),x=-Pi..Pi);`

Statements are keyboard input instructions executed by Maple (e.g., `break`, `by`, `do`, `end`, `for`, `function`, `if`, `proc`, `restart`, `return`, `save`, and `while`).

The statement separators are the semicolon (;) and colon (:). The result of a statement followed with a semicolon (;) will be displayed, and it will not be displayed if it is followed by a colon (:); e.g., `plot(sin(x),x=0..Pi); plot(sin(x),x=0..Pi):`

An expression is a valid statement and is formed as a combination of constants, variables, operators, and functions. Every expression is represented as a tree structure in which each node (and leaf) has a particular data type. For the analysis of any node and branch, the functions `type`, `whattype`, `nops`, and `op` can be used. *A Boolean expression* is formed with *logical operators* and relation operators.

An equation is represented using the binary operator `(=)` and has two operands, the left-hand side, `lhs`, and the right-hand side, `rhs`.

Inequalities are represented using relation operators and have two operands, the left-hand side, `lhs`, and the right-hand side, `rhs`.

A string is a sequence of characters having no value other than itself; it cannot be assigned to, and will always evaluate to itself. For instance, `x:="string"; sqrt(x);` is an invalid function. Names and strings can be used with the `convert` and `printf` functions.

Maple is sensitive to types of brackets and quotes.

Types of brackets: parentheses for grouping expressions, `(x+9)*3`, and for delimiting the arguments of functions, `sin(x)`; square brackets for constructing lists, `[a,b,c]`, vectors, matrices, and arrays; curly brackets for constructing sets, `{a,b,c}`.

Types of quotes: forward-quotes to delay the evaluation of expression, `'x+9+1'`, to clear variables, `x:='x'`; back-quotes to form a symbol or a name, ``` `the name:=7`; k:=5; print(`the value of k is`,k); ``` double quotes to create strings; and a single double quote `"` to delimit strings.

Types of numbers: integer, rational, real, complex, root; e.g., `-55`, `5/6`, `3.4`, `-2.3e4`, `Float(23,-45),3-4*I, Complex(2/3,3); RootOf(_Z^3-2,index=1);`

Predefined constants: symbols for commonly used mathematical constants, `gamma`, `Pi`, `I`, `true`, `false`, `infinity`, `FAIL`, `exp(1)` (see `?ininames`, `?constants`).

Functions or function expressions have the form `f(x)` or `expr(args)` and represent a function call, or an application of a function (or procedure) to arguments (`args`). *Active functions* (beginning with a lowercase letter) are used for computing; e.g., `diff`, `int`, `limit`. *Inert functions* (beginning with a capital letter) are used for showing steps in the problem-solving process; e.g., `Diff`, `Int`, `Limit`.

Library functions (or predefined functions) and user-defined functions.

Predefined functions: most of the well-known functions are predefined by Maple, and they are known to some Maple functions (e.g., `diff`, `evalc`, `evalf`, `expand`, `series`, `simplify`). Numerous special functions are defined (see `?FunctionAdvisor`)*.

User-defined functions: the functional operator (`->`) (see `?->`); for example, the function $f(x) = \sin x$ is defined as `f:=x->sin(x);`

Alternative definitions of functions: `unapply` converts an expression to a function, and a procedure is defined with `proc`.

Evaluation of function $f(x)$ at $x = a$, $\{x = a, y = b\}$; e.g., `f(a); subs(x=a, f(x)); eval(f(x),x=a);`

In Maple language, there are two forms of modularity: *procedures* and *modules*.

*The `FunctionAdvisor` is the computer algebra handbook for mathematical functions.

A procedure (see `?procedure`) is a block of statements which one needs to use repeatedly. A procedure can be used to define a function (if the function is too complicated to be written with the use of the arrow operator) or to create a matrix, graph, or logical value.

A module (see `?module`) is a generalization of the procedure concept. While a procedure groups a sequence of statements into a single statement (block of statements), a module groups related functions and data.

In Maple language, there are essentially *two control structures*, the selection structure `if` and the repetition structure `for`.

Maple objects, sequences, lists, sets, tables, arrays, vectors, and matrices are used for representing more complicated data.

Sequences `a1,a2,a3`, *lists* `[a1,a2,a3]`, and *sets* $\{a1, a2, a3\}$ are groups of expressions. Maple preserves the order and repetitions in sequences and lists and does not preserve them in sets. The order in sets can change during a Maple session.

A *table* is a group of expressions represented in tabular form. Each entry has an index (an integer or any arbitrary expression) and a value (see `?table`).

An *array* is a table with an integer range of indices (see `?Array`). In Maple, arrays can be of any dimension (depending of computer memory).

A *vector* is a one-dimensional array with a positive integer range of indices (for details, see `?vector`, `?Vector`).

A *matrix* is a two-dimensional array with a positive integer range of indices (for details, see `?matrix`, `?Matrix`).

► **Different types of symbolic notation for derivatives.**

In Maple, it is possible to work with derivatives written in the Leibniz, Lagrange, and Arbogast notation. Maple has two different types of symbolic notation for derivatives [for details, see Shingareva and Lizárraga-Celaya (2015)], `D` and `diff`, which can be used for derivatives of single- and multivariable expressions and functions.

- The *functional derivative notation* (or the *differential operator notation*):

```
D(f),        D[1$2](f),    ...,   D[1$n](f), D[1](f)(t),
D[1$2](f)(t), ...,   D[1$n](f)(t)
```

  or

```
D(f),        (D@@2)(f),    ...,   (D@@n)(f), D(f)(t),
(D@@2)(f)(t), ...,   (D@@n)(f)(t)
```

The function `f(t)` can be defined explicitly; e.g., `f:=x->sin(x)`. According to the defined *Maple output*, it is the Arbogast notation. The Arbogast notation can be converted to the Leibniz notation with the aid of `convert`; e.g., `convert(D(f)(x),diff)`. The symbol `f` is the name of a *function*, the symbol `D(f)` is the name of the *derivative function*, and the symbol `D(f)(t)` is the value of the *derivative function*.

- The *expression derivative notation*:

  `diff(y(x),x),   diff(y(x),x$2),   ...,   diff(y(x),x$n)`

The expression `y(x)` can be defined explicitly; e.g., `y:=sin(x)`. According to the defined *Maple output*, it can be the *Leibniz* or *Lagrange notation*, depending on the settings in the statement `interface(typesetting=A)`, where, respectively, `A=standard` or `A=extended`. The symbol `y` is the name of an *expression* in `x`, and the symbolic notation `diff(y(x),x)` denotes the name of the *expression* for its derivative. Also, the Lagrange notation can be displayed in another way if we indicate the following statements: `with(PDEtools),declare(y(x),prime=x)`,* and for returning to the Leibniz notation, we indicate `OFF`.

Remark 19.2. It should be noted that there is a function `convert` that permits switching between the functional and expression derivative notation; e.g., `convert(D(y)(x),diff)`, `convert(diff(y(x),x),D)`.

⊙ *Literature for Section 18.1*: E. Kamke (1977), J. A. van Hulzen and J. Calmet (1983), A. G. Akritas (1989), B. W Char, K. O. Geddes, G. H. Gonnet, M. B. Monagan, and S. M. Watt (1992), S. R. Czapor, K. O. Geddes, and G. Labahn (1992), J. H. Davenport, Y. Siret, and E. Tournier (1993), E. Kreyszig and E. J. Normington (1994), R. M. Corless (1995), D. Zwillinger (1997), M. J. Wester (1999), D. Richards (2002), A. Heck (2003), M. L. Abel and J. P. Braselton (2005), A. D. Polyanin and V. F. Zaitsev (2003), C.-K. Cheung, D. B. Meade, S. J. M. May, and G. E. Keough (2009), I. K. Shingareva and C. Lizárraga-Celaya (2009, 2011, 2015), Maplesoft (2012).

19.2 Analytical Solutions and Their Visualizations

19.2.1 Exact Analytical Solutions in Terms of Predefined Functions

The computer algebra system Maple has various predefined functions based on symbolic algorithms for constructing analytical solutions of ODEs [see a more detailed description in Cheb-Terrab *et al.* (1997)]. Although predefined functions are an implementation of known methods for solving ODEs, this permits solving ordinary differential equations and obtaining solutions automatically (in terms of predefined functions) as well as developing new methods and procedures for constructing new solutions.

Consider the most relevant related functions for finding all possible analytical solutions of a given ODE problem.

```
dsolve(ODE);      dsolve(ODE,y(x),ops);     dsolve(ODE,Lie);
dsolve({ODE,ICs},y(x),ops);  dsolve[interactive](ODE,ops);
dsolve(ODE,y(x),'formal_series','coeffs'=CoeffType);
dsolve(ODE,y(x),'formal_solution','coeffs'=CoeffType,ops);
dsolve(ODE,y(x),method=transform);  odetest(Sol,ODE,y(x));
with(DEtools); odeadvisor(ODE);   particularsol(ODE,y(x));
dchange(rules,ODE);  riccatisol(ODE,y(x)); intfactor(ODE);
infolevel[dsolve]:=L;with(PDEtools);declare(y(x),prime=x);
Solve(ODE,vars,ops);
```

*We declare that `y(x)` will be displayed as `y`, and the derivatives of the expressions with respect to `x` will be displayed in the Lagrange notation.

Remark 19.3. `infolevel[dsolve]:=L` (L∈ ℕ) prints useful information for solving ODEs in Maple (the default value is 1). One can set a value of `L` before working with the solver `dsolve`. Maple conventions are $L = 2, 3$ for general information (including technique or algorithm being used) and $L = 4, 5$ for more detailed information (how the problem is being solved).

- `dsolve`, solving ODEs of various types (the main general ODE solver); for more details see `?dsolve`
- `dsolve` can solve different types of ODE problems, e.g., find analytical solutions:

  `dsolve,ODE`, finding closed-form solutions for a single ODE or a system of ODEs*

  `dsolve,ICs`, solving ODEs or a system of ODEs with given initial or boundary conditions

  `dsolve,formal series`, finding formal power series solutions for a linear ODE with polynomial coefficients

  `dsolve,formal solution`, finding formal solution for a linear ODE with polynomial coefficients

  `dsolve,method`, finding solutions using integral transforms

  `dsolve,Lie`, solving ODEs using the Lie method of symmetries

  `dsolve,series`, finding series solutions for ODE problems

- `dsolve[interactive]`, interactive symbolic and numeric solving of ODEs
- `odetest`, verifying explicit and implicit solutions for ODEs
- `DEtools` (package), a collection of functions for working with ODEs and their solutions; e.g.,

  `odeadvisor`, classifying ODEs and suggesting solution methods

  `riccatisol`, finding solutions of a first-order Riccati ODE

  `particularsol`, finding a particular solution of a nonlinear ODE (or a linear nonhomogeneous ODE) without computing its general solution

  `intfactor`, looking for integrating factors for a given ODE

- `PDEtools` (package), a collection of functions for working with PDEs and ODEs and their solutions; e.g.,

  `declare,prime`, declaring a function for the prime notation

  `dchange`, performing change of variables in ODEs

  `Solve`, finding exact, series, or numerical solutions of equations or systems (of algebraic or differential equations, including inequalities and initial and boundary conditions)

*See Section 19.2.4 for systems of ODEs.

► Verification of exact solutions.

Assume that we have obtained exact solutions and wish to verify whether these solutions are exact solutions of given ODEs.

Example 19.1. *First-order nonlinear ODE. Special Riccati equation. Verification of solutions.* For the first-order nonlinear ODE, the special Riccati equation

$$y'_x = ay^2 + bx^n,$$

we can verify that the solutions

$$y(x) = -\frac{1}{a}\frac{w'_x}{w},$$

where

$$w(x) = \sqrt{x}\left[C_1 J_v\left(\frac{\sqrt{ab}}{k}x^k\right) + C_2 Y_v\left(\frac{\sqrt{ab}}{k}x^k\right)\right], \qquad k = \frac{1}{2}(n+2), \quad v = \frac{1}{2k},$$

are exact solutions of the special Riccati equation as follows:

```
with(PDEtools): declare(y(x),w(x),prime=x);
k:=(n+2)/2; v:=1/(2*k); q:=1/k*sqrt(a*b);
w(x):=sqrt(x)*(C1*BesselJ(v,q*x^k)+C2*BesselY(v,q*x^k));
ODE1:=diff(y(x),x)=a*(y(x))^2+b*x^n;
Sol1:=y(x)=-1/a*diff(w(x),x)/w(x); Test1:=odetest(Sol1,ODE1);
```

Here a, b, and n are real parameters ($ab \neq 0$, $n \neq -2$), $J_v(x)$ and $Y_v(x)$ are the Bessel functions, and C_1 and C_2 are arbitrary constants.

► Finding and verification of exact solutions.

Let us find exact solutions and verify whether these solutions are exact solutions of given ODEs.

Example 19.2. *First-order linear ODE. Finding and verification of the general solution.* For the first-order linear ODE of the general form,

$$g(x)y'_x = f_1(x)y + f_0(x),$$

we can find and verify that the solution

$$y(x) = Ce^F + e^F \int e^{-F}\frac{f_0(x)}{g(x)}\,dx, \quad \text{where} \quad F = \int \frac{f_1(x)}{g(x)}\,dx,$$

is the general solution of the first-order linear ODE as follows:

```
with(PDEtools): declare(y(x),prime=x);
ODE1:=g(x)*diff(y(x),x)=f1(x)*y(x)+f0(x); F:= int(f1(x)/g(x),x);
Sol1:=subs(_C1=C,expand(dsolve(ODE1,y(x))));
Test1:=odetest(Sol1,ODE1);
```

where $f_0(x)$, $f_1(x)$, and $g(x)$ are arbitrary functions, C is an arbitrary constant, and the Maple result reads:

$$Sol1 := y(x) = \mathrm{e}^{\int \frac{f1(x)}{g(x)}\,\mathrm{d}x}\left(\int \frac{f0(x)}{g(x)\,\mathrm{e}^{\int \frac{f1(x)}{g(x)}\,\mathrm{d}x}}\,\mathrm{d}x\right) + \mathrm{e}^{\int \frac{f1(x)}{g(x)}\,\mathrm{d}x}C$$

Example 19.3. *Clairaut's equation. Finding and verification of solutions.*
For Clairaut's equation

$$y = xy'_x + f(y'_x),$$

we can find and verify that

$$y(x) = Cx + f(C) \quad \text{and} \quad x(t) = -f'_t,\ y(t) = -tf'_t + f(t)$$

are, respectively, the general solution and a parametric solution (which is a singular solution) of this equation as follows:

```
with(PDEtools): with(DEtools): declare(y(x),prime=x);
ODE1:=y(x)=x*diff(y(x),x)+f(diff(y(x),x));
Sol1:=y(x)=C*x +f(C); Sol2:=subs({_C1=C,_T=t},[dsolve(ODE1,y(x))]);
Test1:=odetest(Sol1,ODE1); Test2:=odetest(Sol2[2],ODE1);
clairautsol(ODE1, y(x)); parametricsol(ODE1);
eliminate({op(Sol2[2])}, t);
```

Here $f(x)$ is an arbitrary function and C is an arbitrary constant.

Example 19.4. *Nonlinear ODE of the first order. Particular solutions.*
The particular solutions $y(x) = \pm\frac{2}{3}\sqrt{3}\sqrt{x\sqrt{ax}}$ of the nonlinear first-order ODE

$$y^2(y'_x)^2 - ax = 0 \qquad (a > 0)$$

can be found and tested as follows:

```
with(PDEtools): with(DEtools): declare(y(x),prime=x);
ODE1:=y(x)^2*diff(y(x),x)^2-a*x=0;
Sol1:=[particularsol(ODE1,y(x))]; n:=nops(Sol1);
for i from 1 to n do Test[i]:=odetest(Sol1[i],ODE1); od;
```

► Graphical solutions.

Consider the most relevant related functions for plotting solutions of ordinary differential equations.

```
Sol:=rhs(dsolve(ODE,y(x),ops));    S:=unapply(Sol1,x,pars);
plot(S,xR,yR,ops);      SolN:=dsolve(ODE,numeric,vars,ops);
with(plots):                      odeplot(SolN,vars,tR,ops);
plot([x(t),y(t),tR],xR,ops); implicitplot(f(x,y)=c,xR,yR);
with(DETools);             DEplot(ODEs,vars,tR,IC,xR,yR,ops);
DEplot3d(ODEs,vars,tR,IC,xR,yR,zR,ops);
dfieldplot(ODEs,vars,tR,xR,yR,ops);
                              phaseportrait(ODEs,vars,tR,IC,ops);
```

Remark 19.4. Here `tR`, `xR`, and `yR` are the ranges of the independent and dependent variables, `t=t1..t2`, `x=x1..x2`, and `y=y1..y2`.

- `plots,odeplot`, constructing graphs or animations of 2D and 3D solution curves obtained from the numerical solution (`dsolve,numeric`; see Section 19.4)

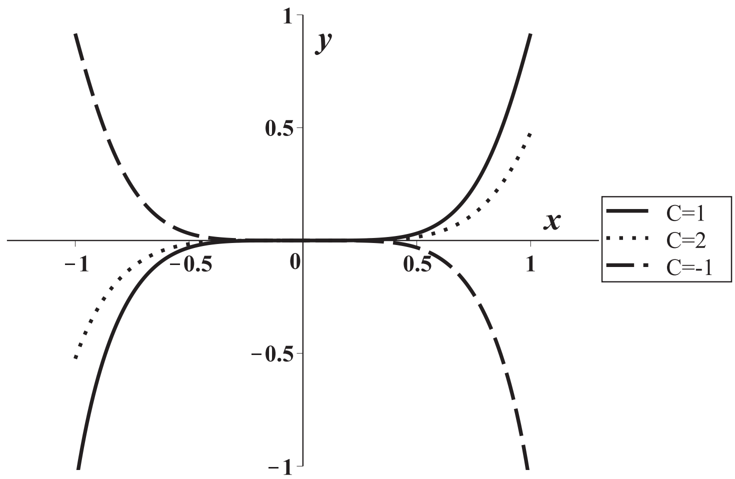


Figure 19.1: Graphical solutions of the Bernoulli equation $y'_x - \frac{5}{x}y = -x^5y^2$.

- DEtools,DEplot,DEplot3d, constructing graphs or animations of 2D and 3D solutions of a single ODE (of any order) or a system of first-order ODEs using numerical methods
- DEtools,dfieldplot, plotting direction field to a system of first-order ODEs
- DEtools,phaseportrait, constructing phase portraits (solutions curves) for a single higher-order ODE or a system of first-order ODEs with initial conditions by numerical methods

Example 19.5. *Nonlinear ODE of the first order. The Bernoulli equation. Graphical solutions.* For the Bernoulli equation

$$y'_x + f(x)y = g(x)y^a,$$

where $a \neq 0, 1$, we can find the general solution (Sol1), and the Maple result reads:

$$Sol1 := \mathrm{e}^{-\int f(x)\,\mathrm{d}x}\left(-a\int g(x)\mathrm{e}^{-(\int f(x)\,\mathrm{d}x)(a-1)}\,\mathrm{d}x + \_C1 + \int g(x)\mathrm{e}^{-(\int f(x)\,\mathrm{d}x)(a-1)}\,\mathrm{d}x\right)^{-\frac{1}{a-1}}$$

We generate graphical solutions of the Bernoulli equation (presented in Fig. 19.1) for a particular case, for example, $f(x) = -5/x$, $g(x) = -x^5$, $a = 2$, as follows:

```
with(DEtools): declare(y(x),prime=x); with(plots):
ODE1:=diff(y(x),x)+f(x)*y(x)=g(x)*y(x)^a;
Sol1:=simplify(rhs(dsolve(ODE1,y(x))));
Sol11:=simplify(value(subs(f(x)=-5/x, g(x)=-x^5,a=2,Sol1)));
S1:=unapply(Sol11,x,_C1); xR:=x=-1..1; yR:=-1..1;
ops:=color=[red,blue,magenta];
plot([S1(x,1),S1(x,2),S1(x,-1)],xR,yR,ops);
```

Remark 19.5. Throughout the book, graphical solutions cannot be presented in color for technical reasons: this would result in an essential increase in the book price.

► ODE classification and solution methods suggestion.

In Maple, there is a variety of functions for solving various classes of ODEs. The function `odeadvisor` of the `DEtools` package permits classifying a given ODE according to standard reference books [e.g., see Boyce and DiPrima (2004), Ince (1956), El'sgol'ts (1961), Murphy (1960), Kamke (1977), Zwillinger (1997), Polyanin and Zaitsev (2003)] and suggests methods for solving it (for more detail, see `?DEtools`, `?odeadvisor`).

Example 19.6. *ODE of the first order. Classification and method suggestion.*
The separable first-order ODE

$$y'_x = \sin^2 x \left(\frac{y}{1-y}\right)$$

can be classified and analyzed and follows:

```
with(PDEtools): declare(y(x),prime=x); with(DEtools):
infolevel[dsolve]:=3;  ODE1:=diff(y(x),x)=y(x)*sin(x)^2/(1-y(x));
odeadvisor(ODE1); odeadvisor(ODE1,[Abel]); odeadvisor(ODE1,[separable]);
Sol11:=dsolve(ODE1,y(x)); Sol12:=separablesol(ODE1,y(x));
```

Here `infolevel[dsolve]:=3` prints useful information for solving an ODE (the technique or algorithm being used).

Example 19.7. *ODE of the first order. Classification and method suggestion.*
The homogeneous first-order ODE

$$(x^2 - xy)y'_x + y^2 = 0$$

can be classified and analyzed as follows:

```
with(PDEtools): declare(y(x),prime=x); with(DEtools):
infolevel[dsolve]:=3; ODE1:=(x^2-y(x)*x)*diff(y(x),x)+y(x)^2=0;
odeadvisor(ODE1); Sol1:=dsolve(ODE1,y(x));
Sol2:=genhomosol(ODE1,y(x)); Sol3:=particularsol(ODE1,y(x));
```

Here the predefined function `genhomosol` determines whether the given ODE is a *general homogeneous ODE of the first order*, and if so, a solution is obtained.

► Order reduction as a result of `dsolve`.

For some second-order and higher-order ODEs, the order of the ODE can be reduced (without obtaining a final solution) with the aid of the predefined function `dsolve`. In this case, the result can be expressed using `ODESolStruc`. Then we can obtain a solution for the reduced ODE in various forms (e.g., as an exact solution using predefined functions available in `DEtools` package or as a series or numerical solution). If we have obtained a solution of the reduced ODE, then a solution of the original problem can be built using `DEtools,buildsol` or constructed numerically or by using analytical approximate methods.

Example 19.8. *Second-order nonlinear ODE. The van der Pol equation. Order reduction.*
Order reduction for the van der Pol equation

$$y''_{xx} + a(y^2 - 1)y'_x + by = 0 \qquad (a, b \in \mathbb{R}),$$

can be found and tested as follows:

```
with(PDEtools): declare(y(x),prime=x);
ODE1:=diff(y(x),x$2)+a*(y(x)^2-1)*diff(y(x),x)+b*y(x)=0;
Sol1:=dsolve(ODE1,y(x));
```

where the Maple result reads:

$$Sol1 := y(x) = \_a\&\text{where}\left[\left\{\left(\frac{d}{d\_a}\_b(\_a)\right)\_b(\_a) + \_b(\_a)\_a^2 a - a\_b(\_a) + b\_a = 0\right\},\right.$$
$$\left.\left\{\_a = y(x), \_b(\_a) = \frac{d}{dx}y(x)\right\}, \left\{x = \int (\_b(\_a))^{-1}\, d\_a + \_C1, y(x) = \_a\right\}\right]$$

This result (in general, the function `ODESolStruc`) has two arguments.

The first argument is the *structure* (written in terms of new variables) introduced by `dsolve` during the solving process. It can be selected using `Struc1:=op([2,1],Sol1)`.

The second argument is a list with three sets: the *reduced ODE* written in terms of new variables, the *transformation of variables* used in the reduction process, and the *inverse transformation*.

The notation of the function `ODESolStruc` uses the symbols `` `&where` `` for displaying the first operand and the second operand with the list including the reduced ODE and the transformation equations.

The structure can be verified using `odetest`; the reduced ODE and the transformation equations can be selected as follows:

```
Test1:=odetest(Sol1,ODE1); ODERed:=op([2,2,1,1],Sol1);
TR:=op([2,2,2],Sol1); TRInv:=op([2,2,3],Sol1);
varsNew:=map(lhs,TR); varsOld:=map(lhs,TRInv);
```

The van der Pol equation is reduced to a first-order ODE of Abel type. This can be verified using `DEtools[odeadvisor](ODERed)`.

The original ODE can be obtained by performing a change of variables (with the function `PDEtools,dchange`) in the reduced equation (`ODERed`) using the transformation (`TR`) and, vice versa, the reduced equation (`ODERed`) can be obtained from the original ODE (`ODE1`) by changing variables using the inverse transformation (`TRInv`) as follows:

```
vdPol:=collect(PDEtools[dchange](TR,ODERed),[diff,a]);
vdPolRed:=expand(PDEtools[dchange](TRInv,vanderPol,[_a,_b(_a)]));
```

► Constructing exact explicit and implicit solutions.

If an exact solution is given as a function of the independent variable, then the solution is said to be *explicit*. For some differential equations, explicit solutions cannot be determined, but we can obtain an *implicit form* of the solution, i.e., an equation that involves no derivatives and relates the dependent and independent variables.

```
dsolve(ODE1,y(x),explicit); dsolve(ODE1,y(x),implicit);
implicitplot(f(x,y)=c,x=x1..x2,y=y1..y2,ops);
```

Example 19.9. *First-order separable ODE. Exact implicit solutions. Graphical solutions.*
For the first-order separable ODE

$$y'_x + \frac{x^2}{y} = 0,$$

we can construct the explicit (`Sol1`) and implicit (`Sol2`) solutions

$$y = \pm\frac{1}{3}\sqrt{-6x^3 + 9C_1}, \qquad y^2 + \frac{2}{3}x^3 - C_1 = 0$$

and plot the graph of the implicit solution as follows:

```
with(PDEtools): declare(y(x),prime=x); with(plots):
ODE1:=diff(y(x),x)+x^2/y(x)=0; Sol1:=dsolve(ODE1,y(x));
Sol2:=dsolve(ODE1,y(x),implicit);
G:=subs({y(x)=y},lhs(Sol2)); Gs:=seq(subs(_C1=i,G),i=-5..5);
contourplot({Gs},x=-5..5,y=-10..10,color=blue);
```

Here C_1 is an arbitrary constant.

Example 19.10. *Second-order nonlinear ODE. Exact implicit solutions.*
For the second-order nonlinear ODE

$$y''_{xx} = Ax\frac{(y'_x)^3}{y^2} - A\frac{(y'_x)^2}{y},$$

we can construct and simplify the implicit (`Sol4`) solution

$$(A-1)x - C_1 y + C_2 y^A = 0 \qquad (A \neq 1)$$

as follows:

```
with(PDEtools): declare(y(x),prime=x); with(plots):
ODE1:=diff(y(x),x$2)=A*x/y(x)^2*diff(y(x),x)^3-A/y(x)*diff(y(x),x)^2;
Sol1:=dsolve(ODE1,y(x),implicit); Sol2:=expand(lhs(Sol1))=0;
Sol3:=expand(Sol2*y(x)); Sol4:=collect(Sol3,x); odetest(Sol4,ODE1);
```

► Constructing exact parametric solutions.

Frequently, differential equations can be solved for the independent variable x in terms of the parameter t, and then $x(t)$ can be used to obtain an equation for the dependent variable $y(t)$. For example, the general solution of the equation $y'_x = f(x,y)$ can be found in implicit form, $\Phi(x,y,C) = 0$, or in parametric form, $x = x(t,C)$, $y = y(t,C)$. In Maple, we can construct and visualize exact parametric solutions with the aid of the following predefined functions:

```
dsolve(ODE,y(x),parametric,implicit);
with(DEtools): parametricsol(ODE,y(x),ops);
plot([x(t),y(t),t=t1..t2],x1..x2,y1..y2,ops);
```

Example 19.11. *First-order nonlinear ODE. Exact parametric solutions. Graphical solutions.*
For the first-order nonlinear ODE

$$(y'_x)^2 + ay'_x + by = 0,$$

we can construct and visualize exact parametric solutions (`Sol1`–`Sol5`), e.g.,

$$x(t) = \frac{Cb - a\ln(t) - 2t}{b}, \quad y(t) = -\frac{t(t+a)}{b},$$

as follows:

```
with(PDEtools): with(DEtools): declare(y(x),prime=x); with(plots):
ODE1:=(diff(y(x),x))^2 +a*diff(y(x),x)+b*y(x)=0;
Sol1:=parametricsol(ODE1,y(x));
Sol2:=dsolve(ODE1,y(x),implicit,parametric);
Sol3:=simplify(subs(_C1=C,_T=t,Sol1),symbolic);
Sol4:=simplify(parametricsol(ODE1,y(x),explicit),symbolic);
Sol5:=combine(parametricsol(ODE1,y(x),Lie),symbolic);
tR:=t=0.1..40; tr1:={a=20,b=10}; P1:=subs(tr1,op(Sol3));
P2:=[rhs(P1[1]),rhs(P1[2]),tR]; valG:={seq(subs(C=i,P2),i=-5..5)}:
plot(valG,view=[-10..10,-10..10],axes=boxed);
```

► Constructing exact solutions of higher-order ODEs.

Consider the most relevant related functions for constructing exact solutions of higher-order ordinary differential equations.

```
dsolve(ODE,y(x),ops);   dsolve(ODE,y(x),output=basis,ops);
with(DEtools); constcoeffsols(ODE,y(x));ratsols(ODE,y(x));
expsols(ODE,y(x)); polysols(ODE,y(x));eulersols(ODE,y(x));
```

- `dsolve,output=basis`, finding a fundamental set of solutions of linear ODEs (homogeneous and nonhomogeneous, with constant and variable coefficients)
- `DEtools,constcoeffsols`, finding a fundamental set of solutions of linear ODEs with constant coefficients
- `DEtools,eulersols,expsols,polysols,ratsols`, finding linearly independent solutions of an appropriate form (Eulerian, exponential, polynomial, or rational)

Example 19.12. *Higher-order linear homogeneous ODEs with constant coefficients.*
For the fourth-order linear homogeneous ODE with constant coefficients

$$y''''_x + a_1 y'''_x + a_2 y''_x + a_3 y' + a_4 y = 0,$$

where the constant coefficients are $a_1 = 1$, $a_2 = -1$, $a_3 = 5$, $a_4 = -2$ and all solutions are of exponential form, we can determine a fundamental set of solutions (`Sol2`)

$$\left\{ e^{(\sqrt{2}-1)x}, e^{-(\sqrt{2}+1)x}, e^{x/2}\cos\left(\frac{\sqrt{7}}{2}x\right), e^{x/2}\sin\left(\frac{\sqrt{7}}{2}x\right) \right\}$$

as follows:

```
with(PDEtools): with(DEtools): with(LinearAlgebra): with(VectorCalculus):
declare(y(x),prime=x); with(plots): ODE1:=diff(y(x),x$4)
 +a[1]*diff(y(x),x$3)+a[2]*diff(y(x),x$2)+a[3]*diff(y(x),x)+a[4]*y(x)=0;
ODE2:=subs({a[1]=1,a[2]=-1,a[3]=5,a[4]=-2},ODE1);
Sol1:=dsolve(ODE2,y(x));  Sol2:=dsolve(ODE2,y(x),output=basis);
Sol3:=expsols(ODE2,y(x)); Sol4:=constcoeffsols(ODE2,y(x));
```

Also, we can verify that these functions are solutions of the given ODE (`Test1`) and that these functions are linearly independent (`Test2`)*:

*Verifying that the Wronskian (the determinant of the Wronskian matrix) has the nonzero value.

```
Test1:=seq(odetest(y(x)=Y,ODE2),Y=Sol2);
A:=Wronskian(Sol2,x); Test2:=simplify(Determinant(A));
```

The superposition principle can be applied for constructing a general solution, since any linear combination of solutions of a homogeneous linear ODE is again a solution of the ODE. The general solution of the nth-order linear ODE is

$$y(x) = \sum_{i=1}^{n} C_i y_i(x),$$

where $y_i(x)$ $(i = 1, \dots, n)$ is a fundamental set of solutions and C_i are arbitrary constants. By applying the superposition principle to the fourth-order linear homogeneous ODE with constant coefficients, we obtain the general solution as follows:

```
SolGen:=y(x)=add(C[i]*Sol2[i],i=1..nops(Sol2));
odetest(SolGen,ODE2);
```

Example 19.13. *Higher-order linear equation with variable coefficients. The Euler equation.*

For the fourth-order linear homogeneous ODE with variable coefficients, the Euler equation

$$a_1 x^4 y''''_x + a_2 x^3 y'''_x + a_3 x^2 y''_x + a_4 x y' + a_5 y = 0,$$

where $a_1 = 1$, $a_2 = 14$, $a_3 = 55$, $a_4 = 65$, $a_5 = 16$, we can determine a fundamental set of solutions (`Sol2`)

$$\left\{ \frac{1}{x^2}, \frac{\ln(x)}{x^2}, \frac{\ln(x)^2}{x^2}, \frac{\ln(x)^3}{x^2} \right\}$$

as follows:

```
with(PDEtools): with(DEtools): with(LinearAlgebra): with(VectorCalculus):
declare(y(x),prime=x); with(plots): ODE1:=a[1]*x^4*diff(y(x),x$4)
 +a[2]*x^3*diff(y(x),x$3)+a[3]*x^2*diff(y(x),x$2)+a[4]*x*diff(y(x),x)
 +a[5]*y(x)=0; ODE2:=subs({a[1]=1,a[2]=14,a[3]=55,a[4]=65,a[5]=16},ODE1);
Sol1:=dsolve(ODE2,y(x));  Sol2:=dsolve(ODE2,y(x),output=basis);
Sol3:=eulersols(ODE2,y(x)); Sol4:=ratsols(ODE2,y(x));
Test1:=seq(odetest(y(x)=Y,ODE2),Y=Sol2);
A:=Wronskian(Sol2,x); Test2:=simplify(Determinant(A));
```

As in the previous example, we verify that these functions are solutions of the given ODE (`Test1`) and that these functions are linearly independent (`Test2`). Since the Wronskian has the nonzero value $12x^{-14}$ for $x \neq 0$, it follows that these four functions are a fundamental set of solutions for this Euler equation on any interval that does not contain the origin.

Example 19.14. *Higher-order linear nonhomogeneous ODEs. General solution.*

The general solution $y(x)$ of a nonhomogeneous linear ODE can be written as the sum of a particular solution $y_{\rm p}(x)$ of the nonhomogeneous equation and the general solution of the corresponding homogeneous equation. The general solution of the homogeneous equation is a linear combination of the solutions in a fundamental set of solutions. The general solution of the nth-order nonhomogeneous linear ODE has the form:

$$y(x) = y_{\rm p}(x) + \sum_{i=1}^{n} C_i y_i(x), \qquad (19.2.1.1)$$

where $y_i(x)$ $(i = 1, \dots, n)$ is a fundamental set of solutions and C_i are arbitrary constants.

Consider the fourth-order linear nonhomogeneous ODE with constant coefficients

$$y''''_x + a_1 y'''_x + a_2 y''_x + a_3 y' + a_4 y = \sin x,$$

where the constant coefficients are $a_1 = 1$, $a_2 = -1$, $a_3 = 5$, $a_4 = -2$. First, we determine a fundamental set of solutions (FundSet1) of the corresponding homogeneous ODE and form the general solution of the homogeneous ODE (SolGenHom). Then we obtain a particular solution of the nonhomogeneous equation (SolPartNonHom) and form the general solution of the nonhomogeneous ODE (SolGenNonHom) according to Eq.(19.2.1.1),

$$y(x) = C_1 e^{x/2} \sin\Big(\frac{\sqrt{7}}{2}x\Big) + C_2 e^{x/2} \cos\Big(\frac{\sqrt{7}}{2}x\Big) + C_3 e^{(\sqrt{2}-1)x} + C_4 e^{-(\sqrt{2}+1)x} - \frac{1}{4}\cos x,$$

as follows:

```
with(PDEtools): with(DEtools): with(LinearAlgebra): with(VectorCalculus):
declare(y(x),prime=x); with(plots):
ODE1:=diff(y(x),x$4)+a[1]*diff(y(x),x$3)+a[2]*diff(y(x),x$2)
 +a[3]*diff(y(x),x)+a[4]*y(x)=sin(x);
ODE2:=subs({a[1]=1,a[2]=-1,a[3]=5,a[4]=-2},ODE1);
FundSet1:=dsolve(eval(ODE2,rhs(ODE2)=0),y(x),output=basis);
SolGenHom:=dsolve(eval(ODE2,rhs(ODE2)=0),y(x));
SolPartNonHom:=particularsol(ODE2,y(x));
SolGenNonHom:=y(x)=rhs(SolGenHom)+rhs(SolPartNonHom);
```

Then we verify that this function is a solution of the given ODE (Test1) and compare the solution SolGenNonHom (as the result of our construction procedure) with the solution Sol1 (which is a 2-element list, where the first element is a fundamental set of solutions and the second element is a particular solution) and the general solution Sol2 (as the result from dsolve). It should be noted that these solutions are the same:

```
Test1:=odetest(SolGenNonHom,ODE2);
Sol1:=dsolve(ODE2,y(x),output=basis); Sol2:=dsolve(ODE2,y(x));
```

19.2.2 Exact Analytical Solutions of Mathematical Problems

► Initial value problems.

In many applications it is required to solve an *initial value problem* or a *Cauchy problem*, i.e., a problem consisting of a differential equation supplemented by one or more initial conditions (which must be satisfied by the solutions). The number of conditions coincides with the order of the equation. Therefore, we have to determine a particular solution that satisfies the given initial conditions.

Consider some initial value problems modeling various processes and phenomena.

Example 19.15. *Malthus model. Cauchy problem. Analytical and graphical solutions.*

A basic *model for population growth* consists of a first-order linear ODE and an initial condition. It has the form

$$y'_t = ky, \quad y(0) = y_0 \quad (k > 0)$$

and was proposed in 1798 by the English economist Thomas Malthus. Here $k > 0$ is a constant representing the rate of growth (the difference between the birth rate and the death rate). The increase in the population is proportional to the total number of people.

We can obtain the solution of this mathematical problem,

$$y(t) = y_0 e^{kt},$$

which predicts exponential growth of the population, as follows:

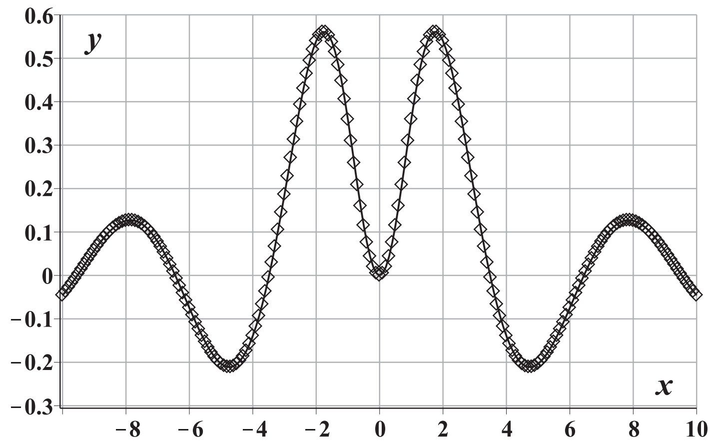


Figure 19.2: Exact solution (solid line) and numerical solution (points) of the Cauchy problem (19.2.2.1).

```
with(PDEtools): declare(y(t),prime=t);
ODE1:=diff(y(t),t)=k*y(t); IC1:=y(0)=y[0];
Sol1:=dsolve({ODE1,IC1},y(t)); Test1:=odetest(Sol1,ODE1);
```

Example 19.16. *Linear ODE. Cauchy problem. Analytical, numerical, and graphical solutions.*
Consider the following second-order linear nonhomogeneous ODE with variable coefficients and with initial conditions:

$$y''_{xx} + xy'_x + y = \cos x, \qquad y(0) = 0, \quad y'_x(0) = 0. \tag{19.2.2.1}$$

Exact analytical and numerical solutions can be constructed as follows:

```
FunctionAdvisor(erf):
Digits:=15: with(PDEtools): declare(y(x),prime=x);
ODE1:=diff(y(x),x$2)+x*diff(y(x),x)+y(x)=cos(x);
IC1:=y(0)=0,D(y)(0)=0; Sol1:=dsolve({ODE1,IC1},y(x));
Test1:=odetest(Sol1,ODE1);
SolN:=dsolve({ODE1,IC1},y(x),type=numeric):
```

The analytical solution has the following form:

$$y(x) = -\frac{\sqrt{2\pi}}{4} e^{-(x-1)(x+1)/2} \left[2\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right) + \mathrm{erf}\left(\frac{I\sqrt{2}(x+I)}{2}\right) + \mathrm{erf}\left(\frac{I\sqrt{2}(I-x)}{2}\right)\right],$$

where erf is a special function (for more details, see `FunctionAdvisor(erf)`) and I is the imaginary unit.*

To obtain real graphical solutions, we make the following additional manipulations with the analytical solution obtained (see the Maple script below, the variable `s[k]`):

1. `eval(rhs(Sol1),x=i)`, evaluating $y(x)$ on a set of points of the interval $[-10, 10]$ (with the predefined function `eval`); e.g., the result at the point $x = -10$ reads:

$$-\tfrac{1}{2}\,\mathrm{e}^{-50}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\tfrac{\sqrt{2}}{2}\right) - \tfrac{1}{4}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}}\sqrt{2}\left(-\mathrm{erf}\left(5\,I\sqrt{2} + \tfrac{\sqrt{2}}{2}\right) - \mathrm{erf}\left(-5\,I\sqrt{2} + \tfrac{\sqrt{2}}{2}\right)\right)\mathrm{e}^{-50}.$$

*The letter I is Maple's notation for the imaginary unit i.

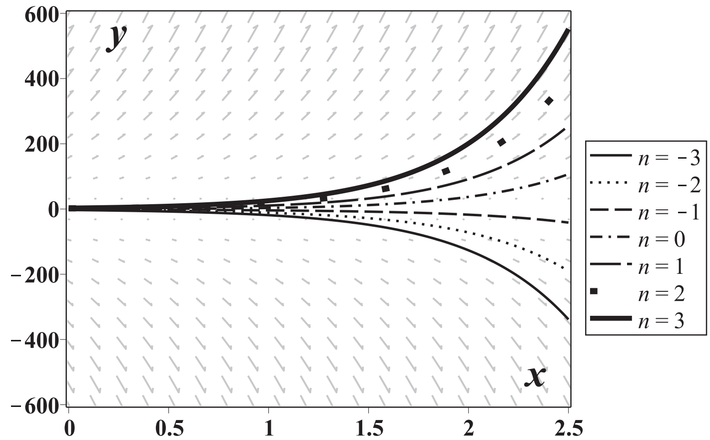


Figure 19.3: Exact solution and the 2-D vector field of the Cauchy problem (19.2.2.2).

2. `evalf(eval(rhs(Sol1),x=i))`, approximating the resulting complex numbers using floating-point arithmetic (with the predefined function `evalf`); e.g., the result at the point $x = -10$ reads: $-0.0458265478887548 - 0.0\,I$.

3. `fnormal(evalf(eval(rhs(Sol1),x=i)))`, carrying out floating-point normalization (with the predefined function `fnormal`).

4. `simplify(fnormal(evalf(eval(rhs(Sol1),x=i))),zero)`, removing a zero imaginary part of the complex floating-point numbers (with the predefined function `simplify`, the option `zero`); e.g., the result at the point $x = -10$ reads: -0.0458265478887548.

Finally, we compare the analytical and numerical solutions as follows:

```
with(plots): k:=0: xR:=x=-10..10;
for i from -10 to 10 by 0.1 do
 k:=k+1: X[k]:=i:
 s[k]:=simplify(fnormal(evalf(eval(rhs(Sol1),x=i))),zero);
od:
N:=k; Seq1:=seq([X[i],s[i]],i=1..N):
G1:=plot([Seq1],style=line,color="MidnightBlue"):
G2:=odeplot(SolN,xR,style=point,color=red,symbolsize=15):
display({G1,G2});
```

As we see in Fig. 19.2, the analytical and numerical solutions are in good agreement.

Example 19.17. *First-order linear ODE. Cauchy problem. Analytical and graphical solutions.* For the first-order linear ODE, with the initial condition,

$$y'_x - 2y = 3x, \qquad y(0) = n, \tag{19.2.2.2}$$

we can determine the exact solution (`Sol1`)

$$y(x) = -\frac{3}{2}x - \frac{3}{4} + e^{2x}\left(n + \frac{3}{4}\right)$$

and construct the direction field (see Fig. 19.3) as follows:

```
with(plots): N:=7; Sols:=Vector(N,0): Gr:=NULL:
setoptions(grid=[30,30]);
IVP1:={D(y)(x)-2*y(x)=3*x,y(0)=n}; Sol1:=dsolve(IVP1,y(x));
for i from 1 to N do Sols[i]:=subs(n=-3+(i-1),Sol1); od:
Sols; SList:=['rhs(Sols[i])' $ 'i'=1..N];
for i from 1 to N do
G||i:=plot(SList[i],x=0..2.5,color=blue): Gr:= Gr,G||i: od:
VField:=fieldplot([1,3*x+2*y],x=0..2.5,y=-600..600):
display({Gr,VField},axes=boxed);
```

► Boundary value problems.

Consider two-point linear boundary value problems that consist of a second-order ODE and boundary conditions at the two endpoints of an interval $[a, b]$. Some (simple) boundary value problems can be solved (with the aid of Maple) analytically as initial value problems except that the value of the function and its derivatives are given at two values of x (the independent variable) instead of one.

We note that an initial value problem has a unique solution, while a boundary value problem may have more than one solution or no solutions at all.

Boundary conditions can be divided into three classes:

(i) the *Dirichlet conditions* (or *first-type boundary conditions*),

$y(a) = g_1, y(b) = g_2$

(ii) the *Neumann boundary conditions* (or *second-type boundary conditions*),

$y'_x(a) = g_1, y'_x(b) = g_2$

(iii) the *Robin boundary conditions* (or *third-type boundary conditions*),

$\alpha_1 y(a) + \beta_1 y'_x(a) = g_1, \alpha_2 y(b) + \beta_2 y'_x(b) = g_2$

Boundary conditions can be homogeneous (if $g_1 = g_2 = 0$) and nonhomogeneous (otherwise).

Example 19.18. *Boundary value problem. Analytical and graphical solutions.*

For a second-order linear homogeneous ODE with constant coefficients and with boundary conditions (the nonhomogeneous Dirichlet conditions),

$$y''_{xx} + a_1 y = 0, \qquad y(a) = g_1, \quad y(b) = g_2, \tag{19.2.2.3}$$

where $a_1 = 2$, $a = 0$, $b = \pi$, $g_1 = 1$, $g_2 = 0$, we can determine the particular analytical solution (`Sol1`)

$$y(x) = -\frac{\cos(\sqrt{2}\pi)\sin(\sqrt{2}x)}{\sin(\sqrt{2}\pi)} + \cos(\sqrt{2}x)$$

and construct the graphical solution as follows:

```
BVP1:={diff(y(x),x$2)+a[1]*y(x)=0,y(a)=g[1],y(b)=g[2]};
BVP2:=subs({a[1]=2,a=0,b=Pi,g[1]=1,g[2]=0},BVP1);
Sol1:=dsolve(BVP2,y(x));
plot(rhs(Sol1),x=0..Pi,axes=boxed,color=blue,thickness=3);
```

Modifying the boundary conditions (the nonhomogeneous Neumann conditions), we obtain the following:

$$y''_{xx} + a_1 y = 0, \qquad y'_x(a) = g_1, \quad y'_x(b) = g_2, \tag{19.2.2.4}$$

where $a_1 = 2$, $a = 0$, $b = \pi$, $g_1 = 1$, $g_2 = 0$, and the particular analytical solution (`Sol2`)

$$y(x) = \frac{1}{2}\sqrt{2}\sin(\sqrt{2}x) + \frac{1}{2}\frac{\sqrt{2}\cos(\sqrt{2}\pi)\cos(\sqrt{2}x)}{\sin(\sqrt{2}\pi)}$$

can be constructed as follows:

```
BVP3:={diff(y(x),x$2)+a[1]*y(x)=0,D(y)(a)=g[1],D(y)(b)=g[2]};
BVP4:=subs({a[1]=2,a=0,b=Pi,g[1]=1,g[2]=0},BVP3);
Sol2:=dsolve(BVP4,y(x));
plot(rhs(Sol2),x=0..Pi,axes=boxed,color=blue,thickness=3);
```

For solving more complicated boundary value problems, we can follow a numerical approach (see Section 19.4.5).

► Eigenvalue problems.

Consider eigenvalue problems, i.e., boundary value problems that include a parameter λ. The parameter values for which the problem is solvable are called eigenvalues of the problem, and for each eigenvalue, the solution $y = y(x)$ ($y \not\equiv 0$) that satisfies the problem is called the corresponding eigenfunction. We will find eigenvalues and eigenfunctions for some eigenvalue problems.

The sufficiently general form of eigenvalue problems reads:

$$a_2(x)y''_{xx} + a_1(x)y'_x + [a_0(x) + \lambda]y = 0, \qquad a < x < b,$$

and the boundary conditions at the endpoints $x = a$ and $x = b$ (see the previous paragraph).

The transformation

$$p(x) = \exp\left[\int \frac{a_1(x)}{a_2(x)}\,dx\right], \quad q(x) = \frac{a_0(x)}{a_2(x)}p(x), \quad s(x) = \frac{p(x)}{a_2(x)}$$

reduces this equation to the differential equation

$$[p(x)y'_x]'_x + [q(x) + \lambda s(x)]y = 0.$$

This form of the equation is called the self-adjoint form.

Example 19.19. *Eigenvalue problem. Dirichlet boundary conditions. Analytical solution.*

Consider a Sturm–Liouville eigenvalue problem consisting of a second-order linear homogeneous ODE with constant coefficients and a parameter λ with homogeneous Dirichlet boundary conditions,

$$y''_{xx} + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0. \qquad (19.2.2.5)$$

If we apply the predefined function `dsolve`,

```
ODE1:=diff(y(x),x$2)+lambda*y(x)=0;
dsolve({ODE1,y(0)=0,y(Pi)=0},y(x));
```

we obtain the trivial solution, $y(x) = 0$, and cannot solve the eigenvalue problem.

However, we can solve such problems step by step with the aid of Maple as follows.

1. We find the characteristic equation (`EqChar`) for the given ODE, $m^2 + \lambda = 0$; the characteristic roots are $m = \pm\sqrt{-\lambda}$ (`RootsChar`):

```
ExpSol:=exp(m*x);
EqChar:=collect(eval(ODE1,y(x)=ExpSol),exp)/ExpSol;
RootsChar:=solve(EqChar,m);
```

2. There are two cases ($\lambda = 0$ and $\lambda \neq 0$). Consider the first case: $\lambda = 0$. The differential equation is $y''_{xx} = 0$ (`Eq1`), and the solution (`Sol1`) of this equation with the first boundary condition is $y(x) = \_C1x$. By applying the second boundary condition to this solution, we obtain the equation $\_C1\pi = 0$ (`Eq2`), so $\_C1 = 0$ and we obtain the trivial solution $y(x) = 0$. Thus, $\lambda = 0$ is not an eigenvalue:

```
Eq1:= eval(ODE1,lambda=0);
Sol1:=dsolve({Eq1,y(0)=0},y(x));
Eq2:=eval(rhs(Sol1),x=Pi)=0;
```

3. Consider the case of $\lambda \neq 0$ and apply the first boundary condition. The resulting solution (`Sol3`) is $y(x) = C\sin(\sqrt{\lambda}x)$. By applying the second boundary condition, we obtain the transcendental equation $C\sin(\sqrt{\lambda}\pi) = 0$ (`Sol4`). For solving this equation correctly, we add the environment variable `_EnvAllSolutions:=true` and finally obtain the eigenvalues

$$\lambda_n = n^2 \quad (n = 1, 2, \ldots)$$

and the eigenfunctions

$$y_n(x) = C\sin(nx)$$

as follows:

```
Sol2:=rhs(dsolve({ODE1,y(0)=0},y(x)));
param:=remove(has,indets(Sol2),{lambda,x})[1];
Sol3:=eval(Sol2,param=C);
Sol4:=eval(Sol3,x=Pi)=0;
_EnvAllSolutions := true;
Sol5:=solve(Sol4,lambda); var:=indets(Sol5)[1];
EVals:=eval(Sol5,var=n);
EFun:=simplify(eval(Sol3,lambda=EVals),symbolic);
```

19.2.3 Different Types of Analytical Solutions

By applying Maple predefined functions, we find exact solutions (general, explicit, implicit, parametric, particular, separable, and symmetric) of various types of ODEs. The results are presented in Table 19.1 below, where

$$p(t) = \sqrt{2\,t^2 - 2\,t + 1}, \quad q(t) = \arctan(2\,t - 1),$$
$$R(x,y) = \sqrt{e^{2x} - y^2 + 2y - 1}, \quad w(x) = \left(\frac{\sqrt{2}}{2}\,x + C_1\right)\frac{\sqrt{2}}{\sqrt{C_2^2 + 1}},$$
$$r_1(t) = t^{\frac{1}{n+1}}(t-n-1)^{-\frac{1}{n+1}}, \quad r_2(t) = t^{-\frac{n}{n+1}}(t-n-1)^{-\frac{1}{n+1}}.$$

JacobiSN(u,k) is the Jacobi elliptic function $\mathrm{sn}(u,k)$. The ordinary differential equations and various types of analytical solutions are defined in Maple as follows:

```
with(PDEtools): with(DEtools): declare(y(x),prime=x,quiet);
alias(y=y(x)); Eq1:=x*diff(y,x)+y=x; dsolve(Eq1,y);
Eq2:=diff(y,x)+y=exp(x); dsolve(Eq2,y,[linear]); linearsol(Eq2,y);
```

Table 19.1.

Maple predefined functions and exact solutions of various types of ODEs

No.	ODE	Exact solution	Maple function
1	$xy'_x + y = x$	$y = \frac{x}{2} + \frac{\_C1}{x}$	`dsolve(Eq1,y)`
2	$y'_x + y = e^x$	$y = \frac{\_C1}{e^x} + \frac{1}{2}e^x$	`dsolve(Eq2,y,[linear])`
3	$yy'_x - y + \frac{1}{2}x = 0$	$x(t) = -2\frac{Ce^{-q(t)}(t-1)}{p(t)}, y(t) = \frac{Ce^{-q(t)}}{p(t)}$	`parametricsol(Eq3,y)`
4	$y'_x = \frac{y^2+xy}{x^2}$	$\_C1 = \frac{x}{y} + \ln(x)$	`dsolve(Eq4,y,implicit)`
5	$y'_x = \frac{1+e^y}{x\ln(x)e^y}$	$y = \ln(-1 + \ln(x)\_C1)$	`separablesol(Eq5,y)`
6	$y'_x(x-y) + y = 0$	$y_{1,2} = \frac{x\_C1 \pm \sqrt{x^2\_C1^2+1}}{\_C1}$	`genhomosol(Eq6,y)`
7	$\ln(xy) + \frac{x}{y}y'_x = 0$	$y = \frac{1}{x}e^{-\frac{\_C1-x}{x}}$	`exactsol(Eq7,y)`
8	$y+(2x-ye^y)y'_x=0$	$xy^2 - \left(y^2 - 2y + 2\right)e^y + \_C1 = 0$	`firint(mu*Eq8)`
9	$y'_x - \frac{1}{4}y = \frac{x}{y}$	$y_{1,2} = \pm\sqrt{e^{x/2}\_C1 - 4x - 8}$	`bernoullisol(Eq9,y)`
10	$y'_x=x^2y^2+\frac{1}{x}y+x^4$	$y=-\frac{x\left(-C\sin\left(\frac{x^4}{4}\right)+\cos\left(\frac{x^4}{4}\right)\right)}{C\cos\left(\frac{x^4}{4}\right)+\sin\left(\frac{x^4}{4}\right)}$	`riccatisol(Eq10,y)`
11	$y^2(y'_x)^2 - x = 0$	$y=\pm\frac{2}{3}\sqrt{3}\sqrt{x^{3/2}}$	`particularsol(Eq11,y)`
12	$xy'_x - (y'_x)^2 = y$	$\left\{y=1/4\,x^2,\ y=-\_C1^2+x\_C1\right\}$	`Solve(Eq14,y)`
13	$yy'_x - y = e^{2x} - 1$	$\_C1 - R(x,y) + \arctan\left(\frac{y-1}{R(x,y)}\right) = 0$	`abelsol(Eq13,y)`
14	$x^2y''_{xx} + xy'_x + y = \frac{1}{x^2}$	$y=\sin(\ln x)\_C2+\cos(\ln x)\_C1+\frac{1}{5x^2}$	`liesol(Eq12,y)`
15	$y''_{xx} + y - y^3 = 0$	$y=C_2\sqrt{2}\sqrt{(C_2{}^2+1)^{-1}}\mathrm{JacobiSN}(w(x),C_2)$	`dsolve(Eq15,y)`
16	$y''_{xx}=yy'_x + \frac{1}{2}y^3$	$y=\frac{\sqrt{5}-1}{-x+C_1},\quad y=-\frac{\sqrt{5}+1}{-x+C_1}$	`particularsol(Eq16)`
17	$y''_{xx} = 2yy'_x$	$C_1\arctan(yC_1)-x-C_2=0$	`dsolve(Eq17,y,implicit)`
18	$y''_{xx} = e^{-y}y'_x$	$y=\ln\left(\frac{e^{C_1C_2}e^{C_1x}+1}{C_1}\right)$	`dsolve(Eq18,y,useint)`
19	$y''_{xx} = \sin(y^2)y'_x$	$\int\frac{1}{\int\sin\left(y^2\right)\,dy + C_1}dy-x-C_2=0$	`dsolve(Eq19,y,useInt)`
20	$y''_{xx} = y^n y'_x$	$y(t)=r_1(t)e^{\frac{C_2}{n}}, x(t)=-\frac{\int r_2(t)\,dt}{e^{C_2}}+C_1$	`parametricsol(Eq20,y)`

```
Eq3:=y*diff(y,x)-y+x/2=0; Sol1:=dsolve(Eq3,y,parametric);
     Sol2:=subs(_T=t,_C1=C,Sol1); parametricsol(Eq3,y);
Eq4:=diff(y,x)=(y^2+x*y)/x^2; dsolve(Eq4,y,implicit); isolate(Sol41,_C1);
Eq5:=diff(y,x)=(1+exp(y))/(exp(y)*x*ln(x)); dsolve(Eq5,y,[separable]);
     separablesol(Eq5,y);
Eq6:=diff(y,x)*(x-y)+y=0; dsolve(Eq6,y,[homogeneous]); genhomosol(Eq6,y);
Eq7:=ln(x*y)+(x/y)*diff(y,x)=0; simplify(dsolve(Eq7,y,[exact]),symbolic);
     exactsol(Eq7,y);
Eq8:=y+(2*x-y*exp(y))*diff(y,x)=0; odeadvisor(Eq8); mu:=intfactor(Eq8);
     odeadvisor(mu*Eq8); dsolve(mu*Eq8,y,[exact]); exactsol(mu*Eq8,y);
     firint(mu*Eq8);
Eq9:=diff(y,x)-y/4=x/y; dsolve(Eq9,y,[Bernoulli]); bernoullisol(Eq9,y);
Eq10:=diff(y,x)-x^2*y^2-1/x*y-x^4=0;
      Sol1:=subs(_C1=C,riccatisol(Eq10,y)); dsolve(Eq10,y,[Riccati]);
Eq11:=y^2*diff(y,x)^2-x=0; particularsol(Eq11,y);
Eq12:=x*diff(y,x)-diff(y,x)^2=y; Solve(Eq14,y); clairautsol(Eq14,y);
Eq13:=y*diff(y,x)-y=exp(2*x)-1; simplify(dsolve(Eq13,y,[Abel]),symbolic);
      abelsol(Eq13,y);
Eq14:=x^2*diff(y,x$2)+x*diff(y,x)+y=x^(-2); odeadvisor(Eq12);
      dsolve(Eq12,y,Lie); liesol(Eq12,y);
Eq15:=diff(y,x$2)+y-y^3=0; subs(_C1=C[1],_C2=C[2],dsolve(Eq15,y));
Eq16:=diff(y,x$2)=y*diff(y,x)+y^3/2;subs(_C1=C[1],[particularsol(Eq16)]);
Eq17:=diff(y,x$2)=2*y*diff(y,x);
      subs(_C1=C[1],_C2=C[2],dsolve(Eq17,y,implicit));
Eq18:=diff(y,x$2)=exp(-y)*(diff(y,x));
      Sol1:=subs(_C1=C[1],_C2=C[2],dsolve(Eq18,y,useint));
Eq19:=diff(y,x$2)=sin(y^2)*(diff(y,x));
Sol1:=[dsolve(Eq19,y,useInt)]; Sol2:=subs(_C1=C[1],_C2=C[2],Sol1[2]);
Eq20:=diff(y,x$2)=y^n*(diff(y,x));
      Sol1:=simplify(subs(_T=t,_C1=C[1],_C2=C[2],parametricsol(Eq20,y)));
```

19.2.4 Analytical Solutions of Systems of ODEs

The computer algebra system Maple has various predefined functions based on symbolic algorithms for constructing analytical solutions of systems of ODEs [see a more detailed description in Cheb-Terrab *et al.* (1997)].

Consider the most relevant related functions for finding analytical solutions of a given ODE system.

```
dsolve({ODEs},{funcs});     dsolve({ODEs,ICs},{funcs},ops);
dsolve({ODEs},{funcs},method=transform);
dsolve({ODEs},{funcs},'series',ops);
odetest({Sols},{ODEs},{funcs});
dsolve[interactive]({ODEs});                        with(DEtools);
autonomous(ODEs,{funcs},x);                    rtaylor(ODEs,ops);
convertsys(ODEs,ICs,{funcs},x,ops);           rifsimp(ODEs,ops);
```

- `DEtools,autonomous`, determining if a set of ODEs is strictly autonomous
- `DEtools,convertsys`, converting a system of one or more ODEs to a system of first-order ODEs

- `DEtools,rifsimp`, simplifying overdetermined polynomially nonlinear systems of ODEs
- `DEtools,rtaylor`, obtaining local Taylor series expansions for all dependent variables in the system of ODEs

► Linear systems of ODEs.

For first-order linear systems of ODEs, it is possible to find the general solution and the particular solution for any initial condition (with the aid of the predefined function `dsolve`). Higher-order linear ODEs or systems can be converted into systems of first-order ODEs (with the aid of the predefined function `convertsys`) and then solved.

Example 19.20. *First-order liner system of two ODEs. Analytical solution.*
Consider the general first-order linear system of two ODEs with constant coefficients

$$u'_x = a_0 + a_1 u + a_2 v, \quad v'_x = b_0 + b_1 u + b_2 v,$$

where $u(x)$ and $v(x)$ are the unknown functions and the values of the coefficients are $a_0 = 1$, $a_1 = 1$, $a_2 = -1$, $b_0 = 1$, $b_1 = 1$, and $b_2 = 1$.

By applying the predefined function `dsolve`, we find the general solution

$$u(x) = -1 + e^x (\_C1 \cos(x) + \_C2 \sin(x)),$$
$$v(x) = -e^x (-\_C1 \sin(x) + \_C2 \cos(x))$$

of this linear system and verify it as follows:

```
ODE1:=diff(u(x),x)=a[0]+a[1]*u(x)+a[2]*v(x):
ODE2:=diff(v(x),x)=b[0]+b[1]*u(x)+b[2]*v(x): Sys1:={ODE1,ODE2};
Coeffs:={a[0]=1,a[1]=1,a[2]=-1,b[0]=1,b[1]=1,b[2]=1};
Sys2:=eval(Sys1,Coeffs);
SolGen1:=dsolve(Sys2,{u(x),v(x)}); odetest(SolGen1,Sys2);
u_x=eval(u(x),SolGen1); v_x=eval(v(x),SolGen1);
```

Example 19.21. *First-order linear system of two ODEs. Cauchy problem. Analytical solution.*
Consider the following first-order linear system of two ODEs with initial conditions:

$$u'_x = a_0 + a_1 u + a_2 v, \quad v'_x = b_0 + b_1 u + b_2 v, \qquad u(x_0) = u_0, \quad v(x_0) = v_0,$$

where $u = u(x)$ and $v = v(x)$ are the unknown functions and the values of the coefficients are $a_0 = -1$, $a_1 = 1$, $a_2 = -1$, $b_0 = 1$, $b_1 = -1$, and $b_2 = 1$. For a first-order two-dimensional system in $u(x)$ and $v(x)$, each initial condition can be specified in the form `IC=`$\{u(x_0)=u_0,\ v(x_0)=v_0\}$ (e.g., $u(0) = 0$, $v(0) = 1$). One solution curve is generated for each initial condition. The solution of the initial value problem (`IVP1`) can be found as follows:

```
ODE1:=diff(u(x),x)=a[0]+a[1]*u(x)+a[2]*v(x):
ODE2:=diff(v(x),x)=b[0]+b[1]*u(x)+b[2]*v(x): Sys1:={ODE1,ODE2};
Coeffs:={a[0]=-1,a[1]=1,a[2]=-1,b[0]=1,b[1]=-1,b[2]=1};
Sys2:=eval(Sys1,Coeffs); SolGen1:=dsolve(Sys2,{u(x),v(x)});
u_x=eval(u(x),SolGen1); v_x=eval(v(x),SolGen1);
odetest(SolGen1,Sys2); IC:={u(0)=0,v(0)=1}; IVP1:=Sys2 union IC;
SolPart1:=dsolve(IVP1,{u(x),v(x)}); odetest(SolPart1,IVP1);
```

Alternatively, the solution of this initial value problem can be found step by step as follows:

```
Eq1:=eval(eval(SolGen1,x=0),IC); Eq2:=solve(Eq1,{_C1,_C2});
SolPart2:=eval(SolGen1,Eq2); odetest(SolPart2,IVP1);
u_xP=eval(u(x),SolPart2); v_xP=eval(v(x),SolPart2);
```

We substitute the initial condition (IC) into the general solution (SolGen1) and obtain equations (Eq1) for the unknowns _C1 and _C2, which can be solved for these constants of integration (Eq2). The particular solution (SolPart2) of this initial value problem reads:

$$u(x) = 1 - e^{2x}, \quad v(x) = e^{2x}$$

This particular solution SolPart2 coincides with the solution SolPart1.

► Nonlinear systems of ODEs.

For more complicated first-order or higher-order nonlinear systems of ODEs, a straightforward application of the predefined function dsolve may give no solutions (general or particular). Therefore, one can introduce some transformations, make some manipulations with the original Cauchy problem, reduce it to a modified Cauchy problem, and finally obtain analytical solutions in terms of the new variables and the original variables. Let us show this in the following example.

Example 19.22. *System of ODEs. Cauchy problem. Analytical and graphical solutions.* Consider the following second-order nonlinear system of two ODEs with initial conditions:

$$u''_{xx} = -au'_x\sqrt{(u'_x)^2 + (v'_x)^2}, \quad v''_{xx} = -av'_x\sqrt{(u'_x)^2 + (v'_x)^2},$$
$$u(0) = 0, \quad v(0) = 0, \quad u'_x(0) = U_0 \sin\phi, \quad v'_x(0) = U_0 \cos\phi,$$

where $u = u(x)$ and $v = v(x)$ are the unknown functions and the parameter values are $a = 5$, $U_0 = 10$, and $\phi = \pi/10$. The solution of the initial value problem (IVP1) can be found step by step as follows:

```
with(PDEtools): declare(u(x),v(x),U(x),prime=x):
xR:=0..20; IC_U:=U(0)=U[0]; Sys1:=
 {diff(u(x),x$2)=-a*diff(u(x),x)*sqrt(diff(u(x),x)^2+diff(v(x),x)^2),
  diff(v(x),x$2)=-a*diff(v(x),x)*sqrt(diff(u(x),x)^2+diff(v(x),x)^2)};
IC:={u(0)=0, v(0)=0, D(u)(0)=U[0]*sin(phi), D(v)(0)=U[0]*cos(phi)};
IVP1:=Sys1 union IC; dsolve(IVP1,{u(x),v(x)});
tr1:=U(x)=sqrt(diff(u(x),x)^2+diff(v(x),x)^2); Eq1:=diff(tr1,x);
Eq2:=simplify(eval(Eq1,Sys1),symbolic);
ODE1:=subs(rhs(tr1)^2=U(x)^2,Eq2); Sol1:=dsolve({ODE1,IC_U},U(x));
Sys2:=subs(rhs(tr1)=rhs(Sol1),Sys1);
IVP2:=Sys2 union IC; Sol2:=simplify(dsolve(IVP2),symbolic);
simplify(odetest(Sol2,IVP1),symbolic); SolG:=[op(Sol2)];
plot(eval([rhs(SolG[1]),rhs(SolG[2])],{U[0]=10,a=5,phi=Pi/10}),x=xR);
```

In this problem, a straightforward application of the predefined function dsolve does not give a solution. Therefore, we introduce the transformation (tr1), $U(x) = \sqrt{(u'_x)^2 + (v'_x)^2}$. Then we find the derivative (Eq1), $U'_x = \dfrac{2u'_x u''_{xx} + 2v'_x v''_{xx}}{2\sqrt{(u'_x)^2 + (v'_x)^2}}$. Substituting the second derivatives u''_{xx}, v''_{xx} of the original system into the expression for U'_x, we obtain the differential equation (ODE1), $U'_x = -aU^2$. Solving this simple differential equation with the initial condition $U(0) = U_0$, we obtain the solution (Sol1), $U(x) = \dfrac{U_0}{1 + aU_0 x}$. Substituting this expression for $U(x)$, which is equal to $\sqrt{(u'_x)^2 + (v'_x)^2}$ (according to `tr1`), into the original system (`Sys1`) and considering the initial conditions, we obtain the modified Cauchy problem (`IVP2`)

$$u''_{xx} = -\frac{aU_0\, u'_x}{aU_0 x + 1}, \quad v''_{xx} = -\frac{aU_0\, v'_x}{aU_0 x + 1},$$
$$u(0) = 0, \quad v(0) = 0, \quad u'_x(0) = U_0 \sin\phi, \quad v'_x(0) = U_0 \cos\phi.$$

Solving this Cauchy problem, we obtain the analytical particular solution (`Sol2`)

$$u(x) = \frac{1}{a}\sin\phi \ln(axU_0 + 1), \quad v(x) = \frac{1}{a}\cos\phi \ln(axU_0 + 1)$$

and then verify that it is an exact particular solution (`Sol2`) of the original Cauchy problem (`IVP1`) and plot the graphs of $u(x)$ and $v(x)$.

19.2.5 Integral Transform Methods for ODEs

In Maple, integral transforms (e.g., Fourier, Hilbert, Laplace, and Mellin integral transforms) can be studied with the aid of the `inttrans` package. Methods of integral transforms can be applied to the solution of many initial value problems. The most important predefined functions for finding analytical solutions of a given Cauchy problem are the following:

```
dsolve({ODE,IC},y(x),method=transform,ops);
dsolve({ODEs,IC},{funcs},method=transform,ops).
```

Here `transform` can be `laplace`, `fourier`, `fouriercos`, or `fouriersin`.

- `DEtools,method`, finding analytical solutions using integral transforms (Laplace or Fourier)

► Linear ODEs and systems of ODEs with constant coefficients.

Methods of integral transforms can be applied for solving the nth-order linear ODE with constant coefficients and with initial conditions,

$$a_n y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \ldots + a_1 y'_x + a_0 y = f(x), \quad x > 0,$$
$$y(0) = y_0, \quad y'_x(0) = y_1, \quad \ldots, \quad y_x^{(n-1)}(0) = y_{n-1},$$

and systems of linear ODEs with constant coefficients coupled by initial conditions. Consider some examples.

Example 19.23. *First-order linear ODE. Initial value problem. Laplace transform.*
For the first-order linear ODE with the initial condition

$$y'_x + ay = e^{-ax}, \qquad y(0) = 1,$$

the exact solution

$$y(x) = (x+1)e^{-ax}$$

of the initial value problem can be obtained (with the aid of the predefined function `dsolve`) and verified as follows:

```
with(PDEtools): declare(y(x),prime=x): with(inttrans);
ODE1:=diff(y(x),x)+a*y(x)=exp(-a*x); IC:=y(0)=1; IVP1:={ODE1,IC};
SolPart1:=dsolve(IVP1,y(x)); SolPart2:=dsolve(IVP1,y(x),method=laplace);
odetest(SolPart1,IVP1); odetest(SolPart2,IVP1);
```

Alternatively, the exact solution of this initial value problem can be found step by step and verified as follows:

```
with(PDEtools): declare(y(x),prime=x): with(inttrans);
ODE1:=diff(y(x),x)+a*y(x)=exp(-a*x); IC:=y(0)=1; IVP1:={ODE1,IC};
Eq1:=laplace(ODE1,x,p); Eq2:=subs(IC,Eq1);
Eq3:=solve(Eq2,laplace(y(x),x,p)); SolPart1:=invlaplace(Eq3,p,x);
```

The graphical solution can be obtained for some value of the parameter a (e.g., $a = 7$) as follows:

```
SolPart2:=subs(a=7,SolPart1); plot(SolPart2,x=0..1,color=blue);
```

Example 19.24. *First-order linear system of ODEs. Initial value problem. Laplace transform.* By applying the Laplace transform, we solve the initial value problem

$$u'_x - 2v = x, \quad 4u + v'_x = 0, \quad u(0) = 1, \quad v(0) = 0$$

and verify the exact solution

$$u(x) = \tfrac{1}{8} + \tfrac{7}{8}\cos(2\sqrt{2}x), \quad v(x) = -\tfrac{7}{8}\sqrt{2}\sin(2\sqrt{2}x) - \tfrac{1}{2}x,$$

as follows:

```
with(PDEtools): declare(u(x),v(x),prime=x): with(inttrans):
IC:={u(0)=1,v(0)=0}; ODESys1:={D(u)(x)-2*v(x)=x,4*u(x)+D(v)(x)=0};
IVP1:=ODESys1 union IC; Sol1:=dsolve(IVP1,{u(x),v(x)});
Sol2:=dsolve(IVP1,{u(x),v(x)},method=laplace); odetest(Sol1,IVP1);
```

Alternatively, the exact and graphical solutions of this initial value problem can be found step by step and verified as follows:

```
Eq1:=laplace(ODESys1,x,s); Eq2:=subs(IC,Eq1);
Eq3:=solve(Eq2,{laplace(u(x),x,s),laplace(v(x),x,s)});
Sol3:=invlaplace(Eq3,s,x); assign(Sol3):
plot([u(x),v(x),x=0..Pi],color=blue,thickness=3);
```

Example 19.25. *First-order linear systems of ODEs. Initial value problem. Laplace transform.* Generalizing the procedure, consider the system of first-order linear ODEs

$$(y_i)'_x = a_{i1}y_1 + a_{i2}y_2 + \dots + a_{in}y_n + f_i(x), \quad x > 0 \quad (i = 1, \dots, n)$$

with the initial conditions

$$y_i(0) = y_{i0} \quad (i = 1, \dots, n),$$

where a_{ij} ($i = 1, \dots, n$; $j = 1, \dots, n$) are constants, $f_i(x)$ are given functions, and the unknown functions $y_1(x), \dots, y_n(x)$ are defined on $x \in [0, \infty]$.

Let $n = 2$. We find the exact solution of the Cauchy problem

$$(y_1)'_x = y_2, \quad (y_2)'_x = -y_1 - 2y_2 + x, \quad x > 0,$$
$$y_1(0) = 1, \quad y_2(0) = 1$$

by applying the integral transform method. Also, we verify and plot this solution on some interval $[a, b]$ as follows:

```
with(PDEtools): declare(u(x),v(x),prime=x): with(inttrans):
with(plots): a:=0: b:=2:
ODESys1:=diff(u(x),x)=v(x),diff(v(x),x)=x-u(x)-2*v(x);
IC:=u(0)=1,v(0)=1; IVP1:={ODESys1,IC};
Sol1:=sort(dsolve(IVP1,{u(x),v(x)},method=laplace));
uEx:=unapply(rhs(Sol1[1]),x); vEx:=unapply(rhs(Sol1[2]),x);
uG1:=plot(uEx(x),x=a..b,color=red):
vG1:=plot(vEx(x),x=a..b,color=blue): display({uG1,vG1});
```

Remark 19.6. If we consider an nth-order ODE ($n > 1$) with n initial conditions,

$$a_n y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \ldots + a_1 y'_x + a_0 y = f(x), \quad x > 0,$$
$$y(0) = y_0, \quad y'_x(0) = y_1, \quad \ldots, \quad y_x^{(n-1)}(0) = y_{n-1},$$

then we can find exact solutions of this higher-order ODE by transforming it into an equivalent system of n first-order equations (with the predefined function `DEtools,convertsys`) and by applying integral transform methods to this system of ODEs.

► Linear ODEs with variable coefficients.

Some initial value problems consisting of linear ODEs with variable coefficients can be solved (in a similar way) by the method of integral transforms. However, integral transforms do not provide a general method for solving ODEs with variable coefficients.

Example 19.26. *Linear ODE with variable coefficients. Cauchy problem. Laplace transform.* For the second-order linear ODE with variable coefficients and initial conditions

$$y''_{xx} + 2x y'_x - 4y = 2, \quad y(0) = 0, \quad y'_x(0) = 0,$$

the exact solution

$$y(x) = x^2$$

of the initial value problem can be obtained (with the aid of the predefined function `dsolve`) and verified as follows:

```
with(PDEtools): declare(y(x),prime=x): with(inttrans):
with(plots): ODE1:=diff(y(x),x,x)+2*x*diff(y(x),x)-4*y(x)=2;
IC:=y(0)=0,D(y)(0)=0; IVP1:={ODE1,IC};
Sol1:=dsolve(IVP1,{y(x)},method=laplace); odetest(Sol1,IVP1);
```

19.2.6 Constructing Solutions via Transformations

Transformation methods are the most powerful analytic tools for studying differential equations. Nowadays, transformations of different types can be computed with the aid of computer algebra systems, such as Maple or Mathematica.

In general, transformations can be divided into two groups: (i) transformations of the independent and dependent variables and (ii) transformations of the independent variables as well as the dependent variables and their derivatives. We will consider various types of transformations relating ODEs, e.g., point and contact transformations.

Transformation methods permit finding transformations under which an ODE is invariant and new variables (independent and dependent) with respect to which the differential equations become simpler, e.g., linear.

Consider the most important functions for performing transformations of ODEs:

```
with(PDEtools):                    dchange(tr1,expr1,ops);
with(DEtools):               convert(ODE1,ODEtype,y(x),ops);
```

- `PDEtools,dchange`, performing transformations (changes of variables) in mathematical expressions (ODEs, PDEs, multiple integrals, integro-differential equations, limits, etc.) or procedures
- `DEtools,convert`, converting ODEs to other ODEs of different type. The parameter `ODEtype` can be one of the following conversion types: `y_x`, `Riccati`, `linearODE`, `NormalForm`, `Abel`, `Abel_RNF`, `FirstKind`, `SecondKind`, `DESol`, `MobiusX`, `MobiusY`, and `MobiusR`

► Point transformations.

Now consider the most important transformations of ODEs, namely, point transformations (transformations of independent and dependent variables). Point transformations can be linear transformations (translation transformations, scaling transformations, rotation transformations, etc.) and nonlinear transformations of independent and dependent variables. These transformations and their combinations can be applied to simplify nonlinear ODEs, linearize them, and reduce them to normal, canonical, or invariant forms.

Example 19.27. *The Bernoulli equation. Transformation and general integral.*
Consider the first-order nonlinear ODE, the Bernoulli equation

$$g(x)y'_x - f_1(x)y - f_n(x)y^n = 0,$$

where $n \neq 0, 1$. By applying the transformation of the dependent variable $X = x$, $Y(X) = [y(x)]^{1-n}$, we obtain the linear equation (`Eq8`)

$$(1-n)\,f_1(X)\,Y + (1-n)\,f_n(X) - g(X)\,Y'_X = 0.$$

Then we obtain the general integral with respect to $Y(X)$ (`Sol1`) and test it (`Test1`).

Finally, we find the general integral of the original equation (with respect to $y(x)$) (`Sol2`) and test it (`Test2`):

```
with(PDETools): declare(y(x),Y(X),prime=x,prime=X);
tr1:={X=x,Y(X)=y(x)^(1-n)}; tr2:=simplify(solve(tr1,{x,y(x)}),symbolic);
Eq1:=g(x)*diff(y(x),x)-f[1](x)*y(x)-f[n](x)*y(x)^n=0;
Eq2:=dchange(tr2,Eq1,[X,Y(X)]); Eq3:=expand(simplify(Eq2,symbolic));
Eq4:=map(simplify,lhs(Eq3),symbolic); Eq5:=map(`*`,Eq4,(n-1));
Eq6:=map(`*`,Eq5,Y(X)^(n/(n-1))); Eq7:=map(simplify,Eq6,symbolic);
Eq8:=collect(Eq7,[f[1](X),f[n](X),Y(X)]); Sol1:=dsolve(Eq8=0,Y(X));
Test1:=odetest(Sol1,Eq8); Sol2:=subs(tr1,Sol1); Test2:=odetest(Sol2,Eq1);
```

Example 19.28. *First-order ODE reducible to a homogeneous ODE. A linear transformation.*
Consider the first-order equation

$$y'_x = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right),$$

where a_i, b_i, c_i $(i = 1, 2)$ are real constants. This equation can be reduced to a homogeneous equation and integrated. Consider the case where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1 b_2 - a_2 b_1 \neq 0.$$

By applying the transformation*

$$x = X + x_0, \quad y(x) = Y(X) + y_0 \tag{19.2.6.1}$$

of the independent and dependent variables, where x_0 and y_0 are constants, which can be uniquely determined (since $D \neq 0$) by solving the linear algebraic system

$$a_1 x_0 + b_1 y_0 + c_1 = 0, \quad a_2 x_0 + b_2 y_0 + c_2 = 0,$$

we obtain the differential equation (Eq2)

$$Y'_X = f\left(\frac{b_1 Y + a_1 X}{b_2 Y + a_2 X}\right)$$

for $Y = Y(X)$, which can be reduced to the homogeneous equation

$$Y'_X = f\left(\frac{b_1 Y/X + a_1}{b_2 Y/X + a_2}\right)$$

and then integrated:

```
with(PDEtools): declare(y(x),Y(X),prime=x,prime=X);
Sol0:=solve({a1*x0+b1*y0+c1=0,a2*x0+b2*y0+c2=0},{x0,y0});
ODE1:=diff(y(x),x)=f((a1*x+b1*y(x)+c1)/(a2*x+b2*y(x)+c2));
tr1:={x=X+x0,y(x)=Y(X)+y0}: Eq1:=dchange(tr1,ODE1,[X,Y(X)]);
Eq2:=simplify(subs(Sol0,Eq1)); Ex1:=op(1,rhs(Eq2));
tr2:=Ex1=expand(numer(Ex1)/X)/expand(denom(Ex1)/X);
Eq3:=subs(tr2,Eq2);
```

Now consider a particular case of point transformations, the hodograph transformation, i.e., a transformation interchanging the roles of the dependent and independent variables in an ODE. The hodograph transformation is used for simplifying nonlinear differential equations and nonlinear systems or converting them into linear ones.

Example 19.29. *First-order nonlinear ODE. Hodograph transformation.*
Consider the nonlinear first-order ODE

$$y + (2x - y\mathrm{e}^y) y'_x = 0.$$

By applying the hodograph transformation $y(x) = X$, $x = Y(X)$ (tr1), we reduce this nonlinear ODE to the linear equation $XY'_X - X\mathrm{e}^X + 2Y = 0$ (Eq2):

```
with(PDEtools): with(DEtools): declare(y(x),Y(X),prime=x,prime=X);
ODE1:=y(x)+(2*x-y(x)*exp(y(x)))*diff(y(x),x)=0; tr1:={y(x)=X,x=Y(X)};
Eq1:=dchange(tr1,ODE1,[X,Y(X)]); Eq2:=numer(lhs(factor(Eq1)))=0;
```

*Equations (19.2.6.1) can be interpreted as a translation of orthogonal coordinate axes to the new origin (x_0, y_0) that is the point of intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ (for the case in which the lines are not parallel).

The original nonlinear equation can be solved in implicit form, and the reduced equation, in explicit form. In Maple's notation (`Sol1`, `Sol2`), we have

$$x-\frac{\left(y^2-2\,y+2\right)\mathrm{e}^{y}+\_C1}{y^2}=0,\qquad \left[Y\left(X\right)=\frac{\left(X^2-2\,X+2\right)\mathrm{e}^{X}+\_C1}{X^2}\right].$$

Alternatively, we can perform the hodograph transformation by applying the predefined function `DEtools,convert,y_x` and rewrite the resulting equation in the same notation (`Eq3`, `Eq4`). Finally, we verify all the solutions obtained:

```
Sol1:=dsolve(ODE1,y(x)); Sol2:=[dsolve(Eq2,Y(X))];
Eq3:=numer(lhs(normal(convert(ODE1,y_x,implicit))))=0;
Eq4:=subs({y=X,x(y)=Y(X)},Eq3);
odetest(Sol1,ODE1); odetest(Sol2,Eq2);
```

► Contact transformations.

For an ODE of general form with the independent variable x and the dependent variable $y=y(x)$, a contact transformation can be represented in the form

$$x=F(X,Y,Y'_x),\quad y=G(X,Y,Y'_x),\quad y'_x=H(X,Y,Y'_X),$$

where the functions $F(X,Y,Y'_x)$ and $G(X,Y,Y'_x)$ are chosen so that the derivative y'_x does not depend on Y''_{xx}.

Consider some examples of contact transformations that reduce complicated nonlinear ODEs to equations of a simpler form.

Example 19.30. *Nonlinear first-order equation. Contact transformation.*
Consider the nonlinear first-order ODE

$$y'_x(y'_x+ax)^n+b\big((y'_x)^2+2ay\big)^m+c=0.$$

By applying the contact transformation (`tr3`)

$$x=\tfrac{1}{a}(X-y'_x),\quad y=\tfrac{1}{2a}\big(Y-(Y'_X)^2\big),\quad y'_x=\tfrac{1}{2}Y'_X,$$

we reduce this nonlinear ODE to the separable ODE (`Eq2`)

$$\tfrac{1}{2}X^nY'_X+bY^m+c=0$$

and obtain the exact solution as follows:

```
with(PDEtools): with(DEtools): declare(y(x),Y(X),prime=x,prime=X);
D1:=diff(y(x),x); ODE1:=D1*(D1+a*x)^n+b*(D1^2+2*a*y(x))^m+c=0;
tr1:={x=(X-diff(y(x),x))/a,y(x)=(Y(X)-(diff(y(x),x))^2)/(2*a)};
tr2:=diff(y(x),x)=diff(Y(X),X)/2; tr3:=subs(tr2,tr1);
Eq1:=dchange(tr3,ODE1,[X,Y(X)]); Eq2:=simplify(Eq1,symbolic);
odeadvisor(Eq2); Sol1:=dsolve(Eq2,Y(X)); odetest(Sol1,Eq2);
```

Example 19.31. *Nonlinear first-order equation. Legendre transformation.*
Consider the nonlinear first-order ODE

$$xf(y'_x)+yg(y'_x)+h(y'_x)=0,$$

where $f(y'_x)$, $g(y'_x)$ and $h(y'_x)$ are arbitrary functions. By applying the Legendre transformation

$$x = Y'_X, \quad y = XY'_X - Y, \quad y'_x = X,$$

we reduce this nonlinear ODE to the linear ODE (Eq2)

$$\big(Xg(X) + f(X)\big)Y'_X - Yg(X) + h(X) = 0$$

and obtain the exact solution as follows:

```
with(PDEtools): with(DEtools): declare(y(x),Y(X),prime=x,prime=X);
D1:=diff(y(x),x); ODE1:=x*f(D1)+y(x)*g(D1)+h(D1)=0;
tr1:={x=diff(Y(X),X),y(x)=-Y(X)+X*diff(Y(X),X)};
Eq1:=dchange(tr1,ODE1,[X,Y(X)]); Eq2:=collect(Eq1,diff);
odeadvisor(Eq2); Sol1:=dsolve(Eq2,Y(X)); odetest(Sol1,Eq2);
```

⊙ *Literature for Section 18.2*: E. L. Ince (1956), G. M. Murphy (1960), L. E. El'sgol'ts (1961), E. Kamke (1977), E. S. Cheb-Terrab, L. G. S. Duarte, and L. A. C. P. da Mota (1997), D. Zwillinger (1997), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004).

19.3 Group Analysis of ODEs

19.3.1 Solution Strategies and Predefined Functions

It is well known that there exist many theoretical methods for solving ODEs. Moreover, nowadays there exist many computational methods for solving ODEs. Among computational methods, there are various computer algebra methods for solving ODEs; an overview of such methods can be found in [MacCallum (1995)]. Some of these methods have been implemented in Maple, and all the methods are provided by the general solver `dsolve`.

The main idea of the Lie symmetry approach is to transform differential equations to standard equations that we know how to solve.

In Maple, one can solve ODEs without specifying a method or specifying one of the embedded methods. In the first case, the system has a proper strategy for testing and analyzing a given ODE and can select an appropriate method depending on some options in `dsolve`. For example, these options include `type=numeric` for numerical solutions, `type=series` for Taylor series solutions, `type=formal_series` for formal power series solutions of linear ODEs with polynomial coefficients, `type=formal_solution` for formal solutions of homogeneous linear differential equations with polynomial coefficients, and `method=integral_transform` for solutions by integral transform methods.

If it is necessary to find an analytical solution of an ODE, Maple's strategy is as follows:

- Solve the ODE by applying *classification methods*, i.e., by verifying whether the ODE matches a recognizable *pattern* for which a solution method is implemented (e.g., quadrature, linear, separable, Bernoulli, Riccati, etc.). If there is a matching pattern, then the corresponding method is applied.
- If the classification methods fail, then apply *Lie symmetry methods* (which suggest finding generators of symmetry groups, integrating the ODE, or reducing its order).

In Maple, it is possible to change this strategy by introducing the option `class` (only classification methods) or `Lie` (only Lie symmetry methods).

Example 19.32. *The Emden–Fowler equation. Exact solutions by the Lie symmetry method.*

By applying the two Maple strategies, i.e., classification methods and Lie symmetry methods, to the Emden–Fowler equation

$$y''_{xx} = Ax^n y^m,$$

where $n = -6$ and $m = 3$, we obtain distinct forms of exact solutions (`SolLie`, `SolClass`):

$$-\frac{1}{x} \pm 2 \int^{\frac{y}{x}} \frac{1}{\sqrt{2\,A\_f^4 - 2\,\_C1}}\,d\_f - \_C2 = 0, \qquad y = \_C2\,\mathrm{JacobiSN}\left(\left(-\frac{1}{2}\,\frac{\sqrt{-2\,A}}{x} + \_C1\right)\_C2, \mathrm{I}\right) x$$

```
with(DEtools): with(PDEtools): declare(y(x),prime=x,quiet);
alias(y=y(x)); ODE1:=diff(y,x$2)=A*x^n*y^m; ODE6:=subs(n=-6,m=3,ODE1);
infolevel[symgen]:=1; odeadvisor(ODE6);
SolLie:=dsolve(ODE6,Lie,implicit); SolClass:=dsolve(ODE6,class);
```

Group analysis of ODEs can be performed in Maple: in *one step* (with `dsolve`), or *step by step* (by applying a specific predefined function to each of the symmetry steps), or *by hand* (by developing appropriate procedures and functions).

In Maple, ODEs can be studied with the aid of the predefined function `dsolve` (which includes symmetry methods for all differential orders) and a collection of predefined functions that implement symmetry methods [for details, see Olver (1986), Bluman and Kumei (1989), Stephani (1989)] in the packages `DEtools` and `PDEtools`. In this section, we study the main symmetry methods with the aid of predefined functions and packages [see Cheb-Terrab, Duarte, and da Mota (1997, 1998), Cheb-Terrab and Roche (1998), and Cheb-Terrab and Kolokolnikov (2003)].

The most relevant predefined functions for performing group analysis of ODEs and finding exact solutions by the Lie symmetry approach are the following:

```
dsolve(ODE,y(x));                      dsolve(ODE,y(x),Lie);
dsolve(ODE,y(x),way=val,HINT=[val],method);
with(DEtools): Sym1:=symgen(ODE,y(x),way=val,HINT=[val]);
symtest(Sym,ODE,y(x));          transinv([Sym],y(x),Y(X));
invariants([Sym],k,y(x),ops);      buildsym(Sol,y(x),ops);
reduce_order(ODE,[Sym],y(x),ops);    equinv([Sym],y(x),n);
buildsol(ODESolStruc,SolRedODE); canoni([Sym],y(x),s(r));
infgen([Sym],k,y(x),ODE);          eta_k([Sym],k,y(x),ODE);
with(PDEtools):          InfinitesimalGenerator(Sym,y(x));
InfinitesimalGenerator(Sym,y(x),prolongation=k,ops);
DeterminingPDE(ODE,typeofsymmetry=val,notation=val,ops);
```

Here `Sym` is a symmetry (or symmetries), `ODESolStructure` is an analytical solution obtained by `dsolve` and expressed in terms of `ODESolStruc` (for details, see Section 19.2.1), and `SolReducedODE` is a solution of the reduced ODE (which is a part of `ODESolStruc`).

- `dsolve,Lie`, computing an analytical solution by applying the Lie method of symmetries
- `DEtools` (package), several functions for performing Lie group analysis; e.g.,

  `symgen`, looking for a symmetry generator for a given ODE

  `symtest`, testing a given symmetry

  `transinv`, looking for the set of transformations of variables that leave the ODE invariant, i.e., the *one-parameter Lie invariance group* of the ODE

  `reduce_order`, reducing the order of an ODE using symmetries (or solutions)

  `equinv`, looking for the most general ODE invariant under a symmetry group (or different symmetry groups)

  `buildsol`, finding a solution of an ODE using order reduction and a solution of the reduced ODE

  `canoni`, determining a pair of canonical coordinates for a given Lie symmetry group

  `invriants`, calculating differential invariants of a given one-parameter Lie group

  `buildsym`, building the symmetry generator given a solution of an ODE

  `infgen`, finding the infinitesimal generator of a one-parameter Lie group (with k-prolongation)

  `eta_k`, determining the k-prolongation of the infinitesimals of a one-parameter Lie group

- `PDEtools` (package), some functions for performing Lie group analysis; e.g.,

  `InfinitesimalGenerator`, finding the infinitesimal generator (default is a differential operator procedure)

  `DeterminingPDE`, computing the *symmetry determining equation* (admitted by a given ODE) that splits into a PDE system

19.3.2 Constructing Point Groups

Consider symmetries that are point transformations, that is, diffeomorphisms of the plane.

Example 19.33. *Simplest ODE. Symmetries. One-parameter Lie group. Infinitesimal generator.* For the first-order ODE

$$y'_x = F(x)$$

with an arbitrary function $F(x)$:

- We obtain the symmetries (`Sym1`) $[\_\xi=0, \_\eta=1]$, i.e., the infinitesimals $\xi(x,y)$ and $\eta(x,y)$ of a *one-parameter Lie point transformation group* that leaves the given ODE invariant, and verify these symmetries (`Test1`).
- We obtain the *one-parameter* ($\_\alpha$) *Lie point transformation group* (in this case, the translation group) $\{X = x,\ Y = y + \_\alpha\}$ (`tr1`) and verify the invariance of the given ODE under this one-parameter Lie group, $Y'_X = F(X)$.
- We obtain the *infinitesimal generator* (or infinitesimal operator) (`G`) of the group, $G{=}f{\to}\dfrac{\partial}{\partial y}$.

```
with(DEtools): with(PDEtools):  declare(Y(X),prime=X);
ODE1:=diff(y(x),x)=F(x); odeadvisor(ODE1);
Sym1:=symgen(ODE1); Test1:=symtest(Sym1,ODE1,y(x));
tr1:=transinv(Sym1,y(x),Y(X)); itr1:=solve(tr1,{x,y(x)});
ODE1Inv:=dchange(itr1,ODE1,[X,Y(X)]);
G:=InfinitesimalGenerator(Sym1,y(x)); G(f(x,y));
```

Example 19.34. *The Blasius equation. One-parameter point transformation group.* Consider a third-order nonlinear ODE, e.g., the Blasius equation

$$y'''_{xxx} = -yy''_{xx}.$$

By applying the predefined function `symgen` with the option `way=formal`, we compute only all point symmetries (`Sym1`) $[\_\xi = 1, \_\eta = 0]$, $[\_\xi = x, \_\eta = -y]$, i.e., the infinitesimals $\xi(x,y)$ and $\eta(x,y)$ of a *one-parameter Lie point transformation group* that leaves the given ODE invariant. Then we obtain the characteristic functions Q_i and the infinitesimal generators G_i $(i = 1, 2)$,

$$Q_1 = -p, \quad Q_2 = -y - px, \qquad G_1 = \partial_x, \quad G_2 = x\partial_x - y\partial_y,$$

of these point symmetries as follows:

```
with(DEtools): with(PDEtools): declare(y(x),prime=x);
ODE1:=diff(y(x),x$3)=-y(x)*diff(y(x),x$2);
Sym1:=[symgen(ODE1,y(x),way=formal)]; N1:=nops(Sym1);
for i from 1 to N1 do
 Q||i:=subs(Sym1[i],_eta-diff(y(x),x)*_xi);
 G||i:=infgen(Sym1[i],0,y(x));
od;
```

19.3.3 Constructing Exact Solutions

Following the Lie symmetry approach, we can construct exact solutions of ODEs (in one step) by applying the predefined function `dsolve,Lie` with various options. For example, the option `method` in `dsolve(ODE,y(x),way=val,HINT=[val],method)` can be one of the types: `fat`, `can`, `can2`, `gon`, `gon2`, `dif`, where

- `fat`, building an integrating factor
- `can`, reducing the order of the ODE by one (canonical coordinates of the invariance group)
- `can2`, reducing a second-order ODE to a quadrature (two pairs of infinitesimals, a 2-D subalgebra)

- `gon`, reducing a second-order ODE to a quadrature (a normal form of the symmetry generator in the space of first integrals)
- `gon2`, reducing a second-order ODE to a quadrature (two pairs of infinitesimals and normal forms of the symmetry generators in the space of first integrals)
- `dif`, applying differential invariants

Remark 19.7. Two methods (`fat, can`) can be applied to first-order ODEs; six methods (`can`—`dif`) can be applied to second-order ODEs; one method (`can`) can be applied to higher-order ODEs. The methods `gon`, `gon2`, and `dif` can work with point and dynamical symmetries.

Example 19.35. *First-order nonlinear ODE. The Riccati equation. General solution.*
For the Riccati equation

$$y'_x = xy^2 - 2\frac{y}{x} - \frac{1}{x^3}, \tag{19.3.3.1}$$

which is a first-order nonlinear ODE, we construct the general solution (`Sol1`, `Sol2`)

$$y = \frac{x^2 + \_C1}{x^2(-x^2 + \_C1)}, \quad y = -\frac{\_C1x^2 + 1}{x^2(\_C1x^2 - 1)}$$

by applying different methods (integration factor and canonical coordinates) as follows:

```
with(DEtools): with(PDEtools): declare(y(x),Y(X),prime=x,prime=X);
ODE1:=diff(y(x),x)=x*y(x)^2-2*y(x)/x-1/x^3;  odeadvisor(ODE1);
Sol1:=dsolve(ODE1,'fat'); Sol2:=dsolve(ODE1,'can');
Test1:=odetest(Sol1,ODE1); Test2:=odetest(Sol2,ODE1);
```

Alternatively, for constructing (step by step) exact solutions of ODEs it is possible to apply various predefined functions (contained in the packages `DEtools` and `PDEtools`) at each step of the solution process.

Example 19.36. *The Riccati equation. Canonical coordinates. General solution.*
For the Riccati equation (19.3.3.1), we construct the general solution as follows:
(i) We determine all the symmetries (`Sym1`)

$$\Bigg[\Big[\_\xi = \frac{1}{x}, \_\eta = -2\,\frac{1}{x^4}\Big], \Big[\_\xi = 0, \_\eta = x^4\left(y + \frac{1}{x^2}\right)^2\Big], \Big[\_\xi = 0, \_\eta = x^2y^2 - \frac{1}{x^2}\Big],$$
$$\Big[\_\xi = -\frac{x}{2}, \_\eta = y\Big], \Big[\_\xi = -\frac{1}{4}\,x^3, \_\eta = yx^2 + \frac{1}{2}\Big]\Bigg]$$

of the one-parameter Lie point transformation group that leaves this ODE invariant and verify these symmetries (`Test1`).

(ii) We select the simplest symmetry `Sym1[4]`, find the corresponding one-parameter Lie point transformation group (`tr1`)

$$\left\{X = x\mathrm{e}^{-\_\alpha/2}, Y = \mathrm{e}^{\_\alpha}y\,(x)\right\},$$

and verify the invariance of the ODE under this one-parameter Lie group (`ODE1Inv`), $Y'_X = XY^2 - 2Y/X - 1/X^3$.

(iii) We find the infinitesimal generator (`G`) of the group, $G = f \to -\frac{1}{2}x\left(\frac{\partial}{\partial x}f\right) + y\left(\frac{\partial}{\partial y}f\right)$.

(iv) We determine the canonical coordinates (`trCan`) $\left\{r = yx^2, s(r) = -2\,\ln{(x)}\right\}$.

(v) We reduce the original Riccati equation to the simpler equation (`ODE3`) $s'_r = -\dfrac{2}{r^2 - 1}$ and obtain its general solution.

(vi) We rewrite the canonical coordinates r and s in terms of x and y and find the general solution (GenSol3) of the Riccati equation, $y = -\dfrac{1}{x^2}\tanh\bigl(\ln(x) + \frac{1}{2}\_C1\bigr)$, which coincides with the result obtained with dsolve, $y = \dfrac{1}{x^2}\tanh\bigl(-\ln(x) + \_C1\bigr)$.

```
with(DEtools): with(PDEtools):  declare(y(x),Y(X),prime=x,prime=X);
ODE1:=diff(y(x),x)=x*y(x)^2-2*y(x)/x-1/x^3; odeadvisor(ODE1);
Sym1:=[symgen(ODE1,way=all)]; Test1:=map(symtest,Sym1,ODE1,y(x));
Sym:=Sym1[4]; tr1:=simplify(transinv(Sym,y(x),Y(X)));
itr1:=simplify(solve(tr1,{x,y(x)}));
ODE1Inv1:=dchange(itr1,ODE1,[X,Y(X)],simplify);
ODE1Inv:=expand(ODE1Inv1/exp(-3/2*(_alpha)));
G:=InfinitesimalGenerator(Sym,y(x)); G(f(x,y));
trCan:=canoni(Sym,y(x),s(r));itrCan:=op(1,[solve(trCan,{x,y(x)})]);
ODE2:=dchange(itrCan,ODE1,[r,s(r)],simplify);
ODE3:=op(solve(ODE2,{diff(s(r),r)})); GenSol1:=dsolve(ODE3,s(r));
GenSol2:=dchange(trCan,GenSol1,{x,y(x)},simplify);
GenSol3:=y(x)=solve(GenSol2,y(x)); Test2:=odetest(GenSol3,ODE1);
GenSol0:=dsolve(ODE1,y(x));
```

19.3.4 Order Reduction of ODE

It is well known that an nth-order ODE

$$y_x^{(n)} = f(x, y, y'_x, \dots, y_x^{(n-1)}) \qquad (n \ge 2)$$

invariant under a one-parameter Lie transformation group (with infinitesimal generator) can be reduced to an $(n-1)$st-order ODE [see Bluman and Kumei (1989)]. This can be done by two methods, order reduction by canonical coordinates or order reduction by differential invariants.

Example 19.37. *Second-order nonlinear ODE. Emden–Fowler equation. Order reduction.* Consider a second-order nonlinear ODE, e.g., the Emden–Fowler equation

$$y''_{xx} = Ax^n y^m \qquad (m = 3,\ n = -6).$$

First, we determine all the symmetries (Sym1) of the one-parameter Lie point transformation groups (tr1, tr2)

$$X = xe^{\_\alpha},\ \ Y = e^{2\_\alpha}y; \qquad X = -\frac{x}{\_\alpha x - 1},\ \ Y = -\frac{y}{\_\alpha x - 1},$$

that leave this ODE invariant and verify these symmetries:

```
with(DEtools): with(PDEtools): declare(y(x),Y(X),prime=x,prime=X);
ODE:=diff(y(x),x$2)=A*x^n*(y(x))^m; ODE1:=subs(m=3,n=-6,ODE);
odeadvisor(ODE1); Sym1:=[symgen(ODE1)]; map(symtest,Sym1,ODE1,y(x));
tr1:=transinv(Sym1[1],y(x),Y(X)); itr1:=solve(tr1,{x,y(x)});
ODE1I:=dchange(itr1, ODE1,[X,Y(X)]);
simplify(isolate(ODE1I,diff(Y(X),X)),symbolic);
tr2:=transinv(Sym1[2],y(x),Y(X)); itr2:=solve(tr2,{x,y(x)});
ODE2I:=dchange(itr2, ODE1,[X,Y(X)]); isolate(ODE2I,diff(Y(X),X));
```

Then, for each symmetry (`Sym1[1]`, `Sym1[2]`)

$$[\_\xi = x,\ \_\eta = 2y], \qquad [\_\xi = x^2,\ \_\eta = xy],$$

we obtain the differential invariants (`Inv11`, `Inv12`, `Inv21`, `Inv22`)*

$$\_I0 = \frac{y}{x^2},\ \_I1 = \frac{x^2}{\_y1\,x - 2y}, \qquad \_I0 = \frac{y}{x^2},\ \_I1 = \frac{\_y1}{x},$$
$$\_I0 = \frac{y}{x},\ \_I1 = \frac{1}{\_y1\,x - y}, \qquad \_I0 = \frac{y}{x},\ \_I1 = -\_y1\,x + y \qquad (19.3.4.1)$$

and the reduced ODEs (`ODE11`, `ODE12`, `ODE21`, `ODE22`) by applying the two methods, reduction of order by canonical coordinates (`can`) and order reduction by differential invariants (`dif`):

$$Y'_X = (-AX^3 + 2X)Y^3 + 3Y^2, \quad Y'_X = \frac{AX^3 - Y}{Y - 2X},$$
$$Y'_X = -Y^3AX^3, \quad Y'_X = A\frac{X^3}{Y}.$$

The second-order Emden–Fowler equation is reduced to first-order ODEs of various types (Abel-type equations and separable equations):

```
for i from 1 to 2 do
  Inv||i||1:=invariants(Sym1[i],y(x),can);
    Inv||i||2:=invariants(Sym1[i],y(x),dif);
RedOr||i||1:=reduce_order(ODE1,Sym1[i],Y(X),can);
RedOr||i||2:=reduce_order(ODE1,Sym1[i],Y(X),dif);
  ODE||i||1:=op([2,1,1],rhs(RedOr||i||1)); odeadvisor(ODE||i||1);
  ODE||i||2:=op([2,1,1],rhs(RedOr||i||2)); odeadvisor(ODE||i||2);
od;
```

Moreover, it is possible to find the most general ODE (in our case, of the second order, `ODE13`)

$$y''_{xx} = \_F1\left((y'_x x - y)\frac{x^2}{y^2}\right)\frac{y^3}{x^6}$$

that is simultaneously invariant under this set of symmetries (`Sym1[1]`, `Sym1[2]`). Here $\_F1$ is an arbitrary function of its argument. Then we obtain the reduced equations (`ODE1331`, `ODE1332`, `ODE2331`, `ODE2332`) by applying the two methods, order reduction by canonical coordinates (`can`) and order reduction by differential invariants (`dif`):

$$(Y1)'_{X1} = -Y1^3\_F1\left(\frac{1}{X1^2Y1}\right)X1^3, \qquad Y2 = -\frac{1}{X2(X2^2\_F1(1/X2) - 2)},$$
$$(Y1)'_{X1} = \_F1\left(-\frac{Y1}{X1^2}\right)\frac{X1^3}{Y1}, \qquad Y2 = \frac{\_F1(-X2)}{X2}.$$

The most general second-order ODE can be reduced to first-order homogeneous ODEs and algebraic equations:

```
ODE13:=equinv(Sym1,y(x),2); odeadvisor(ODE13);
Red131:=reduce_order(ODE13,Sym1[1],Y(X),can);
Red132:=reduce_order(ODE13,Sym1[1],Y(X),can);
Red133:=reduce_order(ODE13,Sym1,Y1(X1),Y2(X2),can,in_sequence);
ODE1331:=op([2,1,1],rhs(Red133[1]));
ODE1332:=op([2,1,1],rhs(Red133[2]));
```

*In Maple's notation, $\_y1$ stands for y'_x.

```
Red231:=reduce_order(ODE13,Sym1[2],Y(X),dif);
Red232:=reduce_order(ODE13,Sym1[2],Y(X),dif);
Red233:=reduce_order(ODE13,Sym1,Y1(X1),Y2(X2),dif,in_sequence);
ODE2331:=op([2,1,1],rhs(Red233[1]));
ODE2332:=op([2,1,1],rhs(Red233[2]));
odeadvisor(ODE1331);  odeadvisor(ODE2331);
```

⊙ *Literature for Section 18.3*: P. J. Olver (1986), G. W. Bluman and S. Kumei (1989), H. Stephani (1989), M. A. H. MacCallum (1995), E. S. Cheb-Terrab, L. G. S. Duarte, and L. A. C. P da Mota (1997, 1998), E. S. Cheb-Terrab and A. D. Roche (1998), E. S. Cheb-Terrab and T. Kolokolnikov (2003).

19.4 Numerical Solutions and Their Visualizations

Although there exist various exact methods for special classes of differential equations, in general one cannot obtain an exact solution of a differential equation in closed form. Moreover, the functions and data in differential equation problems are frequently defined at discrete points. Therefore, we have to study numerical approximation methods for differential equations.

Consider various numerical and approximate analytical methods for initial value problems, boundary value problems, and eigenvalue problems for ordinary differential equations.

19.4.1 Numerical Solutions in Terms of Predefined Functions

Consider the most important functions for finding numerical solutions of a given ODE problem.

```
dsolve(ODEs,numeric,vars,ops);dsolve(numeric,procops,ops);
with(DEtools);          dsolve(ODEs,numeric,method=m,ops);
with(plots):            dsolve(ODEs,numeric,output=n,ops);
dsolve[interactive](ODEs,ops);
NS:=dsolve(ODE,numeric,vars);     odeplot(NS,vars,tR,ops);
```

- `dsolve,numeric`, finding numerical solutions of ODE problems
- `dsolve,method`, finding numerical solutions of ODE problems using one of the numerical methods `rkf45`, `ck45`, `rosenbrock`, `bvp`, `rkf45_dae`, `ck45_dae`, `rosenbrock_dae`, `dverk78`, `lsode`, `gear`, `taylorseries`, `mebdfi`, or `classical`
- `dsolve,procops`, for specifying the input system in procedure form (instead of specifying ODEs)
- `dsolve,output`, for obtaining the output from `dsolve` in various formats, e.g., as a procedure (with the default keyword `procedurelist`), as a list of equations of the form `variable=procedure` (with the keyword `listprocedure`), and as a list of equations of the form `operator=procedure` (with `operator`)

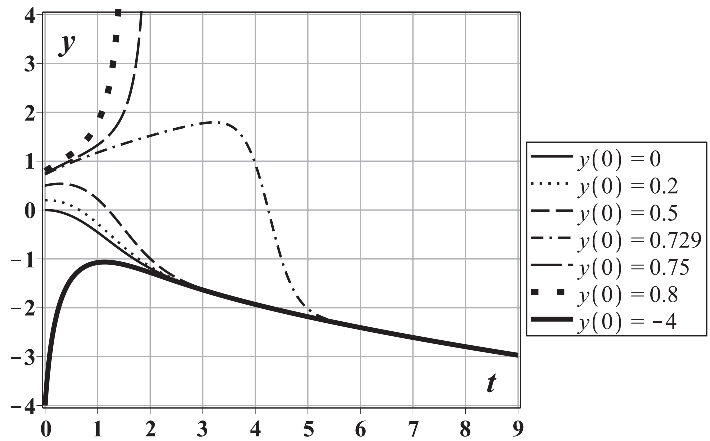


Figure 19.4: Numerical solutions of the Cauchy problem (19.4.1.1).

- `dsolve[interactive]`, interactive numerical solution of ODEs
- `odeplot` (the `plots` package), constructing graphs or animations of two-dimensional and three-dimensional solution curves obtained from the numerical solution
- `DETools` package, for working with graphical presentation of solutions of ODEs, where numerical methods are used for computing trajectories (e.g., `dfieldplot`, `phaseportrait`, `DEplot`, `DEplot3d`)

Remark 19.8. For more comprehensive details on numerical methods embedded in Maple (for solving ODEs) and graphical representation of solutions, we refer to Sections 19.4.2 and 19.2.1.

Example 19.38. *Cauchy problem. Numerical and graphical solutions.*
For the Cauchy problem (with several initial conditions)

$$y'_x = py^m + qx^n, \qquad y(0) = y_0 \tag{19.4.1.1}$$

on the interval $[a, b]$ ($a = 0$, $b = 9$) with parameters $p = 1$, $q = -1$, $m = 2$, $n = 1$, and $y_0 = \{0, 0.2, 0.5, 0.729, 0.75, 0.8 - 4\}$, we find numerical and graphical solutions (see Fig. 19.4) as follows:

```
with(plots): setoptions(scaling=constrained); N:=7;
ODE:=diff(y(x),x)=p*(y(x))^m+q*x^n;
IC:=[y(0)=0,y(0)=0.2,y(0)=0.5,y(0)=0.729,y(0)=0.75,y(0)=0.8,y(0)=-4];
a:=0; b:=9; p:=1; q:=-1; n:=1; m:=2;
Tn:=type=numeric; Op:=output=operator;
for i from 1 to N do
 Sol||i:=dsolve([ODE,IC[i]],Tn,Op):
 G||i:=plot(rhs(Sol||i[2](t)),t=a..b,axes=boxed,gridlines=true):
od:
display([seq(G||i,i=1..N)],view=[a..b,-5..5]);
```

19.4.2 Numerical Methods Embedded in Maple

In Maple, one can obtain numerical solutions of ODEs and systems of ODEs

- without specifying a method (automatically),
- specifying one of the predefined methods (described in Tables 19.2–19.6),
- specifying various other options in `dsolve,numeric`.

In Maple, it is possible to solve different types of problems:

- initial value problems (nonstiff, stiff, and complex-valued with a real-valued independent variable),
- boundary value problems (linear and nonlinear),
- initial value problems for differential algebraic equations (nonstiff and stiff),
- initial value problems for delay differential equations (nonstiff and stiff).

The default result of `dsolve,numeric` is a procedure (which can be used to obtain numerical values and visualizations).

The default methods are:

- `rkf45`, the Runge–Kutta–Fehlberg method (for nonstiff initial value problems), and `rosenbrock`, the Rosenbrock method (for stiff initial value problems);
- `bvp`, the finite difference method with Richardson extrapolation (for linear and nonlinear boundary value problems);
- `rkf45_dae`, the modified Runge–Kutta–Fehlberg method (nonstiff initial value problems for differential algebraic equations), and `rosenbrock_dae`, the modified Rosenbrock method (stiff initial value problems for differential algebraic equations);
- `rkf45`, the Runge–Kutta–Fehlberg method (nonstiff initial value problems for delay differential equations), and `rosenbrock`, the Rosenbrock method (stiff initial value problems for delay differential equations).

The numerical methods for initial value problems embedded in Maple (except for the classical methods[*]) control the discretization error (for more details, see options `abserr`, `relerr`, `minstep`, `maxstep`, and `initstep`).

The classical methods do not estimate the discretization error, and the step size is fixed. However, there is no best numerical method for initial value problems.

The efficiency depends on various parameters (e.g., the order, the step size, the discretization error, tolerances, and accuracy).

For example, for greater accuracy, a higher-order method (`dverk78`) is more appropriate.

---

[*] See Table 19.6 for details.

However, `rkf45` is more efficient than the higher-order method `dverk78` for modest tolerances, and `dverk78` is more efficient for stringent tolerances.

The `lsode` method adapts the order and the step size; therefore, it is efficient over a wide range of tolerances.

More detailed information about numerical methods for initial value problems embedded in Maple is presented in Table 19.2.

Remark 19.9. The following abbreviations in Tables 19.2–19.6 are adopted: `IVP`, initial value problem; `BDF`, backward differentiation formula; `IVP–DAE`, initial value problem for differential algebraic equations, `IVP–DDE`, initial value problem for delay differential equations.

Table 19.2.
Numerical methods for initial value problems embedded in Maple
with brief description and some references

Numerical method	Brief description	References
`rkf45`	The Runge–Kutta–Fehlberg method with 4-degree interpolant. Order: 4-5. Explicit default method for nonstiff IVP without singularities. Adaptive method with a control of the discretization error.	Enright et al. (1986) Fehlberg (1970) Shampine and Corless (2000)
`ck45`	The Cash–Karp Runge–Kutta method with 4-degree interpolant. Order: 4–5. Explicit method for nonstiff IVP. Adaptive method with a control of the discretization error.	Enright et al. (1986) Cash and Karp (1990) Forsythe et al. (1977)
`rosenbrock`	The Rosenbrock Runge–Kutta method with 3-degree interpolant. Order: 3-4. Implicit default method for stiff IVP. Adaptive method with a control of the discretization error.	Hairer and Wanner (1996) Shampine and Corless (2000) Forsythe et al. (1977)
`dverk78`	The continuous Runge–Kutta method. Order: 7-8. Explicit method for nonstiff IVP. High-accuracy solutions can be obtained. Adaptive method with a control of the discretization error.	Enright (1991) Verner (1978) Forsythe et al. (1977)
`lsode`	The Livermore method for stiff IVP. 8 submethods: `adamsfunc`, `adamsfull`, `adamsdiag`, `adamsband`, `backfunc`, `backfull`, `backdiag`, `backband`. Adaptive method (order, step size), high-accuracy, a wide range of tolerances.	Hindmarsh (1983) Forsythe et al. (1977) Shampine and Corless (2000)
`gear`	The Gear single-step extrapolation method for stiff IVP. 2 submethods: `bstoer` (Burlirsch–Stoer rational extrapolation), `polyextr` (polynomial extrapolation). Adaptive method (order, step size), high-accuracy solutions.	Gear (1971) Shampine and Gear (1979) Shampine and Corless (2000)
`taylorseries`	Taylor series method for nonstiff IVP. High-accuracy solutions (takes more time). 2 submethods: `lazyseries` (lazy series expansion), `series` (local series expansion). Adaptive method with a control of the discretization error. The order can be specified.	Barton et al. (1972) Forsythe et al. (1977) Shampine and Corless (2000)
`classical`	Classical numerical methods (for education). 8 submethods: `foreuler`, `heunform`, `impoly`, `rk2`, `rk3`, `rk4`, `adambash`, `abmoulton`. Fixed step size, without error estimation or correction.	Boyce and DiPrima (2004) Conte and de Boor (1980) Fox and Mayers (1987)

There are two forms of introducing numerical methods for solving boundary value problems, `method=bvp` and `method=bvp[submethod]` (a specific submethod for solving boundary value problems).

The available methods are quite general and work on a variety of boundary value problems:

- linear and nonlinear (with fixed, periodic, and nonlinear boundary conditions),
- nonstiff boundary value problems,
- boundary value problems without singularities in higher-order derivatives,
- boundary value problems with undetermined parameters.

The submethods for boundary value problems embedded in Maple are presented in Table 19.3.

Table 19.3.
Numerical methods for boundary value problems embedded in Maple
with brief description and some references

Numerical method	Brief description	References
`traprich`	Trapezoid method with Richardson extrapolation enhancement. More efficient for typical problems. Richardson extrapolation is generally faster.	Ascher et al. (1995) Ascher and Petzold (1998)
`trapdefer`	Trapezoid method with deferred correction enhancement. More efficient for typical problems. Deferred corrections uses less memory on difficult problems.	Ascher et al. (1995) Ascher and Petzold (1998)
`midrich`	Midpoint method with Richardson extrapolation enhancement. Can work with end-point singularities. Richardson extrapolation is generally faster.	Ascher et al. (1995) Ascher and Petzold (1998)
`middefer`	Midpoint method with deferred correction enhancement. Can work with end-point singularities. Deferred corrections uses less memory on difficult problems.	Ascher et al. (1995) Ascher and Petzold (1998)

In general, the extension methods for solving initial value problems for differential algebraic equations are very similar to the standard methods for initial value problems (see Table 19.2).

More detailed information about numerical methods for solving initial value problems for differential algebraic equations embedded in Maple is presented in Table 19.4.

More detailed information about numerical methods for solving initial value problems for delay differential equations embedded in Maple is presented in Table 19.5.

► Classical numerical methods embedded in Maple.

The classical numerical methods embedded in Maple for solving ODEs are

- the forward Euler method,
- the Heun method (the improved Euler method),
- the improved polygon method (the modified Euler method),
- the second-order classical Runge–Kutta method, the third-order classical Runge–Kutta method,

Table 19.4.
Numerical methods for initial value problems for differential-algebraic equations embedded in Maple with description and references

Numerical method	Brief description	References
`rkf45_dae`	The modified Runge–Kutta–Fehlberg method with 4-degree interpolant. Order: 4-5. An extension of `rkf45` method for nonstiff real-valued IVP-DAE.	Enright et al. (1986) Fehlberg (1970) Shampine and Corless (2000)
`ck45_dae`	The modified Cash–Karp Runge–Kutta method with 4-degree interpolant. Order: 4-5. An extension of `ck45` method for nonstiff real-valued IVP-DAE.	Enright et al. (1986) Cash and Karp (1990) Forsythe et al. (1977)
`rosenbrock_dae`	The modified Rosenbrock Runge–Kutta method with 3-degree interpolant. Order: 3-4. An extension of an implicit `rosenbrock` method for stiff real-valued IVP-DAE.	Hairer and Wanner (1996) Shampine and Corless (2000) Forsythe et al. (1977)
`mebdfi`	The modified extended BDF implicit method. For real-valued stiff IVP-DAE and for DAE of index 2 and lower.	Cash (1983) Cash (1992) Forsythe et al. (1977)

Table 19.5.
Numerical methods for initial value problems for delay differential equations embedded in Maple with description and references

Numerical method	Brief description	References
`rkf45`	The Runge–Kutta–Fehlberg method with 4-degree interpolant. Order: 4-5. For nonstiff real-valued IVP-DDE with constant and variable delays.	Enright et al. (1986) Fehlberg (1970) Shampine and Corless (2000)
`ck45`	The Cash–Karp Runge–Kutta method with 4-degree interpolant. Order: 4-5. For nonstiff real-valued IVP-DDE with constant and variable delays.	Enright et al. (1986) Cash and Karp (1990) Forsythe et al. (1977)
`rosenbrock`	The Rosenbrock Runge–Kutta method with 3-degree interpolant. Order: 3-4. The implicit method. For stiff real-valued IVP-DDE with constant and variable delays.	Hairer and Wanner (1996) Shampine and Corless (2000) Forsythe et al. (1977)

- the fourth-order classical Runge–Kutta method, the Adams–Bashforth method (a predictor method),
- the Adams–Bashforth–Moulton method (a predictor-corrector method).

However, there are some restrictions associated with these classical methods: they use a static (fixed) step size and provide no error estimation or correction.

The default classical method is the forward Euler method.

More detailed information about the classical numerical methods embedded in Maple is presented in Table 19.6.

Let us describe the classical methods for the first-order ODE $y'_x = f(x,y)$. We introduce the following notation: Y_i is the value of the solution at point X_i, h is the fixed step size $X_i - X_{i-1}$, and the value Y_{n+1} of the solution at X_{n+1} is being computed.

Table 19.6.
Classical numerical methods embedded in Maple
with brief description and some references

Numerical method	Brief description	References
`foreuler`	The forward Euler method (the default submethod). Order: 1. Single-step explicit method for nonstiff IVP.	Boyce and DiPrima (2004) Conte and de Boor (1980)
`heunform`	The Heun method (the improved Euler method). Order: 2. Single-step explicit method for nonstiff IVP.	Boyce and DiPrima (2004) Conte and de Boor (1980)
`impoly`	The improved polygon method (the modified Euler method). Order: 2. Single-step explicit method for nonstiff IVP.	Boyce and DiPrima (2004) Conte and de Boor (1980)
`rk2`	The second-order classical Runge–Kutta method. Order: 2. Single-step explicit method for nonstiff IVP.	Boyce and DiPrima (2004) Conte and de Boor (1980)
`rk3`	The third-order classical Runge–Kutta method. Order: 3. Single-step explicit method for nonstiff IVP.	Boyce and DiPrima (2004) Conte and de Boor (1980)
`rk4`	The fourth-order classical Runge–Kutta method. Order: 4. Single-step explicit method for nonstiff IVP.	Boyce and DiPrima (2004) Conte and de Boor (1980)
`adambash`	The Adams–Bashforth method (or a predictor method). Order: 4. Explicit 4-step method for nonstiff IVP.	Boyce and DiPrima (2004) Lambert (1973)
`abmoulton`	The Adams–Bashforth–Moulton method. Order: 4. Implicit 3-step predictor-corrector method for nonstiff IVP.	Boyce and DiPrima (2004) Fox and Mayers (1987)

The *forward Euler method* `foreuler` is specified by the equation

$$Y_{n+1} = Y_n + hf(X_n, Y_n).$$

The *Heun method* (or the *improved Euler method*) `heunform` applies the forward Euler method (as a predictor) and the trapezoid rule (as a corrector); it is specified by the equations

$$Y_p = Y_n + hf(X_n, Y_n), \qquad Y_{n+1} = Y_n + \tfrac{1}{2}h\left(f(X_n, Y_n) + f(X_{n+1}, Y_p)\right).$$

The *improved polygon method* (or the *modified Euler method*) `impoly` is specified by the equation

$$Y_{n+1} = Y_n + hf\left(X_n + \tfrac{1}{2}h, Y_n + \tfrac{1}{2}hf(X_n, Y_n)\right).$$

The *second-order classical Runge–Kutta method* `rk2` is specified by the equations

$$k_1 = f(X_n, Y_n), \quad k_2 = f(X_n + h, hk_1 + Y_n), \quad Y_{n+1} = Y_n + \tfrac{1}{2}h(k_1 + k_2).$$

The *third-order classical Runge–Kutta method* `rk3` is specified by the equations

$$\begin{aligned} &k_1 = f(X_n, Y_n), \quad k_2 = f(X_n + \tfrac{1}{2}h, Y_n + \tfrac{1}{2}hk_1), \\ &k_3 = f\left(X_n + h, Yn + h(-k1 + 2k2)\right), \qquad Y_{n+1} = Y_n + \tfrac{1}{6}h(k_1 + 4k_2 + k_3). \end{aligned}$$

The *fourth-order classical Runge–Kutta method* `rk4` is specified by the equations

$$\begin{aligned} &k_1 = f(X_n, Y_n),\ k_2 = f(X_n + \tfrac{1}{2}h, Y_n + \tfrac{1}{2}hk_1),\ k_3 = f(X_n + \tfrac{1}{2}h, Y_n + \tfrac{1}{2}hk_2), \\ &k_4 = f(X_n + h, hk_3 + Y_n) \qquad Y_{n+1} = Y_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

The *Adams–Bashforth method* (a *predictor method*) `adambash` is specified by the equation

$$Y_{n+1}=Y_n+\tfrac{h}{24}(55f(X_n,Y_n)-59f(X_{n-1},Y_{n-1})+37f(X_{n-2},Y_{n-2})-9f(X_{n-3},Y_{n-3})).$$

The *Adams–Bashforth–Moulton method* (a *predictor-corrector method*) `abmoulton` is specified by the equations

$$Y_{n+1}=Y_n+\tfrac{h}{24}\left(9f(X_{n+1},Y_{n+1})+19f(X_n,Y_n)-5f(X_{n-1},Y_{n-1})+f(X_{n-2},Y_{n-2})\right),$$

where $f(X_{n+1},Y_{n+1})$ is found by applying the Adams–Bashforth method (the predictor) and then the Adams–Bashforth–Moulton method (the corrector).

The `adambash` and `abmoulton` are *multistep methods*, which require the initial condition and three other *starting values* (equally spaced). These starting values can be obtained by computing the first 3 steps with the `rk4` method. The final step values (for fixed spacing) use the `rk4` method as well.

Example 19.39. *Cauchy problem. Exact solution. Classical numerical methods.*
For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = y_0 \tag{19.4.2.1}$$

on the interval $[a,b]$ ($a=0$, $b=9$) with parameters $p=1$, $q=-1$, $m=2$, $n=1$, and $y_0=0.729$, we find the exact solution (`SolEx`) and numerical solutions using the Euler method, the improved Euler method (the Heun method), and the Runge–Kutta method and compare the graphical solutions as follows:

```
with(plots): IVP1:={diff(y(x),x)=p*(y(x))^m+q*x^n, y(0)=0.729};
Tn:=type=numeric; S:=stepsize=0.17; a:=0; b:=9; p:=1; q:=-1;
n:=1; m:=2; C:=[color=magenta,color=red,color=green,color=blue]:
SolEx:=unapply(rhs(dsolve(IVP1,y(x))),x);
EulM:=dsolve(IVP1,y(x),Tn,method=classical[foreuler],S);
HeunM:=dsolve(IVP1,y(x),Tn,method=classical[heunform],S);
RK4M:=dsolve(IVP1,y(x),Tn,method=classical[rk4],S);
G1:=plot(SolEx(x),x=a..b,C[1]):  G2:=odeplot(EulM,[x,y(x)],C[2]):
G3:=odeplot(HeunM,[x,y(x)],C[3]):G4:=odeplot(RK4M,[x,y(x)],C[4]):
display({G1,G2,G3,G4},view=[a..b,0..3]);
```

The exact solution (`SolEx`) of this nonlinear Cauchy problem reads:

$$y(x) = -\frac{(r/q)\mathrm{Ai}^{(1)}(x)+\mathrm{Bi}^{(1)}(x)}{(r/q)\mathrm{Ai}(x)+\mathrm{Bi}(x)},$$

where $r=243\,3^{5/6}\pi+500\,\left(\Gamma\left(\frac{2}{3}\right)\right)^2\,3^{2/3}$, $q=500\,\sqrt[6]{3}\left(\Gamma\left(\frac{2}{3}\right)\right)^2-243\,\pi\,\sqrt[3]{3}$, the special functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions, and $\mathrm{Ai}^{(1)}(x)$ and $\mathrm{Bi}^{(1)}(x)$ are their first derivatives.

We can see (Fig. 19.5) that the numerical solution obtained by the Runge–Kutta method is in good agreement with the exact solution. To get a good approximation to the solution, we can modify the step size for the Euler method (e.g., `S1:=stepsize=0.0001`) and the Heun method (e.g., `S2:=stepsize=0.01`) and introduce a new parameter `MF:=maxfun=500000` for increasing the total number of evaluations of the right-hand side of the ODE for any single call to the procedure returned by `dsolve`. For example, we write out the modified version:

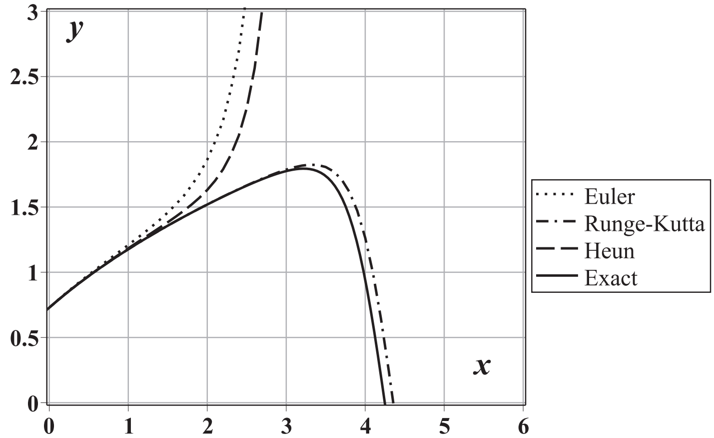


Figure 19.5: Exact solution of the Cauchy problem (19.4.2.1) and numerical approximations (obtained by the Euler, Heun, and Runge–Kutta methods).

```
with(plots): IVP1:={diff(y(x),x)=p*(y(x))^m+q*x^n, y(0)=0.729};
Tn:=type=numeric; S:=stepsize=0.17; S1:=stepsize=0.0001;
S2:=stepsize=0.01; MF:=maxfun=500000; a:=0; b:=9; p:=1; q:=-1;
n:=1; m:=2; C:=[color=magenta,color=red,color=green,color=blue]:
SolEx:=unapply(rhs(dsolve(IVP1,y(x))),x);
EulM:=dsolve(IVP1,y(x),Tn,method=classical[foreuler],S1,MF);
HeunM:=dsolve(IVP1,y(x),Tn,method=classical[heunform],S2,MF);
RK4M:=dsolve(IVP1,y(x),Tn,method=classical[rk4],S,MF);
G1:=plot(SolEx(x),x=a..b,C[1]):  G2:=odeplot(EulM,[x,y(x)],C[2]):
G3:=odeplot(HeunM,[x,y(x)],C[3]):G4:=odeplot(RK4M,[x,y(x)],C[4]):
display({G1,G2,G3,G4},view=[a..b,0..3]);
```

19.4.3 Initial Value Problems: Examples of Numerical Solutions

► **Preliminary remarks.**

In general, the ordinary differential equation $y'_x = f(x,y)$ admits infinitely many solutions $y = y(x)$. To find one of them, we have to add a condition of the form $y(x_0) = y_0$ ($x_0 = a$), where y_0 is a given value called the initial data.

Consider some examples of initial value problems.

► **Linear initial value problems.**

Example 19.40. *First-order linear ODE. Analytical, numerical, and graphical solutions.* For the first-order linear initial value problem

$$y'_x = -y\cos(x^2), \qquad y(0) = 1 \tag{19.4.3.1}$$

on the interval $[a,b]$ ($a = 0$, $b = 10$), we find infinitely many exact solutions

$$y(x) = C\,\mathrm{e}^{\frac{\sqrt{2\pi}}{2}\mathrm{FresnelC}(\sqrt{2}\,x/\sqrt{\pi})}$$

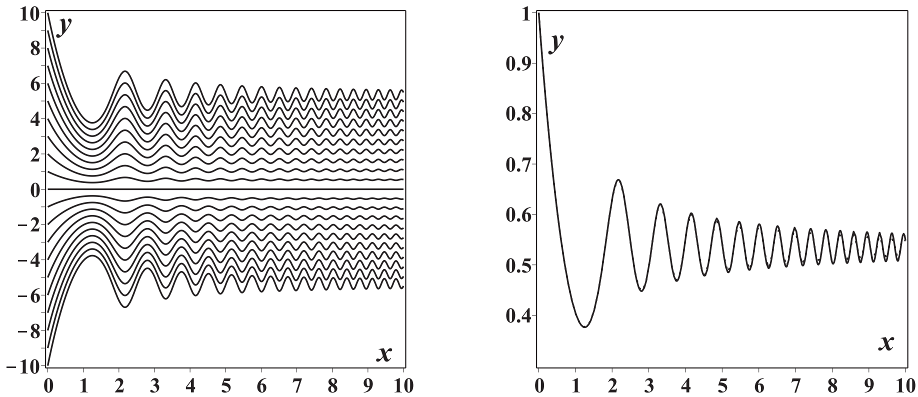


Figure 19.6: Exact solution of the Cauchy problem (19.4.2.1) and numerical approximations (obtained by the Euler, the Heun, and the Runge–Kutta methods).

admitted by this ordinary differential equation and plot some of them (`Sols`). Here the special function $\mathrm{FresnelC}(x)$ is the Fresnel cosine integral. Then we plot the unique exact solution of the Cauchy problem with the vector field (`DEplot`). Finally, we compute the numerical solution (`SolN`) of the Cauchy problem, plot it together with the exact solution (`G1`, `G2`), and compare the results as follows:

```
with(DETools):with(plots):setoptions(axes=boxed,numpoints=200);
ODE1:=D(y)(x)=-y(x)*cos(x^2); IC:=y(0)=1; IVP1:={ODE1,IC};
a:=0; b:=10; SolEx1:=dsolve(ODE1,y(x));
Sols:={seq(subs(_C1=i,rhs(SolEx1)),i=-b..b)}; plot(Sols,x=a..b);
DEplot(ODE1,y(x),x=a..b,[[0,1]],y=0..3);
SolEx2:=dsolve(IVP1,y(x)); SolN:=dsolve(IVP1,numeric,y(x));
G:=array(1..2); G[1]:=odeplot(SolN,[x,y(x)],a..b,color=blue):
G[2]:=plot(rhs(SolEx2),x=a..b,color=red): display(G);
```

We can see (Fig. 19.6) that the numerical solution is in good agreement with the exact solution.

Example 19.41. *Second-order linear ODE. Analytical, numerical, and graphical solutions.* For the second-order linear initial value problem

$$y''_{xx} - y'_x + (x-1)y = 0, \qquad y(0) = 1, \quad y'_x(0) = 0$$

on the interval $[a, b]$ ($a = 0$, $b = 10$), we find the exact solution (`Sol1`)

$$y(x) = \frac{1}{2}\,\frac{e^{x/2}\left(\mathrm{Bi}(x_1)\mathrm{Ai}(c) - \mathrm{Ai}(x_1)\mathrm{Bi}(c) - 2\,\mathrm{Bi}(x_1)\mathrm{Ai}^{(1)}(c) + 2\,\mathrm{Ai}(x_1)\mathrm{Bi}^{(1)}(c)\right)}{\mathrm{Bi}^{(1)}(c)\mathrm{Ai}(c) - \mathrm{Bi}(c)\mathrm{Ai}^{(1)}(c)},$$

where $x_1 = \frac{5}{4} - x$, $c = \frac{5}{4}$, the special functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions, and $\mathrm{Ai}^{(1)}(x)$ and $\mathrm{Bi}^{(1)}(x)$ are their first derivatives. The numerical solution (`Sol2`) and the graphical solutions (array `G`) can be obtained as follows:

```
with(plots): setoptions(axes=boxed,scaling=unconstrained,numpoints=200);
G:=Array(1..3); ODE1:=diff(y(x),x$2)-diff(y(x),x)+(x-1)*y(x)=0;
IC1:=D(y)(0)=0,y(0)=1; a:=0; b:=10; C:=[blue,red,magenta];
```

```
Sol1:=dsolve({ODE1,IC1},y(x)); Sol2:=dsolve({ODE1,IC1},y(x),numeric);
G[1]:=odeplot(Sol2,[x,y(x)],a..b,color=C[1]):
G[2]:=plot(rhs(Sol1),x=a..b,color=C[2]):
G[3]:=odeplot(Sol2,[y(x),diff(y(x),x)],0..10,color=C[3]): display(G);
```

► Nonlinear initial value problems.

Example 19.42. *First-order nonlinear ODE. Numerical and graphical solutions.*
For the nonlinear initial value problem

$$y'_x = -e^{xy}\cos(x^2), \qquad y(0) = p$$

on the interval $[a, b]$ ($a = 0$, $b = 10$), we find the numerical and graphical solutions for various initial conditions $y(0) = p$, where $p = 0.1\,i$ ($i = 1, 2, ..., 5$), as follows:

```
with(plots): R:=0..10;
Ops:=numpoints=100,color=blue,thickness=2,axes=boxed;
SolNG:=proc(IC) local Eq,EqIC,L1,SolN,ICN,i;
 Eq:=D(y)(x)=-exp(y(x)*x)*cos(x^2); L1:=NULL; ICN:=nops(IC);
 for i from 1 to ICN do
  EqIC:=evalf(y(0)=IC[i]); SolN:=dsolve({Eq,EqIC},y(x),type=numeric);
  L1:=L1,odeplot(SolN,[x,y(x)],R,Ops):
 od; display([L1]);
end:
List1:=[seq(0.1*i,i=1..5)]; SolNG(List1);
```

Example 19.43. *First-order nonlinear Cauchy problem. Numerical and graphical solutions.*
Consider the initial value problem for the nonlinear differential equation

$$y'_x = 1 - \sqrt{1 - qx^2y^2}, \qquad y(0) = p, \tag{19.4.3.2}$$

where $p \in \mathbb{R}$ and $q > 0$.

The existence domain of solutions of this differential equation with $q > 0$ is given by the inequality $x^2y^2 \leq 1/q$.

The differential equation in the Cauchy problem (19.4.3.2) has the equilibrium point $y = 0$. The solutions of the Cauchy problem for this equation with the initial conditions $y(0) = p$ behave differently depending on the sign of p.

If $p < 0$, then the solutions are infinitely extendible to the right. If $p > 0$, then the solutions approach the boundary of the existence domain at some x (that is, they are not infinitely extendible to the right). Therefore, the equilibrium position $y = 0$ is unstable, because, in any neighborhood of $y = 0$, there exist solutions that are not infinitely extendible.

For $q = 1$, several numerical solutions of the Cauchy problem (19.4.3.2) for various values of p are presented in Fig. 19.7 (left) for $p > 0$ and in Fig. 19.7 (right) for $p < 0$.

For example, for $p > 0$ we take the values $0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, and for $p < 0$ we take the values $-0.2, -0.4, -0.6, -0.8, -1.0, -1.2$. The solutions are valid for $x \geq 0$ and are presented on the interval $[a, b]$, where $a = 0$ and $b = 3$ or $b = 9$. Also in these figures we draw the boundary of the existence domain of solutions, $xy = \pm 1$.

To generate Fig. 19.7 (left), where $q = 1$ and $p = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, we can write the following program:

```
with(plots): R1:=0..3; R2:=0..9; Q1:=0..3; Q2:=-2..0.1;
Ops:=numpoints=100,color=blue,thickness=2,axes=boxed;
SolNG:=proc(IC,q,R) local Eq,EqIC,L1,SolN,ICN,i;
```

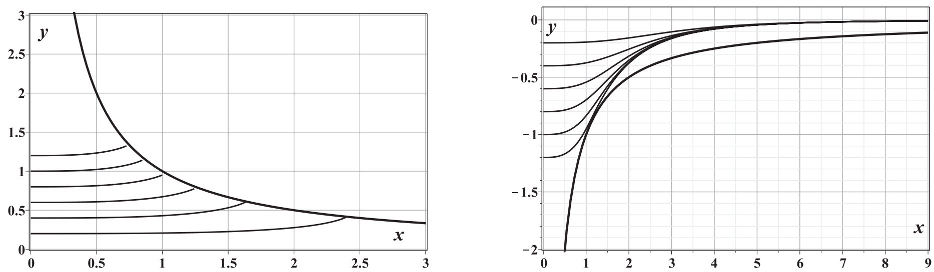


Figure 19.7: Numerical solutions of the Cauchy problem (19.4.3.2) for $q = 1$, $p > 0$ (left) and $p < 0$ (right).

```
 Eq:=D(y)(x)=1-sqrt(1-q*x^2*y(x)^2); L1:=NULL; ICN:=nops(IC);
 for i from 1 to ICN do
  EqIC:=evalf(y(0)=IC[i]); SolN:=dsolve({Eq,EqIC},y(x),type=numeric);
  L1:=L1,odeplot(SolN,[x,y(x)],R,Ops):
 od; display([L1]);
end:
List1:=[seq(0.2*i,i=1..6)]; List2:=[seq(-0.2*i,i=1..6)];
G1:=SolNG(List1,1,R1):  G2:=plot(1/x,x=R1,Q1):  display({G1,G2});
G3:=SolNG(List2,1,R2):  G4:=plot(-1/x,x=R2,Q2): display({G3,G4});
```

19.4.4 Initial Value Problems: Constructing Numerical Methods and Solutions

Alternatively, numerical methods and solutions of initial value problems can be constructed (step by step) and analyzed as follows.

► **Single-step methods.**

First, consider one of the classical methods, the *forward Euler method*, or the *explicit Euler method*. This method belongs to a family of *single-step methods*, which compute the numerical solution Y_{i+1} at the node X_{i+1} knowing the information related only to the previous node X_i.

The strategy of these methods is to divide the integration interval $[a, b]$ into N subintervals of length $h = (b - a)/N$, which is called the discretization step. Then at the nodes X_i ($0 \le i \le N$) we compute the unknown value Y_i, which approximates the exact value $y(X_i)$; i.e., $Y_i \approx y(X_i)$. The set of values $\{Y_0 = y_0, Y_1, \dots, Y_N\}$ is the numerical solution. The formula for the explicit Euler method reads:

$$Y_{i+1} = Y_i + hF(X_i, Y_i), \quad Y_0 = y(X_0), \quad i = 0, \dots, N-1.$$

Example 19.44. *The Euler method. Analytical, numerical, and graphical solutions.*
For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5 \qquad (19.4.4.1)$$

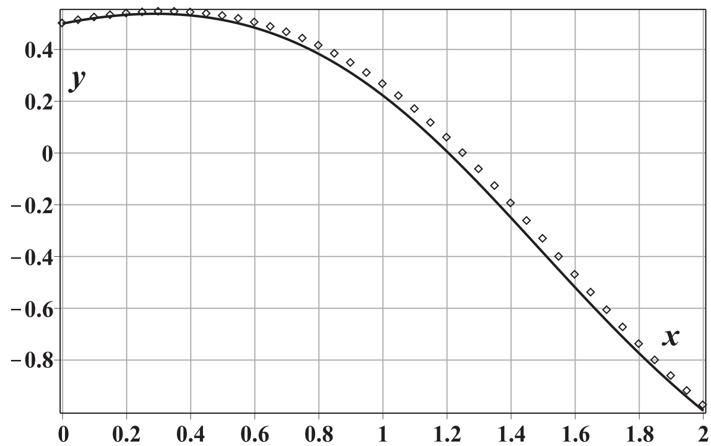


Figure 19.8: Exact solution (solid line) and numerical solution (points, the solution is obtained by the explicit Euler method) of the Cauchy problem (19.4.4.1).

on the interval $[a, b]$ ($a = 0$, $b = 2$) with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we find the exact solution (SolEx) and a numerical solution (F1) using the explicit Euler method, compare the results, plot the exact and numerical solutions (see Fig. 19.8), and determine the absolute computational error at each step as follows:

```
with(plots): ODE1:=diff(y(x),x)=p*(y(x))^m+q*x^n; IC:=y(0)=0.5;
a:=0; b:=2; p:=1; q:=-1; n:=1; m:=2; N:=40;
IVP1:={ODE1,IC}; SolEx:=unapply(rhs(dsolve(IVP1,y(x))),x);
F:=(x,y)->p*y^m+q*x^n; h:=evalf((b-a)/N); X:=x->a+x*h;
Y:=proc(n) option remember; Y(n-1)+h*F(X(n-1),Y(n-1)) end;
Y(0):=0.5; F1:=[seq([X(i),Y(i)],i=0..N)]; Array(F1);
for i from 0 to N do
 print(i,X(i),Y(i),evalf(SolEx(X(i))),evalf(abs(Y(i)-SolEx(X(i)))));
od;
G1:=plot(SolEx(x),x=a..b): G2:=plot(F1,style=point,color=red):
display({G1,G2});
```

There is a general way to determine the order of convergence of a numerical method. If we know the errors E_i ($i = 1, \dots, N$) depending on the values h_i of the discretization parameter (in our case, h_i is the discretization step of the Euler method) and assume that $E_i = Ch_i^p$ and $E_{i-1} = Ch_{i-1}^p$, then

$$p = \frac{\log(E_i/E_{i-1})}{\log(h_i/h_{i-1})}, \qquad i = 2, \dots, N. \tag{19.4.4.2}$$

Example 19.45. *The Euler method. The order of convergence.*
For the same Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5$$

(as in the previous example) on the interval $[a, b]$ ($a = 0$, $b = 2$) with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we obtain a numerical solution by applying the explicit Euler method for various values of the discretization step h and, according to formula (19.4.4.2), verify that the order of convergence of the explicit Euler method is 1:

```
ODE1:=diff(y(x),x)=p*(y(x))^m+q*x^n; IC:=y(0)=0.5;
a:=0; b:=2; p:=1; q:=-1; n:=1; m:=2; N:=40;
IVP1:={ODE1,IC};
Euler:=proc(IVP::set,a,b,p,q,n,m,N) local h,xL,X,Y,F,F1,SolEx,EN;
 h:=(b-a)/N; X:=xL->a+xL*h; F:=(x,y)->p*y^m+q*x^n;
 SolEx:=unapply(rhs(dsolve(IVP,y(x))),x);
 Y:=proc(xL) option remember; Y(xL-1)+h*F(X(xL-1),Y(xL-1)) end;
 Y(0):=0.5; EN:=[seq(abs(Y(i)-evalf(SolEx(X(i)))),i=0..N)];
RETURN(EN); end:
L1:=NULL: N1:=4:
for k from 1 to 12 do
 E||k:=Euler(IVP1,0,2,1,-1,1,2,N1): print(E||k[N1+1]);
 L1:=L1,E||k[N1+1]; N1:=N1*2;
od:  Ers:=[L1]; NErs:=nops(Ers);
p:=[seq(evalf(abs(log(Ers[i]/Ers[i-1])/log(2))),i=2..NErs)];
```

Runge–Kutta methods are single-step methods that involve several evaluations of the function $f(x,y)$ and none of its derivatives on every interval $[X_i, X_{i+1}]$.

In general, explicit or implicit Runge–Kutta methods can be constructed in arbitrary order according to the formulas. Consider the s-stage explicit Runge–Kutta method

$$k_1 = f(x_n, y_n), \quad k_2 = f(x_n + c_2 h, y_n + a_{2,1} k_1 h), \quad \dots,$$
$$k_s = f\Big(x_n + c_s h, y_n + \sum_{i=1}^{s-1} a_{s,j}\, k_j\Big),$$
$$Y_{n+1} = Y_n + h\sum_{i=1}^{s} b_i k_i, \quad Y_0 = y_0, \quad n = 0, \dots N-1.$$

Example 19.46. *Higher-order methods. Derivation of explicit Runge–Kutta methods.* Let us perform analytical derivation of the best-known Runge–Kutta methods.

```
h0:=h=0; alias(F=f(x,y(x)),Fx=D[1](f)(x,y(x)),
 Fy=D[2](f)(x,y(x)),Fxx=D[1,1](f)(x,y(x)),Fxy=D[1,2](f)(x,y(x)),
 Fyy=D[2,2](f)(x,y(x)),Fyyy=D[2,2,2](f)(x,y(x)),
 Fxxx=D[1,1,1](f)(x,y(x)),Fxyy=D[1,2,2](f)(x,y(x)),
 Fxxy=D[1,1,2](f)(x,y(x))); D(y):=x->f(x,y(x));
```

For $s = 1$, we obtain the *Euler method* (`Sol`), where $b_1 = 1$:

```
s:=1; P1:=convert(taylor(y(x+h),h0,s+1),polynom);
P2:=expand((P1-y(x))/h); k1:=taylor(f(x,y(x)),h0,s);
P3:=expand(convert(taylor(add(b[i]*k||i,i=1..s),h0,s),polynom));
Eq1:=P2-P3; Eq2:={coeffs(Eq1,[h,F])};Sol:=solve(Eq2,indets(Eq2));
```

For $s = 2$, we obtain the *2-stage modified Euler method* (`Sol1`), where

$$a_{2,1} = \tfrac{1}{2}, \quad b_1 = 0, \quad b_2 = 1, \quad c_2 = \tfrac{1}{2},$$

the *2-stage improved Euler method* (`Sol2`), where

$$a_{2,1} = 1, \quad b_1 = b_2 = \tfrac{1}{2}, \quad c_2 = 1,$$

and the *2-stage Heun method* (`Sol3`), where

$$a_{2,1} = \tfrac{2}{3}, \quad b_1 = \tfrac{1}{4}, \quad b_2 = \tfrac{3}{4}, \quad c_2 = \tfrac{2}{3}:$$

```
s:=2; P1:=convert(taylor(y(x+h),h0,s+1),polynom);
P2:=expand((P1-y(x))/h); k1:=taylor(f(x,y(x)),h0,s);
k2:=taylor(f(x+c[2]*h,y(x)+h*add(a[2,i]*k||i,i=1..2-1)),h0,s);
P3:=expand(convert(taylor(add(b[i]*k||i,i=1..s),h0,s),polynom));
Eq1:=P2-P3; Eq2:={coeffs(Eq1,[h,F,Fx,Fy])}; Eq3:={}:
for i from 2 to s do Eq3:=Eq3 union{c[i]=add(a[i,j],j=1..i-1)}; od;
Sol1:=solve(Eq2 union Eq3 union {c[2]=1/2},indets(Eq2));
Sol2:=solve(Eq2 union Eq3 union {b[2]=1/2},indets(Eq2));
Sol3:=solve(Eq2 union Eq3 union {b[2]=3/4},indets(Eq2));
```

For $s = 3$, we obtain the *3-stage Heun method* (`Sol`), where

$$a_{2,1} = \tfrac{1}{3},\ \ a_{3,1} = 0,\ \ a_{3,2} = \tfrac{2}{3},\ \ b_1 = \tfrac{1}{4},\ \ b_2 = 0,\ \ b_3 = \tfrac{3}{4},\ \ c_2 = \tfrac{1}{3},\ \ c_3 = \tfrac{2}{3}:$$

```
s:=3; P1:=taylor(y(x+h),h0,s+1);
P2:=expand(convert(expand((P1-y(x))/h),polynom));
k1:=taylor(f(x,y(x)),h0,s):
k2:=taylor(f(x+c[2]*h,y(x)+h*(add(a[2,i]*k||i,i=1..2-1))),h0,s);
k3:=taylor(f(x+c[3]*h,y(x)+h*(add(a[3,i]*k||i,i=1..3-1))),h0,s);
P3:=expand(convert(taylor(add(b[i]*k||i,i=1..s),h0,s),polynom)):
Eq1:=P2-P3: Eq2:={coeffs(Eq1,[h,F,Fx,Fy,Fxx,Fxy,Fyy])}; Eq3:={}:
for i from 2 to s do Eq3:=Eq3 union {c[i]=add(a[i,j],j=1..i-1)}; od;
Sol:=solve(Eq2 union Eq3 union {b[1]=1/4,c[2]=1/3},indets(Eq2));
```

For $s = 4$, we obtain the fourth-order *Runge–Kutta* method (`Sol`), where

$$\begin{gathered}a_{2,1} = \tfrac{1}{2},\ \ a_{3,1} = 0,\ \ a_{3,2} = \tfrac{1}{2},\ \ a_{4,1} = 0,\ \ a_{4,2} = 0,\ \ a_{4,3} = 1,\\ b_1 = \tfrac{1}{6},\ \ b_2 = \tfrac{1}{3},\ \ b_3 = \tfrac{1}{3},\ \ b_4 = \tfrac{1}{6},\ \ c_2 = \tfrac{1}{2},\ \ c_3 = \tfrac{1}{2},\ \ c_4 = 1:\end{gathered}$$

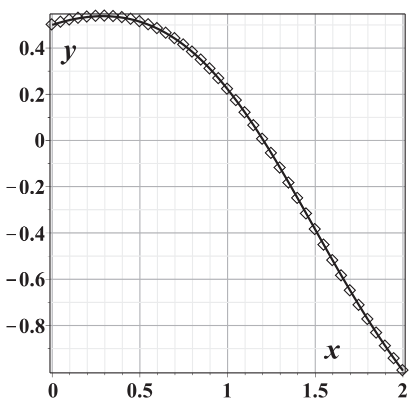
```
s:=4; P1:=taylor(y(x+h),h0,s+1);
P2:=expand(convert(expand((P1-y(x))/h),polynom));
k1:=taylor(f(x,y(x)),h0,s):
k2:=taylor(f(x+c[2]*h,y(x)+h*(add(a[2,i]*k||i,i=1..2-1))),h0,s);
k3:=taylor(f(x+c[3]*h,y(x)+h*(add(a[3,i]*k||i,i=1..3-1))),h0,s);
k4:=taylor(f(x+c[4]*h,y(x)+h*(add(a[4,i]*k||i,i=1..4-1))),h0,s);
P3:=expand(convert(taylor(add(b[i]*k||i,i=1..s),h0,s),polynom)):
Eq1:=P2-P3:
Eq2:={coeffs(Eq1,[h,F,Fx,Fy,Fxx,Fxy,Fyy,Fxxx,Fxxy,Fxyy,Fyyy])};
Eq3:={}:
for i from 2 to s do Eq3:=Eq3 union {c[i]=add(a[i,j],j=1..i-1)}; od;
Sol:=solve(Eq2 union Eq3 union
 {b[1]=1/6,c[2]=1/2,a[3,2]=1/2},indets(Eq2));
```

► Multistep methods.

There are more sophisticated methods that achieve a high order of accuracy by considering several values $(Y_i, Y_{i-1}, \dots)$ to determine Y_{i+1}. One of the most notable methods is the explicit four-step fourth-order Adams–Bashforth method

$$Y_{i+1}=Y_i+\frac{h}{24}\Big(55F(T_i,Y_i)-59F(T_{i-1},Y_{i-1})+37F(T_{i-2},Y_{i-2})-9F(T_{i-3},Y_{i-3})\Big).$$

Example 19.47. *Cauchy problem. The explicit Adams–Bashforth method.*



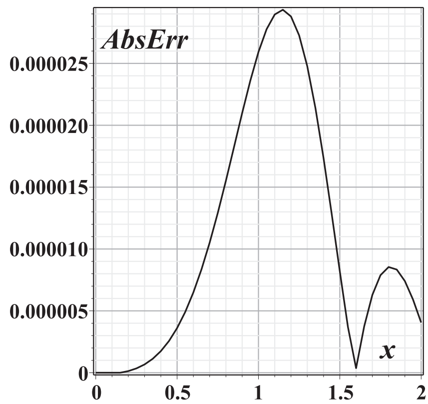


Figure 19.9: (Left) Exact solution (solid line) and numerical solution (points; the solution is obtained by the Adams–Bashforth method) of the Cauchy problem (19.4.4.3). (Right) The absolute computational error (at each step).

For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5 \tag{19.4.4.3}$$

on the interval $[a, b]$ ($a = 0$, $b = 2$) with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we find the exact solution (`SolEx`) and a numerical solution (`F1`) by the explicit Adams–Bashforth method and plot them (see Fig. 19.8). Finally, we compute the absolute computational error on $[a, b]$ (at each step) and plot it (`F2`) as follows:

```
with(plots): a:=0; b:=2; p:=1; q:=-1; n:=1; m:=2; N:=40;
ODE1:=diff(y(x),x)=p*(y(x))^m+q*x^n; ICs:=y(0)=0.5; IVP1:={ODE1,ICs};
SolEx:=unapply(rhs(dsolve(IVP1,y(x))),x); F:=(x,y)->p*y^m+q*x^n;
h:=evalf((b-a)/N); X:=x->a+x*h;
Y_AB:= proc(n) option remember;  Y_AB(n-1)+h/24*(55*F(X(n-1),Y_AB(n-1))
 -59*F(X(n-2),Y_AB(n-2))+37*F(X(n-3),Y_AB(n-3))-9*F(X(n-4),Y_AB(n-4)));
end:
Y_AB(0):=0.5: Y_AB(1):=evalf(SolEx(X(1))); Y_AB(2):=evalf(SolEx(X(2)));
Y_AB(3):=evalf(SolEx(X(3))); F1:=[seq([X(i),Y_AB(i)],i=0..N)];
for i from 0 to N do
 print(X(i),evalf(SolEx(X(i))),Y_AB(i),evalf(abs(Y_AB(i)-SolEx(X(i))))):
od:
G1:=plot(SolEx(x),x=a..b): G2:=plot(F1,style=point,color=red):
F2:=[seq([X(i),abs(Y_AB(i)-evalf(SolEx(X(i))))],i=0..N)];
display({G1,G2});  plot(F2);
```

We can see (Fig. 19.9) that the numerical solution is in good agreement with the exact solution.

Another important example of multistep methods is the implicit three-step fourth-order Adams–Bashforth–Moulton method

$$Y_{i+1} = Y_i + \frac{h}{24}\Big(9F(T_{i+1}, Y_{i+1}) + 19F(T_i, Y_i) - 5F(T_{i-1}, Y_{i-1}) + F(T_{i-2}, Y_{i-2})\Big).$$

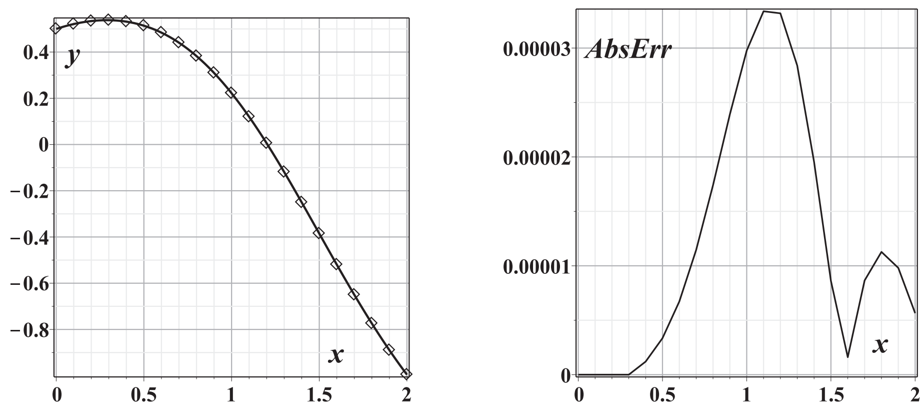


Figure 19.10: (Left) Exact solution (solid line) and numerical solution (points; the solution is obtained by the Adams–Bashforth–Moulton method) of the Cauchy problem (19.4.4.4). (Right) The absolute computational error (at each step).

Example 19.48. *Cauchy problem. The implicit Adams–Bashforth–Moulton method.*
For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5 \tag{19.4.4.4}$$

on the interval $[a, b]$ ($a=0, b=2$) with parameters $p=1, q=-1, m=2$, and $n=1$, we find the exact solution (`SolEx`) and numerical solutions (`F1`) by applying the implicit Adams–Moulton method, compare the results and the graphical solutions, find the absolute computational error on $[a, b]$ (at each step), and plot it (`F2`) as follows:

```
with(plots): with(codegen): a:=0; b:=2; p:=1; q:=-1; n:=1; m:=2; N:=20;
ODE1:=diff(y(x),x)=p*(y(x))^m+q*x^n; ICs:=y(0)=0.5; IVP1:={ODE1,ICs};
SolEx:=unapply(rhs(dsolve(IVP1,y(x))),x); F:=(x,y)->p*y^m+q*x^n;
h:=evalf((b-a)/N); X:=x->a+x*h;
Eq1:=Y_ABM(i)-Y_ABM(i-1)-h/24*(9*F(X(i),Y_ABM(i))+19*F(X(i-1),
 Y_ABM(i-1))-5*F(X(i-2),Y_ABM(i-2))+F(X(i-3),Y_ABM(i-3)));
Eq2:=[solve(Eq1,Y_ABM(i))]; Y_ABM:=makeproc(Eq2[2],i); Y_ABM(0):=0.5;
Y_ABM(1):=evalf(SolEx(X(1))); Y_ABM(2):=evalf(SolEx(X(2)));
Y_ABM(3):=evalf(SolEx(X(3))); F1:=[seq([X(i),Y_ABM(i)],i=0..N)];
for i from 0 to N do
 print(X(i),evalf(SolEx(X(i))),Y_ABM(i),
 evalf(abs(Y_ABM(i)-SolEx(X(i))))): od:
G1:=plot(SolEx(x),x=a..b): G2:=plot(F1,style=point,color=red):
display({G1,G2});
F2:=[seq([X(i),abs(Y_ABM(i)-evalf(SolEx(X(i))))],i=0..N)]; plot(F2);
```

19.4.5 Boundary Value Problems: Examples of Numerical Solutions

A two-point boundary value problem includes an ODE (of order ≥ 2) and the value of the solution at two distinct points. Note a difference between initial value problems and boundary value problems: initial value problems (with well-behaved functions) have unique solutions; i.e., they are "well posed"; but boundary value problems (with well-behaved functions) may have more than one solution or no solution (see Example 19.50).

Consider some examples of boundary value problems applying embedded numerical methods and constructing step-by-step solutions.

► Linear boundary value problems.

Example 19.49. *Boundary value problem. Analytical, numerical, and graphical solutions.*
Consider the following second-order linear nonhomogeneous ODE with variable coefficients and with boundary conditions:

$$y''_{xx} + xy'_x + y = \cos(x), \qquad y(a) = 0, \quad y(b) = 1, \tag{19.4.5.1}$$

where $a = 0$ and $b = 2$. Analytical, numerical, and graphical solutions (Sol1, Sol2, G1, G2) can be constructed as follows:

```
Digits:=15: with(PDEtools): declare(y(x),prime=x);
a:=0; b:=2; ODE1:=diff(y(x),x$2)+x*diff(y(x),x)+y(x)=cos(x);
BC1:=y(a)=0,y(b)=1; Sol1:=dsolve({ODE1,BC1},y(x));
Test1:=odetest(Sol1,ODE1);
Sol2:=dsolve({ODE1,BC1},y(x),type=numeric);
with(plots): k:=0: xR:=x=a..b;
for i from a to b by 0.1 do
 k:=k+1: X[k]:=i:
 s[k]:=simplify(fnormal(evalf(eval(rhs(Sol1),x=i))),zero);
od:
N:=k; Seq1:=seq([X[i],s[i]],i=1..N):
G1:=plot([Seq1],style=line,color="MidnightBlue"):
G2:=odeplot(Sol2,xR,style=point,color=red,symbolsize=15):
display({G1,G2});
```

By comparing the results, we can conclude that the analytical and numerical solutions are in good agreement.

Example 19.50. *Two-point boundary value problem for a linear ODE. No solution.*
Solving a boundary value problem for the second-order linear homogeneous ODE with constant coefficients

$$y''_{xx} + \pi^2 y = 0, \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{19.4.5.2}$$

where $a = 0$, $b = 1$, $\alpha = 1$, and $\beta = 1$, we can find the general solution of the equation. However, the boundary conditions cannot be satisfied (for any choice of the arbitrary constants occurring in the solution). Therefore, there is no solution of this problem:

```
with(PDEtools): declare(y(x),prime=x);
a:=0; b:=1; alpha:=1; beta:=1; ODE1:=diff(y(x),x$2)+Pi^2*y(x)=0;
BC1:=y(a)=alpha,y(b)=beta; Sol1:=dsolve({ODE1,BC1},y(x));
SolGen:=dsolve(ODE1,y(x)); Eq1:=eval(SolGen,x=a);
Eq2:=eval(SolGen,x=b); sys1:={rhs(Eq1)=alpha,rhs(Eq2)=beta};
solve(sys1,{_C1,_C2});
```

Consider the boundary value problem for the second-order ODE

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y(b) = \beta.$$

We assume that the functions $f(x, y, u)$, $f_y(x, y, u)$, and $f_u(x, y, u)$ are continuous in the open domain $D = \{a \le x \le b,\ -\infty < y < \infty,\ -\infty < u < \infty\}$. If $f_y(x, y, u) > 0$

and there exist constants M and K such that $|f_y(x,y,u)| \le M$ and $|f_u(x,y,u)| \le K$ for all $(x,y,u) \in D$, then the boundary value problem has a unique solution.

For the special case in which the function $f(x,y,u)$ is linear, i.e.,

$$f(x,y,y'_x) = p(x)y'_x + q(x)y + r(x),$$

the boundary value problem has a unique solution if $p(x)$, $q(x)$, and $r(x)$ are continuous in $[a,b]$ and $q(x) > 0$.

Linear shooting methods employ the numerical methods (discussed above) for solving initial value problems; e.g.,

$$\begin{aligned} u''_{xx} &= p(x)u'_x + q(x)u + r(x), \quad & u(a) = \alpha, \quad & u'_x(a) = 0; \\ v''_{xx} &= p(x)v'_x + q(x)v, & v(a) = 0, \quad & v'_x(a) = 1, \end{aligned}$$

where $x \in [a,b]$, and the solution of the original boundary value problem is

$$y(x) = u(x) + v(x)\frac{\beta - u(b)}{v(b)}.$$

Example 19.51. *Boundary value problems. Linear shooting methods.*
For the linear boundary value problem

$$y''_{xx} = -\frac{2}{x}y'_x + \frac{2}{x^2}y + x^3, \qquad y(1) = 1, \quad y(2) = 2, \tag{19.4.5.3}$$

we can find the exact solution (`SolEx`), a numerical solution (`F1`) by applying the linear shooting method, compare the results, and plot the exact and numerical solutions (`G1`, `G2`) as follows:

```
with(plots): Fu1:=(x,u1,u2)->u2: Fu2:=(x,u1,u2)->-2/x*u2+2/x^2*u1+x^3:
Fv1:=(x,v1,v2)->v2: Fv2:=(x,v1,v2)->-2/x*v2+2/x^2*v1:
N:=10: a:=1: b:=2: h:=evalf((b-a)/N); X:=i->a+h*i; alpha:=1; beta:=2;
ODE1:=(D@@2)(y)(x)+2/x*D(y)(x)-2/x^2*y(x)-x^3; BCs:=y(a)=alpha,y(b)=beta;
BVP1:={ODE1,BCs}; SolEx:=unapply(rhs(dsolve(BVP1,y(x))),x);
RK41:=proc(i,F1,F2,K) local k1,k2,k3,k4,m1,m2,m3,m4; option remember;
k1:=h*F1(X(i-1),RK41(i-1,F1,F2,RK41),RK41(i-1,F1,F2,RK42));
m1:=h*F2(X(i-1),RK41(i-1,F1,F2,RK41),RK41(i-1,F1,F2,RK42));
k2:=h*F1(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k1/2,RK41(i-1,F1,F2,RK42)+m1/2);
m2:=h*F2(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k1/2,RK41(i-1,F1,F2,RK42)+m1/2);
k3:=h*F1(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k2/2,RK41(i-1,F1,F2,RK42)+m2/2);
m3:=h*F2(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k2/2,RK41(i-1,F1,F2,RK42)+m2/2);
k4:=h*F1(X(i-1)+h,RK41(i-1,F1,F2,RK41)+k3,RK41(i-1,F1,F2,RK42)+m3);
m4:=h*F2(X(i-1)+h,RK41(i-1,F1,F2,RK41)+k3,RK41(i-1,F1,F2,RK42)+m3);
if K=RK41 then evalf(RK41(i-1,F1,F2,RK41)+1/6*(k1+2*k2+2*k3+k4));
 else evalf(RK41(i-1,F1,F2,RK42)+1/6*(m1+2*m2+2*m3+m4)); fi; end;
RK41(0,Fu1,Fu2,RK41):=alpha; RK41(0,Fu1,Fu2,RK42):=0;
RK41(0,Fv1,Fv2,RK41):=0; RK41(0,Fv1,Fv2,RK42):=1;
C:=(beta-RK41(N,Fu1,Fu2,RK41))/RK41(N,Fv1,Fv2,RK41);
Y:=proc(i) option remember;
 evalf(RK41(i,Fu1,Fu2,RK41)+C*RK41(i,Fv1,Fv2,RK41));
end:
array([seq([RK41(i,Fu1,Fu2,RK41),RK41(i,Fv1,Fv2,RK41),Y(i),
 evalf(SolEx(X(i))),abs(Y(i)-evalf(SolEx(X(i))))],i=0..N)]);
F1:=[seq([X(i),Y(i)],i=0..N)]; G1:=plot(SolEx(x),x=a..b,color=red):
G2:=plot(F1,style=point,color=blue): display({G1,G2});
```

Let us apply the finite difference method for approximating the solution of the linear boundary value problem

$$y''_{xx} = p(x)y'_x + q(x)y + r(x), \quad y(a) = \alpha, \quad y(b) = \beta.$$

The basic idea of finite difference methods is to replace the derivatives in differential equations by appropriate finite differences. We choose an equidistant grid $X_i = a + ih$ $(i = 0, \dots, N+1)$ on $[a, b]$ with step size $h = (b-a)/(N+1)$ $(N \in \mathbb{N})$, where $X_0 = a$ and $X_{N+1} = b$. The differential equation must be satisfied at any internal node X_i $(i = 1, \dots, n)$, and by approximating this set of N equations and by replacing the derivatives with appropriate finite differences, we obtain the system of equations

$$\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} = p(X_i)\frac{Y_{i+1} - Y_{i-1}}{2h} + q(X_i)Y_i + r(X_i), \quad Y_0 = \alpha, \;\; Y_{N+1} = \beta$$

for the approximate values Y_i of the exact solution $y(X_i)$. This linear system admits a unique solution, because the matrix of the system is an $N \times N$ symmetric positive definite tridiagonal matrix.

Example 19.52. *Approximations by finite differences.*

For the linear boundary value problem (19.4.5.1), we can find the exact solution (`SolEx`) and a numerical solution (`F1`) by the finite difference method, compare the results, and plot the exact and numerical solutions (`G1`, `G2`) as follows:

```
with(plots): a:=1; b:=2; alpha:=1; beta:=2; N:=10; h:=(b-a)/(N+1);
ODE1:=(D@@2)(y)(x)=-2/x*diff(y(x),x)+2/x^2*y(x)+x^3;
BCs:=y(a)=alpha,y(b)=beta; BVP1:={ODE1,BCs};
SolEx:=unapply(rhs(dsolve(BVP1,y(x))),x); X:=i->a+i*h;
p:=x->-2/x; q:=x->2/x^2; r:=x->x^3; SEq:={}:
for i from 1 to N do
 SEq:=SEq union {-(1+h/2*p(X(i)))*Y(i-1)+(2+h^2*q(X(i)))*Y(i)
 -(1-h/2*p(X(i)))*Y(i+1)=-h^2*r(X(i))};
od: SEq;
Y_DF:=convert(solve(SEq,{'Y(i)'$'i'=1..N}),list);
Y_DFBC:=evalf(subs({Y(0)=alpha,Y(N+1)=beta},Y_DF));
array([seq([rhs(Y_DFBC[i]),evalf(SolEx(X(i))),
 rhs(Y_DFBC[i])-evalf(SolEx(X(i)))],i=1..N)]);
F1:=[seq([X(i),rhs(Y_DFBC[i])],i=1..N),[X(0),alpha],[X(N+1),beta]];
G1:=plot(SolEx(x),x=a..b,color=red):
G2:=plot(F1,style=point,color=blue): display({G1,G2});
```

► Nonlinear boundary value problems.

In addition to the nonlinear boundary value problem

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{19.4.5.4}$$

consider the initial value problem

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y'_x(a) = s, \tag{19.4.5.5}$$

where $x \in [a, b]$. The real parameter s describes the initial slope of the solution curve.

Let $f(x, y, u)$ be a continuous function satisfying the Lipschitz condition with respect to y and u. Then, by the Picard–Lindelöf theorem, for each s there exists a unique solution $y(x, s)$ of the above initial value problem.

To find a solution of the nonlinear boundary value problem, we choose a value of the parameter s such that $y(b, s) = \beta$; i.e., we have to solve the nonlinear equation $F(s) = y(b, s) - \beta = 0$ by applying one of the known numerical methods.

Example 19.53. *Nonlinear boundary value problem. Nonlinear shooting methods.*
For the nonlinear boundary value problem

$$y''_{xx} = -y^2, \qquad y(0) = 0, \quad y(2) = 1, \tag{19.4.5.6}$$

we find a numerical solution by applying the nonlinear shooting method (`ShootNL`) and plot the numerical results obtained with (`ShootNL`) for various values of the parameter s and the numerical solution obtained with the predefined function (`dsolve,numeric`) as follows:

```
with(plots): a:=0.; b:=2.; alpha:=0.; beta:=1.; sR:=0.5..1;
ODE1:=diff(y(x),x$2)+y(x)^2=0; BCs:=y(a)=alpha,y(b)=beta;
Op:=output=listprocedure; Opt:=thickness=2;
IC:=[0.6,0.5,1,0.8,0.85,R]; k:=nops(IC);
ShootNL:=proc(x,s) local yN, ICs;
 ICs:=y(0)=0,D(y)(0)=s; yN:=rhs(dsolve({ODE1,ICs},numeric,Op)[2]);
RETURN(evalf(yN(x))); end; ShootNL(b,0.1); ShootNL(b,0.5);
plot(['ShootNL(b,s)',beta],'s'=sR);
R:=fsolve('ShootNL(2,s)=1','s'=sR); ShootNL(b,R)=beta;
plot('ShootNL(x,R)','x'=a..b,color=red,Opt);
for i from 1 to k do
 G||i:=plot('ShootNL(x,IC[i])','x'=a..b,axes=boxed,Opt,
  color=COLOR(RGB,rand()/10^12,rand()/10^12,rand()/10^12)):
od: display({seq(G||i,i=1..k)});
plot(rhs(dsolve({ODE1,BCs},y(x),numeric,Op)[2]),a..b);
```

Let us apply the finite difference method for approximating the solution of the nonlinear boundary value problem (19.4.5.4). We choose an equidistant grid $X_i = a + ih$ ($i = 0, \dots, N+1$) on $[a, b]$ with step size $h = (b-a)/(N+1)$, where $X_0 = a$ and $X_{N+1} = b$ ($N \in \mathbb{N}$). By approximating the nonlinear boundary value problem, we arrive at the system of nonlinear equations

$$\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} = f\left(X_i, Y_i, \frac{Y_{i+1} - Y_{i-1}}{2h}\right), \quad Y_0 = \alpha, \quad Y_{N+1} = \beta$$

for the approximate values Y_i of the exact solution $y(X_i)$. To solve this system of nonlinear equations, we can apply the Newton method.

Example 19.54. *Nonlinear boundary value problem. Approximations by finite differences.*
For the nonlinear boundary value problem (19.4.5.6), we find the numerical solution by applying the predefined function (`Sol`):

```
with(plots): with(LinearAlgebra): with(codegen): Nmax:=100:
epsilon:=10^(-4); f:=(x,y,dy)->y(x)^2; a:=0: b:=2: N:=20:
h:=evalf((b-a)/(N+1)); alpha:=0: beta:=1: Op:=output=listprocedure;
ODE1:=diff(y(x),x$2)=-y(x)^2; BCs:=y(a)=alpha,y(b)=beta;
BVP1:={ODE1,BCs}; Sol:=rhs(dsolve(BVP1,y(x),numeric,Op)[2]);
```

We find a numerical solution by applying the finite difference method (`F1`):

```
FNewton:=W->W-convert(J(seq(W[i],i=1..N)),Matrix)^(-1).<
 seq(F[k](seq(W[i],i=1..N)),k=1..N)>;  Y[0]:=<seq(0,i=1..N)>:
Z||0:=alpha; Z||(N+1):=beta;
for i from 1 to N do
 X||i:=a+i*h; Eq||i:=(Z||(i+1)-2*Z||i+Z||(i-1))/(h^2)
 +f(X||i,Z||i,(Z||(i+1)-Z||(i-1))/(2*h)); od:
SeqEq:=seq(Eq||i,i=1..N): SeqVar:=seq(Z||i,i=1..N):
for i from 1 to N do F[i]:=unapply(Eq||i,[SeqVar]): od:
J:=JACOBIAN([seq(F[i],i=1..N)],result_type=array):
for i from 1 to Nmax do
 Y[i]:=FNewton(Y[i-1]);
 if max(seq(abs(F[m](seq(Y[i][k],k=1..N))),m=1..N))>=epsilon
  then print(i,seq(Y[i][k],k=1..N)):
 else Iend:=i: lprint(`the results is`); print(Iend);
 for k from 1 to N do
  X:=k->a+k*h; print(X(k),Y[i][k],Sol(X(k)),
     evalf(abs(Y[i][k]-Sol(X(k))))): od: break: fi:
od:
F1:=[seq([X(k),Y[Iend][k]],k=1..N),[X(0),alpha],[X(N+1),beta]]:
```

We compare the results and plot the numerical solutions (`G1` and `G2`) as follows:

```
G1:=plot(Sol(x),x=a..b,color=red):
G2:=plot(F1,style=point,color=blue): display({G1,G2});
```

19.4.6 Eigenvalue Problems: Examples of Numerical Solutions

An eigenvalue problem is a linear boundary value problem with homogeneous boundary conditions where the differential equation depends on a parameter. The homogeneous boundary conditions imply that there exists a trivial solution of the problem. However, there exist nontrivial solutions called *eigenfunctions* (or sometimes *eigenmodes*). The corresponding special values of the parameter are called *eigenvalues* (or sometimes *eigenfrequencies*).

Eigenvalue problems play an important role in the solution of linear PDEs. When exact solutions of difficult eigenvalue problems are unavailable, various approximation methods (e.g., the Rayleigh–Ritz method, the finite element method, the shooting method, the Galerkin method, difference methods, and iteration methods) can be applied for approximating the leading and most significant eigenvalues and eigenfunctions.

In this section, we consider an approximation method, i.e., an iteration method (which is based on applying the Maple predefined function `dsolve,numeric` for solving IVPs for differential equations) for determining the first few eigenvalues and eigenfunctions. In the following examples, we apply the iteration method to the Sturm–Liouville eigenvalue problem (previously considered in Section 19.2.2) for approximating the lowest eigenvalues.

Example 19.55. *Sturm–Liouville eigenvalue problem. Neumann boundary conditions.*
For the Sturm–Liouville eigenvalue problem

$$y''_{xx} + \lambda y = 0, \qquad y'_x(a) = 0, \quad y'_x(b) = 0, \tag{19.4.6.1}$$

i.e., a homogeneous linear two-point boundary value problem with the parameter λ and with the homogeneous Neumann boundary conditions, where $a = 0$ and $b = \pi$, we obtain a numerical approximation (with the aid of `dsolve,numeric`) to the first eigenvalue as follows:

```
a:=0; b:=Pi; c:=1; IC:=y(a)=c,D(y)(a)=0;
for i from 1 to 10 do
 m[i]:=0.9+i/60; ODE[i]:=diff(y(x),x$2)+m[i]*y(x)=0;
 IVP[i]:={ODE[i],IC}; solN[i]:=dsolve(IVP[i],numeric,range=a..b):
 print(i,evalf(m[i]),solN[i](b));
od:
y1:=rhs(solN[6](a)[3]); y2:=rhs(solN[6](b)[3]); lambda[1]:=m[6];
```

Note that, setting the initial conditions $y(a) = c$, $y'_x(a) = 0$ (`IC`), where c is a constant (guessing value) for the additional initial condition $y(a) = c$, we make several iterations guessing the values of m_i and compute numerical values of y'_x at $x = b$.

The idea of this approach is to find a value of m_i at which $y'_x(b) = 0$; we denote such a value by $m_6 = \lambda_1$,

$$i{=}6,\ \lambda_1{=}1.000000000,\ [x{=}3.14159265358979,\ y(x){=}-1.00000032552861,\ y'_x{=}-4.15500041790299e{-}8].$$

Finally, we determine the subsequent eigenvalues as follows:

```
for i from 1 to 10 do
 lambda[i]:=i^2; ODE[i]:=diff(y(x),x$2)+lambda[i]*y(x)=0;
 IVP[i]:={ODE[i],IC}; solN[i]:=dsolve(IVP[i],numeric,range=a..b):
 print(i,evalf(lambda[i]),solN[i](b));
od:
```

One can improve numerical results by applying linear interpolation, which we consider in the next example.

Example 19.56. *Sturm–Liouville eigenvalue problem. Mixed boundary conditions.*

For the Sturm–Liouville eigenvalue problem

$$y''_{xx} + \lambda^2 y = 0, \qquad y(a) = 0, \quad y'_x(b) = 0, \tag{19.4.6.2}$$

i.e., a homogeneous linear two-point boundary value problem with the parameter λ and with the homogeneous mixed boundary conditions, where $a = 0$ and $b = 1$, we obtain numerical approximations (with the aid of `dsolve,numeric`) to the first eigenvalue as in the previous example:

```
a:=0; b:=1; c:=1; IC:=y(a)=0,D(y)(a)=c;
for i from 1 to 10 do
 m[i]:=1+i/10; ODE[i]:=diff(y(x),x$2)+m[i]^2*y(x)=0;
 IVP[i]:={ODE[i],IC}; solN[i]:=dsolve(IVP[i],numeric,range=a..b):
 print(i,evalf(m[i]),solN[i](b));
od:
```

Note that c is a constant (guessing value) for the additional initial condition $y'_x(a) = c$ (see the previous eigenvalue problem).

The idea of this approach is to find a value of m_i for which $y'_x(b) = 0$.
Then, by carrying out linear interpolation for m_5 and m_6,

$$i{=}5,\ m_5{=}1.500000000,\ [x{=}1.,\ y(x){=}.664996771484993,\ y'_x{=}0.707372139709869e{-}1]$$

$$i{=}6,\ m_6{=}1.000000000,\ [x{=}1.,\ y(x){=}.624733611966352,\ y'_x{=}-0.291995039651808e{-}1]$$

we obtain the first eigenvalue $\lambda_1 = 1.56978798301350$,

$$i{=}1,\ \lambda_1{=}1.56978798301350,\ [x{=}1.,\ y(x){=}.637028489521006,\ y'_x{=}0.100836279208973e{-}2]$$

```
y1:=rhs(solN[5](1)[3]); y2:=rhs(solN[6](1)[3]);
Y:=[y1,y2]; L:=[m[5],m[6]];
lambda[1]:=CurveFitting:-ArrayInterpolation(L,Y,0);
```

Finally, we determine the subsequent eigenvalues as follows:

```
for i from 1 to 10 do
 lambda[i]:=lambda[1]+Pi*(i-1);
 ODE[i]:=diff(y(x),x$2)+lambda[i]^2*y(x)=0; IVP[i]:={ODE[i],IC};
 solN[i]:=dsolve(IVP[i],numeric,range=a..b):
 print(i,evalf(lambda[i]),solN[i](b));
od:
```

Also, changing the values of a and b, we can find a numerical approximation to the first eigenvalue as in the previous example (without linear interpolation):

$i{=}5,\ \lambda_1{=}1.500000000,\ [x{=}3.14159265358979,\ y(x){=}{-}.666667014283704,\ y'_x{=}{-}5.28423594247093e{-}8].$

```
a:=0; b:=Pi; c:=1; IC:=y(0)=0,D(y)(0)=c;
for i from 1 to 10 do
 m[i]:=1+i/10; ODE[i]:=diff(y(x),x$2)+m[i]^2*y(x)=0;
 IVP[i]:={ODE[i],IC}; solN[i]:=dsolve(IVP[i],numeric,range=a..b):
 print(i,evalf(m[i]),solN[i](b));
od:
lambda:=m[5];
```

19.4.7 First-Order Systems of ODEs. Higher-Order ODEs. Numerical Solutions

► **First-order systems of ODEs.**

Consider a system of first-order ordinary differential equations with the initial conditions

$$(y_i)'_x = f_i\bigl(x, y_1, \dots, y_n\bigr), \quad y_i(a) = y_{i0} \quad (i = 1, \dots, n).$$

The unknown functions are $y_1(x), \dots, y_n(x)$, and $x \in [a, b]$.

To obtain numerical solutions, we can apply a predefined function or, alternatively, construct solutions step by step by applying one of the known numerical methods (developed for a single equation) to each equation in the system.

Let us numerically solve some first-order linear and nonlinear systems of ODEs.

Example 19.57. *Linear system. Cauchy problem. Exact, numerical, and graphical solutions.*

For the first-order linear system with the initial conditions

$$u'_x = v, \quad v'_x = x - u - 2v, \qquad u(a) = \alpha, \quad v(a) = \beta, \tag{19.4.7.1}$$

where $a = 0$, $b = 2$, $\alpha = 1$, and $\beta = 1$, we find the exact solution (`SolEx`) for $x \in [a, b]$ as follows:

```
with(plots): C:=[color=red, color=blue]; N:=10: a:=0: b:=2:
alpha:=1; beta:=1; h:=evalf((b-a)/N); X:=x->a+x*h;
F1:=(x,u,v)->v; F2:=(x,u,v)->x-u-2*v;
ODEsys:=diff(u(x),x)=v(x),diff(v(x),x)=x-u(x)-2*v(x);
IC:=u(a)=alpha,v(a)=beta; IVP1:={ODEsys,IC};
SolEx:=sort(dsolve(IVP1,{u(x),v(x)},method=laplace));
uEx:=unapply(rhs(SolEx[1]),x); vEx:=unapply(rhs(SolEx[2]),x);
```

Then we find a numerical solution (uF1, vF1) by applying the explicit fourth-order Runge–Kutta method:

```
RK41:=proc(i,F1,F2,K) local k1,k2,k3,k4,m1,m2,m3,m4; option remember;
k1:=h*F1(X(i-1),RK41(i-1,F1,F2,RK41),RK41(i-1,F1,F2,RK42));
m1:=h*F2(X(i-1),RK41(i-1,F1,F2,RK41),RK41(i-1,F1,F2,RK42));
k2:=h*F1(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k1/2,RK41(i-1,F1,F2,RK42)+m1/2);
m2:=h*F2(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k1/2,RK41(i-1,F1,F2,RK42)+m1/2);
k3:=h*F1(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k2/2,RK41(i-1,F1,F2,RK42)+m2/2);
m3:=h*F2(X(i-1)+h/2,RK41(i-1,F1,F2,RK41)+k2/2,RK41(i-1,F1,F2,RK42)+m2/2);
k4:=h*F1(X(i-1)+h,RK41(i-1,F1,F2,RK41)+k3,RK41(i-1,F1,F2,RK42)+m3);
m4:=h*F2(X(i-1)+h,RK41(i-1,F1,F2,RK41)+k3,RK41(i-1,F1,F2,RK42)+m3);
if K=RK41 then evalf(RK41(i-1,F1,F2,RK41)+1/6*(k1+2*k2+2*k3+k4));
 else evalf(RK41(i-1,F1,F2,RK42)+1/6*(m1+2*m2+2*m3+m4)); fi;
end;
RK41(0,F1,F2,RK41):=1: RK41(0,F1,F2,RK42):=1:
array([seq([X(i),RK41(i,F1,F2,RK41),evalf(uEx(X(i))),
       RK41(i,F1,F2,RK42),evalf(vEx(X(i)))],i=0..N)]);
uF1:=[seq([X(i),RK41(i,F1,F2,RK41)],i=0..N)];
vF1:=[seq([X(i),RK41(i,F1,F2,RK42)],i=0..N)];
```

Finally, we compare the results and plot the exact and numerical solutions (uG1, vG1, uG2, and vG2) as follows:

```
uG1:=plot(uEx(x),x=a..b,C[1]):   vG1:=plot(vEx(x),x=a..b,C[2]):
uG2:=plot(uF1,style=point,C[1]): vG2:=plot(vF1,style=point,C[2]):
display({uG1,uG2,vG1,vG2});
```

Example 19.58. *Nonlinear system. Cauchy problem. Numerical and graphical solutions.* For the first-order nonlinear system with the initial conditions

$$u'_x = uv, \quad v'_x = u + v, \qquad u(a) = \alpha, \quad v(a) = \beta, \tag{19.4.7.2}$$

where $a = 0$, $b = 1$, $\alpha = 1$, and $\beta = 1$, we obtain numerical and graphical solutions as follows:

```
with(plots): setoptions(scaling=unconstrained); A:=Array(1..3);
a:=0; b:=1; alpha:=1; beta:=1; IC:={u(a)=alpha,v(a)=beta};
ODE:={D(u)(x)=u(x)*v(x),D(v)(x)=u(x)+v(x)};
Sol:=dsolve(ODE union IC,numeric,output=operator);
A[1]:=plot(rhs(Sol[2](x)),x=a..b): A[2]:=plot(rhs(Sol[3](x)),x=a..b):
A[3]:=plot({rhs(Sol[2](x)),rhs(Sol[3](x))},x=a..b): display(A);
```

► Higher-order ODEs.

If we consider an ordinary differential equation of order n ($n > 1$) with n initial conditions

$$y_x^{(n)} = f(x, y, y'_x, \dots, y_x^{(n-1)}),$$
$$y(a) = y_0, \quad y'_x(a) = y_1, \quad \dots, \quad y_x^{(n-1)}(a) = y_{n-1},$$

then we can always obtain solutions of this higher-order differential equation by transforming it into an equivalent system of n first-order differential equations and by applying an appropriate numerical method to this system of differential equations.

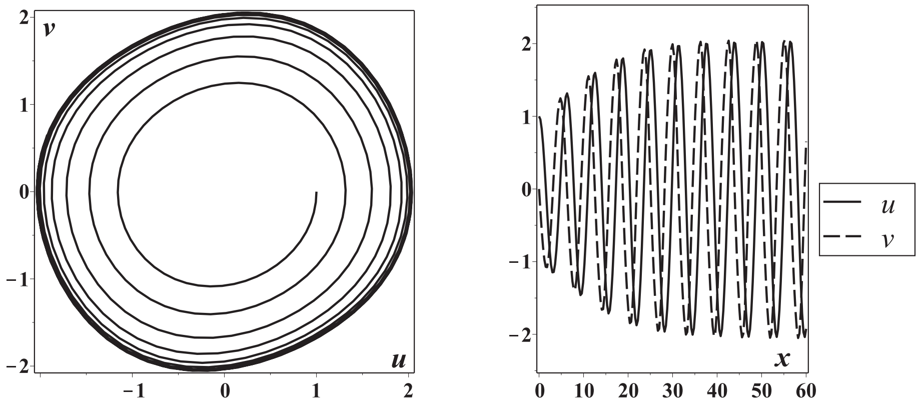


Figure 19.11: Graphical solutions of the van der Pol equation $y''_{xx} + \mu(y^2 - 1)y'_x + y = 0$ (the equivalent system of two first-order ODEs).

Example 19.59. *Van der Pol equation. Cauchy problem. Numerical and graphical solutions.* For the van der Pol equation with the initial conditions

$$y''_{xx} + \mu(y^2 - 1)y'_x + y = 0, \quad y(a) = \alpha, \quad y'_x(a) = \beta, \tag{19.4.7.3}$$

where $x \in [a, b]$, $a = 0$, $b = 60$, $\alpha = 1$, and $\beta = 0$, by applying the function `DEtools,convertsys`, we transform the second-order ODE into an equivalent system of two first-order differential equations (`Sys1`). In Maple's notation, we have

$$\Big[[YP_1 = Y_2, \quad YP_2 = -\mu\,\left(Y_1{}^2 - 1\right) Y_2 - Y_1], \quad [Y_1 = y\,(x)\,, \quad Y_2 = \frac{\mathrm{d}}{\mathrm{d}x} y\,(x)], \quad 0, \quad [1, 0]\Big].$$

The first component of this list is a system of first-order equations, and the second is the definitions of new variables; then the initial point and the initial conditions are presented. Changing the notation, we write

$$\Big[[u'_x = v, \quad v'_x = -\mu(u^2 - 1)v - u], \quad [u = y(x),\ v = y'_x], \quad x_0 = 0, \quad [u(x_0) = 1,\ v(x_0) = 0]\Big].$$

Then, applying a classical numerical method (e.g., Euler's method, by default) to this system of differential equations, we obtain a numerical solution (`SolEuler`) and graphical solutions, a phase portrait of the solution, and a plot of $u(x)$, $v(x)$ (see Fig. 19.11) as follows:

```
with(DEtools): with(plots); a:=0; b:=60; mu:=evalf(1/8);
alpha:=1; beta:=0;
ODE1:=((D@@2)(y))(x)+mu*((y(x))^2-1)*(D(y))(x)+y(x)=0;
IC1:=y(a)=alpha,(D(y))(a)=beta;
Sys1:=convertsys(ODE1,[IC1],y(x),x); Sys1[1];
Sys2:=diff(u(x),x)=v(x),diff(v(x),x)=-mu*((u(x))^2-1)*v(x)-u(x);
IC2:=u(a)=alpha,v(a)=beta; IVP2:={Sys2,IC2};
SolEuler:=dsolve(IVP2,[u(x),v(x)],numeric,method=classical);
SolEuler(0); SolEuler(1);
odeplot(SolEuler,[u(x),v(x)],a..b,numpoints=500);
odeplot(SolEuler,[[x,u(x)],[x,v(x)]],a..b,labels=[`x`,``],
 legend=[`u(x)`,`v(x)`]);
```

⊙ *Literature for Section 18.4*: E. Fehlberg (1970), C. W. Gear (1971), J. R. Ockendon and A. B. Taylor (1971), D. Barton, I. M. Willers and R. V. M. Zahar (1972), J. D. Lambert (1973), G. E. Forsythe, M. A. Malcolm and C. B. Moler (1977), J. H. Verner (1978), L. F. Shampine and C. W. Gear (1979), S. D. Conte and C. de

Boor (1980), J. R. Cash (1983, 1992), A. C. Hindmarsh (1983), W. H. Enright, K. R. Jackson, S. P. Nørsett and P. G. Thomsen (1986), L. Fox and D. F. Mayers (1987), J. R. Cash and A. H. Karp (1990), W. H. Enright (1991), U. Ascher, R. Mattheij and R. Russell (1995), I. K. Shingareva (1995), E. Hairer and G. Wanner (1996), U. Ascher and L. Petzold (1998), L. F. Shampine and R. M. Corless (2000), W. E. Boyce and R. C. DiPrima (2004), D. J. Evans and K. R. Raslan (2005).

Chapter 20

Symbolic and Numerical Solutions of ODEs with Mathematica

20.1 Introduction

20.1.1 Brief Introduction to Mathematica

► **Preliminary remarks.**

Mathematica® is a general-purpose computer algebra system in which symbolic computation can readily be combined with exact, approximate (floating-point), and arbitrary-precision numerical computation. Mathematica provides powerful scientific graphics capabilities [for details, see Bahder (1994), Getz and Helmstedt (2004), Gray (1994), Gray and Glynn (1991), Green, Evans, and Johnson (1994), Ross (1995), Shingareva and Lizárraga-Celaya (2009), Vvedensky (1993), Zimmerman and Olness (1995), etc.].

The first concept of Mathematica and its first versions were developed by Stephen Wolfram in 1979–1988. The Wolfram Research company, which continues to develop Mathematica, was founded in 1987 [Wolfram (2002, 2003)].

In Mathematica, as in Maple, one can find symbolic, numerical, and graphical solutions of ordinary differential equations.

► **Mathematica's conventions and terminology.**

In this chapter, we use the following conventions introduced in Mathematica:

• `C[n]` ($n = 1, 2, \ldots$), for arbitrary constants or arbitrary functions

In general, arbitrary parameters can be specified, e.g., F_1, F_2, …, by applying the option `GeneratedParameters->(Subscript[F,#]&)` of the predefined function `DSolve`.

Also we introduce the following notation for the Mathematica solutions:

• `eqn`, for equations ($n = 1, 2, \ldots$)

• `oden`, for ODEs

• `ivpn`, for initial value problems

- `bvpn`, for boundary value problems
- `soln`, for solutions
- `trn`, for transformations
- `sysn`, for systems
- `icn`, `bcn`, for initial and boundary conditions
- `listn`, `ln`, for lists of expressions
- `gn`, for graphs of solutions
- `ops`, options (various optional arguments) in predefined functions
- `vars`, independent variables
- `funcs`, dependent variables (indeterminate functions)

► Most important features.

The most important features of Mathematica are fast symbolic, numerical, acoustic, and parallel computation; static and dynamic computation, and interactive visualization; it can incorporate new user-defined capabilities; it is available for almost all operating systems; it has a powerful and logical programming language; there is an extensive library of mathematical functions and specialized packages; an interactive mathematical typesetting system is available; and there are numerous free resources (e.g., see the Mathematica Learning Center, `www.wolfram.com/support/learn`; Wolfram Demonstrations Project, `demonstrations.wolfram.com`; Wolfram Library Archive, `library.wolfram.com`; Wolfram Information Center, `library.wolfram.com/infocenter`; Wolfram Community, `community.wolfram.com`, etc.).

► Basic parts.

Mathematica consists of two basic parts: the *kernel*, computational engine, and the *interface*, *front end*. These two parts are separate but communicate with each other via the *MathLink* protocol.

The kernel interprets the user input and performs all computations. The kernel assigns the labels `In[number]` to the input expression and `Out[number]` to the output. These labels can be used for keeping the computation order. In this chapter, we do not include these labels in the examples.

The result of the kernel's work can be viewed with the function `InputForm`. The interface between the user and the kernel is called the *front end* and is used to display the input and the output generated by the kernel. The medium of the front end is the Mathematica *notebook*.

There are significant changes to numerous Mathematica functions incorporated in the new versions of the system. The description of important differences for Ver. < 6 and Ver. ≥ 6 is reported in the literature [e.g., see Shingareva and Lizárraga-Celaya (2009)].*

Mathematica Ver. 10 (launched in 2014) is the first version based on the *complete Wolfram Language* and has more than 700 new functions (e.g., finite element analysis, enhanced PDEs, symbolic delay differential equations, hybrid differential equations, highly

*A complete list of all changes can be found in the Documentation Center and on the Wolfram Web Site `www.wolfram.com`.

automated machine learning, integrated geometric computation, advanced geographic computation, and expanded random process framework); it integrates with the *Wolfram Cloud*, introduces the *Mathematica Online* version, and provides access to the expanded *Wolfram Knowledgebase*.

► **Basic concepts.**

If we type a Mathematica command and press the `RightEnter` key or `Shift+ Enter` (or `Enter` to continue the command on the next line), Mathematica evaluates the command, displays the result, and inserts a horizontal line (for the next input).

Mathematica contains many sources of online help, e.g., the Wolfram Documentation Center, Wolfram Demonstrations Project (for Ver. $\geq$ 6), Mathematica Virtual Book (for Ver. $\geq$ 7), and the `Help` menu; one can mark a function and press F1; to type `?func`, `??func`, `Options[func]`; to use the symbols `(?)` and `(*)`; e.g., `?Inv*`, `?*Plot`, or `?*our*`.

Mathematica notebooks are electronic documents that may contain Mathematica output, text, and graphics (see `?Notebook`). One can work with many notebooks simultaneously. A Mathematica notebook consists of a list of cells. Cells are indicated along the right edge of the notebook by brackets. Cells can contain subcells, and so on. The kernel evaluates a notebook cell by cell. There are *various types of cells:* input cells (for evaluation) and text cells (for comments); Title, Subtitle, Section, Subsection, etc., can be found in the menu `Format` → `Style`.

Previous results (during a session) can be referred to with symbols `%` (the last result), `%%` (the next-to-last result), and so on.

Comments can be included within the characters `(*comments*)`.

Incorrect response: if some functions take an "infinite" computation time, you may have entered or executed the command incorrectly. To terminate a computation, you can use `Evaluation` → `Quit Kernel` → `Local`.

Palettes can be used for building or editing mathematical expressions, texts, and graphics, and allow one to access the most common mathematical symbols by mouse clicks.

In Mathematica, there exist many specialized functions and modules that are not loaded initially. They must be loaded separately from files in the Mathematica directory. These files are of the form `filename.m`. The full name of a package consists of a `context` and a `short` name, and it is written as ``context`short``. To load a package corresponding to a context, type ``<<context` ``. To get a list of the functions in a package, type ``Names["context`*"]``.

Numerical approximations: `N[expr]`, `expr//N` (numerical approximation of `expr` to 6 significant digits); `N[expr,n]`, `NumberForm[expr,n]` (numerical approximation of the expression to n significant digits); `ScientificForm[expr,n]`, scientific notation of numerical approximation of `expr` to n significant digits.

20.1.2 Mathematica Language

Mathematica language is a very powerful programming language based on systems of transformation rules and on functional, procedural, and object-oriented programming techniques [see Maeder (1996)]. This distinguishes it from traditional programming languages. It supports a large collection of data structures, or Mathematica objects (functions, sequences, sets, lists, arrays, tables, matrices, vectors, etc.), and operations on these objects (type-testing, selection, composition, etc.). The library can be extended with custom programs and packages.

In Mathematica Version 10, a new concept is introduced, namely, the *complete Wolfram Language*. It has a vast depth of built-in algorithms and knowledge, all accessible automatically through unified symbolic language. The main idea of the Wolfram Language is to build as much knowledge (about algorithms and the world) as possible into the language.

Symbol refers to a token with a specified name, e.g., an expression, function, object, optional value, result, or argument name. The *name of symbol* is a combination of letters, digits, or certain special characters not beginning with a digit; e.g., `a12new`. Once defined, a symbol retains its value until it is changed or removed.

Expression is a symbol that represents an ordinary Mathematica expression `expr` in readable form. The head of `expr` can be obtained with `Head[expr]`. The structure and various forms of an expression `expr` can be analyzed with the predefined functions: `TreeForm`, `FullForm[expr]`, `InputForm[expr]`.

A Boolean expression is formed with *logical operators* and relation operators.

Basic arithmetic operators and the corresponding functions:
`+ - * / ^`, `Plus`, `Subtract`, `Minus`, `Times`, `Divide`, `Power`.

Logic and relation operators and their equivalent functions: `&&`, `||`, `!`, `=>`, `==`, `!=` `<`, `>`, `<=` `>=`, `And`, `Or`, `Xor`, `Not`, `Implies`, `Equal`, `Unequal`, `Less`, `Greater`, `LessEqual`, `GreaterEqual`.

Mathematica is case sensitive, i.e., distinguishes lowercase and uppercase letters; e.g., `Sin[Pi]` and `sin[Pi]` are different. All Mathematica functions begin with a capital letter. Some functions (e.g., `PlotPoints`) use more than one capital. To avoid conflicts, it is best to begin with a lower-case letter for all user-defined symbols.

The result of each calculation is displayed, but it can be suppressed by using a semicolon (`;`); e.g., `Plot[Sin[x],x,0,2*Pi]; a=9; b=3; c=a*b`.

Patterns: Mathematica language is based on pattern matching. A pattern is an expression that contains an underscore character (`_`). The pattern can stand for any expression. Patterns can be constructed from templates; e.g., `x_`, `x_/;cond`, `pattern?test`, `x_:IniValue`, `x^n_`, `x_^n_`, `f[x_]`, `f_[x_]`.

Basic transformation rules: `->`, `:>`, `=`, `:=`, `^:=`, `^=`.

The rule `lhs->rhs` transforms `lhs` to `rhs`. Mathematica regards the left-hand side as a pattern. The rule `lhs:>rhs` transforms `lhs` to `rhs` evaluating `rhs` only after the rule is actually used. The assignment `lhs=rhs` (or `Set`) specifies that the rule `lhs->rhs` should be used whenever it applies. The assignment `lhs:=rhs` (or `SetDelayed`) specifies that `lhs:>rhs` should be used whenever it applies; i.e., `lhs:=rhs` does not evaluate `rhs` immediately but leaves it unevaluated until the rule is actually called. The rule `lhs^:=rhs` assigns `rhs` to be the delayed value of `lhs` and associates the assignment with symbols that occur at level one in `lhs`. The rule `lhs^=rhs` assigns `rhs` to be the value of `lhs` and associates the assignment with symbols that occur at level one in `lhs`. Transformation rules are useful for making substitutions without making the definitions permanent and are applied to an expression using the operator `/.` (`ReplaceAll`) or `//.` (`ReplaceRepeated`).

The difference between the operators (=) and (==) is as follows: the operator `lhs=rhs` is used to assign `rhs` to `lhs`, and the equality operator `lhs==rhs` indicates equality (not assignment) between `lhs` and `rhs`.

Unassignment of definitions:

`Clear[symb],ClearAll[symb],Remove[symb],symb=.;`

`Clear["Global`*"]; ClearAll["Global`*"]; Remove["`*"];`

(to clear all global symbols defined in a Mathematica session)

`?symb`, `?`*` (to recall a symbol's definition)

`ClearAll["Global`*"];Remove["Global`*"];` is a useful initialization to start working on a problem.

An equation is represented using the binary operator == and has two operands, the left-hand side `lhs` and the right-hand side `rhs`.

Inequalities are represented using relational operators and have two operands, the left-hand side `lhs` and the right-hand side `rhs`.

A string is a sequence of characters having no value other than itself and can be used as labels for graphs, tables, and other displays. The strings are enclosed within double-quotes; e.g., `"abc"`.

Data types: every expression is represented as a tree structure in which each node (and leaf) has a particular data type. A variety of functions can be used for the analysis of any node and branch; e.g., `Length`, `Part`, and a group of functions ending in the letter `Q` (`DigitQ`, `IntegerQ`, etc.).

Types of brackets: parentheses for grouping, `(x+9)*3`; square brackets for function arguments, `Sin[x]`; curly brackets for lists, `{a,b,c}`.

Types of quotes: back-quotes for context mark, format string character, number mark, precision mark, and accuracy mark; double quotes for strings.

Types of numbers: integer, rational, real, complex, and root; e.g., `-5`, `5/6`, `-2.3^-4`, `ScientificForm[-2.3^-4],3-4*I,Root[#^2+#+1&,2]`.

Mathematical constants: symbols for definitions of selected mathematical constants; e.g., `Catalan`, `Degree`, `E`, `EulerGamma`, `I`, `Pi`, `Infinity`, `GoldenRatio`; for example, `{60Degree//N, N[E,30]}`.

Two classes of functions: *pure functions* and functions defined in terms of a variable (*predefined* and *user-defined* functions).

Pure functions are defined without a reference to any specific variable. The arguments are labeled `#1`, `#2`, ..., and an ampersand (`&`) is used at the end of the definition. Most of the mathematical functions are predefined. Mathematica includes all common special functions of mathematical physics.

The names of mathematical functions are complete English words (e.g., `Conjugate`) or, for a few very common functions, the traditional abbreviations (e.g., `Mod`). The names of functions associated with a person's name have the form `PersonSymbol`; for example, the Legendre polynomials $P_n(x)$ is denoted `LegendreP[n,x]`.

User-defined functions are defined using the pattern `x_`; e.g., the function $f(x)$ =`expr` of one variable is defined as `f[x_]:=expr;`

Evaluation of a function or an expression without assigning a value can be performed using the replacement operator `/.`; e.g., `f[a]`, `expr/.x->a`.

Function application: `expr//func` is equivalent to `fun[expr]`.

A module is a local object that consists of several functions which one needs to use repeatedly (see `?Module`). A module can be used to define a function (if the function is too complicated to write by using the notation `f[x_]:=expr`), to create a matrix, a graph, a logical value, etc. *Block* is similar to `Module`; the main difference between them is that `Block` treats the values assigned to symbols as local but the names as global, whereas `Module` treats the names of local variables as local. *With* is similar to `Module`; the important difference between them is that `With` uses local constants that are evaluated only once, but `Module` uses local variables whose values may change many times.

The *Mathematica* language includes the following two types *control structures*: the selection structures (`If`, `Which`, `Switch`) and the repetition structures (`Do`, `While`, `For`).

Mathematica objects: *lists* are the fundamental objects in Mathematica. The other objects (for example, sets, matrices, tables, vectors, arrays, tensors, and objects containing data of mixed type) are represented as lists. A list is an ordered set of objects separated by commas and enclosed in curly braces, {`elements`}, or defined with the function `List[elements]`. *Nested lists* are lists that contain other lists. There are many functions which manipulate lists, and here we review some of the most basic ones. *Sets* are represented as lists. *Vectors* are represented as lists; vectors are simple lists. Vectors can be expressed as single columns with `ColumnForm[list,horiz,vert]`. *Tables, matrices, and tensors* are represented as nested lists. There is no difference between the way they are stored: they can be generated using the functions `MatrixForm[list]`, `TableForm[list]`, or using the nested list functions. Matrices and tables can also be conveniently generated using the *Palettes* or `Insert` menu. *A matrix* is a list of vectors. *A tensor* is a list of matrices with the same dimension.

► **Various types of symbolic notation for derivatives.**

Mathematica differentiates between *functions* and expressions. Therefore, the *differential operator notation* was introduced to denote the derivatives of functions. There exist three forms of representation of derivatives in *Mathematica* [for details, see Shingareva and Lizárraga (2015)]:

1. The *brief form* in terms of *pure functions*:

$$\mathtt{f}', \quad \mathtt{f}'', \quad \dots, \quad \mathtt{f}^{(\mathtt{n})}.$$

2. The *brief form* in terms of *variables of functions*:

$$\mathtt{f}'[\mathtt{x}], \quad \mathtt{f}''[\mathtt{x}], \quad \dots, \quad \mathtt{f}^{(\mathtt{n})}[\mathtt{x}].$$

3. The respective *full forms* of the two previous kinds of derivative notation:

$$\mathtt{Derivative}[1][\mathtt{f}], \quad \mathtt{Derivative}[2][\mathtt{f}], \quad \dots, \quad \mathtt{Derivative}[\mathtt{n}][\mathtt{f}],$$
$$\mathtt{Derivative}[1][\mathtt{f}][\mathtt{x}], \quad \mathtt{Derivative}[2][\mathtt{f}][\mathtt{x}], \quad \dots, \quad \mathtt{Derivative}[\mathtt{n}][\mathtt{f}][\mathtt{x}].$$

According to *Mathematica output*, this notation corresponds to the Lagrange notation.

⊙ *Literature for Section .1*: J. A. van Hulzen and J. Calmet (1983), A. G. Akritas (1989), T. Gray and J. Glynn (1991), J. H. Davenport, Y. Siret, and E. Tournier (1993), D. D. Vvedensky (1993), T. B. Bahder (1994), J. W. Gray (1994), E. Green, B. Evans, and J. Johnson (1994), C. C. Ross (1995), R. L. Zimmerman and F. Olness (1995), R. E. Maeder (1996), M. J. Wester (1999), S. Wolfram (2002, 2003) C. Getz and J. Helmstedt (2004), I. K. Shingareva and C. Lizárraga-Celaya (2009, 2015).

20.2 Analytical Solutions and Their Visualizations

20.2.1 Exact Analytical Solutions in Terms of Predefined Functions

► The predefined function DSolve.

The computer algebra system Mathematica has the unique function `DSolve` that permits obtaining analytical (symbolic) solutions for most classes of ODEs whose solutions are given in standard textbooks and reference books [see Murphy (1960), El'sgol'ts (1961), Hartman (1964), Ince (1956), Matveev (1967), Petrovskii (1970), Simmons (1972), Kamke (1977), Birkhoff and Rota (1978)]. Although the predefined function is an implementation of known methods for solving ODEs [e.g., see. Zwillinger (1997), Polyanin and Zaitsev (2003), Boyce and DiPrima (2004), Polyanin and Manzhirov (2007)], it permits solving ODEs and obtaining solutions automatically as well as developing new methods and procedures for constructing new solutions.

The predefined function `DSolve` is a general analytical differential equation solver. `DSolve` can solve the following types of differential equations: ODEs (ordinary differential equations), PDEs (partial differential equations), and DAEs (differential-algebraic equations).

```
DSolve[ODE,y[x],x]          DSolve[{ODE1,...},{y1[x],...},x]
DSolve[ODE,y[x],x,GeneratedParameters->(Subscript[c,#]&)]
DSolve[ODE,y,x]                  DSolve[{ODE1,...},{y1,...},x]
DSolve[{ODEs,ICs},y[x],x]           DSolve[{ODEs,ICs},y,x]
```

- `DSolve`, finding the general solution `y[x]` or `y` (expressed as a "pure" function) for a single ODE or a system of ODEs
- `DSolve,ICs`, solving an ordinary differential equation or system with given initial or boundary conditions

Remark 20.1. When solving some ODEs, Mathematica generates warning messages. These warning messages can be ignored or suppressed with the `Off` function, or some other alternative methods can be applied.

Example 20.1. *First-order separable and homogeneous ODEs. Analytical solutions.*

The `DSolve` function can use the `Solve` function (for obtaining an analytical solution) that can use inverse functions (when solving transcendental equations). For example, for the separable and homogeneous first-order ODEs

$$y'_x = \frac{y\sin^2 x}{1-y}, \quad (x^2 - xy)y'_x + y^2 = 0,$$

the warning messages

```
Solve::ifun:  Inverse functions are being used by Solve, so some
solutions may not be found; use Reduce for complete solution
information.  >>
```

can be ignored or suppressed as follows:

```
ClearAll["Global`*"]; Remove["Global`*"];
Off[InverseFunction::ifun]; Off[General::stop]; Off[Solve::ifun];
ODE1=D[y[x],x]==y[x]*Sin[x]^2/(1-y[x])
ODE2=(x^2-y[x]*x)*D[y[x],x]+y[x]^2==0
{DSolve[DSolve[ODE1,y[x],x],DSolve[ODE2,y[x],x]}
```

Alternatively, we can obtain the solution of `ODE1` (`sol1`) by applying the function `Reduce` through the function `Solve` with the option `Method->Reduce` and by suppressing another long message via `Off[Solve::useq]`, or we can obtain the solution of `ODE2` (`sol2`) by transforming the original ODE into an ODE in terms of the inverse function `x[y]` as follows:

```
ClearAll["Global`*"]; Remove["Global`*"]; Off[Solve::useq];
ops=Options[Solve]; SetOptions[Solve,Method->Reduce];
ODE1=D[y[x],x]==y[x]*Sin[x]^2/(1-y[x]);sol1=DSolve[ODE1,y[x],x]
SetOptions[Solve,ops];
sol2=DSolve[{(x[y]^2-y[x]*x[y])*D[y[x],x]+y[x]^2==0}/.{y[x]->y,
 y'[x]->1/x'[y]},x[y],y]
```

where the Mathematica result for `sol2` reads:

$$\left\{\left\{x[y]\to\frac{y}{C[1]+\mathrm{Log}[y]}\right\}\right\}$$

► Verification of exact solutions.

Let us assume that we have obtained exact solutions and we wish to verify whether these solutions are exact solutions of given ODEs.

Example 20.2. *First-order nonlinear ODE. Special Riccati equation. Verification of solutions.* For a first-order nonlinear ODE, the *special Riccati equation*

$$y'_x = ay^2 + bx^n,$$

we can verify that the solutions

$$y(x) = -\frac{1}{a}\frac{w'_x}{w},$$

where

$$w(x) = \sqrt{x}\left[C_1 J_v\left(\frac{\sqrt{ab}}{k}x^k\right) + C_2 Y_v\left(\frac{\sqrt{ab}}{k}x^k\right)\right], \qquad k = \frac{1}{2}(n+2), \quad v = \frac{1}{2k},$$

are exact solutions of the special Riccati equation as follows:

```
{k=(n+2)/2, v=1/(2*k), q=1/k*Sqrt[a*b]}
w[X_]:=Sqrt[X]*(c[1]*BesselJ[v,q*X^k]+c[2]*BesselY[v,q*X^k]); w[x]
ode1=D[y[x],x]==a*(y[x])^2+b*x^n
sol1=(y->Function[x,-1/a*D[W,x]/W])/.W->w[x]
test1=(ode1/.sol1)/.{n->1,a->1,b->1}//FullSimplify
```

Here $a, b, n \in \mathbb{R}$ ($ab \neq 0$ and $n \neq -2$) are real parameters, $J_v(x)$ and $Y_v(x)$ are the Bessel functions, and C_1 and C_2 are arbitrary constants.

Example 20.3. *First-order linear ODE. Finding and verification of the general solution.*

For the first-order linear ODE

$$g(x)y'_x = f_1(x)y + f_0(x),$$

we can find and verify that the solution

$$y(x) = Ce^F + e^F \int e^{-F} \frac{f_0(x)}{g(x)}\, dx, \quad \text{where} \quad F = \int \frac{f_1(x)}{g(x)}\, dx,$$

is the general solution of the first-order linear ODE as follows:

```
{ode1=g[x]*D[y[x],x]==f1[x]*y[x]+f0[x]}
{sol1=DSolve[ode1,y[x],x]/.C[1]->C//Flatten, trD=D[sol1,x]}
test1=(ode1/.sol1/.trD)//FullSimplify
```

Here $f_0(x)$, $f_1(x)$, and $g(x)$ are arbitrary functions, and C is an arbitrary constant.
The Mathematica result reads:

$$\left\{ y[x] \to C e^{\int_1^x \frac{\mathrm{f1}[K[1]]}{g[K[1]]} dK[1]} + e^{\int_1^x \frac{\mathrm{f1}[K[1]]}{g[K[1]]} dK[1]} \int_1^x \frac{e^{-\int_1^{K[2]} \frac{\mathrm{f1}[K[1]]}{g[K[1]]} dK[1]}\, \mathrm{f0}[K[2]]}{g[K[2]]} dK[2] \right\}$$

Example 20.4. *Clairaut's equation. Finding and verification of solutions.*

For Clairaut's equation

$$y = xy'_x + f(y'_x),$$

we can find and verify that

$$y(x) = Cx + f(C)$$

is the general solution of this equation as follows:

```
ode1=y[x]==x*D[y[x],x]+f[D[y[x],x]]
{sol1=y[x]->c*x+f[c], trD1=D[sol1,x]}
{sol2=DSolve[ode1,y[x],x]/.C[1]->c//Flatten, trD2=D[sol2,x]}
{test1=ode1/.sol1/.trD1, test2=ode1/.sol2/.trD2}
```

Here $f(x)$ is an arbitrary function and C is an arbitrary constant.

Alternatively, finding and verifying the exact solution can be performed in terms of a pure function as follows:

```
{ode1=y[x]==x*D[y[x],x]+f[D[y[x],x]], sol1=y->Function[x,c*x+f[c]]}
{sol2=DSolve[ode1,y,x]/.C[1]->c//Flatten}
{test1=ode1/.sol1, test2=ode1/.sol2}
```

► Graphical solutions.

Consider the most relevant related functions for plotting solutions of ordinary differential equations.

```
SetOptions[plotFun,ops]          Plot[sol1,{x,x1,x2},ops]
         Plot[Evaluate[{sol1[x],sol2[x],...}/.solSys],ops]
           ParametricPlot[{sol1[x],sol2[x]},{x,x1,x2},ops]
  ContourPlot[Evaluate[sol1,sol2],{x,x1,x2},{y,y1,y2},ops]
               VectorPlot[{vx,vy},{x,x1,x2},{y,y1,y2},ops]
               StreamPlot[{vx,vy},{x,x1,x2},{y,y1,y2},ops]
                        GraphicsGrid[{{g1,g2,...},...,gn}]
```

Here `sol1`, `sol2`, `solSys` are solutions of ODEs and systems of ODEs, and `g1`, ..., `gn` are 2D graphics of solutions constructed by using various predefined functions (e.g., `Plot`).

- `SetOptions`, setting various options for a predefined plot function `plotFun`
- `Plot`, `ParametricPlot`, `ContourPlot`, constructing various types of graphs of solutions of ODEs
- `GraphicsGrid`, aligning various plots (constructed by using various predefined functions `plotFun`)
- `VectorPlot`, generating a vector plot of the vector field $\{v_x, v_y\}$
- `StreamPlot`, generating a stream plot of the vector field $\{v_x, v_y\}$

Example 20.5. *Nonlinear ODE of the first order. The Bernoulli equation. Graphical solutions.* Graphical solutions of the Bernoulli equation

$$y'_x + f(x)y = g(x)y^a,$$

where $a \neq 0, 1$, can be generated for a particular case (e.g., $f(x) = -5/x$, $g(x) = -x^5$, and $a = 2$) as follows:

```
SetOPtions[Plot,PlotStyle->{Thickness[0.01]},PlotRange->All]
{f=-5/x, g=-x^5, a=2, ode1=D[y[x],x]+f*y[x]==g*y[x]^a}
sol1=DSolve[ode1,y[x],x]/.C[1]->c
S1[X_,C_]:=sol1[[1,1,2]]/.{x->X,c->C}; S1[x,c]
Plot[Evaluate[{S1[x,1],S1[x,2],S1[x,-1]}],{x,-1,1},PlotRange->{-1,1},
 PlotStyle->{Blue,Purple,Orange}]
```

In Mathematica, there are many options available for plotting graphs, which can determine the final picture (in more detail, see `Options[Plot]`, `Options[Plot3D]`); e.g., light modeling, legends, axis control, titles, gridlines, colors, etc. The general rule for defining options is

```
Plot[f[x],{x,x1,x2},opName->value,...]
Plot3D[f[x,y],{x,x1,x2},{y,y1,y2},opName->value,...]
```

Here `opName` is the option name.

There are many predefined functions for color graphs. For example, the functions:

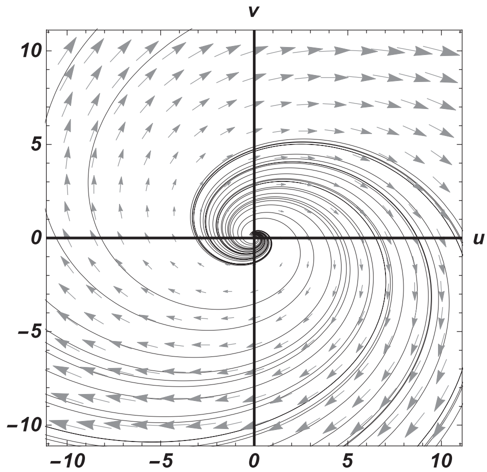


Figure 20.1: Exact solutions together with the direction field associated with system (20.2.1.1).

- `RGBColor`, `Hue`, `GrayLevel`, for defining color models
- `Lighter`, `Darker`, `Blend`, `ColorNegate`, for derived colors
- `ColorData`, for color schemes
- `Red`, `Blue`, `Purple`, `Green`, for named colors
- `ColorSetter`, `ColorSlider`, for interactive color controls
- `Opacity`, `Directive`, `Glow`, for graphics directives
- `Lighting`, `FrameStyle`, `GridLinesStyle`, for graphics options
- `ColorFunction`, `PlotStyle`, `ColorRules`, for plotting options
- `RandomColor`, `ColorReplace`, `ColorConvert`, for color operations

The list of all colors can be obtained by typing `ColorData["Legacy"]["Names"]` (193 predefined named colors), and the RGB formula of a particular color, e.g., `Coral`, by typing `ColorData["Legacy"]["Coral"]`. Additionally, all predefined color schemes can be inserted by using `Palettes->ColorSchemes`, and the RGB formula for color graphs can be computed by using `Insert->Color`.

However, throughout the book, graphical solutions cannot be presented in color for technical reasons: this would result in an essential increase in the book price.

Example 20.6. *First-order constant-coefficients linear system of ODEs. Graphical solutions.* For the first-order linear system of differential equations with constant coefficients

$$u'_x = u + 3v, \quad v'_x = -2u + v, \tag{20.2.1.1}$$

we plot the solutions together with the direction field associated with the system as follows (see Fig. 20.1):

```
sys3={u'[x]==u[x]+3*v[x],v'[x]==-2*u[x]+v[x]};
sol=Table[DSolve[sys3,{u[x],v[x]},x]/.{C[1]->i,C[2]->j},{i,1,5},{j,1,5}];
g1=Table[ParametricPlot[Evaluate[{sol[[i,j,1,1,2]],sol[[i,j,1,2,2]]},
 {x,-3,3}],PlotStyle->Hue[0.7],Frame->True,Background->LightBlue],
 {i,1,5},{j,1,5}];
g2=VectorPlot[{u+3*v,-2*u+v},{u,-10,10},{v,-10,10},
ColorFunction->Function[{x},Hue[x]]];
Show[g1,g2,AspectRatio->1,Frame->True,PlotRange->{{-10,10},{-10,10}}]
```

► Dynamic computation and visualization.

In Mathematica (for Ver. ≥ 6), a new kind of manipulation of Mathematica expressions, *dynamic computation and visualization*, has been introduced allowing to create dynamic and control interfaces of various types. Numerous new functions for producing interactive elements (or various dynamic and control interfaces) have been developed within a Mathematica notebook (for more details, see the Documentation Center, "Introduction to Manipulate," "Introduction to Dynamic," "Dynamic and Control," "Interactive Manipulation," "How to: Build an Interactive Application," etc.).

Let us mention the most important of them:

```
Dynamic[expr]                    DynamicModule[{x=x0,...},expr]
Slider[Dynamic[x]] Slider[x,{x1,x2,xStep}]         Pane[expr]
Manipulate[expr,{x,x1,x2,xStep}]      TabView[{expr1,...}]
Manipulator[expr,{x,x1,x2}]         Animator[x,{x1,x2,dx}]
SlideView[{expr1,expr2,...}]
```

- `Dynamic`, `DynamicModule`, representing an object that displays as the dynamically updated current value of `expr`; the object can be interactively changed or edited
- `Slider`, `Slider,Dynamic`, representing sliders of various configurations
- `Manipulate`, `Manipulator` generating a version of `expr` with controls added to allow interactive manipulations of the value of x etc.

Example 20.7. *The Airy equation. Cauchy problem. Dynamic and control objects.*

Let us create various dynamic and control objects, for example, for the exact solution of the Cauchy problem

$$y''_{xx} - xy = 0, \quad y(0) = \frac{1}{\sqrt{2\pi}}, \quad y'(0) = \frac{q}{\sqrt{2\pi}}, \qquad q \in \mathbb{R},$$

for the Airy equation as follows:

```
r=10; SetOptions[Plot,PlotRange->{{-r,r},{-r,r}},ImageSize->500];
{ODE=y''[x]-x*y[x]==0,ic={y[0]==1/Sqrt[2*Pi],
 y'[0]==q/Sqrt[2*Pi]}, ivp={ODE,ic}//Flatten}
sol=DSolve[ivp,y[x],x]
F[X_,Q_]:=sol[[1,1,2]]/.{x->X,q->Q}; F[x,q]
{Slider[Dynamic[q1]], Dynamic[Plot[F[x1,q1],{x1,-r,r}]]}
TabView[Table[Plot[F[x2,q2],{x2,-r,r}],{q2,0,r}]]
SlideView[Table[Plot[F[x3,q3],{x3,-r,r}],{q3,0,r}]]
Manipulate[N[F[x4,q4]],{q4,3,10,1},{x4,-r,r}]
```

where `ivp` is the abbreviation for initial value problems (see Section 20.1.1).

Example 20.8. *The Lorenz system. Cauchy problem. Dynamic and control objects.*

Let us create dynamic and control objects, for example, for the numerical solution of the Cauchy problem for the nonlinear system of first-order ODEs, e.g., the Lorenz system

$$x'_t = \sigma(y - x), \quad y'_t = \rho x - y - xz, \quad z'_t = xy - \beta z,$$
$$x(0) = 1, \quad y(0) = 15, \quad z(0) = 10, \tag{20.2.1.2}$$

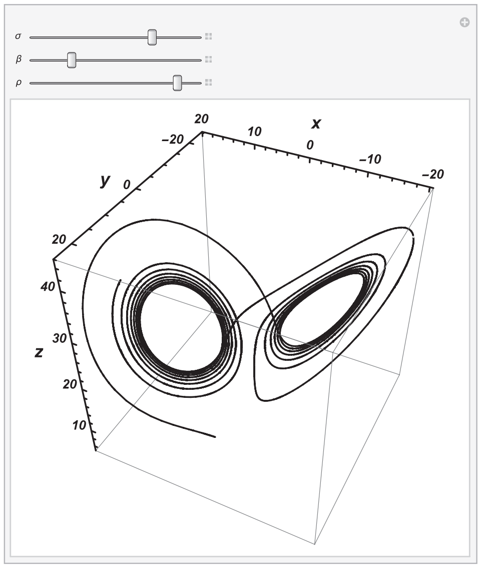


Figure 20.2: Dynamic and control objects for the Lorenz system (20.2.1.2).

where σ, ρ, and β are the system parameters. The Lorenz system is an example of a dissipative chaotic system with a strange attractor. These features can be observed for certain values of the system parameters and initial conditions. This system models an unstable thermally convecting fluid (heated from below) and also arises in other simplified models.

One can study the behavior of the system by varying the system parameters σ, ρ, and β and observe the strange attractor (see Figure 20.2) as follows:

```
Manipulate[
 tN=10; sys1={x'[t]==\[Sigma]*(y[t]-x[t]),
 y'[t]==\[Rho]*x[t]-y[t]-x[t]*z[t], z'[t]==x[t]*y[t]-\[Beta]*z[t]};
 ic={x[0]==1,y[0]==15,z[0]==10}; ivp={sys1,ic}//Flatten;
 solN=NDSolve[ivp,{x,y,z},{t,0,tN},MaxSteps->2000];
 gr={x[t],y[t],z[t]}/.solN;
 ParametricPlot3D[Evaluate[gr],{t,0,tN},PlotRange->All,PlotPoints->500],
{{\[Sigma],15},1,20},{{\[Beta],3},1,10},{{\[Rho],28},10,30}]
```

Example 20.9. *The Legendre equation. Cauchy problem. Dynamic object without controls.*

Let us create a dynamic object without controls and the animation frame, for example, for the exact solution of the boundary value problem

$$(1-x^2)y''_{xx} - 2xy'_x + n(n+1)y = 0, \quad y(-0.9) = -1, \quad y(0.9) = 1 \quad (20.2.1.3)$$

for the Legendre equation, where $-0.9 < x < 0.9$ and $n = 0, 1, 2, \ldots$

```
r=0.9; {ODE=(1-x^2)*y''[x]-2*x*y'[x]+n*(n+1)*y[x]==0,
 ic={y[-r]==-1,y[r]==1}, ivp={ODE,ic}//Flatten}
sol=DSolve[ivp,y[x],x]
F[X_,N_]:=sol[[1,1,2]]/.{x->X,n->N}; F[x1,n1]
Row[{Pane[Animator[Dynamic[n1],{0,7}],{0,20,0.1}],
  Dynamic[Plot[Evaluate[F[x1,n1]],{x1,-r,r},
   PlotRange->{{-1,1},{-20,20}},ImageSize->500]]}]
```

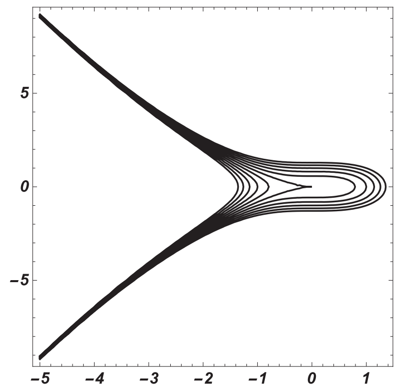


Figure 20.3: Exact implicit solutions for the first-order separable ODE (20.2.1.4).

► Constructing exact explicit and implicit solutions.

In Mathematica, exact solutions of ODEs can be obtained (in one step) with the aid of the predefined function `DSolve` only in explicit or implicit form. The system will display only one of these forms. If an ODE has an implicit form of a solution, the result can contain an unevaluated `Solve` object or `InverseFunction` object.

The design of `DSolve` is *modular* (i.e., the algorithms for various classes of problems work independently) and *internal* (i.e., the solution process performs internally).

```
DSolve[ODE,y[x],x]    DSolve[{ODE1,ODE2,...},{y1[x],...},x]
DSolve[ODE,y,x]          DSolve[{ODE1,ODE2,...},{y1,...},x]
ContourPlot[f[x,y]==c,{x,x1,x2},{y,y1,y2},ops]
ContourPlot[{f1[x,y]==c1,...},{x,x1,x2},{y,y1,y2},ops]
```

Example 20.10. *First-order separable ODE. Exact implicit solutions. Graphical solutions.* For the first-order separable ODE

$$y'_x + \frac{x^2}{y} = 0, \tag{20.2.1.4}$$

we can construct the explicit one-step solution (`sol1`) and implicit step-by-step solution (`sol2`) and plot graphics of the implicit solution (see Fig. 20.3) as follows:

```
{ode1=D[y[x],x]+x^2/y[x]==0, sol1=DSolve[ode1,y[x],x]}
eq1=sol1[[1,1]]/.Rule->Equal
{sol20=Thread[eq1^2,Equal]//Simplify, sol2=sol20/.{6*C[1]->c}}
{g=sol2/.y[x]->Y, gs=Table[g/.c->i,{i,-5, 5}]}
ContourPlot[Evaluate[gs],{x,-5,5},{Y,-10,10},PlotRange->Automatic]
```

The Mathematica results (for `sol1` and `sol2`) read:

$$\left\{\left\{y[x] \to -\sqrt{\frac{2}{3}}\sqrt{-x^3+3C[1]}\right\},\left\{y[x] \to \sqrt{\frac{2}{3}}\sqrt{-x^3+3C[1]}\right\}\right\}, \quad 2x^3+3y[x]^2 == c$$

Here $C[1]$ is an arbitrary constant.

Example 20.11. *Second-order nonlinear ODE. Exact implicit solutions.*
For the second-order nonlinear ODE

$$y''_{xx} = Ax\frac{(y'_x)^3}{y^2} - A\frac{(y'_x)^2}{y},$$

we can construct the implicit (`sol3`) solution

$$(1-A)x - C_1 y + C_2 y^A = 0 \qquad (A \neq 1)$$

as follows:

```
ode1=D[y[x],{x,2}]==A*x/y[x]^2*D[y[x],x]^3-A/y[x]*D[y[x], x]^2
{sol1=DSolve[ode1,y[x],x], sol2=sol1[[1]]}
eq1=Assuming[x>0,FullSimplify[sol2]]
eq2=FullSimplify[Thread[eq1*(A-1),Equal]]
eq3=eq2[[1,1]]==eq2[[2]]-eq2[[1,2]]
{eq4=Thread[Exp[eq3],Equal], eq5=eq4/.eq4[[1]]->c20/.C[1]->C1}
eq6=Thread[eq5/eq5[[2,2]],Equal]//Expand
{eq7=Collect[Thread[eq6/c20,Equal]//Expand,x], sol3=eq7/.c20->1/C2}
```

► Constructing exact solutions of higher-order ODEs.

Exact solutions of higher-order ODEs can be constructed with the aid of the predefined function `DSolve`. The design of `DSolve` has a hierarchical structure, and the solution of complicated problems is reduced to the solution of simpler problems (for which various methods are implemented). For example, higher-order ODEs are solved by reducing their order.

Example 20.12. *Higher-order linear homogeneous ODEs with constant coefficients.*
For the fourth-order linear homogeneous ODE with constant coefficients

$$y''''_x + a_1 y'''_x + a_2 y''_x + a_3 y' + a_4 y = 0,$$

where the constant coefficients are $a_1 = 1$, $a_2 = -1$, $a_3 = 5$, and $a_4 = -2$ and all solutions are of exponential form, we can determine the fundamental set of solutions (`sol2`)

$$\left\{ e^{(\sqrt{2}-1)x}, e^{-(\sqrt{2}+1)x}, e^{x/2}\cos\Big(\frac{\sqrt{7}}{2}x\Big), e^{x/2}\sin\Big(\frac{\sqrt{7}}{2}x\Big) \right\}$$

as follows:

```
{ode1=D[y[x],{x,4}]+a[1]*D[y[x],{x,3}]+a[2]*D[y[x],{x,2}]+a[3]*D[y[x],x]
 +a[4]*y[x]==0, ode2=ode1/.{a[1]->1,a[2]->-1,a[3]->5,a[4]->-2}}
{sol1=DSolve[ode2,y[x],x], sol2=DeleteCases[CoefficientList[
 sol1[[1,1,2]],Table[C[i],{i,1,4}]]//Flatten,0]}
```

Also, we can verify that these functions are solutions of the given ODE (`test1`) and that these functions are linearly independent (`test2`):*

```
trD[y_,s_,x_,n_]:=D[y[x],{x,n}]->D[s,{x,n}];
test1[i_]:=ode2/.y[x]->sol2[[i]]/.Table[trD[y,sol2[[i]],x,j],
 {j,1,4}]//Simplify; Table[test1[i],{i,1,4}]
test2=Wronskian[sol2,x]
```

*By verifying that the Wronskian (the determinant of the Wronskian matrix) has a nonzero value.

The superposition principle can be applied for constructing a general solution, because any linear combination of solutions of a homogeneous linear ODE is again a solution of the ODE. The general solution of the nth-order linear ODE is

$$y(x) = \sum_{i=1}^{n} C_i y_i(x),$$

where $y_i(x)$ $(i = 1, \ldots, n)$ is a fundamental set of solutions and C_i are arbitrary constants. By applying the superposition principle to the fourth-order linear homogeneous ODE with constant coefficients, we obtain the general solution as follows:

```
solGen=y[x]->Sum[C[i]*sol2[[i]],{i,1,4}]
test3=ode2/.solGen/.Table[trD[y,solGen[[2]],x,j],{j,1,4}]//Simplify
```

Example 20.13. *Higher-order linear ODEs with variable coefficients. The Euler equation.*
For a fourth-order linear homogeneous ODE with variable coefficients, the Euler equation

$$a_1 x^4 y''''_x + a_2 x^3 y'''_x + a_3 x^2 y''_x + a_4 x y' + a_5 y = 0,$$

where $a_1 = 1$, $a_2 = 14$, $a_3 = 55$, $a_4 = 65$, and $a_5 = 16$, we can determine the fundamental set of solutions (`sol2`)

$$\left\{ \frac{1}{x^2}, \frac{\ln(x)}{x^2}, \frac{\ln(x)^2}{x^2}, \frac{\ln(x)^3}{x^2} \right\}$$

as follows:

```
ode1=a[1]*x^4*D[y[x],{x,4}]+a[2]*x^3*D[y[x],{x,3}]+a[3]*x^2*D[y[x],
 {x,2}]+a[4]*x*D[y[x],x]+a[5]*y[x]==0
ode2=ode1/.{a[1]->1,a[2]->14,a[3]->55,a[4]->65,a[5]->16}
{sol1=DSolve[ode2,y[x],x], sol2=DeleteCases[CoefficientList[
 sol1[[1,1,2]],Table[C[i],{i,1,4}]]//Flatten,0]}
trD[y_,s_,x_,n_]:=D[y[x],{x,n}]->D[s,{x, n}];
test1[i_]:=ode2/.y[x]->sol2[[i]]/.Table[trD[y,sol2[[i]],x,j],
 {j,1,4}]//Simplify; Table[test1[i],{i,1,4}]
test2=Wronskian[sol2,x]
```

As in the previous example, we verify that these functions are solutions of the given ODE (`test1`) and that these functions are linearly independent (`test2`). Since the Wronskian has the nonzero value $12x^{-14}$ if $x \neq 0$, it follows that these four functions are a fundamental set of solutions for this Euler equation on any interval that does not contain the origin.

Example 20.14. *Constant-coefficient linear nonhomogeneous ODEs. General solution.*
The general solution $y(x)$ of a nonhomogeneous linear ODE can be written as the sum of a particular solution $y_\text{p}(x)$ of the nonhomogeneous equation and the *general solution* of the corresponding homogeneous equation. The general solution of the homogeneous equation is a linear combination of the solutions in a fundamental set of solutions. The general solution of the nth-order nonhomogeneous linear ODE has the form

$$y(x) = y_\text{p}(x) + \sum_{i=1}^{n} C_i y_i(x), \tag{20.2.1.5}$$

where $y_i(x)$ $(i = 1, \ldots, n)$ is a fundamental set of solutions and C_i are arbitrary constants.
Consider the fourth-order linear nonhomogeneous ODE with constant coefficients

$$y''''_x + a_1 y'''_x + a_2 y''_x + a_3 y' + a_4 y = \sin(x),$$

where the constant coefficients are $a_1 = 1$, $a_2 = -1$, $a_3 = 5$, and $a_4 = -2$. First, we determine a fundamental set of solutions (`fundSet1`) of the corresponding homogeneous ODE and form the general solution of the homogeneous ODE (`solGenHom`). Then we obtain a particular solution of the nonhomogeneous equation (`solPartNonHom`) and form the general solution of the nonhomogeneous ODE (`solGenNonHom`) according to Eq. (20.2.1.5),

$$y(x) = C_1 e^{x/2} \sin\Big(\frac{\sqrt{7}}{2}x\Big) + C_2 e^{x/2} \cos\Big(\frac{\sqrt{7}}{2}x\Big) + C_3 e^{(\sqrt{2}-1)x} + C_4 e^{-(\sqrt{2}+1)x} - \frac{1}{4}\cos(x),$$

as follows:

```
ode1=D[y[x],{x,4}]+a[1]*D[y[x],{x,3}]+a[2]*D[y[x],{x,2}]+a[3]*D[y[x],
 x]+a[4]*y[x]==Sin[x]
ode2=ode1/.{a[1]->1,a[2]->-1,a[3]->5,a[4]->-2}
{sol1=DSolve[ode2[[1]]==0,y[x],x], fundSet1=DeleteCases[CoefficientList[
 sol1[[1,1,2]],Table[C[i],{i,1,4}]]//Flatten,0]}
{solGenHom=DSolve[ode2[[1]]==0,y[x],x], solPartNonHom=y[x]->-Cos[x]/4}
solGenNonHom=y[x]->solGenHom[[1,1,2]]+solPartNonHom[[2]]
```

Then we verify that this function is a solution of the given ODE (`test1`) and compare the solution `solGenNonHom` (as a result of our construction procedure) with the solution `sol1` (5-element list) and the general solution `sol2` (as the result from `DSolve`). It should be noted that these solutions are the same:

```
trD[y_,s_,x_,n_]:=D[y[x],{x,n}]->D[s,{x, n}];
test1=ode2/.solGenNonHom/.Table[trD[y,solGenNonHom[[2]],x,j],
 {j,1,4}]//Simplify
sol2=DSolve[ode2,y[x],x]//FullSimplify
sol1=DeleteCases[CoefficientList[sol2[[1,1,2]],Table[C[i],
 {i,1,4}]]//Flatten,0]
```

20.2.2 Analytical Solutions of Mathematical Problems

► Initial value problems (Cauchy problems).

In many applications it is required to solve an initial value problem or a Cauchy problem, i.e., a problem consisting of the differential equation supplemented by one or more initial conditions (which must be satisfied by the solutions). The number of the conditions equals the order of the equation. Therefore, we have to determine a particular solution that satisfies the given initial conditions.

Consider some initial value problems that model various processes and phenomena.

Example 20.15. *Malthus model. Cauchy problem. Analytical and graphical solutions.*

A basic model for population growth, the Malthus model, consists of a first-order linear ODE and an initial condition,

$$y'_t = ky, \quad y(0) = y_0 \quad (k > 0).$$

Here k ($k > 0$) is a constant representing the rate of growth (the difference between the birth rate and the death rate). The increase in the population is proportional to the total number of people.

We can obtain the particular solution

$$y(t) = y_0 e^{kt}$$

of this mathematical problem, which predicts exponential growth of the population, as follows:

```
trD[y_,s_,x_,n_]:=D[y[x],{x,n}]->D[s,{x,n}];
{ode1=D[y[t],t]==k*y[t], ic1=y[0]==y0}
sol1=DSolve[{ode1,ic1},y[t],t]
test1=ode1/.sol1/.trD[y,sol1[[1,1,2]],t,1]
```

Example 20.16. *Linear ODE. Cauchy problem. Analytical, numerical, and graphical solutions.*

Consider a second-order linear nonhomogeneous ODE with variable coefficients and with the initial conditions

$$y''_{xx} + xy'_x + y = \cos(x), \qquad y(0) = 0, \quad y'_x(0) = 0. \tag{20.2.2.1}$$

Analytical and numerical solutions can be constructed as follows:

```
trD[y_,s_,x_,n_]:=D[y[x],{x,n}]->D[s,{x,n}];
{ode1=D[y[x],{x,2}]+x*D[y[x],x]+y[x]==Cos[x], ic1={y[0]==0,y'[0]==0}}
sol1=DSolve[{ode1,ic1},y[x],x]
test1=ode1/.sol1/.Table[trD[y,sol1[[1,1,2]],x,j],{j,1,2}]//Simplify
sol2=NDSolve[{ode1,ic1},y[x],{x,-10,10}]; solN=sol2[[1,1,2]];
```

The analytical solution (`sol1`) has the following form in the Mathematica notation:

$$\left\{\left\{y[x] \to -\frac{1}{2}e^{\frac{1}{2}-\frac{x^2}{2}}\sqrt{\frac{\pi}{2}}\left(2\text{Erf}\left[\frac{1}{\sqrt{2}}\right] - i\text{Erfi}\left[\frac{-i+x}{\sqrt{2}}\right] + i\text{Erfi}\left[\frac{i+x}{\sqrt{2}}\right]\right)\right\}\right\}$$

where i is the imaginary unit, `Erf` is the error function erf(z) (special function), and `Erfi` is the imaginary error function erf$(iz)/i$.

To obtain real graphical solutions, we make the following additional manipulations with the analytical solution obtained (see the Mathematica script below, the variable `s[k]`):

1. We evaluate $y(x)$ on a set of points of the interval $[-10, 10]$ and approximate the resulting complex numbers using floating-point arithmetic (with the predefined function `N`); e.g., the result at the point $x = -10$ reads $-0.0458265 + 0.\,i$.

2. We remove the zero imaginary part of the complex floating point numbers (with the predefined function `Chop`); e.g., the result at the point $x = -10$ reads: -0.0458265.

Finally, we compare the analytical and numerical solutions as follows:

```
k=0; Do[{k=k+1; X[k]=m; s[k]=N[sol1[[1,1,2]]/.x->m]},{m,-10,10,0.1}]; n=k
seq1=Table[{X[m],(s[m]//Chop)},{m,1,n}]
g1=ListLinePlot[seq1,PlotStyle->{Blue,Thickness[0.01]}];
g2=Plot[solN,{x,-10,10},PlotStyle->{Red,Dashed,Thickness[0.007]}];
Show[{g1,g2}]
```

Example 20.17. *First-order linear ODE. Cauchy problem. Analytical and graphical solutions.*

For the first-order linear ODE with the initial condition

$$y'_x - 2y = 3x, \qquad y(0) = n, \tag{20.2.2.2}$$

we can determine the particular analytical solution (`sol1`)

$$y(x) = -\frac{3}{2}x - \frac{3}{4} + e^{2x}\left(n + \frac{3}{4}\right)$$

and construct the direction field as follows:

```
nN=7; SetOptions[Plot,GridLines->{Automatic,Automatic}];
ivp1={y'[x]-2*y[x]==3*x,y[0]==n}
sol1=DSolve[ivp1,y[x],x]//Expand
```

```
sols=Table[sol1/.n->-3+(i-1),{i,1,nN}]//Flatten
sList=Table[sols[[i,2]],{i,1,nN}]
Do[g[i]=Plot[sList[[i]],{x,0,2.5},PlotStyle->Blue],{i,1,nN}];
gr=Table[g[i],{i,1,nN}];
vField=VectorPlot[{1,3*x+2*y},{x,0,2.5},{y,-600,600},
 VectorScale->0.00015,VectorColorFunction->"Rainbow"];
Show[{gr, vField},PlotRange->All,Frame->True,AspectRatio->1]
```

► Boundary value problems.

Consider two-point boundary value problems that consist of a second-order ODE and boundary conditions at the two endpoints of an interval $[a, b]$. Some (simple) boundary value problems can be solved (with the aid of Mathematica) analytically just as initial value problems except that the value of the function and its derivatives are given at two values of x (the independent variable) rather than one.

Example 20.18. *Two-point boundary value problem. Analytical and graphical solutions.*

For the second-order linear homogeneous ODE with constant coefficients with the boundary conditions (the nonhomogeneous Dirichlet conditions)

$$y''_{xx} + a_1 y = 0, \qquad y(a) = g_1, \quad y(b) = g_2, \tag{20.2.2.3}$$

where $a_1 = 2$, $a = 0$, $b = \pi$, $g_1 = 1$, and $g_2 = 0$, we can determine the particular analytical solution (`sol1`)

$$y(x) = \cos(\sqrt{2}x) - \cot(\sqrt{2}\pi)\,\sin(\sqrt{2}x)$$

and construct a graphical solution as follows:

```
bvp1={D[y[x],{x,2}]+a[1]*y[x]==0,y[a]==g[1],y[b]==g[2]}
bvp2=bvp1/.{a[1]->2,a->0,b->Pi,g[1]->1,g[2]->0}
sol1=DSolve[bvp2,y[x],x]
Plot[sol1[[1,1,2]],{x,0,Pi},Frame->True,
 PlotStyle->{Blue,Thickness[0.01]}]
```

where `bvp1` and `bvp2` are the abbreviations for boundary value problems (see Section 20.1.1).

By modifying the boundary conditions (the nonhomogeneous Neumann conditions), we obtain the following:

$$y''_{xx} + a_1 y = 0, \qquad y'_x(a) = g_1, \quad y'_x(b) = g_2, \tag{20.2.2.4}$$

where $a_1 = 2$, $a = 0$, $b = \pi$, $g_1 = 1$, and $g_2 = 0$, and the particular analytical solution (`Sol2`)

$$y(x) = \frac{1}{2}\left(\sqrt{2}\,\cos(\sqrt{2}\,x)\cot(\sqrt{2}\,\pi) + \sqrt{2}\sin(\sqrt{2}\,x)\right)$$

can be constructed as follows:

```
bvp3={D[y[x],{x,2}]+a[1]*y[x]==0,y'[a]==g[1],y'[b]==g[2]}
bvp4=bvp3/.{a[1]->2,a->0,b->Pi,g[1]->1,g[2]->0}
sol2=DSolve[bvp4,y[x],x]
Plot[sol2[[1,1,2]],{x,0,Pi},Frame->True,PlotStyle->{Blue,Thickness[0.01]}]
```

For solving more complicated boundary value problems, we can follow a numerical approach (see Section 20.3.5).

► Eigenvalue problems.

Consider *eigenvalue problems*, i.e., boundary value problems that include a parameter λ. The parameter values that satisfy the differential equation are called *eigenvalues* of the problem, and for each eigenvalue, the solution $y(x)$ ($y(x) \neq 0$) that satisfies the problem is called the corresponding *eigenfunction*. We will find eigenvalues and eigenfunctions for some eigenvalue problems.

A sufficiently general form of linear eigenvalue problems reads:

$$a_2(x)y''_{xx} + a_1(x)y'_x + [a_0(x) + \lambda]y = 0, \qquad a < x < b,$$

and boundary conditions must be posed at the endpoints $x = a$ and $x = b$ (see the previous paragraph).

Example 20.19. *Eigenvalue problem. Dirichlet boundary conditions. Analytical solution.*

Consider a Sturm–Liouville eigenvalue problem consisting of a second-order linear homogeneous ODE with constant coefficients and a parameter λ with the homogeneous Dirichlet boundary conditions,

$$y''_{xx} + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0. \tag{20.2.2.5}$$

If we apply the predefined function `DSolve`,

```
{ode1=D[y[x],{x,2}]+lambda*y[x]==0,
 DSolve[{ode1,y[0]==0,y[Pi]==0},y[x],x]}
```

then we obtain the trivial solution $y(x) = 0$ and cannot solve the eigenvalue problem.

However, we can solve such problems step by step with the aid of Mathematica as follows.

1° We find the characteristic equation (`eqChar`) $m^2 + \lambda = 0$ for the given ODE; the characteristic roots are $m = \pm i\sqrt{\lambda}$ (`rootsChar`):

```
trD[y_,s_,x_,n_]:=D[y[x],{x,n}]->D[s,{x,n}]; expSol=Exp[m*x]
eqChar0=ode1/.y[x]->expSol/.Table[trD[y,expSol,x,j],{j,1,2}]//Simplify
{eqChar=Thread[eqChar0/expSol,Equal], rootsChar=Solve[eqChar,m]}
```

2° There are two cases ($\lambda=0$ and $\lambda\neq 0$). Consider the first case, $\lambda=0$. The differential equation is $y''_{xx} = 0$ (`eq1`), and the solution (`sol1`) of this equation with the first boundary condition is $y(x) = C[2]x$. By applying the second boundary condition to this solution, we obtain the equation $C[2]\pi = 0$ (`eq2`), so $C[2] = 0$ and we arrive at the trivial solution $y(x) = 0$. Thus, $\lambda = 0$ is not an eigenvalue:

```
{eq1=ode1/.lambda->0, sol1=DSolve[{eq1,y[0]==0},y[x],x]}
eq2=(sol1[[1,1,2]]/.x->Pi)==0
```

3° Consider the case of $\lambda \neq 0$ and apply the first boundary condition. The resulting solution (`sol3`) is $y(x) = c\sin(\sqrt{\lambda}x)$. By applying the second boundary condition, we obtain the transcendental equation $c\sin(\sqrt{\lambda}\pi) = 0$ (`sol4`). By solving this equation (using `Reduce`), we determine the eigenvalues $\lambda_n = n^2$ ($n = 1, 2, \ldots$) and the eigenfunctions $y_n(x) = c\sin(nx)$:

```
{sol20=DSolve[{ode1,y[0]==0},y[x],x], sol2=sol20[[1,1,2]]}
{param=Select[sol2,FreeQ[#,lambda]&], sol3=sol2/.param->c}
sol4=(sol3/.x->Pi)==0,
sol50=Reduce[sol4 && lambda>0 && c!=0, lambda, Reals]
{sol51=Thread[sol50[[2,1,3]]/4,Equal], sol5=sol51[[2]]}
{var=Variables[sol5], eVals=sol5/.var[[1]]->n}
eFun=Assuming[n>=1,Simplify[sol3/.lambda->eVals]]
```

20.2.3 Analytical Solutions of Systems of ODEs

The predefined function DSolve can be used for finding analytical solutions of a given ODE system (linear or nonlinear; linear with constant or variable coefficients; homogeneous or nonhomogeneous).

```
DSolve[{odeSys},{y1[x],...},x] DSolve[{odeSys},{y1,...},x]
DSolve[{odeSys,ICs},{y1[x],...},x]
DSolve[{odeSys,ICs},{y1,...},x]
LaplaceTransform[odeSys,x,p]/.{ICs}
ParametricPlot[{xSol,ySol},{x,x1,x2},ops]
```

- DSolve, finding the general solution y1[x], y2[x], ... or y1, y2, ... (expressed as a "pure" function) for a system of ODEs
- DSolve, ICs, solving a system of ODEs with given initial or boundary conditions

► **Linear systems of ODEs.**

For first-order linear systems of ODEs, one can find the general solution and the particular solution for any initial condition (with the aid of the predefined function DSolve). For higher-order linear ODEs or systems of ODEs, one can convert them to a system of first-order ODEs and then solve them.

Example 20.20. *First-order linear system of two ODEs. Analytical solution.*

Consider the general first-order two-dimensional linear system of ODEs with constant coefficients

$$u'_x = a_0 + a_1 u + a_2 v, \quad v'_x = b_0 + b_1 u + b_2 v,$$

where $u(x)$ and $v(x)$ are unknown functions and the coefficients are $a_0 = 1$, $a_1 = 1$, $a_2 = -1$, $b_0 = 1$, $b_1 = 1$, and $b_2 = 1$.

By applying the predefined function DSolve, we find the general solution

$$u(x) = -1 + \mathrm{e}^x \left(C_1 \cos(x) + C_2 \sin(x)\right),$$
$$v(x) = -\mathrm{e}^x \left(C_2 \cos(x) - C_1 \sin(x)\right)$$

of this linear system and verify it as follows:

```
ode1=D[u[x],x]==a[0]+a[1]*u[x]+a[2]*v[x]
ode2=D[v[x],x]==b[0]+b[1]*u[x]+b[2]*v[x]
coeffs={a[0]->1,a[1]->1,a[2]->-1,b[0]->1,b[1]->1,b[2]->1}
{sys1={ode1,ode2}, sys2=sys1/.coeffs}
{solGen1=DSolve[sys2,{u,v},x], test1=sys2/.solGen1//Simplify}
Map[FullSimplify,
 {uX=u[x]->solGen1[[1,1,2,2]],vX=v[x]->solGen1[[1,2,2,2]]}]
```

Example 20.21. *First-order linear system of two ODEs. Cauchy problem. Analytical solution.*

Consider the following first-order two-dimensional linear system of ODEs with initial conditions:

$$u'_x = a_0 + a_1 u + a_2 v, \quad v'_x = b_0 + b_1 u + b_2 v, \qquad u(x_0) = u_0, \quad v(x_0) = v_0,$$

where $u(x)$ and $v(x)$ are unknown functions and the coefficients are $a_0 = -1$, $a_1 = 1$, $a_2 = -1$, $b_0 = 1$, $b_1 = -1$, and $b_2 = 1$. For a first-order two-dimensional system in $u(x)$ and $v(x)$, each initial condition can be specified in the form `ic=` $\{u(x_0) = u_0,\ v(x_0) = v_0\}$ (e.g., $u(0) = 0$, $v(0) = 1$). One solution curve is generated for each initial condition. The solution of the initial value problem (`ivp1`) can be found as follows:

```
ode1=D[u[x],x]==a[0]+a[1]*u[x]+a[2]*v[x]
ode2=D[v[x],x]==b[0]+b[1]*u[x]+b[2]*v[x]
coeffs={a[0]->-1,a[1]->1,a[2]->-1,b[0]->1,b[1]->-1,b[2]->1}
{sys1={ode1,ode2}, sys2=sys1/.coeffs}
{solGen1=DSolve[sys2,{u,v},x], test1=sys2/.solGen1//Simplify}
Map[FullSimplify,
 {uX=u[x]->solGen1[[1,1,2,2]],vX=v[x]->solGen1[[1,2,2,2]]}]
{ic={u[0]==0,v[0]==1}, ivp1=Union[sys2,ic]}
{solPart1=DSolve[ivp1,{u,v},x], test2=ivp1/.solPart1//Simplify}
```

Alternatively, the solution of this initial value problem can be found step by step as follows:

```
ic1=ic/.Equal->Rule
eq1={u[x]==uX[[2]],v[x]==vX[[2]]}/.x->0/.ic1//Simplify
{eq2=Solve[eq1,{C[1],C[2]}], solPart2=Map[Simplify,{uX,vX}/.eq2]}
test21=Map[FullSimplify,sys2/.solPart2/.D[solPart2,x]]
test22=(Flatten[solPart2/.x->0])===ic1
{uXP==solPart2[[1,1,2]],vXP==solPart2[[1,2,2]]}
```

We substitute the initial condition (`ic`) into the general solution (`solGen1`) and obtain equations (`eq1`) for the unknowns `C[1]` and `C[2]`, which can be solved for these constants of integration (`eq2`). The particular solution (`solPart2`) of this initial value problem reads:

$$u\,(x) = 1 - \mathrm{e}^{2\,x}, \quad v\,(x) = \mathrm{e}^{2\,x}.$$

This particular solution `solPart2` is equal to the solution `solPart1`.

► Nonlinear systems of ODEs.

For more complicated first-order or higher-order nonlinear systems of ODEs, a straightforward application of the predefined function `dsolve` may give no solutions (general or particular). Therefore, one can introduce some transformations, make some manipulations with the original Cauchy problem, reduce it to a modified Cauchy problem, and finally obtain analytical solutions in terms of new variables and the original variables. Let us show this in the following example.

Example 20.22. *Second-order nonlinear system of ODEs. Analytical and graphical solutions.*

Consider the following second-order two-dimensional nonlinear system of ODEs subject to initial conditions:

$$u''_{xx} = -a u'_x \sqrt{(u'_x)^2 + (v'_x)^2}, \quad v''_{xx} = -a v'_x \sqrt{(u'_x)^2 + (v'_x)^2},$$
$$u(0) = 0, \quad v(0) = 0, \quad u'_x(0) = U_0 \sin\phi, \quad v'_x(0) = U_0 \cos\phi,$$

where $u(x)$ and $v(x)$ are unknown functions and the parameter values are $a = 5$, $U_0 = 10$, and $\phi = \pi/10$. The solution of the initial value problem (`ivp1`) can be found step by step as follows:

```
sys1={D[u[x],{x,2}]==-a*D[u[x],x]*Sqrt[D[u[x],x]^2+D[v[x],x]^2],
      D[v[x],{x,2}]==-a*D[v[x],x]*Sqrt[D[u[x],x]^2+D[v[x],x]^2]}
ic={u[0]==0,v[0]==0,u'[0]==U0*Sin[phi],v'[0]==U0*Cos[phi]}
{ivp1=Union[sys1,ic], DSolve[ivp1,{u[x],v[x]},x]}
tr1=U[x]->Sqrt[D[u[x],x]^2+D[v[x],x]^2]
{eq1=D[tr1,x], eq2=eq1/.(sys1/.Equal->Rule)//Simplify}
ode1=eq2/.tr1[[2]]^2->U[x]^2
sol1=DSolve[{(ode1/.Rule->Equal),U[0]==U0},U[x],x]
{sys2=sys1/.tr1[[2]]->sol1[[1,1,2]], ivp2=Union[sys2,ic]}
sol2=DSolve[ivp2,{u,v},x]
test1=Assuming[{U0>0,x>0,a>0},ivp1/.sol2//FullSimplify]
solG={sol2[[1,1,2,2]],sol2[[1,2,2,2]]}
Plot[Evaluate[solG/.{U0->10,a->5,phi->Pi/10}],{x,0,20}]
```

In this problem, a straightforward application of the predefined function `DSolve` does not give a solution. Therefore, we introduce the transformation (`tr1`), $U(x) = \sqrt{(u'_x)^2 + (v'_x)^2}$. Then we find the derivative (`eq1`), $U'_x = \dfrac{2u'_x u''_{xx} + 2v'_x v''_{xx}}{2\sqrt{(u'_x)^2 + (v'_x)^2}}$. By substituting the second derivatives u''_{xx} and v''_{xx} from the original system into the expression for U'_x, we obtain the differential equation (`ode1`), $U'_x = -aU^2$. By solving this simple differential equation with the initial condition $U(0) = U_0$, we obtain the solution (`sol1`) $U(x) = \dfrac{U_0}{1 + aU_0 x}$. By substituting this expression for $U(x)$, which is equal to $\sqrt{(u'_x)^2 + (v'_x)^2}$ (according to `tr1`), into the original system (`sys1`) and by taking into account the initial conditions, we obtain the modified Cauchy problem (`ivp2`)

$$
\begin{aligned}
&u''_{xx} = -\frac{aU_0\, u'_x}{aU_0 x + 1}, \quad v''_{xx} = -\frac{aU_0\, v'_x}{aU_0 x + 1}, \\
&u(0) = 0, \quad v(0) = 0, \quad u'_x(0) = U_0 \sin\phi, \quad v'_x(0) = U_0 \cos\phi.
\end{aligned}
$$

By solving this Cauchy problem, we obtain the analytical particular solution (`sol2`)

$$
u(x) = \frac{1}{a} \sin\phi \ln(axU_0 + 1), \quad v(x) = \frac{1}{a} \cos\phi \ln(axU_0 + 1)
$$

and then verify that it is an exact particular solution (`sol2`) of the original Cauchy problem (`ivp1`) and plot the graphs of $u(x)$, $v(x)$.

20.2.4 Integral Transform Methods for ODEs

In Mathematica, integral transforms (Laplace, Fourier, and Z-transforms) can be studied with the aid of several predefined functions.

For example, the Laplace integral transforms are defined by the two predefined functions `LaplaceTransform`, `InverseLaplaceTransform`.

Integral transform methods can be applied to many initial value problems.

► Linear ODEs and systems of ODEs with constant coefficients.

Integral transform methods can be applied for solving the nth-order linear ODE with constant coefficients and with initial conditions

$$
\begin{aligned}
&a_n y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \ldots + a_1 y'_x + a_0 y = f(x), \\
&y(0) = \alpha_0, \quad y'_x(0) = \alpha_1, \quad \ldots, \quad y_x^{(n-1)}(0) = \alpha_{n-1}
\end{aligned}
$$

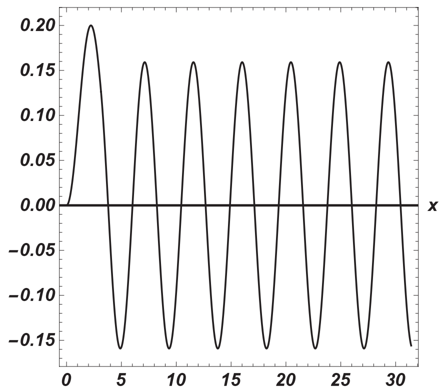


Figure 20.4: Analytical solution of the nonhomogeneous Cauchy problem for the second-order linear ODE (20.2.4.1).

and to systems of linear ODEs with constant coefficients and with initial conditions. Consider some examples.

Example 20.23. *First-order linear ODE. Initial value problem. Laplace transform.*
For the first-order linear ODE with the initial condition

$$y'_x + ay = e^{-ax}, \qquad y(0) = 1,$$

the analytical particular solution

$$y(x) = (x+1)e^{-ax}$$

of the initial value problem can be obtained and verified as follows:

```
ode={y'[x]+a*y[x]==Exp[-a*x]}
eq1=LaplaceTransform[ode,x,s]/.{y[0]->1}
eq2=Solve[eq1,LaplaceTransform[y[x],x,s]]
sol1=Map[InverseLaplaceTransform[#,s,x]&,eq2,{3}]
trD=D[sol1,x]//Flatten
sol2=DSolve[{ode,y[0]==1},y,x]
{test1=ode/.sol2, test2=ode/.sol1/.trD}
```

The graphical solution can be obtained for some value of the parameter a (e.g., $a = 7$) as follows:

```
Plot[Evaluate[y[x]/.sol1/.{a->7}],{x,0,1},
 PlotStyle->{Hue[0.7],Thickness[0.01]}]
```

Example 20.24. *Second-order linear ODE. Cauchy problem. Laplace transform.*
For the second-order linear ODE with the initial conditions

$$ay''_{xx} + by = f(x), \qquad y(0) = 0, \quad y'_x(0) = 0, \tag{20.2.4.1}$$

where $f(x) = H(x) - H(x-\pi)$ is a given function representing a source term,* the analytical particular solution

$$y(x) = \frac{1}{b}\left\{1 - \cos\left[\frac{\sqrt{b}x}{\sqrt{a}}\right] + \left(-1 + \cos\left[\frac{\sqrt{b}(x-\pi)}{\sqrt{a}}\right]\right) H(x-\pi)\right\}$$

and graphical solutions (see Figure 20.4) of the nonhomogeneous Cauchy problem can be obtained and verified as follows:

*Here $H(x)$ is the Heaviside step function.

```
SetOptions[Plot,PlotStyle->{Hue[0.8],Thickness[0.01]}];
f[x_]:=HeavisideTheta[x]-HeavisideTheta[x-Pi]; tr1={a->5,b->10}
ode={a*y''[x]+b*y[x]==f[x]}
eq1=LaplaceTransform[ode,x,s]/.{y[0]->0,y'[0]->0}
eq2=Solve[eq1,LaplaceTransform[y[x],x,s]]
eq3=eq2[[1,1,2]]
sol1=InverseLaplaceTransform[eq3,s,x]
trD=D[sol1,{x,2}]//Flatten//FullSimplify
sol2={y[x]->PiecewiseExpand[sol1/.HeavisideTheta->UnitStep]}
sol3=DSolve[{ode,y[0]==0,y'[0]==0},y[x],x]//Simplify//Flatten
{sol1//FullSimplify,f1=sol1/.tr1}
Plot[f1,{x,0,10*Pi}]
Plot[Evaluate[sol3[[1,2]]/.tr1],{x,0,10*Pi}]
test1=Assuming[x>0,Simplify[ode/.y[x]->sol1/.y''[x]->trD]]
```

Example 20.25. *Constant-coefficient linear system. Cauchy problem. Laplace transform.*
By applying the Laplace transform, we solve the initial value problem

$$u'_x - 2v = x, \quad 4u + v'_x = 0, \quad u(0) = 1, \quad v(0) = 0$$

and verify the analytical solution

$$u(x) = \tfrac{1}{8} + \tfrac{7}{8}\cos(2\sqrt{2}\,x), \qquad v(x) = -\tfrac{1}{2}x - \tfrac{7}{8}\sqrt{2}\sin(2\sqrt{2}\,x)$$

as follows:

```
odeSys={u'[x]-2*v[x]==x,4*u[x]+v'[x]==0}
eq1=LaplaceTransform[odeSys,x,s]
eq2=Solve[eq1,{LaplaceTransform[u[x],x,s],LaplaceTransform[v[x],x,s]}]
sol1=Map[InverseLaplaceTransform[#,s,x]&,eq2,{3}]/.{u[0]->1,v[0]->0}
sol2=DSolve[{odeSys,u[0]==1,v[0]==0},{u[x],v[x]},x]//Simplify
ParametricPlot[Evaluate[{u[x],v[x]}/.sol1],{x,0,Pi},
 PlotStyle->{Hue[0.5],Thickness[0.01]},AspectRatio->1]
```

Example 20.26. *First-order linear systems of ODEs. Initial value problem. Laplace transform.*
Generalizing this procedure, consider the system of first-order linear ODEs

$$(y_i)'_x = a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n + f_i(x), \quad x > 0 \quad (i = 1, \ldots, n)$$

with the initial conditions

$$y_i(0) = y_{i0} \quad (i = 1, \ldots, n),$$

where a_{ij} ($i = 1, \ldots, n$; $j = 1, \ldots, n$) are constants, $f_i(x)$ are given functions, and the unknown functions $y_1(x), \ldots, y_n(x)$ are defined on $x \in [0, \infty]$.

Let $n = 2$. We find the exact solution of the Cauchy problem

$$(y_1)'_x = y_2, \quad (y_2)'_x = -y_1 - 2y_2 + x, \quad x > 0,$$
$$y_1(0) = 1, \quad y_2(0) = 1$$

by applying the integral transform method. Also, we verify and plot this solution on some interval $[a, b]$ as follows:

```
{a=0,b=2, odeSys={D[u[x],x]==v[x],D[v[x],x]==x-u[x]-2*v[x]},
 ICs={u[0]==1,v[0]==1}, IVP1=Flatten[{odeSys,ICs}]}
eq1=LaplaceTransform[odeSys,x,s]
eq2=Solve[eq1,{LaplaceTransform[u[x],x,s],LaplaceTransform[v[x],x,s]}]
sol1=Map[InverseLaplaceTransform[#,s,x]&,eq2,{3}]/.{u[0]->1,v[0]->1}
```

```
sol2=Sort[DSolve[IVP1,{u[x],v[x]},x]]
{sol1,sol2}//Simplify
uExt[x1_]:=sol2[[1,1,2]]/.{x->x1}; vExt[x1_]:=sol2[[1,2,2]]/.{x->x1};
uG1=Plot[uExt[x],{x,a,b},PlotStyle->Red,PlotRange->All];
vG1=Plot[vExt[x],{x,a,b},PlotStyle->Blue,PlotRange->All];
Show[{uG1,vG1}]
```

Remark 20.2. If we consider an nth-order ODE ($n > 1$) with n initial conditions

$$a_n y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \ldots + a_1 y'_x + a_0 y = f(x), \quad x > 0,$$
$$y(0) = y_0, \quad y'_x(0) = y_1, \quad \ldots, \quad y_x^{(n-1)}(0) = y_{n-1},$$

then we can find exact solutions of this higher-order ODE by transforming it into an equivalent system of n first-order equations (with the predefined function `DEtools,convertsys`) and by applying integral transform methods to this system of ODEs.

20.2.5 Constructing Solutions via Transformations

Transformation methods are the most powerful analytic tools for studying differential equations. In general, transformations can be divided into two parts: transformations of the independent variables and dependent variables; transformations of the independent variables, dependent variables, and their derivatives. We will consider various types of transformations, e.g., point and contact transformations, relating ODEs.

Transformation methods permit finding transformations under which an ODE is invariant and new variables (independent and dependent) with respect to which differential equations become simpler, e.g., linear.

In *Mathematica*, transformations of various types can be computed by defining new functions (a predefined function does not exist).

For example, if $f(x)$ is some function, the transformation of the independent variable, $f'_x(x) \to f'_x(g(x))$, i.e., the chain rule, can be computed as follows:

```
D[f[x],x]/.f->(f[g[#]]&)
```

► Point transformations.

Now consider the most important transformations of ODEs, namely, point transformations (transformations of independent and dependent variables). Point transformations can be linear point transformations (translation transformations, scaling transformations, and rotation transformations) and nonlinear transformations of the dependent variables. These transformations and their combinations can be applied to simplify nonlinear ODEs, linearize them, and reduce them to normal, canonical, or invariant form.

Example 20.27. *The Bernoulli equation. Transformation and general integral.*
Consider the first-order nonlinear ODE, the Bernoulli equation

$$g(x)y'_x - f_1(x)y - f_n(x)y^n = 0,$$

where $n \neq 0, 1$. By applying the transformation $X = x$, $Y(X) = [y(x)]^{1-n}$ (`seq2`) of the dependent variable, we obtain the linear equation (`eq4`)

$$(1-n)\, f_1(X)\, Y + (1-n)\, f_n(X) - g(X)\, Y'_X = 0$$

and the general integral with respect to $Y(X)$ (`sol1`). Finally, we find the general integral of the original equation (with respect to $y(x)$) (`sol2`) as follows:

```
Off[Solve::ifun]; $Assumptions={n\[Element]Integers, n!=1, Y[X]!=0}
{tr1={X==x,Y[X]==y[x]^(1-n)}, tr2=Solve[tr1,{x,y[x]}]}
eq1=g[x]*D[y[x],x]-f1[x]*y[x]-fn[x]*y[x]^n==0
eq2=eq1/.{y->((1/y[#])^(1/(n-1))&)}/.x->X/.y->Y
eq3=eq2//PowerExpand
eq4=Thread[eq2*(n-1),Equal]//PowerExpand//FullSimplify
sol1=DSolve[eq4,Y[X],X]
{tr11=tr1/.Equal->Rule, sol2=sol1/.tr11//FullSimplify}
```

Example 20.28. *First-order ODE reducible to a homogeneous ODE. A linear transformation.* Consider the first-order equation

$$y'_x = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right),$$

where a_i, b_i, and c_i ($i = 1, 2$) are real constants. These equations can be reduced to a homogeneous equation and integrated. Consider the case in which

$$D = \det\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1 b_2 - a_2 b_1 \neq 0.$$

By applying the transformations of the independent and dependent variables* (`ode1T`)

$$x = X + x_0, \quad y(x) = Y(X) + y_0, \qquad (20.2.5.1)$$

where x_0 and y_0 are constants and can be uniquely determined (since $D \neq 0$) by solving the linear algebraic system

$$a_1 x_0 + b_1 y_0 + c_1 = 0, \quad a_2 x_0 + b_2 y_0 + c_2 = 0,$$

we obtain the differential equation (`eq2`)

$$Y'_X = f\left(\frac{b_1 Y + a_1 X}{b_2 Y + a_2 X}\right)$$

for $Y = Y(X)$, which can be reduced to the homogeneous equation (`eq3`)

$$Y'_X = f\left(\frac{b_1 Y/X + a_1}{b_2 Y/X + a_2}\right)$$

and integrated:

```
sol0=Solve[{a1*x0+b1*y0+c1==0,a2*x0+b2*y0+c2==0},{x0,y0}]
{tr1={x->X+x0,y[x]->Y[X]+y0},tr11=tr1/.Rule->Equal,
 tr2=Solve[tr11,{X,Y[X]}]}
ode1[X_]:=D[y[X],X]==f[(a1*X+b1*y[X]+c1)/(a2*X+b2*y[X]+c2)];
ode1T[Y_]:=((ode1[x]/.y->Function[{x},y[x-x0]+y0])/.tr1)/.{y->Y};
{ode1[x], eq1=ode1T[Y]//PowerExpand, eq2=eq1/.sol0//Simplify}
ex1=eq2[[1,1,1]]
tr2=ex1->(Numerator[ex1]/X // Expand)/(Denominator[ex1]/X // Expand)
eq3=eq2/.tr2
```

*Equations (20.2.5.1) can be interpreted as a translation of orthogonal coordinate axes to the new origin (x_0, y_0) that is the point of intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ (for the case in which the lines are not parallel).

► Contact transformations.

A contact transformation is a transformation that acts on the space of the independent variable, the dependent variable, and its first derivative. For an ODE of general form with the independent variable x and the dependent variable $y = y(x)$, a contact transformation can be represented in the form

$$x = F(X, Y, Y'_x), \quad y = G(X, Y, Y'_x), \quad y'_x = H(X, Y, Y'_X),$$

where the functions $F(X, Y, Y'_x)$ and $G(X, Y, Y'_x)$ are chosen so that the derivative y'_x does not depend on Y''_{xx}.

Consider some examples of contact transformations reducing complicated nonlinear ODEs to equations of a simpler form.

Example 20.29. *Nonlinear first-order equation. Contact transformation.*
Consider the nonlinear first-order ODE

$$y'_x(y'_x + ax)^n + b\big((y'_x)^2 + 2ay\big)^m + c = 0,$$

where a, b, c, m, and n are arbitrary parameters. By applying the contact transformation (trF)

$$x = \tfrac{1}{a}(X - y'_x), \quad y = \tfrac{1}{2a}\big(Y - (Y'_X)^2\big), \quad y'_x = \tfrac{1}{2}Y'_X,$$

we reduce this nonlinear ODE to the separable ODE (eq2)

$$X^n Y'_X + 2bY^m + 2c = 0$$

and obtain the exact solution as follows:

```
{d1=D[y[x],x], ode1=d1*(d1+a*x)^n+b*(d1^2+2*a*y[x])^m+c==0}
{tr1={x->(X-d1)/a, y[x]->(Y[X]-(d1)^2)/(2*a)},
 tr2=D[y[x],x]->D[Y[X],X]/2,tr3=tr1/.tr2, trF={tr3,tr2}//Flatten}
{eq2=ode1/.trF//FullSimplify, sol1=DSolve[eq2,Y[X],X]}
```

Example 20.30. *Nonlinear first-order equation. Legendre transformation.*
Consider the nonlinear first-order ODE

$$xf(y'_x) + yg(y'_x) + h(y'_x) = 0,$$

where $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions. By applying the Legendre transformation

$$x = Y'_X, \quad y = XY'_X - Y, \quad y'_x = X,$$

we reduce nonlinear ODE to the linear ODE (eq2)

$$\big(Xg(X) + f(X)\big)Y'_X - Yg(X) + h(X) = 0$$

and obtain the exact solution as follows:

```
{d1=D[y[x],x], ode1=x*f[d1]+y[x]*g[d1]+h[d1]==0}
{legTr={x->D[Y[X],X], y[x]->-Y[X]+X*D[Y[X],X]}, legd1={D[y[x],x]->X}}
legendreTr={legTr,legd1}//Flatten
eq2=ode1/.legendreTr//FullSimplify
sol1=DSolve[eq2,Y[X],X]
```

⊙ *Literature for Section .2*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), P. Hartman (1964), E. L. Ince (1956), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and G. C. Rota (1978), D. Zwillinger (1997), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

20.3 Numerical Solutions and Their Visualizations

It is well known that there exist many differential equations for which one cannot find exact solutions in terms of elementary or special functions. For example, *Maple* and *Mathematica* cannot find an analytic solution for the first-order nonlinear ODE

$$y'_x = ax(y+x)^{-n},$$

where $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

```
infolevel[dsolve]:=3; a:=1; n:=-2;
dsolve(diff(y(x),x)=a*x*(y(x)+x)^(-n),y(x));
{a=1, n=2}; DSolve[y'[x]==a*x*(y[x]+x)^(-n),y[x],x]
```

We can study this ODE following other approaches, e.g., the geometric-qualitative approach (performing phase plane analysis to find the qualitative behavior of solutions), the approximate analytical approach (finding approximate analytical solutions), or the numerical approach (finding numerical solutions).

In this section, we follow the numerical approach and consider various numerical approximation methods for initial value problems, boundary value problems, and eigenvalue problems for ordinary differential equations.

20.3.1 Numerical Solutions in Terms of Predefined Functions

The predefined function `NDSolve` is a general numerical differential equation solver. `DSolve` can solve the following types of differential equations: ordinary differential equations, partial differential equations, and differential-algebraic equations.

```
NDSolve[{edo1,edo2,...,edoN},y,{x,x1,x2},ops]
NDSolve[{edo1,edo2,...,edoN},{y1,...,yN},{x,x1,x2},ops]
s=NDSolve[{ODEs,ICs},y,{x,x1,x2},ops] s1=Evaluate[y[x]/.s]
NDSolve[{ODEs,ICs},y[x],{x,x1,x2},ops] Plot[s1,{x,x1,x2}]
NDSolve[{ODEs,ICs},y,{x,x1,x2},Method->m]
NDSolve[{ODEs,ICs},y,{t,t1,t2},Method->{m,Method->subM}]
VectorPlot[{vx,vy},{x,x1,x2},{y,y1,y2},ops]
```

- `NDSolve`, finding numerical solutions of ODE problems
- `NDSolve,Method`, finding numerical solutions of ODE problems using one of the numerical methods embedded in Mathematica
- `VectorFieldPlot`, constructing vector fields

Remark 20.3. For more comprehensive details on numerical methods embedded in Mathematica (for solving ODEs) and graphical representation of solutions, we refer to Sections 20.3.2 and 20.2.1.

Note that the predefined function `NDSolve` represents solutions for the function $y(x)$ (or the functions $y_i(x)$) as `InterpolatingFunction` objects; i.e., the solution is represented as a list of possible solutions for $y(x)$, where the function $y(x)$ is determined

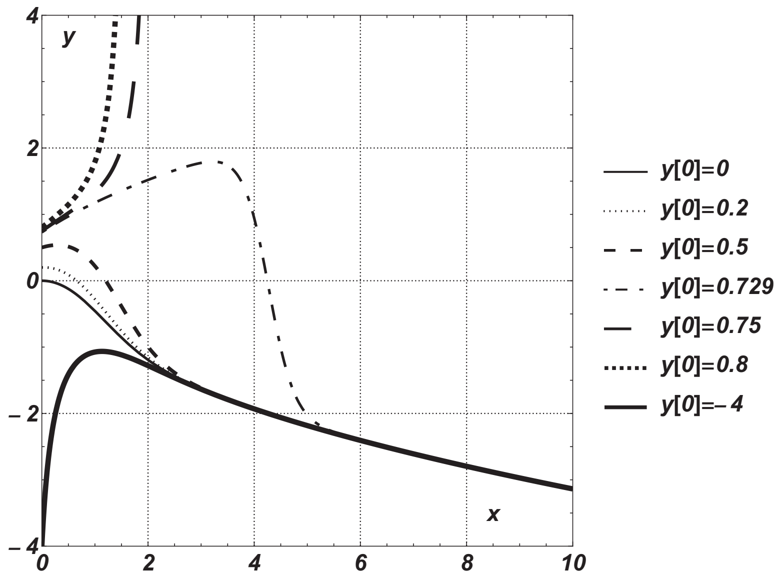


Figure 20.5: Numerical solutions of the Cauchy problem (20.3.1.1).

numerically via `InterpolatingFunction`. `NDSolve` finds numerical solutions iteratively and computes numerical values for $y(x)$ only at specific values of x between x_1 and x_2, and then uses `InterpolatingFunction` to interpolate between these values of x.

Example 20.31. *First-order ODE. Cauchy problem. Numerical and graphical solutions.*
For the Cauchy problem (with several initial conditions)

$$y'_x = py^m + qx^n, \qquad y(0) = y_0 \tag{20.3.1.1}$$

on the interval $[a, b]$ ($a = 0$, $b = 9$) with parameters $p = 1$, $q = -1$, $m = 2$, $n = 1$, and $y_0 = \{0, 0.729, 0.5, 0.2, -4, 0.69, 0.72\}$, we find numerical and graphical solutions (see Fig. 20.5) as follows:

```
SetOptions[Plot,ImageSize->500,AspectRatio->1,Frame->True,
 Axes->False];
{nD=50,nN=7,a=0,b=9,p=1,q=-1,n=1,m=2}
ODE={y'[x]==p*y[x]^m+q*x^n}
IC={y[0]==0,y[0]==729/1000,y[0]==1/2,y[0]==1/5,y[0]==-4,
 y[0]==69/100,y[0]==72/100}
IVP=Union[ODE,IC]
Do[eq[i]=NDSolve[{ODE,IC[[i]]},y,{x,a,b},Method->
 {"StiffnessSwitching",Method->{"ExplicitRungeKutta",Automatic}},
 WorkingPrecision->nD,MaxSteps->500000,AccuracyGoal->15,
 PrecisionGoal->15];  sol[i]=eq[i][[1,1,2]];
 Print[Table[{x,sol[i][x]},{x,a,b,1}]//TableForm];
 g[i]=Plot[Evaluate[sol[i][x]],{x,a,b},PlotStyle->Hue[0.25+0.1*i],
  PlotRange->{{a,b},All},GridLines->{Automatic,Automatic}],{i,1,nN}];
Show[Table[g[i],{i,1,nN}]]
```

Note that in the process of computing numerical solutions for nonlinear ODEs (and systems of ODEs), various complicated phenomena can appear (e.g., round-off errors, singularities, and stiffness properties). In these cases, we can include additional options (for `NDSolve`), e.g., `Method`, `WorkingPrecision`, `MaxSteps`, `AccuracyGoal`, and `PrecisionGoal`.

20.3.2 Numerical Methods Embedded in Mathematica

In Mathematica, one can numerically solve various types of problems:

- Initial value problems (nonstiff, stiff, and complex-valued with a real-valued independent variable)
- Boundary value problems (linear and nonlinear)
- Initial value problems for differential algebraic equations (nonstiff and stiff)
- Initial value problems for delay differential equations (nonstiff and stiff)

Mathematica has a large collection of numerical methods for solving differential equations. These methods permit obtaining numerical solutions for a single differential equation (or a system of ODEs):

- without specifying a method;
- specifying one of the embedded methods (described in Tables 20.1–20.3) and various appropriate options for the selected embedded method, i.e., configuring a method via various options;
- constructing a new special-purpose class of methods by using predefined numerical methods as building blocks;
- adding new additional numerical methods into `NDSolve`.

One can do this owing to a special modular design and unification of the collection of methods.

The methods are hierarchical (i.e., one method can call another).

There exist *classes of methods* (e.g., `ExplicitRungeKutta` class). For a given class of methods, we can specify numerical schemes, orders, the coefficients of the method, etc. Each class of methods has appropriate options that can be obtained, e.g., for the class `ExplicitRungeKutta` methods, as follows:

```
Options[NDSolve`ExplicitRungeKutta]
```

Also there exists an automatic step size selection and a method order selection. For example, we can select the class `ExplicitRungeKutta` methods, and *Mathematica* will automatically select (depending on a given problem) an appropriate order, relative and absolute local error tolerances, and an initial step size estimate.

- `NDSolve,Method->Automatic`, finding numerical solutions of ODE problems without specification of a method (automatically); `NDSolve` will choose a method that should be appropriate for a given differential equation
- `NDSolve,Method->m`, finding numerical solutions of ODE problems specifying one of the basic numerical methods (see Table 20.1)
- `NDSolve,Method->{c,Method->subMeth}`, finding numerical solutions of ODE problems using one of the controller numerical methods (see Table 20.2) and submethods (see Table 20.3)

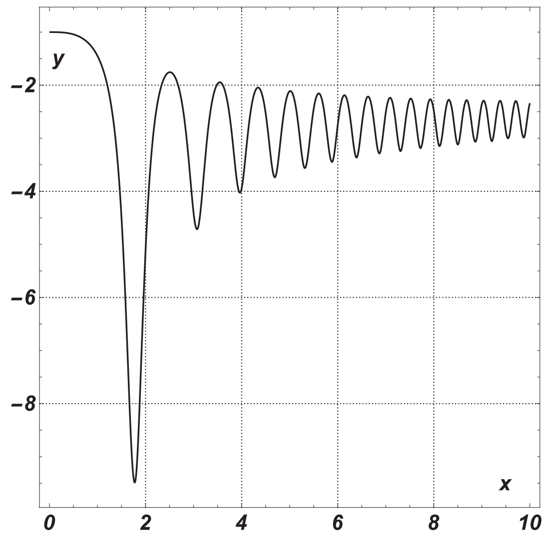


Figure 20.6: Numerical solution of the Cauchy problem (20.3.2.1) calculated by embedding the classical explicit fourth-order Runge–Kutta method.

Moreover, there exist mechanisms for constructing and embedding a new method (or a special-purpose class of methods) by using predefined numerical methods and various options as building blocks and for adding new additional methods.

Example 20.32. *First-order nonlinear ODE. Cauchy problem. Embedding new methods.*
The classical explicit fourth-order Runge–Kutta method can be embedded in Mathematica to obtain numerical solutions of the nonlinear Cauchy problem

$$y'_x = -y^2 \sin(x^2), \qquad y(0) = -1 \tag{20.3.2.1}$$

as follows:

```
{ode1=y'[x]==-y[x]^2*Sin[x^2], ics=y[0]==-1,
 ivp1={ode1,ics}//Flatten}
ClassicalRungeKutta/:NDSolve`InitializeMethod[
 ClassicalRungeKutta,___]:=ClassicalRungeKutta[];
ClassicalRungeKutta[___]["Step"[f_,x_,h_,y_,yp_]]:=
 Block[{ySol,k1,k2,k3,k4},
  k1=yp; k2=f[x+1/2*h,y+1/2*h*k1];
  k3=f[x+1/2*h,y+1/2*h*k2]; k4=f[x+h,y+h*k3];
  ySol=h*(1/6*k1+1/3*k2+1/3*k3+1/6*k4); {h,ySol}];
sol=NDSolve[ivp1,y,{x,0,10},Method->ClassicalRungeKutta,
 StartingStepSize->0.1]
Plot[y[x]/.sol,{x,0,10},PlotRange->All]
```

Also, the implicit second-order Runge–Kutta method can be embedded in Mathematica by using predefined numerical methods and various options as building blocks) to obtain numerical solutions of the same nonlinear Cauchy problem as follows:

```
implicitMidpoint={"FixedStep",
 Method->{"ImplicitRungeKutta",
 "Coefficients"->"ImplicitRungeKuttaGaussCoefficients",
```

```
 "DifferenceOrder"->2,
 "ImplicitSolver"->{"FixedPoint",
 "AccuracyGoal"->MachinePrecision,
 "PrecisionGoal"->MachinePrecision,
 "IterationSafetyFactor"->1}}};
{ode1=y'[x]==-y[x]^2*Sin[x^2],ics=y[0]==-1,
 ivp1={ode1,ics}//Flatten}
eq1=NDSolve[ivp1,y,{x,0,10},Method->implicitMidpoint,
 StartingStepSize->0.01]; solN=eq1[[1,1,2]]
Table[{x,solN[x]},{x,0,10}]//TableForm
Plot[solN[x],{x,0,10},PlotStyle->Blue,PlotRange->All]
```

The default methods are:

- `LSODA` approach, switching between a nonstiff `Adams` (multistep Adams-Moulton) method and a stiff Gear `BDF` (Backward Difference Formula) method
- `Chasing` method, the Gelfand–Lokutsiyevskii for linear boundary value problems
- `IDA`, general purpose solver based on repeated BDF and Newton iteration methods for initial value problems for differential algebraic equations
- `LSODA` approach and the step method for initial value problems for delay differential equations

Remark 20.4. `LSODA` is a version of the `LSODE` (the Livermore Solver for Ordinary Differential Equations). Since the 1980s, `LSODE` has been part of the solver collection `ODEPACK`.

More detailed information about numerical methods embedded in Mathematica is presented in Table 20.1 (basic numerical methods), Table 20.2 (controller methods), and Table 20.3 (submethods).

Table 20.1.
Basic numerical methods embedded in Mathematica

Numerical method	Brief description	References
`Adams`	The predictor-corrector Adams method. Order: 1-12.	Boyce and DiPrima (1992)
`BDF`	The implicit BDF (backward differentiation formulas) methods. Order: 1-5.	Conte and de Boor (1980)
`ExplicitRungeKutta`	The explicit pairs of Runge–Kutta methods Order: 2(1)-9(8). Features: adaptive, FSAL strategy local extrapolation mode, with stiffness detection, proportional-integral step-size controller (stiff ODEs).	Gustafsson (1991) Shampine (1994) Sofroniou & Spaletta (2004) Bogacki & Shampine (1989)
`ImplicitRungeKutta`	Families of implicit Runge–Kutta methods. The Gauss–Legendre methods. Order: arbitrary. Features: self-adjoint, with generic framework, arbitrary order, arbitrary precision.	Golub & Van Loan (1996) Shampine (1994) Sofroniou & Spaletta (2004) Bogacki & Shampine (1989)
`SymplecticPartitioned RungeKutta`	Families of explicit symplectic partitioned Runge–Kutta methods for separable Hamiltonian systems. Order: 1–10.	Sanz-Serna & Calvo (1994) McLachlan & Atela (1992) Sofroniou & Spaletta (2005)

Table 20.2.
Controller methods embedded in Mathematica

Numerical method	Brief description	References
`Composition`	Composing a list of submethods. Features: an arbitrary number of submethods.	Hairer, Lubich, Wanner (2002) Sofroniou & Spaletta (2005,2006).
`Splitting`	Splitting equations and applying submethods. Features: an arbitrary number of submethods, a generalization of the composition method.	Strang (1968), Marchuk (1968) Trotter (1959)
`DoubleStep`	A single application of Richardson's extrapolation. Features: adaptive (step size), a special case of extrapolation.	Deuflhard (1985) Gragg (1965), Shampine (1987) Hairer & Lubich (1988)
`EventLocator`	Methods that respond to specified events. Features: event location for detecting discontinuities, periods, etc.	Brent (2002), Dekker (1969)
`Extrapolation`	Gragg–Bulirsch–Stoer extrapolation method. Features: polynomial extrapolation, adaptive (order and step size).	Bulirsch & Stoer (1964) Hairer, Nørsett, Wanner (1993) Hairer & Wanner (1996)
`FixedStep`	Carrying out numerical integration using a constant step size. Features: for any one-step integration method.	Deuflhard, Hairer, Zugck (1987)
`OrthogonalProjection`	Projecting solutions to fulfill orthogonal constraints. Features: preserving orthonormality of solutions.	Dieci, Russel, Van Vleck (1994) Dieci & Van Vleck (1999) Del Buono & Lopez (2001)
`Projection`	Projecting solutions to fulfill general constraints. Features: invariant-preserving method.	Ascher & Petzold (1991) Hairer (2000) Hairer, Lubich, Wanner (2002)
`StiffnessSwitching`	Switching from explicit to implicit methods if stiffness is detected (in the middle of the integration).	Petzold (1983) Butcher (1990) Cohen & Hindmarsh (1996)

Table 20.3.
Submethods embedded in Mathematica

Numerical method	Brief description	References
`ExplicitEuler`	Explicit forward Euler method.	Boyce & DiPrima (2004)
`ExplicitMidpoint`	One-step explicit midpoint rule method.	Conte & de Boor (1980)
`ExplicitModified` `Midpoint`	Midpoint rule method with Gragg smoothing.	Gragg (1965)
`LocallyExact`	Numerical approximation to locally exact symbolic solution.	Murphy (1960)
`LinearlyImplicit` `Euler`	Linearly implicit Euler method.	Lubich (1989)
`LinearlyImplicit` `Midpoint`	Linearly implicit midpoint rule method.	Bader & Deuflhard (1983)
`LinearlyImplicit` `ModifiedMidpoint`	Linearly implicit Bader-smoothed midpoint rule method.	Shampine & Baca (1983)

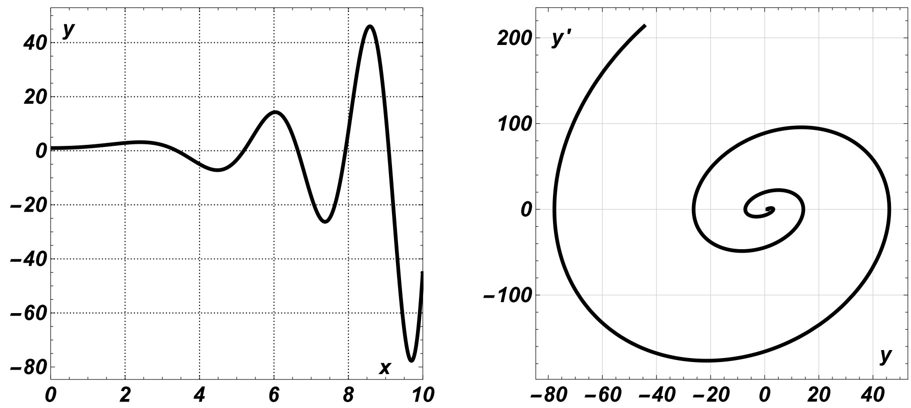


Figure 20.7: Numerical solution in the coordinate plane (left) and in the phase plane (right) for equation (20.3.3.1).

20.3.3 Initial Value Problems: Examples of Numerical Solutions

In general, an ordinary differential equation

$$y'_x = f(x, y)$$

can admit infinitely many solutions $y = y(x)$. To find one of them, we have to add a condition of the form $y(x_0) = y_0$, where y_0 is a given value called the *initial data*. Thus, consider the *Cauchy problem*

$$y'_x = f(x, y(x)), \qquad x_0 < x \le b;$$
$$y(x_0) = y_0.$$

According to the fundamental Picard–Lindelöf existence and uniqueness theorem for initial value problems (with the assumptions that $f(x, y)$ is a given continuous function with respect to (x, y) Lipschitz continuous with respect to y), the initial value problem has a unique solution.

Consider some examples of initial value problems.

► **Linear initial value problems.**

Example 20.33. *Linear initial value problem. Analytical, numerical, and graphical solutions.* For the first-order linear initial value problem

$$y'_x = -y\cos(x^2), \qquad y(0) = 1$$

on the interval $[a, b]$ ($a = 0$, $b = 10$), we find infinitely many solutions (`exSol1`) admitted by this ordinary differential equation and plot some of them (`sols`); then we obtain the unique exact solution (`exSol2`) of the Cauchy problem with the vector field (`VectorPlot`).

In the Mathematica notation, these solutions (`exSol1` and `exSol2`) read:

$$\left\{\left\{y[x] \to e^{-\sqrt{\frac{\pi}{2}}\text{FresnelC}\left[\sqrt{\frac{2}{\pi}}x\right]} C[1]\right\}\right\}, \qquad e^{-\sqrt{\frac{\pi}{2}}\text{FresnelC}\left[\sqrt{\frac{2}{\pi}}x\right]}$$

where FresnelC is the Fresnel integral $C(z) = \int_0^z \cos\left(\frac{1}{2}\pi t^2\right) dt$. Finally, we plot the exact and numerical solutions (`exSol2`, `solN`) of the Cauchy problem and compare the results as follows:

```
SetOptions[Plot,PlotRange->All,PlotStyle->Thickness[0.01],
 ImageSize->300]; a=0; b=10;
{ODE1=y'[x]==-y[x]*Cos[x^2],IC=y[0]==1,IVP1={ODE1,IC}}
exSol1=DSolve[ODE1,y[x],x]
sols=Table[exSol1[[1,1,2]]/.{C[1]->i},{i,-b,b}]
Plot[sols,{x,a,b},PlotRange->Automatic]
eq2=DSolve[IVP1,y[x],x]; exSol2=eq2[[1,1,2]]
eq3=NDSolve[IVP1,y,{x,a,b}]; solN=eq3[[1,1,2]]
Table[{x,solN[x]},{x,a,b}]//TableForm
g1=Plot[exSol2,{x,a,b},PlotStyle->Hue[0.6]];
g2=Plot[solN[x],{x,a,b},PlotStyle->Hue[0.8]];
GraphicsRow[{g1,g2}]
g3=VectorPlot[{1,-Y*Cos[T^2]},{T,a,b},{Y,0,1},Axes->Automatic,
 AspectRatio->1]; Show[g2,g3,PlotRange->{All,{0.3,1}}]
```

Example 20.34. *Second-order linear ODE. Numerical and graphical solutions.*
For the second-order linear initial value problem

$$y''_{xx} - y'_x + (x-1)y = 0, \qquad y(0) = 1, \quad y'_x(0) = 0 \tag{20.3.3.1}$$

on the interval $[a, b]$ ($a = 0$, $b = 10$), we find the numerical solution (`sol1`) and the graphical solutions (presented in Fig. 20.7) as follows:

```
SetOptions[ParametricPlot,AspectRatio->1,PlotRange->All,
 PlotStyle->Thickness[0.01],Frame->True];
SetOptions[Plot,PlotRange->All,PlotStyle->Thickness[0.01],
 ImageSize->300,Frame->True]; a=0; b=10;
{IC1={y'[0]==0,y[0]==1},ODE1=D[y[x],{x,2}]-D[y[x],x]+(x-1)*y[x]==0,
 IVP1={ODE1,IC1},lCol={Blue,Red}}
eq1=NDSolve[IVP1,y,{x,a,b}]; sol1=eq1[[1,1,2]]
g1=Plot[sol1[x],{x,a,b},PlotStyle->lCol[[1]]];
g2=ParametricPlot[Evaluate[{sol1[x],D[sol1[x],x]}],{x,a,b},
 PlotStyle->lCol[[2]]]; GraphicsGrid[{{g1,g2}}]
```

► Nonlinear initial value problems.

Example 20.35. *Nonlinear initial value problem. Numerical and graphical solutions.*
For the initial value problem

$$y'_x = -e^{xy}\cos(x^2), \qquad y(0) = p \tag{20.3.3.2}$$

for a first-order nonlinear ODE on the interval $[a, b]$ ($a = 0$, $b = 10$), we find the numerical and graphical solutions (see Figure 20.8) of the problem for various initial conditions $y(0) = p$, where $p = 0.1\,i$ ($i = 1, 2, ..., 5$), as follows:

```
eq=y'[x]==-Exp[y[x]*x]*Cos[x^2]; a=0; b=4*Pi;
Do[{IC={y[0]==0.1*i}; sN=NDSolve[{eq,IC},y[x],{x,a,b},
 MaxSteps->1000]; solN[x_]:=sN[[1,1,2]];
 g[i]=Plot[solN[x],{x,a,b},PlotStyle->Hue[0.5+i*0.07]];},{i,1,5}]
Show[Table[g[i],{i,1,5}],Axes->False,PlotRange->All]
```

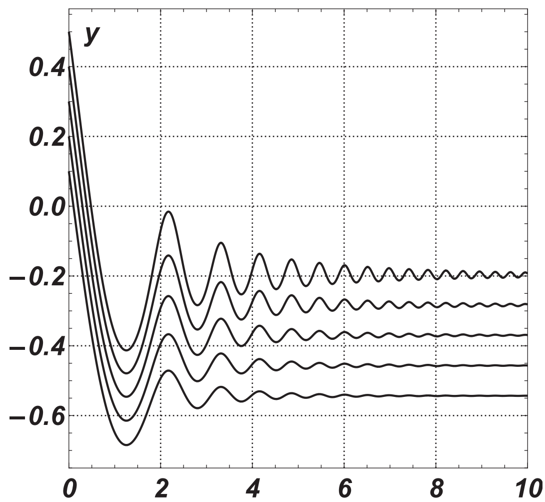


Figure 20.8: Numerical solutions of the nonlinear initial value problem (20.3.3.2).

20.3.4 Initial Value Problems: Constructing Numerical Methods and Solutions

Alternatively, numerical methods and solutions of initial value problems can be constructed (step by step) and analyzed as follows.

► **Single-step methods.**

First, consider one of the classical methods, the *forward Euler method*, or *explicit Euler method*. This method belongs to a family of *single-step methods*, which compute the numerical solution Y_{i+1} at the node X_{i+1} knowing information related to the previous node X_i alone.

The strategy of these methods is to divide the integration interval $[a, b]$ into N subintervals of length $h = (b - a)/N$ called the *discretization step*. Then at the nodes X_i $(0 \le i \le N)$ we compute the unknown values Y_i which approximate the exact values $y(X_i)$; i.e., $Y_i \approx y(X_i)$. The set of values $\{Y_0 = y_0, Y_1, \ldots, Y_N\}$ is the *numerical solution*. The formula for the explicit Euler method reads:

$$Y_{i+1} = Y_i + hF(X_i, Y_i), \quad Y_0 = y(X_0), \quad i = 0, \ldots, N-1.$$

Example 20.36. *The Euler method. Analytical, numerical, and graphical solutions.*
For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5 \tag{20.3.4.1}$$

on the interval $[a, b]$ $(a = 0,\ b = 2)$ with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we find the exact solution (`extSol`) and the numerical solution (`F1`) using the explicit Euler method, compare the results, plot the exact and numerical solutions (`g1`, `g2`), see Fig. 20.9, and determine the absolute computational error at each step as follows:

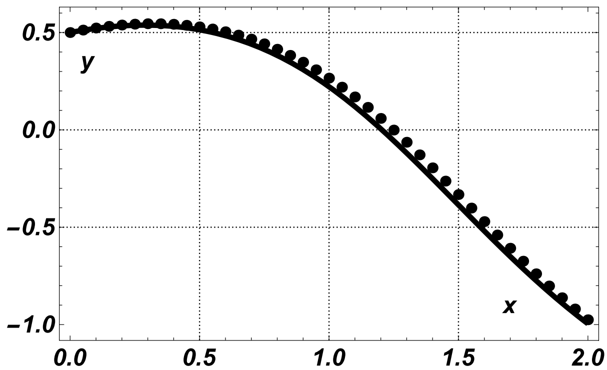


Figure 20.9: Exact solution (solid line) and numerical solution (points; the solution is obtained by the explicit Euler method) of the Cauchy problem (20.3.4.1).

```
SetOptions[Plot,PlotRange->All,PlotStyle->Thickness[0.02],
 ImageSize->300,Frame->True]; nD=10;
{a=0,b=2,p=1,q=-1,n=1,m=2,nN=40,h=N[(b-a)/nN,nD],
 Y=Table[0,{i,1,nN+1}]}
{ODE1=D[y[x],x]==p*y[x]^m+q*x^n,IC=y[0]==1/2,IVP1={ODE1,IC}}
extSol[x1_]:=DSolve[IVP1,y[x],x][[1,1,2]]/.{x->x1}//Simplify;
extSol[x]
F[x_,y_]:=p*y^m+q*x^n; X=Table[a+i*h,{i,0,nN}]; Y[[1]]=N[1/2,nD];
Do[Y[[i+1]]=N[Y[[i]]+h*F[X[[i]],Y[[i]]],nD],{i,1,nN}];
F1=Table[{X[[i+1]],Y[[i+1]]},{i,0,nN}]
Do[Print[PaddedForm[k,3]," ",PaddedForm[X[[k]],{15,10}]," ",
 PaddedForm[Y[[k]],{15,10}]," ",
 PaddedForm[N[extSol[X[[k]]]]//Chop,{15,10}]," ",
 PaddedForm[N[Abs[Y[[k]]-extSol[X[[k]]]],nD],{15,10}]],{k,2,nN+1}];
g1=Plot[extSol[x1],{x1,a,b},PlotStyle->Blue];
g2=ListPlot[F1,PlotStyle->{PointSize[.02],Hue[0.9]}]; Show[{g1,g2}]
```

There is a general way to determine the order of convergence of a numerical method. If we know the errors E_i ($i = 1, \dots, N$) corresponding to the values h_i of the discretization parameter (in our case, h_i is the discretization step of the Euler method) and assume that $E_i = Ch_i^p$ and $E_{i-1} = Ch_{i-1}^p$, then

$$p = \frac{\log(E_i/E_{i-1})}{\log(h_i/h_{i-1})}, \qquad i = 2, \dots, N. \tag{20.3.4.2}$$

Example 20.37. *The Euler method. The order of convergence.*
For the same Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5$$

(as in the previous example) on the interval $[a, b]$ ($a = 0$, $b = 2$) with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we obtain a numerical solution by applying the explicit Euler method for various values of the discretization step h and, according to formula (20.3.4.2), verify that the order of convergence of the explicit Euler method is 1:

```
$RecursionLimit=Infinity; $HistoryLength=0; nD=10;
F[x_,y_]:=p*y^m+q*x^n; a=0; b=2; p=1; q=-1; n=1; m=2; nN=40;
{ODE1=D[y[x],x]==p*y[x]^m+q*x^n, IC=y[0]==1/2, IVP1={ODE1,IC}}
extSol[x1_]:=DSolve[IVP1,y[x],x][[1,1,2]]/.{x->x1}; extSol[x]
Euler[a_,b_,n_]:=Module[{h,x,Y,EN}, h=N[(b-a)/n,nD];
 Y[0]=N[1/2,nD]; X[x_]:=a+x*h;
 Do[Y[i_]:=Y[i]=N[Y[i-1]+h*F[X[i-1],Y[i-1]],nD],{i,1,n}];
 EN=Table[N[Abs[Y[i]-N[extSol[X[i]],nD]],nD],{i,1,n}];EN];
L1={}; n1=4;
Do[Er[k]=Euler[a,b,n1]; Print[Last[Er[k]]]; L1=Append[L1,Last[Er[k]]];
 n1=n1*2,{k,1,12}]; {Ers=L1, NErs=Length[Ers]}
p=Table[N[Abs[Log[Ers[[i]]/Ers[[i-1]]]/Log[2]],nD],{i,2,NErs}]
```

Runge–Kutta methods are single-step methods that involve several evaluations of the function $f(x,y)$ and none of its derivatives on every interval $[X_i, X_{i+1}]$.

In general, explicit or implicit Runge–Kutta methods can be constructed with arbitrary order according to the formulas. Consider the s-stage explicit Runge–Kutta method

$$k_1 = f(x_n, y_n), \quad k_2 = f(x_n + c_2 h, y_n + a_{2,1} k_1 h), \quad \dots,$$
$$k_s = f\Big(x_n + c_s h, y_n + \sum_{i=1}^{s-1} a_{s,j}\, k_j\Big),$$
$$Y_{n+1} = Y_n + h \sum_{i=1}^{s} b_i k_i, \quad Y_0 = y_0, \quad n = 0, \dots N-1.$$

Example 20.38. *Higher-order methods. Derivation of explicit Runge–Kutta methods.* Let us perform analytical derivation of the best-known Runge–Kutta methods.

```
subsF={Dt[F]->Dt[f[x,y[x]]]};
subs1={D[y[x],x]->F};
subs2={D[y[x],{x,2}]->Dt[f[x,y[x]]]};
subs3={D[y[x],{x,3}]->Dt[Dt[f[x,y[x]]]]};
subs4={D[y[x],{x,4}]->Dt[Dt[Dt[f[x,y[x]]]]]};
sD={f[x,y[x]]->F,Dt[x]->1,(D[f[x,x1],x]/.{x1->y[x]})->Fx,
 (D[f[x,x1],x1]/.{x1->y[x]})->Fy,(D[f[x,x1],{x,2}]/.{x1->y[x]})->Fxx,
 (D[f[x,x1],x1,x]/.{x1->y[x]})->Fxy,(D[f[x,x1],{x1,2}]/.{x1->y[x]})->Fyy,
 (D[f[x,x1],{x1,3}]/.{x1->y[x]})->Fyyy,
 (D[f[x,x1],{x,3}]/.{x1->y[x]})->Fxxx,
 (D[f[x,x1],x,x1,x1]/.{x1->y[x]})->Fxyy,
 (D[f[x,x1],x,x,x1]/.{x1->y[x]})->Fxxy};
```

For $s = 1$, we obtain the *Euler method* (`sol`), where $b_1 = 1$:

```
{s=1, l1={h,F},
 p1=Normal[Series[y[x+h],{h,0,s}]]/.subs1,
 p2=Expand[(p1-y[x])/h], k[1]=Series[f[x,y[x]],{h,0,s}]/.sD,
 p3=Expand[Normal[Series[Sum[b[i]*k[i],{i,1,s}],{h,0,s}]]]/.sD,
 eq1=p2-p3, eq2=DeleteCases[Flatten[CoefficientList[eq1,l1]],_0]}
sol=Solve[eq2==0,Variables[eq2]]
```

For $s = 2$, we obtain the *2-stage modified Euler method* (`sol1`), where

$$a_{2,1} = \frac{1}{2}, \quad b_1 = 0, \quad b_2 = 1, \quad c_2 = \frac{1}{2},$$

the *2-stage improved Euler method* (`sol2`), where

$$a_{2,1} = 1, \quad b_1 = b_2 = \frac{1}{2}, \quad c_2 = 1,$$

or the *2-stage Heun method* (`sol3`), where

$$a_{2,1} = \frac{2}{3}, \quad b_1 = \frac{1}{4}, \quad b_2 = \frac{3}{4}, \quad c_2 = \frac{2}{3} :$$

```
{s=2, l2={h,F,Fx,Fy},
 p1=Normal[Series[y[x+h],{h,0,s}]]/.subs2/.subs1/.sD,
 p2=Expand[(p1-y[x])/h]}
{k[1]=Series[f[x,y[x]],{h,0,s}]/.sD,
 k[2]=Series[f[x+c[2]*h,y[x]+h*Sum[a[2,i]*k[i],{i,1,1}]],{h,0,s-1}]/.sD}
 {p3=Expand[Normal[Series[Sum[b[i]*k[i],{i,1,s}],{h,0,s}]]]/.sD,
  eq1=p2-p3, eq21=DeleteCases[Flatten[CoefficientList[eq1,l2]],_0],
  eq2=Map[Thread[#1==0,Equal]&,eq21]}
eq3={}; Do[eq3=Append[eq3,{c[i]==Sum[a[i,j],{j,1,i-1}]}],{i,2,s}];
{s1=Flatten[{eq2,eq3,{c[2]==1/2}}], s2=Flatten[{eq2,eq3,{b[2]==1/2}}],
 s3=Flatten[{eq2,eq3,{b[2]==3/4}}]}
{sol1=Solve[s1,Variables[eq2]], sol2=Solve[s2,Variables[eq2]],
 sol3=Solve[s3,Variables[eq2]]}
```

For $s = 3$, we obtain the *3-stage Heun method* (`sol`), where

$$a_{2,1} = \frac{1}{3},\ a_{3,1} = 0,\ a_{3,2} = \frac{2}{3},\ b_1 = \frac{1}{4},\ b_2 = 0,\ b_3 = \frac{3}{4},\ c_2 = \frac{1}{3},\ c_3 = \frac{2}{3} :$$

```
{s=3, l3={h,F,Fx,Fy,Fxx,Fxy,Fyy},
 p1=Normal[Series[y[x+h],{h,0,s}]]/.subs3/.subs2/.subs1/.sD,
 p2=Expand[Normal[Expand[(p1-y[x])/h]]]}
{k[1]=Series[f[x,y[x]],{h,0,s}]/.sD, k[2]=Series[f[x+c[2]*h,
 y[x]+h*(Sum[a[2,i]*k[i],{i,1,1}])],{h,0,s-1}]/.sD,
 k[3]=Series[f[x+c[3]*h,
 y[x]+h*(Sum[a[3,i]*k[i],{i,1,2}])],{h,0,s-1}]/.sD}
{p3=Expand[Normal[Series[Sum[b[i]*k[i],{i,1,s}],{h,0,s}]]],
 eq1=p2-p3, eq21=DeleteCases[Flatten[CoefficientList[eq1,l3]],_0],
 eq2=Map[Thread[#1==0,Equal]&,eq21]}
eq3={}; Do[eq3=Append[eq3,{c[i]==Sum[a[i,j],{j,1,i-1}]}],{i,2,s}];
s1=Flatten[{eq2,eq3,{b[1]==1/4},{c[2]==1/3}}]
sol1=Solve[s1,Variables[eq2]]
```

For $s = 4$, we obtain the fourth-order *Runge–Kutta* method* (`Sol`), where

$$a_{2,1} = \frac{1}{2},\ a_{3,1} = 0,\ a_{3,2} = \frac{1}{2},\ a_{4,1} = 0,\ a_{4,2} = 0,\ a_{4,3} = 1,$$
$$b_1 = \frac{1}{6},\ b_2 = \frac{1}{3},\ b_3 = \frac{1}{3},\ b_4 = \frac{1}{6},\ c_2 = \frac{1}{2},\ c_3 = \frac{1}{2},\ c_4 = 1 :$$

```
{s=4, l4={h,F,Fx,Fy,Fxx,Fxy,Fyy,Fxxx,Fxxy,Fxyy,Fyyy},
 p1=Normal[Series[y[x+h],{h,0,s}]]/.subs4/.subs3/.subs2/.subs1/.sD,
 p2=Expand[Normal[Expand[(p1-y[x])/h]]]}
{k[1]=Series[f[x,y[x]],{h,0,s}]/.sD, k[2]=Series[f[x+c[2]*h,
 y[x]+h*(Sum[a[2,i]*k[i],{i,1,1}])],{h,0,s-1}]/.sD,
 k[3]=Series[f[x+c[3]*h,
```

*This method was introduced by Runge in 1895.

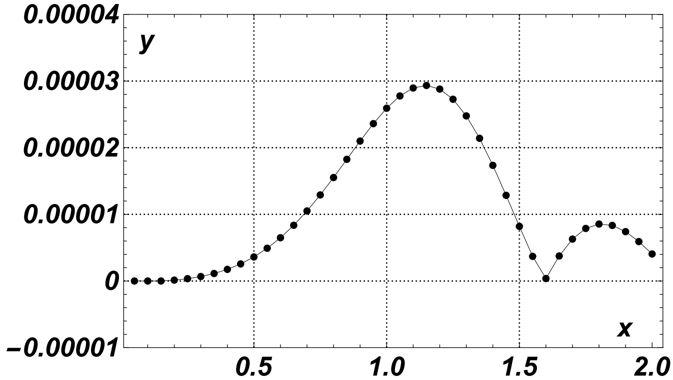


Figure 20.10: The absolute computational error (at each step) for the exact solution and a numerical solution (obtained by the Adams–Bashforth method) of the Cauchy problem (20.3.4.3).

```
 y[x]+h*(Sum[a[3,i]*k[i],{i,1,2}])],{h,0,s-1}]/.sD,
 k[4]=Series[f[x+c[4]*h,
 y[x]+h*(Sum[a[4,i]*k[i],{i,1,3}])],{h,0,s-1}]/.sD}
{p3=Expand[Normal[Series[Sum[b[i]*k[i],{i,1,s}],{h,0,s}]]]/.sD,
 eq1=p2-p3, eq21=DeleteCases[Flatten[CoefficientList[eq1,14]],_0],
 eq2=Map[Thread[#1==0,Equal]&,eq21]}
eq3={}; Do[eq3=Append[eq3,{c[i]==Sum[a[i,j],{j,1,i-1}]}],{i,2,s}];
s1=Flatten[{eq2,eq3,{b[1]==1/6},{c[2]==1/2},{a[3,2]==1/2}}];
sol1=Solve[s1,Variables[eq2]]
```

► **Multistep methods.**

There are more sophisticated methods that achieve a high order of accuracy by considering several values $(Y_i, Y_{i-1}, \dots)$ to determine Y_{i+1}. One of the most notable methods is the explicit four-step fourth-order Adams–Bashforth method

$$Y_{i+1}{=}Y_i{+}\frac{h}{24}\Big(55F(X_i,Y_i){-}59F(X_{i-1},Y_{i-1}){+}37F(X_{i-2},Y_{i-2}){-}9F(X_{i-3},Y_{i-3})\Big).$$

Example 20.39. *Cauchy problem. The explicit Adams–Bashforth method.*
For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5 \tag{20.3.4.3}$$

on the interval $[a, b]$ ($a = 0$, $b = 2$) with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we find the exact solution (`extSol`) and the numerical solution (`F1`) by the explicit Adams–Bashforth method and plot them. Finally, we compute the absolute computational error on $[a, b]$ at each step (`F2`), and plot it (see Fig. 20.10) as follows:

```
nD=20; {a=0,b=2,p=1,q=-1,n=1,m=2,nN=40,h=N[(b-a)/nN,nD],
 YAB=Table[0,{i,1,nN+1}]}
```

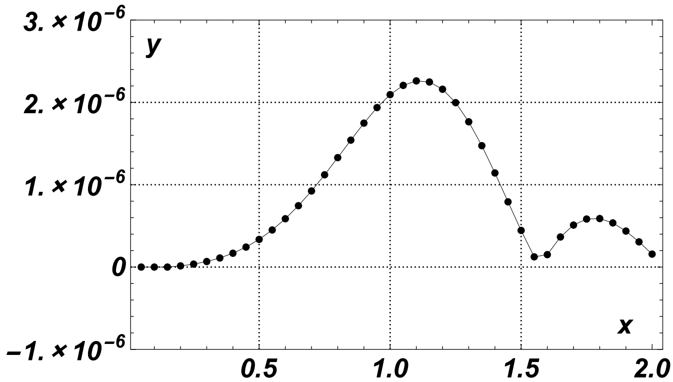


Figure 20.11: The absolute computational error (at each step) for the exact solution and a numerical solution (obtained by the Adams–Bashforth–Moulton method) of the Cauchy problem (20.3.4.4).

```
ODE1=D[y[x],x]==p*y[x]^m+q*x^n
{IC=y[0]==1/2, IVP1={ODE1,IC}}
extSol[x1_]:=DSolve[IVP1,y[x],x][[1,1,2]]/.{x->x1}; extSol[x]
F[x_, y_]:=p*y^m+q*x^n;
X=Table[a+i*h,{i,0,nN}];
YAB[[1]]=N[1/2,nD]; YAB[[2]]=N[extSol[X[[2]]],nD];
YAB[[3]]=N[extSol[X[[3]]],nD]; YAB[[4]]=N[extSol[X[[4]]],nD];
Do[YAB[[i+1]]=N[YAB[[i]]+h/24*(55*F[X[[i]],YAB[[i]]]-59*F[X[[i-1]],
 YAB[[i-1]]]+37*F[X[[i-2]],YAB[[i-2]]]-9*F[X[[i-3]],YAB[[i-3]]]),nD],
{i,4,nN}];
F1=Table[{X[[i+1]],YAB[[i+1]]//Chop},{i,0,nN}]
Do[Print[PaddedForm[i,3]," ",PaddedForm[X[[i]],{15,10}]," ",
 PaddedForm[YAB[[i]]//Chop,{15,10}]," ",
 PaddedForm[N[extSol[X[[i]]],nD]//Chop,{15,10}]," ",
 PaddedForm[N[Abs[YAB[[i]]-extSol[X[[i]]]],nD],{15,10}]],{i,2,nN+1}];
g0=Plot[extSol[x1],{x1,a,b},PlotStyle->Blue];
g1=ListPlot[F1,PlotStyle->{PointSize[.02],Hue[0.9]}]; Show[{g0,g1}]
F2=Table[{X[[i+1]],Abs[YAB[[i+1]]-N[extSol[X[[i+1]]],nD]]},{i,1,nN}];
ListPlot[F2,Joined->True,PlotStyle->Hue[0.99],PlotRange->All]
```

Another important example of multistep methods is the implicit three-step fourth-order Adams–Bashforth–Moulton method

$$Y_{i+1} = Y_i + \frac{h}{24}\Big(9F(X_{i+1}, Y_{i+1}) + 19F(X_i, Y_i) - 5F(X_{i-1}, Y_{i-1}) + F(X_{i-2}, Y_{i-2})\Big).$$

Example 20.40. *Cauchy problem. The implicit Adams–Bashforth–Moulton method.* For the Cauchy problem

$$y'_x = py^m + qx^n, \qquad y(0) = 0.5 \tag{20.3.4.4}$$

on the interval $[a, b]$ ($a = 0$, $b = 2$) with parameters $p = 1$, $q = -1$, $m = 2$, and $n = 1$, we find the exact solution (`extSol`) and a numerical solutions (`F1`) by the implicit Adams–Moulton method, compare the results and the graphical solutions, find the absolute computational error on $[a, b]$ at each step (F2), and plot it (see Fig. 20.11) as follows:

```
nD=20; {a=0,b=2,p=1,q=-1,n=1,m=2,nN=40,h=(b-a)/nN}
YAM=Table[0,{i,1,nN+1}]
{ODE1=D[y[x],x]==p*y[x]^m+q*x^n,IC=y[0]==1/2,IVP1={ODE1,IC}}
extSol[x1_]:=DSolve[IVP1,y[x],x][[1,1,2]]/.{x->x1}; extSol[x]
F[x_,y_]:=p*y^m+q*x^n; X[x_]:=a+x*h;
X1=Table[a+i*h,{i,0,nN}];
{eq1=Y[i]-Y[i-1]-h/24*(9*F[X[i],Y[i]]+19*F[X[i-1],Y[i-1]]
 -5*F[X[i-2],Y[i-2]]+F[X[i-3],Y[i-3]]),
 eq21=Solve[eq1==0,Y[i]][[1,1,2]],eq2=Expand[eq21]}
YAM[[1]]=N[1/2,nD]; YAM[[2]]=N[extSol[X1[[2]]],nD];
YAM[[3]]=N[extSol[X1[[3]]],nD]; YAM[[4]]=N[extSol[X1[[4]]],nD];
Do[YAM[[i+1]]=N[eq2/.{Y[i-1]->YAM[[i]],Y[i-2]->YAM[[i-1]],
 Y[i-3]->YAM[[i-2]]},nD],{i,4,nN}];
F1=Table[{N[X1[[i+1]],nD],YAM[[i+1]]//Chop},{i,0,nN}]
Do[Print[PaddedForm[i,3]," ",PaddedForm[N[X1[[i]],nD],{15,10}]," ",
 PaddedForm[YAM[[i]]//Chop,{15,10}]," ",
 PaddedForm[N[extSol[X1[[i]]],nD]//Chop,{15,10}]," ",
 PaddedForm[N[Abs[YAM[[i]]-extSol[X1[[i]]]],nD],{15,10}]],{i,2,nN+1}];
g0=Plot[extSol[x1],{x1,a,b},PlotStyle->Blue];
g1=ListPlot[F1,PlotStyle->{PointSize[.02],Hue[0.9]}]; Show[{g0,g1}]
F2=Table[{X1[[i+1]],Abs[YAM[[i+1]]-N[extSol[X1[[i+1]]],nD]]},{i,1,nN}];
ListPlot[F2,Joined->True,PlotStyle->Hue[0.99],PlotRange->All]
```

20.3.5 Boundary Value Problems: Examples of Numerical Solutions

► Preliminary remarks.

Let us numerically solve two-point boundary value problems. A two-point boundary value problem includes an ODE (of order ≥ 2) and the value of the solution at two distinct points. Note a difference between initial value problems and boundary value problems: initial value problems (with well-behaved functions) have unique solutions; i.e., they are "well-posed"; but boundary value problems (with well-behaved functions) may have more than one solution or no solution at all (see Example 20.42).

Consider some examples of boundary value problems applying embedded methods and constructing step-by-step solutions.

► Linear boundary value problems.

Example 20.41. *Boundary value problem. Analytical, numerical, and graphical solutions.*
Consider the following second-order linear nonhomogeneous ODE with variable coefficients (see the initial value problem for this ODE in Example 20.16) and with boundary conditions:

$$y''_{xx} + xy'_x + y = \cos(x), \qquad y(a) = 0, \quad y(b) = 1, \tag{20.3.5.1}$$

where $a = 0$, $b = 2$. Analytical, numerical, and graphical solutions (sol1, sol2, solN, g1, g2) can be constructed as follows:

```
{a=0, b=2, h=0.1, ODE1=D[y[x],{x,2}]+x*D[y[x],x]+y[x]==Cos[x],
 BC={y[a]==0,y[b]==1}, BVP1={ODE1,BC}//Flatten}
```

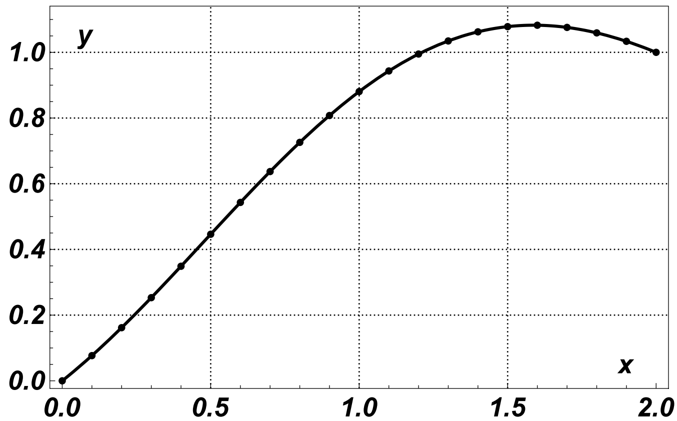


Figure 20.12: Exact and numerical solutions of the boundary value problem (20.3.5.1).

```
{sol1=DSolve[BVP1,y,x], test1=ODE1/.sol1//FullSimplify}
{sol2=DSolve[BVP1,y[x],x], solN=NDSolve[BVP1,y,{x,a,b}][[1,1,2]]}
k=0; Do[{k=k+1; X[k]=m; s[k]=N[sol2[[1,1,2]]/.x->m]},{m,a,b,h}]; n=k
seq1=Table[{X[m],(s[m]//Chop)},{m,1,n}]
g1=ListLinePlot[seq1,PlotStyle->{Blue,Thickness[0.01]}];
g2=Plot[solN[x],{x,a,b},PlotStyle->{Red,Dashed,Thickness[0.03]}];
Show[{g1,g2}]
```

Comparing the results, we conclude that the analytical and numerical solutions (see Fig. 20.12) are in good agreement.

Example 20.42. *Two-point linear boundary value problem. No solution.*

Solving a boundary value problem for the second-order linear homogeneous ODE with constant coefficients

$$y''_{xx} + \pi^2 y = 0, \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{20.3.5.2}$$

where $a = 0$, $b = 1$, $\alpha = 1$, and $\beta = 1$, we can find the general solution of the equation. However, the boundary conditions cannot be satisfied (for any choice of the constants occurring in the general solution). Therefore, there is no solution of this problem:

```
{a=0, b=1, alpha=1, beta=1, ODE1=D[y[x],{x,2}]+Pi^2*y[x]==0,
 BC={y[a]==alpha,y[b]==beta}, BVP1={ODE1,BC}//Flatten}
{sol1=Dsolve[BVP1,y,x], sol2=Dsolve[BVP1,y[x],x]}
{solGen=DSolve[ODE1,y[x],x], eq1=solGen/.x->a, eq2=solGen/.x->b}
sys1={eq1[[1,1,2]]==alpha, eq2[[1,1,2]]==beta}
Solve[sys1, {C[1], C[2]}]
```

Consider the boundary value problem for the second-order linear ODE

$$y''_{xx} = p(x)y'_x + q(x)y + r(x), \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{20.3.5.3}$$

where $a \le x \le b$. This problem has a unique solution if $p(x)$, $q(x)$, and $r(x)$ are continuous and $q(x) > 0$.

Linear shooting methods employ numerical methods (discussed above) for solving two initial value problems

$$u''_{xx} = p(x)u'_x + q(x)u + r(x), \qquad u(a) = \alpha, \quad u'_x(a) = 0,$$
$$v''_{xx} = p(x)v'_x + q(x)v, \qquad v(a) = 0, \quad v'_x(a) = 1,$$

and the solution of the original boundary value problem is

$$y(x) = u(x) + v(x)\frac{\beta - u(b)}{v(b)}, \qquad a \le x \le b.$$

Example 20.43. *Boundary value problems. Linear shooting methods.*
For the linear boundary value problem

$$y''_{xx} = -\frac{2}{x}y'_x + \frac{2}{x^2}y + x^3, \qquad y(1) = 1, \quad y(2) = 2, \tag{20.3.5.4}$$

the exact solution (`extSol`)

$$y[x] \to \frac{30}{49x^2} + \frac{69x}{196} + \frac{x^5}{28}$$

can be obtained as follows:

```
nD=10; Fu1[x_,u1_,u2_]:=u2; Fu2[x_,u1_,u2_]:=-2/x*u2+2/x^2*u1+x^3;
Fv1[x_,v1_,v2_]:=v2; Fv2[x_,v1_,v2_]:=-2/x*v2+2/x^2*v1;
{n=10, a=1, b=2, h=N[(b-a)/n,nD], X=Table[a+h*i,{i,0,n}],
 alpha=1, beta=2, RK41u=Table[0,{i,0,n}], RK42u=Table[0,{i,0,n}],
 RK41v=Table[0,{i,0,n}], RK42v=Table[0,{i,0,n}], Y=Table[0,{i,0,n}]}
{ODE1=D[y[x],{x,2}]+2/x*D[y[x],x]-2/x^2*y[x]-x^3==0,
 BC={y[a]==alpha,y[b]==beta}, BVP1={ODE1,BC}}
extSol[x1_]:=Expand[DSolve[BVP1,y[x],x][[1,1,2]]]/.{x->x1};
extSol[x]
```

Then we find the numerical solution (`F1`) by applying the linear shooting method, compare the results, and plot the exact and numerical solutions (`g1`, `g2`) as follows:

```
{RK41u[[1]]=alpha, RK42u[[1]]=0, RK41v[[1]]=0, RK42v[[1]]=1}
Do[k1=h*Fu1[X[[i]], RK41u[[i]], RK42u[[i]]];
 m1=h*Fu2[X[[i]], RK41u[[i]], RK42u[[i]]];
 k2=h*Fu1[X[[i]]+h/2, RK41u[[i]]+k1/2, RK42u[[i]]+m1/2];
 m2=h*Fu2[X[[i]]+h/2, RK41u[[i]]+k1/2, RK42u[[i]]+m1/2];
 k3=h*Fu1[X[[i]]+h/2, RK41u[[i]]+k2/2, RK42u[[i]]+m2/2];
 m3=h*Fu2[X[[i]]+h/2, RK41u[[i]]+k2/2, RK42u[[i]]+m2/2];
 k4=h*Fu1[X[[i]]+h, RK41u[[i]]+k3, RK42u[[i]]+m3];
 m4=h*Fu2[X[[i]]+h, RK41u[[i]]+k3, RK42u[[i]]+m3];
 RK41u[[i+1]]=N[RK41u[[i]]+1/6*(k1+2*k2+2*k3+k4),nD];
 RK42u[[i+1]]=N[RK42u[[i]]+1/6*(m1+2*m2+2*m3+m4),nD],{i,1,n}];
Do[k1=h*Fv1[X[[i]], RK41v[[i]], RK42v[[i]]];
 m1=h*Fv2[X[[i]], RK41v[[i]], RK42v[[i]]];
 k2=h*Fv1[X[[i]]+h/2, RK41v[[i]]+k1/2, RK42v[[i]]+m1/2];
 m2=h*Fv2[X[[i]]+h/2, RK41v[[i]]+k1/2, RK42v[[i]]+m1/2];
 k3=h*Fv1[X[[i]]+h/2, RK41v[[i]]+k2/2, RK42v[[i]]+m2/2];
 m3=h*Fv2[X[[i]]+h/2, RK41v[[i]]+k2/2, RK42v[[i]]+m2/2];
 k4=h*Fv1[X[[i]]+h, RK41v[[i]]+k3, RK42v[[i]]+m3];
```

```
 m4=h*Fv2[X[[i]]+h, RK41v[[i]]+k3, RK42v[[i]]+m3];
 RK41v[[i+1]]=N[RK41v[[i]]+1/6*(k1+2*k2+2*k3+k4),nD];
 RK42v[[i+1]]=N[RK42v[[i]]+1/6*(m1+2*m2+2*m3+m4),nD],{i,1,n}];
C1=(beta-RK41u[[n+1]])/RK41v[[n+1]]
Do[Y[[i]]=N[RK41u[[i]]+C1*RK41v[[i]],nD],{i,1,n+1}];
Do[Print[PaddedForm[i,3]," ",PaddedForm[RK41u[[i+1]],{12,10}]," ",
 PaddedForm[RK41v[[i+1]],{12,10}]," ",
 PaddedForm[Y[[i+1]],{12,10}]," ",
 PaddedForm[N[extSol[X[[i+1]]],nD],{12,10}]," ",
 PaddedForm[Abs[Y[[i+1]]-N[extSol[X[[i+1]]],nD]],{12,10}]],{i,0,n}];
F1=Table[{X[[i+1]],Y[[i+1]]},{i,0,n}]
g1=Plot[extSol[x1],{x1,a,b},PlotStyle->Hue[0.99]];
g2=ListPlot[F1,PlotStyle->{PointSize[.02],Hue[0.7]}]; Show[{g1,g2}]
```

Let us apply the finite difference method for approximating the solution of the linear boundary value problem (20.3.5.3)

$$y''_{xx} = p(x)y'_x + q(x)y + r(x), \quad y(a) = \alpha, \quad y(b) = \beta.$$

The basic idea of finite difference methods is to replace the derivatives in the differential equations by appropriate finite differences. We choose an equidistant grid $X_i = a + ih$ $(i = 0, \dots, N + 1)$ on $[a, b]$ with step size $h = (b - a)/(N + 1)$ $(N \in \mathbb{N})$, where $X_0 = a$ and $X_{N+1} = b$.

The differential equation must be satisfied at any internal node X_i $(i = 1, \dots, n)$, and by approximating this set of N equations and by replacing the derivatives with appropriate finite differences, we obtain the system of equations

$$\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} = p(X_i)\frac{Y_{i+1} - Y_{i-1}}{2h} + q(X_i)Y_i + r(X_i), \quad Y_0 = \alpha, \quad Y_{N+1} = \beta$$

for the approximate values Y_i of the exact solution $y(X_i)$. This linear system admits a unique solution, because the matrix of the system is an $N \times N$ symmetric positive definite tridiagonal matrix.

Example 20.44. *Approximations by finite differences.*

For the linear boundary value problem (20.3.5.1), we can find the exact solution (`extSol`) and a numerical solution (`F1`) by the finite difference method, compare the results, and plot the exact and numerical solutions (`g1`, `g2`) as follows:

```
sEq={}; {nD=10, a=1, b=2, alpha=1, beta=2, n=10,
 h=(b-a)/(n+1), X=Table[a+i*h,{i,0,n}]}
{ODE1=D[y[x],{x,2}]==-2/x*D[y[x],x]+2/x^2*y[x]+x^3,
 BC={y[a]==alpha,y[b]==beta}, BVP1={ODE1,BC}}
extSol[x1_]:=Expand[(DSolve[BVP1,y[x],x]/.{x->x1})[[1,1,2]]];
p[x_]:=-2/x; q[x_]:=2/x^2; r[x_]:=x^3; extSol[x]
Do[sEq=Append[sEq, {-(1+h/2*p[X[[i+1]]])*Y[i-1]
 +(2+h^2*q[X[[i+1]]])*Y[i]-(1-h/2*p[X[[i+1]]])*Y[i+1]==
 -h^2*r[X[[i+1]]]}],
{i,1,n}]; sEqs=Flatten[sEq]
{Yvars=Table[Y[i],{i,1,n}], YDF=Solve[sEqs,Yvars],
 YDF1=Yvars/.YDF, YDFN=N[YDF1/.{Y[0]->alpha,Y[n+1]->beta},nD]}
Do[Print[PaddedForm[YDFN[[1,i]],{12,10}]," ",
 PaddedForm[N[extSol[X[[i+1]]],nD],{12,10}]," ",
```

```
 PaddedForm[YDFN[[1,i]]-N[extSol[X[[i+1]]],nD],{12,10}]],
{i,1,n}]; F1=Table[{X[[i+1]],YDFN[[1,i]]},{i,1,n}]
F11=Append[Append[F1,{a,alpha}],{b,beta}]
g1=Plot[Evaluate[extSol[x1]],{x1,a,b},PlotStyle->Hue[0.99]];
g2=ListPlot[F11,PlotStyle->{PointSize[.02],Hue[0.7]}];
Show[{g1,g2}]
```

▶ Nonlinear boundary value problems.

In addition to the nonlinear boundary value problem

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{20.3.5.5}$$

consider the initial value problem

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y'_x(a) = s, \tag{20.3.5.6}$$

where $a \le x \le b$. The real parameter s describes the initial slope of the solution curve.

Let $f(x, y, u)$ be a continuous function satisfying the Lipschitz condition with respect to y and u. Then, by the Picard–Lindelöf theorem, for each s there exists a unique solution $y(x, s)$ of the above initial value problem.

To find a solution of the nonlinear boundary value problem, we choose the parameter s such that $y(b, s) = \beta$; i.e., we have to solve the nonlinear equation $F(s) = y(b, s) - \beta = 0$ by applying one of the known numerical methods.

Example 20.45. *Nonlinear boundary value problem. Nonlinear shooting methods.*
For the nonlinear boundary value problem

$$y''_{xx} = -y^2, \qquad y(0) = 0, \quad y(2) = 1, \tag{20.3.5.7}$$

we find a numerical solution by applying the nonlinear shooting method (`ShootNL`) and plot the numerical results obtained with (`ShootNL`):

```
{a=0, b=2, alpha=0, beta=1, epsilon=N[1/50000,10], h1=1/10000}
{ODE1={D[y[x],{x,2}]+y[x]^2==0}, BC={y[a]==alpha,y[b]==beta}}
shootNL[s_]:=Module[{IC}, IC={y[0]==0,(D[y[x],x]/.{x->0})==s};
 IVP1=Flatten[{ODE1,IC}]; yN1=NDSolve[IVP1,y,{x,a,b}];
 yN=y[x]/.yN1];
{N[shootNL[1/10]/.{x->b}],N[shootNL[1/2]/.{x->b}]}
g1=Plot[beta,{x,1/2,1},PlotStyle->Green];
g2=Plot[N[shootNL[s]/.{x->b}],{s,1/2,1},PlotStyle->Red];
Show[{g1,g2},PlotRange->{0.,1.05}]
```

For various values of the parameter s, we have:

```
Do[R=N[shootNL[s]/.{x->b}]; If[Abs[N[R[[1]]-beta]]<epsilon,
 {sN=s,RN=R[[1]],Break[]}],{s,1/2,1,h1}];
Print["\n",PaddedForm[N[sN],{12,9}]," ",PaddedForm[RN,{12,9}]];
N[shootNL[sN]/.{x->b}]==N[beta]
Plot[Evaluate[N[shootNL[RN]]],{x,a,b},PlotStyle->{Red,Thickness[0.01]}]
{ICs={6/10,5/10,1,8/10,85/100,RN}, k=Length[ICs]}
Do[g[i]=Plot[Evaluate[N[shootNL[ICs[[i]]]]],{x,a,b},Frame->True,
 PlotStyle->{Hue[0.15*i],Thickness->0.01},PlotRange->{0.,1.2}],{i,1,k}];
Show[Table[g[i],{i,1,k}]]
```

Alternatively, we present the numerical solution obtained with the predefined function (`NDSolve`) as follows:

```
{yNB1=NDSolve[Flatten[{ODE1,BC}],y,{x,a,b}], yNB=Evaluate[y[x]/.yNB1]}
Plot[yNB,{x,a,b},PlotStyle->Red]
```

Let us apply the finite difference method for approximating the solution of the nonlinear boundary value problem (20.3.5.5). Just as in the linear case, we choose an equidistant grid $X_i = a + ih$ $(i = 0, 1, \ldots, N+1)$ on $[a, b]$ with step size $h = (b-a)/(N+1)$, where $X_0 = a$ and $X_{N+1} = b$. By approximating the nonlinear boundary value problem, we arrive at the system of nonlinear equations

$$\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} = f\left(X_i, Y_i, \frac{Y_{i+1} - Y_{i-1}}{2h}\right), \quad Y_0 = \alpha, \quad Y_{N+1} = \beta$$

for the approximate values Y_i of the exact solution $y(X_i)$. To solve this system of nonlinear equations, we can apply the Newton method.

Example 20.46. *Nonlinear boundary value problem. Approximations by finite differences.* For the nonlinear boundary value problem (considered in Example 20.45)

$$y''_{xx} = -y^2, \qquad y(0) = 0, \quad y(2) = 1, \tag{20.3.5.8}$$

we find the numerical solution (`sol`) by applying the predefined function (`NDSolve`):

```
nD=20; JacobianMatrix[f_List?VectorQ,x_List]:=
 Outer[D,f,x]/;Equal@@(Dimensions/@{f,x});
{a=0, b=2, n=8, h=N[(b-a)/(n+1),nD], alpha=0, beta=1, nMax=100,
 epsilon=N[10^(-4),nD], X=N[Table[a+i*h,{i,0,n+1}],nD]}
ODE1=D[y[x],{x,2}]==-y[x]^2
{BC={y[a]==alpha,y[b]==beta}, BVP1={ODE1,BC}}
sol=NDSolve[BVP1,y,{x,a,b}]
{solN=Evaluate[y[x]/.sol][[1]], N[solN/.{x->2},nD]}
subs0={y[X[[1]]]->alpha, y[X[[n+2]]]->beta}
seqEq1=Expand[Table[(y[X[[i+1]]]-2*y[X[[i]]]+y[X[[i-1]]])/(h^2)
 +(y[X[[i]]])^2,{i,2,n+1}]]
{seqEq=seqEq1/.subs0, seqVar=Variables[seqEq], nV=Length[seqVar],
 seqVar1=Table[Subscript[Z,i],{i,1,nV}]}
subs=Table[seqVar[[i]]->seqVar1[[i]],{i,1,nV}]
F[seqVar1_List]:=Table[seqEq[[i]]/.Table[seqVar[[i]]->seqVar1[[i]],
 {i,1,nV}],{i,1,nV}]; F[seqVar]
J=JacobianMatrix[F[seqVar],seqVar]
JInv[seqVar1_List?VectorQ]:=(Inverse[JacobianMatrix[F[seqVar],
 seqVar]])/.Table[seqVar[[i]]->seqVar1[[i]],{i,1,nV}];
FNewton[W_List?VectorQ]:=W-JInv[Table[W[[i]],{i,1,nV}]].Table[
 F[Table[W[[i]],{i,1,nV}]][[k]],{k,1,nV}];
Y=Table[Table[0,{k,1,nV}],{i,1,nMax}];
Y[[1]]=Table[N[10^(-10),nD],{i,1,nV}];
```

We find the numerical solution (`F1`) by applying the finite difference method:

```
Do[Y[[i]]=FNewton[Y[[i-1]]]; Print[i," ",Y[[i]]];
If[Max[N[Abs[Table[F[Y[[i]]][[m]],{m,1,nV}]],nD]]<epsilon,
 {iEnd=i,Break[]}],{i,2,7}];
```

```
{Print[iEnd],Print["\n The result is"]}
Do[Print[X[[k+1]]," ",Y[[iEnd]][[k]]," ",
 N[solN/.{x->X[[k+1]]},nD]," ",
 N[Abs[Y[[iEnd]][[k]]-N[solN/.{x->X[[k+1]]},nD]],nD]],{k,1,nV}];
F1=Table[{X[[k+1]],Y[[iEnd]][[k]]},{k,1,n}]
F11=Append[Append[F1,{a,alpha}],{b,beta}]
```

We compare the results and plot the numerical solutions (g1, g2) as follows:

```
g1=Plot[Evaluate[solN],{x,a,b},PlotStyle->Hue[0.99]];
g2=ListPlot[F11,PlotStyle->{PointSize[.02],Hue[0.7]}]; Show[{g1,g2}]
```

Note that the above numerical solution obtained with the aid of symbolic-numerical computations in Mathematica can be produced for small values of the partition parameter n; e.g., $n = 8$. For $n > 10$, we have written another version of the solution:

```
nD=20; {a=0, b=2, n=20, h=N[(b-a)/(n+1),nD], alpha=0, beta=1,
 nMax=100, epsilon=N[10^(-4),nD], X=N[Table[a+i*h,{i,0,n+1}],nD]}
{ODE1=D[y[x],{x,2}]==-y[x]^2,BC={y[a]==alpha,y[b]==beta}, BVP1={ODE1,BC}}
{sol=NDSolve[BVP1,y,{x,a,b}], solN=Evaluate[y[x]/.sol][[1]],
 N[solN/.{x->2},nD]}
subs0={y[X[[1]]]->alpha, y[X[[n+2]]]->beta}
seqEq1=Expand[Table[(y[X[[i+1]]]-2*y[X[[i]]]+y[X[[i-1]]])/(h^2)
 +(y[X[[i]]])^2,{i,2,n+1}]]
{seqEq=seqEq1/.subs0, seqVar=Variables[seqEq],
 subsInitial=Table[{seqVar[[i]],0.},{i,1,n}]}
eqs=Map[Thread[#1==0,Equal]&,seqEq]
sol1=FindRoot[eqs,subsInitial,WorkingPrecision->nD]
F1=Table[{sol1[[k,1,1]], sol1[[k,2]]},{k,1,n}]
F11=Append[Append[F1,{a,alpha}],{b,beta}]
g1=Plot[Evaluate[solN],{x,a,b},PlotStyle->Hue[0.99]];
g2=ListPlot[F11,PlotStyle->{PointSize[.02],Hue[0.7]}]; Show[{g1,g2}]
```

20.3.6 Eigenvalue Problems: Examples of Numerical Solutions

It is well known that eigenvalue problems play an important role in various methods for solving linear problems for PDEs (e.g., the method of separation of variables for PDEs). When we cannot find exact solutions of difficult eigenvalue problems, various approximation methods (e.g., the Rayleigh–Ritz method, the finite element method, the shooting method, the Galerkin method, difference methods, and iteration methods) can be applied for approximating the leading and most significant eigenvalues and eigenfunctions [see Akulenko and Nesterov (2005)].

In this section, we consider an elegant and useful approximation method, namely the Rayleigh–Ritz method (which is based on the variational approach) for determining the first few eigenvalues and eigenfunctions.

In particular, for a specific problem (in a given region) it is important to approximate the first (lowest) eigenvalue as accurately as possible.

In the following example, we apply the Rayleigh–Ritz method to the Sturm–Liouville eigenvalue problem (previously considered in Section 20.2.2) for approximating two lowest eigenvalues.

Example 20.47. *Eigenvalue problem. Rayleigh–Ritz method. Dirichlet boundary conditions.* For the Sturm–Liouville eigenvalue problem

$$y''_{xx} + \lambda y = 0, \qquad y(a) = 0, \quad y(b) = 0, \tag{20.3.6.1}$$

i.e., a homogeneous linear two-point boundary value problem with parameter λ and with the homogeneous Dirichlet boundary conditions, where $a \le x \le b$, $a = 0$, $b = \pi$, $p(x) = 1$, $w(x) = 1$, and $q(x) = 0$, we apply the Rayleigh–Ritz method for finding approximations to eigenvalues and eigenfunctions.

First, we define trial functions (`f1`, `f2`, `f3`), the Rayleigh quotients for trial functions (`appr1`, `appr2`), and the exact and approximate eigenfunctions with normalization (`eEF`, `aEF1`, `aEF2`, `aEF3`) as follows:

```
{a=0, b=Pi, n=1}
f1[x_]:=x*(b-x);
f2[x_]:=x*(b-x)+c*(x*(b-x))^2;
f3[x_]:=x*(b-x)*(c+x); f1[x]; f2[x]; f3[x];
appr1[f_,x_]:=Integrate[-f[x]*D[f[x],{x,2}],{x,a,b}]/Integrate[
 (f[x])^2,{x,a,b}];
appr2[f1_,f2_,x_]:=Integrate[f1[x]*f2[x],{x,a,b}];
eEF[x_,n_]:=Sin[n*x]/Sqrt[Integrate[(Sin[n*x])^2,{x,a,b}]];
aEF1[x_]:=f1[x]/Sqrt[Integrate[(f1[x])^2,{x,a,b}]];
aEF2[x_]:=(f2[x]/.sol2)/Sqrt[Integrate[(f2[x]/.sol2)^2,{x,a,b}]];
aEF3[x_]:=(f3[x]/.sol3)/Sqrt[Integrate[(f3[x]/.sol3)^2,{x,a,b}]];
```

Then we estimate the first eigenvalue and find the first and second approximations to the first eigenvalue (`EV11`, `EV12`) and to the second eigenvalue (`EV2`):

```
EV11=N[appr1[f1,x],15]
sol2=FindRoot[D[appr1[f2,x],c]==0//FullSimplify,{c,0}]
EV12=N[appr1[f2,x]/.sol2,15]
sol3=Solve[appr2[f1,f3,x]==0,c]
EV2=N[appr1[f3,x]/.sol3,15]
```

Finally, we compare the exact eigenfunction (`eEF`) and the corresponding approximations `aEF1`, `aEF2`, `aEF3` to eigenfunctions (see Figure 20.13):

```
SetOptions[Plot,AxesLabel->{"x","EF"}, PlotStyle->{Hue[0.8],
 Dashing[{0.02,0.03}],Thickness[0.01]},
 PlotLegends->{"exactEF","approxEF"}];
Plot[Evaluate[{eEF[x,1],aEF1[x]}],{x,a,b}]
Plot[Evaluate[{eEF[x,1],aEF2[x]}],{x,a,b}]
Plot[Evaluate[{eEF[x,2],-aEF3[x]}],{x,a,b}]
```

20.3.7 First-Order Systems of ODEs. Higher-Order ODEs. Numerical Solutions

► First-order systems of ODEs.

Consider the system of first-order ordinary differential equations with the initial conditions

$$(y_i)'_x = f_i\bigl(x, y_1, \dots, y_n\bigr), \qquad y_i(a) = y_{i0} \quad (i = 1, \dots, n).$$

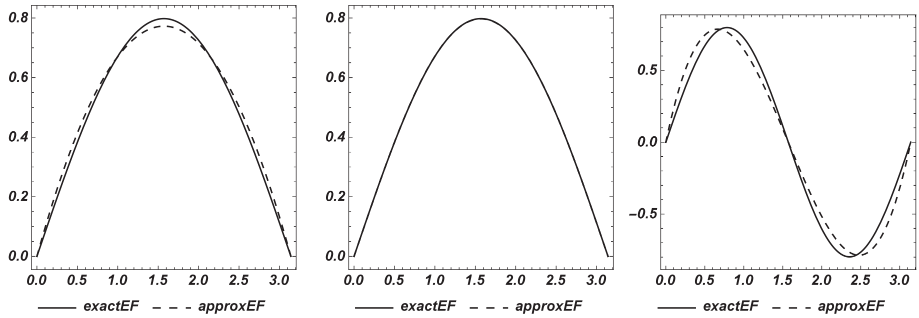


Figure 20.13: The first two eigenfunctions (exact) and their approximations (first, second, and first) for the Sturm–Liouville problem (20.3.6.1).

The unknown functions are $y_1(x), \ldots, y_n(x)$, and $a \le x \le b$.

To obtain numerical solutions, we can apply predefined functions or, alternatively, construct solutions step by step by applying one of the known numerical methods (developed for a single equation) to each equation in the system.

Let us numerically solve some first-order linear and nonlinear systems of ODEs.

Example 20.48. *Linear system. Cauchy problem. Exact, numerical, and graphical solutions.* For the first-order linear nonhomogeneous system with the initial conditions

$$u'_x = v, \quad v'_x = x - u - 2v, \qquad u(a) = \alpha, \quad v(a) = \beta, \tag{20.3.7.1}$$

where $a \le x \le b$, $a = 0$, $b = 2$, $\alpha = 1$, and $\beta = 1$, the exact solution (extSol)

$$\{\{u[x] \to e^{-x}(3 - 2e^x + 3x + e^x x), \qquad v[x] \to e^{-x}(e^x - 3x)\}\}$$

can be obtained as follows:

```
F1[x_,u_,v_]:=v; F2[x_,u_,v_]:=x-u-2*v; nD=10;
{n=10, a=0, b=2, h=N[(b-a)/n,nD], X=Table[a+i*h,{i,0,n}],
 RK41=Table[0,{i,0,n}], RK42=Table[0,{i,0,n}]}
{ODEsys={D[u[x],x]==v[x], D[v[x],x]==x-u[x]-2*v[x]},
 IC={u[0]==1,v[0]==1}, IVP1=Flatten[{ODEsys,IC}]}
extSol=Sort[DSolve[IVP1,{u[x],v[x]},x]]
uExt[x1_]:=extSol[[1,1,2]]/.{x->x1};
vExt[x1_]:=extSol[[1,2,2]]/.{x->x1};
```

Then we find a numerical solution (uF1, vF1) by applying the explicit fourth-order Runge–Kutta method:

```
{RK41[[1]]=1, RK42[[1]]=1}
Do[k1=h*F1[X[[i]],RK41[[i]],RK42[[i]]];
 m1=h*F2[X[[i]],RK41[[i]],RK42[[i]]]; k2=h*F1[X[[i]]+h/2,
 RK41[[i]]+k1/2,RK42[[i]]+m1/2];      m2=h*F2[X[[i]]+h/2,
 RK41[[i]]+k1/2,RK42[[i]]+m1/2];      k3=h*F1[X[[i]]+h/2,
 RK41[[i]]+k2/2,RK42[[i]]+m2/2];      m3=h*F2[X[[i]]+h/2,
 RK41[[i]]+k2/2,RK42[[i]]+m2/2];      k4=h*F1[X[[i]]+h,
 RK41[[i]]+k3,  RK42[[i]]+m3];        m4=h*F2[X[[i]]+h,
 RK41[[i]]+k3,  RK42[[i]]+m3];
```

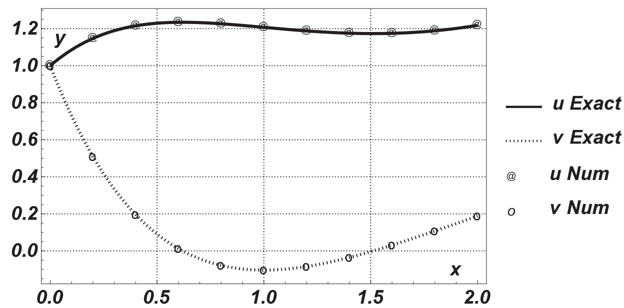


Figure 20.14: Exact and numerical solutions of the Cauchy problem (20.3.7.1) for the first-order linear system.

```
 RK41[[i+1]]=N[RK41[[i]]+1/6*(k1+2*k2+2*k3+k4),nD];
 RK42[[i+1]]=N[RK42[[i]]+1/6*(m1+2*m2+2*m3+m4),nD], {i,1,n}];
Do[Print[PaddedForm[i,3]," ",PaddedForm[X[[i+1]],{12,10}]," ",
 PaddedForm[RK41[[i+1]],{12,10}]," ",
 PaddedForm[N[uExt[X[[i+1]]],nD],{12,10}]," ",
 PaddedForm[RK42[[i+1]],{12,10}]," ",
 PaddedForm[N[vExt[X[[i+1]]],nD],{12,10}]], {i,0,n}];
uF1=Table[{X[[i+1]],RK41[[i+1]]},{i,0,n}]
vF1=Table[{X[[i+1]],RK42[[i+1]]},{i,0,n}]
```

Finally, we compare the results and plot the exact and numerical solutions (`uG1`, `vG1`, `uG2`, `vG2`) as follows:

```
uG1=Plot[uExt[x1],{x1,a,b},PlotStyle->Red,PlotRange->All];
vG1=Plot[vExt[x1],{x1,a,b},PlotStyle->Blue,PlotRange->All];
uG2=ListPlot[uF1,PlotStyle->{PointSize[.02],Hue[0.99]}];
vG2=ListPlot[vF1,PlotStyle->{PointSize[.02],Hue[0.7]}];
Show[{uG1,uG2,vG1,vG2}]
```

Example 20.49. *Nonlinear system. Cauchy problem. Numerical and graphical solutions.* For the first-order nonlinear system with the initial conditions

$$u'_x = uv, \quad v'_x = u + v, \qquad u(a) = \alpha, \quad v(a) = \beta, \tag{20.3.7.2}$$

where $a \le x \le b$, $a = 0$, $b = 1$, $\alpha = 1$, and $\beta = 1$, we obtain numerical and graphical solutions as follows:

```
SetOptions[Plot,ImageSize->300,AspectRatio->1,Frame->True,Axes->False];
{ODE={u'[x]==u[x]*v[x],v'[x]==u[x]+v[x]},IC={u[0]==1,v[0]==1}}
eq1=NDSolve[{ODE,IC},{u,v},{x,0,1}];
{solNu=eq1[[1,1,2]],solNv=eq1[[1,2,2]]}
Table[{x,solNu[x]},{x,0,1,0.1}]//TableForm
Table[{x,solNv[x]},{x,0,1,0.1}]//TableForm
g1=Plot[solNu[x],{x,0,1},PlotStyle->{Hue[0.8],Thickness[0.01]}];
g2=Plot[solNv[x],{x,0,1},PlotStyle->{Hue[0.9],Thickness[0.01]}];
g12=Show[g1,g2,AspectRatio->1,Frame->True,ImageSize->300];
GraphicsGrid[{{g1,g2},{g12}}]
```

► Higher-order ODEs.

For the ordinary differential equation of order n ($n > 1$) with n initial conditions

$$y_x^{(n)} = f(x, y, y'_x, \dots, y_x^{(n-1)}),$$
$$y(a) = y_0, \quad y'_x(a) = y_1, \quad \dots, \quad y_x^{(n-1)}(a) = y_{n-1},$$

we can always obtain solutions by transforming the equation to an equivalent system of n first-order differential equations and by applying an appropriate numerical method to this system of differential equations.

Example 20.50. *Van der Pol equation. Cauchy problem. Numerical and graphical solutions.*

For the van der Pol equation with the initial conditions

$$y''_{xx} + \mu(y^2 - 1)y'_x + y = 0, \qquad y(a) = \alpha, \quad y'_x(a) = \beta, \tag{20.3.7.3}$$

where $a \le x \le b$, $a = 0$, $b = 60$, $\alpha = 1$, and $\beta = 0$, we transform the second-order ODE to the equivalent system of two first-order differential equations (`sys2`)

$$[u'_x = v, \quad v'_x = -\mu(u^2 - 1)v - u], \qquad u(x_0) = 1, \quad v(x_0) = 0,$$

where $u = y(x)$, $v = y'_x$, and $x_0 = 0$. Then, by applying a classical numerical method (e.g., Euler's method) to this system of differential equations, we obtain a numerical solution (`solEuler`), and a graphical solution, a phase portrait of the solution, and a plot of $u(x)$ and $v(x)$ as follows:

```
SetOptions[ParametricPlot,PlotRange->All,AspectRatio->1];
{nD=10, a=0, b=60, mu=N[1/8,nD], alpha=1, beta=0}
{ODE1=D[y[x],{x,2}]+mu*(y[x]^2-1)*D[y[x],x]+y[x]==0,
 IC1={y[a]==alpha,y'[a]==beta}, IVP1={ODE1,IC1}//Flatten}
{sys2={D[u[x],x]==v[x], D[v[x],x]==-mu*(u[x]^2-1)*v[x]-u[x]},
 IC2={u[a]==alpha,v[a]==beta},IVP2={sys2,IC2}//Flatten}
solEuler=NDSolve[IVP2,{u,v},{x,a,b},StartingStepSize->0.01,
 Method->{"FixedStep",Method->"ExplicitEuler"}];
{solNu=solEuler[[1,1,2]], solNv=solEuler[[1,2,2]]}
Table[{x,solNu[x]},{x,a,b}]//TableForm;
Table[{x,solNv[x]},{x,a,b}]//TableForm;
{solNu[0], solNv[0], solNu[1.], solNv[1.]}
ParametricPlot[{solNu[x],solNv[x]},{x,a,b},
 PlotStyle->{Hue[0.8],Thickness[0.007]}]
ParametricPlot[{{x,solNu[x]},{x,solNv[x]}},{x,a,b},PlotStyle->{Hue[0.8],
 Dashing[{0.02,0.03}],Thickness[0.02]},PlotLegends->{"u[x]","v[x]"}]
```

20.3.8 Phase Plane Analysis for First-Order Autonomous Systems

In general, a first-order 2D autonomous system of ODEs has the form

$$u'_t = f(u, v), \qquad v'_t = g(u, v).$$

For a 2D system in $u(t)$ and $v(t)$, each initial condition (IC) (for producing one solution curve) can be specified in two forms, $[t_0, u(t_0), v(t_0)]$ or $[u(t_0) = u_0, v(t_0) = v_0]$.

The *phase portrait* of a first-order autonomous system of ODEs consists of solutions of the system in the *phase space*, where the solutions $u(t)$ and $v(t)$ are presented as parametric equations for the curve $v(u)$.

In Mathematica, the predefined functions `ParametricPlot` and `VectorPlot` can be applied to create the phase portrait.

Consider some examples for performing phase plane analysis for first-order autonomous systems of ODEs (linear and nonlinear).

► First-order linear autonomous systems.

Example 20.51. *First-order linear autonomous system of ODEs. Phase portrait animated.*
For the first-order linear autonomous system of ODEs with the initial conditions

$$u'_t = \pi u - 2v, \quad v'_t = 4u - v, \qquad u(0) = \tfrac{1}{2}C, \quad v(0) = \tfrac{1}{2}C,$$

where $C \in [-1, 0]$, we create an animation (with the aid of the predefined functions `ListAnimate` and `ParametricPlot`) of tracing the trajectory in the phase portrait as follows:

```
{g={}, n=50, subs={u[t]->u,v[t]->v}}
sys1={u'[t]==Pi*u[t]-2*v[t],v'[t]==4*u[t]+v[t]}
IC={u[0]==-1/2*C1,v[0]==1/2*C1}
{fu=sys1[[1,2]]/.subs, fv=sys1[[2,2]]/.subs}
vf=VectorPlot[{fu,fv},{u,-40,40},{v,-40,40},Frame->True,ColorFunction->
 Function[{u},Hue[u]],PlotRange->{{-40,40},{-40,40}}];
ivp1=Table[Flatten[{sys1,IC}],{C1,-1,0,1/n}];
sols=Table[NDSolve[ivp1[[i]],{u[t],v[t]},{t,0,Pi}],{i,1,n}];
l1=Table[{cu=u[t]/.sols[[i,1]],cv=v[t]/.sols[[i,1]]},{i,1,n}];
Do[g=Append[g,Show[vf,Evaluate[ParametricPlot[l1[[i]],{t,0,Pi},
 AspectRatio->1,PlotStyle->{Blue,Thickness[0.01]},PlotRange->
 {{-40,40},{-40,40}}]]]],{i,1,n}]; ListAnimate[g]
```

► First-order nonlinear autonomous systems.

Example 20.52. *First-order nonlinear autonomous system of ODEs. Phase portrait.*

Consider one application of first-order nonlinear autonomous systems of ODEs, namely, the dynamical system that describes the evolution of the amplitude and the slow phase of a fluid under the subharmonic resonance:

$$\begin{aligned} v'_t &= -\nu v + \varepsilon u \left[\delta + \tfrac{1}{4} - \tfrac{1}{2}\phi_2(u^2+v^2) + \tfrac{1}{4}\phi_4(u^2+v^2)^2\right], \\ u'_t &= -\nu u + \varepsilon v \left[-\delta + \tfrac{1}{4} + \tfrac{1}{2}\phi_2(u^2+v^2) - \tfrac{1}{4}\phi_4(u^2+v^2)^2\right]. \end{aligned} \tag{20.3.8.1}$$

This system has been obtained by performing averaging transformations [for more details, see Shingareva (1995)]. Here ν is the fluid viscosity, ε is the small parameter, ϕ_2 and ϕ_4 are the second and the fourth corrections to the nonlinear wave frequency, and δ is the off-resonance detuning.

Choosing the corresponding parameter values (for the six regions where the solution exists), we can obtain phase portraits. For example, here we create a phase portrait (presented in Figure 20.15) for one region (where the solution exists) with the corresponding parameter values as follows:

```
{delta=-1/2, phi2=1, phi4=1, nu=0.005, epsilon=0.1}
eq1=-nu*v[t]+epsilon*u[t]*(delta+1/4-phi2/2*(u[t]^2+v[t]^2)
 +phi4/4*(u[t]^2+v[t]^2)^2)
eq2=-nu*u[t]+epsilon*v[t]*(-delta+1/4+phi2/2*(u[t]^2+v[t]^2)
 -phi4/4*(u[t]^2+v[t]^2)^2)
IC={{0,1.1033},{0,-1.1033},{1.1055,0},{-1.1055,0},{0,1.613},
 {0,-1.613}}
n=Length[IC]
Do[{sys[i]={v'[t]==eq1,u'[t]==eq2,v[0]==IC[[i,1]],u[0]==IC[[i,2]]}};
 sols=NDSolve[sys[i],{u,v},{t,-48,400}]; cv=v/.sols[[1]];
 cu=u/.sols[[1]]; c[i]=ParametricPlot[Evaluate[{cu[t],cv[t]}],
 {t,-48,400},PlotStyle->{Hue[0.1*i+0.2],Thickness[.001]}];},
{i,1,n}]
{fv=eq1/.{v[t]->v,u[t]->u}, fu=eq2/.{v[t]->v,u[t]->u}}
fd=VectorPlot[{fu,fv},{u,-2.2,2.2},{v,-2.2,2.2},Frame->True,
 ColorFunction->Function[{u},Hue[u]]]; Show[fd,Table[c[i],{i,1,6}]]
```

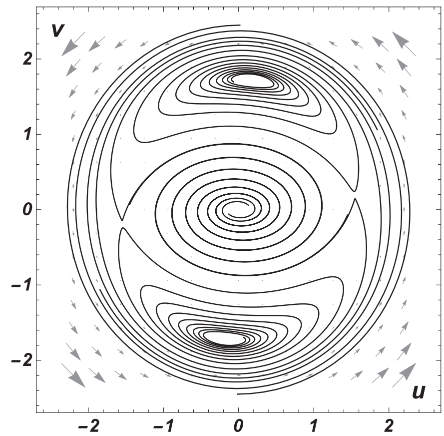


Figure 20.15: Phase portrait of the first-order nonlinear autonomous system (20.3.8.1).

20.3.9 Numerical-Analytical Solutions

In this section, we show the very helpful role of computer algebra systems for analytical derivation of numerical methods, for combining the analytical approach (and methods) with numerical computations (and methods) for solving mathematical problems of various types, and for comparing exact, approximate analytical, and numerical solutions.

Consider some examples of an analytical-numerical approach (variational and projection methods) for constructing exact and approximate analytical and numerical solutions of two-point boundary value problems.

► Variational methods. The Ritz method.

Consider a two-point linear boundary value problem, i.e., a second-order linear nonhomogeneous ODE with the mixed boundary conditions,

$$-[p(x)y'_x]'_x + q(x)y = f(x), \qquad y(a) = \alpha, \quad y'_x(b) = \beta,$$

where $a \le x \le b$. We assume that $p(x)$ is a continuously differentiable function, $q(x)$ and $f(x)$ are continuous functions, $p(x) \ge 0$, and $q(x) \ge 0$. Then there exists a unique twice continuously differentiable solution $y(x)$ if and only if $y(x)$ is the unique function minimizing the functional

$$J[y] = \int_a^b \Big\{ p(x)(y'_x)^2 + q(x)y^2 - 2f(x)y \Big\}\, dx.$$

In the Ritz method (which belongs in *variational methods*), by introducing an approximate analytical solution $y(x)$ in the form of a linear combination of the basis functions $\phi_i(x)$,

$$y(x) = C_1\phi_1(x) + \ldots + C_N\phi_N(x),$$

we obtain a quadratic form in the unknown coefficients C_i. By minimizing this quadratic form, we determine the coefficients C_i and the approximate analytical solution $y(x)$.

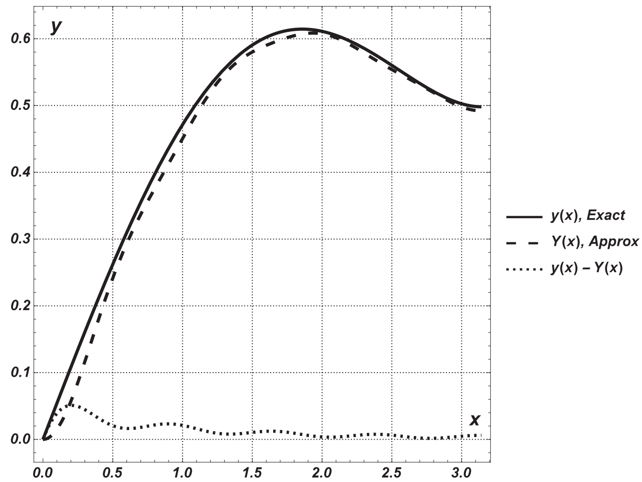


Figure 20.16: Exact solution, approximate analytical solution, and their difference for the two-point linear boundary value problem with mixed boundary conditions (20.3.9.1).

Example 20.53. *Linear boundary value problem. Mixed boundary conditions. The Ritz method.* For the two-point linear boundary value problem with the mixed boundary conditions

$$y''_{xx} - y = \sin(x), \qquad y(a) = \alpha, \quad y'_x(b) = \beta, \tag{20.3.9.1}$$

where $a \le x \le b$, $a = 0$, $b = \pi$, $\alpha = 0$, and $\beta = 0$, we find the exact solution $y(x)$ (`SolEx`) and the approximate analytical solution $Y(x)$ (`SolApp`) by the Ritz method, compare the results, and plot the solutions `SolEx` and `SolApp` and their difference `SolEx-SolApp` (see Figure 20.16) as follows:

```
nD=10; {a=0,b=Pi,n=8,h=N[(b-a)/n,nD], alpha=0, beta=0}
f[x_]:=Sin[x]; p[x_]:=1; q[x_]:=1;
{ODE1={-p[x]*D[y[x],{x,2}]+q[x]*y[x]-f[x]==0},
 BCs={y[a]==alpha,(D[y[x],x]/.{x->b})==beta},BVP1={ODE1,BCs}}
SolEx[x1_]:=FullSimplify[ExpToTrig[(DSolve[BVP1,y[x],x]/.
 {x->x1})[[1,1,2]]]]; SolEx[x]
Do[phi[i]=Cos[i*X]-1,{i,1,n}];
Y[X1_]:=Sum[c[i]*phi[i],{i,1,n}]/.{X->X1}; Y[X]
{J=Integrate[Expand[p[x]*D[Y[X],X]^2+q[x]*Y[X]^2-2*f[X]*Y[X]],{X,a,b}],
 V=D[J,{Variables[J]}]}
CL=Flatten[Solve[Map[Thread[#1==0,Equal]&,V],Variables[J]]]
SolApp[X1_]:=(Y[X]/.CL)/.{X->X1}; SolApp[x]
g1=Plot[Evaluate[SolEx[x]],{x,a,b},PlotStyle->Red];
g2=Plot[Evaluate[SolApp[x]],{x,a,b},PlotStyle->Blue];
g3=Plot[Evaluate[SolEx[x]-SolApp[x]],{x,a,b},PlotStyle->Green];
Show[{g1,g2,g3},PlotRange->{-0.1,0.6}]
```

► Projection methods. The Galerkin method.

Consider the same linear boundary value problem as in the previous section written in the operator form

$$L[y] + f(x) = 0, \qquad y(a) = \alpha, \quad y'_x(b) = \beta,$$

where $y = y(x)$, $a \le x \le b$, and $L[y] = [p(x)y'_x]'_x + q(x)y$ is a linear differential operator. We assume that all the functions occurring in the problem are square integrable. Note that not every differential equation admits the minimization of a functional, and we consider a more powerful and general method for solving differential equations, the *Galerkin method* that belongs to the *projection methods*; i.e., the equation to be approximated is projected onto a finite-dimensional function subspace.

In the Galerkin method (as in the Ritz method), we also introduce the approximate analytical solution $y(x)$ in the form of a linear combination of the basis functions $\phi_i(x)$,

$$y(x) = C_1\phi_1(x) + \ldots + C_N\phi_N(x),$$

and we choose the unknown coefficients C_i such that the residual

$$r(x) = L[y(x)] + f(x)$$

is orthogonal to the space spanned by the basis functions $\phi_i(x)$,

$$\int_a^b r(x)\phi_i(x)\,dx = 0, \quad i = 1, \ldots, N;$$

i.e., the Galerkin equations are reduced to the solution of a system of linear equations.

Example 20.54. *Linear boundary value problem. Mixed boundary conditions. Galerkin method.* For the two-point linear boundary value problem with the mixed boundary conditions

$$y''_{xx} + \sin(x) = y, \qquad y(a) = \alpha, \quad y'_x(b) = \beta,$$

where $a \le x \le b$, $a = 0$, $b = \pi$, $\alpha = 0$, and $\beta = 0$, we find the exact solution (`SolEx`) and the approximate analytical solution (`SolApp`) by the Galerkin method, compare the results, and plot the solutions (`SolEx`, `SolApp`) and their difference (`SolApp-SolEx`) as follows:

```
nD=10; f[x_]:=-Sin[x]; p[x_]:=x; q[x_]:=x^2;
ODE1[x1_]:=p[x]*D[y[x],{x,2}]-q[x]*y[x]-f[x]==0/.{x->x1};
{a=0,b=Pi,n=8,h=N[(b-a)/n,nD],alpha=0,beta=0}
{BCs={y[a]==alpha,(D[y[x],x]/.{x->b})==beta},BVP1={ODE1[x],BCs}}
SolEx[x1_]:=FullSimplify[ExpToTrig[(DSolve[BVP1,y[x],x]/.
 {x->x1})[[1,1,2]]]]; SolEx[x]
Do[phi[i]=Cos[i*t]-1,{i,1,n}];
Y[t1_]:=Sum[c[i]*phi[i],{i,1,n}]/.{t->t1}; Y[t]
Eq1=Expand[ODE1[t]/.{y[t]->Y[t],D[y[t],{t,2}]->D[Y[t],{t,2}]}]
Eqs=Table[Integrate[Expand[Eq1[[1]]*phi[i]],{t,a,b}],{i,1,n}]
CL=Flatten[Solve[Map[Thread[#1==0,Equal]&,Eqs],Variables[Eqs]]]
SolApp[t1_]:=(Y[t]/.CL)/.{t->t1}; Expand[N[SolApp[t],nD]]
g1=Plot[Evaluate[SolEx[t]],{t,a,b},PlotStyle->Red];
g2=Plot[Evaluate[SolApp[t]],{t,a,b},PlotStyle->Blue];
g3=Plot[Evaluate[SolApp[t]-SolEx[t]],{t,a,b},PlotStyle->Green];
Show[{g1,g2,g3},PlotRange->{-0.1,0.6}]
```

⊙ *Literature for Section .3*: H. F. Trotter (1959), G. M. Murphy (1960), R. Bulirsch and J. Stoer (1964), W. B. Gragg (1965), G. Strang (1968), G. I. Marchuk (1968), T. J. Dekker (1969), R. P. Brent (1973), L. R. Petzold (1983), G. Bader and P. Deuflhard (1983), L. F. Shampine and L. S. Baca (1983), P. Deuflhard (1985), P. Deuflhard, E. Hairer and J. Zugck (1987), L. F. Shampine (1987), E. Hairer and Ch. Lubich (1988), P. Bogacki and L. F. Shampine (1989), C. Lubich (1989), J. C. Butcher (1990), K. Gustafsson (1991), J. Candy

and R. Rozmus (1991), U. Ascher and L. Petzold (1991), R. I. McLachlan and P. Atela (1992), E. Hairer, S. P. Norsett, and G. Wanner (1993), L. F. Shampine (1994), L. Dieci, R. D. Russel and E. S. Van Vleck (1994), J. M. Sanz-Serna and M. P. Calvo (1994), I. K. Shingareva (1995), E. Hairer and G. Wanner (1996), G. H. Golub and C. F. van Loan (1996), S. D. Cohen and A. C. Hindmarsh (1996), L. Dieci and E. S. Van Vleck (1999), E. Hairer (2000), N. Del Buono and L. Lopez (2001), E. Hairer, Ch. Lubich and G. Wanner (2002), M. Sofroniou and G. Spaletta (2004, 2005, 2006), L. D. Akulenko and S. V. Nesterov (2005).

Chapter 21

Symbolic and Numerical Solutions of ODEs with MATLAB

21.1 Introduction

21.1.1 Preliminary Remarks

In the previous two chapters, we paid special attention to analytical solutions of ordinary differential equations and systems owing to the availability of the computer algebra systems Maple and Mathematica in modern mathematics.

Frequently, the functions and data in ODE problems are defined at discrete points and equations are too complicated, so it is not possible to construct analytical solutions. Therefore, we have to study and develop numerical approximation methods for ordinary differential equations [e.g., see Gear (1971), Shampine and Gordon (1975), Forsythe et al. (1977), Conte and de Boor (1980), Fox and Mayers (1987), Kahaner et al. (1989), Shampine (1994), Ascher et al. (1995), Shampine and Reichelt (1997), Shampine et al. (2003), Lee and Schiesser (2004)].

Following the most important ideas and methods, we apply and develop numerical methods to obtain numerical and graphical solutions for studying ordinary differential equations.

Nowadays, for this purpose one can use computers and supercomputers extensively applying convenient and powerful computational software, e.g., an interactive programming environment for scientific computing, MATLAB®, which provides integrated symbolic and numerical computation and graphics visualization in a high-level programming language. Additionally, MATLAB's excellent graphics capabilities can help one understand the results and analyze the solution properties.

In this chapter, we turn our attention to numerical methods for solving ordinary differential equations using MATLAB. Since numerical and analytical methods are complementary techniques for investigating solutions of differential equations, we will also consider some essential analytical tools provided in MATLAB.

MATLAB has an extensive library of predefined functions for solving ordinary differential equations. We compute symbolic and numerical solutions using MATLAB's predefined functions (which implement known methods for solving ordinary differential equations) and develop new MATLAB procedures for constructing symbolic and numerical solutions.

Remark 21.1. The numerical methods embedded in MATLAB can solve only first-order ODEs or systems of first-order ODEs. To obtain solutions of nth-order ODEs (where the order is $n > 1$) by applying predefined functions, we have to rewrite higher-order ODEs as an equivalent system of first-order ODEs.

21.1.2 Brief Introduction to MATLAB

► MATLAB's conventions and terminology.

In this chapter, we use the following conventions introduced in MATLAB:

- `Cn` ($n = 1, 2, \ldots$), for arbitrary constants
- the letter `D`, the differential operator (should not be used for symbolic variables)
- `Dc`, for a dependent variable (in differential equations), where `c` is any character
- the letter `t`, the independent variable (by default) for the predefined function `dsolve`

Also we introduce the following notation for the MATLAB solutions:

- `Eqn`, for equations ($n = 1, 2, \ldots$)
- `ODEn`, for ODEs
- `Soln`, for solutions
- `Exprn`, for expressions
- `Strn`, for string expressions
- `ODESysn`, for systems of ODEs
- `ICn`, `BCn`, for initial and boundary conditions
- `IVPn`, `BVPn`, for initial and boundary value problems
- `Ln`, for lists of expressions
- `Gn`, for graphs of solutions
- `ops`, `val`, for various optional arguments in predefined functions and their values
- `vars`, for independent variables
- `funcs`, for dependent variables (indeterminate functions)

► Basic description.

MATLAB (short for "matrix laboratory") is not a general purpose programming language as Maple and Mathematica. MATLAB is an interactive programming environment that provides powerful high-performance numerical computing, excellent graphics visualization, symbolic computing capabilities, and capabilities for writing new software programs using a high-level programming language.

The *Symbolic Math Toolbox* (Ver. $\geq$ 4.9), based on the muPAD symbolic kernel, provides symbolic computations and variable-precision arithmetic. Earlier versions of the *Symbolic Math Toolbox* are based on the Maple symbolic kernel.

Simulink (short for "simulation and link"), also included in MATLAB, offers modeling, simulation, and analysis of dynamical systems (e.g., signal processing, control, communications, etc.) under a *graphical user interface* (GUI) environment.

The first concept of MATLAB and its original version (written in Fortran) was developed by Prof. Cleve Moler at the University of New Mexico in the late 1970s to provide his students with a simple interactive access (without having to learn Fortran) to LINPACK and EISPACK software.* Over the next several years, this original version of MATLAB had spread within the applied mathematics community. In early 1983, Jack Little (an engineer), together with Cleve Moler and Steve Bangert, developed a professional version of MATLAB (written in C and integrated with graphics). The company MathWorks was created in 1984 and headquartered in Natick, Massachusetts, to continue its development.

► Most important features.

The most important features of MATLAB are as follows:
- interactive user interface;
- a combination of comprehensive mathematical and graphics functions with a powerful high-level language in an easy-to-use environment;
- fast numerical computation and visualization, especially for performing matrix operations [e.g., see Higham (2008)];
- easy usability and great flexibility in data manipulation;
- symbolic computing capabilities via the Symbolic Math Toolbox (Ver. < 4.9 or Ver. $\geq$ 4.9), based on the Maple or muPAD symbolic kernel, respectively;
- the basic data element is an array that does not require dimensioning;
- a large library of functions for a wide range of applications;
- it is easy to incorporate new user-defined capabilities (toolboxes consisted of M-files and written for specific applications);
- understandable and available for almost all operating systems;
- powerful programming language, intuitive and concise syntax, and easy debugging;
- Simulink, as an integral part of MATLAB, provides modeling, simulation, and analysis of dynamical systems;
- free resources, such as MathWorks Web Site (`www.mathworks.com`), MathWorks Education Web Site (`www.mathworks.com/education`), MATLAB newsgroup (`comp.soft-sys.matlab`), etc.

► Basic parts.

MATLAB consists of five parts:
- The *Development Environment*, a set of tools that facilitate using MATLAB functions and files (e.g., graphical user interfaces and the workspace).
- The *Mathematical Function Library*, a vast collection of computational algorithms.
- The *MATLAB language*, a high-level matrix/array language (with flow control statements, functions, data structures, input/output, and object-oriented programming features).
- The *MATLAB graphics system*, which includes high-level functions (for 2D/3D data visualization, image processing, animation, etc.) and low-level functions (for fully customizing the graphics appearance and constructing complete graphical user interfaces).
- The *Application Program Interface* (API), a library for writing C and Fortran programs that interact with MATLAB.

*LINPACK and EISPACK is a collection of Fortran subroutines, developed by Cleve Moler and his several colleagues, for solving linear equations and eigenvalue problems, respectively.

► **Basic concepts.**

The *prompt symbol* >> indicates where to type a MATLAB command; typing a statement and pressing `Return` or `Enter` at the end starts the evaluation of the command, displays the result, and inserts a new prompt; the semicolon (`;`) symbol at the end of the command tells MATLAB to evaluate the command but not display any result.

In MATLAB, the cursor cannot be moved to the desired line (unlike Maple and Mathematica) but, for simple problems, corrections can be made by pressing the `up` or `down arrow` key to scroll through the list of (recently used) functions and then the `left` or `right arrow` key to change the text. Also, corrections can be made using `copy/paste` of the previous lines located in the `Command Window` or `Command History`.

The *previous result* (during a session) can be referred to with the variable `ans` (the last result). MATLAB prints the *answer* and assigns the value to `ans`, which can be used for further calculations.

MATLAB has many forms of help: a complete *online help system* with tutorials and reference information for all functions; the command-line help system, which can be accessed by using the `Help` menu, pressing `F1`, selecting `Help->Demos`, or entering `Help` and selecting `Functions->Alphabetical List` or `Index`, `Search`, `MATLAB->Mathematics`; or by typing `helpbrowser`, `lookfor` (e.g., `lookfor plot`) or `help FunctionName`, `doc FunctionName`, etc.

In MATLAB (Ver. 7), a new feature for correctly typing function names has been added. One can type only the first few letters of the function and then press the `TAB` key (to see all available functions and complete typing the function).

MATLAB *desktop* appears, containing tools (graphical user interfaces) for managing files, variables, and applications. The default configuration of desktop includes various tools, e.g., `Command Window`, `Command History`, `Workspace`, `Find Files`, `Current Directory` (for more details, see `demo MATLAB desktop`), etc. One can modify the arrangement of tools and documents.

For a new problem, it is best to begin with the statement `clear all` for cleaning all variables from MATLAB's memory. All examples and problems in the book assume that they begin with `clear all`.

A MATLAB program can be typed at the prompt >> or, alternatively (e.g., for more complicated problems), by creating an *M-file* (with `.m` extension) using MATLAB *editor* (or using another text editor). MATLAB editor is invoked by typing `edit` at the prompt.

M-files are files that contain code in the MATLAB language. There are two kinds of M-files: *script M-files* (which do not accept input arguments or return output data) and *function M-files* (which can accept input arguments and return output arguments).

In the process of working with various M-files, it is necessary to define the path, which can be done by selecting `File->Set Path->Add Folder` or via the `cd` function.

The *structure* of a MATLAB program or source code is as follows: the *main program* or *script* and the necessary *user-defined functions*. The execution starts by typing the file name of the main program.

Incorrect response. If you get no response or an incorrect response, you may have entered or executed the function incorrectly. Correct the function or interrupt the computation by entering debug mode and setting breakpoints: select the following on the Desktop menu:

`Debug->Open M-files when Debugging`
`Debug->Stop if Errors/Warnings`

Also, one can detect erroneous or unexpected behavior in a program with the aid of MATLAB functions, e.g., `break`, `warning`, and `error`.

Palettes can be used, e.g., for building or editing graphs (`Figure Palette`), displaying the names of the GUI components (`Component Palette`), etc.

MATLAB *graphical user interface development environment* (GUIDE) provides a set of tools for creating graphical user interfaces (GUIs). These tools greatly simplify the construction of GUIs, e.g., layout the GUI components (panels, buttons, menus, etc.) and program the GUI.

MATLAB consists of a family of add-on *toolboxes*, which are collections of functions (M-files) and extend the MATLAB environment to solve particular classes of problems.

The toolboxes can be *standard* or *specialized* (see `Contents` in `Help`). Nowadays, many specialized toolboxes are available. MATLAB can be augmented by a number of toolboxes consisting of M-files and written for specific applications.

21.1.3 MATLAB Language

MATLAB language is a high-level procedural dynamic and imperative programming language (similar to Fortran 77, C, and C++), with powerful matrix/array operations, control statements, functions, data structures, input/output, and object-oriented programming features. In addition, it is an interpreted language, similar to Maple and Mathematica [e.g., see Shingareva and Lizárraga-Celaya (2009)]; i.e., the instructions are translated into machine language and executed in real time (one at a time). MATLAB language allows programming-in-the-small (coding or creating programs for performing small-scale tasks) and programming-in-the-large (creating complete large and complex application programs). It supports a large collection of data structures or MATLAB classes and operations among these classes.

In linear algebra, there exist two types of operations with vectors/matrices: operations based on the mathematical structure of vector spaces and element-by-element operations on vectors/matrices as in data arrays. This difference can be made in the name of the operation or the name of the data structure. In MATLAB, separate operations are defined (for matrix and array manipulation), but the data structures `array` and `vector/matrix` are the same. But, for example, in Maple the situation is opposite: the operations are the same, but the data structures are different.

Arithmetic operators: scalar operators (`+ - * / ^`), matrix multiplication/power (`* ^`), array multiplication/power (`.* .^`), matrix left/right division (`\ /`), and array division (`./`).

Logical operators: and (`&`), or (`|`), exclusive or (`xor`), and not (`~`).

Relational operators: less/greater than (< >), less/greater than or equal to (<= >=), and equal/not equal (== ~=).

A variable name is a character string of letters, digits, and underscores such that it begins with a letter and its length is bounded by `N=namelengthmax` (e.g., $N = 63$). Punctuation marks are not allowed (see `genvarname` function). Variable declaration is not necessary in MATLAB, but all variables must be given initial values; e.g., `a12_new=9`. A variable can change in the calculation process, e.g., from integer to real (and vice versa).

MATLAB is case sensitive, and there is a difference between lowercase and uppercase letters, e.g., `pi` and `Pi`.

Various reserved keywords, symbols, names, and functions, for example, reserved keywords and function names, cannot be used as variable names (see `isvarname`, `which -all`, `isreserved`, `iskeyword`).

A string variable is enclosed by single quotes and belongs to the `char` class (e.g., `x='string'`), and the function `sin(x)` is invalid. Strings can be used with converting, formatting, and parsing functions (e.g., see `cellstr`, `char`, `sprintf`, `fprintf`, `strfind`, `findstr`).

MATLAB provides three basic types of variables: *local variables*, *global variables*, and *persistent variables*.

The operator "set equal to" (=). A variable in MATLAB (in contrast to Maple and Mathematica) cannot be "free" (with no assigned value) and must be assigned any initial value by the operator "set equal to" (=).

The difference between the operators "set equal to" (=) and "equal" (==) is that the operator `var=val` is used to assign `val` to the variable `var`, while `val1==val2` compares two values; e.g., `A=3; B=3; A==B`.

Statements are input instructions from the keyboard that are executed by MATLAB (e.g., `for i=1:N s=s+i*2; end`). A MATLAB statement may begin at any position in a line and may continue indefinitely in the same line, or may continue in the next line, by typing by three dots (`...`) at the end of the current line. White spaces between words in a statement are ignored; a number cannot be split into two pieces separated by a space.

The statement separator semicolon (;). The result of a statement followed with a semicolon (;) will not be displayed. If the semicolon is omitted, the results will be printed on the screen; e.g., `x=-pi:pi/3:pi;` and `x=-pi:pi/3:pi`.

Multiple statements in a line: two or more statements may be written in the same line if they are separated with semicolons.

Comments can be included with the percentage sign `%` and all characters following it up to the end of a line. Comments at the start of a code have a special significance: they are used by MATLAB to provide the entry for the help manual for a particular script. The block comment operators, `%{ %}`, can be used for writing comments that require more than one line.

An expression is a valid statement and is formed as a combination of numbers, variables, operators, and functions. The arithmetic operators have different precedences (increasing precedence `+ - * / ^`). Precedence is altered by parentheses (expressions within parentheses are evaluated before expressions outside parentheses).

A Boolean or *logical expression* is formed with *logical* and *relational operators*; e.g., `x>0`. Logical expressions are used in `if`, `switch`, and `while` statements. The logical values, *true* and *false*, are represented by numerical values, *1* and *0*, respectively.

A regular expression is a string of characters that defines a *pattern* (for details, see `help pattern`). For example, `'Math?e\w*'`. Regular and dynamic expressions can be used to search text for a group of words that matches the pattern (e.g., for parsing or replacing a subset of characters within text).

MATLAB is sensitive to types of brackets and quotes (for details, see `help paren`, `help punct`).

Types of brackets:

Square brackets, `[ ]`, for constructing vectors and matrices, for example, `A1=[1 2 3]`, `A2=[1, 2, 3]`, `A3=[1, 2; 4,5]`. For multiple assignment statements, for example `A4=[1,5;2,6]`, `[L,U]=lu(A4)`.

Parentheses, `( )`, for grouping expressions, `(5+9)*3`, for delimiting the arguments of functions, `sin(5)`, for vector and matrix elements, `A1(2)`, `A3(1,1)`, `A2([1 2]);` in logical expressions, `A1(A1>2)`.

Curly brackets, { }, for working with cell arrays; e.g., `C1={int8(3) 2.59 'A'}`, `C1{1}`, `X(2,1)={[1 3; 4 6]}`.

Dot-parentheses, `.( )`, for working with a structure via a dynamic field name; e.g., `S.F1=1; S.F2=2; F='F1'; val1=S.(F)`.

Quotes:

Forward-quotes, (`' '`), for creating strings, for example, `T='the name=7;' k=5; disp('the value of k is'); disp(k)`,

A single forward-quote and dot single forward-quote, (`' .'`), for matrix transposition (the complex conjugate/nonconjugate transpose of a matrix), `A1=[1+i,i;-i,1-i]; A1'; A.'`.

Types of numbers. Numbers are stored (by default) as double-precision floating point (class `double`). To operate with integers, it is necessary to convert from double to the integer type (e.g., classes `int8`, `int16`, `int32`), `x=int16(12.3)`, `str='MATLAB'`, `int8(str)`. Mathematical operations that involve integers and floating-point numbers result in an integer data type. Real numbers can be stored as *double-precision floating point* (by default) or *single-precision floating point*; e.g., `x1=3.25`, `x2=single(x1)`, `x3=double(x2)` (for details, see `whos`, `isfloat`, `class`). Complex numbers can be created as `z1=1+2*i`, `z2=complex(1,2)`. Rational numbers can be formed by setting the `format` to `rational`; e.g., `x=3.25; format rational x format`. To check the current format setting, we type `get(0,'format')`.

Predefined constants: symbols for definitions of commonly used mathematical constants; e.g., `true`, `false`, `pi`, `i`, `j`, `Inf`, `inf`, `NaN` (not a number), `exp(1)`, the Euler constant γ, `-psi(1)`, `eps`.

In MATLAB, there are *predefined functions* and *user-defined functions*. Predefined functions are divided into *built-in functions* and *library functions*:

- *Built-in functions* are precompiled executable programs and run much more efficiently (see `help elfun`, `help elmat`).
- *Library functions* are stored as M-files (in the libraries or toolboxes), which are available in readable form (see `which`, `type`, `exist`). MATLAB can be complemented with locally user-developed M-files and toolboxes.

Many functions are overloaded (i.e., have an additional implementation of an existing function) so that they handle different classes (e.g., `which -all plot`).

Numerous *special functions* are defined; e.g., `help bessel`, `help specfun`.

User-defined functions can be created as M-files (see `help 'function'`) or as anonymous functions.

A *User-defined function* written in an M-file (with the extension `.m`) must contain only one function. It is best to have the same name for the *function name* and the *file name*. The process of creating functions is as follows: create and save an M-file using a text editor, then call the function in the main program (or in `Command Window`).

Functions written in M-files have the following forms:
`function OArg=FunName(IArg); FunBody; end`, or
`function [OArg1,OArg2,...]=FunName(IArg1,IArg2,...); FunBody; end`
where `OArg` and `IArg` are the output arguments and the input arguments, respectively. For example, the function $y = \sin x$ is defined as follows:

```
function f=SinFun(x); f=sin(x); end
```

Evaluation of functions: `FunName(Args)`.
For example, for the sine function we have `cd('c:/mypath'); SinFun(pi/2); type SinFun`.

Anonymous functions create simple functions without storing functions to files. Anonymous functions can be constructed either in the `Command Window` or in any function or script; e.g., the function $f(x) = \sin x$ is defined as `f=@(x) sin(x); f(pi/2)`.

A *function handle*, `@`, is one of the standard MATLAB data types that provides calling functions indirectly, e.g., to call a subfunction when outside the file that defines that function (see class `function_handle`).

Nested functions are allowed in MATLAB; i.e., one or more functions or *subfunctions* within another function can be defined in MATLAB. In this case, the `end` statements are necessary.

MATLAB language has the following *control structures*: the selection structures `if`, `switch`, `try` and the repetition structures `for`, `while`.

MATLAB does not have a *module system* in the traditional form: it has a system based on storing *scripts* and *functions* in M-files and placing them into *directories* (see `cd` function for changing the current directory, `help ..`).

MATLAB *data structures* or *classes*, vectors, matrices, and arrays, are used to represent more complicated data. There are 15 fundamental classes, which are in the form of a matrix or array: `double`, `single`, `int8`, `uint8`, `int16`, `uint16`, `int32`, `uint32`, `int64`, `uint64`, `char`, `logical`, `function_handle`, `struct`, and `cell`. The numerical values are represented (by default) as floating-point double precision (`float double`). One can construct various *composite data types* (e.g., sequences, lists, sets, tables, etc.) using the classes `struct` and `cell`.

Vectors are ordered lists of numbers separated by commas or spaces inside `[ ]`; no dimensioning is required. But *vector and array indices* can only be positive and nonzero. The notation `X=[1:0.1:9]` stands for a vector of numbers from 1 to 9 in increments of 0.1 (see `help colon`).

Matrices are rectangular arrays of numbers (row/column vectors are special cases of matrices).

⊙ *Literature for Section .1*: C. W. Gear (1971), L. F. Shampine and M. K. Gordon (1975), G. E. Forsythe, M. A. Malcolm, and C. B. Moler (1977), S. D. Conte and C. de Boor (1980), L. Fox and D. F. Mayers (1987), D. Kahaner, C. B. Moler, and S. Nash (1989), L. F. Shampine (1994), U. M. Ascher, R. M. M. Mattheij, and R. D. Russell (1995), L. F. Shampine and M. W. Reichelt (1997), L. F. Shampine, I. Gladwell, and S. Thompson (2003), H. J. Lee and W. E. Schiesser (2004), N. J. Higham (2008), I. K. Shingareva and C. Lizárraga-Celaya (2009).

21.2 Analytical Solutions and Their Visualizations

21.2.1 Analytical Solutions in Terms of Predefined Functions

The `Symbolic Toolbox` provides various predefined functions for solving, plotting, and manipulating symbolic mathematical equations. If we solve ordinary differential equations, we can obtain explicit or implicit exact solutions [e.g., see Murphy (1960), Kamke (1977), Zwillinger (1997), Polyanin and Manzhirov (2007)]. Consider the most relevant related functions for finding analytical solutions of a given ODE problem.

```
syms y(x); Sol1=dsolve(ODE)        Sol2=dsolve('ODE','var')
Sol3=dsolve(ODE,ICs)               Sol4=dsolve(ODE,ICs,ops,val)
```

- `x=sym('x')`, `y=sym('y')`, declaring symbolic objects (one at a time)
- `syms y(x)`, declaring symbolic objects (all at once)
- `dsolve`, finding closed-form solutions for a single `ODE`, where `ODE` is a symbolic equation containing `diff` or a string with the letter `D` (for the derivatives); for more details, see `help dsolve`
- `dsolve`, `ODE`, `ICs`, solving an ODE with given initial or boundary conditions
- `dsolve`, `ODE`, `ICs`, `ops`, `val`, specifying additional options and their values for solving ODEs

► **Verification of exact solutions.**

Let us assume that we have obtained exact solutions and we wish to verify whether these solutions are exact solutions of given ODEs.

Example 21.1. *First-order nonlinear ODE. Special Riccati equation. Verification of solutions.* For the first-order nonlinear ODE, the *special Riccati equation*

$$y'_x = ay^2 + bx^n,$$

we can verify that the solutions

$$y(x) = -\frac{1}{a}\frac{w'_x}{w},$$

where

$$w(x) = \sqrt{x}\left[C_1 J_v\left(\frac{\sqrt{ab}}{k}x^k\right) + C_2 Y_v\left(\frac{\sqrt{ab}}{k}x^k\right)\right], \qquad k = \frac{1}{2}(n+2), \quad v = \frac{1}{2k},$$

are exact solutions of the special Riccati equation as follows:

```
syms x y w k n v q a b C1 C2; k=(n+2)/2; v=1/(2*k); q=1/k*sqrt(a*b);
w=sqrt(x)*(C1*besselj(v,q*x^k)+C2*bessely(v,q*x^k));
y=-1/a*diff(w,x)/w; Test1=simplify(diff(y,x)-a*y^2-b*x^n)
```

Here $a, b, n \in \mathbb{R}$ ($ab \neq 0$, $n \neq -2$) are real parameters, $J_v(x)$ and $Y_v(x)$ are the Bessel functions, and C_1 and C_2 are arbitrary constants.

► Finding and verification of exact solutions.

Let us find exact solutions and verify whether these solutions are exact solutions of given ODEs.

Example 21.2. *First-order linear ODE. Finding and verification of the general solution.* For the first-order linear ODE

$$g(x)y'_x = f_1(x)y + f_0(x),$$

we can find and verify that the solution

$$Sol1 = \mathrm{e}^{\int \frac{\mathrm{f1}(x)}{g(x)}\,\mathrm{d}x} \left(\int \frac{\mathrm{e}^{-\int \frac{\mathrm{f1}(x)}{g(x)}\,\mathrm{d}x}\,\mathrm{f0}(x)}{g(x)}\,\mathrm{d}x \right) + \mathrm{C3}\,\mathrm{e}^{\int \frac{\mathrm{f1}(x)}{g(x)}\,\mathrm{d}x}$$

presented here as the MATLAB result (for `Sol1`) is the general solution of this ODE as follows:

```
syms x g(x) y(x) f1(x) f0(x) Sol1(x);
ODE1='g(x)*Dy-f1(x)*y-f0(x)==0';
Sol1=expand(dsolve(ODE1,'x'))
pretty(Sol1)
Test1=simplify(g(x)*diff(Sol1,x)-f1(x)*Sol1-f0(x))
latex(Sol1)
```

Here $f0(x)$, $f1(x)$, and $g(x)$ are arbitrary functions, and $C3$ is an arbitrary constant.

Remark 21.2. It should be noted that in this example and in what follows, when we solve a differential equation using the predefined function `dsolve` (without specifying initial or boundary conditions), we obtain the solution with an *arbitrary parameter name* (in this case, `C3`). Since the solution of this problem has just one parameter, the name of the arbitrary constant should be `C1` (according to standard mathematical notation). We think this is an example of stylistic negligence and should be corrected in the future.

Example 21.3. *Clairaut's equation. Finding and verifying solutions.* For Clairaut's equation

$$y = xy'_x + f(y'_x),$$

we can find and verify that

$$y(x) = Cx + f(C)$$

is the general solution of this equation as follows:

```
syms x y(x) f(x) Sol1(x);
ODE1='y-x*Dy-f(Dy)==0';
Sol1=expand(dsolve(ODE1,'x'))
Test1=simplify(Sol1-x*diff(Sol1,x)-f(diff(Sol1,x))==0)
```

Here $f(x)$ is an arbitrary function and C is an arbitrary constant.

Example 21.4. *Linear ODE of the second order. Exact explicit solution.* The exact explicit solution

$$y(x) = \frac{C_3 M_{k,\mu}(z)\left(\frac{a-2b}{4a}, -\frac{1}{4}, ax - \frac{a}{2} - \frac{ax^2}{2}\right)}{\sqrt{e^{\frac{ax(x-2)}{2}}}\,\sqrt{x-1}} + \frac{C_4 W_{k,\mu}(z)\left(\frac{a-2b}{4a}, -\frac{1}{4}, ax - \frac{a}{2} - \frac{ax^2}{2}\right)}{\sqrt{e^{\frac{ax(x-2)}{2}}}\,\sqrt{x-1}}$$

of the second-order linear ODE

$$y''_{xx} + a(x-1)y'_x + by = 0 \qquad (a, b \in \mathbb{R})$$

can be found and tested as follows:

```
syms a b x y(x) f(x) Sol1(x);
ODE1='D2y+a*(x-1)*Dy+b*y==0';
Sol1=dsolve(ODE1,'x')
Test1=simplify(diff(Sol1,x,x)+a*(x-1)*diff(Sol1,x)+b*Sol1==0)
```

Here $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are the Whittaker M and W functions, denoted by $\text{whittakerM}(k, \mu, z)$ and $\text{whittakerW}(k, \mu, z)$, respectively, in MATLAB.

Remark 21.3. In this example, we have the arbitrary constants `C3` and `C4` instead of `C1` and `C2`; for details, see Remark 21.2 (stylistic negligence of MATLAB).

► Graphical solutions.

Consider the most relevant related functions for plotting solutions of ordinary differential equations.

```
x=linspace(x1,x2,n); Y= eval(vectorize(y)); plot(x,Y,ops);
ezplot(func);                            ezplot(func,[x1,x2]);
ezplot(func,[x1,x2,y1,y2]);    ezplot(funcX,funcY,[t1,t2]);
                     fcontour(func,[x1,x2,y1,y2],ops);
```

- `linspace`, generating a linear space vector
- `vectorize`, converting symbolic objects into strings
- `eval`, evaluating strings (character arrays and symbolic objects)
- `plot`, constructing a 2-D line plot of the data in `Y` versus the corresponding values in `X`
- `ezplot`, constructing plots of the expression `func(x)` over the default domain $-2\pi < x < 2\pi$, where `func(x)` is an explicit function of x

Example 21.5. *Linear ODE of the first order. Graphical solutions.*
Graphical solutions of the linear first-order ODE

$$y'_x = y + \cos(x)x^2$$

can be generated as follows:

```
clear all; close all; echo on; format long;
syms x y(x); ODE1='Dy==y+cos(x)*x^2'; Sol1=dsolve(ODE1,'x')
x = 0:0.01:2; Y = [];
for i=-2:2 C3=i+i*2;
 Y=[Y;C3*exp(x)+((x+1).*(cos(x)+sin(x)-x.*cos(x)+x.*sin(x)))/2]; end
plot(x,Y(1,:),'k-',x,Y(2,:),'k-.',x,Y(3,:),'k--',...
 x,Y(4,:),'k.',x,Y(5,:),'k:','LineWidth',1);
grid on; xlabel('x'); ylabel('y'); title('Solutions of the
first-order linear ODE');
legend('C3=-2','C3=-1','C3=0','C3=1','C3=2');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg
```

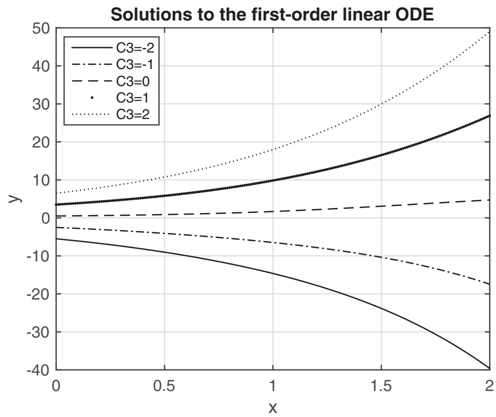


Figure 21.1: Graphical solutions of the linear equation $y'_x = y + \cos(x)x^2$.

Since we obtain the analytical solution (`Sol1`)

```
Sol1 =
C3*exp(x) + ((x + 1)*(cos(x) + sin(x) - x*cos(x) + x*sin(x)))/2,
```

where `C3` is an arbitrary constant, we generate several graphical solutions of this ODE and present them in Fig. 21.1.

Example 21.6. *Linear second-order ODE with constant coefficients. Graphical solutions.*
Graphical solutions of the linear second-order ODE with constant coefficients and with the initial conditions

$$y''_{xx} - 9y'_x + 5y = \sin x, \qquad y(0) = 0, \quad y'_x(0) = -1$$

can be generated (by using the predefined functions `plot` and `ezplot`) as follows:

```
clear all; close all; echo on; format long; syms x y(x);
ODE1='D2y-9*Dy+5*y==sin(x)'; ICs='y(0)=0,Dy(0)=-1';
Sol1=dsolve(ODE1,ICs,'x') x=linspace(0,1,40);
Y=eval(vectorize(Sol1)); figure(1); plot(x,Y) figure(2);
ezplot(Sol1,[0,1]) title('Solution of the second-order linear ODE')
```

► Constructing exact explicit and implicit solutions.

If an exact solution is given as a function of the independent variable, then the solution is said to be *explicit*. For some differential equations, explicit solutions cannot be determined; however, we can obtain an *implicit form* of the solution, i.e., an equation that involves no derivatives and relates the dependent and independent variables.

```
dsolve(ODE1,'x'); ezplot(impSol==0,[x1,x2],ops);
ezcontour(func,[x1,x2,y1,y2],name,value);
```

Example 21.7. *First-order separable ODE. Exact implicit solutions. Graphical solutions.*
For the first-order separable ODE

$$y'_x + \frac{x^2}{y} = 0, \tag{21.2.1.1}$$

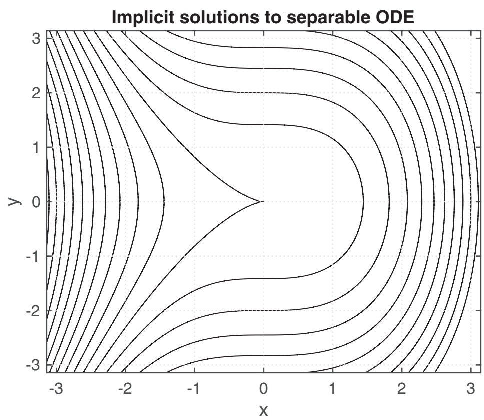


Figure 21.2: Implicit solutions of the first-order separable ODE (21.2.1.1).

we can construct the explicit (`Sol1`) and implicit (`Sol2`) solutions

$$y = \pm\sqrt{2}\sqrt{-\frac{x^3}{3} + C4}, \qquad y^2 + \frac{2}{3}x^3 - 2C4 = 0,$$

respectively, and plot the graphs of the implicit solution as follows:

```
clear all; close all; echo on; format long; syms x y(x) z; Y=[];
ODE1='Dy+x^2/y==0'; Sol1=dsolve(ODE1,'x') Sol2=z^2==Sol1(1)^2 for
i=-10:10 C4=i; Y=[Y;subs(Sol2)]; end for i=1:21
h=ezplot(Y(i),[-pi,pi]); hold on; set(h,'color',[0 0 0]); end grid
on; xlabel('x'); ylabel('y'); title('Implicit solutions of
separable ODE'); set(gca,'FontSize',12);
set(gca,'FontName','Arial'); set(gca,'LineWidth',1); shg; hold off
```

Here `C4` is an arbitrary constant.

Example 21.8. *First-order nonlinear ODE. Exact implicit solutions. Graphical solutions.* For the first-order nonlinear ODE

$$y'_x(1 + y^2) = \sin x,$$

we can construct the implicit solution (`Sol1`)

$$Sol1 = \mathrm{RootOf}(z^3 + 3z - 3C_4 + 3\cos(x), z),$$

where the function `RootOf` represents the symbolic set of roots of the expression $z^3 + 3z - 3C_4 + 3\cos(x)$ with respect to the variable z. Also, we plot the graphs of the implicit solution (see Fig. 21.2) as follows:

```
clear all; close all; echo on; format long; syms x y(x) z; Y = [];
ODE1='Dy*(1+y^2)=sin(x)'; Sol1=dsolve(ODE1,'x') for i=-10:10 C4=i;
Y=[Y;subs(z^3+3*z-3*C4+3*cos(x)==0)]; end for i=1:21
ezplot(Y(i),[-pi,pi]); hold on; end grid on; xlabel('x');
ylabel('y'); title('Implicit solutions of nonlinear ODE'); shg;
hold off
```

Here the arbitrary constant is `C4` (instead of `C1`); see Remark 21.2. In this case (the predefined function `dsolve`), we have the warning: "Explicit solution could not be found; implicit solution returned." This result means that the solution in implicit form reads

$$y^3 + 3y - 3C4 + 3\cos x = 0.$$

► Constructing exact solutions of higher-order ODEs.

One can construct exact solutions of higher-order ordinary differential equations by applying the predefined function `dsolve`.

Example 21.9. *Higher-order linear homogeneous ODEs with constant coefficients.* For the fourth-order linear homogeneous ODE with constant coefficients

$$y''''_x + a_1 y'''_x + a_2 y''_x + a_3 y' + a_4 y = 0,$$

where the constant coefficients are $a_1 = 1$, $a_2 = -1$, $a_3 = 5$, and $a_4 = -2$ and all solutions are of exponential form, we can determine the general solution (`Sol1`)

$$y(x) = C3e^{x/2}\cos\Big(\frac{\sqrt{7}}{2}x\Big) + C4e^{x/2}\sin\Big(\frac{\sqrt{7}}{2}x\Big) + C5e^{(\sqrt{2}-1)x} + C6e^{-(\sqrt{2}+1)x}$$

as follows:

```
clear all; close all; echo on; format long;
syms x y(x); ODE1='D4y+D3y-D2y+5*Dy-2*y==0';
Sol1=dsolve(ODE1,'x'); pretty(Sol1)
```

Example 21.10. *Higher-order linear homogeneous ODEs with nonconstant coefficients.* For the fourth-order linear homogeneous ODE with nonconstant coefficients, the *Euler equation*

$$a_1 x^4 y''''_x + a_2 x^3 y'''_x + a_3 x^2 y''_x + a_4 x y' + a_5 y = 0,$$

where $a_1 = 1$, $a_2 = 14$, $a_3 = 55$, $a_4 = 65$, and $a_5 = 16$, we can determine the general solution (`Sol1`)

$$y(x) = \frac{C3\ln(x)^2}{x^2} + \frac{C4\ln(x)^3}{x^2} + \frac{C5\ln(x)}{x^2} + \frac{C6}{x^2}$$

as follows:

```
clear all; close all; echo on; format long; syms x y(x);
ODE1='x^4*D4y+14*x^3*D3y+55*x^2*D2y+65*x*Dy+16*y==0';
Sol1=dsolve(ODE1,'x'); pretty(Sol1)
```

The general solution $y = y(x)$ of a nonhomogeneous linear ODE can be written as the sum of a particular solution $y_\text{p}(x)$ of the nonhomogeneous equation and the general solution of the corresponding homogeneous equation. The general solution of the homogeneous equation is a linear combination of the solutions in a fundamental set of solutions. The general solution of the nth-order nonhomogeneous linear ODE has the form

$$y = y_\text{p}(x) + \sum_{i=1}^{n} C_i y_i(x), \qquad (21.2.1.2)$$

where $y_i(x)$ $(i = 1, \dots, n)$ is a fundamental set of solutions and C_i are arbitrary constants.

Example 21.11. *Higher-order linear nonhomogeneous ODEs with constant coefficients.*
Consider the fourth-order linear nonhomogeneous ODE with constant coefficients

$$y''''_x + a_1 y'''_x + a_2 y''_x + a_3 y' + a_4 y = \sin(x),$$

where the constant coefficients are $a_1 = 1$, $a_2 = -1$, $a_3 = 5$, $a_4 = -2$.

First, we determine the general solution of the homogeneous ODE (SolGenHom). Then we write out a particular solution of the nonhomogeneous equation (SolPartNonHom) and form the general solution of the nonhomogeneous ODE (SolGenNonHom) according to Eq.(21.2.1.2),

$$y(x) = C3e^{x/2}\cos\Big(\frac{\sqrt{7}}{2}x\Big) + C4e^{x/2}\sin\Big(\frac{\sqrt{7}}{2}x\Big) + C5e^{(\sqrt{2}-1)x} + C6e^{-(\sqrt{2}+1)x} - \frac{1}{4}\cos(x),$$

as follows:

```
clear all; close all; echo on; format long; syms x y(x);
ODE1='D4y+D3y-D2y+5*Dy-2*y==0';
ODE2='D4y+D3y-D2y+5*Dy-2*y==sin(x)';
SolGenHom=dsolve(ODE1,'x'); pretty(SolGenHom)
SolPartNonHom=-cos(x)/4;
SolGenNonHom=SolGenHom+SolPartNonHom; pretty(SolGenNonHom)
```

Finally, we find the general solution of the given ODE and compare the solution SolGenNonHom (as a result of our construction procedure) to the solution SolGenNonHom1 (as a result of dsolve). It should be noted that these solutions (SolGenNonHom and SolGenNonHom1) are the same:

```
SolGenNonHom1=simplify(dsolve(ODE2,'x')); pretty(SolGenNonHom1)
```

21.2.2 Analytical Solutions of Mathematical Problems

▶ Initial value problems.

In many applications, it is required to solve an *initial value problem* or a *Cauchy problem*, i.e., a problem consisting of a differential equation supplemented with one or more initial conditions (which must be satisfied by the solutions). The number of conditions equals the order of the equation. Therefore, we have to determine a *particular solution* that satisfies the given initial conditions.

Consider some initial value problems that model various processes and phenomena [see Lin and Segel (1998)].

Example 21.12. *Malthus model. Cauchy problem. Analytical and graphical solutions.*
A basic model for population growth consists of a first-order linear ODE and an initial condition and has the form

$$y'_t = ky, \quad y(0) = y_0 \quad (k > 0),$$

where k ($k > 0$) is a constant representing the rate of growth (the difference between the birth rate and the death rate). The increase in the population is proportional to the total number of people.

We can obtain the particular solution

$$y(t) = y_0 e^{kt}$$

of this mathematical problem, which predicts exponential growth of the population, as follows:

```
clear all; close all; echo on; format long; syms k t y(t);
ODE1='Dy==k*y'; IC1='y(0)==y0';
Sol1=dsolve(ODE1,IC1,'t')
Test1=simplify(diff(Sol1,t)-k*Sol1)
```

Example 21.13. *First-order linear ODE with nonconstant coefficients. Cauchy problem.*
For the first-order linear ODE with nonconstant coefficients and with the initial condition

$$y'_x - 2y = 3x, \qquad y(0) = n, \tag{21.2.2.1}$$

we can determine the particular analytical solution (`Sol1`)

$$y(x) = -\frac{3}{2}x - \frac{3}{4} + e^{2x}\Big(n + \frac{3}{4}\Big)$$

and construct it for various values of the parameter n as follows:

```
clear all; close all; echo on; format long; syms n x y(x);
N=7; ODE1='Dy-2*y==3*x'; IC1='y(0)==n';
Sol1=dsolve(ODE1,IC1,'x')
for i=1:N n=-3+(i-1); Sols(i)=subs(Sol1); end
Sols
for i=1:N h=ezplot(eval(vectorize(Sols(i))),[0,2.5]); hold on;
 set(h,'color',[0 0 0]);
end grid on; xlabel('x'); ylabel('y'); title('Analytical solutions
of Cauchy problem'); set(gca,'FontSize',12);
set(gca,'FontName','Arial'); set(gca,'LineWidth',1); shg; hold off
```

► Boundary value problems.

Consider the *two-point linear boundary value problems* that consist of the second-order ODE

$$\mathcal{F}(x, y, y'_x, y''_{xx}) = 0$$

and boundary conditions at the two endpoints of an interval $[a, b]$ [e.g., see Bailey et al. (1968)]. Some (simple) boundary value problems can be solved (with the aid of MATLAB) analytically as initial value problems except that the value of the function and its derivatives are given at two values of x (the independent variable) rather than one. Note that an initial value problem has a unique solution, while a boundary value problem may have more than one solution or no solution at all.

Boundary conditions can be *homogeneous* (if the prescribed values are zero) and *nonhomogeneous* (otherwise) and can be divided into three classes, the *Dirichlet conditions*, the *Neumann boundary conditions*, and the *Robin boundary conditions*.

Example 21.14. *Second-order linear homogeneous ODE. Boundary value problem.*
For the second-order linear homogeneous ODE with constant coefficients and with the boundary conditions (the nonhomogeneous Dirichlet conditions)

$$y''_{xx} + a_1 y = 0, \qquad y(a) = g_1, \quad y(b) = g_2, \tag{21.2.2.2}$$

where $a_1 = 2$, $a = 0$, $b = \pi$, $g_1 = 1$, and $g_2 = 0$, we can determine the particular analytical solution (`Sol1`)

$$y(x) = -\frac{\cos(\sqrt{2}\pi)\sin(\sqrt{2}x)}{\sin(\sqrt{2}\pi)} + \cos(\sqrt{2}x)$$

and construct the graphical solution as follows:

```
clear all; close all; echo on; format long; syms x y(x);
ODE1='D2y+2*y==0'; BC1='y(0)==1,y(pi)==0';
Sol1=dsolve(ODE1,BC1,'x') h1=ezplot(Sol1,[0,pi]) set(h1,'color',[0
0 0]); grid on; xlabel('x'); ylabel('y'); title({'Analytical
solution of boundary value problem.',...
       'Dirichlet boundary conditions.'});
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

Modifying the boundary conditions (the nonhomogeneous Neumann conditions), we obtain the following:

$$y''_{xx} + a_1 y = 0, \qquad y'_x(a) = g_1, \quad y'_x(b) = g_2, \tag{21.2.2.3}$$

where $a_1 = 2$, $a = 0$, $b = \pi$, $g_1 = 1$, and $g_2 = 0$, and the particular analytical solution (`Sol2`)

$$y(x) = \frac{1}{2}\sqrt{2}\sin(\sqrt{2}x) + \frac{1}{2}\frac{\sqrt{2}\cos(\sqrt{2}\pi)\cos(\sqrt{2}x)}{\sin(\sqrt{2}\pi)}$$

can be constructed as follows:

```
clear all; close all; echo on; format long; syms x y(x);
ODE2='D2y+2*y==0'; BC2='Dy(0)==1,Dy(pi)==0';
Sol2=dsolve(ODE2,BC2,'x') h2=ezplot(Sol2,[0,pi]) set(h2,'color',[0
0 0]); grid on; xlabel('x'); ylabel('y'); title({'Analytical
solution of boundary value problem.',...
       'Neumann boundary conditions.'});
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

For solving more complicated boundary value problems, we can follow the numerical approach (see Section 21.3.3).

21.2.3 Analytical Solutions of Systems of ODEs

One can find analytical solutions of a given ODE system by applying the predefined function `dsolve`:

```
Y=dsolve(ODESys)                     Y=dsolve(ODESys,ICs,ops,val)
[y1,...,yN]=dsolve(ODESys)   [y1,...,yN]=dsolve(ODESys,ICs)
[y1,...,yN]=dsolve(ODESys,ICs,ops,val)
```

- `Y=dsolve(ODESys)`, solving a system of ODEs with the result being a structure array that contains the solutions
- `Y=dsolve(ODESys,ICs,ops,val)`, solving a system of ODEs with initial (or boundary) conditions and additional options (specified by one or more pair arguments `ops,val`)
- `[y1,...,yN]=dsolve(ODESys)`, solving a system of ODEs and assigning the solutions to the variables `y1,...,yN`
- `[y1,...,yN]=dsolve(ODESys,ICs,ops,val)`, solving a system of ODEs with initial (or boundary) conditions and additional options (specified by one or more pair arguments `ops,val`)

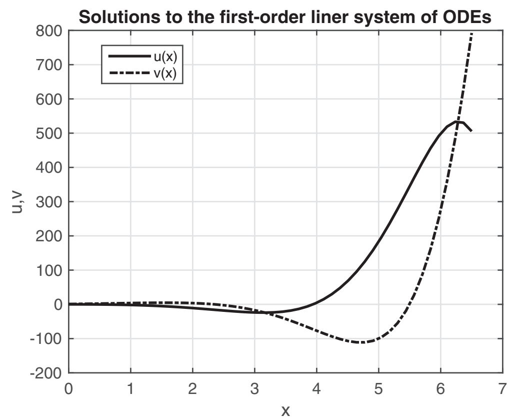


Figure 21.3: Exact solutions of the first-order linear system of ODEs (21.2.3.1).

► Linear systems of ODEs.

For first-order linear systems of ODEs, one can find the general solution and the particular solution for any initial condition (with the aid of the predefined function `dsolve`). For higher-order linear ODEs or systems of ODEs, one can convert them to a system of first-order ODEs and then solve them.

Example 21.15. *First-order two-dimensional linear system of ODEs. Analytical solution.*

Consider the general first-order two-dimensional linear system of ODEs with constant coefficients

$$u'_x = a_0 + a_1 u + a_2 v, \quad v'_x = b_0 + b_1 u + b_2 v, \tag{21.2.3.1}$$

where $u(x)$ and $v(x)$ are unknown functions and the coefficients are $a_0 = 1$, $a_1 = 1$, $a_2 = -1$, $b_0 = 1$, $b_1 = 1$, and $b_2 = 1$.

By applying the predefined function `dsolve`, we find the general solution

$$u\,(x) = -1 + \mathrm{e}^x\,(C2\,\cos(x) + C3\,\sin(x))\,,$$
$$v\,(x) = -\mathrm{e}^x\,(C3\,\cos(x) - C2\,\sin(x))$$

of this linear system; then we verify and plot it for certain values of the parameters $C2$ and $C3$ (see Fig. 21.3) as follows:

```
clear all; close all; echo on; format long; syms x u(x) v(x);
ODE1='Du==1+u-v', ODE2='Dv==1+u+v' [uS,vS]=dsolve(ODE1,ODE2,'x')
Test1=simplify(diff(uS,x)-1-uS+vS)
Test2=simplify(diff(vS,x)-1-uS-vS) C2=1; C3=-1; uSP=subs(uS),
vSP=subs(vS) x=linspace(0,6.5,50); ux=eval(vectorize(uSP));
vx=eval(vectorize(vSP)); plot(x,ux,'k-',x,vx,'k-.','LineWidth',2)
grid on; xlabel('x'); ylabel('u,v'); title('Solutions of the
first-order linear system of ODEs') legend('u(x)','v(x)');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg
```

Example 21.16. *First-order two-dimensional linear system of ODEs. Cauchy problem.*

Consider the following first-order two-dimensional linear system of ODEs with initial conditions:

$$u'_x = a_0 + a_1 u + a_2 v, \quad v'_x = b_0 + b_1 u + b_2 v, \\ u(x_0) = u_0, \quad v(x_0) = v_0, \tag{21.2.3.2}$$

where $u(x)$ and $v(x)$ are unknown functions and the coefficients are $a_0 = -1$, $a_1 = 1$, $a_2 = -1$, $b_0 = 1$, $b_1 = -1$, and $b_2 = 1$. For a first-order two-dimensional system in $u(x)$ and $v(x)$, each initial condition can be specified in the form `IC=` $\{u(x_0) = u_0,\ v(x_0) = v_0\}$ (e.g., $u(0) = 0$, $v(0) = 1$). One solution curve is generated for each initial condition. The solution of the initial value problem (`IVP1`) can be found as follows:

```
clear all; close all; echo on; format long; syms x u(x) v(x);
ODE1='Du==-1+u-v', ODE2='Dv==1-u+v' [uS,vS]=dsolve(ODE1,ODE2,'x')
Test1=simplify(diff(uS,x)-1-uS+vS)
Test2=simplify(diff(vS,x)-1-uS-vS) IC='u(0)==0,v(0)==1';
[uC,vC]=dsolve(ODE1,ODE2,IC,'x') simplify(uC), simplify(vC)
x=linspace(0,6.5,50); ux=eval(vectorize(uC));
vx=eval(vectorize(vC)); plot(x,ux,'k-',x,vx,'k-.','LineWidth',2)
grid on; xlabel('x'); ylabel('u,v'); title('Exact solutions of the
Cauchy problem for ODE system') legend('u(x)','v(x)');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg
```

⊙ *Literature for Section .2*: G. M. Murphy (1960), P. B. Bailey, L. F. Shampine, and P. E. Waltman (1968), C. C.Lin and L. A. Segel (1998), D. Zwillinger (1997), A. D. Polyanin and A. V. Manzhirov (2007).

21.3 Numerical Solutions of ODEs

Since 2000, MATLAB has become one of the most important *problem-solving environments* (PSEs) for scientists, professors, and students.

The first implementation of numerical methods for solving ODEs, `RKF45` [see Shampine and Watts (1977, 1979)], was a FORTRAN program (based on the explicit Runge–Kutta formulas $F(4,5)$ of Fehlberg), which is widely used in *general scientific computation* (GSC). It is the foundation of the predefined functions for solving initial value problems `rkf45` (in Maple), `NDSolve` (in Mathematica), and `ode45` (in MATLAB).

`MATLAB ODE Suite` was then developed [Shampine & Reichelt (1997)] with further evolutions [Shampine et al. (1999), Kierzenka & Shampine (2001), Shampine & Thompson (2001)]. `MATLAB ODE Suite` (replacing `ode45` since *Ver.* 5) is different in many aspects; e.g., it is based on the *explicit Runge–Kutta* $(4,5)$ formulas, the Dormand–Prince pair.

Frequently, it is not possible to solve nonlinear (or complicated) systems of ODEs arising in realistic problems by applying analytical solution methods. In this section, we consider various numerical approximation methods for initial value problems, boundary value problems, and eigenvalue problems for ordinary differential equations.

21.3.1 Numerical Solutions via Predefined Functions

MATLAB has several predefined functions for finding numerical solutions of a given ODE problem. These predefined functions are very effective, and they have a common syntax (i.e., it is easy to use them).

► Numerical methods embedded in MATLAB for initial value problems.

Let us refer to numerical methods embedded in MATLAB or predefined functions for solving some type of problems as *solvers*.

The syntax, common to all solvers (predefined functions for solving initial value problems), is as follows:

```
[outputs]=SolverName(inputs)
[IndVar,DepVar]=SolverName(ODEfun,InInteg,ICs,ops)
```

- `ODEfun`, a given function containing the derivatives (specified as a scalar or vector function)
- `ICs`, initial conditions (specified as a scalar or vector)
- `InInteg`, the interval of integration (specified as a vector)
- `ops`, option structure (specified as a structure array)
- `IndVar`, evaluation points (specified as a column vector)
- `DepVar`, numerical solution (specified as an array)
- `SolverName`, one of the numerical methods embedded in MATLAB

The solvers for initial value problems implement a variety of methods. All the solvers for initial value problems of MATLAB require first-order ODEs or systems of first-order ODEs. More detailed information about numerical methods for initial value problems is presented in Table 21.1 (for variable-step solvers embedded in MATLAB) and in Table 21.2 (for fixed-step solvers available in Simulink or on the internet).

Remark 21.4. The following abbreviations in Tables 21.1–21.3 are adopted: `IVP`, initial value problem; `BVP`, boundary value problem; `BDF`, backward-differentiation formula; `IVP–DAE`, initial value problem for differential-algebraic equations; `IVP–DDE`, initial value problem for delay differential equations.

Fixed-step numerical methods for solving initial value problems are available in Simulink (for modeling and generating code for real-time systems) or on the internet.

Example 21.17. *Cauchy problem with several initial conditions.*

For the Cauchy problem (with several initial conditions)

$$y'_x = y + x^2, \qquad \{y(0) = 0,\ y(0) = 0.5,\ y(0) = 1\} \tag{21.3.1.1}$$

on the interval $[a, b]$ ($a = 0$, $b = 2$), we find the numerical and graphical solutions (see Fig. 21.4) as follows:

```
clear all; close all; echo on; format long; InInteg=[0 2]; y01=0;
y02=0.5; y03=1; [x,y1]=ode45(@(x,y) y+x^2,InInteg,y01);
[x,y2]=ode45(@(x,y) y+x^2,InInteg,y02); [x,y3]=ode45(@(x,y)
y+x^2,InInteg,y03);
plot(x,y1,'k-o',x,y2,'k-.',x,y3,'k--','LineWidth',1) grid on;
xlabel('x'); ylabel('y'); title({'Numerical solutions of Cauchy
problem.',...
       'Several initial conditions.'});
legend('IC1=0','IC2=0.5','IC3=1')
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg
```

Table 21.1.
Variable-step numerical methods for initial value problems embedded in MATLAB with brief description and some references

Numerical method	Brief description	References
`ode45`	Explicit one-step Runge–Kutta $(4,5)$ formula, the Dormand–Prince pair. Variable step. Method for nonstiff IVP. Order of accuracy: medium (4-5). Apply as a "first step" for most problems.	Enright et al. (1986) Fehlberg (1970) Shampine and Corless (2000)
`ode23`	Explicit one-step Runge–Kutta $(2,3)$ formula, the Bogacki–Shampine pair. Variable step. Method for nonstiff IVP. Order of accuracy: low (2-3).	Enright et al. (1986) Cash and Karp (1990) Forsythe et al. (1977)
`ode113`	Multistep Adams–Bashforth–Moulton method. Variable step. Method for nonstiff IVP. Order of accuracy: low to high (1-13).	Hairer and Wanner (1996) Shampine and Corless (2000) Forsythe et al. (1977)
`ode15s`	Implicit multistep BDF formulas (the Gear method). Method for stiff IVP (and IVP-DAEs). Variable step. Order of accuracy: low to medium (1-5). Apply if `ode45` fails (or inefficient).	Enright (1989) Verner (1978) Forsythe et al. (1977)
`ode23s`	Implicit one-step method. The modified Rosenbrock formula. Method for stiff IVP. Order of accuracy: low (2).	Hindmarsh (1983) Forsythe et al. (1977) Shampine and Corless (2000)
`ode23t`	Implicit one-step method. Method for stiff IVP. The trapezoidal rule using a "free" interpolant. Order of accuracy: low (2). IVP-DAEs can be solved with `ode23t`.	Hosea and Shampine (1996) Shampine et. al (1999)
`ode23tb`	Implicit Runge–Kutta formula with 2 stages (TR–BDF2). The first stage: trapezoidal rule (TR), the second stage: BDF of order 2 (BDF2). Method for moderately stiff IVP. Order of accuracy: low (2).	Barton et al. (1971) Forsythe et al. (1977) Shampine and Corless (2000)
`ode15i`	Order: variable (1–5). for fully implicit problems $f(x,y,y'_x)=0$, for IVP-DAE of index 1. Order of accuracy: low.	Boyce and DiPrima (2004) Conte and de Boor (1980) Fox and Mayers (1987)

► **Numerical methods embedded in MATLAB for boundary value problems.**

One can obtain a solution of a given boundary value problem of the form

$$\begin{aligned} &y'_x = f(x,y), \quad g\big(a,b,y(a),y(b)\big)=0, \quad \text{or}\\ &y'_x = f(x,y,p), \quad g\big(a,b,y(a),y(b),p\big)=0, \end{aligned}$$

where p is the vector of unknown parameters and f is a continuous function on $[a,b]$ and a Lipschitz function in y and has a continuous first derivative there.

In MATLAB, there exist two predefined functions for solving boundary value problems, `bvp4c` and `bvp5c`. These solvers have been developed by Kierzenka and Shampine [see Kierzenka and Shampine (2001)] and require first-order ODEs or systems of first-order ODEs.

The solvers `bvp4c` and `bvp4c` can solve boundary value problems with unknown parameters, multi-point boundary value problems, and a class of singular boundary value

Table 21.2.
Fixed-step numerical methods for initial value problems
with brief description and some references

Numerical method	Brief description	References
`ode1`	Explicit one-step Euler's method, Euler1. Method for nonstiff IVP. Order of accuracy: 1. Fixed-step method.	Enright et al. (1986) Fehlberg (1970) Shampine and Corless (2000)
`ode2`	Explicit one-step Heun's method, Euler2. Method for nonstiff IVP. Order of accuracy: 2. Fixed-step method.	Enright et al. (1986) Cash and Karp (1990) Forsythe et al. (1977)
`ode3`	Explicit one-step Bogacki–Shampine formula, RK3. Method for nonstiff IVP. Order of accuracy: 3. Fixed-step method.	Hairer and Wanner (1996) Shampine and Corless (2000) Forsythe et al. (1977)
`ode4`	Explicit fourth-order Runge–Kutta formula, RK4. Method for nonstiff IVP. Order of accuracy: 4. Fixed-step method.	Enright (1989) Verner (1978) Forsythe et al. (1977)
`ode5`	Explicit one-step Dormand–Prince formula, RK5. Method for nonstiff IVP. Order of accuracy: 5. Fixed-step method.	Hindmarsh (1983) Forsythe et al. (1977) Shampine and Corless (2000)
`ode87`	Explicit one-step Dormand–Prince formula, RK8(7). Method for nonstiff IVP. Order of accuracy: 8. Fixed-step method.	Hosea and Shampine (1996) Shampine et. al (1999)
`ode14x`	Implicit one-step Newton's method with extrapolation. Method for stiff IVP. Order of accuracy: variable. Fixed-step method.	Lubich (1989) Deuflhrd et al. (1987) Hairer and Wanner (1996)

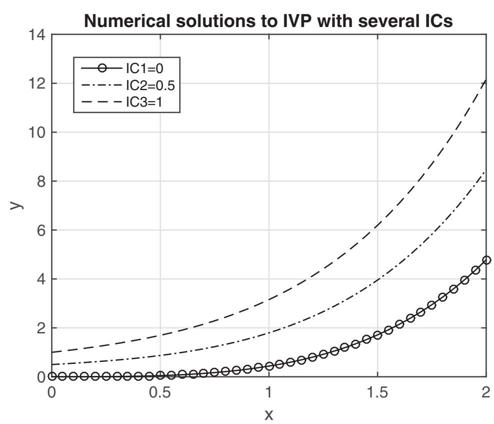


Figure 21.4: Numerical solutions of the initial value problem (21.3.1.1) with several initial conditions (`IC1`, `IC2`, `IC3`).

problems. The difference between the solvers `bvp4c` and `bvp5c` is in the meaning of error tolerances. The function `bvp5c` controls the true error $|y(x) - Y(x)|$* directly, and the function `bvp4c` controls the true error indirectly; i.e., it controls the discrepancy $|Y'_x - f(x, Y(x))|$.

*$Y(x)$ is an approximate solution.

Table 21.3.
Numerical methods for boundary value problems embedded in MATLAB
with brief description and some references

Numerical method	Brief description	References
bvp4c	Implicit three-stage Lobatto IIIa formula. Order of accuracy: 4. Mesh selection, error control are based on the discrepancy. Collocation polynomials provide $C^1[a,b]$-continuous solutions.	Enright et al. (1986) Fehlberg (1970) Shampine and Corless (2000)
bvp5c	Implicit four-stage Lobatto IIIa formula. Order of accuracy: 5. Mesh selection, error control are based on the discrepancy. Collocation polynomials provide $C^1[a,b]$-continuous solutions.	Enright et al. (1986) Cash and Karp (1990) Forsythe et al. (1977)

More detailed information about numerical methods for boundary value problems embedded in MATLAB is presented in Table 21.3.

The syntax, common to these predefined functions for solving boundary value problems, is as follows:

```
Sol=SolverName(ODEfun,BCfun,SolIG,ops)
SolIG=bvpinit(x,yIG,params)                  yk=deval(Sol,xk)
```

- `ODEfun` is a given function $f(x,y,p)$ (specified as a scalar or vector function); it can include unknown parameters p (specified as a scalar or a vector).

- `BCfun` is a function that computes the *discrepancy in the boundary conditions* (BCs). For example, for two-point boundary conditions of the form $g\big(y(a),y(b),p\big)=0$, `BCfun` can have the form `Res=BCfun(ya,yb,params)`, where `ya` and `yb` are column vectors corresponding to $y(a)$ and $y(b)$ and `Res` is a column vector.

- `IG` is a structure containing the *initial guess* for the numerical solution, where `IG.x` are ordered nodes of the initial mesh, `IG.y` is the initial guess for the solution, and `IG.parameters` is a vector for specifying the initial guess for unknown parameters. The boundary conditions are `a=IG.x(1)` and `b=IG.x(end)`. A guess for the solution at the node `IG.x(i)` is `IG.y(:,i)`. It can be formed by using the function `bvpinit` (for specifying the boundary points).

- `ops` is the option structure (specified as a structure array); it can be formed by using the function `bvpset`.

- `Sol` is the numerical solution structure, where `Sol.x` is a mesh (selected by the solver), `Sol.y` is an approximation to $y(x)$ at the mesh points, `Sol.yp` is an approximation to y'_x at the mesh points, and `Sol.parameters` are the resulting values for the unknown parameters.

- `deval` evaluates the solution at specific points `xk` on the interval $[a,b]$.

- `SolverName` is one of the numerical methods for solving boundary value problems.

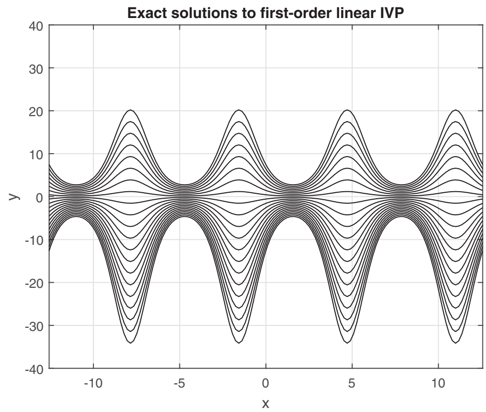


Figure 21.5: Several exact solutions of equation (21.3.2.1).

21.3.2 Initial Value Problems: Examples of Numerical Solutions

Consider some examples of initial value problems.

► Linear initial value problems.

Example 21.18. *First-order linear Cauchy problem. Analytical, numerical, graphical solutions.* For the first-order linear initial value problem

$$y'_x = -y\cos(x), \qquad y(0) = 1 \tag{21.3.2.1}$$

on the interval $[a, b]$ ($a = 0$, $b = 4\pi$), we find infinitely many solutions (`Sols`) of the ordinary differential equation and plot some of them (see Fig. 21.5). Then we obtain the unique exact solution (`Sol1`) and an approximate numerical solution (with `ode23` solver) of the Cauchy problem and plot them (see the first graph in Fig. 21.6) as follows:

```
clear all; close all; echo on; format long;
syms x y(x) z; Z=[]; ODE1='Dy==-y*cos(x)'; IC1='y(0)==1';
Sols=dsolve(ODE1,'x')
for i=-4*pi:4*pi C3=i; Z=[Z;subs(Sols)]; end
figure(1); K=21; for i=1:K h=ezplot(Z(i),[-4*pi,4*pi,-40,40]); hold on;
 set(h,'color',[0 0 0]); end
grid on; xlabel('x'); ylabel('y'); title('Exact solutions of
first-order linear initial value problem');
Sol1=dsolve(ODE1,IC1,'x') figure(2); ezplot(Sol1,[0,4*pi])
title('Exact solution of linear initial value problem'); N=46;
x=linspace(0,4*pi,N); Y=eval(vectorize(Sol1)); InInteg=[0 4*pi];
y0=1; [x,y]=ode23(@(x,y) -y*cos(x),InInteg,y0); figure(3);
plot(x,Y,'k-',x,y,'k-o'); grid on; xlabel('x'); ylabel('y');
title('Exact and numerical solutions of Cauchy problem');
legend('Exact','Numerical'); set(gca,'FontSize',12);
set(gca,'FontName','Arial'); set(gca,'LineWidth',1); shg
```

Remark 21.5. Here the MATLAB notation `'k-'` and `'k-o'` (for the predefined function `plot`) denotes the line styles of the two solutions, the solid line (`-`) of black color (`k`) for the exact solution, and the line with marker type (`-o`) of black color (`k`) for the numerical solution.

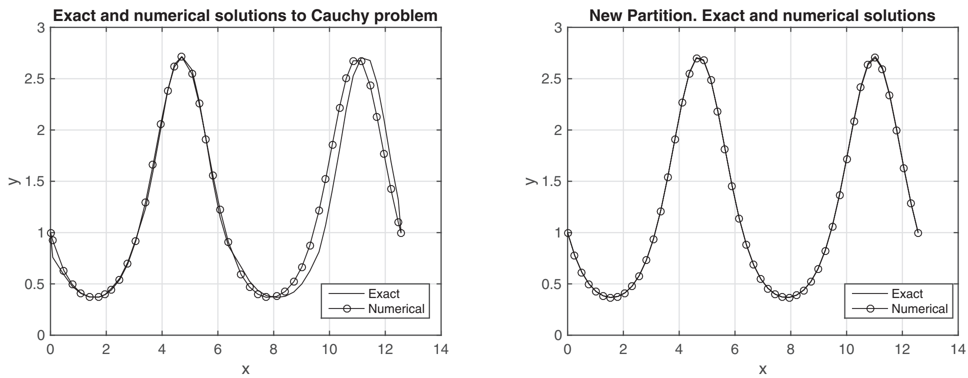


Figure 21.6: New partition. Exact and numerical solutions of the Cauchy problem (21.3.2.1).

Each numerical solver has a certain partition of the interval $[a, b]$, and we obtain a value of y at each point in this partition. For this problem, we choose the solver `ode23`, the interval of integration is $[0, 4\pi]$, and the number of points for this solver is $N = 46$. To plot the numerical and analytical solutions, we have to use the same number of points.

If we would like to increase the accuracy of our approximate solution (see the second graph in Fig. 21.6), we can specify the partition of values, e.g., $N = 50$, as follows:

```
N=50;  x=linspace(0,4*pi,N); Y=eval(vectorize(Sol1));
InInteg=0:(4/49*pi):(4*pi); y0=1;
[x,y]=ode23(@(x,y) -y*cos(x),InInteg,y0);
figure(4); plot(x,Y,'k-',x,y,'k-o'); grid on; xlabel('x'); ylabel('y');
title('New Partition. Exact and numerical solutions');
legend('Exact','Numerical'); set(gca,'FontSize',12);
set(gca,'FontName','Arial'); set(gca,'LineWidth',1); shg
```

► Nonlinear initial value problems.

Example 21.19. *First-order nonlinear Cauchy problem. Numerical and graphical solutions.* For the nonlinear initial value problem

$$y'_x = -e^{xy}\cos(x^2), \qquad y(0) = p$$

on the interval $[a, b]$ ($a = 0$, $b = 4\pi$), we find the numerical and graphical solutions of the problem for various initial conditions $y(0) = p$, where $p = 0.1\, i$ ($i = 1, 2, ..., 5$), as follows:

```
clear all; close all; echo on; format long;
ICs=[0.1:0.1:0.5]; InInteg=[0 4*pi];
for n=1:5 [x,Y]=ode45(@(x,y) -exp(y*x)*cos(x^2),InInteg,ICs); end
for i=1:5
 h=plot(x,Y(:,i),'k-'); hold on; set(h,'color',[0 0 0]);
end grid on; xlabel('x'); ylabel('y'); title('Numerical solutions
of nonlinear Cauchy problem'); set(gca,'FontSize',12);
set(gca,'FontName','Arial'); set(gca,'LineWidth',1); shg; hold off
```

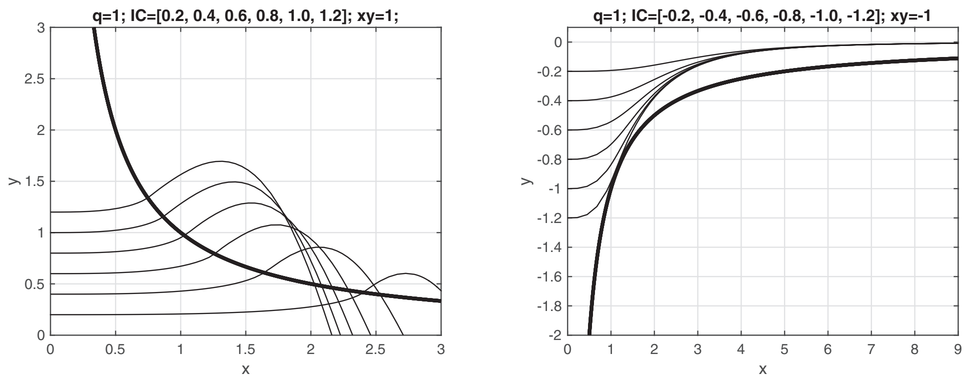


Figure 21.7: Numerical solutions of the Cauchy problem (21.3.2.2) for $q = 1$, $p > 0$ (left) and $p < 0$ (right).

Example 21.20. *First-order nonlinear Cauchy problem. Numerical and graphical solutions.* Consider the initial value problem for the nonlinear differential equation

$$y'_x = 1 - \sqrt{1 - qx^2y^2}, \qquad y(0) = p, \tag{21.3.2.2}$$

where $p \in \mathbb{R}$ and $q > 0$.

The existence domain for the solutions of this differential equation with $q > 0$ is given by the inequality $x^2y^2 \leq 1/q$.

The differential equation in the Cauchy problem (21.3.2.2) has the equilibrium point $y = 0$. The solutions of the Cauchy problem for this equation with the initial condition $y(0) = p$ behave differently depending on the sign of p.

If $p < 0$, then the solutions are infinitely extendible to the right. If $p > 0$, then the solutions approach the boundary of the existence domain at some x (that is, they are not infinitely extendible to the right). Therefore, the equilibrium position $y = 0$ is unstable, because in any neighborhood of $y = 0$ there exist solutions that are not infinitely extendible.

For $q = 1$, several numerical solutions of the Cauchy problem (21.3.2.2) for various values of p are presented in Fig. 21.7 (left) for $p > 0$ and in Fig. 21.7 (right) for $p < 0$.

For example, for $p > 0$ we take the values $0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, and for $p < 0$ we take the values $-0.2, -0.4, -0.6, -0.8, -1.0, -1.2$. The solutions are valid for $x \geq 0$ and are presented on the interval $[a, b]$, where $a = 0$ and $b = 3$ or $b = 9$. In these figures, we also draw the boundary $xy = \pm 1$ of the existence domain of solutions.

To generate Fig. 21.7 (left), where $q = 1$ and $p = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, we can write the following program:

```
clear all; close all; echo on; format long;
a=0; b=3; IC=[0.2:0.2:1.2]; InInteg=[a b]; c=0; d=3; Lstyle=['-'];
for n=1:6
 [x,Y]=ode45(@(x,y) (1.-sqrt(1.-(1.).*x.^2.*y.^2)),InInteg,IC);
end
for i=1:6
  h=plot(x,Y(:,i),Lstyle);  hold on; axis([a b c d]);
  set(h,'color',[0 0 0],'linewidth',1);
    z=ezplot('1/x',[a,b]); set(z,'color',[0 0 0],'linewidth',3);
  hold on; axis([a b c d]);
end
grid on; xlabel('x'); ylabel('y');
```

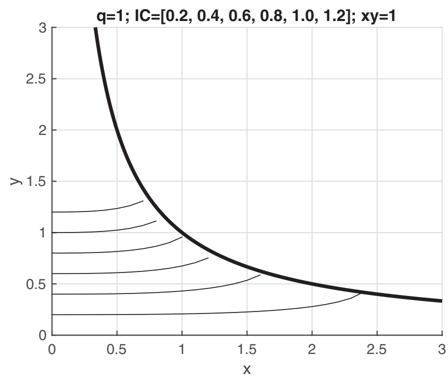


Figure 21.8: Real numerical solutions of the Cauchy problem (21.3.2.2) for $q = 1$, $p > 0$.

```
title('q=1; IC=[0.2, 0.4, 0.6, 0.8, 1.0, 1.2]; xy=1;');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg; hold off
```

However, the solutions presented in Fig. 21.7 (left) are wrong, since the real solutions do not exist if $xy > 1$. The correct solutions for this case are presented in Fig. 21.8. There is no possibility of simple correction of this situation for all solvers for initial value problems embedded in MATLAB. Therefore, we have to write another program for this case, for example, as follows:

```
clear all; close all; echo on; format long;
a=0; b=3; IC=[0.2:0.2:1.2];  c=0; d=3; hold on;
b1=2.4028792; b2=1.6394157; b3=1.2629138; b4=1.0282819;
b5=.86598559; b6=.74669500;
IC1=0.2; IC2=0.4; IC3=0.6; IC4=0.8; IC5=1.0; IC6=1.2;
InInteg1=0:0.1:b1; InInteg2=0:0.1:b2; InInteg3=0:0.1:b3;
InInteg4=0:0.1:b4; InInteg5=0:0.1:b5; InInteg6=0:0.1:b6;
g=@(x,y) 1.-sqrt(1.-(1.).*x.^2.*y.^2);
[x1,Y1]=ode15s(g,InInteg1,IC1); [x2,Y2]=ode15s(g,InInteg2,IC2);
[x3,Y3]=ode15s(g,InInteg3,IC3); [x4,Y4]=ode15s(g,InInteg4,IC4);
[x5,Y5]=ode15s(g,InInteg5,IC5); [x6,Y6]=ode15s(g,InInteg6,IC6);
h1=plot(x1,Y1,'k-'); h2=plot(x2,Y2,'k-'); h3=plot(x3,Y3,'k-');
h4=plot(x4,Y4,'k-'); h5=plot(x5,Y5,'k-'); h6=plot(x6,Y6,'k-');
z1=ezplot('1/x',[a,b]); set(z1,'color',[0 0 0],'linewidth',3);
axis([a b c d]); grid on; xlabel('x'); ylabel('y');
title('q=1; IC=[0.2, 0.4, 0.6, 0.8, 1.0, 1.2]; xy=1');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg; hold off
```

With the aid of Maple, we have evaluated the values b_i ($i = 1, \ldots, 6$) to the right of which the solution becomes complex (see Chapter 18).

To generate Fig. 21.7 (right), where $q = 1$ and $p = -0.2, -0.4, -0.6, -0.8, -1.0, -1.2$, we can write the following program:

```
clear all; close all; echo on; format long;
a=0; b=9; IC=[-2/10 -4/10 -6/10 -8/10, -1, -12/10];
InInteg=[a b]; c=-2; d=0.1;
for n=1:6
```

```
 [x,Y]=ode45(@(x,y) (1-sqrt(1-(1)*(x.*y).^2)),InInteg,IC);
end
for i=1:6
  h=plot(x,Y(:,i),'-');  hold on; axis([a b c d]);
  set(h,'color',[0 0 0],'linewidth',1);
      z=ezplot('-1/x',[a,b]); set(z,'color',[0 0 0],'linewidth',3);
  hold on; axis([a b c d]);
end
grid on; xlabel('x'); ylabel('y');
title('q=1; IC=[-0.2, -0.4, -0.6, -0.8, -1.0, -1.2]; xy=-1');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg; hold off
```

21.3.3 Boundary Value Problems: Examples of Numerical Solutions

Let us numerically solve *two-point boundary value problems*. A two-point boundary value problem includes an ODE (of order $\geq$ 2) and the values of the solution at two distinct points.

Consider some examples of boundary value problems applying embedded methods and constructing step-by-step solutions.

▶ Linear boundary value problems.

Example 21.21. *Second-order linear nonhomogeneous ODE with nonconstant coefficients.*

Consider a second-order linear nonhomogeneous ODE with nonconstant coefficients and with the boundary conditions

$$y''_{xx} + xy'_x + y = \cos(x), \qquad y(a) = 0, \quad y(b) = 1, \tag{21.3.3.1}$$

where $a = 0$ and $b = 2$. Numerical and graphical solutions (`solN`, `figure(1)`, and `figure(2)`) can be constructed as follows:

1. We rewrite the boundary value problem (21.3.3.1) as the first-order system

$$(y_1)'_x = y_2, \qquad (y_2)'_x = \cos x - xy_2 - y_1,$$

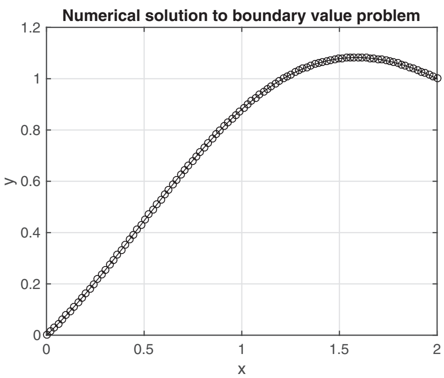
where $y_1 = y$ and $y_2 = y'_x$, and define this system in the M-file (`bvp1.m`) as follows:

```
function dydx=bvp1(x,y); dydx=[y(2); cos(x)-x*y(2)-y(1)]; end
```

2. We write the boundary conditions in the M-file (`bc1.m`) as the residues of the boundary conditions. For the boundary conditions $y(a) = 0$ and $y(b) = 1$, the residues are $ya(1)$ and $yb(1)$. The variables ya and yb represent the solution at $x = a$ and $x = b$ respectively. The symbol 1 in parentheses indicates the first component of the vector (e.g., if we have the boundary condition $y'_x(a) = 1$, we have to write $ya(2) - 1$).

```
function res=bc1(ya,yb); res=[ya(1); yb(1)-1]; end
```

3. We solve the boundary value problem by specifying an initial guess $[y(0), y'_x(0)]$ (where $y(0)$ is known and $y'_x(0)$ is a guess) for the initial value problem and a grid of x values. Thus, a family of initial value problems is solved such that the boundary conditions are satisfied. Finally, we find the numerical solution (see Fig. 21.9) of the boundary value problem with the solver `bvp4c` as follows:



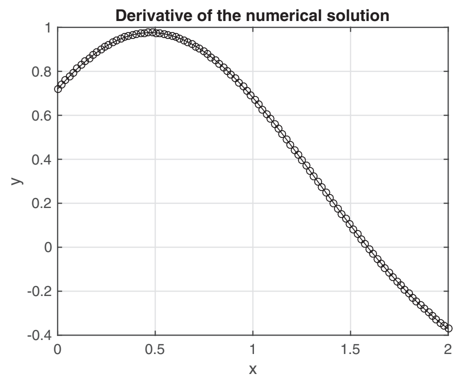


Figure 21.9: Numerical solution of the boundary value problem (21.3.3.1) and its derivative.

```
clear all; close all; echo on; format long;
solInit1=bvpinit(linspace(0,2,5),[0 0]);
solN=bvp4c(@bvp1,@bc1,solInit1); x=linspace(0,2,100);
y=deval(solN,x); figure(1); plot(x,y(1,:),'k-o'); grid on;
xlabel('x'); ylabel('y'); title('Numerical solution of boundary
value problem'); set(gca,'FontSize',12);
set(gca,'FontName','Arial'); set(gca,'LineWidth',1); shg;
figure(2); plot(x,y(2,:),'k.'); grid on; xlabel('x'); ylabel('y');
title('Derivative of the numerical solution');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

Example 21.22. *Second-order linear ODE. Boundary value problem. No solution.*

Solving a boundary value problem for the second-order linear homogeneous ODE with constant coefficients

$$y''_{xx} + \pi^2 y = 0, \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{21.3.3.2}$$

where $a = 0$, $b = 1$, $\alpha = 1$, and $\beta = 1$, we can find the general solution. However, the boundary conditions cannot be satisfied for any choice of the constants (see Chapter 18). Therefore, there exists no solution of this boundary value problem. In MATLAB, this can be observed with the following functions (M-files `bvp2.m` and `bc2.m`):

```
function dydx=bvp2(x,y); dydx=[y(2); -pi^2*y(1)]; end
function res=bc2(ya,yb); res=[ya(1)-1; yb(1)-1]; end
```

and the main program:

```
clear all; close all; echo on; format long;
solInit2=bvpinit(linspace(0,1,5),[0 0]);
solN=bvp4c(@bvp2,@bc2,solInit2); x=linspace(0,1,100);
solN.x, solN.y(1,:), solN.y(2,:)
```

Note that the solution is written as a structure whose first component `sol.x` gives the x values, and the second component `sol.y` of the structure is a matrix, where the first row contains the values of $y(x)$ (at the x grid points) and the second row contains the values of y'_x.

The results `solN.y(1,:)` tell us that the solution $y(x)$ is wrong (with unexpected behavior):

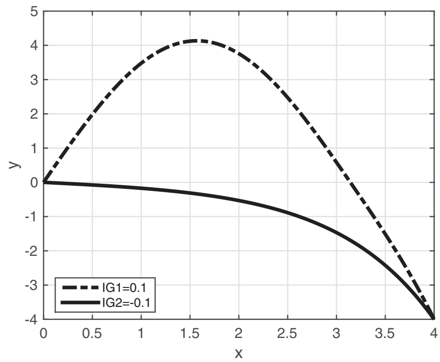


Figure 21.10: Nonuniqueness of numerical solutions of the nonlinear boundary value problem (21.3.3.3).

```
ans = 1.0e+04 *
Columns 1 through 5
0.000100000000000   0.714160289360580   1.400775911995384   2.588166088502601
3.381556897420243
Columns 6 through 10
3.589840740907997   3.660169584376011   3.589840740907997   3.381556897420243
2.588166088502601
Columns 11 through 13
1.400775911995384   0.714160289360580   0.000100000000000
```

► **Nonlinear boundary value problems.**

In addition to the nonlinear boundary value problem

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y(b) = \beta,$$

consider the initial value problem

$$y''_{xx} = f(x, y, y'_x), \qquad y(a) = \alpha, \quad y'_x(a) = s,$$

where $x \in [a, b]$. The real parameter s describes the initial slope of the solution curve.

Let $f(x, y, u)$ be a continuous function satisfying the Lipschitz condition with respect to y and u. Then, by the Picard–Lindelöf theorem, for each s there exists a unique solution $y(x, s)$ of the above initial value problem.

To find a solution of the nonlinear boundary value problem, we choose a value of the parameter s such that $y(b, s) = \beta$; i.e., we have to solve the nonlinear equation $F(s) = y(b, s) - \beta = 0$ by applying one of the known numerical methods.

Example 21.23. *Second-order nonlinear ODE. Boundary value problem. Nonuniqueness.* Solving a boundary value problem for the second-order nonlinear ODE

$$y''_{xx} + k|y| = 0, \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{21.3.3.3}$$

where $a = 0$, $b = 4$, $\alpha = 0$, $\beta = -4$, and $k = 1$, we can find two numerical solutions by applying the `bvp4c` solvers twice with distinct guess functions (`solInit1`, `solInit1`), where `IG1`$=0.1$ and

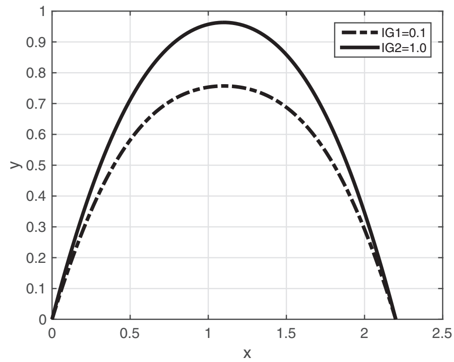


Figure 21.11: Nonuniqueness of numerical solutions of the nonlinear boundary value problem (21.3.3.4).

IG2$= -0.1$ are two distinct initial guesses. In MATLAB, this can be observed with the following functions (M-files `bvp3.m`, `bc3.m`):

```
function dydx=bvp3(x,y); dydx=[y(2); -1*abs(y(1))]; end
function res=bc3(ya,yb); res=[ya(1); yb(1)+4]; end
```

and the main program:

```
clear all; close all; echo on; format long;
a=0; b=4; N=100; IG1=0.1; IG2=-0.1;
solInit1=bvpinit(linspace(a,b,N),[0 IG1]);
solInit2=bvpinit(linspace(a,b,N),[0 IG2]);
solN1=bvp4c(@bvp3,@bc3,solInit1);
solN2=bvp4c(@bvp3,@bc3,solInit2);
x=linspace(a,b,N); y1=deval(solN1,x); y2=deval(solN2,x);
figure(1);
plot(x,y1(1,:),'k-.',x,y2(1,:),'k-','LineWidth',3);
grid on; xlabel('x'); ylabel('y');
legend('IG1=0.1','IG2=-0.1');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

The two numerical solutions of this boundary value problem are presented in Fig. 21.10.

Example 21.24. *Second-order nonlinear ODE. Boundary value problem. Nonuniqueness.* Solving a boundary value problem for the second-order nonlinear ODE

$$y''_{xx} + k(1 + y^2) = 0; \qquad y(a) = \alpha, \quad y(b) = \beta, \tag{21.3.3.4}$$

where $a=0$, $b=3$, $\alpha=0$, $\beta=0$, and $k=2$, we can find two positive numerical solutions by applying the `bvp5c` solvers twice with distinct guess functions (`solInit1` and `solInit1`), where IG1$=$ 0.1 and IG2$= 1.0$ are two distinct initial guesses. In this case, we apply the other MATLAB solver (`bvp5c`), since the solver `bvp4c` does not approach any reasonable accuracy (e.g., $1e-1$). Also, we include some options (see function `odeset`), e.g., `NonNegative`, `RelTol`, and `AbsTol`. In MATLAB, this can be observed with the following functions (M-files `bvp4.m`, `bc4.m`):

```
function dydx=bvp4(x,y); dydx=[y(2); -2*(1+y(1).^2)]; end
function res=bc4(ya,yb); res=[ya(1); yb(1)]; end
```

and the main program:

```
clear all; close all; echo on; format long;
a=0; b=2.2; N=100; IG1=0.1; IG2=1.;
options=odeset('NonNegative',1,'RelTol',1e-1,'AbsTol',1e-1);
solInit1=bvpinit(linspace(a,b,N),[0 IG1]);
solInit2=bvpinit(linspace(a,b,N),[0 IG2]);
solN1=bvp5c(@bvp4,@bc4,solInit1,options);
solN2=bvp5c(@bvp4,@bc4,solInit2,options);
x=linspace(a,b,N); y1=deval(solN1,x); y2=deval(solN2,x);
figure(1);
plot(x,y1(1,:),'k-.',x,y2(1,:),'k-','LineWidth',3);
grid on; xlabel('x'); ylabel('y');
legend('IG1=0.1','IG2=1.0');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

The two numerical solutions of this boundary value problem are presented in Fig. 21.11.

21.3.4 Eigenvalue Problems: Examples of Numerical Solutions

Consider *eigenvalue problems*, i.e., boundary value problems that include a real parameter. Discrete values of the parameter that satisfy the ODE are called *eigenvalues* of the problem. For each eigenvalue λ_n, there exists a *nontrivial* solution $y_n(x)$ that satisfies the problem, and it is called the *eigenfunction* associated with λ_n. The set of real eigenvalues is infinite, and the set of eigenfunctions is complete.

Consider the *Sturm–Liouville system* or *Sturm–Liouville eigenvalue problem*, i.e., the second-order linear homogeneous differential equation

$$\big(p(x)y'_x\big)'_x + \big(q(x) + \lambda w(x)\big)y = 0, \qquad a < x < b,$$

together with the boundary conditions

$$\beta_1 y(a) + \beta_2 y'_x(a) = 0, \quad \beta_3 y(b) + \beta_4 y'_x(b) = 0.$$

If $p(x)$, $q(x)$, and $w(x)$ are continuous functions and if both $p(x)$ and $w(x)$ are positive on $[a, b]$, then the *Sturm–Liouville eigenvalue problem* is called *regular*. Let us find the eigenvalues and eigenfunctions for some regular Sturm–Liouville eigenvalue problems.

Example 21.25. *Sturm–Liouville problem. Homogeneous Dirichlet boundary conditions.*
We solve the Sturm–Liouville eigenvalue problem

$$y''_{xx} + \lambda y = 0, \qquad y(a) = 0, \quad y(b) = 0, \tag{21.3.4.1}$$

i.e., a homogeneous linear two-point boundary value problem with a parameter λ and with the homogeneous Dirichlet boundary conditions,* where $a \le x \le b$, $a = 0$, $b = \pi$, $p(x) = 1$, $w(x) = 1$, and $q(x) = 0$, by applying the predefined function `bvp4c`. In MATLAB, this eigenvalue problem can be solved, e.g., for the first two eigenvalues and the corresponding eigenfunctions with the following functions (M-files `evp1.m`, `evp1bc1.m`, `evp1Guess1`, `evp1bc2.m`, and `evp1Guess2`):

*These boundary conditions are called *separated conditions*, for which there exist a complete set of orthogonal eigenfunctions.

```
function dydx=evp1(x,y,lambda); dydx=[y(2);-lambda*y(1)]; end
function res=evp1bc1(ya,yb,lambda); res=[ya(1);yb(1);yb(2)+1]; end
function res=evp1bc2(ya,yb,lambda); res=[ya(1);yb(1);yb(2)-1]; end
function v=evp1Guess1(x); v=[sin(x);cos(x)]; end
function v=evp1Guess2(x); v=[cos(x);sin(x)]; end
```

and the main program:

```
clear all; close all; echo on; format long;
a=0; b=pi; N=10; lambda=0.5;
solInit1=bvpinit(linspace(a,b,N),@evp1Guess1,lambda);
solInit2=bvpinit(linspace(a,b,N),@evp1Guess2,lambda);
solN1=bvp4c(@evp1,@evp1bc1,solInit1);
solN2=bvp4c(@evp1,@evp1bc2,solInit2);
fprintf('The first eigenvalue = %15.6f.\n',solN1.parameters)
fprintf('The second eigenvalue = %15.6f.\n',solN2.parameters)
x=linspace(a,b); y1=deval(solN1,x); y2=deval(solN2,x);
figure(1); plot(x,y1(1,:),'k-','LineWidth',3);
axis([0 pi 0 1]); grid on; xlabel('x'); ylabel('y');
figure(2); plot(x,y2(1,:),'k-','LineWidth',3);
axis([0 pi -0.6 0.6]); grid on; xlabel('x'); ylabel('y');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

Note that the predefined function `bvp4c` makes it easy to solve Sturm–Liouville problems involving unknown parameters. If there are unknown parameters, we have to include estimates for them (as the third argument of `bvpinit`). Also, we have to include the vector of unknown parameters (as the third argument of the functions for evaluating the ODEs and the discrepancy in the boundary conditions). If there are unknown parameters, then the solution structure (in our problem, `solN1`, `solN2`) has the *parameters* field, which contains the vector of parameters computed by the solver `bvp4c`.

When solving boundary value problems with `bvp4c`, we have to provide a *guess for the solution*. The guess is included in `bvp4c` as a structure formed by the function `bvpinit`. The first argument of `bvpinit` is a *guess for the mesh*. In our problem, we try 10 equally spaced points in $[0, \pi]$. The second argument is a *guess for the solution* on the specified mesh. In our problem, the solution has two components, $y(x)$ and y'_x, and we try a *function guess*, e.g., $[\sin(x), \cos(x)]$. We guess that λ is about 0.5.

As a result, we have the first eigenvalue $\lambda_1 \approx 1.000028$ and the second eigenvalue $\lambda_2 \approx 4.000130$.

The first two eigenfunctions of this Sturm–Liouville problem are presented in Fig. 21.12.

Example 21.26. *Sturm–Liouville problem. Homogeneous Neumann boundary conditions.* We solve the Sturm–Liouville eigenvalue problem

$$y''_{xx} + \lambda y = 0, \qquad y'_x(a) = 0, \quad y'_x(b) = 0, \tag{21.3.4.2}$$

i.e., a homogeneous linear two-point boundary value problem with the parameter λ and with the homogeneous Neumann boundary conditions, where $a \le x \le b$, $a = 0$, $b = \pi$, $p(x) = 1$, $w(x) = 1$, and $q(x) = 0$, by applying the predefined function `bvp4c`. In MATLAB, this eigenvalue problem can be solved, e.g., for the second and third eigenvalues and the corresponding eigenfunctions, with the following functions (M-files `evp2.m`, `evp2bc1.m`, `evp2Guess1`, `evp2bc2.m`, and `evp2Guess2`):

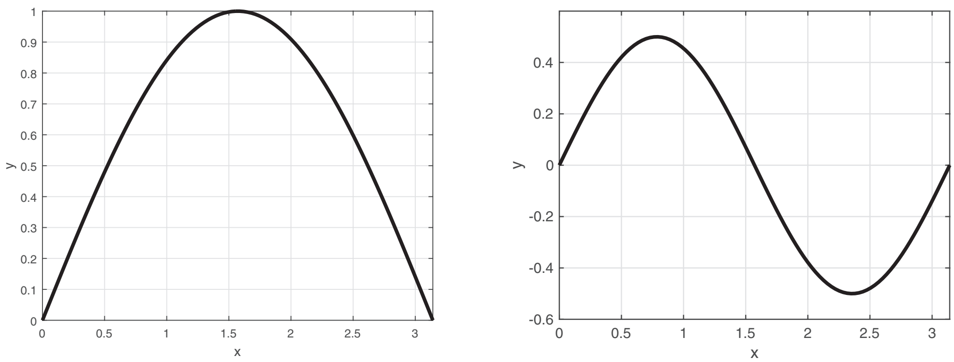


Figure 21.12: The first two eigenfunctions of the Sturm–Liouville problem (21.3.4.1).

```
function dydx=evp2(x,y,lambda); dydx=[y(2);-lambda*y(1)]; end
function res=evp2bc1(ya,yb,lambda); res=[ya(2);yb(2);yb(1)+1]; end
function res=evp2bc2(ya,yb,lambda); res=[ya(2);yb(2);yb(1)-1]; end
function v=evp2Guess1(x); v=[cos(x);sin(x)]; end
function v=evp2Guess2(x); v=[sin(x);cos(x)]; end
```

and the main program:

```
clear all; close all; echo on; format long;
a=0; b=pi; N=10; lambda1=0.5; lambda2=3.5;
solInit1=bvpinit(linspace(a,b,N),@evp2Guess1,lambda1);
solInit2=bvpinit(linspace(a,b,N),@evp2Guess2,lambda2);
solN1=bvp4c(@evp2,@evp2bc1,solInit1);
solN2=bvp4c(@evp2,@evp2bc2,solInit2);
fprintf('The second eigenvalue = %15.6f.\n',solN1.parameters)
fprintf('The third eigenvalue = %15.6f.\n',solN2.parameters)
x=linspace(a,b); y1=deval(solN1,x); y2=deval(solN2,x);
figure(1); plot(x,y1(1,:),'k-','LineWidth',3);
axis([0 pi -1 1]); grid on; xlabel('x'); ylabel('y');
figure(2); plot(x,y2(1,:),'k-','LineWidth',3);
axis([0 pi -1 1]); grid on; xlabel('x'); ylabel('y');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

As a result, we have the second eigenvalue $\lambda_2 \approx 1.000028$ and the third eigenvalue $\lambda_3 \approx 4.000130$. The second and third eigenfunctions of this Sturm–Liouville problem are presented in Fig. 21.13.

⊙ *Literature for Section .3*: E. Fehlberg (1970) D. Barton, I. M. Willer, and R. V. M. Zahar (1971) G. E. Forsythe, M. A. Malcolm, and C. B. Moler (1977), L. F. Shampine and H. A. Watts (1977), J. H. Verner (1978), L. F. Shampine and H. A. Watts (1979), S. D. Conte and C. de Boor (1980), A. C. Hindmarsh (1983), H. W. Enright, K. R. Jackson, S. P. Norsett, P. G. Thomsen (1986), L. Fox and D. F. Mayers (1987), P. Deuflhard, B. Fiedler, P. Kunkel (1987), Ch. Lubich (1989), W. H. Enright (1989), J. R. Cash and A. H. Karp (1990), M. E. Hosea and L. F. Shampine (1996), E. Hairer and G. Wanner (1996), L. F. Shampine and M. W. Reichelt (1997), L. F. Shampine et. al (1999), L. F. Shampine, M. W. Reichelt, and J. Kierzenka (1999), L. F. Shampine and R. M. Corless (2000), J. Kierzenka and L. F. Shampine (2001), L. F. Shampine and S. Thompson (2001), W. E. Boyce and R. C. DiPrima (2004).

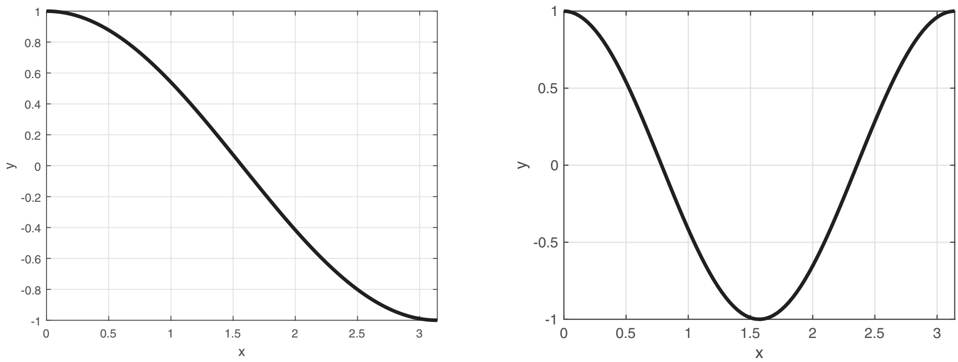


Figure 21.13: The second and third eigenfunctions of the Sturm–Liouville problem (21.3.4.2).

21.4 Numerical Solutions of Systems of ODEs

In this section, we numerically solve initial value problems for systems of differential equations of various classes in MATLAB [e.g., see Murphy (1960), Lapidus et al.(1973), Kamke (1977), MacDonald (1989), Lambert (1991), Zwillinger (1997), Polyanin and Manzhirov (2007)]. We consider the following classes of ODE systems: first-order linear and nonlinear systems of two ODEs, higher-order ODEs with transformations to first-order systems of ODEs, first-order systems of general form, and second-order systems. To this end, we define differential systems in M-files.

21.4.1 First-Order Systems of Two Equations

Consider the system of two first-order ordinary differential equations with the initial conditions

$$u'_x = f_1(x, u, v), \quad v'_x = f_2(x, u, v), \qquad u(a) = u_0, \quad v(a) = v_0.$$

The unknown functions are $u(x)$ and $v(x)$, and $x \in [a, b]$.

To obtain numerical solutions, we can apply predefined functions or, alternatively, construct solutions step by step by applying known numerical methods to each equation of the system.

Let us numerically solve some first-order systems of two differential equations (linear and nonlinear).

► **First-order linear systems.**

Example 21.27. *First-order linear system. Exact, numerical, and graphical solutions.* For the first-order linear system with the initial conditions

$$u'_x = v, \quad v'_x = x - u - 2v, \qquad u(a) = \alpha, \quad v(a) = \beta, \tag{21.4.1.1}$$

where $a \le x \le b$, $a = 0$, $b = 2$, $\alpha = 1$, and $\beta = 1$, we compute the numerical solution `Y` (where `Y(:,1)` and `Y(:,2)` are u and v, respectively) by applying the predefined function `ode45`, com-

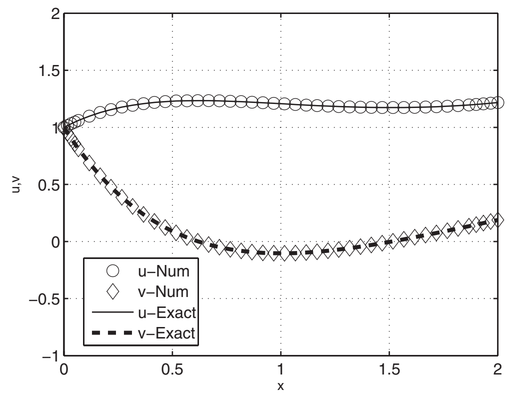


Figure 21.14: The numerical and exact solutions of the linear Cauchy problem (21.4.1.1).

pare the numerical results with the exact solution (uEx, vEx)*, and plot the exact and numerical solutions. We write the MATLAB M-file containing the differential system (sys1.m):

```
function Yprime=sys1(x,Y); Yprime=[Y(2);x-Y(1)-2*Y(2)]; end
```

and the main program:

```
clear all; close all; echo on; format long;
Y0=[1 1]; X=[0,2]; [x,Y]=ode45(@sys1,X,Y0)
plot(x,Y(:,1),'ko','MarkerSize',9);
hold on; grid on; axis([0 2 -1 2]);
plot(x,Y(:,2),'kd','MarkerSize',9);
uEx=x+(3.*x+3).*exp(-x)-2; vEx=-3.*x.*exp(-x)+1;
plot(x,uEx,'k-','LineWidth',1); plot(x,vEx,'k--','LineWidth',3);
grid on; xlabel('x'); ylabel('u,v');
legend('u-Num','v-Num','u-Exact','v-Exact');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg; hold off;
```

► First-order nonlinear systems.

Example 21.28. *First-order nonlinear system. Numerical and graphical solutions.* For the first-order nonlinear system with the initial conditions

$$u'_x = uv, \quad v'_x = u + v, \qquad u(a) = \alpha, \quad v(a) = \beta, \tag{21.4.1.2}$$

where $a = 0$, $b = 1$, $\alpha = 1$, and $\beta = 1$, we obtain the numerical solution Y (where Y(:,1) and Y(:,2) are u and v, respectively) by applying the predefined function ode45 and plot the results. We write the MATLAB M-file containing the differential system (sys2.m):

```
function Yprime=sys2(x,Y); Yprime=[Y(1)*Y(2);Y(1)+Y(2)]; end
```

and the main program:

*For more details, see Chapter 18.

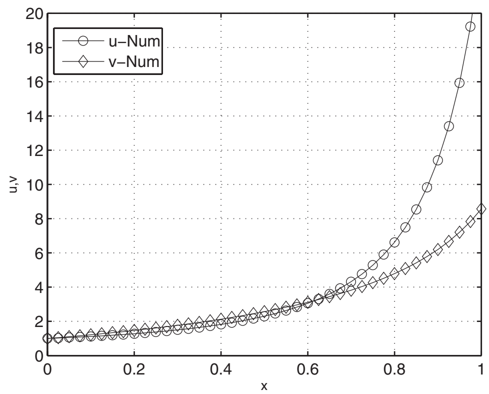


Figure 21.15: The numerical solution of the nonlinear Cauchy problem (21.4.1.2).

```
clear all; close all; echo on; format long;
Y0=[1 1]; X=[0,1]; [x,Y]=ode45(@sys2,X,Y0)
plot(x,Y(:,1),'k-o','MarkerSize',7); hold on;
plot(x,Y(:,2),'k-d','MarkerSize',7); axis([0 1 0 20]);
grid on; xlabel('x'); ylabel('u,v'); legend('u-Num','v-Num');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg; hold off;
```

► **Higher-order ODEs.**

If we consider an ordinary differential equation of order n ($n > 1$) with n initial conditions

$$\begin{aligned} &y_x^{(n)} = f(x, y, y'_x, \dots, y_x^{(n-1)}) \qquad (x \in [a, b]), \\ &y(a) = y_0, \quad y'_x(a) = y_1, \quad \dots, \quad y_x^{(n-1)}(a) = y_n, \end{aligned}$$

then we can always obtain solutions of this higher-order differential equation by transforming it to an equivalent system of n first-order differential equations and by applying an appropriate numerical method (e.g., `ode45`) to this system of differential equations. To this end, we can apply the predefined function `odeToVectorField` and then generate a MATLAB function from the symbolic expression obtained (i.e., the system of first-order differential equations) by applying the predefined function `matlabFunction`:

```
    V=odeToVectorField(eqn1,...,eqnN)
[V,Y]=odeToVectorField(eqn1,...,eqnN)
 mFun=matlabFunction(V,'vars',{'x','Y'})
```

where `V` is a symbolic vector representing the resulting system of first-order differential equations, `Y` is a symbolic vector representing the substitutions made during the transformation, and `M` is a MATLAB function generated from a symbolic expression.

Example 21.29. *Van der Pol equation. Cauchy problem. Numerical and graphical solutions.*

For the van der Pol equation with the initial conditions

$$y''_{xx} + \mu(y^2 - 1)y'_x + y = 0, \quad y(a) = \alpha, \quad y'_x(a) = \beta, \tag{21.4.1.3}$$

where $x \in [a, b]$, $a = 0$, $b = 60$, $\alpha = 1$, $\beta = 0$, and $\mu = \frac{1}{8}$, we can compute a numerical solution as follows:

1. By applying the predefined function `odeToVectorField`, we transform the second-order ODE into an equivalent system of two first-order differential equations (`Sys1`):

```
Sys1=[Y[2]; -Y[1]-Y[2]*(Y[1]^2/8-1/8)].
```

2. By applying the predefined function `matlabFunction`,* we generate a MATLAB function (`mFun`) from this system of first-order differential equations:

```
mFun=@(x,Y)[Y(2); -Y(1)-Y(2).*(Y(1).^2.*(1.0./8.0)-1.0./8.0)].
```

3. By applying the MATLAB numerical solver `ode45`

```
solN=ode45(mFun,[a b],IC)
```

to this system of differential equations, we obtain a numerical solution (`solN`) and graphical solutions, a phase portrait of the solution, and a graph of $u(x)$ and $v(x)$ (see Fig. 21.16) as follows:

```
clear all; close all; echo on; format long;
a=0; b=60; N=500; alpha=1; beta=0; IC=[1 0];
syms x y(x) Sys1
Sys1=odeToVectorField(diff(y,2)+(1/8)*(y^2-1)*diff(y)+y==0)
mFun=matlabFunction(Sys1,'vars',{'x','Y'})
solN=ode45(mFun,[a b],IC);
x=linspace(a,b,N); yN=deval(solN,x);
figure(1);
plot(yN(1,:),yN(2,:),'k-','LineWidth',1);
grid on; xlabel('u'); ylabel('v'); axis([-2.5 2.5 -2.5 2.5]);
figure(2);
plot(x,yN(1,:),'k-','LineWidth',1); hold on;
plot(x,yN(2,:),'k-.','LineWidth',1); axis([a b -2.5 2.5]);
grid on; xlabel('x'); ylabel('u,v'); legend('u(x)','v(x)');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg; hold off;
```

21.4.2 First-Order Systems of General Form

Consider the first-order system of differential equations of general form with the initial conditions

$$(y'_x)_i = f_i(x, y_1, \dots, y_n), \quad y_i(a) = (y_0)_i \quad (i = 1, \dots, n).$$

The unknown functions are $y_1(x), \dots, y_n(x)$, and $x \in [a, b]$. To obtain numerical solutions, we can apply predefined functions or, alternatively, construct solutions step by step by applying known numerical methods.

As an example, consider the *Lorenz system*, which is a dissipative chaotic system with a strange attractor. These features can be observed for certain values of the system parameters

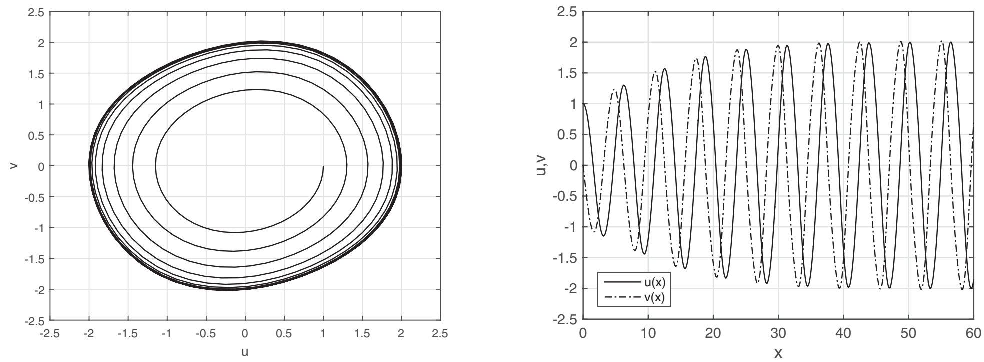


Figure 21.16: Graphical solutions of the van der Pol equation $y''_{xx} + \mu(y^2 - 1)y'_x + y = 0$ (the equivalent system of two first-order ODEs).

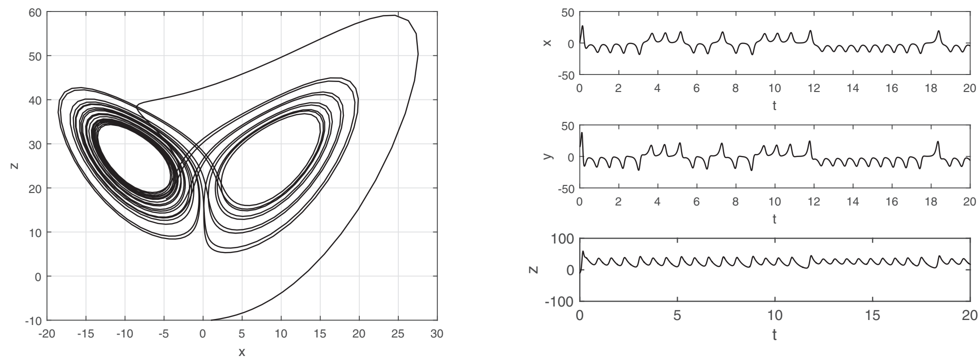


Figure 21.17: Graphical solutions of the Lorenz system (21.4.2.1).

and initial conditions. This system models an unstable thermally convecting fluid (heated from below) and also arises in other simplified models.

Example 21.30. *The Lorenz system. Cauchy problem. Numerical and graphical solutions.* For the nonlinear system of first-order ODEs, e.g., the Lorenz system [see Sparrow (1982)],

$$x'_t = \sigma(y - x), \quad y'_t = \rho x - y - xz, \quad z'_t = xy - \beta z,$$
$$x(0) = 1, \quad y(0) = 15, \quad z(0) = 10, \tag{21.4.2.1}$$

where σ, ρ, and β are the system parameters, one can investigate the behavior of the system by varying the system parameters σ, ρ, and β and observe the strange attractor.

We obtain the numerical solution `W` (where `W(:,1)`, `W(:,2)`, and `W(:,3)` are $x(t)$, $y(t)$, and $z(t)$), respectively, by applying the predefined function `ode45` and plot the results.
We write the MATLAB M-file containing the differential system (`sys3.m`):

```
function Wprime=sys3(t,W); sigma=15; beta=3; rho=28;
 Wprime=[sigma*(W(2)-W(1);rho*W(1)-W(2)-W(1)*W(3);-beta*W(3)+W(1)*W(2)];
end
```

*The MATLAB predefined functions for solving initial value problems do not accept symbolic expressions (as an input), and so we have to convert the system obtained to a MATLAB function.

and the main program:

```
clear all; close all; echo on; format long;
a=0; b=20; W0=[1 15 -10]; t=[a,b];
[t,W]=ode45(@sys3,t,W0)
figure(1);
plot(W(:,1),W(:,3),'k-','LineWidth',1);
grid on; xlabel('x'); ylabel('z');
figure(2);
subplot(3,1,1); plot(t,W(:,1),'k-','LineWidth',1)
xlabel('t'); ylabel('x');
subplot(3,1,2); plot(t,W(:,2),'k-','LineWidth',1)
xlabel('t'); ylabel('y');
subplot(3,1,3); plot(t,W(:,3),'k-','LineWidth',1)
xlabel('t'); ylabel('z');
set(gca,'FontSize',12); set(gca,'FontName','Arial');
set(gca,'LineWidth',1); shg;
```

The Lorenz strange attractor (the plot of z versus x) and each component of the solution (x, y, z as functions of t) are presented in Fig. 21.17.

Remark 21.6. If a given problem (including a single differential equation or a system of differential equations) is of order 2 or higher, we have to convert this problem into an equivalent system of first-order differential equations and apply an appropriate numerical method (e.g., `ode45`) to this system of differential equations. For this conversion, we can apply the predefined function `odeToVectorField` and then generate a MATLAB function from the symbolic expression obtained (i.e., a system of first-order differential equations) by applying the predefined function `matlabFunction`.

⊙ *Literature for Section .4*: . M. Murphy (1960), L. Lapidus, R. C. Aiken, and Y. A. Liu (1973), E. Kamke (1977), C. Sparrow (1982), N. MacDonald (1989), J. D. Lambert (1991), D. Zwillinger (1997), A. D. Polyanin and A. V. Manzhirov (2007).

Part IV

Supplements

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Chapter S1

Elementary Functions and Their Properties

◆ *Throughout Chapter S1, it is assumed that n is a positive integer unless otherwise specified.*

S1.1 Power, Exponential, and Logarithmic Functions

S1.1.1 Properties of the Power Function

Basic properties of the power function:

$$x^\alpha x^\beta = x^{\alpha+\beta}, \quad (x_1x_2)^\alpha = x_1^\alpha x_2^\alpha, \quad (x^\alpha)^\beta = x^{\alpha\beta},$$

for any α and β, where $x > 0$, $x_1 > 0$, $x_2 > 0$.

Differentiation and integration formulas:

$$(x^\alpha)' = \alpha x^{\alpha-1}, \qquad \int x^\alpha\,dx = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1} + C & \text{if } \alpha \neq -1, \\ \ln|x| + C & \text{if } \alpha = -1. \end{cases}$$

The Taylor series expansion in a neighborhood of an arbitrary point:

$$x^\alpha = \sum_{n=0}^{\infty} C_\alpha^n x_0^{\alpha-n}(x-x_0)^n \quad \text{for} \quad |x-x_0| < |x_0|,$$

where $C_\alpha^n = \dfrac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$ are binomial coefficients.

S1.1.2 Properties of the Exponential Function

Basic properties of the exponential function:

$$a^{x_1}a^{x_2} = a^{x_1+x_2}, \qquad a^x b^x = (ab)^x, \qquad (a^{x_1})^{x_2} = a^{x_1x_2},$$

where $a > 0$ and $b > 0$.

Number e, *base of natural (Napierian) logarithms*, and the function e^x:

$$e = \lim_{n\to\infty}\Bigl(1+\frac{1}{n}\Bigr)^n = 2.718281\ldots, \qquad e^x = \lim_{n\to\infty}\Bigl(1+\frac{x}{n}\Bigr)^n.$$

The formula for passing from an arbitrary base a to the base e of natural logarithms:

$$a^x = e^{x\ln a}.$$

The inequality

$$a^{x_1} > a^{x_2} \qquad \Longleftrightarrow \qquad \begin{cases} x_1 > x_2 & \text{if } a > 1, \\ x_1 < x_2 & \text{if } 0 < a < 1. \end{cases}$$

The limit relations for any $a > 1$ and $b > 0$:

$$\lim_{x\to+\infty}\frac{a^x}{|x|^b} = \infty, \qquad \lim_{x\to-\infty} a^x|x|^b = 0.$$

Differentiation and integration formulas:

$$\begin{aligned} (e^x)' &= e^x, & \int e^x\,dx &= e^x + C; \\ (a^x)' &= a^x\ln a, & \int a^x\,dx &= \frac{a^x}{\ln a} + C. \end{aligned}$$

The expansion in power series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

S1.1.3 Properties of the Logarithmic Function

By definition, the logarithmic function is the inverse of the exponential function. The following equivalence relation holds:

$$y = \log_a x \qquad \Longleftrightarrow \qquad x = a^y,$$

where $a > 0$, $a \neq 1$.

Basic properties of the logarithmic function:

$$\begin{aligned} a^{\log_a x} &= x, & \log_a(x_1x_2) &= \log_a x_1 + \log_a x_2, \\ \log_a(x^k) &= k\log_a x, & \log_a x &= \frac{\log_b x}{\log_b a}, \end{aligned}$$

where $x > 0$, $x_1 > 0$, $x_2 > 0$, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$.

The simplest inequality:

$$\log_a x_1 > \log_a x_2 \qquad \Longleftrightarrow \qquad \begin{cases} x_1 > x_2 & \text{if } a > 1, \\ x_1 < x_2 & \text{if } 0 < a < 1. \end{cases}$$

For any $b > 0$, the following limit relations hold:

$$\lim_{x\to+\infty} \frac{\log_a x}{x^b} = 0, \qquad \lim_{x\to+0} x^b \log_a x = 0.$$

The logarithmic function with the base e (*base of natural logarithms* or *Napierian base*) is denoted by

$$\log_e x = \ln x,$$

where $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = 2.718281\ldots$

Formulas for passing from an arbitrary base a to the Napierian base e:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Differentiation and integration formulas:

$$(\ln x)' = \frac{1}{x}, \qquad \int \ln x\,dx = x\ln x - x + C.$$

Expansion in power series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}, \qquad |x| < 1;$$

$$\ln\left(\frac{x+1}{x-1}\right) = \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \cdots = 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)x^{2k-1}}, \qquad |x| > 1;$$

$$\begin{aligned}\ln x &= 2\left(\frac{x-1}{x+1}\right) + \frac{2}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{2}{5}\left(\frac{x-1}{x+1}\right)^5 + \frac{2}{7}\left(\frac{x-1}{x+1}\right)^7 + \cdots \\ &= 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}, \qquad x > 0.\end{aligned}$$

S1.2 Trigonometric Functions

S1.2.1 Simplest Relations

$$\begin{aligned} &\sin^2 x + \cos^2 x = 1, && \tan x \cot x = 1, \\ &\sin(-x) = -\sin x, && \cos(-x) = \cos x, \\ &\tan x = \frac{\sin x}{\cos x}, && \cot x = \frac{\cos x}{\sin x}, \\ &\tan(-x) = -\tan x, && \cot(-x) = -\cot x, \\ &1 + \tan^2 x = \frac{1}{\cos^2 x}, && 1 + \cot^2 x = \frac{1}{\sin^2 x}. \end{aligned}$$

S1.2.2 Reduction Formulas

$$\sin(x \pm 2n\pi) = \sin x, \qquad \cos(x \pm 2n\pi) = \cos x,$$
$$\sin(x \pm n\pi) = (-1)^n \sin x, \qquad \cos(x \pm n\pi) = (-1)^n \cos x,$$
$$\sin\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = \pm(-1)^n \cos x, \qquad \cos\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = \mp(-1)^n \sin x,$$
$$\sin\Bigl(x \pm \frac{\pi}{4}\Bigr) = \frac{\sqrt{2}}{2}(\sin x \pm \cos x), \qquad \cos\Bigl(x \pm \frac{\pi}{4}\Bigr) = \frac{\sqrt{2}}{2}(\cos x \mp \sin x),$$
$$\tan(x \pm n\pi) = \tan x, \qquad \cot(x \pm n\pi) = \cot x,$$
$$\tan\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = -\cot x, \qquad \cot\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = -\tan x,$$
$$\tan\Bigl(x \pm \frac{\pi}{4}\Bigr) = \frac{\tan x \pm 1}{1 \mp \tan x}, \qquad \cot\Bigl(x \pm \frac{\pi}{4}\Bigr) = \frac{\cot x \mp 1}{1 \pm \cot x},$$

where $n = 1, 2, \dots$

S1.2.3 Relations between Trigonometric Functions of Single Argument

$$\sin x = \pm\sqrt{1-\cos^2 x} = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} = \pm\frac{1}{\sqrt{1+\cot^2 x}},$$
$$\cos x = \pm\sqrt{1-\sin^2 x} = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \pm\frac{\cot x}{\sqrt{1+\cot^2 x}},$$
$$\tan x = \pm\frac{\sin x}{\sqrt{1-\sin^2 x}} = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{1}{\cot x},$$
$$\cot x = \pm\frac{\sqrt{1-\sin^2 x}}{\sin x} = \pm\frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\tan x}.$$

The sign before the radical is determined by the quarter in which the argument takes its values.

S1.2.4 Addition and Subtraction of Trigonometric Functions

$$\sin x + \sin y = 2\sin\Bigl(\frac{x+y}{2}\Bigr)\cos\Bigl(\frac{x-y}{2}\Bigr),$$
$$\sin x - \sin y = 2\sin\Bigl(\frac{x-y}{2}\Bigr)\cos\Bigl(\frac{x+y}{2}\Bigr),$$
$$\cos x + \cos y = 2\cos\Bigl(\frac{x+y}{2}\Bigr)\cos\Bigl(\frac{x-y}{2}\Bigr),$$
$$\cos x - \cos y = -2\sin\Bigl(\frac{x+y}{2}\Bigr)\sin\Bigl(\frac{x-y}{2}\Bigr),$$
$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y),$$
$$\sin^2 x - \cos^2 y = -\cos(x+y)\cos(x-y),$$
$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}, \qquad \cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y},$$
$$a\cos x + b\sin x = r\sin(x+\varphi) = r\cos(x-\psi).$$

Here $r = \sqrt{a^2+b^2}$, $\sin\varphi = a/r$, $\cos\varphi = b/r$, $\sin\psi = b/r$, and $\cos\psi = a/r$.

S1.2.5 Products of Trigonometric Functions

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)],$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)],$$
$$\sin x \cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)].$$

S1.2.6 Powers of Trigonometric Functions

$$\cos^2 x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2},$$
$$\cos^3 x = \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x,$$
$$\cos^4 x = \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8},$$
$$\cos^5 x = \tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x,$$
$$\sin^2 x = -\tfrac{1}{2}\cos 2x + \tfrac{1}{2},$$
$$\sin^3 x = -\tfrac{1}{4}\sin 3x + \tfrac{3}{4}\sin x,$$
$$\sin^4 x = \tfrac{1}{8}\cos 4x - \tfrac{1}{2}\cos 2x + \tfrac{3}{8},$$
$$\sin^5 x = \tfrac{1}{16}\sin 5x - \tfrac{5}{16}\sin 3x + \tfrac{5}{8}\sin x,$$
$$\cos^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^{k}\cos[2(n-k)x] + \frac{1}{2^{2n}}C_{2n}^{n},$$
$$\cos^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^{k}\cos[(2n-2k+1)x],$$
$$\sin^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^{n-k} C_{2n}^{k}\cos[2(n-k)x] + \frac{1}{2^{2n}}C_{2n}^{n},$$
$$\sin^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^{n-k} C_{2n+1}^{k}\sin[(2n-2k+1)x].$$

Here $n = 1,\, 2,\, \ldots$ and $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients $(0! = 1)$.

S1.2.7 Addition Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{1 \mp \tan x \tan y}{\tan x \pm \tan y}.$$

S1.2.8 Trigonometric Functions of Multiple Arguments

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \qquad \sin 2x = 2\sin x\cos x,$$
$$\cos 3x = -3\cos x + 4\cos^3 x, \qquad \sin 3x = 3\sin x - 4\sin^3 x,$$
$$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x, \qquad \sin 4x = 4\cos x\,(\sin x - 2\sin^3 x),$$
$$\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x, \qquad \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$$
$$\cos(2nx) = 1 + \sum_{k=1}^{n} (-1)^k 4^k \frac{n^2(n^2-1)\dots[n^2-(k-1)^2]}{(2k)!}\sin^{2k} x,$$
$$\cos[(2n+1)x] = \cos x\left\{1 + \sum_{k=1}^{n}(-1)^k \times \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!}\sin^{2k} x\right\},$$
$$\sin(2nx) = 2n\cos x\left[\sin x + \sum_{k=1}^{n}(-1)^k 4^k \frac{(n^2-1)(n^2-2^2)\dots(n^2-k^2)}{(2k-1)!}\sin^{2k-1} x\right],$$
$$\sin[(2n+1)x] = (2n+1)\left\{\sin x + \sum_{k=1}^{n}(-1)^k \times \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\sin^{2k+1} x\right\},$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}, \quad \tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}, \quad \tan 4x = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x},$$

where $n = 1,\ 2,\ \dots$

S1.2.9 Trigonometric Functions of Half Argument

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}, \qquad \cos^2\frac{x}{2} = \frac{1+\cos x}{2},$$
$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}, \qquad \cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x},$$
$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \qquad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}.$$

S1.2.10 Differentiation Formulas

$$\frac{d\sin x}{dx} = \cos x, \qquad \frac{d\cos x}{dx} = -\sin x, \qquad \frac{d\tan x}{dx} = \frac{1}{\cos^2 x}, \qquad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x}.$$

S1.2.11 Integration Formulas

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C,$$
$$\int \tan x\,dx = -\ln|\cos x| + C, \qquad \int \cot x\,dx = \ln|\sin x| + C,$$

where C is an arbitrary constant.

S1.2.12 Expansion in Power Series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad (|x| < \infty),$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad (|x| < \infty),$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1} + \cdots \qquad (|x| < \pi/2),$$
$$\cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \cdots + \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1} + \cdots \right) \qquad (0 < |x| < \pi),$$
$$\frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} + \cdots \qquad (|x| < \pi/2),$$
$$\frac{1}{\sin x} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \cdots + \frac{(-1)^{n-1} 2(2^{2n-1}-1) B_{2n}}{(2n)!} x^{2n-1} + \cdots \qquad (0 < |x| < \pi),$$

where B_n and E_n are Bernoulli and Euler numbers (see Sections 30.1.3 and 30.1.4).

S1.2.13 Representation in the Form of Infinite Products

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \ldots \left(1 - \frac{x^2}{n^2\pi^2}\right) \ldots$$
$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{9\pi^2}\right)\left(1 - \frac{4x^2}{25\pi^2}\right) \ldots \left(1 - \frac{4x^2}{(2n+1)^2\pi^2}\right) \ldots$$

S1.2.14 Euler and de Moivre Formulas. Relationship with Hyperbolic Functions

$$e^{y+ix} = e^y(\cos x + i \sin x), \quad (\cos x + i \sin x)^n = \cos(nx) + i \sin(nx), \quad i^2 = -1,$$
$$\sin(ix) = i \sinh x, \quad \cos(ix) = \cosh x, \quad \tan(ix) = i \tanh x, \quad \cot(ix) = -i \coth x.$$

S1.3 Inverse Trigonometric Functions

S1.3.1 Definitions of Inverse Trigonometric Functions

Inverse trigonometric functions (*arc functions*) are the functions that are inverse to the trigonometric functions. Since the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$ are periodic, the corresponding inverse functions, denoted by $\operatorname{Arcsin} x$, $\operatorname{Arccos} x$, $\operatorname{Arctan} x$, $\operatorname{Arccot} x$, are multi-valued. The following relations define the multi-valued inverse trigonometric functions:

$$\sin\bigl(\operatorname{Arcsin} x\bigr) = x, \quad \cos\bigl(\operatorname{Arccos} x\bigr) = x,$$
$$\tan\bigl(\operatorname{Arctan} x\bigr) = x, \quad \cot\bigl(\operatorname{Arccot} x\bigr) = x.$$

These functions admit the following verbal definitions: $\operatorname{Arcsin} x$ is the angle whose sine is equal to x; $\operatorname{Arccos} x$ is the angle whose cosine is equal to x; $\operatorname{Arctan} x$ is the angle whose tangent is equal to x; $\operatorname{Arccot} x$ is the angle whose cotangent is equal to x.

The principal (single-valued) branches of the inverse trigonometric functions are denoted by

$$\begin{aligned}
&\arcsin x \equiv \sin^{-1} x \quad \text{(arcsine is the inverse of sine)},\\
&\arccos x \equiv \cos^{-1} x \quad \text{(arccosine is the inverse of cosine)},\\
&\arctan x \equiv \tan^{-1} x \quad \text{(arctangent is the inverse of tangent)},\\
&\operatorname{arccot} x \equiv \cot^{-1} x \quad \text{(arccotangent is the inverse of cotangent)}
\end{aligned}$$

and are determined by the inequalities

$$\begin{aligned}
-\tfrac{\pi}{2} \le \arcsin x \le \tfrac{\pi}{2}, \quad 0 \le \arccos x \le \pi \qquad (-1 \le x \le 1);\\
-\tfrac{\pi}{2} < \arctan x < \tfrac{\pi}{2}, \quad 0 < \operatorname{arccot} x < \pi \qquad (-\infty < x < \infty).
\end{aligned}$$

The following equivalent relations can be taken as definitions of single-valued inverse trigonometric functions:

$$\begin{aligned}
&y = \arcsin x, \quad -1 \le x \le 1 &&\Longleftrightarrow\quad x = \sin y, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2};\\
&y = \arccos x, \quad -1 \le x \le 1 &&\Longleftrightarrow\quad x = \cos y, \quad 0 \le y \le \pi;\\
&y = \arctan x, \quad -\infty < x < +\infty &&\Longleftrightarrow\quad x = \tan y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2};\\
&y = \operatorname{arccot} x, \quad -\infty < x < +\infty &&\Longleftrightarrow\quad x = \cot y, \quad 0 < y < \pi.
\end{aligned}$$

The multi-valued and the single-valued inverse trigonometric functions are related by the formulas

$$\begin{aligned}
\operatorname{Arcsin} x &= (-1)^n \arcsin x + \pi n,\\
\operatorname{Arccos} x &= \pm \arccos x + 2\pi n,\\
\operatorname{Arctan} x &= \arctan x + \pi n,\\
\operatorname{Arccot} x &= \operatorname{arccot} x + \pi n,
\end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$

S1.3.2 Simplest Formulas

$$\begin{aligned}
\sin(\arcsin x) = x, \quad \cos(\arccos x) = x,\\
\tan(\arctan x) = x, \quad \cot(\operatorname{arccot} x) = x.
\end{aligned}$$

S1.3.3 Some Properties

$$\begin{aligned}
\arcsin(-x) = -\arcsin x, \qquad \arccos(-x) = \pi - \arccos x,\\
\arctan(-x) = -\arctan x, \qquad \operatorname{arccot}(-x) = \pi - \operatorname{arccot} x,
\end{aligned}$$

$$\arcsin(\sin x) = \begin{cases} x - 2n\pi & \text{if } 2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2},\\ -x + 2(n+1)\pi & \text{if } (2n+1)\pi - \frac{\pi}{2} \le x \le 2(n+1)\pi + \frac{\pi}{2}, \end{cases}$$

$$\arccos(\cos x) = \begin{cases} x - 2n\pi & \text{if } 2n\pi \le x \le (2n+1)\pi,\\ -x + 2(n+1)\pi & \text{if } (2n+1)\pi \le x \le 2(n+1)\pi, \end{cases}$$

$$\begin{aligned}
\arctan(\tan x) = x - n\pi \quad \text{if} \quad n\pi - \tfrac{\pi}{2} < x < n\pi + \tfrac{\pi}{2},\\
\operatorname{arccot}(\cot x) = x - n\pi \quad \text{if} \quad n\pi < x < (n+1)\pi.
\end{aligned}$$

S1.3.4 Relations between Inverse Trigonometric Functions

$$\arcsin x + \arccos x = \tfrac{\pi}{2}, \quad \arctan x + \operatorname{arccot} x = \tfrac{\pi}{2};$$

$$\arcsin x = \begin{cases} \arccos\sqrt{1-x^2} & \text{if } 0 \le x \le 1, \\ -\arccos\sqrt{1-x^2} & \text{if } -1 \le x \le 0, \\ \arctan\dfrac{x}{\sqrt{1-x^2}} & \text{if } -1 < x < 1, \\ \operatorname{arccot}\dfrac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0; \end{cases}$$

$$\arccos x = \begin{cases} \arcsin\sqrt{1-x^2} & \text{if } 0 \le x \le 1, \\ \pi - \arcsin\sqrt{1-x^2} & \text{if } -1 \le x \le 0, \\ \arctan\dfrac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1, \\ \operatorname{arccot}\dfrac{x}{\sqrt{1-x^2}} & \text{if } -1 < x < 1; \end{cases}$$

$$\arctan x = \begin{cases} \arcsin\dfrac{x}{\sqrt{1+x^2}} & \text{for any } x, \\ \arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if } x \ge 0, \\ -\arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if } x \le 0, \\ \operatorname{arccot}\dfrac{1}{x} & \text{if } x > 0; \end{cases}$$

$$\operatorname{arccot} x = \begin{cases} \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if } x > 0, \\ \pi - \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if } x < 0, \\ \arctan\dfrac{1}{x} & \text{if } x > 0, \\ \pi + \arctan\dfrac{1}{x} & \text{if } x < 0. \end{cases}$$

S1.3.5 Addition and Subtraction of Inverse Trigonometric Functions

$$\arcsin x + \arcsin y = \arcsin\bigl(x\sqrt{1-y^2} + y\sqrt{1-x^2}\,\bigr) \qquad \text{for} \quad x^2 + y^2 \le 1,$$

$$\arccos x \pm \arccos y = \pm\arccos\bigl[xy \mp \sqrt{(1-x^2)(1-y^2)}\,\bigr] \quad \text{for} \quad x \pm y \ge 0,$$

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy} \quad \text{for} \quad xy < 1,$$

$$\arctan x - \arctan y = \arctan\frac{x-y}{1+xy} \quad \text{for} \quad xy > -1.$$

S1.3.6 Differentiation Formulas

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}},$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}.$$

S1.3.7 Integration Formulas

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C,$$
$$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + C,$$
$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C,$$
$$\int \operatorname{arccot} x\,dx = x\operatorname{arccot} x + \frac{1}{2}\ln(1+x^2) + C,$$

where C is an arbitrary constant.

S1.3.8 Expansion in Power Series

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\frac{x^5}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}\frac{x^7}{7} + \cdots + \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{x^{2n+1}}{2n+1} + \cdots \qquad (|x|<1),$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots \qquad (|x|\le 1),$$
$$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots + (-1)^n\frac{1}{(2n-1)x^{2n-1}} + \cdots \qquad (|x|>1).$$

The expansions for $\arccos x$ and $\operatorname{arccot} x$ can be obtained from the relations $\arccos x = \frac{\pi}{2} - \arcsin x$ and $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$.

S1.4 Hyperbolic Functions

S1.4.1 Definitions of Hyperbolic Functions

Hyperbolic functions are defined in terms of the exponential functions as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

S1.4.2 Simplest Relations

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh x \coth x = 1,$$
$$\sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x,$$
$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$
$$\tanh(-x) = -\tanh x, \qquad \coth(-x) = -\coth x,$$
$$1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \coth^2 x - 1 = \frac{1}{\sinh^2 x}.$$

S1.4.3 Relations between Hyperbolic Functions of Single Argument ($x \geq 0$)

$$\sinh x = \sqrt{\cosh^2 x - 1} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} = \frac{1}{\sqrt{\coth^2 x - 1}},$$
$$\cosh x = \sqrt{\sinh^2 x + 1} = \frac{1}{\sqrt{1 - \tanh^2 x}} = \frac{\coth x}{\sqrt{\coth^2 x - 1}},$$
$$\tanh x = \frac{\sinh x}{\sqrt{\sinh^2 x + 1}} = \frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \frac{1}{\coth x},$$
$$\coth x = \frac{\sqrt{\sinh^2 x + 1}}{\sinh x} = \frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x}.$$

S1.4.4 Addition and Subtraction of Hyperbolic Functions

$$\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right),$$
$$\sinh x - \sinh y = 2\sinh\left(\frac{x-y}{2}\right)\cosh\left(\frac{x+y}{2}\right),$$
$$\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right),$$
$$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right),$$
$$\sinh^2 x - \sinh^2 y = \cosh^2 x - \cosh^2 y = \sinh(x+y)\sinh(x-y),$$
$$\sinh^2 x + \cosh^2 y = \cosh(x+y)\cosh(x-y),$$
$$(\cosh x \pm \sinh x)^n = \cosh(nx) \pm \sinh(nx),$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}, \qquad \coth x \pm \coth y = \pm\frac{\sinh(x \pm y)}{\sinh x \sinh y},$$

where $n = 0, \pm 1, \pm 2, \ldots$

S1.4.5 Products of Hyperbolic Functions

$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)],$$
$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)],$$
$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)].$$

S1.4.6 Powers of Hyperbolic Functions

$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$, $\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$,

$\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$, $\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x$,

$\cosh^4 x = \frac{1}{8}\cosh 4x + \frac{1}{2}\cosh 2x + \frac{3}{8}$, $\sinh^4 x = \frac{1}{8}\cosh 4x - \frac{1}{2}\cosh 2x + \frac{3}{8}$,

$\cosh^5 x = \frac{1}{16}\cosh 5x + \frac{5}{16}\cosh 3x + \frac{5}{8}\cosh x$, $\sinh^5 x = \frac{1}{16}\sinh 5x - \frac{5}{16}\sinh 3x + \frac{5}{8}\sinh x$,

$$\cosh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cosh[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^n,$$

$$\cosh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \cosh[(2n-2k+1)x],$$

$$\sinh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^k C_{2n}^k \cosh[2(n-k)x] + \frac{(-1)^n}{2^{2n}} C_{2n}^n,$$

$$\sinh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^k C_{2n+1}^k \sinh[(2n-2k+1)x].$$

Here $n = 1, 2, \ldots$ and C_m^k are binomial coefficients.

S1.4.7 Addition Formulas

$$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y},$$

$$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}.$$

S1.4.8 Hyperbolic Functions of Multiple Argument

$\cosh 2x = 2\cosh^2 x - 1$, $\sinh 2x = 2\sinh x \cosh x$,

$\cosh 3x = -3\cosh x + 4\cosh^3 x$, $\sinh 3x = 3\sinh x + 4\sinh^3 x$,

$\cosh 4x = 1 - 8\cosh^2 x + 8\cosh^4 x$, $\sinh 4x = 4\cosh x(\sinh x + 2\sinh^3 x)$,

$\cosh 5x = 5\cosh x - 20\cosh^3 x + 16\cosh^5 x$, $\sinh 5x = 5\sinh x + 20\sinh^3 x + 16\sinh^5 x$.

$$\cosh(nx) = 2^{n-1}\cosh^n x + \frac{n}{2}\sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}}{k+1} C_{n-k-2}^{k-2} 2^{n-2k-2}(\cosh x)^{n-2k-2},$$

$$\sinh(nx) = \sinh x \sum_{k=0}^{[(n-1)/2]} 2^{n-k-1} C_{n-k-1}^k (\cosh x)^{n-2k-1}.$$

Here C_m^k are binomial coefficients and $[A]$ stands for the integer part of the number A.

S1.4.9 Hyperbolic Functions of Half Argument

$$\sinh\frac{x}{2} = \operatorname{sign} x\sqrt{\frac{\cosh x - 1}{2}}, \qquad \cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}},$$

$$\tanh\frac{x}{2} = \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}, \quad \coth\frac{x}{2} = \frac{\sinh x}{\cosh x - 1} = \frac{\cosh x + 1}{\sinh x}.$$

S1.4.10 Differentiation Formulas

$$\frac{d\sinh x}{dx} = \cosh x, \qquad \frac{d\cosh x}{dx} = \sinh x,$$

$$\frac{d\tanh x}{dx} = \frac{1}{\cosh^2 x}, \qquad \frac{d\coth x}{dx} = -\frac{1}{\sinh^2 x}.$$

S1.4.11 Integration Formulas

$$\int \sinh x\,dx = \cosh x + C, \qquad \int \cosh x\,dx = \sinh x + C,$$

$$\int \tanh x\,dx = \ln\cosh x + C, \qquad \int \coth x\,dx = \ln|\sinh x| + C,$$

where C is an arbitrary constant.

S1.4.12 Expansion in Power Series

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \qquad (|x| < \infty),$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad (|x| < \infty),$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots + (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)|B_{2n}|x^{2n-1}}{(2n)!} + \cdots \qquad (|x| < \pi/2),$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots + (-1)^{n-1}\frac{2^{2n}|B_{2n}|x^{2n-1}}{(2n)!} + \cdots \qquad (|x| < \pi),$$

$$\frac{1}{\cosh x} = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots + \frac{E_{2n}}{(2n)!}x^{2n} + \cdots \qquad (|x| < \pi/2),$$

$$\frac{1}{\sinh x} = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots + \frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots \qquad (0 < |x| < \pi),$$

where B_n and E_n are Bernoulli and Euler numbers (see Sections 30.1.3 and 30.1.4).

S1.4.13 Representation in the Form of Infinite Products

$$\sinh x = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\ldots\left(1 + \frac{x^2}{n^2\pi^2}\right)\ldots$$

$$\cosh x = \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{9\pi^2}\right)\left(1 + \frac{4x^2}{25\pi^2}\right)\ldots\left(1 + \frac{4x^2}{(2n+1)^2\pi^2}\right)\ldots$$

S1.4.14 Relationship with Trigonometric Functions

$$\sinh(ix) = i\sin x, \qquad \cosh(ix) = \cos x, \qquad i^2 = -1,$$
$$\tanh(ix) = i\tan x, \qquad \coth(ix) = -i\cot x.$$

S1.5 Inverse Hyperbolic Functions

S1.5.1 Definitions of Inverse Hyperbolic Functions

Inverse hyperbolic functions are the functions that are inverse to hyperbolic functions. The following notation is used for inverse hyperbolic functions:

$$\begin{aligned}
&\operatorname{arcsinh} x \equiv \sinh^{-1} x && \text{(inverse of hyperbolic sine)},\\
&\operatorname{arccosh} x \equiv \cosh^{-1} x && \text{(inverse of hyperbolic cosine)},\\
&\operatorname{arctanh} x \equiv \tanh^{-1} x && \text{(inverse of hyperbolic tangent)},\\
&\operatorname{arccoth} x \equiv \coth^{-1} x && \text{(inverse of hyperbolic cotangent)}.
\end{aligned}$$

Inverse hyperbolic functions can be expressed in terms of logarithmic functions:

$$\operatorname{arcsinh} x = \ln\bigl(x+\sqrt{x^2+1}\bigr) \quad (x \text{ is any}); \qquad \operatorname{arccosh} x = \ln\bigl(x+\sqrt{x^2-1}\bigr) \quad (x \geq 1);$$
$$\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x} \quad (|x| < 1); \qquad \operatorname{arccoth} x = \frac{1}{2}\ln\frac{x+1}{x-1} \quad (|x| > 1).$$

Here only one (principal) branch of the function $\operatorname{arccosh} x$ is listed, the function itself being double-valued. In order to write out both branches of $\operatorname{arccosh} x$, the symbol $\pm$ should be placed before the logarithm on the right-hand side of the formula.

S1.5.2 Simplest Relations

$$\operatorname{arcsinh}(-x) = -\operatorname{arcsinh} x, \quad \operatorname{arctanh}(-x) = -\operatorname{arctanh} x, \quad \operatorname{arccoth}(-x) = -\operatorname{arccoth} x.$$

S1.5.3 Relations between Inverse Hyperbolic Functions

$$\begin{aligned}
\operatorname{arcsinh} x &= \operatorname{arccosh}\sqrt{x^2+1} = \operatorname{arctanh}\frac{x}{\sqrt{x^2+1}},\\
\operatorname{arccosh} x &= \operatorname{arcsinh}\sqrt{x^2-1} = \operatorname{arctanh}\frac{\sqrt{x^2-1}}{x},\\
\operatorname{arctanh} x &= \operatorname{arcsinh}\frac{x}{\sqrt{1-x^2}} = \operatorname{arccosh}\frac{1}{\sqrt{1-x^2}} = \operatorname{arccoth}\frac{1}{x}.
\end{aligned}$$

S1.5.4 Addition and Subtraction of Inverse Hyperbolic Functions

$$\operatorname{arcsinh} x \pm \operatorname{arcsinh} y = \operatorname{arcsinh}\bigl(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\,\bigr),$$
$$\operatorname{arccosh} x \pm \operatorname{arccosh} y = \operatorname{arccosh}\bigl[xy \pm \sqrt{(x^2-1)(y^2-1)}\,\bigr],$$
$$\operatorname{arcsinh} x \pm \operatorname{arccosh} y = \operatorname{arcsinh}\bigl[xy \pm \sqrt{(x^2+1)(y^2-1)}\,\bigr],$$
$$\operatorname{arctanh} x \pm \operatorname{arctanh} y = \operatorname{arctanh}\frac{x \pm y}{1 \pm xy}, \quad \operatorname{arctanh} x \pm \operatorname{arccoth} y = \operatorname{arctanh}\frac{xy \pm 1}{y \pm x}.$$

S1.5.5 Differentiation Formulas

$$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{x^2+1}}, \qquad \frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}},$$
$$\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{1-x^2} \quad (x^2<1), \qquad \frac{d}{dx}\operatorname{arccoth} x = \frac{1}{1-x^2} \quad (x^2>1).$$

S1.5.6 Integration Formulas

$$\int \operatorname{arcsinh} x\,dx = x\operatorname{arcsinh} x - \sqrt{1+x^2} + C,$$
$$\int \operatorname{arccosh} x\,dx = x\operatorname{arccosh} x - \sqrt{x^2-1} + C,$$
$$\int \operatorname{arctanh} x\,dx = x\operatorname{arctanh} x + \frac{1}{2}\ln(1-x^2) + C,$$
$$\int \operatorname{arccoth} x\,dx = x\operatorname{arccoth} x + \frac{1}{2}\ln(x^2-1) + C,$$

where C is an arbitrary constant.

S1.5.7 Expansion in Power Series

$$\operatorname{arcsinh} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\frac{x^5}{5} - \cdots + (-1)^n \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{x^{2n+1}}{2n+1} + \cdots \qquad (|x|<1),$$

$$\operatorname{arcsinh} x = \ln(2x) + \frac{1}{2}\frac{1}{2x^2} + \frac{1\times 3}{2\times 4}\frac{1}{4x^4} + \cdots + \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{1}{2nx^{2n}} + \cdots \qquad (|x|>1),$$

$$\operatorname{arccosh} x = \ln(2x) - \frac{1}{2}\frac{1}{2x^2} - \frac{1\times 3}{2\times 4}\frac{1}{4x^4} - \cdots - \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\frac{1}{2nx^{2n}} - \cdots \qquad (|x|>1),$$

$$\operatorname{arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots \qquad (|x|<1),$$

$$\operatorname{arccoth} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots + \frac{1}{(2n+1)x^{2n+1}} + \cdots \qquad (|x|>1).$$

⊙ *References for Chapter* S1: M. Abramowitz and I. A. Stegun (1964), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986), D. G. Zill and J. M. Dewar (1990), M. Kline (1998), R. Courant and F. John (1999), I. S. Gradshteyn and I. M. Ryzhik (2000), G. A. Korn and T. M. Korn (2000), C. H. Edwards and D. Penney (2002), D. Zwillinger (2002), E. W. Weisstein (2003), I. N. Bronshtein and K. A. Semendyayev (2004), M. Sullivan (2004), H. Anton, I. Bivens, and S. Davis (2005), R. Adams (2006).

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Chapter S2

Indefinite and Definite Integrals

S2.1 Indefinite Integrals

◆ *Throughout Section S2.1, the integration constant C is omitted for brevity.*

S2.1.1 Integrals Involving Rational Functions

► **Integrals involving $a + bx$**

1. $\displaystyle \int \frac{dx}{a+bx} = \frac{1}{b} \ln|a+bx|.$

2. $\displaystyle \int (a+bx)^n dx = \frac{(a+bx)^{n+1}}{b(n+1)}, \qquad n \neq -1.$

3. $\displaystyle \int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\bigl(a+bx-a\ln|a+bx|\bigr).$

4. $\displaystyle \int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\Bigl[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\Bigr].$

5. $\displaystyle \int \frac{dx}{x(a+bx)} = -\frac{1}{a}\ln\left|\frac{a+bx}{x}\right|.$

6. $\displaystyle \int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right|.$

7. $\displaystyle \int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left(\ln|a+bx| + \frac{a}{a+bx}\right).$

8. $\displaystyle \int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left(a+bx-2a\ln|a+bx| - \frac{a^2}{a+bx}\right).$

9. $\displaystyle \int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right|.$

10. $\displaystyle \int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$

► Integrals involving $a + x$ and $b + x$

1. $\displaystyle\int \frac{a+x}{b+x}\,dx = x + (a-b)\ln|b+x|.$
2. $\displaystyle\int \frac{dx}{(a+x)(b+x)} = \frac{1}{a-b}\ln\left|\frac{b+x}{a+x}\right|, \quad a \neq b.$ For $a = b$, see Integral 2 with $n = -2$ in Section S2.1.
3. $\displaystyle\int \frac{x\,dx}{(a+x)(b+x)} = \frac{1}{a-b}\big(a\ln|a+x| - b\ln|b+x|\big).$
4. $\displaystyle\int \frac{dx}{(a+x)(b+x)^2} = \frac{1}{(b-a)(b+x)} + \frac{1}{(a-b)^2}\ln\left|\frac{a+x}{b+x}\right|.$
5. $\displaystyle\int \frac{x\,dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2}\ln\left|\frac{a+x}{b+x}\right|.$
6. $\displaystyle\int \frac{x^2\,dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(a-b)^2}\ln|a+x| + \frac{b^2-2ab}{(b-a)^2}\ln|b+x|.$
7. $\displaystyle\int \frac{dx}{(a+x)^2(b+x)^2} = -\frac{1}{(a-b)^2}\left(\frac{1}{a+x} + \frac{1}{b+x}\right) + \frac{2}{(a-b)^3}\ln\left|\frac{a+x}{b+x}\right|.$
8. $\displaystyle\int \frac{x\,dx}{(a+x)^2(b+x)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{a+x} + \frac{b}{b+x}\right) + \frac{a+b}{(a-b)^3}\ln\left|\frac{a+x}{b+x}\right|.$
9. $\displaystyle\int \frac{x^2\,dx}{(a+x)^2(b+x)^2} = -\frac{1}{(a-b)^2}\left(\frac{a^2}{a+x} + \frac{b^2}{b+x}\right) + \frac{2ab}{(a-b)^3}\ln\left|\frac{a+x}{b+x}\right|.$

► Integrals involving $a^2 + x^2$

1. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}.$
2. $\displaystyle\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\arctan\frac{x}{a}.$
3. $\displaystyle\int \frac{dx}{(a^2+x^2)^3} = \frac{x}{4a^2(a^2+x^2)^2} + \frac{3x}{8a^4(a^2+x^2)} + \frac{3}{8a^5}\arctan\frac{x}{a}.$
4. $\displaystyle\int \frac{dx}{(a^2+x^2)^{n+1}} = \frac{x}{2na^2(a^2+x^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(a^2+x^2)^n}; \qquad n = 1,\ 2,\ \dots$
5. $\displaystyle\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2).$
6. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^2} = -\frac{1}{2(a^2+x^2)}.$
7. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^3} = -\frac{1}{4(a^2+x^2)^2}.$
8. $\displaystyle\int \frac{x\,dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2n(a^2+x^2)^n}; \qquad n = 1,\ 2,\ \dots$
9. $\displaystyle\int \frac{x^2\,dx}{a^2+x^2} = x - a\arctan\frac{x}{a}.$
10. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^2} = -\frac{x}{2(a^2+x^2)} + \frac{1}{2a}\arctan\frac{x}{a}.$

11. $$\int \frac{x^2\,dx}{(a^2+x^2)^3} = -\frac{x}{4(a^2+x^2)^2} + \frac{x}{8a^2(a^2+x^2)} + \frac{1}{8a^3}\arctan\frac{x}{a}.$$

12. $$\int \frac{x^2\,dx}{(a^2+x^2)^{n+1}} = -\frac{x}{2n(a^2+x^2)^n} + \frac{1}{2n}\int \frac{dx}{(a^2+x^2)^n}; \quad n = 1,\,2,\,\ldots$$

13. $$\int \frac{x^3\,dx}{a^2+x^2} = \frac{x^2}{2} - \frac{a^2}{2}\ln(a^2+x^2).$$

14. $$\int \frac{x^3\,dx}{(a^2+x^2)^2} = \frac{a^2}{2(a^2+x^2)} + \frac{1}{2}\ln(a^2+x^2).$$

15. $$\int \frac{x^3\,dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2(n-1)(a^2+x^2)^{n-1}} + \frac{a^2}{2n(a^2+x^2)^n}; \quad n = 2,\,3,\,\ldots$$

16. $$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\frac{x^2}{a^2+x^2}.$$

17. $$\int \frac{dx}{x(a^2+x^2)^2} = \frac{1}{2a^2(a^2+x^2)} + \frac{1}{2a^4}\ln\frac{x^2}{a^2+x^2}.$$

18. $$\int \frac{dx}{x(a^2+x^2)^3} = \frac{1}{4a^2(a^2+x^2)^2} + \frac{1}{2a^4(a^2+x^2)} + \frac{1}{2a^6}\ln\frac{x^2}{a^2+x^2}.$$

19. $$\int \frac{dx}{x^2(a^2+x^2)} = -\frac{1}{a^2x} - \frac{1}{a^3}\arctan\frac{x}{a}.$$

20. $$\int \frac{dx}{x^2(a^2+x^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(a^2+x^2)} - \frac{3}{2a^5}\arctan\frac{x}{a}.$$

21. $$\int \frac{dx}{x^3(a^2+x^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(a^2+x^2)} - \frac{1}{a^6}\ln\frac{x^2}{a^2+x^2}.$$

22. $$\int \frac{dx}{x^2(a^2+x^2)^3} = -\frac{1}{a^6x} - \frac{x}{4a^4(a^2+x^2)^2} - \frac{7x}{8a^6(a^2+x^2)} - \frac{15}{8a^7}\arctan\frac{x}{a}.$$

23. $$\int \frac{dx}{x^3(a^2+x^2)^3} = -\frac{1}{2a^6x^2} - \frac{1}{a^6(a^2+x^2)} - \frac{1}{4a^4(a^2+x^2)^2} - \frac{3}{2a^8}\ln\frac{x^2}{a^2+x^2}.$$

► Integrals involving $a^2 - x^2$

1. $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|.$$

2. $$\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right|.$$

3. $$\int \frac{dx}{(a^2-x^2)^3} = \frac{x}{4a^2(a^2-x^2)^2} + \frac{3x}{8a^4(a^2-x^2)} + \frac{3}{16a^5}\ln\left|\frac{a+x}{a-x}\right|.$$

4. $$\int \frac{dx}{(a^2-x^2)^{n+1}} = \frac{x}{2na^2(a^2-x^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(a^2-x^2)^n}; \quad n = 1,\,2,\,\ldots$$

5. $$\int \frac{x\,dx}{a^2-x^2} = -\frac{1}{2}\ln|a^2-x^2|.$$

6. $$\int \frac{x\,dx}{(a^2-x^2)^2} = \frac{1}{2(a^2-x^2)}.$$

7. $$\int \frac{x\,dx}{(a^2-x^2)^3} = \frac{1}{4(a^2-x^2)^2}.$$

8. $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^{n+1}} = \frac{1}{2n(a^2-x^2)^n}; \quad n=1,\,2,\,\dots$

9. $\displaystyle\int \frac{x^2\,dx}{a^2-x^2} = -x + \frac{a}{2}\ln\left|\frac{a+x}{a-x}\right|.$

10. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^2} = \frac{x}{2(a^2-x^2)} - \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right|.$

11. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^3} = \frac{x}{4(a^2-x^2)^2} - \frac{x}{8a^2(a^2-x^2)} - \frac{1}{16a^3}\ln\left|\frac{a+x}{a-x}\right|.$

12. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{n+1}} = \frac{x}{2n(a^2-x^2)^n} - \frac{1}{2n}\int\frac{dx}{(a^2-x^2)^n}; \quad n=1,\,2,\,\dots$

13. $\displaystyle\int \frac{x^3\,dx}{a^2-x^2} = -\frac{x^2}{2} - \frac{a^2}{2}\ln|a^2-x^2|.$

14. $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^2} = \frac{a^2}{2(a^2-x^2)} + \frac{1}{2}\ln|a^2-x^2|.$

15. $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^{n+1}} = -\frac{1}{2(n-1)(a^2-x^2)^{n-1}} + \frac{a^2}{2n(a^2-x^2)^n}; \quad n=2,\,3,\dots$

16. $\displaystyle\int \frac{dx}{x(a^2-x^2)} = \frac{1}{2a^2}\ln\left|\frac{x^2}{a^2-x^2}\right|.$

17. $\displaystyle\int \frac{dx}{x(a^2-x^2)^2} = \frac{1}{2a^2(a^2-x^2)} + \frac{1}{2a^4}\ln\left|\frac{x^2}{a^2-x^2}\right|.$

18. $\displaystyle\int \frac{dx}{x(a^2-x^2)^3} = \frac{1}{4a^2(a^2-x^2)^2} + \frac{1}{2a^4(a^2-x^2)} + \frac{1}{2a^6}\ln\left|\frac{x^2}{a^2-x^2}\right|.$

► Integrals involving $a^3 + x^3$

1. $\displaystyle\int \frac{dx}{a^3+x^3} = \frac{1}{6a^2}\ln\frac{(a+x)^2}{a^2-ax+x^2} + \frac{1}{a^2\sqrt{3}}\arctan\frac{2x-a}{a\sqrt{3}}.$

2. $\displaystyle\int \frac{dx}{(a^3+x^3)^2} = \frac{x}{3a^3(a^3+x^3)} + \frac{2}{3a^3}\int\frac{dx}{a^3+x^3}.$

3. $\displaystyle\int \frac{x\,dx}{a^3+x^3} = \frac{1}{6a}\ln\frac{a^2-ax+x^2}{(a+x)^2} + \frac{1}{a\sqrt{3}}\arctan\frac{2x-a}{a\sqrt{3}}.$

4. $\displaystyle\int \frac{x\,dx}{(a^3+x^3)^2} = \frac{x^2}{3a^3(a^3+x^3)} + \frac{1}{3a^3}\int\frac{x\,dx}{a^3+x^3}.$

5. $\displaystyle\int \frac{x^2\,dx}{a^3+x^3} = \frac{1}{3}\ln|a^3+x^3|.$

6. $\displaystyle\int \frac{dx}{x(a^3+x^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{a^3+x^3}\right|.$

7. $\displaystyle\int \frac{dx}{x(a^3+x^3)^2} = \frac{1}{3a^3(a^3+x^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{a^3+x^3}\right|.$

8. $\displaystyle\int \frac{dx}{x^2(a^3+x^3)} = -\frac{1}{a^3x} - \frac{1}{a^3}\int\frac{x\,dx}{a^3+x^3}.$

9. $\displaystyle\int \frac{dx}{x^2(a^3+x^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(a^3+x^3)} - \frac{4}{3a^6}\int\frac{x\,dx}{a^3+x^3}.$

► **Integrals involving $a^3 - x^3$**

1. $\displaystyle\int \frac{dx}{a^3 - x^3} = \frac{1}{6a^2} \ln \frac{a^2 + ax + x^2}{(a-x)^2} + \frac{1}{a^2\sqrt{3}} \arctan \frac{2x + a}{a\sqrt{3}}.$
2. $\displaystyle\int \frac{dx}{(a^3 - x^3)^2} = \frac{x}{3a^3(a^3 - x^3)} + \frac{2}{3a^3} \int \frac{dx}{a^3 - x^3}.$
3. $\displaystyle\int \frac{x\,dx}{a^3 - x^3} = \frac{1}{6a} \ln \frac{a^2 + ax + x^2}{(a-x)^2} - \frac{1}{a\sqrt{3}} \arctan \frac{2x + a}{a\sqrt{3}}.$
4. $\displaystyle\int \frac{x\,dx}{(a^3 - x^3)^2} = \frac{x^2}{3a^3(a^3 - x^3)} + \frac{1}{3a^3} \int \frac{x\,dx}{a^3 - x^3}.$
5. $\displaystyle\int \frac{x^2\,dx}{a^3 - x^3} = -\frac{1}{3} \ln |a^3 - x^3|.$
6. $\displaystyle\int \frac{dx}{x(a^3 - x^3)} = \frac{1}{3a^3} \ln\left|\frac{x^3}{a^3 - x^3}\right|.$
7. $\displaystyle\int \frac{dx}{x(a^3 - x^3)^2} = \frac{1}{3a^3(a^3 - x^3)} + \frac{1}{3a^6} \ln\left|\frac{x^3}{a^3 - x^3}\right|.$
8. $\displaystyle\int \frac{dx}{x^2(a^3 - x^3)} = -\frac{1}{a^3 x} + \frac{1}{a^3} \int \frac{x\,dx}{a^3 - x^3}.$
9. $\displaystyle\int \frac{dx}{x^2(a^3 - x^3)^2} = -\frac{1}{a^6 x} - \frac{x^2}{3a^6(a^3 - x^3)} + \frac{4}{3a^6} \int \frac{x\,dx}{a^3 - x^3}.$

► **Integrals involving $a^4 \pm x^4$**

1. $\displaystyle\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}} \ln \frac{a^2 + ax\sqrt{2} + x^2}{a^2 - ax\sqrt{2} + x^2} + \frac{1}{2a^3\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}.$
2. $\displaystyle\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2}.$
3. $\displaystyle\int \frac{x^2\,dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}} \ln \frac{a^2 + ax\sqrt{2} + x^2}{a^2 - ax\sqrt{2} + x^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}.$
4. $\displaystyle\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \ln\left|\frac{a + x}{a - x}\right| + \frac{1}{2a^3} \arctan \frac{x}{a}.$
5. $\displaystyle\int \frac{x\,dx}{a^4 - x^4} = \frac{1}{4a^2} \ln\left|\frac{a^2 + x^2}{a^2 - x^2}\right|.$
6. $\displaystyle\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a} \ln\left|\frac{a + x}{a - x}\right| - \frac{1}{2a} \arctan \frac{x}{a}.$

S2.1.2 Integrals Involving Irrational Functions

► **Integrals involving $x^{1/2}$**

1. $\displaystyle\int \frac{x^{1/2}\,dx}{a^2 + b^2 x} = \frac{2}{b^2} x^{1/2} - \frac{2a}{b^3} \arctan \frac{b x^{1/2}}{a}.$
2. $\displaystyle\int \frac{x^{3/2}\,dx}{a^2 + b^2 x} = \frac{2x^{3/2}}{3b^2} - \frac{2a^2 x^{1/2}}{b^4} + \frac{2a^3}{b^5} \arctan \frac{b x^{1/2}}{a}.$

3. $\displaystyle\int \frac{x^{1/2}\,dx}{(a^2+b^2x)^2} = -\frac{x^{1/2}}{b^2(a^2+b^2x)} + \frac{1}{ab^3}\arctan\frac{bx^{1/2}}{a}.$

4. $\displaystyle\int \frac{x^{3/2}\,dx}{(a^2+b^2x)^2} = \frac{2x^{3/2}}{b^2(a^2+b^2x)} + \frac{3a^2x^{1/2}}{b^4(a^2+b^2x)} - \frac{3a}{b^5}\arctan\frac{bx^{1/2}}{a}.$

5. $\displaystyle\int \frac{dx}{(a^2+b^2x)x^{1/2}} = \frac{2}{ab}\arctan\frac{bx^{1/2}}{a}.$

6. $\displaystyle\int \frac{dx}{(a^2+b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} - \frac{2b}{a^3}\arctan\frac{bx^{1/2}}{a}.$

7. $\displaystyle\int \frac{dx}{(a^2+b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2+b^2x)} + \frac{1}{a^3b}\arctan\frac{bx^{1/2}}{a}.$

8. $\displaystyle\int \frac{x^{1/2}\,dx}{a^2-b^2x} = -\frac{2}{b^2}x^{1/2} + \frac{2a}{b^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

9. $\displaystyle\int \frac{x^{3/2}\,dx}{a^2-b^2x} = -\frac{2x^{3/2}}{3b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{a^3}{b^5}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

10. $\displaystyle\int \frac{x^{1/2}\,dx}{(a^2-b^2x)^2} = \frac{x^{1/2}}{b^2(a^2-b^2x)} - \frac{1}{2ab^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

11. $\displaystyle\int \frac{x^{3/2}\,dx}{(a^2-b^2x)^2} = \frac{3a^2x^{1/2}-2b^2x^{3/2}}{b^4(a^2-b^2x)} - \frac{3a}{2b^5}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

12. $\displaystyle\int \frac{dx}{(a^2-b^2x)x^{1/2}} = \frac{1}{ab}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

13. $\displaystyle\int \frac{dx}{(a^2-b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} + \frac{b}{a^3}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

14. $\displaystyle\int \frac{dx}{(a^2-b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2-b^2x)} + \frac{1}{2a^3b}\ln\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|.$

► Integrals involving $(a+bx)^{p/2}$

1. $\displaystyle\int (a+bx)^{p/2}\,dx = \frac{2}{b(p+2)}(a+bx)^{(p+2)/2}.$

2. $\displaystyle\int x(a+bx)^{p/2}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{(p+4)/2}}{p+4} - \frac{a(a+bx)^{(p+2)/2}}{p+2}\right].$

3. $\displaystyle\int x^2(a+bx)^{p/2}\,dx = \frac{2}{b^3}\left[\frac{(a+bx)^{(p+6)/2}}{p+6} - \frac{2a(a+bx)^{(p+4)/2}}{p+4} + \frac{a^2(a+bx)^{(p+2)/2}}{p+2}\right].$

► Integrals involving $(x^2+a^2)^{1/2}$

1. $\displaystyle\int (x^2+a^2)^{1/2}\,dx = \frac{1}{2}x(a^2+x^2)^{1/2} + \frac{a^2}{2}\ln\bigl[x+(x^2+a^2)^{1/2}\bigr].$

2. $\displaystyle\int x(x^2+a^2)^{1/2}\,dx = \frac{1}{3}(a^2+x^2)^{3/2}.$

3. $\displaystyle\int (x^2+a^2)^{3/2}\,dx = \frac{1}{4}x(a^2+x^2)^{3/2} + \frac{3}{8}a^2x(a^2+x^2)^{1/2} + \frac{3}{8}a^4\ln\bigl|x+(x^2+a^2)^{1/2}\bigr|.$

4. $\displaystyle\int \frac{1}{x}(x^2+a^2)^{1/2}\,dx = (a^2+x^2)^{1/2} - a\ln\left|\frac{a+(x^2+a^2)^{1/2}}{x}\right|.$

5. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\bigl[x+(x^2+a^2)^{1/2}\bigr].$

6. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+a^2}} = (x^2+a^2)^{1/2}.$

7. $\displaystyle\int (x^2+a^2)^{-3/2}\,dx = a^{-2}x(x^2+a^2)^{-1/2}.$

► Integrals involving $(x^2 - a^2)^{1/2}$

1. $\displaystyle\int (x^2-a^2)^{1/2}\,dx = \frac{1}{2}x(x^2-a^2)^{1/2} - \frac{a^2}{2}\ln\bigl|x+(x^2-a^2)^{1/2}\bigr|.$

2. $\displaystyle\int x(x^2-a^2)^{1/2}\,dx = \frac{1}{3}(x^2-a^2)^{3/2}.$

3. $\displaystyle\int (x^2-a^2)^{3/2}\,dx = \frac{1}{4}x(x^2-a^2)^{3/2} - \frac{3}{8}a^2x(x^2-a^2)^{1/2} + \frac{3}{8}a^4\ln\bigl|x+(x^2-a^2)^{1/2}\bigr|.$

4. $\displaystyle\int \frac{1}{x}(x^2-a^2)^{1/2}\,dx = (x^2-a^2)^{1/2} - a\arccos\left|\frac{a}{x}\right|.$

5. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\bigl|x+(x^2-a^2)^{1/2}\bigr|.$

6. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-a^2}} = (x^2-a^2)^{1/2}.$

7. $\displaystyle\int (x^2-a^2)^{-3/2}\,dx = -a^{-2}x(x^2-a^2)^{-1/2}.$

► Integrals involving $(a^2 - x^2)^{1/2}$

1. $\displaystyle\int (a^2-x^2)^{1/2}\,dx = \frac{1}{2}x(a^2-x^2)^{1/2} + \frac{a^2}{2}\arcsin\frac{x}{a}.$

2. $\displaystyle\int x(a^2-x^2)^{1/2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2}.$

3. $\displaystyle\int (a^2-x^2)^{3/2}\,dx = \frac{1}{4}x(a^2-x^2)^{3/2} + \frac{3}{8}a^2x(a^2-x^2)^{1/2} + \frac{3}{8}a^4\arcsin\frac{x}{a}.$

4. $\displaystyle\int \frac{1}{x}(a^2-x^2)^{1/2}\,dx = (a^2-x^2)^{1/2} - a\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$

5. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}.$

6. $\displaystyle\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -(a^2-x^2)^{1/2}.$

7. $\displaystyle\int (a^2-x^2)^{-3/2}\,dx = a^{-2}x(a^2-x^2)^{-1/2}.$

► **Integrals involving arbitrary powers. Reduction formulas**

1. $\displaystyle\int \frac{dx}{x(ax^n+b)} = \frac{1}{bn}\ln\left|\frac{x^n}{ax^n+b}\right|.$

2. $\displaystyle\int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{2}{an}\ln\left|\frac{x^{n/2}}{\sqrt{x^n+a^2}+a}\right|.$

3. $\displaystyle\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{an}\arccos\left|\frac{a}{x^{n/2}}\right|.$

4. $\displaystyle\int \frac{dx}{x\sqrt{ax^{2n}+bx^n}} = -\frac{2\sqrt{ax^{2n}+bx^n}}{bnx^n}.$

◆ *The parameters* a, b, p, m, *and* n *below in Integrals 5–8 can assume arbitrary values, except for those at which denominators vanish in successive applications of a formula. Notation:* $w = ax^n + b$.

5. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{m+np+1}\Bigl(x^{m+1}w^p + npb\int x^m w^{p-1}\,dx\Bigr).$

6. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{bn(p+1)}\Bigl[-x^{m+1}w^{p+1} + (m+n+np+1)\int x^m w^{p+1}\,dx\Bigr].$

7. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{b(m+1)}\Bigl[x^{m+1}w^{p+1} - a(m+n+np+1)\int x^{m+n} w^{p}\,dx\Bigr].$

8. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{a(m+np+1)}\Bigl[x^{m-n+1}w^{p+1} - b(m-n+1)\int x^{m-n} w^{p}\,dx\Bigr].$

S2.1.3 Integrals Involving Exponential Functions

1. $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax}.$

2. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}.$

3. $\displaystyle\int xe^{ax}\,dx = e^{ax}\Bigl(\frac{x}{a} - \frac{1}{a^2}\Bigr).$

4. $\displaystyle\int x^2e^{ax}\,dx = e^{ax}\Bigl(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\Bigr).$

5. $\displaystyle\int x^ne^{ax}\,dx = e^{ax}\Bigl[\frac{1}{a}x^n - \frac{n}{a^2}x^{n-1} + \frac{n(n-1)}{a^3}x^{n-2} - \cdots + (-1)^{n-1}\frac{n!}{a^n}x + (-1)^n\frac{n!}{a^{n+1}}\Bigr],$
$n = 1,\,2,\,\ldots$

6. $\displaystyle\int P_n(x)e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k}{a^{k+1}}\frac{d^k}{dx^k}P_n(x),$ where $P_n(x)$ is an arbitrary polynomial of degree n.

7. $\displaystyle\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\ln|a+be^{px}|.$

8. $$\int \frac{dx}{ae^{px}+be^{-px}} = \begin{cases} \dfrac{1}{p\sqrt{ab}}\arctan\Bigl(e^{px}\sqrt{\dfrac{a}{b}}\Bigr) & \text{if } ab>0,\\[2ex] \dfrac{1}{2p\sqrt{-ab}}\ln\Bigl(\dfrac{b+e^{px}\sqrt{-ab}}{b-e^{px}\sqrt{-ab}}\Bigr) & \text{if } ab<0.\end{cases}$$

9. $$\int \frac{dx}{\sqrt{a+be^{px}}} = \begin{cases} \dfrac{1}{p\sqrt{a}} \ln \dfrac{\sqrt{a+be^{px}}-\sqrt{a}}{\sqrt{a+be^{px}}+\sqrt{a}} & \text{if } a>0, \\ \dfrac{2}{p\sqrt{-a}} \arctan \dfrac{\sqrt{a+be^{px}}}{\sqrt{-a}} & \text{if } a<0. \end{cases}$$

S2.1.4 Integrals Involving Hyperbolic Functions

► **Integrals involving cosh x**

1. $$\int \cosh(a+bx)\,dx = \frac{1}{b}\sinh(a+bx).$$

2. $$\int x\cosh x\,dx = x\sinh x - \cosh x.$$

3. $$\int x^2\cosh x\,dx = (x^2+2)\sinh x - 2x\cosh x.$$

4. $$\int x^{2n}\cosh x\,dx = (2n)!\sum_{k=1}^{n}\left[\frac{x^{2k}}{(2k)!}\sinh x - \frac{x^{2k-1}}{(2k-1)!}\cosh x\right].$$

5. $$\int x^{2n+1}\cosh x\,dx = (2n+1)!\sum_{k=0}^{n}\left[\frac{x^{2k+1}}{(2k+1)!}\sinh x - \frac{x^{2k}}{(2k)!}\cosh x\right].$$

6. $$\int x^p\cosh x\,dx = x^p\sinh x - px^{p-1}\cosh x + p(p-1)\int x^{p-2}\cosh x\,dx.$$

7. $$\int \cosh^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sinh 2x.$$

8. $$\int \cosh^3 x\,dx = \sinh x + \tfrac{1}{3}\sinh^3 x.$$

9. $$\int \cosh^{2n} x\,dx = C_{2n}^{n}\frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^{k}\frac{\sinh[2(n-k)x]}{2(n-k)}, \qquad n=1,\,2,\,\ldots$$

10. $$\int \cosh^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^{k}\frac{\sinh[(2n-2k+1)x]}{2n-2k+1} = \sum_{k=0}^{n} C_{n}^{k}\frac{\sinh^{2k+1} x}{2k+1},$$
$n=1,\,2,\ldots$

11. $$\int \cosh^p x\,dx = \frac{1}{p}\sinh x\cosh^{p-1} x + \frac{p-1}{p}\int \cosh^{p-2} x\,dx.$$

12. $$\int \cosh ax\cosh bx\,dx = \frac{1}{a^2-b^2}(a\cosh bx\sinh ax - b\cosh ax\sinh bx).$$

13. $$\int \frac{dx}{\cosh ax} = \frac{2}{a}\arctan\bigl(e^{ax}\bigr).$$

14. $$\int \frac{dx}{\cosh^{2n} x} = \frac{\sinh x}{2n-1}\left[\frac{1}{\cosh^{2n-1} x} + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\frac{1}{\cosh^{2n-2k-1} x}\right], \quad n=1,\,2,\,\ldots$$

15. $$\int \frac{dx}{\cosh^{2n+1} x} = \frac{\sinh x}{2n}\left[\frac{1}{\cosh^{2n} x} + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \frac{1}{\cosh^{2n-2k} x}\right] + \frac{(2n-1)!!}{(2n)!!} \arctan \sinh x,$$
$n = 1,\ 2,\ \dots$

16. $$\int \frac{dx}{a + b\cosh x} = \begin{cases} -\dfrac{\operatorname{sign} x}{\sqrt{b^2 - a^2}} \arcsin \dfrac{b + a\cosh x}{a + b\cosh x} & \text{if } a^2 < b^2, \\ \dfrac{1}{\sqrt{a^2 - b^2}} \ln \dfrac{a + b + \sqrt{a^2 - b^2}\, \tanh(x/2)}{a + b - \sqrt{a^2 - b^2}\, \tanh(x/2)} & \text{if } a^2 > b^2. \end{cases}$$

► **Integrals involving $\sinh x$**

1. $$\int \sinh(a + bx)\, dx = \frac{1}{b} \cosh(a + bx).$$

2. $$\int x \sinh x\, dx = x\cosh x - \sinh x.$$

3. $$\int x^2 \sinh x\, dx = (x^2 + 2)\cosh x - 2x\sinh x.$$

4. $$\int x^{2n} \sinh x\, dx = (2n)!\left[\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \cosh x - \sum_{k=1}^{n} \frac{x^{2k-1}}{(2k-1)!} \sinh x\right].$$

5. $$\int x^{2n+1} \sinh x\, dx = (2n+1)! \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} \cosh x - \frac{x^{2k}}{(2k)!} \sinh x\right].$$

6. $$\int x^p \sinh x\, dx = x^p \cosh x - px^{p-1} \sinh x + p(p-1)\int x^{p-2} \sinh x\, dx.$$

7. $$\int \sinh^2 x\, dx = -\tfrac{1}{2}x + \tfrac{1}{4}\sinh 2x.$$

8. $$\int \sinh^3 x\, dx = -\cosh x + \tfrac{1}{3}\cosh^3 x.$$

9. $$\int \sinh^{2n} x\, dx = (-1)^n C_{2n}^n \frac{x}{2^{2n}} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k C_{2n}^k \frac{\sinh[2(n-k)x]}{2(n-k)}, \quad n = 1,\ 2,\ \dots$$

10. $$\int \sinh^{2n+1} x\, dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k C_{2n+1}^k \frac{\cosh[(2n-2k+1)x]}{2n-2k+1} = \sum_{k=0}^{n} (-1)^{n+k} C_n^k \frac{\cosh^{2k+1} x}{2k+1}, \quad n = 1,\ 2,\ \dots$$

11. $$\int \sinh^p x\, dx = \frac{1}{p} \sinh^{p-1} x \cosh x - \frac{p-1}{p} \int \sinh^{p-2} x\, dx.$$

12. $$\int \sinh ax \sinh bx\, dx = \frac{1}{a^2 - b^2}\bigl(a\cosh ax \sinh bx - b\cosh bx \sinh ax\bigr).$$

13. $$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln\left|\tanh \frac{ax}{2}\right|.$$

14. $$\int \frac{dx}{\sinh^{2n} x} = \frac{\cosh x}{2n-1}\left[-\frac{1}{\sinh^{2n-1} x} + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{A_{nk}}{\sinh^{2n-2k-1} x}\right],$$
$$A_{nk} = \frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \qquad n = 1,\,2,\,\dots$$

15. $$\int \frac{dx}{\sinh^{2n+1} x} = \frac{\cosh x}{2n}\left[-\frac{1}{\sinh^{2n} x} + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{A_{nk}}{\sinh^{2n-2k} x}\right]$$
$$+ (-1)^n\frac{(2n-1)!!}{(2n)!!}\ln\tanh\frac{x}{2}, \quad A_{nk} = \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)},$$
$n = 1,\,2,\,\dots$

16. $$\int \frac{dx}{a + b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a\tanh(x/2) - b + \sqrt{a^2+b^2}}{a\tanh(x/2) - b - \sqrt{a^2+b^2}}.$$

17. $$\int \frac{Ax + B\sinh x}{a + b\sinh x}\,dx = \frac{B}{b}x + \frac{Ab - Ba}{b\sqrt{a^2+b^2}}\ln\frac{a\tanh(x/2) - b + \sqrt{a^2+b^2}}{a\tanh(x/2) - b - \sqrt{a^2+b^2}}.$$

► **Integrals involving $\tanh x$ or $\coth x$**

1. $$\int \tanh x\,dx = \ln\cosh x.$$

2. $$\int \tanh^2 x\,dx = x - \tanh x.$$

3. $$\int \tanh^3 x\,dx = -\tfrac{1}{2}\tanh^2 x + \ln\cosh x.$$

4. $$\int \tanh^{2n} x\,dx = x - \sum_{k=1}^{n}\frac{\tanh^{2n-2k+1} x}{2n-2k+1}, \qquad n = 1,\,2,\,\dots$$

5. $$\int \tanh^{2n+1} x\,dx = \ln\cosh x - \sum_{k=1}^{n}\frac{(-1)^k C_n^k}{2k\cosh^{2k} x} = \ln\cosh x - \sum_{k=1}^{n}\frac{\tanh^{2n-2k+2} x}{2n-2k+2},$$
$n = 1,\,2,\,\dots$

6. $$\int \tanh^p x\,dx = -\frac{1}{p-1}\tanh^{p-1} x + \int \tanh^{p-2} x\,dx.$$

7. $$\int \coth x\,dx = \ln|\sinh x|.$$

8. $$\int \coth^2 x\,dx = x - \coth x.$$

9. $$\int \coth^3 x\,dx = -\tfrac{1}{2}\coth^2 x + \ln|\sinh x|.$$

10. $$\int \coth^{2n} x\,dx = x - \sum_{k=1}^{n}\frac{\coth^{2n-2k+1} x}{2n-2k+1}, \qquad n = 1,\,2,\,\dots$$

11. $$\int \coth^{2n+1} x\,dx = \ln|\sinh x| - \sum_{k=1}^{n}\frac{C_n^k}{2k\sinh^{2k} x} = \ln|\sinh x| - \sum_{k=1}^{n}\frac{\coth^{2n-2k+2} x}{2n-2k+2},$$
$n = 1,\,2,\,\dots$

12. $$\int \coth^p x\,dx = -\frac{1}{p-1}\coth^{p-1} x + \int \coth^{p-2} x\,dx.$$

S2.1.5 Integrals Involving Logarithmic Functions

1. $\displaystyle\int \ln ax\,dx = x\ln ax - x.$

2. $\displaystyle\int x\ln x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2.$

3. $$\int x^p \ln ax\,dx = \begin{cases} \dfrac{1}{p+1}x^{p+1}\ln ax - \dfrac{1}{(p+1)^2}x^{p+1} & \text{if } p\neq -1,\\ \tfrac{1}{2}\ln^2 ax & \text{if } p=-1. \end{cases}$$

4. $\displaystyle\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x.$

5. $\displaystyle\int x(\ln x)^2\,dx = \tfrac{1}{2}x^2(\ln x)^2 - \tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2.$

6. $$\int x^p(\ln x)^2\,dx = \begin{cases} \dfrac{x^{p+1}}{p+1}(\ln x)^2 - \dfrac{2x^{p+1}}{(p+1)^2}\ln x + \dfrac{2x^{p+1}}{(p+1)^3} & \text{if } p\neq -1,\\ \tfrac{1}{3}\ln^3 x & \text{if } p=-1. \end{cases}$$

7. $$\int (\ln x)^n\,dx = \frac{x}{n+1}\sum_{k=0}^{n}(-1)^k(n+1)n\ldots(n-k+1)(\ln x)^{n-k}, \qquad n=1,\,2,\,\ldots$$

8. $\displaystyle\int (\ln x)^q\,dx = x(\ln x)^q - q\int(\ln x)^{q-1}\,dx,\ q\neq -1.$

9. $$\int x^n(\ln x)^m\,dx = \frac{x^{n+1}}{m+1}\sum_{k=0}^{m}\frac{(-1)^k}{(n+1)^{k+1}}(m+1)m\ldots(m-k+1)(\ln x)^{m-k},$$
$n,m=1,\,2,\,\ldots$

10. $$\int x^p(\ln x)^q\,dx = \frac{1}{p+1}x^{p+1}(\ln x)^q - \frac{q}{p+1}\int x^p(\ln x)^{q-1}\,dx, \qquad p,\,q\neq -1.$$

11. $\displaystyle\int \ln(a+bx)\,dx = \frac{1}{b}(ax+b)\ln(ax+b) - x.$

12. $$\int x\ln(a+bx)\,dx = \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right)\ln(a+bx) - \frac{1}{2}\left(\frac{x^2}{2} - \frac{a}{b}x\right).$$

13. $$\int x^2\ln(a+bx)\,dx = \frac{1}{3}\left(x^3 - \frac{a^3}{b^3}\right)\ln(a+bx) - \frac{1}{3}\left(\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2}\right).$$

14. $$\int \frac{\ln x\,dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab}\ln\frac{x}{a+bx}.$$

15. $$\int \frac{\ln x\,dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b}\ln\frac{x}{a+bx}.$$

16. $$\int \frac{\ln x\,dx}{\sqrt{a+bx}} = \begin{cases} \dfrac{2}{b}\left[(\ln x-2)\sqrt{a+bx} + \sqrt{a}\,\ln\dfrac{\sqrt{a+bx}+\sqrt{a}}{\sqrt{a+bx}-\sqrt{a}}\right] & \text{if } a>0,\\ \dfrac{2}{b}\left[(\ln x-2)\sqrt{a+bx} + 2\sqrt{-a}\,\arctan\dfrac{\sqrt{a+bx}}{\sqrt{-a}}\right] & \text{if } a<0. \end{cases}$$

17. $\displaystyle\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\arctan(x/a).$

18. $\displaystyle\int x\ln(x^2+a^2)\,dx = \tfrac{1}{2}\big[(x^2+a^2)\ln(x^2+a^2) - x^2\big].$

19. $\displaystyle\int x^2 \ln(x^2+a^2)\,dx = \tfrac{1}{3}\big[x^3\ln(x^2+a^2) - \tfrac{2}{3}x^3 + 2a^2x - 2a^3\arctan(x/a)\big].$

S2.1.6 Integrals Involving Trigonometric Functions

► **Integrals involving $\cos x$ $(n = 1,\ 2,\ \ldots)$**

1. $\displaystyle\int \cos(a+bx)\,dx = \frac{1}{b}\sin(a+bx).$

2. $\displaystyle\int x\cos x\,dx = \cos x + x\sin x.$

3. $\displaystyle\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$

4. $\displaystyle\int x^{2n}\cos x\,dx = (2n)!\bigg[\sum_{k=0}^{n}(-1)^k\frac{x^{2n-2k}}{(2n-2k)!}\sin x + \sum_{k=0}^{n-1}(-1)^k\frac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\bigg].$

5. $\displaystyle\int x^{2n+1}\cos x\,dx = (2n+1)!\sum_{k=0}^{n}\bigg[(-1)^k\frac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x + \frac{x^{2n-2k}}{(2n-2k)!}\cos x\bigg].$

6. $\displaystyle\int x^p\cos x\,dx = x^p\sin x + px^{p-1}\cos x - p(p-1)\int x^{p-2}\cos x\,dx.$

7. $\displaystyle\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$

8. $\displaystyle\int \cos^3 x\,dx = \sin x - \tfrac{1}{3}\sin^3 x.$

9. $\displaystyle\int \cos^{2n} x\,dx = \frac{1}{2^{2n}}C_{2n}^{n}x + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}C_{2n}^{k}\frac{\sin[(2n-2k)x]}{2n-2k}.$

10. $\displaystyle\int \cos^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}C_{2n+1}^{k}\frac{\sin[(2n-2k+1)x]}{2n-2k+1}.$

11. $\displaystyle\int \frac{dx}{\cos x} = \ln\Big|\tan\Big(\frac{x}{2}+\frac{\pi}{4}\Big)\Big|.$

12. $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x.$

13. $\displaystyle\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\Big|\tan\Big(\frac{x}{2}+\frac{\pi}{4}\Big)\Big|.$

14. $\displaystyle\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2} x}, \quad n>1.$

15. $\displaystyle\int \frac{x\,dx}{\cos^{2n} x} = \sum_{k=0}^{n-1}A_{nk}\frac{(2n-2k)x\sin x - \cos x}{\cos^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!}\big(x\tan x + \ln|\cos x|\big),$

$$A_{nk} = \frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)}\,\frac{1}{(2n-2k+1)(2n-2k)}.$$

16. $\displaystyle\int \cos ax\,\cos bx\,dx = \frac{\sin\big[(b-a)x\big]}{2(b-a)} + \frac{\sin\big[(b+a)x\big]}{2(b+a)}, \quad a \neq \pm b.$

17. $$\int \frac{dx}{a+b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}} \arctan \dfrac{(a-b)\tan(x/2)}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\ \dfrac{1}{\sqrt{b^2-a^2}} \ln \left| \dfrac{\sqrt{b^2-a^2}+(b-a)\tan(x/2)}{\sqrt{b^2-a^2}-(b-a)\tan(x/2)} \right| & \text{if } b^2 > a^2. \end{cases}$$

18. $$\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2} \int \frac{dx}{a+b\cos x}.$$

19. $$\int \frac{dx}{a^2+b^2\cos^2 x} = \frac{1}{a\sqrt{a^2+b^2}} \arctan \frac{a\tan x}{\sqrt{a^2+b^2}}.$$

20. $$\int \frac{dx}{a^2-b^2\cos^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \arctan \dfrac{a\tan x}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\ \dfrac{1}{2a\sqrt{b^2-a^2}} \ln \left| \dfrac{\sqrt{b^2-a^2}-a\tan x}{\sqrt{b^2-a^2}+a\tan x} \right| & \text{if } b^2 > a^2. \end{cases}$$

21. $$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b}{a^2+b^2}\sin bx + \frac{a}{a^2+b^2}\cos bx\right).$$

22. $$\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\cos^2 x + 2\sin x\cos x + \frac{2}{a}\right).$$

23. $$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1} x}{a^2+n^2}(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\,dx.$$

► Integrals involving $\sin x$ ($n = 1, 2, \ldots$)

1. $$\int \sin(a+bx)\,dx = -\frac{1}{b}\cos(a+bx).$$

2. $$\int x\sin x\,dx = \sin x - x\cos x.$$

3. $$\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x.$$

4. $$\int x^3\sin x\,dx = (3x^2-6)\sin x - (x^3-6x)\cos x.$$

5. $$\int x^{2n}\sin x\,dx = (2n)!\left[\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k}}{(2n-2k)!}\cos x + \sum_{k=0}^{n-1}(-1)^{k}\frac{x^{2n-2k-1}}{(2n-2k-1)!}\sin x\right].$$

6. $$\int x^{2n+1}\sin x\,dx = (2n+1)!\sum_{k=0}^{n}\left[(-1)^{k+1}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\cos x + (-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x\right].$$

7. $$\int x^p\sin x\,dx = -x^p\cos x + px^{p-1}\sin x - p(p-1)\int x^{p-2}\sin x\,dx.$$

8. $$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$$

9. $$\int x\sin^2 x\,dx = \tfrac{1}{4}x^2 - \tfrac{1}{4}x\sin 2x - \tfrac{1}{8}\cos 2x.$$

10. $$\int \sin^3 x\,dx = -\cos x + \tfrac{1}{3}\cos^3 x.$$

11. $$\int \sin^{2n} x\,dx = \frac{1}{2^{2n}} C_{2n}^{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k C_{2n}^{k} \frac{\sin[(2n-2k)x]}{2n-2k},$$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients $(0! = 1)$.

12. $$\int \sin^{2n+1} x\,dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n+k+1} C_{2n+1}^{k} \frac{\cos[(2n-2k+1)x]}{2n-2k+1}.$$

13. $$\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right|.$$

14. $$\int \frac{dx}{\sin^2 x} = -\cot x.$$

15. $$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|.$$

16. $$\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}, \qquad n > 1.$$

17. $$\int \frac{x\,dx}{\sin^{2n} x} = -\sum_{k=0}^{n-1} A_{nk} \frac{\sin x + (2n-2k)x\cos x}{\sin^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!}\big(\ln|\sin x| - x\cot x\big),$$

$$A_{nk} = \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{1}{(2n-2k+1)(2n-2k)}.$$

18. $$\int \sin ax\,\sin bx\,dx = \frac{\sin[(b-a)x]}{2(b-a)} - \frac{\sin[(b+a)x]}{2(b+a)}, \qquad a \neq \pm b.$$

19. $$\int \frac{dx}{a + b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}} \arctan \dfrac{b + a\tan x/2}{\sqrt{a^2-b^2}} & \text{if } a^2 > b^2, \\[2ex] \dfrac{1}{\sqrt{b^2-a^2}} \ln\left|\dfrac{b - \sqrt{b^2-a^2} + a\tan x/2}{b + \sqrt{b^2-a^2} + a\tan x/2}\right| & \text{if } b^2 > a^2. \end{cases}$$

20. $$\int \frac{dx}{(a + b\sin x)^2} = \frac{b\cos x}{(a^2-b^2)(a + b\sin x)} + \frac{a}{a^2-b^2}\int \frac{dx}{a + b\sin x}.$$

21. $$\int \frac{dx}{a^2 + b^2\sin^2 x} = \frac{1}{a\sqrt{a^2+b^2}} \arctan \frac{\sqrt{a^2+b^2}\,\tan x}{a}.$$

22. $$\int \frac{dx}{a^2 - b^2\sin^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \arctan \dfrac{\sqrt{a^2-b^2}\,\tan x}{a} & \text{if } a^2 > b^2, \\[2ex] \dfrac{1}{2a\sqrt{b^2-a^2}} \ln\left|\dfrac{\sqrt{b^2-a^2}\,\tan x + a}{\sqrt{b^2-a^2}\,\tan x - a}\right| & \text{if } b^2 > a^2. \end{cases}$$

23. $$\int \frac{\sin x\,dx}{\sqrt{1 + k^2\sin^2 x}} = -\frac{1}{k} \arcsin \frac{k\cos x}{\sqrt{1+k^2}}.$$

24. $$\int \frac{\sin x\,dx}{\sqrt{1 - k^2\sin^2 x}} = -\frac{1}{k} \ln\left|k\cos x + \sqrt{1 - k^2\sin^2 x}\,\right|.$$

25. $$\int \sin x\,\sqrt{1 + k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1 + k^2\sin^2 x} - \frac{1+k^2}{2k} \arcsin \frac{k\cos x}{\sqrt{1+k^2}}.$$

26. $$\int \sin x\,\sqrt{1 - k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1 - k^2\sin^2 x} - \frac{1-k^2}{2k} \ln\left|k\cos x + \sqrt{1 - k^2\sin^2 x}\,\right|.$$

27. $\displaystyle\int e^{ax}\sin bx\,dx = e^{ax}\Big(\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\Big).$

28. $\displaystyle\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^2+4}\Big(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\Big).$

29. $\displaystyle\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx.$

► Integrals involving $\sin x$ and $\cos x$

1. $\displaystyle\int \sin ax\cos bx\,dx = -\frac{\cos[(a+b)x]}{2(a+b)} - \frac{\cos\big[(a-b)x\big]}{2(a-b)}, \qquad a\neq\pm b.$

2. $\displaystyle\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc}\arctan\Big(\frac{c}{b}\tan ax\Big).$

3. $\displaystyle\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\ln\Big|\frac{c\tan ax + b}{c\tan ax - b}\Big|.$

4. $\displaystyle\int \frac{dx}{\cos^{2n}x\sin^{2m}x} = \sum_{k=0}^{n+m-1} C^k_{n+m-1}\frac{\tan^{2k-2m+1}x}{2k-2m+1}, \qquad n,m=1,\,2,\,\ldots$

5. $\displaystyle\int \frac{dx}{\cos^{2n+1}x\sin^{2m+1}x} = C^m_{n+m}\ln|\tan x| + \sum_{k=0}^{n+m} C^k_{n+m}\frac{\tan^{2k-2m}x}{2k-2m}, \quad n,m=1,\,2,\,\ldots$

► Reduction formulas

◆ *The parameters p and q below can assume any values, except for those at which the denominators on the right-hand side vanish.*

1. $\displaystyle\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}x\cos^q x\,dx.$

2. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q-1}x}{p+q} + \frac{q-1}{p+q}\int \sin^p x\cos^{q-2}x\,dx.$

3. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p-1}x\cos^{q-1}x}{p+q}\Big(\sin^2 x - \frac{q-1}{p+q-2}\Big) + \frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int \sin^{p-2}x\cos^{q-2}x\,dx.$

4. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q+1}x}{p+1} + \frac{p+q+2}{p+1}\int \sin^{p+2}x\cos^q x\,dx.$

5. $\displaystyle\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p+1}x\cos^{q+1}x}{q+1} + \frac{p+q+2}{q+1}\int \sin^p x\cos^{q+2}x\,dx.$

6. $\displaystyle\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q+2}x\,dx.$

7. $\displaystyle\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q-1}x}{p+1} + \frac{q-1}{p+1}\int \sin^{p+2}x\cos^{q-2}x\,dx.$

► **Integrals involving $\tan x$ and $\cot x$**

1. $\displaystyle\int \tan x\,dx = -\ln|\cos x|.$

2. $\displaystyle\int \tan^2 x\,dx = \tan x - x.$

3. $\displaystyle\int \tan^3 x\,dx = \tfrac{1}{2}\tan^2 x + \ln|\cos x|.$

4. $\displaystyle\int \tan^{2n} x\,dx = (-1)^n x - \sum_{k=1}^{n} \frac{(-1)^k(\tan x)^{2n-2k+1}}{2n-2k+1}, \quad n = 1, 2, \ldots$

5. $\displaystyle\int \tan^{2n+1} x\,dx = (-1)^{n+1}\ln|\cos x| - \sum_{k=1}^{n} \frac{(-1)^k(\tan x)^{2n-2k+2}}{2n-2k+2}, \quad n = 1, 2, \ldots$

6. $\displaystyle\int \frac{dx}{a + b\tan x} = \frac{1}{a^2+b^2}\bigl(ax + b\ln|a\cos x + b\sin x|\bigr).$

7. $\displaystyle\int \frac{\tan x\,dx}{\sqrt{a + b\tan^2 x}} = \frac{1}{\sqrt{b-a}}\arccos\biggl(\sqrt{1 - \frac{a}{b}}\,\cos x\biggr), \quad b > a,\ b > 0.$

8. $\displaystyle\int \cot x\,dx = \ln|\sin x|.$

9. $\displaystyle\int \cot^2 x\,dx = -\cot x - x.$

10. $\displaystyle\int \cot^3 x\,dx = -\tfrac{1}{2}\cot^2 x - \ln|\sin x|.$

11. $\displaystyle\int \cot^{2n} x\,dx = (-1)^n x + \sum_{k=1}^{n} \frac{(-1)^k(\cot x)^{2n-2k+1}}{2n-2k+1}, \quad n = 1, 2, \ldots$

12. $\displaystyle\int \cot^{2n+1} x\,dx = (-1)^n\ln|\sin x| + \sum_{k=1}^{n} \frac{(-1)^k(\cot x)^{2n-2k+2}}{2n-2k+2}, \quad n = 1, 2, \ldots$

13. $\displaystyle\int \frac{dx}{a + b\cot x} = \frac{1}{a^2+b^2}\bigl(ax - b\ln|a\sin x + b\cos x|\bigr).$

S2.1.7 Integrals Involving Inverse Trigonometric Functions

1. $\displaystyle\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2 - x^2}.$

2. $\displaystyle\int \Bigl(\arcsin\frac{x}{a}\Bigr)^2 dx = x\Bigl(\arcsin\frac{x}{a}\Bigr)^2 - 2x + 2\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}.$

3. $\displaystyle\int x\arcsin\frac{x}{a}\,dx = \frac{1}{4}(2x^2 - a^2)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2}.$

4. $\displaystyle\int x^2\arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2}.$

5. $\displaystyle\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2 - x^2}.$

6. $\displaystyle\int \Bigl(\arccos\frac{x}{a}\Bigr)^2 dx = x\Bigl(\arccos\frac{x}{a}\Bigr)^2 - 2x - 2\sqrt{a^2-x^2}\,\arccos\frac{x}{a}.$

7. $\displaystyle\int x \arccos\frac{x}{a}\,dx = \frac{1}{4}(2x^2 - a^2)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2 - x^2}.$

8. $\displaystyle\int x^2 \arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2}.$

9. $\displaystyle\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2 + x^2).$

10. $\displaystyle\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2 + a^2)\arctan\frac{x}{a} - \frac{ax}{2}.$

11. $\displaystyle\int x^2\arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2 + x^2).$

12. $\displaystyle\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} + \frac{a}{2}\ln(a^2 + x^2).$

13. $\displaystyle\int x\operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}(x^2 + a^2)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}.$

14. $\displaystyle\int x^2\operatorname{arccot}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arccot}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2 + x^2).$

S2.2 Tables of Definite Integrals

◆ *Throughout Section S2.2 it is assumed that n is a positive integer, unless otherwise specified.*

S2.2.1 Integrals Involving Power-Law Functions

▶ **Integrals over a finite interval**

1. $\displaystyle\int_0^1 \frac{x^n\,dx}{x+1} = (-1)^n\left[\ln 2 + \sum_{k=1}^{n}\frac{(-1)^k}{k}\right].$

2. $\displaystyle\int_0^1 \frac{dx}{x^2 + 2x\cos\beta + 1} = \frac{\beta}{2\sin\beta}.$

3. $\displaystyle\int_0^1 \frac{(x^a + x^{-a})\,dx}{x^2 + 2x\cos\beta + 1} = \frac{\pi\sin(a\beta)}{\sin(\pi a)\sin\beta}, \qquad |a| < 1,\ \beta \neq (2n+1)\pi.$

4. $\displaystyle\int_0^1 x^a(1-x)^{1-a}\,dx = \frac{\pi a(1-a)}{2\sin(\pi a)}, \qquad -1 < a < 1.$

5. $\displaystyle\int_0^1 \frac{dx}{x^a(1-x)^{1-a}} = \frac{\pi}{\sin(\pi a)}, \qquad 0 < a < 1.$

6. $\displaystyle\int_0^1 \frac{x^a\,dx}{(1-x)^a} = \frac{\pi a}{\sin(\pi a)}, \qquad -1 < a < 1.$

7. $\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \equiv B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p, q > 0.$

8. $\displaystyle\int_0^1 x^{p-1}(1-x^q)^{-p/q}\,dx = \frac{\pi}{q\sin(\pi p/q)}, \qquad q > p > 0.$

9. $\displaystyle\int_0^1 x^{p+q-1}(1-x^q)^{-p/q}\,dx = \frac{\pi p}{q^2\sin(\pi p/q)}, \quad q > p.$

10. $\displaystyle\int_0^1 x^{q/p-1}(1-x^q)^{-1/p}\,dx = \frac{\pi}{q\sin(\pi/p)}, \quad p > 1,\ q > 0.$

11. $\displaystyle\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \pi\cot(\pi p), \quad |p| < 1.$

12. $\displaystyle\int_0^1 \frac{x^{p-1}-x^{-p}}{1+x}\,dx = \frac{\pi}{\sin(\pi p)}, \quad |p| < 1.$

13. $\displaystyle\int_0^1 \frac{x^p-x^{-p}}{x-1}\,dx = \frac{1}{p} - \pi\cot(\pi p), \quad |p| < 1.$

14. $\displaystyle\int_0^1 \frac{x^p-x^{-p}}{1+x}\,dx = \frac{1}{p} - \frac{\pi}{\sin(\pi p)}, \quad |p| < 1.$

15. $\displaystyle\int_0^1 \frac{x^{1+p}-x^{1-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\Bigl(\frac{\pi p}{2}\Bigr) - \frac{1}{p}, \quad |p| < 1.$

16. $\displaystyle\int_0^1 \frac{x^{1+p}-x^{1-p}}{1+x^2}\,dx = \frac{1}{p} - \frac{\pi}{2\sin(\pi p/2)}, \quad |p| < 1.$

17. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1+a^2x)(1-x)}} = \frac{2}{a}\arctan a.$

18. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-a^2x)(1-x)}} = \frac{1}{a}\ln\frac{1+a}{1-a}.$

19. $\displaystyle\int_{-1}^1 \frac{dx}{(a-x)\sqrt{1-x^2}} = \frac{\pi}{\sqrt{a^2-1}}, \quad 1 < a.$

20. $\displaystyle\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}} = \frac{2\,(2n)!!}{(2n+1)!!}, \quad n = 1,\ 2,\ \dots$

21. $\displaystyle\int_0^1 \frac{x^{n-1/2}\,dx}{\sqrt{1-x}} = \frac{\pi\,(2n-1)!!}{(2n)!!}, \quad n = 1,\ 2,\ \dots$

22. $\displaystyle\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\,\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}, \quad n = 1,\ 2,\ \dots$

23. $\displaystyle\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)}, \quad n = 1,\ 2,\ \dots$

24. $\displaystyle\int_0^1 \frac{x^{\lambda-1}\,dx}{(1+ax)(1-x)^\lambda} = \frac{\pi}{(1+a)^\lambda\sin(\pi\lambda)}, \quad 0 < \lambda < 1, \quad a > -1.$

25. $\displaystyle\int_0^1 \frac{x^{\lambda-1/2}\,dx}{(1+ax)^\lambda(1-x)^\lambda} = 2\pi^{-1/2}\Gamma\bigl(\lambda+\tfrac{1}{2}\bigr)\Gamma\bigl(1-\lambda\bigr)\cos^{2\lambda}k\,\frac{\sin[(2\lambda-1)k]}{(2\lambda-1)\sin k},$
$k = \arctan\sqrt{a},\ -\tfrac{1}{2} < \lambda < 1, \quad a > 0.$

► Integrals over an infinite interval

1. $\displaystyle\int_0^\infty \frac{dx}{ax^2+b} = \frac{\pi}{2\sqrt{ab}}.$

2. $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}.$

3. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x+1} = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1.$

4. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(1+ax)^2} = \frac{\pi(1-\lambda)}{a^\lambda \sin(\pi\lambda)}, \quad 0 < \lambda < 2.$

5. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(x+a)(x+b)} = \frac{\pi(a^{\lambda-1}-b^{\lambda-1})}{(b-a)\sin(\pi\lambda)}, \quad 0 < \lambda < 2.$

6. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}(x+c)\,dx}{(x+a)(x+b)} = \frac{\pi}{\sin(\pi\lambda)}\left(\frac{a-c}{a-b}a^{\lambda-1} + \frac{b-c}{b-a}b^{\lambda-1}\right), \quad 0 < \lambda < 1.$

7. $\displaystyle\int_0^\infty \frac{x^\lambda\,dx}{(x+1)^3} = \frac{\pi\lambda(1-\lambda)}{2\sin(\pi\lambda)}, \quad -1 < \lambda < 2.$

8. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi\big(b^{\lambda-2}-a^{\lambda-2}\big)}{2\big(a^2-b^2\big)\sin(\pi\lambda/2)}, \quad 0 < \lambda < 4.$

9. $\displaystyle\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x}\,dx = \pi[\cot(\pi p) - \cot(\pi q)], \quad p, q > 0.$

10. $\displaystyle\int_0^\infty \frac{x^{\lambda-1}\,dx}{(1+ax)^{n+1}} = (-1)^n \frac{\pi C^n_{\lambda-1}}{a^\lambda \sin(\pi\lambda)}, \quad C^n_{\lambda-1} = \frac{(\lambda-1)(\lambda-2)\ldots(\lambda-n)}{n!},$
$0 < \lambda < n+1.$

11. $\displaystyle\int_0^\infty \frac{x^m\,dx}{(a+bx)^{n+1/2}} = 2^{m+1} m!\,\frac{(2n-2m-3)!!}{(2n-1)!!}\,\frac{a^{m-n+1/2}}{b^{m+1}}, \quad a, b > 0, \quad m < b - \tfrac{1}{2},$
$n, m = 1,\ 2,\ \ldots$

12. $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{\pi}{2}\,\frac{(2n-3)!!}{(2n-2)!!}\,\frac{1}{a^{2n-1}}, \quad n = 1,\ 2,\ \ldots$

13. $\displaystyle\int_0^\infty \frac{(x+1)^{\lambda-1}}{(x+a)^{\lambda+1}}\,dx = \frac{1-a^{-\lambda}}{\lambda(a-1)}, \quad a > 0.$

14. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{x^b+1} = \frac{\pi}{b\sin(\pi a/b)}, \quad 0 < a \le b.$

15. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{(x^b+1)^2} = \frac{\pi(a-b)}{b^2\sin[\pi(a-b)/b]}, \quad a < 2b.$

16. $\displaystyle\int_0^\infty \frac{x^{\lambda-1/2}\,dx}{(x+a)^\lambda (x+b)^\lambda} = \sqrt{\pi}\big(\sqrt{a}+\sqrt{b}\big)^{1-2\lambda}\frac{\Gamma(\lambda-1/2)}{\Gamma(\lambda)}, \quad \lambda > 0.$

17. $\displaystyle\int_0^\infty \frac{1-x^a}{1-x^b}\,x^{c-1}\,dx = \frac{\pi\sin A}{b\sin C\sin(A+C)}, \quad A = \frac{\pi a}{b}, \quad C = \frac{\pi c}{b}; \quad a+c < b,$
$c > 0.$

18. $\displaystyle\int_0^\infty \frac{x^{a-1}\,dx}{(1+x^2)^{1-b}} = \tfrac{1}{2}B\big(\tfrac{1}{2}a, 1-b-\tfrac{1}{2}a\big), \quad \tfrac{1}{2}a + b < 1, \quad a > 0.$

19. $\displaystyle\int_0^\infty \frac{x^{2m}\,dx}{(ax^2+b)^n} = \frac{\pi(2m-1)!!\,(2n-2m-3)!!}{2\,(2n-2)!!\,a^m b^{n-m-1}\sqrt{ab}}, \quad a, b > 0, \quad n > m+1.$

20. $\displaystyle\int_0^\infty \frac{x^{2m+1}\,dx}{(ax^2+b)^n} = \frac{m!\,(n-m-2)!}{2(n-1)!a^{m+1}b^{n-m-1}}, \quad ab > 0, \quad n > m+1 \ge 1.$

21. $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+ax^p)^\nu} = \frac{1}{pa^{\mu/p}} B\Big(\frac{\mu}{p}, \nu - \frac{\mu}{p}\Big), \quad p>0, \quad 0<\mu<p\nu.$

22. $\displaystyle\int_0^\infty \big(\sqrt{x^2+a^2}-x\big)^n\,dx = \frac{na^{n+1}}{n^2-1}, \quad n=2,\,3,\,\dots$

23. $\displaystyle\int_0^\infty \frac{dx}{\big(x+\sqrt{x^2+a^2}\big)^n} = \frac{n}{a^{n-1}(n^2-1)}, \quad n=2,\,3,\,\dots$

24. $\displaystyle\int_0^\infty x^m\big(\sqrt{x^2+a^2}-x\big)^n\,dx = \frac{m!\,na^{n+m+1}}{(n-m-1)(n-m+1)\dots(n+m+1)},$
$n, m = 1,\,2,\,\dots, \quad 0\le m\le n-2.$

25. $\displaystyle\int_0^\infty \frac{x^m\,dx}{\big(x+\sqrt{x^2+a^2}\big)^n} = \frac{m!\,n}{(n-m-1)(n-m+1)\dots(n+m+1)a^{n-m-1}},$
$n=2,\,3,\,\dots$

S2.2.2 Integrals Involving Exponential Functions

1. $\displaystyle\int_0^\infty e^{-ax}\,dx = \frac{1}{a}, \quad a>0.$

2. $\displaystyle\int_0^1 x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} - e^{-a}\sum_{k=0}^{n}\frac{n!}{k!}\frac{1}{a^{n-k+1}}, \quad a>0, \ n=1,\,2,\,\dots$

3. $\displaystyle\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}, \quad a>0, \ n=1,\,2,\,\dots$

4. $\displaystyle\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}, \quad a>0.$

5. $\displaystyle\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \frac{\Gamma(\nu)}{\mu^\nu}, \quad \mu,\nu>0.$

6. $\displaystyle\int_0^\infty \frac{dx}{1+e^{ax}} = \frac{\ln 2}{a}.$

7. $\displaystyle\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\Big(\frac{2\pi}{p}\Big)^{2n}\frac{B_{2n}}{4n}, \quad n=1,\,2,\,\dots;$ the B_m are Bernoulli numbers (see Section S4.1.3).

8. $\displaystyle\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = (1-2^{1-2n})\Big(\frac{2\pi}{p}\Big)^{2n}\frac{|B_{2n}|}{4n}, \quad n=1,\,2,\,\dots;$ the B_m are Bernoulli numbers.

9. $\displaystyle\int_{-\infty}^\infty \frac{e^{-px}\,dx}{1+e^{-qx}} = \frac{\pi}{q\sin(\pi p/q)}, \quad q>p>0 \text{ or } 0>p>q.$

10. $\displaystyle\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{bx}+e^{-bx}}\,dx = \frac{\pi}{2b\cos\Big(\dfrac{\pi a}{2b}\Big)}, \quad b>a.$

11. $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\frac{\pi p}{p+q}, \quad p,q>0.$

12. $\displaystyle\int_0^\infty \big(1-e^{-\beta x}\big)^\nu e^{-\mu x}\,dx = \frac{1}{\beta}B\Big(\frac{\mu}{\beta}, \nu+1\Big).$

13. $\displaystyle\int_0^\infty \exp(-ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a>0.$

14. $\displaystyle\int_0^\infty x^{2n+1}\exp(-ax^2)\,dx = \frac{n!}{2a^{n+1}}, \quad a>0, \quad n=1,\,2,\,\ldots$

15. $\displaystyle\int_0^\infty x^{2n}\exp(-ax^2)\,dx = \frac{1\times 3\times\cdots\times(2n-1)\sqrt{\pi}}{2^{n+1}a^{n+1/2}}, \quad a>0, \quad n=1,\,2,\,\ldots$

16. $\displaystyle\int_{-\infty}^\infty \exp(-a^2x^2\pm bx)\,dx = \frac{\sqrt{\pi}}{|a|}\exp\left(\frac{b^2}{4a^2}\right).$

17. $\displaystyle\int_0^\infty \exp\left(-ax^2-\frac{b}{x^2}\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-2\sqrt{ab}\right), \quad a,b>0.$

18. $\displaystyle\int_0^\infty \exp(-x^a)\,dx = \frac{1}{a}\Gamma\left(\frac{1}{a}\right), \quad a>0.$

S2.2.3 Integrals Involving Hyperbolic Functions

1. $\displaystyle\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2|a|}.$

2. $\displaystyle\int_0^\infty \frac{dx}{a+b\cosh x} = \begin{cases} \dfrac{2}{\sqrt{b^2-a^2}}\arctan\dfrac{\sqrt{b^2-a^2}}{a+b} & \text{if } |b|>|a|,\\[2ex] \dfrac{1}{\sqrt{a^2-b^2}}\ln\dfrac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2+b^2}} & \text{if } |b|<|a|. \end{cases}$

3. $\displaystyle\int_0^\infty \frac{x^{2n}\,dx}{\cosh ax} = \left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}|, \quad a>0;$ the E_m are Euler numbers (see Section S4.1.4).

4. $\displaystyle\int_0^\infty \frac{x^{2n}\,dx}{\cosh^2 ax} = \frac{\pi^{2n}(2^{2n}-2)}{|a|(2a)^{2n}}|B_{2n}|;$ the B_m are Bernoulli numbers (see Section S4.1.3).

5. $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b\cos\left(\dfrac{\pi a}{2b}\right)}, \quad b>|a|.$

6. $\displaystyle\int_0^\infty x^{2n}\frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2n}}{da^{2n}}\frac{1}{\cos\left(\frac{1}{2}\pi a/b\right)}, \quad b>|a|, \quad n=1,\,2,\,\ldots$

7. $\displaystyle\int_0^\infty \frac{\cosh ax\cosh bx}{\cosh(cx)}\,dx = \frac{\pi}{c}\,\frac{\cos\left(\dfrac{\pi a}{2c}\right)\cos\left(\dfrac{\pi b}{2c}\right)}{\cos\left(\dfrac{\pi a}{c}\right)+\cos\left(\dfrac{\pi b}{c}\right)}, \quad c>|a|+|b|.$

8. $\displaystyle\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{2a^2}, \quad a>0.$

9. $\displaystyle\int_0^\infty \frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}, \quad ab\neq 0.$

10. $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\left(\frac{\pi a}{2b}\right), \quad b>|a|.$

11. $\displaystyle\int_0^\infty x^{2n}\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2n}}{dx^{2n}}\tan\left(\frac{\pi a}{2b}\right), \quad b>|a|, \quad n=1,\,2,\,\ldots$

12. $\displaystyle\int_0^\infty \frac{x^{2n}}{\sinh^2 ax}\,dx = \frac{\pi^{2n}}{a^{2n+1}}|B_{2n}|, \quad a>0;$ the B_m are Bernoulli numbers.

S2.2.4 Integrals Involving Logarithmic Functions

1. $\displaystyle\int_0^1 x^{a-1}\ln^n x\,dx = (-1)^n n!\,a^{-n-1}, \quad a>0, \quad n=1,\,2,\,\dots$

2. $\displaystyle\int_0^1 \frac{\ln x}{x+1}\,dx = -\frac{\pi^2}{12}.$

3. $\displaystyle\int_0^1 \frac{x^n\ln x}{x+1}\,dx = (-1)^{n+1}\left[\frac{\pi^2}{12}+\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right], \quad n=1,\,2,\,\dots$

4. $\displaystyle\int_0^1 \frac{x^{\mu-1}\ln x}{x+a}\,dx = \frac{\pi a^{\mu-1}}{\sin(\pi\mu)}\bigl[\ln a-\pi\cot(\pi\mu)\bigr], \quad 0<\mu<1.$

5. $\displaystyle\int_0^1 |\ln x|^\mu\,dx = \Gamma(\mu+1), \quad \mu>-1.$

6. $\displaystyle\int_0^\infty x^{\mu-1}\ln(1+ax)\,dx = \frac{\pi}{\mu a^\mu\sin(\pi\mu)}, \quad -1<\mu<0.$

7. $\displaystyle\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}, \quad n=1,\,2,\,\dots$

8. $\displaystyle\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{1}{2n+1}\left[\ln 4+\sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right], \quad n=0,\,1,\,\dots$

9. $\displaystyle\int_0^1 x^{n-1/2}\ln(1+x)\,dx = \frac{2\ln 2}{2n+1}+\frac{4(-1)^n}{2n+1}\left[\pi-\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right], \quad n=1,\,2,\,\dots$

10. $\displaystyle\int_0^\infty \ln\frac{a^2+x^2}{b^2+x^2}\,dx = \pi(a-b), \quad a,b>0.$

11. $\displaystyle\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\,dx = -\frac{\pi^2\cos(\pi p/q)}{q^2\sin^2(\pi p/q)}, \quad 0<p<q.$

12. $\displaystyle\int_0^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}(\mathcal{C}+\ln\mu), \quad \mu>0, \quad \mathcal{C}=0.5772\dots$

S2.2.5 Integrals Involving Trigonometric Functions

► Integrals Over a Finite Interval

1. $\displaystyle\int_0^{\pi/2} \cos^{2n} x\,dx = \frac{\pi}{2}\,\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}, \quad n=1,\,2,\,\dots$

2. $\displaystyle\int_0^{\pi/2} \cos^{2n+1} x\,dx = \frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)}, \quad n=1,\,2,\,\dots$

3. $$\int_0^{\pi/2} x\cos^n x\,dx = -\sum_{k=0}^{m-1}\frac{(n-2k+1)(n-2k+3)\dots(n-1)}{(n-2k)(n-2k+2)\dots n}\,\frac{1}{n-2k} + \begin{cases} \dfrac{\pi}{2}\dfrac{(2m-2)!!}{(2m-1)!!} & \text{if } n=2m-1,\\ \dfrac{\pi^2}{8}\dfrac{(2m-1)!!}{(2m)!!} & \text{if } n=2m,\end{cases} \qquad m=1,\,2,\,\dots$$

4. $$\int_0^{\pi}\frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n}\frac{(2n-2k-1)!!\,(2k-1)!!}{(n-k)!\,k!}\left(\frac{a+b}{a-b}\right)^k,$$ $a>|b|$.

5. $$\int_0^{\pi/2}\sin^{2n}x\,dx = \frac{\pi}{2}\,\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)},\qquad n=1,\,2,\,\dots$$

6. $$\int_0^{\pi/2}\sin^{2n+1}x\,dx = \frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)},\qquad n=1,\,2,\,\dots$$

7. $$\int_0^{\pi} x\sin^{\mu}x\,dx = \frac{\pi^2}{2^{\mu+1}}\,\frac{\Gamma(\mu+1)}{\left[\Gamma\left(\mu+\frac{1}{2}\right)\right]^2},\qquad \mu>-1.$$

8. $$\int_0^{\pi/2}\frac{\sin x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2k}\ln\frac{1+k}{1-k}.$$

9. $$\int_0^{\pi/2}\sin^{2n+1}x\cos^{2m+1}x\,dx = \frac{n!\,m!}{2(n+m+1)!},\qquad n,m=1,\,2,\,\dots$$

10. $$\int_0^{\pi/2}\sin^{p-1}x\cos^{q-1}x\,dx = \tfrac{1}{2}B\left(\tfrac{1}{2}p,\tfrac{1}{2}q\right).$$

11. $$\int_0^{2\pi}(a\sin x+b\cos x)^{2n}\,dx = 2\pi\frac{(2n-1)!!}{(2n)!!}\left(a^2+b^2\right)^n,\qquad n=1,\,2,\,\dots$$

12. $$\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{a^2+1-2a\cos x}} = \begin{cases}2 & \text{if } 0\le a\le 1,\\ 2/a & \text{if } 1<a.\end{cases}$$

13. $$\int_0^{\pi/2}(\tan x)^{\pm\lambda}\,dx = \frac{\pi}{2\cos\left(\frac{1}{2}\pi\lambda\right)},\qquad |\lambda|<1.$$

14. $$\int_0^{a}\frac{\cos(xt)\,dt}{\sqrt{a^2-t^2}} = \frac{\pi}{2}J_0(ax),$$ $J_0(z)$ is the Bessel function (see Section S4.6).

15. $$\int_0^{a}\frac{t\sin(xt)\,dt}{\sqrt{a^2-t^2}} = \frac{\pi}{2}aJ_1(ax),$$ $J_1(z)$ is the Bessel function.

16. $$\int_0^{2\pi}\cos(a\cos x)\,dx = 2\pi J_0(a),$$ $J_0(z)$ is the Bessel function.

17. $$\int_0^{2\pi}\sin(a\cos x)\,dx = 0.$$

▶ Integrals over an infinite interval

1. $$\int_0^{\infty}\frac{\cos ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}},\qquad a>0.$$

2. $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\left|\frac{b}{a}\right|, \quad ab \neq 0.$

3. $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \tfrac{1}{2}\pi(b-a), \quad a, b \geq 0.$

4. $\displaystyle\int_0^\infty x^{\mu-1}\cos ax\,dx = a^{-\mu}\Gamma(\mu)\cos\bigl(\tfrac{1}{2}\pi\mu\bigr), \quad a > 0, \quad 0 < \mu < 1.$

5. $\displaystyle\int_0^\infty \frac{\cos ax}{b^2 + x^2}\,dx = \frac{\pi}{2b}e^{-ab}, \quad a, b > 0.$

6. $\displaystyle\int_0^\infty \frac{\cos ax}{b^4 + x^4}\,dx = \frac{\pi\sqrt{2}}{4b^3}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left[\cos\left(\frac{ab}{\sqrt{2}}\right) + \sin\left(\frac{ab}{\sqrt{2}}\right)\right], \quad a, b > 0.$

7. $\displaystyle\int_0^\infty \frac{\cos ax}{(b^2 + x^2)^2}\,dx = \frac{\pi}{4b^3}(1 + ab)e^{-ab}, \quad a, b > 0.$

8. $\displaystyle\int_0^\infty \frac{\cos ax\,dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi\bigl(be^{-ac} - ce^{-ab}\bigr)}{2bc\bigl(b^2 - c^2\bigr)}, \quad a, b, c > 0.$

9. $\displaystyle\int_0^\infty \cos\bigl(ax^2\bigr)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$

10. $\displaystyle\int_0^\infty \cos\bigl(ax^p\bigr)\,dx = \frac{\Gamma(1/p)}{pa^{1/p}}\cos\frac{\pi}{2p}, \quad a > 0, \quad p > 1.$

11. $\displaystyle\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a.$

12. $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi}{2}|a|.$

13. $\displaystyle\int_0^\infty \frac{\sin ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}, \quad a > 0.$

14. $\displaystyle\int_0^\infty x^{\mu-1}\sin ax\,dx = a^{-\mu}\Gamma(\mu)\sin\bigl(\tfrac{1}{2}\pi\mu\bigr), \quad a > 0, \quad 0 < \mu < 1.$

15. $\displaystyle\int_0^\infty \sin\bigl(ax^2\bigr)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$

16. $\displaystyle\int_0^\infty \sin\bigl(ax^p\bigr)\,dx = \frac{\Gamma(1/p)}{pa^{1/p}}\sin\frac{\pi}{2p}, \quad a > 0, \quad p > 1.$

17. $\displaystyle\int_0^\infty \frac{\sin x\cos ax}{x}\,dx = \begin{cases} \frac{\pi}{2} & \text{if } |a| < 1, \\ \frac{\pi}{4} & \text{if } |a| = 1, \\ 0 & \text{if } 1 < |a|. \end{cases}$

18. $\displaystyle\int_0^\infty \frac{\tan ax}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a.$

19. $\displaystyle\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2 + b^2}, \quad a > 0.$

20. $\displaystyle\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2 + b^2}, \quad a > 0.$

21. $\displaystyle\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2 + b^2)^2}, \quad a > 0.$

22. $\displaystyle\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}, \quad a>0.$

23. $\displaystyle\int_0^\infty x^n e^{-ax}\sin bx\,dx = (-1)^n \frac{\partial^n}{\partial a^n}\left(\frac{b}{a^2+b^2}\right), \quad a>0,\ n=1,\,2,\,\ldots.$

24. $\displaystyle\int_0^\infty x^n e^{-ax}\cos bx\,dx = (-1)^n \frac{\partial^n}{\partial a^n}\left(\frac{a}{a^2+b^2}\right), \quad a>0,\ n=1,\,2,\,\ldots.$

25. $\displaystyle\int_0^\infty \exp(-ax^2)\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\Big(-\frac{b^2}{4a}\Big), \quad a>0.$

26. $\displaystyle\int_0^\infty x\exp(-ax^2)\sin bx\,dx = \frac{\sqrt{\pi}\,b}{4a^{3/2}}\exp\Big(-\frac{b^2}{4a}\Big), \quad a>0.$

27. $\displaystyle\int_0^\infty \cos(ax^2)\cos bx\,dx = \sqrt{\frac{\pi}{8a}}\left[\cos\left(\frac{b^2}{4a}\right)+\sin\left(\frac{b^2}{4a}\right)\right], \quad a,b>0.$

28. $\displaystyle\int_0^\infty \cos(ax^2)\sin bx\,dx = \sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{b^2}{4a}\right)C\left(\frac{b^2}{4a}\right)-\sin\left(\frac{b^2}{4a}\right)S\left(\frac{b^2}{4a}\right)\right],$
$a,b>0$ and $C(z)$ and $S(z)$ are Fresnel integrals.

29. $\displaystyle\int_0^\infty \sin(ax^2)\cos bx\,dx = \sqrt{\frac{\pi}{8a}}\left[\cos\left(\frac{b^2}{4a}\right)-\sin\left(\frac{b^2}{4a}\right)\right], \quad a,b>0.$

30. $\displaystyle\int_0^\infty \sin(ax^2)\sin bx\,dx = \sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{b^2}{4a}\right)C\left(\frac{b^2}{4a}\right)+\sin\left(\frac{b^2}{4a}\right)S\left(\frac{b^2}{4a}\right)\right],$
$a,b>0$ and $C(z)$ and $S(z)$ are Fresnel integrals.

31. $\displaystyle\int_0^\infty \frac{1}{x^2}\sin(ax^2)\cos bx\,dx = \frac{b\pi}{2}\left[S\left(\frac{b^2}{4a}\right)-C\left(\frac{b^2}{4a}\right)+\sqrt{\pi a}\,\sin\left(\frac{b^2}{4a}+\frac{\pi}{4}\right)\right],$
$a,b>0$ and $C(z)$ and $S(z)$ are Fresnel integrals.

32. $\displaystyle\int_0^\infty (\cos ax+\sin ax)\cos(b^2x^2)\,dx = \frac{1}{b}\sqrt{\frac{\pi}{8}}\exp\left(-\frac{a^2}{2b}\right), \quad a,b>0.$

33. $\displaystyle\int_0^\infty \big(\cos ax+\sin ax\big)\sin(b^2x^2)\,dx = \frac{1}{b}\sqrt{\frac{\pi}{8}}\exp\Big(-\frac{a^2}{2b}\Big), \quad a,b>0.$

S2.2.6 Integrals Involving Bessel Functions

► **Integrals over an infinite interval**

1. $\displaystyle\int_0^\infty J_\nu(ax)\,dx = \frac{1}{a}, \quad a>0,\ \operatorname{Re}\nu>-1.$

2. $\displaystyle\int_0^\infty \cos(xu)J_0(tu)\,du = \begin{cases}\dfrac{1}{\sqrt{t^2-x^2}} & \text{if } x<t,\\ 0 & \text{if } x>t.\end{cases}$

3. $\displaystyle\int_0^\infty \sin(xu)J_0(tu)\,du = \begin{cases}0 & \text{if } x<t,\\ \dfrac{1}{\sqrt{x^2-t^2}} & \text{if } x>t.\end{cases}$

4. $\displaystyle\int_0^\infty \cos(xu)J_1(tu)\,du = \begin{cases}\dfrac{1}{t} & \text{if } x<t,\\ -\dfrac{t}{\sqrt{x^2-t^2}(x+\sqrt{x^2-t^2}\,)} & \text{if } x>t.\end{cases}$

5. $\displaystyle\int_0^\infty \frac{\sin(tu)J_0(au)}{u^2+b^2}\,du = \frac{\sinh(bt)}{b}K_0(ab), \quad b>0,\ 0<t<a,$ $K_0(z)$ is the modified Bessel function (see Section S4.7).

6. $\displaystyle\int_0^\infty \frac{u\sin(tu)J_0(au)}{u^2+b^2}\,du = \frac{\pi}{2}e^{-bt}I_0(ab), \quad b>0,\ a<t<\infty,$ $I_0(z)$ is the modified Bessel function.

7. $\displaystyle\int_0^\infty \frac{\sin(tu)J_1(au)}{u^2+b^2}\,du = \frac{\pi}{2b}e^{-bt}I_1(ab), \quad b>0,\ a<t<\infty,$ $I_1(z)$ is the modified Bessel function.

8. $\displaystyle\int_0^\infty \frac{u\sin(tu)J_1(au)}{u^2+b^2}\,du = \sinh(bt)K_1(ab), \quad b>0,\ 0<t<a,$ $K_1(z)$ is the modified Bessel function.

9. $\displaystyle\int_0^\infty \frac{J_1(au)}{\sqrt{u^2+b^2}}\,du = \frac{1-e^{-ab}}{ab}, \quad a>0,\ \operatorname{Re} b>0.$

► Other integrals

1. $\displaystyle\int_0^1 uJ_0(xu)\,du = \frac{J_1(x)}{x}.$

2. $\displaystyle\int_0^a \frac{J_1(bx)\,dx}{\sqrt{a^2-x^2}} = \frac{1-\cos(ab)}{ab}, \quad a>0.$

3. $\displaystyle\int_0^t \frac{uJ_0(xu)\,du}{\sqrt{t^2-u^2}} = \frac{\sin(xt)}{x}.$

4. $\displaystyle\int_t^\infty \frac{J_1(xu)\,du}{\sqrt{u^2-t^2}} = \frac{\sin(xt)}{x}, \quad x>0,\ t>0.$

⊙ *References for Chapter* S2: H. B. Dwight (1961), I. S. Gradshteyn and I. M. Ryzhik (2000), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1986, 1988), D. Zwillinger (2002), I. N. Bronshtein and K. A. Semendyayev (2004).

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Chapter S3

Tables of Laplace and Inverse Laplace Transforms

S3.1 Tables of Laplace Transforms

S3.1.1 General Formulas

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$af_1(x)+bf_2(x)$	$a\widetilde{f}_1(p)+b\widetilde{f}_2(p)$
2	$f(x/a),\;\; a>0$	$a\widetilde{f}(ap)$
3	$\begin{cases}0 & \text{if } 0<x<a,\\ f(x-a) & \text{if } x>a\end{cases}$	$e^{-ap}\widetilde{f}(p)$
4	$x^n f(x);\;\; n=1,\,2,\,\ldots$	$(-1)^n\dfrac{d^n}{dp^n}\widetilde{f}(p)$
5	$\dfrac{1}{x}f(x)$	$\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$
6	$e^{ax}f(x)$	$\widetilde{f}(p-a)$
7	$\sinh(ax)f(x)$	$\frac{1}{2}\big[\widetilde{f}(p-a)-\widetilde{f}(p+a)\big]$
8	$\cosh(ax)f(x)$	$\frac{1}{2}\big[\widetilde{f}(p-a)+\widetilde{f}(p+a)\big]$
9	$\sin(\omega x)f(x)$	$-\frac{i}{2}\big[\widetilde{f}(p-i\omega)-\widetilde{f}(p+i\omega)\big],\;\; i^2=-1$
10	$\cos(\omega x)f(x)$	$\frac{1}{2}\big[\widetilde{f}(p-i\omega)+\widetilde{f}(p+i\omega)\big],\;\; i^2=-1$
11	$f(x^2)$	$\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^\infty \exp\Big(-\frac{p^2}{4t^2}\Big)\widetilde{f}(t^2)\,dt$
12	$x^{a-1}f\Big(\dfrac{1}{x}\Big),\;\; a>-1$	$\displaystyle\int_0^\infty (t/p)^{a/2}J_a\big(2\sqrt{pt}\,\big)\widetilde{f}(t)\,dt$
13	$f(a\sinh x),\;\; a>0$	$\displaystyle\int_0^\infty J_p(at)\widetilde{f}(t)\,dt$
14	$f(x+a)=f(x)$ (periodic function)	$\dfrac{1}{1-e^{ap}}\displaystyle\int_0^a f(x)e^{-px}\,dx$
15	$f(x+a)=-f(x)$ (antiperiodic function)	$\dfrac{1}{1+e^{-ap}}\displaystyle\int_0^a f(x)e^{-px}\,dx$

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
16	$f'_x(x)$	$p\widetilde{f}(p) - f(+0)$
17	$f_x^{(n)}(x)$	$p^n \widetilde{f}(p) - \sum_{k=1}^{n} p^{n-k} f_x^{(k-1)}(+0)$
18	$x^m f_x^{(n)}(x), \quad m \ge n$	$\left(-\frac{d}{dp}\right)^m \left[p^n \widetilde{f}(p)\right]$
19	$\frac{d^n}{dx^n}\left[x^m f(x)\right], \quad m \ge n$	$(-1)^m p^n \frac{d^m}{dp^m} \widetilde{f}(p)$
20	$\int_0^x f(t)\,dt$	$\frac{\widetilde{f}(p)}{p}$
21	$\int_0^x (x-t) f(t)\,dt$	$\frac{1}{p^2}\widetilde{f}(p)$
22	$\int_0^x (x-t)^\nu f(t)\,dt, \quad \nu > -1$	$\Gamma(\nu+1) p^{-\nu-1} \widetilde{f}(p)$
23	$\int_0^x e^{-a(x-t)} f(t)\,dt$	$\frac{1}{p+a}\widetilde{f}(p)$
24	$\int_0^x \sinh\left[a(x-t)\right] f(t)\,dt$	$\frac{a\widetilde{f}(p)}{p^2 - a^2}$
25	$\int_0^x \sin\left[a(x-t)\right] f(t)\,dt$	$\frac{a\widetilde{f}(p)}{p^2 + a^2}$
26	$\int_0^x f_1(t) f_2(x-t)\,dt$	$\widetilde{f}_1(p)\widetilde{f}_2(p)$
27	$\int_0^x \frac{1}{t} f(t)\,dt$	$\frac{1}{p}\int_p^\infty \widetilde{f}(q)\,dq$
28	$\int_x^\infty \frac{1}{t} f(t)\,dt$	$\frac{1}{p}\int_0^p \widetilde{f}(q)\,dq$
29	$\int_0^\infty \frac{1}{\sqrt{t}} \sin\left(2\sqrt{xt}\right) f(t)\,dt$	$\frac{\sqrt{\pi}}{p\sqrt{p}}\widetilde{f}\left(\frac{1}{p}\right)$
30	$\frac{1}{\sqrt{x}}\int_0^\infty \cos\left(2\sqrt{xt}\right) f(t)\,dt$	$\frac{\sqrt{\pi}}{\sqrt{p}}\widetilde{f}\left(\frac{1}{p}\right)$
31	$\int_0^\infty \frac{1}{\sqrt{\pi x}} \exp\left(-\frac{t^2}{4x}\right) f(t)\,dt$	$\frac{1}{\sqrt{p}}\widetilde{f}\left(\sqrt{p}\right)$
32	$\int_0^\infty \frac{t}{2\sqrt{\pi x^3}} \exp\left(-\frac{t^2}{4x}\right) f(t)\,dt$	$\widetilde{f}\left(\sqrt{p}\right)$
33	$f(x) - a\int_0^x f\left(\sqrt{x^2 - t^2}\right) J_1(at)\,dt$	$\widetilde{f}\left(\sqrt{p^2 + a^2}\right)$
34	$f(x) + a\int_0^x f\left(\sqrt{x^2 - t^2}\right) I_1(at)\,dt$	$\widetilde{f}\left(\sqrt{p^2 - a^2}\right)$

S3.1.2 Expressions with Power-Law Functions

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	1	$\frac{1}{p}$
2	$\begin{cases}0 & \text{if } 0<x<a,\\ 1 & \text{if } a<x<b,\\ 0 & \text{if } b<x\end{cases}$	$\frac{1}{p}\bigl(e^{-ap}-e^{-bp}\bigr)$
3	x	$\frac{1}{p^2}$
4	$\frac{1}{x+a}$	$-e^{ap}\,\mathrm{Ei}(-ap)$
5	$x^n,\quad n=1,\,2,\,\ldots$	$\frac{n!}{p^{n+1}}$
6	$x^{n-1/2},\quad n=1,\,2,\,\ldots$	$\frac{1\cdot 3\ldots(2n-1)\sqrt{\pi}}{2^n p^{n+1/2}}$
7	$\frac{1}{\sqrt{x+a}}$	$\sqrt{\frac{\pi}{p}}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
8	$\frac{\sqrt{x}}{x+a}$	$\sqrt{\frac{\pi}{p}}-\pi\sqrt{a}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
9	$(x+a)^{-3/2}$	$2a^{-1/2}-2(\pi p)^{1/2}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
10	$x^{1/2}(x+a)^{-1}$	$(\pi/p)^{1/2}-\pi a^{1/2}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
11	$x^{-1/2}(x+a)^{-1}$	$\pi a^{-1/2}e^{ap}\,\mathrm{erfc}\bigl(\sqrt{ap}\,\bigr)$
12	$x^\nu,\quad \nu>-1$	$\Gamma(\nu+1)p^{-\nu-1}$
13	$(x+a)^\nu,\quad \nu>-1$	$p^{-\nu-1}e^{-ap}\Gamma(\nu+1,ap)$
14	$x^\nu(x+a)^{-1},\quad \nu>-1$	$ke^{ap}\Gamma(-\nu,ap),\quad k=a^\nu\Gamma(\nu+1)$
15	$(x^2+2ax)^{-1/2}(x+a)$	$ae^{ap}K_1(ap)$

S3.1.3 Expressions with Exponential Functions

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	e^{-ax}	$(p+a)^{-1}$
2	xe^{-ax}	$(p+a)^{-2}$
3	$x^{\nu-1}e^{-ax},\quad \nu>0$	$\Gamma(\nu)(p+a)^{-\nu}$
4	$\frac{1}{x}\bigl(e^{-ax}-e^{-bx}\bigr)$	$\ln(p+b)-\ln(p+a)$
5	$\frac{1}{x^2}\bigl(1-e^{-ax}\bigr)^2$	$(p+2a)\ln(p+2a)+p\ln p-2(p+a)\ln(p+a)$
6	$\exp\bigl(-ax^2\bigr),\quad a>0$	$(\pi b)^{1/2}\exp\bigl(bp^2\bigr)\,\mathrm{erfc}\bigl(p\sqrt{b}\,\bigr),\quad a=\frac{1}{4b}$
7	$x\exp\bigl(-ax^2\bigr)$	$2b-2\pi^{1/2}b^{3/2}p\,\mathrm{erfc}\bigl(p\sqrt{b}\,\bigr),\quad a=\frac{1}{4b}$
8	$\exp(-a/x),\quad a\ge 0$	$2\sqrt{a/p}\,K_1\bigl(2\sqrt{ap}\,\bigr)$
9	$\sqrt{x}\exp(-a/x),\quad a\ge 0$	$\frac{1}{2}\sqrt{\pi/p^3}\bigl(1+2\sqrt{ap}\,\bigr)\exp\bigl(-2\sqrt{ap}\,\bigr)$
10	$\frac{1}{\sqrt{x}}\exp(-a/x),\quad a\ge 0$	$\sqrt{\pi/p}\exp\bigl(-2\sqrt{ap}\,\bigr)$

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
11	$\dfrac{1}{x\sqrt{x}}\exp(-a/x),\quad a>0$	$\sqrt{\pi/a}\exp\bigl(-2\sqrt{ap}\,\bigr)$
12	$x^{\nu-1}\exp(-a/x),\quad a>0$	$2(a/p)^{\nu/2}K_\nu\bigl(2\sqrt{ap}\,\bigr)$
13	$\exp\bigl(-2\sqrt{ax}\,\bigr)$	$p^{-1}-(\pi a)^{1/2}p^{-3/2}e^{a/p}\operatorname{erfc}\bigl(\sqrt{a/p}\,\bigr)$
14	$\dfrac{1}{\sqrt{x}}\exp\bigl(-2\sqrt{ax}\,\bigr)$	$(\pi/p)^{1/2}e^{a/p}\operatorname{erfc}\bigl(\sqrt{a/p}\,\bigr)$

S3.1.4 Expressions with Hyperbolic Functions

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$\sinh(ax)$	$\dfrac{a}{p^2-a^2}$
2	$\sinh^2(ax)$	$\dfrac{2a^2}{p^3-4a^2p}$
3	$\dfrac{1}{x}\sinh(ax)$	$\dfrac{1}{2}\ln\dfrac{p+a}{p-a}$
4	$x^{\nu-1}\sinh(ax),\quad \nu>-1$	$\frac{1}{2}\Gamma(\nu)\bigl[(p-a)^{-\nu}-(p+a)^{-\nu}\bigr]$
5	$\sinh\bigl(2\sqrt{ax}\,\bigr)$	$\dfrac{\sqrt{\pi a}}{p\sqrt{p}}e^{a/p}$
6	$\sqrt{x}\sinh\bigl(2\sqrt{ax}\,\bigr)$	$\pi^{1/2}p^{-5/2}\bigl(\frac{1}{2}p+a\bigr)e^{a/p}\operatorname{erf}\bigl(\sqrt{a/p}\,\bigr)-a^{1/2}p^{-2}$
7	$\dfrac{1}{\sqrt{x}}\sinh\bigl(2\sqrt{ax}\,\bigr)$	$\pi^{1/2}p^{-1/2}e^{a/p}\operatorname{erf}\bigl(\sqrt{a/p}\,\bigr)$
8	$\dfrac{1}{\sqrt{x}}\sinh^2\bigl(\sqrt{ax}\,\bigr)$	$\frac{1}{2}\pi^{1/2}p^{-1/2}\bigl(e^{a/p}-1\bigr)$
9	$\cosh(ax)$	$\dfrac{p}{p^2-a^2}$
10	$\cosh^2(ax)$	$\dfrac{p^2-2a^2}{p^3-4a^2p}$
11	$x^{\nu-1}\cosh(ax),\quad \nu>0$	$\frac{1}{2}\Gamma(\nu)\bigl[(p-a)^{-\nu}+(p+a)^{-\nu}\bigr]$
12	$\cosh\bigl(2\sqrt{ax}\,\bigr)$	$\dfrac{1}{p}+\dfrac{\sqrt{\pi a}}{p\sqrt{p}}e^{a/p}\operatorname{erf}\bigl(\sqrt{a/p}\,\bigr)$
13	$\sqrt{x}\cosh\bigl(2\sqrt{ax}\,\bigr)$	$\pi^{1/2}p^{-5/2}\bigl(\frac{1}{2}p+a\bigr)e^{a/p}$
14	$\dfrac{1}{\sqrt{x}}\cosh\bigl(2\sqrt{ax}\,\bigr)$	$\pi^{1/2}p^{-1/2}e^{a/p}$
15	$\dfrac{1}{\sqrt{x}}\cosh^2\bigl(\sqrt{ax}\,\bigr)$	$\frac{1}{2}\pi^{1/2}p^{-1/2}\bigl(e^{a/p}+1\bigr)$

S3.1.5 Expressions with Logarithmic Functions

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$\ln x$	$-\frac{1}{p}(\ln p + \mathcal{C})$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
2	$\ln(1+ax)$	$-\frac{1}{p}e^{p/a}\,\mathrm{Ei}(-p/a)$
3	$\ln(x+a)$	$\frac{1}{p}\bigl[\ln a - e^{ap}\,\mathrm{Ei}(-ap)\bigr]$
4	$x^n \ln x, \quad n = 1, 2, \ldots$	$\frac{n!}{p^{n+1}}\bigl(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln p - \mathcal{C}\bigr)$, $\mathcal{C} = 0.5772\ldots$ is the Euler constant
5	$\frac{1}{\sqrt{x}} \ln x$	$-\sqrt{\pi/p}\,\bigl[\ln(4p) + \mathcal{C}\bigr]$
6	$x^{n-1/2} \ln x, \quad n = 1, 2, \ldots$	$\frac{k_n}{p^{n+1/2}}\bigl[2 + \frac{2}{3} + \frac{2}{5} + \cdots + \frac{2}{2n-1} - \ln(4p) - \mathcal{C}\bigr]$, $k_n = 1 \cdot 3 \cdot 5 \ldots (2n-1)\frac{\sqrt{\pi}}{2^n}, \quad \mathcal{C} = 0.5772\ldots$
7	$x^{\nu-1} \ln x, \quad \nu > 0$	$\Gamma(\nu)p^{-\nu}\bigl[\psi(\nu) - \ln p\bigr]$, $\psi(\nu)$ is the logarithmic derivative of the gamma function
8	$(\ln x)^2$	$\frac{1}{p}\bigl[(\ln x + \mathcal{C})^2 + \frac{1}{6}\pi^2\bigr], \quad \mathcal{C} = 0.5772\ldots$
9	$e^{-ax} \ln x$	$-\frac{\ln(p+a) + \mathcal{C}}{p+a}, \quad \mathcal{C} = 0.5772\ldots$

S3.1.6 Expressions with Trigonometric Functions

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx$
1	$\sin(ax)$	$\frac{a}{p^2 + a^2}$
2	$\lvert\sin(ax)\rvert, \quad a > 0$	$\frac{a}{p^2 + a^2} \coth\Bigl(\frac{\pi p}{2a}\Bigr)$
3	$\sin^{2n}(ax), \quad n = 1, 2, \ldots$	$\frac{a^{2n}(2n)!}{p\bigl[p^2 + (2a)^2\bigr]\bigl[p^2 + (4a)^2\bigr] \ldots \bigl[p^2 + (2na)^2\bigr]}$
4	$\sin^{2n+1}(ax), \quad n = 1, 2, \ldots$	$\frac{a^{2n+1}(2n+1)!}{\bigl[p^2 + a^2\bigr]\bigl[p^2 + 3^2a^2\bigr] \ldots \bigl[p^2 + (2n+1)^2a^2\bigr]}$
5	$x^n \sin(ax), \quad n = 1, 2, \ldots$	$\frac{n!\,p^{n+1}}{\bigl(p^2 + a^2\bigr)^{n+1}} \sum_{0 \le 2k \le n} (-1)^k C_{n+1}^{2k+1} \Bigl(\frac{a}{p}\Bigr)^{2k+1}$
6	$\frac{1}{x} \sin(ax)$	$\arctan\Bigl(\frac{a}{p}\Bigr)$
7	$\frac{1}{x} \sin^2(ax)$	$\frac{1}{4} \ln\bigl(1 + 4a^2p^{-2}\bigr)$

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px} f(x)\,dx$
8	$\dfrac{1}{x^2}\sin^2(ax)$	$a\arctan(2a/p)-\frac{1}{4}p\ln\bigl(1+4a^2p^{-2}\bigr)$
9	$\sin\bigl(2\sqrt{ax}\,\bigr)$	$\dfrac{\sqrt{\pi a}}{p\sqrt{p}}e^{-a/p}$
10	$\dfrac{1}{x}\sin\bigl(2\sqrt{ax}\,\bigr)$	$\pi\,\mathrm{erf}\bigl(\sqrt{a/p}\,\bigr)$
11	$\cos(ax)$	$\dfrac{p}{p^2+a^2}$
12	$\cos^2(ax)$	$\dfrac{p^2+2a^2}{p(p^2+4a^2)}$
13	$x^n\cos(ax),\quad n=1,\,2,\,\ldots$	$\dfrac{n!\,p^{n+1}}{(p^2+a^2)^{n+1}}\displaystyle\sum_{0\le 2k\le n+1}(-1)^k C_{n+1}^{2k}\Bigl(\frac{a}{p}\Bigr)^{2k}$
14	$\dfrac{1}{x}\bigl[1-\cos(ax)\bigr]$	$\frac{1}{2}\ln\bigl(1+a^2p^{-2}\bigr)$
15	$\dfrac{1}{x}\bigl[\cos(ax)-\cos(bx)\bigr]$	$\dfrac{1}{2}\ln\dfrac{p^2+b^2}{p^2+a^2}$
16	$\sqrt{x}\cos\bigl(2\sqrt{ax}\,\bigr)$	$\frac{1}{2}\pi^{1/2}p^{-5/2}(p-2a)e^{-a/p}$
17	$\dfrac{1}{\sqrt{x}}\cos\bigl(2\sqrt{ax}\,\bigr)$	$\sqrt{\pi/p}\,e^{-a/p}$
18	$\sin(ax)\sin(bx)$	$\dfrac{2abp}{\bigl[p^2+(a+b)^2\bigr]\bigl[p^2+(a-b)^2\bigr]}$
19	$\cos(ax)\sin(bx)$	$\dfrac{b\bigl(p^2-a^2+b^2\bigr)}{\bigl[p^2+(a+b)^2\bigr]\bigl[p^2+(a-b)^2\bigr]}$
20	$\cos(ax)\cos(bx)$	$\dfrac{p\bigl(p^2+a^2+b^2\bigr)}{\bigl[p^2+(a+b)^2\bigr]\bigl[p^2+(a-b)^2\bigr]}$
21	$\dfrac{ax\cos(ax)-\sin(ax)}{x^2}$	$p\arctan\dfrac{a}{x}-a$
22	$e^{bx}\sin(ax)$	$\dfrac{a}{(p-b)^2+a^2}$
23	$e^{bx}\cos(ax)$	$\dfrac{p-b}{(p-b)^2+a^2}$
24	$\sin(ax)\sinh(ax)$	$\dfrac{2a^2p}{p^4+4a^4}$
25	$\sin(ax)\cosh(ax)$	$\dfrac{a\bigl(p^2+2a^2\bigr)}{p^4+4a^4}$
26	$\cos(ax)\sinh(ax)$	$\dfrac{a\bigl(p^2-2a^2\bigr)}{p^4+4a^4}$
27	$\cos(ax)\cosh(ax)$	$\dfrac{p^3}{p^4+4a^4}$

S3.1.7 Expressions with Special Functions

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
1	$\operatorname{erf}(ax)$	$\frac{1}{p}\exp\bigl(b^2p^2\bigr)\operatorname{erfc}(bp), \quad b=\frac{1}{2a}$
2	$\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)$	$\frac{\sqrt{a}}{p\sqrt{p+a}}$
3	$e^{ax}\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)$	$\frac{\sqrt{a}}{\sqrt{p}\,(p-a)}$
4	$\operatorname{erf}\bigl(\frac{1}{2}\sqrt{a/x}\,\bigr)$	$\frac{1}{p}\bigl[1-\exp\bigl(-\sqrt{ap}\,\bigr)\bigr]$
5	$\operatorname{erfc}\bigl(\sqrt{ax}\,\bigr)$	$\frac{\sqrt{p+a}-\sqrt{a}}{p\sqrt{p+a}}$
6	$e^{ax}\operatorname{erfc}\bigl(\sqrt{ax}\,\bigr)$	$\frac{1}{p+\sqrt{ap}}$
7	$\operatorname{erfc}\bigl(\frac{1}{2}\sqrt{a/x}\,\bigr)$	$\frac{1}{p}\exp\bigl(-\sqrt{ap}\,\bigr)$
8	$\operatorname{Ci}(x)$	$\frac{1}{2p}\ln(p^2+1)$
9	$\operatorname{Si}(x)$	$\frac{1}{p}\operatorname{arccot} p$
10	$\operatorname{Ei}(-x)$	$-\frac{1}{p}\ln(p+1)$
11	$J_0(ax)$	$\frac{1}{\sqrt{p^2+a^2}}$
12	$J_\nu(ax), \quad \nu>-1$	$\frac{a^\nu}{\sqrt{p^2+a^2}\,\bigl(p+\sqrt{p^2+a^2}\,\bigr)^\nu}$
13	$x^nJ_n(ax), \quad n=1,\,2,\,\ldots$	$1\cdot3\cdot5\ldots(2n-1)a^n\bigl(p^2+a^2\bigr)^{-n-1/2}$
14	$x^\nu J_\nu(ax), \quad \nu>-\frac{1}{2}$	$2^\nu\pi^{-1/2}\Gamma\bigl(\nu+\frac{1}{2}\bigr)a^\nu\bigl(p^2+a^2\bigr)^{-\nu-1/2}$
15	$x^{\nu+1}J_\nu(ax), \quad \nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma\bigl(\nu+\frac{3}{2}\bigr)a^\nu p\bigl(p^2+a^2\bigr)^{-\nu-3/2}$
16	$J_0\bigl(2\sqrt{ax}\,\bigr)$	$\frac{1}{p}e^{-a/p}$
17	$\sqrt{x}J_1\bigl(2\sqrt{ax}\,\bigr)$	$\frac{\sqrt{a}}{p^2}e^{-a/p}$
18	$x^{\nu/2}J_\nu\bigl(2\sqrt{ax}\,\bigr), \quad \nu>-1$	$a^{\nu/2}p^{-\nu-1}e^{-a/p}$
19	$I_0(ax)$	$\frac{1}{\sqrt{p^2-a^2}}$
20	$I_\nu(ax), \quad \nu>-1$	$\frac{a^\nu}{\sqrt{p^2-a^2}\,\bigl(p+\sqrt{p^2-a^2}\,\bigr)^\nu}$
21	$x^\nu I_\nu(ax), \quad \nu>-\frac{1}{2}$	$2^\nu\pi^{-1/2}\Gamma\bigl(\nu+\frac{1}{2}\bigr)a^\nu\bigl(p^2-a^2\bigr)^{-\nu-1/2}$
22	$x^{\nu+1}I_\nu(ax), \quad \nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma\bigl(\nu+\frac{3}{2}\bigr)a^\nu p\bigl(p^2-a^2\bigr)^{-\nu-3/2}$

No	Original function, $f(x)$	Laplace transform, $\widetilde{f}(p)=\int_0^\infty e^{-px}f(x)\,dx$
23	$I_0\big(2\sqrt{ax}\,\big)$	$\dfrac{1}{p}e^{a/p}$
24	$\dfrac{1}{\sqrt{x}}I_1\big(2\sqrt{ax}\,\big)$	$\dfrac{1}{\sqrt{a}}\big(e^{a/p}-1\big)$
25	$x^{\nu/2}I_\nu\big(2\sqrt{ax}\,\big),\quad \nu>-1$	$a^{\nu/2}p^{-\nu-1}e^{a/p}$
26	$Y_0(ax)$	$-\dfrac{2}{\pi}\dfrac{\operatorname{arcsinh}(p/a)}{\sqrt{p^2+a^2}}$
27	$K_0(ax)$	$\dfrac{\ln\big(p+\sqrt{p^2-a^2}\,\big)-\ln a}{\sqrt{p^2-a^2}}$

⊙ Literature for Section S3.1: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), V. A. Ditkin and A. P. Prudnikov (1965), F. Oberhettinger and L. Badii (1973), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, Vol. 4).

S3.2 Tables of Inverse Laplace Transforms

S3.2.1 General Formulas

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}\widetilde{f}(p)\,dp$
1	$\widetilde{f}(p+a)$	$e^{-ax}f(x)$
2	$\widetilde{f}(ap),\quad a>0$	$\dfrac{1}{a}f\Big(\dfrac{x}{a}\Big)$
3	$\widetilde{f}(ap+b),\quad a>0$	$\dfrac{1}{a}\exp\Big(-\dfrac{b}{a}x\Big)f\Big(\dfrac{x}{a}\Big)$
4	$\widetilde{f}(p-a)+\widetilde{f}(p+a)$	$2f(x)\cosh(ax)$
5	$\widetilde{f}(p-a)-\widetilde{f}(p+a)$	$2f(x)\sinh(ax)$
6	$e^{-ap}\widetilde{f}(p),\quad a\ge 0$	$\begin{cases}0 & \text{if } 0\le x<a,\\ f(x-a) & \text{if } a<x.\end{cases}$
7	$p\widetilde{f}(p)$	$\dfrac{df(x)}{dx}$ if $f(+0)=0$
8	$\dfrac{1}{p}\widetilde{f}(p)$	$\int_0^x f(t)\,dt$
9	$\dfrac{1}{p+a}\widetilde{f}(p)$	$e^{-ax}\int_0^x e^{at}f(t)\,dt$
10	$\dfrac{1}{p^2}\widetilde{f}(p)$	$\int_0^x (x-t)f(t)\,dt$
11	$\dfrac{\widetilde{f}(p)}{p(p+a)}$	$\dfrac{1}{a}\int_0^x \big[1-e^{a(x-t)}\big]f(t)\,dt$
12	$\dfrac{\widetilde{f}(p)}{(p+a)^2}$	$\int_0^x (x-t)e^{-a(x-t)}f(t)\,dt$
13	$\dfrac{\widetilde{f}(p)}{(p+a)(p+b)}$	$\dfrac{1}{b-a}\int_0^x \big[e^{-a(x-t)}-e^{-b(x-t)}\big]f(t)\,dt$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
14	$\dfrac{\widetilde{f}(p)}{(p+a)^2+b^2}$	$\dfrac{1}{b}\displaystyle\int_0^x e^{-a(x-t)}\sin\bigl[b(x-t)\bigr]f(t)\,dt$
15	$\dfrac{1}{p^n}\widetilde{f}(p), \quad n=1,\,2,\,\ldots$	$\dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1}f(t)\,dt$
16	$\widetilde{f}_1(p)\widetilde{f}_2(p)$	$\displaystyle\int_0^x f_1(t)f_2(x-t)\,dt$
17	$\dfrac{1}{\sqrt{p}}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty \frac{\cos\bigl(2\sqrt{xt}\,\bigr)}{\sqrt{\pi x}}f(t)\,dt$
18	$\dfrac{1}{p\sqrt{p}}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty \frac{\sin\bigl(2\sqrt{xt}\,\bigr)}{\sqrt{\pi t}}f(t)\,dt$
19	$\dfrac{1}{p^{2\nu+1}}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty (x/t)^\nu J_{2\nu}\bigl(2\sqrt{xt}\,\bigr)f(t)\,dt$
20	$\dfrac{1}{p}\widetilde{f}\Bigl(\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^\infty J_0\bigl(2\sqrt{xt}\,\bigr)f(t)\,dt$
21	$\dfrac{1}{p}\widetilde{f}\Bigl(p+\dfrac{1}{p}\Bigr)$	$\displaystyle\int_0^x J_0\bigl(2\sqrt{xt-t^2}\,\bigr)f(t)\,dt$
22	$\dfrac{1}{p^{2\nu+1}}\widetilde{f}\Bigl(p+\dfrac{a}{p}\Bigr), \quad -\dfrac{1}{2}<\nu\le 0$	$\displaystyle\int_0^x \Bigl(\frac{x-t}{at}\Bigr)^\nu J_{2\nu}\bigl(2\sqrt{axt-at^2}\,\bigr)f(t)\,dt$
23	$\widetilde{f}\bigl(\sqrt{p}\bigr)$	$\displaystyle\int_0^\infty \frac{t}{2\sqrt{\pi x^3}}\exp\Bigl(-\frac{t^2}{4x}\Bigr)f(t)\,dt$
24	$\dfrac{1}{\sqrt{p}}\widetilde{f}\bigl(\sqrt{p}\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\displaystyle\int_0^\infty \exp\Bigl(-\frac{t^2}{4x}\Bigr)f(t)\,dt$
25	$\widetilde{f}\bigl(p+\sqrt{p}\bigr)$	$\dfrac{1}{2\sqrt{\pi}}\displaystyle\int_0^x \frac{t}{(x-t)^{3/2}}\exp\Bigl[-\frac{t^2}{4(x-t)}\Bigr]f(t)\,dt$
26	$\widetilde{f}\bigl(\sqrt{p^2+a^2}\,\bigr)$	$f(x)-a\displaystyle\int_0^x f\bigl(\sqrt{x^2-t^2}\,\bigr)J_1(at)\,dt$
27	$\widetilde{f}\bigl(\sqrt{p^2-a^2}\,\bigr)$	$f(x)+a\displaystyle\int_0^x f\bigl(\sqrt{x^2-t^2}\,\bigr)I_1(at)\,dt$
28	$\dfrac{\widetilde{f}\bigl(\sqrt{p^2+a^2}\,\bigr)}{\sqrt{p^2+a^2}}$	$\displaystyle\int_0^x J_0\bigl(a\sqrt{x^2-t^2}\,\bigr)f(t)\,dt$
29	$\dfrac{\widetilde{f}\bigl(\sqrt{p^2-a^2}\,\bigr)}{\sqrt{p^2-a^2}}$	$\displaystyle\int_0^x I_0\bigl(a\sqrt{x^2-t^2}\,\bigr)f(t)\,dt$
30	$\widetilde{f}\bigl(\sqrt{(p+a)^2-b^2}\,\bigr)$	$e^{-ax}f(x)+be^{-ax}\displaystyle\int_0^x f\bigl(\sqrt{x^2-t^2}\,\bigr)I_1(bt)\,dt$
31	$\widetilde{f}(\ln p)$	$\displaystyle\int_0^\infty \frac{x^{t-1}}{\Gamma(t)}f(t)\,dt$
32	$\dfrac{1}{p}\widetilde{f}(\ln p)$	$\displaystyle\int_0^\infty \frac{x^t}{\Gamma(t+1)}f(t)\,dt$
33	$\widetilde{f}(p-ia)+\widetilde{f}(p+ia), \ i^2=-1$	$2f(x)\cos(ax)$
34	$i\bigl[\widetilde{f}(p-ia)-\widetilde{f}(p+ia)\bigr], \ i^2=-1$	$2f(x)\sin(ax)$
35	$\dfrac{d\widetilde{f}(p)}{dp}$	$-xf(x)$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
36	$\dfrac{d^n\widetilde{f}(p)}{dp^n}$	$(-x)^n f(x)$
37	$p^n\dfrac{d^m\widetilde{f}(p)}{dp^m}, \quad m \geq n$	$(-1)^m\dfrac{d^n}{dx^n}\big[x^m f(x)\big]$
38	$\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	$\dfrac{1}{x}f(x)$
39	$\dfrac{1}{p}\displaystyle\int_0^p \widetilde{f}(q)\,dq$	$\displaystyle\int_x^\infty \frac{f(t)}{t}\,dt$
40	$\dfrac{1}{p}\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	$\displaystyle\int_0^x \frac{f(t)}{t}\,dt$

S3.2.2 Expressions with Rational Functions

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{p}$	1
2	$\dfrac{1}{p+a}$	e^{-ax}
3	$\dfrac{1}{p^2}$	x
4	$\dfrac{1}{p(p+a)}$	$\dfrac{1}{a}\big(1-e^{-ax}\big)$
5	$\dfrac{1}{(p+a)^2}$	xe^{-ax}
6	$\dfrac{p}{(p+a)^2}$	$(1-ax)e^{-ax}$
7	$\dfrac{1}{p^2-a^2}$	$\dfrac{1}{a}\sinh(ax)$
8	$\dfrac{p}{p^2-a^2}$	$\cosh(ax)$
9	$\dfrac{1}{(p+a)(p+b)}$	$\dfrac{1}{a-b}\big(e^{-bx}-e^{-ax}\big)$
10	$\dfrac{p}{(p+a)(p+b)}$	$\dfrac{1}{a-b}\big(ae^{-ax}-be^{-bx}\big)$
11	$\dfrac{1}{p^2+a^2}$	$\dfrac{1}{a}\sin(ax)$
12	$\dfrac{p}{p^2+a^2}$	$\cos(ax)$
13	$\dfrac{1}{(p+b)^2+a^2}$	$\dfrac{1}{a}e^{-bx}\sin(ax)$
14	$\dfrac{p}{(p+b)^2+a^2}$	$e^{-bx}\Big[\cos(ax)-\dfrac{b}{a}\sin(ax)\Big]$
15	$\dfrac{1}{p^3}$	$\frac{1}{2}x^2$
16	$\dfrac{1}{p^2(p+a)}$	$\dfrac{1}{a^2}\big(e^{-ax}+ax-1\big)$
17	$\dfrac{1}{p(p+a)(p+b)}$	$\dfrac{1}{ab(a-b)}\big(a-b+be^{-ax}-ae^{-bx}\big)$
18	$\dfrac{1}{p(p+a)^2}$	$\dfrac{1}{a^2}\big(1-e^{-ax}-axe^{-ax}\big)$
19	$\dfrac{1}{(p+a)(p+b)(p+c)}$	$\dfrac{(c-b)e^{-ax}+(a-c)e^{-bx}+(b-a)e^{-cx}}{(a-b)(b-c)(c-a)}$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
20	$\dfrac{p}{(p+a)(p+b)(p+c)}$	$\dfrac{a(b-c)e^{-ax}+b(c-a)e^{-bx}+c(a-b)e^{-cx}}{(a-b)(b-c)(c-a)}$
21	$\dfrac{p^2}{(p+a)(p+b)(p+c)}$	$\dfrac{a^2(c-b)e^{-ax}+b^2(a-c)e^{-bx}+c^2(b-a)e^{-cx}}{(a-b)(b-c)(c-a)}$
22	$\dfrac{1}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\big[e^{-ax}-e^{-bx}+(a-b)xe^{-bx}\big]$
23	$\dfrac{p}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\big\{-ae^{-ax}+\big[a+b(b-a)x\big]e^{-bx}\big\}$
24	$\dfrac{p^2}{(p+a)(p+b)^2}$	$\dfrac{1}{(a-b)^2}\big[a^2e^{-ax}+b(b-2a-b^2x+abx)e^{-bx}\big]$
25	$\dfrac{1}{(p+a)^3}$	$\frac{1}{2}x^2e^{-ax}$
26	$\dfrac{p}{(p+a)^3}$	$x\big(1-\frac{1}{2}ax\big)e^{-ax}$
27	$\dfrac{p^2}{(p+a)^3}$	$\big(1-2ax+\frac{1}{2}a^2x^2\big)e^{-ax}$
28	$\dfrac{1}{p(p^2+a^2)}$	$\dfrac{1}{a^2}\big[1-\cos(ax)\big]$
29	$\dfrac{1}{p\big[(p+b)^2+a^2\big]}$	$\dfrac{1}{a^2+b^2}\left\{1-e^{-bx}\left[\cos(ax)+\dfrac{b}{a}\sin(ax)\right]\right\}$
30	$\dfrac{1}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\left[e^{-ax}+\dfrac{a}{b}\sin(bx)-\cos(bx)\right]$
31	$\dfrac{p}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\big[-ae^{-ax}+a\cos(bx)+b\sin(bx)\big]$
32	$\dfrac{p^2}{(p+a)(p^2+b^2)}$	$\dfrac{1}{a^2+b^2}\big[a^2e^{-ax}-ab\sin(bx)+b^2\cos(bx)\big]$
33	$\dfrac{1}{p^3+a^3}$	$\dfrac{e^{-ax}-e^{ax/2}}{3a^2}\big[\cos(kx)-\sqrt{3}\sin(kx)\big]$, $k=\frac{1}{2}a\sqrt{3}$
34	$\dfrac{p}{p^3+a^3}$	$-\dfrac{e^{-ax}-e^{ax/2}}{3a}\big[\cos(kx)+\sqrt{3}\sin(kx)\big]$, $k=\frac{1}{2}a\sqrt{3}$
35	$\dfrac{p^2}{p^3+a^3}$	$\frac{1}{3}e^{-ax}+\frac{2}{3}e^{ax/2}\cos(kx)$, $\quad k=\frac{1}{2}a\sqrt{3}$
36	$\dfrac{1}{(p+a)\big[(p+b)^2+c^2\big]}$	$\dfrac{e^{-ax}-e^{-bx}\cos(cx)+ke^{-bx}\sin(cx)}{(a-b)^2+c^2}$, $k=\dfrac{a-b}{c}$
37	$\dfrac{p}{(p+a)\big[(p+b)^2+c^2\big]}$	$\dfrac{-ae^{-ax}+ae^{-bx}\cos(cx)+ke^{-bx}\sin(cx)}{(a-b)^2+c^2}$, $k=\dfrac{b^2+c^2-ab}{c}$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
38	$\dfrac{p^2}{(p+a)\big[(p+b)^2+c^2\big]}$	$\dfrac{a^2e^{-ax}+(b^2+c^2-2ab)e^{-bx}\cos(cx)+ke^{-bx}\sin(cx)}{(a-b)^2+c^2}$, $k=-ac-bc+\dfrac{ab^2-b^3}{c}$
39	$\dfrac{1}{p^4}$	$\frac{1}{6}x^3$
40	$\dfrac{1}{p^3(p+a)}$	$\dfrac{1}{a^3}-\dfrac{1}{a^2}x+\dfrac{1}{2a}x^2-\dfrac{1}{a^3}e^{-ax}$
41	$\dfrac{1}{p^2(p+a)^2}$	$\dfrac{1}{a^2}x\big(1+e^{-ax}\big)+\dfrac{2}{a^3}\big(e^{-ax}-1\big)$
42	$\dfrac{1}{p^2(p+a)(p+b)}$	$-\dfrac{a+b}{a^2b^2}+\dfrac{1}{ab}x+\dfrac{1}{a^2(b-a)}e^{-ax}+\dfrac{1}{b^2(a-b)}e^{-bx}$
43	$\dfrac{1}{(p+a)^2(p+b)^2}$	$\dfrac{1}{(a-b)^2}\Big[e^{-ax}\Big(x+\dfrac{2}{a-b}\Big)+e^{-bx}\Big(x-\dfrac{2}{a-b}\Big)\Big]$
44	$\dfrac{1}{(p+a)^4}$	$\frac{1}{6}x^3e^{-ax}$
45	$\dfrac{p}{(p+a)^4}$	$\frac{1}{2}x^2e^{-ax}-\frac{1}{6}ax^3e^{-ax}$
46	$\dfrac{1}{p^2(p^2+a^2)}$	$\dfrac{1}{a^3}\big[ax-\sin(ax)\big]$
47	$\dfrac{1}{p^4-a^4}$	$\dfrac{1}{2a^3}\big[\sinh(ax)-\sin(ax)\big]$
48	$\dfrac{p}{p^4-a^4}$	$\dfrac{1}{2a^2}\big[\cosh(ax)-\cos(ax)\big]$
49	$\dfrac{p^2}{p^4-a^4}$	$\dfrac{1}{2a}\big[\sinh(ax)+\sin(ax)\big]$
50	$\dfrac{p^3}{p^4-a^4}$	$\dfrac{1}{2}\big[\cosh(ax)+\cos(ax)\big]$
51	$\dfrac{1}{p^4+a^4}$	$\dfrac{1}{a^3\sqrt{2}}\big(\cosh\xi\sin\xi-\sinh\xi\cos\xi\big)$, $\xi=\dfrac{ax}{\sqrt{2}}$
52	$\dfrac{p}{p^4+a^4}$	$\dfrac{1}{a^2}\sin\Big(\dfrac{ax}{\sqrt{2}}\Big)\sinh\Big(\dfrac{ax}{\sqrt{2}}\Big)$
53	$\dfrac{p^2}{p^4+a^4}$	$\dfrac{1}{a\sqrt{2}}\big(\cos\xi\sinh\xi+\sin\xi\cosh\xi\big)$, $\xi=\dfrac{ax}{\sqrt{2}}$
54	$\dfrac{1}{(p^2+a^2)^2}$	$\dfrac{1}{2a^3}\big[\sin(ax)-ax\cos(ax)\big]$
55	$\dfrac{p}{(p^2+a^2)^2}$	$\dfrac{1}{2a}x\sin(ax)$
56	$\dfrac{p^2}{(p^2+a^2)^2}$	$\dfrac{1}{2a}\big[\sin(ax)+ax\cos(ax)\big]$
57	$\dfrac{p^3}{(p^2+a^2)^2}$	$\cos(ax)-\frac{1}{2}ax\sin(ax)$
58	$\dfrac{1}{[(p+b)^2+a^2]^2}$	$\dfrac{1}{2a^3}e^{-bx}\big[\sin(ax)-ax\cos(ax)\big]$
59	$\dfrac{1}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{1}{a^2-b^2}\Big[\dfrac{1}{a}\sinh(ax)-\dfrac{1}{b}\sinh(bx)\Big]$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
60	$\dfrac{p}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{\cosh(ax)-\cosh(bx)}{a^2-b^2}$
61	$\dfrac{p^2}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{a\sinh(ax)-b\sinh(bx)}{a^2-b^2}$
62	$\dfrac{p^3}{(p^2-a^2)(p^2-b^2)}$	$\dfrac{a^2\cosh(ax)-b^2\cosh(bx)}{a^2-b^2}$
63	$\dfrac{1}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{1}{b^2-a^2}\Big[\dfrac{1}{a}\sin(ax)-\dfrac{1}{b}\sin(bx)\Big]$
64	$\dfrac{p}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{\cos(ax)-\cos(bx)}{b^2-a^2}$
65	$\dfrac{p^2}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{-a\sin(ax)+b\sin(bx)}{b^2-a^2}$
66	$\dfrac{p^3}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{-a^2\cos(ax)+b^2\cos(bx)}{b^2-a^2}$
67	$\dfrac{1}{p^n},\quad n=1,\,2,\,\dots$	$\dfrac{1}{(n-1)!}x^{n-1}$
68	$\dfrac{1}{(p+a)^n},\quad n=1,\,2,\,\dots$	$\dfrac{1}{(n-1)!}x^{n-1}e^{-ax}$
69	$\dfrac{1}{p(p+a)^n},\quad n=1,\,2,\,\dots$	$a^{-n}\big[1-e^{-ax}e_n(ax)\big]$, $e_n(z)=1+\dfrac{z}{1!}+\cdots+\dfrac{z^n}{n!}$
70	$\dfrac{1}{p^{2n}+a^{2n}},\quad n=1,\,2,\,\dots$	$-\dfrac{1}{na^{2n}}\displaystyle\sum_{k=1}^{n}\exp(a_kx)\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\ \ b_k=a\sin\varphi_k,\ \ \varphi_k=\dfrac{\pi(2k-1)}{2n}$
71	$\dfrac{1}{p^{2n}-a^{2n}},\quad n=1,\,2,\,\dots$	$\dfrac{1}{na^{2n-1}}\sinh(ax)+\dfrac{1}{na^{2n}}\displaystyle\sum_{k=2}^{n}\exp(a_kx)$ $\times\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\ \ b_k=a\sin\varphi_k,\ \ \varphi_k=\dfrac{\pi(k-1)}{n}$
72	$\dfrac{1}{p^{2n+1}+a^{2n+1}},\quad n=0,\,1,\,\dots$	$\dfrac{e^{-ax}}{(2n+1)a^{2n}}-\dfrac{2}{(2n+1)a^{2n+1}}\displaystyle\sum_{k=1}^{n}\exp(a_kx)$ $\times\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\ \ b_k=a\sin\varphi_k,\ \ \varphi_k=\dfrac{\pi(2k-1)}{2n+1}$
73	$\dfrac{1}{p^{2n+1}-a^{2n+1}},\quad n=0,\,1,\,\dots$	$\dfrac{e^{ax}}{(2n+1)a^{2n}}+\dfrac{2}{(2n+1)a^{2n+1}}\displaystyle\sum_{k=1}^{n}\exp(a_kx)$ $\times\big[a_k\cos(b_kx)-b_k\sin(b_kx)\big]$, $a_k=a\cos\varphi_k,\ \ b_k=a\sin\varphi_k,\ \ \varphi_k=\dfrac{2\pi k}{2n+1}$
74	$\dfrac{Q(p)}{P(p)}$, $P(p)=(p-a_1)\dots(p-a_n)$; $Q(p)$ is a polynomial of degree $\le n-1$; $a_i\ne a_j$ if $i\ne j$	$\displaystyle\sum_{k=1}^{n}\dfrac{Q(a_k)}{P'(a_k)}\exp\big(a_kx\big)$ (prime stands for differentiation)

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
75	$\dfrac{Q(p)}{P(p)}$, $P(p)=(p-a_1)^{m_1}\dots(p-a_n)^{m_n}$; $Q(p)$ is a polynomial of degree $< m_1+m_2+\dots+m_n-1$; $a_i\neq a_j$ if $i\neq j$	$\displaystyle\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\Phi_{kl}(a_k)}{(m_k-l)!\,(l-1)!}x^{m_k-l}\exp\bigl(a_k x\bigr)$, $\Phi_{kl}(p)=\dfrac{d^{l-1}}{dp^{l-1}}\left[\dfrac{Q(p)}{P_k(p)}\right]$, $P_k(p)=\dfrac{P(p)}{(p-a_k)^{m_k}}$
76	$\dfrac{Q(p)+pR(p)}{P(p)}$, $P(p)=(p^2+a_1^2)\dots(p^2+a_n^2)$; $Q(p)$ and $R(p)$ are polynomials of degree $\le 2n-2$; $a_l\neq a_j$, $l\neq j$	$\displaystyle\sum_{k=1}^{n}\frac{Q(ia_k)\sin(a_k x)+a_k R(ia_k)\cos(a_k x)}{a_k P_k(ia_k)}$, $P_m(p)=\dfrac{P(p)}{p^2+a_m^2}$, $i^2=-1$

S3.2.3 Expressions with Square Roots

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}$
2	$\sqrt{p-a}-\sqrt{p-b}$	$\dfrac{e^{bx}-e^{ax}}{2\sqrt{\pi x^3}}$
3	$\dfrac{1}{\sqrt{p+a}}$	$\dfrac{1}{\sqrt{\pi x}}e^{-ax}$
4	$\sqrt{\dfrac{p+a}{p}}-1$	$\frac{1}{2}ae^{-ax/2}\bigl[I_1\bigl(\frac{1}{2}ax\bigr)+I_0\bigl(\frac{1}{2}ax\bigr)\bigr]$
5	$\dfrac{\sqrt{p+a}}{p+b}$	$\dfrac{e^{-ax}}{\sqrt{\pi x}}+(a-b)^{1/2}e^{-bx}\operatorname{erf}\bigl[(a-b)^{1/2}x^{1/2}\bigr]$
6	$\dfrac{1}{p\sqrt{p}}$	$2\sqrt{\dfrac{x}{\pi}}$
7	$\dfrac{1}{(p+a)\sqrt{p+b}}$	$(b-a)^{-1/2}e^{-ax}\operatorname{erf}\bigl[(b-a)^{1/2}x^{1/2}\bigr]$
8	$\dfrac{1}{\sqrt{p}\,(p-a)}$	$\dfrac{1}{\sqrt{a}}e^{ax}\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)$
9	$\dfrac{1}{p^{3/2}(p-a)}$	$a^{-3/2}e^{ax}\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)-2a^{-1}\pi^{-1/2}x^{1/2}$
10	$\dfrac{1}{\sqrt{p}+a}$	$\pi^{-1/2}x^{-1/2}-ae^{a^2x}\operatorname{erfc}\bigl(a\sqrt{x}\,\bigr)$
11	$\dfrac{a}{p\bigl(\sqrt{p}+a\bigr)}$	$1-e^{a^2x}\operatorname{erfc}\bigl(a\sqrt{x}\,\bigr)$
12	$\dfrac{1}{p+a\sqrt{p}}$	$e^{a^2x}\operatorname{erfc}\bigl(a\sqrt{x}\,\bigr)$
13	$\dfrac{1}{\bigl(\sqrt{p}+\sqrt{a}\bigr)^2}$	$1-\dfrac{2}{\sqrt{\pi}}(ax)^{1/2}+(1-2ax)e^{ax}\bigl[\operatorname{erf}\bigl(\sqrt{ax}\,\bigr)-1\bigr]$
14	$\dfrac{1}{p\bigl(\sqrt{p}+\sqrt{a}\bigr)^2}$	$\dfrac{1}{a}+\Bigl(2x-\dfrac{1}{a}\Bigr)e^{ax}\operatorname{erfc}\bigl(\sqrt{ax}\,\bigr)-\dfrac{2}{\sqrt{\pi a}}\sqrt{x}$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
15	$\dfrac{1}{\sqrt{p}\,\left(\sqrt{p}+a\right)^2}$	$2\pi^{-1/2}x^{1/2}-2axe^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\,\right)$
16	$\dfrac{1}{\left(\sqrt{p}+a\right)^3}$	$\dfrac{2}{\sqrt{\pi}}(a^2x+1)\sqrt{x}-ax(2a^2x+3)e^{a^2x}\operatorname{erfc}\left(a\sqrt{x}\,\right)$
17	$p^{-n-1/2}, \quad n=1,\,2,\,\dots$	$\dfrac{2^n}{1\cdot 3\dots(2n-1)\sqrt{\pi}}\,x^{n-1/2}$
18	$(p+a)^{-n-1/2}$	$\dfrac{2^n}{1\cdot 3\dots(2n-1)\sqrt{\pi}}\,x^{n-1/2}e^{-ax}$
19	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(ax)$
20	$\dfrac{1}{\sqrt{p^2-a^2}}$	$I_0(ax)$
21	$\dfrac{1}{\sqrt{p^2+ap+b}}$	$\exp\left(-\frac{1}{2}ax\right)J_0\left[\left(b-\frac{1}{4}a^2\right)^{1/2}x\right]$
22	$\left(\sqrt{p^2+a^2}-p\right)^{1/2}$	$\dfrac{1}{\sqrt{2\pi x^3}}\sin(ax)$
23	$\dfrac{1}{\sqrt{p^2+a^2}}\left(\sqrt{p^2+a^2}+p\right)^{1/2}$	$\dfrac{\sqrt{2}}{\sqrt{\pi x}}\cos(ax)$
24	$\dfrac{1}{\sqrt{p^2-a^2}}\left(\sqrt{p^2-a^2}+p\right)^{1/2}$	$\dfrac{\sqrt{2}}{\sqrt{\pi x}}\cosh(ax)$
25	$\left(\sqrt{p^2+a^2}+p\right)^{-n}$	$na^{-n}x^{-1}J_n(ax)$
26	$\left(\sqrt{p^2-a^2}+p\right)^{-n}$	$na^{-n}x^{-1}I_n(ax)$
27	$\left(p^2+a^2\right)^{-n-1/2}$	$\dfrac{(x/a)^nJ_n(ax)}{1\cdot 3\cdot 5\dots(2n-1)}$
28	$\left(p^2-a^2\right)^{-n-1/2}$	$\dfrac{(x/a)^nI_n(ax)}{1\cdot 3\cdot 5\dots(2n-1)}$

S3.2.4 Expressions with Arbitrary Powers

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$(p+a)^{-\nu}, \quad \nu>0$	$\dfrac{1}{\Gamma(\nu)}x^{\nu-1}e^{-ax}$
2	$\left[(p+a)^{1/2}+(p+b)^{1/2}\right]^{-2\nu}, \quad \nu>0$	$\dfrac{\nu}{(a-b)^\nu}x^{-1}\exp\left[-\frac{1}{2}(a+b)x\right]I_\nu\left[\frac{1}{2}(a-b)x\right]$
3	$\left[(p+a)(p+b)\right]^{-\nu}, \quad \nu>0$	$\dfrac{\sqrt{\pi}}{\Gamma(\nu)}\left(\dfrac{x}{a-b}\right)^{\nu-1/2}\exp\left(-\dfrac{a+b}{2}x\right)I_{\nu-1/2}\left(\dfrac{a-b}{2}x\right)$
4	$\left(p^2+a^2\right)^{-\nu-1/2}, \quad \nu>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+\frac{1}{2})}x^\nu J_\nu(ax)$
5	$\left(p^2-a^2\right)^{-\nu-1/2}, \quad \nu>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+\frac{1}{2})}x^\nu I_\nu(ax)$
6	$p\left(p^2+a^2\right)^{-\nu-1/2}, \quad \nu>0$	$\dfrac{a\sqrt{\pi}}{(2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}x^\nu J_{\nu-1}(ax)$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
7	$p\bigl(p^2-a^2\bigr)^{-\nu-1/2},\ \ \nu>0$	$\dfrac{a\sqrt{\pi}}{(2a)^\nu\Gamma\bigl(\nu+\frac{1}{2}\bigr)}x^\nu I_{\nu-1}(ax)$
8	$\bigl[(p^2+a^2)^{1/2}+p\bigr]^{-\nu}=$ $a^{-2\nu}\bigl[(p^2+a^2)^{1/2}-p\bigr]^\nu,\ \ \nu>0$	$\nu a^{-\nu}x^{-1}J_\nu(ax)$
9	$\bigl[(p^2-a^2)^{1/2}+p\bigr]^{-\nu}=$ $a^{-2\nu}\bigl[p-(p^2-a^2)^{1/2}\bigr]^\nu,\ \ \nu>0$	$\nu a^{-\nu}x^{-1}I_\nu(ax)$
10	$p\bigl[(p^2+a^2)^{1/2}+p\bigr]^{-\nu},\ \ \nu>1$	$\nu a^{1-\nu}x^{-1}J_{\nu-1}(ax)-\nu(\nu+1)a^{-\nu}x^{-2}J_\nu(ax)$
11	$p\bigl[(p^2-a^2)^{1/2}+p\bigr]^{-\nu},\ \ \nu>1$	$\nu a^{1-\nu}x^{-1}I_{\nu-1}(ax)-\nu(\nu+1)a^{-\nu}x^{-2}I_\nu(ax)$
12	$\dfrac{\bigl(\sqrt{p^2+a^2}+p\bigr)^{-\nu}}{\sqrt{p^2+a^2}},\ \ \nu>-1$	$a^{-\nu}J_\nu(ax)$
13	$\dfrac{\bigl(\sqrt{p^2-a^2}+p\bigr)^{-\nu}}{\sqrt{p^2-a^2}},\ \ \nu>-1$	$a^{-\nu}I_\nu(ax)$

S3.2.5 Expressions with Exponential Functions

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$p^{-1}e^{-ap},\quad a>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ 1 & \text{if } a<x\end{cases}$
2	$p^{-1}\bigl(1-e^{-ap}\bigr),\quad a>0$	$\begin{cases}1 & \text{if } 0<x<a,\\ 0 & \text{if } a<x\end{cases}$
3	$p^{-1}\bigl(e^{-ap}-e^{-bp}\bigr),\quad 0\le a<b$	$\begin{cases}0 & \text{if } 0<x<a,\\ 1 & \text{if } a<x<b,\\ 0 & \text{if } b<x\end{cases}$
4	$p^{-2}\bigl(e^{-ap}-e^{-bp}\bigr),\quad 0\le a<b$	$\begin{cases}0 & \text{if } 0<x<a,\\ x-a & \text{if } a<x<b,\\ b-a & \text{if } b<x\end{cases}$
5	$(p+b)^{-1}e^{-ap},\quad a>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ e^{-b(x-a)} & \text{if } a<x\end{cases}$
6	$p^{-\nu}e^{-ap},\quad \nu>0$	$\begin{cases}0 & \text{if } 0<x<a,\\ \dfrac{(x-a)^{\nu-1}}{\Gamma(\nu)} & \text{if } a<x\end{cases}$
7	$p^{-1}\bigl(e^{ap}-1\bigr)^{-1},\quad a>0$	$f(x)=n\ \text{ if } na<x<(n+1)a;\ \ n=0,1,2,\dots$
8	$e^{a/p}-1$	$\sqrt{\dfrac{a}{x}}\,I_1\bigl(2\sqrt{ax}\,\bigr)$
9	$p^{-1/2}e^{a/p}$	$\dfrac{1}{\sqrt{\pi x}}\cosh\bigl(2\sqrt{ax}\,\bigr)$
10	$p^{-3/2}e^{a/p}$	$\dfrac{1}{\sqrt{\pi a}}\sinh\bigl(2\sqrt{ax}\,\bigr)$
11	$p^{-5/2}e^{a/p}$	$\sqrt{\dfrac{x}{\pi a}}\cosh\bigl(2\sqrt{ax}\,\bigr)-\dfrac{1}{2\sqrt{\pi a^3}}\sinh\bigl(2\sqrt{ax}\,\bigr)$
12	$p^{-\nu-1}e^{a/p},\quad \nu>-1$	$(x/a)^{\nu/2}I_\nu\bigl(2\sqrt{ax}\,\bigr)$
13	$1-e^{-a/p}$	$\sqrt{\dfrac{a}{x}}\,J_1\bigl(2\sqrt{ax}\,\bigr)$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
14	$p^{-1/2}e^{-a/p}$	$\dfrac{1}{\sqrt{\pi x}}\cos\bigl(2\sqrt{ax}\,\bigr)$
15	$p^{-3/2}e^{-a/p}$	$\dfrac{1}{\sqrt{\pi a}}\sin\bigl(2\sqrt{ax}\,\bigr)$
16	$p^{-5/2}e^{-a/p}$	$\dfrac{1}{2\sqrt{\pi a^3}}\sin\bigl(2\sqrt{ax}\,\bigr) - \sqrt{\dfrac{x}{\pi a}}\cos\bigl(2\sqrt{ax}\,\bigr)$
17	$p^{-\nu-1}e^{-a/p}, \quad \nu > -1$	$(x/a)^{\nu/2}J_\nu\bigl(2\sqrt{ax}\,\bigr)$
18	$\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{\sqrt{a}}{2\sqrt{\pi}}\,x^{-3/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
19	$p\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{\sqrt{a}}{8\sqrt{\pi}}\,(a-6x)x^{-7/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
20	$\dfrac{1}{p}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \geq 0$	$\operatorname{erfc}\Bigl(\dfrac{\sqrt{a}}{2\sqrt{x}}\Bigr)$
21	$\sqrt{p}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a > 0$	$\dfrac{1}{4\sqrt{\pi}}\,(a-2x)x^{-5/2}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
22	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \geq 0$	$\dfrac{1}{\sqrt{\pi x}}\exp\Bigl(-\dfrac{a}{4x}\Bigr)$
23	$\dfrac{1}{p\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr), \quad a \geq 0$	$\dfrac{2\sqrt{x}}{\sqrt{\pi}}\exp\Bigl(-\dfrac{a}{4x}\Bigr) - \sqrt{a}\operatorname{erfc}\Bigl(\dfrac{\sqrt{a}}{2\sqrt{x}}\Bigr)$
24	$\dfrac{\exp\bigl(-k\sqrt{p^2+a^2}\,\bigr)}{\sqrt{p^2+a^2}}, \quad k > 0$	$\begin{cases} 0 & \text{if } 0 < x < k, \\ J_0\bigl(a\sqrt{x^2-k^2}\,\bigr) & \text{if } k < x \end{cases}$
25	$\dfrac{\exp\bigl(-k\sqrt{p^2-a^2}\,\bigr)}{\sqrt{p^2-a^2}}, \quad k > 0$	$\begin{cases} 0 & \text{if } 0 < x < k, \\ I_0\bigl(a\sqrt{x^2-k^2}\,\bigr) & \text{if } k < x \end{cases}$

S3.2.6 Expressions with Hyperbolic Functions

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{p\sinh(ap)}, \quad a > 0$	$f(x) = 2n \;$ if $\; a(2n-1) < x < a(2n+1)$; $n = 0, 1, 2, \ldots \; (x > 0)$
2	$\dfrac{1}{p^2\sinh(ap)}, \quad a > 0$	$f(x) = 2n(x-an) \;$ if $\; a(2n-1) < x < a(2n+1)$; $n = 0, 1, 2, \ldots \; (x > 0)$
3	$\dfrac{\sinh(a/p)}{\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi x}}\bigl[\cosh\bigl(2\sqrt{ax}\,\bigr) - \cos\bigl(2\sqrt{ax}\,\bigr)\bigr]$
4	$\dfrac{\sinh(a/p)}{p\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi a}}\bigl[\sinh\bigl(2\sqrt{ax}\,\bigr) - \sin\bigl(2\sqrt{ax}\,\bigr)\bigr]$
5	$p^{-\nu-1}\sinh(a/p), \quad \nu > -2$	$\tfrac{1}{2}(x/a)^{\nu/2}\bigl[I_\nu\bigl(2\sqrt{ax}\,\bigr) - J_\nu\bigl(2\sqrt{ax}\,\bigr)\bigr]$
6	$\dfrac{1}{p\cosh(ap)}, \quad a > 0$	$f(x) = \begin{cases} 0 & \text{if } a(4n-1) < x < a(4n+1), \\ 2 & \text{if } a(4n+1) < x < a(4n+3), \end{cases}$ $n = 0, 1, 2, \ldots \; (x > 0)$
7	$\dfrac{1}{p^2\cosh(ap)}, \quad a > 0$	$x - (-1)^n(x-2an) \;$ if $\; 2n-1 < x/a < 2n+1$; $n = 0, 1, 2, \ldots \; (x > 0)$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
8	$\dfrac{\cosh(a/p)}{\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi x}}\bigl[\cosh\bigl(2\sqrt{ax}\,\bigr)+\cos\bigl(2\sqrt{ax}\,\bigr)\bigr]$
9	$\dfrac{\cosh(a/p)}{p\sqrt{p}}$	$\dfrac{1}{2\sqrt{\pi a}}\bigl[\sinh\bigl(2\sqrt{ax}\,\bigr)+\sin\bigl(2\sqrt{ax}\,\bigr)\bigr]$
10	$p^{-\nu-1}\cosh(a/p), \quad \nu>-1$	$\frac{1}{2}(x/a)^{\nu/2}\bigl[I_\nu\bigl(2\sqrt{ax}\,\bigr)+J_\nu\bigl(2\sqrt{ax}\,\bigr)\bigr]$
11	$\dfrac{1}{p}\tanh(ap), \quad a>0$	$f(x)=(-1)^{n-1}$ if $2a(n-1)<x<2an$; $n=1,\,2,\,\dots$
12	$\dfrac{1}{p}\coth(ap), \quad a>0$	$f(x)=(2n-1)$ if $2a(n-1)<x<2an$; $n=1,\,2,\,\dots$
13	$\operatorname{arccoth}(p/a)$	$\dfrac{1}{x}\sinh(ax)$

S3.2.7 Expressions with Logarithmic Functions

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{1}{p}\ln p$	$-\ln x-\mathcal{C}, \quad \mathcal{C}=0.5772\dots$ is the Euler constant
2	$p^{-n-1}\ln p$	$\bigl(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln x-\mathcal{C}\bigr)\dfrac{x^n}{n!}$, $\mathcal{C}=0.5772\dots$ is the Euler constant
3	$p^{-n-1/2}\ln p$	$k_n\bigl[2+\frac{2}{3}+\frac{2}{5}+\cdots+\frac{2}{2n-1}-\ln(4x)-\mathcal{C}\bigr]x^{n-1/2}$, $k_n=\dfrac{2^n}{1\cdot3\cdot5\dots(2n-1)\sqrt{\pi}}, \quad \mathcal{C}=0.5772\dots$
4	$p^{-\nu}\ln p, \quad \nu>0$	$\dfrac{1}{\Gamma(\nu)}x^{\nu-1}\bigl[\psi(\nu)-\ln x\bigr]$, $\psi(\nu)$ is the logarithmic derivative of the gamma function
5	$\dfrac{1}{p}(\ln p)^2$	$(\ln x+\mathcal{C})^2-\frac{1}{6}\pi^2, \quad \mathcal{C}=0.5772\dots$
6	$\dfrac{1}{p^2}(\ln p)^2$	$x\bigl[(\ln x+\mathcal{C}-1)^2+1-\frac{1}{6}\pi^2\bigr]$
7	$\dfrac{\ln(p+b)}{p+a}$	$e^{-ax}\bigl\{\ln(b-a)-\operatorname{Ei}\bigl[(a-b)x\bigr]\bigr\}$
8	$\dfrac{\ln p}{p^2+a^2}$	$\dfrac{1}{a}\cos(ax)\operatorname{Si}(ax)+\dfrac{1}{a}\sin(ax)\bigl[\ln a-\operatorname{Ci}(ax)\bigr]$
9	$\dfrac{p\ln p}{p^2+a^2}$	$\cos(ax)\bigl[\ln a-\operatorname{Ci}(ax)\bigr]-\sin(ax)\operatorname{Si}(ax)\bigr]$
10	$\ln\dfrac{p+b}{p+a}$	$\dfrac{1}{x}\bigl(e^{-ax}-e^{-bx}\bigr)$
11	$\ln\dfrac{p^2+b^2}{p^2+a^2}$	$\dfrac{2}{x}\bigl[\cos(ax)-\cos(bx)\bigr]$
12	$p\ln\dfrac{p^2+b^2}{p^2+a^2}$	$\dfrac{2}{x}\bigl[\cos(bx)+bx\sin(bx)-\cos(ax)-ax\sin(ax)\bigr]$
13	$\ln\dfrac{(p+a)^2+k^2}{(p+b)^2+k^2}$	$\dfrac{2}{x}\cos(kx)(e^{-bx}-e^{-ax})$
14	$p\ln\Bigl(\dfrac{1}{p}\sqrt{p^2+a^2}\Bigr)$	$\dfrac{1}{x^2}\bigl[\cos(ax)-1\bigr]+\dfrac{a}{x}\sin(ax)$
15	$p\ln\Bigl(\dfrac{1}{p}\sqrt{p^2-a^2}\Bigr)$	$\dfrac{1}{x^2}\bigl[\cosh(ax)-1\bigr]-\dfrac{a}{x}\sinh(ax)$

S3.2.8 Expressions with Trigonometric Functions

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\dfrac{\sin(a/p)}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}\sinh\bigl(\sqrt{2ax}\,\bigr)\sin\bigl(\sqrt{2ax}\,\bigr)$
2	$\dfrac{\sin(a/p)}{p\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi a}}\cosh\bigl(\sqrt{2ax}\,\bigr)\sin\bigl(\sqrt{2ax}\,\bigr)$
3	$\dfrac{\cos(a/p)}{\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi x}}\cosh\bigl(\sqrt{2ax}\,\bigr)\cos\bigl(\sqrt{2ax}\,\bigr)$
4	$\dfrac{\cos(a/p)}{p\sqrt{p}}$	$\dfrac{1}{\sqrt{\pi a}}\sinh\bigl(\sqrt{2ax}\,\bigr)\cos\bigl(\sqrt{2ax}\,\bigr)$
5	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr)\sin\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\sin\Bigl(\dfrac{a}{2x}\Bigr)$
6	$\dfrac{1}{\sqrt{p}}\exp\bigl(-\sqrt{ap}\,\bigr)\cos\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\cos\Bigl(\dfrac{a}{2x}\Bigr)$
7	$\arctan\dfrac{a}{p}$	$\dfrac{1}{x}\sin(ax)$
8	$\dfrac{1}{p}\arctan\dfrac{a}{p}$	$\operatorname{Si}(ax)$
9	$p\arctan\dfrac{a}{p} - a$	$\dfrac{1}{x^2}\bigl[ax\cos(ax) - \sin(ax)\bigr]$
10	$\arctan\dfrac{2ap}{p^2+b^2}$	$\dfrac{2}{x}\sin(ax)\cos\bigl(x\sqrt{a^2+b^2}\,\bigr)$

S3.2.9 Expressions with Special Functions

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
1	$\exp\bigl(ap^2\bigr)\operatorname{erfc}\bigl(p\sqrt{a}\,\bigr)$	$\dfrac{1}{\sqrt{\pi a}}\exp\Bigl(-\dfrac{x^2}{4a}\Bigr)$
2	$\dfrac{1}{p}\exp\bigl(ap^2\bigr)\operatorname{erfc}\bigl(p\sqrt{a}\,\bigr)$	$\operatorname{erf}\Bigl(\dfrac{x}{2\sqrt{a}}\Bigr)$
3	$\operatorname{erfc}\bigl(\sqrt{ap}\,\bigr),\quad a>0$	$\begin{cases} 0 & \text{if } 0<x<a, \\ \dfrac{\sqrt{a}}{\pi x\sqrt{x-a}} & \text{if } a<x \end{cases}$
4	$e^{ap}\operatorname{erfc}\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{\sqrt{a}}{\pi\sqrt{x}\,(x+a)}$
5	$\dfrac{1}{\sqrt{p}}e^{ap}\operatorname{erfc}\bigl(\sqrt{ap}\,\bigr)$	$\dfrac{1}{\sqrt{\pi(x+a)}}$
6	$\operatorname{erf}\bigl(\sqrt{a/p}\,\bigr)$	$\dfrac{1}{\pi x}\sin\bigl(2\sqrt{ax}\,\bigr)$
7	$\dfrac{1}{\sqrt{p}}\exp(a/p)\operatorname{erf}\bigl(\sqrt{a/p}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\sinh\bigl(2\sqrt{ax}\,\bigr)$
8	$\dfrac{1}{\sqrt{p}}\exp(a/p)\operatorname{erfc}\bigl(\sqrt{a/p}\,\bigr)$	$\dfrac{1}{\sqrt{\pi x}}\exp\bigl(-2\sqrt{ax}\,\bigr)$
9	$p^{-a}\gamma(a,bp),\quad a,b>0$	$\begin{cases} x^{a-1} & \text{if } 0<x<b, \\ 0 & \text{if } b<x \end{cases}$

No	Laplace transform, $\widetilde{f}(p)$	Inverse transform, $f(x) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} e^{px}\widetilde{f}(p)\,dp$
10	$\gamma(a, b/p), \quad a > 0$	$b^{a/2}x^{a/2-1}J_a\bigl(2\sqrt{bx}\,\bigr)$
11	$a^{-p}\gamma(p, a)$	$\exp\bigl(-ae^{-x}\bigr)$
12	$K_0(ap), \quad a > 0$	$\begin{cases} 0 & \text{if } 0 < x < a, \\ (x^2 - a^2)^{-1/2} & \text{if } a < x \end{cases}$
13	$K_\nu(ap), \quad a > 0$	$\begin{cases} 0 & \text{if } 0 < x < a, \\ \dfrac{\cosh\bigl[\nu \operatorname{arccosh}(x/a)\bigr]}{\sqrt{x^2 - a^2}} & \text{if } a < x \end{cases}$
14	$K_0\bigl(a\sqrt{p}\bigr)$	$\dfrac{1}{2x}\exp\left(-\dfrac{a^2}{4x}\right)$
15	$\dfrac{1}{\sqrt{p}}K_1\bigl(a\sqrt{p}\,\bigr)$	$\dfrac{1}{a}\exp\left(-\dfrac{a^2}{4x}\right)$

⊙ Literature for Section S3.2: G. Doetsch (1950, 1956, 1958), H. Bateman and A. Erdélyi (1954), I. I. Hirschman and D. V. Widder (1955), V. A. Ditkin and A. P. Prudnikov (1965), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, Vol. 5).

Chapter S4

Special Functions and Their Properties

◆ *Throughout Chapter S4, it is assumed that n is a positive integer unless otherwise specified.*

S4.1 Some Coefficients, Symbols, and Numbers

S4.1.1 Binomial Coefficients

► **Definitions**

$$C_n^k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad \text{where } k = 1, \dots, n;$$

$$C_a^0 = 1, \quad C_a^k = \binom{a}{k} = (-1)^k \frac{(-a)_k}{k!} = \frac{a(a-1)\dots(a-k+1)}{k!}, \quad \text{where } k = 1, 2, \dots$$

Here a is an arbitrary real number.

► **Generalization. Some properties**

General case:

$$C_a^b = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, \quad \text{where } \Gamma(x) \text{ is the gamma function}$$

Properties:

$$C_a^0 = 1, \quad C_n^k = 0 \quad \text{for } k = -1, -2, \dots \text{ or } k > n,$$

$$C_a^{b+1} = \frac{a}{b+1} C_{a-1}^b = \frac{a-b}{b+1} C_a^b, \quad C_a^b + C_a^{b+1} = C_{a+1}^{b+1},$$

$$C^n_{-1/2} = \frac{(-1)^n}{2^{2n}} C^n_{2n} = (-1)^n \frac{(2n-1)!!}{(2n)!!},$$
$$C^n_{1/2} = \frac{(-1)^{n-1}}{n2^{2n-1}} C^{n-1}_{2n-2} = \frac{(-1)^{n-1}}{n} \frac{(2n-3)!!}{(2n-2)!!},$$
$$C^{2n+1}_{n+1/2} = (-1)^n 2^{-4n-1} C^n_{2n}, \quad C^n_{2n+1/2} = 2^{-2n} C^{2n}_{4n+1},$$
$$C^{1/2}_n = \frac{2^{2n+1}}{\pi C^n_{2n}}, \quad C^{n/2}_n = \frac{2^{2n}}{\pi} C^{(n-1)/2}_n.$$

S4.1.2 Pochhammer Symbol

▶ Definition

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}.$$

▶ Some properties ($k = 1, 2, \dots$)

$$(a)_0 = 1, \quad (a)_{n+k} = (a)_n (a+n)_k, \quad (n)_k = \frac{(n+k-1)!}{(n-1)!},$$
$$(a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}, \quad \text{where } a \neq 1, \dots, n;$$
$$(1)_n = n!, \quad (1/2)_n = 2^{-2n} \frac{(2n)!}{n!}, \quad (3/2)_n = 2^{-2n} \frac{(2n+1)!}{n!},$$
$$(a+mk)_{nk} = \frac{(a)_{mk+nk}}{(a)_{mk}}, \quad (a+n)_n = \frac{(a)_{2n}}{(a)_n}, \quad (a+n)_k = \frac{(a)_k (a+k)_n}{(a)_n}.$$

S4.1.3 Bernoulli Numbers

▶ Definition

The *Bernoulli numbers* are defined by the recurrence relation

$$B_0 = 1, \quad \sum_{k=0}^{n-1} C^k_n B_k = 0, \quad n = 2, 3, \dots$$

Numerical values:

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30},$$
$$B_{10} = \tfrac{5}{66}, \quad \dots; \quad B_{2m+1} = 0 \quad \text{for} \quad m = 1, 2, \dots$$

All odd-numbered Bernoulli numbers but B_1 are zero; all even-numbered Bernoulli numbers have alternating signs.

The Bernoulli numbers are the values of Bernoulli polynomials at $x = 0$: $B_n = B_n(0)$.

► **Generating function**

Generating function:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x| < 2\pi.$$

This relation may be regarded as a definition of the Bernoulli numbers.

The following expansions may be used to calculate the Bernoulli numbers:

$$\tan x = \sum_{n=1}^{\infty} |B_{2n}| \frac{2^{2n}(2^{2n} - 1)}{(2n)!} x^{2n}, \qquad |x| < \frac{\pi}{2};$$

$$\cot x = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{2^{2n}}{(2n)!} x^{2n-1}, \qquad |x| < \pi.$$

S4.1.4 Euler Numbers

► **Definition**

The *Euler numbers* E_n are defined by the recurrence relation

$$\sum_{k=0}^{n} C_{2n}^{2k} E_{2k} = 0 \qquad \text{(even numbered)},$$

$$E_{2n+1} = 0 \qquad \text{(odd numbered)},$$

where $n = 0, 1, \ldots$

Numerical values:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \quad E_{10} = -50251, \quad \ldots,$$
$$E_{2n+1} = 0 \quad \text{for} \quad n = 0, 1, \ldots$$

All Euler numbers are integers, the odd-numbered Euler numbers are zero, and the even-numbered Euler numbers have alternating signs.

The Euler numbers are expressed via the values of Euler polynomials at $x = 1/2$: $E_n = 2^n E_n(1/2)$, where $n = 0, 1, \ldots$

► **Generating function. Integral representation**

Generating function:

$$\frac{e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}, \qquad |x| < 2\pi.$$

This relation may be regarded as a definition of the Euler numbers.

Representation via a definite integral:

$$E_{2n} = (-1)^n 2^{2n+1} \int_0^{\infty} \frac{t^{2n}\,dt}{\cosh(\pi t)}.$$

S4.2 Error Functions. Exponential and Logarithmic Integrals

S4.2.1 Error Function and Complementary Error Function

► Integral representations

Definitions:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt \quad \text{(error function, also called the probability integral)},$$
$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)\,dt \quad \text{(complementary error function)}.$$

Properties:

$$\operatorname{erf}(-x) = -\operatorname{erf} x; \quad \operatorname{erf}(0) = 0, \quad \operatorname{erf}(\infty) = 1; \quad \operatorname{erfc}(0) = 1, \quad \operatorname{erfc}(\infty) = 0.$$

► Expansions as $x \to 0$ and $x \to \infty$. Definite integral

Expansion of $\operatorname{erf} x$ into series in powers of x as $x \to 0$:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{k!\,(2k+1)} = \frac{2}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!}.$$

Asymptotic expansion of $\operatorname{erfc} x$ as $x \to \infty$:

$$\operatorname{erfc} x = \frac{1}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \left[\sum_{m=0}^{M-1} (-1)^m \frac{\bigl(\frac{1}{2}\bigr)_m}{x^{2m+1}} + O\bigl(|x|^{-2M-1}\bigr) \right], \qquad M = 1, 2, \ldots$$

Integral:

$$\int_0^x \operatorname{erf} t\,dt = x \operatorname{erf} x - \frac{1}{2} + \frac{1}{2} \exp(-x^2).$$

S4.2.2 Exponential Integral

► Integral representations

Definition:

$$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt = -\int_{-x}^\infty \frac{e^{-t}}{t}\,dt \qquad \text{for} \quad x < 0,$$
$$\operatorname{Ei}(x) = \lim_{\varepsilon \to +0} \left(\int_{-\infty}^{-\varepsilon} \frac{e^t}{t}\,dt + \int_\varepsilon^x \frac{e^t}{t}\,dt \right) \quad \text{for} \quad x > 0.$$

Other integral representations:

$$\operatorname{Ei}(-x) = -e^{-x}\int_0^\infty \frac{x\sin t + t\cos t}{x^2+t^2}\,dt \qquad \text{for} \quad x>0,$$
$$\operatorname{Ei}(-x) = e^{-x}\int_0^\infty \frac{x\sin t - t\cos t}{x^2+t^2}\,dt \qquad \text{for} \quad x<0,$$
$$\operatorname{Ei}(-x) = -x\int_1^\infty e^{-xt}\ln t\,dt \qquad \text{for} \quad x>0,$$
$$\operatorname{Ei}(x) = \mathcal{C} + \ln x + \int_0^x \frac{e^t-1}{t}\,dt \qquad \text{for} \quad x>0,$$

where $\mathcal{C} = 0.5772\ldots$ is the Euler constant.

► Expansions as $x \to 0$ and $x \to \infty$

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Ei}(x) = \begin{cases} \mathcal{C} + \ln(-x) + \displaystyle\sum_{k=1}^\infty \frac{x^k}{k!\,k} & \text{if } x<0, \\ \mathcal{C} + \ln x + \displaystyle\sum_{k=1}^\infty \frac{x^k}{k!\,k} & \text{if } x>0. \end{cases}$$

Asymptotic expansion as $x \to \infty$:

$$\operatorname{Ei}(-x) = e^{-x}\sum_{k=1}^n (-1)^k \frac{(k-1)!}{x^k} + R_n, \qquad R_n < \frac{n!}{x^n}.$$

S4.2.3 Logarithmic Integral

► Integral representations

Definition:

$$\operatorname{li}(x) == \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln t} & \text{if } 0<x<1, \\ \displaystyle\lim_{\varepsilon\to+0}\left(\int_0^{1-\varepsilon}\frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t}\right) & \text{if } x>1. \end{cases}$$

► Limiting properties. Relation to the exponential integral

For small x,

$$\operatorname{li}(x) \approx \frac{x}{\ln(1/x)}.$$

For large x,

$$\operatorname{li}(x) \approx \frac{x}{\ln x}.$$

Asymptotic expansion as $x \to 1$:

$$\operatorname{li}(x) = \mathcal{C} + \ln|\ln x| + \sum_{k=1}^{\infty} \frac{\ln^k x}{k!\,k}.$$

Relation to the exponential integral:

$$\begin{aligned} \operatorname{li} x &= \operatorname{Ei}(\ln x), \qquad && x < 1; \\ \operatorname{li}(e^x) &= \operatorname{Ei}(x), \qquad && x < 0. \end{aligned}$$

S4.3 Sine Integral and Cosine Integral. Fresnel Integrals

S4.3.1 Sine Integral

► Integral representations. Properties

Definition:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \operatorname{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = \operatorname{Si}(x) - \frac{\pi}{2}.$$

Specific values:

$$\operatorname{Si}(0) = 0, \quad \operatorname{Si}(\infty) = \frac{\pi}{2}, \quad \operatorname{si}(\infty) = 0.$$

Properties:

$$\operatorname{Si}(-x) = -\operatorname{Si}(x), \quad \operatorname{si}(x) + \operatorname{si}(-x) = -\pi, \quad \lim_{x\to-\infty} \operatorname{si}(x) = -\pi.$$

► Expansions as $x \to 0$ and $x \to \infty$

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Si}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)\,(2k-1)!}.$$

Asymptotic expansion as $x \to \infty$:

$$\begin{aligned} \operatorname{si}(x) = &-\cos x \left[\sum_{m=0}^{M-1} \frac{(-1)^m (2m)!}{x^{2m+1}} + O\big(|x|^{-2M-1}\big) \right] \\ &+ \sin x \left[\sum_{m=1}^{N-1} \frac{(-1)^m (2m-1)!}{x^{2m}} + O\big(|x|^{-2N}\big) \right], \end{aligned}$$

where $M, N = 1, 2, \ldots$

S4.3.2 Cosine Integral

► Integral representation

Definition:

$$\operatorname{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt = \mathcal{C} + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt,$$

where $\mathcal{C} = 0.5772\ldots$ is the Euler constant.

► Expansions as $x \to 0$ and $x \to \infty$

Expansion into series in powers of x as $x \to 0$:

$$\operatorname{Ci}(x) = \mathcal{C} + \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2k\,(2k)!}.$$

Asymptotic expansion as $x \to \infty$:

$$\operatorname{Ci}(x) = \cos x \left[\sum_{m=1}^{M-1} \frac{(-1)^m (2m-1)!}{x^{2m}} + O\big(|x|^{-2M}\big)\right] + \sin x \left[\sum_{m=0}^{N-1} \frac{(-1)^m (2m)!}{x^{2m+1}} + O\big(|x|^{-2N-1}\big)\right],$$

where $M, N = 1, 2, \ldots$

S4.3.3 Fresnel Integrals

► Integral representation

Definitions:

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\,dt = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \sin t^2\,dt,$$

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\,dt = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \cos t^2\,dt.$$

► Expansions as $x \to 0$ and $x \to \infty$

Expansion into series in powers of x as $x \to 0$:

$$S(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(4k+3)\,(2k+1)!},$$

$$C(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(4k+1)\,(2k)!}.$$

Asymptotic expansion as $x \to \infty$:

$$S(x) = \frac{1}{2} - \frac{\cos x}{\sqrt{2\pi x}} P(x) - \frac{\sin x}{\sqrt{2\pi x}} Q(x),$$
$$C(x) = \frac{1}{2} + \frac{\sin x}{\sqrt{2\pi x}} P(x) - \frac{\cos x}{\sqrt{2\pi x}} Q(x),$$
$$P(x) = 1 - \frac{1 \times 3}{(2x)^2} + \frac{1 \times 3 \times 5 \times 7}{(2x)^4} - \cdots, \quad Q(x) = \frac{1}{2x} - \frac{1 \times 3 \times 5}{(2x)^3} + \cdots.$$

S4.4 Gamma Function, Psi Function, and Beta Function

S4.4.1 Gamma Function

► Integral representations. Simplest properties

The gamma function, $\Gamma(z)$, is an analytic function of the complex argument z everywhere except for the points $z = 0, -1, -2, \ldots$

For $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For $-(n+1) < \operatorname{Re} z < -n$, where $n = 0, 1, 2, \ldots$,

$$\Gamma(z) = \int_0^\infty \left[e^{-t} - \sum_{m=0}^{n} \frac{(-1)^m}{m!} \right] t^{z-1}\, dt.$$

Simplest properties:

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!, \quad \Gamma(1) = \Gamma(2) = 1.$$

Fractional values of the argument:

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = \sqrt{\pi}, \qquad \Gamma\Bigl(n + \frac{1}{2}\Bigr) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,$$
$$\Gamma\Bigl(-\frac{1}{2}\Bigr) = -2\sqrt{\pi}, \quad \Gamma\Bigl(\frac{1}{2} - n\Bigr) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}.$$

► Euler, Stirling, and other formulas

Euler formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\ldots(z+n)} \qquad (z \neq 0, -1, -2, \ldots).$$

Symmetry formulas:

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin(\pi z)}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$
$$\Gamma\Bigl(\frac{1}{2} + z\Bigr)\Gamma\Bigl(\frac{1}{2} - z\Bigr) = \frac{\pi}{\cos(\pi z)}.$$

Multiple argument formulas:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big),$$

$$\Gamma(3z) = \frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma\Big(z+\frac{1}{3}\Big)\Gamma\Big(z+\frac{2}{3}\Big),$$

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma\Big(z+\frac{k}{n}\Big).$$

Asymptotic expansion (*Stirling formula*):

$$\Gamma(z) = \sqrt{2\pi}\, e^{-z} z^{z-1/2} \big[1 + \tfrac{1}{12} z^{-1} + \tfrac{1}{288} z^{-2} + O(z^{-3})\big] \qquad (|\arg z| < \pi).$$

S4.4.2 Psi Function (Digamma Function)

► Definition. Integral representations

Definition:

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'_z(z)}{\Gamma(z)}.$$

The psi function is the logarithmic derivative of the gamma function and is also called the *digamma function*.

Integral representations ($\operatorname{Re} z > 0$):

$$\psi(z) = \int_0^\infty \big[e^{-t} - (1+t)^{-z}\big] t^{-1}\, dt,$$

$$\psi(z) = \ln z + \int_0^\infty \big[t^{-1} - (1-e^{-t})^{-1}\big] e^{-tz}\, dt,$$

$$\psi(z) = -\mathcal{C} + \int_0^1 \frac{1-t^{z-1}}{1-t}\, dt,$$

where $\mathcal{C} = -\psi(1) = 0.5772\ldots$ is the Euler constant.

Values for integer argument:

$$\psi(1) = -\mathcal{C}, \qquad \psi(n) = -\mathcal{C} + \sum_{k=1}^{n-1} k^{-1} \quad (n = 2, 3, \ldots).$$

► Properties. Asymptotic expansion as $z \to \infty$

Functional relations:

$$\psi(z) - \psi(1+z) = -\frac{1}{z},$$

$$\psi(z) - \psi(1-z) = -\pi \cot(\pi z),$$

$$\psi(z) - \psi(-z) = -\pi \cot(\pi z) - \frac{1}{z},$$

$$\psi\big(\tfrac{1}{2} + z\big) - \psi\big(\tfrac{1}{2} - z\big) = \pi \tan(\pi z),$$

$$\psi(mz) = \ln m + \frac{1}{m} \sum_{k=0}^{m-1} \psi\Big(z + \frac{k}{m}\Big).$$

Asymptotic expansion as $z \to \infty$ ($|\arg z| < \pi$):

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots = \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}},$$

where the B_{2n} are Bernoulli numbers.

S4.4.3 Beta Function

► Integral representation. Relationship with the gamma function

Definition:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

where $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$.

Relationship with the gamma function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

► Some properties

$$B(x, y) = B(y, x);$$
$$B(x, y+1) = \frac{y}{x} B(x+1, y) = \frac{y}{x+y} B(x, y);$$
$$B(x, 1-x) = \frac{\pi}{\sin(\pi x)}, \quad 0 < x < 1;$$
$$\frac{1}{B(n, m)} = mC_{n+m-1}^{n-1} = nC_{n+m-1}^{m-1},$$

where n and m are positive integers.

S4.5 Incomplete Gamma and Beta Functions

S4.5.1 Incomplete Gamma Function

► Integral representations. Recurrence formulas

Definitions:

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1}\,dt, \qquad \operatorname{Re} \alpha > 0,$$
$$\Gamma(\alpha, x) = \int_x^{\infty} e^{-t} t^{\alpha-1}\,dt = \Gamma(\alpha) - \gamma(\alpha, x).$$

Recurrence formulas:

$$\gamma(\alpha+1, x) = \alpha\gamma(\alpha, x) - x^{\alpha} e^{-x},$$
$$\gamma(\alpha+1, x) = (x+\alpha)\gamma(\alpha, x) + (1-\alpha)x\gamma(\alpha-1, x),$$
$$\Gamma(\alpha+1, x) = \alpha\Gamma(\alpha, x) + x^{\alpha} e^{-x}.$$

Special cases:

$$\gamma(n+1,x) = n!\left[1 - e^{-x}\left(\sum_{k=0}^{n}\frac{x^k}{k!}\right)\right], \qquad n = 0,\ 1,\ \ldots;$$

$$\Gamma(n+1,x) = n!\,e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}, \qquad n = 0,\ 1,\ \ldots;$$

$$\Gamma(-n,x) = \frac{(-1)^n}{n!}\left[\Gamma(0,x) - e^{-x}\sum_{k=0}^{n-1}(-1)^k\frac{k!}{x^{k+1}}\right], \qquad n = 1,\ 2,\ \ldots$$

► **Expansions as $x \to 0$ and $x \to \infty$. Relation to other functions**

Asymptotic expansions as $x \to 0$:

$$\gamma(\alpha,x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)},$$

$$\Gamma(\alpha,x) = \Gamma(\alpha) - \sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!\,(\alpha+n)}.$$

Asymptotic expansions as $x \to \infty$:

$$\gamma(\alpha,x) = \Gamma(\alpha) - x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(1-\alpha)_m}{(-x)^m} + O\big(|x|^{-M}\big)\right],$$

$$\Gamma(\alpha,x) = x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(1-\alpha)_m}{(-x)^m} + O\big(|x|^{-M}\big)\right] \quad \big(-\tfrac{3}{2}\pi < \arg x < \tfrac{3}{2}\pi\big).$$

Asymptotic formulas as $\alpha \to \infty$:

$$\gamma(x,\alpha) = \Gamma(\alpha)\Big[\Phi\big(2\sqrt{x} - \sqrt{\alpha-1}\big) + O\Big(\frac{1}{\sqrt{\alpha}}\Big)\Big], \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\Big(-\frac{1}{2}t^2\Big)\,dt;$$

$$\gamma(x,\alpha) = \Gamma(\alpha)\Big[\Phi\big(3\sqrt{\alpha}\,z\big) + O\Big(\frac{1}{\alpha}\Big)\Big], \quad z = \Big(\frac{x}{\alpha}\Big)^{1/3} - 1 + \frac{1}{9\alpha}.$$

Representation of the error function, complementary error function, and exponential integral in terms of the gamma functions:

$$\operatorname{erf} x = \frac{1}{\sqrt{\pi}}\gamma\Big(\frac{1}{2},\ x^2\Big), \quad \operatorname{erfc} x = \frac{1}{\sqrt{\pi}}\Gamma\Big(\frac{1}{2},\ x^2\Big), \quad \operatorname{Ei}(-x) = -\Gamma(0,x).$$

S4.5.2 Incomplete Beta Function

► **Integral representation**

Definitions:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \qquad I_x(a,b) = \frac{B_x(a,b)}{B(a,b)},$$

where $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$, and $B(a,b) = B_1(a,b)$ is the beta function.

► **Some properties**

Symmetry:

$$I_x(a,b) + I_{1-x}(b,a) = 1.$$

Recurrence formulas:

$$\begin{aligned}
&I_x(a,b) = xI_x(a-1,b) + (1-x)I_x(a,b-1),\\
&(a+b)I_x(a,b) = aI_x(a+1,b) + bI_x(a,b+1),\\
&(a+b-ax)I_x(a,b) = a(1-x)I_x(a+1,b-1) + bI_x(a,b+1).
\end{aligned}$$

S4.6 Bessel Functions (Cylindrical Functions)

S4.6.1 Definitions and Basic Formulas

► **Bessel functions of the first and the second kind**

The *Bessel function of the first kind*, $J_\nu(x)$, and the *Bessel function of the second kind*, $Y_\nu(x)$ (also called the *Neumann function*), are solutions of the Bessel equation

$$x^2 y''_{xx} + xy'_x + (x^2 - \nu^2)y = 0$$

and are defined by the formulas

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \quad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}. \tag{1}$$

The formula for $Y_\nu(x)$ is valid for $\nu \neq 0, \pm1, \pm2, \ldots$ (the cases $\nu \neq 0, \pm1, \pm2, \ldots$ are discussed in what follows).

The general solution of the Bessel equation has the form $Z_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$ and is called the *cylinder function*.

► **Some formulas**

$$\begin{gathered}
2\nu Z_\nu(x) = x[Z_{\nu-1}(x) + Z_{\nu+1}(x)],\\
\frac{d}{dx} Z_\nu(x) = \frac{1}{2}[Z_{\nu-1}(x) - Z_{\nu+1}(x)] = \pm\Big[\frac{\nu}{x} Z_\nu(x) - Z_{\nu\pm1}(x)\Big],\\
\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \quad \frac{d}{dx}[x^{-\nu} Z_\nu(x)] = -x^{-\nu} Z_{\nu+1}(x),\\
\left(\frac{1}{x}\frac{d}{dx}\right)^n [x^\nu J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x), \quad \left(\frac{1}{x}\frac{d}{dx}\right)^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x),\\
J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, \ldots
\end{gathered}$$

► **Bessel functions for $\nu = \pm n \pm \frac{1}{2}$, where $n = 0, 1, 2, \dots$**

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{1}{x}\sin x - \cos x\Big), \qquad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(-\frac{1}{x}\cos x - \sin x\Big),$$

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\sin\Big(x - \frac{n\pi}{2}\Big)\sum_{k=0}^{[n/2]}\frac{(-1)^k(n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}}\right.$$
$$\left. + \cos\Big(x - \frac{n\pi}{2}\Big)\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$J_{-n-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\Big(x + \frac{n\pi}{2}\Big)\sum_{k=0}^{[n/2]}\frac{(-1)^k(n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}}\right.$$
$$\left. - \sin\Big(x + \frac{n\pi}{2}\Big)\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x, \qquad Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$Y_{n+1/2}(x) = (-1)^{n+1}J_{-n-1/2}(x), \qquad Y_{-n-1/2}(x) = (-1)^n J_{n+1/2}(x),$$

where $[A]$ is the integer part of the number A.

► **Bessel functions for $\nu = \pm n$, where $n = 0, 1, 2, \dots$**

Let $\nu = n$ be an arbitrary integer. The relations

$$J_{-n}(x) = (-1)^n J_n(x), \qquad Y_{-n}(x) = (-1)^n Y_n(x)$$

are valid. The function $J_n(x)$ is given by the first formula in (1) with $\nu = n$, and $Y_n(x)$ can be obtained from the second formula in (1) by proceeding to the limit $\nu \to n$. For nonnegative n, $Y_n(x)$ can be represented in the form

$$Y_n(x) = \frac{2}{\pi}J_n(x)\ln\frac{x}{2} - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(\frac{2}{x}\Big)^{n-2k}$$
$$- \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\Big(\frac{x}{2}\Big)^{n+2k}\frac{\psi(k+1)+\psi(n+k+1)}{k!\,(n+k)!},$$

where $\psi(1) = -\mathcal{C}$, $\psi(n) = -\mathcal{C} + \sum\limits_{k=1}^{n-1}k^{-1}$, $\mathcal{C} = 0.5772\dots$ is the Euler constant, and $\psi(x) = [\ln\Gamma(x)]'_x$ is the logarithmic derivative of the gamma function, also known as the digamma function.

► **Wronskians and similar formulas**

$$W(J_\nu, J_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \quad W(J_\nu, Y_\nu) = \frac{2}{\pi x},$$
$$J_\nu(x)J_{-\nu+1}(x) + J_{-\nu}(x)J_{\nu-1}(x) = \frac{2\sin(\pi\nu)}{\pi x},$$
$$J_\nu(x)Y_{\nu+1}(x) - J_{\nu+1}(x)Y_\nu(x) = -\frac{2}{\pi x}.$$

Here, the notation $W(f,g) = fg'_x - f'_x g$ is used.

S4.6.2 Integral Representations and Asymptotic Expansions

► **Integral representations**

The functions $J_\nu(x)$ and $Y_\nu(x)$ can be represented in the form of definite integrals (for $x > 0$):

$$\pi J_\nu(x) = \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu \int_0^\infty \exp(-x\sinh t - \nu t)\,dt,$$
$$\pi Y_\nu(x) = \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu)\,e^{-x\sinh t}\,dt.$$

For $|\nu| < \frac{1}{2}$, $x > 0$,

$$J_\nu(x) = \frac{2^{1+\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\sin(xt)\,dt}{(t^2-1)^{\nu+1/2}},$$
$$Y_\nu(x) = -\frac{2^{1+\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\cos(xt)\,dt}{(t^2-1)^{\nu+1/2}}.$$

For $\nu > -\frac{1}{2}$,

$$J_\nu(x) = \frac{2(x/2)^\nu}{\pi^{1/2}\Gamma(\frac{1}{2}+\nu)} \int_0^{\pi/2} \cos(x\cos t)\sin^{2\nu} t\,dt \quad (\textit{Poisson's formula}).$$

For $\nu = 0$, $x > 0$,

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh t)\,dt, \qquad Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt.$$

For integer $\nu = n = 0, 1, 2, \dots,$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\,dt \quad (\textit{Bessel's formula}),$$
$$J_{2n}(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin t)\cos(2nt)\,dt,$$
$$J_{2n+1}(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin t)\sin[(2n+1)t]\,dt.$$

► **Asymptotic expansions as $|x| \to \infty$**

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m)(2x)^{-2m} + O(|x|^{-2M})\right]\right.$$
$$\left. - \sin\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\right]\right\},$$
$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\sin\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m)(2x)^{-2m} + O(|x|^{-2M})\right]\right.$$
$$\left. + \cos\left(\frac{4x - 2\nu\pi - \pi}{4}\right)\left[\sum_{m=0}^{M-1}(-1)^m(\nu, 2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\right]\right\},$$

where $(\nu, m) = \dfrac{1}{2^{2m}m!}(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2] = \dfrac{\Gamma(\frac{1}{2}+\nu+m)}{m!\,\Gamma(\frac{1}{2}+\nu-m)}$.

For nonnegative integer n and large x,

$$\sqrt{\pi x}\, J_{2n}(x) = (-1)^n(\cos x + \sin x) + O(x^{-2}),$$
$$\sqrt{\pi x}\, J_{2n+1}(x) = (-1)^{n+1}(\cos x - \sin x) + O(x^{-2}).$$

► **Asymptotic for large ν ($\nu \to \infty$)**

$$J_\nu(x) \simeq \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^\nu, \quad Y_\nu(x) \simeq -\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu},$$

where x is fixed,

$$J_\nu(\nu) \simeq \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)}\frac{1}{\nu^{1/3}}, \quad Y_\nu(\nu) \simeq -\frac{2^{1/3}}{3^{1/6}\Gamma(2/3)}\frac{1}{\nu^{1/3}}.$$

► **Integrals with Bessel functions**

Let $F(a, b, c; x)$ be the hypergeometric series (see Section S4.10.1). Then

$$\int_0^x x^\lambda J_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\,\Gamma(\nu+1)}\,F\left(\frac{\lambda+\nu+1}{2}, \frac{\lambda+\nu+3}{2}, \nu+1; -\frac{x^2}{4}\right),$$

where $\operatorname{Re}(\lambda+\nu) > -1$, and

$$\int_0^x x^\lambda Y_\nu(x)\,dx = -\frac{\cos(\nu\pi)\Gamma(-\nu)}{2^\nu\pi(\lambda+\nu+1)}x^{\lambda+\nu+1}\,F\left(\frac{\lambda+\nu+1}{2}, \nu+1, \frac{\lambda+\nu+3}{2}; -\frac{x^2}{4}\right)$$
$$- \frac{2^\nu\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}\,F\left(\frac{\lambda-\nu+1}{2}, 1-\nu, \frac{\lambda-\nu+3}{2}; -\frac{x^2}{4}\right),$$

where $\operatorname{Re}\lambda > |\operatorname{Re}\nu| - 1$.

S4.6.3 Zeros and Orthogonality Properties of Bessel Functions

► Zeros of Bessel functions

Each of the functions $J_\nu(x)$ and $Y_\nu(x)$ has infinitely many real zeros (for real ν). All zeros are simple, except possibly for the point $x = 0$.

The zeros γ_m of $J_0(x)$, i.e., the roots of the equation $J_0(\gamma_m) = 0$, are approximately given by

$$\gamma_m = 2.4 + 3.13\,(m-1) \qquad (m = 1, 2, \dots),$$

with a maximum error of 0.2%.

► Orthogonality properties of Bessel functions

1°. Let $\mu = \mu_m$ be positive roots of the Bessel function $J_\nu(\mu)$, where $\nu > -1$ and $m = 1,\ 2,\ 3,\ \dots$ Then the set of functions $J_\nu(\mu_m r/a)$ is orthogonal on the interval $0 \le r \le a$ with weight r:

$$\int_0^a J_\nu\Big(\frac{\mu_m r}{a}\Big) J_\nu\Big(\frac{\mu_k r}{a}\Big) r\,dr = \begin{cases} 0 & \text{if } m \ne k, \\ \frac{1}{2}a^2\big[J'_\nu(\mu_m)\big]^2 = \frac{1}{2}a^2 J^2_{\nu+1}(\mu_m) & \text{if } m = k. \end{cases}$$

2°. Let $\mu = \mu_m$ be positive zeros of the Bessel function derivative $J'_\nu(\mu)$, where $\nu > -1$ and $m = 1,\ 2,\ 3,\ \dots$ Then the set of functions $J_\nu(\mu_m r/a)$ is orthogonal on the interval $0 \le r \le a$ with weight r:

$$\int_0^a J_\nu\Big(\frac{\mu_m r}{a}\Big) J_\nu\Big(\frac{\mu_k r}{a}\Big) r\,dr = \begin{cases} 0 & \text{if } m \ne k, \\ \dfrac{1}{2}a^2\Big(1 - \dfrac{\nu^2}{\mu_m^2}\Big) J^2_\nu(\mu_m) & \text{if } m = k. \end{cases}$$

3°. Let $\mu = \mu_m$ be positive roots of the transcendental equation $\mu J'_\nu(\mu) + sJ_\nu(\mu) = 0$, where $\nu > -1$ and $m = 1,\ 2,\ 3,\ \dots$ Then the set of functions $J_\nu(\mu_m r/a)$ is orthogonal on the interval $0 \le r \le a$ with weight r:

$$\int_0^a J_\nu\Big(\frac{\mu_m r}{a}\Big) J_\nu\Big(\frac{\mu_k r}{a}\Big) r\,dr = \begin{cases} 0 & \text{if } m \ne k, \\ \dfrac{1}{2}a^2\Big(1 + \dfrac{s^2 - \nu^2}{\mu_m^2}\Big) J^2_\nu(\mu_m) & \text{if } m = k. \end{cases}$$

4°. Let $\mu = \mu_m$ be positive roots of the transcendental equation

$$J_\nu(\lambda_m b) Y_\nu(\lambda_m a) - J_\nu(\lambda_m a) Y_\nu(\lambda_m b) = 0 \qquad (\nu > -1,\ m = 1,\ 2,\ 3,\ \dots).$$

Then the set of functions

$$Z_\nu(\lambda_m r) = J_\nu(\lambda_m r) Y_\nu(\lambda_m a) - J_\nu(\lambda_m a) Y_\nu(\lambda_m r), \qquad m = 1,\ 2,\ 3,\ \dots\ ;$$

satisfying the conditions $Z_\nu(\lambda_m a) = Z_\nu(\lambda_m b) = 0$ is orthogonal on the interval $a \le r \le b$ with weight r:

$$\int_a^b Z_\nu(\lambda_m r) Z_\nu(\lambda_k r) r\,dr = \begin{cases} 0 & \text{if } m \ne k, \\ \dfrac{2}{\pi^2 \lambda_m^2} \dfrac{J^2_\nu(\lambda_m a) - J^2_\nu(\lambda_m b)}{J^2_\nu(\lambda_m b)} & \text{if } m = k. \end{cases}$$

5°. Let $\mu = \mu_m$ be positive roots of the transcendental equation

$$J'_\nu(\lambda_m b)Y'_\nu(\lambda_m a) - J'_\nu(\lambda_m a)Y'_\nu(\lambda_m b) = 0 \qquad (\nu > -1,\ m = 1,\ 2,\ 3,\ \ldots).$$

Then the set of functions

$$Z_\nu(\lambda_m r) = J_\nu(\lambda_m r)Y'_\nu(\lambda_m a) - J'_\nu(\lambda_m a)Y_\nu(\lambda_m r), \qquad m = 1,\ 2,\ 3,\ \ldots\ ;$$

satisfying the conditions $Z'_\nu(\lambda_m a) = Z'_\nu(\lambda_m b) = 0$ is orthogonal on the interval $a \le r \le b$ with weight r:

$$\int_a^b Z_\nu(\lambda_m r)Z_\nu(\lambda_k r)r\,dr = \begin{cases} 0 & \text{if } m \ne k, \\ \dfrac{2}{\pi^2\lambda_m^2}\left[\left(1-\dfrac{\nu^2}{b^2\lambda_m^2}\right)\dfrac{\left[J'_\nu(\lambda_m a)\right]^2}{\left[J'_\nu(\lambda_m b)\right]^2}-\left(1-\dfrac{\nu^2}{a^2\lambda_m^2}\right)\right] & \text{if } m = k. \end{cases}$$

S4.6.4 Hankel Functions (Bessel Functions of the Third Kind)

► Definition

The *Hankel functions of the first kind and the second kind* are related to Bessel functions by

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z),$$

where $i^2 = -1$.

► Expansions as $z \to 0$ and $z \to \infty$

Asymptotics for $z \to 0$:

$$H_0^{(1)}(z) \simeq \frac{2i}{\pi}\ln z, \qquad H_\nu^{(1)}(z) \simeq -\frac{i}{\pi}\frac{\Gamma(\nu)}{(z/2)^\nu} \qquad (\operatorname{Re}\nu > 0),$$
$$H_0^{(2)}(z) \simeq -\frac{2i}{\pi}\ln z, \quad H_\nu^{(2)}(z) \simeq \frac{i}{\pi}\frac{\Gamma(\nu)}{(z/2)^\nu} \qquad (\operatorname{Re}\nu > 0).$$

Asymptotics for $|z| \to \infty$:

$$H_\nu^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}}\exp\left[i\left(z - \tfrac{1}{2}\pi\nu - \tfrac{1}{4}\pi\right)\right] \qquad (-\pi < \arg z < 2\pi),$$
$$H_\nu^{(2)}(z) \simeq \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z - \tfrac{1}{2}\pi\nu - \tfrac{1}{4}\pi\right)\right] \quad (-2\pi < \arg z < \pi).$$

S4.7 Modified Bessel Functions

S4.7.1 Definitions. Basic Formulas

► Modified Bessel functions of the first and the second kind

The *modified Bessel functions of the first kind*, $I_\nu(x)$, and the *modified Bessel functions of the second kind*, $K_\nu(x)$ (also called the *Macdonald function*), of order ν are solutions of the modified Bessel equation

$$x^2 y''_{xx} + x y'_x - (x^2 + \nu^2) y = 0$$

and are defined by the formulas

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}, \qquad K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)},$$

(see below for $K_\nu(x)$ with $\nu = 0, 1, 2, \dots$).

► Some formulas

The modified Bessel functions possess the properties

$$K_{-\nu}(x) = K_\nu(x), \quad I_{-n}(x) = (-1)^n I_n(x) \quad (n = 0, 1, 2, \dots),$$

$$2\nu I_\nu(x) = x[I_{\nu-1}(x) - I_{\nu+1}(x)], \quad 2\nu K_\nu(x) = -x[K_{\nu-1}(x) - K_{\nu+1}(x)],$$

$$\frac{d}{dx} I_\nu(x) = \frac{1}{2}[I_{\nu-1}(x) + I_{\nu+1}(x)], \quad \frac{d}{dx} K_\nu(x) = -\frac{1}{2}[K_{\nu-1}(x) + K_{\nu+1}(x)].$$

► Modified Bessel functions for $\nu = \pm n \pm \frac{1}{2}$, where $n = 0, 1, 2, \dots$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x, \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x,$$

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(-\frac{1}{x}\sinh x + \cosh x\Big), \quad I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(-\frac{1}{x}\cosh x + \sinh x\Big),$$

$$I_{n+1/2}(x) = \frac{1}{\sqrt{2\pi x}}\left[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} - (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}\right],$$

$$I_{-n-1/2}(x) = \frac{1}{\sqrt{2\pi x}}\left[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} + (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}\right],$$

$$K_{\pm 1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}, \qquad K_{\pm 3/2}(x) = \sqrt{\frac{\pi}{2x}}\Big(1 + \frac{1}{x}\Big) e^{-x},$$

$$K_{n+1/2}(x) = K_{-n-1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}.$$

► Modified Bessel functions for $\nu = n$, where $n = 0, 1, 2, \dots$

If $\nu = n$ is a nonnegative integer, then

$$K_n(x) = (-1)^{n+1} I_n(x) \ln\frac{x}{2} + \frac{1}{2}\sum_{m=0}^{n-1} (-1)^m \Big(\frac{x}{2}\Big)^{2m-n} \frac{(n-m-1)!}{m!}$$
$$+ \frac{1}{2}(-1)^n \sum_{m=0}^{\infty} \Big(\frac{x}{2}\Big)^{n+2m} \frac{\psi(n+m+1) + \psi(m+1)}{m!\,(n+m)!},$$

where $\psi(z)$ is the logarithmic derivative of the gamma function; for $n = 0$, the first sum is dropped.

► **Wronskians and similar formulas**

$$W(I_\nu, I_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \quad W(I_\nu, K_\nu) = -\frac{1}{x},$$
$$I_\nu(x)I_{-\nu+1}(x) - I_{-\nu}(x)I_{\nu-1}(x) = -\frac{2\sin(\pi\nu)}{\pi x},$$
$$I_\nu(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_\nu(x) = \frac{1}{x},$$

where $W(f,g) = fg'_x - f'_x g$.

S4.7.2 Integral Representations and Asymptotic Expansions

► **Integral representations**

The functions $I_\nu(x)$ and $K_\nu(x)$ can be represented in terms of definite integrals:

$$I_\nu(x) = \frac{x^\nu}{\pi^{1/2}2^\nu\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}\exp(-xt)(1-t^2)^{\nu-1/2}\,dt \quad (x>0,\ \nu>-\tfrac{1}{2}),$$
$$K_\nu(x) = \int_0^\infty \exp(-x\cosh t)\cosh(\nu t)\,dt \quad (x>0),$$
$$K_\nu(x) = \frac{1}{\cos\left(\frac{1}{2}\pi\nu\right)}\int_0^\infty \cos(x\sinh t)\cosh(\nu t)\,dt \quad (x>0,\ -1<\nu<1),$$
$$K_\nu(x) = \frac{1}{\sin\left(\frac{1}{2}\pi\nu\right)}\int_0^\infty \sin(x\sinh t)\sinh(\nu t)\,dt \quad (x>0,\ -1<\nu<1).$$

For integer $\nu = n$,

$$I_n(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos t)\cos(nt)\,dt \quad (n=0,1,2,\dots),$$
$$K_0(x) = \int_0^\infty \cos(x\sinh t)\,dt = \int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}}\,dt \quad (x>0).$$

► **Asymptotic expansions as $x \to \infty$**

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left\{1 + \sum_{m=1}^{M}(-1)^m\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m}\right\},$$
$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\left\{1 + \sum_{m=1}^{M}\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m}\right\}.$$

The terms of the order of $O(x^{-M-1})$ are omitted in the braces.

► **Integrals with modified Bessel functions**

Let $F(a,b,c;x)$ be the hypergeometric series (see Section S4.10.1). Then

$$\int_0^x x^\lambda I_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\Gamma(\nu+1)}\,F\left(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ \frac{x^2}{4}\right),$$

where $\operatorname{Re}(\lambda+\nu) > -1$, and

$$\int_0^x x^\lambda K_\nu(x)\,dx = \frac{2^{\nu-1}\Gamma(\nu)}{\lambda-\nu+1} x^{\lambda-\nu+1} F\biggl(\frac{\lambda-\nu+1}{2},\ 1-\nu,\frac{\lambda-\nu+3}{2};\ \frac{x^2}{4}\biggr) + \frac{2^{-\nu-1}\Gamma(-\nu)}{\lambda+\nu+1} x^{\lambda+\nu+1} F\biggl(\frac{\lambda+\nu+1}{2},\ 1+\nu,\frac{\lambda+\nu+3}{2};\ \frac{x^2}{4}\biggr),$$

where $\operatorname{Re}\lambda > |\operatorname{Re}\nu| - 1$.

S4.8 Airy Functions

S4.8.1 Definition and Basic Formulas

► Airy functions of the first and the second kind

The *Airy function of the first kind*, $\operatorname{Ai}(x)$, and the *Airy function of the second kind*, $\operatorname{Bi}(x)$, are solutions of the Airy equation

$$y''_{xx} - xy = 0$$

and are defined by the formulas

$$\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\bigl(\tfrac{1}{3}t^3 + xt\bigr)\,dt,$$
$$\operatorname{Bi}(x) = \frac{1}{\pi}\int_0^\infty \bigl[\exp\bigl(-\tfrac{1}{3}t^3 + xt\bigr) + \sin\bigl(\tfrac{1}{3}t^3 + xt\bigr)\bigr]\,dt.$$

Wronskian: $W\{\operatorname{Ai}(x), \operatorname{Bi}(x)\} = 1/\pi$.

► Relation to the Bessel functions and the modified Bessel functions

$$\operatorname{Ai}(x) = \tfrac{1}{3}\sqrt{x}\,\bigl[I_{-1/3}(z) - I_{1/3}(z)\bigr] = \pi^{-1}\sqrt{\tfrac{1}{3}x}\,K_{1/3}(z), \qquad z = \tfrac{2}{3}x^{3/2},$$
$$\operatorname{Ai}(-x) = \tfrac{1}{3}\sqrt{x}\,\bigl[J_{-1/3}(z) + J_{1/3}(z)\bigr],$$
$$\operatorname{Bi}(x) = \sqrt{\tfrac{1}{3}x}\,\bigl[I_{-1/3}(z) + I_{1/3}(z)\bigr],$$
$$\operatorname{Bi}(-x) = \sqrt{\tfrac{1}{3}x}\,\bigl[J_{-1/3}(z) - J_{1/3}(z)\bigr].$$

S4.8.2 Power Series and Asymptotic Expansions

► Power series expansions as $x \to 0$

$$\operatorname{Ai}(x) = c_1 f(x) - c_2 g(x),$$
$$\operatorname{Bi}(x) = \sqrt{3}\,[c_1 f(x) + c_2 g(x)],$$
$$f(x) = 1 + \frac{1}{3!}x^3 + \frac{1\times 4}{6!}x^6 + \frac{1\times 4\times 7}{9!}x^9 + \cdots = \sum_{k=0}^\infty 3^k \bigl(\tfrac{1}{3}\bigr)_k \frac{x^{3k}}{(3k)!},$$
$$g(x) = x + \frac{2}{4!}x^4 + \frac{2\times 5}{7!}x^7 + \frac{2\times 5\times 8}{10!}x^{10} + \cdots = \sum_{k=0}^\infty 3^k \bigl(\tfrac{2}{3}\bigr)_k \frac{x^{3k+1}}{(3k+1)!},$$

where $c_1 = 3^{-2/3}/\Gamma(2/3) \approx 0.3550$ and $c_2 = 3^{-1/3}/\Gamma(1/3) \approx 0.2588$.

► **Asymptotic expansions as $x \to \infty$**

For large values of x, the leading terms of asymptotic expansions of the Airy functions are

$$\begin{aligned}
\operatorname{Ai}(x) &\simeq \tfrac{1}{2}\pi^{-1/2}x^{-1/4}\exp(-z), \qquad z = \tfrac{2}{3}x^{3/2},\\
\operatorname{Ai}(-x) &\simeq \pi^{-1/2}x^{-1/4}\sin\bigl(z+\tfrac{\pi}{4}\bigr),\\
\operatorname{Bi}(x) &\simeq \pi^{-1/2}x^{-1/4}\exp(z),\\
\operatorname{Bi}(-x) &\simeq \pi^{-1/2}x^{-1/4}\cos\bigl(z+\tfrac{\pi}{4}\bigr).
\end{aligned}$$

S4.9 Degenerate Hypergeometric Functions (Kummer Functions)

S4.9.1 Definitions and Basic Formulas

► **Degenerate hypergeometric functions $\Phi(a,b;x)$ and $\Psi(a,b;x)$**

The *degenerate hypergeometric functions* (*Kummer functions*) $\Phi(a,b;x)$ and $\Psi(a,b;x)$ are solutions of the degenerate hypergeometric equation

$$xy''_{xx} + (b-x)y'_x - ay = 0.$$

In the case $b \neq 0, -1, -2, -3, \ldots$, the function $\Phi(a,b;x)$ can be represented as Kummer's series:

$$\Phi(a,b;x) = 1 + \sum_{k=1}^{\infty}\frac{(a)_k}{(b)_k}\frac{x^k}{k!},$$

where $(a)_k = a(a+1)\ldots(a+k-1)$, $(a)_0 = 1$.

Table S4.1 presents some special cases where Φ can be expressed in terms of simpler functions.

The function $\Psi(a,b;x)$ is defined as follows:

$$\Psi(a,b;x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\Phi(a,b;x) + \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}\Phi(a-b+1,\ 2-b;\ x).$$

Table S4.2 presents some special cases where Ψ can be expressed in terms of simpler functions.

► **Kummer transformation and linear relations**

Kummer transformation:

$$\begin{aligned}
\Phi(a,b;x) &= e^x\Phi(b-a,b;-x),\\
\Psi(a,b;x) &= x^{1-b}\Psi(1+a-b,2-b;x).
\end{aligned}$$

TABLE S4.1
Special cases of the Kummer function $\Phi(a, b; z)$

a	b	z	Φ	Conventional notation
a	a	x	e^x	
1	2	$2x$	$\frac{1}{x}e^x \sinh x$	
a	$a+1$	$-x$	$ax^{-a}\gamma(a, x)$	Incomplete gamma function $\gamma(a, x) = \int_0^x e^{-t}t^{a-1}\,dt$
$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{\sqrt{\pi}}{2}\operatorname{erf} x$	Error function $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt$
$-n$	$\frac{1}{2}$	$\frac{x^2}{2}$	$\frac{n!}{(2n)!}\Bigl(-\frac{1}{2}\Bigr)^{-n} H_{2n}(x)$	Hermite polynomials $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\bigl(e^{-x^2}\bigr)$, $n = 0, 1, 2, \ldots$
$-n$	$\frac{3}{2}$	$\frac{x^2}{2}$	$\frac{n!}{(2n+1)!}\Bigl(-\frac{1}{2}\Bigr)^{-n} H_{2n+1}(x)$	
$-n$	b	x	$\frac{n!}{(b)_n}L_n^{(b-1)}(x)$	Laguerre polynomials $L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}\bigl(e^{-x}x^{n+\alpha}\bigr)$, $\alpha = b-1$, $(b)_n = b(b+1)\ldots(b+n-1)$
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\Gamma(1+\nu)e^x\Bigl(\frac{x}{2}\Bigr)^{-\nu} I_\nu(x)$	Modified Bessel functions $I_\nu(x)$
$n+1$	$2n+2$	$2x$	$\Gamma\Bigl(n+\frac{3}{2}\Bigr)e^x\Bigl(\frac{x}{2}\Bigr)^{-n-\frac{1}{2}} I_{n+\frac{1}{2}}(x)$	

TABLE S4.2
Special cases of the Kummer function $\Psi(a, b; z)$

a	b	z	Ψ	Conventional notation
$1-a$	$1-a$	x	$e^x\Gamma(a, x)$	Incomplete gamma function $\Gamma(a, x) = \int_x^\infty e^{-t}t^{a-1}\,dt$
$\frac{1}{2}$	$\frac{1}{2}$	x^2	$\sqrt{\pi}\,\exp(x^2)\operatorname{erfc} x$	Complementary error function $\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt$
1	1	$-x$	$-e^{-x}\operatorname{Ei}(x)$	Exponential integral $\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt$
1	1	$-\ln x$	$-x^{-1}\operatorname{li} x$	Logarithmic integral $\operatorname{li} x = \int_0^x \frac{dt}{t}$
$\frac{1}{2}-\frac{n}{2}$	$\frac{3}{2}$	x^2	$2^{-n}x^{-1}H_n(x)$	Hermite polynomials $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\bigl(e^{-x^2}\bigr)$, $n = 0, 1, 2, \ldots$
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\pi^{-1/2}(2x)^{-\nu}e^x K_\nu(x)$	Modified Bessel functions $K_\nu(x)$

Linear relations for Φ:

$$(b-a)\Phi(a-1,b;x)+(2a-b+x)\Phi(a,b;x)-a\Phi(a+1,b;x)=0,$$
$$b(b-1)\Phi(a,b-1;x)-b(b-1+x)\Phi(a,b;x)+(b-a)x\Phi(a,b+1;x)=0,$$
$$(a-b+1)\Phi(a,b;x)-a\Phi(a+1,b;x)+(b-1)\Phi(a,b-1;x)=0,$$
$$b\Phi(a,b;x)-b\Phi(a-1,b;x)-x\Phi(a,b+1;x)=0,$$
$$b(a+x)\Phi(a,b;x)-(b-a)x\Phi(a,b+1;x)-ab\Phi(a+1,b;x)=0,$$
$$(a-1+x)\Phi(a,b;x)+(b-a)\Phi(a-1,b;x)-(b-1)\Phi(a,b-1;x)=0.$$

Linear relations for Ψ:

$$\Psi(a-1,b;x)-(2a-b+x)\Psi(a,b;x)+a(a-b+1)\Psi(a+1,b;x)=0,$$
$$(b-a-1)\Psi(a,b-1;x)-(b-1+x)\Psi(a,b;x)+x\Psi(a,b+1;x)=0,$$
$$\Psi(a,b;x)-a\Psi(a+1,b;x)-\Psi(a,b-1;x)=0,$$
$$(b-a)\Psi(a,b;x)-x\Psi(a,b+1;x)+\Psi(a-1,b;x)=0,$$
$$(a+x)\Psi(a,b;x)+a(b-a-1)\Psi(a+1,b;x)-x\Psi(a,b+1;x)=0,$$
$$(a-1+x)\Psi(a,b;x)-\Psi(a-1,b;x)+(a-c+1)\Psi(a,b-1;x)=0.$$

► Differentiation formulas and Wronskian

Differentiation formulas:

$$\frac{d}{dx}\Phi(a,b;x)=\frac{a}{b}\Phi(a+1,b+1;x),\qquad \frac{d^n}{dx^n}\Phi(a,b;x)=\frac{(a)_n}{(b)_n}\Phi(a+n,b+n;x),$$
$$\frac{d}{dx}\Psi(a,b;x)=-a\Psi(a+1,b+1;x),\qquad \frac{d^n}{dx^n}\Psi(a,b;x)=(-1)^n(a)_n\Psi(a+n,b+n;x).$$

Wronskian:

$$W(\Phi,\Psi)=\Phi\Psi'_x-\Phi'_x\Psi=-\frac{\Gamma(b)}{\Gamma(a)}x^{-b}e^x.$$

► Degenerate hypergeometric functions for $n=0,1,2,\ldots$

$$\Psi(a,n+1;x)=\frac{(-1)^{n-1}}{n!\,\Gamma(a-n)}\biggl\{\Phi(a,n+1;x)\ln x+\frac{(n-1)!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r}{(1-n)_r}\frac{x^{r-n}}{r!}$$
$$+\sum_{r=0}^{\infty}\frac{(a)_r}{(n+1)_r}\bigl[\psi(a+r)-\psi(1+r)-\psi(1+n+r)\bigr]\frac{x^r}{r!}\biggr\},$$

where $n=0,\ 1,\ 2,\ \ldots$ (the last sum is dropped for $n=0$), $\psi(z)=[\ln\Gamma(z)]'_z$ is the logarithmic derivative of the gamma function,

$$\psi(1)=-\mathcal{C},\quad \psi(n)=-\mathcal{C}+\sum_{k=1}^{n-1}k^{-1},$$

where $\mathcal{C}=0.5772\ldots$ is the Euler constant.

If $b < 0$, then the formula

$$\Psi(a, b; x) = x^{1-b}\Psi(a - b + 1,\ 2 - b;\ x)$$

is valid for any x.

For $b \neq 0, -1, -2, -3, \ldots$, the general solution of the degenerate hypergeometric equation can be represented in the form

$$y = C_1\Phi(a, b; x) + C_2\Psi(a, b; x),$$

and for $b = 0, -1, -2, -3, \ldots$, in the form

$$y = x^{1-b}\big[C_1\Phi(a - b + 1,\ 2 - b;\ x) + C_2\Psi(a - b + 1,\ 2 - b;\ x)\big].$$

S4.9.2 Integral Representations and Asymptotic Expansions

► Integral representations

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b - a)} \int_0^1 e^{xt} t^{a-1}(1 - t)^{b-a-1}\,dt \qquad (\text{for } b > a > 0),$$

$$\Psi(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1}(1 + t)^{b-a-1}\,dt \qquad (\text{for } a > 0,\ x > 0),$$

where $\Gamma(a)$ is the gamma function.

► Asymptotic expansion as $|x| \to \infty$

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left[\sum_{n=0}^{N} \frac{(b - a)_n(1 - a)_n}{n!} x^{-n} + \varepsilon\right], \quad x > 0,$$

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(b - a)} (-x)^{-a} \left[\sum_{n=0}^{N} \frac{(a)_n(a - b + 1)_n}{n!} (-x)^{-n} + \varepsilon\right], \quad x < 0,$$

$$\Psi(a, b; x) = x^{-a} \left[\sum_{n=0}^{N} (-1)^n \frac{(a)_n(a - b + 1)_n}{n!} x^{-n} + \varepsilon\right], \quad -\infty < x < \infty,$$

where $\varepsilon = O(x^{-N-1})$.

► Integrals with degenerate hypergeometric functions

$$\int \Phi(a, b; x)\,dx = \frac{b - 1}{a - 1}\Psi(a - 1, b - 1; x) + C,$$

$$\int \Psi(a, b; x)\,dx = \frac{1}{1 - a}\Psi(a - 1, b - 1; x) + C,$$

$$\int x^n\Phi(a, b; x)\,dx = n! \sum_{k=1}^{n+1} \frac{(-1)^{k+1}(1 - b)_k x^{n-k+1}}{(1 - a)_k(n - k + 1)!}\Phi(a - k, b - k; x) + C,$$

$$\int x^n\Psi(a, b; x)\,dx = n! \sum_{k=1}^{n+1} \frac{(-1)^{k+1} x^{n-k+1}}{(1 - a)_k(n - k + 1)!}\Psi(a - k, b - k; x) + C.$$

S4.9.3 Whittaker Functions

The *Whittaker functions* $M_{k,\mu}(x)$ and $W_{k,\mu}(x)$ are linearly independent solutions of the Whittaker equation:

$$y''_{xx} + \left[-\tfrac{1}{4} + kx^{-1} + \left(\tfrac{1}{4} - \mu^2\right)x^{-2}\right]y = 0.$$

The Whittaker functions are expressed in terms of degenerate hypergeometric functions as

$$M_{k,\mu}(x) = x^{\mu+1/2}e^{-x/2}\Phi\left(\tfrac{1}{2} + \mu - k,\ 1 + 2\mu;\ x\right),$$
$$W_{k,\mu}(x) = x^{\mu+1/2}e^{-x/2}\Psi\left(\tfrac{1}{2} + \mu - k,\ 1 + 2\mu;\ x\right).$$

S4.10 Hypergeometric Functions

S4.10.1 Various Representations of the Hypergeometric Function

► **Representations of the hypergeometric function via hypergeometric series**

The *hypergeometric function* $F(\alpha, \beta, \gamma; x)$ is a solution of the Gaussian hypergeometric equation

$$x(x-1)y''_{xx} + [(\alpha + \beta + 1)x - \gamma]y'_x + \alpha\beta y = 0.$$

For $\gamma \neq 0, -1, -2, -3, \ldots$, the function $F(\alpha, \beta, \gamma; x)$ can be expressed in terms of the hypergeometric series:

$$F(\alpha, \beta, \gamma; x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}, \quad (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1),$$

which certainly converges for $|x| < 1$.

If γ is not an integer, then the general solution of the hypergeometric equation can be written in the form

$$y = C_1F(\alpha, \beta, \gamma; x) + C_2x^{1-\gamma}F(\alpha - \gamma + 1,\ \beta - \gamma + 1,\ 2 - \gamma;\ x).$$

Table S4.3 shows some special cases where F can be expressed in terms of elementary functions.

► **Integral representation**

For $\gamma > \beta > 0$, the hypergeometric function can be expressed in terms of a definite integral:

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt,$$

where $\Gamma(\beta)$ is the gamma function.

TABLE S4.3
Some special cases where the hypergeometric function $F(\alpha,\beta,\gamma;z)$ can be expressed in terms of elementary functions.

α	β	γ	z	F
$-n$	β	γ	x	$\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}$, where $n=1,2,\dots$
$-n$	β	$-n-m$	x	$\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(-n-m)_k}\frac{x^k}{k!}$, where $n=1,2,\dots$
α	β	β	x	$(1-x)^{-\alpha}$
α	$\frac{1}{2}\alpha+1$	$\frac{1}{2}\alpha$	x	$(1+x)(1-x)^{-\alpha-1}$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	x	$\left(\frac{1+\sqrt{1-x}}{2}\right)^{-2\alpha}$
α	$\alpha+\frac{1}{2}$	2α	x	$\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	x^2	$\frac{(1+x)^{1-2\alpha}-(1-x)^{1-2\alpha}}{2x(1-2\alpha)}$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	$-\tan^2 x$	$\cos^{2\alpha}x\cos(2\alpha x)$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	x^2	$\frac{1}{2}\left[(1+x)^{-2\alpha}+(1-x)^{-2\alpha}\right]$
α	$\alpha-\frac{1}{2}$	2α	x	$2^{2\alpha-1}\left(1+\sqrt{1-x}\right)^{1-2\alpha}$
α	$2-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\frac{\sin[(2\alpha-2)x]}{(\alpha-1)\sin(2x)}$
α	$1-\alpha$	$\frac{1}{2}$	$-x^2$	$\frac{\left(\sqrt{1+x^2}+x\right)^{2\alpha-1}+\left(\sqrt{1+x^2}-x\right)^{2\alpha-1}}{2\sqrt{1+x^2}}$
α	$1-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\frac{\sin[(2\alpha-1)x]}{(\alpha-1)\sin(2x)}$
α	$1-\alpha$	$\frac{1}{2}$	$\sin^2 x$	$\frac{\cos[(2\alpha-1)x]}{\cos x}$
α	$-\alpha$	$\frac{1}{2}$	$-x^2$	$\frac{1}{2}\left[\left(\sqrt{1+x^2}+x\right)^{2\alpha}+\left(\sqrt{1+x^2}-x\right)^{2\alpha}\right]$
α	$-\alpha$	$\frac{1}{2}$	$\sin^2 x$	$\cos(2\alpha x)$
1	1	2	$-x$	$\frac{1}{x}\ln(x+1)$
$\frac{1}{2}$	1	$\frac{3}{2}$	x^2	$\frac{1}{2x}\ln\frac{1+x}{1-x}$
$\frac{1}{2}$	1	$\frac{3}{2}$	$-x^2$	$\frac{1}{x}\arctan x$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	x^2	$\frac{1}{x}\arcsin x$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{1}{x}\operatorname{arcsinh} x$
$n+1$	$n+m+1$	$n+m+l+2$	x	$\frac{(-1)^m(n+m+l+1)!}{n!\,l!\,(n+m)!\,(m+l)!}\frac{d^{n+m}}{dx^{n+m}}\left\{(1-x)^{m+l}\frac{d^lF}{dx^l}\right\}$, $F=-\frac{\ln(1-x)}{x}$, $n,m,l=0,1,2,\dots$

S4.10.2 Basic Properties

► Linear transformation formulas

$$
\begin{aligned}
F(\alpha,\beta,\gamma;x) &= F(\beta,\alpha,\gamma;x),\\
F(\alpha,\beta,\gamma;x) &= (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),\\
F(\alpha,\beta,\gamma;x) &= (1-x)^{-\alpha}F\Big(\alpha,\ \gamma-\beta,\ \gamma;\ \frac{x}{x-1}\Big),\\
F(\alpha,\beta,\gamma;x) &= (1-x)^{-\beta}F\Big(\beta,\ \gamma-\alpha,\ \gamma;\ \frac{x}{x-1}\Big).
\end{aligned}
$$

► Gauss's linear relations for contiguous functions

$$
\begin{aligned}
&(\beta-\alpha)F(\alpha,\beta,\gamma;x)+\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x) = 0,\\
&(\gamma-\alpha-1)F(\alpha,\beta,\gamma;x)+\alpha F(\alpha+1,\beta,\gamma;x)-(\gamma-1)F(\alpha,\beta,\gamma-1;x) = 0,\\
&(\gamma-\beta-1)F(\alpha,\beta,\gamma;x)+\beta F(\alpha,\beta+1,\gamma;x)-(\gamma-1)F(\alpha,\beta,\gamma-1;x) = 0,\\
&(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma;x)+\alpha(1-x)F(\alpha+1,\beta,\gamma;x)-(\gamma-\beta)F(\alpha,\beta-1,\gamma;x) = 0,\\
&(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma;x)-(\gamma-\alpha)F(\alpha-1,\beta,\gamma;x)+\beta(1-x)F(\alpha,\beta+1,\gamma;x) = 0.
\end{aligned}
$$

► Differentiation formulas

$$
\begin{aligned}
&\frac{d}{dx}F(\alpha,\beta,\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\ \beta+1,\ \gamma+1;\ x),\\
&\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\ \beta+n,\ \gamma+n;\ x),\\
&\frac{d^n}{dx^n}\big[x^{\gamma-1}F(\alpha,\beta,\gamma;x)\big] = (\gamma-n)_n x^{\gamma-n-1}F(\alpha,\ \beta,\ \gamma-n;\ x),\\
&\frac{d^n}{dx^n}\big[x^{\alpha+n-1}F(\alpha,\beta,\gamma;x)\big] = (\alpha)_n x^{\alpha-1}F(\alpha+n,\ \beta,\ \gamma;\ x),
\end{aligned}
$$

where $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$.

See Abramowitz and Stegun (1964) and Bateman and Erdélyi (1953, Vol. 1) for more detailed information about hypergeometric functions.

S4.11 Legendre Polynomials, Legendre Functions, and Associated Legendre Functions

S4.11.1 Legendre Polynomials and Legendre Functions

► Implicit and recurrence formulas for Legendre polynomials and functions

The *Legendre polynomials* $P_n(x)$ and the *Legendre functions* $Q_n(x)$ are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx} - 2xy'_x + n(n+1)y = 0.$$

The *Legendre polynomials* $P_n(x)$ and the Legendre functions $Q_n(x)$ are defined by the formulas

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n,$$

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{m=1}^{n}\frac{1}{m}P_{m-1}(x)P_{n-m}(x).$$

The polynomials $P_n = P_n(x)$ can be calculated using the formulas

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1),$$

$$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4-30x^2+3),$$

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x).$$

The first five functions $Q_n = Q_n(x)$ have the form

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1,$$

$$Q_2(x) = \frac{1}{4}(3x^2-1)\ln\frac{1+x}{1-x} - \frac{3}{2}x, \quad Q_3(x) = \frac{1}{4}(5x^3-3x)\ln\frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3},$$

$$Q_4(x) = \frac{1}{16}(35x^4-30x^2+3)\ln\frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x.$$

The polynomials $P_n(x)$ have the explicit representation

$$P_n(x) = 2^{-n}\sum_{m=0}^{[n/2]}(-1)^m C_n^m C_{2n-2m}^n x^{n-2m},$$

where $[A]$ stands for the integer part of a number A.

► Integral representation. Useful formulas

Integral representation of the Legendre polynomials (*Laplace integral*):

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \left(x \pm \sqrt{x^2-1}\,\cos t\right)^n dt, \qquad x > 1.$$

Integral representation of the Legendre polynomials (*Dirichlet–Mehler integral*):

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos\left[(n+\frac{1}{2})\psi\right]d\psi}{\sqrt{\cos\psi - \cos\theta}}, \qquad 0 < \theta < \pi, \quad n = 0,\,1,\,\ldots$$

Integral representation of the Legendre functions:

$$Q_n(x) = 2^n\int_x^\infty \frac{(t-x)^n}{(t^2-1)^{n+1}}\,dt, \qquad x > 1.$$

Properties:

$$P_n(-x) = (-1)^n P_n(x),$$
$$Q_n(-x) = (-1)^{n+1} Q_n(x).$$

Recurrence relations:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$
$$(x^2-1)\frac{d}{dx}P_n(x) = n\big[xP_n(x) - P_{n-1}(x)\big] = \frac{n(n+1)}{2n+1}\big[P_{n+1}(x) - P_{n-1}(x)\big].$$

Values of the Legendre polynomials and their derivatives at $x = 0$:

$$P_{2m}(0) = (-1)^m \frac{(2m-1)!!}{2^m m!}, \quad P_{2m+1}(0) = 0,$$
$$P'_{2m}(0) = 0, \quad P'_{2m+1}(0) = (-1)^m \frac{(2m+1)!!}{2^m m!}.$$

Asymptotic formula as $n \to \infty$:

$$P_n(\cos\theta) \approx \left(\frac{2}{\pi n \sin\theta}\right)^{1/2} \sin\left[\left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right], \qquad 0 < \theta < \pi.$$

► **Zeros and orthogonality of the Legendre polynomials**

The polynomials $P_n(x)$ (with natural n) have exactly n real distinct zeros; all zeros lie on the interval $-1 < x < 1$. The zeros of $P_n(x)$ and $P_{n+1}(x)$ alternate with each other. The function $Q_n(x)$ has exactly $n+1$ zeros, which lie on the interval $-1 < x < 1$.

The functions $P_n(x)$ form an orthogonal system on the interval $-1 \le x \le 1$, with

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{2}{2n+1} & \text{if } n = m. \end{cases}$$

► **Generating functions**

The generating function for Legendre polynomials is

$$\frac{1}{\sqrt{1-2sx+s^2}} = \sum_{n=0}^{\infty} P_n(x)s^n \qquad (|s| < 1).$$

The generating function for Legendre functions is

$$\frac{1}{\sqrt{1-2sx+s^2}} \ln\left[\frac{x - s + \sqrt{1-2sx+s^2}}{\sqrt{1-x^2}}\right] = \sum_{n=0}^{\infty} Q_n(x)s^n \qquad (|s| < 1,\ x > 1).$$

S4.11.2 Associated Legendre Functions with Integer Indices and Real Argument

► Formulas for associated Legendre functions. Differential equation

The *associated Legendre functions* $P_n^m(x)$ of order m are defined by the formulas

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x), \qquad n = 1,\, 2,\, 3,\, \ldots, \qquad m = 0,\, 1,\, 2,\, \ldots$$

It is assumed by definition that $P_n^0(x) = P_n(x)$.

Properties:

$$P_n^m(x) = 0 \quad \text{if} \quad m > n, \qquad P_n^m(-x) = (-1)^{n-m}P_n^m(x).$$

The associated Legendre functions $P_n^m(x)$ have exactly $n-m$ real zeros, which and lie on the interval $-1 < x < 1$.

The associated Legendre functions $P_n^m(x)$ with low indices:

$$P_1^1(x) = (1-x^2)^{1/2}, \quad P_2^1(x) = 3x(1-x^2)^{1/2}, \quad P_2^2(x) = 3(1-x^2),$$
$$P_3^1(x) = \tfrac{3}{2}(5x^2-1)(1-x^2)^{1/2}, \quad P_3^2(x) = 15x(1-x^2), \quad P_3^3(x) = 15(1-x^2)^{3/2}.$$

The associated Legendre functions $P_n^m(x)$ with $n > m$ are solutions of the linear ordinary differential equation

$$(1-x^2)y''_{xx} - 2xy'_x + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0.$$

► Orthogonality of the associated Legendre functions

The functions $P_n^m(x)$ form an orthogonal system on the interval $-1 \le x \le 1$, with

$$\int_{-1}^{1} P_n^m(x)P_k^m(x)\,dx = \begin{cases} 0 & \text{if } n \neq k, \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{if } n = k. \end{cases}$$

The functions $P_n^m(x)$ (with $m \neq 0$) are orthogonal on the interval $-1 \le x \le 1$ with weight $(1-x^2)^{-1}$, that is,

$$\int_{-1}^{1} \frac{P_n^m(x)P_n^k(x)}{1-x^2}\,dx = \begin{cases} 0 & \text{if } m \neq k, \\ \dfrac{(n+m)!}{m(n-m)!} & \text{if } m = k. \end{cases}$$

S4.11.3 Associated Legendre Functions. General Case

► Definitions. Basic formulas

In the general case, the associated Legendre functions of the first and the second kind, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, are linearly independent solutions of the Legendre equation

$$(1-z^2)y''_{zz} - 2zy'_z + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]y = 0,$$

where the parameters ν and μ and the variable z can assume arbitrary real or complex values.

For $|1-z|<2$, the formulas

$$P_\nu^\mu(z)=\frac{1}{\Gamma(1-\mu)}\Big(\frac{z+1}{z-1}\Big)^{\mu/2}F\Big(-\nu,\ 1+\nu,\ 1-\mu;\ \frac{1-z}{2}\Big),$$
$$Q_\nu^\mu(z)=A\Big(\frac{z-1}{z+1}\Big)^{\frac{\mu}{2}}F\Big(-\nu,\ 1+\nu,\ 1+\mu;\ \frac{1-z}{2}\Big)$$
$$+B\Big(\frac{z+1}{z-1}\Big)^{\frac{\mu}{2}}F\Big(-\nu,\ 1+\nu,\ 1-\mu;\ \frac{1-z}{2}\Big),$$
$$A=e^{i\mu\pi}\frac{\Gamma(-\mu)\,\Gamma(1+\nu+\mu)}{2\,\Gamma(1+\nu-\mu)},\qquad B=e^{i\mu\pi}\frac{\Gamma(\mu)}{2},\qquad i^2=-1,$$

are valid, where $F(a,b,c;z)$ is the hypergeometric series (see Section S4.10).

For $|z|>1$,

$$P_\nu^\mu(z)=\frac{2^{-\nu-1}\Gamma(-\frac{1}{2}-\nu)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)}z^{-\nu+\mu-1}(z^2-1)^{-\mu/2}F\Big(\frac{1+\nu-\mu}{2},\ \frac{2+\nu-\mu}{2},\ \frac{2\nu+3}{2};\ \frac{1}{z^2}\Big)$$
$$+\frac{2^{\nu}\Gamma(\frac{1}{2}+\nu)}{\Gamma(1+\nu-\mu)}z^{\nu+\mu}(z^2-1)^{-\mu/2}F\Big(-\frac{\nu+\mu}{2},\ \frac{1-\nu-\mu}{2},\ \frac{1-2\nu}{2};\ \frac{1}{z^2}\Big),$$
$$Q_\nu^\mu(z)=e^{i\pi\mu}\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}z^{-\nu-\mu-1}(z^2-1)^{\mu/2}F\Big(\frac{2+\nu+\mu}{2},\ \frac{1+\nu+\mu}{2},\ \frac{2\nu+3}{2};\ \frac{1}{z^2}\Big).$$

The functions $P_\nu(z)\equiv P_\nu^0(z)$ and $Q_\nu(z)\equiv Q_\nu^0(z)$ are called the *Legendre functions*.

For $n=1,2,\dots,$

$$P_\nu^n(z)=(z^2-1)^{n/2}\frac{d^n}{dz^n}P_\nu(z),\qquad Q_\nu^n(z)=(z^2-1)^{n/2}\frac{d^n}{dz^n}Q_\nu(z).$$

► Relations between associated Legendre functions

$$P_\nu^\mu(z)=P_{-\nu-1}^\mu(z),\qquad P_\nu^n(z)=\frac{\Gamma(\nu+n+1)}{\Gamma(\nu-n+1)}P_\nu^{-n}(z),\qquad n=0,1,2,\dots,$$
$$P_{\nu+1}^\mu(z)=\frac{2\nu+1}{\nu-\mu+1}\,zP_\nu^\mu(z)-\frac{\nu+\mu}{\nu-\mu+1}P_{\nu-1}^\mu(z),$$
$$P_{\nu+1}^\mu(z)=P_{\nu-1}^\mu(z)+(2\nu+1)(z^2-1)^{1/2}P_\nu^{\mu-1}(z),$$
$$(z^2-1)\frac{d}{dz}P_\nu^\mu(z)=\nu zP_\nu^\mu(z)-(\nu+m)P_{\nu-1}^\mu(z),$$
$$Q_\nu^\mu(z)=\frac{\pi}{2\sin(\mu\pi)}e^{i\pi\mu}\Big[P_\nu^\mu(z)-\frac{\Gamma(1+\nu+\mu}{\Gamma(1+\nu-\mu)}P_\nu^{-\mu}(z)\Big],$$
$$Q_\nu^\mu(z)=e^{i\pi\mu}\Big(\frac{\pi}{2}\Big)^{1/2}\Gamma(\nu+\mu+1)(z^2-1)^{-1/4}P_{-\mu-1/2}^{-\nu-1/2}\Big(\frac{z}{\sqrt{z^2-1}}\Big),\qquad \operatorname{Re}z>0.$$

► Integral representations

For $\operatorname{Re}(-\mu)>\operatorname{Re}\nu>-1$,

$$P_\nu^\mu(z)=\frac{2^{-\nu}(z^2-1)^{-\mu/2}}{\Gamma(\nu+1)\Gamma(-\mu-\nu)}\int_0^\infty(z+\cosh t)^{\mu-\nu-1}(\sinh t)^{2\nu+1}\,dt,$$

where z does not lie on the real axis between -1 and ∞.

For $\mu < 1/2$,

$$P_\nu^\mu(z) = \frac{2^\mu (z^2-1)^{-\mu/2}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\mu)} \int_0^\pi \bigl(z + \sqrt{z^2-1}\,\cos t\bigr)^{\nu+\mu} (\sin t)^{-2\mu}\,dt,$$

where z does not lie on the real axis between -1 and 1.

For $\operatorname{Re}\nu > -1$ and $\operatorname{Re}(\nu+\mu+1) > 0$,

$$Q_\nu^\mu(z) = e^{\pi\mu i} \frac{\Gamma(\nu+\mu+1)(z^2-1)^{-\mu/2}}{2^{\nu+1}\Gamma(\nu+1)} \int_0^\pi \bigl(z + \cos t\bigr)^{\mu-\nu-1} (\sin t)^{2\nu+1}\,dt,$$

where z does not lie on the real axis between -1 and 1.

For $n = 0, 1, 2, \dots,$

$$P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\pi\Gamma(\nu+1)} \int_0^\pi \bigl(z + \sqrt{z^2-1}\,\cos t\bigr)^\nu \cos(nt)\,dt, \qquad \operatorname{Re} z > 0;$$

$$Q_\nu^n(z) = (-1)^n \frac{\Gamma(\nu+n+1)}{2^{\nu+1}\Gamma(\nu+1)} (z^2-1)^{-n/2} \int_0^\pi (z + \cos t)^{n-\nu-1} (\sin t)^{2\nu+1}\,dt.$$

Note that $z \neq x$, $-1 < x < 1$, and $\operatorname{Re}\nu > -1$ in the latter formula for $Q_\nu^n(z)$.

► Modified associated Legendre functions

The *modified associated Legendre functions*, on the cut $z = x$, $-1 < x < 1$, of the real axis are defined by the formulas

$$\mathsf{P}_\nu^\mu(x) = \tfrac{1}{2}\bigl[e^{\frac{1}{2}i\mu\pi} P_\nu^\mu(x+i0) + e^{-\frac{1}{2}i\mu\pi} P_\nu^\mu(x-i0)\bigr],$$

$$\mathsf{Q}_\nu^\mu(x) = \tfrac{1}{2}e^{-i\mu\pi}\bigl[e^{-\frac{1}{2}i\mu\pi} Q_\nu^\mu(x+i0) + e^{\frac{1}{2}i\mu\pi} Q_\nu^\mu(x-i0)\bigr].$$

Notation:

$$\mathsf{P}_\nu(x) = \mathsf{P}_\nu^0(x), \qquad \mathsf{Q}_\nu(x) = \mathsf{Q}_\nu^0(x).$$

► Trigonometric expansions

For $-1 < x < 1$, the modified associated Legendre functions can be represented in the form of the trigonometric series:

$$\mathsf{P}_\nu^\mu(\cos\theta) = \frac{2^{\mu+1}}{\sqrt{\pi}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\sin\theta)^\mu \sum_{k=0}^\infty \frac{(\frac{1}{2}+\mu)_k (1+\nu+\mu)_k}{k!\,(\nu+\frac{3}{2})_k} \sin[(2k+\nu+\mu+1)\theta],$$

$$\mathsf{Q}_\nu^\mu(\cos\theta) = \sqrt{\pi}\,2^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\sin\theta)^\mu \sum_{k=0}^\infty \frac{(\frac{1}{2}+\mu)_k (1+\nu+\mu)_k}{k!\,(\nu+\frac{3}{2})_k} \cos[(2k+\nu+\mu+1)\theta],$$

where $0 < \theta < \pi$.

► **Some relations for the modified associated Legendre functions**

For $0 < x < 1$,

$$\mathsf{P}_\nu^\mu(-x) = \mathsf{P}_\nu^\mu(x)\cos[\pi(\nu+\mu)] - 2\pi^{-1}\mathsf{Q}_\nu^\mu(x)\sin[\pi(\nu+\mu)],$$
$$\mathsf{Q}_\nu^\mu(-x) = -\mathsf{Q}_\nu^\mu(x)\cos[\pi(\nu+\mu)] - \tfrac{1}{2}\pi\mathsf{P}_\nu^\mu(x)\sin[\pi(\nu+\mu)].$$

For $-1 < x < 1$,

$$\mathsf{P}_{\nu+1}^\mu(x) = \frac{2\nu+1}{\nu-\mu+1}x\mathsf{P}_\nu^\mu(x) - \frac{\nu+\mu}{\nu-\mu+1}\mathsf{P}_{\nu-1}^\mu(x),$$
$$\mathsf{P}_{\nu+1}^\mu(x) = \mathsf{P}_{\nu-1}^\mu(x) - (2\nu+1)(1-x^2)^{1/2}\mathsf{P}_\nu^{\mu-1}(x),$$
$$\mathsf{P}_{\nu+1}^\mu(x) = x\mathsf{P}_\nu^\mu(x) - (\nu+\mu)(1-x^2)^{1/2}\mathsf{P}_\nu^{\mu-1}(x),$$
$$\frac{d}{dx}\mathsf{P}_\nu^\mu(x) = \frac{\nu x}{x^2-1}\mathsf{P}_\nu^\mu(x) - \frac{\nu+\mu}{x^2-1}\mathsf{P}_{\nu-1}^\mu(x).$$

Wronskian:

$$\mathsf{P}_\nu^\mu(x)\frac{d}{dx}\mathsf{Q}_\nu^\mu(x) - \mathsf{Q}_\nu^\mu(x)\frac{d}{dx}\mathsf{P}_\nu^\mu(x) = \frac{k}{1-x^2}, \qquad k = 2^{2\mu}\frac{\Gamma\bigl(\frac{\nu+\mu+1}{2}\bigr)\Gamma\bigl(\frac{\nu+\mu+2}{2}\bigr)}{\Gamma\bigl(\frac{\nu-\mu+1}{2}\bigr)\Gamma\bigl(\frac{\nu-\mu+2}{2}\bigr)}.$$

For $n = 1, 2, \ldots,$

$$\mathsf{P}_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}\mathsf{P}_\nu(x), \quad \mathsf{Q}_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}\mathsf{Q}_\nu(x).$$

S4.12 Parabolic Cylinder Functions

S4.12.1 Definitions. Basic Formulas

► **Differential equation**

Formulas for the parabolic cylinder functions.

The *Weber parabolic cylinder function* $D_\nu(z)$ is a solution of the linear ordinary differential equation:

$$y''_{zz} + \bigl(-\tfrac{1}{4}z^2 + \nu + \tfrac{1}{2}\bigr)y = 0,$$

where the parameter ν and the variable z can assume arbitrary real or complex values. Another linearly independent solution of this equation is the function $D_{-\nu-1}(iz)$; if ν is noninteger, then $D_\nu(-z)$ can also be taken as a linearly independent solution.

The parabolic cylinder functions can be expressed in terms of degenerate hypergeometric functions as

$$D_\nu(z) = \exp\bigl(-\tfrac{1}{4}z^2\bigr)\left[2^{1/2}\frac{\Gamma\bigl(\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}-\frac{\nu}{2}\bigr)}\Phi\bigl(-\tfrac{\nu}{2},\tfrac{1}{2};\tfrac{1}{2}z^2\bigr) + \frac{\Gamma\bigl(-\frac{1}{2}\bigr)}{\Gamma\bigl(-\frac{\nu}{2}\bigr)}z\Phi\bigl(\tfrac{1}{2}-\tfrac{\nu}{2},\tfrac{3}{2};\tfrac{1}{2}z^2\bigr)\right].$$

► Special cases

For nonnegative integer $\nu = n$, we have

$$D_n(z) = \frac{1}{2^{n/2}} \exp\Big(-\frac{z^2}{4}\Big) H_n\Big(\frac{z}{\sqrt{2}}\Big), \qquad n = 0,\ 1,\ 2,\ \ldots;$$
$$H_n(z) = (-1)^n \exp\big(z^2\big) \frac{d^n}{dz^n} \exp\big(-z^2\big),$$

where $H_n(z)$ is the Hermitian polynomial of order n.

Connection with the error function:

$$D_{-1}(z) = \sqrt{\frac{\pi}{2}}\, \exp\Big(\frac{z^2}{4}\Big) \operatorname{erfc}\Big(\frac{z}{\sqrt{2}}\Big),$$
$$D_{-2}(z) = \sqrt{\frac{\pi}{2}}\, z \exp\Big(\frac{z^2}{4}\Big) \operatorname{erfc}\Big(\frac{z}{\sqrt{2}}\Big) - \exp\Big(-\frac{z^2}{4}\Big).$$

S4.12.2 Integral Representations, Asymptotic Expansions, and Linear Relations

► Integral representations and the asymptotic expansion

Integral representations:

$$D_\nu(z) = \sqrt{2/\pi}\, \exp\big(\tfrac{1}{4}z^2\big) \int_0^\infty t^\nu \exp\big(-\tfrac{1}{2}t^2\big) \cos\big(zt - \tfrac{1}{2}\pi\nu\big)\, dt \qquad \text{for} \quad \operatorname{Re}\nu > -1,$$
$$D_\nu(z) = \frac{1}{\Gamma(-\nu)} \exp\big(-\tfrac{1}{4}z^2\big) \int_0^\infty t^{-\nu-1} \exp\big(-zt - \tfrac{1}{2}t^2\big)\, dt \qquad \text{for} \quad \operatorname{Re}\nu < 0.$$

Asymptotic expansion as $|z| \to \infty$:

$$D_\nu(z) = z^\nu \exp\big(-\tfrac{1}{4}z^2\big) \Bigg[\sum_{n=0}^{N} \frac{(-2)^n \big(-\frac{\nu}{2}\big)_n \big(\frac{1}{2} - \frac{\nu}{2}\big)_n}{n!} \frac{1}{z^{2n}} + O\big(|z|^{-2N-2}\big)\Bigg],$$

where $|\arg z| < \frac{3}{4}\pi$ and $(a)_0 = 1$, $(a)_n = a(a+1)\ldots(a+n-1)$ for $n = 1,\ 2,\ 3,\ \ldots$

► Recurrence relations

$$D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0,$$
$$\frac{d}{dz}D_\nu(z) + \frac{1}{2}zD_\nu(z) - \nu D_{\nu-1}(z) = 0,$$
$$\frac{d}{dz}D_\nu(z) - \frac{1}{2}zD_\nu(z) + D_{\nu+1}(z) = 0.$$

S4.13 Elliptic Integrals

S4.13.1 Complete Elliptic Integrals

► Definitions. Properties. Conversion formulas

Complete elliptic integral of the first kind:

$$\mathsf{K}(k)=\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Complete elliptic integral of the second kind:

$$\mathsf{E}(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\,d\alpha=\int_0^1\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx.$$

The argument k is called the *elliptic modulus* ($k^2<1$).

Notation:

$$k'=\sqrt{1-k^2},\qquad \mathsf{K}'(k)=\mathsf{K}(k'),\qquad \mathsf{E}'(k)=\mathsf{E}(k'),$$

where k' is the *complementary modulus*.

Properties:

$$\begin{aligned}&\mathsf{K}(-k)=\mathsf{K}(k),\qquad \mathsf{E}(-k)=\mathsf{E}(k);\\&\mathsf{K}(k)=\mathsf{K}'(k'),\qquad \mathsf{E}(k)=\mathsf{E}'(k');\\&\mathsf{E}(k)\mathsf{K}'(k)+\mathsf{E}'(k)\mathsf{K}(k)-\mathsf{K}(k)\mathsf{K}'(k)=\frac{\pi}{2}.\end{aligned}$$

Conversion formulas for complete elliptic integrals:

$$\begin{aligned}\mathsf{K}\left(\frac{1-k'}{1+k'}\right)&=\frac{1+k'}{2}\mathsf{K}(k),\\\mathsf{E}\left(\frac{1-k'}{1+k'}\right)&=\frac{1}{1+k'}\big[\mathsf{E}(k)+k'\mathsf{K}(k)\big],\\\mathsf{K}\left(\frac{2\sqrt{k}}{1+k}\right)&=(1+k)\mathsf{K}(k),\\\mathsf{E}\left(\frac{2\sqrt{k}}{1+k}\right)&=\frac{1}{1+k}\big[2\mathsf{E}(k)-(k')^2\mathsf{K}(k)\big].\end{aligned}$$

► Representation of complete elliptic integrals in series form

Representation of complete elliptic integrals in the form of series in powers of the modulus k:

$$\begin{aligned}\mathsf{K}(k)&=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\times3}{2\times4}\right)^2k^4+\cdots+\left[\frac{(2n-1)!!}{(2n)!!}\right]^2k^{2n}+\cdots\right\},\\\mathsf{E}(k)&=\frac{\pi}{2}\left\{1-\left(\frac{1}{2}\right)^2\frac{k^2}{1}-\left(\frac{1\times3}{2\times4}\right)^2\frac{k^4}{3}-\cdots-\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{k^{2n}}{2n-1}-\cdots\right\}.\end{aligned}$$

Representation of complete elliptic integrals in the form of series in powers of the complementary modulus $k' = \sqrt{1-k^2}$:

$$\mathsf{K}(k) = \frac{\pi}{1+k'}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-k'}{1+k'}\right)^2+\left(\frac{1\times 3}{2\times 4}\right)^2\left(\frac{1-k'}{1+k'}\right)^4 +\cdots+\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\left(\frac{1-k'}{1+k'}\right)^{2n}+\cdots\right\},$$

$$\mathsf{K}(k) = \ln\frac{4}{k'}+\left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'}-\frac{2}{1\times 2}\right)(k')^2+\left(\frac{1\times 3}{2\times 4}\right)^2\left(\ln\frac{4}{k'}-\frac{2}{1\times 2}-\frac{2}{3\times 4}\right)(k')^4 +\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2\left(\ln\frac{4}{k'}-\frac{2}{1\times 2}-\frac{2}{3\times 4}-\frac{2}{5\times 6}\right)(k')^6+\cdots;$$

$$\mathsf{E}(k) = \frac{\pi(1+k')}{4}\left\{1+\frac{1}{2^2}-\left(\frac{1-k'}{1+k'}\right)^2+\frac{1^2}{(2\times 4)^2}\left(\frac{1-k'}{1+k'}\right)^4 +\cdots+\left[\frac{(2n-3)!!}{(2n)!!}\right]^2\left(\frac{1-k'}{1+k'}\right)^{2n}+\cdots\right\},$$

$$\mathsf{E}(k) = 1+\frac{1}{2}\left(\ln\frac{4}{k'}-\frac{1}{1\times 2}\right)(k')^2+\frac{1^2\times 3}{2^2\times 4}\left(\ln\frac{4}{k'}-\frac{2}{1\times 2}-\frac{1}{3\times 4}\right)(k')^4 +\frac{1^2\times 3^2\times 5}{2^2\times 4^2\times 6}\left(\ln\frac{4}{k'}-\frac{2}{1\times 2}-\frac{2}{3\times 4}-\frac{1}{5\times 6}\right)(k')^6+\cdots.$$

► Differentiation formulas. Differential equations

Differentiation formulas:

$$\frac{d\mathsf{K}(k)}{dk} = \frac{\mathsf{E}(k)}{k(k')^2}-\frac{\mathsf{K}(k)}{k},\qquad \frac{d\mathsf{E}(k)}{dk} = \frac{\mathsf{E}(k)-\mathsf{K}(k)}{k}.$$

The functions $\mathsf{K}(k)$ and $\mathsf{K}'(k)$ satisfy the second-order linear ordinary differential equation

$$\frac{d}{dk}\left[k(1-k^2)\frac{d\mathsf{K}}{dk}\right]-k\mathsf{K}=0.$$

The functions $\mathsf{E}(k)$ and $\mathsf{E}'(k)-\mathsf{K}'(k)$ satisfy the second-order linear ordinary differential equation

$$(1-k^2)\frac{d}{dk}\left(k\frac{d\mathsf{E}}{dk}\right)+k\mathsf{E}=0.$$

S4.13.2 Incomplete Elliptic Integrals (Elliptic Integrals)

► Definitions. Properties

Elliptic integral of the first kind:

$$F(\varphi,k) = \int_0^{\varphi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Elliptic integral of the second kind:

$$E(\varphi,k) = \int_0^{\varphi}\sqrt{1-k^2\sin^2\alpha}\,d\alpha = \int_0^{\sin\varphi}\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx.$$

Elliptic integral of the third kind:

$$\Pi(\varphi, n, k) = \int_0^{\varphi} \frac{d\alpha}{(1 - n\sin^2\alpha)\sqrt{1 - k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{(1 - nx^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

The quantity k is called the *elliptic modulus* ($k^2 < 1$), $k' = \sqrt{1 - k^2}$ is the *complementary modulus*, and n is the *characteristic parameter*.

Complete elliptic integrals:

$$\mathsf{K}(k) = F\Big(\frac{\pi}{2}, k\Big), \qquad \mathsf{E}(k) = E\Big(\frac{\pi}{2}, k\Big),$$
$$\mathsf{K}'(k) = F\Big(\frac{\pi}{2}, k'\Big), \qquad \mathsf{E}'(k) = E\Big(\frac{\pi}{2}, k'\Big).$$

Properties of elliptic integrals:

$$F(-\varphi, k) = -F(\varphi, k), \qquad F(n\pi \pm \varphi, k) = 2n\mathsf{K}(k) \pm F(\varphi, k);$$
$$E(-\varphi, k) = -E(\varphi, k), \qquad E(n\pi \pm \varphi, k) = 2n\mathsf{E}(k) \pm E(\varphi, k).$$

► Conversion formulas

Conversion formulas for elliptic integrals (first set):

$$F\left(\psi, \frac{1}{k}\right) = kF(\varphi, k),$$
$$E\left(\psi, \frac{1}{k}\right) = \frac{1}{k}\big[E(\varphi, k) - (k')^2F(\varphi, k)\big],$$

where the angles φ and ψ are related by $\sin\psi = k\sin\varphi$, $\cos\psi = \sqrt{1 - k^2\sin^2\varphi}$.

Conversion formulas for elliptic integrals (second set):

$$F\left(\psi, \frac{1-k'}{1+k'}\right) = (1+k')F(\varphi, k),$$
$$E\left(\psi, \frac{1-k'}{1+k'}\right) = \frac{2}{1+k'}\big[E(\varphi, k) + k'F(\varphi, k)\big] - \frac{1-k'}{1+k'}\sin\psi,$$

where the angles φ and ψ are related by $\tan(\psi - \varphi) = k'\tan\varphi$.

Transformation formulas for elliptic integrals (third set):

$$F\left(\psi, \frac{2\sqrt{k}}{1+k}\right) = (1+k)F(\varphi, k),$$
$$E\left(\psi, \frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}\left[2E(\varphi, k) - (k')^2F(\varphi, k) + 2k\frac{\sin\varphi\,\cos\varphi}{1+k\sin^2\varphi}\sqrt{1-k^2\sin^2\varphi}\right],$$

where the angles φ and ψ are related by $\sin\psi = \dfrac{(1+k)\sin\varphi}{1+k\sin^2\varphi}$.

► Trigonometric expansions

Trigonometric expansions for small k and φ:

$$F(\varphi,k) = \frac{2}{\pi}\mathsf{K}(k)\varphi - \sin\varphi\cos\varphi\biggl(a_0 + \frac{2}{3}a_1\sin^2\varphi + \frac{2\times 4}{3\times 5}a_2\sin^4\varphi + \cdots\biggr),$$

$$a_0 = \frac{2}{\pi}\mathsf{K}(k) - 1, \qquad a_n = a_{n-1} - \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n};$$

$$E(\varphi,k) = \frac{2}{\pi}\mathsf{E}(k)\varphi - \sin\varphi\cos\varphi\biggl(b_0 + \frac{2}{3}b_1\sin^2\varphi + \frac{2\times 4}{3\times 5}b_2\sin^4\varphi + \cdots\biggr),$$

$$b_0 = 1 - \frac{2}{\pi}\mathsf{E}(k), \qquad b_n = b_{n-1} - \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{k^{2n}}{2n-1}.$$

Trigonometric expansions for $k \to 1$:

$$F(\varphi,k) = \frac{2}{\pi}\mathsf{K}'(k)\ln\tan\biggl(\frac{\varphi}{2}+\frac{\pi}{4}\biggr) - \frac{\tan\varphi}{\cos\varphi}\biggl(a_0' - \frac{2}{3}a_1'\tan^2\varphi + \frac{2\times 4}{3\times 5}a_2'\tan^4\varphi - \cdots\biggr),$$

$$a_0' = \frac{2}{\pi}\mathsf{K}'(k) - 1, \qquad a_n' = a_{n-1}' - \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 (k')^{2n};$$

$$E(\varphi,k) = \frac{2}{\pi}\mathsf{E}'(k)\ln\tan\biggl(\frac{\varphi}{2}+\frac{\pi}{4}\biggr) + \frac{\tan\varphi}{\cos\varphi}\biggl(b_0' - \frac{2}{3}b_1'\tan^2\varphi + \frac{2\times 4}{3\times 5}b_2'\tan^4\varphi - \cdots\biggr),$$

$$b_0' = \frac{2}{\pi}\mathsf{E}'(k) - 1, \qquad b_n' = b_{n-1}' - \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{(k')^{2n}}{2n-1}.$$

S4.14 Elliptic Functions

An *elliptic function* is a function that is the inverse of an elliptic integral. An elliptic function is a doubly periodic meromorphic function of a complex variable. All its periods can be written in the form $2m\omega_1 + 2n\omega_2$ with integer m and n, where ω_1 and ω_2 are a pair of (primitive) half-periods. The ratio $\tau = \omega_2/\omega_1$ is a complex quantity that may be considered to have a positive imaginary part, $\operatorname{Im}\tau > 0$.

Throughout the rest of this section, the following brief notation will be used: $\mathsf{K} = \mathsf{K}(k)$ and $\mathsf{K}' = \mathsf{K}(k')$ are complete elliptic integrals with $k' = \sqrt{1-k^2}$.

S4.14.1 Jacobi Elliptic Functions

► Definitions. Simple properties. Special cases

When the upper limit φ of the incomplete elliptic integral of the first kind

$$u = \int_0^\varphi \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = F(\varphi,k)$$

is treated as a function of u, the following notation is used:

$$u = \operatorname{am}\varphi.$$

Naming: φ is the *amplitude* and u is the *argument*.

Jacobi elliptic functions:

$$\operatorname{sn} u = \sin\varphi = \sin\operatorname{am} u \qquad (\textit{sine amplitude}),$$
$$\operatorname{cn} u = \cos\varphi = \cos\operatorname{am} u \qquad (\textit{cosine amplitude}),$$
$$\operatorname{dn} u = \sqrt{1 - k^2\sin^2\varphi} = \frac{d\varphi}{du} \qquad (\textit{delta amplitude}).$$

Along with the brief notations $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, the respective full notations are also used: $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, $\operatorname{dn}(u,k)$.

Simple properties:

$$\operatorname{sn}(-u) = -\operatorname{sn} u, \qquad \operatorname{cn}(-u) = \operatorname{cn} u, \qquad \operatorname{dn}(-u) = \operatorname{dn} u;$$
$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \qquad k^2\operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, \qquad \operatorname{dn}^2 u - k^2\operatorname{cn}^2 u = 1 - k^2,$$

where $i^2 = -1$.

Jacobi functions for special values of the modulus ($k = 0$ and $k = 1$):

$$\operatorname{sn}(u,0) = \sin u, \qquad \operatorname{cn}(u,0) = \cos u, \qquad \operatorname{dn}(u,0) = 1;$$
$$\operatorname{sn}(u,1) = \tanh u, \qquad \operatorname{cn}(u,1) = \frac{1}{\cosh u}, \qquad \operatorname{dn}(u,1) = \frac{1}{\cosh u}.$$

Jacobi functions for special values of the argument:

$$\operatorname{sn}(\tfrac{1}{2}\mathsf{K},k) = \frac{1}{\sqrt{1+k'}}, \qquad \operatorname{cn}(\tfrac{1}{2}\mathsf{K},k) = \sqrt{\frac{k'}{1+k'}}, \qquad \operatorname{dn}(\tfrac{1}{2}\mathsf{K},k) = \sqrt{k'};$$
$$\operatorname{sn}(\mathsf{K},k) = 1, \qquad \operatorname{cn}(\mathsf{K},k) = 0, \qquad \operatorname{dn}(\mathsf{K},k) = k'.$$

▶ Reduction formulas

$$\operatorname{sn}(u\pm\mathsf{K}) = \pm\frac{\operatorname{cn} u}{\operatorname{dn} u}, \qquad \operatorname{cn}(u\pm\mathsf{K}) = \mp k'\frac{\operatorname{sn} u}{\operatorname{dn} u}, \qquad \operatorname{dn}(u\pm\mathsf{K}) = \frac{k'}{\operatorname{dn} u};$$
$$\operatorname{sn}(u\pm 2\mathsf{K}) = -\operatorname{sn} u, \qquad \operatorname{cn}(u\pm 2\mathsf{K}) = -\operatorname{cn} u, \qquad \operatorname{dn}(u\pm 2\mathsf{K}) = \operatorname{dn} u;$$
$$\operatorname{sn}(u+i\mathsf{K}') = \frac{1}{k\operatorname{sn} u}, \qquad \operatorname{cn}(u+i\mathsf{K}') = -\frac{i}{k}\frac{\operatorname{dn} u}{\operatorname{sn} u}, \qquad \operatorname{dn}(u+i\mathsf{K}') = -i\frac{\operatorname{cn} u}{\operatorname{sn} u};$$
$$\operatorname{sn}(u+2i\mathsf{K}') = \operatorname{sn} u, \qquad \operatorname{cn}(u+2i\mathsf{K}') = -\operatorname{cn} u, \qquad \operatorname{dn}(u+2i\mathsf{K}') = -\operatorname{dn} u;$$
$$\operatorname{sn}(u+\mathsf{K}+i\mathsf{K}') = \frac{\operatorname{dn} u}{k\operatorname{cn} u}, \qquad \operatorname{cn}(u+\mathsf{K}+i\mathsf{K}') = -\frac{ik'}{k\operatorname{cn} u}, \qquad \operatorname{dn}(u+\mathsf{K}+i\mathsf{K}') = ik'\frac{\operatorname{sn} u}{\operatorname{cn} u};$$
$$\operatorname{sn}(u+2\mathsf{K}+2i\mathsf{K}') = -\operatorname{sn} u, \qquad \operatorname{cn}(u+2\mathsf{K}+2i\mathsf{K}') = \operatorname{cn} u, \qquad \operatorname{dn}(u+2\mathsf{K}+2i\mathsf{K}') = -\operatorname{dn} u.$$

▶ Periods, zeros, poses, and residues

TABLE S4.4

Periods, zeros, poles, and residues of the Jacobian elliptic functions ($m,\ n = 0,\ \pm1,\ \pm2,\ \dots;\ i^2 = -1$)

Functions	Periods	Zeros	Poles	Residues
$\operatorname{sn} u$	$4m\mathsf{K}+2n\mathsf{K}'i$	$2m\mathsf{K}+2n\mathsf{K}'i$	$2m\mathsf{K}+(2n+1)\mathsf{K}'i$	$(-1)^m\frac{1}{k}$
$\operatorname{cn} u$	$(4m+2n)\mathsf{K}+2n\mathsf{K}'i$	$(2m+1)\mathsf{K}+2n\mathsf{K}'i$	$2m\mathsf{K}+(2n+1)\mathsf{K}'i$	$(-1)^{m-1}\frac{i}{k}$
$\operatorname{dn} u$	$2m\mathsf{K}+4n\mathsf{K}'i$	$(2m+1)\mathsf{K}+(2n+1)\mathsf{K}'i$	$2m\mathsf{K}+(2n+1)\mathsf{K}'i$	$(-1)^{n-1}i$

▶ Double-argument formulas

$$\operatorname{sn}(2u) = \frac{2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^4u} = \frac{2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u}{\operatorname{cn}^2u+\operatorname{sn}^2u\operatorname{dn}^2u},$$
$$\operatorname{cn}(2u) = \frac{\operatorname{cn}^2u-\operatorname{sn}^2u\operatorname{dn}^2u}{1-k^2\operatorname{sn}^4u} = \frac{\operatorname{cn}^2u-\operatorname{sn}^2u\operatorname{dn}^2u}{\operatorname{cn}^2u+\operatorname{sn}^2u\operatorname{dn}^2u},$$
$$\operatorname{dn}(2u) = \frac{\operatorname{dn}^2u-k^2\operatorname{sn}^2u\operatorname{cn}^2u}{1-k^2\operatorname{sn}^4u} = \frac{\operatorname{dn}^2u+\operatorname{cn}^2u\,(\operatorname{dn}^2u-1)}{\operatorname{dn}^2u-\operatorname{cn}^2u\,(\operatorname{dn}^2u-1)}.$$

▶ Half-argument formulas

$$\operatorname{sn}^2\frac{u}{2} = \frac{1}{k^2}\frac{1-\operatorname{dn}u}{1+\operatorname{cn}u} = \frac{1-\operatorname{cn}u}{1+\operatorname{dn}u},$$
$$\operatorname{cn}^2\frac{u}{2} = \frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{dn}u} = \frac{1-k^2}{k^2}\frac{1-\operatorname{dn}u}{\operatorname{dn}u-\operatorname{cn}u},$$
$$\operatorname{dn}^2\frac{u}{2} = \frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{cn}u} = (1-k^2)\frac{1-\operatorname{cn}u}{\operatorname{dn}u-\operatorname{cn}u}.$$

▶ Argument addition formulas

$$\operatorname{sn}(u\pm v) = \frac{\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v\pm\operatorname{sn}v\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$
$$\operatorname{cn}(u\pm v) = \frac{\operatorname{cn}u\operatorname{cn}v\mp\operatorname{sn}u\operatorname{sn}v\operatorname{dn}u\operatorname{dn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$
$$\operatorname{dn}(u\pm v) = \frac{\operatorname{dn}u\operatorname{dn}v\mp k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}u\operatorname{cn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v}.$$

▶ Conversion formulas

Table S4.5 presents conversion formulas for Jacobi elliptic functions. If $k > 1$, then $k_1 = 1/k < 1$. Elliptic functions with real modulus can be reduced, using the first set of conversion formulas, to elliptic functions with a modulus lying between 0 and 1.

▶ Descending Landen transformation (Gauss's transformation)

Notation:

$$\mu = \left|\frac{1-k'}{1+k'}\right|, \qquad v = \frac{u}{1+\mu}.$$

Descending transformations:

$$\operatorname{sn}(u,k) = \frac{(1+\mu)\operatorname{sn}(v,\mu^2)}{1+\mu\operatorname{sn}^2(v,\mu^2)}, \qquad \operatorname{cn}(u,k) = \frac{\operatorname{cn}(v,\mu^2)\operatorname{dn}(v,\mu^2)}{1+\mu\operatorname{sn}^2(v,\mu^2)},$$
$$\operatorname{dn}(u,k) = \frac{\operatorname{dn}^2(v,\mu^2)+\mu-1}{1+\mu-\operatorname{dn}^2(v,\mu^2)}.$$

TABLE S4.5
Conversion formulas for Jacobi elliptic functions. Full notation is used: $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$

u_1	k_1	$\mathrm{sn}(u_1,k_1)$	$\mathrm{cn}(u_1,k_1)$	$\mathrm{dn}(u_1,k_1)$
ku	$\dfrac{1}{k}$	$k\,\mathrm{sn}(u,k)$	$\mathrm{dn}(u,k)$	$\mathrm{cn}(u,k)$
iu	k'	$i\dfrac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$	$\dfrac{1}{\mathrm{cn}(u,k)}$	$\dfrac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}$
$k'u$	$i\dfrac{k}{k'}$	$k'\dfrac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1}{\mathrm{dn}(u,k)}$
iku	$i\dfrac{k'}{k}$	$ik\dfrac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1}{\mathrm{dn}(u,k)}$	$\dfrac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$
$ik'u$	$\dfrac{1}{k'}$	$ik'\dfrac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$	$\dfrac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}$	$\dfrac{1}{\mathrm{cn}(u,k)}$
$(1+k)u$	$\dfrac{2\sqrt{k}}{1+k}$	$\dfrac{(1+k)\,\mathrm{sn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$	$\dfrac{\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$	$\dfrac{1-k\,\mathrm{sn}^2(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$
$(1+k')u$	$\dfrac{1-k'}{1+k'}$	$\dfrac{(1+k')\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1-(1+k')\,\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}$	$\dfrac{1-(1-k')\,\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}$

► Ascending Landen transformation

Notation:

$$\mu = \frac{4k}{(1+k)^2}, \qquad \sigma = \left|\frac{1-k}{1+k}\right|, \qquad v = \frac{u}{1+\sigma}.$$

Ascending transformations:

$$\mathrm{sn}(u,k) = (1+\sigma)\frac{\mathrm{sn}(v,\mu)\,\mathrm{cn}(v,\mu)}{\mathrm{dn}(v,\mu)}, \qquad \mathrm{cn}(u,k) = \frac{1+\sigma}{\mu}\frac{\mathrm{dn}^2(v,\mu)-\sigma}{\mathrm{dn}(v,\mu)},$$
$$\mathrm{dn}(u,k) = \frac{1-\sigma}{\mu}\frac{\mathrm{dn}^2(v,\mu)+\sigma}{\mathrm{dn}(v,\mu)}.$$

► Series representation

Representation Jacobi functions in the form of power series in u:

$$\begin{aligned}
\mathrm{sn}\,u &= u - \frac{1}{3!}(1+k^2)u^3 + \frac{1}{5!}(1+14k^2+k^4)u^5 \\
&\quad - \frac{1}{7!}(1+135k^2+135k^4+k^6)u^7 + \cdots, \\
\mathrm{cn}\,u &= 1 - \frac{1}{2!}u^2 + \frac{1}{4!}(1+4k^2)u^4 - \frac{1}{6!}(1+44k^2+16k^4)u^6 + \cdots, \\
\mathrm{dn}\,u &= 1 - \frac{1}{2!}k^2u^2 + \frac{1}{4!}k^2(4+k^2)u^4 - \frac{1}{6!}k^2(16+44k^2+k^4)u^6 + \cdots, \\
\mathrm{am}\,u &= u - \frac{1}{3!}k^2u^3 + \frac{1}{5!}k^2(4+k^2)u^5 - \frac{1}{7!}k^2(16+44k^2+k^4)u^7 + \cdots.
\end{aligned}$$

These functions converge for $|u| < |\mathsf{K}(k')|$.

Representation Jacobi functions in the form of trigonometric series:

$$\operatorname{sn} u = \frac{2\pi}{k\mathsf{K}\sqrt{q}} \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n-1}} \sin\left[(2n-1)\frac{\pi u}{2\mathsf{K}}\right],$$

$$\operatorname{cn} u = \frac{2\pi}{k\mathsf{K}\sqrt{q}} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n-1}} \cos\left[(2n-1)\frac{\pi u}{2\mathsf{K}}\right],$$

$$\operatorname{dn} u = \frac{\pi}{2\mathsf{K}} + \frac{2\pi}{\mathsf{K}} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos\left(\frac{n\pi u}{\mathsf{K}}\right),$$

$$\operatorname{am} u = \frac{\pi u}{2\mathsf{K}} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^{2n}} \sin\left(\frac{n\pi u}{\mathsf{K}}\right),$$

where $q = \exp(-\pi\mathsf{K}'/\mathsf{K})$, $\mathsf{K} = \mathsf{K}(k)$, $\mathsf{K}' = \mathsf{K}(k')$, and $k' = \sqrt{1-k^2}$.

► Derivatives and integrals

Derivatives:

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \qquad \frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \qquad \frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

Integrals:

$$\int \operatorname{sn} u\, du = \frac{1}{k} \ln(\operatorname{dn} u - k \operatorname{cn} u) = -\frac{1}{k} \ln(\operatorname{dn} u + k \operatorname{cn} u),$$

$$\int \operatorname{cn} u\, du = \frac{1}{k} \arccos(\operatorname{dn} u) = \frac{1}{k} \arcsin(k \operatorname{sn} u),$$

$$\int \operatorname{dn} u\, du = \arcsin(\operatorname{sn} u) = \operatorname{am} u.$$

The arbitrary additive constant C in the integrals is omitted.

S4.14.2 Weierstrass Elliptic Function

► Infinite series representation. Some properties

The Weierstrass elliptic function (or Weierstrass $\wp$-function) is defined as

$$\wp(z) = \wp(z|\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{m,n} \left[\frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2}\right],$$

where the summation is assumed over all integer m and n, except for $m = n = 0$. This function is a complex, double periodic function of a complex variable z with periods $2\omega_1$ and $2\omega_1$:

$$\wp(-z) = \wp(z),$$

$$\wp(z+2m\omega_1+2n\omega_2) = \wp(z),$$

where $m,\ n = 0,\ \pm1,\ \pm2,\ \dots$ and $\operatorname{Im}(\omega_2/\omega_1) \neq 0$. The series defining the Weierstrass $\wp$-function converges everywhere except for second-order poles located at $z_{mn} = 2m\omega_1 + 2n\omega_2$.

Argument addition formula:

$$\wp(z_1+z_2)=-\wp(z_1)-\wp(z_2)+\frac{1}{4}\left[\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right]^2.$$

► Representation in the form of a definite integral

The Weierstrass function $\wp=\wp(z,g_2,g_3)=\wp(z|\omega_1,\omega_2)$ is defined implicitly by the elliptic integral:

$$z=\int_{\wp}^{\infty}\frac{dt}{\sqrt{4t^3-g_2t-g_3}}=\int_{\wp}^{\infty}\frac{dt}{2\sqrt{(t-e_1)(t-e_2)(t-e_3)}}.$$

The parameters g_2 and g_3 are known as the *invariants*.

The parameters e_1, e_2, e_3, which are the roots of the cubic equation $4z^3-g_2z-g_3=0$, are related to the half-periods ω_1, ω_2 and invariants g_2, g_3 by

$$e_1=\wp(\omega_1),\qquad e_2=\wp(\omega_1+\omega_2),\qquad e_1=\wp(\omega_2),$$
$$e_1+e_2+e_3=0,\qquad e_1e_2+e_1e_3+e_2e_3=-\tfrac{1}{4}g_2,\qquad e_1e_2e_3=\tfrac{1}{4}g_3.$$

Homogeneity property:

$$\wp(z,g_2,g_3)=\lambda^2\wp(\lambda z,\lambda^{-4}g_2,\lambda^{-6}g_3).$$

► Representation as a Laurent series. Differential equations

The Weierstrass $\wp$-function can be expanded into a Laurent series:

$$\wp(z)=\frac{1}{z^2}+\frac{g_2}{20}z^2+\frac{g_3}{28}z^4+\frac{g_2^2}{1200}z^6+\frac{3g_2g_3}{6160}z^8+\cdots=\frac{1}{z^2}+\sum_{k=2}^{\infty}a_kz^{2k-2},$$

$$a_k=\frac{3}{(k-3)(2k+1)}\sum_{m=2}^{k-2}a_ma_{k-m}\quad\text{for}\quad k\ge 4,\qquad 0<|z|<\min(|\omega_1|,|\omega_2|).$$

The Weierstrass $\wp$-function satisfies the first-order and second-order nonlinear differential equations:

$$(\wp'_z)^2=4\wp^3-g_2\wp-g_3,$$
$$\wp''_{zz}=6\wp^2-\tfrac{1}{2}g_2.$$

► Connection with Jacobi elliptic functions

Direct and inverse representations of the Weierstrass elliptic function via Jacobi elliptic functions:

$$\wp(z)=e_1+(e_1-e_3)\frac{\operatorname{cn}^2 w}{\operatorname{sn}^2 w}=e_2+(e_1-e_3)\frac{\operatorname{dn}^2 w}{\operatorname{sn}^2 w}=e_3+\frac{e_1-e_3}{\operatorname{sn}^2 w};$$
$$\operatorname{sn} w=\sqrt{\frac{e_1-e_3}{\wp(z)-e_3}},\qquad \operatorname{cn} w=\sqrt{\frac{\wp(z)-e_1}{\wp(z)-e_3}},\qquad \operatorname{dn} w=\sqrt{\frac{\wp(z)-e_2}{\wp(z)-e_3}};$$
$$w=z\sqrt{e_1-e_3}=\mathsf{K}z/\omega_1.$$

The parameters are related by

$$k=\sqrt{\frac{e_2-e_3}{e_1-e_3}},\qquad k'=\sqrt{\frac{e_1-e_2}{e_1-e_3}},\qquad \mathsf{K}=\omega_1\sqrt{e_1-e_3},\qquad i\mathsf{K}'=\omega_2\sqrt{e_1-e_3}.$$

S4.15 Jacobi Theta Functions

S4.15.1 Series Representation of the Jacobi Theta Functions. Simplest Properties

► Definition of the Jacobi theta functions

The *Jacobi theta functions* are defined by the following series:

$$\begin{aligned}
\vartheta_1(v)=\vartheta_1(v,q)=\vartheta_1(v|\tau)&=2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin[(2n+1)\pi v]\\
&=i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2}e^{i\pi(2n-1)v},\\
\vartheta_2(v)=\vartheta_2(v,q)=\vartheta_2(v|\tau)&=2\sum_{n=0}^{\infty}q^{(n+1/2)^2}\cos[(2n+1)\pi v]=\sum_{n=-\infty}^{\infty}q^{(n-1/2)^2}e^{i\pi(2n-1)v},\\
\vartheta_3(v)=\vartheta_3(v,q)=\vartheta_3(v|\tau)&=1+2\sum_{n=0}^{\infty}q^{n^2}\cos(2n\pi v)=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2i\pi nv},\\
\vartheta_4(v)=\vartheta_4(v,q)=\vartheta_4(v|\tau)&=1+2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos(2n\pi v)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2i\pi nv},
\end{aligned}$$

where v is a complex variable and $q=e^{i\pi\tau}$ is a complex parameter (τ has a positive imaginary part).

► Simplest properties

The Jacobi theta functions are periodic entire functions that possess the following properties:

$[l]$	$\vartheta_1(v)$	odd,	has period 2,	vanishes at $v=m+n\tau$;
	$\vartheta_2(v)$	even,	has period 2,	vanishes at $v=m+n\tau+\frac{1}{2}$;
	$\vartheta_3(v)$	even,	has period 1,	vanishes at $v=m+(n+\frac{1}{2})\tau+\frac{1}{2}$;
	$\vartheta_4(v)$	even,	has period 1,	vanishes at $v=m+(n+\frac{1}{2})\tau$.

Here, $m,\ n=0,\ \pm1,\ \pm2,\ \dots$

Remark S4.1. The theta functions are not elliptic functions. The very good convergence of their series allows the computation of various elliptic integrals and elliptic functions using the relations given above in Section S4.15.1.

S4.15.2 Various Relations and Formulas. Connection with Jacobi Elliptic Functions

► Linear and quadratic relations

Linear relations (first set):

$$\vartheta_1\Big(v+\frac{1}{2}\Big)=\vartheta_2(v),\qquad \vartheta_2\Big(v+\frac{1}{2}\Big)=-\vartheta_1(v),$$
$$\vartheta_3\Big(v+\frac{1}{2}\Big)=\vartheta_4(v),\qquad \vartheta_4\Big(v+\frac{1}{2}\Big)=\vartheta_3(v),$$
$$\vartheta_1\Big(v+\frac{\tau}{2}\Big)=ie^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_4(v),\qquad \vartheta_2\Big(v+\frac{\tau}{2}\Big)=e^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_3(v),$$
$$\vartheta_3\Big(v+\frac{\tau}{2}\Big)=e^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_2(v),\qquad \vartheta_4\Big(v+\frac{\tau}{2}\Big)=ie^{-i\pi\left(v+\frac{\tau}{4}\right)}\vartheta_1(v).$$

Linear relations (second set):

$$\vartheta_1(v|\tau+1)=e^{i\pi/4}\vartheta_1(v|\tau),\qquad \vartheta_2(v|\tau+1)=e^{i\pi/4}\vartheta_2(v|\tau),$$
$$\vartheta_3(v|\tau+1)=\vartheta_4(v|\tau),\qquad \vartheta_4(v|\tau+1)=\vartheta_3(v|\tau),$$
$$\vartheta_1\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big)=\frac{1}{i}\sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_1(v|\tau),\qquad \vartheta_2\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big)=\sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_4(v|\tau),$$
$$\vartheta_3\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big)=\sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_3(v|\tau),\qquad \vartheta_4\Big(\frac{v}{\tau}\Big|-\frac{1}{\tau}\Big)=\sqrt{\frac{\tau}{i}}\,e^{i\pi v^2/\tau}\vartheta_2(v|\tau).$$

Quadratic relations:

$$\vartheta_1^2(v)\vartheta_2^2(0)=\vartheta_4^2(v)\vartheta_3^2(0)-\vartheta_3^2(v)\vartheta_4^2(0),$$
$$\vartheta_1^2(v)\vartheta_3^2(0)=\vartheta_4^2(v)\vartheta_2^2(0)-\vartheta_2^2(v)\vartheta_4^2(0),$$
$$\vartheta_1^2(v)\vartheta_4^2(0)=\vartheta_3^2(v)\vartheta_2^2(0)-\vartheta_2^2(v)\vartheta_3^2(0),$$
$$\vartheta_4^2(v)\vartheta_4^2(0)=\vartheta_3^2(v)\vartheta_3^2(0)-\vartheta_2^2(v)\vartheta_2^2(0).$$

► Representation of the theta functions in the form of infinite products

$$\vartheta_1(v)=2q_0q^{1/4}\sin(\pi v)\prod_{n=1}^{\infty}\big[1-2q^{2n}\cos(2\pi v)+q^{4n}\big],$$
$$\vartheta_2(v)=2q_0q^{1/4}\cos(\pi v)\prod_{n=1}^{\infty}\big[1+2q^{2n}\cos(2\pi v)+q^{4n}\big],$$
$$\vartheta_3(v)=q_0\prod_{n=1}^{\infty}\big[1+2q^{2n-1}\cos(2\pi v)+q^{4n-2}\big],$$
$$\vartheta_4(v)=q_0\prod_{n=1}^{\infty}\big[1-2q^{2n-1}\cos(2\pi v)+q^{4n-2}\big],$$

where $q_0=\prod\limits_{n=1}^{\infty}(1-q^{2n})$.

► Connection with Jacobi elliptic functions

Representations of Jacobi elliptic functions in terms of the theta functions:

$$\operatorname{sn} w = \frac{\vartheta_3(0)}{\vartheta_2(0)}\frac{\vartheta_1(v)}{\vartheta_4(v)}, \quad \operatorname{cn} w = \frac{\vartheta_4(0)}{\vartheta_2(0)}\frac{\vartheta_2(v)}{\vartheta_4(v)}, \quad \operatorname{dn} w = \frac{\vartheta_4(0)}{\vartheta_3(0)}\frac{\vartheta_3(v)}{\vartheta_4(v)}, \quad w = 2\mathsf{K}v.$$

The parameters are related by

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}, \quad k' = \frac{\vartheta_4^2(0)}{\vartheta_3^2(0)}, \quad \mathsf{K} = \frac{\pi}{2}\vartheta_3^2(0), \quad \mathsf{K}' = -i\tau\mathsf{K}.$$

S4.16 Mathieu Functions and Modified Mathieu Functions

S4.16.1 Mathieu Functions

► Mathieu equation and Mathieu functions

The Mathieu functions $\operatorname{ce}_n(x,q)$ and $\operatorname{se}_n(x,q)$ are periodic solutions of the Mathieu equation

$$y''_{xx} + (a - 2q\cos 2x)y = 0.$$

Such solutions exist for definite values of parameters a and q (those values of a are referred to as eigenvalues). The Mathieu functions are listed in Table S4.6.

TABLE S4.6
The Mathieu functions $\operatorname{ce}_n = \operatorname{ce}_n(x,q)$ and $\operatorname{se}_n = \operatorname{se}_n(x,q)$ (for odd n, functions ce_n and se_n are 2π-periodic, and for even n, they are π-periodic); definite eigenvalues $a = a_n(q)$ and $a = b_n(q)$ correspond to each value of parameter q

Mathieu functions	Recurrence relations for coefficients	Normalization conditions
$\operatorname{ce}_{2n} = \sum_{m=0}^{\infty} A_{2m}^{2n}\cos 2mx$	$qA_2^{2n} = a_{2n}A_0^{2n}$; $qA_4^{2n} = (a_{2n}-4)A_2^{2n} - 2qA_0^{2n}$; $qA_{2m+2}^{2n} = (a_{2n}-4m^2)A_{2m}^{2n} - qA_{2m-2}^{2n}$, $m\ge 2$	$(A_0^{2n})^2 + \sum_{m=0}^{\infty}(A_{2m}^{2n})^2 = \begin{cases} 2 & \text{if } n=0, \\ 1 & \text{if } n\ge 1 \end{cases}$
$\operatorname{ce}_{2n+1} = \sum_{m=0}^{\infty} A_{2m+1}^{2n+1}\cos(2m+1)x$	$qA_3^{2n+1} = (a_{2n+1}-1-q)A_1^{2n+1}$; $qA_{2m+3}^{2n+1} = [a_{2n+1}-(2m+1)^2]A_{2m+1}^{2n+1} - qA_{2m-1}^{2n+1}$, $m\ge 1$	$\sum_{m=0}^{\infty}(A_{2m+1}^{2n+1})^2 = 1$
$\operatorname{se}_{2n} = \sum_{m=0}^{\infty} B_{2m}^{2n}\sin 2mx$, $\operatorname{se}_0 = 0$	$qB_4^{2n} = (b_{2n}-4)B_2^{2n}$; $qB_{2m+2}^{2n} = (b_{2n}-4m^2)B_{2m}^{2n} - qB_{2m-2}^{2n}$, $m\ge 2$	$\sum_{m=0}^{\infty}(B_{2m}^{2n})^2 = 1$
$\operatorname{se}_{2n+1} = \sum_{m=0}^{\infty} B_{2m+1}^{2n+1}\sin(2m+1)x$	$qB_3^{2n+1} = (b_{2n+1}-1-q)B_1^{2n+1}$; $qB_{2m+3}^{2n+1} = [b_{2n+1}-(2m+1)^2]B_{2m+1}^{2n+1} - qB_{2m-1}^{2n+1}$, $m\ge 1$	$\sum_{m=0}^{\infty}(B_{2m+1}^{2n+1})^2 = 1$

► Properties of the Mathieu functions

The Mathieu functions possess the following properties:

$$\operatorname{ce}_{2n}(x,\,-q) = (-1)^n \operatorname{ce}_{2n}\Bigl(\frac{\pi}{2}-x,\,q\Bigr), \qquad \operatorname{ce}_{2n+1}(x,\,-q) = (-1)^n \operatorname{se}_{2n+1}\Bigl(\frac{\pi}{2}-x,\,q\Bigr),$$
$$\operatorname{se}_{2n}(x,\,-q) = (-1)^{n-1} \operatorname{se}_{2n}\Bigl(\frac{\pi}{2}-x,\,q\Bigr), \quad \operatorname{se}_{2n+1}(x,\,-q) = (-1)^n \operatorname{ce}_{2n+1}\Bigl(\frac{\pi}{2}-x,\,q\Bigr).$$

Selecting a sufficiently large number m and omitting the term with the maximum number in the recurrence relations (indicated in Table S4.6), we can obtain approximate relations for eigenvalues a_n (or b_n) with respect to parameter q. Then, equating the determinant of the corresponding homogeneous linear system of equations for coefficients A_m^n (or B_m^n) to zero, we obtain an algebraic equation for finding $a_n(q)$ (or $b_n(q)$).

For fixed real $q \neq 0$, eigenvalues a_n and b_n are all real and different, while

$$\begin{aligned} &\text{if}\quad q>0 \quad \text{then}\quad a_0<b_1<a_1<b_2<a_2<\cdots;\\ &\text{if}\quad q<0 \quad \text{then}\quad a_0<a_1<b_1<b_2<a_2<a_3<b_3<b_4<\cdots. \end{aligned}$$

The eigenvalues possess the properties

$$a_{2n}(-q) = a_{2n}(q), \quad b_{2n}(-q) = b_{2n}(q), \quad a_{2n+1}(-q) = b_{2n+1}(q).$$

Tables of the eigenvalues $a_n = a_n(q)$ and $b_n = b_n(q)$ can be found in Abramowitz and Stegun (1964, Chapter 20).

The solution of the Mathieu equation corresponding to eigenvalue a_n (or b_n) has n zeros on the interval $0 \le x < \pi$ (q is a real number).

► Asymptotic expansions as $q \to 0$ and $q \to \infty$

Listed below are two leading terms of the asymptotic expansions of the Mathieu functions $\operatorname{ce}_n(x,q)$ and $\operatorname{se}_n(x,q)$, as well as of the corresponding eigenvalues $a_n(q)$ and $b_n(q)$, as $q \to 0$:

$$\operatorname{ce}_0(x,q) = \frac{1}{\sqrt{2}}\Bigl(1-\frac{q}{2}\cos 2x\Bigr), \quad a_0(q) = -\frac{q^2}{2}+\frac{7q^4}{128};$$
$$\operatorname{ce}_1(x,q) = \cos x-\frac{q}{8}\cos 3x, \quad a_1(q) = 1+q;$$
$$\operatorname{ce}_2(x,q) = \cos 2x+\frac{q}{4}\Bigl(1-\frac{\cos 4x}{3}\Bigr), \quad a_2(q) = 4+\frac{5q^2}{12};$$
$$\operatorname{ce}_n(x,q) = \cos nx+\frac{q}{4}\biggl[\frac{\cos(n+2)x}{n+1}-\frac{\cos(n-2)x}{n-1}\biggr], \quad a_n(q) = n^2+\frac{q^2}{2(n^2-1)} \quad (n \ge 3);$$
$$\operatorname{se}_1(x,q) = \sin x-\frac{q}{8}\sin 3x, \quad b_1(q) = 1-q;$$
$$\operatorname{se}_2(x,q) = \sin 2x-q\frac{\sin 4x}{12}, \quad b_2(q) = 4-\frac{q^2}{12};$$
$$\operatorname{se}_n(x,q) = \sin nx-\frac{q}{4}\biggl[\frac{\sin(n+2)x}{n+1}-\frac{\sin(n-2)x}{n-1}\biggr], \quad b_n(q) = n^2+\frac{q^2}{2(n^2-1)} \quad (n \ge 3).$$

Asymptotic results as $q \to \infty$ $(-\pi/2 < x < \pi/2)$:

$$\begin{aligned} a_n(q) &\approx -2q + 2(2n+1)\sqrt{q} + \tfrac{1}{4}(2n^2+2n+1), \\ b_{n+1}(q) &\approx -2q + 2(2n+1)\sqrt{q} + \tfrac{1}{4}(2n^2+2n+1), \\ \mathrm{ce}_n(x,q) &\approx \lambda_n q^{-1/4} \cos^{-n-1} x \big[\cos^{2n+1}\xi \exp(2\sqrt{q}\sin x) + \sin^{2n+1}\xi \exp(-2\sqrt{q}\sin x)\big], \\ \mathrm{se}_{n+1}(x,q) &\approx \mu_{n+1} q^{-1/4} \cos^{-n-1} x \big[\cos^{2n+1}\xi \exp(2\sqrt{q}\sin x) - \sin^{2n+1}\xi \exp(-2\sqrt{q}\sin x)\big], \end{aligned}$$

where λ_n and μ_n are some constants independent of the parameter q, and $\xi = \frac{1}{2}x + \frac{\pi}{4}$.

The Mathieu functions are discussed in the books by McLachlan (1947), Whittaker & Watson (1952), Bateman & Erdélyi (1955, vol. 3), and Abramowitz & Stegun (1964) in more detail.

S4.16.2 Modified Mathieu Functions

The modified Mathieu functions $\mathrm{Ce}_n(x,q)$ and $\mathrm{Se}_n(x,q)$ are solutions of the modified Mathieu equation

$$y''_{xx} - (a - 2q\cosh 2x)y = 0,$$

with $a = a_n(q)$ and $a = b_n(q)$ being the eigenvalues of the Mathieu equation (see Section S4.16.1).

The modified Mathieu functions are defined as

$$\mathrm{Ce}_{2n+p}(x,q) = \mathrm{ce}_{2n+p}(ix,q) = \sum_{k=0}^{\infty} A_{2k+p}^{2n+p} \cosh[(2k+p)x],$$

$$\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q) = \sum_{k=0}^{\infty} B_{2k+p}^{2n+p} \sinh[(2k+p)x],$$

where p may be equal to 0 and 1, and coefficients A_{2k+p}^{2n+p} and B_{2k+p}^{2n+p} are indicated in Section S4.16.1.

S4.17 Orthogonal Polynomials

All zeros of each of the orthogonal polynomials $\mathcal{P}_n(x)$ considered in this section are real and simple. The zeros of the polynomials $\mathcal{P}_n(x)$ and $\mathcal{P}_{n+1}(x)$ alternate.

For Legendre polynomials, see Section S4.11.1.

S4.17.1 Laguerre Polynomials and Generalized Laguerre Polynomials

► **Laguerre polynomials**

The Laguerre polynomials $L_n = L_n(x)$ satisfy the second-order linear ordinary differential equation

$$xy''_{xx} + (1-x)y'_x + ny = 0$$

and are defined by the formulas

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\bigl(x^n e^{-x}\bigr) = \frac{(-1)^n}{n!}\biggl[x^n - n^2 x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} + \cdots\biggr].$$

The first four polynomials have the form

$$L_0(x) = 1, \quad L_1(x) = -x + 1, \quad L_2(x) = \tfrac{1}{2}(x^2 - 4x + 2),$$
$$L_3(x) = \tfrac{1}{6}(-x^3 + 9x^2 - 18x + 6).$$

To calculate $L_n(x)$ for $n \ge 2$, one can use the recurrence formulas

$$L_{n+1}(x) = \frac{1}{n+1}\bigl[(2n+1-x)L_n(x) - nL_{n-1}(x)\bigr].$$

The functions $L_n(x)$ form an orthonormal system on the interval $0 < x < \infty$ with weight e^{-x}:

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \begin{cases} 0 & \text{if } n \ne m, \\ 1 & \text{if } n = m. \end{cases}$$

The generating function is

$$\frac{1}{1-s} \exp\Bigl(-\frac{sx}{1-s}\Bigr) = \sum_{n=0}^\infty L_n(x) s^n, \qquad |s| < 1.$$

► Generalized Laguerre polynomials

The generalized Laguerre polynomials $L_n^\alpha = L_n^\alpha(x)$ $(\alpha > -1)$ satisfy the equation

$$xy''_{xx} + (\alpha + 1 - x)y'_x + ny = 0$$

and are defined by the formulas

$$\begin{aligned} L_n^\alpha(x) &= \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}\bigl(x^{n+\alpha} e^{-x}\bigr) \\ &= \sum_{m=0}^n C_{n+\alpha}^{n-m} \frac{(-x)^m}{m!} = \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)}{\Gamma(m+\alpha+1)} \frac{(-x)^m}{m!\,(n-m)!}. \end{aligned}$$

Notation: $L_n^0(x) = L_n(x)$.
Special cases:

$$L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = \alpha + 1 - x, \quad L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}.$$

To calculate $L_n^\alpha(x)$ for $n \ge 2$, one can use the recurrence formulas

$$L_{n+1}^\alpha(x) = \frac{1}{n+1}\bigl[(2n+\alpha+1-x)L_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x)\bigr].$$

Other recurrence formulas:

$$L_n^\alpha(x) = L_{n-1}^\alpha(x) + L_n^{\alpha-1}(x), \qquad \frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x),$$
$$x\frac{d}{dx} L_n^\alpha(x) = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x).$$

The functions $L_n^\alpha(x)$ form an orthogonal system on the interval $0 < x < \infty$ with weight $x^\alpha e^{-x}$:

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{\Gamma(\alpha+n+1)}{n!} & \text{if } n = m. \end{cases}$$

The generating function is

$$(1-s)^{-\alpha-1}\exp\Bigl(-\frac{sx}{1-s}\Bigr) = \sum_{n=0}^\infty L_n^\alpha(x)s^n, \qquad |s| < 1.$$

S4.17.2 Chebyshev Polynomials and Functions

► Chebyshev polynomials of the first kind

The *Chebyshev polynomials of the first kind* $T_n = T_n(x)$ satisfy the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx} - xy'_x + n^2 y = 0 \qquad \text{(S4.17.2.1)}$$

and are defined by the formulas

$$T_n(x) = \cos(n \arccos x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}\bigl[(1-x^2)^{n-\frac{1}{2}}\bigr]$$
$$= \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m} \quad (n = 0, 1, 2, \dots),$$

where $[A]$ stands for the integer part of a number A.

An alternative representation of the Chebyshev polynomials:

$$T_n(x) = \frac{(-1)^n}{(2n-1)!!}(1-x^2)^{1/2}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}.$$

The first five Chebyshev polynomials of the first kind are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1.$$

The recurrence formulas:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 2.$$

The functions $T_n(x)$ form an orthogonal system on the interval $-1 < x < 1$, with

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{2}\pi & \text{if } n = m \neq 0, \\ \pi & \text{if } n = m = 0. \end{cases}$$

The generating function is

$$\frac{1-sx}{1-2sx+s^2} = \sum_{n=0}^{\infty} T_n(x)s^n \qquad (|s| < 1).$$

The functions $T_n(x)$ have only real simple zeros, all lying on the interval $-1 < x < 1$.

The normalized Chebyshev polynomials of the first kind, $2^{1-n}T_n(x)$, deviate from zero least of all. This means that among all polynomials of degree n with the leading coefficient 1, it is the maximum of the modulus $\max\limits_{-1\le x\le 1} |2^{1-n}T_n(x)|$ that has the least value, the maximum being equal to 2^{1-n}.

► Chebyshev polynomials of the second kind

The *Chebyshev polynomials of the second kind* $U_n = U_n(x)$ satisfy the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx} - 3xy'_x + n(n+2)y = 0$$

and are defined by the formulas

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}} = \frac{2^n(n+1)!}{(2n+1)!}\frac{1}{\sqrt{1-x^2}}\frac{d^n}{dx^n}(1-x^2)^{n+1/2}$$
$$= \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m)!}{m!\,(n-2m)!}(2x)^{n-2m} \qquad (n = 0,\ 1,\ 2,\ \ldots).$$

The first five Chebyshev polynomials of the second kind are

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1,$$
$$U_3(x) = 8x^3 - 4x, \quad U_4(x) = 16x^4 - 12x^2 + 1.$$

The recurrence formulas:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \geq 2.$$

The generating function is

$$\frac{1}{1-2sx+s^2} = \sum_{n=0}^{\infty} U_n(x)s^n \qquad (|s| < 1).$$

The Chebyshev polynomials of the first and second kind are related by

$$U_n(x) = \frac{1}{n+1}\frac{d}{dx}T_{n+1}(x).$$

► Chebyshev functions of the second kind

The *Chebyshev functions of the second kind*,

$$\mathsf{U}_0(x) = \arcsin x,$$
$$\mathsf{U}_n(x) = \sin(n \arccos x) = \frac{\sqrt{1-x^2}}{n}\frac{dT_n(x)}{dx} \quad (n = 1, 2, \dots),$$

just as the Chebyshev polynomials, also satisfy the differential equation (S4.17.2.1).

The first five Chebyshev functions are

$$\mathsf{U}_0(x) = 0, \quad \mathsf{U}_1(x) = \sqrt{1-x^2}, \quad \mathsf{U}_2(x) = 2x\sqrt{1-x^2},$$
$$\mathsf{U}_3(x) = (4x^2-1)\sqrt{1-x^2}, \quad \mathsf{U}_5(x) = (8x^3-4x)\sqrt{1-x^2}.$$

The recurrence formulas:

$$\mathsf{U}_{n+1}(x) = 2x\mathsf{U}_n(x) - \mathsf{U}_{n-1}(x), \qquad n \geq 2.$$

The functions $\mathsf{U}_n(x)$ form an orthogonal system on the interval $-1 < x < 1$, with

$$\int_{-1}^{1} \frac{\mathsf{U}_n(x)\mathsf{U}_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } n \neq m \quad \text{or} \quad n = m = 0, \\ \frac{1}{2}\pi & \text{if } n = m \neq 0. \end{cases}$$

The generating function is

$$\frac{\sqrt{1-x^2}}{1-2sx+s^2} = \sum_{n=0}^{\infty} \mathsf{U}_{n+1}(x)s^n \qquad (|s| < 1).$$

S4.17.3 Hermite Polynomials

► Various representations of the Hermite polynomials

The *Hermite polynomials* $H_n = H_n(x)$ satisfy the second-order linear ordinary differential equation

$$y''_{xx} - 2xy'_x + 2ny = 0$$

and is defined by the formulas

$$H_n(x) = (-1)^n \exp\bigl(x^2\bigr)\frac{d^n}{dx^n}\exp\bigl(-x^2\bigr) = \sum_{m=0}^{[n/2]} (-1)^m \frac{n!}{m!\,(n-2m)!}(2x)^{n-2m}.$$

The first five polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2,$$
$$H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12.$$

Recurrence formulas:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad n \geq 2;$$
$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x).$$

Integral representation:

$$H_{2n}(x) = \frac{(-1)^n 2^{2n+1}}{\sqrt{\pi}} \exp\bigl(x^2\bigr) \int_0^\infty \exp\bigl(-t^2\bigr) t^{2n} \cos(2xt)\,dt,$$
$$H_{2n+1}(x) = \frac{(-1)^n 2^{2n+2}}{\sqrt{\pi}} \exp\bigl(x^2\bigr) \int_0^\infty \exp\bigl(-t^2\bigr) t^{2n+1} \sin(2xt)\,dt,$$

where $n = 0,\ 1,\ 2,\ \ldots$

► Orthogonality. The generating function. An asymptotic formula

The functions $H_n(x)$ form an orthogonal system on the interval $-\infty < x < \infty$ with weight e^{-x^2}:

$$\int_{-\infty}^{\infty} \exp\bigl(-x^2\bigr) H_n(x) H_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \sqrt{\pi}\, 2^n n! & \text{if } n = m. \end{cases}$$

Generating function:

$$\exp\bigl(-s^2 + 2sx\bigr) = \sum_{n=0}^{\infty} H_n(x) \frac{s^n}{n!}.$$

Asymptotic formula as $n \to \infty$:

$$H_n(x) \approx 2^{\frac{n+1}{2}} n^{\frac{n}{2}} e^{-\frac{n}{2}} \exp\bigl(x^2\bigr) \cos\Bigl(\sqrt{2n+1}\, x - \tfrac{1}{2}\pi n\Bigr).$$

► Hermite functions

The *Hermite functions* $h_n(x)$ are introduced by the formula

$$h_n(x) = \exp\Bigl(-\frac{1}{2}x^2\Bigr) H_n(x) = (-1)^n \exp\Bigl(\frac{1}{2}x^2\Bigr) \frac{d^n}{dx^n} \exp\bigl(-x^2\bigr), \qquad n = 0, 1, 2, \ldots$$

The Hermite functions satisfy the second-order linear ordinary differential equation

$$h''_{xx} + (2n + 1 - x^2)h = 0.$$

The functions $h_n(x)$ form an orthogonal system on the interval $-\infty < x < \infty$ with weight 1:

$$\int_{-\infty}^{\infty} h_n(x) h_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \sqrt{\pi}\, 2^n n! & \text{if } n = m. \end{cases}$$

S4.17.4 Jacobi Polynomials and Gegenbauer Polynomials

► Jacobi polynomials

The *Jacobi polynomials*, $P_n^{\alpha,\beta}(x)$, are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx} + \big[\beta - \alpha - (\alpha+\beta+2)x\big]y'_x + n(n+\alpha+\beta+1)y = 0$$

and are defined by the formulas

$$\begin{aligned} P_n^{\alpha,\beta}(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big] \\ &= 2^{-n}\sum_{m=0}^{n} C_{n+\alpha}^{m} C_{n+\beta}^{n-m}(x-1)^{n-m}(x+1)^m, \end{aligned}$$

where the C_b^a are binomial coefficients.

The generating function:

$$2^{\alpha+\beta}R^{-1}(1-s+R)^{-\alpha}(1+s+R)^{-\beta} = \sum_{n=0}^{\infty} P_n^{\alpha,\beta}(x)s^n, \quad R = \sqrt{1-2xs+s^2}, \quad |s|<1.$$

The Jacobi polynomials are orthogonal on the interval $-1 \le x \le 1$ with weight $(1-x)^\alpha(1+x)^\beta$:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\,\dfrac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} & \text{if } n = m. \end{cases}$$

For $\alpha > -1$ and $\beta > -1$, all zeros of the polynomial $P_n^{\alpha,\beta}(x)$ are simple and lie on the interval $-1 < x < 1$.

► Gegenbauer polynomials

The *Gegenbauer polynomials* (also called *ultraspherical polynomials*), $C_n^{(\lambda)}(x)$, are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''_{xx} - (2\lambda+1)xy'_x + n(n+2\lambda)y = 0$$

and are defined by the formulas

$$\begin{aligned} C_n^{(\lambda)}(x) &= \frac{(-2)^n}{n!}\,\frac{\Gamma(n+\lambda)\,\Gamma(n+2\lambda)}{\Gamma(\lambda)\,\Gamma(2n+2\lambda)}(1-x^2)^{-\lambda+1/2}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2} \\ &= \sum_{m=0}^{[n/2]}(-1)^m\frac{\Gamma(n-m+\lambda)}{\Gamma(\lambda)\,m!\,(n-2m)!}(2x)^{n-2m}. \end{aligned}$$

Recurrence formulas:

$$C_{n+1}^{(\lambda)}(x) = \frac{2(n+\lambda)}{n+1}\,xC_n^{(\lambda)}(x) - \frac{n+2\lambda-1}{n+1}C_{n-1}^{(\lambda)}(x);$$
$$C_n^{(\lambda)}(-x) = (-1)^n C_n^{(\lambda)}(x), \qquad \frac{d}{dx}C_n^{(\lambda)}(x) = 2\lambda C_{n-1}^{(\lambda+1)}(x).$$

The generating function:

$$\frac{1}{(1-2xs+s^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)s^n.$$

The Gegenbauer polynomials are orthogonal on the interval $-1 \le x \le 1$ with weight $(1-x^2)^{\lambda-1/2}$:

$$\int_{-1}^{1}(1-x^2)^{\lambda-1/2}C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{\pi\Gamma(2\lambda+n)}{2^{2\lambda-1}(\lambda+n)n!\,\Gamma^2(\lambda)} & \text{if } n = m. \end{cases}$$

S4.18 Nonorthogonal Polynomials

S4.18.1 Bernoulli Polynomials

► Definition. Basic properties

The *Bernoulli polynomials* $B_n(x)$ are introduced by the formula

$$B_n(x) = \sum_{k=0}^{n} C_n^k B_k x^{n-k} \qquad (n = 0,\ 1,\ 2,\ \dots),$$

where C_n^k are the binomial coefficients and B_n are Bernoulli numbers (see Section S4.1.3).

The Bernoulli polynomials can be defined using the recurrence relation

$$B_0(x) = 1, \qquad \sum_{k=0}^{n-1} C_n^k B_k(x) = nx^{n-1}, \qquad n = 2,\ 3,\ \dots$$

The first six Bernoulli polynomials are given by

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}, \quad B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x.$$

Basic properties:

$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad B'_{n+1}(x) = (n+1)B_n(x),$$
$$B_n(1-x) = (-1)^n B_n(x), \qquad (-1)^n E_n(-x) = E_n(x) + nx^{n-1},$$

where the prime denotes a derivative with respect to x, and $n = 0,\ 1,\ \dots$

Multiplication and addition formulas:

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\Big(x + \frac{k}{m}\Big),$$

$$B_n(x+y) = \sum_{k=0}^{n} C_n^k B_k(x) y^{n-k},$$

where $n = 0,\ 1,\ \dots$ and $m = 1,\ 2,\ \dots$

► Generating function. Fourier series expansions. Integrals

The generating function is expressed as

$$\frac{te^{xt}}{e^t - 1} \equiv \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi).$$

This relation may be used as a definition of the Bernoulli polynomials.

Fourier series expansions:

$$B_n(x) = -2\frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - \frac{1}{2}\pi n)}{k^n} \qquad (n = 1,\ 0 < x < 1;\ \ n > 1,\ 0 \le x \le 1);$$

$$B_{2n-1}(x) = 2(-1)^n \frac{(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n-1}} \qquad (n = 1,\ 0 < x < 1;\ \ n > 1,\ 0 \le x \le 1);$$

$$B_{2n}(x) = 2(-1)^n \frac{(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}} \qquad (n = 1,\ 2,\ \dots,\ \ 0 \le x \le 1).$$

Integrals:

$$\int_a^x B_n(t)\,dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1},$$

$$\int_0^1 B_m(t) B_n(t)\,dt = (-1)^{n-1} \frac{m!\,n!}{(m+n)!} B_{m+n},$$

where m and n are positive integers and B_n are Bernoulli numbers.

S4.18.2 Euler Polynomials

► Definition. Basic properties

Definition:

$$E_n(x) = \sum_{k=0}^{n} C_n^k \frac{E_k}{2^n} \Big(x - \frac{1}{2}\Big)^{n-k} \qquad (n = 0,\ 1,\ 2,\ \dots),$$

where C_n^k are the binomial coefficients and E_n are Euler numbers.

The first six Euler polynomials are given by

$$E_0(x) = 1, \quad E_1(x) = x - \tfrac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{4},$$
$$E_4(x) = x^4 - 2x^3 + x, \quad E_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{2}x^2 - \tfrac{1}{2}.$$

Basic properties:

$$E_n(x+1)+E_n(x)=2x^n, \qquad E'_{n+1}=(n+1)E_n(x),$$
$$E_n(1-x)=(-1)^nE_n(x), \qquad (-1)^{n+1}E_n(-x)=E_n(x)-2x^n,$$

where the prime denotes a derivative with respect to x, and $n=0,\,1,\,\ldots$

Multiplication and addition formulas:

$$E_n(mx)=m^n\sum_{k=0}^{m-1}(-1)^kE_n\Bigl(x+\frac{k}{m}\Bigr), \qquad n=0,\,1,\,\ldots,\ \ m=1,\,3,\,\ldots;$$
$$E_n(mx)=-\frac{2}{n+1}m^n\sum_{k=0}^{m-1}(-1)^kE_{n+1}\Bigl(x+\frac{k}{m}\Bigr), \qquad n=0,\,1,\,\ldots,\ \ m=2,\,4,\,\ldots;$$
$$E_n(x+y)=\sum_{k=0}^{n}C_n^kE_k(x)y^{n-k}, \qquad n=0,\,1,\,\ldots$$

► Generating function. Fourier series expansions. Integrals

The generating function is expressed as

$$\frac{2e^{xt}}{e^t+1}\equiv\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \qquad (|t|<\pi).$$

This relation may be used as a definition of the Euler polynomials.

Fourier series expansions:

$$E_n(x)=4\frac{n!}{\pi^{n+1}}\sum_{k=0}^{\infty}\frac{\sin\bigl((2k+1)\pi x-\frac{1}{2}\pi n\bigr)}{(2k+1)^{n+1}} \qquad (n=0,\ 0<x<1;\ \ n>0,\ 0\le x\le 1);$$
$$E_{2n}(x)=4(-1)^n\frac{(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\frac{\sin\bigl((2k+1)\pi x\bigr)}{(2k+1)^{2n+1}} \qquad (n=0,\ 0<x<1;\ \ n>0,\ 0\le x\le 1);$$
$$E_{2n-1}(x)=4(-1)^n\frac{(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\frac{\cos\bigl((2k+1)\pi x\bigr)}{(2k+1)^{2n}} \qquad (n=1,\,2,\,\ldots,\ \ 0\le x\le 1).$$

Integrals:

$$\int_a^x E_n(t)\,dt=\frac{E_{n+1}(x)-E_{n+1}(a)}{n+1},$$
$$\int_0^1 E_m(t)E_n(t)\,dt=4(-1)^n(2^{m+n+2}-1)\frac{m!\,n!}{(m+n+2)!}B_{m+n+2},$$

where $m,\,n=0,\,1,\,\ldots$ and B_n are Bernoulli numbers. The Euler polynomials are orthogonal for even $n+m$.

Connection with the Bernoulli polynomials:

$$E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] = \frac{2}{n}\left[B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right],$$

where $n = 1, 2, \ldots$

⊙ *References for Chapter* S4: H. Bateman and A. Erdélyi (1953, 1955), N. W. McLachlan (1955), M. Abramowitz and I. A. Stegun (1964), W. Magnus, F. Oberhettinger, and R. P. Soni (1966), I. S. Gradshteyn and I. M. Ryzhik (2000), G. A. Korn and T. M. Korn (2000), S. Yu. Slavyanov and W. Lay (2000), D. Zwillinger (2002), A. D. Polyanin and V. F. Zaitsev (2003), E. W. Weisstein (2003), A. D. Polyanin and A. V. Manzhirov (2007), F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (2010).

REFERENCES

Abel, M. L. and Braselton, J. P., *Maple by Example*, 3rd ed., AP Professional, Boston, MA, 2005.

Abramowitz, M. and Stegun, I. A. (Editors), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics, Washington, 1964.

Acosta, G., Durán, G., and Rossi, J. D., An adaptive time step procedure for a parabolic problem with blow-up, *Computing*, Vol. 68, pp. 343–373, 2002.

Akô, K., Subfunctions for ordinary differential equations, II, *Funkcialaj Ekvacioj (International Series)*, Vol. 10, No. 2, pp. 145–162, 1967.

Akô, K., Subfunctions for ordinary differential equations, III, *Funkcialaj Ekvacioj (International Series)*, Vol. 11, No. 2, pp. 111–129, 1968.

Akritas, A. G., *Elements of Computer Algebra with Applications*, Wiley, New York, 1989.

Akulenko, L. D. and Nesterov, S. V., Accelerated convergence method in the Sturm–Liouville problem, *Russ. J. Math. Phys.,* Vol. 3, No. 4, pp. 517–521, 1996.

Akulenko, L. D. and Nesterov, S. V., Determination of the frequencies and forms of oscillations of non-uniform distributed systems with boundary conditions of the third kind, *Appl. Math. Mech.* (PMM), Vol. 61, No. 4, pp. 531–538, 1997.

Akulenko, L. D. and Nesterov, S. V., *High Precision Methods in Eigenvalue Problems and Their Applications*, Chapman & Hall/CRC Press, Boca Raton, 2005.

Alexandrov, A. Yu., Platonov, A. V., Starkov, V. N., Stepenko, N. A., *Mathematical Modeling and Stability Analysis of Biological Communities*, Lan', St. Petersburg, 2016.

Alexeeva, T. A., Zaitsev, V. F., and Shvets, T. B., On discrete symmetries of the Abel equation of the 2nd kind [in Russian]. In: *Applied Mechanics and Mathematics*, MIPT, Moscow, pp. 4–11, 1992.

Alshina, E. A., Kalitkin, N. N., and Koryakin, P. V., Diagnostics of singularities of exact solutions in computations with error control [in Russian], *Zh. Vychisl. Mat. Mat. Fiz.,* Vol. 45, No. 10, pp. 1837–1847, 2005; http://eqworld.ipmnet.ru/ru/solutions/interesting/alshina2005.pdf.

Anderson, R. L. and Ibragimov, N. H., *Lie–Bäcklund Transformations in Applications*, SIAM Studies in Applied Mathematics, Philadelphia, 1979.

Andronov, A. A., Vitt, A. A., and Khaikin, S. E., *Theory of Oscillators*, Dover Publ., New York, 2011.

Andronov, A. A., Leontovich, E. A., and Gordon, I. I., *Theory of Bifurcations of Dynamic Systems on a Plane*, Israel Program for Scientific Translations (IPST), Jerusalem, 1971.

Antimirov, M. Ya., *Applied Integral Transforms*, American Mathematical Society, Providence, Rhode Island, 1993.

Arnold, V. I., *Additional Chapters of Ordinary Differential Equation Theory* [in Russian], Nauka, Moscow, 1978.

Arnold, V. I. and Afraimovich, V. S., *Bifurcation Theory and Catastrophe Theory*, Springer-Verlag, New York, 1999.

Arnold, V. I., Kozlov, V. V., and Neishtadt, A. I., *Mathematical Aspects of Classical and Celestial Mechanics, Dynamical System III*, Springer, Berlin, 1993.

Ascher, U., Mattheij, R. and Russell, R., *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Ser. SIAM Classics in Applied Mathematics, Vol. 13, 1995.

Ascher, U. and Petzold, L., *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, SIAM, Philadelphia, 1998.

Ascher, U. and Petzold, L.,. Projected implicit Runge–Kutta methods for differential algebraic equations, *SIAM J. Numer. Anal.*, Vol. 28, pp. 1097–1120, 1991.

Bader, G. and Deuflhard, P., A semi-implicit mid-point rule for stiff systems of ordinary differential equations, *Numer. Math*, Vol. 41, pp. 373–398, 1983.

Bahder, T. B., *Mathematica for Scientists and Engineers*, Addison-Wesley, Redwood City, CA, 1994.

Bailey, P. B., Shampine, L. F., and Waltman, P. E., *Nonlinear Two Point Boundary Value Problems*, Academic Press, New York, 1968.

Baker, G. A. (Jr.) and Graves–Morris, P., *Padé Approximants*, Addison–Wesley, London, 1981.

Bakhvalov, N. S., *Numerical Methods: Analysis, Algebra, Ordinary Differential Equations*, Mir Publishers, Moscow, 1977.

Barton, D., Willer, I. M. and Zahar, R. V. M., Taylor series method for ordinary differential equations. In: *Mathematical Software* (J. R. Rice, editor), Academic Press, New York, 1972.

Barton, D., Willer, I. M., and Zahar, R. V. M., The automatic solution of systems of ordinary differential equations by the method of Taylor series, *Comput. J.*, Vol. 14, pp. 243–248, 1971.

Bateman, H. and Erdélyi, A., *Higher Transcendental Functions*, Vols. 1 and 2, McGraw-Hill, New York, 1953.

Bateman, H. and Erdélyi, A., *Higher Transcendental Functions*, Vol. 3, McGraw-Hill, New York, 1955.

Bateman, H. and Erdélyi, A., *Tables of Integral Transforms. Vols. 1 and 2*, McGraw-Hill, New York, 1954.

Bazykin, A. D., *Mathematical Biophysics of Interacting Populations* [in Russian], Nauka, Moscow, 1985.

Bebernes, J. and Gaines, R., Dependence on boundary data and a generalized boundary value problem, *J. Differential Equations*, Vol. 4, pp. 359–368, 1968.

Bekir, A., New solitons and periodic wave solutions for some nonlinear physical models by using the sine-cosine method, *Physica Scripta*, Vol. 77, No. 4, 2008.

Bekir, A., Application of the G'/G-expansion method for nonlinear evolution equations, *Phys. Lett. A,* Vol. 372, pp. 3400–3406, 2008.

Bellman, R. and Roth, R., *The Laplace Transform*, World Scientific Publishing Co., Singapore, 1984.

Berezin, I. S. and Zhidkov, N. P., *Computational Methods, Vol. II* [in Russian], Fizmatgiz, Moscow, 1960.

Berkovich, L. M., *Factorization and Transformations of Ordinary Differential Equations* [in Russian], Saratov University Publ., Saratov, 1989.

Birkhoff, G. and Rota, G. C., *Ordinary Differential Equations*, John Wiley & Sons, New York, 1978.

Blaquiere, A., *Nonlinear System Analysis*, Academic Press, New York, 1966.

Bluman, G. W. and Anco, S. C., *Symmetry and Integration Methods for Differential Equations*, Springer, New York, 2002.

Bluman, G. W. and Cole, J. D., *Similarity Methods for Differential Equations*, Springer, New York, 1974.

Bluman, G. W. and Kumei, S., *Symmetries and Differential Equations*, Springer, New York, 1989.

Bogacki, P. and Shampine, L. F., *An Efficient Runge–Kutta* $(4, 5)$ *Pair*, Report 89–20, Math. Dept. Southern Methodist University, Dallas, Texas, 1989.

Bogolyubov, N. N. and Mitropol'skii, Yu. A., *Asymptotic Methods in the Theory of Nonlinear Oscillations* [in Russian], Nauka, Moscow, 1974.

Boltyanskii, V. G. and Vilenkin, N. Ya., *Symmetry in Algebra*, 2nd ed. [in Russian], Nauka, Moscow, 2002.

Boor, C. and Swartz, B., *SIAM J. Numerical Analysis*, Vol. 10, No. 4, pp. 582–606, 1993.

Borisov, A. V. and Mamaev, I. S., *Rigid Body Dynamics* [in Russian], Regular and Chaotic Dynamics, Izhevsk, 2001.

Boyce, W. E. and DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems, 8th Edition*, John Wiley & Sons, New York, 2004.

Bratus, A. S., Novozhilov, A. S., and Platonov, A. P., *Dynamic Models and Models in Biology* [in Russian], Fizmatlit, Moscow, 2009.

Braun, M., *Differential Equations and Their Applications, 4th Edition*, Springer, New York, 1993.

Brenan, K. E., Campbell, S. L., and Petzold, L. R., *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, SIAM, Philadelphia, PA, 1996.

Brent, R. P., *Algorithms for Minimization without Derivatives*, Dover, 2002 (Original edition 1973).

Bulirsch, R. and Stoer, J., Fehlerabschätzungen und extrapolation mit rationalen funktionen bei verfahren vom Richardson–typus, *Numer. Math.*, Vol. 6, pp. 413–427, 1964.

Butcher, J. C., Modified multistep method for numerical integration of ordinary differential equations, *J. Assoc. Comput. Mach.,* Vol. 12, No. 1, pp. 124–135, 1965.

Butcher, J. C., Order, stepsize and stiffness switching, *Computing*, Vol. 44, No. 3, pp. 209–220, 1990.

Butcher, J. C., *The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods*, Wiley-Interscience, New York, 1987.

Candy, J. and Rozmus, R., A symplectic integration algorithm for separable Hamiltonian functions, *J. Comput. Phys.*, Vol. 92, pp. 230–256, 1991.

Cash, J. R., The integration of stiff IVP in ODE using modified extended BDF, *Computers and Mathematics with Applications*, Vol. 9, pp. 645–657, 1983.

Cash, J. R. and Considine, S., An MEBDF code for stiff initial value problems, *ACM Trans. Math. Software (TOMS)*, Vo. 18, No. 2, pp. 142–155, 1992.

Cash, J. R. and Karp, A. H., A variable order Runge–Kutta method for initial value problems with rapidly varying right-hand sides, *ACM Transactions on Mathematical Software*, Vol. 16, No. 3, pp. 201–222, 1990.

Chapra, S. C. and Canale, R. P., *Numerical Methods for Engineers, 6th Edition*, McGraw-Hill, Boston, 2010.

Char, B. W., Geddes, K. O., Gonnet, G. H., Monagan, M. B., and Watt, S. M., *A Tutorial Introduction to Maple V*, Springer, Wien, New York, 1992.

Cheb-Terrab, E. S., Duarte, L. G. S., and da Mota, L. A. C. P., Computer algebra solving of first order ODEs using symmetry methods, *Computer Physics Communications*, Vol. 101, pp.254–268, 1997.

Cheb-Terrab, E. S. and von Bulow, K., A computational approach for the analytical solving of partial differential equations, *Computer Physics Communications*, Vol. 90, pp. 102–116, 1995.

Cheb-Terrab, E. S. and Kolokolnikov, T., First-order ordinary differential equations, symmetries and linear transformations, *European Journal of Applied Mathematics*, Vol. 14, pp. 231–246, 2003.

Cheb-Terrab, E. S. and Roche, A. D., Symmetries and first order ODE patterns, *Computer Physics Communications*, Vol. 113, pp. 239–260, 1998.

Cheb-Terrab, E. S., Duarte, L. G. S., and da Mota, L. A. C. P., Computer algebra solving of second order ODEs using symmetry methods, *Computer Physics Communications*, Vol. 108, pp. 90–114, 1998.

Chicone, C., *Ordinary Differential Equations with Applications*, Springer, Berlin, 1999.

Chowdhury, A. R., *Painlevé Analysis and Its Applications*, Chapman & Hall/CRC Press, Boca Raton, 2000.

Clarkson, P. A., The third Painlevé equation and associated special polynomials, *J. Phys. A*, Vol. 36, pp. 9507–9532, 2003.

Clarkson, P. A., Painlevé Equations — Nonlinear Special Functions: Computation and Application. In: *Orthogonal Polynomials and Special Functions* (Eds. Marcell, F. and van Assche, W.), Vol. 1883 of Lecture Notes in Math., pp. 331–411. Springer, Berlin , 2006.

Cohen, S. D. and Hindmarsh, A. C., CVODE, a Stiff/Nonstiff ODE Solver in C, *Computers in Physics*, Vol. 10, No. 2, pp. 138–143, 1996.

Cole, G. D., *Perturbation Methods in Applied Mathematics*, Blaisdell Publishing Company, Waltham, MA, 1968.

Collatz, L., *Eigenwertaufgaben mit Technischen Anwendungen*, Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1963.

Conte, S. D. and de Boor, C., *Elementary Numerical Analysis. An Algorithmic Approach*, McGraw-Hill, 1980.

Conte, R., The Painlevé approach to nonlinear ordinary differential equations. In: *The Painlevé Property* (ed. by R. Conte), pp. 77–180, CRM Series in Mathematical Physics, Springer, New York, 1999.

Corless, R. M., *Essential Maple*, Springer, Berlin, 1995.

Crawford, J. D., Introduction to bifurcation theory, *Rev. Mod. Phys.,* Vol. 63, No. 4, 1991.

Davenport, J. H., Siret, Y., and Tournier, E., *Computer Algebra Systems and Algorithms for Algebraic Computation*, Academic Press, London, 1993.

Dekker, T. J., Finding a zero by means of successive linear interpolation. In: *Constructive Aspects of the Fundamental Theorem of Algebra* (Dejon, B. and Henrici, P., eds.), Wiley-Interscience, London, 1969.

Del Buono, N. and Lopez, L., Runge–Kutta type methods based on geodesics for systems of ODEs on the Stiefel manifold, *BIT*, Vol. 41, No. 5, pp. 912–923, 2001.

Deuflhard, P., Recent progress in extrapolation methods for ordinary differential equations, *SIAM Rev.*, Vol. 27, pp. 505–535, 1985.

Deuflhard, P., Hairer, E. and Zugck, J., One-step and extrapolation methods for differential-algebraic systems, *Numer. Math.*, Vol. 51, pp. 501–516, 1987.

Deuflhard, P., Fiedler, B., and Kunkel, P., Efficient numerical path following beyond critical points, *SIAM Journal on Numerical Analysis*, Society for Industrial and Applied Mathematics, Vol. 24, pp. 912–927, 1987.

Dieci, L., Russel, R. D., and van Vleck, E. S., Unitary integrators and applications to continuous orthonormalization techniques, *SIAM J. Num. Anal.*, Vol 31, pp. 261–281, 1994.

Dieci, L. and van Vleck, E. S., Computation of orthonormal factors for fundamental solution matrices, *Numer. Math.*, Vol. 83, pp. 599–620, 1999.

Ditkin, V. A. and Prudnikov, A. P., *Integral Transforms and Operational Calculus*, Pergamon Press, New York, 1965.

Dlamini, P.G. and Khumalo, M., On the computation of blow-up solutions for semilinear ODEs and parabolic PDEs, *Math. Problems in Eng.,* Vol. 2012, Article ID 162034, 15 p., 2012.

Dobrokhotov, S. Yu., Integration by quadratures of $2n$-dimensional linear Hamiltonian systems with n known skew-orthogonal solutions, *Russ. Math. Surveys*, Vol. 53, No. 2, pp. 380–381, 1998.

Doetsch, G., *Handbuch der Laplace-Transformation. Theorie der Laplace-Transformation*, Birkhäuser, Basel–Stuttgart, 1950.

Doetsch, G., *Handbuch der Laplace-Transformation. Anwendungen der Laplace-Transformation*, Birkhäuser, Basel–Stuttgart, 1956.

Doetsch, G., *Introduction to the Theory and Application of the Laplace Transformation*, Springer, Berlin, 1974.

Dormand, J. R., *Numerical Methods for Differential Equations: A Computational Approach*, CRC Press, Boca Raton, 1996.

Ebaid, A., Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method, *Phys. Letters A*, Vol. 365, No. 3, pp. 213–219, 2007.

Elkin, V. I., Reduction of underdetermined systems of ordinary differential equations: I, *Differential Equations*, Vol. 45, No. 12, pp. 1721–1731, 2009.

Elkin, V. I., Classification of certain types of underdetermined systems of ordinary differential equations, *Doklady Mathematics*, Vol. 81, No. 3, pp. 362–363, 2010.

El'sgol'ts, L. E., *Differential Equations*, Gordon & Breach Inc., New York, 1961.

Enright, W. H., *The Relative Efficiency of Alternative Defect Control Schemes for High Order Continuous Runge–Kutta Formulas*, Technical Report 252/91, Dept. of Computer Science, University of Toronto, 1991.

Enright, W. H., Jackson, K. R., Nørsett, S. P. and Thomsen, P. G., Interpolants for Runge–Kutta formulas, *ACM TOMS*, Vol. 12, pp. 193–218, 1986.

Enright, W. H., Jackson, K. R., Nørsett, S. P. and Thomsen, P. G., Effective solution of discontinuous IVPS using a RKF pair with interpolants, *Proceed. ODE Conference held at Sandia National Lab.*, Albuquerque, New Mexico, 1986.

Enright, W. H., A new error-control for initial value solvers, *Appl. Math. Comput.*, Vol. 31, pp. 588–599, 1989.

Erbe, L. H., Hu, S., and Wang, H., Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.,* Vol. 184, pp. 640–648, 1994.

Erbe, L. H. and Wang, H.,, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.,* Vol. 120, pp. 743–748, 1994.

Evans, D. J and Raslan, K. R., The Adomian decomposition method for solving delay differential equations, *Int. J. Comput. Math.*, Vol. 82, pp. 49–54, 2005.

Faddeev, S. I. and Kogan, V. V., *Nonlinear Boundary Value Problems for Systems of Ordinary Differential Equations on Finite Interval* [in Russian], Novosibirsk State University, Novosibirsk, 2008.

Fan, E., Extended tanh-function method and its applications to nonlinear equations, *Phys. Letters A*, Vol. 277, No. 4–5, pp. 212–218, 2000.

Fedoryuk, M. V., *Asymptotic Analysis. Linear Ordinary Differential Equations*, Springer, Berlin, 1993.

Fehlberg, E., Klassische Runge–Kutta–Formeln vierter und niedrigerer ordnung mit schrittweitenlontrolle und ihre anwendung auf waermeleitungsprobleme, *Computing*, Vol. 6, pp 61–71, 1970.

Finlayson, B. A., *The Method of Weighted Residuals and Variational Principles*, Academic Press, New York, 1972.

Fokas, A. S. and Ablowitz M. J., On unified approach to transformation and elementary solutions of Painlevé equations, *J. Math. Phys.,* Vol. 23, No. 11, pp. 2033–2042, 1982.

Forsythe, G. E., Malcolm, M. A., and Moler, C. B., *Computer Methods for Mathematical Computations*, Prentice Hall, New Jersey, 1977.

Fox, L. and Mayers, D. F., *Numerical Solution of Ordinary Differential Equations for Scientists and Engineers*, Chapman & Hall, 1987.

Frank-Kamenetskii, D. A., *Diffusion and Heat Transfer in Chemical Kinetics, 2nd Edition* [in Russian], Nauka, Moscow, 1987 (English edition: Plenum Press, 1969).

Gaisaryan, S. S., Differential equations, ordinary, approximate methods of solution. In: *Encyclopedia of Mathematics*, Kluwer, 2002. URL: http://www.encyclopediaofmath.org/index.php?title=Differential_equations,_ordinary,_approximate_methods_of_solution_of&oldid=11532

Galaktionov, V. A. and Svirshchevskii, S. R., *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*, Chapman & Hall/CRC Press, Boca Raton, 2006.

Gambier, B., Sur les équations differentielles du second ordre et du premier degré dont l'integrale générale est à points critiques fixes, *Acta Math.,* Vol. 33, pp. 1–55, 1910.

Gantmakher, F. R., *Lectures on Analytical Mechanics* [in Russian], Fizmatlit, Moscow, 2002.

Gear, C.W., *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice Hall, Upper Saddle River, NJ, 1971.

Geddes, K. O., Czapor, S. R., and Labahn, G., *Algorithms for Computer Algebra*, Kluwer Academic Publishers, Boston, 1992.

Getz, C. and Helmstedt, J., *Graphics with Mathematica: Fractals, Julia Sets, Patterns and Natural Forms*, Elsevier Science & Technology Book, Amsterdam, Boston, 2004.

Giacaglia, G. E. O., *Perturbation Methods in Non-Linear Systems*, Springer, New York, 1972.

Godunov, S. K. and Ryaben'kii, V. S., *Differences Schemes* [in Russian], Nauka, Moscow, 1973.

Golub, G. H. and van Loan, C. F., *Matrix Computations*, 3rd ed., Johns Hopkins University Press, 1996.

Golubev, V. V., *Lectures on Analytic Theory of Differential Equations* [in Russian], GITTL, Moscow, 1950.

Goriely, A. and Hyde, C., Necessary and sufficient conditions for finite time singularities in ordinary differential equations, *J. Differential Equations*, Vol. 161, pp. 422–448, 2000.

Gradshteyn, I. S. and Ryzhik, I. M., *Tables of Integrals, Series, and Products*, Academic Press, New York, 1980.

Gragg, W. B., On extrapolation algorithms for ordinary initial value problems, *SIAM J. Num. Anal.*, Vol. 2, pp. 384–403, 1965.

Gray, T. and Glynn, J., *Exploring Mathematics with Mathematica: Dialogs Concerning Computers and Mathematics*, Addison-Wesley, Reading, MA, 1991.

Gray, J. W., *Mastering Mathematica: Programming Methods and Applications*, Academic Press, San Diego, 1994.

Green, E., Evans, B. and Johnson, J., *Exploring Calculus with Mathematica*, Wiley, New York, 1994.

Grimshaw, R., *Nonlinear Ordinary Differential Equations*, CRC Press, Boca Raton, 1991.

Gromak, V. I., *Painlevé Differential Equations in the Complex Plane*, Walter de Gruyter, Berlin, 2002.

Gromak, V. I. and Lukashevich, N. A., *Analytical Properties of Solutions of Painlevé Equations* [in Russian], Universitetskoe, Minsk, 1990.

Guckenheimer J. and Holms P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

Gustafsson, K., Control theoretic techniques for stepsize selection in explicit Runge–Kutta methods, *ACM Trans. Math. Soft.*, No. 17, pp. 533–554, 1991.

Hairer, E. and Lubich, C., On extrapolation methods for stiff and differential-algebraic equations, *Teubner Texte zur Mathematik*, Vol. 104, pp. 64–73, 1988.

Hairer, E., Symmetric projection methods for differential equations on manifolds, *BIT*, Vol. 40, No. 4, pp. 726–734, 2000.

Hairer, E., Lubich, C., and Roche, M., *The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta Methods*, Springer, Berlin, 1989.

Hairer, E., Lubich, C., and Wanner, G., *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer, Berlin, 2002.

Hairer, E., Nørsett, S. P., and Wanner, G., *Solving Ordinary Differential Equations I: Nonstiff Problems*, 2nd ed., Springer, Berlin, 1993.

Hairer, E. and Wanner, G., *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems*, 2nd ed., Springer, New York, 1996.

Hartman, P., *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

He, J.-H. and Wu, X.-H., Exp-function method for nonlinear wave equations, *Chaos, Solitons & Fractals*, Vol. 30, No. 3, pp. 700–708, 2006.

He, J.-H. and Abdou, M. A., New periodic solutions for nonlinear evolution equations using Exp-function method, *Chaos, Solitons & Fractals*, Vol. 34, No. 5, pp. 1421–1429, 2007.

Heck, A., *Introduction to Maple*, 3rd ed., Springer, New York, 2003.

Higham, N. J., *Functions of Matrices. Theory and Computation*, SIAM, Philadelphia, 2008.

Hill, J. M., *Solution of Differential Equations by Means of One-Parameter Groups*, Pitman, Marshfield, MA, 1982.

Hindmarsh, A. C., Odepack, a systemized collection of ODE solvers. In: *Scientific Computing* (Stepleman, R. S. et al., eds.), pp. 55–64, North-Holland, Amsterdam, 1983.

Hirota, C. and Ozawa, K., Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations: An application to the blow-up problems of partial differential equations, *J. Comput. & Applied Math.*, Vol. 193, No. 2, pp. 614-637, 2006.

Hosea, M. E. and Shampine, L. F., Analysis and implementation of TR-BDF2, *Appl. Numer. Math.*, Vol. 20, pp. 21–37, 1996.

Hubbard J. H. and West, B. H., *Differential Equations: A Dynamical Systems Approach. Part I. One Dimensional Equations*, Springer, New York, 1990.

Hull, T. E., Enright, W. H., Fellen, B. M., and Sedgwick, A. E., Comparing numerical methods for ordinary differential equations, *SIAM J. Numer. Anal.*, Vol. 9, pp. 603–637, 1972.

Hydon, P. E., *Symmetry Methods for Differential Equations: A Beginner's Guide*, Cambridge Univ. Press, Cambridge, 2000.

Ibragimov, N. H., *A Practical Course in Differential Equations and Mathematical Modelling*, Higher Education Press – World Scientific Publ., Beijing – Singapore, 2010.

Ibragimov, N. H. (Editor), *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 1, CRC Press, Boca Raton, 1994.

Ibragimov, N. H., *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley, Chichester, 1999.

Ibragimov, N. H., *Transformation Groups Applied to Mathematical Physics*, (English translation published by D. Reidel), Dordrecht, 1985.

Ince, E. L., *Ordinary Differential Equations*, Dover Publications, New York, 1956.

Iooss G. and Joseph D. D., *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York, 1997.

Its, A. R. and Novokshenov, V. Yu., *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*, Springer, Berlin, 1986.

Izhikevich, E. M., *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting*, MIT Press, 2007.

Jackson, L. K. and Palamides, P. K.,, An existence theorem for a nonlinear two-point boundary value problem, *J. Differential Equations*, Vol. 53, pp. 48–66, 1984.

Jacobi, C. G. J., *Vorlesungen über Dynamik*, G. Reimer, Berlin, 1884.

Kahaner, D., Moler, C., and Nash, S., *Numerical Methods and Software*, Prentice-Hall, New Jersey, 1989.

Kalitkin, N. N., *Numerical Methods* [in Russian], Nauka, Moscow, 1978.

Kalitkin, N. N., Alshin, A. B., Alshina, E. A., and Rogov, B. V., *Computation on Quasiuniform Meshes* [in Russian], Fizmatlit, Moscow, 2005.

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.

Kantorovich, L. V. and Akilov, G. P., *Functional Analysis in Normed Spaces* [in Russian], Fizmatgiz, Moscow, 1959.

Kantorovich, L. V. and Krylov, V. I., *Approximate Methods of Higher Analysis* [in Russian], Fizmatgiz, Moscow, 1962.

Karakostas, G. L., Nonexistence of solutions to some boundary-value problems for second-order ordinary differential equations, *Electron. J. Differential Equations*, No. 20, pp. 1–10, 2012.

Keller, H. B., *Numerical Solutions of Two Point Boundary Value Problems*, SIAM, Philadelphia, 1976.

Keller, J. B., The shape of the strongest column, *Archive for Rational Mechanics and Analysis*, Vol. 5, 275–285, 1960.

Keller J. and Antman S. (Eds.), *Bifurcation Theory and Nonlinear Eigenvalue Problems*, W. A. Benjamin Publ., New York, 1969.

Kevorkian, J. and Cole, J. D., *Perturbation Methods in Applied Mathematics*, Springer, New York, 1981.

Kevorkian, J. and Cole, J. D., *Multiple Scale and Singular Perturbation Methods*, Springer, New York, 1996.

Kierzenka, J. and Shampine, L. F., A BVP solver based on residual control and the MATLAB PSE, *ACM Trans. Math. Software*, Vol. 27, pp. 299–316, 2001.

Klein, F. and Sommerfeld, A., *Über die Theorie des Kreisels*, Johnson Reprint corp., New York, 1965.

Klimov, D. M. and Zhuravlev, V. Ph., *Group-Theoretic Methods in Mechanics and Applied Mathematics*, Taylor & Francis, London, 2002.

Korman, P. and Li, Y., Computing the location and the direction of bifurcation for sign changing solutions, *Dif. Equations & Applications*, Vol. 2, No. 1, pp. 1–13, 2010.

Korman, P. and Li, Y., On the exactness of an S-shaped bifurcation curve, *Proc. Amer. Math. Soc.,* Vol. 127, No. 4, pp. 1011–1020, 1999.

Korman, P., Li, Y., and Ouyang, T., Computing the location and the direction of bifurcation, *Math. Research Letters*, Vol. 12, pp. 933–944, 2005.

Korman, P., Global solution branches and exact multiplicity of solutions for two point boundary value problems. In: *Handbook of Differential Equations, Ordinary Differential Equations, Vol 3* (Canada A., Drabek P., and Fonda, A., eds.), Elsevier Science, North Holland, pp. 547–606, 2006.

Korn, G. A. and Korn, T. M., *Mathematical Handbook for Scientists and Engineers, 2nd Edition*, Dover Publications, New York, 2000.

Kostyuchenko, A. G. and Sargsyan, I. S., *Distribution of Eigenvalues (Self-Adjoint Ordinary Differential Operators)* [in Russian], Nauka, Moscow, 1979.

Kowalewsky, S., Sur le proble'me de la rotation d'un corps solide autor d'un point fixe, *Acta. Math.,* Vol. 12, No. 2, pp. 177–232, 1889.

Kowalewsky, S., Me'moires sur un cas particulies du proble'me de la rotation d'un point fixe, cu' l'integration s'effectue a' l'aide de fonctions ultraelliptiques du tems, *Me'moires pre'sente's par divers savants a' l'Acade'mie des seiences de l'Institut national de France*, Paris, Vol. 31, pp. 1–62, 1890.

Koyalovich, B. M., *Studies on the differential equation* $y\,dy - y\,dx = R\,dx$ [in Russian], Akademiya Nauk, St. Petersburg, 1894.

Krasnosel'skii, M. A., Vainikko, G. M., Zabreiko, P. P., et al., *Approximate Solution of Operator Equations* [in Russian], Nauka, Moscow, 1969.

Kreyszig, E., and Normington, E. J., *Maple Computer Manual for Advanced Engineering Mathematics*, Wiley, New York, 1994.

Kudryashov, N. A., Symmetry of algebraic and differential equations, *Soros Educational Journal* [in Russian], No. 9, pp. 104–110, 1998.

Kudryashov, N. A., Nonlinear differential equations with exact solutions expressed via the Weierstrass function, Zeitschrift fur Naturforschung, Vol. 59, pp. 443–454, 2004.

Kudryashov, N. A., *Analytical Theory of Nonlinear Differential Equations* [in Russian], Institut kompjuternyh issledovanii, Moscow–Izhevsk, 2004.

Kudryashov, N. A., Simplest equation method to look for exact solutions of nonlinear differential equations, *Chaos, Solitons and Fractals*, Vol. 24, No. 5, pp. 1217–1231, 2005.

Kudryashov, N. A., A note on the G'/G-expansion method, *Applied Math. & Computation*, Vol. 217, No. 4, pp. 1755–1758, 2010.

Kudryashov, N. A., One method for finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simulat.*, Vol. 17, pp. 2248–2253, 2012.

Kudryashov, N. A., Polynomials in logistic function and solitary waves of nonlinear differential equations, *Applied Math. & Computation*, Vol. 219, pp. 9245–9253, 2013.

Kudryashov, N. A., Logistic function as solution of many nonlinear differential equations, *Applied Math. & Modelling*, Vol. 39, No. 18, pp. 5733–5742, 2015.

Kudryashov, N. A. and Loguinova, N. V., Extended simplest equation method for nonlinear differential equations, *Applied Math. & Computation*, Vol. 205, pp. 396–402, 2008.

Kudryashov, N. A. and Sinelshchikov, D. I., Nonlinear differential equations of the second, third and fourth order with exact solutions, *Applied Math. & Computation*, Vol. 218, pp. 10454–10467, 2012.

Kuznetsov, Y. A., *Elements of Applied Bifurcation Theory, 3rd Edition*, Springer, New York, 2004.

Laetsch, T., The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.,* Vol. 20, pp. 1–13, 1970.

Lagerstrom, P. A., *Matched Asymptotic Expansions. Ideas and Techniques*, Springer, New York, 1988.

Lagrange, J. L., *Me'canique analyticque. Oeuvres de Lagrange*, Vol. 12, Gauthier–Villars, Paris, 1889.

Lambert, J. D., *Computational Methods in Ordinary Differential Equations*, Wiley, New York, 1973.

Lambert, J. D., *Numerical Methods for Ordinary Differential Systems*, Wiley, New York, 1991.

Lapidus, L., Aiken, R. C., and Liu, Y. A., The occurrence and numerical solution of physical and chemical systems having widely varying time constants. In: *Stiff Differential Systems* (R. A. Willoughby, editor), pp. 187–200, Plenum, New York, 1973.

Lee, H. J. and Schiesser, W. E., *Ordinary and Partial Differential Equation Routines in C, C++, Fortran, Java, Maple, and MATLAB*, Chapman & Hall/CRC Press, Boca Raton, 2004.

LePage, W. R., *Complex Variables and the Laplace Transform for Engineers*, Dover Publications, New York, 1980.

Levitan, B. M. and Sargsjan, I. S., *Sturm–Liouville and Dirac Operators*, Kluwer Academic, Dordrecht, 1990.

Lian, W.-C., Wong, F.-H., and Yen, C.-C., On the existence of positive solutions of nonlinear second order differential equations, *Proc. American Math. Society*, Vol. 124, No. 4, pp. 1117–1126, 1996.

Lin, C. C. and Segel, L. A., *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM, Philadelphia, PA, 1998.

Linchuk, L. V., On group analysis of functional differential equations [in Russian]. In: *Proc. of the Int. Conf MOGRAN-2000 "Modern Group Analysis for the New Millennium,"* pp. 111–115, USATU Publishers, Ufa, 2001.

Linchuk, L. V. and Zaitsev, V. F., Searching for first integrals and alternative symmetries [in Russian]. In: *Some Topical Problems of Modern Mathematics and Mathematical Education*, Proc. LXVIII International Conference "Herzen Readings – 2015" (13–17 April 2015, St. Petersburg, Russia), A. I. Herzen Russian State Pedagogical University, 2015, pp. 50–53.

Lubich, C., Linearly implicit extrapolation methods for differential-algebraic systems, *Numer. Math.*, Vol. 55, pp. 197–211, 1989.

MacCallum, M. A. H., Using computer algebra to solve ordinary differential equations. In: *Studies for Computer Algebra in Industry*, (Cohen, A.M., van Gastel L. J. and Verduyn Lunel, S. M., eds.) Vol. 2, John Wiley and Sons, Chichester, 1995.

MacDonald, N., *Biological Delay Systems: Linear Stability Theory*, Cambridge University Press, Cambridge, 1989.

Maeder, R. E., *Programming in Mathematica*, 3rd ed., Addison-Wesley, Reading, MA, 1996.

Malfliet, W., Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.,* Vol. 60, No. 7, pp. 650–654, 1992.

Malfliet, W. and Hereman, W., The tanh method: exact solutions of nonlinear evolution and wave equations, *Phys. Scripta,* Vol. 54, pp. 563–568, 1996.

Maplesoft, *Differential Equations in Maple 15*, Maplesoft, Waterloo, 2012.

Marchenko, V. A., *Sturm–Liouville Operators and Applications*, Birkhäuser Verlag, Basel–Boston, 1986.

Marchuk, G., Some applications of splitting-up methods to the solution of mathematical physics problems, *Aplikace Matematiky*, Vol. 13, pp. 103–132, 1968.

Markeev, A. P., *Theoretical Mechanics* [in Russian], Regular and Chaotic Dynamics, Moscow–Izhevsk, 2001.

Marsden J. E. and McCracken M., *Hopf Bifurcation and Its Applications*, Springer-Verlag, New York, 1976.

Matveev, N. M., *Methods of Integration of Ordinary Differential Equations* [in Russian], Vysshaya Shkola, Moscow, 1967.

McGarvey, J. F., Approximating the general solution of a differential equation, *SIAM Review*, Vol. 24, No. 3, pp. 333–337, 1982.

McLachlan, N. W., *Theory and Application of Mathieu Functions*, Clarendon Press, Oxford, 1947.

McLachlan, R. I. and Atela, P., The accuracy of symplectic integrators, *Nonlinearity*, Vol. 5, pp. 541–562, 1992.

Meade, D. B., May, M. S. J., Cheung, C-K., and Keough, G. E., *Getting Started with Maple*, 3rd ed., Wiley, Hoboken, NJ, 2009.

Mickens, R.E., *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.

Mikhlin, S. G., *Variational Methods in Mathematical Physics* [in Russian], Nauka, Moscow, 1970.

Meyer-Spasche, R., Difference schemes of optimum degree of implicitness for a family of simple ODEs with blow-up solutions, *J. Comput. and Appl. Mathematics*, Vol. 97, pp. 137–152, 1998.

Mikhlin, S. G. and Smolitskii, Kh. L., *Approximate Solution Methods for Differential and Integral Equations* [in Russian], Nauka, Moscow, 1965.

Miles, J. W., *Integral Transforms in Applied Mathematics*, Cambridge University Press, Cambridge, 1971.

Milne, E., Clarkson, P. A., and Bassom, A. P., Bäcklund transformations and solution hierarchies for the third Painlevé equation., *Stud. Appl. Math.,* Vol. 98, No. 2, pp. 139–194, 1997.

Moriguti, S., Okuno, C., Suekane, R., Iri, M., and Takeuchi, K., *Ikiteiru Suugaku — Suuri Kougaku no Hatten* [in Japanese], Baifukan, Tokyo, 1979.

Moussiaux, A., *CONVODE: Un Programme REDUCE pour la Résolution des Équations Différentielles*, Didier Hatier, Bruxelles, 1996.

Murdock, J. A., *Perturbations. Theory and Methods*, John Wiley & Sons, New York, 1991.

Murphy, G. M., *Ordinary Differential Equations and Their Solutions*, D. Van Nostrand, New York, 1960.

Nayfeh, A. H., *Perturbation Methods*, John Wiley & Sons, New York, 1973.

Nayfeh, A. H., *Introduction to Perturbation Techniques*, John Wiley & Sons, New York, 1981.

Nikiforov, A. F. and Uvarov, V. B., *Special Functions of Mathematical Physics. A Unified Introduction with Applications,* Birkhäuser Verlag, Basel–Boston, 1988.

Oberhettinger, F. and Badii, L., *Tables of Laplace Transforms*, Springer, New York, 1973.

Ockendon, J. R. and Taylor, A. B., The dynamics of a current collection system for an electric locomotive, *Proc. Royal Society of London A*, Vol. 322, pp. 447–468, 1971.

Olver, F. W. J., *Asymptotics and Special Functions*, Academic Press, New York, 1974.

Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W. (Editors), *NIST Handbook of Mathematical Functions*, NIST and Cambridge Univ. Press, Cambridge, 2010.

Olver, P. J., *Application of Lie Groups to Differential Equations*, Springer, New York, 1986.

Olver, P. J., *Equivalence, Invariants, and Symmetry*, Cambridge Univ. Press, Cambridge, 1995.

Olver, P. J., *Nonlinear Ordinary Differential Equations*, 2012 http://www-users.math.umn.edu/~olver/am_/odz.pdf

Ovsiannikov, L. V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

Painlevé, P., Mémoire sur les équations differentielles dont l'integrale générale est uniforme, *Bull. Soc. Math. France*, Vol. 28, pp. 201–261, 1900.

Paris, R. B., *Asymptotic of High Order Differential Equations*, Pitman, London, 1986.

Parkes, E. J., Observations on the tanh-coth expansion method for finding solutions to nonlinear evolution equations, *Appl. Math. Comp.*, Vol. 217, No. 4, pp. 1749–1754, 2010.

Pavlovskii, Yu. N. and Yakovenko, G. N., Groups admitted by dynamical systems [in Russian]. In: *Optimization Methods and Applications*, Nauka, Novosibirsk, pp. 155–189, 1982.

Petrovskii, I. G., *Lectures on the Theory of Ordinary Differential Equations* [in Russian], Nauka, Moscow, 1970.

Petzold, L. R., Automatic selection of methods for solving stiff and nonstiff systems of ordinary differential equations, *SIAM J. Sci. Stat. Comput.*, Vol. 4, pp. 136–148, 1983.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC Press, Boca Raton, 2002.

Polyanin, A. D., *Systems of Ordinary Differential Equations*, From Website *EqWorld—The World of Mathematical Equations*, 2006; http://eqworld.ipmnet.ru/en/solutions/sysode.htm.

Polyanin, A. D., Elementary theory of using invariants for solving mathematical equations, *Vestnik Samar. Gos. Univ., Estestvennonauchn. Ser.* [in Russian], No. 6(65), pp. 152–176, 2008.

Polyanin, A. D., Overdetermined systems of nonlinear ordinary differential equations with parameters and their applications, *Bulletin of the National Research Nuclear University MEPhI* [in Russian], Vol. 5, No. 2, pp. 122–136, 2016.

Polyanin, A. D. and Chernoutsan, A. I. (Eds.) *A Concise Handbook of Mathematics, Physics, and Engineering Sciences*, CRC Press, Boca Raton, 2011.

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations, 2nd Edition*, Chapman & Hall/CRC Press, Boca Raton, 2008.

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Mathematics for Engineers and Scientists*, Chapman & Hall/CRC Press, Boca Raton, 2007.

Polyanin, A. D. and Shingareva, I. K., Nonlinear blow-up problems: Numerical integration based on differential and nonlocal transformations, *Bulletin of the National Research Nuclear University MEPhI* [in Russian], Vol. 6, No. 4, pp. 282–297, 2017a.

Polyanin, A. D. and Shingareva, I. K., The use of differential and non-local transformations for numerical integration of non-linear blow-up problems, *Int. J. Non-Linear Mechanics*, Vol. 95, pp. 178–184, 2017b.

Polyanin, A. D. and Shingareva, I. K., Numerical integration of blow-up problems on the basis of non-local transformations and differential constraints, *arXiv:1707.03493 [math.NA]*, 2017c.

Polyanin, A. D. and Shingareva, I. K., Nonlinear problems with non-monotonic blow-up solutions: The method of non-local transformations, test problems, and numerical integration, *Bulletin of the National Research Nuclear University MEPhI* [in Russian], Vol. 6, No. 5, pp. 405–424, 2017d.

Polyanin, A. D. and Shingareva, I. K., Non-monotonic blow-up problems: Test problems with solutions in elementary functions, numerical integration based on non-local transformations, *Applied Mathematics Letters* (in press, doi: 10.1016/j.aml.2017.08.009), 2017e.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, Boca Raton, 1995.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition*, Chapman & Hall/CRC Press, Boca Raton, 2003.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Mathematical Physics Equations* [in Russian], Fizmatlit, Moscow, 2002.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC Press, Boca Raton, 2004.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations, 2nd Edition*, CRC Press, Boca Raton, 2012.

Polyanin, A. D., Zaitsev, V. F., and Moussiaux, A., *Handbook of First Order Partial Differential Equations*, Taylor & Francis, London, 2002.

Polyanin, A. D. and Zhurov, A. I., Parametrically defined nonlinear differential equations and their applications in boundary layer theory, *Bulletin of the National Research Nuclear University MEPhI* [in Russian], Vol. 5, No. 1, pp. 23–31, 2016a.

Polyanin, A. D. and Zhurov, A. I., Parametrically defined nonlinear differential equations and their solutions: Application in fluid dynamics, *Appl. Math. Lett.,* Vol. 55, pp. 72–80, 2016b.

Polyanin, A. D. and Zhurov, A. I., Functional and generalized separable solutions to unsteady Navier–Stokes equations, *Int. J. Non-Linear Mechanics*, Vol. 79, pp. 88–98, 2016c.

Polyanin, A. D. and Zhurov, A. I., Parametrically defined nonlinear differential equations, differential-algebraic equations, and implicit ODEs: Transformations, general solutions, and integration methods, *Appl. Math. Lett.,* Vol. 64, pp. 59–66, 2017a.

Polyanin, A. D. and Zhurov, A. I., Parametrically defined differential equations, *Journal of Physics: IOP Conf. Series*, Vol. 788, 2017b, 012078; http://iopscience.iop.org/article/10.1088/1742-6596/788/1/012028/pdf.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 1, Elementary Functions*, Gordon & Breach Sci. Publ., New York, 1986.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 4, Direct Laplace Transform*, Gordon & Breach, New York, 1992*a*.

Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series, Vol. 5, Inverse Laplace Transform*, Gordon & Breach, New York, 1992*b*.

Puu, T., *Attractions, Bifurcations, and Chaos: Nonlinear Phenomena in Economics*, Springer-Verlag, New York, 2000.

Rabier, P. J. and Rheinboldt, W. C., Theoretical and numerical analysis of differential-algebraic equations, *Handbook of Numerical Analysis*, Elsevier, Vol. 8, pp. 183–540, North-Holland, Amsterdam, 2002.

Ray J. R. and Reid, J. L., More exact invariants for the time dependent harmonic oscillator, *Phys. Letters*, Vol. 71, pp. 317–319, 1979.

Reid, W. T., *Riccati Differential Equations*, Academic Press, New York, 1972.

Reiss, E. L., *Bifurcation Theory and Nonlinear Eigenvalue Problems.*, W. A. Benjamin Publ., New York, 1969.

Reiss, E. L., Column buckling: An elementary example of bifurcation. In: *Bifurcation Theory and Nonlinear Eigenvalue Problems* (J. B. Keller and S. Antman, eds.), pp. 1–16, W. A. Benjamin Publ., New York, 1969.

Reiss, E. L. and Matkowsky, B. J., Nonlinear dynamic buckling of a compressed elastic column, *Quart. Appl. Math.,* Vol. 29, 245–260, 1971.

Richards, D., *Advanced Mathematical Methods with Maple*, Cambridge University Press, Cambridge, 2002.

Ronveaux, A. (Editor), *Heun's Differential Equations*, Oxford University Press, Oxford, 1995.

Rosen, G., Alternative integration procedure for scale-invariant ordinary differential equations, *Intl. J. Math. & Math. Sci.,* Vol. 2, pp. 143–145, 1979.

Rosenbrock, H. H., Some general implicit processes for the numerical solution of differential equations, *Comput. J.,* 1963, Vol. 5, No. 4, pp. 329–330.

Ross, C. C., *Differential Equations: An Introduction with Mathematica*, Springer, New York, 1995.

Russell, R. D. and Shampine, L. F., *Numerische Mathematik*, Bd. 19, No. 1, S. 1–28, 1972.

Sachdev, P. L., *Nonlinear Ordinary Differential Equations and Their Applications*, Marcel Dekker, New York, 1991.

Sagdeev, R. Z., Usikov, D. A., and Zaslavsky, G. M., *Nonlinear Physics: From the Pendulum to Turbulence and Chaos*, Harwood Academic Publ., New York, 1988.

Saigo, M. and Kilbas, A. A., Solution of one class of linear differential equations in terms of Mittag-Leffler type functions [in Russian], *Dif. Uravneniya*, Vol. 38, No. 2, pp. 168–176, 2000.

Sanz-Serna, J. M. and Calvo, M. P., *Numerical Hamiltonian Problems: Applied Mathematics and Mathematical Computation*, Chapman & Hall, London, 1994.

Schiesser, W. E., *Computational Mathematics in Engineering and Applied Science: ODEs, DAEs, and PDEs*, CRC Press, Boca Raton, 1994.

Shampine, L. F. and M. K. Gordon, M. K., *Computer Solution of Ordinary Differential Equations: the Initial Value Problem*, W. H. Freeman, San Francisco, 1975.

Shampine, L. F. and Watts, H. A., The art of writing a Runge–Kutta code I. In: *Mathematical Software III* (J. R. Rice, editor), Academic Press, New York, 1977.

Shampine, L. F. and Watts, H. A., The art of writing a Runge–Kutta code II, *Appl. Math. Comput.*, Vol. 5, pp. 93–121, 1979.

Shampine, L. F. and Gear, C. W., A user's view of solving stiff ordinary differential equations, *SIAM Review*, Vol. 21, pp.1–17, 1979.

Shampine, L. F. and Baca, L. S., Smoothing the extrapolated midpoint rule, *Numer. Math.*, Vol. 41, pp. 165–175, 1983.

Shampine, L. F., Control of step size and order in extrapolation codes, *J. Comp. Appl. Math.*, Vol. 18, pp. 3–16, 1987.

Shampine, L. F., *Numerical Solution of Ordinary Differential Equations*, Chapman & Hall/CRC Press, Boca Raton, 1994.

Shampine, L. F. and Reichelt, M. W., The MATLAB ODE Suite, *SIAM Journal on Scientific Computing*, Vol. 18, pp. 1–22, 1997.

Shampine, L. F., Reichelt, M. W., and Kierzenka, J., Solving index-1 DAEs in MATLAB and Simulink, *SIAM Rev.*, Vol. 41, No. 3, pp. 538552, 1999.

Shampine, L. F. and Corless, R. M., Initial value problems for ODEs in problem solving environments, *Journal of Computational and Applied Mathematics*, Vol. 125, No. 1–2, pp.31–40, 2000.

Shampine, L. F. and Thompson, S., Solving DDEs in MATLAB, *Appl. Numer. Math.*, Vol. 37, pp. 441–458, 2001.

Shampine, L. F., Gladwell, I., and Thompson, S., *Solving ODEs with MATLAB*, Cambridge University Press, Cambridge, UK, 2003.

Shingareva, I. K., *Investigation of Standing Surface Waves in a Fluid of Finite Depth by Computer Algebra Methods*, PhD thesis, Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 1995.

Shingareva, I. K. and Lizárraga-Celaya, C., *Maple and Mathematica. A Problem Solving Approach for Mathematics*, 2nd ed., Springer, Wien, New York, 2009.

Shingareva, I. K. and Lizárraga-Celaya, C., *Solving Nonlinear Partial Differential Equations with Maple and Mathematica*, Springer, Wien, New York, 2011.

Shingareva, I. K. and Lizárraga-Celaya, C., On different symbolic notations for derivatives, *The Mathematical Intelligencer*, Vol. 37, No. 3, pp. 33–38, 2015.

Simmons, G. F., *Differential Equations with Applications and Historical Notes*, McGraw-Hill, New York, 1972.

Sintsov, D. M., *Integration of Ordinary Differential Equations* [in Russian], Kharkov, 1913.

Slavyanov, S. Yu., Lay, W., and Seeger, A., *Heun's Differential Equation*, University Press, Oxford, 1955.

Sofroniou, M. and Spaletta, G., Construction of explicit Runge–Kutta pairs with stiffness detection, *Mathematical and Computer Modelling* (Special Issue on the Numerical Analysis of Ordinary Differential Equations), Vol. 40, No. 11–12, pp. 1157–1169, 2004.

Sofroniou, M. and Spaletta, G., Derivation of symmetric composition constants for symmetric integrators, *Optimization Methods and Software*, Vol. 20, No. 4–5, pp. 597–613, 2005.

Sofroniou, M. and Spaletta, G., Hybrid solvers for splitting and composition methods, *J. Comp. Appl. Math.* (Special Issue from the International Workshop on the Technological Aspects of Mathematics), Vol. 185, No. 2, pp. 278–291, 2006.

Stepanov, V. V., *A Course of Differential Equations, 7th Edition* [in Russian], Gostekhizdat, Moscow, 1958.

Sparrow, C., *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer, New York, 1982.

Stephani, H., *Differential Equations: Their Solution Using Symmetries* (edited by M. A. H. MacCallum), Cambridge University Press, New York, 1989.

Strang, G., On the construction of difference schemes, *SIAM J. Num. Anal.*, Vol. 5, pp. 506–517, 1968.

Stuart, M. and Floater, M. S., On the computation of blow-up, *European J. Applied Math.,* Vol. 1, No. 1, pp. 47–71, 1990.

Stephani, H., *Differential Equations: Their Solutions Using Symmetries*, Cambridge University Press, Cambridge, 1989.

Strang, G., On the construction of difference schemes, *SIAM J. Num. Anal.*, Vol. 5, pp. 506–517, 1968.

Svirshchevskii, S. R., Lie–Bäcklund symmetries of linear ODEs and generalized separation of variables in nonlinear equations, *Phys. Letters A*, Vol. 199, pp. 344–348, 1995.

Takayasu, A., Matsue, K., Sasaki, T., Tanaka, K., Mizuguchi, M., and Oishi, S., Numerical validation of blow-up solutions of ordinary differential equations, *J. Comput. Applied Mathematics*, Vol. 314, pp. 10–29, 2017.

Temme, N. M., *Special Functions. An Introduction to the Classical Functions of Mathematical Physics*, Wiley-Interscience, New York, 1996.

Tenenbaum, M. and Pollard, H., *Ordinary Differential Equations*, Dover Publications, New York, 1985.

Teodorescu, P. P., *Mechanical Systems, Classical Models: Volume II: Mechanics of Discrete and Continuous Systems*, Springer, Berlin, 2009.

Tikhonov, A. N., Vasil'eva, A. B., and Sveshnikov, A. G., *Differential Equations* [in Russian], Nauka, Moscow, 1985.

Trotter, H. F., On the product of semi-group operators, *Proc. Am. Math. Soc.*, Vol. 10, pp. 545–551, 1959.

Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.

Van Hulzen, J. A. and Calmet, J., Computer algebra systems. In: *Computer Algebra, Symbolic and Algebraic Manipulation*, (Buchberger, B., Collins, G. E., and Loos, R., eds.), 2nd ed., pp. 221–243, Springer, Berlin, 1983.

Vasil'eva, A. B. and Nefedov, H. H., *Nonlinear Boundary Value Problems* [in Russian], Lomonosov MSU, Moscow, 2006; http://math.phys.msu.ru/data/57/Nefmaterial.pdf.

Verner, J. H., Explicit Runge–Kutta methods with estimates of the local truncation error, *SIAM Journal of Numerical Analysis*, Vol. 15, No. 4, pp. 772–790, 1978.

Vinogradov, I. M. (Editor), *Mathematical Encyclopedia* [in Russian], Soviet Encyclopedia, Moscow, 1979.

Vinokurov, V. A. and Sadovnichii, V. A., Arbitrary-order asymptotic relations for eigenvalues and eigenfunctions in the Sturm–Liouville boundary-value problem on an interval with summable potential [in Russian], *Izv. RAN, Ser. Matematicheskaya*, Vol. 64, No. 4, pp. 47–108, 2000.

Vitanov, N. K. and Dimitrova, Z. I., Application of the method of simplest equation for obtaining exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics, *Commun. Nonlinear Sci. Numer. Simulat.*, Vol. 15, No. 10, pp. 2836–2845, 2010.

Vvedensky, D. D., *Partial Differential Equations with Mathematica*, Addison-Wesley, Wokingham, 1993.

Wang, D.-S., Ren, Y.-J., and Zhang, H.-Q., Further extended sinh-cosh and sin-cos methods and new non traveling wave solutions of the (2+1)-dimensional dispersive long wave equations, *Appl. Math. E-Notes*, Vol. 5, pp. 157–163, 2005.

Wang, M. L., Li, X., Zhang, J., The G'/G-expansion method and evolution equations in mathematical physics, *Phys. Lett. A*, Vol. 372, pp. 417–421, 2008.

Wang, S.-H., On S-shaped bifurcation curves, *Nonlinear Anal.,* Vol. 22, No. 12, pp. 1475–1485, 1994.

Wang, S.-H., On the evolution and qualitative behaviors of bifurcation curves for a boundary value problem, *Nonlinear Analysis*, Vol. 67, pp. 1316–1328, 2007.

Wasov, W., *Asymptotic Expansions for Ordinary Differential Equations,* John Wiley & Sons, New York, 1965.

Wazwaz, A. M., A sine-cosine method for handling nonlinear wave equations, *Math. and Computer Modelling,* Vol. 40, No. 5-6, pp. 499–508, 2004.

Wazwaz, A. M., The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations, *Appl. Math. Comput.,* Vol. 188, No. 2, pp. 1467–1475, 2007a.

Wazwaz, A. M., Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh-coth method, *Appl. Math. Comput.,* Vol. 190, No. 1, pp. 633–640, 2007b.

Wazwaz, A. M., Solitary wave solutions of the generalized shallow water wave (GSWW) equation by Hirota's method, tanh-coth method and Exp-function method, *Appl. Math. Comput.,* Vol. 202, No. 1, pp. 275–286, 2008.

Wazzan, L., A modified tanh-coth method for solving the KdV and the KdV–Burgers' equations, *Commun. Nonlinear Sci. and Numer. Simulation*, Vol. 14, No. 2, pp. 443–450, 2009.

Wester, M. J., *Computer Algebra Systems: A Practical Guide*, Wiley, Chichester, UK, 1999.

Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis*, Vols. 1–2, Cambridge University Press, Cambridge, 1952.

Wiens, E. G., *Bifurcations and Two Dimensional Flows*. In: Egwald Mathematics: http://www.egwald.ca/nonlineardynamics/bifurcations.php

Wiggins, S., *Global bifurcations and Chaos: Analytical Methods*, Springer-Verlag, New York, 1988.

Wolfram, S., *A New Kind of Science*, Wolfram Media, Champaign, IL, 2002.

Wolfram, S., *The Mathematica Book*, 5th ed., Wolfram Media, Champaign, IL, 2003.

Yanenko, N. N., The compatibility theory and methods of integration of systems of nonlinear partial differential equations. In: *Proceedings of All-Union Math. Congress*, Vol. 2, pp. 613–621, Nauka, Leningrad, 1964.

Yermakov, V. P., Second-order differential equations. Integrability conditions in closed form [in Russian], *Universitetskie Izvestiya*, Kiev, No. 9, pp. 1–25, 1880.

Zaitsev, O. V. and Khakimova, Z. N., Classification of new solvable cases in the class of polynomial differential equations [in Russian], *Topical Issues of Modern Science [Aktual'nye voprosy sovremennoi nauki]*, No. 3, pp. 3–11, 2014.

Zaitsev, V. F., Universal description of symmetries on a basis of the formal operators, *Proc. Intl. Conf. MOGRAN-2000 "Modern Group Analysis for the New Millennium,"* USATU Publishers, Ufa, pp. 157–160, 2001.

Zaitsev, V. F. and Huan, H. N., Analogues of variational symmetry of third order ODE, *Izvestia: Herzen University Journal of Humanities & Sciences*, No. 154, pp. 33–41, 2013.

Zaitsev, V. F. and Huan, H. N., Analogues of variational symmetry of the equation of the type $y''' = F(y, y', y'')$, *Izvestia: Herzen University Journal of Humanities & Sciences*, No. 163, pp. 7–17, 2014.

Zaitsev, V. F. and Linchuk, L. V., *Differential Equations (Structural Theory), Part I* [in Russian], A. I. Herzen Russian State Pedagogical University, St. Petersburg, 2015.

Zaitsev, V. F. and Linchuk, L. V., *Differential Equations (Structural Theory), Part II* [in Russian], A. I. Herzen Russian State Pedagogical University, St. Petersburg, 2009.

Zaitsev, V. F. and Linchuk, L. V., *Differential Equations (Structural Theory), Part III* [in Russian], A. I. Herzen Russian State Pedagogical University, St. Petersburg, 2014.

Zaitsev, V. F. and Linchuk, L. V., Six new factorizable classes of 2nd-order ODEs [in Russian], *Some Topical Problems of Modern Mathematics and Mathematical Education*, Proc. LXVIII International Conference "Herzen Readings – 2016" (11–15 April 2016, St. Petersburg, Russia), A. I. Herzen Russian State Pedagogical University, 2016, pp. 82–89.

Zaitsev, V. F., Linchuk, L. V., and Flegontov, A. V., *Differential Equations (Structural Theory), Part IV* [in Russian], A. I. Herzen Russian State Pedagogical University, St. Petersburg, 2014.

Zaitsev, V. F. and Polyanin, A. D., *Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics*, CRC Press, Boca Raton, 1994.

Zaitsev, V. F. and Polyanin, A. D., *Handbook of Nonlinear Differential Equations: Exact Solutions and Applications in Mechanics* [in Russian], Nauka, Moscow, 1993.

Zaitsev, V. F. and Polyanin, A. D., *Handbook of Ordinary Differential Equations* [in Russian], Fizmatlit, Moscow, 2001.

Zaslavsky, G. M., *Introduction to Nonlinear Physics* [in Russian], Fizmatlit, Moscow, 1988.

Zayed, E. M. E., The G'/G-method and its application to some nonlinear evolution equations, *J. Appl. Math. Comput.*, Vol. 30, pp. 89–103, 2009.

Zhang, H., New application of the G'/G-expansion method, *Commun. Nonlinear Sci. Numer. Simulat.,* Vol. 14, pp. 3220–3225, 2009.

Zhang, S., Application of Exp-function method to a KdV equation with variable coefficients, *Phys. Letters A,* Vol. 365, No. 5–6, pp. 448–453, 2007.

Zhuravlev, V. Ph., The solid angle theorem in rigid body dynamics, *J. Appl. Math. Mech.,* Vol. 60, No. 2, pp. 319–322, 1996.

Zhuravlev, V. Ph., *Foundations of Theoretical Mechanics* [in Russian], Fizmatlit, Moscow, 2001.

Zhuravlev, V. Ph. and Klimov, D. M., *Applied Methods in Oscillation Theory* [in Russian], Nauka, Moscow, 1988.

Zimmerman, R. L. and Olness, F., *Mathematica for Physicists*, Addison-Wesley, Reading, MA, 1995.

Zinchenko, N. S., *A Lecture Course on Electron Optics* [in Russian], Kharkov State University, Kharkov, 1958.

Zwillinger, D., *Handbook of Differential Equations, 3rd Edition*, Academic Press, New York, 1997.

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