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Course Guidebook

Arthur T. Benjamin
Harvey Mudd College

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Smallwood Family Professor of Mathematics

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Arthur T. Benjamin is the Smallwood Family Professor of Mathematics at Harvey Mudd College, where he has also served as the chair of the Department of Mathematics. He earned his BS in Applied Mathematics from Carnegie Mellon University and his PhD in Mathematical Sciences from Johns Hopkins University.

From the Mathematical Association of America (MAA), Professor Benjamin has received the Southern California-Nevada Section Award for Distinguished College or University Teaching of Mathematics and the national Deborah and Franklin Tepper Haimo Award. He was also named the George Pólya Lecturer by the MAA from 2006 to 2008. Professor Benjamin was chosen by The Princeton Review as one of its Best 300 Professors. He was also selected as an inaugural Fellow of the American Mathematical Society. *Reader's Digest* named him "America's Best Math Whiz."

Professor Benjamin's research interests include combinatorics and number theory, with a special fondness for Fibonacci numbers. Many of these ideas appear in his book (coauthored with Jennifer Quinn) *Proofs That Really Count: The Art of Combinatorial Proof*, which received the MAA's Beckenbach Book Prize. From 2004 to 2008, Professors Benjamin and Quinn served as the coeditors of *Math Horizons* magazine. Professor Benjamin's mental calculation techniques are explained in his book *Secrets* *of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*. Prolific math and science writer Martin Gardner called the book "the clearest, simplest, most entertaining, and best book yet on the art of calculating in your head." Professor Benjamin's most recent book, *The Magic of Math: Solving for x and Figuring Out Why*, was a *New York Times* Best Seller.

In addition to his academic career, Professor Benjamin is also a professional magician who performs his mixture of math and magic to audiences all over the world, including at the Magic Castle in Hollywood, California. He has appeared on dozens of television and radio programs, including the *TODAY* show and *The Colbert Report* as well as CNN and National Public Radio. He has been featured in *Scientific American*, *Omni*, *Discover*, *People*, *Esquire*, *The New York Times*, the *Los Angeles Times*, and *Reader's Digest*. Professor Benjamin has given numerous wide-reaching TED Talks that have been viewed more than 20 million times.

Professor Benjamin's other Great Courses are *The Joy of Mathematics*, *Discrete Mathematics*, *The Secrets of Mental Math*, and *The Mathematics of Games and Puzzles: From Cards to Sudoku*. ♦

ACKNOWLEDGMENTS

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Naturally, there are many mathematicians and magicians to thank, many of whose ideas and suggestions made their way into this course.

I am especially grateful to the following mathematicians: Barry Balof, Steve Butler, Jay Cordes, Joyati Debnath, Doug Ensley, Allison Henrich, John Harris, Jon Jacobsen, Jason Lee, Mario Micheli, Brent Morris, Colm Mulcahy, Sam Nelson, Mohamed Omar, Michael Orrison, Matt Parker, Lizard Reiland, Sarah Rundell, Ricardo Teixeira, Tadashi Tokieda, Robert Vallin, and Glen Whitney.

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For displaying their special talents on camera, I am grateful for the poker prowess of Lizard Reiland, the perfect shuffling of Brent Morris, and the beguiling bagel carving of Michelle Pellatt (with a shout-out to Glen Whitney for providing us with expert instruction). Special thanks to whiz kids Lucy Ingram, Nikita Lugovoy, and Stephan Lugovoy for showing off their calculating cleverness and to the many staff members from The Great Courses who helped out as unsuspecting volunteers for my magic tricks: Randy Kraemer, Tammy Cornelious, Kevin Barnhill, Art Jaruphaiboon, Michelle Pellatt, Torrance Hill, Amy Derr, Mark Ginesi, Leslie Meade, Andrew Volpe, Ines Perez Thompson, Kenya Hughes, Courtney Westphal, Josh Rodriguez, James Nidel, Peter Dwyer, and Erin O'Riley. Thanks also to the studio crew for this course: Roberto de Moraes (director), Justin Thomas (camera), Kelly Tagliaferri (camera), and Sherri Macdonald-Hagelen (production assistant). It is always a pleasure to work with the staff of The Great Courses, both on camera and off. I especially want to thank Maureen Faley and Michelle Pellatt for their tireless devotion to this project.

Last but not least (indeed, most!), I thank my wife, Deena, and daughters Laurel and Ariel, who supported me and assisted me throughout this project. Thank you for bringing so much magic to my life! ♦

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INTRODUCTION

GUIDES

SUPPLEMENTARY MATERIAL

MATH AND MAGIC

On the surface, mathematics and magic seem to be very different, yet they have an awful lot in common. In both subjects, you have a problem to solve: What is the value of x? How did the magician find my card? Both seem mysterious at first, but with a little bit of practice (okay, sometimes a lot of practice), you can become quite good at both. And once mastered, both subjects are a great deal of fun. Most people enjoy magic tricks, especially if they are performed well, and one of the goals of this course is that by the end of it you'll enjoy math more, too—or at least see how it can be applied in magical ways.

By the time you finish this course, you'll learn to dazzle your friends and family with some impressive feats designed to make you look like a card shark, a psychic, a genius, or just a very talented magician.

None of the magic will involve any difficult sleight of hand. If there is any physical dexterity required, it'll be so easy to learn that you should be able learn it in just a few minutes.

All the magic tricks in this course use ordinary, everyday objects, such as cards, coins, and paper. Some of the tricks use numbers visibly, but with some of them, the math is pretty well hidden.

The first 6 lessons focus on mathematical card tricks. In lesson 1, the presence of mathematics is noticeably present, as cards are being counted, added, or spelled out. In lesson 2, the mathematics is much more concealed, as cards are being mixed and dealt, culminating in the appearance of 4 aces and a royal flush. You'll also learn how to create the illusion of being able to memorize a well-mixed deck of cards. In lesson 3, you'll learn how to handle cards like a professional gambler—whether that involves cutting to the 4 aces, always winning at high-card poker, or determining the facedown card in a game of stud poker.

Imagine being given any number of cards from a deck and being able to determine the exact number of cards by seemingly weighing the cards in your hand. One way of doing this is shown in lesson 4, where you'll learn how to perform miracles with a stacked deck. In lesson 5, you'll learn the mathematical properties of various kinds of shuffles, including the perfect shuffle, in which the cards are interleaved so precisely that, despite outward appearances, the location of the cards is still known to the magician. In lesson 6—the final card lesson—you'll discover that even when the cards are mixed imperfectly after being cut and given a riffle shuffle, there are still many astonishing miracles that can be performed using the Gilbreath principle, a surprisingly beautiful application of mathematics.

In the second half of the course, you'll turn your attention away from cards and explore magical topics designed to make you look like a psychic or a genius—or a mathemagician. In lesson 7, you'll discover magic tricks based solely on numbers, including the Fibonacci numbers and the golden ratio. You'll also briefly explore the underlying algebra that makes the magic happen. Imagine being able to add, divide, and multiply numbers faster than a calculator; in lesson 8, you'll learn the tricks of the trade that are so easy even children can do them with just a small amount of practice. In lesson 9, special attention will be given to the number 9, which has a magic all its own, including a way to determine someone's age by performing a few simple calculations.

In lesson 10, you'll use everyday objects—such as coins, dice, and calculators—to give you the appearance of possessing psychic powers. You'll have fun with geometrical and topological magic in lesson 11, where you'll learn how to make a (paper) rabbit disappear by the power of mathematics and explore the marvelously magical properties of Möbius strips. You'll even learn how to carve a bagel into 2 identical halves that are completely linked together. Finally, in lesson 12, you'll explore a topic that has been the object of fascination of mathematicians and magicians alike: magic squares. You'll even learn the instructor's own method for creating a magic square based on anyone's birthday.

Now go get yourself a deck of cards, prepare to be amazed, and soon you'll be amazing others! ♦

MATHEMATICAL CARD TRICKS

This book includes explanations for the tricks performed in the video by mathemagician Arthur T. Benjamin.

This first lesson will focus on card tricks in which there is clearly some mathematics being used but the secret is not so easy to figure out. To get started, all you need is a complete deck of 52 cards (no jokers).

EXPLANATIONS

SPELLER

- In this trick, you mix up 9 cards and then look at the third card from the top.
- When you spell your card's value, it will have at least 3 letters.

A-C-E	F-O-U-R	S-E-V-E-N	T-E-N	K-I-N-G
T-W-O	F-I-V-E	E-I-G-H-T	J-A-C-K	
T-H-R-E-E	S-I-X	N-I-N-E	Q-U-E-E-N	

- When this trick was performed for you, the third card was the 10 of clubs. But no matter what it is, after you spell it, it will be third from the bottom. After spelling the word "of," the card is now fifth from the bottom (which is also fifth from the top). And after spelling any suit—spades, hearts, diamonds, or clubs—because they all have at least 5 letters, your card will still be fifth from the top.

- Finally, because the word "magic" has 5 letters, you are guaranteed to end on the chosen card.

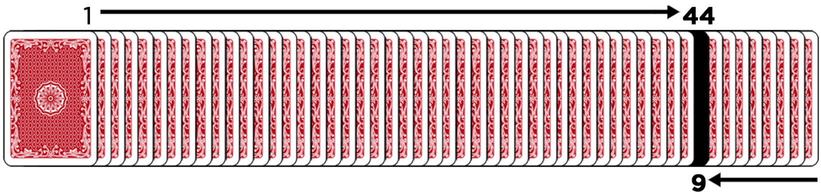
MATHEMATICAL DECK

- The final trick that was performed in the video lesson has a more complicated secret. The advantage of doing this one after the spelling trick is that you already have 9 cards in use. If you don't do the spelling trick, then just have the cards shuffled and deal out 9 cards. Then, have a volunteer pick one of them and put it on top of the pile.

- After placing the deck on top, where is the person's card now? It is ninth from the bottom, which means there are 43 cards above it. Thus, the card is in position 44.

Magicians are not supposed to reveal their secrets. But by taking this course, you are considered a serious student of magic, so sharing this information is allowed. Also, nearly all of the tricks in this course are based on mathematics, and math should be open to everyone.

However, if you perform these tricks for your family or friends, you'll find that they will be much more amazed if you keep the secrets to yourself.



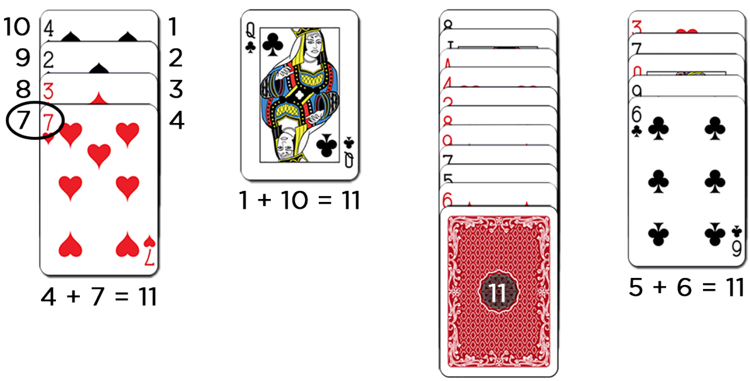
- Now for the sneaky part: You lose the card by giving the deck a few cuts. But unless they pay very close attention, most people won't notice that the cards are still in the exact same order. If you didn't notice the first time, watch this portion of the video again. This is called a false cut, and it's extremely deceptive.

- So, the person's card is still in position 44. Believe it or not, no matter what cards appear, you will always arrive at the 44th card. Why is that?

- Each of the 4 piles will contribute 11 to the final total.

  - **EXAMPLE:** Suppose that you start counting backwards from 10 and get a match at 7. So far, you have dealt 4 cards, but when you use the 7 later as part of the total, you will deal 7 more cards, for a total of 11.

  - If your next card is a face card, then you have dealt 1 card and will later deal 10 more cards, for a contribution of 11.



- If you go 10 cards without a match, then you kill the pile with 1 more card—so that pile has 11 cards, and it will not contribute any more to the total.
- And when you do your fourth pile, you get 6, so you have dealt 5 cards and your total will tell you to contribute 6 more.

- Because each of the 4 piles contributes 11 cards being dealt, you are guaranteed to reach the 44th card.

4-CARD TRICK

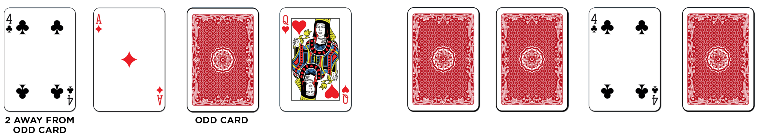
- Finally, the secret of the first trick from the lesson, which uses 4 cards of different suits, is based on a subtle idea that was originally attributed to magician Bob Hummer. When you start the trick, the cards are in the alternating color order, and then the spade card is turned upside down.
- So, the spade is the odd card at the moment. Notice that the club card is 2 away from the spade suit, so the club is 2 away from the odd card, no matter how the cards were originally ordered.
- Now here's the neat observation: No matter where the club card is, it will never be the odd card, and it will always be 2 away from the odd card, no matter what you do.
- Notice that as you transfer cards from the top to the bottom, the club will stay 2 away from the odd card. Even if you turn the deck upside down, the club is still 2 away from the odd card. This is also true when you turn the top 2 cards over as 1. What about when you turn the top 2 cards over as 1?
- Let's assume that the club is facedown and the odd card is faceup, but the logic still works if the situation is reversed. In all 4 possible places where the club can be, if the club is 2 away from the odd card, it will still be 2 away from the odd card after you turn the top 2 cards over as 1.

- This is called the **INVARIANT PRINCIPLE**, which means that even though the order of the cards is changing, there is some quality that stays the same. In this case, the invariant is that the club will always be 2 away from the odd card.

- Finally, you will go through your final flipping procedure at the end of the trick, flipping 1 card, then 2, then 3, and then 4. Let's look at all 4 situations where the club could be and see what happens after you do your final flipping procedure. You will start with the club faceup, but the same logic applies when you start with the club facedown.

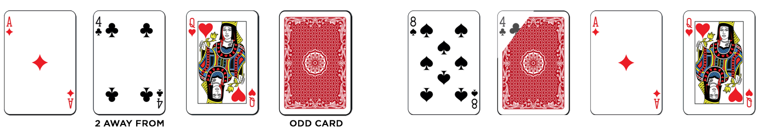
    - Suppose the club is faceup in the first position, so the odd card is facedown in the third position. When you go through your final flipping procedure, the club is the only faceup card, so it's the odd card.

BEFORE FINAL FLIP **AFTER FINAL FLIP**



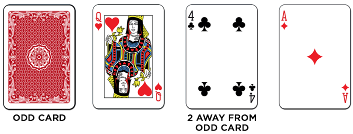
    - If the club is faceup in position 2, then the odd card will be facedown in position 4, and after the final flipping, the club will be the only facedown card.

BEFORE FINAL FLIP **AFTER FINAL FLIP**

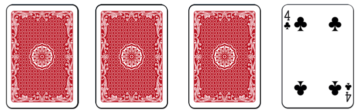


- If the club is in position 3, then after flipping 1, 2, 3, and 4, it's the only faceup card.

BEFORE FINAL FLIP

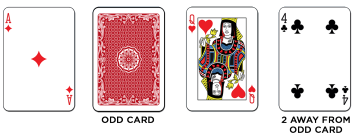


AFTER FINAL FLIP

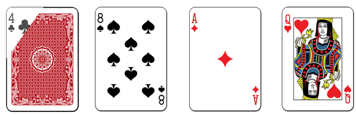


- Finally, if the club is in position 4, then after flipping 1, 2, 3, and 4, it's the only facedown card.

BEFORE FINAL FLIP



AFTER FINAL FLIP



- Thus, no matter where the club begins, it will be the odd card in the end.

REFERENCES

Diaconis and Graham, *Magical Mathematics.*

Gardner, *Mathematics, Magic and Mystery.*

Mulcahy, *Mathematical Card Magic.*

EXERCISES

1 One of the first mathematical card tricks that people learn is known as the 21-card trick or the princess card trick. It is an easy trick to do, and the magician never has to look at the cards at any point in the trick. The magician begins by showing 21 cards, displayed in 3 columns of cards, each with 7 cards. The volunteer is asked to think of a card and to indicate which column it is in. The cards are gathered, one column at a time, with the selected pile sandwiched between the other 2 columns. The 21 cards are redealt into 3 columns, one card at a time, with the first card going to column 1, the second card going to column 2, and so on, and the spectator is again asked which column the card is in. Again, the columns are gathered with the selected column between the other 2. This process is repeated one more time—after which the magician deals all the cards into a single pile, and after a bunch of cards are dealt, he or she claims that the next card that is turned over will be the selected card. Although the volunteer is often skeptical, the magician turns over the selected card. What's the secret?

2 Explain the "tearable" card trick at the end of the video lesson.

SOLUTIONS CAN BE FOUND ON PAGE 126.

CLICK the buttons to navigate.

To go back to the page you came from, **PRESS Alt + ←** on a PC or **⌘ + ←** on a Mac. On a tablet, use the bookmarks panel.

Before moving on to the next lesson, practice what you've learned in this one. Show off the tricks to your friends and family and then come back for more.

WHAT'S YOUR DEAL?

In this lesson, you will explore some magic tricks where the cards are shuffled, dealt, and turned around in interesting ways, leading to some very surprising outcomes.

EXPLANATIONS

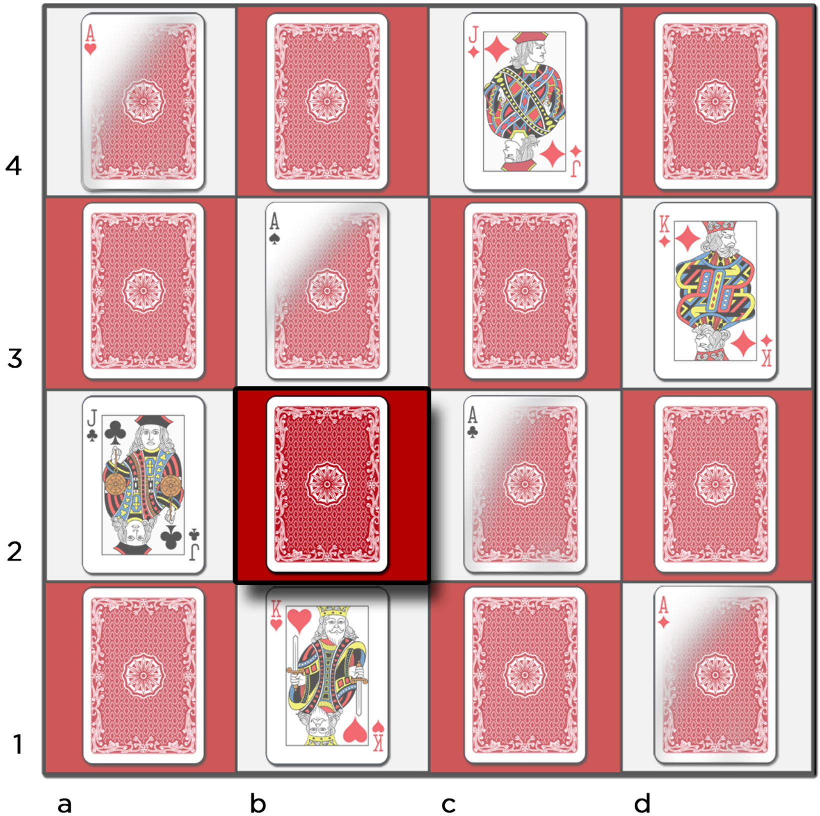
4-ACE SURPRISE

- For this trick, imagine that the cards are dealt onto a 4-by-4 checkerboard with red and white squares.

- To do this trick, you begin with the 4 aces on top of your deck and you deal them facedown on the white diagonal. You say, "The cards can go anywhere, and they can come from any part of the deck—the top, the bottom, the middle. It doesn't matter." But of course it does matter, and you've just dealt the 4 aces on the white diagonal. Then, you turn the 4 cards on the other white squares faceup.

- Next, you do your folding process until you have one big pile on top of a single square. Let's say that you end up at square **B2**.

- Now consider the ace in square **C2**. How many flips will it take to get to the dark square? It could happen in 1 flip. Or it might take 3 flips. It could even take 5 flips. Whatever the number of flips, it has to be an odd number. Why?



- Because on each flip, the color of the square changes from light to dark or from dark to light. And because your ace starts on a light square, then if you end up on a dark square, then it had to make an odd number of flips. And because your ace starts facedown, then after an odd number of flips, it will have to be faceup. The same is true for all of the other aces: They start facedown on light squares, so after an odd number of flips, they will all be faceup.

- What about the other cards that started on light squares? They also make an odd number of flips, but because they started faceup, when they get to the dark square, they will all be facedown.

- As for the cards that started on dark squares, they will all make an even number of flips, and because they all started facedown, then by the time they get to the destination square, they will all be facedown, no matter how you fold your cards.

- So, in the end, all the aces will be faceup and all of the non-aces will be facedown. On the other hand, if all of your cards end up on a pile on top of a light square, then all of your aces will be facedown and your non-aces will all be faceup. Either way, you have a surprise ending.

HEART'S CONTENT

- Let your volunteer shuffle the cards to his or her heart's content. Then, as you go through the cards, 2 at a time, you will either put the cards face to face, back to back, or facing the same direction, depending on the number of hearts you have.

- All you have to remember is that you are a "heart lover." As you go through the cards, 2 at a time, notice how many hearts there are. Arrange the cards so that the only suit you can see is hearts.

  - If you have no hearts, turn the cards face to face. This way, because there are no hearts, there's nothing to see, no matter how the cards are flipped.

  - If both cards are hearts, turn the cards back to back. This way, you get to see a heart no matter how the cards are flipped.

  - If just 1 card is a heart, then arrange the cards so that the heart is faceup. For example, if you see 2 cards with a heart on top of a non-heart, then just leave them in that order and ask your volunteer: "Keep or Flip?" But if the cards show up with the non-heart on top, then do a very slight amount of sleight of hand: Turn to another person and ask, "How about these 2 cards? Keep or flip?" It's a very subtle move, and nobody notices it.

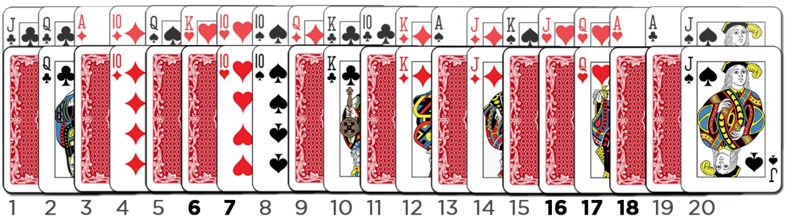
- If you can remember these rules, you can perform the trick. It works by itself. To understand why this works, look at the first 2 cards.

- All 3 rules produce the same situation: Where will the hearts be? Either they are faceup on top or facedown on the bottom. This will be true even if the cards are flipped.

- The same is true with the next pair of cards—3 and 4.

- After you're done going through the cards, where will the hearts be?

- If you number the card positions from 1 to 20, then the hearts will be faceup in an odd position or facedown in an even position. (And the non-hearts are the opposite.)

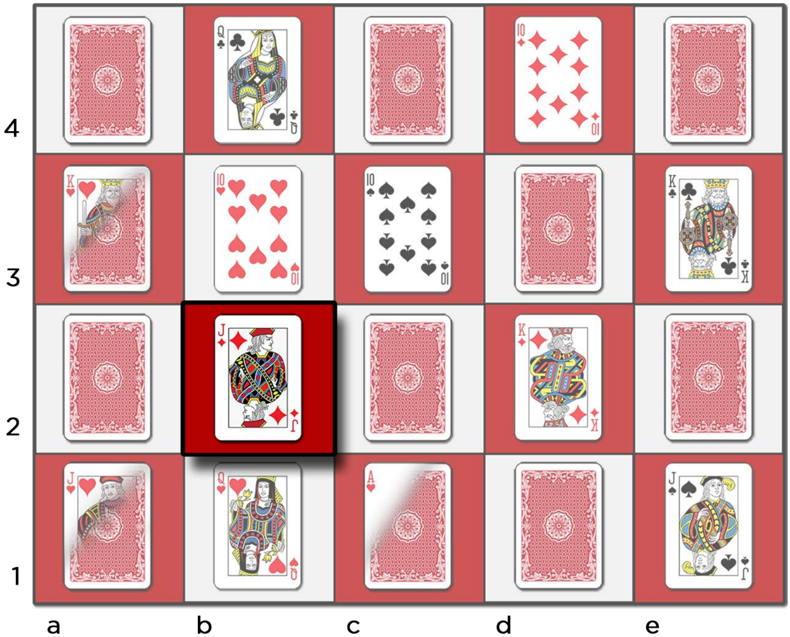


- Now deal the deck into 4 rows, with 5 cards in each row. It's helpful to imagine dealing the cards on a checkerboard with light and dark squares.

- The first 5 cards are dealt left to right, starting with a light square. The next 5 cards will start with a dark square whether you start from the left or from the right.

- So, after you deal the cards, the cards in the odd positions will be on light squares and the cards in even positions will be on dark squares. So, where are the hearts? They are either faceup on a light square or facedown on a dark square. And the non-hearts are just the opposite.

- So, if the cards all end up at **B2**, then what will happen to the hearts? Now it becomes like the 4-Ace Surprise. A faceup heart is on a light square, so it will take an odd number of flips and be facedown. A facedown heart starts on a dark square, so it will take an even number of flips and still be facedown.



- For the non-hearts, it's just the opposite: If it's faceup, then it must start on a dark square and make an even number of flips, so it will end faceup. If it's facedown, then it must start on a light square and make an odd number of flips and end faceup. Thus, when you've consolidated to a single pile, all your hearts will be facedown and the rest will be faceup.
- By the same reasoning, if all the cards ended up on a light square, then you would have all of your hearts faceup and the rest facedown. Either way, the trick ends in dramatic fashion.

This idea has been explored and exploited by many magicians, including Martin Gardner, Lennart Green, Steve Freeman, and John Bannon, but the credit is often given to Bob Hummer, and tricks of this nature are said to use the Hummer principle.

CARD MEMORY

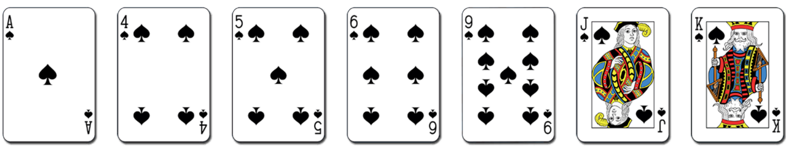
- Believe it or not, despite all of the shuffling, cutting, flipping, and transferring of cards in this trick, the magician knew exactly where all the cards would end up. This is sort of like the 4-Ace Surprise on steroids! As you'll see, the cards that Tammy and Randy end up with are exactly the same cards that they started with.

- The magician placed the jokers in the very middle of the deck. When he removed the jokers, he gave the cards below the jokers to Randy and the top cards to Tammy.

- The top cards are the ones that the magician "memorized." He chose these cards to make them easy to remember. And once you know the pattern, you'll never forget them!

- Let's start with the spades, which might be the trickiest. The magician wanted the pointing back and forth to look pretty random, especially at the beginning when he takes things slowly.

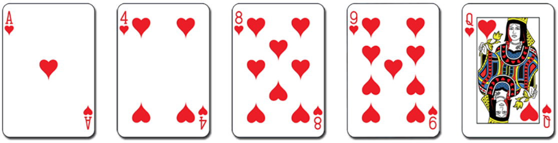
MEMORIZED SPADES



- What makes this easy to remember is that it's physical. Just remember 1-2-3-2-1. The first person drops 1 card (the ace), the next person drops 2 cards (the 2 and 3 of spades). The next person drops 3 cards (the 4, 5, and 6 of spades), and then you go back to 2 cards (the 7 and 8), then 1 card (the 9), and then you alternate back and forth for the 10, jack, queen, and king. It looks very random and dramatic.

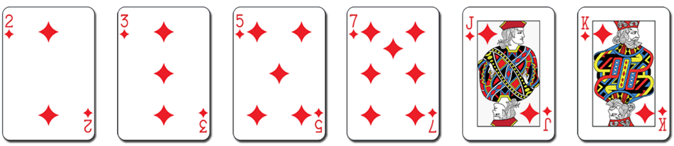
- For the hearts, the magician loves numbers that are perfect squares and perfect cubes, so he chose to memorize the ace (1), 4, 8, 9, and also the supposedly "loveliest" of cards, the queen of hearts.

MEMORIZED HEARTS



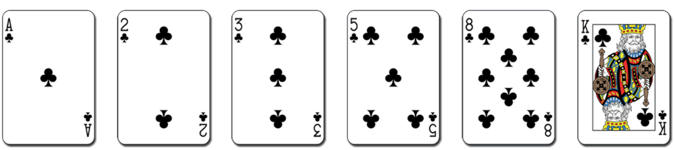
- Next, the diamonds are the "prime ones" (which almost rhymes!), so these are the numbers 2, 3, 5, 7, 11, and 13. Just remember that the number 1 is technically not a prime number. (You could put it on the list, but then the same person would have all 4 aces, and it's preferable to avoid that.)

MEMORIZED DIAMONDS



- For the clubs, do you recognize the numbers 1, 2, 3, 5, 8, and 13? These are the Fibonacci numbers. (1 + 2 = 3; 2 + 3 = 5; 3 + 5 = 8; 5 + 8 = 13). Think of the Fibonacci Association, which studies the amazing properties of these numbers, as a "club" so that the clubs become easy to recall.

MEMORIZED CLUBS



- Feel free to modify the list of memorized cards however you like; whatever will make it easiest for you to remember is what you should use.

- Now that you've memorized all the suits, how do you know they will all end back together? Initially, all the memorized cards are facedown on your right (and the other cards are facedown on your left). Remember that every time you move a pile from one side to the other, you turn it over. That's important.

- The ace of spades starts facedown on your right, and if it stays there, then it will stay facedown. But if you move it, then it becomes faceup on your left, where it will stay unless it gets moved back, in which case it will be facedown on your right.

- In other words, no matter how much transferring of cards you do, the ace of spades will either be facedown on your right or faceup on your left. So, when the process finishes, the ace of spades—and all the cards that you have memorized—will be facedown on your right or facedown on your left.

- Next comes the sneaky part, when you perform a secret move: Before you give the final riffle shuffle, you turn the right pile over. And look what happens. Your situation goes from this …

LEFT PILE	RIGHT PILE
memorized (up)	memorized (down)
unmemorized (down)	unmemorized (up)

… to this!

LEFT PILE	RIGHT PILE	=	RIGHT PILE
memorized (up)	memorized (down)		memorized (up)
unmemorized (down)	unmemorized (up)		unmemorized (down)

- Now all of your memorized cards are faceup on the right or faceup on the left. In other words, the memorized cards are faceup everywhere. And by the same logic, the unmemorized cards are facedown everywhere. So, when you separate the faceup and facedown cards, everyone gets the cards they started with. The rest is just acting!

REFERENCES

Diaconis and Graham, *Magical Mathematics*.

Gardner, *Mathematics, Magic and Mystery*.

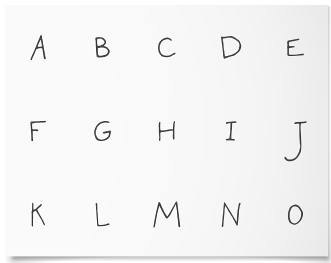
Mulcahy, *Mathematical Card Magic*.

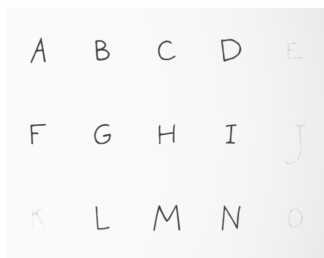
EXERCISES

1 Consider the following "magic joke." Take a half sheet of paper and write the letters A through O, as shown at right.

Then, erase the letters J-O-K-E.

Next, fold the paper like a map until it's all folded into one giant square. Then, using scissors, cut the edges of the paper until you have 15 separate squares. Some letters will be faceup and some will be facedown. Do the faceup or facedown letters form a word?





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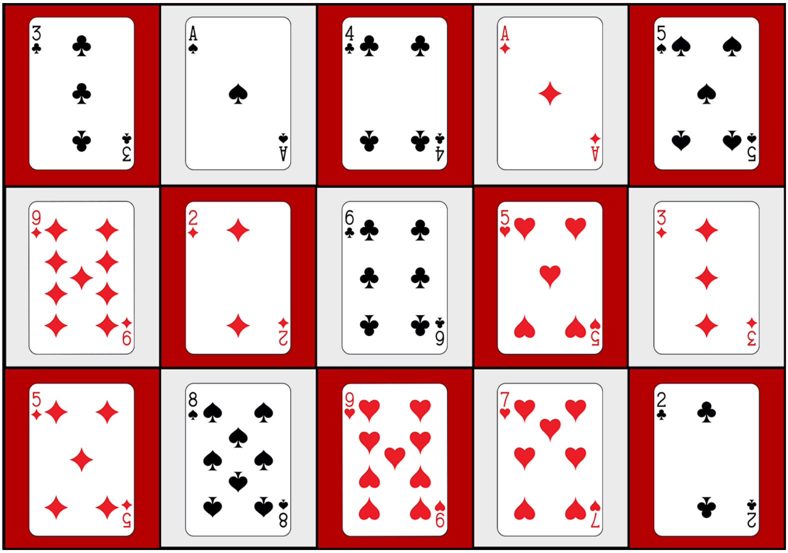
2 This time, let's use number cards. Deal the following 15 number cards (inspired by the digits of pi) on a table.



Next, fold the cards in any order you wish, folding the left, right, top, and bottom edges until you are left with a single pile of cards, some of which are faceup and some of which are facedown. What will be the sum of the faceup cards?

SOLUTIONS CAN BE FOUND ON PAGE 127.

LOOK LIKE A CARD SHARK

People assume that magicians must be skillful card players, yet we all have the ability to perform card miracles—if we allow our inner magic to flow. In this lesson, you will learn some magic tricks that are designed to make you look like a professional gambler or card shark.

EXPLANATIONS

SPECTATOR CUTS TO THE ACES

- This is a fun and easy trick to perform. You start with the aces on the top of the deck. If you want, you can even give the deck a riffle shuffle that keeps the 4 aces on the top.

- After your spectator has cut into 4 piles, make sure you remember which pile has the 4 aces on top. For example, have the person first cut the cards to your right and then cut each pile toward you. This way, the pile of aces is always the pile closest to you on the right. (You'll turn those aces faceup so that you can watch what happens to them.)

- With the first pile on your left, you deal 3 cards down and then 3 across the tops of the other piles. This adds 1 card to the pile with the aces.

- Then, you let the spectator do the same with the other 3 piles, ending with the pile on your right. Now, when your spectator gets to the last pile, he or she deals 3 cards down and then 3 cards across and end up with an ace on each pile! It's a very easy trick to do, but it always gets a big reaction.

ALWAYS WIN AT POKER

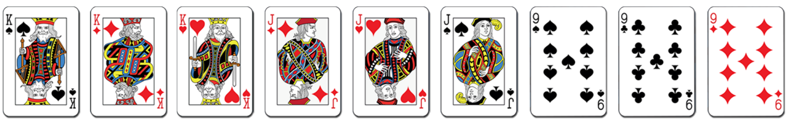
- The secret to learning how to never lose in poker is really quite clever. In the video, the magician only played with the 9s through aces, but what he didn't tell you was that only 3 of each of these cards were used—9s, 10s, jacks, queens, kings, and aces—for a total of 18 cards.

- Originally, the packet that the magician shuffled had 9 cards, consisting of all of the 10s, queens, and aces. Let's call that group A, as in "aces."

- The 9 cards that the volunteer shuffled—let's call it group B—consisted of the 9s, jacks, and kings.

GROUP A

GROUP B



- Now here is what makes it all work. Magicians call it the **JONAH PRINCIPLE**. Suppose you play with 10 cards, consisting of all 9 cards from group B and one extra card from group A—for example, the ace of diamonds. The ace of diamonds would be the Jonah card—the odd card, the stranger card, the card that doesn't belong. The Jonah principle says that when these 10 cards are dealt into two 5-card poker hands, whoever gets the Jonah card is guaranteed to lose.

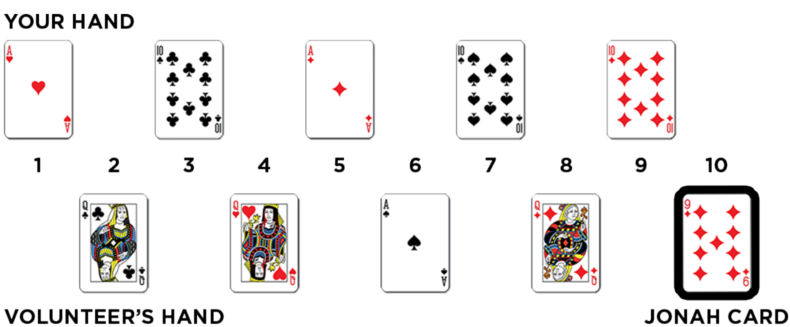
- If you try every possibility, you will discover that

  - if your volunteer has 1 pair, then you have 2 pairs;

- if your volunteer has 2 pairs, then you have 3 of a kind; and
- if your volunteer has 3 of a kind, then you have a full house.

Straights and flushes and 4 of a kinds are impossible in this game.

- The upshot is that if your volunteer ends up with the Jonah card, then he or she is guaranteed to lose the hand. The challenge for you, as a magician, is to give your volunteer the Jonah card without him or her realizing it. Here's how to do it. It's pretty sneaky.
- You begin by shuffling the 9 cards of group A while your volunteer shuffles the 9 cards of group B. Then, you let him or her shuffle the cards of group A, and you place one pile on top of the other.
- It doesn't matter which pile goes on top, so let's say that pile B is on the top. You now give your volunteer the cards and ask him or her to deal 10 cards. As is the custom when dealing, he or she deals the cards one at a time, back and forth, starting with you.
- This means that you will get cards 1, 3, 5, 7, and 9 and your volunteer will get cards 2, 4, 6, 8, and 10. But the first 9 cards are all from group B, so the volunteer will get card 10, which is the lone card from group A, and that will be the Jonah card, so you are guaranteed to win the hand.



- Conveniently, when you flip over your volunteer's cards, the Jonah card will be at the bottom of his or her pile. You place your cards faceup on top of your volunteer's, and when you turn that pile facedown, the Jonah card will be on top.
- So, these cards have a card from group A on top, followed by 9 cards from group B. When your volunteer shuffles the remaining 8 cards, which are all from group A, and then places them on top, then the first 9 cards will be from group A and the 10th card will be from group B, so when your volunteer deals the cards back and forth, you will win again.
- Once again, you place your cards faceup on top of your volunteer's and turn the pile over, so the top card is from group B and the rest are from group A. When he or she shuffles the remaining 8 cards and puts them on top, you are back to the setup with 9 cards from group B and then 9 cards from group A.
- In the last phase of the trick, you tell your volunteer that you will deal the cards. Then, ask him or her, "Do you want us to use the top 10 cards or the bottom 10 cards?" If your volunteer wants the bottom half, then you quickly remove the top 8 cards and you will have 10 cards left over, with the Jonah card on top.

At no point in the trick do you tell the audience that you have exactly 18 cards. Many people will assume that you are using 20 or 24 cards, which is fine. That will make it much harder to figure out later.

- If instead your volunteer asks for the top 10 cards, then you deal the cards 1 at a time on the table, counting 1 through 10, putting the other cards aside, and again the Jonah card will be on top.
- Either way, the Jonah card is on top. So, when you deal the 2 cards facedown, starting with your volunteer, he or she is guaranteed to get the Jonah card, so you are guaranteed to win the hand, no matter what choices your volunteer makes afterward.

5-CARD STUD

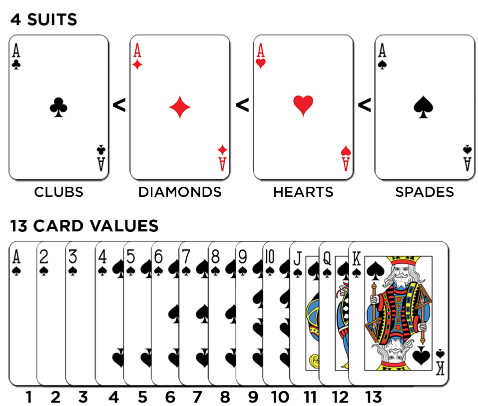
This trick was invented by mathematician Fitch Cheney and then published in a booklet for magicians called Math Miracles *by Wallace Lee.*

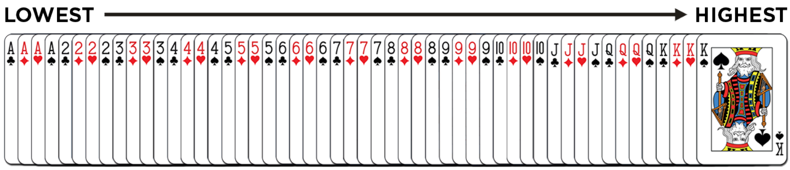
- This trick, known as Cheney's 5-card trick, is a perfect blend of mathematics and magic. It involves teaching your assistant a special code so that when he or she looks at 4 faceup cards, he or she can figure out the facedown card.

- Keep in mind that when your volunteer gives you the cards, you choose which card goes facedown and how to arrange the 4 other cards. No matter what 5 cards he or she gives you, there will always be a way to arrange them that will tell your assistant exactly what the facedown card is.

- Before you learn how to do this trick, remember that there are 13 card values, from ace to king. Let's consider the ace to be the lowest card, with a value of 1, and the king to be the highest card, with a value of 13. There is also an ordering to the suits, which you don't usually need, but here it is: ♣ ♦ ♥ ♠

- This is the ordering for playing bridge, but it also happens to be alphabetical order. This would make the lowest card the ace of clubs, followed by the ace of diamonds, and so on—all the way up to the king of spades.





- Because your volunteer gives you 5 cards and there are just 4 suits in the deck, then at least one of the suits has to be repeated. For example, let's say your volunteer gives you these 5 cards:

3♥ Q♠ 5♣ 8♦ K♠

- You have 2 spades, so you will turn one of the spade cards facedown. But which one goes facedown? If the card values are within 6 of each other, then the higher card goes facedown. Otherwise, the lower card goes facedown.

- In this case, because the queen of spades and the king of spades are just 1 apart, then the higher card—the king of spades—will go facedown, and the queen of spades will go next to it as the fourth card in your arrangement. But if instead of a king of spades you had a 2 of spades, then because the queen of spades and the 2 of spades are 10 apart, the 2 of spades would go facedown and the queen of spades would go next to it. The card that goes next to the facedown card—in this case, the queen of spades—is called the **ANCHOR CARD**.

- Next, you arrange the other 3 cards in such a way that will tell your assistant how much to add to the anchor card to get the facedown card.

- Here are the details. Let's call the other 3 cards **L**, **M**, and **H**, where **L** denotes the lowest-valued card, **M** is the middle-valued card, and **H** is the high card.

- With these 3 cards—the 3 of hearts, the 5 of clubs, and the 8 of diamonds—the low card is the 3 of hearts, the middle card is the 5 of clubs, and the high card is the 8 of diamonds.

- How many ways can these 3 cards be arranged?

3♥ 5♣ 8♦

- They can be arranged in exactly 6 ways:

LMH	MLH	HLM
LHM	MHL	HML

- And each arrangement tells us what to add to the anchor card:

LMH = ADD 1	MLH = ADD 3	HLM = ADD 5
LHM = ADD 2	MHL = ADD 4	HML = ADD 6

- The way to remember this is if the first card is low, add a low number, 1 or 2; if the first card is middle, add a medium number, 3 or 4; and if the first card is high, add a high number, 5 or 6. To determine which of the 2 numbers to add, you look at the next 2 cards and compare them.

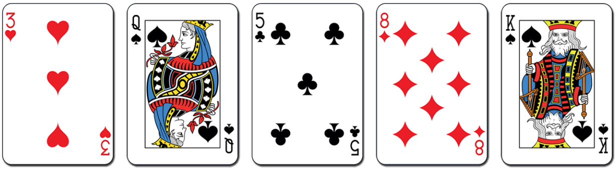
- One is bigger and the other is smaller. If the card in position 2 is smaller than the card in position 3, then you add the smaller number; if the card in position 2 is bigger than the card in position 3, then you add the bigger number.

- For example, suppose you want to arrange these 3 cards—3 of hearts, 5 of clubs, and 8 of diamonds—to send the signal to add the number 5? How would you do that?

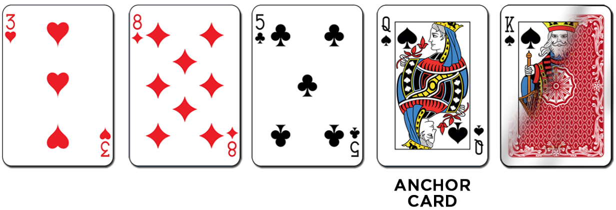
- Because 5 is a high number, you start with the high card—namely, the 8 of diamonds. That says that you are adding either 5 or 6. Because you want to add 5, which is smaller than 6, you follow it up with the smaller card—namely, the 3 of hearts.

- How would you arrange the cards to add 4? That's a middle number, so you start with the middle card, which is the 5 of clubs. That signals that you are adding 3 or 4. Because you want 4, your next card is the bigger one—namely, the 8 of diamonds.

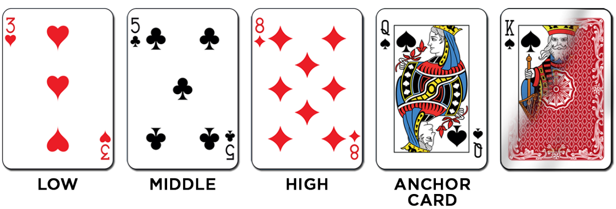
♦ Let's do a full example with your original 5 cards:



♦ You have 2 spades—the queen and king of spades—which are 1 apart, so you turn the king facedown and put the queen next to it:

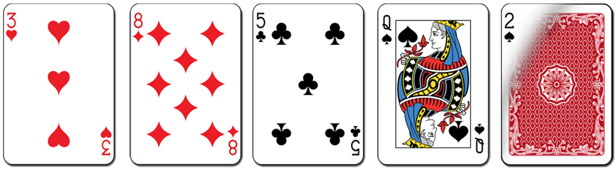


♦ How do you arrange the other 3 cards? Because you want to add 1 to the queen of spades, you arrange the cards in **LMH** order:

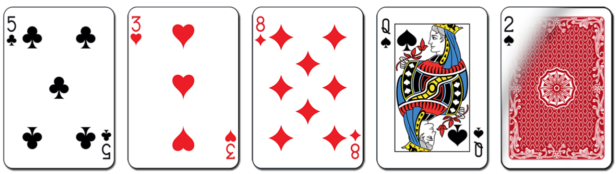


♦ So, your assistant would look at the 358, see **LMH**, and then add 1 to the queen of spades to get the king of spades!

- But now suppose instead of having the king of spades, you have the 2 of spades. Because the 2 and queen are 10 apart, you put the lower card facedown, which is the 2, and put the queen next to it.



- How do you arrange the other 3 cards? What do you need to add to the queen to get 2? (king, ace, 2!) You need to add 3, so you need to arrange the cards in the order of **MLH**.



- So, when your assistant sees these 4 cards, he or she knows that the suit is a spade. From the first 3 cards (**MLH**), she knows to add 3, making the facedown card the 2 of spades!

- If you just perform this trick once or twice, nobody will see the pattern of the cards, and they might even suspect that you are sending a signal to your assistant through words or physical gestures. But you can perform this trick silently or over the phone if you'd like.

- If you do this trick more than a few times, people might start to see that the last 2 cards always have the same suit. You can mix it up a little bit, such as changing the location of the anchor card. For example, you can tell your assistant that on the third and final time, the anchor card will be first instead of last.

- If you plan to do this trick many times, then maybe the first time the anchor card could be in the fourth position. The next time, it could be in the third position; then, it could be in the second position, and so on. If you do it this way, you can keep your audience puzzled for a very long time.

WEIGHING THE CARDS

- This trick involves using what magicians call a **KEY CARD**, which is marked with a light pencil dot in the upper-right and lower-left corners. For the setup, the key card is the 26^{th} card from the top. To make it easier for you to see in the video, the magician uses a card with a different-colored back.

If you don't want to mark one of your cards on the front or back, you can replace one of your cards with a marked joker and make that your key card.

- You might be wondering how marking a card should matter to you if you don't even look at the cards. For this trick, you also put some clear tape on the front of the key card so that you can feel that card when you get to it.

- Suppose you cut off less than half of the cards. You don't know how many cards are in the pile—yet. You'll hold this smaller pile of cards in your right hand and then thumb off cards from the larger pile with your left hand, mentally counting the cards that you drop until you drop the key card.

- How many cards are in your left hand? There were 26 cards below the key card, which means that the number of cards in your right hand plus the dropped cards is also 26. And because you have dropped x cards onto the table, you know that the pile in your right hand has exactly $26 - x$ cards in it.

- You could say the number now, but to make the trick more convincing (and to keep the key card from showing), you count another x cards onto the pile. Because your left hand also has 26 cards, then when you count x more cards, it will also have $26 - x$ cards. Now you can count the piles together and they will have the same number of cards.

- In general, when you take the larger pile, if the key card is the n^{th} card that you drop, then the smaller pile has $26 - n$ cards in it. If you drop n more cards on top of the key card, then both piles will match.

- When you are done with this trick, the key card has n cards on top of it, so it is in position $n + 1$. If you take the top of these cards and scoop all of the cards below it, the key card will be in position n, so when you put it underneath either of the counted piles, it will be back in the 26^{th} position and you're ready to perform the trick again.

REFERENCES

Fulves, *Self-Working Card Tricks.*

Mulcahy, *Mathematical Card Magic.*

Scarne, *Scarne on Card Tricks.*

EXERCISES

1 Consider the following card arrangements in Cheney's 5-card trick. In each example, the anchor card will be the fourth card. Determine the facedown card.

a)	5♥	9♠	10♠	6♦	
b)	8♥	J♦	3♠	8♠	
c)	K♦	4♥	A♠	J♦	
d)	6♦	9♠	9♥	4♣	

2 Naturally, if the anchor card is always the fourth card, then people will discover part of the secret pretty quickly. To make it harder to figure out, it's a good idea to rotate the anchor card. One particularly subtle way to do so is to let the location of the anchor card be determined by the sum of the 4 faceup values. Specifically, let's let picture cards count as 0 and the number cards (A through 10) count as 1 through 10. If the total is a multiple of 4, then the anchor card goes in the fourth position, as normal. If the sum is 1, 2, or 3 larger than a multiple of 4, then the anchor card is in the first, second, or third position, respectively.

For example, with cards 2♣, 3♣, 5♦, 8♠, and K♥, you know that the facedown card will be 3♣. Adding the other values, you get 2 + 5 + 8 + 0 = 15, which is 3 bigger than a multiple of 4. Hence, the anchor card will go in the third position, and you would order the cards as follows:

5♦ 8♠ 2♣ K♥

Your assistant would add the values (5 + 8 + 2 + 0 = 15) and determine that the anchor card is 2♣. The other 3 cards are in **LMH** order, which signals to add 1 to 2♣, for 3♣.

Use the above system to determine the facedown card for the following arrangements:

a) 2♠ 7♥ A♣ 8♠

b) K♠ Q♥ 9♣ 4♠

SOLUTIONS CAN BE FOUND ON PAGE 128.

THE DECK IS STACKED

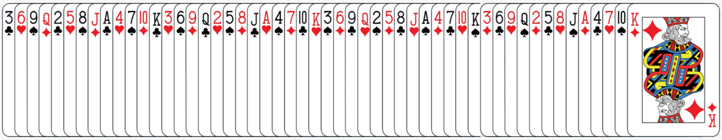
In the previous lesson, you learned a way to determine the number of cards in a deck by using a key card. In this lesson, you will learn another way to accomplish the same effect using an entirely different method that you might find superior. You will learn a very useful and mathematically interesting way to stack a deck of cards as well as a way to give a deck a false cut and a false shuffle.

EXPLANATIONS

SI STEBBINS STACK

- The secret to the "pick a card" trick performed in the lesson lies in how the cards are arranged beforehand; magicians say that a deck like this is **STACKED**. The name of this stack is called Si Stebbins, named after a magician who popularized it in the early 20^{th} century, yet it was written about in the 16^{th} century in one of the first books on magic ever published.
- In the video, the magician briefly showed you the cards as he looked for jokers. That was his way of showing the cards to you so that you saw that they were different and in no apparent order, without drawing too much attention to the cards.

- But if you look closely at the cards, you will see that there is a pattern to them. First, the colors alternate from black to red.



- In addition, the suits are arranged in a very specific order: clubs, hearts, spades, diamonds. We say that the suits are in CHaSeD (clubs, hearts, spades, diamonds) order.

- Next, the values of the cards start at 3, 6, 9, and 12, where 12 is represented by the queen, and they continue to grow by 3 as you go. If you add 3 to queen, you get 2 (going past king and then ace), so the next card is 2, followed by 5, 8, 11 (jack), 14 (ace), 4, 7, 10, and 13 (king)—and the pattern repeats.

PICK A CARD

- Let's use the Si Stebbins pattern to figure out this trick, where the magician had a volunteer pick a card and then seemingly read her thoughts. The volunteer pointed to a card and then looked at it, showed it to others, and put it back in its original location.

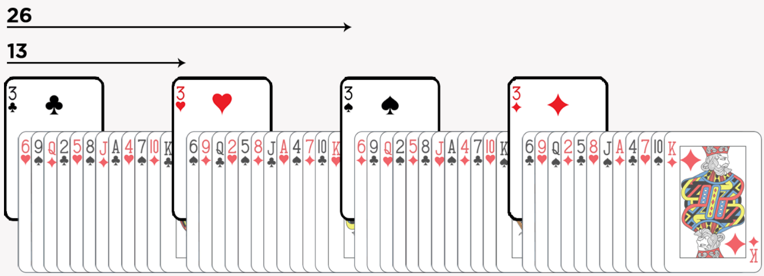
- If somehow the magician could see the card above the chosen card, then he could figure out the chosen card using the pattern. In this case, the card above the chosen card is 9♣, so what would the chosen card, the one that follows, have to be? The suit would have to be hearts and the value would have to be 12; therefore, the chosen card would be Q♥.

- How does the magician spot the bottom card? After the volunteer takes her card, while she shows her card to others, the magician squares up the cards above and below her card. Then, the magician tells the volunteer to put the card back in the deck and points to the cards below her selection. In that brief moment, as he points downward in a very natural gesture, the card above her card is staring the magician in the face.

- Once the magician see that the card is the 9♣, he simply adds 3 to the value and shifts the suit, and he knows your card. Then, he can reveal the card in any dramatic fashion that he chooses.

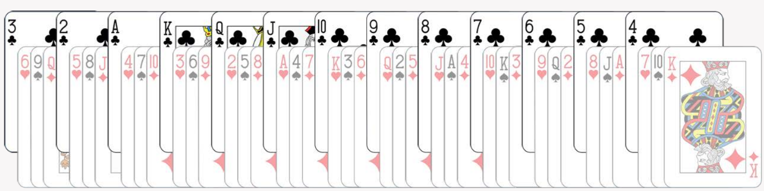
OTHER SI STEBBINS PATTERNS

- Here's another useful Si Stebbins pattern. Notice that the card values repeat every 13 cards. And those repeated values are in CHaSeD order, too. For example, if you start with the 3 of clubs, then 13 cards later, you have the 3 of hearts, then the 3 of spades, and then the 3 of diamonds.



- An even simpler pattern is that the black 3s are 26 apart, the red 6s are 26 apart, and so on. For every card in the deck, you can find its **MATE** (same color and value) 26 cards later.

- Here's another pattern. You know that every 4 cards, the suit stays the same. But what's also true is that every 4 cards, the value goes down by 1.



- For example, if you take any card—for example, a 6—then 4 cards later you have a 5, and then 4 cards later you have a 4, and so on. The reason for this is because every 4 cards is like adding 12. Think of it this way: If you look at the clock on your wall and it says 7, then 12 hours from now it will still say 7. Thus, if you add 11 hours to it, it will say 6. But if you think of your card values on a clock with 13 hours, then adding 12 is the same as going back 1. So, for example, adding 12 to the 3 of clubs, you would get the 2 of clubs.

NO WEIGH

- The magician exploited Si Stebbins patterns when he pretended to weigh the deck of cards in the beginning of the lesson. A volunteer holds the cards faceup in her hand. It's easiest when the faceup card is a king, so let's do that example first. Then, the volunteer cuts any number of cards from the magician's pile.

- How many cards are left? Look at the value of the card that was cut to. Suppose you see a value of v. Based on the earlier pattern, you can exploit the following handy formula:

$$n = (13 - v) \times 4$$

- In this case, the value of the card that was cut to is $v = 2$. When you plug this into the formula, you get

$$n = (13 - 2) \times 4 = 11 \times 4 = 44.$$

- This tells you that there is a card with value of 2 in the 44^{th} position. But does the magician have 44 cards in his hand? Definitely not. But you know that the card values repeat every 13 cards, so you can subtract 13 from 44 to get 31. Does he have 31 cards now? No, he definitely has less than half of the deck. So, how about $31 - 13 = 18$? That feels more like it. If you subtract 13 again, you'd get 5, and the magician definitely has more than 5 cards in his hand. So, you can say with certainty that the 2 of hearts is the 18^{th} card.

- You can verify this by counting 18 cards, starting with the 3 of clubs. Hold the deck facedown and flip each card faceup while counting, and put either pile on top. This way, the cards are still in Si Stebbins order and the trick can be done again.

- Let's do the trick again with a card that is not a king on the bottom—with a value of b (as in "bottom").

- When you look at the faceup card, you subtract it from the bottom card and then multiply it by 4. In other words, you calculate

  $$n = (b - v) \times 4,$$

  where b is the bottom card and v is the card that is cut to.

It's okay if you need some time to do the calculation. You have plenty of time to do it while you are pretending to weigh the cards.

- That tells you the number of cards remaining in your hand, plus or minus a multiple of 13. So, in this case, because the magician has a 9 on the bottom and, after cutting off a large number of cards, has $v = 3$ on top, when you subtract 3 from 9 and multiply by 4, you get 24.

- What if v has a value that is larger than the bottom card?

- Let's say the deck is cut to the jack of diamonds.

  $$(b - v) = 9 - 11 = -2$$

- Before you multiply by 4, you add 13 to it to get 11.

- Then, you multiply by 4 to get 44. So, there is a jack in position 44, and that seems to be what the magician has. (He certainly has more than 31 cards). You can check this by seeing that there are 8 cards on the table, so there must be 44 cards remaining.

- When v is bigger than b, the general formula is:

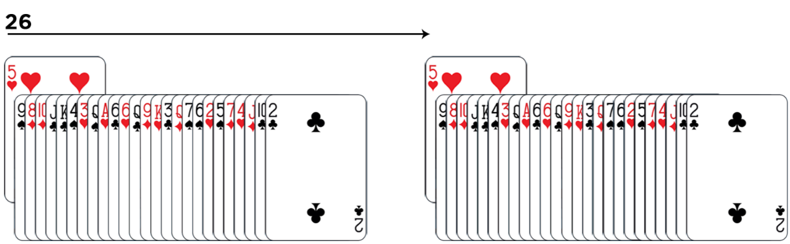
  $$n = (13 + b - v) \times 4.$$

ARRANGING CARDS INTO SI STEBBINS ORDER

- Separate your cards into 4 piles, one for each suit.
- Arrange each pile from ace to king, with a king at the top of the pile.
- Put the piles in CHaSeD (clubs, hearts, spades, diamonds) order.
- Cut the piles so that the club pile starts with the 3 of clubs, the hearts pile starts with the 6 of hearts, the spades pile starts with the 9 of spades, and the diamonds pile starts with the queen of diamonds.
- Now you simply deal the cards one at a time faceup into a pile.
- Notice that every 4 cards, your suit is repeated and that the value of each suit goes down by 1. When you're done, the cards will be in Si Stebbins order.

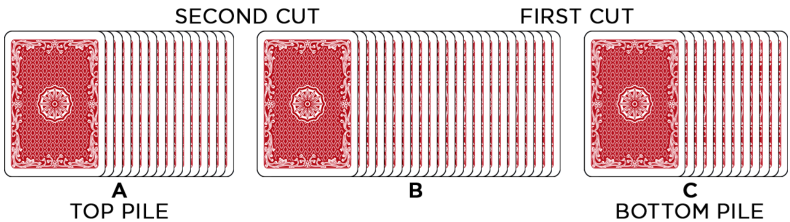
THE NAME OF YOUR CARD IS…

- For this trick, although it may appear that the magician has 52 different cards, he actually has the same 26 cards repeated twice. The first set of 26 cards can be in any order. The second set of 26 cards then matches the first set of 26 exactly. The key point here is that for every card, its twin is located exactly 26 cards later.

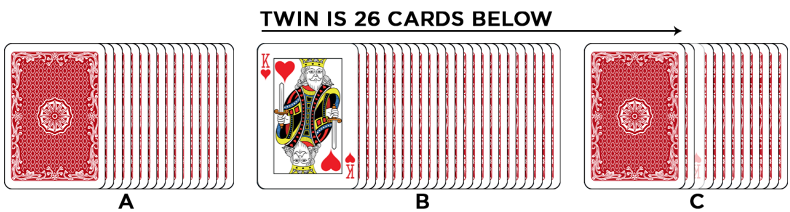


- Let's suppose that the deck initially had this order, where **A** denotes the top pile and **C** denotes the bottom pile.

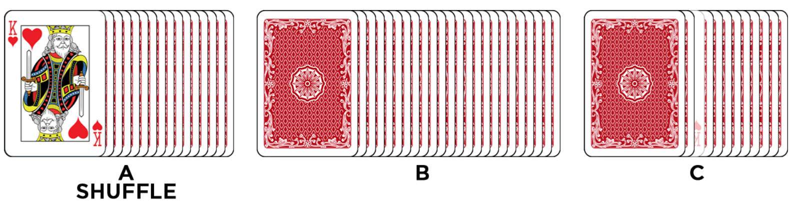
- The volunteer's first cut leaves less than half of the deck in pile **C**. After the second cut, the remaining cards are divided into piles **B** and **A**, with with pile **A** containing less than half the remaining cards. The deck will look something like this:



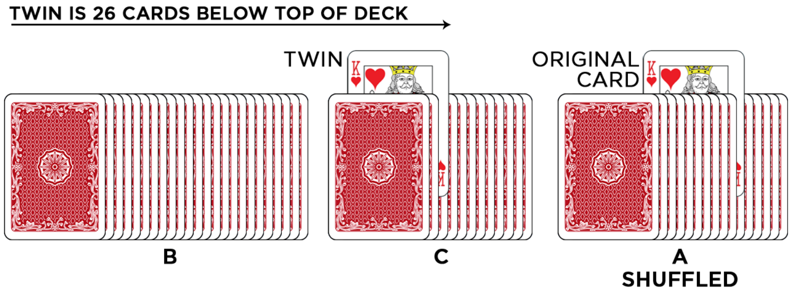
- After the cards are cut into 3 piles, you take the card that is on top of **B**. Where is **B**'s twin located? It's 26 cards below **B**, so it is somewhere in pile **B** or **C**. As long as pile **A** doesn't have 26 or more cards, the twin must be in pile **B** or **C**.



- You put the selected card on top of pile **A** and shuffle it, but it doesn't matter, because that's not the card you're going to find!



- You now put **C** on top of **A** and then **B** on top of **C**, resulting in an order of **BCA**. Now all you have to do is count 26 cards and your card's twin is guaranteed to be there.



FALSE SHUFFLE

- You can do a **CHARLIER FALSE SHUFFLE** to make it look like your cards are really being mixed, but in fact, it's just an elaborate way to cut the cards in such a way that it looks like a shuffle.
    - Hold the cards in your left hand and thumb a few cards into your right hand.
    - Push off a few cards from the bottom of the left pile onto the top of the right pile.
    - Thumb a few cards from the top of the left onto the bottom of the right pile.
    - Repeat this process.
- Notice that the cards are still in the same relative order as they started.
- If you practice this move, you can eventually do it without even looking at your hands, which makes this false shuffle very convincing.

FALSE CUT

- With a **SWIVEL CUT**, you hold the cards in your right hand. Then, with your left index finger, you push the top half of the cards against your right fingers and push slightly up so that the cards swivel around your right fingers and you catch the cards in your left hand. Then, place the remaining cards in your right hand on top of those cards.
- With a **FALSE SWIVEL CUT**, instead of the cards in your right hand going on top of the cards on your left hand, you put them on the table and then take the cards from your left hand and put them on top.
- What makes a false swivel cut so deceptive is that it's the combination of 2 legitimate cuts: the swivel cut and the normal cut. But when you combine these 2 true cuts, you get a false cut.

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Scarne, *Scarne on Card Tricks.*

Simon, *Mathematical Magic.*

Trost, *The Card Magic of Nick Trost.*

EXERCISES

1. Suppose a deck in Si Stebbins order is cut with J♠ on the bottom of the deck. What is the 20th card in the deck? What is the 23rd card in the deck?

2. Explain the following magic trick. The magician invites you to cut the deck and complete the cut. You then remove the top card and put it in your pocket. The magician then asks the deck for the color of the selected card by spelling C-O-L-O-R and looking at the next card,

which happens to be red and thus concludes that the chosen card is red. The magician then asks the deck for the number of the selected card by spelling N-U-M-B-E-R and looking at the next card, which happens to be a black queen and thus concludes that the chosen card is also a queen. But which red queen? The magician then spells M-A-T-C-H-I-N-G C-A-R-D, and the next card is the queen of diamonds and therefore concludes that the selected card is the queen of hearts, which proves to be correct. How did the trick work?

SOLUTIONS CAN BE FOUND ON PAGE 128.

PERFECT SHUFFLES

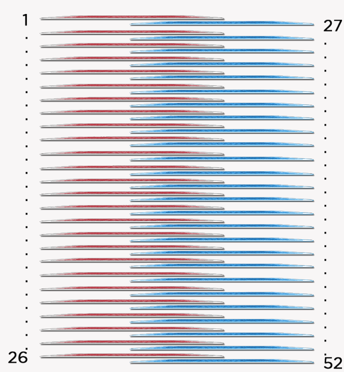
In this lesson, you will discover various ways to mix up the cards in a deck that are not the same as cutting the cards. But the mixing isn't quite as random as you may think. The focus of the lesson is on the perfect shuffle and some of its mathemagical properties.

PERFECT SHUFFLES

- When you do a **RIFFLE SHUFFLE**, you are cutting the cards approximately in half and then merging those halves together so that the cards interlace together in approximately alternating fashion.

- In what mathematicians call a **PERFECT SHUFFLE**, the cards are cut exactly in half—a deck of 52 cards is cut into 2 piles of 26—and then the cards are interlaced perfectly. Magicians call this the **FARO SHUFFLE**, named after an old casino game, faro, where the cards were mixed in a not-so-random way.

- Let's use this diagram to describe perfect shuffles mathematically.



- First, the cards are cut into exactly 2 halves. Then, the cards are interweaved perfectly. So, in the new order, card 1 is followed by card 27, then card 2, then card 28, and so on.

- After one perfect shuffle, that top card stays on top. In other words, card 1 is still in position 1, but card 2 is now in position 3, card 3 is in position 5, card 4 is in position 7, and so on down to card 26, which is in position 51.

- The rule is that for the first 26 cards, card n is sent to position $2n - 1$.

- For the next 26 cards, we see that card 27 is in position 2, card 28 is in position 4, card 29 is in position 6, and so own down to card 52 which is still on the bottom in position 52.

- The general formula for these cards is that card n ends up in position $2n - 52$.

- We can keep track of what happens to every card with something called **CYCLE NOTATION**. For the perfect shuffle, your cycles look like this:

(1)	(52)	(18	35)				
(2	3	5	9	17	33	14	27)
(4	7	13	25	49	46	40	28)
(6	11	21	41	30	8	15	29)
(10	19	37	22	43	34	16	31)
(12	23	45	38	24	47	42	32)

- This table says that card 1 stays in position 1 and card 52 stays in position 52. Next, it says that cards 18 and 35 swap positions. You can check that:

  - Card 18 goes to $(2 \times 18 - 1) = 35$.

  - Card 35 goes to $(2 \times 35 - 52) = 18$.

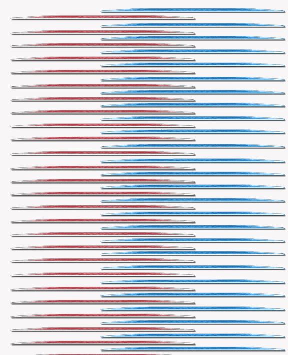
- The next cycle says that card 2 goes to position 3, card 3 goes to position 5, 5 goes to 9, 9 goes to 17, 17 goes to 33, 33 goes to 14, 14 goes to 27, and finally card 27 goes to position 2.

- The other cycles can be read in the same way. Using this table, you can see where every card goes after a perfect shuffle. In fact, you can also see where every card goes after 2 shuffles, just by moving 2 numbers to the right.

- For example, after 2 shuffles, the card in position 17 will be in position 14. The card in position 29 will be in position 11. Where will the card in position 42 be after 3 shuffles? Just go 3 spaces to the right and it will be in position 23.

- Now here's a surprise: Where will each card be after 8 shuffles? Notice that because all of your cycles have length 1, 2, or 8, every card will be back in its original position after 8 perfect shuffles.

- There are actually 2 different kinds of perfect shuffles. Notice that in the shuffle that was just described, after cutting the cards exactly in half, the cards are then interweaved perfectly, with the top card staying on top and the bottom card staying on the bottom. Because the outermost cards—1 and 52—stay on the outside, this is called a perfect out-shuffle. But if instead you let the 27^{th} card go on top, now cards 1 and 52 head into the deck, so this is called an in-shuffle. When you do an in-shuffle, the cards get rearranged as shown here.



- With an in-shuffle, cards 1 through 26 end up in positions 2, 4, 6, 8, … 52. In other words, card n ends up in position $2n$. Cards 27 through 52 end up in the odd positions (1, 3, 5, …), with the formula that card n is sent to position $2n - 53$.

- The cycle notation for the in-shuffle is not as nice as the out-shuffle. The in-shuffle has just one giant cycle that contains all 52 numbers:

(1	2	4	8	16	32	11	22	44	35	17	34	15
30	7	14	28	3	6	12	24	48	43	33	13	16
52	51	49	45	37	21	42	31	9	18	36	19	38
23	46	39	25	50	47	41	29	5	10	20	40	27)

- When you combine in-shuffles with out-shuffles, they create a magic all their own.

- Suppose you want to bring the top card to a certain position of the deck—for example, position 40. What is the fastest way to do it using perfect shuffles?

- The answer is pretty amazing. First, you subtract 1 from the position:

  $40 - 1 = 39.$

- Then, you write that number as a sum of powers of 2. In this case, 39 is expressed as the sum of powers of 2. The relevant powers of 2 are these:

  32 16 8 **4** **2** **1**

- Notice that

  $39 = 32 + 4 + 2 + 1.$

- Now you use that to write the number in **BINARY NOTATION**, where you have a 1 for every power of 2 that is used and a 0 for every power of 2 that is not used. In this case, 39 has this binary representation:

  1 0 0 1 1 1

- Each 1 represents an "1"n-shuffle and each 0 represents an "0"ut-shuffle. This tells us that to bring the top card to position 40, you perform the following:

1	0	0	1	1	1
n	u	u	n	n	n
	t	t			

You can find the reason why this method works in S. Brent Morris's book or from the Great Course Discrete Mathematics.

- Let's check to see that this will work. Remember that for cards in the top half of the deck, in-shuffles double the position, so card n goes to position $2n$; out-shuffles send card n to position $2n - 1$.

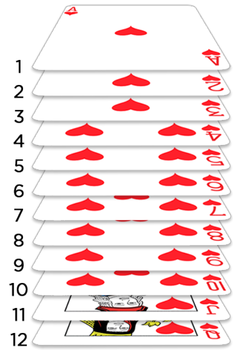
- After the first in-shuffle, card 1 goes to position 2. After an out-shuffle, card 2 goes to position 3; the next out-shuffle sends it to position 5. And the next 3 in-shuffles each doubles the location, so it goes to position 10, then 20, and then 40—as promised!

EXPLANATIONS

LEFT OR RIGHT?

- It takes a lot of practice to be able to perform a perfect shuffle. There are tricks that use other kinds of shuffles that require no skill to perform but, like the perfect shuffle, don't mix the cards as much as they appear.

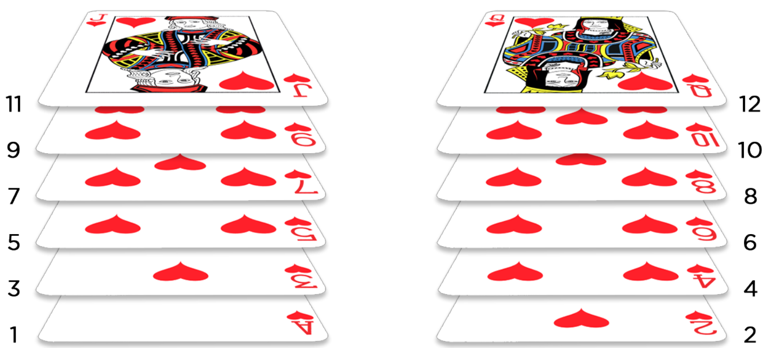
- In this trick, when dealing the cards into 2 piles, the magician is doing a different sort of shuffle, and depending on which pile goes on top, it is called either a left shuffle or a right shuffle.



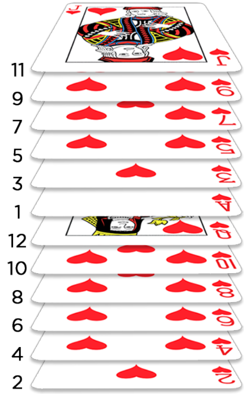
- Suppose that the cards are initially in the order shown at right.

- Notice that the top and bottom numbers add to 13 (1 + 12) and so do the next cards in (2 + 11), and the next in (3 + 10), and so on. This order is said to have **MIRROR SYMMETRY**.

- Now look what happens when the cards are dealt into 2 piles.



- Notice that cards 1 and 2 are now on the bottom of their piles. If the magician places the left pile on top, it is a left shuffle, and the new ordering looks like the image at right.



- Notice that the new order still has mirror symmetry. The top and bottom cards still add to 13 (11 + 2), as do all the other pairs: 9 + 4, 7 + 6, and so on.

- There is also mirror symmetry with a right shuffle. The top and bottom pairs, 12 and 1, still add to 13, as do the other pairs: 10 + 3, 8 + 5, and so on.

- The moral is that if the cards begin with mirror symmetry, then every time you do a left or right shuffle, they will still have mirror symmetry. Magicians call this the **STAY-STACK PRINCIPLE**.

- How does this explain the magic trick? At the beginning, you can have the cards in order—ace, 2, 3, through queen of hearts—but you can also give the cards a few left shuffles beforehand so that the order of the cards looks more random.

- After you've given the cards a few left (or right) shuffles, have a volunteer look at the top card and show it to others. When you pick up the cards and ask the volunteer to place his or her card in the middle of the pack, you look at the bottom card. And once you see the bottom card, you know what the volunteer's card—the top card—must have been, because their total is 13.

- For example, if you glance at the bottom card and see a 9 of hearts, then you know that the volunteer's card must have been the 4 of hearts. The rest is just acting.

The mirror symmetry property does not depend on having 12 cards; it could be any even number of cards. And this property is also satisfied when perfect shuffles are done with an even number of cards, regardless of whether they're out-shuffles or in-shuffles.

MATHEMATICIANS VS. MAGICIANS

- To do this trick, you ask a volunteer to think of a number between 1 and 8. To make things easier to understand, let's use the ace through 8 of hearts, in that order, although you can use any 8 cards.

- Once the volunteer has thought of his or her number, and the corresponding card, you show the volunteer half the cards—specifically, the cards that are in positions 1, 3, 5, and 7. You then ask the volunteer if he or she sees his or her card among these cards.

This trick was invented by Alex Elmsley, who was a mathematician, magician, and pioneering computer scientist.

- If the volunteer says yes, then you put these cards on the top, which means that his or her card is now in position 1, 2, 3, or 4. If the volunteer says no, then you put these cards on the bottom, which *also* means that his or her card must now be in position 1, 2, 3, or 4. After 3 rounds of questions, the volunteer's card is guaranteed to be on top. That explains the first half of the trick.

- What about the second half of the trick? How do you know what card the volunteer is thinking of the second time around? Suppose that your volunteer thought of the number 7. The card that is now in the seventh position is the ace of hearts, which was the original top card of the deck. It turns out that this is not a coincidence.

- After going through the 3-question procedure, not only will the volunteer's original card be at the top of the pack, but his or her card will have traded places with the original top card of the deck.

- In general, when the volunteer thinks of a number between 1 and 8, then the top card will always end up in the corresponding position. So, when you perform this trick, give the volunteer 8 cards to shuffle and then when you show everyone that the cards are well mixed, just remember the top card and the trick works by itself.

- ♦ Naturally, the trick will work with any 8 cards, but perhaps the hardest part of the trick is remembering the top card for the entire trick. But if you perform this trick after the last one and just use 8 hearts, then you know what the suit will be, so you only need to remember the card's value.

This trick will also work with 16 cards; you just have to ask 4 questions instead of 3. But using 8 cards makes the trick goes much faster, and there's less of a chance for you or the volunteer to make a mistake.

WHAT ARE THE ODDS?

- ♦ This trick always gets a great reaction, and it's entirely self-working. You set up the deck so that the first, second, fourth, and eighth cards are aces, and your prediction card—for example, the 3 of clubs—is the 16$^{\text{th}}$ card. If you want extra security, you can put a joker in the 32$^{\text{nd}}$ position. Give the deck a false cut and then ask a volunteer to cut approximately half of the cards.

- ♦ You would think that the final card would depend on where the cards were cut, but it essentially doesn't matter as long as the volunteer cuts at least 16 cards. Once he or she has cut the cards, put the bottom half of the cards aside and then out-jog every other card.

- ♦ For example, if the volunteer cuts 25 cards, then you are removing cards 1, 3, 5, 7, … , 25, so the first ace will be on top of this discard pile. The remaining cards are the cards originally in positions 2, 4, 6, 8, … , 24, so now when you out-jog every other card, you are removing cards 2, 6, 10, 14, … , so card 2 (another ace) is on top.

- ♦ The remaining cards are the cards originally in positions 4, 8, 12, 16, … . After out-jogging again, you remove cards 4, 12, 20, … , giving you the third ace and leaving cards that were in positions 8, 16, and possibly 24 or 32.

- Now if the volunteer cuts 31 cards or fewer (which will usually be the case because you asked him or her to cut about half of the cards), the last pile will have the eighth card on top, another ace, and the only card remaining will be the 16th card, which was the card you predicted.
- But what if the volunteer actually cuts 32 cards or more? Then you say, "That's strange! There was only supposed to be one card left over, the 3 of clubs," which you reveal. Then you say, "Oh, you know what went wrong? It was that darned joker. He's always messing me up!"
- This trick is guaranteed to work as long as the volunteer cuts between 16 and 47 cards, which is practically guaranteed. The main thing to be careful about is that during your first deal, you carefully out-jog every other card, and be sure to grab all of the out-jogged cards, or else you might not get the 16th card at the end. With just a little practice, you'll get it right all the time.

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Craven and Gordon, *The Second 16th Card Book*.

Kaufman, *Paul Gertner's Steel and Silver.*

Minch, *The Collected Works of Alex Elmsley.*

Morris, *Magic Tricks, Card Shuffling and Dynamic Computer Memories.*

EXERCISES

1 Give a sequence of perfect shuffles (in-shuffles and out-shuffles) that will bring the top card to position 29. How about position 42?

2 After a deck is given 2 in-shuffles followed by 4 out-shuffles, where does the top card go? Where does the 17th card go?

SOLUTIONS CAN BE FOUND ON PAGE 129.

RIFFLE SHUFFLES

In the previous lesson, you learned that there are ways of mixing cards, such as with perfect shuffles, that are not quite as random as they appear. This lesson will explore what happens when you do an ordinary riffle shuffle, where the cards are cut approximately in half (but not necessarily exactly in half) and the cards interlace approximately (but don't have to interlace perfectly).

EXPLANATIONS

SUIT SENSING

- The secret to this trick, and many others presented in this lesson, is known as the **GILBREATH PRINCIPLE**, discovered by amateur magician and professional engineer Norman Gilbreath in the 1960s.
- Gilbreath's original application was in a trick called Magnetic Colors, where a deck was cut and then riffle shuffled, yet the magician could pull out cards 2 at a time in such a way that they always had opposite colors.
- To begin, the cards are set up so that the colors alternate perfectly—red-black-red-black-red-black all the way through. Once your deck is set up like this, notice that you can cut the cards as often as you'd like and the colors would still alternate.

- Now cut about half the cards and pick up both piles. Before you or your assistant does a riffle shuffle, check to make sure that the bottom cards of the 2 piles have different colors. If they don't, then just move the bottom card from the right pile to the top of the left pile and say, "Here, let's make the piles a little more even."

- Now give the cards a riffle shuffle (or let your volunteer do it) and watch what happens. No matter how good or bad the riffle shuffle is, it will *always* be the case that—starting from the top or the bottom—every 2 cards will contain a red card and a black card. Even when one pile runs out of cards, the other pile will still continue to put cards down in alternating colors.

- When you cut the cards, if both bottoms have the same color, you can still do a riffle shuffle. Then, if you just move the bottom card to the top or the top card to the bottom, or cut the cards any place where 2 matching colors are next to each other, the trick will still work.

- This principle, which has been in the magic literature for about 60 years, has been extended in many interesting ways. In fact, it is often called **GILBREATH'S FIRST PRINCIPLE**. The general principle is even more amazing.

- For the trick where you find one card of each suit—just like for the color-matching trick—you begin with the deck set up in a special way. But instead of just having the colors alternate, you have the suits alternate in a regular pattern—clubs, hearts, spades, diamonds—all the way through the deck. Notice that you can cut the cards as often as you'd like and the suits will still have a cyclic pattern.

- Before you do the riffle shuffle, you need to reverse the cards in one of the piles. Ask your volunteer for a number between 15 and 30, although truly any number will work. (It's just easier to do a riffle shuffle when the piles are close to the same size.) After dealing any number of cards down, the first pile is reversed and the cards can be riffle shuffled.

- Even though the cards are no longer in cyclic order, it is still the case that every group of 4, starting from the top and going all the way to the bottom, still has one card of each suit (in some order). So, as you go through the deck, you just deal 4 cards at a time into piles. Sometimes you can bring 4 cards to the bottom of the pile, or put 4 cards in a pile on the side, but that's just to make the trick look harder.

- In general, the Gilbreath principle says this: Suppose your cards start with any repeated cyclic pattern—whether it be red and black; club, heart, spade, diamond; or even ace to king throughout. Deal any number of cards into a pile and then riffle shuffle the 2 piles together. The Gilbreath principle says that if your original pattern had *n* cards in cyclic order, then after the riffle shuffle, it will still be the case that, starting from the top, every group of *n* cards will contain one card of each type.

- So, for example, if you start with cards ace through king and deal into 1 pile with any number of cards and then give the 2 piles a riffle shuffle, then starting from the top or bottom, every 13 cards will have 1 card of each value—no matter how the cards fall.

7-CARD STUD

- The cards in this trick are set up to exploit the Gilbreath principle. You could start with the Si Stebbins stack (lesson 4), in which the suits appear in cyclic fashion every 4 cards (CHSD, CHSD, ...) and the values also appear in cyclic fashion every 13 cards.

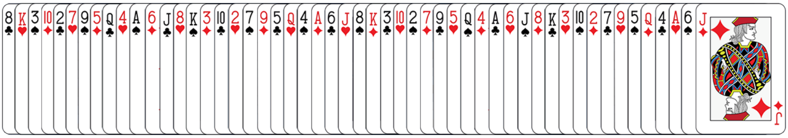
- Any arrangement of the 13 values is acceptable. In fact, this stack—called the 8-kings stack—has the advantage that the cards do not form a mathematical pattern:

  8 K 3 10 2 7 9 5 Q 4 A 6 J

In games like 7-card stud and Texas Hold'em, there are 2 facedown cards and 5 faceup cards.

- It can be memorized by this rhyming mnemonic: **8 K**ings **Thr**ea**Ten** **T**o **S**ave **95** **Ladies** **F**or **One** **S**ick **Knave**!

- In the 8-kings stack, the 52 cards would be arranged like this:



- The cards can be cut as often as you'd like.

- After the cards have been cut, then dealt, and then riffle shuffled, you deal 14 cards, 7 cards each, where the first 4 cards are facedown. You know that the first 4 cards will contain one of each suit, so as soon as you glance at your facedown cards, you know exactly what the other 2 suits are.

- In the video, the magician's down cards were the ace of spades and the queen of clubs, so he knew that the other facedown cards had to have hearts and diamonds as their suits.

- But you don't jump out and name the suits immediately. First say the colors, as if you're getting a vague impression and it's coming into sharper focus. This also gives you time to look at the other faceup cards to figure out the values of your volunteer's cards.

- How do you determine that? Because of the Gilbreath principle, you know that all of the card values will be represented among the first 13 cards. Conveniently, with 7-card stud, 14 cards are dealt and you can safely ignore the 14th card (the last faceup card in your hand).

- Because the facedown cards in the video are the ace of spades and the queen of clubs, the magician just has to look at the other 9 faceup cards to figure out which 2 values are missing. He can see every card value except for 4 and 6, so he knows that these have to be the person's values.

- And he still knows that the person's 2 suits are hears and diamonds. The only thing he can't be sure of is whether the person has the 4 of hearts and the 6 of diamonds, or the 4 of diamonds and the 6 of hearts.

- At this point, you can cover yourself by saying something like this: "I see that one of your values is lower than the other. Is that the heart suit?" You have at least a 50% chance of being right, and it won't matter if you're wrong, because it doesn't seem like your volunteer has given you much information, and then you name the 2 cards exactly.

Remember when you perform this to ignore the 14th card, which is the last card faceup. The good news is that this card is guaranteed to have the same value as one of the other 13 cards, so there is at least one pair on the table, and it provides good camouflage for the secret.

2 PREDICTIONS

- This trick uses 2 predictions—a double application of the Gilbreath principle. The cards are set up with 16 red cards (4 of which are picture cards) and 8 black cards (2 of which are picture cards) so that every third card is black and every fourth card is a picture card.

- As usual, the cards retain their cyclic structure even after being cut. Then, you can use the smoosh shuffle, which involves dealing about half the cards and smooshing them together. The smoosh shuffle is really just a riffle shuffle in disguise, even though they don't look the same. Or you can use what is sometimes called the rosary shuffle, made popular by Swedish magician Lennart Green.

This trick was inspired by a card trick called the Schizoid Rosary and was invented by Max Maven, one of the world's greatest mentalists and a leading authority on the Gilbreath principle.

- After any of these shuffles, the Gilbreath principle guarantees that, starting from the top, every 3 cards will contain 1 black and 2 red cards. Thus, the top 12 cards and the bottom 12 cards are each guaranteed to contain 4 black cards and 8 red cards, as predicted. Also, starting from the top, every 4 cards will have one picture card, so the top 12 cards and bottom 12 cards will have 3 picture cards. Thus, your prediction will be accurate, no matter which pile they take.

REFERENCES

Diaconis and Graham, *Magical Mathematics.*

Mulcahy, *Mathematical Card Magic.*

EXERCISES

1 Suppose that you are performing the 7-card stud trick using the 8-kings stack. After dealing 14 cards to you and your volunteer, your volunteer has these cards, from left to right: J♠ K♣ A♥ Q♦ 10♠ ?? ??.

Your cards are 6♥ 8♦ 6♠ 4♣ 3♥ ?? ??.

Your facedown cards are 2♦ and 9♥.

What are the volunteer's facedown cards?

2 Consider the following psychic challenge. You display a deck of cards that looks randomly mixed and ask the spectator to cut the cards. (This person can give the deck as many complete cuts as he or she wishes.) Then have the volunteer give the cards a riffle shuffle. You deal a few cards faceup from the top to show that there is no pattern to the cards and ask him or her to guess the color of the next card. Then, you'll do the same with the following card, and you keep doing so for the first 10 cards or so. The volunteer guesses right about half the time, but your guess is right every time. How do you do it?

SOLUTIONS CAN BE FOUND ON PAGE 129.

This completes the portion of the course that addresses mathematical card magic, but there is still plenty of magic remaining, most of which has nothing to do with cards.

MAGIC WITH NUMBERS

Perhaps the most natural mathemagical objects are numbers themselves, and they are the topic of this lesson. Even though the tricks in this lesson are all based on very simple algebra, they will still fool and impress most people. A few of the tricks involve are so-called nature's numbers, which show up everywhere—in nature, mathematics, art, and even magic.

EXPLANATIONS

2n + 10 TRICK

- In this trick, you start by asking participants to think of any number between 1 and 10. (In fact, your participants can think of any number—small or large—and it could even be negative or a fraction.)

- You don't know the number they are thinking of, so let's call it n, as in "number." Then, you ask them to double their original number. Now they are thinking of the number

  $$2n.$$

The notion of letting an unknown quantity be represented by a letter—called a **VARIABLE***—is probably the most important idea in algebra.*

- Next, you ask the participants to add 10:

  $$2n + 10.$$

- Then, you tell them to divide by 2, and when both terms are cut in half, this is the result:

  $n + 5.$

- Finally, you ask your participants to subtract the number they started with—which was *n*! When *n* is subtracted,

  $n + 5 - n,$

  which is 5, as predicted.

- Naturally, you can modify this trick to get any number you want. For example, if you wanted to reach 100, you can go through the same steps to reach 5, but don't end the trick there. You could have your participants double the number, which you know to be 10, and then multiply that number by itself, which is 100.

- Instead, if you want to reach a number other than 5, you can just have your participants add a different number at the second step. For example, if they add the number 8 instead of 10, then you have the number $2n + 8$, which—when cut in half and the original number subtracted—is 4.

BLACK AND RED TRICK

- For this trick, you begin by dealing any 30 cards on the table. Let's call the number of black cards in the pile *B*.

- If there are *B* black cards in that pile, that means there are $30 - B$ red cards in that same pile.

- Now consider the other pile of cards. How many red cards will it contain? There is a total of 26 red cards between both piles, and the first pile has $30 - B$ of them, so the second pile must have $B - 4$:

  $26 - (30 - B) = B - 4.$

- So, even though you don't how many black cards or red cards that you have in either pile, you do know that when you take the number of black cards in pile 1 and subtract the number of red cards in pile 2, you are guaranteed to get

  $$B - (B - 4),$$

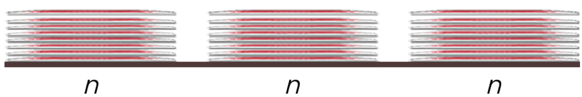
  which equals 4.

- Let's hear it for algebra!

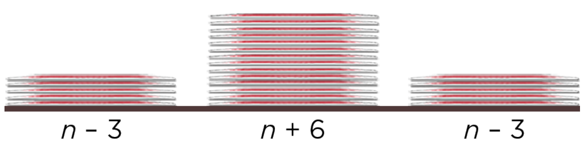
3 HEAPS

Instead of cards, you can use any objects—such as coins or paperclips—for this trick.

- This trick begins with every pile having the same number of cards. How many cards does each pile have? *n*.

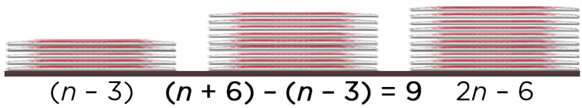


- Then, you take 3 cards from the left pile and 3 cards from the right pile and place them in the middle pile. Your situation now looks like this, where the left and right pile each have $n - 3$ cards and the middle pile has $n + 6$ cards.



- Next, you count the cards on the left pile and then remove that many cards from the middle pile and place those cards on the right pile. When you remove the $n - 3$ cards from the middle pile and place them on top of the $n - 3$ cards on the right pile, then the number of cards in the right pile becomes $2n - 6$.

- That's fine. But here's the surprising part: How many cards are now in the middle pile?



- Finally, when you add 1 more card to the middle pile from either pile, it is guaranteed to have exactly 10 cards, as predicted.

1089 TRICK

- The algebra behind this trick is just a little more complicated than the previous tricks, but you can do it.

- Originally, you choose a 3-digit number, abc, where a is the largest digit and c is the smallest digit. Notice that a represents the hundreds digit, b represents the tens digit, and c is the ones digit. Therefore, the number abc really represents $100a + 10b + c$.

- Next, you reverse the number to get cba, which represents $100c + 10b + a$.

- When you subtract this from the first number, you get this:

$$(100a + 10b + c) - (100c + 10b + a).$$

- Notice that the $10b$ numbers cancel and you are left with this:

$$99a - 99c = 99(a - c).$$

- In other words, after you do the first subtraction, the answer has to be one of these multiples of 99:

99 × 2 = 198	99 × 6 = 594
99 × 3 = 297	99 × 7 = 693
99 × 4 = 396	99 × 8 = 792
99 × 5 = 495	99 × 9 = 891

- When you add any of these numbers to its reversal, you always get 1089. You can verify this with algebra, but it's easier just to verify the 4 possible situations:

198	297	396	495
+ 891	+ 792	+ 693	+ 594
1089	1089	1089	1089

- Thus, no matter what number you have at this step, when you add its reversal, you are guaranteed to end up with 1089.

EASY AS PHI

- One of the most interesting numbers in mathematics is phi. This trick uses the first 16 digits of phi: 1.618033988749894…
- You need 4 volunteers; let's call them *a*, *b*, *c*, and *d*. You give the last 4 digits (9894) to *d*; the 3 digits before those last 4 digits, along with the number 6, to *c* (6874); the middle digits (8033) to *a*; and the remaining digits (1198) to *b*.
- Notice that when you create the four 4-digit numbers, *a* always gives the first digit, *b* gives the second digit, and so on.

- So, your final 4 numbers will have to look like this, where the first column contains the digits 8 0 3 3 in some order, the second column has 1 1 9 8 in some order, and so on.

- The last column of numbers—no matter what the order of numbers is—will always add to 30, so the total will end in 0. Putting down the 0 and carrying the 3, the next column must add to 28. Writing the 8 and carrying the 2, the next column will add to 21. Putting down the 1 and carrying the 2, the first column will add to 16, producing the predicted total of 16,180. And when you throw in the decimal points, you are guaranteed to get 1.6180, which are the first 5 digits of phi.

$$
\begin{array}{rrrrr}
 & {}^{2}8 & {}^{2}1 & {}^{3}6 & 9 \\
 & 0 & 1 & 8 & 8 \\
 & 3 & 9 & 7 & 9 \\
+ & 3 & 8 & 4 & 4 \\
\hline
 & 16 & 1 & 8 & 0
\end{array}
$$

- As it turns out, there is nothing particularly special about the number phi here. Once you give your 4 volunteers any group of numbers, you know what their final total will be, regardless of how each individual arranges his or her cards.

PHI AND FIBONACCI

- Often referred to as the golden ratio, phi has been deliberately used by artists, architects, and advertisers to produce appealing designs.

- The number of petals on a flower or the number of spirals on a pinecone or pineapple often turns out to be a Fibonacci number.

- Phi is intimately connected with the Fibonacci numbers—1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … ,—which start with 1 and 1, and then every number is the sum of the 2 numbers before it. For example, 3 + 5 = 8, 5 + 8 = 13, and so on.
- The ratios of consecutive Fibonacci numbers get closer and closer to 1.618—closer and closer to phi.

$1/1$	1
$2/1$	2
$3/2$	1.5
$5/3$	1.666…
$8/5$	1.6
$13/8$	1.625
$21/13$	1.6153846153862
$34/21$	1.61904761904762
$55/34$	1.61764705882353
$89/55$	1.6181818181818182
φ	**1.618033988749894…**

FIBONACCI SUM AND QUOTIENT

- The numbers in this trick are created like the Fibonacci numbers.
- Let's call the numbers you start with x and y. Thus, the number in row 3 will be $x + y$.
- Row 4 is the sum of rows 2 and 3, so when you add y to $x + y$, you get $x + 2y$.

- Continuing in this way, the rest of the rows will appear as shown at right. (It's no coincidence that all of these coefficients are Fibonacci numbers.)

1	x
2	y
3	$x + y$
4	$x + 2y$
5	$2x + 3y$
6	$3x + 5y$
7	$5x + 8y$
8	$8x + 13y$
9	$13x + 21y$
10	$21x + 34y$

- If you add all 10 of these numbers together, you get a grand total of $55x + 88y$

- What makes that number special is that

$$\begin{aligned} &55x + 88y \\ &= 11 \times (5x + 8y) \\ &= 11 \times \text{row } 7. \end{aligned}$$

- Thus, to add all the numbers, as soon as you see the number in row 7, you just multiply it by 11. For example, if the number in row 7 is 81, then the grand total is guaranteed to be 891.

- How can you guarantee that the final ratio of row 10 divided by row 9 would start with 1.61? The secret has to do with a surprising property of fractions that isn't taught in school. When you first learned about fractions, you were probably taught that to add 2 fractions, such as $\frac{1}{2} + \frac{1}{3}$, you add them by putting them over a common denominator. Here you have

$$\tfrac{3}{6} + \tfrac{2}{6} = \tfrac{5}{6}.$$

- But what happens if you take 2 fractions and add the numerators together and add the denominators together? This has a special name called the **MEDIANT**. For example, the mediant of $\frac{1}{2}$ and $\frac{1}{3}$ is

$$\frac{1+1}{2+3} = \frac{2}{5}.$$

You will learn a quick way to multiply by 11 in the next lesson.

- The mediant has the interesting property that it always lies in between the original 2 fractions. For example, with the fractions $\frac{1}{3}$ and $\frac{1}{2}$, their median, $\frac{2}{5}$, lies between them:

  $$\tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2},$$

  which you can verify, because $\frac{1}{3}$ is about 0.333, $\frac{2}{5}$ is 0.4, and $\frac{1}{2}$ is 0.5.

- In general, for any positive numbers *a*, *b*, *c*, and *d*, if

  $$\frac{a}{b} < \frac{c}{d},$$

  then the mediant $(a+b)/(c+d)$ will always lie between them:

  $$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

- Let's see how this applies to the magic trick. When you divide row 10 by row 9, you are looking at the quotient

  $$\frac{21x+34y}{13x+21y},$$

  but that's the mediant of the fractions $21x/13x$ and $34y/21y$, so it must lie in between them:

  $$\frac{21x}{13x} < \frac{21x+34y}{13x+21y} < \frac{34y}{21y}.$$

- And because $21/13$ begins with 1.615 and $34/21$ begins with 1.619, then the ratio $(21x+34y)/(13x+21y)$ must lie between 1.615 and 1.619:

  $$1.615\ldots = \frac{21}{13} = \frac{21x}{13x} < \frac{21x+34y}{13x+21y} < \frac{34y}{21y} = \frac{34}{21} = 1.619\ldots,$$

  so the first 3 digits of the quotient must begin with 1.61.

- If you continue the process of creating more and more numbers in leapfrog Fibonacci fashion, the ratio will get closer and closer to the golden ratio, phi: 1.618….

PINPIN

- The secret behind this trick is pretty simple, yet it usually gets a very big reaction. It's based on the fact that if you take any 4-digit number—let's say your bank account PIN is 2358—and write it twice so that it appears as an 8-digit number, such as 23582358, that's the same as the 4-digit PIN multiplied by 10,001.

  $$2358 \times 10001 = 23582358$$

- Thus, when you divide this 8-digit number by the 4-digit PIN, you are guaranteed to get back to 10,001. And if you divide this number by 9 and then 9 again, your answer begins with 123.469.

- What makes this trick so deceptive is that you divide the 8-digit number in the opposite order. First, you divide by 9, then 9 again, and then by the 4-digit PIN. But either way, you are guaranteed to get 123.469.

- This trick gets such a good reaction that you might be asked to do it a second time. If so, then have your audience take their 4-digit PIN, enter it twice, then divide by 7, and then divide by 7 again, and then divide by the original PIN. They'll get 10,001 divided by 49, which begins 204.102….

- You can either memorize this number or bring out a calculator as if it will help you figure out the audience's random answer. Just calculate 10,001 divided by 49 and their answer, 204.102, will be staring you in the face.

- It turns out that 10,001 = 73 × 137, so if you ask someone to take the 8-digit number, divide it by 73, and then divide that by his or her PIN, you are guaranteed to get exactly 137—which is easier to remember but seems suspicious. After all, if you take a large random number and divide it by a bunch of numbers, you expect to see an answer with a bunch of digits after the decimal point.

REFERENCES

Benjamin, *The Magic of Math.*

Benjamin and Shermer, *Secrets of Mental Math.*

Gardner, *Mathematics, Magic and Mystery.*

Gardner, *Mental Magic.*

EXERCISES

1 Take any 3-digit PIN and enter it twice so that it reads as a 6-digit number. (For example, the PIN 123 would be entered as 123123.) Using a calculator, divide that number by 13, then divide by 11, and then divide by the original PIN. You should now be looking at the number 7. Why does that work? What would you get if you first divided by 23, then by 16, and then by the PIN?

2 Start with any 3 different positive numbers from 1 to 9 and create all possible 2-digit numbers that you can from them, listing them in a column. (There should be 6 numbers.) For example, if you started with 358, you would create the numbers as shown at right.

$$\begin{array}{r} 35 \\ 38 \\ 53 \\ 58 \\ 83 \\ +\ 85 \\ \hline 352 \end{array}$$

Next, divide this total by the sum of the original 3 numbers. Here, 3 + 5 + 8 = 16, and when you divide 352 by this you get 352 ÷ 16 = 22. Explain why you will always get an answer of 22.

SOLUTIONS CAN BE FOUND ON PAGE 130.

LOOK LIKE A GENIUS

In this lesson, you will learn skills that will make you look like a mathematical genius. Some of these take some practice, but some can be learned immediately. You will learn methods that are easy to do, and you will get the answers so fast that it will seem like magic.

For more methods that will make you look like a mathematical genius, check out the Great Course Secrets of Mental Math *by Arthur T. Benjamin.*

EXPLANATIONS

MULTIPLYING BY 11S

- Let's start with a small 2-digit number: 23. Multiplying 23 by 11 is as easy as **2** + **3**, which is **5**. And there's your answer: **2 5 3**.

  23 × 11 = 253

- Let's do a bigger problem: 63 × 11. The sum of **6** and **3** is **9**, so the answer is **6 9 3**:

  63 × 11 = 693.

- But suppose the numbers add up to a number that is higher than 9. For example, what is 86 × 11? The sum of **8** and **6** is **14**, but the answer is not **8 14 6**. The 1 makes the 8 carry, and the answer is **9 4 6**.

WHY DOES THIS WORK?

- If you were doing a problem like 23 × 11 on paper, the calculation would look like what is shown at right.
- The answer begins with 2, ends in 3, and in between is 2 + 3 (which is 5).
- This method even works for larger numbers. Take, for example, the first 3 digits of pi and multiply that by 11:

    3 1 4 × 11.

- The answer begins with a 3 and ends with a 4. But in between is 3 + 1 (which is 4) and 1 + 4 (which is 5), so here's the answer:

    3 4 5 4.

```
    2 3
  × 1 1
  -----
    2 3
  2 3 0
  -----
  2 5 3
```

DIVIDING BY 91

- Once you've learned how to multiply by 11s, it is practically as easy to divide numbers by 91. Take any number less than 91—for example, 34.

    34 ÷ 91

- **STEP 1** | Multiply your number by 11:

    34 × 11 = 374.

- **STEP 2** | Subtract 1:

    374 − 1 = 373.

- **STEP 3** | These are the first 3 digits of your answer after the decimal point. In this case, 34 ÷ 91 begins like this:

    0.373.

- **STEP 4** | For the next 3 digits of the answer, subtract the first 3 digits from 999:

    999 − 373 = 626.

- Hence, 34 ÷ 91 begins like this:

  0.373626....

- In fact, the exact answer is a repeating decimal with those 6 digits repeating indefinitely:

  34 ÷ 91 = 0.373626 373626 373626....

WHY DOES THIS WORK?

- This method is based on the fact that 91 × 11 = 1001. So, the fraction

  $$\frac{34}{91},$$

  when you multiply the top and bottom by 11, is equal to

  $$\frac{34}{91} = \frac{34}{91} \times \frac{11}{11} = \frac{374}{1001}.$$

- As it happens,

  $^1/_{1001}$ = 0.000999 000999 000999...,

  so when each group of 6 gets multiplied by 374, then each group of 6 becomes

  374 × 999 = 374 × (1000 − 1),

  which equals 374,000 − 374, which is 373,626.

$$\begin{array}{r} 374000 \\ -\quad 374 \\ \hline 373626 \end{array}$$

MULTIPLYING NEAR 100

- Let's now look at how to multiply numbers that are close to 100. Let's begin with 2 numbers that are a little bit above 100. For example:

$$\begin{array}{r} 106 \\ \times 111 \\ \hline \end{array}$$

- First note how far each number is above 100; let's call these distance numbers.
- In this case, 106 is 6 above 100 and 111 is 11 above 100, which you will denote as follows.

$$\begin{array}{rl} 106 & (6) \\ \times 111 & (11) \\ \hline \end{array}$$

- Next, add 106 + 11 or 111 + 6; either way, you'll get 117, which you can write down.

$$\begin{array}{rl} 106 & \mathbf{(6)} \\ \times \mathbf{111} & (11) \\ \hline \mathbf{117} & \end{array} \quad 111+6=\mathbf{117} \qquad \text{or} \qquad \begin{array}{rl} \mathbf{106} & (6) \\ \times 111 & \mathbf{(11)} \\ \hline \mathbf{117} & \end{array} \quad 106+11=\mathbf{117}$$

- Finally, multiply your distance numbers together; in this case, 6 × 11 = 66, which you write down as follows.

$$\begin{array}{rl} 106 & \mathbf{(6)} \\ \times 111 & \mathbf{(11)} \\ \hline 117 & \mathbf{66} \end{array} \quad 6\times 11=\mathbf{66}$$

- And that's the answer!

106 × 111 = 11,766

- The same method works when multiplying numbers below 100, except the distance numbers are negative. For example, to do 96 × 93, your distance numbers are −4 and −7, giving you what follows.

$$\begin{array}{rr} 96 & (-4) \\ \times 93 & (-7) \\ \hline \end{array}$$

- Next, you subtract either 96 − 7 or 93 − 4. Either way, you get 89.

$$\begin{array}{rr} 96 & (-4) \\ \times 93 & (-7) \\ \hline \mathbf{89} & \end{array}$$

- Then, you multiply (−4)(−7) = 28 to get the answer.

$$\begin{array}{rr} 96 & (-4) \\ \times 93 & (-7) \\ \hline 89 & \mathbf{28} \end{array}$$

WHY DOES THIS WORK?

- Here's the algebra behind this method:

$$\begin{aligned} &(z + a)(z + b) \\ &= z^2 + za + zb + ab \\ &= z(z + a + b) + ab \end{aligned}$$

- This says that if you take the number $(z + a)$ times the number $(z + b)$, then after 2 lines of algebra, you get z times the quantity $(z + a + b)$ plus ab.
- The variables z, a, and b can be any numbers, but typically z is some number with lots of zeros in it. When $z = 100$,

$$(100 + a)(100 + b) = 100(100 + a + b) + ab.$$

- The numbers a and b are the distance numbers.

- But suppose you had larger numbers, such as 109 × 112. Your distance numbers are 9 and 12.

$$\begin{array}{rl} 109 & (9) \\ \times 112 & (12) \\ \hline \end{array}$$

- Adding 109 + 12 (or 112 + 9), you get 121, but that really represents 121 × 100, which is 12,100.

$$\begin{array}{rrl} & 109 & (9) \\ \times & 112 & (12) \\ \hline & 121 & \\ \times & 100 & \\ \hline & \mathbf{12100} & \end{array}$$

- Now when you multiply your distance numbers,

$$\begin{array}{rrl} & 109 & (9) \\ \times & 112 & (12) \\ \hline & 121 & \mathbf{108} \\ \times & 100 & \\ \hline & 12100 & \end{array}$$

you add that to 12,100, resulting in what follows.

$$\begin{array}{rrl} & 109 & (9) \\ \times & 112 & (12) \\ \hline & 121 & 108 \\ \times & 100 & \\ \hline & \mathbf{12100} & \\ \mathbf{+} & \mathbf{108} & \\ \hline & \mathbf{12208} & \end{array}$$

- Note that the answer is *not* 121 108.

- You can even use this idea to multiply numbers where one is above 100 and the other is below 100.
- Let's try 116 × 97. You start by doing 116 − 3 (or 97 + 16) to get 113, which you multiply by 100 to get 11,300.

$$\begin{array}{lrl} & 116 & (16) \\ \times & 97 & (-3) \\ \hline & 113 & \\ \times & 100 & \\ \hline & \mathbf{11300} & \end{array}$$

- Next, because 16 × (− 3) = −48, you subtract 48.

$$\begin{array}{lrl} & 116 & (16) \\ \times & 97 & (-3) \\ \hline & 113 & -48 \\ \times & 100 & \\ \hline & \mathbf{11300} & \\ - & \mathbf{48} & \\ \hline & \mathbf{11252} & \end{array}$$

- Once you understand the algebra, you can apply this method to all kinds of multiplication problems whenever the 2 numbers being multiplied are close together.

For more details, check out the Great Course Secrets of Mental Math.

MENTAL LOGS

- How do you add big numbers quickly?

```
   9563
   8459
   6837
+  1976
  26835
```

- The answer is actually staring you in the face.
- The answer appears in the second row from the bottom.
- To get the answer, take the third number, 6837; place a 2 in front of it, 26,837; and then subtract 2 from the last digit:

  26,835.

- Now try it with these special numbers:

```
   5763
   4939
   8533
+  9296
```

- From the third number, 8533, you get the answer (put 2 in front and then subtract 2 from the end):

  28,531.

WHY DOES THIS WORK?

- The numbers on each side are chosen so that the first, second, and fourth numbers of each log always add up to 18.

- For example, in the last example, you have what follows.

	5	**7**	**6**	**3**	5 + 4 + 9 = 18
	4	**9**	**3**	**9**	7 + 9 + 2 = 18
	8	5	3	3	6 + 3 + 9 = 18
+	**9**	**2**	**9**	**6**	3 + 9 + 6 = 18

- As a result, the ones digits contribute 18, the tens digits contribute 180, the hundreds digits contribute 1800, and the thousands digits contribute 18,000 for a grand total of 19,998.

$$
\begin{array}{r}
18 \\
180 \\
1800 \\
+\ 18000 \\
\hline
19998
\end{array}
$$

- Now, when you add the third number—for example, 8533—it gets added to 19,998 (which is 20,000 − 2).

- So, all you need to do is add 20,000 to 8533 (by placing a 2 in front) and then subtract 2 to get 28,531.

DO IT YOURSELF

◊ You can do the mental logs trick at home using popsicle sticks. Create 10 popsicle sticks, each with 5 numbers on the front and back. You can use any numbers that you like; all that is needed is that with each number, the first, second, third, and fifth digits add to 18.

◊ For example, on the first stick, you can put

7 + 6 + 5 + 0 = 18.

◊ On the second stick, you can put

3 + 7 + 6 + 2 = 18.

◊ Thus, no matter which popsicle sticks are chosen, regardless of what side is showing or what order is used, to get their total, all you have to do is take the second number from the bottom, attach 2 in the front, and subtract 2 from the end.

◊ For example, what's the total if your volunteer chooses 6 sticks that give you these five 6-digit numbers shown at right.

9	3	8	6	4	9
1	7	0	3	0	7
2	6	7	4	9	2
5	**7**	**8**	**7**	**7**	**3**
6	2	3	5	5	0

◊ It will be a 7-digit number, beginning with 2.

◊ When you attach that to your key number, 578773, and subtract 2 from the end, you get the answer:

2,**578,77**1.

◊ Try not to use more than 5 or 6 digits because the more digits you use, the longer it takes for your volunteer to check your answer with a calculator, and there is the increased chance that he or she will make a mistake themselves. Also, the more digits that you have in your answer, the more likely it is that your volunteer might notice that it has a lot in common with the fourth row.

SQUARING

- Squaring is the act of multiplying a number with itself. To square a number that ends in 5, you just have to remember 2 things: The answer will always end in 25, and it begins by taking the first digit and multiplying it by the next-higher digit.

- For example, for the square of 35, the first digit of 35 is 3, so the answer begins with 3 times 4,

  $$3 \times 4 = 12,$$

  and ends with 25. Therefore, the answer is

  $$35^2 = 1225.$$

- It even works with larger numbers. For example, let's do the square of 115. You start with 11 and its next-higher number:

  $$11 \times 12 = 132.$$

- Therefore,

  $$115^2 = 13{,}225.$$

WHY DOES THIS WORK?

- The reason this works is based on the algebra:

  $$a^2 = (a - 5)(a + 5) + 25.$$

- For example, this says that the square of 35 is

  $$(30 \times 40) + 25 = 1200 + 25 = 1225.$$

- Let's extend this in 2 different ways. If you multiply both sides by *a*, you get a fast way to **CUBE**—or take to the third power—a 2-digit number that ends in 5 (or at least get a quick approximation):

  $$a^3 = (a - 5)a(a + 5) + 25a.$$

- For example, 35 cubed is approximately

  $$35^3 \approx 30 \times 35 \times 40 = 42{,}000.$$

- To get the exact answer, add $25 \times 35 = 875$:

  42,875.

- A more useful generalization is that with practice, you can square any 2-digit number (or higher) based on this formula:

  $$a^2 = (a - d)(a + d) + d^2.$$

- For example, you may already know that 12 squared is 144, but here's how you would calculate that with this method. Just let $d = 2$ to get

  $$\begin{aligned} 12^2 &= (12 - 2)(12 + 2) + 2^2 \\ &= 10 \times 14 + 4 = 144. \end{aligned}$$

- You don't have to think about the algebra when you square numbers. For example, when you square 12, you can go down 2 to 10 and then to balance it go up 2 to 14. Multiply those together to get 140. And because you went up and down 2, you add 2^2, which is 4, to get 144.

REFERENCES

Benjamin, *The Magic of Math.*

Benjamin and Shermer, *Secrets of Mental Math.*

Gardner, *Mental Magic.*

Gardner, *Martin Gardner's Mathematical Games.*

EXERCISE

1 Using the skills from this lesson, quickly mentally compute the answers to the following problems:

a) 54×11

b) 76×11

c) $76 \div 91$

d) 85^2

e) 54^2

f) 106×109

g) 98×95

h) 112×94

SOLUTIONS CAN BE FOUND ON PAGE 130.

THE MAGIC OF NINE

In this lesson, you will learn how to do some impressive calculations and feats of mind that are based on the number 9. You will also learn how to apply cube roots in magic tricks as well as how to figure out how old someone is from just a few seemingly random calculations.

EXPLANATIONS

MISSING DIGIT

- People usually assume that this trick requires superhuman calculating abilities, yet it's all based on the magic property of the number 9. What does 9 have to do with it?

- Let's look at the first few multiples of 9.

    9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, ...

- What do they all have in common? If you add the digits, you get 9.

    1 + 8 = 9; 2 + 7 = 9; 5 + 4 = 9; 1 + 2 + 6 = 9

- An exception is 99, which adds to 18. But that's okay, because 18 is a multiple of 9.

- Here's the deal: A number is a multiple of 9 if and only if when you add the digits, you either get

  9 or 18 or 27 or 36 …

- The digits have to add up to a multiple of 9.

WHY DOES THIS WORK?

- Let's take the number 3456.

$$\begin{aligned} 3456 &= (3 \times 1000) + (4 \times 100) + (5 \times 10) + 6 \\ &= 3 \times (999 + 1) + 4 \times (99 + 1) + 5 \times (9 + 1) + 6 \\ &= (3 \times 999) + (4 \times 99) + (5 \times 9) + 3 + 4 + 5 + 6 \end{aligned}$$

- Because 999 is a multiple of 9, then so is 3×999. And so is 4×99 and 5×9. And when you add these multiples of 9 together, you still have a multiple of 9:

$$\begin{aligned} 3456 &= (\text{a multiple of } 9) + 3 + 4 + 5 + 6 \\ &= (\text{a multiple of } 9) + 18, \end{aligned}$$

  which is still a multiple of 9.

- By the same argument, if you take any number and add its digits, it tells you how far it is above a multiple of 9.

- For example, the number 64 has

$$6 + 4 = 10,$$

  so it's 10 bigger than a multiple of 9. That can be verified, because

$$64 - 10 = 54$$

  is a multiple of 9.

- In this trick, you ask your audience to choose a 3-digit number, mentally add the digits, and then subtract that number from the total. You do that to force them to start with a multiple of 9.
- For example, if they start with the number

  352,

  sum the digits,

  3 + 5 + 2 = 10,

  and then subtract that from their original number, they get

  352 − 10 = 342,

  which is a multiple of 9. You can tell that it's a multiple of 9 because the digits of 342 add up to 9.
- Another way to start the trick without doing the subtraction step is to just give them a multiple of 9 to start with, such as

  3456 or 567,

  which add to 18.
- To make this number more personal, you can use a year that is meaningful to you, or you can just use the first 4 digits of pi: 3141, which adds to 9.
- Next, have your audience multiply their number by any 3-digit number. The number can be of any length, but it's probably most impressive if the answer is 6 or 7 digits long. (If it's any longer than that, then there is too much of a chance that the person you choose to call out his or her digits will make a mistake.)
- Let's suppose that the person calls out 6 of his or her 7 digits in random order, and they happen to be

  235711.

- You don't know what that person's 7-digit number is, but you do know that his or her answer has to be a multiple of 9. This is because the person started with a multiple of 9 and multiplied it by something, so his or her answer still has to be a multiple of 9. So, as the person is calling out his or her numbers, you add them up. In this case,

  2 + 3 + 5 + 7 + 1 + 1 = 19.

- So, what do you need to add to reach a multiple of 9?

  19 + 8 = 27,

  so the missing digit has to be 8.

- Remember that the person can call out his or her 6 digits in any order and you can still figure it out.

- Let's do a trickier example. Suppose the person calls out the digits

  442557,

  which adds to 27. What did they leave out? You may be tempted to call out 0—and it might be 0, but it could also be 9. You don't know if the 7-digit numbers add to 27 or if they add to 36. In that situation, you guess!

- But as a magician, you can do it in such a way that it does not look like you're guessing. For example, you can say you are having trouble with one of the person's numbers—that's because he or she left out a 0 or 9. When this happens, ask the person to concentrate on his or her number and say, "It seems like you're thinking of nothing. You didn't leave out a 0, did you?" If the person says yes, then you say, "Ah, that's why I was getting a lot of nothing!" and you take your bow. If the person says no, then you say, "I didn't think so, but it seemed like you were thinking of nothing, so really concentrate this time. Then say, "You're thinking of the number 9" and take a bow.

- If you want to avoid the 0 or 9 issue altogether, there's another approach that some magicians use. When the person looks at his or her 7-digit number, have him or her mentally circle any one of his or her numbers. But tell the person, "Just don't circle a 0, because that's already a circle." That seems reasonable, or at least humorous. Then have the person call out his or her other digits, and if they add to a multiple of 9, then because the person agreed to not think of 0, you know he or she left out the number 9.

DIGITAL ROOTS

- If you take a number, such as

  1776,

  and add its digits,

  1+ 7 + 7 + 6 = 21,

  and if you get a 2-digit number, add those digits,

  2 + 1 = 3,

  then the 1-digit number that you end up with is called the **DIGITAL ROOT**. So, in this example, 1776 has a digital root of 3.

- What does the digital root tell you about the number?

- If the digital root is 9, then your number is definitely a multiple of 9.

- If the digital root is not 9, then the number that you get tells you the remainder that you get when you divide that number by 9. In the case of 1776, because it has a digital root of 3, then you know that it is 3 bigger than a multiple of 9.

- Incidentally, if the digital root is 3, 6, or 9, then your original number is a multiple of 3. So, 1776, though not a multiple of 9, is a multiple of 3.

CASTING OUT 9S

- There is another surprising way to create a multiple of 9: Take any number, such as 1618, and scramble the digits to create a new number, such as 8116, and when you subtract the smaller number from the larger one, you are guaranteed to get a multiple of 9.

- In this case, you have

  $$8116 - 1618 = 6498,$$

  which (because the digits add to 27) is a multiple of 9.

- The reason this works is because when you scramble the digits, you don't change the digital root. In this case, 8116 and 1618 both add to 16, so they both have a digital root of 7. So, both of these numbers are 7 bigger than a multiple of 9. In this case,

  $$8116 = 9x + 7$$

  for some integer x, and

  $$1618 = 9y + 7$$

  for some integer y.

- Hence, their difference will be

  $$(9x + 7) - (9y + 7) = 9x - 9y = 9(x - y),$$

  which is definitely a multiple of 9.

- The process of simplifying and reducing numbers to their digital roots and doing arithmetic with them is sometimes called **CASTING OUT 9S** because every number is reduced to a single digit, casting out—or subtracting—a multiple of 9. This can be used as a useful way for checking your work or even solving tricky math problems.

CUBE ROOTS

- Let's do another calculation that seems extremely hard but is in fact very easy to do. This time, instead of multiplying, squaring, or even cubing, you will work with cube roots.

- In this trick, you ask a volunteer to cube a 2-digit or 3-digit number. You know that the number that he or she is giving you is a perfect cube. If the number is not a perfect cube—for example, if the person asked for the cube root of 10—then the answer is not a whole number (the cube root of 10 is about 2.154).

- This method only works when the person's answer is a perfect cube, and it's very easy to do. All you need to know are the cubes of the numbers from 1 to 10.

Recall that cubing a number is taking a number to the third power. For example, 5 cubed is $5^3 = 5 \times 5 \times 5 = 125$.

The cube root is the reverse of this process. For example, the cube root of 125 is 5.

$1^3 = 1$	$6^3 = 216$
$2^3 = 8$	$7^3 = 343$
$3^3 = 27$	$8^3 = 512$
$4^3 = 64$	$9^3 = 729$
$5^3 = 125$	$10^3 = 1000$

- Look at the last digit of each number:

  1, 8, 7, 4, 5, 6, 3, 2, 9, 0.

- Notice that all of the digits are different. There's only one cube that ends in 1—namely, 1. There's only one cube that ends in 2—namely, 8. In fact, the last digit of the cube is either the same as the original digit or is 10 minus the original digit. For example, 3 cubed, or 27, has a last digit of 7, and 3 + 7 = 10.

- How does this help you? Let's suppose you ask a volunteer to cube a 2-digit number and the answer is 74,088. To figure out his or her original 2-digit number, you start by finding the cube root of 74,088.

- When you hear this answer, you listen for 2 things. The first thing you listen for are the thousands—that is, the numbers before the comma. In this case, that would be 74.

  74,088

- Now 74 lies between 4 cubed and 5 cubed.

  $4^3 = 64$

  $5^3 = 125$

- This means that the first digit of the cube root must be 4. Why? Well,

  $40^3 = 64{,}000$

  $50^3 = 125{,}000$

- Hence, the cube root of 74,088 must lie between 40 and 50, so the answer must be in the 40s.

- To find the last digit of the cube root, you just look at the last digit of the cube. The number ends in 8,

  74,08**8**,

- and only one number when cubed ends in 8—namely, 2. Thus, the original 2-digit number must have been

  42.

- Let's try another example. Suppose that a 3-digit number is cubed and the 9-digit answer is

  197,137,36**8**.

- How can you figure out the original 3-digit number? Again, you look at the number before the first comma, representing the millions, which is 197. From the table of cubes, you see that 197 lies between 125 and 216.

  $5^3 = 125$

  $6^3 = 216$

- So, the first digit has to be 5. And because the 9-digit number ends in 8, the original 3-digit number ends in 2. Hence, the cube root looks like this:

  5 **?** 2.

- How do you find the middle digit? Notice that if you add the digits of the 9-digit number, you get 45.

  $1 + 9 + 7 + 1 + 3 + 7 + 3 + 6 + 8 = 45$

- This means that the 9-digit number is a multiple of 9 and therefore the original 3-digit number has to be a multiple of 3. This means that the digits of the original number have to add up to a multiple of 3. Thus, the 3-digit number has to be either

  522 or 552 or 582.

- Which one is it? You saw that the millions number, 197, is between 125 and 216.

  $5^3 = 125$

  $6^3 = 216$

- And it's much closer to 216 than it is to 125. Thus, your answer should be the largest of the 3 candidates, so the cube root is

  582.

If the cube root is not a multiple of 3, then the situation is a little trickier. The details are provided in the Exercises section at the end of this lesson.

AGE DETERMINATION

- This trick applies the ideas of this lesson to figure out how old someone is from just a few seemingly random calculations. Let's go through the steps of the first calculation of this trick.

- Your volunteer starts with his or her age, but it can be any random 2-digit number. Next, he or she adds 7, which creates another random number. Nothing has happened yet.

- Then, the volunteer multiplies by 24, which creates a multiple of 3. When the person adds or subtracts 60, he or she still has a multiple of 3. Then, when the volunteer multiplies by 12, this creates a multiple of 9. In addition, it has to be an even number. Then, the person multiplies by another number, so it's still an even multiple of 9.

- But after the volunteer adds his or her age, 2 things happen. First, if the person's age is even, then his or her answer will still be even, and if the person's age is odd, then his or her answer will be odd. So, by looking at the last digit of the answer, you can tell whether the person's age is even or odd.

- Second, when the volunteer adds his or her age to a multiple of 9, the digital root changes from 9 to whatever the digital root of his or her age is. For example, if the person is 43, which has a digital root of 7, when he or she adds that to a multiple of 9, the answer will have a digital root of 7. So, by adding up the digits of the answer, you know the digital root of the person's age.

- For example, suppose that after asking your questions, your volunteer produces an answer of

  314159.

- How do you determine their age?

- First, by looking at the last digit, you know that his or her age is an odd number.

- Next, you add the digits of their answer. In this case, the numbers add to 23:

  $$3 + 1 + 4 + 1 + 5 + 9 + 2 = 23.$$

- This person's age has the same digital root as 23. And because his or her age is an odd number, the answer might be 23. If it's not, then all you have to do is add or subtract 18 until you reach his or her age.

- Remember that the digital root stays the same as you add and subtract multiples of 9, and because you know that the last digit of the person's age is odd, you can just add multiples of 18.

- So, starting with 23, and adding or subtracting 18, the person's age must be one of these numbers:

  5, 23, 41, 59, 77, or 95.

- And that should be enough information for you.
- Let's do another example. Suppose the person shows you the number

  1618033.

- You can tell from the last digit that this person's age must be an odd number.
- Next, add the digits of this number:

  1 + 6 + 1 + 8 + 0 + 3 + 3 = 22.

- Can this person's age be 22? No, because his or her age is odd. And because 22 has a digital root of 4, if the person is younger than 22, then subtract 9 and his or her age must be 13, which also has a digital root of 4. If the person is older than 22, then you add 9 to get 31. So, the person's age must be 31 or be larger by a multiple of 18. So, his or her age is either

  31 or 49 or 67 or 85 or 103.

- If you're still not sure, then play it safe and guess young! For example, suppose you're not sure if the person is 49 or 67. Guess 49 and if the person says, "I wish," you say, "Well, the numbers were saying that 67 was most likely, but I didn't believe it, so I went with the second-most-likely option. Are you really 67? You look much younger!" So, even if your first guess is wrong, the person will be happy with your answer.
- If you don't want to take any chances, then the second calculation gives you the answer without any guesswork.
- This time, the first few steps are random, but as soon as your volunteer multiplies by 12 and then by 30, he or she has a multiple of 360, which is a multiple of 9. In addition, you know that it ends in 0. And that's still true, even after the person multiplies by another 2-digit number.

- For example, at this point, the person's number could be

  17,357,760.

  (Notice that it's a multiple of 9 because the digits add to 36.)

- Then, you have the person subtract 3141, which is a multiple of 9, so the answer is still a multiple of 9. Moreover, the last digit will now be 9, which is less conspicuous than the 0.

- Subtracting 3141 from the last answer gives this:

  $17{,}357{,}760 - 3141 = 17{,}354{,}619.$

- Next, when the person adds the age of his or her friend, 2 things happen. First, as before, the digital root of the answer becomes the digital root of the age. And second, because the number used to end in 9, as soon as you see the final answer, you will know the last digit of the age, by simply increasing the last digit by 1.

- For example, if the friend's age was 42, the new answer would be

  $17{,}354{,}619 + 42 = 17{,}354{,}661.$

- When the person shows you this big number, you know 2 things. By looking at the last digit, which is 1, you know immediately that the age the person is thinking of ends in 2. Second, when you add the digits of the big number, you get 33:

  $1 + 7 + 3 + 5 + 4 + 6 + 6 + 1 = 33,$

  which has a digital root of

  $3 + 3 = 6.$

- So, the person's secret 2-digit number has a digital root of 6. This means that the sum of the digits must be 6 or 9 bigger, which is 15. And because the person's final digit was 2, you know that his or her secret age must be 42.

REFERENCES

Benjamin, *The Magic of Math.*

Benjamin and Shermer, *Secrets of Mental Math.*

Gardner, *Mathematics, Magic and Mystery.*

Gardner, *Mental Magic.*

EXERCISES

Here's another interesting pattern about cubes. Look at the table of cubes from 1 to 10 and notice the digital root of each number:

	DIGITAL ROOT		DIGITAL ROOT
$1^3 = 1$	1	$6^3 = 216$	9
$2^3 = 8$	8	$7^3 = 343$	1
$3^3 = 27$	9	$8^3 = 512$	8
$4^3 = 64$	1	$9^3 = 729$	9
$5^3 = 125$	8	$10^3 = 1000$	1

What's going on here? When you cube a multiple of 3 (such as 3, 6, 9, etc.), then it makes sense that the result will have a digital root of 9 (because it will be a multiple of 9). What this table also shows is that if you cube a number that is 1 or 2 more than a multiple of 3, then the digital root will be 1 or 8, respectively. This pattern can be useful when doing cube roots of 3-digit cubes, as described in some of the following exercises.

1 Find the cube roots of the following perfect cubes. The first 2 have 2-digit answers and the last 3 have 3-digit answers.

a) 148,877

b) 704,969

c) 34,012,224

d) 273,359,449

e) 656,234,909

2 Here's another way to determine anyone's age between 10 and 99 with no guesswork required at all. Suppose you ask a volunteer to perform the following steps, making sure that he or she presses the equals (=) button after each step.

STEP 1 | Enter a random 3-digit number on your calculator.

STEP 2 | Multiply by 12 (number of months in a year).

STEP 3 | Add or subtract 24 (hours in a day).

STEP 4 | Multiply by 30 (average number of days per month).

STEP 5 | Multiply by any 2-digit number.

STEP 6 | Subtract 3141 (digits of pi: circular reasoning).

STEP 7 | Add any age.

When your volunteer shows you the answer, you tell him or her the age with just a little bit of thought. How do you do it?

SOLUTIONS CAN BE FOUND ON PAGE 131.

10

LOOK LIKE A PSYCHIC

In this lesson, you will learn some magic tricks that will give the illusion of your ability to read someone's mind or predict the future—all using the power of mathematics.

EXPLANATIONS

EXACT BIRTHDATE CALCULATION

- The secret to figuring out someone's birthday exactly is a little special algebra.
- Let's say that your volunteer's birthday has month m, date d (written as a 2-digit number), and year y (also written as a 2-digit number).
- You start with the month and add 16, producing

  $m + 16.$

- Then, you multiply this by 25, which is

  $25(m + 16) = 25m + 400.$

- Then, you subtract 36, giving you

  $25m + 364.$

- Then, you multiply this by 8, which is

  $200m + 2912.$

- Then, you subtract 64, giving you

  $$200m + 2848.$$

- Next, you divide by 2, giving you

  $$100m + 1424.$$

- Then, you add the date, leaving you with

  $$100m + d + 1424.$$

- If your volunteer doesn't want to give you the year, you could end the trick now by asking them to subtract

  $$1424,$$

  which will produce

  $$100m + d,$$

  which is the person's birthday.

- For example, if the birthday is March 19, then $m = 3$ and $d = 19$, so the number would be

  $$319,$$

  which is the birthday

  $$03/19.$$

- Assuming that your volunteer is willing to enter his or her year of birth, then instead of subtracting 1424, multiply by 5, producing

  $$5(100m + d + 1424) = 500m + 5d + 7120.$$

- Then, you add 314, giving you

  $$500m + 5d + 7434.$$

- And when you multiply this by 20, you get

  $$20(500m + 5d + 7434) = 10{,}000m + 100d + 148680$$

- Next, you subtract your fudge factor of 5823, giving you

  $$10{,}000m + 100d + 142{,}857.$$

- Finally, when you add the year y,

  $$10{,}000m + 100d + 142{,}857 + y,$$

  and then subtract 142,857, you get what you want, which is

  $$10{,}000m + 100d + y.$$

- So, if you take March 19, 1961, where $m = 3$, $d = 19$ and $y = 61$, this would give you what follows.

$$\begin{array}{r} 30000 \\ 1900 \\ +\quad 61 \\ \hline 31961 \end{array}$$

- And this is the birthday 03/19/61.

MAGIC NUMBER 142,857

- The number 142857 is a magical one. When you write the fraction $^1/_7$ in decimal notation, you get

  $$^1/_7 = 0.142857\ 142857\ 142857\ \ldots.$$

- As a consequence of that, it has a curious property: If you multiply 142857 by 1, you get the same number! Okay, that's not so amazing. But if you multiply it by 2, you get

  $$2 \times 142857 = 285714,$$

  which are the same numbers as before, starting with the number 2.

- Furthermore,

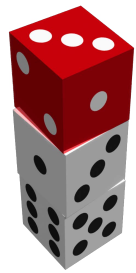
$$3 \times 142857 = 428571$$
$$4 \times 142857 = 571428$$
$$5 \times 142857 = 714285$$
$$6 \times 142857 = 857142,$$

which are the same digits in the same circular order.

- And if you multiply this magical number by 7, then you get

$$7 \times 142857 = 999999.$$

STACKED DICE



- This trick is very simple, but it still fools most people, even those who play with dice a lot. The secret of this trick is to look at the number on top of the top die and subtract it from 21. That's it.
- So, if the dice were stacked like this, with a 3 on top, then the total must be 21 − 3 = 18.

WHY DOES THIS WORK?

- This trick exploits the well-known feature that the numbers on the opposite sides of a die always sum to 7. If you were to sum the top and bottom number of each stacked die, the total would come to 21. But you are only summing the numbers on 5 of the 6 sides, so you subtract the top side.
- In general, you can let your volunteer roll any number of dice, and if he or she uses n dice, then the sum of all the "hidden numbers" is

$$7n - \text{the top number.}$$

ALPHABET CARDS

- The secret to this trick is almost staring you in the face, and it has a lot in common with some of the card tricks in lesson 2, which were based on **PARITY**. Imagine that the letters of the alphabet were placed on a 5-by-5 checkerboard, plus 1 extra square at the bottom for the letter Z.

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y
		Z		

- When you do so, you'll notice that all of the vowels happen to be on light squares. That's because the vowels A, E, I, O, U, W, and Y are in odd positions of the alphabet: 1, 5, 9, 15, 21, 23, and 25. In other words, all of the vowels have the same parity, because they correspond to odd numbers. As a result, you are guaranteed to start on a light square.

- Every time you make a move—whether it is up, down, left, or right—you will always move to a square of opposite color: either from light to dark or from dark to light. More specifically, if you take 1 step, or 3 steps, or 5 steps, or any odd number of steps, the ending square will be of opposite color of the starting square.

- So, because you know that your volunteer started off on a light square, then after he or she has taken 3 steps, (by going 1 step and then 2 steps more), you know that the person is now on a dark square.

- At this point, you can safely remove some light squares because you know that your volunteer can't be on them. After that, you have the person move an odd number of steps again and then you remove some dark squares because you know he or she won't be on any of those.

- Eventually, you are led to the only dark square that remains on the board at the end (which was the letter P in the video).

PILE-SPLITTING PREDICTION

- This is a trick where the secret might be more interesting than the trick itself.

For this trick, you can use coins, buttons, poker chips, etc.

- If you start with 10 chips and split them into piles and record the pile sizes as you go along, then somehow, when you multiply the pile sizes at each step and add them together, you always get a total of 45. That's pretty surprising, because there are almost 100 ways that you can split these piles up into smaller and smaller piles, yet they all result in the number 45. Why is that?

- If you start with a pile that contains *n* chips, then when you finish the calculation, the final result will be

$$\frac{n^2-n}{2}.$$

- So, with 10 chips, the final number will be

$$\frac{10^2-10}{2}=\frac{90}{2}=45.$$

- Let's check out smaller piles and see what they contribute.

- Notice that a pile of 2 chips can only be split into piles of size 1 and 1, so its total, after 1 split, will be

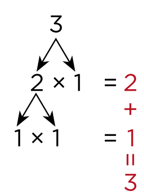
$$1\times 1=1,$$

which agrees with the formula because

$$\frac{2^2-2}{2}=\frac{2}{2}=1.$$

- A pile of 3 chips must be split into a 2 1 pile, and then a 1 1 pile.

- This will lead to products of 2 × 1 = 2 and 1 × 1 = 1, with a total of 3.



- This also agrees with the calculation because

$$\frac{3^2-3}{2}=\frac{6}{2}=3.$$

- As the number of chips grows, there are more possibilities for how the chips will be split.

PROOF BY STRONG INDUCTION

- Suppose you verified the formula whenever you had less than n chips. Here's a quick proof that the formula will continue to work when you have exactly n chips. (This is known as a proof by strong induction.)
- Suppose that you split your first pile of n chips into piles with size x and y, where $x + y = n$.
- This means that this split will directly contribute a product of xy to your total.
- Also, because x is smaller than n, then by the assumed formula, no matter how you split it into smaller piles, it will produce a grand total of

$$\frac{x^2-x}{2}.$$

- Likewise, the y pile will produce a grand total of

$$\frac{y^2-y}{2}.$$

- So, altogether, the grand total starting from n chips will be

$$xy+\frac{x^2-x}{2}+\frac{y^2-y}{2}.$$

which algebraically simplifies to

$$\frac{x^2+2xy+y^2-(x+y)}{2}=\frac{(x+y)^2-(x+y)}{2}.$$

- And because $x + y = n$, this is none other than

$$\frac{n^2 - n}{2}.$$

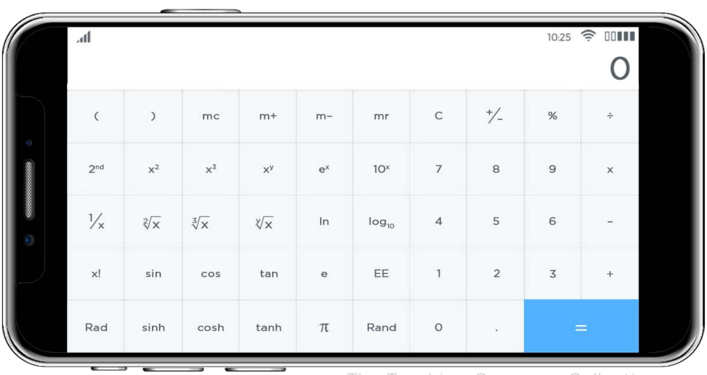
which is what you wanted to see.

- If you haven't done a proof by induction before, don't sweat it. Luckily, the trick will always work whether you completely understand the theory or not.

TOXIC PREDICTION

- At the end of this trick, the fact is that you can produce any number on your calculator. It could be 1.618033 (as it was in the video), or pi, or even your phone number. You just have to set things up with your calculator in advance.

- This is how to force an answer of pi—but you can use any number that you want. To do this, you will need a scientific calculator. In fact, if you have a smartphone, you probably have a scientific calculator on it. If you bring up the calculator and turn the phone on its side, the simple calculator turns into a scientific one.



The Teaching Company Collection.

- First, enter the number pi (or press the p button). Then, follow these simple instructions:
    - Press the + sign.
    - Press 0.
    - Press the multiplication button, ×.
    - Press the left parenthesis button, (.

You can even perform this trick on someone else's calculator if you can quickly get to the scientific mode for a few seconds to set things up. That will make the effect even more amazing.

- It's easy to remember these steps because the 4 symbols

    + 0 × (

  seem to look like the letters

    t o x c

  and this is why magicians call it the **TOXIC CALCULATION**.

- What does this accomplish? When you add 0 times anything, you are just adding 0, so your answer won't change. Everything you type after the left parenthesis will eventually be multiplied by 0, so it doesn't matter what numbers your volunteer gives you.
- Note: Don't press the equals button until you are done with the calculation. When you press the equals button, that acts like a right parenthesis, and everything before it will get multiplied by 0 and then added to your original number and the trick will be over.
- Let's go through the trick so that you learn some important details.
- On your smartphone, bring up the calculator and then turn the phone on its side so that you are working with the scientific calculator. Enter your force number, pi (or press the p button), and then press the toxic buttons.

    + 0 × (

- Next, turn your phone upright so that the simple calculator is in the display.

- When your phone is on its side, the multiplication button might still be lit up in the scientific mode, and that might cause suspicion, so don't turn the phone upright—and into the simple calculator mode—until after the volunteer gives you his or her first number.

- Let's say that the person chooses the number 5. Once you press 5, the multiplication button is no longer lit up and you'll show that to your audience.

- Then, ask for a 1- or 2-digit number to multiply. Let's say the volunteer chooses 7. You hit ×, then 7, and then + (don't press =). This will show the answer of 5 × 7, which is 35, so the audience will see that the calculator is working properly.

- Next, ask the volunteer to add a 2- or 3-digit number. (You've already pressed +, but there's no harm in pressing it again.) When the person gives you his or her number, you enter it and then press − (not =). Again, the calculation will display the right answer.

- Then, have your volunteer subtract a smaller number. Enter it and then press ÷. When you do this, the calculator will not yet show an answer because it's waiting for the next number to complete its calculation. Most people won't seem to notice this, and they will stop thinking hard about the numbers on the display when they hear the word "division" anyway.

- Finally, ask the volunteer to divide by another number. You can ask for a number that will be somewhere in the range of the answer you want, but it's not that important. You can just ask for a random 2-digit number if you want. After the person gives you his or her number, you press = and, voilà, out pops your force number, which in this case is pi.

- If you keep the method of this trick secret—and please do—then you will amaze a whole lot of people!

REFERENCES

Gardner, *Mathematics, Magic and Mystery.*

Gardner, *Mental Magic.*

Scarne, *Scarne on Card Tricks.*

Simon, *Mathematical Magic.*

EXERCISES

1 Think of a number between 1 and 30. The magician shows you the following 5 cards and asks you which cards contain your number.

Card 1: 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

Card 2: 2 3 6 7 10 11 14 15 18 19 22 23 26 27 30

Card 3: 4 5 6 7 12 13 14 15 20 21 22 23 28 29 30

Card 4: 8 9 10 11 12 13 14 15 24 25 26 27 28 29 30

Card 5: 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

After indicating which cards have your number, the magician immediately reveals the number. How?

2 The following curiosity can be turned into a feat of psychic power or supersensitivity. Take any number of coins—for example, 10—and notice how many of them are showing heads. Suppose that h of the coins are showing heads. Divide the coins into 2 piles by choosing any h coins and putting them in a pile on your right, and the rest of the coins go to the left. Then, turn over every coin in the right pile so that it shows the opposite side. It should now be the case that the right pile and the left pile have an equal number of heads. Why?

SOLUTIONS CAN BE FOUND ON PAGE 133.

GEOMETRIC AND TOPOLOGICAL MAGIC

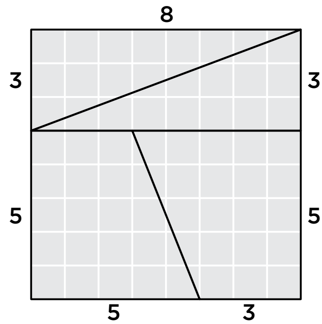
This lesson focuses on things that seem physically impossible—that seem to violate the laws of geometry and topological intuition. You will be exposed to geometric and topological mysteries that are based on mathematics.

EXPLANATIONS

VANISHING RABBIT

- Every course on magic should include the illusion of making a rabbit disappear. This trick is a mathemagical version. It is an example of what mathemagicians call a **GEOMETRICAL VANISH**. But before this trick is explained, let's look at a few other puzzling geometrical vanishes—starting with one that wouldn't fool anybody but illustrates the point.
- In the video, there are 5 long lines on the screen.
- With a wave of the magician's hands, the 5 lines become 4 lines. Which line disappeared?
- Technically, none of them did. Instead, he switched the 5 lines for 4 lines that are each slightly longer than they were before. That's the basic premise of most geometrical vanishes.

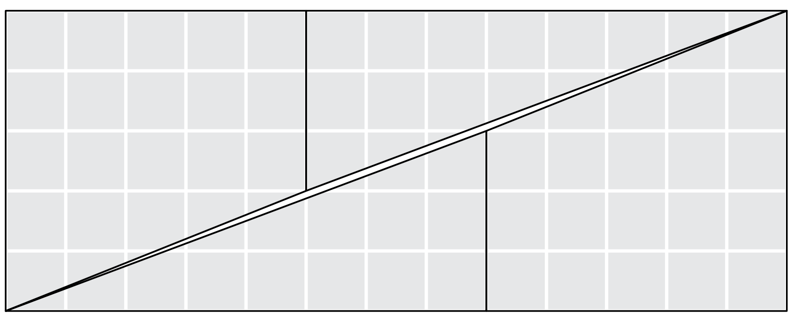
- Let's start with an 8-by-8 square, which has an area of exactly 64. Cut out a 3-by-8 rectangle; then cut that along the diagonal to form 2 right triangles. From the other rectangle, cut 2 trapezoids with sides of length 5 and 3.



- When you rearrange the pieces in the way they are displayed in the video, you get a 5-by-13 rectangle. Note that the sides really do have lengths 5 and 13, but this figure has an area of 65. Where does the extra area come from?

- The secret, which you can see if you look closely at the screen on the video, is that the diagonal from the bottom left to the upper right is not really a straight line. If it were a straight line, then it would have a slope of $5/13$, which is approximately 0.38.

$$5/13 \approx 0.385$$



- But note that the bottom triangle has slope $3/8$, which is

$$3/8 = 0.375,$$

which is slightly lower.

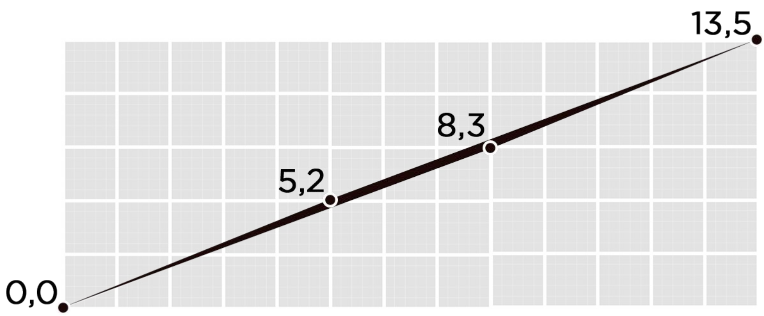
- And the upper part of the diagonal that comes from the other trapezoid has slope $2/5$, which is

  $2/5 = 0.4,$

  which is slightly higher. So, what looks like a diagonal line actually has a little bit of a dent in it.

- Similarly, if you look above the diagonal, you see that the trapezoid has a slope of 0.4 and then the triangle has a slope of 0.375. The result is that there is a little bit of open space inside the rectangle. In fact, it's a tiny parallelogram that goes through the points (0, 0), (5, 2), (8,3), and (13, 5).

- The area of that parallelogram turns out to be exactly 1.



Area = 1

FURTHER EXPLANATION

- The area of a parallelogram that goes through the points (0, 0), (a, b), (c, d), and $(a + c, b + d)$ is the absolute value of $ad - bc$.

- In this example, because the parallelogram goes from (0,0) to the points (5, 2) and (8, 3), that gives you an absolute value of

  $(5 \times 3) - (8 \times 2) = 15 - 16 = -1.$

- So, the area of the parallelogram is 1.

- Incidentally, you may have noticed that the numbers that appeared in the parallelogram—2, 3, 5, 8, and 13—are all Fibonacci numbers! Even the dimensions of the original square (8 by 8) and final rectangle (5 by 13) are Fibonacci numbers. That is not a coincidence.

THE FIBONACCI NUMBERS

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

- The Fibonacci numbers have many amazing mathematical properties. One of them is that if you take a Fibonacci number, such as 8, and multiply its immediate neighbors—5 and 13—you get 65, which is just 1 away from 8^2, or 64.

- The Fibonacci numbers also have the parallelogram property. If you take any 4 consecutive Fibonacci numbers a, b, c, and d, then it can be shown that $ad - bc$ is always 1 or −1.

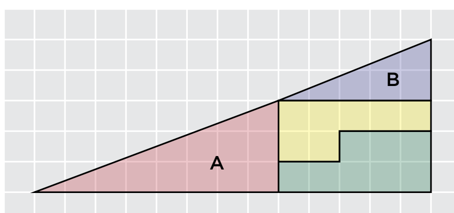
- For example, with 2, 3, 5, and 8, you have

$$(2 \times 8) - (3 \times 5) = 16 - 15 = 1.$$

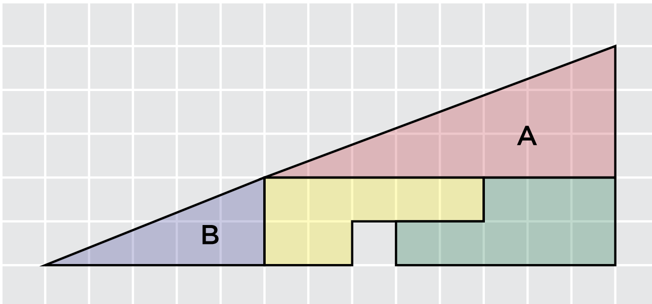
- You can use the Fibonacci numbers to create larger geometrical vanishes where the tiny parallelogram becomes even more invisible. In the case of a 13-by-13 square, the new 8-by-21 rectangle loses 1 unit of area because the diagonals just barely overlap to form a parallelogram of area 1.

- Here's a variation on the last geometrical vanish. This idea is attributed to Paul Curry, an amateur magician who created some amazing card tricks.

- Start with the 5-by-13 triangle shown at right.



- Notice how you can rearrange the pieces to obtain a new 5-by-13 triangle but with an entire 1-by-1 square missing from it. Essentially, what's happening here is the same idea as before, but it's more difficult to see the missing parallelogram.



- You start with a right triangle with sides of length 8 and 3. Let's call it triangle A, and it has an area of 12. Triangle B is a 5-by-2 right triangle, which has an area of 5. There are also figures that look like a comb and a toothbrush, with 7 and 8 squares, respectively. The total area of these pieces is

$$12 + 5 + 7 + 8 = 32.$$

- When you put them together, it looks like a triangle. But it's not, because triangle A has a slope of $3/8 = 0.375$ and triangle B has a slope of $2/5 = 0.4$. So, the figure, which looks like a right triangle with a hypotenuse going from (0, 0) to (13, 5), actually has a tiny dent in it at the point (8, 3) that is hard for the eye to detect.

- After the pieces are rearranged, again it looks like a right triangle with a hypotenuse going from (0, 0) to (13, 5), but this time there is a tiny bulge at the point (5, 2). So, once again, the difference in the area of this new triangle minus the difference of the old triangle is the parallelogram from before, which has an area of 1.

- Hence, this bulging triangle would have an area of 33, which allows you to squeeze in an extra 1-by-1 square.

- In the original disappearing rabbit trick, instead of using a 5-by-13 triangle, a 6-by-13 rectangle is used. Here, there's an extra legitimate 6-by-13 right triangle that doesn't move. Also, there are 13 extra rabbits that are attached to the 2 triangles so that they appear as trapezoids. But otherwise, the same mathematics shows that the bulge of the rearranged object (which is partly hidden by the upper triangle) has an area of 1, which allows you to have an extra space for the unit square.

There are many fascinating variations of geometric vanishes, and the principle goes back centuries.

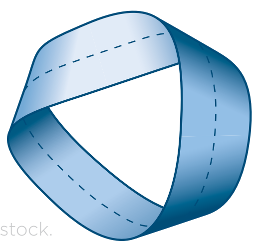
In the video, there is an Alice in Wonderland*–themed vanish created by ThinkFun, a company that produces games and puzzles based on mathematics and logic. The puzzle is in honor of Martin Gardner, who was the world's expert on mathematical magic, geometrical vanishes, and* Alice in Wonderland*. See if you can figure it out!*

MÖBIUS MAGIC

- When you tape the ends of a simple strip of paper together, they form a loop. Naturally, if you cut through the loop, you end up with 2 loops. But suppose you take your strip and give the end a half twist before taping the ends together. Mathematicians call this a Möbius strip, named after the 19th-century mathematician August Ferdinand Möbius, who discovered many of its properties.

- The Möbius strip only has 1 side. If you run your finger around the "outside," you eventually reach the "inside." This is in contrast with the previous loop, which is 2-sided, because if you run your finger on the outside, then it stays on the outside, and if your finger starts on the inside, then it stays on the inside.

- The video shows what happens if you cut through Möbius strips in various ways—leading to mysterious and surprising results! Try these experiments yourself and explore variations to see what you can discover.



BAGEL CARVING

This lesson requires the careful and skillful use of a knife and is potentially dangerous. Please use extreme caution in attempting this or any tricks involving knives.

- This trick applies the Möbius ideas presented in the video. It involves taking a regular bagel and carving it in just the right way to create 2 linked bagels—a Möbius bagel.
- The steps that follow outline the method of Glen Whitney, founder of the National Museum of Mathematics in New York. You just have to learn 1 move and repeat it 4 times. The key points to note on your bagel are the north pole (in the top center), the south pole (at the bottom center), and the points that represent 3 o'clock and 9 o'clock on the front and back of the bagel.
- First, you carve a curve from the north pole to 3 o'clock. As you carve, your knife should always be perpendicular to the surface and go all the way through. In other words, when your knife is at the north pole, it is pointing down vertically, and when it reaches 3 o'clock, it's pointing horizontally. It's helpful to go back through the same path again, just to make sure that the knife went all the way through.
- Next, you flip the bagel over so that the north and south poles switch places; then, you do the same move again, carving from the north pole to 3 o'clock, where you join up with the previous cut. Again, it's good to go through it a second time.
- Next, twirl the bagel around so that 9 o'clock and 3 o'clock switch position. Then, perform the same move again, carving from the north pole to 3 o'clock. Finally, you switch the north and south poles and do the same move one last time, joining up with the previous cut.
- Then, carefully pull the bagel apart, and if all has gone well, you will have linked bagels!

REFERENCES

Gardner, *Mathematics, Magic and Mystery.*

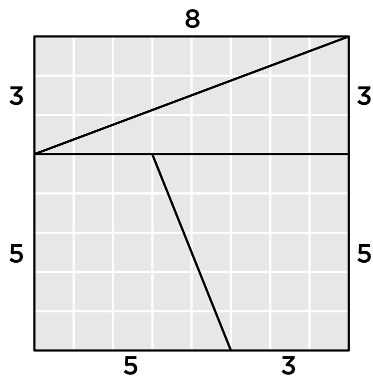
Pickover, *The Möbius Strip.*

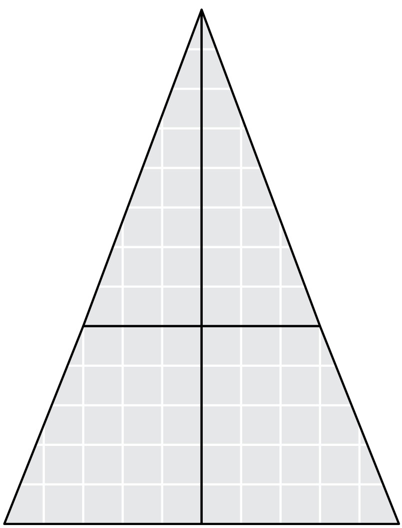
Prevos, *The Möbius Strip in Magic.*

Simon, *Mathematical Magic.*

EXERCISES

1 Using a sheet of graph paper take an 8-by-8 square and cut it into 2 right triangles (with legs of length 3 and 8) and 2 trapezoids (each with a height of 5 and parallel sides with lengths of 3 and 5). Clearly, the 8-by-8 square has an area of 64.





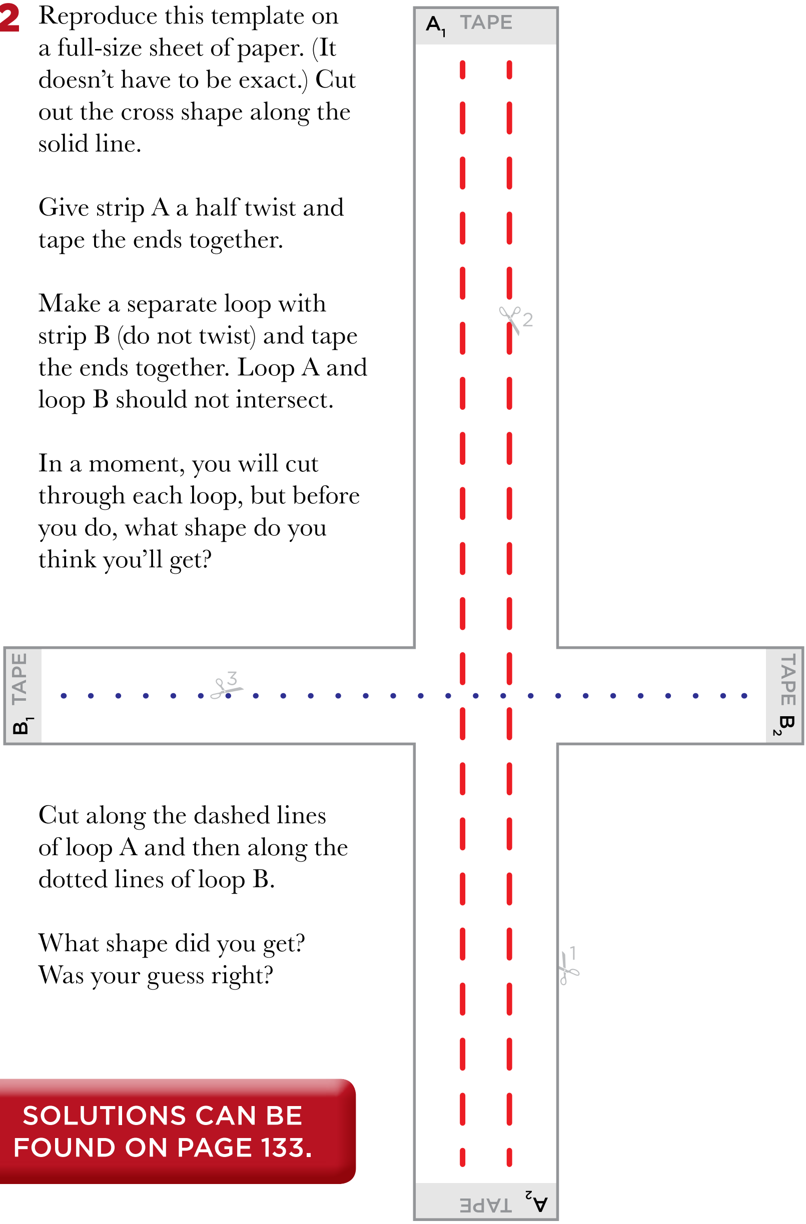
Next, rearrange the pieces to form an object that looks like the triangle shown at left. Because the triangle has a width of 10 and a height of 13, then its area is 65. What's going on?

2 Reproduce this template on a full-size sheet of paper. (It doesn't have to be exact.) Cut out the cross shape along the solid line.

Give strip A a half twist and tape the ends together.

Make a separate loop with strip B (do not twist) and tape the ends together. Loop A and loop B should not intersect.

In a moment, you will cut through each loop, but before you do, what shape do you think you'll get?



Cut along the dashed lines of loop A and then along the dotted lines of loop B.

What shape did you get? Was your guess right?

SOLUTIONS CAN BE FOUND ON PAGE 133.

12

MAGIC SQUARES

This final lesson explores the magic and mystery of what mathemagicians refer to as magic squares. Although there are many ways to create magic squares, the video focuses on 3 methods: one that is fast, one that is "smooth," and one that is personal.

EXPLANATIONS

FAST MAGIC SQUARE

- There is a quick way to come up with a magic square that uses all the numbers from 1 through 16. To make it, start with the numbers 1 through 16 in their natural order.
- Draw an X through the square, which goes through 8 of the numbers.
- Swap each number that is on the X with the number that is diagonally opposite it. So, you swap 1 with 16, 4 with 13, 6 with 11, and 7 with 10.
- Removing the X gives you a magic square where every row, column, and diagonal add to 34.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	2	3	**13**
5	**11**	**10**	8
9	**7**	**6**	12
4	14	15	**1**

- It turns out that there are 880 different ways to arrange the numbers 1 through 16 to create a magic square. Such a magic square must always add to 34 because the numbers 1 through 16 add up to 136, which is 4 times 34.
- The magic square shown at right has even more symmetries. In this magic square, not only does every row, column, and both diagonals add to 34, but so does every 2-by-2 square inside of it, as well as the wraparound diagonals.

34

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

- How do you create a magic square for any total, besides 34? To use the fastest method, all you have to remember is the magic square shown above. In fact, you only need to remember 12 of the 16 numbers: the numbers 1 through 12.
- If you're given a total of *t*, then all you have to do is subtract 20 from that total and put it in the first missing spot. For example, if the total was 76, you would subtract 20 and put 56 in the higlighted square.

76

8	11	$\boxed{\textbf{56}}$	1
	2	7	12
3		9	6
10	5	4	

- For the other 3 squares, you just subtract 1, resulting in 55; then add 3, resulting in 58; and subtract 1 again, resulting in 57. You end up with the square shown at right.
- You can check that every row, column, diagonal, and most symmetrically placed groups of 4 squares will add to 76.

76

8	11	**56**	1
55	2	7	12
3	**58**	9	6
10	5	4	**57**

- In general, if you want a total of *t*, you get the square shown here.

- As you can check, every row, every column, both diagonals, and many other groups of 4 add up to the total *t*. This method is fast, because you always write the same 12 numbers down and the other 4 numbers are easy to remember. But this method tends to produce a very lopsided magic square, where 4 of the numbers can be much larger than the others.

t

8	11	***t* − 20**	1
***t* − 21**	2	7	12
3	***t* − 18**	9	6
10	5	4	***t* − 19**

"SMOOTH" MAGIC SQUARE

- In this version, all 16 numbers are close together. In fact, they are almost consecutive, with no large gaps to be seen. Also, this version gives your assistant something to do because he or she can point to the squares in any order and you can fill in the square immediately.

- First, show the original magic square with the numbers 1 through 16. As you'll see, that's not just for the audience's benefit, but for yours as well. You will use this magic square to create a magic square for your assistant's total.

- Then, you ask for any 2-digit number larger than 30—prefereably bigger than 40. Let's say your assistant chose 74. If you can somehow add 40 the first magic square, which adds to 34, then you can create a magic square with a total of 74. The easiest way to do that is to add 10 to each number in the original square, because that will add 40 to every group of 4. Thus, all the groups of 4 that add to 34 in the first magic square will now add to 74 in the square you create.

- So, as your assistant points to a square, all you do is look at the corresponding square and add 10 to it. The resulting magic square would look like the one shown at right, which adds to 74 in dozens of different ways.

74

18	21	24	11
23	12	17	22
13	26	19	16
20	15	14	25

- But now suppose that instead of a total of 74, your assistant asked for a total of 76. What do you do in that situation? (You're certainly not going to add 12.5 to every square because that would be ugly.) You pretty much do the same thing as before, but now you have to squeeze in an extra 2 to each row, column, and diagonal.

- To do this, you take the 4 largest numbers of the original square: 13, 14, 15, and 16 (circled). And for these numbers, instead of adding 10, you add 12, which is 2 more than 10.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

- Notice that every row, every column, and both diagonals have one circled number in it. This means that they will all increase by

$$10 + 10 + 10 + 12 = 42.$$

- So, the new total is guaranteed to add to 76.

$$34 + 42 = 76$$

- In other words, if this were the magic square that added to 74, then by adding 2 more to the circled numbers, you get the magic square for 76.

76

18	21	26	11
25	12	17	22
13	28	19	16
20	15	14	27

- A word of caution: It is no longer the case that every 2-by-2 block that added to 74 will add to 76. To get a total of 76, you need exactly 1 circled number. But luckily that does include every row, every column, both diagonals, the 4 corners, and all the other 2-by-2 blocks (except for one that has 2 circled numbers and one that has no circled numbers).

The reason that the 4 circled numbers were chosen to correspond to the 13, 14, 15, and 16 in the original square is that because you are adding 12 to the 4 largest numbers of the original square, the new magic square will still consist of 16 different numbers. No numbers will be repeated, which is aesthetically nice.

- In general, to create a magic square with total t, you need to add $t - 34$ to the original magic square. If this number is divisible by 4—that is, if

  $$t - 34 = 4q,$$

  where q stands for quotient—then you simply add q to every number.

- For example, if the requested number was 62, then because

  $$62 - 34 = 28 = 4 \times 7,$$

  then you simply add 7 to every entry of the first magic square to get the new magic square.

- On the other hand, if your assistant chose the number 63, which is 1 more than 28, then you still add 7 to most of the squares, but whenever he or she points to one of the squares corresponding to the 13, 14, 15, or 16, then you add 1 more than 7—namely, 8—giving you the square shown at right.

63

15	18	22	8
21	9	14	19
10	24	16	13
17	12	11	23

- The general rule is this: Given your total t, if $t - 34$ is not a multiple of 4—for example,

$$r - 34 = 4q + r,$$

- where the remainder r is equal to 1, 2, or 3—then when your assistant chooses a square that corresponds to one of the 12 small numbers, you add q to it. But when your assistant chooses a square that corresponds to 13, 14, 15, or 16, you add $q + r$.

After you create the new magic square, say to your assistant, "Now we noticed that the first magic square adds to 34; let's see how we did with your magic square." Then, put the first magic square away. That's called ditching the evidence—so nobody has the time to compare the 2 magic squares.

DOUBLE-BIRTHDAY MAGIC SQUARE

- Although this method's resulting magic square is not as "smooth" as the one you were just introduced to, it looks much more challenging, has a surprise ending, and is much more personalized because you base it on your assistant's birthday.

- To start, your assistant's birthday goes in the top row. In the video, the magician used his assistant's daughter's birthday, which is 12/3/2013.

- The most important step is to add these 4 numbers—12, 3, 20, and 13—correctly, because if you make a mistake here, there's no recovery.

$$12 + 3 + 20 + 13 = 48$$

People really enjoy this magic square routine, especially if you base it on someone's birthday. And if you do it on the back of your business card, you can be sure that the person will keep that card as a souvenir for a very long time.

- Because you've seen this trick before, you know that the person's birthday not only appears in the top row but also in the 4 corners, so write down a 3 in the bottom left corner. (Save the 20 from the year for last to keep it a surprise.)

48

12	3	20	13
	$\boxed{?}$		
3			

- Next, you start heading up the diagonal. The secret to the magic square is this: You can essentially place any number you want in the highlighted square and the rest of the square is essentially forced. The only number you don't want to put in this square is the number that appears in the third square in the top row, which is 20 in this case. If you do, you'll still get a magic square, but every row and column will have the same 4 numbers—12, 3, 20, and 13—and that won't look very magical.

- On the other hand, the farther you get away from this number (in this case, 20), the more likely it is that you will end up with negative numbers, which you should try to avoid if possible. So, all you do is take this number (again, 20) and add 1 to it. In this case, 20 + 1 = 21, and you write that number in the highlighted square.

48

12	3	20	13
19	14	11	4
14	$\boxed{21}$	2	11
3	10	15	20

- Now this magic square becomes like the easiest Sudoku you've ever done.

- As you can verify, there are dozens of groups of 4 that add to 48.

- But how can you be sure they will always work? The answer is through algebra.
- For any birthdate with numbers a, b, c, and d, the grand total will be

$$a + b + c + d.$$

- The end result is the magic square shown here, where adding 1 to a number is denoted by +, adding 2 to a number is denoted by ++, subtracting 1 from a number is denoted by −, and subtracting 2 from a number is denoted by −−.

a	b	c	d
$c-$	$d+$	$a-$	$b+$
$d+$	$c+$	$b-$	$a-$
b	$a--$	$d++$	c

- Notice that every row, every column, and both diagonals contain an a, a b, a c, and a d as well as an equal number of pluses and minuses. Hence, they will add up to the total: $a + b + c + d$.
- If your assistant is born between 2000 and 2009, avoid using $c = 0$ in the top row because that forces a $c- = -1$ in the second row and you want to avoid negative numbers if possible. Instead, for a date like 2005, put $c = 20$ in the top row and $d = 05$ next to it so that the entire year is spelled out. When this happens, it's worth noting that you will always have $c+ = 21$ and $c- = 19$ in your magic square, which saves calculating time.

The reason most magicians don't explain their tricks is because they've all made that mistake before and haven't gotten the reaction they want.

As a magician, you want people to say, "Wow, that's clever. You're good!" But they never do once you've shared the secret of how the trick is done. Instead, they say, "Oh, that's simple! I thought you were good, but I just wasn't paying attention!"

So, if a magician shares his or her method with you, remember how impressed you were before he or she explained it.

- Finally, although you might try to avoid negative numbers in your square, sometimes they are unavoidable. For example, any birthdays in January will have $a = 1$ and will therefore have an $a - - = -1$ in the bottom row. But you can live with that.

- If you look for them, there are more than 3 dozen different neatly arranged groups of 4 that are guaranteed to add to your total. In addition to the 10 main rows and diagonals, there are 12 other 2-by-2 squares (including the 4 corners), 2 broken diagonals, 2 pentagons, 4 corners of 3-by-3 squares, 4 stalagmite-stalactite pairs, and 4 L-shaped regions. See if you can find more!

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EXERCISES

1 **a)** Create a fast magic square for the total 38.

b) Create a "smooth" magic square for the total 38.

c) Create a double-birthday magic square for the date 11/18/36.

d) Use the algebraic magic square chart to verify that the magic square will give the proper total for at least 40 different groups of 4.

2 What is the secret to the magic matrix trick at the end of the video lesson?

SOLUTIONS CAN BE FOUND ON PAGE 134.

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EXERCISE SOLUTIONS

CLICK to go back to the lesson.

LESSON 1

1 After performing the procedure described above, the selected card will be the 11th card from the top. The way you should perform it is to deal about 15 cards faceup, mentally noting the 11th card, and then say to your volunteer, "Would you be surprised if the next card I turned over was your card?" Because he or she has seen you deal past his or her card, the volunteer will usually say that he or she would be very surprised. Your volunteer expects you to deal the 16th card, but instead you simply turn the 11th card from faceup to facedown!

Why does the selected card go to the 11th position? After the first deal, the card's column is picked up second, so it will be in position 8, 9, 10, 11, 12, 13, or 14. After the second deal, the card will be in the third, fourth, or fifth position in the column. Thus, after the cards are gathered, it will be in the 10th, 11th, or 12th position. After the third deal, the card will be in the fourth position of its column, and after those cards are gathered, it will be the 11th card of the deck.

2 The trick begins with 4 cards, and after they are torn, you have 8 half cards arranged in some order *abcdabcd*. Let's see what happens after each step.

STEP 1 | Transfer the top card to the bottom: *bcdabcdab*.

STEP 2 | Transfer the top 2 cards to the bottom: *dabcdabc*.

STEP 3 | Bury the top 3 cards in the middle: *c*(6 other cards)*c*.

STEP 4 | Put the top card in your pocket: (6 other cards)*c*.

At this point, you have taken card *c*, and its mate is at the bottom of the packet.

STEP 5 | Take the top 1, 2, or 3 cards and put them in the middle of the packet. Card *c* is still at the bottom.

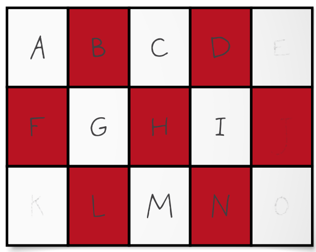
STEP 6 | Take 1, 2, or 3 cards and either put them in the middle or throw them away. You now have 4, 5, 6, or 7 cards, and card *c* remains at the bottom.

STEP 7 | Transfer the top card to the bottom 7 times. Depending on whether you have 4, 5, 6, or 7 cards, your new arrangement will now be one of these 4 arrangements, where *x* represents any card that is not *c*: *cxxx*, *xxcxx*, *xxxxcx*, or *xxxxxxc*.

STEP 8 | Play She Loves Me, She Loves Me Not, where every other card is eliminated until you are left with a single card. Notice that in all 4 arrangements, the card that survives is card *c*. For example, with 6 cards, numbered 123456, you first eliminate numbers 2, 4, and 6 and then card 3 and then 1, so the surviving card would be the fifth card, which is card *c*.

LESSON 2

1 Either the faceup or facedown letters will spell the word "MAGIC." The reason is that if you imagine that the letters are written on a checkerboard with alternating light and dark squares, then the light-colored squares consist only of the letters M, A, G, I, C, and some blank squares.



2 The faceup cards (and the facedown cards) will sum to 35. That's because the cards on the light squares and dark squares each add to 35. Note that the first 14 numbers of the table come from the digits of pi but the 15th digit was chosen to make the trick work. (The 15th digit of pi would be 9 instead of 2.)

LESSON 3

1 **a) LMH** (low, medium, high) adds 1 to 6♦, so the facedown card is 7♦.

b) MHL (medium, high, low) adds 4 to 8♠, yielding Q♠.

c) HML (high, medium, low) adds 6 to J♦, yielding 4♦.

d) LHM adds 2 to 4♣, yielding 6♣.

2 **a)** 2 + 7 + 1 + 8 = 18 is 2 greater than 16, so the anchor card is in the second position—namely, 7♥. The other 3 cards, 2A8, are in MLH order, so you add 3 to 7♥ to obtain 10♥.

b) 0 + 0 + 9 + 4 = 13 is 1 greater than 12, so the anchor card is in the first position—namely, K♠. The other 3 cards, Q94, are in HML order, so the facedown card is 6♠.

LESSON 4

1 Recall that every 4 cards is the same suit and causes the value to go down by 1. Because 20 is a multiple of 4 (4 × 5), then 20 cards after the J♠ the card must still be a spade, and the value has decreased by 5, so the card would be 6♠. To find the 23rd card, just count 3 cards forward from 6♠. Adding 9 and shifting 3 suits farther gives you 15♥, which is 2♥.

2 The cards are in Si Stebbins order, and cutting the cards will not disturb that. After spelling "COLOR" and looking at the next card, the magician has counted 6 cards, which will be the same color (and different suit) from the chosen card. After spelling "NUMBER," the following card would be 13 cards after the original card, which will have the same value. (Incidentally, it will also be of the opposite color.) Then, after spelling "MATCHING CARD," the next card will be another 13 cards and therefore will be 26 cards away from the original card, which will be the mate of the selected card.

LESSON 5

1 $28 = 16 + 8 + 4$ has binary representation 11100. So, the proper sequence would be in-in-in-out-out. For position 42, you note that $41 = 32 + 8 + 1 = 101001$, so you would perform in-out-in-out-out-in.

2 The binary number 110000 represents the number $32 + 16 = 48$, so the top card will go to position 49. For the 17th card, recall that for cards in the top half of the deck, an in-shuffle sends a card in position n to position $2n$, whereas an out-shuffle sends the card to position $2n - 1$. For cards in the bottom half, the in-shuffle sends the card in position n to position $2n - 53$ and the out-shuffle sends it to position $2n - 52$. Hence, the 2 in-shuffles will send the card at position 17 to position 34 and then to position $68 - 53 = 15$. The next 4 out-shuffles will send the card to position 29, then to position 6, then to 11, and finally to position 21.

LESSON 6

1 Because of the Gilbreath principle, the first 4 cards consist of all 4 suits. Because your facedown cards are a diamond and a heart, then you know that your volunteer's facedown cards must be a club and a spade. Also by the Gilbreath principle, the first 13 cards will consist of all 13 different values. Because you can see all values except for the 5 and 7 among these 13 cards (ignoring 6♥), you know that your volunteer's facedown cards must be 5 and 7. Thus, his or her cards must either be 5♣ and 7♠, or 5♠ and 7♣. You can ask (or guess) which suit has the larger number, but you can do better than guessing. Look at your facedown cards. In the 8-kings stack, 9♥ comes before 5♠, so the chances are very, very good that the 5 is 5♠ instead of 5♣. Also, 2♦ and 7♣ are adjacent in the 8-kings stack, so you can say with confidence that the facedown cards are 5♠ and 7♣.

2 Begin with the cards in alternating color order, which won't be disturbed when the cards are cut. After the volunteer gives the cards a riffle shuffle, deal a few cards to show that there is no pattern to the cards. Deal until you get 2 cards of the same color next to each other;

then, deal one more card. The cards will now have the property that every pair of cards will have a black card and a red card. So, when the volunteer guesses the color of the card, you turn up the card to see if his or her guess is right or wrong. Look at the turned-up card and then guess the opposite color for your card. You will be right every time.

LESSON 7

1 Starting with PIN *abc*, the number *abcabc* is equal to *abc* × 1001. And because 1001 = 7 × 11 × 13, when you divide the 6-digit number by 11, 13, and the PIN, then you must end up with 7. But it's more impressive to end up with something that is not a whole number. By dividing by 23, then 16, and then the PIN, you will end up with 1001 ÷ 368 = 2.72010869…. Provide as many digits as you want to.

2 If you start with the numbers a, b, and c, then when you create the numbers ab, ac, ba, bc, ca, and cb, you note that each letter appears twice in the ones column and twice in the tens column. Hence, the ones digits will contribute $2a + 2b + 2c$ to the total, and the tens digits will contribute $20a + 20b + 20c$ to the total, so the numbers must add to $22a + 22b + 22c = 22(a + b + c)$. When this number is divided by the sum of the digits, $a + b + c$, you must get 22.

LESSON 8

1 **a)** 594

b) 836

c) 0.835164…

d) 7225

e) 2916

f) 11,554

g)

$$\begin{array}{rl} 98 & (-2) \\ \times 95 & (-5) \\ \hline \end{array}$$

In this case, you compute 98 − 5 = 93 and then multiply by 100 to obtain 9300. Adding (−2) × (−5) = 10 to this gives you 9310.

h)

$$\begin{array}{rl} 112 & (12) \\ \times\ \ 94 & (-6) \\ \hline \end{array}$$

In this case, you compute 112 − 6 = 106 and then multiply by 100 to obtain 10,600. Adding 12 × (−6) = −72 gives you 10,528.

LESSON 9

1 **a)** The cube root is 53, because 148 lies between 5^3 and 6^3, and 3^3 ends in 7.

b) The cube root is 89, because 704 lies between 8^3 and 9^3, and 9^3 ends in 9.

c) The cube root is 324. Because 34 lies between $3^3 = 27$ and $4^3 = 64$, the first digit of the cube root must be 3. And because 4^3 ends in 4, the last digit must be 4. Thus, the cube root has the form 3 ? 4.

To find the middle digit, you add the digits of the cube:

$$3 + 4 + 0 + 1 + 2 + 2 + 2 + 4 = 18,$$

which has a digital root of 9, which means that the original 3-digit number must be a multiple of 3. Thus, the digits of 3 ? 4 must sum to a multiple of 3, so the middle digit is either 2 or 5 or 8. And because the 9-digit number is *much* closer to 27 than it is to 64, you choose the smallest of the middle numbers for a cube root of 324.

d) Comparing 273 with $6^3 = 216$ and $7^3 = 343$, you see that the first digit is 6, and because 9^3 ends in 9, the last digit is 7. So, the cube root has the form 6 ? 9. For the middle digit, you sum the digits

$$2 + 7 + 3 + 3 + 5 + 9 + 4 + 4 + 9 = 46,$$

which has a digital root of 1. Thus, 6 ? 7 must be 1 more than a multiple of 3, so the middle digit is either 1 or 4 or 7. Because 273 is squarely in between 216 and 343, you choose the middle candidate for a cube root of 649. To verify this, notice that 650^3 is approximately (and a little larger than) $600 \times 650 \times 700 = 273{,}000{,}000$.

e) As in the previous 2 problems, because 656 lies between $8^3 = 512$ and $9^3 = 729$, you know that the first digit is 8, and because the cube ends in 9, you know that the last digit is 9. Hence, your cube root is of the form 8 ? 8. To find the middle digit, you see that the digits of 656,234,909 sum to 44, so it has a digital root of 8. This means that 8 ? 9 must be 2 bigger than a multiple of 3, so the middle digit must be 0, 3, 6, or 9. Because 656 is not especially close to 512 or 729, you can rule out 0 or 9 as the middle digit, and because 656 is a bit closer to 729 than it is to 512, you choose the larger candidate for a cube root of 869. (Also, 850^3 is about $800 \times 850 \times 900 = 612{,}000{,}000$, so the middle digit should be greater than 5.)

2 After step 5, the number on the volunteer's calculator will be a multiple of 9 that ends in 0. After step 6, the number is a multiple of 9 that ends in 9. After adding the age, the final answer will have the same digital root as the age, and the last digit of the answer will be 1 less than the last digit of the age. This will completely determine the age, because you know the last digit and you know what the sum must be. For example, suppose you see the final answer 5,545,981. By looking at the last digit, you know that the chosen age ends in 2. Also, by summing the digits of the final answer,

$$5 + 5 + 4 + 5 + 9 + 8 + 1 = 37,$$

and 37 sums to 10. So, you know that the age has a digital root of 1 and therefore sums to 1 or 10. Because the last digit is 2, the only possible first digit is 8, so the chosen age was 82.

LESSON 10

1 Every number from 1 to 30 can be uniquely expressed as the sum of some of these powers of 2:

1, 2, 4, 8, and 16.

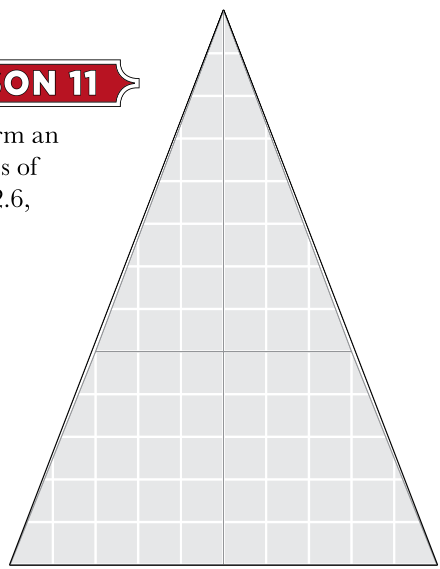
The numbers on cards 1, 2, 3, 4, and 5 are the numbers that use the numbers 1, 2, 4, 8, and 16, respectively, in their representation. For example, the number $19 = 16 + 2 + 1$ appears on cards 1, 2, and 5. All the magician needs to do is add the first numbers that appear on each card that is used. For example, if your number is on cards 3, 4, and 5, the magician knows that the chosen number must be $4 + 8 + 16 = 28$.

2 Suppose you start with n coins, h of which are heads. You move any h coins to the pile on the right. Let's say that among these h coins, x of them are currently heads and $h - x$ of them are tails. How many coins in the left pile are heads? $h - x$. When you flip the coins in the right pile, you will have x tails and $h - x$ heads, so the number of heads in each pile will now be equal.

LESSON 11

1 The 4 rearranged pieces don't form an exact triangle—because the slopes of the triangles, $8/3 = 2.67$ and $13/5 = 2.6$, are close but not equal.

2 You get a small Möbius loop interlocked with a big square. Try it!



LESSON 12

1

a) **38**

8	11	18	1
17	2	7	12
3	20	9	6
10	5	4	19

b) **38**

9	12	15	2
14	3	8	13
4	17	10	7
11	6	5	16

c) **38**

11	18	3	6
2	7	10	19
7	4	17	10
18	9	8	3

d) Every group of 4 that contains an *a*, a *b*, a *c*, and a *d* and an equal number of plus signs as minus signs will add to the magic total of 38. It's easy to check that all 4 rows, 4 columns, and 2 diagonals have this property. But there are many others. For example, there are seven 2-by-2 squares like these 4 in the middle that add to $a + b + c + d$ (at right).

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

See below for more examples.

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

There are 2 broken diagonals:

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

There are 4 that are shaped like a pentagon:

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

The corners of the 3-by-3 squares sum to the proper total:

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

There are 4 stalagmite-stalactite pairs:

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

There are 4 L-shaped patterns:

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

a	b	c	d
c−	d+	a−	b+
d+	c+	b−	a−
b	a−−	d++	c

And last but not least (and by design), there are the 4 corners.

2 Let's look more closely at the numbers in the matrix.

It's true that the first 5 squares come from the digits of pi, but that's just a red herring. What's really going on is that this is an addition table.

31	41	59	26
5	15	33	0
17	27	45	12
14	24	42	9

Notice what happens when you put numbers alongside the rows and columns of the matrix like this:

	0	10	28	−5
31	31	41	59	26
5	5	15	33	0
17	17	27	45	12
14	14	24	42	9

Every entry in the matrix is the sum of the corresponding row and column numbers. For example, the number 42 in the fourth row and third column is equal to 14 + 28. When you find the matching colors, you will have 4 entries that each lie in different rows and columns.

For example, consider what happens if you end up like this:

	0	10	28	−5
31	31	41	59	26
5	5	15	33	0
17	17	27	45	12
14	14	24	42	9

Each of the 4 selected numbers will be the sum of one of the row numbers and one of the column numbers. So, the grand total must be the sum of the 8 row and column numbers, specifically. Any 4 numbers from different rows and columns must add to

31 + 5 + 17 + 14 + 0 + 10 + 28 − 5 = 100.