Correspondence between continuous-variable and discrete quantum systems of arbitrary dimensions

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We establish a mapping between a continuous-variable (CV) quantum system and a discrete quantum system of arbitrary dimension. This opens up the general possibility to perform any quantum information task with a CV system as if it were a discrete system. The Einstein-Podolsky-Rosen state is mapped onto the maximally entangled state in any finite-dimensional Hilbert space and thus can be considered as a universal resource of entanglement. An explicit example of the map and a proposal for its experimental realization are discussed.

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Quantum information processing enables performance of communication and computational tasks beyond the limits that are achievable on the basis of laws of classical physics [1,2]. While most of the quantum information protocols were initially developed for quantum systems with finite dimensions they have also been proposed for the quantum systems with continuous variables (CV).

With the exception of two-mode bipartite Gaussian states [3] there are no general criteria to test separability of a general state in infinite-dimensional Hilbert space. Similarly, the demonstration of the violation of Bell's inequalities for CV systems is based predominantly on the phase-space formalism [4] and the generalization to CV systems of various Bell's inequalities derived for discrete systems is still open. It is thus highly desirable to find a mapping between CV and discrete systems. This would allow us to apply all criteria known for discrete systems for the classification of states (e.g., for separability or for violation of local realism) to CV systems. It would also open up the possibility for the CV systems to be exploited to perform quantum information tasks as if they were qu*n*its, by applying protocols which are already developed for discrete *n*-dimensional systems.

Recently, a mapping between CV systems and qubits (two-dimensional systems) was introduced [5,6], which enables us to construct a Clauser-Horne-Shimony-Holt (CHSH) inequality [7] for CV systems without relying on the phase-space formalism and to analyze the separability of the infinite-dimensional Werner states. Other Bell's inequalities for CV systems and dichotomic observables were derived [8,9]. It was shown [5] that the Einstein-Podolsky-Rosen (EPR) [10] state

$$|\mathrm{EPR}\rangle = \int dq |q\rangle_1 \otimes |q\rangle_2, \tag{1}$$

where $|q\rangle_1 \otimes |q\rangle_2$ denotes a product state of two subsystems of a composite system, maximally violates the CHSH inequality (for maximal violation of Bell's inequalities in the vacuum states see Ref. [11]), a question which remained unanswered within the phase-space formalism. This is important because the EPR state is considered to be the natural resource of entanglement in CV quantum information processing.

Similar mappings between two discrete systems with Hilbert spaces of different dimensions were proposed for constructing quantum error resistant codes [12], certain Bell's inequality for higher-dimensional systems [13], and in entanglement concentration procedure [14].

It is intuitively clear that the potentiality of an infinite-dimensional system as a resource for quantum information processing goes beyond that of the qubit system. In particular, as it will be shown below, the CHSH inequality for CV systems [5,6] can be maximally violated even with nonmaximally entangled states. Thus to show the *full* potential of infinite-dimensional systems it will be important to find a mapping between CV and discrete quantum systems of *arbitrarily high dimensions*. As an example of the use of this mapping one may check the violation of Bell's inequalities for arbitrarily high-dimensional systems [15,16]. Such a mapping is also necessary if one wants to implement those quantum information tasks developed for discrete systems to CV systems, which exclusively *requires* higher-dimensional Hilbert spaces. These are, for example, the quantum key distribution based on higher alphabets [17] and the quantum solutions of the coin-flipping problem [18], of the Byzantine agreement problem [19], and of a certain communication complexity problem [20].

Here we give a general mapping between systems with Hilbert spaces of different dimensions. This opens up a possibility not only to use a quantum system of one dimension as having another dimension, but also to study an efficient quantum interface between two different dimensional systems. Taking an infinite-dimensional Hilbert space as one of the two Hilbert spaces we obtain a map between a CV and discrete quantum system of arbitrary dimension (the map "embeds" an arbitrary qu*n*it state in a CV system). This allows to exploit a CV system in *any* quantum information procedure developed for discrete systems, even though the different procedures may require systems of different dimensions. The EPR state is mapped onto the maximally entangled state in any finite-dimensional Hilbert space. Thus, it can be considered as an *universal* resource of entanglement.

We use a quantum-mechanical description based on generators of SU(n) algebra, as introduced in Ref. [21]. One can introduce transition-projection operators

$$\hat{P}_{jk}=|j\rangle\langle k|, \tag{2}$$

where $|j\rangle$ with $j=1,\dots,n$ are orthonormal basis vectors in the Hilbert space of dimension n. The operators $\hat{P}_{jk}$ will next be used to define another set of n^2-1 operators, which are formed in the following three groups:

$$\hat{u}_{jk}=\hat{P}_{jk}+\hat{P}_{kj}, \tag{3}$$

$$\hat{v}_{jk}=i(\hat{P}_{jk}-\hat{P}_{kj}), \tag{4}$$

$$\hat{w}_l=-\sqrt{\frac{2}{l(l+1)}}(\hat{P}_{11}+\hat{P}_{22}+\cdots+\hat{P}_{ll}-l\hat{P}_{l+1,l+1}), \tag{5}$$

where $1\leqslant l\leqslant n-1$ and $1\leqslant j<k\leqslant n$.

For $n=2$, these operators are the ordinary Pauli (spin) operators along the x, y, and z directions. In general, the operators in Eqs. (3)–(5) generate the SU(n) algebra. That is, the vector $\vec{\hat{s}}=(\hat{u}_{12},\dots,\hat{v}_{12},\dots,\hat{w}_1,\dots,\hat{w}_{n-1})$ has components $\hat{s}_j$ ($j=1,\dots,n^2-1$) that satisfy the algebraic relation

$$[\hat{s}_j,\hat{s}_k]=2if_{jkl}\hat{s}_l, \tag{6}$$

where repeated indices are summed from 1 to n^2-1, and f_{jkl} is the completely antisymmetric structure constant of the SU(n) group. The operators $\hat{s}_j$ fulfill the relations $\mathrm{Tr}(\hat{s}_j)=0$ and $\mathrm{Tr}(\hat{s}_i\hat{s}_j)=2\delta_{ij}$.

Any Hermitian operator in an n-dimensional Hilbert space can be expended into a linear sum of $\hat{s}_j$ [21]. We use this to expend the general quantum state $\hat{\rho}$ of a composite system consisting of L systems each with dimension n as given by

$$\hat{\rho}=\sum_{x_1,\dots,x_L=0}^{n^2-1} t_{x_1\dots x_L}\hat{s}_{x_1}\otimes\cdots\otimes\hat{s}_{x_L}, \tag{7}$$

where $\hat{s}_0=\mathbb{1}_n$ is the identity operator in n-dimensional Hilbert space. The vector with components $t_{x_1\dots x_L}$ is the "generalized Bloch vector" and has real components due to the hermiticity of $\hat{\rho}$. Specifically, $t_{0\dots 0}=1/n^L$ due to the normalization and $t_{x_1\dots x_L}=1/2^L\mathrm{Tr}(\hat{\rho}\cdot\hat{s}_{x_1}\otimes\cdots\otimes\hat{s}_{x_L})$ for $x_1,\dots,x_L\in\{1,\dots,n^2-1\}$.

We consider an observable $\hat{a}$ in the Hilbert space of the composite system as given by

$$\hat{a}=\sum_{x_1,\dots,x_L=1}^{n^2-1} a_{x_1\dots x_L}\hat{s}_{x_1}\otimes\cdots\otimes\hat{s}_{x_L}. \tag{8}$$

The expectation value of the observable $\hat{a}$ in the state $\hat{\rho}$ is given by $\langle\hat{a}\rangle_\rho\equiv\mathrm{Tr}(\hat{\rho}\hat{a})=2^L\Sigma_{x_1,\dots,x_L=1}^{n^2-1}t_{x_1\dots x_L}a_{x_1\dots x_L}$.

We now establish a mapping between Hilbert spaces of different dimensions. It will embed a n-dimensional quantum system in a Hilbert space of dimension $N\geqslant n$. Roughly speaking, it consists of dividing the Hilbert space of dimension N into a direct sum of "boxes" of dimension n. In the limit $N\to\infty$ we then obtain a mapping that embeds n-dimensional system in a CV system.

We first introduce the transition-projection operators

$$\hat{P}_{jk}(m)=|nm+j\rangle\langle nm+k|, \tag{9}$$

where $0\leqslant m\leqslant[N/n]-1$ and $1\leqslant j<k\leqslant n$. Here $[N/n]$ denotes the integer part of N/n. For each m one constructs the n^2-1 operators

$$\hat{u}_{jk}(m)=\hat{P}_{jk}(m)+\hat{P}_{kj}(m), \tag{10}$$

$$\hat{v}_{jk}(m)=i[\hat{P}_{jk}(m)-\hat{P}_{kj}(m)], \tag{11}$$

$$\hat{w}_l(m)=-\sqrt{\frac{2}{l(l+1)}}[\hat{P}_{11}(m)+\hat{P}_{22}(m)+\cdots+\hat{P}_{ll}(m)-l\hat{P}_{l+1,l+1}(m)], \tag{12}$$

where $1\leqslant l\leqslant n-1$. For any given m the set of operators

$$\vec{\hat{s}}(m)=[\hat{u}_{12}(m),\dots,\hat{v}_{12}(m),\dots,\hat{w}_1(m),\dots,\hat{w}_{n-1}(m)]$$

represents generators of the SU(n) algebra as they satisfy the algebraic relation (6) by the definition. Thus we have decomposed the original N-dimensional Hilbert space into a series of subspaces (indexed by m) that each is made isomorphic to the n-dimensional Hilbert space.

Next, we define the operators

$$\hat{U}_{jk}=\oplus\sum_{m=0}^{[N/n]-1}\hat{u}_{jk}(m), \tag{13}$$

$$\hat{V}_{jk}=\oplus\sum_{m=0}^{[N/n]-1}\hat{v}_{jk}(m), \tag{14}$$

$$\hat{W}_l=\oplus\sum_{m=0}^{[N/n]-1}\hat{w}_l(m), \tag{15}$$

where $\oplus$ denotes the direct sum of operators. The central point in the construction of the mapping is the introduction of the set of operators $\vec{\hat{S}}\equiv(\hat{S}_1,\dots,\hat{S}_{n^2-1})=(\hat{U}_{12},\dots,\hat{V}_{12},\dots,\hat{W}_1,\dots,\hat{W}_{n-1})$. The elements $\hat{S}_j$ of this set also satisfy the algebraic relation (6). This follows from $[\hat{S}_j,\hat{S}_k]=[\oplus\Sigma_m\hat{s}_j(m),\oplus\Sigma_r\hat{s}_k(r)]=\oplus\Sigma_{m,r}[\hat{s}_j(m),\hat{s}_k(r)]=\oplus\Sigma_m 2if_{jkl}\hat{s}_l(m)=2if_{jkl}\hat{S}_l$, where one uses $[\hat{s}_j(m),\hat{s}_k(r)]=0$ if $m\neq r$. Thus, the operators $\vec{\hat{S}}$ generate the SU(n) algebra as well. Importantly, in contrast to $\vec{\hat{s}}(m)$, the operators $\vec{\hat{S}}$ act on the full N-dimensional Hilbert space. However, only if N is exactly divisible by n, all N dimensions of the Hilbert space will be exploited; otherwise those less than N will be exploited. Note that $\mathrm{Tr}\hat{S}_i=0$ and $\mathrm{Tr}(\hat{S}_i\hat{S}_j)=2[N/n]\delta_{ij}$.

Our analysis so far concerns an algebraic relation between Hilbert spaces of different dimensions. In what follows, we shall give a concrete correspondence between quantum states and observables of two systems, one with dimension n and the other with dimension $N>n$.

With any operator $\hat{a}$ acting in a Hilbert space of L n-dimensional systems and having coefficients $a_{x_1\ldots x_L}$ in the expansion (8), we associate the operator

$$\hat{A}=\sum_{x_1,\ldots,x_L=1}^{n^2-1} a_{x_1\ldots x_L}\hat{S}_{x_1}\otimes\cdots\otimes\hat{S}_{x_L} \tag{16}$$

in a Hilbert space of L N-dimensional systems, which has the *same* coefficients $a_{x_1,\ldots,x_L}$ in the expansion (16). This maps the full set of observables in the n-dimensional Hilbert space onto a specific *subset* of observables in the N-dimensional Hilbert space.

From the physical perspective two quantum systems can be considered as equivalent *if the probabilities for outcomes of all possible future experiments performed on one and on the other system are the same.* This suggests to map the quantum states of the two Hilbert spaces as follows. With any state $\hat{\rho}$ [as given in Eq. (7)] of L n-dimensional systems we associate a class $[\Omega]$ of states of L N-dimensional systems with the property that the expectation value $\langle\hat{a}\rangle_\rho$ of *any* observable $\hat{a}$ measured on $\hat{\rho}$ is equal to the expectation value $\langle\hat{A}\rangle_\Omega$ of the observable $\hat{A}$ [Eq. (16)] measured on *any* state from the class $[\hat{\Omega}]$. The mapping is established by the requirement

$$\langle\hat{a}\rangle_\rho\equiv\mathrm{Tr}(\hat{\rho}\hat{a})=\mathrm{Tr}(\hat{\Omega}\hat{A})\equiv\langle\hat{A}\rangle_\Omega \tag{17}$$

for any $\hat{a}$ and associated $\hat{A}$ and for any state $\hat{\Omega}$ from the class $[\hat{\Omega}]$. Since the measurements are constrained to the type (16), the proper expectation value $\mathrm{Tr}(\hat{\Omega}\hat{A})$ can be obtained if one represents the class $[\hat{\Omega}]$ by

$$[\hat{\Omega}]:=\sum_{x_1,\ldots,x_L=0}^{n^2-1} T_{x_1\ldots x_L}\hat{S}_{x_1}\otimes\cdots\otimes\hat{S}_{x_L}, \tag{18}$$

with $T_{x_1\ldots x_L}=(1/[N/n]^L)\ t_{x_1\ldots x_L}$. The qu*n*it embedded in the N-dimensional Hilbert space has the correct inner product.

We establish a mapping between a n-dimensional quantum system and a CV quantum system by requiring condition (17) also to be satisfied in the limit $N\to\infty$. Note that here one first takes the expectation value $\mathrm{Tr}(\hat{\Omega}\hat{A})$ and then the limit, which differs from the value which is obtained if one first takes the limits of the state $\hat{\Omega}$ and observable $\hat{A}$ separately and then builds the expectation value (in this case one may obtain unphysical states; see Ref. [22] for some discussion of this issue). Here it is important to note that strictly speaking, only expectation values (probabilities) have an operational meaning neither states alone nor observables alone.

In order to give an example of different infinite-dimensional states that all belong to the same class $[\Omega]$, consider the maximally entangled state $|\psi\rangle=\lim_{N\to\infty}(1/\sqrt{N})\Sigma_{i=0}^{N-1}|j\rangle_1\otimes|j\rangle_2$ and the mixture $\hat{w}=\oplus\Sigma_{m=0}^{\infty}p(m)|\psi(m)\rangle\langle\psi(m)|$ [with $\Sigma_{m=0}^{\infty}p(m)=1$] of maximally entangled states $|\psi(m)\rangle=(1/\sqrt{n})\Sigma_{j=0}^{n-1}|nm+j\rangle_1\otimes|nm+j\rangle_2$, in different $(n\times n)$-dimensional subspaces of the original Hilbert space. Both of them are mapped onto the maximally entangled state $|\psi\rangle=(1/\sqrt{n})\Sigma_{j=0}^{n-1}|j\rangle_1\otimes|j\rangle_2$ in an $(n\times n)$-dimensional space. This shows that even nonmaximally entangled states can be considered as resources of maximal entanglement in lower-dimensional Hilbert spaces. For example, the mixture $\hat{w}$ introduced above for $n=2$ can maximally violate the CHSH inequality of Ref. [5].

The EPR state is the only state which is mapped onto the maximally entangled state in any finite-dimensional Hilbert space. Thus the violation of Bell's inequalities for arbitrarily high dimensional systems [16] or various quantum protocols which use maximally entangled states of different dimensions [17,19,20] can *all* be demonstrated by the EPR state.

Experimentally, a state produced by nondegenerate optical parametric amplifier (the NOPA state) can be considered as the "regularized" EPR state [note that the original EPR state (1) is un-normalized] [23]. The NOPA state is given by

$$|\mathrm{NOPA}\rangle=\sum_{k=0}^{\infty}\frac{(\tanh r)^k}{\cosh r}|k\rangle_1\otimes|k\rangle_2, \tag{19}$$

where $r>0$ is the squeezing parameter and $|k\rangle_1\otimes|k\rangle_2$ is a product of the Fock states of the two modes, each containing k photons. It becomes the optical analog of the EPR state in the limit of high squeezing [23].

In order to give an explicit example for the application of our method we will map the NOPA state onto an entangled state of two qutrits. This is important if one wants to use the NOPA state in quantum information processes which are developed for entangled qutrits (see, for example, Ref. [20]). We will analyze the violation of Bell's inequality for two qutrits [15,16] by the NOPA state. This Bell inequality is given by $B\leqslant 2$, where B (Bell's expression) is a certain combination of probabilities for the measurements of two qutrits and 2 is the limit imposed by local realistic models. In Ref. [24] the violation of this inequality is analyzed for the states of the form $|\psi\rangle=\Sigma_{k=0}^{2}a_k|k\rangle_1\otimes|k\rangle_2$ and a class of observables constructed by unbiased symmetric beam splitters [25]. Here a_k are real coefficients and $|k\rangle_1\otimes|k\rangle_2$ are product states of two qutrits. The maximal value of Bell's expression was found to be $B_{max}=4|a_1a_2|+4/\sqrt{3}(|a_1a_3|+|a_2a_3|)$ (if $|a_1|\geqslant|a_2|\geqslant|a_3|$ and $\max\{|a_1|,|a_2|,|a_3|\}\leqslant\sqrt{6+3\sqrt{3}}/2$, which is in our case).

Bell's expression in quantum mechanics is given by the expectation value of a certain operator (Bell's operator). The general method for establishing the correspondence between CV and discrete systems implies that the entangled two-qutrit state onto which the NOPA is mapped is of the form as

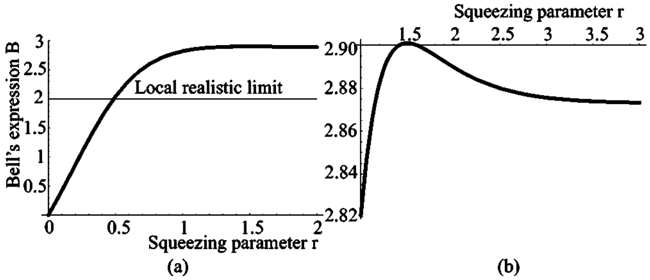

FIG. 1. Bell's expression B for the NOPA state as a function of the squeezing parameter r for different ranges of r. The NOPA state is mapped onto a state of two entangled qutrits for which Bell's inequality $B>2$ is analyzed. In the interval $[0,0.5]$ of r there is no violation (a). For $r>0.5$ the amount of violation increases with an increase of r, until it reaches the maximal value at $r=1.4998$ ($B=2.901$). With further increase of r, B begins to decrease reaching asymptotically the value of 2.8729 (b).

given above with the coefficients $a_k=(\tanh r)^k/\sqrt{1+(\tanh r)^2+(\tanh r)^4}$, $k\in\{0,1,2\}$. These states can be obtained by projecting the NOPA state onto any of the (3×3)-dimensional subspaces spanned by the states $|nm+j\rangle_1\otimes|nm+k\rangle_2$ for a given m and $j,k\in\{0,1,2\}$.

The amount of violation of Bell's inequality as a function of the squeezing parameter r is given in Figs. 1(a) and 1(b) for different ranges of r. Interestingly, in the interval $r\in[0,0.5]$ there is no violation. This explicitly shows that for the set of observables considered in [15,16,24] not even all pure entangled states violate Bell's inequality. Further, the maximal violation ($B=2.9011$) is at $r=1.4998$; not for $r\to\infty$ which one would expect. This again explicitly confirms the result of Ref. [26] that nonmaximally entangled states can violate Bell's inequality more strongly than the maximally entangled one. Finally, Bell's expression for $r\to\infty$ reaches asymptotically the value 2.872 93 which is also the value obtained for the maximally entangled two-qutrit state. In that limit the NOPA state becomes the EPR state and thus is mapped onto the maximally entangled two-qutrit state.

We give a concrete proposal for an experimental method to realize the measurement (16) in a realistic physical system, which is the key ingredient in our map between CV and discrete quantum systems. The method, though still experimentally challenging, might offer a realistic possibility to perform these measurements. It follows the idea of Refs. [12,27], where a state of the radiation field is measured by a coupling of the field with a single atom. The coupling is described by the perturbation: $H'=\lambda\hat{a}^\dagger\hat{a}\hat{\sigma}_z$, where $\hat{a}^\dagger$ is the creation and $\hat{a}$ the annihilation operator and $\hat{\sigma}_z$ is (formally) the spin operator along z and has eigenvalues -1 in the atomic ground state $|g\rangle$ and 1 in the atomic excited state $|e\rangle$. By turning on this coupling for a time $t=\pi/(2\lambda)$, we execute the unitary transformation $\hat{U}=\exp[-i(\pi/2)\hat{a}^\dagger\hat{a}\hat{\sigma}_z]$. Evolving the state $(|g\rangle+|e\rangle)/\sqrt{2}$ under this transformation and then measuring it in the basis $(|g\rangle\pm|e\rangle)/\sqrt{2}$, we project an arbitrary state of the radiation field onto a subspace spanned by the states with an even number of photons (the photon number operator: $\hat{a}^\dagger\hat{a}=2n,\ n\in\mathbb{N}$) and a subspaces spanned by the states with an odd number of photons ($\hat{a}^\dagger\hat{a}=2n+1$).

Since this is a nondemolition measurement, it can always be repeated not only to improve reliability, but also to perform further measurements. By turning again the coupling, but now for a time $t=\pi/(4\lambda)$ and repeating the procedure, we can project the component of radiation field with an even number of photons onto the subspaces spanned by $4n$ and $4n+2$ photon number states. One can project the components just obtained even further by a suitable choice of the time duration of the coupling and repeating the procedure again and again. The component of the radiation field with an odd number of photons can also be further projected by first removing a single photon, e.g., by adoptive absorption as suggested in Ref. [28], and then repeating the procedure as given above. The method can be applied successively with (theoretically) perfect accuracy and with only a small number of atoms, however, it is not general (e.g., one cannot project the field onto subspaces with $3n$, $3n+1$, and $3n+2$ photons). In those cases the method results in different but nonorthogonal states, which can be distinguished probabilistically by the use of a positive-operator-valued measure.

For completeness we note that a method was proposed [5,29], to measure *any* observable on the radiation field by measuring the appropriate observable of atoms interacting with the field in the limit of a large number of atoms. With current technology atomic states can be measured with good accuracy [2].

In conclusion, we have establish a general mapping between CV and discrete systems of arbitrary dimension. This allows construction of *all* quantum information protocols known for discrete systems also to CV systems.

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