Springer Texts in Education

Opher Liba
Bat-Sheva Ilany

From the Golden Rectangle to the Fibonacci Sequences

With Contributions by Isaac Native

Springer

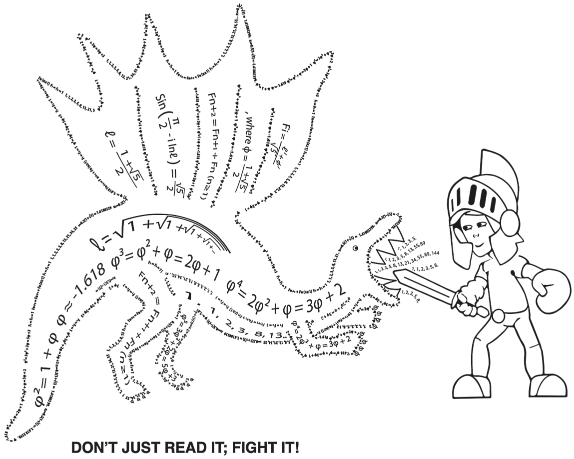
Springer Texts in Education

Springer Texts in Education delivers high-quality instructional content for graduates and advanced graduates in all areas of Education and Educational Research. The textbook series is comprised of self-contained books with a broad and comprehensive coverage that are suitable for class as well as for individual self-study. All texts are authored by established experts in their fields and offer a solid methodological background, accompanied by pedagogical materials to serve students such as practical examples, exercises, case studies etc. Textbooks published in the Springer Texts in Education series are addressed to graduate and advanced graduate students, but also to researchers as important resources for their education, knowledge and teaching. Please contact Yoka Janssen at Yoka.Janssen@springer.com or your regular editorial contact person for queries or to submit your book proposal.

Opher Liba · Bat-Sheva Ilany

From the Golden Rectangle to the Fibonacci Sequences

With Contributions by Isaac Nativ



DON'T JUST READ IT; FIGHT IT!
Paul R. Halmos

(► https://oritgigo6209.myportfolio.com)

Springer

Opher Liba
Jerusalem, Israel

Dr. Bat-Sheva Ilany
Academit Hemdat College
Netivot, Israel

Hebrew University of Jerusalem
Jerusalem, Israel

ISSN 2366-7672 ISSN 2366-7680 (electronic)
Springer Texts in Education
ISBN 978-3-030-97599-9 ISBN 978-3-030-97600-2 (eBook)
https://doi.org/10.1007/978-3-030-97600-2

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Opher Liba of blessed memory, 1953–2016

Dear readers,

Beyond the interest and pleasure that this book will surely produce, especially for Opher's family, the actual publication is a very moving event. Opher worked on the manuscript for many years but did not live to see it published.

Opher was a distinguished teacher of mathematics at all levels. Opher organized and convened several mathematics conferences and wrote numerous books and papers for both teachers and students of mathematics.

Opher was particularly interested in the Fibonacci sequences, researched it in depth and organized conferences that dealt with this topic.

Opher gave the Hebrew manuscript to Bat-Sheva several years ago. After Opher's death, Bat-Sheva decided to finish the Fibonacci manuscript, and add material, exercises and explanations. Later on, Bat-Sheva arranged the publication of the manuscript both in Hebrew and in English.

A number of friends volunteered to help with the work, led by Dr. Bat-Sheva Ilany: Isaac Nativ, Dr. Anatoly Starkman, Arie Rokach and Gidi Shenholz. Many other who read the manuscript gave invaluable feedback.

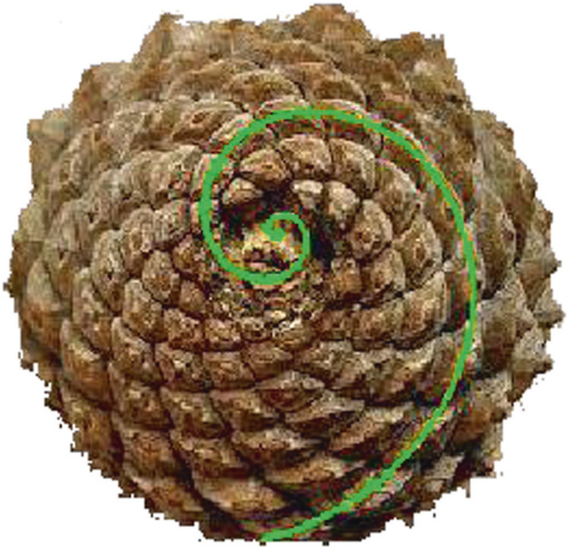
Special thanks to Anatoly Starkman for writing with Bat Sheva Ilany: Background on the Golden Rectangle and the Fibonacci Sequences, in the introduction and Chapter 6: In Opher's Footsteps—Challenges for Exploration.

Linda Yechiel and Isaac Nativ translated the manuscript to English.

Special thanks to Natalie Rieborn and to Lay Peng Ang of Springer for their infinite patience and dedication.

The book is dedicated to Opher's memory, and hopefully, it will inspire many students and lovers of mathematics.

Introduction



Fibonacci sequence of a pinecone
(Daniel Briskman)

The **Fibonacci sequence** is a sequence whose first two elements are 1,1, and each subsequent number equals the sum of its two predecessors (1, 1, 2, 3, 5, 8, 12, 21, 34, 55, …). This will be expanded later in the book.

This **sequence** is a mathematical **sequence** that miraculously can be discovered in many places in nature.

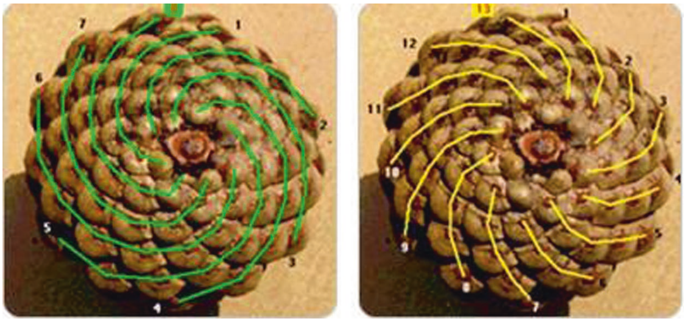
Patterns based on the Fibonacci sequence appear in nature in many and varied objects, such as snail shells, pinecones, the arrangement of leaves and branches on a stem, and more.

One of the arguments that has been suggested as a possible explanation is that the scales of a pinecone are arranged spirally so as to allow optimal exposure to the sun. If the angle between two adjacent scales would be 180 degrees for instance, the second scale would grow in the opposite direction, but the third scale would grow on top of the first and hide it from the sun. On the other hand, if the angle between the scales would be too small, the scales would overlap and shade one another.

The ratio between Fibonacci numbers approaches the golden ratio. If we divide 360 degrees by this number (which is approximately 1.618), we obtain an angle of about 222.5 degrees or 137.5 degrees in the opposite direction. This angle allows new scales to grow in the spaces between the scales below them. It seems that for this reason, the Fibonacci sequence often appears in plants. And this is the kind of reason to why it is associated with so many natural phenomena.

Note that the Fibonacci sequence can progress in either clockwise or counter-clockwise direction in the pinecone.

For more advanced discussion, read the following source: ► https://awkward-botany.com/2019/12/25/pine-cones-and-the-fibonacci-sequence



The two directions of the Fibonacci sequence in a pinecone (► https://papaitaly.files.wordpress.com/2014/02/blogpinecone2-e1391960557180.jpg)

This chapter includes:

A. How to read this book
B. Background information about the golden rectangle and the Fibonacci sequence by Dr. Bat-sheva Ilany and Dr. Anatoly Shtarkman
C. Foreword by Dr. Uzi Armon
D. Introductory problem: The Lewis Carol Paradox: Does $0 = 1$?

A. How to Read This Book

Opher Liba

This book is based on ongoing, independent research and is aimed at a wide range of readers: gifted high school students; undergraduate and graduate students, pre-service mathematics teachers in universities and colleges; practicing mathematics teachers; and, in general, anyone who loves mathematics. I would now like to address the two main groups of readers: learners and teachers.

A few words for the readers

This book has many educational objectives. Some of them I list here:

- to raise awareness of one of the more beautiful mathematical topics and to develop awareness of the beauty of mathematics in general,
- to reinforce and expand secondary-school mathematical knowledge and to allow a glimpse of some academic concepts,
- to demonstrate and instill values of mathematical research: observation, hypothesis, proof, application, and raising new questions,
- to deal with specific issues in more than one way (see, for example: sum of Fibonacci sequences in the books' chapters and exercises),
- to stimulate curiosity and the desire to explore and develop beyond that which appears in this book.

This book has some special didactic characteristics:

- The procession of the topics is from the tangible, to the abstract, to the general.
- The book includes a large number of exercises, stemming from the belief that varied practice is extremely valuable in learning mathematics, just as it is in learning to play an instrument. The aims of the exercises are for the learner to internalize the material taught in the various chapters, to refine and expand on selected issues, and to prepare a solid basis for future topics. **Most of the exercises were written especially for this book**. At the end of each of the exercise chapters, answers, hints, and partial solutions are offered.
- The book is meant to encourage self-study to develop the learner's knowledge. The learner must invest energy to elucidate subsequent stages and actively develop the proofs set forth in the book. This is done by asking the reader **to fill in plenty of details and complete lots of missing steps**. They were left out on purpose!

In general, it should be emphasized that reading a mathematical text is unlike reading a novel. It must be "accompanied by paper and pencil." As the famous mathematician Paul Halmos said: "Don't just read it; fight it!" (Halmos, 1985). Ask questions, look for patterns, discover your own proofs! Persistence and effort will lead you to more profound understanding, satisfaction, and pleasure!

- In the text and in some of the exercises, you will find many formulas. Do your best to try to visualize these formulas numerically, to "see" what they stand for. You can make use of the table at the end of the book ("Fibonacci and Lucas Numbers") as well as a calculator or mathematical software.
- There are many references to definitions and formulas in both the chapters and the exercises. These are all numbered and gathered together at the end of the book for your convenience.

Some comments for teachers/lecturers

In this address to teachers, I would like to turn the spotlight on the following aspects of the book:

- This book links most of the topics taught in advanced mathematics classes in high school.
- This book contains quite a number of original proofs (e.g. a simple, concise proof for Binet's formula using analytical geometry, a proof of some Fibonacci sequence properties by comparing rational coefficients, and more). These include φ-numbers, $\delta(a, b)$, $M(a, b)$ matrices, the Lucas-like sequence of the Fibonacci-like sequence, the "meta" sequence of the Fibonacci-like series, and other original results.
- Special emphasis has been placed on the close relationship between the sequences discussed and between the golden ratio and its powers. The analysis of the Lucas sequence and the Fibonacci-like sequences (► Chaps. 4 and 5) is systematically extended.

When presenting this book to your students, you should:

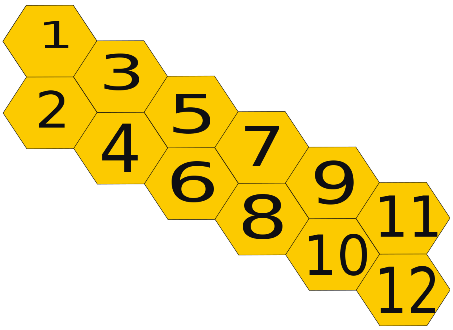
- Try to encourage your students to tackle as many exercises as possible, as well as to complete the details and steps missing in the text.
- Be sure to praise and encourage original proofs and novel approaches to dealing with the same problem.
- If this is your students' first experience in studying post-secondary school material, this is a wonderful opportunity to insist that they write their proofs properly, both mathematically and linguistically. This will make them aware of the importance of the matter and how proper writing exhibits respect toward mathematics. It is also extremely beneficial for the learners themselves, since precise, lucid writing reinforces proper, clear thinking.

This book is structured so that the topics can be taught in various formats:

- A comprehensive course on the golden ratio and the Fibonacci and Lucas sequences: ► Chaps. 1–4.
- A more comprehensive course: this will include ► Chap. 5, which requires special effort.
- An interdisciplinary cultural course: This will include featured content from this book plus specific applications from the natural and life sciences, technology, art and architecture, and perhaps even some historical background. Some sources for such content appear in the bibliography and you can find many sites on the Internet (a selected list of which appears at the end of the book).

Reference

Halmos, P. R. (1985). *I want to be a mathematician*. Washington: Mathematical Association of America Spectrum.



Quiz: Starting from the 1 at the top, in how many ways can you reach the number 6, always moving downward? (► https://he.wikipedia.org/wiki/%D7%A1%D7%93%D7%A8%D7%AA_%D7%A4%D7%99%D7%91%D7%95%D7%A0%D7%90%D7%A6%27%D7%99#/media/File:FibHive.svg)

B. Background on the Golden Rectangle and the Fibonacci Sequences

Bat-sheva Ilany and Anatoly Starkman

Much has been written about the connection between the golden rectangle (often described as the golden ratio or the golden mean, the golden section, the divine ratio, etc.) and the Fibonacci sequence.

The Fibonacci numbers have many interesting properties. Entire books have been written about them and there is even a mathematical journal, *The Fibonacci Quarterly* (► http://www.fq.math.ca), that is entirely devoted to discoveries and generalizations of the Fibonacci numbers. In addition, the Fibonacci Association was established (► http://www.mathstat.dal.ca/fibonacci), whose goal is to discover new results, problems, and proofs pertaining to the Fibonacci sequence.

This book presents a unique mathematical perspective on the golden ratio and Fibonacci numbers. It begins with the golden ratio and goes on from that to the Fibonacci sequences and series.

The book begins with a brief glimpse at the history of Fibonacci and the definitions of Fibonacci numbers and the golden section.

Leonardo Fibonacci was born circa 1170 in Pisa, Italy. He received a broad education in the subjects of arithmetic, geometry, astronomy, logic, etc. Fibonacci traveled with his father, a tax and commission collector for imported goods. During those travels, Fibonacci attended universities in Bagdad, Egypt, Syria, and Spain, studied Greek mathematics, and learned about Arabic culture and its important contributions to mathematics. In 1192, while in Algeria, he learned about the Hindu-Arab numeral system invented by Muḥammad ibn Mūsā al-Khwārizmī that was not yet recognized in Europe. (al-Khwārizmī is sometimes called Algoritmi or Algorismi. He is considered the father of modern algebra.) His system used the Arabic numeral digits 0–9 and introduced a place numbering system (up till then, numbers were written using Roman numerals). On Fibonacci's return to Pisa in approximately 1200, he taught mathematics until about 1230s.

Around 1200, Fibonacci presented a problem about breeding rabbits from which the Fibonacci sequence was derived. Fibonacci did it as a stimulating mathematical exercise, but eventually it was discovered that the Fibonacci numbers occur in far ranging aspects of nature, such as in the dimensions of the DNA helix, in plants and flowers such as the sunflower, and so on. The average relative distances between the planets in the solar system and the sun itself are approximately equal to the golden section. More information can be found in the site: ► https://www.goldennumber.net/solar-system/

Fibonacci, who wrote five books about mathematics, was primarily interested in number theory. He was the first in Europe to describe the sequence that is named after him, which he did in his book *Liber Abacci* (The Book of Calculation), published in 1202. (The sequences had already been discovered in India as early as the sixth century).

The riddle regarding the breeding of rabbits appears in the book and goes like this:

A pair of young rabbits, which we call “bunnies,” are introduced into a closed cage. Rabbits reach maturity at two months and at the end of the second month, a pair will begin to produce offspring, producing a pair (male and female) of bunnies at the end of each month. Each pair of rabbits, at two months old, also begins producing a pair of bunnies at the end of each month, which, at the end of their second month, also begin to reproduce, and so on and so forth. Assuming the rabbits never die, how many pairs of rabbits will be in the cage at the beginning of every month?

The answer is as follows: At the beginning of the first and second months, there will be one pair of young rabbits in the cage. At the beginning of the third month, there will be two pairs (the original pair plus the pair that was born at the end of the second month). At the beginning of the fourth month, there will be three pairs (the original pair has produced another pair at the end of the third month, but the second pair has not yet begun to reproduce). At the beginning of the fifth month, there will be five pairs and so on.

If we write the number of pairs of rabbits in the cage as a sequence of numbers, we get 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

As Fibonacci pointed out in his book, this is a unique sequence because every number in it is the sum of the two preceding numbers.

The number of rabbits in successive generations (Surprising patterns could be seen in the table. For example: 3 + 3 = 6; 15 + 6 = 21 as highlighted in the table)

Month	Generation							Total no. of pairs of rabbits (Results in Fibonacci sequence)
	1	2	3	4	5	6	7	
1	1							1
2	1							1
3	1	1						2
4	1	2						3
5	1	3	1					5
6	1	4	3					8
7	1	5	6	1				13
8	1	6	10	4				21
9	1	7	15	10	1			34
10	1	8	21	20	5			55
11	1	9	28	35	15	1		89
12	1	10	36	56	35	6		144
13	1	11	45	84	70	21	1	233
14	1	12	55	120	126	56	7	377

In other words, the **Fibonacci sequence** is a sequence whose first two elements are 1,1, and each subsequent number equals the sum of its two predecessors.

The recursive definition of the sequence appears in Chap. 3, The Fibonacci Sequence. Although the sequence was already known to Indian mathematicians, the French mathematician Édouard Anatole Lucas (1891–1842) named it the Fibonacci sequence. Lucas also discovered some interesting properties of the sequence (see Chap. 4: The Lucas Sequence).

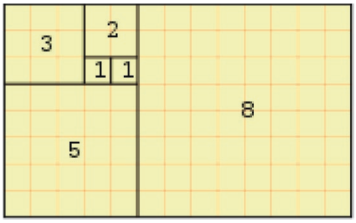
The ratio between two consecutive elements of the Fibonacci sequence approaches the golden ratio, which is an irrational number with value 1.61803398…, as first shown by Johannes Kepler. The golden ratio is already closely approximated for the 11th element of the Fibonacci sequence: 155/89 = 1.6181818, and with the 16th element, the accuracy improves to 5 decimal places: 987/610 = 1.61803 ....

The Fibonacci sequence can be represented geometrically. One begins with a square of side length 1 for the first element, and squares are added to the sequence as follows:

Another 1×1 square is placed next to the first one. These two squares create a rectangle that is 1 square high and two squares wide (the elements 1, 1, and 2 in the sequence).

We next add a 2×2 square to get a 2×3 rectangle, and then the addition of a 3×3 square leads to a 3×5 rectangle. The construction continues thus, each time adding a square, the side lengths of which are equal to the next element in the Fibonacci sequence, and producing a rectangle whose sides represent two consecutive Fibonacci elements. As the construction continues, the rectangles approach the exact proportions of the golden rectangle, which is a rectangle where the ratio of its sides produce the golden ratio—which is approximately 1.618.

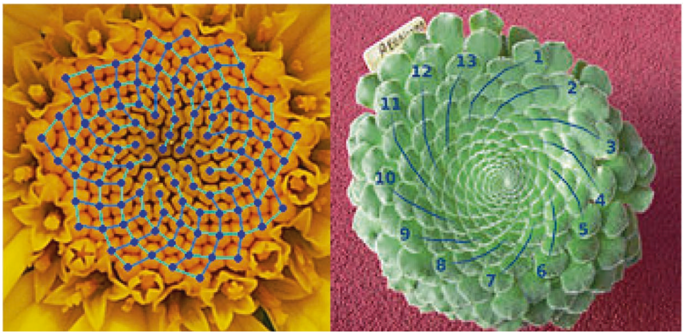
Both the golden ratio and the golden rectangle appear in nature, architecture, and art.



Sequence rectangles
(► https://he.wikipedia.org/wiki/%D7%A1%D7%93%D7%A8%D7%AA_%D7%A4%D7%99%D7%91%D7%95%D7%A0%D7%90%D7%A6%27%D7%99#/media/File:FibonacciBlocks.svg)

Fibonacci cubes
(► https://he.wikipedia.org/wiki/%D7%A1%D7%93%D7%A8%D7%AA_%D7%A4%D7%99%D7%91%D7%95%D7%A0%D7%90%D7%A6%27%D7%99#/media/File:Diepholz_Skulpturenpfad_Fibonacci.JPG)

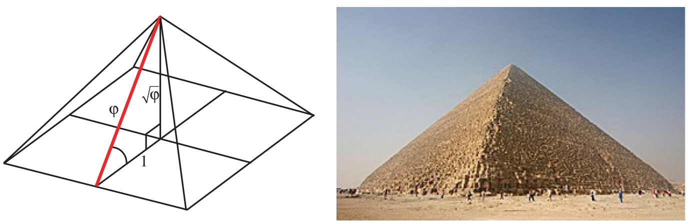
The ratios of successive cube lengths approximate the Fibonacci sequence.



Fibonacci sequence in nature
(► https://en.wikipedia.org/wiki/Fibonacci_number#/media/File:FibonacciChamomile.PNG; ► https://he.wikipedia.org/wiki/%D7%A1%D7%93%D7%A8%D7%AA_%D7%A4%D7%99%D7%91%D7%95%D7%A0%D7%90%D7%A6%27%D7%99#/media/File:Aeonium_tabuliforium_2_spirals_13.jpg)

It has been claimed that the use of the golden ratio was already apparent in the architecture of ancient Greece (► https://en.wikipedia.org/wiki/Golden_ratio). Some experts believe that the proportions of several classical structures, such as the Parthenon in Athens, were deliberately designed to represent the golden ratio. However, other experts have challenged that claim, stating that measurements can be taken in many ways, and that the golden ratio was artificially super-imposed on those ancient structures.

There are also some who claim that the golden ratio can be observed in the Great Pyramid of Giza, and the ratio of its height to its base is 1.618. Furthermore, they claim, the ratio of the distance from the head to the feet and between the distance from the navel to the feet is 1.618. Some also have brought evidence of the golden ratio in the Ark of the Covenant, whose ratio of length to breadth is 1.66 (2.5 amot × 1.5 amot).

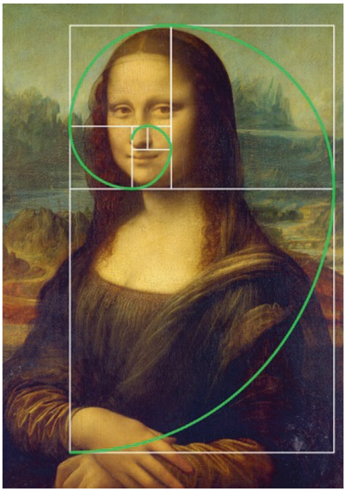


The Great Pyramid of Giza
(► https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza#/media/File:Kheops-Pyramid.jpg)

In the Middle Ages, the golden ratio was used in Islamic architecture. For example, it is believed that the Dome of the Rock in Jerusalem was built in proportions that approach the golden ratio, although, when speaking about the golden ratio, it is difficult to prove.

In art, the golden ratio is considered to be the most perfect and pleasing proportion imaginable. Renaissance painters, sculptors, and architects utilized the golden mean to arrive at what was considered perfect beauty. Golden ratios can be found in abundance in the paintings of Leonardo da Vinci and other great artists. Renaissance scholars Piero della Francesca and Fra Luca Bartolomeo de Pacioli both wrote books about the golden ratio. Pacioli's book, *De divina proportione* (The Divine Proportion), was written in Milan in 1496–1498 and published in Venice in 1509. Leonardo da Vinci did the illustrations.

In Leonardo da Vinci's most famous painting, the *Mona Lisa*, the golden ratio can be precisely measured on the Mona Lisa's face and over the entire painting. Also, da Vinci used the golden ratio in his 1483 painting *Saint Jerome in the Wilderness.* The painting depicts a lion at the feet of a sitting Saint Jerome. Saint Jerome himself is enclosed in a golden rectangle.



Mona Lisa
(Michael Paukner/substudio.com)

Today, the golden rectangle is still abundant in architecture and art. Several modern buildings have been constructed according to the golden ratio. For example, the ratio of the height (152 ft) of the UN building in New York to its breadth (95 m) is 1.621, which is very close to the golden ratio. The dimensions of credit cards and other magnetic cards also approach those of the golden rectangle.

Fibonacci in architecture—a spiral structure in the Vatican's staircase
(► http://www.goldenmuseum.com/index_engl.html)

It is interesting to point out that there many unsolved questions arising from the Fibonacci sequence. For example, are there infinitely many primes in the Fibonacci sequence?

You can find a full annotated bibliography at the end of the book.

Further Reading

- Herz-Fischler, Roger (1987). *A Mathematical History of Division in Extreme and Mean Ratio.* Reprint, 1998 by Dover. Mineola, New York: Dover Publications.
- Livio M. (2002). *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*. Broadway Books.

C. Foreword by Dr. Uzi Armon

Opher Liba's book takes us on a fascinating journey through one of the most beautiful and fascinating areas of mathematics. It presents a wealth of information about the golden ratio and the Fibonacci sequence. It begins by introducing the golden ratio, $\varphi = \frac{1}{2}(1+\sqrt{5})$ and its properties, such as $\varphi^2 = \varphi + 1$ or $1 - \varphi = \frac{-1}{\varphi}$, $\varphi + \frac{1}{\varphi} = \sqrt{5}$. From there, he introduces us to the golden triangles (isosceles triangles in where the ratio between two different sides is equal to φ) and their properties, such as their existence in a regular pentagon. Continuing on, we arrive at topics which range from the Fibonacci and Lucas sequences to the Fibonacci-like and Lucas-like sequences. These sequences have diverse properties, such as the Cassini formula, $F_{n+1}F_{n-1} = F_n^2 + (-1)^n$, or the formulas

$L_n = F_{n+1} + F_{n-1}$ and $F_{2n} = L_nF_n$. They also have a number of surprising relationships with the golden ratio, such as $\varphi^n = F_n\varphi + F_{n-1}$.

Additionally, this book offers a fascinating way of looking at mathematics as it is taught in schools, with emphasis on the fundamental principles of genuine mathematics, which include aesthetics (for example, in geometrical shapes), problems that invite a variety of approaches and solutions, research approach (discovering laws, generalizations, and relationships), cerebral challenges, and problem solving, with an emphasis on reasoning. Opher always believed that beauty and elegance are essential properties of mathematics.

Beyond its presentation of the abundance and assortment of properties related to the golden ratio and the Fibonacci sequence, this book's great contribution is in creating an environment for learning mathematics that is rich with activities and challenges. An extensive learning environment is a powerful educational concept that improves overall learning and enhances and refreshes the learning of mathematics in particular. Such an environment is necessary to achieve two ultimate and interrelated educational goals. One is the **freedom to learn**, as the title of a book by Carl Rogers. A learning environment with numerous activities provides each individual learner the freedom to choose the way that best suits his or her learning, the topics of most interest, and the problems that intrigue the learner, all according to their own personal inclinations. Freedom to learn is vital, both to foster the creativity inherent in each and every student, to improve their motivation, and to make real learning enjoyable. **Real learning** is the second educational goal. Real learning is expressed by the specific, independent activities that the student takes personal responsibility to perform. Real learning is achieved through a myriad of activities that include reading relevant material, investigating the topic under study, coping with cognitive tasks, comparing their solution to prepared ones, and more. Real learning in an activity-rich learning environment allows internalizing knowledge and making it understandable and meaningful.

Indeed, this book has the potential to help transform mathematics education and learning. It encourages basing the learning of a mathematical topic on reading a mathematical text, an approach that is not very common. Yet such an approach can lead to a real understanding of the mathematical concepts involved. For example, investigating sequences that are related to the Fibonacci sequence can improve understanding the concepts of sequences and series. Today many consider a "sequence" to be an abstract concept (an ordered set of numbers), while the concept of a "series" is one that deals with the sum of the sequence. In fact, the term "sum of the sequence" refers to the sequence of partial sums of the sequence, and this sequence is a special case of a sequence that is formed from some sequence. All sequences produce many "offspring": a sequence of differences (in an arithmetic sequence this is a sequence of constants), a sequence of quotients (in a geometric sequence, it is a sequence of constants, and in a Fibonacci sequence it is a sequence that tends to a limit—the golden ratio), a sequence of squares, a sequence of the product of two adjacent values, and more.

In a manner befitting the topic of "golden," Opher Liba's book excels in its richness. This wealth is expressed both in the variety of mathematical topics woven into the book, in the multitude of activities offered (some described in the chapters and some presented as exercises), and also by the relationships between various areas of mathematics as expressed in one of the most beautiful, enchanting, and astounding mathematical subjects.

The late Uzi Armon was a faculty member of Kinneret College, an extension of Bar Ilan University.

D. Introductory Problem: The Lewis Carol Paradox: Does 0 = 1?

Let us cut a square with sides of 8 units into two identical triangles and two identical trapezoids, as illustrated in Fig. 1a. Let us then recombine them to form a 5×13 rectangle, as illustrated in Fig. 1b.

Now, the area of the square is 64 square units but the area of the rectangle is 65 square units, yet both shapes are constructed from exactly the same pieces. This seems to imply that $64 = 65$, or, in other words, that $0 = 1$! A contradiction.

This visual paradox has two aspects:

1. What's the trick? This can be answered relatively easily.
2. Why does this paradox work so well? The answer to this requires some deep understanding of the Fibonacci sequence (see ► Chap. 3).

Notice that the lengths of the sides adjacent to the right angles, in all the shapes in the illustrations—the triangles, the trapezoids, the square, and the rectangle—are 3, 5, 8, or 13 units. Notice, too, that this is a quadruplet of sequential numbers in the Fibonacci sequence (4th to 7th). Moreover, the further up the sequence the quadruplet of Fibonacci numbers are, the more "perfect" the illusion.

There is an interesting connection between the paradox and the Cassini formula:

$$\mathbf{F_{n+1}F_{n-1} - F_n{}^2 = (-1)^n}$$

F_n is the n-th element of the Fibonacci sequence, which we shall meet in ► Chap. 3. The paradox also stems from the fact that:

$$\mathbf{lim_{n\to\infty}F_{n+1}/F_n} = \varphi$$

where φ is the golden ratio, which we shall meet already in ► Chap. 1.

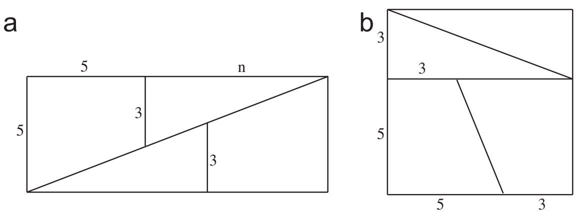


Fig. 1 The Lewis Carol Paradox

Contents

The Golden Rectangle and the Golden Ratio

Contents

Algebra is but written geometry and geometry is but figured algebra.
Sophie Germain (1776–1831)

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*,
Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2_1

The Parthenon

(► https://en.wikipedia.org/wiki/Parthenon#/media/File:The_Parthenon_in_Athens.jpg)

A Greek temple that stands on the Acropolis of ancient Athens. It is considered the most famous building of ancient Greece. The Parthenon's design is based on the golden ratio.

■ Introduction to Chapter 1

At the beginning of our journey, let us introduce the golden rectangle and the golden ratio, both of which will have enormous importance throughout this book. In ► Sect. 1.3, we will define several mathematical tools—commutative group, matrices, isomorphism, and norms—for more in-depth analysis and to prepare the background for Chaps. 3, 4 and 5.

The exercises that follow each chapter are meant to develop mathematical skills related to the golden ratio. We shall also apply this ratio to various topics taught in high school mathematics: algebra, analysis, geometry, analytic geometry, complex numbers, and so forth. (We shall leave trigonometry for ► Chap. 2.)

1.1 The Golden Rectangle and the Golden Ratio

When we analyze a rectangle numerically, we are generally looking at the lengths of its sides, the length of its perimeter, and its area.

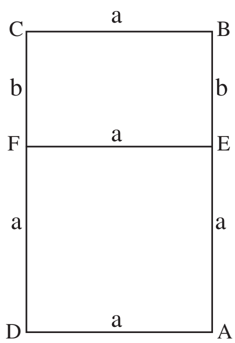


Fig. 1.1

In this section, we shall focus on the ratio of the sides.

We shall attempt to find a rectangle, ABCD, such that if we remove the square AEFD from it, we will be left with rectangle EBCF such that the ratio of the length to width of the original rectangle, ABCD, will be equal to the ratio of the length and width of the resulting rectangle, EBCF. In other words, the two rectangles are similar:

$\frac{AB}{AD} = \frac{BC}{EB}$ (See ◘ Fig. 1.1)

We denote: $AE = EF = FD = DA = BC = a$

$EB = FC = b$

As stated above, we want the following to hold: $\frac{a+b}{a} = \frac{a}{b}$

We rearrange the equation to highlight the $\frac{a}{b}$ ratio: $ab + b^2 = a^2$

Divide both sides by b^2

$$\frac{a}{b} + 1 = \left(\frac{a}{b}\right)^2$$

By denoting $x = \frac{a}{b}$, we have attained the **golden equation**:

$$\boldsymbol{x^2 = x + 1} \tag{1.1a}$$

This is a quadratic equation whose solutions are:

$$x_1 = \tfrac{1}{2}(1 + \sqrt{5}) \approx 1.61803399$$
$$x_2 = \tfrac{1}{2}(1 - \sqrt{5}) \approx -0.61803399$$

We denote the positive solution (the one relevant to our task at this point) by the symbol φ (the Greek letter "phi", which is the first letter of the name of Greek sculptor Phidias, who lived between 490 and 431 BC). Hence:

$$\boldsymbol{\varphi = \frac{1+\sqrt{5}}{2}} \tag{1.2}$$

We call φ the **golden ratio**, (or the **golden ratio**), and φ satisfies the golden equation, i.e.:

$$\boldsymbol{\varphi^2 = \varphi + 1} \tag{1.1b}$$

According to the Vieta's formula, the product of the solutions is −1 and their sum is 1. Therefore, the negative solution is $\frac{-1}{\varphi}$ or $1 - \varphi$.

$1 - \varphi$ is the other solution of the golden equation. Accordingly, we can write:

$$\boldsymbol{\frac{-1}{\varphi} = 1 - \varphi = \frac{1}{2}(1 - \sqrt{5})} \tag{1.3}$$

This number also satisfies the golden equation, i.e.:

$$\boldsymbol{\left(\frac{-1}{\varphi}\right)^2 = \frac{-1}{\varphi} + 1} \tag{1.4a}$$

1

or:

$$\mathbf{(1-\varphi)^2 = (1-\varphi) + 1} \tag{1.4b}$$

Rectangle ABCD (as well as all rectangles that are similar to it) is defined as the **golden rectangle**.

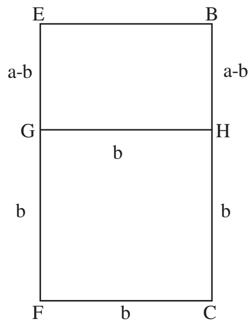


Fig. 1.2

To further test our results, we note that rectangle EBCF is also a golden rectangle (which is what we sought from the beginning). If we divide it into a square and a rectangle, as shown in Fig. 1.2, and calculate the ratio of length to width of rectangle EBHG, we find that:

$$\frac{EB}{BH} = \frac{b}{a-b} = \frac{1}{\frac{a}{b}-1} = \frac{1}{\varphi - 1} = \varphi$$

We shall now see how to construct a golden rectangle out of a square (using compass and straightedge).

Let point M be the midpoint of side DC in the square ABCD. With the point of the compass on M, draw an arc that passes through vertex B until it meets the continuation of side DC at point F. Complete the construction by constructing a perpendicular to DF through point F until it meets the continuation of AB at point E (Fig. 1.3).

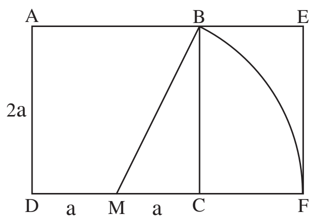


Fig. 1.3

We shall now prove that rectangle AEFD is, indeed, a golden rectangle. We denote $2a = AB = BC = CD = DA$, and use the Pythagoras theorem for triangle MBC:

$$MB^2 = MC^2 + BC^2 = a^2 + (2a)^2 = 5a^2.$$

From here we get:

$$MB = MF = a\sqrt{5}.$$

The length of the rectangle is:

$$DF = DM + MF = a + a\sqrt{5} = 2a \cdot \tfrac{1}{2}(1 + \sqrt{5}) = 2a\varphi,$$

and therefore the ratio between the length and the width of the rectangle is:

$$\frac{DF}{AD} = \frac{2a\varphi}{2a} = \varphi.$$

Q.E.D.

An accepted alternative definition of the golden ratio (ratio) is that it is the ratio of lengths obtained by cutting away a given portion of a line segment in such a way that the ratio between the length of the entire line segment to the length of the longest segment (created by the cut) will be equal to the ratio between the lengths of the longer and shorter segments. In other words (see ◘ Fig. 1.4):

$$\frac{AB}{AE} = \frac{AE}{EB}$$

◘ **Fig. 1.4**

If we denote $AE = a,\ EB = b$, we obtain: $\frac{a+b}{a} = \frac{a}{b}$. Therefore, $\frac{a}{b} = \varphi$.

We say that point E "divides (cuts) segment AB into the golden ratio." (This alternative definition is equivalent to the first definition; it already appeared in the writings of Euclid.)

Based on the fact that $\sqrt{5}$ is known to be an irrational number it is possible to deduce that the golden ratio is **irrational**. Nevertheless, we are now going to present a direct proof of the irrationality of φ.

Notice that according to ◘ Figs. 1.1 and 1.2, that the following conditions hold:

$$a > b$$

$$b > a - b \quad (\Leftrightarrow a < 2b).$$

In other words:

$$b < a < 2b.$$

Now, assume that one can write $\varphi = \frac{a}{b}$ where a and b are integers, and also assume that this fraction has been reduced "all the way" (in other words, it has the lowest possible numerator and denominator).
From our construction of the golden rectangle, we have:

$$\frac{a}{b} = \frac{b}{a-b},$$

but (as previously noted) $b < a$ and also $a - b < b$. In other words: the fraction on the right has a **smaller** numerator and denominator than that on the left, contradicting the assumption.

1.2 Quantitative Attributes of φ

We will first present two interesting methods of denoting the golden ratio. We begin with the equation:

$$\varphi^2 = \varphi + 1 \tag{1.1b}$$

and obtain the square root of each side (remember that φ is a positive number):

$$\varphi = \sqrt{\varphi + 1} = \sqrt{1 + \varphi}.$$

If we repeat this step recursively (repeat it over and over), we obtain:

$$\varphi = \sqrt{1+\varphi} = \sqrt{1+\sqrt{1+\varphi}} = \sqrt{1+\sqrt{1+\sqrt{1+\varphi}}} = \dots$$

Now, we define a sequence (a_n) as recursive as follows:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{1+a_n} \quad (n \geq 1) \end{cases}$$

and:

$$\begin{aligned} a_2 &\approx 1.414214 \\ a_3 &\approx 1.553774 \\ a_4 &\approx 1.598053 \\ a_5 &\approx 1.611848 \\ a_6 &\approx 1.616121 \\ a_7 &\approx 1.617443 \\ a_8 &\approx 1.617851. \end{aligned}$$

Note that:

$$|a_8 - \varphi| \approx 0.000183$$

It is possible to formally prove that the sequence (a_n) has a limit (it is a rising monotonic that is bounded at the upper level), and that this limit is indeed φ. This can be commonly written as:

$$\lim_{n\to\infty} a_n = \varphi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$

This last equation is called **the infinite nested radicals.** It uses only the number 1 and two arithmetic functions: addition and the square root. Divinely elegant!
We begin again with the equation

$$\varphi^2 = \varphi + 1 \tag{1.1b}$$

Now, let us divide both sides by φ. We obtain:

$$\varphi = 1 + \frac{1}{\varphi}$$

By performing recursive substitutions (over and over), we obtain:

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \frac{1}{1+\frac{1}{\varphi}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}} = \ldots$$

Let us define the sequence (b_n) recursively:

$$\begin{cases} b_1 = 1 \\ b_{n+1} = 1 + \frac{1}{b_n} \quad (n \geq 1) \end{cases}$$

Thus:

$$b_2 = \frac{2}{1} \approx 2.000000$$

$$b_3 = \frac{3}{2} \approx 1.500000$$

$$b_4 = \frac{5}{3} \approx 1.666667$$

$$b_5 = \frac{8}{5} \approx 1.600000$$

$$b_6 = \frac{13}{8} \approx 1.625000$$

$$b_7 = \frac{21}{13} \approx 1.615385$$

$$b_8 = \frac{34}{21} \approx 1.619048$$

We note that : $|b_8 - \varphi| \approx 0.001014$.

1

It is possible to formally prove that sequence (b_n) approaches a limit as *n* approaches infinity, and that the limit is φ, and therefore write the sequence as:

$$\lim_{n\to\infty} b_n = \varphi = 1 + \frac{1}{1+\frac{1}{1+\cdots}}$$

The last entry is called the **simple continued fraction.** It uses only the number 1 and two arithmetic operations: addition and division.

Also, a number is rational if and only if it has a representation as a finite continued fraction: the fact that φ has this representation as an infinite simple continued fraction proves that it is irrational. More about **continued fraction** can be read in: Olds, C. (1963). *Continued Fractions*. Mathematical Association of America.

At the beginning of this chapter, we noted three fundamental relationships between φ and "itself" and between φ and $\sqrt{5}$:

$$\varphi^2 = \varphi + 1 \tag{1.1b}$$

$$\varphi = \tfrac{1}{2}\,(1 + \sqrt{5}) \tag{1.2}$$

$$\frac{-1}{\varphi} = 1 - \varphi = \tfrac{1}{2}\,(1 - \sqrt{5}) \tag{1.3}$$

We shall now introduce six other arithmetical relationships that are also quite useful.

$$\mathbf{2\varphi - 1 = \sqrt{5}} \tag{1.5}$$

Hint: This can be directly derived (in two steps beginning with the numeric value of φ).

$$\boldsymbol{\varphi^2 = \varphi + 1 = \tfrac{1}{2}\,(3 + \sqrt{5})} \tag{1.6a}$$

Proof: The first equality is already known. To arrive at the second, we use (1.2):

$$\varphi + 1 = \tfrac{1}{2}\,(1 + \sqrt{5}) + 1 = \tfrac{1}{2}\,(3 + \sqrt{5})$$

* * *

$$\boldsymbol{\left(\frac{1}{\varphi}\right)^2 = (\varphi - 1)^2 = 2 - \varphi = \tfrac{1}{2}\,(3 - \sqrt{5})} \tag{1.6b}$$

Proof: The first equality is already known. This can be expanded to arrive at the second and third:

$$\begin{aligned}(\varphi - 1)^2 &= \varphi^2 - 2\varphi + 1 = (\varphi + 1) - 2\varphi + 1\\ &= 2 - \varphi = 2 - \tfrac{1}{2}(1 + \sqrt{5}) = \tfrac{1}{2}(3 - \sqrt{5})\end{aligned}$$

The proofs for the following equations we leave to the reader:

$$\boldsymbol{\varphi^2 + 1 = \varphi + 2 = \varphi\sqrt{5}} \tag{1.7a}$$

$$\left(\frac{1}{\varphi}\right)^2 + 1 = 3 - \varphi = \frac{\sqrt{5}}{\varphi} = (\varphi - 1)\sqrt{5} \quad (1.7b)$$

$$\varphi^3 = 2\varphi + 1 = 2 + \sqrt{5} \quad (1.8)$$

1.3 The Group of φ-Numbers

First, we define a **commutative group**:
A non-empty set, G, together **with a binary operation** (represented here by "∘") is deemed a **commutative group** if the following properties hold:

1. Closure

$\forall a \in G,\ \forall b \in G\ :\ a \circ b \in G$

That is to say, the value obtained as a result of the operation is also an element in G. We point out that in some of the literature, closure is an integral part of the concept of the "operation," and is not presented as a property in itself.

2. Commutativity

$\forall a \in G,\ \forall b \in G\ :\ a \circ b = b \circ a$

3. Associativity

$\forall a \in G,\ \forall b \in G,\ \forall c \in G:\ a \circ (b \circ c) = (a \circ b) \circ c$

This is commonly expressed as:

$a \circ (b \circ c) = (a \circ b) \circ c = a \circ b \circ c$

4. An Identity or a Neutral Element Exists (Denoted Here as "e"):

$\exists e \in G,\ \forall a \in G\ :\ a \circ e = e \circ a = a$

If the operation resembles addition, the neutral element is called the "**zero element**," since 0 is the neutral element in addition. If the operation resembles multiplication, the neutral element is termed the "unit element" since 1 is the neutral element in multiplication.

5. An Inverse Element Exists (The Element That "Neutralizes" a is Denoted Here as a′)

$\forall a \in G,\ \exists a' \in G\ :\ a \circ a' = a' \circ a = e$

(Since the set is commutative, it will suffice to use only one equality: $a' \circ a = e$.)
If the operation resembles addition, the inverse element is called the "**negative element**," and $(-a)$ is the accepted denotation. If the operation resembles multiplication, the inverse element is termed the "**reciprocal**" and a^{-1} is the accepted denotation.

We will now look at the set of φ-numbers.
A number of the form:

$$a + b\varphi \tag{1.9}$$

where a and b are **rational numbers**, is called a "φ-number."
We will begin with an equality between two φ-numbers:

$$a + b\varphi = c + d\varphi$$

It is easy to see that a = c and b = d must hold. (This would be the case for any irrational number that replaces φ).
This is because if $b \neq d$ we get:

$$\varphi = \frac{c - a}{b - d},$$

which is impossible because the fraction on the RHS, $\frac{c - a}{b - d}$, is rational while φ is irrational. It thus follows that b = d and a = c must hold.
We have obtained an important and useful rule that we shall call **the principle of equating (rational) coefficients**.

We now shall prove that the set of all φ-numbers other than 0 together with the standard multiplication operation will produce a commutative group.

1. **Closure**
   For any two numbers $a + b\varphi$ and $c + d\varphi$

$$\begin{aligned}(a + b\varphi)(c + d\varphi) &= ac + ad\varphi + bc\varphi + bd\varphi^2\\ &= ac + ad\varphi + bc\varphi + bd\varphi + bd\\ &= (ac + bd) + (ad + bc + bd)\,\varphi\end{aligned}$$

It remains to be shown that the result cannot be 0, since the product of two real, non-zero numbers is different from zero.
A special case of the result should be pointed out: $(a + b\varphi)^2 = (a^2 + b^2) + b\,(2a + b)\,\varphi$.

2. **Commutativity:** It is Not Necessary to Prove This, Since Multiplication in R is Commutative

.

3. **Associativity:** Ditto
   The Unit Element is $1 + 0\ \varphi$

4. **An Inverse Element Exists**
   We first observe the following:

$$\begin{aligned}(a + b\varphi)\,[a + b\,(1 - \varphi)] &= a^2 + ab\,(1 - \varphi) + ab\varphi + b^2\varphi\,(1 - \varphi)\\ &= a^2 + ab - ab\varphi + ab\varphi + b^2 \cdot (-1)\\ &= a^2 + ab - b^2\end{aligned}$$

We define the function $\delta(a, b)$ as follows:

$$\boldsymbol{\delta(a, b) = a^2 + ab - b^2} \tag{1.10}$$

Hence:

$$\boldsymbol{(a + b\varphi)^{-1} = \frac{a + b\,(1 - \varphi)}{\delta(a,b)} = \frac{a + b - b\varphi}{\delta(a,b)}} \tag{1.11a}$$

It possible to check that this equation also holds for the case where $a = 0\ ,\ b \neq 0$ and also for the case where $a \neq 0,\ b = 0$. We now just have to prove that, except for the case where $(a, b) = (0, 0)$, **there are no other values of** a and b for which $\delta\,(a, b) = 0$.
To prove it, we assume that the above statement is true, that is:

$$a^2 + ab - b^2 = 0.$$

In this case:
If $b = 0$, then $a^2 = 0$, and hence $a = 0$, which contradicts our assumption.
If $b \neq 0$, we can divide each side of the equation by b^2 to get: $\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 1 = 0$, and thus $\frac{a}{b} = -\varphi$ or $\frac{a}{b} = \frac{1}{\varphi}$, neither of which is possible since a and b were defined to be rational numbers and φ is irrational.

We shall now examine the special case where $\delta\,(a, b) = 1$. Using expression (1.11a) we write the following:

$$(a + b\varphi)^{-1} = a + b\,(1 - \varphi) = a + b - b\varphi.$$

Therefore, we define (in a general sense: not specifically for the case where $\delta\,(a, b) = 1$):

$$\boldsymbol{(a + b\varphi)^{*} = a + b\,(1 - \varphi)} \tag{1.12a}$$

This shall call this number the "**companion of a+bφ.**" "Companion" indicates a "mutual relationship", namely:

$$\boldsymbol{[a + b\,(1 - \varphi)\ ^{*} = a + b\varphi} \tag{1.12b}$$

$$\boldsymbol{(a + b\varphi)^{**} = a + b\varphi} \tag{1.12c}$$

Therefore, we can write Eq. (1.11a) as:

$$\mathbf{(a + b\varphi)^{-1} = \frac{(a + b\varphi)^*}{\delta(a,b)}} \tag{1.1b}$$

or alternatively as:

$$\mathbf{(a + b\varphi)\,(a + b\varphi)^* = \delta\,(a, b)} \tag{1.11b$'$}$$

As we pointed out above, if $\delta\,(a, b) = 1$ then:

$$(a + b\varphi)^{-1} = (a + b\varphi)^*$$

The inverse equations give us a way to **rationalize the denominator** (that is to say to "eliminate" φ from the denominator). In other words, if φ appears in the denominator of any fraction, we multiply both the numerator and the denominator by its companion.

Here are three examples:

$$\frac{1}{1+2\varphi} = \frac{1+2\,(1-\varphi)}{\delta(1,2)} = \frac{3-2\varphi}{-1} = -3 + 2\varphi$$

$$\frac{2-\varphi}{3-2\varphi} = \frac{(2-\varphi)\,[3-2\,(1-\varphi)]}{\delta(3,-2)} = \frac{\varphi}{-1} = -\varphi$$

$$\frac{1+2\varphi}{-1+2\varphi} = \frac{(1+2\varphi)\,[-1+2\,(1-\varphi)]}{\delta(-1,2)} = \frac{-3-4\varphi}{-5} = \frac{3}{5} + \frac{4}{5}\varphi$$

For the sake of completeness, we will now show that the set of φ-numbers is a field. First, recall that a "field" is a non-empty set, F, **together with two binary operations** ("addition," denoted here by ⊕, and "multiplication," denoted by ⊗), provided the following properties hold true:

a. The set F with operation ⊕ only constitutes a **commutative group**. The zero element is denoted by "0" and the inverse of a is denoted as (−a).
b. The non-empty set F\{0} together with operation ⊗ only constitutes a **commutative group**. The unit element is denoted as 1, and the inverse of a is denoted as a^{-1}.
c. The "multiplication" operation is **distributive** over the "addition" operation:

$$\forall a \in F\ ,\ \ \forall b \in F\ ,\ \ \forall c \in F\ :\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

We shall now see that a set of φ-numbers, along with the usual operations of addition and multiplication between the numbers constitutes a field.

a. A set of φ-numbers, along with addition, constitutes a commutative group:
   1. **Closure: For any two numbers** $a + b\varphi$ **and** $c + d\varphi$:

      $$(a + b\varphi) + (c + d\varphi) = (a + c) + (b + d)\,\varphi$$

   2. **Commutativity**: It is clear that addition in R is commutative.
   3. **Associativity**: It is clear that addition in R is associative.
   4. **The zero element is** $0 + 0\varphi$
   5. **The negative element** of $a + b\varphi$ is $(-a) + (-b)\,\varphi$
b. A set of φ-numbers without $0 + 0\varphi$ and with multiplication, constitutes a commutative group.

The proof for this was presented at the beginning of the current section.

c. Multiplication is distributive over addition: It is clear that this property exists in R.

1.4 Matrices and Isomorphism

The group described in the previous section closely corresponds to the set of matrices of type:

$$\mathbf{M}(\mathbf{a},\mathbf{b}) = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{a}+\mathbf{b} \end{pmatrix} \tag{1.13}$$

where a and b are **rational numbers,** neither of which are not both 0.

We shall see that this set, combined with the standard multiplication operation between matrices, also constitutes a commutative group.

1. **First, closure**:

$$\mathrm{M}(a,b)\,\mathrm{M}(c,d) = \begin{pmatrix} ac+bd & ad+bc+bd \\ bc+ad+bd & bd+ac+ad+bc+bd \end{pmatrix}$$

It is obvious that closure exists due to the commutative property of addition in R.

2. **Commutativity**:

$$\mathrm{M}(c,d)\,\mathrm{M}(a,b) = \begin{pmatrix} ca+db & cb+da+db \\ da+cb+db & db+ca+bc+da+db \end{pmatrix}$$

Hence commutativity exists (due to the commutativity of addition and multiplication in R).

3. **Associativity: It is not necessary to prove this because multiplication over of matrixes is (in general) associative.**

4. **The neutral element is:**

$$\mathrm{M}(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1+0 \end{pmatrix} = \mathrm{I}$$

"I" is conventionally used to indicate the unit matrix.

5. **Inverse element**

Before considering the inverse matrix for each matrix in the set, we shall remind ourselves how to calculate inverse matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where

$$\Delta = ad - cb \neq 0$$

is the determinant of the matrix.
Since the determinant of M(a, b) is:

$$a\,(a + b) - b^2 = a^2 + ab - b^2 = \delta(a, b),$$

this yields

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{a+b} \end{pmatrix}^{-1} = \frac{\mathbf{1}}{\boldsymbol{\delta}(\mathbf{a,b})} \begin{pmatrix} \mathbf{a+b} & -\mathbf{b} \\ -\mathbf{b} & \mathbf{a} \end{pmatrix} \tag{1.14a}$$

which can be written as:

$$\mathbf{M}^{-1}(\mathbf{a}, \mathbf{b}) = \boldsymbol{\delta}^{-1}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{M}(\mathbf{a} + \mathbf{b}, -\mathbf{b}) \tag{1.14b}$$

Hence, it is clear that the inverse matrix belongs to our set.
Q.E.D.

Now, we are ready to establish the precise relation between the two groups.
Let us define the following function:

$$a + b\varphi \mapsto M(a, b)$$

To begin with, we will show that it is a **one-to-one function:**
If M(a, b) = M(c, d), then according to the definition of equality between matrices, the following must occur:

$$(a, b) = (c, d),$$

from which it follows:

$$a + b\varphi = c + d\varphi.$$

In addition, we can see that this is a **surjective function** because the "natural" origin of each matrix M(a, b) is $a + b\varphi$.

Finally, we show that the multiplication operation preserves the one to one correspondence between the functions:

$$(a + b\varphi)(c + d\varphi) \mapsto M(a, b)M(c, d)$$

(The multiplication operation on the left is the usual multiplication on R, and the multiplication operation on the right is the standard multiplication operation between matrices.)
In other words, the image of the product is the product of the images.
Proof

$$(a + b\varphi)\,(c + d\varphi) = (ac + bd) + (ad + bc + bd)\,\varphi,$$

and

$$M(a, b)\, M(c, d) = \begin{pmatrix} ac + bd & ad + bc + bd \\ bc + ad + bd & bd + ac + ad + bc + bd \end{pmatrix}$$

In other words, the multiplication operation preserves the one to one correspondence between the functions.

A function mapping one group to another that satisfies the three properties just proved is said to be **isomorphic** between groups, and hence the groups themselves are denoted as **isomorphic.**

To illustrate the operation preserving property, we shall take the φ-numbers $3 - \varphi$ and $1 + 2\varphi$. Their product is:

$$(1 + 2\varphi)\,(3 - \varphi) = 3 - \varphi + 6\varphi - 2\varphi - 2 = 1 + 3\varphi.$$

Now:

$$1 + 2\varphi \mapsto M(1, 2)$$
$$3 - \varphi \mapsto M(3, -1)$$

We multiply the two matrices:

$$M(1, 2)\, M(3, -1) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = M(1, 3).$$

As stated, the following is true:

$$(1 + 2\varphi)(3 - \varphi) \mapsto M(1, 2)M(3, -1)$$

The isomorphism that we have presented here brings us naturally to a matrix that corresponds to φ. Since $\varphi = 0 + 1\varphi$, the corresponding matrix is:

$$M(0, 1) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

We can check that it satisfies the golden ratio in the form of a matrix. In other words:

$$M^2(0, 1) = M(0, 1) + I$$

(This is not really surprising!)
Thus, we can now define the **golden matrix** as:

$$\mathbf{\Phi} = \mathbf{M(0, 1)} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \tag{1.15}$$

Φ is a simple variation of the Fibonacci Q-Matrix: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

So far, this matrix behaves like the **φ-number,** since it satisfies the equality:

$$\mathbf{\Phi^2} = \mathbf{\Phi} + \mathbf{1} \tag{1.16}$$

which is the matrix version of the equality:

$$\varphi^2 = \varphi + 1.$$

(In ► Chap. 3 we shall return to the above equations and develop them further.)
We can check that the following equality is true:

$$\mathbf{\Phi}^{-1} = \mathbf{\Phi} - \mathbf{I} = \mathbf{M}(-\mathbf{1}, \mathbf{1}) \tag{1.17}$$

Notice that this is the matrix version of the equality:

$$\frac{1}{\varphi} = \varphi - 1.$$

Another matrix that may be of interest to us is the one that "applies" to $\sqrt{5}$.
Since $\sqrt{5} = 2\varphi - 1$, the "natural" candidate would be:

$$\mathrm{M}(-1, 2) = 2\Phi - \mathrm{I}.$$

Indeed,

$$\mathrm{M}^2(-1, 2) = \mathrm{M}(5, 0) = 5\mathrm{I}$$

which is exactly what we expect.
Therefore, we can define the **root-five matrix** to be:

$$\mathrm{R}(5) = \mathrm{M}(-1, 2) = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \tag{1.18}$$

(R stands for "root". It is interesting to observe that the $\sqrt{\ }$ symbol is a stylized "r".)
We shall meet these two matrices again in the chapters to come.

1.5 The Norm of φ-Numbers

Let us define the following number:
Define:

$$\boldsymbol{\mu}(\mathbf{a}, \mathbf{b}) = \boldsymbol{\mu}(\mathbf{a} + \mathbf{b}\boldsymbol{\varphi}) = \sqrt{|\boldsymbol{\delta}(\mathbf{a}, \mathbf{b})|} \tag{1.19}$$

where $a + b\varphi$ is a φ-number.
This number shall be termed the "norm of **a**+**bφ**."
(We could define the norm without the square root, as is customary in some places. However, the definition we use here allows us to do the normalization easily. We will discuss normalization later on.)
We first point out some basic properties of the norm (x denotes any φ-number):

$$\boldsymbol{\mu}(\boldsymbol{x}) = \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{x} = \mathbf{0} \tag{1.20}$$

$$\boldsymbol{\mu}(-\boldsymbol{x}) = \boldsymbol{\mu}(\boldsymbol{x}^*) = \boldsymbol{\mu}(\boldsymbol{x}) \tag{1.21a}$$

$$\mu(\boldsymbol{x}^{-1}) = [\mu(\boldsymbol{x})]^{-1} \tag{1.21b}$$

$$\mu(\mathbf{k}\boldsymbol{x}) = |\mathbf{k}|\mu(\boldsymbol{x}) \quad (\forall \mathbf{k} \in \mathbb{Q}) \tag{1.22}$$

(We shall leave the details to the reader. With respect to (1.20), recall that a and b are rational.)

We shall now move on to another essential property of the norm, one that is associated with multiplication:

$$\mu(\boldsymbol{xy}) = \mu(\boldsymbol{x})\mu(\boldsymbol{y}) \tag{1.23}$$

where x and y are φ-numbers.

(Notice that (1.22) is actually a special case of (1.23)).

Proof of (1.23):
We denote $x = \mathrm{a} + \mathrm{b}\varphi$, $y = \mathrm{c} + \mathrm{d}\varphi$.

$$xy = (\mathrm{a} + \mathrm{b}\varphi)(\mathrm{c} + \mathrm{d}\varphi) = (\mathrm{ac} + \mathrm{bd}) + (\mathrm{ad} + \mathrm{bc} + \mathrm{bd})\,\varphi$$

$$\begin{aligned}
\delta(\mathrm{ac} + \mathrm{bd}\,,\ \mathrm{ad} + \mathrm{bc} + \mathrm{bd}) &= (\mathrm{ac} + \mathrm{bd})^2 + (\mathrm{ac} + \mathrm{bd})(\mathrm{ad} + \mathrm{bc} + \mathrm{bd}) \\
&\quad - (\mathrm{ad} + \mathrm{bc} + \mathrm{bd})^2 \\
&= \dots\dots\dots\dots\dots \quad \text{(complete the missing steps)} \\
&= \mathrm{a}^2(\mathrm{c}^2 + \mathrm{cd} - \mathrm{d}^2) + \mathrm{ab}(\mathrm{c}^2 + \mathrm{cd} - \mathrm{d}^2) \\
&\quad - \mathrm{b}^2(\mathrm{c}^2 + \mathrm{cd} - \mathrm{d}^2) \\
&= (\mathrm{a}^2 + \mathrm{ab} - \mathrm{b}^2)(\mathrm{c}^2 + \mathrm{cd} - \mathrm{d}^2) \\
&= \delta(\mathrm{a}, \mathrm{b})\delta(\mathrm{c}, \mathrm{d}).
\end{aligned}$$

Hence (1.23) is true.

An important inference from Formula (1.23) is that:

$$\begin{aligned}
\mu[(\mathbf{a} + \mathbf{b}\boldsymbol{\varphi})^2] &= \mu[(\mathbf{a}^2 + \mathbf{b}^2) + \mathbf{b}(2\mathbf{a} + \mathbf{b})\,\boldsymbol{\varphi} \\
&= [\mu\,(\mathbf{a} + \mathbf{b}\boldsymbol{\varphi})^2] = |\boldsymbol{\delta}(\mathbf{a}, \mathbf{b})|
\end{aligned} \tag{1.24}$$

An additional inference is:

$$\mu\left(\boldsymbol{x}^{\boldsymbol{k}}\right) = [\mu(\boldsymbol{x})]^{\boldsymbol{k}}(\forall \boldsymbol{k} \in \mathbb{N}) \tag{1.25}$$

This can be easily proved by induction.

We shall now move to the special case where the norm is equal to 1.
A φ-number whose norm is equal to 1 is designated as a **"normalized φ-number."**
If x and y are normalized φ-numbers, then (as a result of the properties mentioned earlier):

$$\mu(-\boldsymbol{x}) = \mu(\boldsymbol{x}^*) = \mu(\boldsymbol{x}^{-1}) = \mathbf{1} \tag{\textbf{1.21}$'$}$$

$$\mu(\boldsymbol{k}x) = |\boldsymbol{k}|(\forall \boldsymbol{k} \in \mathbb{Q}) \tag{1.22$'$}$$

$$\boldsymbol{\mu(xy) = 1} \qquad (\mathbf{1.23'})$$

$$\mu\left(x^k\right) = \mathbf{1}(\forall k \in \mathbb{N}) \qquad (1.24')$$

We can prove that the entire set of normalized φ-numbers presents a commutative group in itself (i.e., it is a subset of the set of φ-numbers).

Closure: $\mu(xy) = 1$ is true.

Commutative: is satisfied in any case.

Associative: same as above.

The unit element, which *must* be the same as the unit element for "general" φ-numbers, that is to say, $1 + 0 \cdot \varphi$ is, in fact, a member of the set, since $\mu(1 + 0 \cdot \varphi) = \sqrt{|\delta(1,0)|} = 1$.

The inverse of x is x^* since it is true that $\mu(x^*) = 1$.

It is natural to ask ourselves if there is a way to "normalize" any given φ-number. That is to say, given a φ-number whose norm is ***not*** 1, is it possible to "do something to it" to obtain a normalized φ-number?

The answer is in the affirmative, as can be deduced by noting (1.22) and by observing the following:

$$\mu[\mu^{-1}(\mathrm{a},\mathrm{b})(\mathrm{a}+\mathrm{b}\varphi)] = \mu^{-1}(\mathrm{a},\mathrm{b})\mu(\mathrm{a},\mathrm{b}) = 1.$$

In other words, in order to normalize any given φ-number, x, it is sufficient to multiply it by $\mu^{-1}(x)$.

We present now the theorem that summarizes the above discussion.

Given any φ-number, x, it will be true that:

$$\mu\left(\frac{x}{\mu(x)}\right) = \mathbf{1} \qquad (1.26)$$

As an example, let us use $x = 3 - \varphi$. In this case,

$$\mu(x) = \sqrt{|\delta(3,-1)|} = \sqrt{|9-3-1)|} = \sqrt{5}$$

$$\frac{x}{\mu(x)} = \frac{3-\varphi}{\sqrt{5}}$$

$$\mu\left[\frac{x}{\mu(x)}\right] = \delta\left(\frac{3}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) = \ldots = 1$$

Q.E.D.

In the following section we will utilize the principles and definitions developed in this chapter in order to solve related exercises and theorems.

Exercises for Chapter 1

Note

When given an isosceles triangle, the first letter (on the left) specifies the vertex angle (opposite the base).

Exercise 1.1: Identifying Numerical Values with φ

Establish the following equalities. Try to express the results in terms of the algebraic equations $\varphi^2 = \varphi + 1$, $\varphi - 1 = 1/\varphi$, not the numeric value of φ:

1. $(\varphi + 1)(\varphi - 1) = \varphi$
2. $(\varphi + \sqrt{\varphi})(\varphi - \sqrt{\varphi}) = 1$
3. $\varphi - \frac{1}{\varphi} = 1$
4. $\varphi + \frac{1}{\varphi} = \sqrt{5}$
5. $\varphi^2 + \frac{1}{\varphi} = 2\varphi$ reference
6. $\varphi^2 - \frac{1}{\varphi} = 2$
7. $\sqrt{\varphi} + \frac{1}{\sqrt{\varphi}} = \varphi\sqrt{\varphi}$
8. $(\varphi^2 - 1)^2 = \varphi + 1$
9. $\varphi^4 = (\varphi + 1)^2 = 3\varphi + 2$
10. $1 - \frac{1}{\varphi^2} = \frac{1}{\varphi} = \varphi - 1$

Exercise 1.2: Equations and Inequalities

Solve in **R**:

1. $x^2 + x - 1 = 0$
2. $x^2 - |x| - 1 = 0$
3. $x^2 - |x| - 1 < 0$
4. $x^2 + |x| - 1 = 0$
5. $x^2 + |x| - 1 < 0$
6. $x^4 - 3x^2 + 1 = 0$
7. $x^4 - 3x^2 + 1 < 0$
8. $x - \sqrt{x} - 1 = 0$
9. $x^4 - x^2 - 2x - 1 = 0$
10. $x^2 - \sqrt{5}x + 1 = 0$
11. $x^2 - ax - a^2 = 0$ $(a \neq 0)$
12. $x^2 - 3x + 1 = 0$
13. $x^2 - 2x - 4 = 0$
14. $4x^2 - 2x - 1 = 0$

Exercise 1.3: An Equation System

Solve in $\mathbf{R}^2$:

1. $x^2 = y + 1$
   $y^2 = x + 1$
2. $x^2 = -y + 1$
   $y^2 = -x + 1$

Exercise 1.4: Equations to Powers of Three

Solve in $\mathbf{R}$:

1. $x^3 - 2x - 1 = 0$
2. $x^3 - 2x^2 + 1 = 0$
3. $x^3 - 2x + 1 = 0$
4. $x^3 + 2x^2 - 1 = 0$

Exercise 1.5: Equations with a Complex Variable

Solve in $\mathbf{C}$:

1. $z^2 + iz + 1 = 0$
2. $z^2 - iz + 1 = 0$
3. $z^4 + 3z^2 + 1 = 0$
4. $z^4 - z^2 - 1 = 0$
5. $z^4 + z^2 - 1 = 0$

Exercise 1.6: Constructing Quadratic Equations

Construct quadratic equations whose solutions are:

1. $\varphi \pm \sqrt{\varphi}$
2. $\varphi^2, \frac{1}{\varphi}$
3. $\varphi^2, -1$
4. $\varphi, 1$
5. $\varphi, -1$
6. $1, -\frac{1}{\varphi^2}$
7. $\varphi^2, -\frac{1}{\varphi}$
8. $\varphi, \frac{1}{\varphi^2}$
9. $\varphi, -\frac{1}{\varphi^2}$

Exercise 1.7: Relationship Between Reciprocals

In both sections: $a > b > 0$. Prove:

A. If

$$\frac{1}{a+b} = \frac{1}{b} - \frac{1}{a}$$

Then $a/b = \varphi$.

B. If

$$\frac{1}{a-b}=\frac{1}{b}-\frac{1}{a}$$

Then $a/b = \varphi^2$.

- **Exercise 1.8: Right-Angled Triangles**

Given the lengths of the hypotenuse and one of the perpendiculars of a right-angled triangle, calculate the length of the second perpendicular.

1. $\varphi^2, 1/\varphi$
2. $\varphi, \sqrt{\varphi}$
3. φ^2, φ
4. $\varphi^2 + a, \varphi^2 - a \quad (0 < a < \sqrt{\varphi})$

- **Exercise 1.9: Geometrical Sequences of Right-Angled Triangles**

The lengths of the sides of a right-angled triangle form a geometric sequence. What is the common ratio of the sequence?

- **Exercise 1.10: Geometric Triangle Sequence**

The lengths of the sides of a triangle present a geometric sequence with a common ratio of q. Prove that $1/\varphi < q < \varphi$.

- **Exercise 1.11: Triangles with "Inversed" Sides**

The following is true for side lengths a, b, and c of a triangle:

$a > b > c$.

The side lengths of another triangle are 1/a, 1/b and 1/c.

A. Justify the following inequalities:

$0 < a - c < b$
$0 < 1/c - 1/a < 1/b$

B. Prove:

$(a/c)^2 - 3(a/c) + 1 < 0$

C. Deduce that:

$1/\varphi^2 < a/c < \varphi^2$

- **Exercise 1.12: Geometric Series**

Calculate:

1. $1 + 1/\varphi + 1/\varphi^2 + 1/\varphi^3 + \cdots$
   $1 - 1/\varphi + 1/\varphi^2 - 1/\varphi^3 + \cdots$
2. $1 + 1/\varphi^2 + 1/\varphi^4 + 1/\varphi^6 + \cdots$
   $1 - 1/\varphi^2 + 1/\varphi^4 - 1/\varphi^6 + \cdots$
3. $1 + 1/\varphi^3 + 1/\varphi^6 + 1/\varphi^9 + \cdots$
   $1 - 1/\varphi^3 + 1/\varphi^6 - 1/\varphi^9 + \cdots$
4. $1 + (\varphi/2) + (\varphi/2)^2 + (\varphi/2)^3 + \cdots$
5. $(1 + 1/\varphi - 1/\varphi^2) + (1/\varphi^3 + 1/\varphi^4 - 1/\varphi^5) + \cdots$

1

- **Exercise 1.13: Constant Sequences**

In both sections, the sequence is comprised of **positive numbers**. Prove that:

A. Sequence (a_n) where $a_{n+1}{}^2 = 1 + a_n$, is a constant sequence if and only if $a_1 = \varphi$.

B. Sequence (b_n) where $b_{n+1}{}^2 = 1 - b_n$, is a constant sequence if and only if $b_1 = \varphi - 1$.

- **Exercise 1.14: Geometric Sequences (A)**

(a_n) is a geometric sequence where $a_{n+2} = a_{n+1} + a_n$ (for all natural n).
What is the common ratio of the sequence?

- **Exercise 1.15: Introducing the Fibonacci Sequence and Locus**

The sequence (L_n) is defined by

$$L_n = \varphi^n + (1 - \varphi)^n$$

A. Calculate the first two elements of the sequence.
B. Prove that $L_{n+2} = L_{n+1} + L_n$ exists for all natural n.
C. Determine if all the elements of the sequence are natural numbers.
D. Repeat the above (A–C) for the sequence (F_n) that is defined by

$$F_n\sqrt{5} = \varphi^n - (1 - \varphi)^n$$

- **Exercise 1.16: Power Series**

A. Prove: A necessary and sufficient condition for the convergence of the series

$$1 + x(x - 1) + x^2(x - 1)^2 + x^3(x - 1)^3 + \cdots$$

is $1 - \varphi < x < \varphi$.
What is the sum of the column for $x = 1/\varphi$?

B. Prove: A necessary and sufficient condition for the convergence of the series

$$1 + x(x + 1) + x^2(x + 1)^2 + x^3(x + 1)^3 + \cdots$$

is $-\varphi < x < \varphi - 1$.
What is the sum of the series for $x = -1/\varphi$?

- **Exercise 1.17: Power Series**

The function f is defined by:

$$f(x) = 1 + x + x^2 + x^3 + \cdots$$

Where $0 < |x| < 1$.

A. Use two different methods to find the derivative function.
B. Find the sums of the following series (use the previous section to help you):

$$1/\varphi + 2/\varphi^2 + 3/\varphi^3 + 4/\varphi^4 + \cdots$$
$$1 - 2/\varphi + 3/\varphi^2 - 4/\varphi^3 + \cdots$$

- **Exercise 1.18: Complex Numbers**

Given two numbers

$A = \frac{1}{2}\left(\sqrt{\varphi} + \frac{i}{\sqrt{\varphi}}\right)$

$B = \frac{1}{2}\left(\sqrt{\varphi} - \frac{i}{\sqrt{\varphi}}\right)$

Calculate:
1. AB
2. $A^2 - B^2$
3. $A^2 + B^2$
4. $(A + B)/(A - B)$
5. $(A + B)^2 + (A - B)^2$
6. $(A + B)^2 - (A - B)^2$

- **Exercise 1.19: Complex Roots**

Solve in **c:**
1. $z^2 = 1 + 2i$
2. $z^2 = \sqrt{5} + 2i$

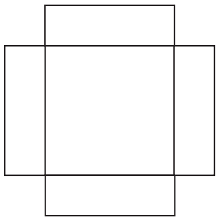
- **Exercise 1.20: Ratio of Areas**

The rays of angle A connect two parallel sections BD and CE, such that $\frac{AC}{AB} = \frac{AE}{AD} = \varphi$.

Prove: $\frac{S_{BDEC}}{S_{BAD}} = \varphi$

- **Exercise 1.21: Two Interlocking Golden Rectangles**

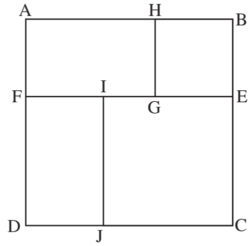
Two golden rectangles of dimensions φ by 1 are arranged to form a "plus" shape (see illustration).



Calculate the area and perimeter of the shape.

- **Exercise 1.22: Golden Rectangle Inside a Square**

A golden rectangle, FECD, is cut out from square ABCD. From the remaining rectangle ABEF, square HBEG is cut, and from rectangle FECD, square IECJ is cut (see illustration, below).



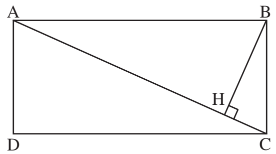
A. Prove: AHGF is a golden rectangle. (You may assume, without loss of generality, that: $AB=\varphi$.)

B. Prove:

$$\frac{S_{IECJ}}{S_{AHGF}}=\frac{S_{FIJD}}{S_{HBEG}}=\varphi$$

- **Exercise 1.23: Golden Ratio in a Rectangle**

ABCD is a rectangle with side lengths: $AB=DC=\sqrt{\varphi}$, $AD=BC=1$.
H is a point on diagonal AC such that $\angle AHB=90°$.



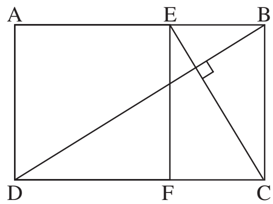
A. Calculate the lengths of segments AH, HC, and BH.

B. Prove:

$$\frac{S_{ABC}}{S_{ABH}}=\varphi$$

- **Exercise 1.24: Golden Rectangle and Square**

ABCD is a golden rectangle ($AB=\varphi AD$).
From point C, a perpendicular is connected to diagonal DB, which intersects side AB at point E. From point E, a perpendicular is connected to DC, which intersects it at point F.



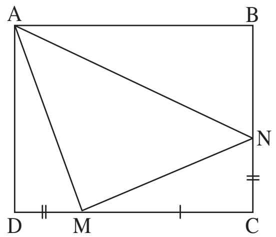
A. Prove: $\Delta DBC \sim \Delta ECF$.

B. Determine whether: AEFD is square.

(You may assume, without loss of generality: $AD = 1$.)

- **Exercise 1.25: Right-Angled Triangles in a Golden Rectangle**

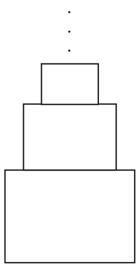
ABCD is a golden rectangle with dimensions $AB = \varphi$, $AD = 1$.
M and N are points on sides DC and BC (respectively) such that $\Delta ADM \cong \Delta MCN$ (see illustration).



A. Prove: $S_{ABN} = S_{ADM}$
B. Prove: $\frac{S_{AMN}}{S_{MCN}} = \sqrt{5}$

- **Exercise 1.26: A Tower of Golden Rectangles**

Given a "horizontal" golden rectangle whose dimensions are φ by 1. On top of it there is another "horizontal" golden rectangle whose dimensions are 1 by $1/\varphi$. On top of that, there is another "horizontal" golden rectangle of dimensions $1/\varphi$ by $1/\varphi^2$. This continues ad infinitum. In other words, the dimensions of each subsequent rectangle are $1/\varphi$ times the previous rectangle.

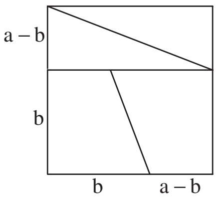


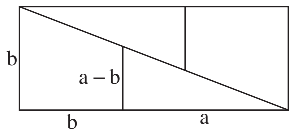
Calculate the height and area of the resulting "tower."

- **Exercise 1.27: There is No Paradox**

A. Explain how the paradox presented in the introduction ("is $1 = 0$?") came about and where "the hole" (in the rectangle) with an area equal to 1 can be found.
B. From a square with side length a, two identical right-angled triangles and two identical right-angled trapezoids are removed, according to the dimensions specified in the drawing on the left. The four sections are joined together to produce a "complete" rectangle (that is to say, unlike the rectangle of the paradox that has a "hole"), as detailed in the drawing on the right:

1

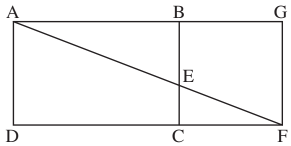




Prove that the area of the square is equal to the area of the rectangle if, and only if, a = bφ.

- **Exercise 1.28: Gold Rectangles and Squares**

In a golden rectangle ABCD, E is the point on BC (the short side) such that BE = φ·EC.
The lines AE and DC intersect at point F. A perpendicular is drawn from F to DF, and this line meets line AB at point G.



A. Prove:

$$\frac{S_{AGFD}}{S_{ABCD}} = \frac{S_{ABCD}}{S_{BGFC}} = \varphi$$

(You can assume, without loss of generality: AB = φ^2.)

B. Show that the result of the question above is also valid when BC is the longer side.

(You can assume, without loss of generality: AB = φ.)

C. Show that the result is also valid if ABCD is a **square**.

(You can assume, without loss of generality: AB = φ^2).

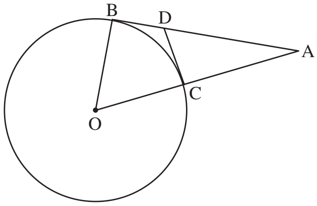
- **Exercise 1.29: In Preparation for Golden Triangles**

ABC is an isosceles triangle whose measurements are AB = AC = φ, BC = 1.
D is a point on side AC such that BC = BD.

A. Prove that: ΔABC ~ ΔBCD
B. Calculate the lengths of segments DC and AD, and show that triangle DAB is also isosceles.
C. Determine the angles of triangle ABC.

Exercise 1.30: A Triangle, a Circle and Tangents

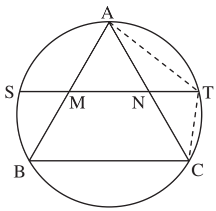
A is a point that lies outside a circle with center O. Segment AB is tangent to the circle at point B, and line AO intersects the circle at point C. A tangent is drawn at point C that intersects segment AB at point D.
Given: BD = 1, AD = φ.



Calculate the radius of the circle.

Exercise 1.31: Equilateral Triangle and Circumcircle

ABC is an equilateral triangle inscribed in a circle. Points M and N are the midpoints of lines AB and AC (respectively). Line MN intersects the circle at points S and T.



Given: AB = 2a.

A. Prove: $SN \cdot NT = a^2$.
B. Determine that: MN/NT = φ.
C. Prove: AT/TC = φ.

Exercise 1.32: Bisector of an Angle in a Right-Angled Triangle

In the right-angled triangle ABC (A = 90°), AB = 2CA. M is a point on AB such that CM bisects angle C.
Prove: AC/MA = φ. (You may assume, without loss of generality: AC = 1.)

Exercise 1.33: The Golden Ratio in a Right-Angled Triangle

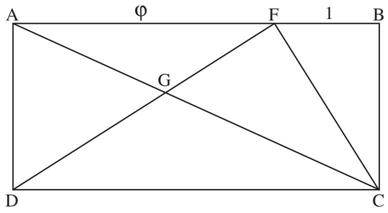
In the right-angled triangle ABC, AC = 2AB, D is a point on BC such that BD = BA, and E is a point on AC such that CE = CD.
Prove:

$$\frac{AC}{EC} = \frac{EC}{AE} = \varphi$$

$$\frac{BD}{CD} = \frac{\varphi}{2}$$

- **Exercise 1.34: The Golden Ratio in a Rectangle**

ABCD is a rectangle. F is a point on side AB such that FB = 1, AF = φ, and therefore DFC = 90°.



A. Calculate the lengths of line segments DF, AD, FC.
B. Calculate cos ∠AFC.
C. G is the intersection of sections DF and AC. Calculate the length of section DG.
D. (Part D does not depend on A, B, or C.) Prove:

$$\frac{S_{ADC}}{S_{FAC}} = \frac{S_{FAC}}{S_{BFC}} = \varphi$$

- **Exercise 1.35: Golden Rectangle That Circumscribes an Isosceles Triangle**

ABCD is a golden rectangle (AB = φ, AD = 1). F is a point on side AB such that DC = DF.
A. Calculate the length of section AF.
B. Prove:

$$\tan \angle FDC = \cos \angle FDC = 1/\sqrt{\varphi}$$

- **Exercise 1.36: Trapezoid**

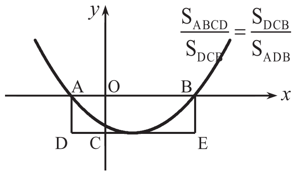
ABCD is an isosceles trapezoid in which the larger base is DC and the smaller base is AB.
Given: DC = a, DA = AB = BC = b.
Prove that if the following is true

$$\frac{S_{ABCD}}{S_{DCB}} = \frac{S_{DCB}}{S_{ADB}}$$

Then a/b = φ.

- **Exercise 1.37: The Golden Function**

The graph of $y = x^2 - x - 1$ intersects the x-axis at points A and B, and the y-axis at point C. Straight lines perpendicular to the x-axis are drawn through points A and B, and a straight line parallel to the x-axis and which intersects the vertical lines at point D (the line extending from A) and point E (the line extending from B) is drawn through point C.



O is the origin point.
Prove that rectangles OBEC, OCDA are golden rectangles.

- **Exercise 1.38: Ratio Function**

The function f is defined by:

$$f(x) = \frac{x^2 - x - 1}{x^2 + x - 1}$$

A. Find the domain of the function, the intersection points with the axes, the asymptotes, and the increasing and decreasing intervals of the function.
B. Show that the following holds:
$f'(1 - \varphi) + f'(\varphi) = 2f'(1 - \varphi)f'(\varphi)$

- **Exercise 1.39: Logarithmic Function**

The function f is defined by:

$$f(x) = \ln(x + 1) + 1/x$$

Show that the following holds:

$$f'(\varphi) = f'(1 - \varphi) = 0$$

- **Exercise 1.40: Functions to the Fifth Power**

The function f is defined by:

$$f(x) = x^5 - 5x^3 + 5x$$

A. Find the fixed points of the function.
B. What are the minimum and maximum points of the graph of the function?
C. Prove that the rectangle whose vertices rest on the extreme points of the graph, is a parallelogram with area $4\sqrt{5}$.

- **Exercise 1.41: The Family of Functions (A)**

Function f_n is defined by

$$f_n(x) = \frac{x^2 - x - 1}{\left(x^2 + 1\right)^n}$$

where n is a natural number.
A. Prove that the sequence $f_n(\varphi)$ is geometric.
B. Calculate:

$$f_1'(\varphi) + f_2'(\varphi) + f_3'(\varphi) + f_4'(\varphi) + \cdots$$

C. Function g_n is defined by

$$g_n(x) = \frac{x^2 - x - 1}{x^n}$$

Calculate:

$$g_1'(\varphi) + g_2'(\varphi) + g_3'(\varphi) + g_4'(\varphi) + \cdots$$

- **Exercise 1.42: The Family of Functions (B)**

The function f is defined by:

$$f(x) = x^n\left(x^2 - x - 1\right)$$

A. Prove that the following holds for the function and its derivative:

$$xf'(x) - (n+1)f(x) = x^n\left(x^2 + 1\right)$$

B. Determine:

$$f'(\varphi) = \sqrt{5}\,\varphi^n$$
$$f'(1-\varphi) = -\sqrt{5}\,(1-\varphi)^n$$

- **Exercise 1.43: The Multiple of Golden Functions**

g is a differentiable function and f(x) is defined as follows:

$$f(x) = \left(x^2 - x - 1\right)g(x)$$

A. Prove:

$$f'(\varphi) = g(\varphi)\sqrt{5}$$

B. Prove:

$$g(\varphi)f'(1-\varphi) + g(1-\varphi)f'(\varphi) = 0$$

- **Exercise 1.44: Integral of an Exponential Function**

Calculate:

$$\int_{1-\varphi}^{\varphi} (2x - 1)e^{x^2 - x - 1}dx$$

- **Exercise 1.45: Integral with Logarithms**

Show that the following holds:

$$\int_1^{\varphi} \frac{\varphi}{\varphi x - 1}dx = 2\int_0^1 \frac{dx}{x + \varphi}$$

- **Exercise 1.46: Hyperbolic Functions**

The hyperbolic sine function is defined as $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, and the hyperbolic cosine function is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

A. Calculate the area limited by the graph of $y = \sinh(x)$, the axis x, and the straight lines $x = \ln\varphi$ and $x = 2\ln\varphi$.

B. Calculate the area limited by the graph of $y = \cosh(x)$, the axis x and the straight lines $x = \ln\varphi$ and $x = 2\ln\varphi$.

- **Exercise 1.47: Golden Rectangle Circumscribed by a Circle.**

A circle whose equation is $x^2 + y^2 = \varphi\sqrt{5}$ circumscribes a "horizontal" golden rectangle (the long sides are parallel to the x-axis). Find the equation of the tangent to the circle at the vertex of the rectangle in the first quarter.

- **Exercise 1.48: Hyperbolas**

A. A hyperbola whose equation is $x^2 - y^2 = 1$ is bisected at four points by the straight lines $x = \sqrt{\varphi}$ and $x = -\sqrt{\varphi}$. Show that the rectangle created by the four points is a golden rectangle.

B. The hyperbola whose equation is $x^2 - y^2 = \varphi$ is bisected at four points by the straight lines $y = \pm 1$. Show that the rectangle created by the four points is a golden rectangle. Find the equations of the tangents to the hyperbola at the vertices of the rectangle and the area of the diamond created by the tangents.

- **Exercise 1.49: Circle of Apollonius**

Points A (–1, 0) and B (1, 0) rest on a planar axis system.
Find the geometric position of points P on the plane for which are $PA = \varphi \cdot PB$.

- **Exercise 1.50: Circle and Golden Ellipse**

Recall that the area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ $(a > b)$ is πab, and the foci are $(\pm\sqrt{a^2 - b^2}, 0)$.
A circle whose center is on the main axis passes through the foci of the ellipse.
Prove that the area of a circle equals the area of the ellipse if and only if $a = b\varphi$.

- **Exercise 1.51: Properties of δ(a, b)**

Prove the following (a and b are not necessarily rational numbers):

1. $\delta(a, 0) = \delta(-a, 0) = \delta(a, a) = \delta(-a, -a) = a^2$
2. $\delta(0, b) = -\delta(b, 0) = -\delta(b, b) = -b^2$
3. $\delta(-a, a) = \delta(a, -a) = -a^2$
4. $\delta(a, b) + \delta(b, a) = 2ab$
5. $\delta(a, a - b) = \delta(a + b, -b) = \delta(-a, -b) = \delta(a, b)$
6. $\delta(a, -b) = \delta(-a, b) = \delta(a, a + b) = -\delta(b, a)$
7. $\delta(a + b, a) = \delta(a + b, b)$
8. $\delta(ka, kb) = k^2\delta(a, b)$
9. $\delta(a^n, 0) = \delta(a^n, a^n) = \delta^n(a, 0)$
10. $a \geq b > 0 \Rightarrow \delta(a, b) > 0$

11. $a^2b^2\delta(1/a, 1/b) = \delta(b, a)$
12. $\delta(ia, ib) = -\delta(a, b)$
13. $\delta(ia, 0)\delta(bi, 0) = \delta(a, 0)\delta(b, 0) = a^2b^2$
14. $\delta(1, \varphi) = \delta(\varphi - 1, 1) = \delta(\varphi, \varphi + 1) = 0$
15. $d(a^2 + b^2, 2ab + b^2) = d^2(a, b)$

▪ **Exercise 1.52: Equations with δ**

A. Given the equations:

1. $\delta(1, x) = 0$
2. $\delta(x, x+1) = 0$
3. $\delta(x - 1, 1) = 0$
4. $\delta(1/x, 1) = 0$

_ Solve the equations in **R**.
_ Solve the equations in **Q**.

B. Given the equations:

1. $\delta(k, x) = 0$
2. $\delta(x, x+k) = 0$
3. $\delta(x - k, k) = 0$
4. $\delta(k/x, 1) = 0$

where x is a real variable, and k is a real parameter different than 0.
Prove that for all the equations the solution set is $S = \{k\varphi, k(1-\varphi)\}$.

C. Prove:

_ The graph of equation $\delta(x, y) = 0$ consists of two straight lines perpendicular to each other and which are cut by the main axis.
_ The vertical distance between the two straight lines is $|x|\sqrt{5}$.

▪ **Exercise 1.53: Linear System with Parameters**

Given the system

$$ax + by = a + b$$
$$bx + (a + b)y = a + 2b,$$

where a and b are rational parameters both of which are not zero.
Prove that the system has a single solution and that it is not dependent on the values of the parameters.

▪ **Exercise 1.54: Multiples of φ-Numbers**

a, b, c, d are rational numbers different than 0.

A. Prove that the product $(a + b\varphi)(c + d\varphi)$ is a rational number if and only if

$$\frac{a}{b} + \frac{c}{d} + 1 = 0$$

B. Prove that the power $(a + b\varphi)^2$ is a rational number if and only if $b = -2a$.

Exercise 1.55: Quadratic Equation

In this exercise, a and b are rational numbers, and $b \neq 0$.

A. Construct a quadratic equation with rational coefficients, where one of its solutions is $a + b\varphi$.

B. Show that we can write the equation as follows:

$$x^2 - (2a + b)x + \delta(a, b) = 0$$

C. What is the second solution for the equation?

Exercise 1.56: Geometric Sequence (b)

A. The sequence (a_n) satisfies $\delta(a_n, a_{n+1}) = 0$. Prove that it is a geometric sequence with a common ratio of φ or $-1/\varphi$.

B. The sequence (b_n) satisfies $\delta(b_{n+1}, b_n) = 0$. Prove that it is a geometric sequence with a common ratio of $-\varphi$ or $1/\varphi$.

Exercise 1.57: Properties of M (a, b)

Prove:

1. $M(a, 0) = a\mathbf{I}$
2. $M(0, b) = b\boldsymbol{\Phi}$
3. $M(a, b) = a\mathbf{I} + b\boldsymbol{\Phi}$
4. $M(a, a) = a\mathbf{I} + a\boldsymbol{\Phi} = aM(1, 1) = a\boldsymbol{\Phi}^2$
5. $M(ka, kb) = kM(a, b)$
6. $M(\alpha a, \beta b) = \alpha M(a, 0) + \beta M(0, b) = \alpha a\mathbf{I} + \beta b\boldsymbol{\Phi}$
7. $M(a, b) + M(c, d) = M(a + c, b + d)$
8. $\alpha M(a, b) + \beta M(c, d) = M(\alpha a + \beta c, \alpha b + \beta d)$
9. $\boldsymbol{\Phi}M(a, b) = M(b, a + b)$
10. $\boldsymbol{\Phi}^{-1}M(a, b) = M(b - a, a)$
11. $M(a, b)M(-b, a) = \delta(a, b)\boldsymbol{\Phi}$
12. $M(a, b)M(a + b, -b) = \delta(a, b)\mathbf{I}$

Exercise 1.58: Identities in Matrix Versions

In the text we saw:

$$\varphi = \tfrac{1}{2}(1 + \sqrt{5}) \tag{1.2}$$

$$1 - \varphi = \tfrac{1}{2}(1 - \sqrt{5}) \tag{1.3}$$

$$\varphi^2 + 1 = \varphi + 2 = \varphi\sqrt{5} \tag{1.7a}$$

$$\varphi^3 = 2\varphi + 1 = 2 + \sqrt{5} \tag{1.8}$$

Ascertain that the above equations are true for matrix versions, namely:

$$\Phi = \tfrac{1}{2}[I + R(5)]$$

$$I - \Phi = \frac{1}{2}[I - R(5)]$$

$$\Phi^2 + I = \Phi + 2I = \Phi R(5)$$

$$\Phi^3 = 2\Phi + I = 2I + R(5)$$

- **Exercise 1.59: Groups and Isomorphism**

In this exercise, a and b are rational numbers, neither of which are 0.

A. Prove: The set of numbers of the form $a + b\sqrt{5}$ along with the operation of multiplication is a commutative group.
B. Prove: The set of matrices of the form M(a − b, 2b) along with the standard multiplication operation between matrices is a commutative group.
C. Prove that the two groups are isomorphic.
D. Prove: The set of numbers of the form $a + b\sqrt{5}$ together with the standard addition and multiplication operations between numbers forms a field.

- **Exercise 1.60: Norms**

A. a and b are rational numbers. We denote: $x = a + b\varphi$. Show/prove:
1. $\mu(1) = 1$
2. $\mu(\varphi) = \mu(1-\varphi) = 1$
3. $\mu(a) = |a|$
4. $\mu(b\varphi) = |b|$
5. $\mu(\varphi^n) = \mu((1-\varphi)^n) = 1$

B. Prove: The set of numbers of the form $a + b\varphi$, where a and b are **integer** numbers (neither of which are 0) and whose norm is 1, is a commutative group.

Answers, Hints and Partial Solutions

- **Exercise 1.2**

1. $-\varphi$, $1/\varphi$
2. $\pm\varphi$
3. $-\varphi < x < \varphi$
4. $\pm 1/\varphi$
5. $-1/\varphi < x < 1/\varphi$
6. 4 solutions: $\pm\varphi$,$\pm 1/\varphi$
7. $-\varphi < x < -1/\varphi$, $1/\varphi < x < \varphi$
8. φ^2
9. φ , $-1/\varphi$
10. φ, $1/\varphi$
11. $a\varphi$, $-a/\varphi$
12. $1/\varphi^2$, φ^2

13. $2\varphi, -2/\varphi$
14. $\varphi/2, -1/2\varphi$

Exercise 1.3

Solving this "systematically" is not necessary, since the solutions are obvious.

1. $(-1, 0), (0, -1), (\varphi, \varphi), \left(-\frac{1}{\varphi}, -\frac{1}{\varphi}\right)$
2. $(1, 0), (0, 1), (-\varphi, -\varphi), \left(\frac{1}{\varphi}, \frac{1}{\varphi}\right)$

Exercise 1.4

Try using "prominent" solutions (± 1) and reducing their factors:

1. $(x+1)(x^2 - x - 1) = 0$
2. $(x - 1\ x^2 - x - 1) = 0$
3. $(x - 1)(x^2 + x - 1) = 0$
4. $(x+1)(x^2 + x - 1) = 0$

Exercise 1.5

1. $-i\varphi, \frac{i}{\varphi}$
2. $i\varphi, \frac{-i}{\varphi}$
3. 4 solutions: $\pm i\varphi, \pm\frac{i}{\varphi}$
4. 4 solutions: $\pm\sqrt{\varphi}, \pm\frac{i}{\sqrt{\varphi}}$
5. 4 solutions: $\pm i\sqrt{\varphi}, \pm\frac{1}{\sqrt{\varphi}}$

Exercise 1.6

Using Vieta's theorem:

1. $x^2 - 2\varphi x + 1 = 0$
2. $x^2 - 2\varphi x + \varphi = 0$
3. $x^2 + \varphi x + \varphi^2 = 0$
4. $x^2 - \varphi^2 x + \varphi = 0$
5. $\varphi x^2 - x - \varphi^2 = 0$ or $x^2 + (1-\varphi)x - \varphi = 0$
6. $\varphi x^2 - x + (1-\varphi) = 0$
7. $x^2 - 2x - \varphi = 0$
8. $x^2 - 2x + \varphi - 1 = 0$ or $\varphi x^2 - 2\varphi x + 1 = 0$
9. $\varphi x^2 - 2x - 1 = 0$

Exercise 1.8

1. $2\sqrt{\varphi}$
2. 1
3. $\varphi\sqrt{\varphi}$
4. $2\varphi\sqrt{a}$

Exercise 1.9

$\sqrt{\varphi}$ or $\frac{1}{\sqrt{\varphi}}$ (Use Pythagoras theorem.)

- **Exercise 1.10**

Use the triangle inequality thrice and solve the resulting system.

- **Exercise 1.11**

A. The solution is based on the triangle inequality and what is given.
B. Multiply the corresponding sides of the inequalities.

- **Exercise 1.12**

1. First: φ^2, second: $\frac{1}{\varphi}$
2. First: φ , second: $\frac{\varphi}{\sqrt{5}}$
3. First: $\frac{\varphi^2}{2}$, second: $\frac{\varphi}{2}$.
4. $2\varphi^2$.
5. φ

- **Exercise 1.14**

φ or $1-\varphi$.

- **Exercise 1.15**

A. $L_1 = 1, L_2 = 3$.
D. $F_1 = 1, F_2 = 1$.

- **Exercise 1.16**

Both are $\varphi/2$.

- **Exercise 1.17**

A. $f(x) = (1 - x)^{-1} \Rightarrow f'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = (x - 1)^{-2}$
B. First: φ^3, second: $1/\varphi^2$.

- **Exercise 1.18**

1. $\frac{\sqrt{5}}{4}$
2. i
3. 1/2
4. $-i\varphi$
5. 1
6. $\sqrt{5}$

- **Exercise 1.19**

1. $\pm\left(\sqrt{\varphi} + \frac{i}{\sqrt{\varphi}}\right)$
2. $\pm\left(\sqrt{\varphi} + \frac{i}{\sqrt{\varphi}}\right)$

- **Exercise 1.21**

The area is $\sqrt{5}$ and the circumference is 4φ.

- **Exercise 1.23**

  A. $AH = 1, HC = \frac{1}{\varphi}, BH = \frac{1}{\sqrt{\varphi}}$

- **Exercise 1.26**

  Both are φ^2.

- **Exercise 1.27**

  A. The diagonal of the rectangle is actually a long, narrow parallelogram with area = 1.

- **Exercise 1.29**

  B. $DC = 1/\varphi$, $AD = 1$.
  C. $A = 36°$, $B = C = 72°$.

- **Exercise 1.30**

  $\varphi\sqrt{\varphi}$

- **Exercise 1.31**

  A. $SN \cdot NT = AN \cdot NC$.
  B. Mark $NT = x$.
  C. Use the law of cosines for triangles ANT, CNT.

- **Exercise 1.32**

  Mark $MA = x$ and use the angle bisector theorem.
  Alternatively, you can write $\mathrm{tg} C = 2$, and prove: $\mathrm{tg}\,(C/2) = 1/\varphi$.

- **Exercise 1.33**

  Mark $BD = a$, $DC = x$ to calculate x using the Pythagorean theorem.

- **Exercise 1.34**

  A. φ , $\sqrt{\varphi}$, $\varphi\sqrt{\varphi}$ (respectively).
  B. $-1/\varphi$. You can use the law of cosines or first calculate $\sin \angle AFD$.
  C. $DG = \frac{1}{\sqrt{\varphi}}$ (You can use similar triangles.)
  D. They have a common height, making it easier to perform the calculations.

- **Exercise 1.35**

  A. $AF = \sqrt{\varphi}$.

- **Exercise 1.38**

  A. Area of definition: $\mathbf{R} \setminus \{-\varphi, 1/\varphi\}$.
  Intersections with the axes: $(\varphi, 0)$, $(-1/\varphi, 0)$, $(0, 1)$.

Asymptotes: $x = -\varphi$, $x = 1/\varphi$, $y = 1$.
Increasing in intervals: $(1/\varphi, \infty)$, $(-\varphi, 1/\varphi)$, $(-\infty, -\varphi)$.
B. Both sides are equal to 5/2.

- **Exercise 1.40**

A. $0, \pm 1, \pm 2$
B. Minimum points: $(\varphi, -2)$, $(-1/\varphi, -2)$
Maximum points: $(-\varphi, 2)$, $(1/\varphi, 2)$

- **Exercise 1.41**

B. $\frac{\sqrt{5}}{\varphi^2}$
C. $\sqrt{5}\,\varphi$

- **Exercise 1.44**

0

- **Exercise 1.45**

Both sides are equal to $2\ln\varphi$.

- **Exercise 1.46**

A. $1/\varphi^2$
B. $1/\varphi$

- **Exercise 1.47**

$y = -\varphi x + \varphi\sqrt{5}$

- **Exercise 1.48**

B. Four points of intersection: $(\pm\varphi, \pm 1)$.
The four tangents: $y = \pm\varphi x, \pm \varphi$. Area of the diamond: 2φ.

- **Exercise 1.49**

A circle with center at $(\sqrt{5}, 0)$ and radius 2.

- **Exercise 1.51**

15. Work on each side separately.

- **Exercise 1.52**

A. $S = \{\varphi, 1-\varphi\}$, $S = \emptyset$.
B. The equation obtained (for all) is $x^2 - kx - k^2 = 0$.
C. The lines are $y = \varphi x$, $y = (1-\varphi)x$.

- **Exercise 1.53**

The solution is (1,1). You don't have to solve it systematically, because the solution is obvious and it is unique because the determinant of the system is $\delta(a, b)$, and is different than 0 according to the conditions.

- **Exercise 1.54**

A. Compare the φ coefficient (in the result) to 0.
B. Use the result of section A, or directly.

- **Exercise 1.55**

A. Mark $x = a + b\varphi$, show that $2x - 2a - b = b\sqrt{5}$, and square the two sides.
C. $a + b\,(1-\varphi)$.

- **Exercise 1.59**

A. Closure: $(a + b\sqrt{5})\,(c + d\sqrt{5}) = (ac + 5bd) + (ad + bc)\,\sqrt{5}$
Associativity: Occurs in **R**.
The identity element is $1 + 0 \cdot \sqrt{5}$
The inverse of $a + b\sqrt{5}$ is $\frac{a}{a^2 - 5b^2} + \frac{-b}{a^2 - 5b^2}\sqrt{5}$
(Also true when $a=0$ or $b=0$, but this must be examined separately!)
It is important to first show that for all a and b (both not zero): $a^2 - 5b^2 \neq 0$

- **Exercise 1.60**

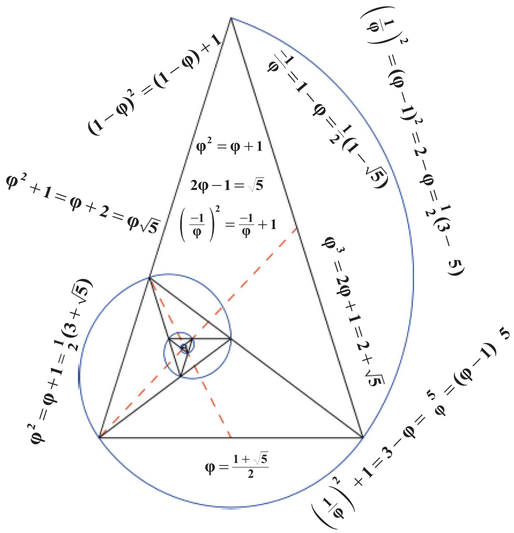
A. Question 5: Solve in a similar manner as Eq. (1.25).
B. The proof is identical to that in the text, where a and b are rational.

Introducing Golden Triangles

Contents

The golden proportion is a scale of proportions which makes the bad difficult [to produce] and the good easy.
Albert Einstein (1879–1955)

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*, Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2_2



The Golden Triangle
(Daniel Briskman)

A golden triangle is an isosceles triangle in which the ratio of the side to the base is equal to the golden ratio.

■ Introduction to Chapter 2

In this chapter we shall become acquainted with a pair of golden triangles and, naturally, combine it with plane trigonometry. The golden ratio will be expressed in terms of the trigonometric functions of angles that are multiples of 18°.

The chapter will conclude with a general analysis of the regular 5-sided polygon (pentagon) and the pentagram (5-pointed star).

2.1 Wide and Narrow Golden Triangles

Note: When given an isosceles triangle, the first letter specifies the vertex angle (opposite the base). This convention also applies to the exercises.

Let us attempt to find an obtuse isosceles triangle, ABC, such that if we remove a similar triangle, DAC, from it, the remaining triangle, BAD, will also be isosceles (see ◘ Fig. 2.1).

Assuming that such a triangle exists (we shall prove its existence in the next section), we first determine what the ratios of their sides will be.

We denote: $a = AB = AC = BD$, $b = DA = DC$

We assume, as noted above, that $\Delta ABC \sim \Delta DAC$, and thus obtain:

$$\frac{BC}{AC} = \frac{AC}{DC}$$

$$\frac{a+b}{a} = \frac{a}{b}$$

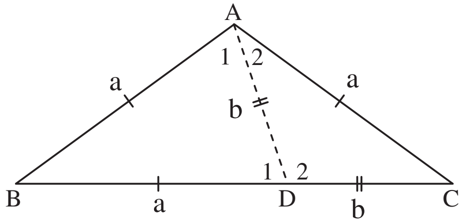


Fig. 2.1 The golden triangle

This equation was already solved in Chap. 1 : $\frac{a}{b} = \varphi$.

From a geometrical aspect, this result points to a number of facts:

- Point D divides segment BC in the golden ratio.
- The ratio of the length of the base to the side lengths of triangle ABC and triangle DAC is φ.
- The ratio of the length of the side to the length of the base of triangle BAD is φ.

From this point on, we shall call triangle ABC (or any similar obtuse triangle) a **wide golden triangle**, and we shall call triangle BAD (or any similar acute triangle) a **narrow golden triangle**.
Thus, we can define the following:

A wide golden triangle is an obtuse isosceles triangle in which the ratio of the lengths of the base to the side is φ. A narrow golden triangle is an acute isosceles triangle in which the ratio of the lengths of the side to the base is φ.

Another way to define a wide golden triangle**:**
ABC is a wide golden triangle if it is an obtuse isosceles trinangle such that when we remove from it an isosceles triangle ABD the remaining triangle DAC is similar to the triangle ABC (Fig. 2.2).

Triangle BAD can also be similarly divided so as to produce two new golden triangles (one narrow and the other wide) DEA and EBD. We shall here make do with the equality: $\frac{b}{a-b} = \varphi$.

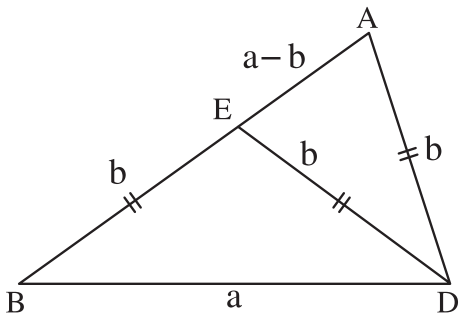


Fig. 2.2 The golden triangle

2

2.2 The Angles in Golden Triangles

In the previous section we assumed that golden triangles exist without actually being certain that they do, because we carried out a construction that was supposed to achieve two things simultaneously: one of the inner triangles was should be similar to the original, and the second should be isosceles. By determining the various angles in ◘ Fig. 2.1, we can show that such triangles exist. We shall therefore return to ◘ Fig. 2.1 and label it as follows: $B = C = \alpha$. Hence:

$$A_2 = C = \alpha$$
$$D_1 = A_2 + C = 2\alpha$$
$$A_1 = D_1 = 2\alpha$$

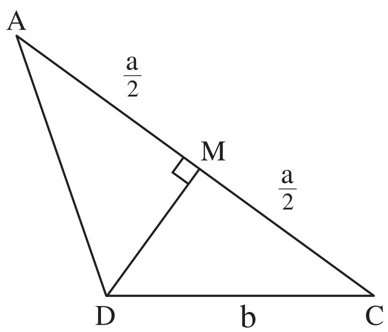
Now, in triangle ABD:

$$A_1 + B + D_1 = 180^\circ$$
$$2\alpha + \alpha + 2\alpha = 180^\circ$$
$$\alpha = 36^\circ$$

To summarize:
The angles of a wide golden triangle are 36°, 108°, 36°, and the angles of a narrow golden triangle are 72°, 36°, 72°.

Note that all the angles are multiples of 18°.

Now, we shall divide triangle ADC into two right-angled triangles: DMC and DMA (◘ Fig. 2.3).



◘ **Fig. 2.3**

In triangle DMC:

$$\cos C = \frac{MC}{DC} = \frac{a}{2b}$$

Therefore:

$$\cos 36^\circ = \frac{\varphi}{2} \tag{2.1a}$$

Hence we can conclude that:

$$\sin 54^\circ = \frac{\varphi}{2} \tag{2.1b}$$

Also:

$$\cos 72^\circ = 2\cos^2 36^\circ - 1 = \frac{1}{2}\left(\varphi^2 - 2\right)$$

$$\cos 72^\circ = \sin 18^\circ = \frac{1}{2}\left(\varphi^2 - 2\right) \tag{2.2}$$

2.3 Area Ratios

Let us again examine ◘ Fig. 2.1, this time focusing on the relation between the areas of the triangles in the figure.

All three triangles have a common height (that which extends from angle A, and denoted by h). We can take advantage of this fact when doing the calculations. Thus we have:

$$S_{ABC} = \frac{1}{2}(a + b)h$$
$$S_{BAD} = \frac{1}{2}ah$$
$$S_{DAC} = \frac{1}{2}bh$$

Which gives us:

$$\frac{S_{ABC}}{S_{BAD}} = \frac{a+b}{a} = \varphi$$
$$\frac{S_{BAD}}{S_{DAC}} = \frac{a}{b} = \varphi$$

We explain these results in simple words:

- The ratio between the area of a wide golden triangle and the area of a narrow golden triangle contained therein (as in ◘ Fig. 2.1) is **φ**.
- The ratio between the area of a narrow golden triangle and the area of a wide golden triangle that combine to form a wide golden triangle (as in ◘ Fig. 2.1) is **φ**.

Now, we repeat the process described above, this time referring to ◘ Fig. 2.2, where the common height extends from vertex D. We have:

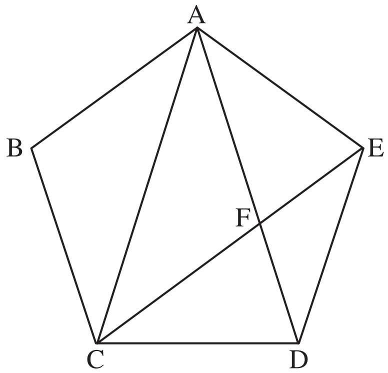
$$\frac{S_{BAD}}{S_{EBD}} = \frac{a}{b} = \varphi$$
$$\frac{S_{EBD}}{S_{DAE}} = \frac{b}{a-b} = \frac{1}{\varphi - 1} = \varphi$$

In words:

- The ratio between the area of a narrow golden triangle and the area of a wide golden triangle contained therein (as in ◘ Fig. 2.2) is **φ**.
- The ratio between the area of a wide golden triangle and the area of a narrow golden triangle that combine to form a narrow golden triangle (as in ◘ Fig. 2.2) is **φ**.

2.4 Pentagons and Pentagrams

In regular pentagon ABCDE, we draw diagonals AC, EC, and AD (◘ Fig. 2.4).
It can be shown that triangles DEC, EAD, and BCA are isosceles and that they are congruent.
We shall use the formula to calculate the angles of a regular polygon with n sides:



◘ **Fig. 2.4**

$\frac{180° (n-2)}{n}$ and use n = 5. We obtain:

$$\angle AED = \angle EDC = 108°$$

Therefore:

$$\angle EAD = \angle EDA = \angle ECD = 36°$$

Hence, triangles EAD and DEC are wide golden triangles.
Similarly, by calculating the angles in triangle ACD, we see that it is a narrow golden triangle.
We now draw the remaining diagonals to obtain a five-pointed star inscribed in the pentagon.
In the illustration that we obtained, there is a large quantity of golden triangles:

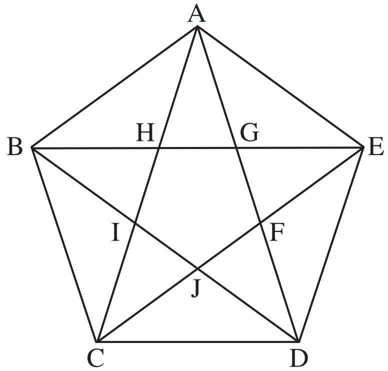


Fig. 2.5

- narrow and small (such as EGF, DFJ, total 5)
- narrow and medium in size (such as EJD, CDF, total 10)
- narrow and large in size (such as ACD, BDE, total 5)
- wide and small (such as JCD, FDE, total 5)
- wide and medium in size (such as JEB, DEC, total 10)

A total of 35 golden triangles!
The "heart" of pentagram HGFJI is also a regular pentagon, therefore one can repeat the process to obtain a smaller star ad infinitum.

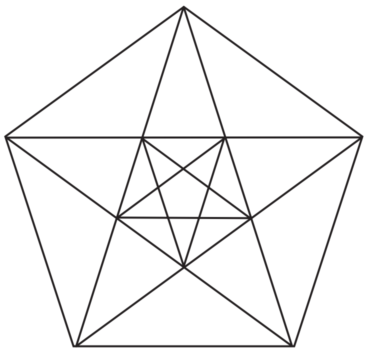


Fig. 2.6

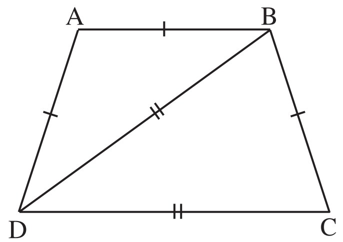
Exercises for Chapter 2

Note: when given an isosceles triangle, the first letter (on the left) specifies the vertex angle (opposite the base).

The exercises in chapter two aim to improve the understanding of the role of golden triangles in geometry.

- **Exercise 2.1: Isosceles Trapezoid**

ABCD is an isosceles trapezoid with large base DC and small base AB. Diagonal BD divides the trapezoid into two golden triangles: ADB (wide) and DCB (narrow).



A. Prove that:

$$\frac{S_{ABCD}}{S_{DCB}} = \frac{S_{DCB}}{S_{ADB}} = \varphi$$

(You can assume without loss of generality that AB = 1).

B. The sides of the trapezoid are extended until they meet at point M. Prove that:

$$\frac{S_{MDC}}{S_{ABCD}} = \varphi$$

C. The diagonals of the trapezoid meet at point P. Prove that:

$$\Delta ADB \cong \Delta PDC$$

- **Exercise 2.2: Deltoid**

ABCD is a convex deltoid golden triangle composed of narrow golden triangle ADB and wide golden triangle CDB (common base DB). Prove that:

$$\frac{S_{ABD}}{S_{ABCD}} = \frac{\varphi}{2}$$

$$\frac{S_{ABCD}}{S_{CBD}} = 2\varphi^2$$

(You can assume without loss of generality: DB = 1.)

- **Exercise 2.3: Narrow Golden Triangle and Triangle with 54°**

ABC is a narrow golden triangle. D is a point on the continuation of base CB such that $\angle ADB = 54°$

A. Prove:

$R_{ADC} = R_{ADB} = BC$

B. Prove:

$$\frac{S_{ADC}}{S_{ABC}} = \frac{S_{ABC}}{S_{ABD}} = \varphi$$

(You can assume without loss of generality: BC = 1.)

- **Exercise 2.4: Wide Golden Triangle and Narrow Golden Triangle**

ABC is a golden triangle. D is a point on BC such that ADC = 54°.

A. Prove:

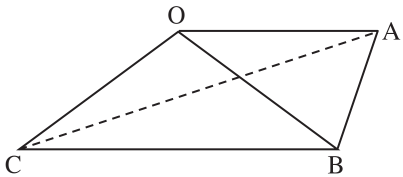
$R_{ADC} = R_{ABD}$

B. Prove:

$$\frac{S_{ADC}}{S_{ABD}} = 2\varphi$$

- **Exercise 2.5: "Adjacent" Golden Triangles**

OCB and OBA are two golden triangles (the first is wide, the second narrow) with common side OB.



A. Prove:

$S_{OCB} = S_{ABC}$

$S_{OBA} = S_{OCA}$

B. Prove:

$$\frac{S_{OCB}}{S_{OBA}} = \frac{S_{ABC}}{S_{OCA}} = \varphi$$

C. Prove:

$$\frac{S_{OCBA}}{S_{OCB}} = \varphi$$

(You can assume without loss of generality: AB = 1.)

- **Exercise 2.6: Recognizing Golden Triangles**

In acute isosceles triangle ABC, D is a point on side AC such that BD bisects angle B.

Prove: If AD = BC, then triangles ABC, BCD, and DAB are golden triangles.

- **Exercise 2.7: Bisector of Wide Golden Triangle**

In wide golden triangle ABC, the length of base BC is φ, AH is the height to the base, and M is a point on side AC such that BM bisects angle B and intersects AH at point T. Prove that:

1. $AM = \frac{1}{\varphi^2}, \quad MC = \frac{1}{\varphi}$
2. $\frac{BT}{TM} = \varphi^2$
3. $\frac{S_{MBC}}{S_{ABM}} = \varphi$
4. $\frac{S_{BTH}}{S_{BAT}} = \frac{\varphi}{2}$

- **Exercise 2.8: Circumcircle and Incircle**

Prove that in a wide golden triangle, the ratio of the radius of the circumcircle (the circumradius) to the radius of the incircle is 2φ.

- **Exercise 2.9: Golden Triangle Inscribed in a Circle (A)**

ABC is a golden triangle. D is a point on side AC such that BD bisects angle B. The extension of segment BD intersects the circle at point E. Lines AE and BC intersect at point F, which is outside the circle.

A. Prove:

$$\frac{FC}{CB} = \frac{FB}{FC} = \varphi$$

B. Prove that triangles ABC and BEA are congruent golden triangles.

C. Prove that triangles ADE and BCD are congruent golden triangles.

- **Exercise 2.10: Golden Triangle Inscribed in a Circle (B)**

Narrow golden triangle ABC is inscribed in a circle with center O.

A. Prove:

$$\frac{S_{OBC}}{S_{ABOC}} = \frac{\varphi}{2}$$

B. Deduce:

$$\frac{S_{ABC}}{S_{OBC}} = \sqrt{5}$$

$$\frac{S_{ABC}}{S_{ABOC}} = \frac{\varphi\sqrt{5}}{2}$$

- **Exercise 2.11: Golden Triangle Inscribed in a Circle (C)**

ABC is a narrow golden triangle. DC is the diameter, which intercepts side AB at point E.

Prove:

$$\frac{S_{ABC}}{S_{BCE}} = \frac{S_{BCE}}{S_{AEC}} = \varphi$$

- **Exercise 2.12: Golden Triangle Intersecting a Circle**
  Base BC of a narrow golden triangle is the diameter of a circle with radius length ½. Sides AB and AC of the triangle intersect the circle at points E and F, respectively.
  Calculate the lengths of segments BE, EA, EF.

- **Exercise 2.13: Circle Inscribed in a Golden Triangle**
  A circle with radius 1 is inscribed in a narrow golden triangle.
  Prove: The distance between the center of the circle and the top vertex is 2φ.

- **Exercise 2.14: A Semicircle Inscribed In a Golden Triangle**
  ABC is a narrow golden triangle. A "semi-circle" with center is inscribed in the triangle such that it is tangent to base BC at point E and to side AC at point D.
  A. Prove:

  $$AD = DC = 1/2\ AC$$

  B. Prove:

  $$\frac{S_{ABC}}{S_{ODEC}} = \varphi$$

  (You can assume without loss of generality: BC = 1.)

- **Exercise 2.15: Golden Triangles and a Circle**
  Triangle ABC is inscribed in a circle. A line tangent to the circle at point C intersects the continuation of segment AB at point D.
  Given: AC = AB = CD
  Show that triangles ABC, BDC, and ACD are golden triangles. (You must make a distinction between two cases!)

- **Exercise 2.16: A Geometric Series**
  Calculate the following:

  $$1 + \cos 36^\circ + \cos^2 36^\circ + \cos^3 36^\circ + \ldots$$

  $$1 - \sin 18^\circ + \sin^2 18^\circ - \sin^3 18^\circ + \ldots$$

- **Exercise 2.17: Trigonometric Equations**
  A. Solve the following equation

  $$4\cos^2\alpha - 2\cos\alpha - 1 = 0$$

  where $0^\circ < \alpha < 180^\circ$.
  B. Solve the following equation

  $$4\cos^2\alpha + 2\cos\alpha - 1 = 0$$

  where $0^\circ < \alpha < 180^\circ$.

2

- **Exercise 2.18: Multiples of 18°**

Prove that

$$\left\{\cos\left(36^{\circ}k\right) \mid k \in \mathbf{Z}\right\} = \left\{\sin\left(18^{\circ} + 36^{\circ}k\right) \mid k \in \mathbf{Z}\right\} = \{\varphi/2, -\varphi/2, 1/2\varphi, -1/2\varphi, 1, -1\}$$

- **Exercise 2.19: Calculations of Sine and Cosine**

A. Prove that:

$$\sin 36^{\circ} = \cos 54^{\circ} = \frac{1}{2}\sqrt{3-\varphi}$$

$$\sin 72^{\circ} = \cos 18^{\circ} = \frac{1}{2}\sqrt{2+\varphi}$$

B. Show that it is impossible to write $\sin 36^{\circ} = a + b\varphi$ if a and b are rational numbers. Show the same for $\sin 72^{\circ}$.

- **Exercise 2.20: A Regular Decagon**

Given a regular decagon with side length 1 and radius of its circumcircle R, prove that $\mathbf{R} = \varphi$.

Answers, Hints and Partial Solutions

- **Exercise 2.1**

A. Make use of the common height.

- **Exercise 2.3**

B. Pay attention to the following two points:
– Triangle CAD is isosceles.
– The three triangles have the same height.

- **Exercise 2.5**

Note the common heights.

- **Exercise 2.6**

Mark AB = AC = a, BC = AD = b, and use the angle bisector theorem.

- **Exercise 2.7**

1. You Can Use the Angle Bisector Theorem or the Law of Sines.

- **Exercise 2.9**

Calculate the angles.

- **Exercise 2.10**

A. Note that OA = OB = OC = R.
B. $S_{ABC} = S_{ABOC} + S_{OBC}$.

- **Exercise 2.11**
  Calculate all the angles.

- **Exercise 2.12**

  $$EF = \frac{\varphi}{2} \ , \ \ EA = \frac{\varphi^2}{2} \ , \ \ BE = \frac{1}{2\varphi}$$

- **Exercise 2.14**
  A. Calculating the angles will lead to the conclusion that triangle OCA is isosceles.

- **Exercise 2.16**
  The first: $2\varphi^2$, the second: $2/\varphi^2$.

- **Exercise 2.17**
  A. 36°, 108°
  B. 72°, 144°

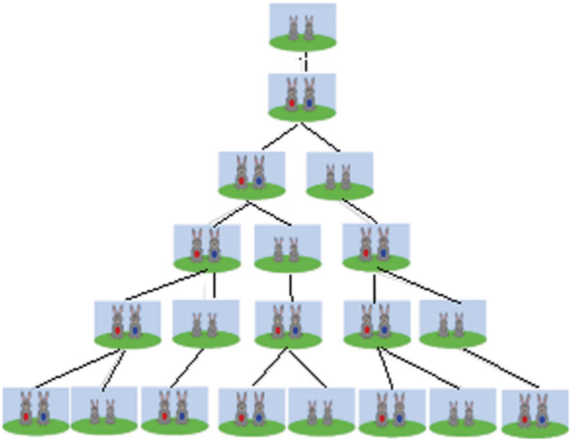
Check for updates

The Fibonacci Sequence

Contents

God created the integers, all else is the work of man.
Leopold Kronecker (1823–1891)

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*,
Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2_3



Fibonacci reproduction in rabbits
(▸https://en.wikipedia.org/wiki/Fibonacci_number#/media/File:FibonacciRabbit.svg)

Introduction to Chapter 3

In this chapter we shall part from the geometric aspects of the golden ratio and start exploring the Fibonacci sequence.

The proofs of many beautiful relationships between the members of the sequence are based on the linearization of the powers of golden ratio, and the principle of comparing the rational coefficients equation shown in the previous chapter. Developing formulas for the sums of the sequence will be based on "telescopic cancellation." (Other methods will be offered in the exercises.)

Expanding the sequence for negative indices (which is interesting in itself) will enable us to discuss isomorphism in a new context. (The two last sections can be skipped in the first reading.)

The exercises will include both technical aspects and additional theoretical developments to those in the text.

We will also extend the golden ratios and geometrical figures to of Fibonacci and Lucas numbers. The reason for this is that the techniques applied to the golden ratios turn out to be very useful for this related topic.

3.1 The Fibonacci Sequence and the Exponents of the Golden Ratio

The sequence (F_n), named after Leonardo Fibonacci (c, 1180-c. 1250), is recursively defined as follows

$$\mathbf{F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).} \tag{3.1}$$

Thus, the beginning of the sequence looks like:

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Below are some sequential powers of φ, repeatedly using the equality of $\varphi^2 = \varphi + 1$:

$$\varphi^1 = 1 \cdot \varphi + 0$$
$$\varphi^2 = 1 \cdot \varphi + 1 \text{ (just shown)}$$
$$\varphi^3 = \varphi^2 + \varphi = 2\varphi + 1 \text{ (An easy exercise)}$$
$$\varphi^4 = 2\varphi^2 + \varphi = 3\varphi + 2$$
$$\varphi^5 = 3\varphi^2 + 2\varphi = 5\varphi + 3$$
$$\varphi^6 = 5\varphi^2 + 3\varphi = 8\varphi + 5$$

From looking at the sequence up to this point, it seems that successive powers of φ can be written as linear expressions ofφ, where the coefficients of φ are sequential members of the Fibonacci sequence. It is interesting to note that the "free numbers," which are not coefficients ofφ, are also members of the Fibonacci sequence. We define $F_0 = 0$ in order to apply mathematical induction. It is consistent since $F_2 = F_1 + F_0$ holds.

We shall use induction to prove that our hypothesis is correct, in other words, that the following holds for each natural n:

$$\boldsymbol{\varphi^n = F_n \varphi + F_{n-1}} \tag{3.2}$$

Proof It is obvious that the following holds: $\varphi^1 = F_1 \varphi + F_0$.
We assume: $\varphi^k = F_k \varphi + F_{k-1}$ and shall prove: $\varphi^{k+1} = F_{k+1}\varphi + F_k$:

$$\begin{aligned}
\varphi^{k+1} &= \varphi\varphi^k \\
&= \varphi\,(F_k\varphi + F_{k-1}) \\
&= F_k\varphi^2 + F_{k-1}\varphi \\
&= F_k\,(\varphi + 1) + F_{k-1}\varphi \\
&= (F_k\,\varphi + F_{k-1})\varphi + F_k \\
&= F_{k+1}\varphi + F_k
\end{aligned}$$

Now, we notice that in the equations above, we only make use of the fact that φ satisfies the golden equation. The expression (1 − φ) also satisfies the golden equation. Hence, we can conclude:

$$\boldsymbol{(1-\varphi)^n = F_n(1-\varphi) + F_{n+1} = -F_n\varphi + F_{n+1}} \tag{3.2b}$$

It is worth observing the following:

A. φ^n and $(1-\varphi)^n$ are normalized φ -numbers since they are products of normalized ϕ-numbers, c.f (1.23) or (1.25).
B. $(1-\varphi)^n$ is the *companion* of φ^n, denoted by: $(\varphi^n)^* = (1-\varphi)^n$. As defined in ► Chapter 1 (1.12a)
C. φ^n and $(1-\varphi)^n$ are normalized φ^n-numbers, that is to say: $\mu(\varphi^n) = \mu[(1-\varphi)^n] = 1$ (see Exercise 1.60 in ► Chap. 1).

3

3.2 Binet's Formula

Our goal in this section is to find a formula for the sequence F_n.

This formula is named after Jacques Binet (1786–1856).

First Proof (Using Powers of the Golden Ratio)

We write the two results obtained in the previous section adjacent to each other:

$$\varphi^n = F_n\varphi + F_{n+1} \tag{3.2a}$$

$$(1-\varphi)^n = F_n(1-\varphi) + F_{n+1} \tag{3.2b}$$

We subtract one equation from the other to obtain:

$$\varphi^n - (1-\varphi)^n = F_n(2\varphi - 1)$$

$$\varphi^n - (1-\varphi)^n = F_n\sqrt{5}$$

Hence:

$$\mathbf{F_n = \frac{1}{\sqrt{5}}\left[\varphi^n - (1-\varphi)^n\right]} \tag{3.3a}$$

(Also for $n = 0$).

Since $(-a)^{2n-1} = -a^{2n-1}$, $(-a)^{2n} = a^{2n}$ for all real a and all natural n, we obtain:

$$F_{2n} = \frac{1}{\sqrt{5}}\left[\varphi^{2n} - (\varphi - 1)^{2n}\right]$$

$$F_{2n-1} = \frac{1}{\sqrt{5}}\left[\varphi^{2n-1} - (\varphi - 1)^{2n-1}\right]$$

We now substitute φ in (3.3a) with its numeric value and obtain:

$$\mathbf{F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]} \tag{3.3b}$$

It is quite surprising that such a complicated looking formula yields only natural numbers!

Second Proof (Using Analytical Geometry)

We consider the straight line whose equation is

$$y = F_n x + F_{n-1}.$$

According to (3.2), this line passes through the two points (φ, φ^n) and $(1-\varphi, (1-\varphi)^n)$.

Therefore, the slope of the line is:

$$\frac{\varphi^n - (1-\varphi)^n}{2\varphi - 1} = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

On the other hand, the slope is F_n (the coefficient of x in the equation for the straight line), therefore:

$$F_n = \frac{1}{\sqrt{5}}\left[\varphi^n - (1-\varphi)^n\right]$$

- **Third Proof (Classic Proof)**

We shall look for a geometric sequence (x^n) where the following recursive equation holds:

$$f_{n+2} = f_{n+1} + f_n.$$

Substituting into the equation gives:

$$x^{n+2} = x^{n+1} + x^n$$

$$x^2 = x^1 + 1$$

$$x = \varphi, 1 - \varphi$$

It is possible to validate that for **the entire sequence** (f_n) with formula $f_n = \alpha\varphi^n + \beta(1-\varphi)^n$, and where coefficients α and β are constants the above recursive equation holds. **The converse** (in this case, f_0 and f_1 must be given) is also true and this can be validated.

We now use $(f_0, f_1) = (F_0, F_1) = (0, 1)$ and arrive at the following system of equations:

$$\alpha\varphi^0 + \beta(1-\varphi)^0 = 0$$

$$\alpha\varphi^1 + \beta(1-\varphi)^1 = 1$$

From the first equation, $\alpha = -\beta$. We substitute this in the second equation:

$$\alpha\varphi - \alpha(1-\varphi) = 1$$

$$\alpha(2\varphi - 1) = 1$$

$$\alpha = \frac{1}{\sqrt{5}}$$

Hence:

$$\beta = -\frac{1}{\sqrt{5}}$$

From here we obtain:

$$F_n = \frac{1}{\sqrt{5}}\left[\varphi^n - (1-\varphi)^n\right] \quad \text{Q.E.D.}$$

3

3.3 Key Relationships Between Members of the Sequence

The Fibonacci sequence has many lovely features, and in this section, we shall point out the main ones. (Additional features will become apparent in the exercises.)

For the proofs, we shall avail ourselves of Eqs. (3.2):

$$\varphi^n = F_n\varphi + F_{n+1}$$

$$(1-\varphi)^n = F_n(1-\varphi) + F_{n+1} = -F_n\varphi + F_{n+1}$$

- **Property 1: Cassini Formula**

(Jean-Dominique Cassini, 1625–1712)

$$\mathbf{F_{n+1}F_{n-1} - F_n^2 = (-1)^n} \tag{3.4a}$$

Proof We first write both equalities:

$$\varphi^n = F_n\varphi + F_{n+1} \tag{3.2a}$$

$$(1-\varphi)^n = F_n(1-\varphi) + F_{n-1} \tag{3.2b}$$

We then multiply the respective sides of each equation, and obtain, using $\varphi(1-\varphi) = -1$

$$\varphi^n(1-\varphi)^n = (F_n\varphi + F_{n-1})[F_n(1-\varphi) + F_{n-1}]$$
$$(-1)^n = -F_n^2 + F_nF_{n-1}\varphi + F_nF_{n-1}(1-\varphi) + F_{n-1}^2$$
$$(-1)^n = -F_n^2 + F_nF_{n-1} + F_{n-1}^2$$
$$(-1)^n = -F_n^2 + F_{n-1}(F_n + F_{n-1})$$
$$(-1)^n = -F_n^2 + F_{n-1}F_{n+1}$$

The next relationship is important: In (1.10), δ was defined as:

$$\begin{aligned}\delta(F_{n-1}, F_n) &= F_{n-1}^2 + F_nF_{n-1} - F_n^2\\ &= F_{n-1}(F_{n-1} + F_n) - F_n^2\\ &= F_{n-1}F_{n+1} - F_n^2\\ &= (-1)^n\end{aligned}$$

Hence, Cassini's formula may be written as follows:

$$\delta(\mathbf{F_{n-1}, F_n}) = (-\mathbf{1})^{\mathbf{n}} \tag{3.4b}$$

The last equation corresponds nicely with the fact that $\mu(\varphi^n) = 1$, as mentioned at the end of ► Chap. 1 (see (1.24')).

- **Property 2: Even and Odd Indices**

$$\mathbf{F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n(F_{n+1} + F_{n-1})} \tag{3.5a}$$

$$\mathbf{F_{2n-1} = F_n^2 + F_{n+1}^2} \tag{3.5b}$$

Proof On the one hand:

$$(\varphi^n)^2 = \varphi^{2n} = F_{2n}\varphi + F_{2n-1}$$

On the other hand:

$$\begin{aligned}(\varphi^n)^2 &= (F_n\varphi + F_{n-1})^2 \\ &= F_n^2\varphi^2 + 2F_nF_{n-1}\varphi + F_{n-1}^2 \\ &= F_n^2\varphi + F_n^2 + 2F_nF_{n-1}\varphi + F_{n-1}^2 \\ &= (F_n^2 + 2F_nF_{n-1})\,\varphi + (F_n^2 + F_{n-1}^2)\end{aligned}$$

Based on the principle of comparing the rational coefficients of an equation (since this is about two φ-numbers), the following must be true:

$$F_{2n} = F_n^2 + 2F_nF_{2n-1}$$

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$

- **Property 3: "Index Sum"**

$$\mathbf{F_{n+m} = F_nF_{m+1} + F_{n-1}F_m = F_{n+1}F_m + F_nF_{m-1}} \tag{3.6}$$

It is obvious that if the first equality is true, then the second must automatically hold true because $F_{n+m} = F_{m+n}$.

Therefore, we shall prove the first.

On one hand:

$$\varphi^n\varphi^m = \varphi^{n+m} = F_{n+m}\varphi + F_{n+m-1}.$$

On the other hand:

$$\begin{aligned}\varphi^n\varphi^m &= (F_n\varphi + F_{n-1})\,(F_m\varphi + F_{m-1}) \\ &= F_nF_m\varphi + F_nF_m + F_nF_{m-1}\varphi + F_{n-1}F_m\varphi + F_{n-1}F_{m-1} \\ &= (F_nF_m + F_nF_{m-1} + F_{n-1}F_m)\,\varphi + F_nF_m + F_{n-1}F_{m-1}\end{aligned}$$

Therefore, the following must hold:

$$\begin{aligned}F_{n+m} &= F_nF_m + F_nF_{m-1} + F_{n-1}F_m \\ &= F_n\,(F_m + F_{m-1}) + F_{n-1}F_m \\ &= F_nF_{m+1} + F_{n-1}F_m\end{aligned}$$

By substituting $m = n$, we get "back" equation (3.5a).

(In ► Chap. 5 we present a simpler proof for (3.5a)).

3

3.4 Sums

The formulas for the sum of the first n elements, the first n elements in the even positions and the first n elements in the odd positions, can be obtained using "telescopic cancellation":

$$\begin{aligned}&F_1 + F_2 + F_3 + \cdots + F_n \\ &= (F_3 - F_2) + (F_4 - F_3) + \cdots + (F_{n+2} - F_{n-1}) \\ &= F_{n+2} - F_2 \\ &= F_{n+2} - 1\end{aligned}$$

$$\begin{aligned}&F_2 + F_4 + F_6 + \cdots + F_{2n} \\ &= (F_3 - F_1) + (F_5 - F_3) + \cdots + (F_{2n+1} - F_{2n-1}) \\ &= F_{2n+2} - F_1 \\ &= F_{2n+1} - 1\end{aligned}$$

$$\begin{aligned}&F_1 + F_3 + F_5 + \cdots + F_{2n-1} \\ &= (F_2 - F_0) + (F_4 - F_2) + \cdots + (F_{2n} - F_{2n-2}) \\ &= F_{2n} - F_0 \\ &= F_{2n}\end{aligned}$$

To summarize the results:

$$\mathbf{F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1} \tag{3.7a}$$

$$\mathbf{F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1} \tag{3.7b}$$

$$\mathbf{F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}} \tag{3.7c}$$

Now we proceed to calculate the sum of squares of elements. We shall prove:

$$\mathbf{F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_nF_{n+1}} \tag{3.8}$$

Notice first that:

$$F_n^2 = F_nF_n = F_n\,(F_{n+1} - F_{n-1}) = F_nF_{n+1} - F_nF_{n-1}$$

Therefore:

$$\begin{aligned}&F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 \\ &= (F_1F_2 - F_1F_0) + (F_2F_3 - F_2F_1) + \cdots + (F_nF_{n+1} - F_nF_{n-1}) \\ &= F_nF_{n+1} - F_1F_0 \\ &= F_nF_{n+1}\end{aligned}$$

Here is a neat geometric depiction of the last equalities (■ Fig. 3.1):

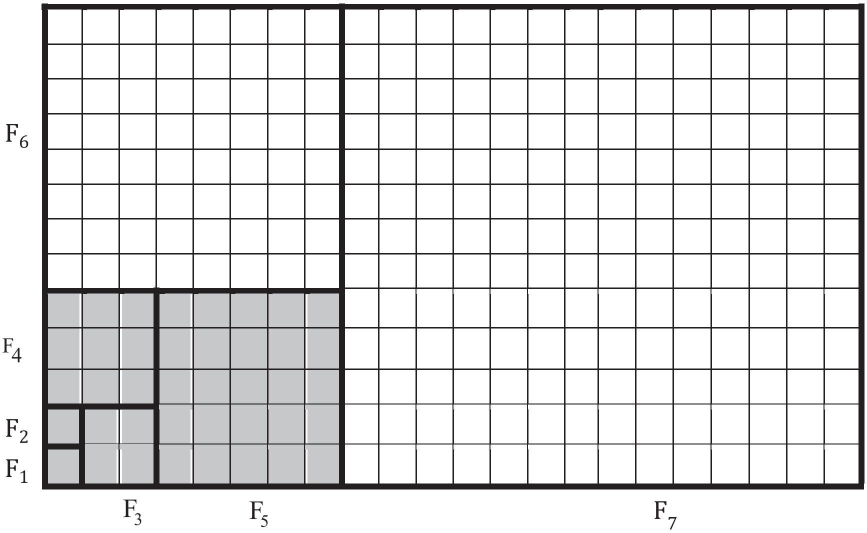


Fig. 3.1 Geometric depiction

$F_1=1, \quad F_2=1, \quad F_3=2, \quad F_4=3, \quad F_5=5, \quad F_6=8, \quad F_7=13$

For example, for the equation (3.8), the gray rectangle is:

$$F_5F_6 = F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2$$

For the equation (3.7c): $F_1 + F_3 + F_5 = F_6$.

3.5 Extending the Sequences

The purpose of this section is to extend the (F_n) sequences to use negative integer indices.

We can approach this extension in two ways:

- As a **new** sequence that is denoted as (F_{-n}). ("Separate" from the (F_n) sequences.)
- As a **function** F, defined over $\mathbb{Z}$ (more interesting!).

Because sequences (F_n) is defined by:

$$F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \ \ (n \geq 1) \tag{3.1}$$

We must, first and foremost, ensure that the following equation,

$$F_{m+2} = F_{m+1} + F_m,$$

holds even when m is assigned negative values (integers), namely:

$$F_{-n+2} = F_{-n+1} + F_{-n}$$

(n are natural numbers.)
(The case for m = 0 is covered in the previous sections, i.e., $F_0 = 0$.)
We substitute n = 1 and get:

$$F_1 = F_0 + F_{-1} \quad \Rightarrow \quad 1 = 0 + F_{-1} \quad \Rightarrow \quad F_{-1} = 1$$

We substitute n = 2 and get:

$$F_0 = F_{-1} + F_{-2} \quad \Rightarrow \quad 0 = 1 + F_{-2} \quad \Rightarrow \quad F_{-2} = -1$$

We substitute n = 3 and get:

$$F_{-1} = F_{-2} + F_{-3} \quad \Rightarrow \quad 1 = -1 + F_{-3} \quad \Rightarrow \quad F_{-3} = 2$$

We substitute n = 4 and get:

$$F_{-2} = F_{-3} + F_{-4} \quad \Rightarrow \quad -1 = 2 + F_{-4} \quad \Rightarrow \quad F_{-4} = -3$$

$$\mathbf{F_{-n} = (-1)^{n+1} F_n} \tag{3.9}$$

for every natural n.
While it is possible to establish that the following is indeed true for every natural n (which is what we wanted):

$$F_{-n+2} = F_{-n+1} + F_{-n},$$

we can take it further:
At the beginning of this chapter, we showed that for every natural n:

$$\varphi^n = F_n \varphi + F_{n-1}$$

$$(1-\varphi)^n = -F_n \varphi + F_{n+1}.$$

Therefore, what we would expect is that the following hold for every natural n:

$$\varphi^{-\mathbf{n}} = \mathbf{F_{-n}} \varphi + \mathbf{F_{-n-1}} \tag{3.10a}$$

$$(\mathbf{1} - \varphi)^{-\mathbf{n}} = -\mathbf{F_{-n}} \varphi + \mathbf{F_{-n+1}} \tag{3.10b}$$

Hence (here we leave what is known and make some suitable changes):

$$\varphi^n = F_n \varphi + F_{n-1}$$

$$(-1)^n (-\varphi)^n = F_n \varphi + F_{n-1}$$

$$(1-\varphi)^{-n} = (-1)^n (F_n \varphi + F_{n-1})$$

$$(1-\varphi)^{-n} = -F_{-n} \varphi + F_{-n+1}$$

We define recursively, for every natural number n that

$$F_{-n} = F_{-(n-2)} - F_{-(n-1)} = F_{-n+2} - F_{-n+1}$$

and this makes the equality

$$F_{n+2} = F_n + F_{n+1}$$

hold also for negative n, and then prove by induction on the natural numbers n that:

$$(1-\varphi)^n = -F_n\varphi + F_{n+1}$$

$$(-1)^n(\varphi-1)^n = -F_n\varphi + F_{n+1}$$

$$(\varphi-1)^n = (-1)^n(-F_n\varphi + F_{n+1})$$

$$\varphi^{-n} = F_{-n}\varphi + F_{-n-1}$$

For both developments, we made use of Eq. (3.9).

The result is the following:

$$\boldsymbol{\varphi^m = F_m\varphi + F_{m-1}} \tag{3.11a}$$

$$\boldsymbol{(1-\varphi)^m = -F_m\varphi + F_{m+1}} \tag{3.11b}$$

for each integer m!

Now, observe the following interesting point. We proved Binet formula and the resultant relationships using:

$$\varphi^n = F_n\varphi + F_{n+1} \tag{3.2a}$$

$$(1-\varphi)^n = F_n(1-\varphi) + F_{n-1} \tag{3.2b}$$

And:

$$F_{n+2} = F_{n+1} + F_n \tag{3.1}$$

where n is a natural number, but **we did not use this fact in any of the proofs!** However, since the above three equalities also hold for negative indices (and exponential), we can conclude that all the proofs are **also valid for integer negative indices (and exponential)**. Hence, the results are respectively valid. In particular, the following:

- Binet's formula (for **integer** m):

$$\boldsymbol{F_m = \frac{1}{\sqrt{5}}\left[\varphi^m - (1-\varphi)^m\right]} \tag{3.12}$$

- Cassini's formula (for **integer** m):

$$\boldsymbol{\delta(F_{m-1}, F_m) = F_{m+1}F_{m-1} - F_m^2 = (-1)^m} \tag{3.13}$$

Similarly:
A. φ^m and $(1-\varphi)^m$ are φ-numbers.

B. $(1-\varphi)^m$ is the companion of φ^m, that is to say $(\varphi^m)^* = (1-\varphi)^m$.
C. φ^m and $(1-\varphi)^m$ are normalized φ-numbers, i.e.: $\mu(\varphi^m) = \mu[(1-\varphi)^m] = 1$.

(This is exactly the same as for the natural numbers only.)

Now, it is obvious that the set $\{\varphi^m | m \in \mathbb{Z}\}$ along with multiplication is a (commutative) group.
In particular, the unit element is φ^0 and the inverse element of φ^m is φ^{-m}. Therefore, since

$$\varphi^m = F_m\varphi + F_{m-1} \tag{3.11a}$$

the set $\{F_m\varphi + F_{m-1} | m \in \mathbb{Z}\}$ with multiplication is a commutative group. In particular, the unit element is $F_0\varphi + F_{-1}$ and the inverse element of $F_m\varphi + F_{m-1}$ is $F_{-m}\varphi + F_{-m-1}$. (This holds even if m is negative. Please check!).

This group is actually a **subgroup of the group of φ-numbers.**

3.6 Matrices and the Fibonacci Sequence

In ► Chap. 1 we saw that the set of matrices of type $M(a, b)$ (a and b are rational numbers and at least one of them is not zero), along with standard multiplication between matrices is a commutative group that is isomorphic to the group of φ-numbers.

It is possible to check that the set $\{M(F_{m-1}, F_m) | m \in \mathbb{Z}\}$ (note that m must be a **integer**) is a commutative group in itself, meaning that the subgroup of the set of matrices of type $M(a, b)$ is also a commutative group. In particular:

- The unit element is:

$$M(F_{-1}, F_0) = M(1, 0) = I$$

- The inverse element of $M(F_{m-1}, F_m)$, which we shall calculate using the formula

$$M^{-1}(a, b) = \delta^{-1}(a, b)\, M(a + b, -b) \tag{1.14b}$$

is:

$$M^{-1}(F_{m-1}, F_m) = \delta^{-1}(F_{m-1}, F_m)\, M(F_{m+1}, -F_m)$$

$$= (-1)^m\, M(F_{m+1}, -F_m) \quad \text{(based on(3.13))}$$

$$= M(F_{-m-1}, F_{-m}) \quad \text{(based on (3.9))}$$

(even if m is negative.)

Now, we shall give special attention to the golden matrix which was defined at the end of ► Chap. 1:

$$\Phi = M(0, 1) = M(F_0, F_1) \tag{1.15}$$

We calculate the first powers of the matrix and obtain:

$$\Phi^2 = M^2(0,1) = M(1,1)$$
$$\Phi^3 = M^3(0,1) = M(1,2)$$
$$\Phi^4 = M^4(0,1) = M(2,3)$$

Therefore, we can assume that:

$$\boldsymbol{\Phi^n = M^n(0,1) = M(F_{n-1}, F_n)} \tag{3.14a}$$

Or explicitly:

$$\boldsymbol{\Phi^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}} \tag{3.14b}$$

It is possible to prove this conjecture by using induction.
From this we conclude:

$$\Phi^{-n} = (\Phi - I)^n = M^{-1}(F_{n-1}, F_n) = (-1)^n (F_{n+1}, -F_n) = M(F_{-n-1}, F_{-n})$$

for every natural n. Similarly:

$$\boldsymbol{(I - \Phi)^n = M(F_{n+1}, -F_n)} \tag{3.15}$$

We point out that if A is an **invertible** square matrix, we can show that

$$(A^{-1})^n = (A^n)^{-1}$$

and that both can be represented by A^{-n}. This is not being overly pedantic, since the multiplication operation (and hence the exponential functions) between matrices does not necessarily uphold the attributes of multiplication between numbers. For example, matrix multiplication is not commutative.
Since

$$\Phi^0 = I = M(1,0) = M(F_{-1}, F_0)$$

We can add together the three equalities **to form one**:

$$\boldsymbol{\Phi^m = M^m(0,1) = M(F_{m-1}, F_m)} \tag{3.16a}$$

Or, more precisely:

$$\Phi^m = \begin{pmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{pmatrix} \tag{3.16b}$$

where m is an integer.
If we connect what we have just written and what we observed in the previous section and at the beginning of this section, we can conclude that the group of **matrices** $\{\Phi^m | m \in \mathbb{Z}\}$ is isomorphic to the group of **numbers** $\{\varphi^m | m \in \mathbb{Z}\}$, in the same way that the group of matrices of type $M(a,b)$ when a and b are rational numbers, is isomorphic to the group of φ -numbers.

Since this is a chapter about the Fibonacci sequence, we shall develop this nexus further.
At the end of ▶ Chap. 1, we pointed out that the golden matrix behaves like φ, as the following holds:

$$\Phi^2 = \Phi + I \tag{1.16}$$

(This is the matrix version of $\varphi^2 = \varphi + 1$)

Since

$$\varphi^n = F_n\varphi + F_{n-1}$$

$$\left(\frac{-1}{\varphi}\right)^n = (1-\varphi)^n = -F_n\varphi + F_{n+1}$$

(n remains natural), we would like the following to hold:

$$\mathbf{\Phi^n = F_n F + F_{n-1} I} \tag{3.17a}$$

$$(\Phi^{-1})^{\mathbf{n}} = (\mathbf{I} - \Phi)^{\mathbf{n}} = -\mathbf{F_n \Phi + F_{n+1} I} \tag{3.17b}$$

The first equality we can write as:

$$M(F_{n-1}, F_n) = F_n\, M(0,1) + F_{n-1}\, M(1,0)$$

(Check!)

The second equality can be written as:

$$M(F_{n+1}, -F_n) = -F_n\, M(0,1) + F_{n+1}\, M(1,0)$$

(Check!)

We now subtract (3.17b) from (3.17a) to get:

$$\begin{aligned}\Phi^n - (I - \Phi)^n &= 2F_n\Phi + (F_{n-1} - F_{n+1})\, I \\ &= 2F_n\Phi - F_n I \\ &= F_n\,(2\Phi - I)\end{aligned}$$

$$\mathbf{\Phi^n - (I - \Phi)^n = F_n R} \tag{3.18}$$

R(5) is the fifth-root matrix that was defined at the end of the ▶ Chap. 1.
We have finally derived the matrix version of Binet's formula!
We could have proven those relationships by using the golden ratio and its exponents and by using the golden matrix and its exponents instead. This is not surprising in light of the isomorphism that we have proved previously.

For example, we shall use matrices to prove this relationship, with which we are already familiar:

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m = F_{n+1} F_m + F_n F_{m-1} \tag{3.6}$$

On the one hand:

$$\Phi^n \Phi^m = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \begin{pmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{pmatrix} =$$

$$\Phi^n \Phi^m = \begin{pmatrix} F_{n-1}F_{m-1} + F_nF_m & F_{n-1}F_m + F_nF_{m+1} \\ F_nF_{m-1} + F_{n+1}F_m & F_nF_m + F_{n+1}F_{m+1} \end{pmatrix}$$

On the other hand:

$$\Phi^n \Phi^m = \Phi^{n+m} = \begin{pmatrix} F_{n+m-1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{pmatrix}$$

Comparing the results proves the relationship (3.6).
It can also be shown that:

$$M(F_{m-1}, F_m)M(F_{n-1}, F_n) = M(F_{m+n-1}, F_{m+n})$$

A special case of it is:

$$M^{-1}(F_{m-1}, F_m) = M(F_{-m-1}, F_{-m})$$

Exercises for Chapter 3

- **Note**
  n, m, p, and k are usually natural numbers. If not, it will be clear from the context.

- **Exercise 3.1: The Sequences of Numerators of $(\mathbf{F_n})$**
  Calculate: $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$.

- **Exercise 3.2: Recursive Sequence**
  The sequence (a_n) is defined by

  $$a_1 = 1\,,\ a_{n+1} = \frac{a_n + 1}{a_n}$$

  (This sequence was first discussed in the second section of the ► Chap. 1).
  A. Calculate a_2, a_3, a_4. Suggest an appropriate hypothesis and prove it.
  B. Deduce $\lim_{n\to} a_n$.
  C. Repeat the previous problems using the sequences (b_n) and (c_n) that are defined by

  $$b_1 = 1\,,\ b_{n+1} = \frac{1}{1 + b_n}$$

$c_1 = 1\,,\ c_{n+1} = \frac{2c_n + 1}{c_n + 1}$

- **Exercise 3.3: Difference of 4th-Power Exponents**
  Prove: $F_{n+1}^4 - F_n^4 = F_{2n+1}F_{n+2}F_{n-1}$.

- **Exercise 3.4: Exponential Functions**
  The number $R(n) = Fn/F_{n+1}$ and the function f_n is defined by $f_n(x) = X^{R(n)}$.
  A. Show that the equation of the tangent to the graph of the function at the point where $x = 1$ is $F_n x - F_{n+1} y + F_{n-1} = 0$
  B. Calculate $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx$

- **Exercise 3.5: Polynomials with Coefficients that are Fibonacci Numbers**
  Given the polynomial

  $f_n(x) = F_n + F_{n-1}x + F_{n-2}x^2 + \cdots + F_1 x^{n-1}$

  A. Prove:

  $x^{n+1} - F_{n+1}x - F_n = (x^2 - x - 1) f_n(x)$

  B. Prove that the following holds (for all x and all n):

  $f_{n+1}(x) = x f_n(x) + F_{n+1}$

- **Exercise 3.6: Relationship Between Two Powers of ϕ**
  Prove For all n and for all k:

  $F_n\varphi^k - F_k\varphi^n = F_n(1-\varphi)^k - F_k(1-\varphi)^n = F_nF_{k+1} - F_kF_{n+1}$

- **Exercise 3.7: Analogy (a)**
  The sequence (a_n) exists for all $n \geq 0$:

  $p^n = a_n p + a_{n-1}$

  $(1-p)^n = a_n\,(1-p) + a_{n-1}$

  $p \neq 0, \frac{1}{2}, 1.$

  A. By subtracting the respective sides of the two equations find the formula for the sequence.
  B. Calculate the first and second elements.
  C. Substitute $n = 2$ into the equations and calculate p.
  D. What do you conclude?

- **Exercise 3.8: Analogy (b)**
  The series (a_n) exists for all $n \geq 0$:
  $p^n = a_n p + a_{n-1}$

  $\left(\frac{-1}{p}\right)^n = a_n \left(\frac{-1}{p}\right) + a_{n-1}$

where $p \neq 0, \pm i, \pm 1$.

A. By subtracting the respective sides of the equations find the formula for the sequence.
B. Calculate a_0, a_1, a_2, and a_3.
C. Prove: If $a_2 = 1$, then $p = \varphi, -1/\varphi$.
D. Calculate α and β so that the following holds for all n:

$$a_{n+2} = \alpha a_{n+1} + \beta a_n$$

Prove: If $1 = \alpha$, then $p = \varphi, \frac{-1}{\varphi}$.

- **Exercise 3.9: Additional Sequence?**

In this chapter we showed that:

$$\varphi^n = F_n \varphi + F_{n-1} \tag{3.2a}$$

$$(1-\varphi)^n = -F_n \varphi + F_{n+1} \tag{3.2b}$$

The purpose of this exercise is to find and characterize all the sequences a_n and b_n in which the following hold:

$$\varphi^n = a_n \varphi + a_{n-1}$$

$$(1-\varphi)^n = -b_n \varphi + b_{n+1}$$

A. Prove:

$$a_n - F_n = (1-\varphi)(a_{n-1} - F_{n-1})$$

$$b_n - F_n = \varphi(b_{n-1} - F_{n-1})$$

B. Deduce that the sequences $(a_n - F_n)$ and $(b_n - F_n)$ are geometric.
C. Deduce:

$$a_n = F_n + a_0(1-\varphi)^n$$

$$b_n = F_n + b_0 \varphi^n$$

D. Prove:
If $a_n = b_n$ for all n, then $a_n = b_n = F_n$
E. Prove:
If $a_n\ b =_n$ for all n, then $a_n = b_n = F_n$

- **Exercise 3.10: Cassini Formula**

Prove Cassini's formula:
A. Using Binet's formula.
B. By induction.

- **Exercise 3.11: Identity for Odd Indices**

Prove:

$$F_{n+1}F_{n-1} = (F_n + 1)(F_n - 1)$$

If and only if n is odd.

- **Exercise 3.12: Quadratic Function**

A. Given the equation $x^2 - 2F_nX + F_{n+1}F_{n-1} = 0$
Show that if n is **odd**, then the solutions of the equation are: $F_n + 1$, $F_n - 1$.
B. The function f_n is defined by $f_n(x) = x^2 - 2F_nx + F_{n+1}F_{n-1}$ when n **is odd**.
Prove: The tangents to the graph of the functions at the points of intersection with the x-axis form, along with the x-axis, a triangle whose area does not depend on n.
C. Prove: The lines that connect those same intersection points (of the graph with x-axis) with the minimum point of the graph (the "apex"), produce, along with the x-axis, a triangle whose area does not depend on n.

- **Exercise 3.13: Linear System**

Prove that the solution of the system

$$F_{n-1}x + F_ny = F_{n+1}$$

$$F_nx + F_{n+1}y = F_{n+2}$$

Does not depend on n.

- **Exercise 3.14: A Straight Comparison**

Prove that the equation of the straight line that passes through the points (F_{n+1}, F_{n+2}) and (F_n, F_{n+1}) is

$$F_nx - F_{n-1}y = (-1)^{n+1}$$

(Test the case of $n = 1$ separately).

- **Exercise 3.15: Even-Odd**

Prove:

A. If n and m are even, then

$$F_{n+1}F_{n-1} - F_{m+1}F_{m-1} = F_n^2 - F_m^2$$

B. If n and m are odd, then

$$F_{n+1}F_{n-1} - F_{m+1}F_{m-1} = F_n^2 - F_m^2$$

C. If n is even and m is odd (or, vice versa, m is even and n is odd), then

$$F_{n+1}F_{n-1} + F_{m+1}F_{m-1} = F_n^2 + F_m^2$$

- **Exercise 3.16: Generalizations of Cassini's Formula**

In this exercise, you may assume that $m \geq n$.

A. Justify the identities:

$$\varphi^m (1-\varphi)^n = (-1)^n \varphi^{m-n}$$
$$\varphi^n (1-\varphi)^m = (-1)^n (1-\varphi)^{m-n}$$

B. Deduce:

$$F_{n-1}F_m - F_nF_{m-1} = (-1)^n F_{m-n}$$
$$F_{n+1}F_m - F_nF_{m+1} = (-1)^n F_{m-n}$$

- **Exercise 3.17: F_{n+1} as an Expression of F_n Only**

A. Prove:

$$F_{n+1^2} - F_{n+1}F_n - F_n^2 = (-1)^n$$

B. Deduce:

$$F_{n+1} = \frac{1}{2}\left(F_n + \sqrt{5F_n^2 + 4(-1)^n}\right)$$

C. Deduce that the expression under the square root sign is always (for all n) a square of a natural number.

- **Exercise 3.18: Even/Odd Indices**

Prove what we obtained in the chapter:

$$\mathbf{F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n\,(F_{n+1} + F_{n-1})} \qquad (3.5a)$$

$$\mathbf{F_{2n-1} = F_n^2 + F_{n+1}^2} \qquad (3.5b)$$

By using the identity $\varphi^{2n} - 1 = (\varphi^n + 1)(\varphi^n - 1)$.

- **Exercise 3.19: Relationship with Two Parameters**

Use the equation $\varphi^{n-p}\varphi^{n+p} = \varphi^{2n}$ to prove that:

$$F_{2n} = F_{n-p+1}F_{n+p} + F_{n-p}F_{n+p-1}$$

$$F_{2n-1} = F_{n-p}F_{n+p} + F_{n-p-1}F_{n+p-1}$$

For all $p < n$.

- **Exercise 3.20: Relationships for F_{3n}**

A. Prove:

$$F_{3n} = F_{n+1^3} + F_n^3 - F_{n-1^3}$$

B. Find a formula for the sum

$F_3 + F_6 + F_9 + \cdots + F_{3n}$

(This question is not connected to the previous one.)

Exercise 3.21: General Properties of Division
Prove that F_{pn} is divisible by F_p for all n.

Exercise 3.22: Additional Proofs for the Formulas of Sums (a)
Prove what we obtained in this chapter:

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1 \quad (3.7a)$$

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1 \quad (3.7b)$$

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} \quad (3.7c)$$

A. Using Binet's formula.
B. By induction.

Exercise 3.23: Additional Proofs for the Formulas of Sums (b)
A. Prove:

$$\varphi + \varphi^2 + \varphi^3 + \varphi^4 + \cdots + \varphi^n = \varphi^{n+2} - \varphi - 1$$

B. Deduce:

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1 \quad (3.7a)$$

C. Prove:

$$\varphi^2 + \varphi^4 + \varphi^6 + \cdots + \varphi^{2n} = \varphi^{2n+1} - \varphi$$

and deduce that

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1 \quad (3.7b)$$

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} \quad (3.7c)$$

Exercise 3.24: Sum of totals/Sum of Sums
A. SF_n is defined as followed:

$$SF_n = F_1 + F_3 + F_5 + \cdots + F_{2n-1}$$

Find a formula for the following sum:

$$SF_1 + SF_2 + SF_3 + SF_4 + \cdots + SF_n$$

B. Deduce:

$$n F_1 + (n-1)F_3 + (n-2)F_5 + \cdots + 2F_{2n-3} + F_{2n-1} = F_{2n+1} - 1$$

- **Exercise 3.25: Sum of Products**

A. Prove for all **n even**:

$$F_0F_2 + F_1F_3 + F_2F_4 + \cdots + F_{n-1}F_{n+1} = F_1^2 + F_2^2 + F_3^2 + F_4^2 + \cdots + F_{n^2}$$
$$= F_nF_{n+1}$$

B. Deduce for all **n even**:

$$F_0F_1 + F_1F_2 + F_2F_3 + \cdots + F_{n-1}F_n = F_n^2$$

C. Deduce for all **n odd**:

$$F_0F_1 + F_1F_2 + F_2F_3 + \cdots + F_{n-1}F_n = F_n^2 - 1$$

- **Exercise 3.26: Formula for the Sum of F_{4n}**

A. Justify the equality:

$$F_{4n} = F_{2\,n+1^2} - F_{2\,n-1^2}$$

B. Find a formula for the sum

$$F_4 + F_8 + F_{12} + \cdots + F_{4n}$$

- **Exercise 3.27: Sum with Alternating Signs**

Find a formula for the sum:

$$-F_1 + F_2 - F_3 + F_4 - \cdots + (-1)^n F_n$$

- **Exercise 3.28: Sum of Products with Alternating Signs**

Prove for all **even n:**

$$-F_1F_2 + F_2F_3 - F_3F_4 + \cdots + F_nF_{n+1} = F_2^2 + F_4^2 + F_6^2 + \cdots + F_n^2$$

- **Exercise 3.29: The Sum of Fractions**

A. Justify the equality:

$$\frac{F_n}{F_{n+1}} - \frac{F_{n-1}}{F_n} = \frac{(-1)^{n+1}}{F_{n+1}F_n}$$

B. Find a formula for the sum:

$$\frac{1}{F_1F_2} - \frac{1}{F_2F_3} + \frac{1}{F_3F_4} - \cdots + \frac{(-1)^{n+1}}{F_nF_{n+1}}$$

Deduce the limit of the above sum, where $n \to \infty$.

- **Exercise 3.30: Equality Between the Sums of Fractions**
  Prove that for any **even n,** the following holds:

$$\frac{F_1}{F_2} - \frac{F_2}{F_3} + \frac{F_3}{F_4} - \frac{F_4}{F_5} + \cdots + \frac{F_{n-1}}{F_n} - \frac{F_n}{F_{n+1}}$$
$$= \frac{1}{F_2F_3} + \frac{1}{F_4F_5} + \cdots + \frac{1}{F_nF_{n+1}}$$

- **Exercise 3.31: Infinite Series**
  A. Prove:

$$\frac{1}{F_nF_{n+2}} = \frac{1}{F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}$$

  B. Deduce:

$$\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \frac{1}{F_3F_5} + \frac{1}{F_4F_6} + \cdots = 1$$

- **Exercise 3.32: "Mixed" Sum**
  A. Prove: If **n is even**, then

$$F_1 + F_2\varphi + F_3\varphi^2 + \cdots + F_n\varphi^{n-1} = F_n\varphi^n$$

  B. Deduce: If **n is odd,** then

$$F_1 + F_2\varphi + F_3\varphi^2 + \cdots + F_n\varphi^{n-1} = F_{n+1}\varphi^{n-1}$$

- **Exercise 3.33: Product**
  We define:

$$P_n = \left(1 + \frac{1}{F_2^2}\right)\left(1 - \frac{1}{F_3^2}\right)\left(1 + \frac{1}{F_4^2}\right)\cdots\left(1 + \frac{(-1)^{n+1}}{F_{n+1}^2}\right)$$

  A. Prove:

$$P_n = \frac{F_{n+2}}{F_{n+1}}$$

  B. Deduce $\lim_{n\to\infty} Pn$.

- **Exercise 3.34: Combinatorics**
  We first recall the formula (special case of the development of Newton's binomial theorem):

$$(1 + x)^n = 1 + C_n^1x + C_n^2x^2 + \cdots + C_n^nx^n$$

A. By substituting $x = \varphi$, prove:

$$C_n^1F_1 + C_n^2F_2 + C_n^3F_3 + \cdots + C_n n_n = F_{2n}$$
$$C_n^2F_1 + C_n^3F_2 + C_n^4F_3 + \cdots + C_n nF_{n-1} = F_{2n-1} - 1$$

B. By substituting $x = -\varphi$, prove:

$$C_n^1F_1 - C_n^2F_2 + C_n^3F_3 - C_n^4F_4 + \cdots + (-1)^{n+1}C_n^nF_n = F_n$$

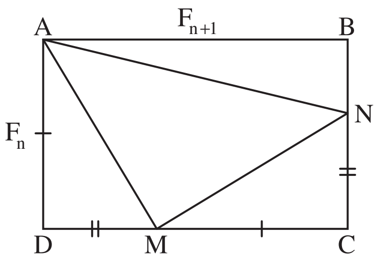
C. By substituting $x = 2\,\varphi$, prove:

$$2^1C_n^1F_1 + 2^2C_n^2F_2 + 2^3C_n^3F_3 + \cdots + 2^nC_n n_n = F_{3n}$$

- **Exercise 3.35: Triangles Within an "Almost Golden" Rectangle**

ABCD is a rectangle with sides $AB = F_{n+1}, AD = F_n (n > 1)$.
M and N are points on sides DC and BC (respectively) so that the following holds:

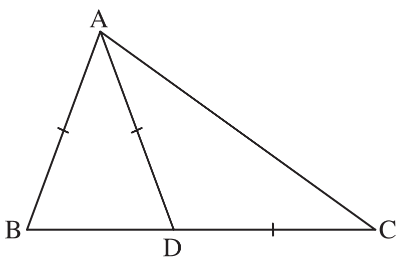
$\Delta\, ADM \cong MCN$



Prove: $S_{AMN} = 1/2\, F_{2n-1}$.

- **Exercise 3.36: "Almost Golden" Triangles (a)**

In this exercise, $n > 1$.
In triangle ABC, D is a point on side BC such that $CD = AD = AB = F_n$.
Similarly: $BC = F_{n+1}$



A. Prove:

$$\cos B = \frac{F_{n-1}}{2F_n}$$

B. Deduce:

$$AC^2 = F_n F_{n+2}$$

- **Exercise 3.37: "Almost Golden" Triangles (b)**

Isosceles triangle ABC intersects isosceles triangle BCD (D is on AC) such that: $\Delta ABC \sim \Delta BCD$. Similarly, given is: $AB = F_{n+1}$, $BC = F_n$.

A. Prove: $CD = \frac{F_n^2}{F_{n+1}}$

B. Deduce: $\lim_{n\to\infty} \frac{AD}{CD} = \varphi$

- **Exercise 3.38: Vectors and Inequalities**

Given are vectors (F_{n+1}, F_n) and (F_{m+1}, F_m).
A. Calculate the scalar products and their lengths.
B. Deduce for all n and for all m: $F_{n+m+1^2} \le F_{2n+1} F_{2m+1}$

- **Exercise 3.39: Ellipse**

Recall that the foci of the ellipse $b^2x^2 + a^2y^2 = a^2b^2 (a > b)$ are $((\pm\sqrt{a^2 - b^2}, 0))$.
Given is the following ellipse:

$$F_{n-1}^2 x^2 + F_{n+1}^2 y^2 = F_{n+1}^2 F_{n-1}^2$$

where $n > 1$
Prove: The equation of the canonical circle (that is to say, a circle with center on the origins of the axes) that passes through the foci of the ellipse is

$$x^2 + y^2 = F_{2n}$$

- **Exercise 3.40: Sums of Negative Indices**

n is a natural number. Prove:

1. $F_{-1} + F_{-2} + F_{-3} + F_{-4} + \cdots + F_{-n} = (-1)^{n-1} F_{n-1} + 1$
   $= -F_{-n+1} + 1$
2. $F_{-2} + F_{-4} + F_{-6} + \cdots + F_{-2n} = -F_{2n+1} + 1 = -F_{-2n-1} + 1.$
3. $F_{-1} + F_{-3} + F_{-5} + \cdots + F_{-2n+1} = F_{2n} = -F_{-2n}.$
4. $F_{-1}^2 + F_{-2}^2 + F_{-3}^2 + F_{-4}^2 + \cdots + F_{-n}^2 = F_n F_{n+1} = -F_{-n} F_{-n-1}$

- **Exercise 3.41: "Double" Amount**

Find a formula, without powers of (-1) for the sum:

$$F_{-n} + F_{-n+1} + \cdots + F_{-2} + F_{-1} + F_0 + F_1 + F_2 + \cdots + F_n$$

when n (natural) is even.
when n (natural) is odd.

- **Exercise 3.42: Proof Using Matrices**

A. Multiply the respective sides of each equation together

$$\mathbf{\Phi}^n = M(F_{n-1}, F_n) \tag{3.14a}$$

$$(\mathbf{I} - \mathbf{\Phi})^n = M(F_{n+1}, -F_n) \tag{3.15}$$

to obtain Cassini's formula.
(Note: $\mathbf{\Phi}^n(\mathbf{I} - \mathbf{\Phi})^n = [\mathbf{\Phi}(\mathbf{I} - \mathbf{\Phi})]^n$ for all n, because the set of matrices M(a, b) is a commutative group.)
B. By using the equation $\mathbf{\Phi}^{2n} = \mathbf{\Phi}^n\mathbf{\Phi}^n$
obtain:

$$\mathbf{F_{2n} = F_{n+1}^2 - F_{n-1}^2} \tag{3.5a}$$

$$\mathbf{F_{2n-1} = F_n^2 + F_{n+1}^2} \tag{3.5b}$$

- **Exercise 3.43: In Anticipation of the LUCAS Sequence**

The (L_n) sequence is defined by:

$$L_0 = 2, L_1 = 1$$

$$L_{n+2} = L_{n+1} + L_n$$

Prove:

$$L_n = \varphi^n + (1 - \varphi)^n$$

- **Exercise 3.44: In anticipation of the Fibonacci sequence**

The sequence (f_n) is defined by:

$$f_0 = a, f_1 = b$$

$$f_{n+2} = f_{n+1} + f_n$$

A. Calculate f_2, f_3, f_4, f_5, suggest an appropriate hypothesis, and prove it.
B. Deduce:

$$f_n\sqrt{5} = (a\varphi + b - a)\,\varphi^n + (a\varphi - b)\,(1 - \varphi)^n$$

Answers, Clues and Partial Solutions

- **Exercise 3.1**

φ. The easiest proof uses Binet's formula.

- **Exercise 3.2**

A. $a_n = \frac{F_{n+1}}{F_n}$

B. φ

C. $b_n = \frac{F_n}{F_{n+1}}$ The limit is $1/\varphi$

D. $c_n = \frac{F_{2n}}{F_{2n-1}}$. The limit is φ

- **Exercise 3.3**

You can use the formula: $a^4 - b^4 = \left(a^2 + b^2\right)(a+b)(a-b)$

- **Exercise 3.4**

B. $1/\varphi$

- **Exercise 3.7**

A. $a_n = \frac{p^n-(1-p)^n}{2p-1}$

B. $a_1 = a_2 = 1$.

C. $p^2 = p + 1$ is obtained, so therefore, $p = \varphi, 1 - \varphi$.

D. The Fibonacci sequence is the only one that simultaneously holds for both equalities at the beginning of the exercise.

- **Exercise 3.8**

A. $a_n = \frac{p}{p^2+1}\left[p^n - (-1/p)^n\right]$

B. $a_0 = 0,\ a_1 = 1,\ a_2 = \left(p^2 - 1\right)/p,\ a_3 = \left(p^4 - p^2 + 1\right)/p^2$

D. $\alpha = \left(p^2 - 1\right)/p,\ \beta = 1$

- **Exercise 3.9**

A. By subtracting the respective sides from each other.

D. Conclusion from section C.

E. Conclusion from section C.

- **Exercise 3.10**

B.
$$\begin{aligned} F_{k+2}F_k - F_{k+1}^2 &= (F_{k+1} + F_k)F_k - F_{k+1}^2 \\ &= F_{k+1}(F_k - F_{k+1}) + F_k^2 = F_{k+1}(-F_{k-1}) + F_k^2 \\ &= -\left(F_{k+1}F_{k-1} - F_k^2\right) = -(-1)^k = (-1)^{k+1} \end{aligned}$$

- **Exercise 3.11**

Use Cassini's formula.

- **Exercise 3.12**

A. $\Delta = 4F_n^2 - 4F_{n+1}F_{n-1} = 4(-1)^{n+1} = 4$

B. The tangents are $y = 2(x - F_n - 1)$, and $y = -2(x - F_n + 1)$. The point of intersection is $(F_n, -2)$, the area is 2.

C. The vertex is $(F_n, -1)$, the area is 1.

- **Exercise 3.13**

The solution is (1,1). This should not be solved systematically because this is an "obvious" solution and it is unique because the determinant of the system equals $(-1)^n$ (in accordance with Cassini's formula) and therefore is different from 0.

- **Exercise 3.14**

Use Cassini's formula.

- **Exercise 3.15**

You can use Cassini's formula.

- **Exercise 3.16**

B. Use equation (3.2).

- **Exercise 3.17**

A. Use Cassini's formula.
B. Refer to the equality in section a as a quadratic equation where the missing variable is F_{n+1}.

- **Exercise 3.20**

A. Use $m = 2n$ in Eq. (3.6).
B. $1/2(F_{3n+2} - 1)$. You can use Binet's formula and find the sum of the two geometric sequences.

- **Exercise 3.21**

If the proof is in the induction (on n), the last step is:

$$F_{p(k+1)} = F_{pk+p} = F_p F_{pk+1} + F_{p-1} F_{pk}$$

- **Exercise 3.23**

A. The left side is the sum of a geometric sequence.
B. Use equation (3.2a).
C. Use equation (3.2a).

- **Exercise 3.24**

A. $F_{2n+1} - 1$
B. $SF_1 + SF_2 + \cdots + SF_n = F_1 + (F_1 + F_3) + \cdots + (F_1 + F_3 + F_5 + \cdots + F_{2n-1})$

- **Exercise 3.25**

A. You can use Cassini's formula.
B. $F_{n-1}F_n = F_{n-1}(F_{n+1} - F_{n-1}) = F_{n-1}F_{n+1} - F_{n-1}^2$

- **Exercise 3.26**

A. $4n = 2(2n)$
B. $F_{2n+1}^2 - 1$. Use the equality given in section A.

3

- **Exercise 3.27**

$(-1)^n F_{n-1} - 1$

(Calculate even n and odd n separately.)

- **Exercise 3.28**

$-F_{n-1}F_n + F_nF_{n+1} = F_n(-F_{n-1} + F_{n+1}) = F_n^2$

- **Exercise 3.29**

A. Derived from Cassini's formula.
B. $\frac{F_n}{F_{n+1}}$. (Based on the previous section.)
C. $\frac{1}{\varphi}$

- **Exercise 3.30**

$$\frac{F_{n-1}}{F_n} - \frac{F_n}{F_{n+1}} = \frac{F_{n+1}F_{n-1} - F_n^2}{F_nF_{n+1}} = \frac{1}{F_nF_{n+1}}$$

- **Exercise 3.31**

A. Start from the right side.

- **Exercise 3.32**

A. Connect the addends two by two.
(Or by induction.)

- **Exercise 3.33**

A. You can use Cassini's formula.
B. φ

- **Exercise 3.34**

A. $(1 + \varphi)^n = \varphi^{2n} = F_{2n}\varphi + F_{2n-1}$
B. $(1 - \varphi)^n = -F_n\varphi + F_{n+1}$
C. $(1 + 2\varphi)^n = \varphi^{3n} = F_{3n}\varphi + F_{3n-1}$

- **Exercise 3.35**

Note that triangle MNA is a right-angled isosceles triangle.

- **Exercise 3.36**

A. Observe (isosceles) triangle ABD.
B. Use the law of cosines in triangle ADC and the fact that $\cos\angle ADC = -\cos B$

- **Exercise 3.38**

  A. Use formulas (3.5) and (3.6).
  B. Calculate the cosine of the angle between the vectors.

- **Exercise 3.40**

  It is preferable to use the definition of F_{-n}, but induction can also be used.

- **Exercise 3.41**

  $2F_n$ when n is even; $2F_{n+1}$ when n is odd.

- **Exercise 3.43**

  Work as with the "classic proof" for Binet's formula for the Fibonacci sequence.

- **Exercise 3.44**

  A. $f_n = aF_{n-1} + bF_n$
  B. Substitute (twice) Binet's formula in the result of section A.

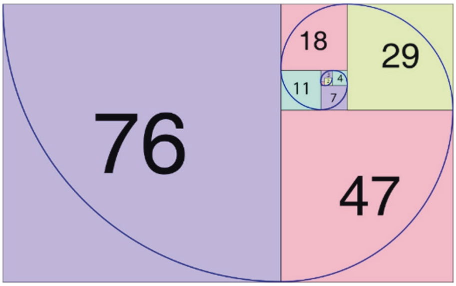
The Lucas Sequence

Contents

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.
G. H. Hardy (1877–1947)

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*,
Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2_4

4



The Lucas sequence spiral[1]

The Lucas spiral, made with quarter-arcs, is a good approximation of the golden spiral when its terms are large. A **golden spiral** is a logarithmic spiral whose growth factor is φ, the golden ratio.

■ Introduction to Chapter 4

The Lucas sequence is based upon the same recursive equation as the Fibonacci sequence, but with different initial conditions (the first two elements). The combination of the two series, including sums, demonstrates a number of beautiful relationships.

Note that some theoretical developments do not appear in the text; instead, they are dealt with in the exercises.

4.1 Three Definitions of the Lucas Sequence

By using the "classic" proof of Binet's formula for the Fibonacci sequence, we demonstrated that the **entire** sequence (f_n) with formula

$$f_n = \alpha\varphi^n + \beta(1-\varphi)^n$$

satisfies the recursive equation:

$$f_{n+2} = f_{n+1} + f_n.$$

In the Fibonacci sequence, $\alpha = -\beta = \frac{1}{\sqrt{5}}$.

1 ► https://en.wikipedia.org/wiki/Lucas_number

It is natural to wonder "what happens" when α and β are replaced by 1, ($\alpha = \beta = 1$), which are the simplest coefficients. (If either of them would be 0, the sequence would be even "simpler" but then it would be geometric, which does not interest us here.)

$$\mathbf{L_n = \varphi^n + (1 - \varphi)^n} \tag{4.1}$$

The first two elements in the sequence are:

$$L_1 = \varphi^1 + (\varphi - 1)^1 = 1$$
$$L_2 = \varphi^2 + (\varphi - 1)^2 = 3$$

And the sequence continues as follows:

$$1, 3, 4, 7, 11, 18, 29, 47, \ldots$$

For consistency, we shall define $L_0 = 2$ so that $L_2 = L_1 + L_0$ will hold, and which is $L_0 = \varphi^0 + (1 - \varphi)^0$.
We thus have another definition:

$$\begin{aligned} &L_1 = 1,\ L_2 = 3 \\ &\mathbf{L_{n+2} = L_{n+1} + L_n (n \geq 1)} \end{aligned} \tag{4.2}$$

We shall now introduce a basic, fundamental relationship between the Lucas sequence and the Fibonacci sequence:

$$\begin{aligned} &L_n = \varphi^n + (1 - \varphi)^n = F_n\varphi + F_{n-1} - F_n\varphi + F_{n+1} = F_{n-1} + F_{n+1} \\ &\mathbf{L_n = F_{n-1} + F_{n+1}} \end{aligned} \tag{4.3}$$

In summary, we have three definitions (it is possible to show that they are equivalent) for the Lucas sequence: one is in the form of Binet's formula (for convenience and consistency, we shall henceforth call it the **Binet's formula for the Lucas sequence**), the second is recursive, and the third uses the Fibonacci sequence.

4.2 Connections Between the Fibonacci and Lucas Sequences

There are many relationships between the elements of the Lucas sequence themselves, but particularly between them and elements of the Fibonacci sequence. We shall explore some of them here.

- **Relationships Between Elements of the Lucas Sequence**

We start out using Binet's formula for the sequence and square both sides:

$$L_n^2 = [\varphi^n + (1 - \varphi)^n]^2 = \varphi^{2n} + (1 - \varphi)^{2n} + 2\varphi^n (1 - \varphi)^n = L_{2n} + 2(-1)^n$$

This gives:

$$\mathbf{L_{2n} = L_n^2 - 2(-1)^n} \tag{4.4}$$

Similarly:

$$\begin{aligned} L_nL_{n-1} &= [\varphi^n + (1-\varphi)^n]\,[\varphi^{n-1} + (1-\varphi)^{n-1}] \\ &= \varphi^{2n-1} + (1-\varphi)^{2n-1} + \varphi^{n-1}(1-\varphi)^{n-1}\,[\varphi + (1-\varphi)] \\ &= L_nL_{n-1} = \varphi^{2n-1} + (1-\varphi)^{2n-1} + (-1)^{n-1} \\ &= L_{2n-1} + (-1)^{n-1} \end{aligned}$$

which gives:

$$\mathbf{L_{2n-1} = L_nL_{n-1} + (-1)^n} \tag{4.5}$$

Using the same method, we also obtain:

$$\mathbf{L_{2n} = L_{n-1}L_{n+1} + 3(-1)^n} \tag{4.6}$$

We now develop the relationship between the two parameters (the indices m and n). Let us assume that $m \geq n$. In this case:

$$\begin{aligned} L_nL_m &= [\varphi^n + (1-\varphi)^n][\varphi^m + (1-\varphi)^m] \\ &= \varphi^{n+m} + (1-\varphi)^{n+m} + \varphi^n(1-\varphi)^n[(1-\varphi)^{m-n} + \varphi^{m-n}] \\ &= L_{n+m} + (-1)^nL_{m-n} \end{aligned}$$

Which leads to:

$$\mathbf{L_{n+m} = L_nL_m + (-1)^n\,L_{m-n}} \tag{4.7}$$

[Note that if we substitute $m = n$ into the equation, it leads "back" exactly to Eq. (4.4)].

Relationships Between the Fibonacci Sequence and the Lucas Sequence

We will now restate some previous result in order to clarify the relationships between the Fibonacci and the Lucas sequences:

$$L_n = F_{n-1} + F_{n+1}$$

$$F_n = F_{n+1} - F_{n-1}$$

By adding (or subtracting) each of the sides, we arrive at the following two equalities:

$$\mathbf{L_n + F_n = 2F_{n+1}} \tag{4.8a}$$

$$\mathbf{L_n - F_n = 2F_{n-1}} \tag{4.8b}$$

We now write down Binet's formula for the two sequence:

$$L_n = \varphi^n + (1-\varphi)^n \tag{4.1}$$

$$F_n\sqrt{5} = \varphi^n + (1-\varphi)^n \tag{3.3a}$$

Multiplying both sides of the equations yields:

$$L_nF_n\sqrt{5} = [\varphi^n + (1-\varphi)^n][\varphi^n - (1-\varphi)^n]$$
$$= \varphi^{2n} - (1-\varphi)^{2n} = F_{2n}\sqrt{5}$$

Therefore:

$$\mathbf{L_nF_n = F_{2n}} \tag{4.9}$$

This is the simplest relationship and between the two sequence.

Based on this last relationship, it is natural to observe some other "multiplicative" combinations between the elements of the two sequences. We shall do this by using their Binet's formulas.

$$L_nF_{n+1}\sqrt{5} = [\varphi^n + (1-\varphi)^n][\varphi^{n+1} - (1-\varphi)^{n+1}]$$
$$= \varphi^{2n+1} - (1-\varphi)^{2n+1} + \varphi^n(1-\varphi)^n(2\varphi-1)$$
$$= F_{2n+1}\sqrt{5} + (-1)^n\sqrt{5}$$

Hence:

$$\mathbf{L_nF_{n+1} = F_{2n+1} + (-1)^n} \tag{4.10a}$$

Similarly, we can obtain:

$$\mathbf{L_{n+1}F_n = F_{2n+1} - (-1)^n} \tag{4.10b}$$

Adding the two equalities together yields:

$$\mathbf{L_nF_{n+1} + L_{n+1}F_n = 2F_{2n+1}} \tag{4.10c}$$

The same method will also yield:

$$\mathbf{L_{n+1}F_{n-1} = F_{2n} + (-1)^n} \tag{4.11a}$$

$$\mathbf{L_{n-1}F_{n+1} = F_{2n} - (-1)^n} \tag{4.11b}$$

Again, by adding together each side of the two equalities, we obtain:

$$\mathbf{L_{n+1}F_{n-1} + L_{n-1}F_{n+1} = 2F_{2n}} \tag{4.11c}$$

Furthermore (either by using (4.10a, 4.10b, 4.10c) or by using Binet's formulas directly, as we have until now):

$$\mathbf{L_{2n} = L_nF_{n+1} + L_{n-1}F_n = L_nF_{n-1} + L_{n+1}F_n} \tag{4.12}$$

Relationships Between the Two Sequences, Using Two Indices

We begin again by assuming that $m \geq n$, and then, by using Binet's formulas, obtain:

$$\mathbf{L_nF_m = F_{n+m} + (-1)^nF_{m-n}} \tag{4.13a}$$

$$\mathbf{L_mF_n = F_{n+m} - (-1)^nF_{m-n}} \tag{4.13b}$$

Adding each side of the equations yields:

$$\mathbf{L_nF_m + L_mF_n = 2F_{n+m}} \tag{4.13c}$$

and also:

$$\mathbf{L_{n+m} = L_nF_{m-1} + L_{n+1}F_m = L_mF_{n-1} + L_{m+1}F_n} \tag{4.14}$$

(Note that if we substitute m = n into the equation, it leads back exactly to Eq. 4.12.)

A simpler proof for the latter is given in ▸ Chap. 5.

4.3 The Powers of the Golden Ratio

We write Binet's formulas for the two sequences consecutively:

$$L_n = \varphi^n + (1-\varphi)^n \tag{4.1}$$

$$F_n\sqrt{5} = \varphi^n - (1-\varphi)^n \tag{3.3a}$$

Adding (or subtracting) each of the sides, yields the following two equalities:

$$L_n + F_n\sqrt{5} = 2\varphi^n \text{ (result from adding)}$$

$$L_n - F_n\sqrt{5} = 2(1-\varphi)^n \text{ (result from subtracting)}$$

Thus we have obtained new formulas for the powers of φ and $(1-\varphi)$:

$$\boldsymbol{\varphi^n = \tfrac{1}{2}(L_n + F_n\sqrt{5})} \text{ (result from addingthe two equalities)} \tag{4.15a}$$

$$\mathbf{(1-\varphi)^n = \tfrac{1}{2}(L_n - F_n\sqrt{5})} \text{ (result from subtracting the two equalities)} \tag{4.15b}$$

Additional relationships can be obtained from these formulas. One of them will be demonstrated below. The others will be derived in the exercises.

Multiplying the respective sides of the two equalities gives:

$$\mathbf{L_n^2 - 5F_n^2 = 4(-1)^n} \tag{4.16}$$

But we are not done yet! At the beginning of ▸ Chap. 3 we saw that:

$$F_n\varphi + F_{n-1} = \varphi^n \tag{3.2a}$$

$$-F_n\varphi + F_{n+1} = (1-\varphi)^n \tag{3.2b}$$

We shall now investigate the relationship between the analoguous expressions:

$$L_n\varphi + L_{n-1}$$

$$-L_n\varphi + L_{n+1}$$

and between the powers of φ and of $(1-\varphi)$. It emerges that:

$$\begin{aligned} L_n\varphi + L_{n-1} &= (F_{n+1}+F_{n-1})\varphi + F_n + F_{n-2} \\ &= F_{n+1}\varphi + F_n + F_{n-1}\varphi + F_{n-2} \\ &= \varphi^{n+1} + \varphi^{n-1} \\ &= \varphi^{n-1}(\varphi^2+1) \\ &= \varphi^{n-1}\varphi\sqrt{5} \\ &= \varphi^n\sqrt{5} \end{aligned}$$

In other words,

$$\boldsymbol{\varphi^n = \frac{1}{\sqrt{5}}(L_n\varphi + L_{n-1})} \tag{4.17a}$$

Similarly, we obtain:

$$\boldsymbol{(1-\varphi)^n = \frac{1}{\sqrt{5}}(L_n\varphi - L_{n+1})} \tag{4.17b}$$

From this, we observe three interesting conclusions:
Combining this with

$$\varphi^n = F_n\varphi + F_{n-1} \tag{3.2a}$$

$$(1-\varphi)^n = -F_n\varphi + F_{n+1} \tag{3.2b}$$

Gives:

$$\boldsymbol{\frac{L_n\varphi + L_{n+1}}{F_n\varphi + F_{n-1}} = \sqrt{5}} \tag{4.18a}$$

$$\boldsymbol{\frac{L_n\varphi - L_{n+1}}{F_n\varphi - F_{n-1}} = -\sqrt{5}} \tag{4.18b}$$

Subtracting the two sides of each gives:

$$\boldsymbol{L_{n+1} + L_{n-1} = 5F_n} \tag{4.19}$$

Note the analogy with:

$$F_{n+1} + F_{n-1} = L_n \tag{4.3}$$

Multiplying the both sides of Eq. (4.19) by L_{n-1} gives:

$$\boldsymbol{L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}} \text{ (Exercise 4.9 below)} \tag{4.20a}$$

Or alternatively:

$$\delta(L_{n-1}, L_n) = 5(-1)^{n+1} \tag{4.20b}$$

This is the "Cassini-like" formula for the Lucas sequence.

Also:

$$\mu(\mathbf{L_{n-1}} + \mathbf{L_n}\varphi) = \mu(-\mathbf{L_{n+1}} + \mathbf{L_n}\varphi) = \sqrt{5} \tag{4.20c}$$

(We leave it to the reader to work out the detailed proofs to these formulas.)

4.4 Sums

Using the telescopic cancellation method through which we arrived at the formulas for various types of sums in the Fibonacci sequence, we similarly obtain:

$$\mathbf{L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3} \tag{4.21a}$$

$$\mathbf{L_2 + L_4 + L_6 + \cdots + L_{2n} = L_{2n+1} - 1} \tag{4.21b}$$

$$\mathbf{L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_{2n} - 2} \tag{4.21c}$$

$$\mathbf{{L_1}^2 + {L_2}^2 + {L_3}^2 + \cdots + {L_n}^2 = L_nL_{n+1} - 2} \tag{4.22}$$

Below is a geometric depiction of the last three equalities:

Explanation of ◘ Fig. 4.1:

$$L_1 = 1, \quad L_2 = 3, \quad L_3 = 4, \quad L_4 = 7, \quad L_5 = 11, \quad L_6 = 18, \quad L_7 = 29$$

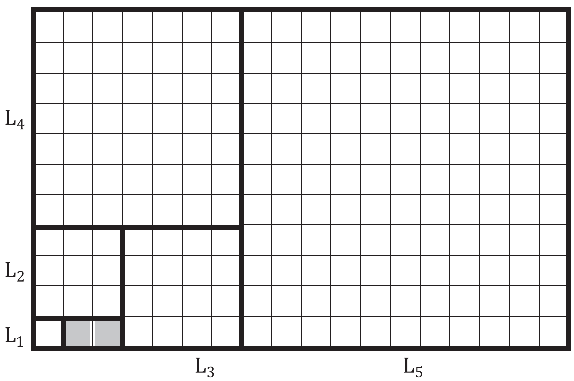
In the left lower figure, there are 3 squares with the following areas:

$$L_1 \times L_1 = 1 \times 1 = 1$$
$$L_2 \times L_2 = 3 \times 3 = 9$$
$$L_3 \times L_3 = 4 \times 4 = 16$$

The total area is 26.

There is also a rectangle which its area is

$$1 \times 2 = 2.$$



◘ **Fig. 4.1** A geometric depiction

The sum is $28 = 4 \times 7 = L_3 L_4$.
In this case: (4.22)

$$L_1^2 + L_2^2 + L_3^2 = L_3 L_4 - 2$$
$$1 + 9 + 16 = 26$$

We now will develop some formulas for the sums of "mixed" equations.
A. From the relationship:

$$L_n F_{n+1} = F_{2n+1} + (-1)^n \qquad (4.10a)$$

We obtain for even n:

$$\begin{aligned} L_1F_2 + L_2F_3 + L_3F_4 + \cdots + L_nF_{n+1} \\ &= F_3 + F_5 + \cdots + F_{2n+1} \\ &= F_{2n+2} - F_1 \\ &= F_{2n+2} - 1 \end{aligned}$$

and for odd n (using the previous result):

$$\begin{aligned} L_1F_2 + L_2F_3 + L_3F_4 + \cdots + L_{n-1}F_n + L_nF_{n+1} \\ &= F_{2n} - 1 + F_{2n+1} - 1 \\ &= F_{2n+2} - 2 \end{aligned}$$

In summary:

A. $$\mathbf{L_1F_2 + L_2F_3 + L_3F_4 + \cdots + L_nF_{n+1} = F_{2n+2} - 1} \quad \text{(even n)} \qquad (4.23a)$$

$$\mathbf{L_1F_2 + L_2F_3 + L_3F_4 + \cdots + L_nF_{n+1} = F_{2n+2} - 2} \quad \text{(odd n)} \qquad (4.23b)$$

B. Similarly, from the relationship:

$$L_{n+1}F_n = F_{2n+1} - (-1)^n \qquad (4.10b)$$

We obtain:

$$\mathbf{L_2F_1 + L_3F_2 + L_4F_3 + \cdots + L_{n+1}F_n = F_{2n+2} - 1} \quad \text{(even n)} \qquad (4.23c)$$

$$\mathbf{L_2F_1 + L_3F_2 + L_4F_3 + \cdots + L_{n+1}F_n = F_{2n+2}} \quad \text{(odd n)} \qquad (4.23d)$$

C. From the relationship:

$$L_{n+1}F_{n-1} = F_{2n} + (-1)^n \qquad (4.11a)$$

We obtain:

$$\mathbf{L_2F_0 + L_3F_1 + L_4F_2 + \cdots + L_{n+1}F_{n-1} = F_{2n+1} - 1} \quad \text{(even n)} \qquad (4.24a)$$

$$\mathbf{L_2F_0 + L_3F_1 + L_4F_2 + \cdots + L_{n+1}F_{n-1} = F_{2n+1} - 2} \quad \text{(odd n)} \qquad (4.24b)$$

D. And finally, from the relationship:

$$L_{n-1}F_{n+1} = F_{2n} - (-1)^n \tag{4.11b}$$

We obtain:

$$\mathbf{L_0F_2 + L_1F_3 + L_2F_4 + \ldots + L_{n+1}F_{n+1} = F_{2n+1} - 1} \quad \text{(even n)} \tag{4.24c}$$

$$\mathbf{L_0F_2 + L_1F_3 + L_2F_4 + \ldots + L_{n+1}F_{n+1} = F_{2n+1}} \quad \text{(odd n)} \tag{4.24d}$$

Exercises for Chapter 4

■ Note

The variables n, m, p, k, usually represent natural numbers. In the cases where they do not, it will be clear from the context.

■ Exercise 4.1: Limits

A. Calculate: $\lim_{n\to\infty} L_{n+1}/L_n$.
B. Calculate: $\lim_{n\to\infty} L_n/F_n$.

■ Exercise 4.2: Recursive Sequence

The sequence (a_n) is defined by

$$a_1 = 1, \; a_{n+1} = \frac{3a_n + 1}{a_n + 2}$$

A. Prove:

$a_n = F_{n+1}/F_n$ for odd n,

$a_n = L_{n+1}/L_n$ for even n.

B. Deduce: $\lim_{n\to\infty} a_n$.

■ Exercise 4.3: Relationships Between Squares

Prove:

$$L_n^2 + F_n^2 = 2\left(F_{n+1}^2 + F_{n-1}^2\right)$$

$$L_n^2 - F_n^2 = 4F_{n+1}F_{n-1}$$

■ Exercise 4.4: Relationships Between the Two Parameters (Indices)

Prove:

$$L_{n-p}L_{n+p} = L_{2n} + (-1)^{n-p}L_{2p}$$

for each $p < n$.

■ Exercise 4.5: Mixed Relationships (a)

Prove:

1. $L_{2n} = 5F_{n+1}F_{n-1} + 3(-1)^{n+1}$
2. $L_{2n} = 5F_{n+p}F_{n-p} + (-1)^{n-p}L_{2p} (n \geq p)$

3. $L_{2n} = 5F_n^2 + 2(-1)^n$
4. $L_{n+m} = 5F_mF_n + (-1)^nL_{m-n}(m \geq n)$

- **Exercise 4.6: Mixed Relationships (b)**
  (In this exercise, you can assume that $n \geq p$).
  A. Prove:

$$L_{n+p}F_{n-p} = F_{2n} - (-1)^{n-p}F_{2p}$$
$$L_{n-p}F_{n+p} = F_{2n} + (-1)^{n-p}F_{2p}$$

  B. Deduce:

$$L_{n+p}F_{n-p} + L_{n-p}F_{n+p} = 2F_{2n}$$

- **Exercise 4.7: The Differences**
  Prove:
  1. $L_nF_{n+1} - L_{n+1}F_n = 2(-1)^n$
  2. $L_{n+1}F_{n-1} - L_{n-1}F_{n+1} = 2(-1)^n$
  3. $L_nF_m - L_mF_n = 2(-1)^nF_{m-n}$

  (You may assume that $m \geq n$).

- **Exercise 4.8: Even/Odd**
  Prove:
  A. If n and m are even, then

$$L_{2n} - L_{2m} = L_n^2 - L_m^2$$

  B. If n and m are odd, then

$$L_{2n} - L_{2m} = L_n^2 - L_m^2$$

  C. If n is even and m is odd (or: m is even and n is odd), then

$$L_{2n} + L_{2m} = L_n^2 + L_m^2$$

- **Exercise 4.9: Cassini's Formula for (L_n)**
  Prove:

$$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}$$

  (According to Definition 4.1).

- **Exercise 4.10: The Golden Function**
  f is the golden function, i.e.: $f(x) = x^2 - x - 1$.
  A. Prove:

$$f\left(\frac{F_{n+1}}{F_n}\right) = \frac{(-1)^n}{F_n^2}$$
$$f'\left(\frac{F_{n+1}}{F_n}\right) = \frac{L_n}{F_n}$$

B. Prove that the equation for the tangent at the point $x = \frac{F_{n+1}}{F_n}$ on the function's curve is

$$F_{2n}x - F_n^2 y = F_{2n+1}$$

C. Prove that the equation for the tangent at the point $x = -\frac{F_{n+1}}{F_n}$ on the function's curve is

4

$$F_{2n+2}x + F_{n+1}^2 y = -F_{2n+1}$$

- **Exercise 4.11: Quadratic Function (a)**

A. Verify that F_{n+1}, F_{n-1} solves equation

$$x^2 - L_n x + F_n^2 + (-1)^n = 0$$

(You can use Viète's formula).

B. Function f_n is defined by:

$$f_n(x) = x^2 - L_n x + F_n^2 + (-1)^n$$

Find the equations of the tangents to the function's curve at the points where it intersects the x-axis. Also, find the intersection point of the two tangents.

- **Exercise 4.12: Quadratic Function (b)**

In this exercise, $n > 1$.

A. Verify that the solutions of the equation

$$x^2 - 2F_{n+1}x + F_{2n} = 0$$

are L_n, F_n (use Viète's formula).

B. The function f_n is defined by:

$$f_n(x) = x^2 - 2F_{n+1}x + F_{2n}$$

Find the equations of the tangents to the function's curve at the points where it intersects the x-axis. Also, find the intersection point of the two tangents.

- **Exercise 4.13: Quadratic Function (a)**

A. Solve the equation.

$$x^2 - L_n x + (-1)^n = 0$$

(You should use Viète's formula).

B. The function f_n is defined by:

$$f_n(x) = x^2 - L_n x + (-1)^n$$

Find the equations of the tangents to the function's curve at the points where it intersects the x-axis. Also, find the intersection point of the two tangents.

- **Exercise 4.14: Hyperbola**
  Given is the hyperbola defined by $xy = F_{2n}$.
  A. A tangent meets the curve at point (F_n, L_n). Show that the equation of the tangent is:

$$L_n x + F_n y = 2F_{2n}$$

  Show that the area of the triangle formed by the tangent and the two axes is $2F_{2n}$.
  B. A tangent meets the hyperbola at (L_n, F_n). Show that the equation of the tangent is:

$$F_n x + L_n y = 2F_{2n}$$

  Show that the area of the triangle formed by the tangent and the two axes is $2F_{2n}$.

- **Exercise 4.15: Tangent**
  A. Verify that point (F_{n-1}, F_n) exists on the curve of the equation $\delta(x, y) = (-1)^n$.
  B. Prove: The equation for the tangent to the curve at this point is:

$$L_n x - L_{n-1} y = 2(-1)^n$$

  (You can use the first equality in Exercise 4.7).

- **Exercise 4.16: Integrals**
  In this exercise, $n > 1$.
  The function f_n is defined by:

$$f_n(x) = \frac{F_{2n}}{x^2}$$

  Show that the area bounded by the curve of the function, the x-axis and the straight lines $x = F_n$ and $x = L_n$ is $2F_{n-1}$.

- **Exercise 4.17: Trapezium**
  In this exercise, $n > 1$.
  In an isosceles trapezium with larger base angles equal to 45°, the lengths of the bases are F_{n+1} and F_{n-1},
  Prove that the area of the trapezium is $\frac{1}{4}F_{2n}$.

- **Exercise 4.18: The Powers of φ**
  We have shown that:

$$\varphi^n = \tfrac{1}{2}(L_n + F_n\sqrt{5}) \tag{4.15a}$$

$$(1-\varphi)^n = \tfrac{1}{2}(L_n - F_n\sqrt{5}) \tag{4.15b}$$

  using Binet's formulas for the two sequences.
  Prove these formulas (again) using:

$$\varphi^n = F_n\varphi + F_{n-1} \tag{3.2a}$$

$$(1-\varphi)^n = -F_n\varphi + F_{n-1} \tag{3.2b}$$

4

- **Exercise 4.19: Additional Formulas to the Powers of** φ

Prove:

$$\varphi^n = F_n\sqrt{5}\varphi - L_{n-1}$$

$$\varphi^n = -F_{n-1}\sqrt{5}\varphi + L_n\varphi$$

- **Exercise 4.20: Mixed Relationships (c)**

In this exercise we use the formula $\varphi^n = \frac{1}{2}(L_n + F_n\sqrt{5})$.
(Some of the equalities we shall obtain were already proven using another method).

A. Based on the equality $(\varphi^n)^2 = \varphi^{2n}$, deduce that

$$F_{2n} = F_nL_n$$
$$L_{2n} = 1/2\left(L_n^2 + 5F_n^2\right)$$

B. Based on the equality $\varphi^n\varphi^m = \varphi^{n+m}$, deduce that

$$F_{n+m} = 1/2(L_nF_m + L_mF_n)$$
$$L_{n+m} = 1/2(L_nL_m + 5\,F_nF_m)$$

C. Based on the equality $\varphi^{n+p}\varphi^{n-p} = \varphi^{2n}$, deduce that

$$F_{2n} = 1/2\left(L_{n+p}F_{n-p} + L_{n-p}F_{n+p}\right)$$
$$L_{2n} = 1/2\left(L_{n+p}L_{n-p} + 5\,F_{n+p}F_{n-p}\right)$$

(You can assume that $n \geq p$).

D. Based on the equality $\varphi^n = \varphi^{n-p}\varphi^p$, deduce that

$$F_n = 1/2\left(L_pF_{n-p} + L_{n-p}F_p\right)$$
$$L_n = 1/2\left(L_pL_{n-p} + 5\,F_{n-p}F_p\right)$$

(You can assume that $n \geq p$).

- **Exercise 4.21: Additional Proofs for the Formulas of Sums**

Prove:

1. $L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3$
2. $L_2 + L_4 + L_6 + \cdots + L_{2n} = L_{2n+1} - 1$
3. $L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_{2n} - 2$

A. Using (4.3), (3.7).
B. By induction.

- **Exercise 4.22: Equality Between Sums**

Prove:

If n is even, then

1. $L_2 + L_4 + L_6 + \cdots + L_{2n} = L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2$
2. $L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_1L_0 + L_2L_1 + L_3L_2 + \cdots + L_nL_{n-1}$
3. $L_2 + L_4 + L_6 + \cdots + L_{2n} = L_0L_2 + L_1L_3 + L_2L_4 + \cdots + L_{n-1}L_{n+1}$

- **Exercise 4.23: Sum with Alternating Signs**

Find a formula for the sum of:

$$-L_1 + L_2 - L_3 + L_4 - \cdots + (-1)^n L_n$$

- **Exercise 4.24: The Fibonacci and Lucas Sums**

A. Prove:

$$\varphi + \varphi^2 + \varphi^3 + \varphi^4 + \ldots + \varphi^n = \varphi^{n+2} - \varphi - 1$$

using the formula for the sum of a geometric sequence.
B. Using the formula $\varphi^n = ½(L_n + F_n\sqrt{5})$, deduce:

$$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - L_1 - 2 = L_{n+2} - 3$$
$$F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - F_1 = F_{n+2} - 1$$

- **Exercise 4.25: Sum with Powers of 2**

The purpose of this exercise is to find a formula for the following sum:

$$S_n = L_1 + 2L_2 + 4L_3 + 8L_4 + \cdots + 2^{n-1}L_n$$

A. Multiply the two sides by 2, subtract the two sides and arrive at:

$$S_n = -L_1 - 2L_0 - 4\left(L_1 + 2L_2 + 4L_3 + \cdots + 2^{n-3}L_{n-2}\right) + 2^n L_n$$

B. Deduce:

$$S_n = -L_1 - 2L_0 - 4\left(S_n - 2^{n-2}L_{n-1} - 2^{n-1}L_n\right) + 2^n L_n$$

C. Deduce:

$$S_n = 2^n F_{n+1} - 1$$

- **Exercise 4.26: Extension to Negative Indices**

A. Justify the definition:

$$L_{-n} = (-1)^n L_n$$

For each natural n.
B. Verify that for every integer m, the following is true:

$$L_{m+2} = L_{m+1} + L_m$$
$$L_m = \varphi^m + (1 - \varphi)^m$$
$$L_m = F_{m+1} + F_{m-1}$$

(It is, of course, sufficient to test only for negative m).

4

- **Exercise 4.27: A Consistency About Negative Indices**
  In this exercise, n is a natural number. Also, $L_{-n}=(-1)^n L_n$ (see the previous exercise).
  A. We proved:
  $$L_{2n-1} = L_n L_{n-1} + (-1)^n$$
  Verify:
  $$L_{-2n-1} = L_{-n}L_{-n-1} + (-1)^{-n}$$
  B. We proved:
  $$L_nF_{n+1} = F_{2n+1} + (-1)^n$$
  $$L_{n+1}F_n = F_{2n+1} - (-1)^n$$
  Verify that:
  $$L_{-n}F_{-n+1} = F_{-2n+1} + (-1)^{-n}$$
  $$L_{-n+1}F_{-n} = F_{-2n+1} - (-1)^{-n}$$

- **Exercise 4.28: Sum for Negative Indices**
  Prove:
  A. If natural n is even:
  $$L_{-1} + L_{-2} + L_{-3} + L_{-4} + \cdots + L_{-n} = L_{n-1} + 1 = -L_{-n+1} + 1$$
  B. If natural n is odd:
  $$L_{-1} + L_{-2} + L_{-3} + L_{-4} + \cdots + L_{-n} = -L_{n-1} + 1 = -L_{-n+1} + 1$$
  C. For all natural n, deduce that
  $$L_{-1} + L_{-2} + L_{-3} + L_{-4} + \cdots + L_{-n} = (-1)^n L_{n-1} + 1 = -L_{-n+1} + 1$$

- **Exercise 4.29: "Half" Indices**
  In this chapter, we saw that
  $$L_nF_n = F_{2n}$$
  We shall define:
  $$F_{\frac{n}{2}}\sqrt{5} = (\sqrt{\varphi})^n - \left(\frac{i}{\sqrt{\varphi}}\right)^n$$
  $$L_{\frac{n}{2}} = (\sqrt{\varphi})^n + \left(\frac{i}{\sqrt{\varphi}}\right)^n$$
  Show that the following is true:
  $$\frac{L_{\frac{n}{2}}}{F_{\frac{n}{2}}} = F_n$$

- **Exercise 4.30: Matrices and the Lucas Sequence**

Prove:

1. $M^2(L_0, L_1) = 5\Phi^2$
2. $M^2(L_{-1}, L_0) = 5I$
3. $\Phi M^2(L_{n-1}, L_n) = M(L_n, L_{n+1})$
4. $\Phi^{-1} M(L_n, L_{n+1}) = M(L_{n-1}, L_n)$
5. $\Phi^n M(L_{n-1}, L_n) = M(L_{2n-1}, L_{2n})$
6. $\Phi^n M(L_{-1}, L_n) = M(L_{n-1}, L_n)$

- **Exercise 4.31: Matrix Versions**

In this chapter we have demonstrated that

$$\varphi^n = \tfrac{1}{2}(L_n + F_n\sqrt{5})$$
$$(1-\varphi)^n = \tfrac{1}{2}(L_n - F_n\sqrt{5})$$
$$\varphi^n\sqrt{5} = L_n\varphi + L_{n-1}$$
$$(1-\varphi)^n\sqrt{5} = L_n\varphi - L_{n+1}$$

Verify that the above equations also hold for matrix versions, namely:

$$\Phi^n = \tfrac{1}{2}[L_n I + F_n R(5)]$$
$$(I-\Phi)^n = \tfrac{1}{2}[L_n I - F_n R(5)]$$
$$\Phi^n R(5) = L_n\Phi + L_{n-1}I = M(L_{n-1}, L_n)$$
$$(I-\Phi)^n R(5) = L_n\Phi - L_{n+1}I = M(-L_{n+1}, L_n)$$

Answers, Hints and Partial Solutions

- **Exercise 4.1**

A. φ. The easiest way is by using Binet's formula.
B. $\sqrt{5}$.

- **Exercise 4.3**

Square both sides of $F_n = F_{n+1} - F_{n-1}$ and of $L_n = F_{n+1} + F_{n-1}$.

- **Exercise 4.5**

1. Calculate the product $F_{n+1}F_{n-1}$ using Binet's formula.
2. Similar to 1.
3. Square both sides of Binet's formula (of the Fibonacci sequence).
4. Calculate the product $F_m F_n$ using Binet's formula.

- **Exercise 4.6**

A. Start from the left side and use Binet's formulas for both sequences.

- **Exercise 4.7**

Subtract (4.10a and 4.10b, (4.11a, (4.13a) (respectively) from each side.

- **Exercise 4.8**

Use (4.4).

- **Exercise 4.9**

Use Definition (4.1).

- **Exercise 4.10**

A. Use Cassini's formula.
B. Use (4.10a).
C. $f'\left(\frac{-F_n}{F_{n+1}}\right) = \frac{-L_n}{F_{n+1}}$
$f\left(\frac{-F_n}{F_{n+1}}\right) = \frac{(-1)^{n+1}}{F_{n+1}^2}$

- **Exercise 4.11**

B. The tangents are $y = F_n(x - F_{n+1})$ and $y = -F_n(x - F_{n-1})$.
The point of intersection is $\left(\frac{L_n}{2}, \frac{-F_n^2}{2}\right)$.

- **Exercise 4.12**

B. The tangents are $y = -2F_{n-1}(x - F_n)$ and $y = 2F_{n-1}(x - L_n)$
The point of intersection is $(F_{n+1}, -2F_{n-1}^2)$.

- **Exercise 4.13**

A. φ^n, $(1 - \varphi)^n$
B. The slopes of the tangents are $\pm F_n\sqrt{5}$, the point of intersection: $\left(\frac{L_n}{2}, \frac{-5F_n^2}{2}\right)$.

- **Exercise 4.15**

A. Use Cassini's formula.
B. Differentiate as an explicit function.

- **Exercise 4.18**

Replace φ with its numeric value.

- **Exercise 4.19**

For both, begin from the right side.

- **Exercise 4.22**

  According to (4.4–4.6) respectively.

- **Exercise 4.23**

$(-1)^n L_{n-1} + 1$

  (It's best to calculate separately for even n and for odd n).

- **Exercise 4.24**

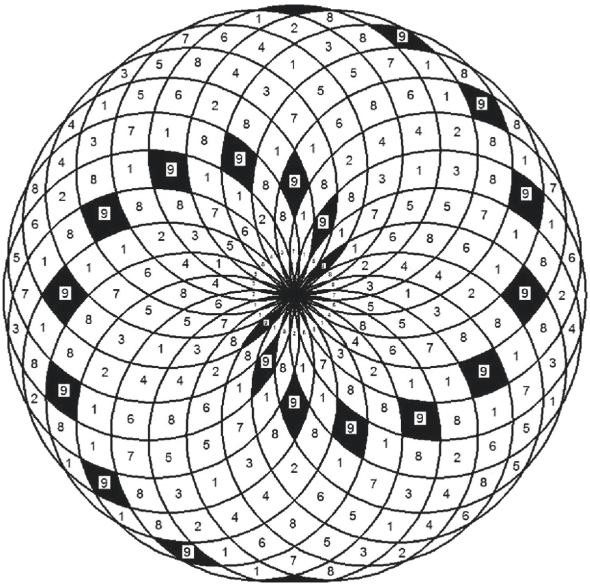
  B. Compare the rational coefficients.

The General Fibonacci-Like Sequences

Contents

Pure mathematics is, in its way, the poetry of logical ideas.
Albert Einstein (1879–1955) (► https://www.brainyquote.com/search_results?q=Albert+Einstein+pure+mathematic).

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*, Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2_5



The recurring sequences in Fibonacci numbers
(Uploaded by Rhuben Nealon)

This is the recurring, 24-digit compressed Fibonacci sequence on a torus skin. The 9 s are highlighted and create approximate Phi spiraling arms… (Posted by Rhuben Nealon FB).

By calculating the Final Digit Sums* (FDS) for the Fibonacci numbers, we get a sequence with cycle length 24. The digit 9 occurs in cycle length 12.

1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, **8**, **8**, **7**, **6**, **4**, **1**, **5**, **6**, **2**, **8**, **1**, **9**, **1**, **1**, **2**, **3** . . .

8, **8**, **7**, **6**, **4**, **1**, **5**, **6**, **2**, **8**, **1**, **9**, 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, **8**, **8**, **7**, **6**, . . .

FDS : 9, 9, 9, 9, ..., 9, 9, 9, 9 . . .

When you add a number from the regular sequence with the corresponding number in the **highlighted sequence**, you get a **Final Digit Sum** 9, for example: $9 + 9 = 18$ and $1 + 8 = 9$.

*The **Final Digit Sum** of 89 is 8: $8 + 9 = 17$ and $1 + 7 = 8$.

■ Introduction to Chapter 5

Analyzing Fibonacci and Lucas sequences naturally evokes the desire to examine in general all the Fibonacci-like sequences that hold to the same sort of recursive equations that both these sequences do.

After dealing with Fibonacci-like sequences in a general sense, and so that we may analyze in depth other aspects of the sequence, we shall define and make use of a Lucas-like sequence and a generalized sequence. The interactions between the various sequences—Fibonacci, Lucas, Fibonacci-like, Lucas-like are extremely fascinating. It also leaves room for more research along various other venues. More experienced readers and those with motivation will be able to identify the "loose ends" and proceed accordingly.

Some of these theoretical developments will appear in the exercises.

5.1 Definitions, Binet's Formulas and Relationships

Using the classic proof of Binet's formula (for the Fibonacci sequence), we saw that any sequence (f_n) whose formula is

$$\boldsymbol{f_n = \alpha\varphi^n + \beta\,(1-\varphi)^n} \tag{5.1}$$

satisfies the recursive equation:

$$\boldsymbol{f_{n+2} = f_{n+1} + f_n} \tag{5.2}$$

This also works vice versa, provided f_0 and f_1 are given. For ease of notation and reading, we shall occasionally (in this section and those following) use $(f_0, f_1) = (a, b)$. We shall call all such sequences **Fibonacci-like sequences.**

Throughout the discussion, we shall assume that when we have α and β, at least one of them is not 0, so that the sequence will not be equivalent to zero. Examining (5.1) leads us to observe the existence of four basic cases:

- If $\alpha = 0$, then $f_n = \beta(1-\varphi)^n$, in other words, the sequence is geometric with quotient $(1-\varphi)$.
- If $\beta = 0$, then $f_n = \beta(1-\varphi)^n$, in other words, the sequence is geometric with quotient φ.
- If $\alpha = \beta$ then $f_n = \alpha\left[\varphi^n + (1-\varphi)^n\right] = \alpha L_n$, meaning that (f_n) is a multiple of the Lucas sequence.
- If $\alpha = -\beta$ then $f_n = \alpha\left[\varphi^n - (1-\varphi)^n\right] = \alpha\sqrt{5}F_n$, meaning that (f_n) is a multiple of the Fibonacci sequence.

We are not interested in investigating the first two cases for two reasons:

- They are geometric sequence (or equivalent to zero if both α and β are 0), and we are already well familiar with geometric series.
- As we shall soon see, in this case, $\delta(a, b) = 0$, and this fact will not allow us to "add it to the denominator".

Therefore, for the rest of the discussion the following shall hold:

$$(\alpha, \beta) \neq (0, 0)$$
$$(\alpha, \beta) \neq (\alpha, 0)$$
$$(\alpha, \beta) \neq (0, \beta)$$

(unless otherwise specified).

Hence, the (f_n) sequence may be defined by using $f_n = \alpha\varphi^n + \beta(1-\varphi)^n$ and then determined through (α, β).

Alternatively, it can be defined by $f_{n+2} = f_{n+1} + f_n$, using (f_0, f_1). Then, it seems natural to ask what the relationship is between α and β on one hand and between f_1 and f_2 on the other?

To determine the value of coefficients α and β using a and b, we must solve the following system of equations:

$$f_0 = \alpha\varphi^0 + \beta(1-\varphi)^0$$
$$f_1 = \alpha\varphi^1 + \beta(1-\varphi)^1$$

That is to say,

$$\alpha + \beta = \mathrm{a}$$
$$\alpha\varphi + \beta(1-\varphi) = \mathrm{b}$$

By multiplying each side of the first equation by φ, and the subtracting the second equation from the first, we obtain:

$$\beta(2\varphi - 1) = \mathrm{a}\varphi - \mathrm{b}, \text{ hence:}$$

$$\boldsymbol{\beta\sqrt{5} = \mathrm{a}\varphi - \mathrm{b}} \tag{5.3b}$$

Substituting this into the original equation produces

$$\boldsymbol{\alpha\sqrt{5} = \mathrm{a}\varphi + \mathrm{b} - \mathrm{a} = \mathrm{a}(\varphi - 1) + \mathrm{b}} \tag{5.3a}$$

and therefore:

$$\boldsymbol{f_{\mathrm{n}} = \frac{1}{\sqrt{5}}[(\mathrm{a}\varphi + \mathrm{b} - \mathrm{a})\,\varphi^{\mathrm{n}} + (\mathrm{a}\varphi - \mathrm{b})\,(1 - \varphi^{\mathrm{n}})]} \tag{5.4a}$$

(We arrived at this result in exercise 3.44 using a different method.)
If we "borrow" 1 from each of the exponents, we obtain:

$$f_{\mathbf{n}} = \frac{1}{\sqrt{5}}\left[(\mathbf{a} + \mathbf{b}\varphi)\varphi^{\mathbf{n}-1} - (\mathbf{a} + \mathbf{b}\varphi)(1-\varphi)^{\mathbf{n-1}}\right] \tag{5.4a'}$$

In particular (the substitutions are made in (5.4):

- if $b = a\varphi$, then $f_{\mathrm{n}} = \mathrm{a}\varphi^{\mathrm{n}}$, that is to say it is geometric sequence with quotient φ.
- If $\mathrm{b} = \mathrm{a}(1-\varphi)$, then $f_{\mathrm{n}} = \mathrm{a}(1-\varphi)^{\mathrm{n}}$, and the sequence is geometric with quotient $(1-\varphi)$.

As we pointed out earlier, we are not interested in dealing with these generalized cases. Therefore, for the rest of the discussion the following shall hold:

$$(\mathrm{a}, \mathrm{b}) \neq (0, 0)$$
$$(\mathrm{a}, \mathrm{b}) \neq (\mathrm{a}, \mathrm{a}\varphi)$$
$$(\mathrm{a}, \mathrm{b}) \neq (\mathrm{a}, \mathrm{a}(1-\varphi))$$

(Unless otherwise specified).

Now we substitute $\varphi = \frac{1}{2}(1+\sqrt{5})$ into 5.4a and obtain:

$$\boldsymbol{f_{\mathrm{n}} = \frac{1}{2\sqrt{5}}\,[\mathrm{a}\sqrt{5} + (2\mathrm{b} - \mathrm{a})\,\varphi^{\mathrm{n}} + [\mathrm{a}\sqrt{5} - (2\mathrm{b} - \mathrm{a})]\,(1 - \varphi)^{\mathrm{n}}]} \tag{5.4b}$$

which can be written as:

$$\boldsymbol{f}_{\mathbf{n}}=\frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\left[\left(\mathbf{c}\sqrt{\mathbf{5}}+\mathbf{d}\right)\varphi^{\mathbf{n}}+\left(\mathbf{c}\sqrt{\mathbf{5}}-\mathbf{d}\right)\left(\mathbf{1}-\varphi\right)^{\mathbf{n}}\right] \tag{5.4c}$$

where

$$(c,d)=\left(\frac{a}{2},\frac{2b-a}{2}\right)=\left(\frac{f_0}{2},\frac{2f_1-f_0}{2}\right)$$

(recall that we indicated that: $(f_0, f_1) = (a, b)$),

and therefore also:

$$\boldsymbol{f}_{\mathbf{n}}=\mathbf{cL_n}+\mathbf{dF_n}=\frac{\mathbf{1}}{\mathbf{2}}\left[\mathbf{aL_n}+(\mathbf{2b}-\mathbf{a})\,\mathbf{F_n}\right] \tag{5.5a}$$

We have arrived at a formula that connects the element in the nth place in a Fibonacci-like sequence to the elements in the same places in the of Fibonacci and Lucas sequences. This formula implies that every Fibonacci-like sequence is in fact a linear combination of the Fibonacci and Lucas sequences. Simpler versions of (5.4c) and (5.5a) will appear below.

We shall now substitute $L_n = 2F_{n-1} + F_n$ (from (4.3)) into (5.5a). We obtain

$$\begin{aligned} f_n &= \frac{1}{2}\left[a\left(2F_{n-1}+F_n\right)+(2b-a)\,F_n\right] \\ &= \frac{1}{2}\left[2aF_{n-1}+aF_n+2bF_n-aF_n\right] \end{aligned}$$

$$\boldsymbol{f}_{\mathbf{n}}=\mathbf{a\,F_{n-1}}+\mathbf{b\,F_n}=\mathbf{f_0F_{n-1}}+\mathbf{f_1F_n} \tag{5.6}$$

(We arrived at this result in Exercise (3.44), by combining observation and proof by induction. Study the answers to that exercise.)

Specifically:
If $a = 0$, then $f_n = bF_n$.
If $b = 0$, then $f_n = aF_{n-1}$.
If $a = b$, then $f_n = aF_{n+1}$.
If $a = -b$, then $f_n = bF_{n-2}$.
If $a = 2b$, then $f_n = bL_n$.

Therefore, we have four definitions for a Fibonacci-like sequence (and we can establish that they are equivalent):

- One uses a "Binet-like" formula (actually two: one using α and β and the other using a and b. For convenience and consistency, henceforth we shall call them the Binet formula for the Fibonacci-like sequence).
- The second is recursive (when a and b are given).
- The third uses the Fibonacci and Lucas sequences.
- The fourth uses the Fibonacci sequence.

We shall now closely observe coefficients α and β. We already know that:

$$\boldsymbol{\alpha}+\boldsymbol{\beta}=\mathbf{a}=\boldsymbol{f}_{\mathbf{0}} \tag{5.7a}$$

What about the product? In fact:

$$
\begin{aligned}
(\alpha\sqrt{5})\,(\beta\sqrt{5}) &= [a\,(\varphi-1)+b]\,(a\varphi-b)\\
&= a^2\,(\varphi-1)\,\varphi - ab\,(\varphi-1) + ab\varphi - b^2\\
&= a^2 - ab\varphi + ab + ab\varphi - b^2\\
&= a^2 + ab - b^2\\
&= \delta\,(a,b)
\end{aligned}
$$

In other words:

$$\mathbf{5\alpha\beta = \delta\,(a,b) = \delta\,(\mathit{f}_0,\mathit{f}_1)} \tag{5.7b}$$

We shall now develop the Cassini formula for (f_n):

$$
\begin{aligned}
\delta(f_{n-1},f_n) &= f_{n+1}f_{n-1} - f_n^2\\
&= \left(\alpha\varphi^{n+1} + \beta(1-\varphi)^{n+1}\right)\left(\alpha\varphi^{n-1} + \beta(1-\varphi)^{n-1}\right) - \left(\alpha\varphi^{n} + \beta(1-\varphi)^{n}\right)^2\\
&= \alpha\beta\varphi^{n+1}(1-\varphi)^{n-1} + \alpha\beta(1-\varphi)^{n+1}\varphi^{n-1} - 2\alpha\beta\varphi^n(1-\varphi)^n\\
&= \alpha\beta\left[\varphi^{n-1}(1-\varphi^{n-1}\left(\varphi^2 + (1-\varphi)^2\right) - 2(-1)^n\right]\\
&= \alpha\beta\left[3(-1)^{n-1} - 2(-1)^n\right]\\
&= 5\alpha\beta(-1)^{n+1}
\end{aligned}
$$

Therefore, according to (5.7b):

$$\boldsymbol{\delta(\mathit{f}_{n-1},\mathit{f}_n) = \mathit{f}_{n+1}\mathit{f}_{n-1} - \mathit{f}_n^2 = \delta(a,b)\,(-1)^{n+1}} \tag{5.8}$$

5.2 Comparing the Fibonacci and the Fibonacci-Like Sequences

In the previous section we established the basic relationship between the Fibonacci sequence and the Fibonacci-like sequence:

$$f_n = aF_{n-1} + bF_n = f_0F_{n-1} + f_1F_n \tag{5.6}$$

In this section, we shall focus on even/odd indices and the sum of indices in the Fibonacci-like sequence. The equalities that we shall prove are the following:

$$\boldsymbol{f_{2n} = f_nF_{n-1} + f_{n+1}F_n = f_{n-1}F_n + f_nF_{n+1}} \tag{5.9a}$$

$$\boldsymbol{f_{2n-1} = f_{n-1}F_{n-1} + f_nF_n} \tag{5.9b}$$

$$\boldsymbol{f_{n+m} = f_mF_{n-1} + f_{m+1}F_n = f_nF_{m-1} + f_{n+1}F_m} \tag{5.10}$$

We begin by proving (5.10), which consists of two equalities. (The second equality arises from the first equality, and thus we shall satisfy ourselves with proving the first):

$$\begin{aligned} f_m F_{n-1} + f_{m+1} F_n &= F_{n-1}(aF_{m-1} + bF_m) + F_n(aF_m + bF_{m+1}) \\ &= a(F_{n-1}F_{m-1} + F_n F_m) + b(F_{n-1}F_m + F_n F_{m+1}) \\ &= aF_{n+m-1} + bF_{n+m} \\ &= f_{n+m} \end{aligned}$$

Now, if we substitute $m = n$, we obtain (5.9a); if we substitute $m = n - 1$, (5.9b).

We shall now present some briefer proofs. To this purpose, we define a new Fibonacci-like sequence, (g_n) by:

$$g_n = f_{n+m}$$

and therefore,

$$(g_0, g_1) = (f_m, f_{m+1})$$

Following (5.6), we can hence write

$$g_n = g_0 F_{n-1} + g_1 F_n = f_m F_{n-1} + f_{m+1} F_n$$

thus arriving at:

$$\boldsymbol{f_{n+m} = f_m F_{n-1} + f_{m+1} F_n} \tag{5.10}$$

And again, substituting $m = n$, gives (5.9a) and substituting $m = n - 1$ gives (5.9b).

In the last two chapters, we promised that we would present simpler proofs for the following two relationships in this chapter:

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m \tag{3.6}$$

$$L_{n+m} = L_m F_{n-1} + L_{m+1} F_n \tag{4.14}$$

In fact, substituting $F_n = f_n$ into (5.10) immediately leads to (3.6) and substituting $L_n = f_n$, we immediately leads to (4.14).

5.3 The Lucas-Like Sequence of the Fibonacci-Like Sequence

Given—**and only if given!**—a Fibonacci-like sequence (f_n) where $(f_0, f_1) = (a, b)$, we can define a Lucas-like sequence (l_n) of (f_n) as follows:

$$\boldsymbol{\ell_n = f_{n+1} + f_{n-1}} \tag{5.11}$$

Specifically:

- (L_n) is the Lucas-like sequence of (F_n) because $L_n = F_{n+1} + F_{n-1}$.
- (F_n) is the Lucas-like sequence of $F_n = (L_{n+1} + L_{n-1})/5$, and for a similar reason, (f_n) is the Lucas-like sequence of $(1_n/5)$.

Since.

$$\mathbf{f_n = f_{n+1} - f_{n-1}}$$

5

we obtain (by adding or subtracting the respective sides)

$$\ell_n + f_n = 2f_{n+1} \tag{5.12a}$$

$$\ell_n - f_n = 2f_{n-1} \tag{5.12b}$$

Exactly as with the Fibonacci and Lucas sequences.

A Lucas-like sequence is a Fibonacci-like sequence **in every respect** (because $l_{n+2} = l_{n+1} + l_n$), and therefore we can automatically write:

$$\ell_n = \ell_0 F_{n-1} + \ell_1 F_n \tag{5.13a}$$

We can check that

$$(l_0, l_1) = (2b - a, 2a + b) = (2f_1 - f_0, 2f_0 + f_1)$$

and hence

$$\ell_n = (2b - a)\,F_{n-1} + (2a + b)\,F_n = (2f_1 - f_0)\,F_{n-1} + (2f_0 + f_1)\,F_n \tag{5.13b}$$

Similarly,

$$\begin{aligned} l_n &= f_{n+1} + f_{n-1} \\ &= aF_n + bF_{n+1} + aF_{n-2} + bF_{n-1} \\ &= a(F_n + F_{n-2}) + b(F_{n+1} + F_{n-1}) \end{aligned}$$

$$\ell_n = aL_n - 1 + bL_n = f_0 L_{n-1} + f_1 L_n \tag{5.14}$$

Note that the similarity between this equality and between

$$f_n = aF_{n-1} + bF_n = f_0 F_{n-1} + f_1 F_n \tag{5.6}$$

allows us to develop a Binet formula for (l_n):

$$\begin{aligned} l_n &= aL_{n-1} + bL_n \\ &= a\left[\varphi^{n-1} + (1-\varphi)^{n-1}\right] + b\left[\varphi^n + (1-\varphi)^n\right] \\ &= \varphi^n[a(\varphi - 1) + b] + (1-\varphi)^n[a(-\varphi) + b] \end{aligned}$$

We obtained:

$$\ell_n = (a\varphi + b - a)\,\varphi^n - (a\varphi - b)\,(1-\varphi)^n \tag{5.15a}$$

$$\ell_n = \sqrt{5}\,[\alpha\varphi^n - \beta\,(1-\varphi)^n] \tag{5.15b}$$

If we "borrow" 1 from each of the exponents, we obtain:

$$\ell_n = (a + b\varphi)\,\varphi^{n-1} - (a + b\varphi)\,(1-\varphi)^{n-1} \tag{5.15a'}$$

In particular:

- If $b = a\varphi$, then $1_n = -a\sqrt{5}(1-\varphi)^n$, meaning that the sequence is geometric with quotient φ.
- If $b = a(1-\varphi)$, then $1_n = -a\sqrt{5}(1-\varphi)^n$ and the sequence is geometric with quotient $(1-\varphi)$.

We now substitute $\varphi = 1/2(1 + \sqrt{5})$ and obtain (after the substitution in 5.15a):

$$\ell_{\mathbf{n}} = \tfrac{1}{2}\left([\mathbf{a}\sqrt{\mathbf{5}} + (\mathbf{2b} - \mathbf{a})]\, \varphi^{\mathbf{n}} - [\mathbf{a}\sqrt{\mathbf{5}} - (\mathbf{2b} - \mathbf{a})]\, (\mathbf{1} - \varphi)^{\mathbf{n}} \right) \tag{5.16a}$$

or

$$\ell_{\mathbf{n}} = (\mathbf{c}\sqrt{\mathbf{5}} + \mathbf{d})\, \varphi^{\mathbf{n}} - (\mathbf{c}\sqrt{\mathbf{5}} - \mathbf{d})\, (\mathbf{1} - \varphi)^{\mathbf{n}} \tag{5.16b}$$

where

$$(c, d) = \left(\frac{a}{2}, \frac{2b - a}{2}\right) = \left(\frac{f_0}{2}, \frac{2f_1 - f_0}{2}\right) = \left(\frac{f_0}{2}, \frac{l_0}{2}\right)$$

We can now write (5.4c) and (5.16b) as follows:

$$f_{\mathbf{n}} = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{5}}}\, [(f_{\mathbf{0}}\sqrt{\mathbf{5}} + \ell_{\mathbf{0}})\, \varphi^{\mathbf{n}} + (f_{\mathbf{0}}\sqrt{\mathbf{5}} - \ell_{\mathbf{0}})\, (\mathbf{1} - \varphi)^{\mathbf{n}}] \tag{5.4d}$$

$$\ell_{\mathbf{n}} = \tfrac{1}{2}[(f_{\mathbf{0}}\sqrt{\mathbf{5}} + \ell_{\mathbf{0}})\, \varphi^{\mathbf{n}} - (f_{\mathbf{0}}\sqrt{\mathbf{5}} - \ell_{\mathbf{0}})\, (\mathbf{1} - \varphi)^{\mathbf{n}}] \tag{5.16c}$$

Similarly, we can simplify (5.5a) as follows:

$$f_{\mathbf{n}} = \tfrac{1}{2}\, (f_{\mathbf{0}}\mathbf{L_n} + \ell_{\mathbf{0}}\mathbf{F_n}) \tag{5.5b}$$

We now return to the formulas for the even/odd indices and the sum of the indices in the Fibonacci-like sequence:

$$f_{\mathbf{2n}} = f_{\mathbf{n}}\mathbf{F_{n-1}} + f_{\mathbf{n+1}}\mathbf{F_n} \tag{5.9a}$$

$$f_{\mathbf{2n-1}} = f_{\mathbf{n-1}}\mathbf{F_{n-1}} + f_{\mathbf{n}}\mathbf{F_n} \tag{5.9b}$$

$$f_{\mathbf{n+m}} = f_{\mathbf{m}}\mathbf{F_{n-1}} + f_{\mathbf{m+1}}\mathbf{F_n} \tag{5.10}$$

Substituting $F_{n-1} = 1/2(L_n - F_n)$ in each, yields:

$$f_{\mathbf{2n}} = \tfrac{1}{2}\, (f_{\mathbf{n}}\mathbf{L_n} + \ell_{\mathbf{n}}\mathbf{F_n}) \tag{5.17a}$$

$$f_{\mathbf{2n-1}} = \tfrac{1}{2}\, (f_{\mathbf{n-1}}\mathbf{L_n} + \ell_{\mathbf{n-1}}\mathbf{F_n}) = \tfrac{1}{2}\, (f_{\mathbf{n}}\mathbf{L_{n-1}} + \ell_{\mathbf{n}}\mathbf{F_{n-1}}) \tag{5.17b}$$

$$f_{\mathbf{n+m}} = \tfrac{1}{2}\, (f_{\mathbf{m}}\mathbf{L_n} + \ell_{\mathbf{m}}\mathbf{F_n}) = \tfrac{1}{2}\, (f_{\mathbf{n}}\mathbf{L_m} + \ell_{\mathbf{n}}\mathbf{F_m}) \tag{5.17c}$$

We have thus obtained a number of lovely simple relationships between the four sequences (Fibonacci, Lucas, Fibonacci-like and Lucas-like). This naturally leads us to inspect the even/odd indices and the index sum for the Lucas-like sequence. The formulas are:

$$\ell_{\mathbf{2n}} = f_{\mathbf{n}}\mathbf{L_{n-1}} + f_{\mathbf{n+1}}\mathbf{L_n} = f_{\mathbf{n-1}}\mathbf{L_n} + f_{\mathbf{n}}\mathbf{L_{n+1}} \tag{5.18a}$$

$$\boldsymbol{\ell_{2n-1} = f_{n-1}L_{n-1} + f_nL_n} \tag{5.18b}$$

$$\boldsymbol{\ell_{n+m} = f_mL_{n-1} + f_{m+1}L_n = f_nL_{m-1} + f_{n+1}L_m} \tag{5.18c}$$

Based on (5.10), and with the aid of (5.12) and (4.8), we can write:

$$\begin{aligned} l_{n+m} &= l_mF_{n-1} + l_{m+1}F_n \\ &= (2f_{m+1} - f_m)F_{n-1} + (2f_m + f_{m+1})F_n \\ &= 2f_{m+1}F_{n-1} - f_mF_{n-1} + 2f_mF_n + f_{m+1}F_n \\ &= f_m(2F_n - F_{n-1}) + f_{m+1}(2F_{n-1} + F_n) \\ &= f_mL_{n-1} + f_{m+1}L_n \end{aligned}$$

5

Substituting m = n gives (5.18a), and substituting m = n – 1 gives (5.18b).

5.4 Comparing the Fibonacci-Like and Lucas-Like Sequences

This section is devoted to developing some **direct** relationships between the Fibonacci-like sequence and its Lucas-like sequence without any "intermediaries" (that is to say, the Fibonacci and/or Lucas sequence). To this effect, we shall look for "inspiration" from ► Chap. 4, where relationships between the Fibonacci and Lucas sequences were presented.

At the beginning of the previous section, we presented the following relationships:

$$\ell_n = f_{n+1} + f_{n-1} \tag{5.11}$$

$$\ell_n + f_n = 2f_{n+1} \tag{5.12a}$$

$$\ell_n - f_n = 2f_{n-1} \tag{5.12b}$$

These are identical to the relationships between the Fibonacci and Lucas sequences.

Before proceeding further, we shall demonstrate an important equivalence that will simplify some later calculations, and that is:

$$\boldsymbol{af_{n-1} + bf_n = \sqrt{5}\,[\alpha^2\varphi^n - \beta^2\,(1 - \varphi)^n]} \tag{5.19}$$

Proof:

$$\begin{aligned} af_{n-1} + bf_n &= a\,[\alpha\varphi^{n-1} + \beta\,(1 - \varphi)^{n-1}] + b\,[\alpha\varphi^n + \beta\,(1 - \varphi)^n] \\ &= a\alpha\varphi^{n-1} + b\alpha\varphi^n + a\beta\,(1 - \varphi)^{n-1} + b\beta\,(1 - \varphi)^n \\ &= \alpha\varphi^{n-1}\,[a\,(\varphi - 1) + b] + \beta\,(1 - \varphi)^n\,[a\,(-\varphi) + b] \\ &= \alpha\varphi^n\alpha\sqrt{5} - \beta\,(1 - \varphi)^n\beta\sqrt{5} \\ &= \sqrt{5}\,[\alpha^2\varphi^n - \beta^2\,(1 - \varphi)^n] \end{aligned}$$

Now, we shall consider some of the multiplicative combinations between the two sequences (the same as what we did in ► Chap. 4 with respect to the Fibonacci and Lucas sequences):

$$\begin{aligned}\ell_n f_n &= \sqrt{5}\,[\alpha\varphi^n - \beta\,(1-\varphi)^n]\,[\alpha\varphi^n + \beta\,(1-\varphi)^n] \\ &= \sqrt{5}\,[\alpha^2\varphi^{2n} - \beta^2\,(1-\varphi)^{2n}]\end{aligned}$$

$$\boldsymbol{\ell_n f_n = a f_{2n-1} + b f_{2n}} \tag{5.20}$$

By substituting $(f_n, l_n)=(F_n, L_n)$, the well-known relationship $L_nF_n=F_{2n}$ is "returned."

$$\begin{aligned}\ell_n f_{n+1} &= \sqrt{5}\,[\alpha\varphi^n - \beta\,(1-\varphi)^n]\,[\alpha\varphi^{n+1} + \beta\,(1-\varphi)^{n+1}] \\ &= \sqrt{5}\,[\alpha^2\varphi^{2n+1} - \beta^2\,(1-\varphi)^{2n+1} + \alpha\beta\,(-1)^n\,(1-\varphi-\varphi)] \\ &= \sqrt{5}\,[\alpha^2\varphi^{2n+1} - \beta^2\,(1-\varphi)^{2n+1}] - 5\alpha\beta\,(-1)^n\end{aligned}$$

$$\boldsymbol{\ell_n f_{n+1} = a f_{2n} + b f_{2n+1} - \delta\,(a,b)\,(-1)^n} \tag{5.21a}$$

Similarly we get:

$$\boldsymbol{\ell_{n+1} f_n = a f_{2n} + b f_{2n+1} + \delta\,(a,b)\,(-1)^n} \tag{5.21b}$$

$$\boldsymbol{\ell_{n-1} f_{n+1} = a f_{2n-1} + b f_{2n} + \delta\,(a,b)\,(-1)^n} \tag{5.21c}$$

$$\boldsymbol{\ell_{n+1} f_{n-1} = a f_{2n-1} + b f_{2n} - \delta\,(a,b)\,(-1)^n} \tag{5.21d}$$

Hence we can conclude:

$$\boldsymbol{\ell_n f_{n+1} + \ell_{n+1} f_n = 2\,(a f_{2n} + b f_{2n+1})} \tag{5.22a}$$

$$\boldsymbol{\ell_n f_{n+1} - \ell_{n+1} f_n = 2\delta\,(a,b)\,(-1)^{n+1}} \tag{5.22b}$$

$$\boldsymbol{\ell_{n-1} f_{n+1} + \ell_{n+1} f_{n-1} = 2\,(a f_{2n-1} + b f_{2n})} \tag{5.22c}$$

$$\boldsymbol{\ell_{n-1} f_{n+1} - \ell_{n+1} f_{n-1} = 2\delta\,(a,b)\,(-1)^n} \tag{5.22d}$$

5.5 The General Sequence of the Fibonacci-Like Sequence

■ **Motivation and definition**

At the beginning of the previous section we proved that:

$$a f_{n-1} + b f_n = \sqrt{5}\,[\alpha^2\varphi^n - \beta^2\,(1-\varphi)^n] \tag{5.19}$$

In this section, we shall take this even further.

Given—**and only if given!**—a Fibonacci-like sequence (f_n) where $(f_0, f_1) = (a, b)$, we can define general sequence (g_n) of (f_n) as follows:

$$\boldsymbol{g_n = af_{n-1} + bf_n = f_0 f_{n-1} + f_1 f_n} \tag{5.23a}$$

By using (5.19) we get:

$$\boldsymbol{g_n = \sqrt{5}\,[\alpha^2 \varphi^n - \beta^2 (1-\varphi)^n]} \tag{5.23b}$$

First, we calculate g_0 and g_1:

$$g_0 = af_{-1} + bf_0 = a(b-a) + ba = a(2b-a) = f_0 \ell_0$$
$$g_1 = af_0 + bf_1 = a^2 + b^2 = f_0^2 + f_1^2$$

Now, according to (5.23a), this sequence is clearly Fibonacci-like for all intents and purposes, so we can automatically write:

$$\boldsymbol{g_n = f_0 \ell_0 F_{n-1} + (f_0^2 + f_1^2)\, F_n} \tag{5.24}$$

Specifically:
If $f_n = F_n$, then $g_n = F_n$ (that is to say: F_n is the generalized sequence of itself!).

- **Even and Odd Indices**

Observing (5.11) and (5.20) and that $f_n = f_{n+1} - f_{n-1}$, allows us to immediately write:

$$\boldsymbol{g_{2n} = f_n \ell_n = f_{n+1}^2 - f_{n-1}^2} \tag{5.25a}$$

Now, we shall prove (using induction over $n \geq 0$) that:

$$\boldsymbol{g_{2n-1} = f_{n-1}^2 + f_n^2} \tag{5.25b}$$

(This hypothesis stems from the value of g_1.)
We saw that this formula is correct for 1. We shall assume that it is correct for $k \geq 1$, and prove it correct for $k+1$:

$$g_{2k+1} = g_{2k-1} + g_{2k} = \left(f_{k-1}^2 + f_k^2\right) + \left(f_{k+1}^2 - f_{k-1}^2\right) = f_k^2 + f_{k+1}^2$$

Earlier we saw that if $f_n = F_n$, then $g_n = F_n$. Therefore, we can substitute $g_n = F_n$ in (5.25), leading back to the relationships we are familiar with from ► Chaps. 3 and 4:

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 \tag{3.5a}$$

$$F_{2n} = L_n F_n \tag{4.9}$$

$$F_{2n-1} = F_n^2 - F_{n-1}^2 \tag{3.5b}$$

Now, we can write the equalities of (5.21) as follows:

$$\boldsymbol{\ell_n f_{n+1} = g_{2n+1} - \delta(a,b)(-1)^n} \tag{5.21a'}$$

$$\boldsymbol{\ell_{n+1} f_n = g_{2n+1} + \delta(a,b)(-1)^n} \tag{5.21b'}$$

$$\boldsymbol{\ell_{n-1} f_{n+1} = g_{2n} + \delta(a,b)(-1)^n} \tag{5.21c'}$$

$$\boldsymbol{\ell_{n+1} f_{n-1} = g_{2n} - \delta(a,b)(-1)^n} \tag{5.21d'}$$

- **Special Cases**

We return to one of the definitions for (g_n):

$$g_n = \sqrt{5}\,[\alpha^2 \varphi^n - \beta^2 (1-\varphi)^n] \tag{5.23b}$$

We shall examine two specific cases:

1. if $\alpha = \pm\beta$, then

$$g_n = \sqrt{5}\,[\alpha^2 \varphi^n - \alpha^2 (1-\varphi)^n] = \sqrt{5}\,\alpha^2 [\varphi^n - (1-\varphi)^n] = \sqrt{5}\,\alpha^2 F_n$$

In other words, (g_n) is a multiple of the Fibonacci sequence.
Recall that in the first section, we saw that if $\alpha = \beta$, then (f_n) is a multiple of the Lucas sequence, and if $\alpha = -\beta$, then (f_n) is a multiple of the Fibonacci sequence.

2. if $\alpha = \pm i\beta$, then

$$g_n = \sqrt{5}\,[\alpha^2 \varphi^n + \alpha^2 (1-\varphi)^n] = \sqrt{5}\,\alpha^2 [\varphi^n + (1-\varphi)^n] = \sqrt{5}\,\alpha^2 L_n$$

in other words, (g_n) is a multiple of the Lucas sequence.

5.6 The Powers of the Golden Ratio

In ► Chaps. 3 and 4 we saw that:

$$F_{n-1} + F_n \varphi = \varphi^n \tag{3.5a}$$

$$F_{n-1} - F_n \varphi = (1-\varphi)^n \tag{3.5b}$$

$$L_{n-1} + L_n \varphi = \sqrt{5}\,\varphi^n \tag{4.17a}$$

$$L_{n-1} - L_n \varphi = \sqrt{5}\,(1-\varphi)^n \tag{4.17b}$$

It seems natural to now address the analogue expressions

$$f_{n-1} + f_n \varphi$$
$$f_{n+1} - f_n \varphi$$

5

In fact:

$$\begin{aligned} f_{n-1} + f_n\varphi &= (aF_{n-2} + bF_{n-1}) + (aF_{n-1} + bF_n)\varphi \\ &= a(F_{n-2} + F_{n-1}\varphi) + b(F_{n-1} + F_n\varphi) \\ &= a\varphi^{n-1} + b\varphi^n \\ &= (a + b\varphi)\varphi^{n-1} \end{aligned}$$

therefore:

$$\boldsymbol{\varphi^{n-1} = \frac{(a + b\varphi) \cdot (f_{n-1} + f_n\varphi)}{\delta(a,b)}} \tag{5.26a}$$

Similarly, we obtain:

$$f_{n-1} + f_n\varphi = (a + b\varphi)\,(1 - \varphi)^{n-1}$$

$$\boldsymbol{(1 - \varphi)^{n-1} = \frac{(a + b\varphi) \cdot (f_{n-1} - f_n\varphi)}{\delta(a,b)}} \tag{5.26b}$$

Here we preferred (n – 1) as the exponent instead of n due to the simplicity of the results.

If we multiply together the respective sides of each equality, we return to the Cassini formula for the Fibonacci sequence:

$$\delta(f_{n-1}, f_n) = f_{n+1}f_{n-1} - f_n^2 = \delta\,(a, b)\,(-1)^{n+1} \tag{5.8}$$

We shall now continue on to another aspect. From the previous chapter we know that

$$\varphi^n = \tfrac{1}{2}\,(L_n + F_n\sqrt{5}) \tag{4.15a}$$

$$(1 - \varphi)^n = \tfrac{1}{2}\,(L_n - F_n\sqrt{5}) \tag{4.15b}$$

If we begin with (5.26), replace φ with its numeric value, and substitute (5.12), we obtain:

$$\boldsymbol{\varphi^{n-1} = \frac{(a + b\varphi) \cdot (\ell_n + f_n\sqrt{5})}{2\delta(a,b)}} \tag{5.27a}$$

$$\boldsymbol{(1 - \varphi)^{n-1} = \frac{(a + b\varphi) \cdot (\ell_n - f_n\sqrt{5})}{2\delta(a,b)}} \tag{5.27b}$$

5.7 Ordering Fibonacci-Like Sequences

A Fibonacci-like sequence may have “strange and different” forms depending on the types of numbers that it is made up of: natural numbers, integers, rational numbers, φ-values, and even complex numbers. In the analysis up to now, there was no restriction on the type of number.

In this section, we shall discuss special cases of Fibonacci-like sequences and describe the properties of some special ones.

- **Multiples of Fibonacci**

In the first section of this chapter we saw that if $\alpha = -\beta$, that is to say $a = 0$, then:

$$f_n = \alpha\sqrt{5}\,F_n = bF_n$$

In other words, (f_n) is a multiple of the Fibonacci sequence.
In this case,

$$\delta(a, b) = \delta(0, b) = -b^2$$

and therefore:

$$\delta(f_{n-1}, f_n) = \delta(0, b)(-1)^{n+1} = b^2(-1)^n$$

$$\mu(f_{n-1} + f_n\varphi) = \mu(f_{n+1} - f_n\varphi) = |b|$$

(Note that the first equality is a generalization of Cassini's formula.)

- **Multiples of the Lucas Sequence**

In the first section of this chapter we saw that if $\alpha = \beta$, meaning that $a = 2b$, then:

$$f_n = \alpha L_n = bL_n$$

In other words, (f_n) is a multiple of the Lucas sequence.
In this case:

$$\delta(a, b) = \delta(2b, b) = 5b^2$$

and therefore:

$$\delta(f_{n-1}, f_n) = \delta(2b, b)\,(-1)^{n+1} = 5b^2\,(-1)^{n+1}$$

$$\mu(f_{n-1} + f_n\varphi) = \mu(f_{n+1} - f_n\varphi) = |\,b\,|\sqrt{5}$$

- **Multiples of "Shifted" Fibonacci Sequences**

A multiple of the Fibonacci sequence that is "shifted" (to the right) is defined by:

$$f_n = cF_{n+k}$$

where k is a natural number, and $c \neq 0$.
Therefore:

$$(a, b) = (cF_k, cF_{k+1})$$

In this case:

$$\delta(a, b) = \delta(cF_k, cF_{k+1}) = c^2\delta(F_k, F_{k+1}) = c^2(-1)^{k+1}$$

and therefore:

$$\delta(f_{n-1}, f_n) = \delta(cF_k, cF_{k+1})(-1)^{n+1} = c^2(-1)^{k+1}(-1)^{n+1} = c^2(-1)^{n+k}$$
$$\mu(f_{n-1} + f_n\varphi) = \mu(f_{n+1} - f_n\varphi) = |c|$$

- **Multiples of "Shifted" Lucas Sequences**

A multiple of a Lucas sequence that is "shifted" (to the right) is defined by:

$$f_n = cL_{n+k}$$

where k is a natural number, and $c \neq 0$.

Therefore:

5

$$(a, b) = (cL_k, cL_{k+1})$$

In this case:

$$\delta(a, b) = \delta(cL_k, cL_{k+1}) = c^2\delta(L_k, L_{k+1}) = 5c^2(-1)^k$$

and therefore:

$$\delta(f_{n-1}, f_n) = \delta(cL_k, cL_{k+1})(-1)^{n+1} = 5c^2(-1)^k(-1)^{n+1} = 5c^2(-1)^{n+k+1}$$
$$\mu(f_{n-1} + f_n\varphi) = \mu(f_{n+1} - f_n\varphi) = |c|\sqrt{5}$$

- **Rational Fibonacci-Like Sequences**

If a and b are rational numbers (and then all the elements of the sequence will be rational numbers), the sequence will be termed a rational Fibonacci-like sequence.

Also:

$$\delta(f_{n-1}, f_n) = \delta(cL_k, cL_{k+1})\,(-1)^{n+1} = 5c^2\,(-1)^k\,(-1)^{n+1} = 5c^2\,(-1)^{n+k+1}$$

$$\mu\,(f_{n-1} + f_n\varphi) = \mu\,(f_{n+1} - f_n\varphi) = \,|\,c\,|\,\sqrt{5}$$

We shall now formulate a theorem that confers a more exact property for a rational Fibonacci-like sequence.

Theorem:

A Fibonacci-like sequence is rational if and only if.

$$\boldsymbol{f_n = (A + B\sqrt{5})\,\varphi^n + (A - B\sqrt{5})\,(1 - \varphi)^n} \tag{5.28a}$$

where A and B are rational numbers.

Proof:

We begin from formula (5.4c):

$$f_n = \frac{1}{\sqrt{5}}\left[\left(c\sqrt{5} + d\right)\varphi^n + \left(c\sqrt{5} - d\right)(1 - \varphi)^n\right]$$

and rewrite it as follows:

$$f_n = \left(c + \frac{d\sqrt{5}}{5}\right)\varphi^n + \left(c - \frac{d\sqrt{5}}{5}\right)(1-\varphi)^n$$

If (f_n) is a rational Fibonacci-like sequence, then c and d are rational numbers, and therefore if we write

$$c = A, d/5 = B$$

we shall arrive at (5.28).

In the opposite direction: If (5.28) holds, then

$$\begin{aligned} f_0 &= (A + B\sqrt{5})\,\varphi^0 + (A - B\sqrt{5})\,(1-\varphi)^0 = 2A \in \mathbf{Q} \\ f_1 &= (A + B\sqrt{5})\,\varphi^1 + (A - B\sqrt{5})\,(1-\varphi)^1 \\ &= A\varphi + B\sqrt{5}\,\varphi + A - B\sqrt{5} - A\varphi + B\sqrt{5}\,\varphi \\ &= B\sqrt{5}\,(2\varphi - 1) + A \\ &= B\sqrt{5}\sqrt{5} + A \\ &= 5B + A \in \mathbf{Q} \end{aligned}$$

If f_0 and f_1 are rational, then according to (5.2), every sequence must necessarily be rational.

Note that you can also write (5.28a) as follows:

$$\boldsymbol{f_n} = \mathbf{AL_n + BF_n} \tag{5.28b}$$

and then the proof of the theorem is easier: If A and B are rational numbers, then clearly, the sequence (f_n) is rational, and if the sequence is rational, then (5.28b) can be derived from (5.5a).

Exercises for Chapter 5

Please note:

- n, m, k, are usually natural numbers (or 0), unless indicated otherwise.
- (f_n) is a Fibonacci-like sequence. ((f_n) is not geometric and is not a (0) identity).
- (l_n) is Lucas-like sequence of (f_n).
- (g_n) is a surjective sequence of (f_n).
- (α, β), (a,b), and (c, d) are as defined in the text in the chapter, and conform to the restrictions noted in the first section.

■ **Exercise 5.1: Cassini's Formula**

In the text we proved that:

$$\delta(f_{n-1}, f_n) = f_{n+1} f_{n-1} - f_n^2 = \delta\,(a, b)\,(-1)^{n+1} \tag{5.8}$$

Prove the formula (again) using:

$$f_n = aF_{n-1} + bF_n \tag{5.6}$$

- **Exercise 5.2: The Sequence of Numerators of (f_n)**

Calculate: $\lim_{n\to\infty} f_{n+1}/f_n$.

- **Exercise 5.3: Sums of a Fibonacci-Like Sequence**

Prove:

1. $f_1 + f_2 + f_3 + \ldots + f_n = f_{n+2} - f_2$
2. $f_2 + f_4 + f_6 + \ldots + f_{2n} = f_{2n+1} - f_1$
3. $f_1 + f_3 + f_5 + \ldots + f_{2n-1} = f_{2n} - f_0$
4. $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n f_{n+1} - f_1 f_0$

- **Exercise 5.4: Linear System**

Prove that the solution of the following system does not depend on n.

$$f_{n-1}x + f_n y = f_{n+1}$$
$$f_n x + f_{n+1} y = f_{n+2}$$

- **Exercise 5.5: Non-Infinite Series**

In this exercise, assume that $f_n \neq 0$ for all n.

A. Prove:

$$\frac{1}{f_n f_{n+2}} = \frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1} f_{n+2}}$$

B. Deduce:

$$\frac{1}{f_1 f_3} + \frac{1}{f_2 f_4} + \frac{1}{f_3 f_5} + \frac{1}{f_4 f_6} + \ldots + \frac{1}{f_n f_{n+2}} = \frac{1}{f_1 f_2} - \frac{1}{f_{n+1} f_{n+2}}$$

C. Deduce:

$$\frac{1}{f_1 f_3} + \frac{1}{f_2 f_4} + \frac{1}{f_3 f_5} + \frac{1}{f_4 f_6} + \ldots = \frac{1}{f_1 f_2}$$

- **Exercise 5.6: Even/Odd**

Prove:

A. If n and m are even, then

$$f_{n+1}f_{n-1} - f_{m+1}f_{m-1} = f_n^2 - f_m^2$$

B. If n and m are odd, then

$$f_{n+1}f_{n-1} - f_{m+1}f_{m-1} = f_n^2 - f_m^2$$

C. If n is even and m is odd (or m is even and n is odd) then

$$f_{n+1}f_{n-1} + f_{m+1}f_{m-1} = f_n^2 + f_m^2$$

- **Exercise 5.7: Extending the Fibonacci-Like Sequence**

A. Calculate f_{-1}, f_{-2}, f_{-3}, f_{-4}, and propose and propose a definition for f_{-n} (where n is natural)

B. Deduce (n is natural):

$$f_{-n} = aF_{-n-1} + bF_{-n}$$

C. Deduce (for each **whole/integer** m)

$$f_m = aF_{m-1} + bF_m$$

D. Prove:

$$f_{-1} + f_{-2} + f_{-3} + f_{-4} + \ldots + f_{-n} = (-1)^n(aF_n - bF_{n-1}) + b = -f_{-n+1} + b$$

- **Exercise 5.8: F_n as an Expression of f_n**

A. Prove that for all natural n and for every Fibonacci-like sequence (f_n):

$$F_n = \delta^{-1}(a,b)(aff_{n+1} - bf_n)$$

B. Deduce that for every natural n sequence and for every Fibonacci-like sequence (f_n), the expression on the right side of the above equality is a natural (!) number.

C. Prove that if a and b are integers, then $(af_{n+1} - bf_n)$ and $\delta(a, b)$ are integers, and that $\delta(a, b)$ is a divisor of $(af_{n+1} - bf_n)$ for all natural n.

- **Exercise 5.9: Comparing the Fibonacci-Like Sequence and the Lucas Sequence**

Prove:

1. $f_nL_n = f_{2n} + a(-1)^n$
2. $f_{n-1}L_n = f_{2n-1} + (a-b)(-1)^{n+1}$
3. $f_nL_{n-1} = f_{2n-1} + b(-1)^{n+1}$

- **Exercise 5.10: Comparing the Lucas-Like Sequence and the Fibonacci Sequence**

Prove:

1. $1_nF_n = f_{2n} + a(-1)^{n+1}$
2. $l_{n-1}F_n = f_{2n-1} + (a-b)(-1)^n$
3. $l_nF_{n-1} = f_{2n-1} + b(-1)^n$

- **Exercise 5.11: All Together**

In this chapter we proved that

$$f_{2n} = 1/2[f_nL_n + 1_nF_n] \quad (5.17a)$$

$$f_{2n-1} = 1/2\left[f_{n-1}L_n + 1_{n-1}F_n\right] = 1/2\left[f_nL_{n-1} + 1_nF_{n-1}\right] \quad (5.17b)$$

Prove (again) the equalities using the results of exercises 5.9 and 5.10.

- **Exercise 5.12: Sums**

Prove:

1. $f_1L_1 + f_2L_2 + f_3L_3 + \ldots + f_nL_n = f_{2n+1} - b$ (n even)

2. $f_1L_1 + f_2L_2 + f_3L_3 + \ldots + f_nL_n = f_{2n+1} - a - b$ (n odd)
3. $1_1F_1 + 1_2F_2 + 1_3F_3 + \ldots + 1_nF_n = f_{2n+1} - b$ (n even)
4. $1_1F_1 + 1_2F_2 + 1_3F_3 + \ldots + 1_nF_n = f_{2n+1} + a - b$ (n odd)

5

- **Exercise 5.13: Inequalities**

A. Given are the vectors (φ^n, $(1 - \varphi)^n$) and (α, β), where α and β are real numbers. Calculate their scalar product and their lengths, and deduce that for all n the following holds:

$$f_n^2 \leq \left(\alpha^2 + \beta^2\right)L_{2n}$$

B. Given are the vectors (F_{n-1}, F_n) and (a, b), where a and b are real numbers. Calculate their scalar product and their lengths, and deduce that for all n the following holds:

$$f_n^2 \leq \left(a^2 + b^2\right)F_{2n-1}$$

C. Given are the vectors (L_n, F_n) and (c, d), where c and d are real numbers. Calculate their scalar product and their lengths, and deduce that for all n the following holds:

$$f_n^2 \leq \left(c^2 + d^2\right)\left(L_n^2 + F_n^2\right)$$

- **Exercise 5.14: Units**

Prove: $1_nf_n = f_{2n}$ if and only if $f_n = F_n$ (and then also $l_n = L_n$).
(Reminder: In the first section, we established that f_n is not geometric.)

- **Exercise 5.15: Between δ and δ**

A. Prove:

$$\delta(l_0, l_1) = -5\delta(f_0, f_1)$$

B. Deduce:

$$\delta(l_{n-1}, l_n) = 5\delta(f_0, f_1)(-1)^n = -5\delta(f_{n-1}, f_n)$$

- **Exercise 5.16: Powers of the Golden Ratio**

A. In this chapter we proved that:

$$\varphi^n = \delta^{-1}(a, b)\,(a + b\varphi) \cdot (f_{n-1} + f_n\varphi) \tag{5.26a}$$

$$(1 - \varphi)^n = \delta^{-1}(a, b)\,(a + b\varphi) \cdot (f_{n+1} - f_n\varphi) \tag{5.26b}$$

Prove (again) the above formulas using:

$$f_n = \alpha\varphi^n + \beta\,(1 - \varphi)^n \tag{5.1}$$

B. Prove:

$$\frac{1}{2}\left(l_n + f_n\sqrt{5}\right) = \frac{1}{2}\left(l_0 + f_0\sqrt{5}\right)\varphi^n$$
$$\frac{1}{2}\left(l_n - f_n\sqrt{5}\right) = \frac{1}{2}\left(l_0 - f_0\sqrt{5}\right)\varphi^n$$

C. Prove:

$$\frac{l_{n-1} + l_n\varphi}{f_{n-1} + f_n\varphi} = \frac{l_0 + l_1\varphi}{f_0 + f_1\varphi} = \sqrt{5}$$

- **Exercise 5.17: Generalized Sequence**
  Prove:
  1. $g_o f_{n-1} + g_1 f_n = f_o g_{n-1} + f_1 g_n$
  2. $a g_{n+1} - b g_n = \delta^{-1}(a, b) f_n$
  3. $g_{n-1} + g_{n+1} = 1_o f_{n-1} + 1_1 f_n$

- **Exercise 5.18: Mixed Sums**
  Prove:

$$l_1 f_1 + l_2 f_2 + l_3 f_3 + \ldots + 1_n f_n = a_{2n} + b f_{2n+1} - a^2 - b^2 = g_{2n+1} - g_1$$

- **Exercise 5.19: Squares**
  A. Prove:

$$(a + b\varphi)^2 = (a^2 + b^2) + b\,(2a + b)\,\varphi$$

B. Deduce:

$$(f_0 + f_1\varphi)^2 = g_1 + g_2\varphi$$

C. Prove:

$$(f_{n-1} + f_n\varphi)^2 = g_{2n-1} + g_{2n}\varphi$$

- **Exercise 5.20: Composite Sequences**
  A. Define:

$$f_n = F_{n-1} + iF_n$$

Prove:
B. In this section, α and β are real numbers.

$$|f_n|^2 = F_{2n-1}$$

We define:

$$f_n = \alpha\varphi^n + i\beta(1 - \varphi)^n$$

Prove:

$$|f_n|^2\sqrt{5} = g_{2n}$$

- **Exercise 5.21: Matrices and Fibonacci-Like Sequences**

In this exercise, (fn) is a rational Fibonacci-like sequence. Prove:

1. $\Phi M(f_{n-1}, f_n) = M(f_n, f_{n+1})$
2. $\Phi^{-1} M(f_n, f_{n+1}) = M(f_{n-1}, f_n)$
3. $\Phi^n M(f_{n-1}, f_n) = M(f_{2n-1}, f_{2n})$
4. $\Phi^k M(f_{n-1}, f_n) = M(f_{n+k-1}, f_{n+k})$
5. $\Phi^n M(f_{-1}, f_0) = M(f_{n-1}, f_n)$
6. $M(f_m - 1, f_m) = M(a, b)\Phi^{n-1}$
7. $M(f_{n-1}, f_n)R(5) = M(l_{n-1}, l_n)$
8. $M(f_{n-1}, f_n)M(f_{n+1}, -f_n) = \delta(f_{n-1}, f_n)\boldsymbol{I} = (-1)^{n+1}\delta(a, b)\boldsymbol{I}$

5

- **Exercise 5.22: Matrix version for Binet's Formula**

In this exercise, (f_n) is a rational Fibonacci-like sequence.
In this chapter, we saw that

$$f_n\sqrt{5} = (a + b\varphi)\, \varphi^{n-1} - (a + b\varphi) \cdot (1 - \varphi)^{n-1} \tag{5.4a'}$$

Verify the following:

$$f_n R(5) = M(a, b)\, \Phi^{n-1} - M(a + b, -b)\, (I - \Phi)^{n-1}$$

- **Exercise 5.23: "Hyperbolic-Fibonacci" Functions**

The functions f and g are defined for all real x by:
$f(x) = \varphi^x + (\varphi - 1)^x$
$g(x) = \varphi^x - (\varphi - 1)^x$

A. Establish the following facts (x is a real variable):
1. $f(x) + g(x) = 2\varphi^x$
2. $f(x) - g(x) = 2(\varphi - 1)^x$
3. $f^2(x) + g^2(x) = 2f(2x)$
4. $f(x)g(x) = g(2x)$
5. $f(x) = 2\cosh(x \ln \varphi)$
6. $g(x) = 2\sinh(x \ln \varphi)$

B. Establish the following facts (x and y are real variables):
1. $f(x + y) = \varphi^x g(y) + (\varphi - 1)^y f(x) = \varphi^y g(x) + (\varphi - 1)^x f(y)$
2. $g(x + y) = \varphi^x g(y) + (\varphi - 1)^y g(x) = \varphi^y g(x) + (\varphi - 1)^x g(y)$
3. $f(x)g(y) + f(y)g(x) = 2g(x + y)$
4. $f(x)f(y) + g(x)g(y) = 2f(x + y)$
5. $f'(x)f(y) + f'(y)f(x) = g'(x)g(y) + g(x)g'(y)$

C. Establish the following facts (n is a natural number):
1. $L_{2n} = f(2n)$
2. $L_{2n-1} = g(2n - 1)$
3. $\sqrt{5}\, F_{2n} = g(2n)$
4. $\sqrt{5}\, F_{2n-1} = f(2n - 1)$
5. $2f_{2n} = (\alpha + \beta)f(2n) + (\alpha - \beta)g(2n)$
6. $2f_{2n-1} = (\alpha + \beta)g(2n - 1) + (\alpha - \beta)f(2n - 1)$

(Be careful! Make sure to distinguish between f_n and (f(n)!).

- **Exercise 5.24: Fibonacci-like triplets (a)**

A triplet (a, b, c) will be called a Fibonacci-like triplet if c = a + b.

A. Show that the following triplets are Fibonacci-like triplets:

1. $\left(a^2 + b^2, 2ab, (a + b)^2\right)$
2. $\left((a - b)^2, 4ab, (a + b)^2\right)$
3. $\left(a^2 - b^2, ab, \delta(a, b)\right)$
4. $\left(\cos^2 \alpha, \sin^2 \alpha, 1\right)$
5. $\left(\cos^2 \alpha, -\sin^2 \alpha, \cos 2\alpha\right)$
6. $[\cos \alpha, \sin \alpha, \sqrt{2} \sin(\alpha + 45°)]$
7. $(\cot \alpha, -\text{tg}\alpha, 2 \cot 2\alpha)$
8. $\left(e^x, e^{-x}, 2 \cosh x\right)$
9. $(\cosh(x), \sinh(x), e^x)$
10. $\left(\cosh^2(x), \sinh^2(x), \cosh(2x)\right)$
11. $\left(\cosh^2(x), -\sinh^2(x), 1\right)$
12. $(\ln(a), \ln(b), \ln(ab))$

B. Show that the following triplets are Fibonacci-like triplets:

1. $(1, \sqrt{5}, 2\varphi)$
2. $(\varphi, \varphi - 1, \sqrt{5})$
3. $(F_{n-1}, F_n\varphi, \varphi^n)$
4. $(F_{n+1}, -F_n\varphi, (1 - \varphi)^n)$
5. $(F_{n+1}, -F_{n-1}, F_n)$
6. $\left(F_{n+1}F_{n-1}, -F_n^2, (-1)^n\right)$
7. $\left(F_n^2, F_{n-1}^2, F_{2n-1}\right)$
8. (F_{n+1}, F_{n-1}, L_n)
9. (L_n, F_n, F_{2n+1})
10. $(f_1F_n, f_0F_{n-1}, f_n)$

- **Exercise 5.25: Fibonacci-Like Triplets (b)**

A. The triplet (a, b, c) is a Fibonacci-like triplet. Prove that the following triplets are Fibonacci-like triplets:

1. (b, a, c)
2. $(-a, c, b)$
3. $(a - b, b, c - b)$
4. $(c, -b, a)$
5. (ka, kb, kc)
6. $(a + k, b + k, c + 2k)$

B. The triplet (a, b, c) is a Fibonacci-like triplet. Prove:

1. $\delta(a, b) = ac - b^2$
2. $\delta(a, b) + \delta(b, c) = 0$
3. $\delta(c, a) = \delta(c, b)$

C. Prove:

- If $(a, a + d, a + 2d)$, is a Fibonacci-like triplet, then $d = a$.
- If $\left(a, aq, aq^2\right)$ is a a Fibonacci-like triplet, then $q = \varphi, -1/\varphi$.

- If $(1/a, 1/b, 1/c)$ is a Fibonacci-like triplet, then $c^2 = ab$.
- If $(a, b, k(a + b))$ is a Fibonacci-like triplet, then $k = 1$ or $b = -a$.

D. Given the equation $x^2 - cx + ab = 0$,

Prove that if $a \neq b$, then $S = \{a, b\}$, and if $a = b$, then $S = \{a\}$.

Answers, Hints and Partial Solutions

5

Exercise 5.2

φ. The simplest proof uses (5.1).

Exercise 5.3

The simplest way: Using the telescopic cancellation method (as we did for finding the sums of (F_n).)

Exercise 5.4

The solution is (1,1). It is not worth trying to solve this systematically. This is an "obvious" solution, and it is unique since the determinants of the system is equal to $\delta(f_{n-1}, f_n)$ and therefore is different from 0. (Reminder: In the first section, we established that fn is not geometric.)

Exercise 5.5

A. Begin with the right side.
C. The conclusion was established in the previous section.

Exercise 5.6

B. This is in accordance to (5.8).

Exercise 5.7

A. $f_n = (-1)^n(aF_{n+1} - bF_n)$
C. Do not forget f_0.

Exercise 5.8

A. Begin with the right side and substitute (5.6).
B. F_n is a natural number for all n.

Exercise 5.12

The first two: According to the first equality in Exercise 5.9 and the second equality in Exercise 5.3.
The latter two: According to the first equality in Exercise 5.10 and the second equality in Exercise 5.3.

- **Exercise 5.13**
  Calculate the cosine of the angle between the vectors.

- **Exercise 5.14**
  $1_n f_n = f_{2n}$ can be transformed into an equality.
  $\sqrt{5}\,[\alpha^2 \varphi^{2n} - \beta^2 (1-\varphi)^{2n}] = a\varphi^{2n} + \beta (1-\varphi)^{2n}$

- **Exercise 5.16**
  This is derived from (5.26a). (Reminder: (l_n) is a Fibonacci-like sequence in every sense of the word.)

- **Exercise 5.19**
  A. See ► Chap. 1, Section C.

In Opher's Footsteps—Challenges for Exploration

The only way to learn mathematics is to do mathematics. That tenet is the foundation of the do-it-yourself, Socratic, or Texas method … (self-learning through exploration).
Paul Richard Halmos (1916–2006) (From ► https://www.azquotes.com/quote/739718*)*

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*, Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2_6

Stamp series, Macau, China 2007

(photographed from the collection of stamps by Dr. Johnny Oberman)

Many stamps around the world have been issued in honor of Fibonacci.

Introduction to Chapter 6

This chapter was written by Anatoly Shtarkman and Bat-Sheva Ilany

This chapter includes 13 challenging tasks intended to supplement Chaps. 1–5. They are mainly explorative exercises that can lead to additional interesting discoveries. They can also serve as a source for investigative projects offered in mathematical courses or seminars.

Task 1: Functions of the 1st, 2nd, and 3rd Degrees—With a Parameter

(1) Given a straight line $y = F_n x + F_{n-1}$.

- (a) Show that this line passes through both points (φ, φ^n) and $(1-\varphi, (1-\varphi)^n)$.
- (b) Determine its intersection points with the x and y axes and also the points of intersection of any two consecutive straight lines in the family.
- (c) Calculate the area bounded by the two lines, the x-axis and line $x = \mathrm{k}$ (natural k).

(2) Given a function $f_n(x) = F_n(x^2 - 1) + F_{n-1}$.

- (a) Show that the graph of the function passes through both points (φ, φ^n) and $(1-\varphi, (1-\varphi)^n)$.
- (b) Determine its intersection points with the x- and y-axes and calculate the value of the vertex.

(c) Determine the general equation of a tangent to the graph at any point (one at a point of ascent and another at a point of descent) and calculate the area of the triangle bounded by the tangent, the x-axis, and straight line $x = \mathrm{k}$ (natural k).
(d) Calculate the area bounded by the graph, the x- and y-axes and the vertical line passing through the point of origin.

(3) Given a function $g_n(x) = \frac{1}{2}F_n\,(x^3 - 1) + F_{n-1}$.
(a) Show that the graph of the function passes through both points (φ, φ^n) and $(1 - \varphi, (1 - \varphi)^n)$.
(b) Show that no extrema points exist but that there is a point of inflection. Calculate the value of the inflection point.
(c) Determine the general equation of the tangent to the graph at any given point (one to the right of the inflection point and one to the left), and calculate the area of the triangle bounded by the tangent, the x-axis, and straight line $x = \mathrm{k}$ (natural k).
(d) Calculate the area bounded by the graph, the x- and y-axes and the vertical line passing through the point of origin.

■ Task 2: An Analogy

(1) Given that $t = 1 + \sqrt{3}$
Construct a quadratic equation with integer coefficients for which t is one solution. Determine the second solution.
(2) Carry out a linearization of the first five powers of tand prove that:
$t^n = a_n t + 2a_{n-1}$, When:

$$a_1 = 1, \quad a_2 = 2$$
$$a_{n+2} = 2a_{n+1} + 2a_n$$

(3) Find an explicit formula for the sequence a_n
(4) Determine formulas for different sums similar to the Fibonacci sequence.

■ Task 3: The Family of Functions with Fibonacci Coefficients

Given a family of functions defined by:

$$f_n(x) = x^n - F_n x - F_{n-1}, \quad n > 1.$$

(1) Show that the graph of each such function intersects the x-axis at both $(\varphi, 0)$ and $(1 - \varphi, 0)$
(2) Prove that if $1 - \varphi < x < \varphi$, then $f_n(x) < 0$
(3) Show that $f_n(x) = (x^2 - x - 1)\,g_n(x), \quad n > 2$, where $g_n(x)$ is a polynomial to the power of $n - 2$. Find a formula for the polynomial's coefficients.
(4) Prove that if n is an even number, the graph will not intersect the x-axis at any other point, but if n is odd, the graph will intersect the x-axis at a third point. For the latter case, find a closed interval that includes the point.

■ Task 4: The Sum of Two Geometric Sequences with Conjugate Complex Ratios

Given that $z = \frac{1}{2}\left(1 + i\sqrt{3}\right)$.

(1) Construct a quadratic equation with integer coefficients for which z is one solution. Determine the second solution.
(2) Express the first eight exponents of z in the form of $a + bi$ (a and b are real numbers), and generalize each one for any exponent.
(3) The sequence (a_n) is defined by $a_n = \alpha\,(z)^n + \beta\,(z^*)^n$ where α and β are real numbers (z^* is the conjugate of z).
   (a) Show that the sequences satisfy the recursive equation: $a_{n+2} = a_{n+1} - a_n$,
   (b) and calculate a_0 and a_1.
   (c) Examine the sequence when $a_0 = 0$.
   Find equations, properties, and equations for the sums.
   (d) Examine the sequence when $a_0 = a, a_1 = b$ (both are natural numbers). Determine relationships, properties, and formulas for the totals.

■ Task 5: Exploring a Family of Functions

A family of functions, f_A, is defined by: $f_A(x) = \frac{x^2 - x - 1}{x^2 - A}$.

where A is a real number.

(1) Explore the function systematically, differentiating between various values of A. Examine the domain, intersection points with the x- and y-axes, asymptotes, ascending and descending areas, extrema, points of inflection, graphs.
(2) Determine the general equations of the tangents at the intersection points with the x- and y-axes and at any point.

■ Task 6: Sequences Defined by the Root of the Golden Ratio

Background: Fibonacci and Lucas Sequences (Chap. 3 without matrices and Chap. 4).

Two sequences, (a_n) and (b_n), are defined by:

$$a_n\sqrt{5} = \left(\sqrt{\varphi}\right)^n - \left(\frac{i}{\sqrt{\varphi}}\right)^n$$

$$b_n = \left(\sqrt{\varphi}\right)^n + \left(\frac{i}{\sqrt{\varphi}}\right)^n$$

(1) Derive formulas for odd and even indexes, the sum of the indexes, and the sums of each of the sequences separately. Find the relationship between the Fibonacci and Lucas sequences.
(2) Find relationships **between** the two sequences, formulas for mixed sums, and the relationship between the sums of the Fibonacci and Lucas sequences.

■ Task 7: Linear Combinations of Two Sequences

Background: Fibonacci and Lucas Sequences (Chap. 3 without matrices and Chap. 4).

Sequence f_n is defined by $f_n = AL_n + BF_n$, where A and B are integers.

Derive formulas for odd and even indexes, the sum of indexes, and the products and sums. Point out instances that are fundamentally dissimilar.

Task 8: The Sum of Two Real Geometric Progressions

Background: The Fibonacci sequence (Chap. 3 without matrices).

(1) Calculate the solutions for x_1 and x_2 for the equation $x^2 - Sx + P = 0$, where S and P are **integers** and $\Delta = S^2 - 4P > 0$.

(2) Two sequences, (a_n) and (b_n), are defined by:

$$a_n\sqrt{\Delta} = (x_1)^n - (x_2)^n$$
$$b_n = (x_1)^n + (x_2)^n$$

where $x_1 > x_2$.

(a) Derive formulas for odd and even indexes and the sums of the indexes. Point out cases that are fundamentally different.

(b) Find relationships between the two sequences, the formulas for mixed sums, and the relationship between the sums of the Fibonacci and Lucas sequences.

Task 9: Complex Series Based on Exponents of the Golden Ratio

Background Chap. 3 (first four sections) and Chap. 4.

Sequences (f_n) and (g_n) are defined by

$$f_n = \varphi^n + i(1-\varphi)^n$$
$$g_n = (1-\varphi)^n + i\varphi^n,$$

and "conjugate" sequences (f_n^*) and (g_n^*) by

$$f_n^* = \varphi^n - i(1-\varphi)^n$$
$$g_n^* = (1-\varphi)^n - i\varphi^n$$

(We use either an asterisk or a line above the number (function) to denote the conjugate of that number of function).

(1) Explore each of the sequences separately and derive formulas. For example, derive the formulas for

$f_n^2, f_n f_{n+1}, f_{n-1} f_{n+1}, f_n f_m, |f_n|$ and so on.

In some cases, the results will involve Fibonacci and Lucas sequences. These cases should be pointed out.

(2) Explore "combinations" of sequences, such as

$f_n g_n, f_n + g_n, f_n f_n^*, f_n g_n^*$, and so on.

In some cases, the results will be identical. For example: $f_n g_n^* = f_n^* g_n = L_{2n}$. Can you identify such cases? Also, sometimes the result will always be a real number, or always pure imaginary result, or always a complex result. Find such cases.

■ Task 10: "Hyperbolic-Fibonacci" Functions

Background: The Fibonacci sequence and the Binet formula (beginning of Chap. 3), Lucas sequences (beginning of Chap. 4).

(1) Explore the sine-hyperbolic and cosine-hyperbolic functions:

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

6

Show that each one is the derivative of the other. Relate this to the equations of tangents and areas in general.

(2) Explore the families:

$$f_m(x) = \sinh(mx)$$
$$g_m(x) = \cosh(mx)$$

Pay particular attention to the general equations for tangents and areas.

(3) Show that the following are true:

$$F_{2n}\sqrt{5} = \varphi^{2n} - (\varphi - 1)^{2n}$$
$$F_{2n-1}\sqrt{5} = \varphi^{2n-1} - (\varphi - 1)^{2n-1}$$
$$L_{2n-1} = \varphi^{2n-1} - (\varphi - 1)^{2n-1}$$
$$L_{2n} = \varphi^{2n} + (\varphi - 1)^{2n}$$

(4) Explore the functions defined by:

$$f_1(x) = \frac{\varphi^{2x-1} + (\varphi - 1)^{2x-1}}{\sqrt{5}}$$
$$f_2(x) = \frac{\varphi^{2x} - (\varphi - 1)^{2x}}{\sqrt{5}}$$
$$\ell_1(x) = \varphi^{2x-1} - (\varphi - 1)^{2x-1}$$
$$\ell_2(x) = \varphi^{2x} + (\varphi - 1)^{2x}$$

Pay particular attention to the general equations for tangents and areas.

(5) For each function, find the general equation for the tangent associated with the sequence at any point; calculate the area bounded by the graph, the x- and y-axes, and the vertical line that passes through this point; and calculate the area of the triangle bounded by the tangent, the x-axis and the vertical line through the point.

Task 11: Partitioning of a Rectangle

Background: The golden rectangle and the first section of ► Chap. 1.

(1) Section a rectangle into a square and a rectangle. Repeat this action once more. The terminology is defined as follows: step 0 is the original rectangle; step 1 is the rectangle that remains after the first square has been removed; step 2 is the rectangle that remains after removing the square from the rectangle formed in step 1.
Classify each rectangle as follows:
   (a) A rectangle whose dimensions are Fibonacci numbers. Find three such rectangles.
   (b) A rectangle whose dimensions are Lucas numbers. Find three such rectangles.
   (c) A rectangle with one side that is more than half the length of the other side. Form three such rectangles.
   (d) A rectangle whose length is greater than twice its width. Form three such rectangles.

(2) All the dimensions in this task should be natural numbers.

(3) Draw all the rectangles obtained and prepare a table using an appropriate computer program. The columns in the table should include: Length, width, and ratio of length to width at step 0, step 1 and step 2. In total, there will be 9 columns (three times three)

Precision: Rounded to 6 digits after the decimal point.
Prepare a dynamic presentation.

Task 12: A "Catalogue" of Triangles

Background: Golden triangles (first three sections in Chap. 2).

(1) Prepare a catalogue of as many triangles that you can whose angles (all three) are multiples of 18° (including both types of golden triangles). Sort them into the following categories: equilateral triangles, right-angled triangles, others. For each triangles, let A designate the largest angle and C the smallest (in some instances, of course, two angles may be equal).

(2) Assuming that the length of shortest side is 1 unit, calculate the lengths of the other sides, the radius of the ccircumscribed circle, the radius of the iinscribed circle, and the ratio between these two radii. Also, calculate the ratios between the sides and determine the values of the angles' trigonometric functions.

(3) Try to divide each type of triangle into two triangles whose angles are also multiples of 18°. Clarify when this is possible and when it is not. Calculate the ratios between the areas of the triangles (three ratios).

(4) Each final result should be written as a φ -number. If this is impossible, write it as the square root of a φ-number. In the latter case, prove that it is impossible to write it in the first form (see Exercises 2.18, 2.19).

Task 13: Partitioning of Squares

Background: Lewis Carroll's paradox: Is $1 = 0$ (Introductory chapter).

Note: Only natural numbers should be use in this task

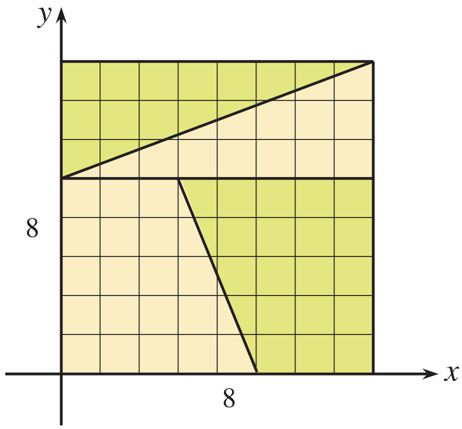


Fig. 6.1 Lewis Carroll's paradox

(1) Position a square on the Cartesian coordinate system such that the lower left vertex of the square lies at (0, 0) and two sides lie on the axes. Divide the square into two identical right-angled triangles and two identical right-angled trapezoids (see Fig. 6.1).

The solution to the apparent paradox lies in the fact that a very narrow section of the resulting rectangle either remains uncovered or is covered twice by the sections cut out from the square.

(a) Prove that the uncovered part or the part that is covered twice (by the two triangles and the two trapezoids) is a parallelogram (see Fig. 6.2).

(b) Determine the equations for the sides of the missing/doubled parallelograms, calculate the distance between the sides of the parallelogram, and calculate the small angle of the parallelogram. Also, calculate the area of the parallelogram, the difference in areas between the rectangle and the square and the relative difference.

Repeat this for the following cases:

(i) The side length of a square is a Fibonacci number and the dimensions of the triangles and the trapezoids are also Fibonacci numbers (as in Fig. 6.1). Investigate at least three different such squares.

(ii) The side length of the square is a Fibonacci number but the dimensions of the triangles and the trapezoids are *not* Fibonacci numbers. Investigate at least three different such squares.

(iii) The side length of the square is a Lucas number and the dimensions of the triangles and the trapezoids are also Lucas numbers. Investigate at least three different such squares.

(iv) The side length of the square is a Lucas number but the dimensions of the triangles and the trapezoids are *not* Lucas numbers. Investigate at least three different such squares.

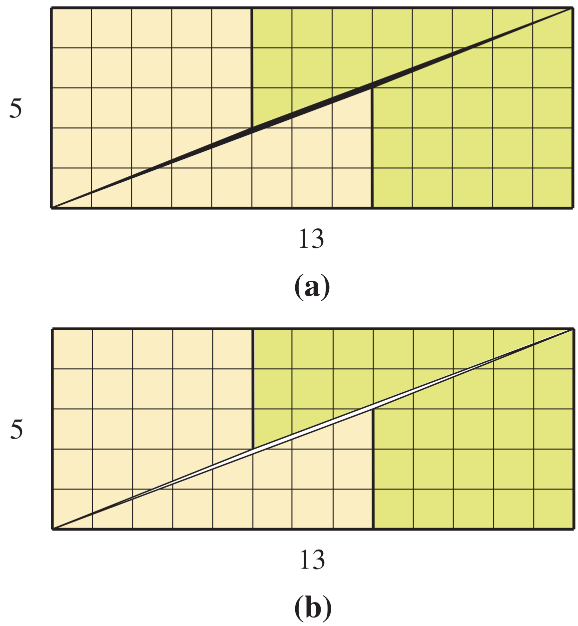


Fig. 6.2 Lewis Carroll's paradox

(2) Is it possible to generalize a square with a side of length *n*? Explain your answer.
(3) Is there any case where the area of the parallelogram equals zero? If so, when does this occur?

Finally

We have reached the end of our long journey, but as we mentioned at the beginning of this book, there are still quite a few "loose ends." Thus, we conclude our discussion on Fibonacci with:

This may be the end of our book, but the journey itself has just begun.

Sadly, Opher Liba passed away on February 2016, and never saw the published book.

Supplementary Information

O. Liba and B.-S. Ilany, *From the Golden Rectangle to the Fibonacci Sequences*, Springer Texts in Education, https://doi.org/10.1007/978-3-030-97600-2

List of Formulas, Theorems, and Definitions

▶ Chapter 1: The Golden Rectangle and the Golden Ratio

The Golden ratio

$$\boldsymbol{\varphi = \frac{1+\sqrt{5}}{2}} \tag{1.2}$$

$$\boldsymbol{\frac{-1}{\varphi} = 1 - \varphi = \frac{1}{2}(1 - \sqrt{5})} \tag{1.3}$$

$$\boldsymbol{\varphi^2 = \varphi + 1} \tag{1.1b}$$

$$\boldsymbol{\left(\frac{-1}{\varphi}\right)^2 = \frac{-1}{\varphi} + 1} \tag{1.4a}$$

$$\boldsymbol{(1 - \varphi)^2 = (1 - \varphi) + 1} \tag{1.4b}$$

The Golden Equation and The Golden Function

$$\boldsymbol{x^2 = x + 1} \tag{1.1a}$$

$x \mapsto x^2 - x - 1$

In geometry, a **golden rectangle** is a rectangle that if we remove from it a square then the ratio between between the length and the width of the remaining rectangle equal to the analogous ratio in the original rectangle. This ratio equals the golden ratio φ.

Identifying Basic Numerical Values with φ

$$\boldsymbol{2\varphi - 1 = \sqrt{5}} \tag{1.5}$$

$$\boldsymbol{\varphi^2 = \varphi + 1 = \frac{1}{2}(3 + \sqrt{5})} \tag{1.6a}$$

$$\left(\frac{1}{\varphi}\right)^2 = (\varphi - 1)^2 = 2 - \varphi = \frac{1}{2}(3 - \sqrt{5}) \tag{1.6b}$$

$$\varphi^2 + 1 = \varphi + 2 = \varphi\sqrt{5} \tag{1.7a}$$

$$\left(\frac{1}{\varphi}\right)^2 + 1 = 3 - \varphi = \frac{\sqrt{5}}{\varphi} = (\varphi - 1)\sqrt{5} \tag{1.7b}$$

$$\varphi^3 = 2\varphi + 1 = 2 + \sqrt{5} \tag{1.8}$$

Commutative Group

$\langle G, \circ \rangle$ is a Commutative Group if:

$\forall a \in G, \forall b \in G \quad : \quad a \circ b \in G$

$\forall a \in G, \forall b \in G \quad : \quad a \circ b = b \circ a$

$\forall a \in G, \forall b \in G \quad , \quad \forall c \in G \quad : \quad a \circ (b \circ c) = (a \circ b) \circ c$

$\exists e \in G, \forall a \in G \quad : \quad a \circ e = e \circ a = a$

$\forall a \in G, \exists a' \in G \quad : \quad a \circ a' = a' \circ a = e$

Field

$\langle F, \oplus, \otimes \rangle$ is a Field if $\langle F, \oplus \rangle$ and $\langle F\backslash\{0\}, \otimes \rangle$ are Commutative Groups, and if:

$\forall a \in F, \forall b \in F, \forall c \in F \quad : \quad a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

A number of the form:

$$a + b\varphi \tag{1.9}$$

where a and b are **rational numbers**, is called a "φ-number."

$$\delta(a, b) = a^2 + ab - b^2 \tag{1.10}$$

$$(a + b\varphi)^{-1} = \frac{a + b(1 - \varphi)}{\delta(a,b)} = \frac{a + b - b\varphi}{\delta(a,b)} \tag{1.11a}$$

$$\delta(a, b) = a^2 + ab - b^2 \tag{1.10}$$

The companion of **a+bφ**:

$$(a + b\varphi)^* = a + b(1 - \varphi) \tag{1.12a}$$

$$[a + b(1 - \varphi)\ ^* = a + b\varphi \tag{1.12b}$$

$$(a + b\varphi)^{**} = a + b\varphi \tag{1.12c}$$

$$(a + b\varphi)^{-1} = \frac{(a + b\varphi)^*}{\delta(a,b)} \tag{1.11b}$$

$$(\mathbf{a}+\mathbf{b}\boldsymbol{\varphi})(a+\mathbf{b}\boldsymbol{\varphi})^{*}=\boldsymbol{\delta}(\mathbf{a},\mathbf{b}) \qquad (1.11b')$$

Theorem *A set of φ-numbers different from zero, along with multiplication, constitutes a commutative group:*

Multiplication between two φ-numbers:

$$(a+b\varphi)(c+d\varphi)=(ac+bd)+(ad+bc+bd)\varphi$$

The square of a φ-number:

$$(a+b\varphi)^2=(a^2+b^2)+b\,(2a+b)\,\varphi$$

The inverse element of a φ-number:

$$(\mathbf{a}+\mathbf{b}\boldsymbol{\varphi})^{-1}=\frac{(\mathbf{a}+\mathbf{b}\boldsymbol{\varphi})^{*}}{\boldsymbol{\delta}(\mathbf{a},\mathbf{b})} \qquad (1.11b)$$

$$(\mathbf{a}+\mathbf{b}\boldsymbol{\varphi})(\mathbf{a}+\mathbf{b}\boldsymbol{\varphi})^{*}=\boldsymbol{\delta}(\mathbf{a},\mathbf{b}) \qquad (1.11b')$$

The matrix $\mathbf{M}(\mathbf{a},\mathbf{b})$

$$\mathbf{M}(\mathbf{a},\mathbf{b})=\begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{b} & \mathbf{a}+\mathbf{b}\end{pmatrix} \qquad (1.13)$$

where a and b are rational numbers, neither of which is 0.

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{b} & \mathbf{a}+\mathbf{b}\end{pmatrix}^{-1}=\frac{\mathbf{1}}{\boldsymbol{\delta}(\mathbf{a},\mathbf{b})}\begin{pmatrix}\mathbf{a}+\mathbf{b} & -\mathbf{b}\\ -\mathbf{b} & \mathbf{a}\end{pmatrix} \qquad (1.14a)$$

$$\mathbf{M}^{-1}(\mathbf{a},\mathbf{b})=\boldsymbol{\delta}^{-1}(\mathbf{a},\mathbf{b})\cdot\mathbf{M}(\mathbf{a}+\mathbf{b},-\mathbf{b}) \qquad (1.14b)$$

Theorem *The set of matrices of the form* M(a, b), *combined with the standard multiplication operation between matrices, constitutes a commutative group, which is* ***isomorphic*** *to the set the non zero φ-numbers.*

The Golden Matrix

$$\boldsymbol{\Phi}=\mathbf{M}(\mathbf{0},\mathbf{1}) \qquad (1.15)$$

$$\boldsymbol{\Phi}^{\mathbf{2}}=\boldsymbol{\Phi}+\mathbf{1} \qquad (1.16)$$

$$\boldsymbol{\Phi}^{-\mathbf{1}}=\boldsymbol{\Phi}-\mathbf{I}=\mathbf{M}(-\mathbf{1},\mathbf{1}) \qquad (1.17)$$

The Root-Five Matrix

$$\mathbf{R}(5) = \mathbf{M}(-1, 2) = 2\boldsymbol{\Phi} - \mathbf{I} \tag{1.18}$$

$$\mathbf{R}^2(5) = 5\mathbf{I}$$

The Norm of φ-Numbers

$$\mu(\mathbf{a} + \mathbf{b}\varphi) = \sqrt{|\delta(\mathbf{a}, \mathbf{b})|} \tag{1.19}$$

$$\mu(x) = 0 \quad \Leftrightarrow \quad x = 0 \tag{1.20}$$

$$\mu(-x) = \mu(x^*) = \mu(x) \tag{1.21a}$$

$$\mu(x^{-1}) = [\mu(x)]^{-1} \tag{1.21b}$$

$$\mu(\mathbf{k}x) = |\mathbf{k}|\mu(x) \quad (\forall \mathbf{k} \in \mathbb{Q}) \tag{1.22}$$

$$\mu(xy) = \mu(x)\mu(y) \tag{1.23}$$

$$\begin{aligned} \mu[(\mathbf{a} + \mathbf{b}\varphi)^2] &= \mu[(\mathbf{a}^2 + \mathbf{b}^2) + \mathbf{b}(2\mathbf{a} + \mathbf{b})\varphi] \\ &= [\mu(\mathbf{a} + \mathbf{b}\varphi)^2] = |\delta(\mathbf{a}, \mathbf{b})| \end{aligned} \tag{1.24}$$

$$\mu(x^{\mathbf{k}}) = [\mu(x)]^{\mathbf{k}} \quad (\forall \mathbf{k} \in \mathbb{N}) \tag{1.25}$$

The Norm of φ-Numbers

Definition:

$x = \mathrm{a} + \mathrm{b}\varphi$ **shall be termed the "norm of a+bφ", if** $\mu(x) = 1$**.**

Attribute if x and y are **φ-numbers**, then:

$$\mu(-x) = \mu(x^*) = \mu(x^{-1}) = 1 \tag{1.21′}$$

$$\mu(\mathbf{k}x) = |\mathbf{k}| \quad (\forall \mathbf{k} \in \mathbb{Q}) \tag{1.22′}$$

$$\mu(xy) = 1 \tag{1.23′}$$

$$\mu(x^{\mathbf{k}}) = 1 \quad (\forall \mathbf{k} \in \mathbb{N}) \tag{1.24′}$$

Theorem *The entire set of normalized φ-numbers constitutes a commutative group.*

Theorem *Given any φ-number,* $x = \mathrm{a} + \mathrm{b}\varphi$*, it will be true thats*:

$$\mu\left(\frac{x}{\mu(x)}\right) = 1 \tag{1.26}$$

▸ Chapter 2: Introducing Golden Triangles

Golden Triangles

A wide golden triangle is an obtuse isosceles triangle in which the ratio of the lengths of the base to the side is φ. Its angels are 36°, 108°, 36°.

A narrow golden triangle is a isosceles triangle that the ratio of the length of the side to the length of the base of triangle is φ. Its angels are 72°, 36°, 72°.

$$\cos 36^{\circ} = \sin 54^{\circ} = \frac{\varphi}{2} \quad (2.1)$$

$$\cos 72^{\circ} = \sin 18^{\circ} = \frac{1}{2}(\varphi^2 - 2) \quad (2.2)$$

▸ Chapter 3: The Fibonacci Sequence

The Fibonacci Sequence
Definition:

$$F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \geq 1) \quad (3.1)$$

The powers of φ^n and $(1-\varphi)^n$:

$$\varphi^n = F_n\varphi + F_{n+1} \quad (3.2)$$

$$(1-\varphi)^n = F_n(1-\varphi) + F_{n+1} = -F_n\varphi + F_{n+1} \quad (3.2b)$$

Relationships

Binet's formula:

$$F_n = \frac{1}{\sqrt{5}}[\varphi^n - (1-\varphi)^n] \quad (3.3a)$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad (3.3b)$$

Cassini formula:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad (3.4a)$$

$$\delta(F_{n-1}, F_n) = (-1)^n \quad (3.4b)$$

Even and odd indices and the sum of indices:

$$\mathbf{F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n(F_{n+1} + F_{n-1})} \tag{3.5a}$$

$$\mathbf{F_{2n-1} = F_n^2 + F_{n+1}^2} \tag{3.5b}$$

$$\mathbf{F_{n+m} = F_nF_{m+1} + F_{n-1}F_m = F_{n+1}F_m + F_nF_{m-1}} \tag{3.6}$$

Sums:

$$\mathbf{F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1} \tag{3.7a}$$

$$\mathbf{F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1} \tag{3.7b}$$

$$\mathbf{F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}} \tag{3.7c}$$

$$\mathbf{F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_nF_{n+1}} \tag{3.8}$$

Expanding the Sequence

Negative indices (n natural):

$$\mathbf{F_{-n} = (-1)^{n+1}F_n} \tag{3.9}$$

$$\boldsymbol{\varphi^{-n} = F_{-n}\varphi + F_{-n-1}} \tag{3.10a}$$

$$\boldsymbol{(1-\varphi)^{-n} = -F_{-n}\varphi + F_{-n+1}} \tag{3.10b}$$

Integer indices (m an integer):

$$\boldsymbol{\varphi^{m} = F_m\varphi + F_{m-1}} \tag{3.11a}$$

$$\boldsymbol{(1-\varphi)^{m} = -F_m\varphi + F_{m+1}} \tag{3.11b}$$

$$\boldsymbol{F_m = \frac{1}{\sqrt{5}}[\varphi^m - (1-\varphi)^m]} \tag{3.12}$$

$$\boldsymbol{\delta(F_{m-1}, F_m) = F_{m+1}F_{m-1} - F_m^2 = (-1)^m} \tag{3.13}$$

Theorem *The set* $\{\varphi^m | m \in \mathbb{Z}\}$ *with multiplication operation is a commutative group.*

Conclusion The set $\{F_m\varphi + F_{m-1} | m \in \mathbb{Z}\}$ with multiplication is a commutative group. This group is a **subgroup of the group of φ-numbers.**

Matrices and the Fibonacci Sequence

$$\boldsymbol{\Phi}^{\mathbf{n}} = \mathbf{M}^{\mathbf{n}}(\mathbf{0},\mathbf{1}) = \mathbf{M}(\mathbf{F}_{\mathbf{n-1}},\mathbf{F}_{\mathbf{n}}) \tag{3.14a}$$

$$\boldsymbol{\Phi}^{\mathbf{n}} = \begin{pmatrix} \mathbf{F}_{\mathbf{n-1}} & \mathbf{F}_{\mathbf{n}} \\ \mathbf{F}_{\mathbf{n}} & \mathbf{F}_{\mathbf{n+1}} \end{pmatrix} \tag{3.14b}$$

$$(\mathbf{I}-\boldsymbol{\Phi})^{\mathbf{n}} = \mathbf{M}(\mathbf{F}_{\mathbf{n+1}},-\mathbf{F}_{\mathbf{n}}) \tag{3.15}$$

$$\boldsymbol{\Phi}^{\mathbf{m}} = \mathbf{M}^{\mathbf{m}}(\mathbf{0},\mathbf{1}) = \mathbf{M}(\mathbf{F}_{\mathbf{m-1}},\mathbf{F}_{\mathbf{m}}) \tag{3.16a}$$

$$\boldsymbol{\Phi}^{\mathbf{m}} = \begin{pmatrix} \mathbf{F}_{\mathbf{m-1}} & \mathbf{F}_{\mathbf{m}} \\ \mathbf{F}_{\mathbf{m}} & \mathbf{F}_{\mathbf{m+1}} \end{pmatrix} \quad (\mathrm{m} \in \mathbb{Z}) \tag{3.16b}$$

Theorem *The group of* ***matrices*** $\{\Phi^{\mathrm{m}}|\mathrm{m} \in \mathbb{Z}\}$ *with the regular multiplication is isomorphic to the group of* ***numbers*** $\{\Phi^{\mathrm{m}}|\mathrm{m} \in \mathbb{Z}\}$*, with the regular multiplication between numbers.*

$$\boldsymbol{\Phi}^{\mathbf{n}} = \mathbf{F}_{\mathbf{n}}\mathbf{F} + \mathbf{F}_{\mathbf{n-1}}\mathbf{I} \tag{3.17a}$$

$$(\boldsymbol{\Phi}^{-\mathbf{1}})^{\mathbf{n}} = (\mathbf{I}-\boldsymbol{\Phi})^{\mathbf{n}} = -\mathbf{F}_{\mathbf{n}}\boldsymbol{\Phi} + \mathbf{F}_{\mathbf{n+1}}\mathbf{I} \tag{3.17b}$$

$$\boldsymbol{\Phi}^{\mathbf{n}} - (\mathbf{I}-\boldsymbol{\Phi})^{\mathbf{n}} = \mathbf{F}_{\mathbf{n}}\mathbf{R}(\mathbf{5}) \tag{3.18}$$

▸ Chapter 4: The Lucas Sequence

Definition:

$$\mathbf{L}_{\mathbf{n}} = \boldsymbol{\varphi}^{\mathbf{n}} + (\mathbf{1}-\boldsymbol{\varphi})^{\mathbf{n}} \tag{4.1}$$

$$\mathbf{L}_{\mathbf{n+2}} = \mathbf{L}_{\mathbf{n+1}} + \mathbf{L}_{\mathbf{n}} \quad (\mathbf{n} \geq \mathbf{1}) \tag{4.2}$$

$$\mathbf{L}_{\mathbf{n}} = \mathbf{F}_{\mathbf{n-1}} + \mathbf{F}_{\mathbf{n+1}} \tag{4.3}$$

Connections:

Connections between elements of the Lucas Sequence:

$$\mathbf{L}_{\mathbf{2n}} = \mathbf{L}_{\mathbf{n}}^{\mathbf{2}} - \mathbf{2}(-\mathbf{1})^{\mathbf{n}} \tag{4.4}$$

$$\mathbf{L}_{\mathbf{2n-1}} = \mathbf{L}_{\mathbf{n}}\mathbf{L}_{\mathbf{n-1}} + (-\mathbf{1})^{\mathbf{n-1}} \tag{4.5}$$

$$\mathbf{L}_{\mathbf{2n}} = \mathbf{L}_{\mathbf{n-1}}\mathbf{L}_{\mathbf{n+1}} + \mathbf{3}(-\mathbf{1})^{\mathbf{n-1}} \tag{4.6}$$

$$\mathbf{L}_{\mathbf{n+m}} = \mathbf{L}_{\mathbf{n}}\mathbf{L}_{\mathbf{m}} + (-\mathbf{1})^{\mathbf{n}}\mathbf{L}_{\mathbf{m-n}} \tag{4.7}$$

Combining the Lucas sequence and the Fibonacci sequence—with one index:

$$L_n + F_n = 2F_{n+1} \tag{4.8a}$$

$$L_n - F_n = 2F_{n-1} \tag{4.8b}$$

$$L_n F_n = F_{2n} \tag{4.9}$$

$$L_n F_{n+1} = F_{2n+1} + (-1)^n \tag{4.10a}$$

$$L_{n+1} F_n = F_{2n+1} - (-1)^n \tag{4.10b}$$

$$L_n F_{n+1} + L_{n+1} F_n = 2F_{2n+1} \tag{4.10c}$$

$$L_{n+1} F_{n-1} = F_{2n} + (-1)^n \tag{4.11a}$$

$$L_{n-1} F_{n+1} = F_{2n} - (-1)^n \tag{4.11b}$$

$$L_{n+1} F_{n-1} + L_{n-1} F_{n+1} = 2F_{2n} \tag{4.11c}$$

$$L_{2n} = L_n F_{n+1} + L_{n-1} F_n = L_n F_{n-1} + L_{n+1} F_n \tag{4.12}$$

Combining the Lucas sequence and the Fibonacci sequence—with two indices:

$$L_n F_m = F_{n+m} + (-1)^n F_{m-n} \tag{4.13a}$$

$$L_m F_n = F_{n+m} - (-1)^n F_{m-n} \tag{4.13b}$$

$$L_n F_m + L_m F_n = 2F_{n+m} \tag{4.13c}$$

$$L_{n+m} = L_n F_{m-1} + L_{n+1} F_m = L_m F_{n-1} + L_{m+1} F_n \tag{4.14}$$

The powers of the golden ratio

$$\varphi^n = \frac{1}{2}(L_n + F_n\sqrt{5}) \tag{4.15a}$$

$$(1-\varphi)^n = \frac{1}{2}(L_n - F_n\sqrt{5}) \tag{4.15b}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{4.16}$$

$$\varphi^n = \frac{1}{\sqrt{5}}(L_n\varphi + L_{n-1}) \tag{4.17a}$$

$$(1-\varphi)^n = \frac{1}{\sqrt{5}}(L_n\varphi - L_{n+1}) \tag{4.17b}$$

More connections:

$$\frac{L_n\varphi+L_{n+1}}{F_n\varphi+F_{n-1}}=\sqrt{5} \tag{4.18a}$$

$$\frac{L_n\varphi-L_{n+1}}{F_n\varphi-F_{n-1}}=-\sqrt{5} \tag{4.18b}$$

$$L_{n+1}+L_{n-1}=5F_n \tag{4.19}$$

$$L_{n+1}L_{n-1}-L_n^2=5(-1)^{n+1} \tag{4.20a}$$

$$\delta(L_{n-1},L_n)=5(-1)^{n+1} \tag{4.20b}$$

$$\mu(L_{n-1}+L_n\varphi)=\mu(-L_{n+1}+L_n\varphi)=\sqrt{5} \tag{4.20c}$$

Sums

Sums of Lucas sequence:

$$L_1+L_2+L_3+\cdots+L_n=L_{n+2}-3 \tag{4.21a}$$

$$L_2+L_4+L_6+\cdots+L_{2n}=L_{2n+1}-1 \tag{4.21b}$$

$$L_1+L_3+L_5+\cdots+L_{2n-1}=L_{2n}-2 \tag{4.21c}$$

$$L_1^2+L_2^2+L_3^2+\cdots+L_n^2=L_nL_{n+1}-2 \tag{4.22}$$

The Fibonacci and Lucas sums—even n:

$$L_1F_2+L_2F_3+L_3F_4+\cdots+L_nF_{n+1}=F_{2n+2}-1 \tag{4.23a}$$

$$L_2F_1+L_3F_2+L_4F_3+\cdots+L_{n+1}F_n=F_{2n+2}-1 \tag{4.23c}$$

$$L_2F_0+L_3F_1+L_4F_2+\cdots+L_{n+1}F_{n-1}=F_{2n+1}-1 \tag{4.24a}$$

$$L_0F_2+L_1F_3+L_2F_4+\cdots+L_{n+1}F_{n+1}=F_{2n+1}-1 \tag{4.24c}$$

The Fibonacci and Lucas sums—odd n:

$$L_1F_2+L_2F_3+L_3F_4+\cdots+L_nF_{n+1}=F_{2n+2}-2 \tag{4.23b}$$

$$L_2F_1+L_3F_2+L_4F_3+\cdots+L_{n+1}F_n=F_{2n+2} \tag{4.23d}$$

$$L_2F_0+L_3F_1+L_4F_2+\cdots+L_{n+1}F_{n-1}=F_{2n+1}-2 \tag{4.24b}$$

$$L_0F_2+L_1F_3+L_2F_4+\cdots+L_{n+1}F_{n+1}=F_{2n+1} \tag{4.24d}$$

▸ Chapter 5: The General Fibonacci-Like Sequences

Definitions, Binet's Formulas and Relationships

$$f_n = \alpha\,\varphi^n + \beta(1-\varphi)^n \tag{5.1}$$

$$f_{n+2} = f_{n+1} + f_n \tag{5.2}$$

$$(f_0, f_1) = (a, b)$$

$$\beta\sqrt{5} = a\varphi - b \tag{5.3b}$$

$$\alpha\sqrt{5} = a\varphi + b - a = a(\varphi - 1) + b \tag{5.3a}$$

$$(a, b) \neq (0, 0),\ (\alpha, \beta) \neq (\alpha, 0),\ (\alpha, \beta) \neq (0, \beta)$$

$$f_n = \frac{1}{\sqrt{5}}[(a\varphi + b - a)\varphi^n + (a\varphi - b)(1 - \varphi^n)] \tag{5.4a}$$

$$(a, b) \neq (0, 0),\ (a, b) \neq (a, a\varphi),\ (a, b) \neq [a, a(1-\varphi)]$$

$$f_n = \frac{1}{\sqrt{5}}[(a + b\varphi)\varphi^{n-1} - (a + b\varphi)\cdot(1 - \varphi)^{n-1}] \tag{5.4a$'$}$$

$$f_n = \frac{1}{2\sqrt{5}}[a\sqrt{5} + (2b - a)\varphi^n + [a\sqrt{5} - (2b - a)](1 - \varphi)^n] \tag{5.4b}$$

$$f_n = \frac{1}{\sqrt{5}}[(c\sqrt{5} + d)\varphi^n + (c\sqrt{5} - d)(1 - \varphi)^n] \tag{5.4c}$$

$$(c, d) = \left(\frac{a}{2}, \frac{2b - a}{2}\right) = \left(\frac{f_0}{2}, \frac{2f_1 - f_0}{2}\right)$$

$$f_n = cL_n + dF_n = \frac{1}{2}[aL_n + (2b - a)F_n] \tag{5.5a}$$

$$f_n = a\,F_{n-1} + b\,F_n = f_0 F_{n-1} + f_1 F_n \tag{5.6}$$

$$\alpha + \beta = a = f_0 \tag{5.7a}$$

$$5\alpha\,\beta = \delta(a, b) = \delta(f_0, f_1) \tag{5.7b}$$

$$\delta(f_{n-1}, f_n) = f_{n+1}f_{n-1} - f_n^2 = \delta(a, b)(-1)^{n+1} \qquad (5.8)$$

$$f_{2n} = f_n F_{n-1} + f_{n+1}F_n = f_{n-1}F_n + f_n F_{n+1} \qquad (5.9a)$$

$$f_{2n-1} = f_{n-1}F_{n-1} + f_n F_n \qquad (5.9b)$$

$$f_{n+m} = f_m F_{n-1} + f_{m+1}F_n = f_n F_{m-1} + f_{n+1}F_m \qquad (5.10)$$

The Lucas-Like Sequence of the Fibonacci-Like Sequence

$$\ell_n = f_{n+1} + f_{n-1} \qquad (5.11)$$

$$\ell_n + f_n = 2f_{n+1} \qquad (5.12a)$$

$$\ell_n - f_n = 2f_{n-1} \qquad (5.12b)$$

$$\ell_n = \ell_0 F_{n-1} + \ell_1 F_n \qquad (5.13a)$$

$$(\ell_0, \ell_1) = (2b - a, 2a + b) = (2f_1 - f_0, 2f_0 + f_1)$$

$$\begin{aligned}\ell_n &= (2b - a)F_{n-1} + (2a + b)F_n \\ &= (2f_1 - f_0)F_{n-1} + (2f_0 + f_1)F_n\end{aligned} \qquad (5.13b)$$

$$\ell_n = aL_{n-1} + bL_n = f_0 L_{n-1} + f_1 L_n \qquad (5.14)$$

$$\ell_n = (a\varphi + b - a)\varphi^n - (a\varphi - b)(1 - \varphi)^n \qquad (5.15a)$$

$$(a, b) \neq (0, 0), (a, b) \neq (a, a\varphi), (a, b) \neq [a, a(1 - \varphi)]$$

$$\ell_n = (a + b\varphi)\varphi^{n-1} - (a + b\varphi)(1 - \varphi)^{n-1} \qquad (5.15\,a')$$

$$\ell_n = \sqrt{5}[\alpha\varphi^n - \beta(1 - \varphi)^n] \qquad (5.15b)$$

$$(a, b) \neq (0, 0), (\alpha, \beta) \neq (\alpha, 0), (\alpha, \beta) \neq (0, \beta)$$

$$\ell_n = \tfrac{1}{2}\left([a\sqrt{5} + (2b - a)]\varphi^n - [a\sqrt{5} - (2b - a)](1 - \varphi)^n\right) \qquad (5.16a)$$

$$\ell_n = (c\sqrt{5} + d)\varphi^n - (c\sqrt{5} - d)(1 - \varphi)^n \qquad (5.16b)$$

$$(c, d) = \left(\tfrac{a}{2}, \tfrac{2b - a}{2}\right) = \left(\tfrac{f_0}{2}, \tfrac{\ell_0}{2}\right)$$

$$f_n = \frac{1}{2\sqrt{5}}[(f_0\sqrt{5} + \ell_0)\varphi^n + (f_0\sqrt{5} - \ell_0)(1 - \varphi)^n] \tag{5.4d}$$

$$\ell_n = \frac{1}{2}[(f_0\sqrt{5} + \ell_0)\varphi^n - (f_0\sqrt{5} - \ell_0)(1 - \varphi)^n] \tag{5.16c}$$

$$f_n = \frac{1}{2}(f_0 L_n + \ell_0 F_n] \tag{5.5b}$$

The Even/Odd Indices and the Sum of Indices

$$f_{2n} = \frac{1}{2}(f_n L_n + \ell_n F_n) \tag{5.17a}$$

$$f_{2n-1} = \frac{1}{2}(f_{n-1} L_n + \ell_{n-1} F_n) = \frac{1}{2}(f_n L_{n-1} + \ell_n F_{n-1}) \tag{5.17b}$$

$$f_{n+m} = \frac{1}{2}(f_m L_n + \ell_m F_n) = \frac{1}{2}(f_n L_m + \ell_n F_m) \tag{5.17c}$$

$$\ell_{2n} = f_n L_{n-1} + f_{n+1} L_n = f_{n-1} L_n + f_n L_{n+1} \tag{5.18a}$$

$$\ell_{2n-1} = f_{n-1} L_{n-1} + f_n L_n \tag{5.18b}$$

$$\ell_{n+m} = f_m L_{n-1} + f_{m+1} L_n = f_n L_{m-1} + f_{n+1} L_m \tag{5.18b}$$

Comparing the Fibonacci and the Lucas Sequence

$$a f_{n-1} + b f_n = \sqrt{5}[\alpha^2 \varphi^n - \beta^2 (1 - \varphi)^n] \tag{5.19}$$

$$\ell_n f_n = a f_{2n-1} + b f_{2n} \tag{5.20}$$

$$\ell_n f_{n+1} = a f_{2n} + b f_{2n+1} - \delta(a, b)(-1)^n \tag{5.21a}$$

$$\ell_{n+1} f_n = a f_{2n} + b f_{2n+1} + \delta(a, b)(-1)^n \tag{5.21b}$$

$$\ell_{n-1} f_{n+1} = a f_{2n-1} + b f_{2n} + \delta(a, b)(-1)^n \tag{5.21c}$$

$$\ell_{n+1} f_{n-1} = a f_{2n-1} + b f_{2n} - \delta(a, b)(-1)^n \tag{5.21d}$$

$$\ell_n f_{n+1} + \ell_{n+1} f_n = 2(a f_{2n} + b f_{2n+1}) \tag{5.22a}$$

$$\ell_n f_{n+1} - \ell_{n+1} f_n = 2\delta(a,b)(-1)^{n+1} \tag{5.22b}$$

$$\ell_{n-1} f_{n+1} + \ell_{n+1} f_{n-1} = 2(af_{2n-1} + bf_{2n}) \tag{5.22c}$$

$$\ell_{n-1} f_{n+1} - \ell_{n+1} f_{n-1} = 2\delta(a,b)(-1)^n \tag{5.22d}$$

The Lucas-Like Sequence of the Fibonacci-Like Sequence

$$g_n = af_{n-1} + bf_n = f_0 f_{n-1} + f_1 f_n \tag{5.23a}$$

$$g_n = \sqrt{5}[\alpha^2 \varphi^n - \beta^2 (1-\varphi)^n] \tag{5.23b}$$

$$(g_0, g_1) = (f_0 \ell_0, f_0^2 + f_1^2)$$

$$g_n = f_0 \ell_0 F_{n-1} + (f_0^2 + f_1^2) F_n \tag{5.24}$$

$$g_{2n} = f_n \ell_n = f_{n+1}^2 - f_{n-1}^2 \tag{5.25a}$$

$$g_{2n-1} = f_{n-1}^2 + f_n^2 \tag{5.25b}$$

$$\ell_n f_{n+1} = g_{2n+1} - \delta(a,b)(-1)^n \tag{5.21a$'$}$$

$$\ell_{n+1} f_n = g_{2n+1} + \delta(a,b)(-1)^n \tag{5.21b$'$}$$

$$\ell_{n-1} f_{n+1} = g_{2n} + \delta(a,b)(-1)^n \tag{5.21c$'$}$$

$$\ell_{n+1} f_{n-1} = g_{2n} - \delta(a,b)(-1)^n \tag{5.21d$'$}$$

The Powers of the Golden Ratio

$$\varphi^{n-1} = \frac{(a+b\varphi)\cdot(f_{n-1} + f_n \varphi)}{\delta(a,b)} \tag{5.26a}$$

$$(1-\varphi)^{n-1} = \frac{(a+b\varphi)\cdot(f_{n-1} - f_n \varphi)}{\delta(a,b)} \tag{5.26b}$$

$$\varphi^{n-1} = \frac{(a+b\varphi)\cdot(\ell_n + f_n \sqrt{5})}{2\delta(a,b)} \tag{5.27a}$$

$$(1-\varphi)^{n-1} = \frac{(a+b\varphi)\cdot(\ell_n - f_n\sqrt{5})}{2\delta(a,b)} \tag{5.27b}$$

Fibonacci-Like sequence Rational

If a and b are rational numbers, the sequence is called **Rational Fibonacci-like sequence**

Theorem Fibonacci-like sequence (f_n) is rational if and only if:

$$f_n = (A + B\sqrt{5})\varphi^n + (A - B\sqrt{5})(1-\varphi)^n \tag{5.28a}$$

$$f_n = AL_n + BF_n \tag{5.28b}$$

When A and B are rational numbers.

The Fibonacci and Lucas Numbers

n	F_n	L_n	F_{2n}	F_{2n+1}	F_n^2	L_n^2
0	0	2	0	1	0	4
1	1	1	1	2	1	1
2	1	3	3	5	1	9
3	2	4	8	13	4	16
4	3	7	21	34	9	49
5	5	11	55	89	25	121
6	8	18	144	233	64	324
7	13	29	377	610	169	841
8	21	47	987	1597	441	2209
9	34	76	2584	4181	1156	5776
10	55	123	6765	10,946	3025	15,129
11	89	199	17,711	28,657	7921	39,601
12	144	322	46,368	75,025	20,736	103,684
13	233	521	121,393	196,418	54,289	271,441
14	377	843	317,811	514,229	142,129	710,649
15	610	1364	832,040	1,346,269	372,100	1,860,496
16	987	2207	2,178,309	3,524,578	974,169	4,870,849

n	F_n	L_n	F_{2n}	F_{2n+1}	F_n^2	L_n^2
17	1597	3571	5,702,887	9,227,465	2,550,409	12,752,041
18	2584	5778	14,930,352	24,157,817	6,677,056	33,385,284

In the following table, all decimal numbers are approximations.

n	F_n	L_n	$\frac{F_{n+1}}{F_n}$	$\frac{L_{n+1}}{L_n}$	$\left\lvert\frac{F_{n+1}}{F_n} - \varphi\right\rvert$	$\left\lvert\frac{L_{n+1}}{L_n} - \varphi\right\rvert$
0	0	2	–	0.500000	–	1.118034
1	1	1	1.000000	3.000000	0.618034	1.381966
2	1	3	2.000000	1.333333	0.381966	0.284701
3	2	4	1.500000	1.750000	0.118034	0.131966
4	3	7	1.666667	1.571429	0.048633	0.046605
5	5	11	1.600000	1.636364	0.018034	0.018330
6	8	18	1.625000	1.611111	0.006966	0.006923
7	13	29	1.615385	1.620690	0.002649	0.002657
8	21	47	1.619048	1.617021	0.001014	0.001013
9	34	76	1.617647	1.618421	0.000387	0.000387
10	55	123	1.618182	1.617886	0.000148	0.000148
11	89	199	1.617978	1.618090	0.000056	0.000056
12	144	322	1.618056	1.618012	0.000022	0.000022
13	233	521	1.618026	1.618042	0.000008	0.000008
14	377	843	1.618037	1.618031	0.000003	0.000003
15	610	1364	1.618033	1.618035	0.000001	0.000001
16	987	2207	1.618034	1.618034	0.000000	0.000000
17	1597	3571	1.618034	1.618034	0.000000	0.000000
18	2584	5778	1.618034	1.618034	0.000000	0.000000

Annotated Bibliography

Books

- Bicknell, Marjorie and Verner Hoggatt. *A Primer for the Fibonacci Numbers.* The Fibonacci Association 1973a.

  A collection of articles that appeared between 1963–1973 in the Fibonacci Organization Quarterly.

- Bicknell, Marjorie and Verner Hoggatt. *Fibonacci's Problem Book.* The Fibonacci Association 1973b.

  A collection of about 150 exercises on different levels, along with their solutions.

- Cleyet-Michaud, Marius. *Le Nombre d'Or.* PUF 1973.

  "A bit of everything": The golden ratio and its properties, history and mysticism, geometry, art, and nature.

- Dunlap, Richard. *The Golden Ratio and Fibonacci Numbers.* World Scientific 1997.

  The mathematics of the Fibonacci and Lucas sequences, and their application in nature.

- Garland, Trudi Hammel. *Fascinating Fibonaccis.* Dale Seymour 1987

  Fibonacci numbers in nature, art, and music.

- Ghyka, Matila. *The Geometry of Art and Life.* Dover 1977.

  The Golden section in geometric bodies, biology, the plastic arts, and architecture.

- Herz-Fischler, Roger. *The Mathematical History of the Golden ratio.* Dover 1998.

  The history of the golden ratio from Euclid to the present.

- Huntley, H.E. *The Divine Proportion – A Study in Mathematical Beauty.* Dover 1970.

  The golden section, the Fibonacci sequence, and the beauty of mathematics in general.

- Jarden, Dov. *Recurring Sequences – A Collection of Papers*. Riveon Lematematika 1973.

  Four articles on the Fibonacci and Lucas sequences and on recursive sequences of the second degree.

- Koshy, Thomas. *Fibonacci and Lucas Numbers with Applications.* Wiley 2001.

  An all-encompassing book that presents different perspectives on the Fibonacci and Lucas sequences and on the golden ratio.

- Livio, Mario. *The Golden Ratio – The Story of Phi, the World's Most Astonishing Number.* Broadway 2002.

  The history of the golden section and the Fibonacci sequence.

- Neveux, Marguerite. *Le Nombre d'Or – Radiographie d'un Mythe.* Seuil 1995.

  The myth of the golden section in the plastic arts. Includes a translation of Huntley's book.

- Posamentier, Alfred and Ingmar Lehmann. *The Fabulous Fibonacci Numbers.* Prometheus Books 2007.

  The Fibonacci numbers in mathematics, botany, art and architecture, and economics.

- Runion, Garth. *The Golden Section.* Dale Seymour 1990.

  Geometric perspectives of the golden section.

- Vajda, Steven. *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications.* Dover 2008.

  About the Fibonacci and Lucas numbers and their applications. (An updated printing of the 1989 edition.)

- Vorobyov, Nikolai. *The Fibonacci Numbers.* Heath and Company 1963.

  A classic booklet that presents the main properties of the Fibonacci sequence, including divisibility properties.

- Walser, Hans. *The Golden Section.* MAA 2001.

  A modern book that connects two theories: the golden section and fractals. Many illustrations.

- Willard, Claude-Jacques. *Le Nombre d'Or.* Magnard 1987.

  The golden ratio and the Fibonacci sequence, the geometry of the golden ratio, and the golden section in art and architecture.

Chapters in Books

Brousseau, Alfred. "Fibonacci Sequences", in Sobel, Max (ed.). *Readings for Enrichment in Secondary School Mathematics.* NCTM 1988.

Cadwell, J.H. "The Fibonacci Sequence", in *Topics in Recreational Mathematics.* Cambridge University Press 1966.

Conway, John and Richard Guy. "Famous Families of Numbers", in *The Book of Numbers.* Springer 1996.

Coxeter, H. S. M. "The Golden Section and Phyllotaxis", in *Introduction to Geometry (2nd edition).* Wiley 1989.

Dudley, Underwood. "Phi", in *Mathematical Cranks.* MAA 1992a.

Hilton, Peter, Derek Holton and Jean Pederson. "Fibonacci and Lucas Numbers", in *Mathematical Reflections – In a Room with Many Mirrors.* Springer 1997.

Hilton, Peter, Derek Holton and Jean Pederson. "Fibonacci and Lucas Numbers – Their Connections and Divisibility Properties", in *Mathematical Vistas – From a Room with Many Windows.* Springer 2002.

Kantor, Jean-Michel. "Le Nombre d'Or", in *Mathématiques Venues d'Ailleurs.* Belin 1982.

Maor, Eli. "Spira Mirabilis", in *e – The Story of a Number.* Princeton University Press 1994.

Movshovitz-Hadar, Nitsa and John Webb. “Congruency Paradox” and “The Lost Square”, in *One Equals Zero, and other Mathematical Surprises.* Key Kurriculum Press 1988.
Salem, Lionel, Frédéric Testard and Coralie Salem. “The Golden Ratio” and “The Fibonacci Numbers”, in *The Most Beautiful Mathematical Formulas.* Wiley 1992.
Young, Robert. “Fibonacci Numbers: Function and Form”, in *Excursions in Calculus – An Interplay of the Continuous and the Discrete.* MAA 1992.
Warusfel, André. “De la Métaphysique... aux Beaux-Arts” in *Les Nombres et leurs Mystères.* Seuil 1961.
Wells, David. “The Divine Proportion”, “The Fibonacci Sequence” and “The Lucas Numbers”, in *The Penguin Dictionary of Curious and Interesting Numbers.* Penguin Books 1987.

Articles

Rokach, Arieh: “Optimal Computation of Fibonacci Numbers”, in The Fibonacci Quarterly, November 1996.

A Selection of Internet Sites

(The sites were active in January 2018.)

The Fibonacci Association

► http://www.mscs.dal.ca/fibonacci/
The Fibonacci Association website. The Association publishes a quarterly journal dedicated to the Fibonacci sequence and accompanying topics. The site gives information on the association, its members, and its publications.

Fibonacci Numbers: From Wolfram MathWorld

► http://mathworld.wolfram.com/FibonacciNumber.html
Detailed mathematical analysis of the Fibonacci sequence.

Fibonacci Numbers, the Golden Section and the Golden String

► http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html
One of the most famous sites that has won many prizes. A vast amount of information on the golden ratio and Fibonacci and Lucas sequences.

(The) Fib-Phi Link Page

► http://www.goldenratio.org/info/
A nicely sorted collection of links connected to the golden ratio and the Fibonacci sequence.

Museum of Harmony and Golden Section

► http://www.goldenmuseum.com/index_engl.html
One of the most comprehensive websites: an extraordinary amount of documents, articles, and papers on both the theoretical and practical aspects, including special attention to applications in computer science.

The Fibonacci Quarterly

► https://www.goldennumber.net/
As the primary publication of the Fibonacci Association, The Fibonacci Quarterly provides the focus for worldwide interest in the Fibonacci number sequence and related mathematics. New results, research proposals, challenging problems and new proofs of known relationships are encouraged. The Quarterly seeks intelligible, well-motivated, university-level articles. Illustrations and tables should be included to the extent that they clarify main ideas of the text. A well-developed list of references is required.

The Golden Ratio and the Fibonacci Sequence in Wikipedia

► https://en.wikipedia.org/wiki/Phyllotaxis
► https://en.wikipedia.org/wiki/Fibonacci_number

London Einav on How the Golden Section Relates to the Environment (in Hebrew)

► https://sites.google.com/site/einavloondon/home/act/gold2/gold3

Eureka: A Website (in Hebrew) that Includes Some Video Clips.

► https://tinyurl.com/ydz2u5vg

The Golden Section and Fibonacci Numbers in Art and Architecture

► http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html

The Golden Section and Fibonacci Numbers on Postage Stamps

► https://www.goldennumber.net/golden-ratio-fibonacci-postage-stamps/

Fibonacci Numbers and Nature

► http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html#petals

Or Simply Type any of the Terms into Google to Search for More Information

► https://educateinspirechange.org/spirituality/learn-magic-fibonacci-nature-math-god/
Learn about the magic of Fibonacci in nature – the math of God.
► https://slideplayer.com.br/slide/11727812/