József Sándor

GEOMETRIC THEOREMS, DIOPHANTINE EQUATIONS, AND ARITHMETIC FUNCTIONS

****

AB/AC=(MB/MC)(sin *u* / sin *v*)

1/z + 1/y = 1/z

Z(n) is the smallest integer m such that 1+2+…+m is divisible by n

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American Research Press
Rehoboth
2002

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ISBN: 1-931233-51-9

Standard Address Number 297-5092
Printed in the United States of America

"...It is just this, which gives the higher arithmetic that magical charm which has made it the favourite science of the greatest mathematicians, not to mention its inexhaustible wealth, wherein it so greatly surpasses other parts of mathematics..."

(K.F. Gauss, Disquisitiones arithmeticae, Göttingen, 1801)

Preface

This book contains short notes or articles, as well as studies on several topics of Geometry and Number theory. The material is divided into five chapters: Geometric theorems; Diophantine equations; Arithmetic functions; Divisibility properties of numbers and functions; and Some irrationality results. Chapter 1 deals essentially with geometric inequalities for the remarkable elements of triangles or tetrahedrons. Other themes have an arithmetic character (as 9-12) on number theoretic problems in Geometry. Chapter 2 includes various diophantine equations, some of which are treatable by elementary methods; others are partial solutions of certain unsolved problems. An important method is based on the famous Euler-Bell-Kalmár lemma, with many applications. Article 20 may be considered also as an introduction to Chapter 3 on Arithmetic functions. Here many papers study the famous Smarandache function, the source of inspiration of so many mathematicians or scientists working in other fields. The author has discovered various generalizations, extensions, or analogues functions. Other topics are connected to the composition of arithmetic functions, arithmetic functions at factorials, Dedekind's or Pillai's functions, as well as semigroup-valued multiplicative functions. Chapter 4 discusses certain divisibility problems or questions related especially to the sequence of prime numbers. The author has solved various conjectures by Smarandache, Bencze, Russo etc.; see especially articles 4,5,7,8,9,10. Finally, Chapter 5 studies certain irrationality criteria; some of them giving interesting results on series involving the Smarandache function. Article 3.13 (i.e. article 13 in Chapter 3) is concluded also with a theorem of irrationality on a dual of the pseudo-Smarandache function.

A considerable proportion of the notes appearing here have been earlier published in

journals in Romania or Hungary (many written in Hungarian or Romanian).

We have corrected and updated these English versions. Some papers appeared already in the Smarandache Notions Journal, or are under publication (see Final References).

The book is concluded with an author index focused on articles (and not pages), where the same author may appear more times.

Finally, I wish to express my warmest gratitude to a number of persons and organizations from whom I received valuable advice or support in the preparation of this material. These are the Mathematics Department of the Babeş-Bolyai University, the Domus Hungarica Foundation of Budapest, the Sapientia Foundation of Cluj and also Professors M.L. Perez, B. Crstici, K. Atanassov, P. Haukkanen, F. Luca, L. Panaitopol, R. Sivaramakrishnan, M. Bencze, Gy. Berger, L. Tóth, V.E.S. Szabó, D.M. Milošević and the late D.S. Mitrinović. My appreciation is due also to American Research Press of Rehoboth for efficient handling of this publication.

József Sándor

Contents

Chapter 1. Geometric theorems

"Recent investigations have made it clear that there exists a very intimate correlation between the Theory of numbers and other departments of Mathematics, not excluding geometry..."

(Felix Klein, Evanston Colloquium Lectures, p.58)

1 On Smarandache's Podaire Theorem

Let A', B', C' be the feet of the altitudes of an acute-angled triangle ABC ($A' \in BC$, $B' \in AC$, $C' \in AB$). Let a', b', b' denote the sides of the podaire triangle $A'B'C'$. Smarandache's Podaire theorem [2] (see [1]) states that

$$\sum a'b' \leq \frac{1}{4} \sum a^2 \tag{1}$$

where a, b, c are the sides of the triangle ABC. Our aim is to improve (1) in the following form:

$$\sum a'b' \leq \frac{1}{3} \left(\sum a' \right)^2 \leq \frac{1}{12} \left(\sum a \right)^2 \leq \frac{1}{4} \sum a^2. \tag{2}$$

First we need the following auxiliary proposition.

Lemma. *Let p and p' denote the semi-perimeters of triangles ABC and $A'B'C'$, respectively. Then*

$$p' \leq \frac{p}{2}. \tag{3}$$

Proof. Since $AC' = b \cos A$, $AB' = c \cos A$, we get

$$C'B' = AB'^2 + AC'^2 - 2AB' \cdot AC' \cdot \cos A = a^2 \cos^2 A,$$

so $C'B' = a \cos A$. Similarly one obtains

$$A'C' = b \cos B, \quad A'B' = c \cos C.$$

Therefore

$$p' = \frac{1}{2} \sum A'B' = \frac{1}{2} \sum a \cos A = \frac{R}{2} \sum \sin 2A = 2R \sin A \sin B \sin C$$

(where R is the radius of the circumcircle). By $a = 2R \sin A$, etc. one has

$$p' = 2R \prod \frac{a}{2R} = \frac{S}{R},$$

where $S = area(ABC)$. By $p = \dfrac{S}{r}$ ($r =$ radius of the incircle) we obtain

$$p' = \frac{r}{R} p. \tag{4}$$

Now, Euler's inequality $2r \leq R$ gives relation (3).

For the proof of (2) we shall apply the standard algebraic inequalities

$$3(xy + xz + yz) \leq (x + y + z)^2 \leq 3(x^2 + y^2 + z^2).$$

Now, the proof of (2) runs as follows:

$$\sum a'b' \leq \frac{1}{3}\left(\sum a'\right)^2 = \frac{1}{3}(2p')^2 \leq \frac{1}{3}p^2 = \frac{1}{3}\frac{\left(\sum a\right)^2}{4} \leq \frac{1}{4}\sum a^2.$$

Remark. Other properties of the podaire triangle are included in a recent paper of the author ([4]), as well as in his monograph [3].

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2 On a Generalized Bisector Theorem

In the book [1] by Smarandache (see also [2]) appears the following generalization of the well-known bisector theorem.

Let AM be a cevian of the triangle which forms the angles u and v with the sides AB and AC, respectively. Then

$$\frac{AB}{AC} = \frac{MB}{MC} \cdot \frac{\sin v}{\sin u}. \tag{1}$$

We wish to mention here that relation (1) appears also in my book [3] on page 112, where it is used for a generalization of Steiner's theorem. Namely, the following result holds true (see Theorem 25 in page 112):

Let AD and AE be two cevians ($D, E \in (BC)$) forming angles α, β with the sides AB, AC, respectively. If $\widehat{A} \leq 90^\circ$ and $\alpha \leq \beta$, then

$$\frac{BD \cdot BE}{CD \cdot CE} \leq \frac{AB^2}{AC^2}. \tag{2}$$

Indeed, by applying the area resp. trigonometrical formulas of the area of a triangle, we get

$$\frac{BD}{CD} = \frac{A(ABD)}{A(ACD)} = \frac{AB \sin \alpha}{AC \sin(A - \alpha)}$$

(i.e. relation (1) with $u = \alpha$, $v = \beta - \alpha$). Similarly one has

$$\frac{BE}{CE} = \frac{AB \sin(A - \beta)}{AC \sin \beta}.$$

Therefore

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \frac{\sin \alpha}{\sin \beta} \cdot \frac{\sin(A - \beta)}{\sin(A - \alpha)}. \tag{3}$$

Now, identity (3), by $0 < \alpha \leq \beta < 90^\circ$ and $0 < A - \beta \leq A - \alpha < 90^\circ$ gives immediately relation (2). This solution appears in [3]. For $\alpha = \beta$ one has

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \tag{4}$$

which is the classical Steiner theorem. When $D \equiv E$, this gives the well known bisector theorem.

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3 Some inequalities for the elements of a triangle

In this paper certain new inequalities for the angles (in radians) and other elements of a triangle are given. For such inequalities we quote the monographs [2] and [3].

1. Let us consider the function $f(x) = \dfrac{x}{\sin x}$ $(0 < x < \pi)$ and its first derivative

$$f'(x) = \frac{1}{\sin x}(\sin x - x\cos x) > 0.$$

Hence the function f is monotonous nondecreasing on $(0, \pi)$, so that one can write $f(B) \leq f(A)$ for $A \leq B$, i.e.

$$\frac{B}{b} \leq \frac{A}{a}, \tag{1}$$

because of $\sin B = \dfrac{b}{2R}$ and $\sin A = \dfrac{a}{2R}$. Then, since $B \leq A$ if $b \leq a$, (1) implies the relation

(i) $\dfrac{A}{B} \geq \dfrac{a}{b}$, if $a \geq b$.

2. Assume, without loss of generality, that $a \geq b \geq c$. Then in view of (i),

$$\frac{A}{a} \geq \frac{B}{b} \geq \frac{C}{c},$$

and consequently

$$(a-b)\left(\frac{A}{a} - \frac{B}{b}\right) \geq 0, \quad (b-c)\left(\frac{B}{b} - \frac{C}{c}\right) \geq 0, \quad (c-a)\left(\frac{C}{c} - \frac{A}{a}\right) \geq 0.$$

Adding these inequalities, we obtain

$$\sum (a-b)\left(\frac{A}{a} - \frac{B}{b}\right) \geq 0,$$

i.e.

$$2(A+B+C) \geq \sum (b+c)\frac{A}{a}.$$

Adding $A+B+C$ to both sides of this inequality, and by taking into account of $A+B+C = \pi$, and $a+b+c = 2s$ (where s is the semi-perimeter of the triangle) we get

(ii) $\displaystyle\sum \frac{A}{a} \leq \frac{3\pi}{2s}$.

This may be compared with Nedelcu's inequality (see [3], p.212)

(ii)' $\displaystyle\sum \frac{A}{a} < \frac{3\pi}{4R}$.

Another inequality of Nedelcu says that

(ii)" $\sum \frac{1}{A} > \frac{2s}{\pi r}$.

Here r and R represent the radius of the incircle, respectively circumscribed circle of the triangle.

3. By the arithmetic-geometric inequality we have

$$\sum \frac{A}{a} \geq 3\left(\frac{ABC}{abc}\right)^{\frac{1}{3}}. \tag{2}$$

Then, from (ii) and (2) one has $\frac{ABC}{abc} \leq \left(\frac{\pi}{2s}\right)^3$, that is

(iii) $\frac{abc}{ABC} \geq \left(\frac{2s}{\pi}\right)^3$.

4. Clearly, one has

$$\left(\frac{\sqrt{x}}{b\sqrt{By}} - \frac{\sqrt{y}}{a\sqrt{Ax}}\right)^2 + \left(\frac{\sqrt{y}}{c\sqrt{Cz}} - \frac{\sqrt{z}}{b\sqrt{By}}\right)^2 + \left(\frac{\sqrt{z}}{a\sqrt{Ax}} - \frac{\sqrt{x}}{c\sqrt{Cz}}\right)^2 \geq 0,$$

or equivalently,

$$\frac{y+z}{x}\cdot\frac{bc}{aA} + \frac{z+x}{y}\cdot\frac{ca}{bB} + \frac{x+y}{z}\cdot\frac{ab}{cC} \geq 2\left(\frac{a}{\sqrt{BC}} + \frac{b}{\sqrt{CA}} + \frac{c}{\sqrt{AB}}\right). \tag{3}$$

By using again the A.M.-G.M. inequality, we obtain

$$\frac{a}{\sqrt{BC}} + \frac{b}{\sqrt{CA}} + \frac{c}{\sqrt{AB}} \geq 3\left(\frac{abc}{ABC}\right)^{\frac{1}{3}}.$$

Then, on base of (iii), one gets

$$\frac{a}{\sqrt{BC}} + \frac{b}{\sqrt{CA}} + \frac{c}{\sqrt{AB}} \geq \frac{6s}{\pi}. \tag{4}$$

Now (4) and (3) implies that

(iv) $\frac{y+z}{x}\cdot\frac{bc}{aA} + \frac{z+x}{y}\cdot\frac{cA}{bB} + \frac{x+y}{z}\cdot\frac{ab}{cC} \geq \frac{12s}{\pi}$.

By putting $(x, y, z) = (s-a, s-b, s-c)$ or $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ in (iv), we can deduce respectively

$$\frac{bc}{A(s-a)} + \frac{ca}{B(s-b)} + \frac{ab}{C(s-c)} \geq \frac{12s}{\pi}, \quad \frac{b+c}{A} + \frac{c+a}{B} + \frac{a+b}{C} \geq \frac{12s}{\pi},$$

which were proved in [1].

5. By applying Jordan's inequality $\sin x \geq \frac{2}{\pi}x$, ($x \in \left[0, \frac{\pi}{2}\right]$, see [3], p.201) in an acute-angled triangle, we can deduce, by using $a = 2R\sin A$, etc. that

(v) $\sum \frac{a}{A} > \frac{12}{\pi} R.$

By (ii) and the algebraic inequality $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$, clearly, one can obtain the analogous relation (in every triangle)

(v)' $\sum \frac{a}{A} \geq \frac{6}{\pi} s.$

Now, Redheffer's inequality (see [3], p.228) says that $\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}$ for $x \in (0, \pi)$. Since $\sum \sin A \leq \frac{3\sqrt{3}}{2}$, an easy calculation yields the following interesting inequality

(vi) $\sum \frac{A^3}{\pi^2 + A^2} > \pi - \frac{3\sqrt{3}}{4}.$

Similarly, without using the inequality on the sum of sin's one can deduce

(vii) $\sum \frac{a}{A} > 2R \sum \frac{\pi^2 - A^2}{\pi^2 + A^2}.$

From this other corollaries are obtainable.

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4 On a geometric inequality for the medians, bisectors and simedians of an angle of a triangle

The simedian AA_2 of a triangle ABC is the symmetrical of the median AA_0 to the angle bisector AA_1. By using Steiner's theorem for the points A_1 and A_0, one can write

$$\frac{A_2B}{A_2C} \cdot \frac{A_0B}{A_0C} = \frac{AB^2}{AC^2}.$$

Since $A_2B + A_2C = a$, this easily implies

$$A_2B = \frac{ac^2}{b^2 + c^2}, \quad A_2C = \frac{ab^2}{b^2 + c^2}.$$

Applying now Stewart's theorem to the point B, A_2, C and A:

$$c^2 A_2C - a \cdot AA_2^2 + b^2 \cdot A_2B = A_2B \cdot A_2C \cdot a;$$

with the notation $AA_2 = s_a$, the following formula can be deduced:

$$s_a^2 = \frac{b^2c^2}{(b^2 + c^2)^2}[2(b^2 + c^2) - a^2] \tag{1}$$

This gives the simedian corresponding to the angle A of a triangle ABC. Let $AA_0 = m_a$ be the median of A. Then, as it is well-known,

$$m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2},$$

so by (1) one can deduce that

$$s_a = \frac{2bc}{b^2 + c^2} m_a \tag{2}$$

Clearly, this implies

$$s_a \leq m_a \tag{3}$$

with equality only for $b = c$, i.e. for an isosceles triangle. Let $AA_1 = l_a$ be the bisector of angle A. It is well-known that $l_a \leq m_a$, but the following refinement holds also true (see [2], p.112).

$$\frac{m_a}{l_a} \geq \frac{b^2 + c^2}{4bc} \geq 1 \tag{4}$$

We shall use in what follows this relation, but for the sake of completeness, we give a sketch of proof: it is known that

$$m_a \cdot l_a \geq p(p-a)$$

(see [2], pp.1001-101), where $p = \dfrac{a+b+c}{2}$ denotes the semiperimeter. Therefore

$$\frac{m_a}{l_a} = \frac{m_a l_a}{l_a^2} \geq \frac{p(p-a)}{4} \cdot \frac{(b+c)^2}{bcp(p-a)} = \frac{(b+c)^2}{4bc},$$

giving (4). We have used also the classical formula

$$l_a = \frac{2}{b+c}\sqrt{bcp(p-a)}.$$

Now, OQ.591, [1] asks for all $\alpha > 0$ such that

$$\left(\frac{l_a}{m_a}\right)^{\alpha} + \left(\frac{l_a}{s_a}\right)^{\alpha} \leq 2 \tag{5}$$

In view of (2), this can be written equivalently as

$$\frac{l_a}{m_a} \leq f(\alpha) = k\left(\frac{2}{k^{\alpha}+1}\right)^{1/\alpha} \tag{6}$$

where $k = \dfrac{2bc}{b^2+c^2}$. Here $\left(\frac{k^{\alpha}+1}{2}\right)^{1/\alpha} = M_{\alpha}(k,1)$ is the well-known Hölder mean of arguments k and 1. It is known, that M_{α} is a strictly increasing, continuous function of α, and

$$\lim_{\alpha\to 0} M_{\alpha} < \sqrt{k} < M_{\alpha} < \lim_{\alpha\to\infty} M_{\alpha} = 1$$

(since $0 < k < 1$). Thus f is a strictly decreasing function with values between $k \cdot \dfrac{1}{\sqrt{k}} = \sqrt{k}$ and k. For $\alpha \in (0,1]$ one has

$$f(\alpha) \geq f(1) = \frac{2k}{k+1} = \frac{4bc}{(b+c)^2}.$$

On view (4) this gives $\dfrac{l_a}{m_a} \leq f(\alpha)$, i.e. a solution of (6) (and (5)). So, one can say that for all $\alpha \in (0,1]$, inequality (5) is true for all triangles. Generally speaking, however $\alpha_0 = 1$ is not the greatest value of α with property (5). Clearly, the equation

$$f(\alpha) = \frac{l_a}{m_a} \tag{7}$$

can have at most one solution. If $\alpha = \alpha_0$ denotes this solution, then for all $\alpha \leq \alpha_0$ one has $\dfrac{l_a}{m_a} \leq f(\alpha)$. Here $\alpha_0 \geq 1$. Remark that $\alpha > \alpha_0$, relaton (6) is not true, since $f(\alpha) < f(\alpha_0) = \dfrac{l_a}{m_a}$.

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5 On Emmerich's inequality

Let ABC be a right triangle of legs $AB = c$, $AC = b$ and hypotenuse a.

Recently, Arslanagić and Milosević have considered (see [1]) certain inequalities for such triangles. A basic result, applied by them is the following inequality of Emmerich (see [2])

$$\frac{R}{r} \geq \sqrt{2} + 1 \tag{1}$$

where R and r denote the radius of circumcircle, respectively incircle of the right triangle. Since $\sqrt{2}+1 > 2$, (1) improves the Euler inequality $R \geq 2r$, which is true in any triangle. Our aim is to extend Emmerich's inequality (1) to more general triangles. Since $R = \dfrac{a}{2}$ and $r = b + c - a$, it is immediate that in fact, (1) is equivalent to the following relation:

$$b + c \leq a\sqrt{2}. \tag{2}$$

Now, (2) is true in any triangle ABC, with $\widehat{A} \geq 90^\circ$, (see [3], where ten distinct proofs are given, see also [4], pp.47-48). First we extend (2) in the following manner:

Lemma 1. *In any triangle ABC holds the following inequality:*

$$b + c \leq \frac{a}{\sin\dfrac{A}{2}}. \tag{3}$$

Proof. Let l_a denote the angle bisector of A. This forms two triangles with the sides AB and AC, whose area sum is equal to area(ABC). By using the trigonometric form of the area, one can write

$$cl_a \sin\frac{A}{2} + bl_a \sin\frac{A}{2} = cb \sin A. \tag{4}$$

Now, since $l_a \geq h_a = \dfrac{bc\sin A}{a}$, ($h_a$ = altitude), (4) immediately gives (3). One has equality only if $l_a = h_a$, i.e. when ABC is an isosceles triangle.

Remark. When $\widehat{A} \geq 90^\circ$, then (3) implies (2). One has equality only when $\widehat{A} = 90^\circ$; $b = c$.

Lemma 2. *In any triangle ABC one has*

$$r \begin{cases} \geq \dfrac{b+c-a}{2} & \text{if} \quad \widehat{A} \geq 90^\circ \\ \leq \dfrac{b+c-a}{2} & \text{if} \quad \widehat{A} \leq 90^\circ \end{cases} \tag{5}$$

Proof. Let I be the centre of incircle and let $IB' \perp AC$, $IC \perp AB$, $IA' \perp BC$, ($B' \in AC$, $C' \in AB$, $A' \in BC$).

Let $AB' = AC' = x$. Then $CB' = b - x = CA'$, $BC' = c - x = BA'$. Since $BA' + A'C = a$, this immediately gives $x = \dfrac{b+c-a}{2}$. In $\triangle AIB'$, $r \geq x$ only if $\widehat{A} \geq 90^\circ - \dfrac{\widehat{A}}{2}$, i.e. $\widehat{A} \geq 90^\circ$. This proves (4).

Theorem 1. *Let $\widehat{A} \leq 90^\circ$. Then:*

$$\frac{R}{r} \geq \frac{1}{\sin A \left(\dfrac{1}{\sin \dfrac{A}{2}} - 1 \right)} \geq \frac{1}{\dfrac{1}{\sin \dfrac{A}{2}} - 1}. \tag{6}$$

Proof. By $R = \dfrac{a}{2 \sin A}$ and Lemma 2, one can write

$$\frac{R}{r} \geq \frac{a}{2 \sin A \left(\dfrac{a}{2 \sin \dfrac{A}{2}} - \dfrac{a}{2} \right)}$$

which gives the first part of (5). The second inequality follows by or $\sin A \leq 1$.

Remark. For $\widehat{A} = 90^\circ$, the right side of (5) gives

$$\frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1,$$

i.e. we obtain Emmerich's inequality (1). Another result connecting r and R is:

Theorem 2. *Let $\widehat{A} \geq 90^\circ$. Then:*

$$R + r \geq \frac{b+c}{2} \geq \frac{h_b + h_c}{2} \geq \min\{h_b, h_c\}. \tag{7}$$

Proof. By (5) one has $r \geq \dfrac{b+c-a}{2}$. On the other hand, in all triangles ABC,

$$R = \frac{a}{2 \sin A} \geq \frac{a}{2}$$

since $0 < \sin A \leq 1$. By adding these two inequalities, we get the first part of (6). Now, if $\widehat{A} \geq 90^\circ$, then $b \geq h_b$, $c \geq h_c$, and the remaining inequalities of (6) follow at once.

Remark. For another proof of $R+r \geq \min\{h_b, h_c\}$, see [4], pp.70-72. When $\widehat{A} \leq 90^\circ$, then $R+r \leq \max\{h_a, h_b, h_c\}$ a result due to P. Erdös (see [4]).

Theorem 3. *Let $\widehat{A} \leq 90^\circ$, Then:*

$$2r + a \leq b + c \leq 4R\cos\frac{A}{2}. \tag{8}$$

Proof. The left side of (8) follows by (5), while for the right side, we use (3):

$$b + c \leq \frac{a}{\sin\frac{A}{2}} = \frac{2R\sin A}{\sin\frac{A}{2}} = \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}} = 4R\cos\frac{A}{2}.$$

Theorem 4. *Let $\widehat{A} \geq 90^\circ$. Then:*

$$l_a \leq \frac{2\sqrt{bc}}{b+c}\sqrt{T} \leq \sqrt{T} \tag{9}$$

where $T = area(ABC)$. One has equality for $\widehat{A} = 90^\circ$ and $\widehat{A} = 90^\circ$, $b = c$, respectively.

Proof. By $r = \dfrac{T}{p} \geq \dfrac{b+c-a}{2} = p - a$, in view of (4), (here p = semiperimeter) one gets $T \geq p(p-a)$. On the other hand, by $l_d = \dfrac{2}{b+c}\sqrt{bcp(p-a)}$ and $2\sqrt{bc} \leq b + c$ the above inequality yields at once (8).

Finally, we extend a result from [1] as follows:

Theorem 5. *Let $\widehat{A} \geq 90^\circ$. Then:*

$$h_a \leq (a^2 + 2bc\sin A)^{\frac{1}{2}} - \left(\sqrt{2} - \frac{1}{2}\right)a. \tag{10}$$

Proof. If $\widehat{A} \geq 90^\circ$, it is known that $m_a \leq \dfrac{a}{2}$ (see e.g. [4], p.17). Since $h_a \leq m_a \leq \dfrac{a}{2}$, we have $h_a \leq \dfrac{a}{2}$. Now, we use the method of [1], first proof: Let $h_a = h$. Then

$$(a - 2h)\left[\left(\sqrt{2} - \frac{5}{4}\right)a + \frac{1}{2}h\right] \geq 0,$$

since $a \geq 2h$ and $\sqrt{2} > \dfrac{5}{4}$. This can be written as

$$a^2 + 2ha \geq \left(\frac{9}{4} - \sqrt{2}\right)a^2 + (2\sqrt{2} - 1)ah + h^2 = \left[\left(\sqrt{2} - \frac{1}{2}\right)a + h\right]^2.$$

Therefore $\sqrt{a^2+2bc\sin A} \geq \left(\sqrt{2}-\frac{1}{2}\right)a+h$, giving (9).

Remark 1. When $\widehat{A}=90^\circ$, then $a^2+2bc\sin A=(b+c)^2$, and we reobtain the result from [1].

Remark 2. The proof shows that one can take l_a in place of h_a in (9). Therefore, when $\widehat{A}=90^\circ$, we get

$$l_n \leq b+c-\left(\sqrt{2}-\frac{1}{2}\right)a. \tag{11}$$

Since $h_a \leq l_a$, this offers an improvement of Theorem A of [1]. When $\widehat{A}>90^\circ$, one can take m_a in place of h_a in (9).

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6 On a geometric inequality of Arslanagić and Milošević

Let ABC be a right triangle with legs b, c and hypothenuse a.

Recently Arslanagić and Milošević have proved the following result:

$$h_a \leq b + c - \left(\sqrt{2} - \frac{1}{2}\right) a \tag{1}$$

where h_a is the altitude corresponding to the hypothenuse. Though, in their paper [1] they give two distinct proofs of (1), the given proofs are not illuminating, and the geometrical meanings are hidden. Our aim is to obtain a geometric proof of (1), in an improvement form (in fact, the best possible result). First remark that since $h_a = \dfrac{bc}{a}$, (1) can be written equivalently as

$$bc \leq (b + c)a - \sqrt{2}a^2 + \frac{b^2 + c^2}{2},$$

or

$$b + c \geq a\sqrt{2} - \frac{(b - c)^2}{2a} \tag{2}$$

This inequality is interesting, since it is complementary to $b + c \leq a\sqrt{2}$, which is an immediate consequence of the identity

$$a^2 = \left(\frac{b + c}{\sqrt{2}}\right)^2 + \left(\frac{b - c}{\sqrt{2}}\right)^2 \tag{3}$$

Indeed, (3) gives $a^2 \geq \left(\dfrac{b + c}{\sqrt{2}}\right)^2$, yielding $b+c \leq a\sqrt{2}$, with equality only for $\dfrac{b - c}{\sqrt{2}} = 0$, i.e. $b = c$. In what follows, we shall prove the following improvement of (2):

$$b + c \geq a\sqrt{2} - \frac{(b - c)^2}{a(\sqrt{2} + 1)} \tag{4}$$

If $b = c$, then $b + c = a\sqrt{2}$, as we have seen above, so there is equality in (4). Let us suppose now that $b > c$. We make the following geometrical construction: Let $AB' = AC = b$ and $BK \perp B'C$, $(K \in B'C)$. Then $BK = \dfrac{b + c}{\sqrt{2}}$ and $CK = \dfrac{b - c}{\sqrt{2}}$ (see (3)). Remark that in $\triangle BKC$ one has $BK < BC$, so we get a geometrical proof of

$b + c < a\sqrt{2}$, many other proofs are given in [3]. But we can obtain a geometric proof of (4), too. In fact, (4) is equivalent to

$$BC - BK < \frac{\sqrt{2}}{\sqrt{2}+1}\frac{KC^2}{BC} \tag{5}$$

Since BC, BK, KC all are sides of a right triangle, we shall prove this result separately, as follows:

Lemma. *Let XYZ be a right triangle of legs y, z and hypothenuse x. Suppose that $y > z$. Then*

$$x - y < \frac{\sqrt{2}}{\sqrt{2}+1} \cdot \frac{z^2}{x} \tag{6}$$

Proof. Let $XT \perp YZ$. Then $y^2 = ZT \cdot YZ$, giving $y^2 = (x-p)x$, where $p = YT$. Thus $y^2 = x^2 - px$, so $x^2 - y^2 = px$ implying $x - y = \dfrac{px}{x+y}$. Here $z^2 = px$, thus we must prove that $x + y > \dfrac{\sqrt{2}+1}{\sqrt{2}}x$, or equivalently, $x < \sqrt{2}y$. Now, $\dfrac{y}{x} = \sin Y > \sin 45^\circ = \dfrac{1}{\sqrt{2}}$, and we are done. Applying the Lemma to the particular triangle KBC, we obtain the inequality (5). ($X \equiv K$, $Y \equiv C, Z \equiv B$).

Remark 1. For other improvement of (2) see [2].

Remark 2. By (4) we get

$$h_a \leq (b + c - a)\left(\frac{\sqrt{2}+1}{2}\right) \tag{7}$$

Now, the right side of (7) is less than the right side of (1) since this is equivalent to $b + c \leq a\sqrt{2}$.

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7 A note on the Erdös-Mordell inequality for tetrahedrons

Let $ABCD$ be a tetrahedron and P an arbitrary point in the interior of this tetrahedron. If p_a, p_b, p_c, p_d denote the distances of P to the faces, then the inequality

$$PA + PB + PC + PD \geq 3(p_a + p_b + p_c + p_d) \tag{1}$$

is called the Erdös-Mordell inequality for tetrahedron. Though for triangles a such inequality is always true (and this is the famous Erdös-Mordell inequality for a triangle), (1) is not valid for all tetrahedrons (see Kazarinoff [1]). It is conjectured that for all tetrahedrons holds true the following weaker inequality:

$$PA + PB + PC + OD \geq 2\sqrt{2}(p_a + p_b + p_c + p_d) \tag{2}$$

This is known to be true for all tetrahedrons having three two-by-two perpendicular faces, or all tetrahedrons which contain in interior the centre of circumscribed sphere. However, (1) is true for certain particular tetrahedrons, as we can see in notes by Dincă [3] or Dincă and Bencze [4]. Our aim is to prove a weaker inequality than (1), but which holds for all tetrahedrons. For a particular case this will give another result of type (1).

Let $S_A = Area(BCD)$, etc. Then in [2] the following is proved: (see pp. 136-137).

$$S_A \cdot PA + S_B \cdot PB + S_C \cdot PC + S_D \cdot PD \geq 3(S_A \cdot p_a + S_B \cdot p_b + S_C \cdot p_c + S_D \cdot p_d) \tag{3}$$

Let now $m(ABCD) = m := \min\{S_A, S_B, S_C, S_D\}$ and $M(ABCD) = M := \max\{S_A, S_B, S_C, S_D\}$. Then (3) gives the following result:

$$PA + PB + PC + PD \geq \frac{3m}{M}(p_a + p_b + p_c + p_d) \tag{4}$$

When $m = M$, i.e. all faces have equal area, we get again an Erdös-Mordell type inequality (1). This is due to Kazarinoff [1].

Remark. The multiplicative analogue of the Erdös-Mordell inequality is

$$PA \cdot PB \cdot PC \cdot PD \geq 3^4 \cdot p_a \cdot p_b \cdot p_c \cdot p_d \tag{5}$$

This is always true, see [2], pp. 127-128.

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8 On certain inequalities for the distances of a point to the vertices and the sides of a triangle

1. Let P be a point in the plane of a triangle ABC. As usual, we shall denote by a, b, c the (lengths of) sides BC, CA, AB; by h_a, h_b, h_c the altitudes; by m_a, m_b, m_c the medians, and by l_a, l_b, l_c the angle bisectors of the triangle. R will be the radius of circumcircle, r - the radius of incircle, p - the semi-perimeter of the triangle. Let p_a, p_b, p_c denote the distances of P to the sides BC, CA, AB of the triangle. We will denote by $T = T(ABC)$ the area; by O - the circumcentre; I - the incentre; H - ortocentre; G - centroid, of the triangle ABC. These notations are standard, excepting that of $p(= s)$; of $T = (S$ or $F)$ and $l_a = (w_a)$; see the monograph [1], and our monograph [2]. The following basic inequalities are well known

$$\sum PA^2 \geq \frac{1}{2}\sum a^2, \tag{1}$$

$$\sum PA \geq 6r \tag{2}$$

$$\sum PA \geq 2\sum p_a, \tag{3}$$

$P \in int(ABC)$ (i.e. when P is an interior-point of the triangle). For questions of priority (which is very difficult to decide in this field of Elementary Geometry - Algebra - Analysis) we note that (1) appears in [R. Sturm: Maxima und minima in der elementaren Geometrie, Leipzig, Berlin 1910, p.17], and also in [T. Lalesco: La géometrie du triangle, Paris, 1937, p.41]. Inequality (2) could be attributed to [M. Schreiber, Aufgabe 196, Jber. Deutsch. Math.-Verein, 45(1935), 63]. It appears also in [J.M. Child, Math. Gaz. 23(1939), 138-143], etc. Relation (3) is the famous Erdös-Mordell inequality [P. Erdös - L.J. Mordell, Problem 3740, American Math. Monthly, 42(1935), 396, and 44(1937), 252-254]. There exist many consequences and applications for these three inequalities. The aim of this paper is to obtain certain new proofs, new applications and inequalities of type (1)-(3).

2. Inequality (1) is proved usually be means of vectors or complex numbers (or Stewart's theorem). For proofs, see [2] p.158, p.189. A new, simple proof can be deduced as follows. Let A_1 be the midpoint of the segment BC. By the triangle inequality in triangle

APA_1, and the algebraic inequality $(x+y)^2 \leq \frac{3}{2}x^2 + 3y^2$ (equivalent with $(2y-z)^2 \geq 0$), we can write

$$m_a^2 \leq (AP + PA_1)^2 \leq \frac{3}{2}AP^2 + 3PA_1^2 = \frac{3}{2}(AP^2 + BP^2 + CP^2) - \frac{3}{4}a^2,$$

if we use the formula for medians in triangle PBC. Thus

$$\sum AP^2 \geq m_a^2 + \frac{3}{4}a^2 = \frac{1}{3}\sum a^2,$$

by application of the known formula for m_a^2. One has equality only if A, P, A_1 are collinear, and $AP = 2PA_1$, i.e. when $P \equiv G$ - the centroid of ABC. For a simple application, let $P \equiv O$ in (1). Then we get

$$a^2 + b^2 + c^2 \leq 9R^2.$$

If we apply the well known algebraic inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ we can deduce that

$$a + b + c \leq \sqrt{3(a^2+b^2+c^2)} \leq 3\sqrt{3}R.$$

The result $a + b + c \leq 3\sqrt{3}R$ is another source for inequalities. For example, by the law of sinus, $a = 2R\sin A$, etc., we get the classical trigonometric inequality

$$\sum \sin A \leq \frac{3\sqrt{3}}{2}.$$

If we use the known identity $abc = 3Rrp$ and the algebraic relation $(x+y+z)^3 \geq 27xyz$ the above results implies also Euler's inequality $R \geq 2r$. Indeed, by using semi-perimeters, we have $8p^3 \geq 24 \cdot 4Rrp$; but $2p^2 \leq \frac{27R^2}{2}$, so $27Rr \leq \frac{27R^2}{2}$, giving $r \leq \frac{R}{2}$. The above proved famous inequality by Euler follows directly from (2) if we put $P \equiv O$. (For a proof of (2) see [2], p.122). By $\sum PA^2 \geq \frac{1}{3}\left(\sum PA\right)^2$, relation (2) implies that

$$\sum PA^2 \geq 12r^2. \tag{4}$$

However this relation can be proved only via (1), by taking into account of the inequality $\sum a^2 \geq 36r^2$ (see [2], p.120). In fact in [2], p.120 it is proved that

$$\sum a^2 \geq \frac{4}{3}p^2 \geq \frac{4}{3}\sum l_a^2 \geq \frac{4}{3}\sum h_a^2 \geq 36r^2,$$

thus by (1) we can write that

$$\sum PA^2 \geq \frac{4}{9}p^2 \tag{5}$$

which is stronger than (4). We want now to show that another refinements of inequality $\sum a^2 \geq 36r^2$ (due to N.A. Edwards and J.M. Child, see [1], p.52) hold true. First let $P \equiv H$ in (2), and assume that ABC is acute angled triangle. Then, since $\sum AH \leq \sqrt{\sum a^2}$ (see [2], p.109), by (2) we get

$$6r \leq \sum AH \leq \sqrt{\sum a^2}. \tag{6}$$

Let now $P \equiv I$. Since $\sum AI \leq \sqrt{\sum ab}$ (see [2], p.109), we obtain from (2) that

$$6r \leq \sum AI \leq \sqrt{\sum ab}. \tag{7}$$

The weaker inequality (i.e. $6r \leq \sqrt{\sum ab}$) is due to F. Leunenberger [Elem. Math. 13(1958), 121-126]. Since

$$AI = \sqrt{\frac{bc(p-a)}{p}}$$

(see [2], p.156), inequality (7) can be written also as

$$6r\sqrt{p} \leq \sum \sqrt{bc(p-a)} \leq \sqrt{p\sum ab}.$$

Let $P \equiv I$ in (3). Since $p_a = p_b = p_c = r$ in this case, we reobtain the left side of (7). In fact, for this choice of P, inequalities (2) and (3) give the same result.

There are many known proofs for (3). For a proof by D.K. Kazarinoff, see [2], p.72. This proof shows in fact the more precise relation

$$\sum PA \geq \sum \left(\frac{b}{c} + \frac{c}{b}\right) p_a \quad (P \in int(ABC)). \tag{8}$$

Thus, by applying (8) for $P \equiv I$, we can deduce the following improvement for the left side of (7):

$$6r \leq \frac{1}{abc}\sum ab(a+b) \leq \sum AI. \tag{9}$$

D.F. Barrow (see [2], p.108) obtained another generalization of (3), by proving that

$$\sum PA \geq 2\sum PA' \quad (P \in int(ABC)), \tag{10}$$

where PA' is the bisector of the angle APB. Let $T_a = T(PBC)$. Since the angle bisector PA' is greater than the corresponding altitude PA_0, and $a \cdot PA_0 = 2T_a$, from (10) we get the inequality

$$\sum PA \geq 4 \sum \frac{T_a}{a}.$$

Here

$$\sum \frac{T_a}{a} \sum a = T + \sum T_a \left(\frac{b+c}{a} \right)$$

(by $\sum T_a = T$), finally we get

$$\sum PA \geq \frac{2}{p} \left[T + \sum T_a \left(\frac{b+c}{a} \right) \right], \tag{11}$$

where, as we noticed, $T = T(ABC)$, $T_a = T(PBC)$. For example, when $P \equiv G$, we have $T_a = T_b = T_c = \dfrac{T}{3}$, and (11) reduces to the inequality $\sum m_a \geq 9r$ due to E.G. Gotman (see [1], p.74). Let the triangle ABC be acute-angled, and let $P \equiv O$ in (10). Then OA' is an altitude in triangle OBC, so

$$OA' = \sqrt{R^2 - \frac{a^2}{4}}.$$

We obtain the curious inequality $\sum \sqrt{4R^2 - a^2} \leq 3R$. But we can remark that by $a = 2R \sin A$, $1 - \sin^2 A = \cos A$, this reduces in fact to the trigonometric inequality $\sum \cos A \leq \dfrac{3}{2}$ (see [2], p.98).

Finally, we give two applications of (2). Let ABC be acute-angled, and let AA', BB', CC' be the altitudes, and O_1, O_2, O_3 the midpoints of the segments BC, AC, AB - respectively. Then it is well known that

$$OO_1 = \frac{AH}{2}, \quad OO_2 = \frac{BH}{2}, \quad OO_3 = \frac{c}{2}.$$

First apply (2) for $p \equiv H$. Then it results

$$2(HA' + HB' + HC') \leq HA + HB + HC. \tag{12}$$

By writing $HA' = h_a = HA$ etc., from (12) we get

$$2 \sum h_a \leq 3 \sum HA. \tag{13}$$

This refines again the left side of (6), since $\sum h_a \geq 9r$. On the other hand, by letting $P \equiv O$ in (2), by $OO_1 = \dfrac{AH}{2}$, we obtain $2(OO_1 + OO_2 + OO_3) \leq 3R$, i.e.

$$\sum HA \leq 3R. \tag{14}$$

Combining (13), (14) we can write

$$9r \leq \sum h_a \leq \frac{3}{2} \sum HA \leq \frac{9}{2} R. \tag{15}$$

Clearly, this refines again Euler's inequality.

3. We now will consider two new inequalities, valid for arbitrary points in the plane, namely

$$\sum a \cdot PA^2 \geq abc \tag{16}$$

and

$$\sum PA^2 \cdot PB^2 \geq \frac{16}{9} T^2. \tag{17}$$

For inequality (16), see [2], p.55 or p.153 (distinct proofs), and for inequality (17) see [J. Sándor, Problem 20942*, Mat. Lapok 11-12/1986, p.486]. We give here a proof of (17), which has not been appeared elsewhere. First suppose that $P \in int(ABC)$ and let $\alpha = \widehat{BPA}$, $\beta = \widehat{BPC}$, $\gamma = \widehat{CPA}$. Clearly $\alpha + \beta + \gamma = 360^\circ$. Now, writing that

$$T = \sum T(ABC) = \frac{1}{2} \sum xy \sin \alpha$$

with $x = AP$, $y = BP$, $z = CP$ and applying the classical Cauchy-Buniakovski inequality, we have

$$T^2 \leq \frac{1}{4} \left(\sum x^2 y^2 \right) \left(\sum \sin^2 \alpha \right).$$

Here $\sin^2 \alpha = \dfrac{1 - \cos 2\alpha}{2}$ and by the formula

$$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2},$$

we easily get

$$\sum \sin^2 \alpha = \frac{3}{2} - \frac{1}{2}(2 \cos(\alpha + \beta) \cos(\alpha - \beta) + \cos 2\gamma) = -t^2 - \theta t + 2$$

where $t = \cos\gamma$ and $\theta = \cos(\alpha - \beta)$. The trinomial $f(t) = -t^2 - \theta t + 2$ has maximum and

$$\frac{-\Delta}{4a} = \frac{\theta^2 + 8}{4} \leq \frac{9}{4}.$$

This finishes the proof in this case, since

$$T^2 \leq \frac{9}{16} \sum x^2 y^2.$$

Equality occurs when

$$\frac{xy}{\sin\alpha} = \frac{xz}{\sin\beta} = \frac{yz}{\sin\gamma}$$

and $\cos(\alpha - \beta) = \pm 1$, i.e. when O coincides with ther Fermat-point of the triangle. Then $\triangle PAB = \triangle PBC = \triangle PAC$, thus ABC is equilateral, and P is its center. When P is situated on the sides or outside the triangle, then by area considerations, we have for example, in a region

$$T = \frac{yz\sin(\alpha+\beta)}{2} - \frac{xy\sin\alpha}{2} - \frac{xz\sin\beta}{2}$$

or

$$T = \frac{xy\sin\alpha}{2} + \frac{yz\sin\beta}{2} - \frac{xz\sin(\alpha+\beta)}{2}$$

in another region. But this doesn't affects the inequality

$$T^2 \leq \frac{1}{4}\left(\sum x^2 y^2\right)(\sin^2\alpha + \sin^2\beta + \sin^2(\alpha+\beta)).$$

As above, we can show that

$$\sin^2\alpha + \sin^\beta + \sin^2(\alpha+\beta) \leq \frac{9}{4}.$$

The only difference is that for exterior points we can't have equality.

As a first application of (17), let $P \equiv O$. This selection gives

$$T \leq \frac{3\sqrt{3}}{4} R^2. \tag{18}$$

For other proofs, see [2], p.107. This inequality contains the fact that in a circle of radius R, the equilateral triangle is the one inscribed, which has a maximal area. If we take $P \equiv G$, then since $GA = \frac{2}{3} m_a$ we can derive

$$9T^2 \leq \sum m_a^2 m_b^2 (\leq \sum m_a^4). \tag{19}$$

For the inequality (16) an immediate application is for $P \equiv O$, in which case

$$R^2 \geq \frac{abc}{a+b+c} = 2Rr \tag{20}$$

giving again Euler's inequality. We note that for $P \equiv I$ there is equality in (16), while for $P \equiv H$, since $AH = 2R\cos A$, we obtain the trigonometric inequality

$$\sum \sin A \cos^2 A \geq \prod \sin A,$$

i.e.

$$\sum \sin A - \prod \sin A \geq \sum \sin^3 A. \tag{21}$$

This inequality is due to J. Sándor. For a vectorial argument, see [2], p.157. The application with $P \equiv G$ (which appears also in [2], p.158) gives

$$\sum am_a^2 \geq \frac{9}{a}abc. \tag{22}$$

4. Finally, we consider some simple, but nice inequalities. Let $\widehat{A} \geq 90^\circ$ and P an arbitrary point in the plane (or space). Then one has

$$PA^2 \leq PB^2 + PC^2. \tag{23}$$

Let A_1 be the midpoint of BC. Then

$$PA^2 \leq (PA_1 + AA_1)^2 \leq 2(PA_1^2 + AA_1^2) = PB^2 + PC^2 - 2\left(\frac{a^2}{4} - AA_1^2\right),$$

where we have applied the known formula for the median PA_1. It is well known that:

$$\text{for } A \geq 90^\circ \text{ we have } AA_1 \leq \frac{a}{2} \tag{24}$$

(see [2], p.17). Thus, (23) follows, with equality only if $\widehat{A} = 90^\circ$ and P is the forth vertex of the rectangle $PBAC$. This inequality has some resemblance with

$$a \cdot PA \leq b \cdot PB + c \cdot PC, \tag{25}$$

valid for all triangles ABC. This is a consequence of Ptolemy's inequality for the quadrilateral $PBAC$ (for three distinct proofs see [2] p.51, 142, 176). The generalized form of Ptolemy's inequality ([2], p.186), gives two analogous inequalities

$$b \cdot PB \leq a \cdot PA + c \cdot PC, \quad c \cdot PC \leq a \cdot PA + b \cdot PB.$$

For various selections of the point P one can obtain various inequalities between the elements of a triangle. For example, when $P \equiv G$, one gets

$$a \cdot m_a \leq b \cdot m_b + c \cdot m_c.$$

If $\widehat{A} \geq 90^\circ$, (23) gives $m_a^2 \leq m_b^2 + m_c^2$. We wish to note that inequalities (1), (16), (25) can be extended to the space, i.e. these results are valid if one considers arbitrary points P in the space. We invite the interested reader to deduce certain other applications of the considered inequalities, i.e. when P is in the space. For many other relations connecting elements in a tetrahedron, we quote the monograph [2].

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9 On certain constants in the geometry of equilateral triangle

The equilateral (or regular) triangle seems to be a very simple triangle. However, there are certain geometrical positions when surprising properties do appear. For example, the well-known Viviani theorem states that if an interior point P is projected to the sides BC, AC, AB in P_1, P_2, resp. P_3, then

$$PP_1 + PP_2 + PP_3 = \text{constant} \tag{1}$$

The simplest proof of (1) is via area considerations. By the additivity of the area function, $A(BPC) + A(APC) + A(APB) = A(ABC)$ (where $A(ABC) = area(ABC)$) we get

$$PP_1 + PP_2 + PP_3 = 2A(ABC)/a = h,$$

where a and h denote the lengths of sides and heights in the triangle.

1. Now let $A_1B_1C_1$ be a new triangle constructed in such a way that $A_1B_1 \perp AC$, $C \in A_1B_1$; $A_1C_1 \perp BC$, $B \in A_1C_1$ and $B_1C_1 \perp AB$, $A \in B_1C_1$. It is immediate that this new triangle is regular, too, and $BP_1 = PB'$, $CP_1 = PC'$, $AP_1 = PA'$ (where A', B', C' are the projections of P to B_1C_1, A_1C_1, resp. A_1B_1). Applying Viviani's theorem (1) in this new triangle $A_1B_1C_1$ one gets $PA' + PB' + PC' = \text{constant}$, implying:

$$BP_1 + CP_2 + AP_3 = \text{constant} \tag{2}$$

This surprising constant is similar to (1), but quite different. We note that a less elegant proof of (2) follows if one applies the generalized Pythagorean theorem in triangle BPC and its analogues. One obtains

$$PC^2 = BP^2 + a^2 - 2aBP_1,$$

$$PA^2 = CP^2 + a^2 - 2aCP_2,$$

$$PB^2 = AP^2 + a^2 - 2aAP_3.$$

By addition, (2) follows.

2. Let now P be an arbitrary point of the circle inscribed in ABC. Applying Stewart's theorem for the points C', O, C (O is the centre of circle, C' is the tangent point of the circle with the side AB) and P, one has

$$PC^2 \cdot C'O - PO^2 \cdot C'C + PC'^2 \cdot OC = C'O \cdot OC \cdot C'C,$$

or, since $OC' = r = \dfrac{a\sqrt{3}}{6}$, after some simple transformations

$$PC^2 + 2PC'^2 = 9r^2 \tag{3}$$

PC' being the median of triangle PAB, with the median formula one has $2PC'^2 = PA^2 + PB^2 - 3r^2$, so (3) gives

$$PA^2 + PB^2 + PC^2 = 15r^2 (= \text{constant}) \tag{4}$$

This is well-known, but we note that by generalizing the above ideas for an arbitrary triangle ABC, the constant

$$a \cdot PA^2 + b \cdot PB^2 + c \cdot PC^2 = \text{constant} \tag{5}$$

can be deduced. Usually, (4) (or (5)) is proved via Analytical Geometry.

3. Let now P_1, P_2, P_3 be the projections of interior point P to the sides (as in 1.), but suppose that P is **on** the inscribed circle (as in 2.). Let $BP_1 = x$, $CP_2 = y$, $AP_3 = z$. From the obtained right triangles one obtains

$$x^2 + y^2 + z^2 = (a-x)^2 + (a-y)^2 + (a-z)^2 \tag{6}$$

(this may give again a proof of (2)). Applying the generalized Pythagorean theorem in triangle P_1P_2C:

$$P_1P_2^2 = y^2 + (a-x)^2 - y(a-x) \tag{7}$$

In the inscriptible quadrilateral PP_1CP_2 the segment PC is a diameter, so by the sin-theorem

$$PB = \frac{2}{\sqrt{3}} \cdot P_1P_2 \tag{8}$$

By writing two similar relations, after addition the equality

$$\frac{3}{4}(PA^2+PB^2+PC^2)=x^2+y^2+z^2+(a-x)^2+(a-y)^2+(a-z)^2-$$

$$-x(a-z)-y(a-x)-z(a-y) \tag{9}$$

will be deduced. By (4), (6), (2) this implies that

$$2(x^2+y^2+z^2)+xy+xz+yz=\text{constant} \tag{10}$$

By (1) and (10) follows

$$xy+xz+yz=\text{constant} \tag{11}$$

$$x^2+y^2+z^2=\text{constant} \tag{12}$$

Since $PP_1^2+PP_2^2+PP_3^2=PA^2+PB^2+PC^2-(x^2+y^2+z^2)$, (4) and (12) gives

$$PP_1^2+PP_2^2+PP_3^2=\text{constant}, \tag{13}$$

known as Gergonne's theorem. The proof of (13) was quite long, but in the course of solution, many "hidden" constants were discovered, e.g.

$$BP_1^2+CP_2^2+AP_3^2=\text{constant} \tag{14}$$

$$P_1P_2^2+P_1P_3^2+P_2P_3^2=\text{constant} \tag{15}$$

$$BP_1\cdot CP_2+BP_1\cdot AP_2+CP_2\cdot AP_3=\text{constant} \tag{16}$$

$$PP_1\cdot PP_2+PP_2\cdot PP_3+PP_2\cdot PP_3=\text{constant} \tag{17}$$

Remarks. 1) If P is on the circumscribed circle (to ABC), relations (4) and (13) remain true. For the proof the geometrical construction of the first paragraph can be repeated, with the difference that in place of $A_1B_1\perp AC$ one considers $A_1B_1\perp AA'$ (where AA' is a heigh of the triangle ABC), etc. Then the circumscribed circle becomes an inscribed circle of triangle $A_1B_1C_1$, and the above proved relations may be applied.

In fact, one can consider any circle which is concentrical with the inscribed circle.

2) The constant (1) has important applications in Geometric inequalities (see [1]). Other connections appear in [2].

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10 The area of a Pythagorean triangle, as a perfect power

Let ABC be a Pythagorean triangle of legs b, c and hypothenuse a. Then it is well know that a, b, c are given generally by

$$\begin{cases} a = d(u^2 + v^2) \\ b = d(v^2 - u^2) \\ c = 2duv \end{cases} \tag{1}$$

where $(u, v) = 1$, $v > u$ and u, v have distinct parity; d is arbitrary (or $b \to c$, $c \to b$). The area of ABC is

$$A = \frac{bc}{2} = d^2uv(v^2 - u^2).$$

One has:

Theorem 1. (Fermat) *The area of a Pythagorean triangle cannot be a perfect square.*

Proof. Let $d^2uv(v^2 - u^2) = t^2$ ($t > 0$ integer). Then $d^2|t^2$, so $d|t$. Let $t = dw$, giving

$$uv(v^2 - u^2) = w^2. \tag{2}$$

Since $(u, v) = (u, v^2 - u^2) = (v, v^2 - u^2) = 1$, clearly (2) gives $u = y^2$, $v = x^2$, $v^2 - u^2 = z^2$. This implies the equation

$$x^4 - y^4 = z^2. \tag{3}$$

Here $(x, y) = (x, z) = (y, z) = 1$, $x > y$ have distinct parities. We shall prove by Fermat's descent that (3) cannot have solutions. First observe that x cannot be even (indeed, if x would be even, then since y is odd, $y^2 \equiv 1 \pmod 8$ so $x^4 - y^4 \equiv -1 \pmod 8 \not\equiv 1 \pmod 8 = z^2$ (since z is odd)). Therefore x must be odd and y even. By (1) (with $d = 1$) one can write

$$x^2 = a^2 + b^2, \quad y^2 = 2ab, \quad z = a^2 - b^2 \tag{4}$$

with $a > b$, $(a, b) = 1$ having distinct parities. Let $a =$ even, $b =$ odd. Since $y^2 = 2ab$, y is even, let $y = 2Y$. From $ab = 2Y^2$ and $a = 2a'$ one gets $a'b = Y^2$ so $a' = p^2$, $b = q^2$, with

$(p,q)=1$. We have obtained: $a=2p^2$, $b=q^2$, so

$$x^2=(2p^2)^2+(q^2)^2, \quad y=2pq. \tag{5}$$

Applying (1) once again to (5) we get

$$x=r^2+s^2, \quad 2p^2=2rs, \quad q^2=r^2-s^2 \tag{6}$$

with $r>s$, $(r,s)=1$ having distinct parities. From $p^2=rs$ one obtains $r=t^2$, $s=n^2$ with $(t,n)=1$, so

$$q^2=t^4-n^4, \tag{7}$$

with $(t,n)=1$ having distinct parities.

Remark moreover that $t=\sqrt{r}\leq p<\sqrt{x}$ since $p^2=rs<2rs\leq r^2+s^2=x$. Therefore we have obtained equation (7) which is completely analogous to (3), but with component $t<x$. By the infinite descent method of Fermat, equation (3) cannot have solutions in positive integers (thus in non-zero integers). This proves Theorem 1.

Remark 1. Theorem 1 was based on the fact that eq. (3) is not solvable when x,y,z are mutually prime and $x>y$ have distinct parities. But eq. (3) has no solutions in positive integers in all cases. Indeed, we may suppose $(x,y)=1$ (otherwise, let $(x,y)=d$, so $x=dX$, $y=dY$, $(X,Y)=1$, implying $d^4(X^2-Y^2)=z^2$. Hence $d^2|z$, i.e. $z=d^2Z$. This now yields $X^2-Y^2=Z^2$, where $(X,Y)=1$. Now this implies similarly $(X,Z)=1$ and $(Y,Z)=1$ (indeed, if δ is a prime divisor of X and Z, then $\delta|Y^2$ so $\delta|Y$, contradiction)). Now, remark that x cannot be even (as above, see the proof of Theorem 1!), and for y we can have two possibilities:

a) y even

b) y odd.

The case a) has been studied above (see eq. (3)). In case b), let $x^2=a^2+b^2$, $y^2=a^2-b^2$, $z=2ab$ with $a>b$, $(a,b)=1$ having distinct parities. It results $x^2y^2=a^4-b^4$, so $a^4-b^4=(xy)^2$ with $a<x$. The descent methods yields the impossibility of existence of solutions of this equation.

Theorem 2. *The area of a Pythagorean triangle cannot be the double of a perfect square.*

Proof. Let $A = 2t^2$. Then $d^2uv(v^2 - u^2) = 2t^2$, where $(u, v) = 1$, $v > u$ have distinct parities. Let $v = 2V$, $u =$ odd, $(u, V) = 1$. Then $d^2uV(4V^2 - u^2) = t^2$. Thus $t = dw$, i.e. $uV(4V^2 - u^2) = w^2$. Here $(u, V) = (u, 4V^2 - u^2) = (V, 4V^2 - u^2) = 1$, so $u = x^2$, $V = y^2$, $4V^2 - u^2 = z^2$. By conclusion: $4y^4 - x^4 = z^2$. Here x is odd, so $x^2 \equiv 1 \pmod 8$. Hence y cannot be even (since, then $4y^4 - x^4 \equiv -1 \pmod 8 \not\equiv 1 \pmod 8 = z^2$). If y is odd then $4y^4 - x^4 \equiv 4 \cdot 1 - 1 \pmod 8 \equiv 3 \pmod 8 \not\equiv 1 \pmod 8 = z^2$.

Remark 2. Let $(3d, 4d, 5d)$ be a Pythagorean triangle. Then $area = 6d^2$. For $d = 6$ this is a perfect cube: $area = 6 \cdot 6^2 = 6^3$, while for $d = 6^2$ one obtains $area = 6 \cdot 6^4 = 6^5$. Thus, the area of a Pythagorean triangle may be a perfect nth power, for $n \geq 3$. However, the following can be proved:

Theorem 3. *The area of a primitive Pythagorean triangle cannot be an nth power (for any $n \geq 2$).*

Proof. Let in (1) $d = 1$. Then $area = uv(v^2 - u^2)$. Let us suppose that

$$uv(v^2 - u^2) = t^n \quad (n \geq 2) \tag{8}$$

Then, by $(u, v) = (u, v^2 - u^2) = (v, v^2 - u^2) = 1$ one has $u = x^n$, $v = y^n$, $v^2 - u^2 = z^n$, implying

$$y^{2n} - x^{2n} = z^n \text{ with } (x, y) = (x, z) = (y, z) = 1. \tag{9}$$

This in fact is

$$X^n + z^n = Y^n \tag{10}$$

where $X = x^2$, $Y = y^2$. Since in 1995 A. Wiles has proved that (10) has no non-trivial solutions, the proof of Theorem 3 is completed.

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11 On Heron Triangles, III

1. Let ABC be a triangle with lengths of sides $BC = a$, $AC = b$, $AB = c$ positive integers. Then ABC is called a Heron triangle (or simply, H-triangle) if its area $\Delta = Area(ABC)$ is an integer number. The theory of H-triangles has a long history and certain results are many times rediscovered. On the other hand there appear always some new questions in this theory, or even there are famous unsolved problems. It is enough (see e.g. [2]) to mention the difficult unsolved problem on the existence of a H-triangle having **all** medians integers. The simplest H-triangle is the Pythagorean triangle (or P-triangle, in what follows). Indeed, by supposing AB as hypothenuse, the general solution of the equation

$$a^2 + b^2 = c^2 \tag{1}$$

(i.e. the so-called **Pythagorean numbers**) are given by

$$a = \lambda(m^2 - n^2), \quad b = 2\lambda mn, c = \lambda(m^2 + n^2) \quad (\text{if } b \text{ is even}) \tag{2}$$

where λ is arbitrary positive integer, while $m > n$ are coprime of different parities (i.e. $(m, n) = 1$ and m and n cannot be both odd or even). Clearly $\Delta = \dfrac{ab}{2} = \lambda^2 mn(m^2 - n^2)$, integer.

Let p be the **semiperimeter** of the triangle. From (2) $p = \lambda(m^2 + mn)$; and denoting by r the **inradius** of a such triangle, it is well known that

$$r = p - c \tag{3}$$

implying that r is always integer.

On the other hand, the radius R of the **circumscribed circle** in this case is given by the simple formula

$$R = \frac{c}{2} \tag{4}$$

which, in view of (2) is integer only if λ is even, $\lambda = 2\Lambda$ ($\Lambda > 0$). The **heights** of a P-triangle are given by

$$h_a = b, \quad h_a = b, \quad h_c = \frac{ab}{c}; \tag{5}$$

therefore **all** heights are integers only if $c|ab$, which, by (2) can be written as $(m^2+n^2)|2\lambda mn(m^2-n^2)$. Since $(m,n)=1$, of different parity, it is immediate that $(m^2+n^2, 2mn(m^2-n^2))=1$, giving $(m^2+n^2)|\lambda$; i.e. $\lambda = K(m^2+n^2)$ $(K>0)$.

By summing, in a P-triangle the following elements: $\Delta, h_a, h_b, h_c, r, R$ are integers at the same time if and only if a, b, c are given by

$$a = 2d(m^4-n^4), \quad b = 4dmn(m^2+n^2), \quad c = 2d(m^2+n^2)^2, \tag{6}$$

where we have denoted $K=2d$ (as by (4), λ is even and m^2+n^2 is odd).

In fact this contains the particular case of the P-triangles with $a=30n$, $b=40n$, $c=50n$ in a problem [7] by F. Smarandache, and in fact gives **all** such triangles.

2. An interesting example of a H-triangle is that which has as sides **consecutive integers**. Let us denote by CH such a H-triangle (i.e. "consecutive Heron"). The CH-triangles appear also in the second part [6] of this series, where it is proved that r is always integer. Since in a H-triangle p is always integer (see e.g. [3], [4]), if $x-1, x, x+1$ are the sides of a CH-triangle, by $p=\dfrac{3x}{2}$, we have that x is even, $x=2y$. Therefore the sides are $2y-1, 2y, 2y+1$, when $p=3y$, $p-a=y+1$, $p-b=y$, $p-c=y-1$ giving $\Delta = \sqrt{3y(y+1)(y-1)} = y\sqrt{3(y^2-1)}$, by Heron's formula of area. This gives $\dfrac{\Delta}{y} = \sqrt{3(y^2-1)}$ = rational. Since $3(y^2-1)$ is integer, it must be a perfect square, $3(y^2-1)=t^2$, where

$$\Delta = yt. \tag{7}$$

Since the prime 3 divides t^2, clearly $3|t$, let $t=3u$. This implies

$$y^2 - 3u^2 = 1 \tag{8}$$

which is a "Pell-equation". Here $\sqrt{3}$ is an irrational number, and the theory of such equations (see e.g. [5]) is well-known. Since $(y_1, u_1)=(2,1)$ is a basic solution (i.e. with y_1 the smallest), **all** other solutions of this equations are provided by

$$y_n + u_n\sqrt{3} = \left(2+\sqrt{3}\right)^n \quad (n \geq 1). \tag{9}$$

By writting

$$y_{n+1} + u_{n+1}\sqrt{3} = \left(2+\sqrt{3}\right)^{n+1} = \left(y_n + u_n\sqrt{3}\right)\left(2+\sqrt{3}\right) = 2y_n + 3u_n + \sqrt{3}(y_n + 2u_n),$$

we get the recurrence relations

$$\begin{cases} y_{n+1} = 2y_n + 3u_n \\ u_{n+1} = y_n + 2u_n \end{cases} \quad (n = 1, 2, \ldots) \tag{10}$$

which give all solution of (8); i.e. all CH-triangles (**all** such triangles have as sides $2y_n - 1$, $2y_n$, $2y_n + 1$). By $y_n = 2, 7, 26, 97, \ldots$ we get the CH-triangles $(3, 4, 5)$; $(13, 14, 15)$; $(51, 52, 53)$; $(193, 194, 195)$; ....

Now, we study certain particular elements of a CH-triangle. As we have remarked, r is **always** integer, since

$$r = \frac{\Delta}{p} = \frac{\Delta}{3y} = \frac{t}{3} = u$$

(in other words, in (10) u_n represents the inradius of the nth CH-triangle). If one denotes by h_{2y} the **height** corresponding to the (single) even side of this triangle, clearly

$$h_{2y} = \frac{2\Delta}{2y} = \frac{\Delta}{y} = 3r.$$

Therefore we have the interesting fact that h_{2y} is integer, and even more, r is the **third part** of this height. On the other hand, in a CH-triangle, which is **not** a P-triangle (i.e. **excluding** the triangle $(3, 4, 5)$), all other heights cannot be integers. (11)

Indeed, $\frac{(2y-1)x}{2} = \Delta = yt$ gives $(2y-1)x = 2yt$ (here $x = h_{2y-1}$ for simplicity). Since $(u, y) = 1$ and $t = 3u$ we have $(t, y) = 1$, so $x = \frac{2yt}{2y-1}$ is integer only if $(2y-1)|t = 3u$. Now, by $y^2 - 3u^2 = 1$ we get $4y^2 - 1 = 12u^2 + 3$, i.e. $(2y-1)(2y+1) = 3(4u^2+1) = 4(3u^2) \nmid 3$. Therefore $(2y-1)|3u$ implies $(2y-1)|3u^2$, so we must have $(2y-1)|3$, implying $y = 2$ $(y > 1)$. For h_{2y+1} we have similarly $\frac{(2y+1)z}{2} = \Delta = yt$, so $z = \frac{2yt}{2y+1}$, where $(2y+1)|t = 3u \Leftrightarrow (2y+1)|3 \Leftrightarrow y = 1$ (as above). Therefore $z = h_{2y+1}$ **cannot** be integer in **all** CH-triangles. (Remember that $x = h_{2y-1}$ is integer **only** in the P-triangle $(3, 4, 5)$).

For R the things are immediate:

$$R = \frac{abc}{4\Delta} = \frac{2y(4y^2-1)}{4yt} = \frac{4y^2-1}{2t} = \frac{\text{odd}}{\text{even}} \neq \text{integer}. \tag{12}$$

Let now r_a denote the radius of the **exscribed** circle corresponding to the side of length a. It is well-known that

$$r_a = \frac{\Delta}{p-a}.$$

By $r_{2y} = \dfrac{yt}{y}$ $(= 3r$, in fact), we get that r_{2y} is **integer**. Now $r_{2y-1} = \dfrac{yt}{y+1}$, $r_{2y+1} = \dfrac{yt}{y-1}$. Here $(y+1, y) = 1$, so r_{2y-1} is integer only when $(y+1)|t = 3u$. Since $y^2 - 3u^2 = 1$ implies $(y-1)(y+1) = 3u^2 = u(3u)$, by $3u = (y+1)k$ one has $3(y-1) = 3uk = (y+1)k^2$ and $y - 1 = uk$. By $k^2 = \dfrac{3(y-1)}{y+1} = 3 - \dfrac{6}{y+1}$ we get that $(y+1)|6$, i.e. $y \in \{1, 2, 5\}$. We can have only $y = 2$, when $k = 1$.

Therefore r_{2y-1} **is integer only in the P-triangle** $(3, 4, 5)$. (13)

In this case (and only this) $r_{2y+1} = \dfrac{2 \cdot 3}{2-1} = 6$ is integer, too.

Remarks 1. As we have shown, in all CH-triangle, which is not a P-triangle, we can exactly one height, which is integer. Such triangles are all acute-angled. (Since $(2y-1)^2 + (2y)^2 > (2y+1)^2$). In [4] it is stated as an open question if in all acute-angled H-triangles there exists **at least** an integer (-valued) height. This is not true, as can be seen from the example $a = 35$, $b = 34$, $c = 15$. (Here $34^2 + 15^2 = 1156 + 225 = 1381 > 35^2 = 1225$, so ABC is acute-angled). Now $p = 42$, $p - a = 7$, $p - b = 8$, $p - c = 27$, $\Delta = 252 = 2^2 \cdot 3^2 \cdot 7$ and $35 = 7 \cdot 5 \nmid 2\Delta$, $34 = 2 \cdot 17 \nmid 2\Delta$, $15 = 3 \cdot 5 \nmid 2\Delta$. We note that $h_a = \dfrac{2\Delta}{a}$ is integer only when a divides 2Δ. Let n be an integer such that $5 \cdot 17 \nmid n$. Then $35n, 34n, 15n$ are the sides of a H-triangle, which is acute-angled, and no height is integer. The H-triangle of sides $39, 35, 10$ is **obtuse-angled**, and no height is integer.

3. Let now ABC be an **isosceles** triangle with $AB = AC = b$, $BC = a$. Assuming that the heights $AA' = x$ and $BB' = y$ are integers (clearly, the third height $CC' = BB'$), by $b^2 = x^2 + \left(\dfrac{a}{2}\right)^2$ we have $\dfrac{a^2}{4} = b^2 - x^2 =$ integer, implying a=even. Let $a = 2u$. Thus

$$b^2 = x^2 + u^2. \tag{14}$$

We note that if x is integer, then $a = 2u$, so ABC is a H-triangle, since $\Delta = \dfrac{xa}{2} = xu$. The general solutions of (14) (see (2)) can be written as one of the followings:

(i) $b = \lambda(m^2 + n^2)$, $x = \lambda(m^2 - n^2)$, $u = 2\lambda mn$;

(ii) $b = \lambda(m^2 + n^2)$, $x = 2\lambda mn$, $u = \lambda(m^2 - n^2)$.

We shall consider only the case (i), the case (ii) can be studied in a completely analogous way.

Now $a = 4\lambda mn$, $b = \lambda(m^2+n^2)$; so $\Delta = 2\lambda^2 mn(m^2-n^2)$. Thus $y = \dfrac{2\Delta}{b}$ is integer only when $\lambda(m^2+n^2)|4\lambda^2 mn(m^2-n^2)$. Thus $y = \dfrac{2\Delta}{b}$ is integer only when $\lambda(m^2+n^2)|4\lambda^2 mn(m^2-n^2)$. Since $(m^2+n^2, 4mn(m^2-n^2)) = 1$ (see **1.**, where the case of P-triangles has been considered), this is possible only when $(m^2+n^2)|\lambda$, i.e.

$$\lambda = s(m^2+n^2). \tag{15}$$

Therefore, in an **isosceles H-triangle**, having **all heights** integers, we must have (in case (ii) $a = 4smn(m^2+n^2)$; $b = s(m^2+n^2)^2$ (where a is the **base** of the triangle) or (in case (ii))

$$a = 2sm(m^4-n^4), \quad b = sm(m^2+n^2)^2. \tag{16}$$

We note here that case (ii) can be studied similarly to the case (i) and we omit the details.

In fact, if an isosceles triangle ABC with integer sides a, b (base a) is H-triangle, then $p = b + \dfrac{a}{2}$ = integer, so $a = 2u$ = even. So $p = b+u$ and $p-b = u$, $p-a = b-\dfrac{a}{2} = b-u$, implying $\Delta = \sqrt{p(p-a)(p-b)^2} = u\sqrt{b^2-u^2}$. This is integer only when $b^2-u^2 = q^2$, when $\Delta = uq$. Now b^2-u^2 is in fact x^2 (where x is the height corresponding to the base a), so $q = x$. In other words, if an isosceles triangles ABC is H-triangle, then its **height** x **must be integer**, and we recapture relation (14). Therefore, in an isosceles H-triangle a height is always integer (but the other ones only in case (16)). In such a triangle, $r = \dfrac{\Delta}{p} = \dfrac{uq}{b+u}$, where $b^2 = u^2+q^2$. By (2) we can write the following equations:

i) $b = \lambda(m^2+n^2)$, $u = 2\lambda mn$, $q = \lambda(m^2-n^2)$;

ii) $b = \lambda(m^2+n^2)$, $u = \lambda(m^2-n^2)$, $q = 2\lambda mn$.

In case i) $b+u = \lambda(m+n)^2 | uq = 2\lambda^2 mn(m^2-n^2)$ only iff $(m+n)^2|2\lambda mn(m^2-n^2)$, i.e. $(m+n)|2\lambda mn(m-n)$; and since $(m+n, 2mn(m-n)) = 1$. This is possible only when $(m+n)|\lambda$, i.e.

$$\left.\begin{array}{l} \lambda = s(m+n) \\ \text{case (ii) we get } m|\lambda, \text{ so } \lambda = sm \end{array}\right\} \tag{17}$$

Therefore in an isosceles triangle r **is integer** only when

i) $b = s(m+n)(m^2+n^2)$, $a = 2n = 4mns(m+n)$; or

ii) $b = sm(m^2+n^2)$, $a = sm(m^2-n^2)$.

For $R = \dfrac{abc}{4\Delta} = \dfrac{ab^2}{4\Delta} = \dfrac{2nb^2}{4nq} = \dfrac{b^2}{2q}$ we have that R is integer only when $2q|b^2$, where $b^2 = n^2+q^2$. In case i) we get $2\lambda(m^2-n^2)|\lambda^2(m^2+n^2)^2$, which is possible only when $2(m^2-n^2)|\lambda$ or in case ii) $4\lambda mn|\lambda^2(m^2+n^2)^2$ i.e. $4mn|\lambda$. By summing, R is integer only if in i) $\lambda = 2s(m^2-n^2)$, while in ii), $\lambda = 4smn$. Then the corresponding sides a, b can be written explicitely.

From the above considerations we can determine **all** isosceles H-triangles, in which **all heights** and r, R are integers. These are one of the following two cases:

$$\begin{aligned} &1) \quad a = 4kmn(m^4-n^4), \quad b = 2k(m^2-n^2)(m^2+n^2)^2; \\ &2) \quad a = 4kmn(m^4-n^4), \quad b = 2kmn(m^2+n^2) \end{aligned} \tag{18}$$

where $k \geq 1$ is arbitrary and $(m,n) = 1$, $m > n$ are of different parity.

In the same manner, by $r_a = \dfrac{\Delta}{p-a} = \dfrac{uq}{b-u}$ in case i) $b-u = \lambda(m-n)^2|uq = 2\lambda^2 mn(m^2-n^2)$ only if $(m-n)|\lambda$ i.e. $\lambda = s(m-n)$, while in case ii) $b-u = 2\lambda n^2|uq = 2\lambda^2 mn(m^2-n^2)$ iff $n|\lambda$, i.e. $\lambda = sn$. We can say that r_a is integer only if $\lambda = s(m-n)$ in i) and $\lambda = sn$ in ii). We omit the further details.

4. As we have seen in Remarks 1 there are infinitely many H-triangles having none of its heights integers (though, they are of course, rationals). Clearly, if at least a height of an **integral triangle** (i.e. whose sides are all integers) is integer, or rational its area is rational. We now prove that in this case the triangle is Heron. More precisely if a height of an integral triangle is rational, then this is a H-triangle. Indeed, by $\Delta = \dfrac{xa}{2} =$ rational, we have that Δ is rational.

On the other hand, by Heron's formula we easily can deduce

$$16\Delta^2 = 2(a^2b^2+a^2c^2+b^2c^2) - (a^4+b^4+c^4). \tag{19}$$

Therefore $(4\Delta)^2$ is integer. Since $\Delta =$ rational, we must have $4\Delta =$ integer. If we can prove that $4|4\Delta$ then clearly Δ will be integer. For this it is sufficient to show $(4\Delta)^2 = 8k$ (since, then $4\Delta = 2l$ so $(4\Delta)^2 = 4l^2 \vdots$ implies $l =$ even). It is an aritmetic fact that $2(a^2b^2+a^2c^2+b^2c^2) - (a^4+b^4+c^4)$ is **always** divisible by 8 (which uses that for x odd $x^2 \equiv 1(\text{mod}4)$, while for x even, $x^2 \equiv 0(\text{mod}4)$).

Let now ABC be a H-triangle with $BC = a =$ odd. We prove that the height

$$AA' = h_a \text{ is integer only if } a|(b^2 - c^2). \tag{20}$$

Indeed, let Δ be integer, with a, b, c integers. Then h_a is integer iff $a|2\Delta$. But this is equivalent to $a^2|4\Delta^2$ or $4a^2|16\Delta^2$. Now, by (19)

$$4a^2|[2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)] \;\Leftrightarrow\; a^2|(2b^2c^2 - b^4 - c^4) = -(b^2 - c^2)^2$$

(since the paranthesis in bracket is divisible by 8 and $(a^2, 4) = 1$). Or, $a^2|(b^2 - c^2)^2$ is equivalent to $a|(b^2 - c^2)$.

Clearly, (19) implies $a^2|(b^2 - c^2)^2$ for all a, therefore if

$$a \nmid (b^2 - c^2) \tag{21}$$

h_a **cannot be** integer. But (19) is **not equivalent** with (20) for all a (especially, for $a =$ even). In fact (19) is the **exact** condition on the integrality of h_a in a H-triangle.

For general H-triangle, the conditions on the integrality of heights on r, R are not so simple as shown in the preceeding examples of P, CH or isosceles H-triangles.

Sometimes we can give simple negative results of type (21). One of these is the following:

Suppose that in an integral triangle of sides a, b, c we have

$$2(a + b + c) \nmid abc. \tag{22}$$

Then r, R **cannot be both** integers.

Indeed, suppose a, b, c, r, R integers. Since $r = \dfrac{\Delta}{p}$, clearly Δ is rational, so by the above argument, Δ is integer. So p is integer, too, $\dfrac{a + b + c}{2}|\Delta \;\Leftrightarrow\; a + b + c|2\Delta$. Now $R = \dfrac{abc}{4\Delta}$, so $4\Delta|abc$. Therefore $2(a + b + c)|4\Delta|abc$ if all the above are satisfied. But this is impossible, by assumption.

Certain direct results follow from the elementary connections existing between the elements of a triangle.

For example, from $R = \dfrac{b}{2\sin B}$ and $\sin B = \dfrac{h_a}{c}$ we get $R = \dfrac{bc}{2h_a}$, implying the following assertion:

If in an integral triangle of sides a, b, c we have $h_a =$ integer, then R is integer only if

$$2h_a | bc. \tag{23}$$

This easily implies the following negative result:

If in an integral triangle of sides a, b, c all heights h_a, h_b, h_c are integers, but one of a, b, c is not even; then R cannot be integer.

Indeed, by (23) $2h_a|bc$, $2h_b|ac$, $2h_c|ab$ so bc, ac, ab are all even numbers. Since $a+b+c = 2p$ is even, clearly all of a, b, c must be even.

5. The characterization of the above general problems (related to an arbitrary H-triangle) can be done if one can give general formulae for the most general case. Such formulae for a H-triangle have been suggested by R.D. Carmichael [1], and variants were many times rediscovered. We wish to note on advance that usually such general formulae are quite difficult to handle and apply in particular cases because the many parameters involved. The theorem by Carmichael can be stated as follows:

An integral triangle of sides a, b, c is a H-triangle if and only if a, b, c can be represented in the following forms

$$a = \frac{(m-n)(k^2+mn)}{d}, \quad b = \frac{m(k^2+n^2)}{d}, \quad c = \frac{n(k^2+m^2)}{d} \tag{24}$$

where d, m, n, k are positive integers; $m > n$; and d is an arbitrary common divisor of $(m-n)(k^2+mn)$, $m(k^2+n^2)$, $n(k^2+m^2)$.

For a complete proof we quote [3].

Now, from (24) we can calculate $p = \dfrac{m(k^2+mn)}{d}$ and

$$\Delta = \frac{kmn(m-n)(k^2+mn)}{d^2}.$$

In fact, the proof of (24) involves that p and Δ are integers for all k, m, n, d as given above. By simple transformations, we get

$$h_a = \frac{2\Delta}{a} = \frac{2kmn}{d}, \quad h_b = \frac{2\Delta}{b} = \frac{2kn(m-n)(k^2+mn)}{d(k^2+n^2)},$$

$$h_c = \frac{2kmn(m-n)(k^2+mn)}{d(k^2+m^2)},$$

$$r = \frac{\Delta}{p} = \frac{kn(m-n)}{d}, \quad R = \frac{abc}{4\Delta} = \frac{(k^2+m^2)(k^2+n^2)}{4kd}. \tag{25}$$

These relations enables us to deduce various conditions on the integer values of the above elements.

Particularly, we mention the following theorem:

All integral triangles of sides a, b, c which are H-triangles, and where r is integer are given by formulae (24), where d is **any common divisor** of the following expressions:

$$(m-n)(k^2+mn); \quad m(k^2+n^2); \quad n(k^2+m^2); \quad kn(m-n). \tag{26}$$

6. As we have considered before, among the CH-triangles in which **all** of r, r_a, r_b, r_c are integers are in fact the P-triangles.

In what follows we will determine all H-triangles having r, r_a, r_b, r_c integers.

Therefore, let

$$r = \frac{\Delta}{p} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}, \quad r_a = \frac{\Delta}{p-a} = \sqrt{(p(p-b)(p-c)},$$

$$r_b = \frac{\Delta}{p-b} = \sqrt{p(p-a)(p-c)}, \quad r_c = \sqrt{p(p-a)(p-b)}$$

be integers.

Put $p-a = x$, $p-b = y$, $p-c = z$, when $3p - 2p = x+y+z = p$.

Then $\sqrt{yz(x+y+z)}$, $\sqrt{xz(x+y+z)}$, $\sqrt{xy(x+y+z)}$, $\sqrt{xyz/(x+y+z)}$ are all integers, and since x, y, z are integer, the expressions on radicals must be perfect squares of integers. Let

$$xy(x+y+z) = t^2, \quad xz(x+y+z) = p^2, \quad yz(x+y+z) = q^2, \quad \frac{xyz}{x+y+z} = u^2. \tag{27}$$

Then by multiplication $x^2y^2z^2(x+y+z)^3 = t^2p^2q^2$, so

$$x+y+z = \left[\frac{tpq}{xyz(x+y+z)}\right]^2 = v^2,$$

where $tpq = vxyz(x+y+z) = v^3xyz$.

This gives $\frac{xyz}{v^2} = u^2$ so $xyz = u^2v^2$ and $tpq = u^2v^5$. Now $xyv^2 = t^2$, $xzv^2 = p^2$, $yzv^2 = q^2$ give $xy = \left(\frac{t}{v}\right)^2$, where $t = vn_1$, $xz = \left(\frac{p}{v}\right)^2$, where $p = vn_2$, $yz = \left(\frac{q}{v}\right)^2$, where

$q = vn_3$ (n_1, n_2, n_3 integers). By $v^3 n_1 n_2 n_3 = u^2 v^5$ we get $n_1 n_2 n_3 = u^2 v^2$. By $xy = n_1^2$, $xz = n_2^2$, $yz = n_3^2$, $xyz = u^2 v^2$, $x + y + z = v^2$, we get $x = d_1 X^2$, $y = d_1 Y^2$ (with $(X, Y) = 1$), $n_1 = d_1 XY$; $x = d_2 U^2$, $z = d_2 V^2$, $n_2 = d_2 UV$, $(U, V) = 1$; $y = d_3 W^2$, $z = d_3 \Omega^2$, where $(W, \Omega) = 1$, $n_3 = d_3 W\Omega$. From $xyz = d_1^2 X^2 Y^2 d_2 V^2 = u^2 v^2$ it follows that d_2 is a perfect square. So x is a square, implying that d_1 is a square, implying $y =$ perfect square. Thus $n_3 =$ square, giving $z =$ perfect square. All in all, x, y, z are all perfect squares. Let $x = \alpha^2$, $y = \beta^2$, $z = \gamma^2$. Then

$$\alpha^2 + \beta^2 + \gamma^2 = v^2. \tag{28}$$

From $p - a = \dfrac{b + c - a}{2} = \alpha^2$, $p - b = y = \dfrac{a + c - b}{2} = \beta^2$, $p - c = z = \dfrac{a + b - c}{2} = \gamma^2$ we can easily deduce

$$a = \beta^2 + \gamma^2, \quad b = \alpha^2 + \gamma^2, \quad c = \alpha^2 + \beta^2. \tag{29}$$

Now, the primitive solutions of (28) (i.e. those with $(\alpha, \beta, \gamma) = 1$) are given by (see e.g. [1])

$$\alpha = mk - ns, \quad \beta = ms + nk, \quad \gamma = m^2 + n^2 - k^2 - s^2,$$
$$v = m^2 + n^2 + k^2 + s^2 \tag{30}$$

where m, k, n, s ($mk > ns$, $m^2 + n^2 > k^2 + s^2$) are integers. By supposing $(\alpha, \beta, \gamma) = d$, clearly $\alpha = d\alpha_1$, $b = d\beta_1$, $e = d\gamma_1$ and $d^2(\alpha_1^2 + \beta_1^2 + \gamma_1^2) = v^2$ implies $d^2 | v^2$, so $d | v$. Let $v = dv_1$, giving $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = v_1^2$. Thus the general solutions of (28) can be obtained from (30), by multiplying each term of (30) by a common factor d.

These give all H-triangles with the required conditions.

Remarks 2. Many generalized or extensions of Heron triangles or arithmetic problems in geometry were included in paper [6]. The part IV of this series (in preparation) will contain other generalized arithmetic problems in plane or space (e.g. "Heron trapeziums").

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Note added in proof. After completing this paper, we learned that **Problem CMJ 354** (College Math. J. **18**(1987), 248) by **Alvin Tirman** asks for the determination of Pythagorean triangles with the property that the triangle formed by the altitude and median corresponding to the hypothenuse is also Pythagorean. It is immediate that the solution of this problem follows from paragraph 1. of our paper.

12 An arithmetic problem in geometry

1. The following old problem in its complete generality was studied in 1891 by Gy. Vályi (see [2] or [3]):

"Determine all triangles with integer sides for which the area and perimeter are measured by the same number."

By applying the Heron formula for the area of a triangle, the above problem reduces to the following diophantine equation:

$$xyz = 4(x + y + z) \tag{1}$$

Vályi deduced a very elegant solution of this equation and obtained that all solutions of the above geometric problem are the following (a, b, c - the sides of the triangle):

$$(a, b, c) = (6, 25, 29); (7, 15, 20); (9, 10, 17); (5, 12, 13); (6, 8, 10).$$

The case of right triangles has been rediscovered also by I. Trifon [5]. Then only the last two solution can be accepted.

2. The analogues problem for quadrilaterals can be stated as follows:

"Determine all inscriptible quadrilaterals with integer sides having area and perimeter measured by the same number." Therefore, instead of general quadrilaterals we consider only inscriptible ones. This case is enough difficult - as we will see - to study. Let a, b, c, d be the lengths of sides and $p = (a + b + c + d)/2$ the semiperimeter. Then (see [1])

$$Area = \sqrt{(p - a)(p - b)(p - c)(p - d)} = a + b + c + d$$

with the notations $p-a = x$, $p-b = y$, $p-c = z$, $p-d = t$ gives the following diophantine equation:

$$xyzt = (x + y + z + t)^2 \tag{2}$$

In what follows first we shall study equation (2) in its most generality, without any geometrical background.

First remark that unlike equation (1), the equation (2) has infinitely many solutions. Indeed, if (x, y, z, t) is a particular solution, then it is easy to verify that $(x, y, z, xyz-2(x+$

$y+z)-t)$ is also a solution. The equation being symmetric, with use of permutations, the following will be solutions, too: $(x, y, xyt-2(x+y+t)-z, t)$; $(x, xzt-2(x+z+t)-y, z, t)$; $(yzt-2(y+z+t)-x, y, z, t)$. Thus, from the solution (4,4,4,4) step-by-step, the following chain of solutions can be obtained: $(4,4,4,4) \to (4,4,4,36) \to (4,4,484,36) \to \ldots$ It can be shown, that in fact all solutions of (2) can be obtained by the aid of the transformation $(x, y, z, t) \to (x, y, z, xyz-2(x+y+z)-t)$ and its permutations from those "fundamental solutions" which satisfy

$$x \le y \le z \le t, \quad x+y+z \ge t \tag{3}$$

(see [2]). Now we shall determine all these fundamental solutions. If (3) is valid and $xy \le 4$, then $xyzt \le 4zt < (x+y+z+t)^2$, so (2) cannot be true. If $xy > 48$, then since $t \le 3z$ it follows

$$xyz > 48\frac{t}{3}t \ge 16t^2 \ge (x+y+z+t)^2,$$

so (2) is again impossible.

Let now $5 \le xy \le 48$. Then from (3) and by (2) one can write

$$4(x+y+z+t)^2 = 4xyzt = xy[(z+t)^2-(t-z)^2] \ge xy[(z+t)^2-(x+y)^2],$$

implying

$$(xy-4)(z+t)^2 - 8(x+y)(z+t) - (xy+4)(x+y)^2 \le 0 \tag{4}$$

This second order inequality gives at once

$$z+t \le \frac{4(x+y)+\sqrt{16(x+y)^2+(x+y)^2(xy-4)(xy+4)}}{xy-4} = \frac{xy+4}{xy-4}(x+y) \tag{5}$$

which shows that z, t can have at most a finite number of values. By taking into account the above considerations, one can say that all fundamental solutions must satisfy the following system of inequalities-equalities:

$$\begin{cases} 5 \le xy \le 48 \\ x \le y \le z \le t,\ x+y+z \ge t \\ z+t \le \dfrac{xy+4}{xy-4}(x+y) \\ (x+y+z+t)^2 = xyzt \end{cases} \tag{6}$$

By using a Basic Program, the following solutions can be deduced:

$$(x, y, z, t) = (1, 5, 24, 30); (1, 6, 14, 21); (1, 8, 9, 18); (1, 9, 10, 10);$$

$$(2, 3, 10, 15); (2, 4, 6, 12); (2, 5, 5, 8); (3, 3, 6, 6); (4, 4, 4, 4).$$

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Chapter 2. Diophantine equations

"...The higher arithmetic, or the theory of numbers, is concerned with the properties of the natural numbers 1,2,3... These numbers must have exercised human curiosity from a very early period; and in all the records of ancient civilizations there is evidence of some preoccupation with arithmetic over and above the needs of everyday life. But as a systematic and independent science, the higher arithmetic is entirely a creation of modern times, and can be said to date from the discoveries of Fermat (1601-1665)..."

(H. Davenport, The higher arithmetic, London, 1962)

1 On the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ in integers

It is sufficient to consider $x, y, z > 0$. Indeed, if $x, y, z < 0$ one can take $(-x, -y, -z)$ in place of (x, y, z). When $x < 0$, $y > 0$, $z > 0$, let $x = -X$. Then $\frac{1}{y} = \frac{1}{z} + \frac{1}{X}$, which is the same, with other notations, as the equation in the title.

We shall use, among others the two lemmas of another article [4] (see also [3]).

Lemma 1. *If $x > 0$, $y > 0$, $(x, y) = 1$, then $(x + y, xy) = 1$.*

Lemma 2. *If $k > 0$ is not a perfect square, then $\sqrt{k} \notin \mathbb{Q}$.*

Another solution will be deduced by the application of the so-called Euler-Bell-Kalmár equation.

The first solution is based on a correction of an incomplete solution due to Marian L. Caines from [5].

1. It is immediate that (1) has the equivalent form

$$(x - z)(y - z) = z^2, \tag{1}$$

or

$$uv = z^2 \tag{2}$$

where $u = x - z > 0$, $v = y - z > 0$. (Clearly $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ implies $z < x$, $z < y$). Let $(u, v) = d$, so $u = du_1$, $v = dv_1$ with $(u_1, v_1) = 1$. Then from (2) $d^2u_1v_1 = z^2$, and Lemma 2 implies $d|z$ (otherwise $\sqrt{u_1v_1}$ would be irrational) so $z = dz_1$. From $u_1v_1 = z^2$ with $(u_1, v_1) = 1$ one has $u_1 = a^2$, $v_1 = b^2$, thus $z = ab$. These relations give $x = dab + ds^2 = da(a + b)$, $y = dab + db^2 = db(a + b)$, $z = dab$, with $(a, b) = 1$, $d \geq 1$ arbitrary. This is the most general solution of the stated equation.

2. Another method is based on Lemma 1. Indeed, writing the equation in the form

$$z(x + y) = xy \tag{3}$$

and letting $(x, y) = k$, i.e. $x = ka$, $y = kb$, $(a, b) = 1$, one has from (3) $z(a + b) = kab$. Now Lemma 1 gives $ab|z$, i.e. $z = dab$ $(d \geq 1)$. Therefore $d(a + b) = k$, giving $d|k$, i.e. $k = dk_1$. Since $a + b = k_1$, it results $x = dk_1a = da(a + b)$, $y = dk_1b = dba(a + b)$, $z = dab$, where $(a, b) = 1$.

3. The equation $XY = ZT$, first studied by Euler, then Bell and Kalmár (see [1], [2]) can be solved as follows: Let $(X, Z) = d$, i.e. $X = dm$, $Z = dn$, $(m, n) = 1$. Then $mY = nT$, so $m|nT$ giving $m|T$, i.e. $T = km$. Thus $mY = nkm$, or $Y = nk$. By summing, $X = dm$, $Y = kn$, $Z = dn$, $T = km$ (this is the famous "four-number theorem"). Now, from (3) one can write $z = dm$, $x + y = kn$, $x = dn$, $y = km$, $(m, n) = 1$. Since $dn + km = kn$, one has $n|km$ so $n|k$, yielding $k = ns$. Finally, one obtains

$$x = sn(n - m), \quad y = snm, \quad z = sm(n - m),$$

which give in fact (by using a change of variables) the general solutions.

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2 On the equation $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}$ in integers

There appeared many incorrect solutions of this equation along the years. For example, J.D. Thérond [5] stated that all solutions in positive integers are given by

$$x=4m(4m^2+1),\quad y=(4m^2-1)(4m^2+1),\quad z=4m(4m^2-1),\quad (m\geq 1) \tag{1}$$

I. Safta [2] "proved" that all integer solutions are given by

$$x=\pm n\left(\frac{n^2+1}{2}\right),\quad y=\pm\left(\frac{n^2-1}{2}\right)\left(\frac{n^2+1}{2}\right),\quad z=\pm\frac{n(n^2-1)}{2} \tag{2}$$

where $n=2k+1$, $k\geq 1$.

In 1981 the author [3], [4] obtained a simple method of determining all solutions, which include - as particular cases - also the families (1) and (2). This appeared in Romanian [3] (with certain missprints, too) and Hungarian [4]. Now we present here an improved version. The proof is based on two simple lemmas, of independent interest.

Lemma 1. *Let $x,y\geq 1$, $(x,y)=1$ (i.e. g.c.d. of x and y is $=1$). Then*

$$(x+y,xy)=1 \tag{3}$$

Proof. Let $p\geq 2$ be a divisor of $(x+y,xy)$. Then $p|xy$ and by $(x,y)=1$, Euclid's theorem implies $p|x$, $p\nmid y$ (or $p\nmid x$, $p|y$). Since $p|(x+y)$ we get $p|y$ - a contradiction.

Lemma 2. *If $k\geq 1$ is not a perfect square, then $\sqrt{k}$ is irrational.*

Proof. In the obvious identity

$$\sqrt{k}=\frac{k-b\sqrt{k}}{\sqrt{k}-b} \tag{4}$$

put $b=[\sqrt{k}]$. Since k is not a perfect square, one has $b<\sqrt{k}<b+1$, i.e. $0<\sqrt{k}-b<1$. Let us suppose that $\sqrt{k}$ is irrational, and let $\sqrt{k}=\frac{u}{v}$, with the smallest possible nominator v. Then, by (2)

$$\sqrt{k}=\frac{kv-bu}{u-bv}$$

which has a nominator $u-bv=v(\sqrt{k}-b)>0$ and $u-bv<v$ since $\sqrt{k}-b<1$. Since v was supposed to be the smallest nominator of $\sqrt{k}$, we have arrived at a contradiction.

Theorem. *All solutions in positive integers of the equation in the title can be written in the form*

$$x = t(u^2+v^2)(u^2-v^2), \quad y = t(u^2+v^2)\cdot 2uv, \quad z = t(u^2-v^2)\cdot 2uv, \tag{5}$$

where $u > v$, $(u,v) = 1$ *are positive integers of the different parity, and* $t \geq 1$ *is arbitrary positive integer, (and* $x \to y$, $y \to x$, $z \to z$*).*

Proof. The equation can be written also as

$$z^2(x^2+y^2) = x^2y^2 \tag{6}$$

Now let $(x,y) = d$, i.e. $x = x_1d$, $y = y_1d$ with $(x_1,y_1) = 1$. Replacing in (4) we get

$$z^2(x_1^2+y_1^2) = d^2x_1^2y_1^2 \tag{7}$$

By $(x_1,y_1) = 1 \;\Rightarrow\; (x_1^2,y_1^2) = 1$, so by Lemma 1 we have $(x_1^2y_1^2, x_1^2+y_1^2) = 1$. Therefore (5) implies $z^2 = kx_1^2y_1^2$ and $k(x_1^2+y_1^2) = d^2$ for some positive integer k. The first equality - via Lemma 2 - implies that k must be a perfect square, otherwise $\sqrt{k}$ would be irrational, a contradiction. Let therefore $k = t^2$ $(t \geq 1)$. Then $z = tx_1y_1$, $t^2(x_1^2+y_1^2) = d^2$, so with the notation $l = \dfrac{d}{t}$ (which is integer again by Lemma 2) one has $x_1^2+y_1^2 = l^2$. It is well-known (see e.g. [1]) that the general solutions of this equation are

$$x_1 = M(u^2-v^2), \quad y_1 = M\cdot 2uv, \quad l = M(u^2+v^2)$$

(or $x_1 \to y_1$, $y_1 \to x_1$, $l \to l$, i.e. x_1 and y_1 may be changed), with $u > v$ having different parities; $(u,v) = 1$. Since $(x_1,y_1) = 1$, we get $M = 1$. By summing

$$\begin{cases} x = dx_1 = ltx_1 = t(u^2+v^2)(u^2-v^2) \\ y = dy_1 = lty_1 = t(u^2+v^2)\cdot 2uv \\ z = tx_1y_1 = t(u^2-v^2)\cdot 2uv \end{cases} \tag{8}$$

(and $x \to y$, $y \to x$, $z \to z$).

Remark that for $t = 1$, $u = \dfrac{n+1}{2}$, $v = \dfrac{n-1}{2}$ $(n = 2k+1)$, i.e. $(u,v) = (k+1,k) = 1$; and u, v with distinct parities, so the solutions (2) (with t) are reobtained. The set of solutions, however is much wider, as can be seen by other selections of u and v.

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3 On the equations $\frac{a}{x}+\frac{b}{y}=\frac{c}{d}$ and $\frac{a}{x}+\frac{b}{y}=\frac{c}{z}$

1. The equation

$$\frac{a}{x}+\frac{b}{y}=\frac{c}{d} \tag{1}$$

where a, b, c, d are given positive integers, and x, y unknown positive integers occurs many times for particular values of a, b, c, d. In order to obtain all solutions of (1), remark that (1) can be written as

$$bdx = y(cx - ad) \tag{2}$$

Let $(x, y) = k$, i.e. $x = kX$, $y = kY$ with $(X, Y) = 1$. Then $bdX = Y(ckX - ad)$, implying $Y|bdX$. Since $(X, Y) = 1$, Euclid's theorem yields $Y|bd$. The numbers b, d being given, there exist a finite number of such values of Y. Let Y_0 be a such value, i.e. $bd = Y_0Z_0$ ($Z_0 \geq 1$). From the above equation one gets

$$X = \frac{ad}{ck - Z_0} \tag{3}$$

This is possible only when $(ck - Z_0)|ad$. Since c and Z_0 are given, there exist a finite number of such k. By taking into account of all values of Z_0, we will obtain all possible values of k; therefore on base of $x = kX$, $y = kY_0$, and $(X, Y_0) = 1$, all solutions have been obtained (theoretically).

For example, when $a = 1$, $b = 2$, $c = 3$, $d = 4$ we have $bd = 8$ so $Y|8$, i.e. $Y_0 \in \{1, 2, 4, 8\}$. Then $Z_0 \in \{8, 4, 2, 1\}$ (in this order!). By $ad = 4$, by analyzing $(3k - 8)|4$, $(3k - 4)|4$, $(3k - 2)|4$, $(3k - 1)|4$ and by taking into account of $(X, Y_0) = 1$,w we get the solutions (12,3); (4,4); (2,8).

2. The equation

$$\frac{a}{x}+\frac{b}{y}=\frac{c}{z} \tag{4}$$

where a, b, c are given, while x, y, z - unknown positive integers has the well-known and important particular case $a = b = c = 1$ (with applications in geometry, physics, nomograms, etc.).

We shall use the Euler-Bell-Kalmár lemma. All solutions of the equation $XY = ZT$ in positive integers are given by $X = mk$, $Y = nt$, $Z = nk$, $T = mt$, where $(m, n) = 1$

and k, t are arbitrary. By writing (4) as

$$z(ay + bx) = cxy \tag{5}$$

we have: $z = mk$, $ay + bx = nt$, $c = nk$, $xy = mt$. Since c is given, the equation $c = nk$ can have a finite number of solutions, let (n_0, k_0) be one of them.

Applying once again the above lemma for the equation $xy = mt$, one can write $x = ps$, $y = qr$, $m = qs$, $t = pr$, where $(p, q) = 1$. By $ay + bx = nt$ one has $aqr + bps = n_0pr$. This implies $p|aqr$, and since $(p, q) = 1$, we have $p|ar$. Let $ar = pR$. Replacing and simplifying one gets $bs = R(n_0p - q)$, so

$$s = \frac{R(n_0p - q)}{b} \tag{6}$$

By the equalities $x = ps$, $y = qr$, $z = mk_0$, one can write the most general solutions in the form

$$x = \frac{pR(n_0p - q)}{b} \quad y = pqR \quad z = \frac{qRk_0(n_0p - q)}{b} \quad (n_0p > q) \tag{7}$$

where $(p, q) = 1$, $n_0k_0 = c$ and all variables are arbitrary, but satisfy the following two conditions:

(i) $b|pR(n_0p - q)$

(ii) $b|qRk_0(n_0p - q)$.

Remarks. 1) When $b = 1$, the conditions (i), (ii) are valid. If even $a = c = 1$, then $n_0 = k_0 = 1$ and we reobtain the well-known solutions $x = pR(p - q)$, $y = pqR$, $z = qR(p - q)$, where $(p, q) = 1$, $p > q$. (Put $p - q = v$, $q = u$ to deduce the more familiar forms $x = Rv(u + v)$, $y = Ru(u + v)$, $z = Ruv$, with $(u, v) = 1$.)

2) Let $b = 2$. Then (7) give the general solutions with the condition that at least one of $p, R, n_0p - q$ resp. $q, R, k_0, n_0p - q$ is even.

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4 The Diophantine equation $x^n + y^n = x^p y^q z$ (where $p + q = n$)

Let $n \geq 2,\ p, q \geq 1$ be positive integers such that $p + q = n$. Our purpose is to obtain all nontrivial solutions of the given equation in positive integers. In fact, the used method permits to obtain all solutions in the set of integers $\mathbb{Z}$. We shall use the following auxiliary results:

Lemma 1. *Let $x, y > 0$ be integers such that $(x, y) = 1$. Then $(x + y, xy) = 1$.*

Lemma 2. *If $a, b > 0$ are integers and $(a, b) = 1$, then $(a^n, b) = 1$ for any integer $n > 0$.*

Let now denote $(x, y) = d$, i.e. $x = dX$, $y = dY$, with $(X, Y) = 1$. From the proposed equation one gets

$$X^n + Y^n = X^p Y^q z \tag{1}$$

From Newton's binomial theorem the following identity can be written:

$$X^n + Y^n = (X + Y)^n - \left[\binom{n}{1} X^n Y + \ldots + \binom{n}{n-1} X Y^{n-1}\right] \tag{2}$$

and, since $n \geq 2$ and by (1) it follows

$$\frac{(X+Y)^n}{XY} = \binom{n}{1} X^{n-2} + \ldots + \binom{n}{n-1} Y^{n-2} = X^{p-q} Y^{q-1} z \in \mathbb{N}^*, \tag{3}$$

since $p \geq 1,\ q \geq 1$ ($\mathbb{N}^* = \{n \in \mathbb{Z} :\ n \geq 1\}$).

By Lemma 1 and Lemma 2, the left side of (3) contains an irreducible fraction. This can be an integer only in the case $XY = 1$. Hence $X = Y = 1$, implying $x = y = d$, $z = 2$. This gives the (infinitely) set of solutions of the given equation.

Remark. Particular cases of our equation are for example:

$$x^2 + y^2 = xyz, \quad x^3 + y^3 = xy^2 z, \quad x^3 + y^3 = x^2 yz, \quad x^4 + y^4 = xy^3 z,$$

$$x^4 + y^4 = x^2 y^2 z, \quad x^5 + y^5 = xy^4 z, \quad x^5 + y^5 = x^2 y^3 z, \quad \text{etc.}$$

5 On the diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}=\frac{1}{x_{n+1}}$$

Let x_i $(i=\overline{1,n+1})$ be positive integers. For $n=2$ we get the equation

$$\frac{1}{x_1}=\frac{1}{x_2},\ \text{i.e. } x_1=x_2.$$

For $n=2$ we get the equation

$$\frac{1}{x_1}+\frac{1}{x_2}=\frac{1}{x_3} \tag{1}$$

This equation is very important, since occurs in geometry, number-theory, optics, physics, etc. For three distinct solutions, see [1]. Let now, $n\geq 3$. Let us note for simplicity by $\sum x_1\ldots\widehat{x}_i\ldots x_n$ the sum of all products $x_1\ldots\widehat{x}_i\ldots x_n$, where $\widehat{x}_i$ means that the term x_i is missing (e.g. $n=3$ this sum is $x_1x_3+x_2x_3+x_1x_2$). Then

$$\frac{1}{x_1}+\ldots+\frac{1}{x_n}=\frac{1}{x_{n+1}} \tag{2}$$

means that

$$\sum x_1\ldots\widehat{x}_i\ldots x_n|x_1x_2\ldots x_n \tag{3}$$

Let $d=(x_1,x_2,\ldots,x_n)=$ greatest common divisor of $x_1,\ldots,x_n$. Then

$$x=dy_1,\ldots x_n=dy_n,$$

where $(y_1,\ldots,y_n)=1$. By replacing in (3) after simplifying with d^{n-1}, we get

$$\sum y_1\ldots\widehat{y}_i\ldots y_n|y_1y_2\ldots y_n \tag{4}$$

Let $D=\left(y_1y_2\ldots,y_n,\sum y_1\ldots\widehat{y}_i\ldots y_n\right)$. Then (4) implies that $\sum y_1\ldots\widehat{y}_i\ldots y_n|dD$, i.e. there exists a positive integer k, such that:

$$dD=k\sum y_1y_2\ldots\widehat{y}_i\ldots y_n.$$

Therefore

$$x_1=\frac{k\sum y_1\ldots\widehat{y}_i\ldots y_n}{\left(y_1\ldots y_n,\sum y_1\ldots\widehat{y}_i\ldots y_n\right)}y_1,x_2=\frac{k\sum y_1\ldots\widehat{y}_i\ldots y_n}{\left(y_1\ldots y_n,\sum y_1\ldots\widehat{y}_i\ldots y_n\right)}y_2,\ldots, \tag{5}$$

$$x_n = \frac{k\sum y_1 \dots \widehat{y_i} \dots y_n}{\left(y_1 \dots y_n, \sum y_1 \dots \widehat{y_i} \dots y_n\right)} y_n$$

Here $\dfrac{k\sum y_1 \dots \widehat{y_i} \dots y_n}{\left(y_1 \dots y_n, \sum y_1 \dots \widehat{y_i} \dots y_n\right)} = A$ is a positive integer, so x_i $(i = \overline{1,n})$ are positive integers, satisfying the equation. Reciprocally, from (5) we have

$$\sum x_1 \dots \widehat{x_i} \dots x_n = A^{n-1} \sum y_1 \dots \widehat{y_i} \dots y_n \tag{6}$$

and

$$x_1 x_2 \dots x_n = A^n y_1 y_2 \dots y_n$$

Since $Ay_1 \dots y_n = k\sum y_1 \dots \widehat{y_i} \dots y_n \cdot y_1 y_2 \dots y_n | \left(y_1 y_2 \dots y_n, \sum y_1 \dots \widehat{y_i} \dots y_n\right)$ and $\left(y_1 \dots y_n, \sum y_1 \dots \widehat{y_i} \dots y_n\right) | y_1 \dots y_n$, we have $\sum y_1 \dots \widehat{y_i} \dots y_n | Ay_1 \dots y_n$, so (6) implies (3). All in all, the general solutions of equation (2) are given by (5) and

$$x_{n+1} = \frac{x_1 \dots x_n}{\sum x_1 \dots \widehat{x_i} \dots x_n} = \frac{Ay_1 \dots y_n}{\sum y_1 \dots \widehat{y_i} \dots y_n},$$

where $y_1, \dots, y_n$ are arbitrary positive integers, satisfying $(y_1, \dots, y_n) = 1$, while k is a positive integer.

Remark 1. For $n = 2$ we get from (5)

$$x_1 = ky_1(y_1 + y_2), \quad x_2 = ky_2(y_1 + y_2), \quad x_3 = ky_1y_2,$$

since $(y_1y_2, y_1 + y_2) = 1$ for $(y_1, y_2) = 1$.

Remark 2. For $n = 3$ the general solutions can be written as follows:

$$x_1 = ky_1(y_1y_2 + y_1y_3 + y_2y_3)/(y_1y_2y_3, y_1y_2 + y_1y_3 + y_2y_3),$$

$$x_2 = ky_2(y_1y_2 + y_1y_3 + y_2y_3)/(y_1y_2y_3, y_1y_2 + y_1y_3 + y_2y_3),$$

$$x_3 = ky_3(y_1y_2 + y_1y_3 + y_2y_3)/(y_1y_2y_3, y_1y_2 + y_1y_3 + y_2y_3),$$

$$x_4 = \frac{ky_1y_2y_3}{(y_1y_2y_3, y_1y_2 + y_1y_3 + y_2y_3)},$$

where $k > 0$ is arbitrary and $(y_1, y_2, y_3) = 1$.

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6 On the diophantine equation

$$x_1! + x_2! + \ldots + x_n! = x_{n+1}!$$

Since, by convention $0! = 1! = 1$, we may suppose that all $x_i \geq 1$ ($i = \overline{1, n+1}$). We need the following:

Lemma. $(a+b)! \geq a! + b!$ *for all* $a, b \geq 1$. *One has equality only for* $a = b = 1$.

Proof. $(a+b)! - a! = 1 \cdot 2 \cdot 3 \ldots a[(a+1)(a+2)\ldots(a+b)-1] \geq 1[2 \cdot 3 \ldots (1+b)-1] \geq 2 \cdot 3 \ldots b = b!$ with equality only for $a = 1$, $b = 1$.

Corollary. $(x_1 + x_2 + \ldots + x_n)! > x_1! + x_2! + \ldots + x_n!$ *for* $n \geq 3$, $x_i \geq 1$.

By induction, from Lemma. For $n > 2$ we cannot have equality since e.g. $x_1 + x_2 \geq 2$.

Theorem. *For all fixed* $n \geq 2$, *the equation in the title has at least a solution in positive integers. The number of solutions is finite.*

Proof. Clearly, $x_1 = x_2 = \ldots = x_n = (n-1)$, $x_{n+1} = n$ give a solution, since $n(n-1)! = n!$. Now, remark that by the Corollary, for $n \geq 3$, $x_{n+1} < x_1 + \ldots + x_n$. Let us suppose that $x_1 \leq x_2 \leq \ldots \leq x_n$. Then $x_{n+1} < nx_n$. In the same manner, by

$$x_1! + \ldots + x_{n-1}! = x_{n+1}! - x_n! = x_n! \left[\frac{x_{n+1}!}{x_n!} - 1\right]$$

and

$$x_{n+1}! > x_n!, \quad x_n! | x_{n+1}!$$

we get

$$x_n! | (x_1! + \ldots + x_{n-1}!),$$

thus

$$x_n! \leq x_1! + \ldots + x_{n-1}! \leq (x_1 + \ldots + x_{n-1})!,$$

so

$$x_n \leq x_1 + \ldots + x_{n-1} \leq (n-1)x_{n-1}.$$

Since

$$x_1! + \ldots + x_k! = x_{k+1}! \left[\frac{x_{n+1}!}{x_{k+1}!} - \frac{x_n!}{x_{k+1}!} - \ldots - 1\right]$$

and $x_{k+1}!|x_i!$ for all $i \leq k+1 \leq n+1$, we get $x_{k+1}!|(x_1! + \ldots + x_k!)$, implying

$$x_{k+1} \leq x_1 + \ldots + x_k \leq kx_k.$$

From $x_2 \leq 2x_1$, $x_3 \leq 3x_2, \ldots$, $x_{n+1} < nx_n$ we get $x_{n+1} < n!x_1$, so x_{n+1} is bounded above, and thus can take only a finite number of values.

Remark 1. For $n = 2$ one has $x_1! + x_2! = x_3!$, with $x_1 \leq x_2 \leq x_3$, implying $x_1!|x_2!$, $x_1!|x_3!$. Similarly $x_2!|x_3!$, so $x_2!|x_1!$. This in turn implies $x_1 = x_2$. The equation $2x_1! = x_3!$ is possible only for $x_1 = 1$, $x_3 = 2$.

Remark 2. For $n = 1$, the equation becomes $x_1! = x_2!$, and this one has infinitely many solutions.

7 The diophantine equation $xy = z^2 + 1$

Problem 18* of [2] asks for the determination of all matrices $A \in M_2[\mathbb{Z}]$ (i.e. second order matrices with integer coefficients) such that $A^2 = -E$, where E is the unity matrix. By letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one obtains the following system of equations:

$$a^2 + bc = -1, \quad ab + bd = 0, \quad ca + cd = 0, \quad cb + d^2 = -1.$$

Thus $b \neq 0$, $c \neq 0$, $d = -a$ and $a^2 + 1 = -bc$. Let $b = -B$ with $B > 0$ and let $c > 0$. We get: $a^2 + 1 = Bc$. In other words we must solve in positive integers the equation $xy = z^2 + 1$.

The following two auxiliary results are well-known:

Lemma 1. *Let x, z be positive integers. If $x|(z^2+1)$, then x can be written as the sum of two squares (i.e. $x = a^2 + b^2$, where $a, b \geq 0$) (see e.g. [1], [3]).*

Lemma 2. (Lagrange identity)

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

Therefore, by Lemma 1, $x = a^2 + b^2$, $y = c^2 + d^2$ $(a, b, c, d \in \mathbb{N})$. By Lemma 2, if $|ad - bc| = 1$, then $z = ac + bd$ provide solutions of the given equation. For example, it is well-known that if $(a, b) = 1$, there exist positive integers (c, d) such that $ab - dc = 1$. Therefore, the given equation can have infinitely many solutions. We now prove the following assertion:

Theorem. *All positive solutions of the equation $xy = z^2 + 1$ can be written as $x = a^2 + b^2$, $y = c^2 + d^2$, $z = ac + bd$, where $a, b, c, d \in \mathbb{N}$ and $|ad - bc| = 1$.*

Proof. Let us suppose that there exist triples (x, y, z), solutions of the equation, which cannot be written as above. Let (x_0, y_0, z_0) be such a triple with z_0 the least possible. Let us suppose $x_0 \leq y_0$ (otherwise we may change the roles of x_0 and y_0). We have thus $x_0 y_0 = z_0^2 + 1$, $z_0, y_0, z_0 \geq 1$ and there are no natural numbers a, b, c, d such that $x_0 = a^2 + b^2$, $y_0 = c^2 + d^2$, $z_0 = ac + bd$ with $|ad - bc| = 1$. We shall construct another triple (x, y, z), solution of the equation. Let $z = u + x$; then the equation becomes $xy - x^2 - 2xu - u^2 = 1$

or $x(y-x-2u)=u^2+1$. By $u=z-x$ it results $y-x-2u=y+x-2z$. In other words, $(x_0, x_0+y_0-2z_0, z_0-x_0)$ satisfies the equation $xy=z^2+1$. Now, verify that each term is ≥ 1. One has $z_0^2=x_0y_0-1<x_0y_0\leq\left(\frac{x_0+y_0}{2}\right)^2$, so $z_0<\frac{x_0+y_0}{2}$, implying $x_0+y_0-2z_0>0$.

On the other hand, if $z_0-x_0\leq 0$ we have $z_0\leq x_0\leq y_0$. We must distinguish two cases:

i) If $z_0=x_0$; then $x_0(y_0-x_0)=1$, so $x_0=1=1^2+0^2$, $y_0=2=1^2+1^2$, $z_0=1=1\cdot 1+1\cdot 0$ and $(1\cdot 1-0\cdot 1)=1$, in contradiction with the states assumption.

ii) If $z_0<x_0$, then $1=x_0y_0-z_0^2\geq x_0^2-z_0^2\geq (z_0+1)^2-z_0^2=2z_0+1$, thus $z_0\leq 0$, impossible.

From $z_0>x_0$ clearly follows $z_0-x_0\geq 1$.

Now the new triple has the component $z_0-x_0<z_0$, by the made assumption there exist natural numbers m,n,p,q such that $x_0=m^2+n^2$, $x_0+y_0-2z_0=p^2+q^2$, $z_0-x_0=mp+nq$, where $|mq-np|=1$. This implies $z_0=m^2+n^2+mp+nq=m(m+p)+n(n+q)$ and $y_0=p^2+q^2+2z_0-x_0=p^2+q^2+2mp+2nq+m^2+n^2=(m+p)^2+(n+q)^2$. Therefore, we can write: $x_0=m^2+n^2$, $y_0=(m+p)^2+(n+q)^2$, $z_0=m(m+p)+n(n+q)$, where $|m(n+q)-n(n+p)|=1$. We thus obtained a contradiction (selectiong $a=m$, $b=n$, $c=m+p$, $d=n+q$).

For other connections of this equation to Elementary Number Theory, see [3].

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8 A note on the equation $y^2 = x^3 + 1$

It is sufficient to solve this equation in positive integers, since if $x < 0$, put $x = -X$ $(X > 0)$, so $1 - X^3 = y^2 \geq 0$ implying $0 \leq X \leq 1$ so $X = 0$ or 1 and $x = 0$, $x = -1$ which give $y = \pm 1$, $y = 0$. If $y > 0$ is a solution, then $-y$ is acceptable, too.

1. In the famous book by Charles W. Trigg [3] this equation is solved as follows:

$$x^3 + 1 = (x+1)(x^2 - x + 1) = (x+1)^2 \left[x - 2 + \frac{3}{x+1} \right] = y^2. \tag{1}$$

Since 0 and 2 are the possible values for which $\dfrac{3}{x+1}$ is integer, then y^2 being non-negative, one must have $y \in \{0, \pm 1, \pm 3\}$. The fallacy in this proof is that if $x - 2 + \dfrac{3}{x+1}$ is integer, then y^2 is divisible by $x + 1$ implies y^2 divisible by $(x+1)^2$. In other words, the implication $b|a^2 \ \Rightarrow \ b^2|a^2$ is applied. This implication is clearly false (e.g. $4|6^2$ but $4^2 = 16 \nmid 6^2$, etc.).

The Romanian translators of this book have observed this mistake and have proposed another solution. Unfortunately, this solution contains a calculation fallacy, and this method cannot be changed for our purpose. In fact, the following "solution" is given. Since $x^2 - x + 1 = (x+1)^2 - 3x$, the terms $x+1$ and $x^2 - x + 1$ are relatively prime, or their g.c.d. is 3. In the first case $x+1 =$ perfect square $= a^2$, $x^2 - x + 1 =$ perfect square $= b^2$ $(a, b \geq 0)$. They conclude with (the erroneous) equality: $a^4 - a^2 + 1 - b^2 = 0$. Fortunately, this error has no great importance, since the correct relation $a^4 - 3a^2 + 3 - b^2 = 0$ can be shown to be impossible. Indeed, $a^4 - 4a^2 + 1 = (a^2 - 2)^2 < a^4 - 3a^2 + 3 < (a^2 - 1)^2 = a^4 - 2a^2 + 1$, so $s^4 - 3a^2 + 3$ is between two consecutive square, which is impossible.

In the second case, when $x + 1$ is multiple of 3, i.e. $x + 1 = 3k$, the new form of the equation becomes $9k(3k^2 - 3k + 1) = y^2$, and since $(k, 3k^2 - 3k + 1) = 1$, then $k = a^2$, $3k^2 - 3k + 1 = b^2$ $(a, b \geq 1)$. This gives $3a^4 - 3a^2 + 1 - b^2 = 0$. The erroneous equality $a^2 = \dfrac{3 \pm \sqrt{4b^2 + 5}}{6}$ is deduced. In fact, we must have $a^2 = \dfrac{3 \pm \sqrt{12b^2 - 3}}{6}$. Here one must have $12b^2 - 3 = c^2$. Since $3(4b^2 - 1) = c^2$, then $c = 3C$, implying the equation of Pell's type $4b^2 - 3C^2 = 1$. Here $a^2 = \dfrac{3 \pm 3C}{6} = \dfrac{1 \pm C}{2}$ and in case $C > 1$ one must have $C = 2a^2 - 1$. The demonstration of the fact that the equation $4b^2 - 3c^2 = 1$, where C has this form, has only the solutions $b = 1$, $a = 1$ seems a very difficult problem!

2. The equation in the title has its origins in the times of Fermat and Euler, and the first acceptable proof appeared only with the algebraic apparatus of the 20^{th} century. For this purpose the arithmetic of the field $Q(i\sqrt{3})$ must be considered, more precisely the unique prime factorization in this field, the existence of unities, etc.

In 1958 A. Wakulicz [2] found an elementary, but quite complicated proof of the fact that the equation

$$x^3 + y^3 = 2z^3$$

has no solutions in positive integers if $x \neq y$ and $z \neq 0$. This is enough to show that the equation of the title admits the single nontrivial solution (2,3). Indeed, from $(y-1)(y+1) = x^3$ it follows that if $y = 2k$ (even), then $2k - 1 = a^3$, $2k + 1 = b^3$, so $b^3 - a^3 = 2$, giving that $b^2 + ba + a^2$ divides 2, which is impossible. If, on the other hand $y = 2k - 1$ is odd, then $x = 2x_1$ (even) and $(k-1)k = 2x_1^3$. Here $(k-1, k) = 1$, yielding $k = y_1^3$, $k - 1 = 2y_2^3$ or $k = 2y_2^3$ and $k - 1 = y_1^3$. In both cases $y_1^3 - 2y_2^3 = \pm 1$. In other words, we have deduced the equations $u^3 - 2v^3 = 1$ and $u^3 - 2v^3 = -1$. Since the form of the first equation is $u^3 + (-1)^3 = 2v^3$, while of the second one is $u^3 + 1^3 = 2v^3$, from the above result of Wakulicz only $u = v = 1$ can be accepted, so $x = 2$, $y = 3$.

3. A generalization of the equation in the title is

$$y^2 = x^p + 1 \quad (p > 3, \text{ prime}) \tag{2}$$

In 1964 Chao Ko [1], by using congruence-theory has shown that this equation hasn't solutions in positive integers. The equation

$$y^p = x^2 + 1 \quad (p \geq 3, \text{ prime}) \tag{3}$$

was solved by Lebesque in 1850, who showed that for $y > 1$ this is impossible. However, equation (3) is much easier than (4).

A variant of this note has been published in 1996 in Hungarian [5], as remarks on a proposed problem for elementary grades. The students were expected to solve such a difficult problem! ([4])

The above considerations come in fact as illustation of how in certain cases small mistakes can lead to difficulties which cannot be overcomed in a simple way.

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9 On the equation $x^3 - y^2 = z^3$

The diophantine equation $x^3 - y^2 = z^3$ may be written also as

$$x^3 - z^3 = y^2. \tag{1}$$

This equation written as $x^3 + (-z)^3 = y^2$ is a particular case of the equation $u^3 + v^3 = w^3$ in integers. This equation (though the application of general theory involves considerable numerical details) is known to be completely solved by algebraic number theoretical methods. (Class number for binary quartic forms with given invariants is used). For example, when v is odd and is prime to u, then u and v are given by one of the following expression:

$$u = -4p^3q + 4q^4, \quad v = p^4 + 8pq^3,$$

$$u = -p^4 + 6p^2q^2 + 3q^4, \quad v = p^4 + 6p^2q^2 - 3q^4,$$

$$u = p^4 + 6p^2q^2 - 3q^4, \quad v = -p^4 + 6p^2q^2 + 3q^4,$$

$$u = 2p^4 - 4p^3q - 4pq^3 + 2q^4, \quad v = p^4 + 4p^3q - 6p^2q^2 + 4pq^3 + q^4,$$

$$u = 4p^3q + 24p^2q^2 + 48pq^3 + 36q^4, \quad v = p^4 + 8p^3q + 246p^2q^2 + 24pq^3.$$

Here p, q take such integer values such that v is odd and is prime to u. See [1], [2]. An elementary approach for (1) can be given formally, to a given point. This is based on the following:

Lemma. *All positive integral solutions of the equation*

$$XY = y^2 \tag{2}$$

can be written as $X = du^2$, $Y = dv^2$, $y = duv$ *where* $d \in N^*$ *is arbitrary and* $(u, v) = 1$, $(u, v \in N^*)$.

Proof. Let $(X, Y) = d$. Then $X = da$, $Y = db$ with $(a, b) = 1$. From (2) we get $d^2ab = y^2$ so $d^2|y^2$, implying $d|y$. Let $y = dl$. Then $ab = l^2$, where $(a, b) = 1$. It is well-known that we must have $a = u^2$, $b = v^2$ with $(u, v) = 1$. This proves the Lemma.

By writing (1) as

$$(x - z)(x^2 + xz + z^2) = y^2,$$

from Lemma we get:

$$x - z = du^2, \quad x^2 + xz + z^2 = dv^2, \quad y = duv, \quad (u, v) = 1 \tag{3}$$

Clearly $x = z + du^2$, $y = duv$ and $(z + du^2)^2 + (z + du^2)z + z^2 = dv^2$ gives the equation

$$3z^2 + 3zdu^2 + d^2u^4 - dv^2 = 0 \tag{4}$$

By solving this quadratic equation in z, one has $z = \dfrac{-3du^2 + t}{6}$, where $12dv^2 - 3d^2u^4 = t^2$. Here $3|t^2$, so $3|t$. Let $t = 3T$. One obtains $-d^2u^4 + 4dv^2 = 3T^2$ and $z = \dfrac{T - du^2}{2}$. Therefore T and du^2 have the same parity. Now, by solving the equation $d^2u^4 - 4dv^2 + 3T = 0$ one obtains

$$d = \frac{2v^2 + s}{u^4}, \text{ where } 4v^4 - 3Tu^4 = s^2 \tag{5}$$

Therefore, all solutions of (1) are given by (3), i.e. $x = z + du^2$, $y = duv$, $z = \dfrac{T - du^2}{2}$ with $(u, v) = 1$ and d, T satisfying the following conditions: $4v^4 - 3Tu^4$ is a perfect square, and s^2 and u^4 divides $2v^2 + s$, where $d = \dfrac{2v^2 + s}{u^4}$. Remark that from (5) we have:

$$y^4 = \frac{4v^4 - s^2}{3T} = \frac{(2v^2 - s)(2v^2 + s)}{3T},$$

so

$$\frac{2v^2 + s}{u^4} = \frac{3T}{2v^2 - s}.$$

Thus $2v^2 - s$ divides $3T$ and $d = \dfrac{3T}{2v^2 - s}$.

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10 On the sum of two cubes

The aim of this note is to describe all solutions in positive integers of the equation

$$x^3 + y^3 = z^3 + u^3. \tag{1}$$

We shall use the following lemma, discovered by Euler, but rediscovered (and showing it's importance) by Bell and Kalmár (see e.g. [1]).

Lemma. (Euler-Bell-Kalmár) *All solutions in positive integers of the equation*

$$XY = ZT \tag{2}$$

can be written as $X = ab$, $Y = cd$, $Z = ad$, $T = bc$, *where* $(b, d) = 1$, $(a, b, c, d \in N^*)$.

Proof. Let $(X, Z) = a$. Then $X = ab$, $Z = ad$, where $(b, d) = 1$. From (2) it results $bY = dT$, so $b|dT$ and Euclid's theorem yields $b|T$ (since $(b, d) = 1$). Therefore $T = bc$. This gives $X = ab$.

Remark. L. Kalmár and J. Surányi call this lemma as the "four-number theorem". Now, let us suppose that $x > z$ (for $(x = z)$ clearly we get $y = u$). Then $w > y$ and (1) can be written equivalently as

$$x - z = ab, \quad x^2 + xz + z^2 = cd, \quad w - y = ad, \quad w^2 + wy + y^2 = bc. \tag{3}$$

Therefore

$$\begin{cases} x = z + ab \\ w = y + ad \end{cases}$$

and

$$(z + ab)^2 + (z + ab)z + z^2 = cd$$

and

$$(y + ad)^2 + (y + ad)y + y^2 = bc.$$

By simple computations we obtain:

$$\begin{cases} 3z^2 + 3zab + a^2b^2 - cd = 0 \\ 3y^2 + 3yad + a^2d^2 - bc = 0 \end{cases} \tag{4}$$

By resolving these quadratic equations we easily get

$$z = \frac{-3ab + t}{6}, \quad y = \frac{-3ad + p}{6},$$

where

$$\begin{cases} 12cd - 3a^2b^2 = t^2 \\ 12bc - 3a^2d^2 = p^2 \end{cases} \tag{5}$$

Here $3|t^2,\ 3|p^2$, i.e. $t = 3T$, $p = 3P$, yielding

$$z = \frac{-ab + T}{2}, \quad y = \frac{-ad + P}{2}, \quad 4cd - a^2b^2 = 3T^2, \quad 4bc - a^2d^2 = 3P^2. \tag{6}$$

Therefore we must solve an equation of type

$$4m - n^2 = 3s^2 \tag{7}$$

where n and s have the same parity (clearly z and y to be integers, it is necessary that ab and T respectively ad and P have the same parity).

i) n is even, $n = 2N$. Then s must be even, too $s = 2S$. From (2) we get $m = N^2 + 3S^2$. Therefore, when n is even, the general solution of (7) can be written as

$$m = N^2 + 3S^2, \quad n = 2N, \quad s = 2S. \tag{8}$$

ii) n is odd, $n = 2N + 1$. Let $s = 2S + 1$. Then from (7) we get

$$m = N^2 + N + 3S^2 + 3S + 1. \tag{9}$$

Now, we return to the original equation.

In case i) $n = ab$ and $n = ad$ must be even. Since $(b, d) = 1$ this is possible only if a is even, $a = 2A$. Then $ab = 2N$, $ad = 2M$ give $N = Ab$, $M = Ad$. The other conditions (8) give

$$cd = N^2 + 3S^2 = A^2b^2 + 3S^2,$$

where $T = 2S$. Similarly,

$$bc = M^2 + 3R^2 = A^2d^2 + 3R^2,$$

where

$$P = 2R. \tag{10}$$

Therefore

$$x = z + ab, \quad w = y + ad, \quad z = \frac{-ab+T}{2} = \frac{-2N+2S}{2} = -N + S,$$

$$y = \frac{-ad+P}{2} = \frac{-2M+2R}{2} = -M + R,$$

where b, c, d satisfy (10) and $(b, d) = 1$.

In case ii) $n = ab$ and $n = ad$ are odd, therefore a, b, d are all odd and $(b, d) = 1$. Thus $ab = 2N + 1$, $ad = 2M + 1$, $T = 2S + 1$, $P = 2Q + 1$. Then

$$cd = N^2 + N + 3S^2 + 3S + 1 \tag{11}$$

and

$$bc = M^2 + M + 3Q^2 + 3Q + 1$$

$$cd = N^2 + N + 3S^2 + 3S + 1$$

and

$$bc = M^2 + M + 3Q^2 + 3Q + 1.$$

Now

$$z = \frac{-ab+T}{2} = S - N, \quad y = \frac{-ad+P}{2} = Q - M$$

where b, c, d satisfy (11) and $(b, d) = 1$.

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11 On an inhomogeneous diophantine equation of degree 3

The Diophantine equation which we will consider here is the following inhomogeneous equation of degree 3:

$$x^2 + y^2 = z(1 + xy). \tag{1}$$

We will study (1) in the set N^* of positive integers. First remark that for $x = y$ from (1) we get $2x^2 = z(1 + x^2)$ i.e. $(1 + x^2)|2x^2 = (1 + x^2) + (x^2 - 1)$, so $1 + x^2|(x^2 - 1)$ which is impossible for $x > 1$ since $1 + x^2|x^2 - 1$. For $x = 1$ however we obtain the solution $z = 1$. Thus (1,1,1) is a solution of (1). Therefore, in what follows we may suppose that $x < y$. For $z = 1$ by $x^2 + y^2 \geq 2xy$ one has $1 + xy \geq 2xy$ giving $xy \leq 1$ i.e. $x = y = 1$. For $z = 2$ relation (1) implies $(x - y)^2 = 2$, impossible by the irrationality of $\sqrt{2}$. Thus we may suppose in what follows $z > 2$. Let us consider first the set

$$A = \{(x, y, z) : 1 < x < y, z > 2, x \nmid y\} \tag{2}$$

where $x \nmid y$ denotes the fact that x doesn't divide y. We will show that the equation (1) has no solutions in the set A. The solutions with $x|y$ will be obtained in the above Lemma. Let us suppose that $(a, b, z) \in A$ is a solution of (1). Then

$$a^2 - zab + b^2 = z \tag{$*$}$$

Let $r = b - za$. Then $b = za + r$, so $a^2 - za(za + r) + (za + r)^2 = z$, i.e. $a^2 + zar + r^2 = z$. Remark here that $a^2 + r^2 = -zar + z > 0$ gives $ar < 1$ i.e. $ar \leq 0$ implying $r \leq 0$. But $r \neq 0$ since then we would have $b = za$, i.e. $a|b$. On the other hand $-r < a$ since this is equivalent to $za < b + a$, i.e. $z < 1 + \dfrac{b}{a}$. This is true since

$$z = \frac{a^2 + b^2}{1 + ab} < \frac{a^2 + b^2}{ab} = \frac{a}{b} + \frac{b}{a} < 1 + \frac{b}{a}.$$

Denoting $-r = a'$, we get the relation

$$a^2 - za'a + a'^2 = z. \tag{$**$}$$

Here $a' < a,\ z > 2$. If $a' = 1$ then we get as above $a = z = 1$, contradiction.

i) $a'|a$

ii) $a' \nmid a$.

In case ii) $(a, b, z) \in A$ and $(a', a, z) \in A$ with $a' < a$. So if we consider a solution A with a minimal, we obtain a contradiction. We will prove that the case i) is impossible. We need the following

Lemma. *Let us suppose that (x, y, z) satisfy (1), where $1 < x < y$, $z > 2$ and $x|y$. Then $y = x^3$, $z = x^2$.*

Proof. Let $y = kx$, where $k \geq 2$. Then (1) implies $x^2(1 + k^2) = z(1 + kx^2)$. Since $(x^2, 1 + kx^2) = 1$, clearly $x^2|z$, i.e. $z = x^2m$. We deduce the equation

$$1 + k^2 = m(1 + kx^2). \tag{3}$$

For $1 + kx^2 > 1 + k^2$, this is impossible since $(1 + kx^2)|(2 + k^2)$. Thus we must have $kx^2 \leq k^2$, i.e. $k \geq x^2$. For $k = x^2$ we get $m = 1$, so $z = x^2$, $y = kx = x^3$. Let us suppose now that $k > x^2$. Then divide k by x^2, so $k = x^2M + h$, where $m \geq 1$, $h < x^2$. Then (3) becomes equivalent to

$$1 + x^4M^2 + h^2 + 2x^2Mh = m(1 + x^4M + x^2h) \tag{4}$$

or

$$(1 + x^4M + x^2h)M + h^2 + x^2Mh - x^4M^2 - M + 1 = m(1 + x^4M + x^2h).$$

Since $1 + x^4M + x^2h$ divides the ledt side, it must divide $h^2 + x^2Mh - x^4M^2 - M$. However, we will show that

$$1 + x^4M + x^2h > h^2 + x^2Mh - x^4M^2 - M + 1,$$

or equivalently

$$z^4(M + M^2) > x^2h(M - 1) + h^2 - M.$$

Here

$$x^2h(M - 1) + h^2 - M \leq x^4(M - 1) + x^4 - 1 = x^4M - 1 < x^4(M + M^2).$$

Therefore, (4) is impossible.

Remark. The Lemma shows also that (1) has infinitely many solutions: (x, x^3, x^2) for $x > 1$. Now, if $a'|a$, then by the Lemma we would have $a = a'^3$, $z = a'^2$. Thus $b = za - a' = a'^5 - a'$. But (a, b, z) with $a = a'^3$, $b = a'^5 - a'$ and $z = a'^2$ do not give a solution of equation (1). It is immediate that

$$(a'^3 + (a'^5 - a')) \neq a'^2[1 + a'^{3(a'^5 - a')}].$$

12 On two equal sums of mth powers

Mihály Bencze in OQ.510 ([1]) proposed the following problem: Solve in integer numbers, the following Diophantine equation:

$$\sum_{k=1}^{n_1} x_k^m = \sum_{k=1}^{n_2} y_k^m,$$

where m is an integer and $n_1+n_2 = m$. This is a very difficult problem, and only particular cases are known. Indeed, this can be written also as

$$x_1^m + x_2^m + \ldots + x_{n_1}^m = y_1^m + y_2^m + \ldots + y_{n_2}^m, \quad n_1 + n_2 = m. \tag{1}$$

1. Let $m = 2$. Then $n_1 = n_2 = 1$, so $x_1^2 = y_1^2 \Leftrightarrow |x_1| = |y_1|$.

2. Let $m = 3$. There are two (essentially not distinct) cases:

a) $n_1 = 1$, $n_2 = 2$;

b) $n_1 = 2$, $n_2 = 1$.

Then we have to solve $x_1^3 = y_1^3 + y_2^3$ (or $x_1^3 + x_2^3 = y_1^3$). This is Fermat's equation with exponent 3, so the only trivial solutions are $y_1 = -y_2$, $x_1 = 0$ or $x_1 = y_1$, $y_2 = 0$ etc.

3. Let $m = 4$. We have two essentially distinct cases, namely:

a) $n_1 = 1,\ n_2 = 3$;

b) $n_1 = 2,\ n_2 = 2$.

The case b) gives

$$x_1^2 + x_2^2 = y_1^2 + y_2^2 \tag{2}$$

and this equation has infinitely many solutions and these can be completely determined. Indeed by $x_1^2 - y_1^2 = y_2^2 - x_2^2$ one has

$$(x_1 - y_1)(x_1 + y_1) = (y_2 - x_2)(y_2 + x_2).$$

Let us suppose $x_1 > y_1$. Then $y_2 > x_2$. The Euler-Bell equation $XY = ZT$ has the general solution $X = ab$, $Y = cd$, $Z = ad$, $T = bc$, where $(b, d) = 1$, so $x_1 - y_1 = ab$, $x_1 + y_1 = cd$, $y_2 - x_2 = ad$, $y_2 + x_2 = bc$, implying

$$x_1 = \frac{ab + cd}{2}, \quad y_1 = \frac{cd - ab}{2}, \quad y_2 = \frac{ad + bc}{2}, \quad x_2 = \frac{bc - ad}{2}.$$

These are the general solutions, where ab and cd, respectively ad and bc have the same parity, with $(b,d)=1$. So, (2) can be solved.

In case a), however we obtain Euler's equation

$$x_1^4 = y_1^4 + y_2^4 + y_3^4. \tag{3}$$

Euler conjectured that (3) has nontrivial solutions, but N. Elkies in 1988 disproved this conjecture by showing that (see [4])

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

Clearly, this generates infinitely many solutions for (3) (if (y_1, y_2, y_3, x_1), is a particular solution, then (ky_1, ky_2, ky_3, kx_1) is a solution, too). However, the general solution of (3) is not known.

4. Let $m = 5$. We have essentially two cases:

a) $n_1 = 1,\ n_2 = 4$;

b) $n_1 = 2,\ n_2 = 3$.

a):

$$x_1^5 = y_1^5 + y_2^5 + y_3^5 + y_4^5 \tag{4}$$

In 1966 L. Lander and T. Parkin [5] found the solution

$$144^5 = 27^5 + 84^5 + 110^5 + 133^5.$$

S. Brudno [3] asks for a parametric solution of (4). We note that for $x_1 \leq 765$ there is no other solution than the one obtained by Lander and Parkin.

b):

$$x_1^2 + x_2^2 = y_1^3 + y_2^3 + y_3^3 \tag{5}$$

For $x_1 = 0$ one has the parametric solution for (5)

$$x_2 = u^6 + 7u^3v^2 + v^6, \quad y_1 = u(u^3 + 2v^3), \quad y_2 = v(2u^3 + v^3), \quad y_3 = 3u^2v^2,$$

see [7]. Finally, we consider:

5. Let $m = 6$. Then essentially three cases may appear:

a) $n_1 = 1,\ n_2 = 5$;

b) $n_1 = 2,\ n_2 = 4$;

c) $n_1 = 3,\ n_2 = 3$.

a):

$$x_1^6 = y_1^6 + y_2^6 + y_3^6 + y_4^6 + y_5^6 \tag{6}$$

I cannot decide if (6) has at least a non-trivial solution.

b):

$$x_1^6 + x_2^6 = y_1^6 + y_2^6 + y_3^6 + y_4^6 \tag{7}$$

The same as above.

c):

$$x_1^6 + x_2^6 + x_3^6 = y_1^6 + y_2^6 + y_3^6 \tag{8}$$

A. Moessner ([6]) obtained the solution

$$3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6.$$

For (8) also parametric solutions are available (see A. Bremner [2]). The general solution of (8) however, is not known.

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13 On the equation $\sum_{k=1}^{n}(x+k)^m = y^{m+1}$

We will consider the equation in integers

$$\sum_{k=1}^{n}(x+k)^m = y^{m+1}. \tag{1}$$

Let us consider first the case $m = 1$. Then

$$\sum_{k=1}^{n}(x+k) = nx + \frac{n(n+1)}{2},$$

so (1) gives

$$n\left(\frac{2x+n+1}{2}\right) = y^2. \tag{2}$$

Case i): $n = 2N$ (even). Then $N(2x+2N+1) = y^2$. It is known that if $XY = y^2$, then $X = du^2$, $Y = dv^2$, $y = duv$, where $(u, v) = 1$, so $N = du^2$, $2x + 2N + 1 = dv^2$, $y = duv$, which gives:

$$x = \frac{dv^2 - 2du^2 - 1}{2}, \quad y = duv \tag{3}$$

where dv^2 is odd, i.e. d and v are odd and $(u, v) = 1$.

Case ii): $n = 2N + 1$ (odd). Then $(2N + 1)(x + N + 1) = y^2$ so we get

$$x = \frac{2dv^2 - du^2 - 1}{2}, \quad y = duv \tag{4}$$

where du^2 is odd, i.e. d and v are odd and $(u, v) = 1$. Hence, the equation has infinitely many solutions.

Let now $m = 2$. From (1) we get the equation

$$ax^2 + bx + c = y^3 \tag{5}$$

where $a = n,\ b = n(n+1),\ c = \frac{n(n+1)(2n+1)}{6}$. When $n = 1$ we have $x^2 + 2x + 1 = y^3$, i.e. $(x + 1)^2 = y^3$. Thus $y^2|(x + 1)^2$, implying $y|x + 1$ i.e. $x + 1 = yk$, so $y^2k^2 = y^3$ gives $y = k^2$, $x + 1 = k^3$. So $x = k^3 - 1$, $y = k^2$, and again we have infinitely many solutions. For $n > 1$ we have $k = b^2 - 4ac \neq 0$, which assures that (5) can be written also as

$$X^2 + k = 4ay^3, \tag{6}$$

where $X = 2ax + b$, $k = -(b^2 - 4ac) = \dfrac{n^2(n^2-1)}{3} > 0$. Now, it is a classical result by Landau and Ostrowski [2] and Thue [1] that (6) has a finite number of integer solutions. For general $m \geq 3$ probably there are only a finite number of solutions, as can be seen from a paper by Voorhoeve, Györy and Tijdeman [3]. However, a method to determine these solutions is not known to the author.

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14 On the diophantine equation $3^x + 3^y = 6^z$

This equation is the particular case $n = 2$ of equation 3) of OQ.490 proposed by Mihály Bencze (see [1]). In fact equations of type $a^x + b^y = c^z$ have been extensively studied in certain cases. For example, when $b > a$ and $\max(a, b, c) > 13$, Z. Cao [2], [3] proved that this equation can have at most one solution with $z > 1$. Another result (see [4]) says that if a, b, c are not powers of two, then $a^x + b^y = c^z$ can have at most a finite number of solutions. Since 3 and 6 are of this type, we can state that the equation in the title can have at most a finite number of solutions. Our aim is to prove elementary that this equation has essentially two distinct solutions, namely $x = y = z = 1$ and $x = 2$, $y = 3$, $z = 2$.

1. When $x = y$, we get $3^{x-z} = 2^{z-1}$, so $x - z = z - 1 = 0$, giving $x = y = z = 1$.

2. When $x = z$, we have $3^{y-x} = 2^x - 1$. Put $y - x = t$. First we prove the following:

Lemma. *The equation* $2^x - 3^t = 1$ *can have exactly two solutions in nonnegative integers, namely* $x = 1$, $t = 0$ *and* $x = 2$, $t = 1$.

Proof. For $t = 0$ we have $x = 1$. We cannot have $x = 0$. Suppose $x, t \geq 1$. Since $2^x = (3-1)^x \equiv (-1)^x \pmod 3$ and $3^t + 1 \equiv 1 \pmod 3$, x must be even. Put $x = 2a$. Then $2^{2a} - 1 = 3^t$, so $(2^a - 1)(2^a + 1) + 3^t$. This is possible only when $2^a - 1 = 3^u$, $2^a + 1 = 3^v$, $(u, v \geq 0)$. By substraction this implies $3^v - 3^u = 2$. If $u, v \geq 1$, the left side is $\equiv 0 \pmod 3$, contradiction, since $3 \nmid 2$. Therefore $u = 0$ and $v = 1$. Since $u + v = t$, one has $t = 1$, $a = 1$ so $x = 2$.

By this lemma one can say that $y - x = 0$, $x = 1$ or $y - x = 1$, $x = 2$. Therefore $y = x = 1$ or $y = 3$, $x = 2$, $z = 2$.

3. For $x > z$ we have two cases:

a) $y > z$

b) $y < z$.

In case a) we can write $3^{x-z} + 3^{y-z} = 2^z$, contradiction since the left side is $\equiv 0 \pmod 3$.

In case b) we have $x > y > z$, so we get $3^{x-y} + 1 = 2^z 3^{x-y}$ and $1 \equiv 0 \pmod 3$.

4. When $x < z$, we have the cases

a) $y < x$

b) $y > x$.

For $y < x$ one has $y < x < z$, contradiction, as above. For $y > x$ we can have $y < z$ or $y > z$. (We can remark that $x = y$ or $x = z$ were studied in **1** and **2**). When $y < z$, then $x < y < z$ so $1 + 3^{y-x} = 2^z 3^{z-x}$ gives the desired contradiction. For $y > z$ we have $x < z < y$, contradiction as above.

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15 On the diophantine equation $4^x + 18^y = 22^z$

Clearly $(1, 1, 1)$ is a solution of the considered equation. Though the general theory of diophantine equations of type $a^x + b^y = c^z$ (see e.g. [1]) says that this equation has at most one solution with $z > 1$, this particular equation needs a special treatment. We now prove the following result:

Theorem. *The diophantine equation*

$$4^x + 18^y = 22^z \tag{1}$$

has exactly one solution, namely $(x, y, z) = (1, 1, 1)$.

Proof. Equation (1) can be written equivalently as

$$2^{2x} + 2^y \cdot 3^{2y} = 2^z \cdot 11^z \tag{2}$$

We can distinguish three cases:

i) $2x > y$. Then $2^y(2^{2x-y} + 3^{2y}) = 2^z \cdot 11^z$ implies necessarily $y = z$ (since $2^{2x-y} + 3^{2y}$ is always odd). Therefore $2^{2x-z} + 3^{2z} = 11^z$. Let $2x - z = u$. Then we get the equation

$$2^u + 9^z = 11^z \tag{3}$$

It is known (see [1] for References) that the only solution of this equation is $(u, z) = (1, 1)$. Therefore $2x - z = 1$ and $z = 1$, giving $x = y = z = 1$.

ii) $2x = y$. Then $2^y(1 + 3^{2y}) = 2^z \cdot 11^z$. Here

$$1 + 3^{2y} = 1 + 3^{4x} = 1 + (20 \cdot 4 + 1)^x \equiv 2 \pmod 4,$$

which means that $1+3^{2y}$ is divisible by 2, but not a higher power of 2. Therefore $z = y+1$ and we get

$$1 + 3^{2y} = 2 \cdot 11^z \tag{4}$$

By denoting $3^y = a$, this gives $a^2 \equiv -1 \pmod{11}$, i.e. -1 is a quadratic residue modulo 11. Since $11 \not\equiv 1 \pmod 4$, it is well known that this cannot be true (see e.g. [3]).

iii) $2x < y$. As above we can write $z = 2x$, yielding the equation

$$1 + 2^{y-2x} \cdot 3^{2y} = 11^{2x} \tag{5}$$

Since $11^{2x} \equiv 6^{2x} \pmod 5$ and $36^x = (35+1)^x \equiv 1 \pmod 5$, so (5) implies $2^{y-2x} \cdot 3^{2y} \equiv 0 \pmod 5$, which is impossible. By concluding, the equation (1) can have solutions only in case i), when $x = y = z = 1$.

Equation $3^x + 3^y = 6^z$ appears in [2].

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16 On certain exponential diophantine equations

1. It is well-known that the equation $x^y = y^x$ has the most general solutions in positive integers as $x = y$ or $x = 2$, $y = 4$ for $x < y$. This equation can be studied by considering the function $t \mapsto t^{\frac{1}{t}}$, $(t > 0)$. We present here an arithmetical solution. Unfortunately, there appear in the literature sometimes incomplete or wrong arithmetic solutions (see e.g. [2]). In the proof the following Lemma will be used:

Lemma 1. *If $a^n|b^n$, then $a|b$ (i.e. a divides b) for all a, b, n positive integers.*

Proof. Let us suppose that in the prime factorizations of a and b the prime p appears at powers k and s respectively (i.e. $p^k \| a$, $p^s \| b$). Then in the prime factorizations of a^n and b^n, the prime p appears to the powers kn, respectively sn. We have $kn \leq sn$, since $a^n|b^n$. So $k \leq n$, which gives $a|b$, since p was selected arbitrary.

Now let us suppose that in the equation

$$x^y = y^x \tag{1}$$

one has $x < y$. Let $y = x + t$ $(t > 0)$. Then (1) can be written also as $x^{z+t} = y^x$ or $x^x x^t = y^x$ which implies $x^x|y^x$. So, by Lemma 1 $(a = x, b = y, n = x)$ we must have $x|y$. Let $y = kx$ $(k > 1)$. Then the equation becomes $x^{kx} = y^x$, yielding $kx = x^k$ or $x^{k-1} = k$. For $k = 2$ we get $x = 2$. For $k \geq 3$, we prove (by induction e.g.) that $x^{k-1} > k$ (for $x \geq 2$), so the equation is impossible. Therefore $k = 2$, $x = 2$, $y = 2 \cdot 2 = 4$.

2. The OQ.58 [1] proposed by Mihály Bencze in the particular case $n = 2$ gives the equation

$$x^y + y = y^x + x. \tag{2}$$

We shall prove that the general solutions of this equation are $x = y$ or for $x \leq y$ we have two possibilities:

i) $x = 1$, y arbitrary

ii) $x = 2,\ y = 3$.

Clearly for $x = y$ or for $x = 1$ we easily get the above solutions. Let us suppose that $x < y$. Then (2) can be written also as $y^x - x^y = y - x$. Therefore $y^x > x^y$ or $y^{1/y} > x^{1/x}$. Let us consider the function $f(t) = t^{1/t}$ $(t > 0)$. Since $f'(t) = f(t)\left[\dfrac{1 - \ln t}{t^2}\right]$, this function

attains a maximum at $t = e$, is strictly increasing for $0 < t \leq e$, and strictly decreasing for $t \geq e$. So we can have two possibilities:

a) $x, y \in [1, e]$

b) $x \in [1, e],\ y \geq 3$.

In case a) $y = 2,\ x = 1$ which is a solution. In case b) we have $x = 1$ or $x = 2$, $y = 3$. For $x = 2$ we obtain a solution for $y = 3$. For $y \geq 5$ we have $2^y > y^2$ (e.g., by induction), see we cannot have such solution. Clearly $y = 4$ doesn't give a solution. In conclusion, the above assertion is proved.

3. The case $n = 3$ of OQ.58, proposed by Mihály Bencze [1] gives the system of equations:

$$x^{yz} + y + z = y^{xz} + x + z = z^{xy} + x + y \tag{3}$$

or equivalently:

$$\begin{cases} x^{yz} + y = y^{xz} + x \\ y^{xz} + z = z^{xy} + y \\ x^{yz} + z = z^{xy} + x \end{cases} \tag{4}$$

Let us suppose that $x \geq y \geq z$. For $y = z$ we get

$$x^{y^2} + y = y^{xy} + x, \quad y^{xy} + y = y^{xy} + y, \quad x^{y^2} + y = y^{xy} + x.$$

From $x^{y^2} - y^{xy} = x - y$ we get an equation of type $a^n - b^n = a - b$. Since

$$a^n - b^n = (a - b)[a^{n-1} + \ldots + b^{n-1}] > a - b$$

for $a > b,\ n > 1$ we can have only $a = b$ or $n = 1$. In case $x = y$ we obtain $x = y = z$ which gives a general solution. When $x \neq y$ we can have $y = 1$ giving $x - 1 = x - 1$. Thus $x, y = z = 1$ is another solution. Now we can admit $x > y > z$. From $x^{yz} - y^{xz} = x - y$ we obtain $x^y > y^x$, i.e. $x^{1/x} > y^{1/y}$. By using again the function from **2**, we can have only $x = 2$, $y = 1$ or $x \geq 3$, $y \in \{1, 2\}$. For $x = 2$, $y = 1$ we get $z = 1$, so $x = 2$, $y = 1$ which is a particular case of the above solution $x, y = z = 1$. For $y = 1$ we have $x^z + 1 = 1 + x$, so $z = 1$. For $y = 2$ we can deduce $2^{xz} - x^{2z} = 2 - x$ so $(2^x)^z - (x^2)^z = 2 - x < 0$. But $2^x \geq x^2$ for $x \geq 4$ thus $x = 3$ is the only possibility. But for $x = 3$, $y = 2$ we do not obtain

a solution. By concluding all solutions of the system (4) are $x = y = z$ or x arbitrary, $y = z = 1$ for $x \geq y \geq z$.

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17 On a diophantine equation involving arctangents

The diophantine equation in the title is the following one:

$$\text{arctg}\ \frac{1}{x_1} + \text{arctg}\ \frac{1}{x_2} + \ldots + \text{arctg}\ \frac{1}{x_n} = \frac{\pi}{4} \tag{1}$$

where the unknowns x_i $(i = \overline{1,n})$ are positive integers. This is in fact OQ.20 (see [1]). The aim of this note is to prove that equation (1) admits at least a solution for all n, and that for fixed n the number of solutions is finite. For particular values of n all solutions can be obtained.

1. First we prove two lemmas:

Lemma 1. *For all $n \geq 2$ one has the identity:*

$$\text{arctg}\ \frac{1}{1+1+1^2} + \text{arctg}\ \frac{1}{1+2+2^2} + \ldots + \tag{2}$$

$$+\text{arctg}\ \frac{1}{1+(n-1)+(n-1)^2} + \text{arctg}\ \frac{1}{n} = \frac{\pi}{4}.$$

Proof. We use the well-known formula

$$\text{arctg}\ u + \text{arctg}\ v = \text{arctg}\ \frac{u+v}{1-uv} \text{ for } u, v > 0. \tag{$*$}$$

Then

$$\text{arctg}\ \frac{1}{3} + \text{arctg}\ \frac{1}{2} = \text{arctg}\ 1 = \frac{\pi}{4},$$

so (2) holds true for $n = 2$. Now, proceeding by induction, let us suppose that (2) is true for n, and try to prove it for $n+1$. It is easy to see that it is necessary to prove that

$$\text{arctg}\ \frac{1}{1+n+n^2} + \text{arctg}\ \frac{1}{n+1} = \text{arctg}\ \frac{1}{n},$$

and this follows at once by the above formula (we omit the simple computations).

Lemma 2. *For all $x > 0$ one has the inequality* $\text{arctg}\ x < x$. *If $0 < x < 1$, then* $\text{arctg}\ x \geq \frac{\pi}{4}x$ *(with equality only for $x = 1$).*

Proof. Since $\text{tg}\ \alpha > \alpha$ for $\alpha > 0$, put $\alpha = \text{arctg}\ x$ $(x > 0)$, giving the first inequality. For the second one remark that the functions $\alpha \mapsto \text{tg}\ \alpha$ is strictly convex on $\left[0, \frac{\pi}{4}\right]$,

therefore it's graph is belong the segment line $y = \frac{y}{\pi}\alpha$ passing on the points (0,0) and $\left(\frac{\pi}{4}, 1\right)$. Therefore tg $\alpha \leq \frac{\pi}{4}\alpha$, giving with $\alpha = \text{arctg } x$, the second inequality. For such trigonometric inequalities see e.g. [2]. Now we prove:

Theorem. *Equation (1) admits for all n at least a solution. The number of solution for a given n is finite.*

Proof. By Lemma 1,

$$x_1 = 1 + 1 + 1^2, \ldots, x_{n-1} = 1 + (n-1) + (n-1)^2,\ x_n = n$$

are solutions of (1) for $n \geq 2$. (For $n = 1$ the single solution is $x_1 = 1$). Now, let $(x_1, \ldots, x_n)$ be a solution of (1). By Lemma 2

$$\text{arctg } \frac{1}{x_i} < \frac{1}{x_i} \text{ and arctg } \frac{1}{x_i} \geq \frac{\pi}{4}\frac{1}{x_i}, \quad (i = \overline{1, n}).$$

In fact, here one has strong inequality for $n \geq 2$, since $x_i > 1$. (If $x_1 = 1$, then $x_i = 1$, $i \geq 2$). Therefore all solutions $(x_1, \ldots, x_n)$ satisfy the double-inequality:

$$\frac{\pi}{4} < \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} < 1 \quad (n \geq 2). \tag{3}$$

This implies that the number of solutions of (1) is finite. Let us consider for simplicity e.g. $n = 3$ and put $x_1 = x$, $x_2 = y$, $x_3 = z$. As we will see, the general case can be treated in the similar way. Thus

$$\frac{\pi}{4} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1. \tag{4}$$

Suppose $x \leq y \leq z$. Then $\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}$ so $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{x}$ and $\frac{\pi}{4} < \frac{3}{x}$ gives $x < \frac{12}{\pi}$, i.e. there are a finite number of values of x. Now, by $\frac{\pi}{4} - \frac{1}{x} < \frac{1}{y} + \frac{1}{z} < 1 - \frac{1}{x}$, since $y \leq z$ we get $\frac{1}{y} + \frac{1}{z} \leq \frac{2}{y}$ and $\frac{\pi}{4} - \frac{1}{x} < \frac{2}{y}$ gives $y < \frac{2}{\frac{\pi}{4} - \frac{1}{x}}$. For each value of x we get a finite number of values of y. Finally, from $\frac{1}{z} > \frac{\pi}{4} - \frac{1}{x} - \frac{1}{y}$ we have $z < \frac{1}{\frac{\pi}{4} - \frac{1}{x} - \frac{1}{y}}$, so the possible values of z are finite, in number. The general case for (3) follows at once by induction.

2. We now study certain particular cases of equation (1).

a) $n = 2$. Then we get the equation:

$$\operatorname{arctg} \frac{1}{x} + \operatorname{arctg} \frac{1}{y} = \frac{\pi}{4}. \tag{5}$$

We will show that (5) has a single solution, namely $x = 2$, $y = 3$ ($x = 3$, $y = 2$ is essentially the same). By formula $(*)$, after a little calculus, from (5) we get

$$x + y = xy - 1. \tag{6}$$

Equation (6) can be written also as $(x-1)(y-1) = 2$, and so $x - 1 = 1$, $y - 1 = 2$ giving $x = 2$, $y = 3$.

b) $n = 3$, i.e.

$$\operatorname{arctg} \frac{1}{x} + \operatorname{arctg} \frac{1}{y} + \operatorname{arctg} \frac{1}{z} = \frac{\pi}{4}. \tag{7}$$

By using again $(*)$, after computations, we get

$$\operatorname{arctg} \frac{1}{x} + \operatorname{arctg} \frac{1}{y} + \operatorname{arctg} \frac{1}{z} = \operatorname{arctg} \frac{z(x+y) + zy - 1}{z(xy-1) - x - y},$$

therefore (7) is equivalent to

$$z(x+y) + xy - 1 = z(xy-1) - x - y. \tag{8}$$

This can be transformed into

$$(x+y)(z+1) = (xy-1)(z-1). \tag{9}$$

Here one can consider two cases:

i) z is even. Then

$$(z_1, z-1) = (z-1+2, z-1) = (2, z-1) = 1,$$

so $(z-1)|(z+y)$, i.e. $x + y = k(z-1)$, and from (9) we obtain $xy - 1 = k(z+1)$. By substraction we have $xy - 1 - x - y = 2k$, i.e.

$$(x-1)(y-1) = 2(k+1).$$

Let d be an arbitrary divisor of $2(k+1)$. Then

$$x - 1 = d, \quad y - 1 = \frac{2(k+1)}{d}, \quad z = \frac{x+y}{k} + 1$$

is the general solution, where

$$k| \left[d + \frac{2(k+1)}{d} + 1 \right]. \tag{10}$$

Clearly (10) is satisfied by a finite number of values of k and we obtain in this way all solutions. For example, when $d = 1$, (10) means that $k|[3 + 2(k + 1)]$, i.e. $k|5$, implying $k \in \{1, 5\}$. For $k = 1$ we obtain the solutions $x = 2$, $y = 5$, $z = 8$; for $j = 5$ we get $x = 2$, $y = 1$, $z = 4$. When $d = 2$, we can deduce $x = 3$, $y = 3$, $z = 7$, $x = 3$, $y = 7$, $z = 3$ which are not solutions since z must be even.

ii) z is odd. Then $z = 2m + 1$ giving $z + 1 = 2(m + 1)$, $z - 1 = 2m$, so (9) gives

$$(x + y)(m + 1) = (xy - 1)m. \tag{11}$$

This yields $m|(x + y)$, i.e. $x + y = mA$ and hence, from (11), $xy - 1 = (m + 1)A$. By substraction $xy - 1 - x - y = A$, giving $(x - 1)(y - 1) = A + 2$, and we can repeat the procedure shown in i).

c) $n = 4$, i.e.

$$\operatorname{arctg} \frac{1}{x} + \operatorname{arctg} \frac{1}{y} + \operatorname{arctg} \frac{1}{z} + \operatorname{arctg} \frac{1}{t} = \frac{\pi}{4}. \tag{12}$$

By $(*)$ we have

$$\operatorname{arctg} \frac{1}{x} + \operatorname{arctg} \frac{1}{y} = \operatorname{arctg} \frac{x + y}{xy - 1},$$
$$\operatorname{arctg} \frac{1}{z} + \operatorname{arctg} \frac{1}{t} = \operatorname{arctg} \frac{z + t}{zt - 1},$$

so form (12), by applying one again $(*)$ we arrive at

$$\frac{x + y}{xy - 1} + \frac{z + t}{zt - 1} = 1 - \left(\frac{x + y}{xy - 1} \right) \left(\frac{z + t}{zt - 1} \right),$$

i.e.

$$(x + y)(zt - 1) + (xy - 1)(z + t) = (xy - 1)(zt - 1) - (x + y)(z + t),$$

implying

$$(x + y)(zt + z + t - 1) = (xy - 1)(zt - z - t - 1). \tag{13}$$

Since the most general solution of $XY = ZT$ is $X = ab$, $Y = cd$, $Z = bd$, $T = ac$, where $(a, d) = 1$, we obtain from (13):

$$x + y = ab, \quad zt + z + t - 1 = cd, \quad xy - 1 = bd, \quad zt - z - t - 1 = ac,$$

where $(a, d) = 1$, $d > a$. These give the pair of equations

$$\begin{cases} x + y = ab \\ xy = bd + 1 \end{cases}$$

and

$$\begin{cases} 2(z + t) = c(d - a) \\ 2(zt - 1) = c(d + a) \end{cases}$$

with $d > 1$, $(d, a) = 1$. These equations provide all solutions by trial. For example when $a = 4$, $d = 7$ we get the solution $x = 3$, $y = 5$, $z = 7$, $t = 8$.

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18 A sum equal to a product

Here appear equations of type

$$x_1^k + x_2^k + \ldots + x_n^k = (x_1 x_2 \ldots x_n)^k \tag{1}$$

where $x_i \geq 0$ $(i = \overline{1,n})$ are integers and $k \in N$. Particularly, it is conjectured [1] that for $k = \varphi(m)$, where $m = p^\alpha$ (p prime, $\alpha \geq 1$) and φ is Euler's totient, the equation (1) has the only solution $x_1 = \ldots = x_n = 0$. We can infirm this conjecture by the equation

$$x_1^{\varphi(4)} + x_2^{\varphi(4)} + \ldots + x_8^{\varphi(4)} + x_{10}^{\varphi(4)} = x_1^{\varphi(4)} \ldots x_8^{\varphi(4)} x_9^{\varphi(4)} x_{10}^{\varphi(4)} \tag{2}$$

where $x_1 = x_2 = \ldots = x_8 = 1$, $x_9 = x_{10} = 2$. Indeed $\varphi(4) = 2$, (where $m = 2^2$) and $1 \cdot 8 + 2 \cdot 2^2 = 1 \cdot 2^2 \cdot 2^2$, i.e. $16 = 16$. For another counterexample consider (1) for $n = 226$, $k = \varphi(8) = 4$ and $x_1 = \ldots = x_{224} = 1$, $x_{225} = x_{226} = 16$. Then $1 \cdot 224 + 2 \cdot 16 = 1 \cdot 16 \cdot 16$, i.e. $256 = 256$. In fact, by putting $x_i^k = y_i$, equation (1) becomes

$$y_1 + y_2 + \ldots + y_n = y_1 y_2 \ldots y_n. \tag{3}$$

Clearly, if one of y's is 0, then all $y = 0$. So, let us suppose $y_i \geq 1$ $(i = \overline{1,n})$. Let $y_i = a_i + 1$ $(a_i \geq 0)$. Then (3) becomes

$$\sum a_i + n = \prod (a_i + 1) = \prod a_i + \sum a_{i_1} a_{i_2} \ldots a_{i_{n-1}} + \ldots + \sum a_{i_1} a_{i_2} + \sum a_i + 1,$$

i.e.

$$\prod a_i + \sum a_{i_1} a_{i_2} \ldots a_{i_{n-1}} + \ldots + \sum a_{i_1} a_{i_2} = n - 1 \tag{4}$$

For fixed n, clearly $\sum a_{i_1} a_{i-2} \ldots a_{i_k} \leq n - 1$, which can be true for only finitely many a's. Thus, the equation (4), i.e. equation (3) can have at most a finite number of solutions. Equation (3) always has at least a nontrivial solution. Let $n \geq 3$ and put $y_1 = y_2 = \ldots = y_{n-2} = 1$, $y_{n-1} = 2$, $y_n = n$. Then

$$y_1 + y_2 + \ldots + y_{n-2} + y_{n-1} + y_n = n - 2 + 2 + n = 2n,$$

and

$$y_1 y_2 \ldots y_{n-2} y_n = 1 \cdot 2 \cdot n = 2n.$$

Therefore, we have obtained a solution $(y_1, \ldots, y_n)$ with $y_1 \leq y_2 \leq \ldots \leq y_n$.

For particular n, all solutions can be obtained. For example, for $n = 3$, all solutions satisfying $y_1 \leq y_2 \leq y_3$ are $y_1 = 1$, $y_2 = 2$, $y_3 = 3$. This follows easily from (4), which in this case is

$$a_1a_2a_3 + \sum a_1a_2 = 2 \tag{5}$$

Now, this is impossible if all $a_i \geq 1$. Let $a_1 = 0$. Then $a_2a_3 = 2$, so $a_2 = 1$, $a_3 = 2$, implying $y_1 = 1$, $y_2 = 2$, $y_3 = 3$. For $n = 4$, (4) is

$$a_1a_2a_3a_4 + \sum a_1a_2a_3 + \sum a_1a_2 = 3 \tag{6}$$

Now, let $a_1 \leq a_2 \leq a_3 \leq a_4$. Then $a_1 = 0$, $a_2 = 0$, $a_3a_4 = 3$, i.e. $a_3 = 1$, $a_4 = 3$, giving $y_1 = 1$, $y_2 = 1$, $y_3 = 1$, $y_4 = 4$, which is the single solution of (3) for $n = 4$, satisfying $y_1 \leq y_2 \leq y_3 \leq y_4$. For $n = 5$, one has two distinct solution with $y_1 \leq y_2 \leq y_3 \leq y_4 \leq y_5$, namely $y_1 = 1$, $y_2 = 1$, $y_3 = 2$, $y_4 = 2$, $y_5 = 2$, $y_1 = 1$, $y_2 = 1$, $y_3 = 1$, $y_4 = 3$, $y_5 = 3$. Now, for equation (1) with $k \geq 2$, one can see that for $n = 2, 3, 4, 5$ the only solutions are $x_i = 0$ $(i = \overline{1, n})$. However, for great of values n, one can obtain solutions, as are shown in the two examples at the beginning of this note.

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19 On certain equations involving $n!$

The aim of this paper is to solve some diophantine equations in which appears the factorial of a natural number. Our method is based on the theory of Gamma function, which is a generalization of the factorial ([1], [5], [6], [7], [8]).

The idea of this method has its origin in the equation $(p!)^k = (k!)^p$, used in [2] in order to study an extremal problem. However the solution from [2] is elementary, while by writing the equation in the form $(p!)^{1/p} = (k!)^{1/k}$, it is natural to consider the function

$$f(x) = \Gamma(x+1)^{1/x}, \quad x > 0,$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

denotes the Euler Gamma function. It is known ([6]) that $f(x)$ is strictly increasing function for $x > 1$, implying $f(x) = f(y)$ only if $x = y$, thus the above diophantine equation has the solutions $p = k$, for $p, k > 1$. We note that for $p = 1$ one has $k = 1$. Start with some introductory lemmas.

Lemma 1. (see [6], [7], [8]) *For $x > 0$ one has:*

$$\psi'(x) = \left(\frac{\Gamma'(x)}{\Gamma(x)}\right) = \sum_{n=0}^{\infty} (x+n)^{-2} \tag{1}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is Euler's "digamma" function ([1], [8]).

Lemma 2. ([6]) *For $x > 1$ one has:*

$$\ln \Gamma(x) > \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12(x+1)} \tag{2}$$

$$\ln \Gamma(x) < \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12(x-1)} \tag{3}$$

Proof. Suppose f has a second order derivative on (a, b). Using the "trapezoidal formula" ([3]), we can write:

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(a) + f(b)] - \frac{(b-a)^3}{12} f''(\xi),$$

where $\xi \in (a,b)$. Take $f(t) = \ln \Gamma(t)$, $a = x$, $b = x+1$. In view of the equality

$$\int_a^{a+1} \ln \Gamma(t)dt = a \ln a - a + \ln \sqrt{2\pi}$$

(see [8]), we have:

$$x \ln x - x + \ln \sqrt{2\pi} = \frac{1}{2} \ln(\Gamma(x)\Gamma(x+1)) - \frac{1}{2}\psi'(\xi).$$

But $\Gamma(x+1) = x\Gamma(x)$, and on the other hand it is well-known that if g is a strictly decreasing positive function with the property that

$$\lim_{x\to\infty} g(x) = 0,$$

then

$$\int_0^{\infty} g(t)dt < \sum_{n=0}^{\infty} g(n) < g(0) + \int_0^{\infty} g(t)dt$$

(see [5] p.359). Letting $g(x) = (x+t)^{-2}$, the above inequalities easily imply

$$\frac{1}{x} < \psi''(x) < \frac{1}{x} + \frac{1}{x^2} < \frac{1}{x-1} \text{ for } x > 1,$$

so by (1) we deduce at one (2) and (3). Using the same method we can prove (see [7]):

Lemma 3. *For $x > 1$ one has:*

$$\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12(x-1)^2} \tag{4}$$

$$\psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12(x+1)^2} \tag{5}$$

As a consequence of this lemma, we obtain from (6) and a simple integration

Corollary 1. *For $x > 1$ we have:*

$$\ln(x-1) < \psi(x) < \ln x \tag{6}$$

Corollary 2. *For $x > 1$ we have:*

$$x \ln x - x + 1 < \ln \Gamma(x+1) < (x+1)\ln(x+1) - x \tag{7}$$

Now we prove:

Theorem 1. *The solutions of the diophantine equations*

$$(k!)^p \cdot p^{kp} = (p!)^k \cdot k^{kp} \tag{8}$$

$$\left(\frac{k^{k-1}}{(k-1)!}\right)^{p(p-1)} = \left(\frac{p^{p-1}}{(p-1)!}\right)^{k(k-1)} \tag{9}$$

are $k = p$.

Proof. Take $f(x) = \Gamma(x+1)^{1/x}$ and $g(x) = f(x+1)/f(x)$. It is immediate that (8) is equivalent with $f(k)/k = f(p)/p$ and (9) with $g(k) = g(p)$. Thus we have to study the monotonicity of $f(x)/x$ and $g(x)$. By a simple computation we get

$$(f(x)/x)' = (f(x)/x)(h(x) - 1/x)$$

and

$$g'(x) = g(x)[h(x+1) - h(x)]$$

where

$$h(x) = \frac{1}{x}\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\ln\Gamma(x+1)}{x} \tag{10}$$

First we show that $h(x) < 1/x$, implying that $f(x)/x$ is strictly increasing so (8) has the only solutions $k = p$. Indeed, by (6) and (7) the above expression of $h(x)$ gives:

$$h(x) - 1/x < (x\ln(x+1) - x\ln x + 1)x^{-2} < 0$$

by the well-known relation $(1+1/x)^x < e$. Twofold, we show that $h(x)$ is strictly decreasing which give the same property for the function $g(x)$, proving the theorem for (9). We have:

$$h'(x) = \left(\frac{2}{x}\ln\Gamma(x+1) - 2\frac{\Gamma'(x+1)}{\Gamma(x+1)} + 1\right)x^{-2} = A(x)x^{-2}.$$

Using (3) and (4) for $A(x)$ we get:

$$A(x) < -\ln(x+1)/x + 1/3x^2 - 1 + 1/(x+1) < -\ln(x+1)/x < 0.$$

Theorem 2. *The solutions of the diophantine equation*

$$\left(\frac{k^{k^2-1}}{(k-1)!}\right)^{p(p-1)} = \left(\frac{p^{p^2-1}}{(p-1)!}\right)^{k(k-1)} \tag{11}$$

are $k = p$.

Proof. (11) is equivalent with $q(k) = q(p)$, where $q(x) = xg(x)$, g being defined in the proof of Theorem 1. One finds:

$$q'(x) = g(x)\{1 + x[h(x+1) - h(x)]\},$$

where $h(x)$ is defined in (10). By the relations (6) and (7) is not difficult to show that

$$1 + x[h(x+1) - h(x)] = -\frac{\Gamma'(x+1)}{\Gamma(x+1)} \cdot \frac{1}{x+1} + \frac{x}{(x+1)^2} +$$

$$+(\ln \Gamma(x+1))\frac{2x+1}{x(x+1)} - x\ln(x+1)(x+1)^{-2} + 1 > (x^3 + x^2)/x(x+1)^2 > 0.$$

Thus $q(x)$ is a strictly increasing function, proving (11).

Theorem 3. *The solutions of the diophantine equations*

$$(k+1)!^{1/(k+2)} - (p+1)!^{1/(p+2)} = k!^{1/(k+1)} - p!^{1/(p+1)} \tag{12}$$

$$(k+2)!^{1/(k+2)} - (p+2)!^{1/(p+2)} = (k+1)!^{1/(k+1)} - (p+1)!^{1/(p+1)} \tag{13}$$

are $k = p$.

Proof. (12) is equivalent with

$$f_1(k+2) - f_1(k+1) = f_1(p+2) - f_1(p+1),$$

where $f_1(x) = \Gamma(x)^{1/x}$, while (13) is equivalent with the same relation for the function $f(x) = \Gamma(x+1)^{1/x}$. For the monotonicity of the sequences $\{f_1(k+1) - f_1(x)\}$, $\{f(k+1) - f(k)\}$ it is sufficient to study the convexity or concavity of the above functions. We prove that $f_1(x)$ is a convex function for $x > 0$. By successive differentiation we obtain:

$$f_1''(x) = f_1(x)\{-x^{-2}\ln\Gamma(x) + x^{-1}\Gamma'(x)/\Gamma(x)]^2 +$$

$$+2x^{-3}\ln\Gamma(x) - 2x^{-2}\Gamma'(x)/\Gamma(x) + x^{-1}(\Gamma'(x)/\Gamma(x))'.$$

Here $(\Gamma'(x)/\Gamma(x))' > 1/x$ (see the proof of Lemma 2), so:

$$f_1''(x) > f_1(x)x^{-2}(\Gamma'(x)/\Gamma(x) - x^{-1}\ln\Gamma(x) - 1)^2 \geq 0.$$

This method cannot be applied for the function $f(x)$, because of $\Gamma'(x+1)/\Gamma(x+1) < 1/x$ (see the proof of Lemma 2). However, by a more complicated argument can be proved ([6]) that for $x \geq 7$, the function $f(x)$ is strictly concave. Thus the sequence $(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!})$ is strictly decreasing (answering in this way an open problem [4]) and this finishes the proof of (13) for $k, p \geq 7$. The remaining cases may be examined by direct verifications.

Remark. For related problems and results on the monotonicity, convexity or logarithmic convexity of functions connected with Euler's Gamma function, see [6].

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20 On certain diophantine equations for particular arithmetic functions

Let $n \geq 1$ be a positive integer, and let $\varphi(n), \sigma(n), d(n), \omega(n), \Omega(n), S(n)$ denote Euler's totient, the sum of divisors, the number of divisors, the number of distinct divisors, the number of all divisors, and the Smarandache function of n, respectively. These function have a very irregular behaviour and their study is very difficult sometimes, or even impossible with the momentary state of the science. For example, the study of the equation $\sigma(n) = 2n$ is nothing else than the study of perfect numbers, and it is known that ([2]) the odd numbers of this kind are greater than 10^{300} and there are few chances to decide practically (or even theoretically) the existence of such numbers. It is not known if the equation $\sigma(n) = \sigma(n+1)$ has infinitely many solutions (though such a result is true for the function d, see [2]); and the same can be said for the equation $\varphi(n) = \varphi(n+1)$. Other equations as $\sigma(\varphi(n)) = n$ or $S(n+1) + S(n+2) = S(n+3) + S(n+4)$ are also unsolved ([2], [6], [15]).

In what follows we will solve, by using an elementary argument based generally on inequalities, certain equation for the above arithmetic functions. The applied methods permit us to obtain, in most cases all solutions. Though the equations which will follow are much simpler than the above stated ones, they have an interest, maybe suggesting ideas for attacking more complex problems, too.

The equation $\sigma(n) = n + 3$

By $\sum_{d|n} d = \sigma(n)$ we can write the equation in the form

$$\sum_{\substack{d|n \\ 1<d<n}} d = 2.$$

This equality hods true only for $d = 2$, and then clearly $n = 4$. Therefore $n = 4$ is the single solution of the equation.

The equation $\sigma(n) = n + k$, $k =$ fixed

From $\sigma(n) \geq n + 1$ (with equality only for $n =$ prime) we get that for $k = 1$ the equation has infinitely many solutions. Let us suppose thus $k > 1$. In this case n must be composite number, so it must have at least a divisor $d \notin \{1, n\}$. Here one can admit $d \geq \sqrt{n}$ for if one writes $n = dq$ with $d < \sqrt{n}$, then clearly $q > \sqrt{n}$ and we may select q instead of d. From the definition of σ one has

$$\sigma(n) \geq 1 + n + d \geq 1 + n + \sqrt{n} \geq 1 + n + k > n + k$$

for sufficiently large n. Therefore, for $k > 1$ the considered equation can have at most a finite number of solutions.

Generally speaking we cannot say more on the solutions of this equations, for special values of k we can apply "special treatments" (for $k = 3$ see above).

The equation $2d(n^2) = 3d(n)$

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the prime factorization of $n > 1$ ($n = 1$ is not a solution). First we prove that

$$\frac{d(n^2)}{d(n)} \geq \left(\frac{3}{2}\right)^{\omega(n)} \quad \text{for all } n \geq 1. \tag{1}$$

Indeed, since

$$d(n) = \prod_{i=1}^{r}(\alpha_i + 1), \quad d(n^2) = \prod_{i=1}^{r}(2\alpha_i + 1),$$

by $2(2\alpha_i + 1) \geq 3(\alpha_i + 1)$, after multiplication we get (1) with $\omega(n) = r$. We can have equality in (1) only for $\alpha_i = 1$ ($i = \overline{1, r}$) thus for $n = p_1 p_2 \dots p_r$, i.e. $n =$squarefree number.

Thus the stated equation implies by (1) $\frac{3}{2} \geq \left(\frac{3}{2}\right)^r$, giving $r = 1$, i.e. the proposed equation has the solutions $n = p$ (prime).

The equation $\sigma(n) = n + d(n)$

First we show that if n is not a prime of a square of a prime, then

$$\sigma(n) \geq (\sqrt{n} + 1)^2 \tag{2}$$

Indeed, let p be the least prime divisor of n. Then $1, p, \frac{n}{p}, n$ all are distinct divisors of n, so

$$\sigma(n) \geq 1 + p + \frac{n}{p} + p \geq 1 + n + 2\sqrt{n} = (\sqrt{n} + 1)^2,$$

since $p + \frac{n}{p} \geq 2\sqrt{n}$.

In what follows, we will use the known inequality

$$d(n) \leq 2\sqrt{n} \tag{3}$$

(see W. Sierpinski [12]). For the sake of completeness, we give the short proof of (3). If d is a divisor of n, the same is true for $\frac{n}{d}$ and we can form couples $\left(d, \frac{n}{d}\right)$. (E.g. for $n = 72$ one has: (1,72), (2,36), (3,24), (4,18), (6,12), (8,9)). Since the number of such couples is at most $\sqrt{n}$, we deduce $d(n) \leq 2\sqrt{n}$. In fact we note that the inequality is strict. Indeed, when n is not a square, then $\sqrt{n}$ is irrational, so $\frac{d(n)}{2}$ = rational $\neq \sqrt{n}$. When $n = m^2$ is a square, then $d(n)$ = odd number (this follows from the formula for $d(n)$), while $2\sqrt{m^2} = 2m$ is even. So $d(n) < 2\sqrt{n}$.

Now (1), (3) yield $\sigma(n) - d(n) \geq n + 1$ when $n \neq p$, p^2, so the proposed equation may have solutions only for $n = p$ or $n = p^2$. Since $p + 1 \neq p + 2$ and $p^2 + p + 1 = p^2 + 3$ only for $p = 2$, the single solution of the given equation is $n = 4$.

The equation $d(n) + \varphi(n) = n + 1$

One can remark that $n =$ prime and $n = 1$ are solutions. Let $n > 1$ be composite and let $i \neq 1$ be a divisor of n. Then g.c.d. $(i, n) \neq 1$. Therefore, clearly $d(n) + \varphi(n) < n + 1$ from the definitions of d and φ (which is the number of couples (i, n) such that g.c.d. $(i, n) = 1$, $i < n$). Therefore $n \neq 1$, $n \neq p$ cannot be solutions.

The equation $\varphi(n) = d(n)$

$n = 1$ is a solution, let $n > 1$, $n = \prod p^{\alpha}$ (for simplicity we do not use indices), where $p =$ prime, $\alpha \geq 1$. Then

$$\frac{\varphi(p^{\alpha})}{d(p^{\alpha})} = \frac{p^{\alpha-1}(p-1)}{\alpha + 1}$$

and for $p \geq 3$ one has $3^{\alpha-1} \cdot 2 \geq \alpha + 1$ for all α (which can be proved easily by induction on α) with equality only for $\alpha = 1$, $p = 3$. One gets

$$\varphi(n) \geq d(n) \text{ for all } n = \text{odd}, \tag{4}$$

with equality for $n \in \{1, 3\}$.

Let now be n even, i.e. $n = 2^{\alpha}m$ with $m = \text{odd}$, $\alpha \geq 1$. For $\alpha \geq 3$ one can write

$$\varphi(n) = \varphi(2^{\alpha})\varphi(m) \geq 2^{\alpha-1}d(m)$$

on base of (4). But $2^{\alpha-1} \geq \alpha + 1$, with equality for $\alpha = 3$, so

$$\varphi(n) \geq d(n) \text{ for } n = \text{even},\ 8|n \tag{5}$$

In the above inequality we must have $m = 1$ or $m = 5$, so in (5) we can have equality only for $n = 1 \cdot 8 = 8$, $n = 3 \cdot 8 = 24$. We have to study the remaining cases $\alpha = 1$, $\alpha = 2$.

For $\alpha = 1$ one obtains the equation

$$\varphi(m) = 2d(m), \quad m = \text{odd}; \tag{6}$$

while for $\alpha = 3$ we have

$$2\varphi(m) = 3d(m), \quad m = \text{odd} \tag{7}$$

Let $m = \prod_{p \geq 3} p^{\beta}$. Then (6) becomes

$$\prod_{p \geq 3} \frac{p^{\beta-1}(p-1)}{\beta+1} = 2$$

with equality only for $\beta = 1$, thus $m = 5$ or $m = 3 \cdot 5$ are the single possibilities. From here, as solutions we get $n = 2 \cdot 5 = 10$; $n = 2 \cdot 3 \cdot 5 = 30$.

In the same manner, (7) becomes

$$\prod_{p \geq 3} \frac{p^{\beta-1}(p-1)}{\beta+1} = \frac{3}{2}.$$

But $\dfrac{3^{\beta-1} \cdot 2}{\beta+1} \geq 1$, $\dfrac{5^{\beta-1} \cdot 4}{\beta+1} > \dfrac{3}{2}$ and we cannot have equality. Therefore, this case doesn't provide solutions.

By summing, all solutions of the initial equations are $n \in \{1, 3, 8, 10, 24, 30\}$. As a consequence, we can write:

$$\varphi(n) > d(n) \text{ for } n > 30. \tag{8}$$

The equation $\varphi(n)d(n) = \varphi(n) + n - 1$

We shall use the well-known Gauss relation ([1], [3]):

$$\sum_{i|n} \varphi(i) = n \tag{9}$$

and the divisibility property $i|n \ \Rightarrow \ \varphi(i) \leq \varphi(n)$ (in fact $\varphi(i)|\varphi(n)$). By (9) and this divisibility property, one can write

$$1 < n = 1 + \sum_{i|n, i>1} \varphi(i) \leq 1 + \varphi(n) \sum_{i|n, i>1} 1 = 1 + \varphi(n)(d(n) - 1),$$

with equality only for $n =$ prime. Therefore

$$\varphi(n)d(n) \geq \varphi(n) + n - 1 \tag{10}$$

improving $\varphi(n)d(n) \geq \varphi(n)$ due to R. Sivaramakrishnan [13]. See also [9]. In (10) one has equality for $n =$ prime or $n = 1$, so these are the general solutions.

On the equation $\varphi(n)(\omega(n) + 1) = n$

$n = 1$ is a solution, since by definition $\omega(1) = 0$, where $\omega(n)$ denotes the number of distinct prime divisors of n.

Let $1 < n = \prod_{i=1}^{r} p_i^{\alpha_i}$, with $r = \omega(n)$. Suppose $p_1 < \ldots < p_r$. Then clearly $p_1 \geq 2$, $p_2 \geq 3, \ldots, p_r \geq r + 1$ (with equality only for $p_1 = 2$, $p_2 = 3$), thus

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_r}\right) \geq m\left(1 - \frac{1}{2}\right)\ldots\left(1 - \frac{1}{r+1}\right) =$$

$$= n \cdot \frac{1}{2} \cdot \frac{2}{3} \ldots \frac{r}{r+1} = \frac{n}{r+1}.$$

This implies

$$\varphi(n)(\omega(n) + 1) \geq n \text{ for all } n, \tag{11}$$

with equality only when n has at most two distinct solutions, namely 2 and 3. By taking into account this fact, the set of solutions is

$$n \in \{1\} \cup \{2^i \cdot 3^j : \ i > 0, \ j \geq 0\} \tag{12}$$

Reciprocally, the numbers from (12) are indeed solutions.

The equation $\sigma(n) = n(\omega(n) + 1)$

We show first that (on the light of (11))

$$\varphi(n)\sigma(n) \leq n^2 \tag{13}$$

with equality for $n = 1$, This is well-known (see e.g. [3]), but the simple proof may be inserted here.

Let $1 < n = \prod_{i=1}^{r} p_i^{\alpha_i}$. Then

$$\varphi(n)\sigma(n) = \prod_{i=1}^{r} p_i^{\alpha_i} \left(1 - \frac{1}{p_i}\right) \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} =$$

$$= \prod_{i=1}^{r} p_i^{2\alpha_i} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{\alpha_i+1}}\right) \leq \prod_{i=1}^{r} p_i^{2\alpha_i} = n^2,$$

giving (13).

Now, (11) and (13) imply

$$\sigma(n) \leq n(\omega(n) + 1) \tag{14}$$

with equality only for $n = 1$, which is the solution of our equation.

The equation $d(n) = 2^{\omega(n)}$

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i} > 1$ be the canonical factorization of n. From

$$d(n) = \prod_{i=1}^{r} (\alpha_i + 1) \geq \prod_{i=1}^{r} 2 = 2^{\omega(n)}$$

it results

$$d(n) \geq 2^{\omega(n)} \tag{15}$$

where one has equality only when all $\alpha_i = 1$, i.e. when n is squarefree. Thus all solutions are: $n = 1$ and $n =$ squarefree (i.e. product of distinct primes).

The equation $d(n) = 2^{\Omega(n)}$

As above, one can write

$$d(n) = \prod_{i=1}^{r}(\alpha_i + 1) \leq \prod_{i=1}^{r} 2^{\alpha_i},$$

by the elementary inequality $\alpha + 1 \leq 2^{\alpha}$. Since

$$\sum_{i=1}^{r} \alpha_i = \Omega(n),$$

we get

$$d(n) \leq 2^{\Omega(n)} \tag{16}$$

with equality for $n =$ squarefree (and $n = 1$).

Remark. By (15) and (16) one can deduce that the normal order of magnitude of $\log d(n)$ is $\log 2 \log\log n$.

The equation $\sigma(n) = \varphi(n) + d(n)(n - \varphi(n))$

We shall apply the following known identity ([1], [3]):

$$\sum_{i|n} \varphi\left(\frac{n}{i}\right) d(i) = \sigma(n) \tag{17}$$

We note here that this identity can be proved for example by using the multiplicativity of φ and d, which implies their Dirichlet product is also multiplicative. Hence, it is sufficient to consider only $n = p^{\alpha}$ (prime powers). In this case a simple algebraic calculation shows the valability of (17).

We now apply the method from paragraph 7, by remarking that $i|n \ \Rightarrow \ d(i) \leq d(n)$. This follows without difficulty from the definition of d. Therefore:

$$\sigma(n) = \varphi(n) + \sum_{\substack{i|n \\ i>1}} \varphi\left(\frac{n}{i}\right) d(i) \leq \varphi(n) + d(n) \sum \varphi\left(\frac{n}{i}\right),$$

implying:

$$\sigma(n) \leq \varphi(n) + d(n)(n - \varphi(n)), \tag{18}$$

a better inequality than

$$nd(n) \geq \varphi(n) + \sigma(n), \tag{19}$$

due to C.A. Nicol [4].

From the demonstration of (18) one can see that one has equality for $n = 1$ and $n =$ prime.

The equation $\varphi\left(n\left[\frac{\sigma(n)}{n}\right]\right) = n$

Though the equation $\varphi(\sigma(n)) = n$ is extremely difficult to study (see e.g. [2], [6]), the equation in the title can be solved relatively easily. (Here $[x]$ notes the integer part of x). Along with relation (13) we need another property of the function φ, namely

$$\varphi(ab) \leq a\varphi(b) \text{ for all } a, b \geq 1 \tag{20}$$

See e.g. [7], [8] for many similar inequalities. If

$$a = \prod p^{\alpha} \prod q^{\beta}, \quad b = \prod p^{\alpha'} \prod t^{\gamma}$$

(where $(p, q) = (p, t) = (q, t) = 1$ are primes) then

$$\frac{\varphi(ab)}{\varphi(b)} = a \prod \left(1 - \frac{1}{t}\right) \leq a,$$

with equality if doesn't exist t, i.e. all prime factor of b is a prime factor of a, too. By (20) and (13) one can write

$$\varphi\left(n\left[\frac{\sigma(n)}{n}\right]\right) \leq \left[\frac{\sigma(n)}{n}\right]\varphi(n) \leq \frac{\sigma(n)}{n}\varphi(n) \leq n.$$

Therefore

$$\varphi\left(n\left[\frac{\sigma(n)}{n}\right]\right) \leq n \tag{21}$$

with equality only for $n = 1$ (see (13)!). Relation (21) is due to A. Oppenheim [5]. Similar inequalities for Dedekind's or other functions appear in [6], [7], [8].

The equations $\varphi(n) = \dfrac{n}{2}$ and $\varphi(n^\alpha) = n$ $(\alpha > 1)$

The first equation is solvable only for $n =$ even; let $n = 2^k m$ with $m \geq 1$, odd. From the multiplicativity of φ results

$$\varphi(n) = \varphi(2^k)\varphi(m) \leq 2^{k-1}m = \frac{n}{2}.$$

Thus:

$$\varphi(n) \leq \frac{n}{2} \text{ for } n = \text{even}. \tag{22}$$

One has equality only for $n = 1$, i.e. $n = 2^k$ (the powers of 2).

For the study of the second equation, we shall use relation (14):

$$\varphi(n) \geq \frac{n}{\omega(n) + 1}.$$

From (15) and (3) we get

$$\omega(n) \leq \frac{\ln 2\sqrt{n}}{\ln 2} = 1 + \frac{1}{2}\frac{\ln n}{\ln 2} < 1 + \frac{5}{6}\ln n$$

from $\ln 2 \approx 0,6931\ldots > \dfrac{3}{5}$. Thus

$$\varphi(n) > \frac{6n}{12 + 5\ln n},$$

or

$$\varphi(n) > \frac{6n}{12 + 5\ln n} \tag{23}$$

Now $n = \varphi(n^\alpha) > \dfrac{6n^\alpha}{12 + 5\alpha \ln n}$ by $\dfrac{n^{\alpha-1}}{\ln n} \to \infty$ gives a contradiction for sufficiently large n $(n \to \infty)$, so the equation $\varphi(n^\alpha) = n$ can have at most a finite number of solutions.

Let us consider the case $\alpha = 2$. From (23) follows the weaker relation

$$\varphi(n) \geq \sqrt{n} \text{ for } n \neq 2, 6 \tag{24}$$

due to A.M. Vaidya [14]. Thus $\varphi(n^2) \geq n$ and the equation $\varphi(n^2) = n$ has the only solutions $n = 1$ and $n = 2$.

The equation $\Lambda(n)(d(n)-1)=\dfrac{d(n)\ln n}{2}$

Let Λ be the von Mangoldt function, i.e. $\Lambda(n)=\ln p$ for $n=p^{\alpha}$ (p prime, $\alpha\geq 1$); $=0$, in other cases (see e.g. [1], [3]). The following identity is well known:

$$\sum_{i|n}\Lambda(i)=\ln n. \tag{25}$$

The identity

$$2\sum_{i|n}\Lambda\left(\frac{n}{i}\right)d(i)=d(n)\ln n \tag{26}$$

can be proved via similar arguments. Now

$$\frac{d(n)\ln n}{2}=\Lambda(n)+\sum_{\substack{i|n\\ i>1}}\Lambda\left(\frac{n}{i}\right)d(i)\leq\Lambda(n)+d(n)(\ln n-\Lambda(n))$$

and on base of property (26), as well as $i|n\ \Rightarrow\ d(i)\leq d(n)$ one gets

$$\Lambda(n)(d(n)-1)\leq\frac{d(n)\ln n}{2} \tag{27}$$

with equality for $n=1$ and $n=$ prime; which provide the most general solutions of the proposed equation.

The equation $S(n)=n$

Let S be the Smarandache function defined by $S(n)=$ least positive integer m such that $n|m!$ (see [10], [11], [15]). We shall prove that for all n one has

$$S(n)\leq n \tag{28}$$

with equality only for $n=1,4$ or prime. Therefore, all solutions of the equation in the title are $n\in\{1,4\}\cup\{p:\ p \text{ prime}\}$. Inequality (28) is trivial from the definition: $n=1,4,p$ are solutions.

Let $n>4$ be composite, another solution i.e. $n=ab$ with $a\geq b\geq 2$. Then $a!=1\cdot 2\cdot 3\ldots b\ldots a$ is divisible by ab, so $ab|a!$. Therefore, by the definition of S one has

$$S(ab)=S(n)\leq a=\frac{n}{b}<n,$$

by contradicting $n = S(n)$.

If $a = b$, then $n = a^2$ and clearly $a^2|(1 \cdot 2 \cdot 3 \ldots a \ldots 2a) = (2a)!$ implies $S(n) \leq 2a$. Here $a > 2$ (by $n > 4$) so $S(n) \leq 2a < a^2 = n$, contradiction.

The equation $S(n^2) = n$

We will prove that

$$S(n^2) \leq n - 1 \text{ if } n \neq 1, p, 2p, 8, 9 \ (p \text{ prime}) \tag{29}$$

If $n \neq p, 2p, p^2, 8, 16$, then $n = ab$, $(a \neq b,\ a, b \geq 3)$. If $n \neq 16$, then $n = ab$ with $a \geq 3$, $b \geq 5$. Let $n = ab$, where $b > a$, $a \geq 3$. Then $a, b, 2a, 2b, 3a < n-1$ and $a, b, 2b$ are distinct, and at least one of $2a, 3a$ is different from $a, b, 2b$. Therefore $(n-1)!$ contains $a, b, 2b, 2a$ or $a, b, 2b$ and $3a$. In all cases $(n-1)!$ is divisible by $a^2b^2 = n^2$.

If $n = p^2$ then $n-1 > 4p$ and $(n-1)!$ contains the terms $p, 2p, 3p, 4p$; implying $(n-1)!$ divisible by $p^4 = n^2$. If $n = 2p$ then $p^2 \nmid (n-1)!$. If $n = 8, 9$ then $n^2 \nmid (n-1)!$. if $n = 16$, then $n^2|(n-1)!$. By summing $n^2 \nmid (n-1)!$ iff $n = p$, $n = 2p$ or $n \in \{8, 9\}$, giving relation (29).

It is immediate that $S(8^2) = 8$ and $S(9^2) = 9$. On the other hand $S(p^2) = 2p = p^2 \ \Leftrightarrow \ p = 2$ and $S(4p^2) = \max\{S(4), S(p^2)\} = \max\{4, 2p\} = 2p$ for $p \geq 3$ and for $p = 2$, $S(16) = 6 \neq 4$. By collecting the above results, all solutions of the equation in the title are

$$n \in \{1, 2\} \cup \{2p : \ p \text{ prime}, \ p \geq 3\} \cup \{8, 9\} \tag{30}$$

See also [11].

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21 On the diophantine equation $a^2 + b^2 = 100a + b$

The numbers 1233 and 8833 have the curious property that $1233 = 12^2 + 33^2$ and $8833 = 88^2 + 33^2$. Let $\overline{xyzt}$ be a four-digit number satisfying this property, i.e. $\overline{xyzt} = \overline{xy}^2 + \overline{zt}^2$. By putting $a = \overline{xy}$, $b = \overline{zt}$, since $\overline{xyzt} = 100\overline{xy} + \overline{zt} = 100a + b$, we are led to the following diophantine equation:

$$a^2 + b^2 = 100a + b. \tag{1}$$

The above problem required a and b to have two digits, but we generally will solve this equation for all positive integers a and b.

By considering (1) as a quadratic equation in a, we can write

$$a_{1,2} = 50 \pm \sqrt{2500 + b - b^2}. \tag{2}$$

To have integer solutions, we must suppose that

$$2500 + b - b^2 = x^2 \tag{3}$$

for certain positive integer x, giving $a_{1,2} = 50 \pm x$.

By multiplying with 4 both sides of equation (3) we can remark that this transforms equation (3) into

$$(2x)^2 + (2b - 1)^2 = 10001. \tag{4}$$

It is well known that an equation of type $u^2 + v^2 = n$ $(n > 1)$ has the number of solutions $4(\tau_1 - \tau_2)$, where τ_1 and τ_2 denote the number of divisors of n having the forms $4k+1$ and $4k+3$, respectively. Since $10001 = 137 \cdot 73$ and $137 = 4 \cdot 34 + 1$, $73 = 4 \cdot 18 + 1$, clearly $\tau_1 = 4$, $\tau_2 = 0$. Thus $u^2 + v^2 = 10001$ can have exactly $16 : 4 = 4$ positive solutions, giving two distinct solutions. Remarking that $73 = 3^2 + 8^2$, $137 = 11^2 + 4^2$, by the identities

$$(\alpha^2 + \beta^2)(u^2 + v^2) = (\alpha u - \beta v)^2 + (u\beta + \alpha v)^2 = (\beta u - \alpha v)^2 + (\alpha u + \beta v)^2,$$

we can deduce the relations $76^2 + 65^2 = 10001$, $100^2 + 1^2 = 10001$; implying $2x = 76$, $2b - 1 = 65$; $2x = 100$, $2b - 1 = 1$ respectively. For $x = 38$ and $b = 33$ we get the values $a_1 = 50 + 38 = 88$, $a_2 = 50 - 38 = 12$. For $x = 50$, $b = 1$ one has $a_1 = 100$,

$a_2 = 0$. Therefore, all solutions in positive integers of equation (1) are $(a, b) = (12, 33)$; $(a, b) = (88, 33)$. These are exactly the numbers stated at the beginning of this note.

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Chapter 3. Arithmetic functions

"... what one would like, of course, is to translate it into number theory, the bedrock of mathematics..."

(Gregory J. Chaitin, Conversations with a mathematician, Springer Verlag, 2002)

1 A note on $S(n)$

Let S be the Smarandache function. We will prove that the inequality

$$S(n) \leq S(n - S(n)) \tag{1}$$

is valid for infinitely many n. Let $p < q$ be prime numbers and put $n = pq$. We shall prove that (1) is valid for such numbers. Indeed, since $S(pq) = q$, in this case (1) becomes

$$q \leq S(pq - q) = S(q(p - 1)). \tag{2}$$

Since $p < q$, clearly $(p - 1, q) = 2$, so by a well known theorem, we have

$$S(q(p - 1)) = \max\{S(q), S(p - 1)\} = \max\{q, S(p - 1)\}.$$

Thus, inequality (2) becomes equivalent with the following obvious relation:

$$q \leq \max\{q, S(p - 1)\}. \tag{3}$$

In the same manner can be proved that (1) is valid for $n = p_1 p_2 \dots p_k$, where $p_1 < p_2 < \dots < p_k$ are primes and $p_k \nmid (p_1 \dots p_{k-1} - 1)$.

Remark. (1) answers an Open Question [1].

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2 On certain inequalities involving the Smarandache function

1. The Smarandache function satisfies certain elementary inequalities which have importance in the deduction of properties of this (or related) functions. We quote here the following relations which have appeared in the Smarandache Functions Journal:

Let p be a prime number. Then

$$S(p^x) \leq S(p^y) \text{ for } x \leq y \tag{1}$$

$$\frac{S(p^a)}{p^a} \geq \frac{S(p^{a+1})}{p^{a+1}} \text{ for } a \geq 0 \tag{2}$$

where x, y, a are nonnegative integers;

$$S(p^a) \leq S(q^a) \text{ for } p \leq q \text{ primes;} \tag{3}$$

$$(p-1)a + 1 \leq S(p^a) \leq pa. \tag{4}$$

If $p > \frac{a}{2}$ and $p \leq a-1$ $(a \geq 2)$, then

$$S(p^a) \leq p(a-1). \tag{5}$$

For inequalities (3), (4), (5), see [2], and for (1), (2), see [1].

We have also the result ([4]):

For composite $n \neq 4$,

$$\frac{S(n)}{n} \leq \frac{2}{3}. \tag{6}$$

Clearly,

$$1 \leq S(n) \text{ for } n \geq 1 \text{ and } 1 < S(n) \text{ for } n \geq 2 \tag{7}$$

and

$$S(n) \leq n \tag{8}$$

which follow easily from the definition $S(n) = \min\{k \in N^* : \ n \text{ divides } k!\}$.

2. Inequality (2), written in the form $S(p^{a+1}) \leq pS(p^a)$, gives by successive application

$$S(p^{a+2}) \leq pS(p^{a+1}) \leq p^2 S(p^a), \ldots,$$

that is

$$S(p^{a+c}) \leq p^c \cdot S(p^a) \tag{9}$$

where a and c are natural numbers. (For $c = 0$ there is equality, and for $a = 0$ this follows by (8)).

Relation (9) suggests the following result:

Theorem 1. *For all positive integers m and n holds true the inequality*

$$S(mn) \leq mS(n). \tag{10}$$

Proof. For a general proof, suppose that m and n have a canonical factorization

$$m = p_1^{a_1} \dots p_r^{a_r} \cdot q_1^{b_1} \dots q_s^{b_s}, \quad n = p_1^{c_1} \dots p_r^{c_r} \cdot t_1^{d_1} \dots t_k^{d_k},$$

where p_i $(i = \overline{1,r})$, q_j $(j = \overline{1,s})$, t_p $(p = \overline{1,k})$ are distinct primes and $a_i \geq 0$, $c_j \geq 0$, $b_j \geq 1$, $d_p \geq 1$ are integers.

By a well known result (see [3]) we can write

$$S(mn) = \max\{S(p_1^{a_1+c_1}), \dots, S(p_r^{a_r+c_r}), S(q_1^{b_1}), \dots, S(q_s^{b_s}), S(t_1^{d_1}), \dots, S(t_k^{d_k})\} \leq$$

$$\leq \max\{p_1^{a_1} S(p_1^{c_1}), \dots, p_r^{a_r} S(p_r^{c_r}), S(q_1^{b_1}), \dots, S(q_s^{b_s}), \dots, S(t_k^{d_k})\}$$

by (9). Now, a simple remark and inequality (8) easily give

$$S(mn) \leq p_1^{a_1} \dots p_r^{a_r} q_1^{b_1} \dots q_s^{b_s} \max\{S(p_1^{c_1}), \dots, S(p_r^{c_r}), S(t_1^{d_1}), \dots, S(t_k^{d_k})\} = mS(n)$$

proving relation (10).

Remark. For $(m, n) = 1$, inequality (10) appears as

$$\max\{S(m), S(n)\} \leq mS(n).$$

This can be proved more generally, for all m and n

Theorem 2. *For all m, n we have:*

$$\max\{S(m), S(n)\} \leq mS(n). \tag{11}$$

Proof. The proof is very simple. Indeed, if $S(m) \geq S(n)$, then $S(m) \leq mS(n)$ holds, since $S(n) \geq 1$ and $S(m) \leq m$, see (7), (8). For $S(m) \leq S(n)$ we have $S(n) \leq mS(n)$ which is true by $m \geq 1$. In all cases, relation (11) follows.

This proof has an independent interest. As we shall see, Theorem 2 will follow also from Theorem 1 and the following result:

Theorem 3. *For all m, n we have*

$$S(mn) \geq \max\{S(m), S(n)\}. \tag{12}$$

Proof. Inequality (1) implies that $S(p^a) \leq S(p^{a+c})$, $S(p^c) \leq S(p^{a+c})$, so by using the representations of m and n, as in the proof of Theorem 1, by Smarandache's theorem and the above inequalities we get relation (12).

We note that, equality holds in (12) only when all $a_i = 0$ or all $c_i = 0$ $(i = \overline{1, r})$, i.e. when m and n are coprime.

3. As an application of (10), we get:

Corollary 1. *a)* $\dfrac{S(a)}{a} \leq \dfrac{S(b)}{b}$, *if* $b|a$ (13)

b) If a has a composite divisor $b \neq 4$, then

$$\frac{S(a)}{a} \leq \frac{S(b)}{b} \leq \frac{2}{3}. \tag{14}$$

Proof. Let $a = bk$. Then $\dfrac{S(bk)}{bk} \leq \dfrac{S(b)}{b}$ is equivalent with $S(kb) \leq kS(b)$, which is relation (10) for $m = k$, $n = b$.

Relation (14) is a consequence of (13) and (6). We note that (14) offers an improvement of inequality (6).

We now prove:

Corollary 2. *Let m, n, r, s be positive integers. Then:*

$$S(mn) + S(rs) \geq \max\{S(m) + S(r), S(n) + S(s)\}. \tag{15}$$

Proof. We apply the known relation:

$$\max\{a+c, b+d\} \leq \max\{a,b\} + \max\{c,d\}. \tag{16}$$

By Theorem 3 we can write $S(mn) \geq \max\{S(m), S(n)\}$ and $S(rs) \geq \max\{S(r), S(s)\}$, so by consideration of (16) with

$$a \equiv S(m), \quad b \equiv S(r), \quad c \equiv S(n), \quad d \equiv S(s)$$

we get the desired result.

Remark. Since (16) can be generalized to n numbers ($n \geq 2$), Theorem 1-3 do hold for the general case (which follow by induction; however these results are based essentially on (10)-(15), we can obtain extensions of these theorems to n numbers).

Corollary 3. *Let a, b be composite numbers, $a \neq 4$, $b \neq 4$. Then*

$$\frac{S(ab)}{ab} \leq \frac{S(a)+S(b)}{a+b} \leq \frac{2}{3};$$

and

$$S^2(ab) \leq ab[S^2(a) + S^2(b)]$$

where $S^2(a) = (S(a))^2$, etc.

Proof. By (10) we have $S(a) \geq \dfrac{S(ab)}{b}$, $S(b) \geq \dfrac{S(ab)}{a}$, so by addition

$$S(a) + S(b) \geq S(ab)\left(\frac{1}{a} + \frac{1}{b}\right),$$

giving the first part of (16).

For the second, we have by (6):

$$S(a) \leq \frac{2}{3}a, \quad S(b) \leq \frac{2}{3}b,$$

so

$$S(a) + S(b) \leq \frac{2}{3}(a+b),$$

yielding the second part of (16).

For the proof of (17), remark that by $2(n^2+r^2) \geq (n+r)^2$, the first part of (16), as well as the inequality $2ab \leq (a+b)^2$ we can write successively:

$$S^2(ab) \leq \frac{a^2b^2}{(a+b)^2}[S(a)+S(b)]^2 \leq \frac{2a^2b^2}{(a+b)^2}[S^2(a)+S^2(b)] \leq ab[S^2(a)+S^2(b)].$$

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3 On certain new inequalities and limits for the Smarandache function

I. **Inequalities**

1) If $n > 4$ is an even number, then $S(n) \leq \frac{n}{2}$.

Indeed, $\frac{n}{2}$ is integer, $\frac{n}{2} > 2$, so in $\left(\frac{n}{2}\right)! = 1 \cdot 2 \cdot 3 \ldots \frac{n}{2}$ we can simplify with 2, so $n|\left(\frac{n}{2}\right)!$. This implies clearly that $S(n) \leq \frac{n}{2}$.

2) If $n > 4$ is an even number, then $S(n^2) \leq n$.

By $n! = 1 \cdot 2 \cdot 3 \ldots \frac{n}{2} \ldots n$, since we can simplify with 2, for $n > 4$ we get that $n^2|n!$. This clearly implies the above stated inequality. For factorials, the above inequality can be much improved, namely one has:

3) $S((m!)^2) \leq 2m$ and more generally, $S((m!)^n) \leq nm$ for all positive integers m and n.

First remark that

$$\frac{(mn)!}{(m!)^n} = \frac{(mn)!}{m!(mn-m)!} \cdot \frac{(mn-m)!}{m!(mn-2m)!} \cdots \frac{(2m)!}{m! \cdot m!} = C_{2m}^m \cdot C_{3m}^m \ldots C_{nm}^m,$$

where $C_n^k = \binom{n}{k}$ denotes a binomial coefficient. Thus $(m!)^n$ divides $(mn)!$, implying the stated inequality. For $n = 2$ one obtains the first part.

4) Let $n > 1$. Then $S((n!)^{(n-1)!}) \leq n!$.

We will use the well-known result that the product of n consecutive integers is divisible by $n!$. By

$$(n!)! = 1 \cdot 2 \cdot 3 \ldots n((n+1)(n+2)\ldots 2n)\ldots((n-1)!-1)\ldots(n-1)!n$$

each group is divisible by $n!$, and there are $(n-1)!$ groups, so $(n!)^{(n-1)!}$ divides $(n!)!$. This gives the stated inequality.

5) For all m and n one has $[S(m), S(n)] \leq S(m)S(n) \leq [m, n]$, where $[a, b]$ denotes the $l \cdot c \cdot m$ of a and b.

If $m = \prod {p_i}^{d_i}$, $n = \prod {q_j}^{b_j}$ are the canonical representations of m, resp. n, then it is well-known that

$$S(m) = S(p_i^{a_i})$$

and

$$S(n) = S(q_j^{b_j}),$$

where

$$S(p_i^{a_i}) = \max\{S(p_i^{a_i}) : \ i = 1, \ldots, r\};$$

$$S(q_j^{b_j}) = \max\left\{\{S(q_j^{b_j}) : \ j = 1, \ldots, h\right\},$$

with r and h the number of prime divisors of m, resp. n. Then clearly

$$[S(m), S(n)] \leq S(m)S(n) \leq p_i^{a_i} \cdot q_j^{b_j} \leq [m, n]$$

6) $(S(m), S(n)) \geq \dfrac{S(m)S(n)}{mn} \cdot (m, n)$ for all m and n

Since

$$(S(m), S(m)) = \frac{S(m)S(n)}{[S(m)S(n)]} \geq \frac{S(m)S(n)}{[m, n]} = \frac{S(m)S(n)}{nm} \cdot (m, n)$$

by 5) and the known formula $[m, n] = \dfrac{mn}{(m, n)}$.

7) $\dfrac{(S(m), S(n))}{(m, n)} \geq \left(\dfrac{S(mn)}{mn}\right)^2$ for all m and n.

Since $S(mn) \leq mS(n)$ and $S(mn) \leq nS(m)$ (see [1]), we have

$$\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)S(n)}{mn},$$

and the result follows by 6).

8) We have $\left(\dfrac{S(mn)}{mn}\right)^2 \leq \dfrac{S(m)S(n)}{mn} \leq \dfrac{1}{(m, n)}$.

This follows by 7) and the stronger inequality from 6), namely

$$S(m)S(n) \leq [m, n] = \frac{mn}{(m, n)}$$

Corollary. $S(mn) \leq \dfrac{mn}{\sqrt{(m, n)}}$.

9) $\max\{S(m), S(n)\} \geq \dfrac{S(mn)}{(m, n)}$ for all m, n, where (m, n) denotes the $g \cdot c \cdot d$ of m and n.

We apply the known result: $\max\{S(m), S(n)\} = S([m, n])$. On the other hand, since $[m, n] | mn$, by Corollary 1 from our paper [1] we get

$$\frac{S(mn)}{mn} \leq \frac{S([m, n])}{[m, n]}$$

Since $[m,n]=\dfrac{mn}{(m,n)}$, the result follows.

Remark. Inequality 9) complements Theorem 3 from [1], namely that

$$\max\{S(m),S(n)\}\leq S(mn)$$

1) Let $d(n)$ be the number of divisors of n. Then

$$\frac{S(n!)}{n!}\leq\frac{S(n^{d(n)/2})}{n^{d(n)/2}}.$$

We will use the known relation

$$\prod_{k|n}k=n^{d(n)/2},$$

where the product is extended over all divisors of n. Since this product divides $\prod_{k\leq n}k=n!$, by Corollary 1 from [1] we can write

$$\frac{S(n!)}{n!}\leq\frac{S\left(\prod_{k|n}k\right)}{\prod_{k|n}k},$$

which gives the desired result.

Remark. If n is of the form m^2, then $d(n)$ is odd, but otherwise $d(n)$ is even. So, in each case $n^{d(n)/2}$ is a positive integer.

11) For infinitely many n we have $S(n+1)<S(n)$, but for infinitely many m one has $S(m+1)>S(m)$.

This a simple application of 1). Indeed, let $n=p-1$, where $p\geq 5$ is a prime. Then, by 1) we have

$$S(n)=S(p-1)\leq\frac{p-1}{2}<p.$$

Since $p=S(p)$, we have $S(p-1)<S(p)$. Let is the same manner $n=p+1$. Then, as above,

$$S(p+1)\leq\frac{p+1}{2}<p=S(p).$$

12) Let p be a prime. Then $S(p!+1)>S(p!)$ and $S(p!-1)>S(p!)$.

Clearly, $S(p!) = p$. Let $p! + 1 = \prod q_j^{\partial j}$ be the prime factorization of $p! + 1$. Here each $q_j > p$, thus $S(p! + 1) = S(q_j^{\partial j})$ (for certain j)$\geq S(p^{\partial j}) \geq S(p) = p$. The same proof applies to the case $p! - 1$.

Remark. This offers a new proof for 11).

13) Let p_k be the kth prime number. Then

$$S(p_1 p_2 \dots p_k + 1) > S(p_1 p_2 \dots p_k)$$

and

$$S(p_1 p_2 \dots p_k - 1) > S(p_1 p_2 \dots p_k)$$

Almost the same proof as in 12) is valid, by remarking that $S(p_1 p_2 \dots p_k) = p_k$ (since $p_1 < p_2 < \dots < p_k$).

14) For infinitely many n one has $(S(n)) < S(n-1)S(n+1)$ and for infinitely many m,

$$(S(m))^2 > S(m-1)S(m+1)$$

By $S(p+1) < p$ and $S(p-1) < p$ (see the proof in 11) we have

$$\frac{S(p+1)}{S(p)} < \frac{S(p)}{S(p)} < \frac{S(p)}{S(p-1)}.$$

Thus

$$(S(p))^2 > S(p-1)S(p+1).$$

On the other hand, by putting $x_n = \dfrac{S(n+1)}{S(n)}$, we shall see in part II, that $\limsup\limits_{n\to\infty} x_n = +\infty$. Thus $x_{n-1} < x_n$ for infinitely many n, giving

$$(S(n))^2 < S(n-1)S(n+1).$$

II. **Limits**

1) $\liminf\limits_{n\to\infty} \dfrac{S(n)}{n} = 0$ and $\limsup\limits_{n\to\infty} \dfrac{S(n)}{n} = 1$.

Clearly, $\dfrac{S(n)}{n} > 0$. Let $n = 2^m$. Then, since $S(2^m) \leq 2m$, and $\lim\limits_{m\to\infty} \dfrac{2m}{2m} = 0$, we have $\lim\limits_{m\to\infty} \dfrac{S(2^m)}{2^m} = 0$, proving the first part. On the other hand, it is well known that $\dfrac{S(n)}{n} \leq 1$. For $n = p_k$ (the kth prime), one has $\dfrac{S(p_k)}{p_k} = 1 \to 1$ as $k \to \infty$, proving the second part.

Remark. With the same proof, we can derive that $\lim\limits_{n\to\infty} \inf \dfrac{S(n^r)}{n} = 0$ for all integers r.

As above $S(2^{kr}) \leq 2kr$, and $\dfrac{2kr}{2^k} \to 0$ as $k \to \infty$ (r fixed), which gives the result.

2) $\lim\limits_{n\to\infty} \inf \dfrac{S(n+1)}{S(n)} = 0$ and $\lim\limits_{n\to\infty} \sup \dfrac{S(n+1)}{S(n)} = +\infty$.

Let p_r denote the rth prime. Since $(p_1 \dots p_r, 1) = 1$, Dirichlet's theorem on arithmetical progressions assures the existence of a prime p of the form $p = ap_1 \dots p_r - 1$. Then

$$S(p+1) = S(ap_1 \dots p_r) \leq aS(p_1 \dots p_r)$$

by $S(mn) \leq mS(n)$ (see [1]).

But $S(p_1 \dots p_r) = \max\{p_1, \dots, p_r\} = p_r$. Thus

$$\frac{S(p+1)}{S(p)} \leq \frac{ap_r}{ap_1 \dots p_r - 1} \leq \frac{p_r}{p_1 \dots p_r - 1} \to 0 \text{ as } r \to \infty.$$

This gives the first part.

Let now p be a prime of the form $p = bp_1 \dots p_r + 1$. Then

$$S(p-1) = S(bp_1 \dots p_r) \leq bS(p_1 \dots p_r) = bp_r$$

and

$$\frac{S(p-1)}{S(p)} \leq \frac{bp_r}{bp_1 \dots p_r + 1} \leq \frac{p_r}{p_1 \dots p_r} \to 0 \text{ as } r \to \infty.$$

3) $\lim\limits_{n\to\infty} \inf[S(n+1) - S(n)] = -\infty$ and $\lim\limits_{m\to\infty} [S(n+1) - S(n)] = +\infty$.

We have

$$S(p+1) - S(p) \leq \frac{p+1}{2} - p = \frac{-p+1}{2} \to -\infty$$

for an odd prime p (see 1) and 11)). On the other hand,

$$S(p) - S(p-1) \geq p - \frac{p-1}{2} = \frac{p+1}{2} \to \infty$$

(Here $S(p) = p$), where $p - 1$ is odd for $p \geq 5$. This finishes the proof.

4) Let $\sigma(n)$ denote the sum of divisors of n. Then

$$\lim_{n\to\infty} \inf \frac{S(\sigma(n))}{n} = 0.$$

This follows by the argument of 2) for $n = p$. Then $\sigma(p) = p + 1$ and $\dfrac{S(p+1)}{p} \to 0$, where $\{p\}$ is the sequence constructed there.

5) Let $\varphi(n)$ be the Enter totient function. Then

$$\lim_{n\to\infty} \inf \frac{S(\varphi(n))}{n} = 0.$$

Let the set of primes $\{p\}$ be defined as in 2). Since $\varphi(n) = p - 1$ and

$$\frac{S(p-1)}{p} = \frac{S(p-1)}{S(p)} \to 0,$$

the assertion is proved. The same result could be obtained by taking $n = 2^k$. Then, since $\varphi(2^k) = 2^{k-1}$, and

$$\frac{S(2^{k-1})}{2^k} \leq \frac{2(k-1)}{2^k} \to 0 \text{ as } k \to \infty,$$

the assertion follows.

6) $\lim\limits_{n\to\infty} \inf \dfrac{S(S(n))}{n} = 0$ and $\max\limits_{n\in N} \dfrac{S(S(n))}{n} = 1$.

Let $n = p!$ (p prime). Then, since $S(p!) = p$ and $S(p) = p$, from $\dfrac{p}{p!} \to 0$ ($p \to \infty$) we get the first result. Now, clearly $\dfrac{S(S(n))}{n} \leq \dfrac{S(n)}{n} \leq 1$. By letting $n = p$ (prime), clearly one has $\dfrac{S(S(p))}{p} = 1$, which shows the second relation.

7) $\lim\limits_{n\to\infty} \inf \dfrac{\sigma(S(n))}{S(n)} = 1$.

Clearly, $\dfrac{\sigma(k)}{k} > 1$. On the other hand, for $n = p$ (prime),

$$\frac{\sigma(S(p))}{S(p)} = \frac{p+1}{p} \to 1 \text{ as } p \to \infty.$$

8) Let $Q(n)$ denote the greatest prime power divisor of n. Then

$$\lim_{n\to\infty} \inf \frac{\varphi(S(n))}{Q(n)} = 0.$$

Let $n = p_1^k \dots p_r^k$ ($k > 1$, fixed). Then, clearly $Q(n) = p_r^k$.

By $S(n) = S(p_r^k)$ (since $S(p_r^k) > S(p_i^k)$ for $i < k$) and $S(p_r^k) = jp_r$, with $j \leq k$ (which is known) and by

$$\varphi(jp_k) \leq j\varphi(p_r) \leq k(p_r - 1),$$

we get

$$\frac{\varphi(S(n))}{Q(n)} \leq \frac{k(p_r-1)}{p_r^k} \to 0$$

as $r \to \infty$ (k fixed).

9) $\lim\limits_{\substack{m\to\infty \\ m\ even}} \dfrac{S(m^2)}{m^2} = 0.$

By 2) we have $\dfrac{S(m^2)}{m^2} \leq \dfrac{1}{m}$ for $m > 4$, even. This clearly implies the above remark.

Remark. It is known that $\dfrac{S(m)}{m} \leq \dfrac{2}{3}$ if $m \neq 4$ is composite. From $\dfrac{S(m^2)}{m^2} \leq \dfrac{1}{m} < \dfrac{2}{3}$ for $m > 4$, for the composite numbers of the perfect squares we have a very strong improvement.

10) $\lim\limits_{n\to\infty} \inf \frac{\sigma(S(n))}{n} = 0.$

By $\sigma(n) = \sum\limits_{d/n} d = n \sum\limits_{d/n} \dfrac{1}{d} \leq n \sum\limits_{d=1}^{n} \dfrac{1}{d} < n(2\log n)$, we get $\sigma(n) < 2n \log n$ for $n > 1$. Thus

$$\frac{\sigma(S(n))}{n} > \frac{2S(n)\log S(n)}{n}.$$

For $n = 2^k$ we have $S(2^k) \leq 2k$, and since $\dfrac{4k \log 2k}{2^k} \to 0$ ($k \to \infty$), the result follows.

11) $\lim\limits_{n\to\infty} \sqrt[n]{S(n)} = 1.$

This simple relation follows by $1 \leq S(n) \leq n$, so $1 \leq \sqrt[n]{S(n)} \leq \sqrt[n]{n}$; and by $\sqrt[n]{n} \to 1$ as $n \to \infty$. However, 11) is one of a (few) limits, which exists for Smarandache function. Finally, we shall prove that:

12) $\lim\limits_{n\to\infty} \sup \dfrac{\sigma(nS(n))}{nS(n)} = +\infty.$

We will use the facts that

$$S(p!) = p, \quad \frac{\sigma(p!)}{p!} = \sum_{d|p!} \frac{1}{d} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{p} \to \infty$$

as $p \to \infty$, and the inequality $\sigma(ab) \geq a\sigma(b)$ (see [2]).

Thus

$$\frac{\sigma(S(p!))p!}{p!S(p!)} \geq \frac{S(p!)\sigma(p!)}{p!p} = \frac{\sigma(p!)}{p!} \to \infty.$$

Thus, for the sequence $\{n\} = \{p!\}$, the results follows.

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4 On two notes by M. Bencze

In vol.10 of SNJ M. Bencze has published two notes on certain inequalities for the Smarandache function. In [2] is it proved that

$$S\left(\prod_{k=1}^{m} m_k\right) \leq \sum_{k=1}^{m} S(m_k) \tag{1}$$

This, in other form is exactly inequality (2) from our paper [5], and follows easily from Le's inequality $S(ab) \leq S(a) + S(b)$.

In [1] it is proved that

$$S\left(\prod_{k=1}^{n} (a_k!)^{b_k}\right) \leq \sum_{k=1}^{n} a_k b_k \tag{2}$$

The proof follows the method of the problem from [3], i.e.

$$S(m!^n) \leq mn \tag{3}$$

This appears also in [4], [5]. We note here that relation (2) is a direct consequence of (1) and (3), since

$$S(a_1!^{b_1} \ldots a_n!^{b_n}) \leq S(a_1!^{b_1}) + \ldots + S(a_n!^{b_n}) \leq b_1 a_1 + \ldots + b_n a_n.$$

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4. J. Sándor, *On certain new inequalities and limits for the Smarandache function*, SNJ, **9**(1998), 63-69.

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5 A note on $S(n^2)$

In the paper [1], it is shown that

$$S(n^2) \leq n \text{ for } n > 4 \text{ and even.} \tag{1}$$

In this note, we will prove that (1) holds for all $n > 4$, $n \neq$ prime.

Let p be a prime. Then:

Lemma. *For $n \neq p,\ 2p,\ 8$ or 9, we have*

$$n^2 | (n-1)! \tag{2}$$

Proof. If $n \neq p,\ 2p,\ p^2,\ 8$, or 16, then n can be written as $n = xy$ ($x \neq y$; $x, y \geq 3$). If $n \neq 16$. then $n = xy$ with $x \geq 3$, $y \geq 5$. Let $n = xy$ with $y > x$; $x \geq 3$. Now $x, y, 2x, 2y, 3x < n - 1$; $x, y, 2y$ are different and one of $2x, 3x$ is different from $x, y, 2y$. Therefore, $(n-1)!$ contains $x, y, 2y$ and $2x$ or $x, y, 2y$ and $3x$. In any case one has $(n-1)! | x^2y^2 = n^2$.

If $n = p^2$, then $n-1 > 4p$, thus $(n-1)!$ contains the factors $p, 2p, 3p, 4p$, so $(n-1)! | p^4 = n^2$. For $n = 2p$, clearly p^2 does not divide $(n-1)!$. For $n = 8$ or 9, n^2 does not divide $(n-1)!$, but for $n = 16$, this holds true by a simple verification.

As a corollary of (2), we can write

$$S(n^2) \leq n - 1 \text{ for } n \neq p,\ 2p,\ 8 \text{ or } 9 \tag{3}$$

Since $2p$ and 8 are even and $S(9^2) = 9$, on the basis of (3), (1) holds true for $n \neq p$, $n > 4$.

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6 Non-Jensen convexity of S

We shall study the inequality

$$S\left(\frac{x+y}{2}\right) \leq \frac{S(x)+S(y)}{2}, \tag{1}$$

where x, y are positive integers with the same parity, and S is the Smarandache function. We will prove that (1) is not generally true, but there are certain cases when this inequality is valid. Let $x = 2a$, $y = 2b$, where $a, b \geq 3$ are odd numbers. Then, since $S(2a) = S(a)$, $S(2b) = S(b)$, (1) is transformed into:

$$S(a+b) \leq \frac{S(a)+S(b)}{2}, \tag{2}$$

where $a, b \geq 3$ are certain odd numbers.

When $a = p$, $b = q$ are odd primes, this inequality is valid. Indeed, since $a + b \geq 4$ is even, it is immediate that (see [1])

$$S(a+b) \leq \frac{a+b}{2} = \frac{p+q}{2} = \frac{S(a)+S(b)}{2}.$$

So, inequality (1) holds true for $x = 2p$, $y = 2q$, where p, q are odd primes.

On the other hand, relation (2) is not generally true. This is the case when a, b have small prime factors, while $a + b$ has a large prime factor. Take e.g. $a = 3$, $b = 15$. Then $S(a + b) = S(18) = 6$, while

$$\frac{S(3)+S(15)}{2} = \frac{3+5}{2} = 4.$$

For another example, take $a = 5$, $b = 20$. Then

$$S(a+b) = S(25) = 10 \not\leq \frac{5+5}{2}.$$

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7 A note on $S(n)$, where n is an even perfect nunber

In a recent paper [1] the following result is proved:

If $n = 2^{k-1}(2^k - 1)$, $2^k - 1 =$ prime, is an even perfect number, then $S(n) = 2^k - 1$, where $S(n)$ is the well-known Smarandache function.

Since $S(ab) = \max\{S(a), S(b)\}$ for $(a, b) = 1$, and $S(a) \leq a$ with equality for $a = 1$, 4, and $a =$ prime (see [3]), we have the following one-line proof of this result:

$$S(2^{k-1}(2^k - 1)) = \max\{S(2^{k-1}), S(2^k - 1)\} = 2^k - 1,$$

since $S(2^{k-1}) \leq 2^{k-1} < 2^k - 1$ for $k \geq 2$.

On the other hand, if $2^k - 1$ is prime, then we have $S(2^k - 1) \equiv 1 \pmod{k}$; an interesting table is considered in [2]. Indeed, k must be a prime too, $k = p$; while Fermat's little theorem gives $2^p - 1 \equiv 1 \pmod{p}$. From $2^{2p-1} = (2^p - 1)(2^p + 1)$ and $(2^p - 1, 2^p + 1) = 1$ we can deduce $S(2^{2p-1}) = \max\{S(2^p - 1), S(2^p + 1)\} = 2^p - 1$ since $2^p + 1$ being composite, $S(2^p + 1) < 2/3(2^p + 1) < 2^p - 1$ for $p \geq 3$. Thus, if $2^k - 1$ is a Mersenne prime, then $S(2^k - 1) \equiv S(2^{2k} - 1) \equiv 1 \pmod{k}$. If $2^p - 1$ and $2^{2p} + 1$ are both primes, then

$$S(2^{4p} - 1) = \max\{S(2^{2p} - 1), S(2^{2p} + 1)\} = 2^{2p} + 1 \not\equiv 1 \pmod{4p}.$$

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8 On certain generalizations of the Smarandache function

1. The famous Smarandache function is defined by $S(n) := \min\{k \in \mathbb{N} : \ n|k!\}$, $n \geq 1$ positive integer. This arithmetical function is connected to the number of divisors of n, and other important number theoretic functions (see e.g. [6], [7], [9], [10]). A very natural generalization is the following one: Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetical function which satisfies the following property:

(P_1) For each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$F_f(n) = \min\{k \in \mathbb{N} : \ n|f(k)\}. \tag{1}$$

Since every subset of natural numbers is well-ordered, the definition (1) is correct, and clearly $F_f(n) \geq 1$ for all $n \in \mathbb{N}^*$.

Examples. 1) Let $id(k) = k$ for all $k \geq 1$. Then clearly (P_1) is satisfied, and

$$F_{id}(n) = n. \tag{2}$$

2) Let $f(k) = k!$. Then $F_!(n) = S(n)$ - the Smarandache function.

3) Let $f(k) = p_k!$, where p_k denotes the kth prime number. Then

$$F_f(n) = \min\{k \in \mathbb{N}^* : \ n|p_k!\}. \tag{3}$$

Here (P_1) is satisfied, as we can take for each $n \geq 1$ the least prime greater than n.

4) Let $f(k) = \varphi(k)$, Euler's totient. First we prove that (P_1) is satisfied. Let $n \geq 1$ be given. By Dirichlet's theorem on arithmetical progressions ([1]) there exists a positive integer a such that $k = an + 1$ is prime (in fact for infinitely many a's). Then clearly $\varphi(k) = an$, which is divisible by n.

We shall denote this function by F_φ. (4)

5) Let $f(k) = \sigma(k)$, the sum of divisors of k. Let k be a prime of the form $an - 1$, where $n \geq 1$ is given. Then clearly $\sigma(n) = an$ divisible by n. Thus (P_1) is satisfied. One obtains the arithmetical function F_σ. (5)

2. Let $A \subset \mathbb{N}^*$, $A \neq \emptyset$ a nonvoid subset of $\mathbb{N}$, having the property:

(P_2) For each $n \geq 1$ there exists $k \in A$ such that $n|k!$.

Then the following arithmetical function may be introduced:

$$S_A(n) = \min\{k \in A : \ n|k!\}. \tag{6}$$

Examples. 1) Let $A = \mathbb{N}^*$. Then $S_{\mathbb{N}}(n) \equiv S(n)$ - the Smarandache function.

2) Let $A = \mathbb{N}_1 =$ set of odd positive integers. Then clearly (P_2) is satisfied. (7)

3) Let $A = \mathbb{N}_2 =$ set of even positive integers. One obtains a new Smarandache-type function. (8)

4) Let $A = P =$ set of prime numbers. Then $S_P(n) = \min\{k \in P : \ n|k!\}$. We shall denote this function by $P(n)$, as we will consider more closely this function. (9)

3. Let $g : \mathbb{N}^* \to \mathbb{N}^*$ be a given arithmetical function. Suppose that g satisfies the following assumption:

(P_3) For each $n \geq 1$ there exists $k \geq 1$ such that $g(k)|n$. (10)

Let the function $G_g : \mathbb{N}^* \to \mathbb{N}^*$ be defined as follows:

$$G_g(n) = \max\{k \in \mathbb{N}^* : \ g(k)|n\}. \tag{11}$$

This is not a generalization of $S(n)$, but for $g(k) = k!$, in fact one obtains a "dual"-function of $S(n)$, namely

$$G_!(n) = \max\{k \in \mathbb{N}^* : \ k!|n\}. \tag{12}$$

Let us denote this function by $S_*(n)$.

There are many other particular cases, but we stop here, and study in more detail some of the above stated functions.

4. The function $P(n)$

This has been defined in (9) by: the least prime p such that $n|p!$. Some values are: $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11, \ldots$

Proposition 1. For each prime p one has $P(p) = p$, and if n is squarefree, then $P(n) =$ greatest prime divisor of n.

Proof. Since $p|p!$ and $p \nmid q!$ with $q < p$, clearly $P(p) = p$. If $n = p_1 p_2 \dots p_r$ is squarefree, with $p_1, \dots, p_r$ distinct primes, if $p_r = \max\{p_1, \dots, p_r\}$, then $p_1 \dots p_r | p_r!$. On the other hand, $p_1 \dots p_r \nmid q!$ for $q < p_r$, since $p_r \nmid q!$. Thus p_r is the least prime with the required property.

The calculation of $P(p^2)$ is not so simple but we can state the following result:

Proposition 2. One has the inequality $P(p^2) \geq 2p + 1$. If $2p + 1 = q$ is prime, then $P(p^2) = q$. More generally, $P(p^m) \geq mp + 1$ for all primes p and all integers m. There is equality, if $mp + 1$ is prime.

Proof. From $p^2 | (1 \cdot 2 \dots p)(p+1) \dots (2p)$ we have $p^2 | (2p)!$. Thus $P(p^2) \geq 2p+1$. One has equality, if $2p+1$ is prime. By writing $p^m | \underbrace{1 \cdot 2 \dots p}\underbrace{(p+1) \dots 2p} \dots \underbrace{[(m-1)p+1] \dots mp}$, where each group of p consecutive terms contains a member divisible by p, one obtains $P(p^m) \geq mp + 1$.

Remark. If $2p + 1$ is not a prime, then clearly $P(p^2) \geq 2p + 3$.

It is not known if there exist infinitely many primes p such that $2p + 1$ is prime too (see [4]).

Proposition 3. The following double inequality is true:

$$2p + 1 \leq P(p^2) \leq 3p - 1 \tag{13}$$

$$mp + 1 \leq P(p^m) \leq (m + 1)p - 1 \tag{14}$$

if $p \geq p_0$.

Proof. We use the known fact from the prime number theory ([1], [8]) tha for all $a \geq 2$ there exists at least a prime between $2a$ and $3a$. Thus between $2p$ and $3p$ there is at least a prime, implying $P(p^2) \leq 3p - 1$. On the same lines, for sufficiently large p, there is a prime between mp and $(m + 1)p$. This gives the inequality (14).

Proposition 4. For all $n, m \geq 1$ one has:

$$S(n) \leq P(n) \leq 2S(n) - 1 \tag{15}$$

and

$$P(nm) \leq 2[P(n) + P(m)] - 1 \tag{16}$$

where $S(n)$ is the Smarandache function.

Proof. The left side of (15) is a consequence of definitions of $S(n)$ and $P(n)$, while the right-hand side follows from Chebyshev's theorem on the existence of a prime between a and $2a$ (where $a = S(n)$, when $2a$ is not a prime).

For the right side of (16) we use the inequality $S(mn) \leq S(n) + S(m)$ (see [5]): $P(nm) \leq 2S(nm) - 1 \leq 2[S(n) + S(m)] - 1 \leq 2[P(n) + P(m)] - 1$, by (15).

Corollary.

$$\lim_{n\to\infty} \sqrt[n]{P(n)} = 1. \tag{17}$$

This is an easy consequence of (15) and the fact that $\lim_{n\to\infty} \sqrt[n]{S(n)} = 1$. (For other limits, see [6]).

5. The function $S_*(n)$

As we have seen in (12), $S_*(n)$ is in certain sense a dual of $S(n)$, and clearly $(S_*(n))!|n|(S(n))!$ which implies

$$1 \leq S_*(n) \leq S(n) \leq n \tag{18}$$

thus, as a consequence,

$$\lim_{n\to\infty} \sqrt[n]{\frac{S_*(n)}{S(n)}} = 1. \tag{19}$$

On the other hand, from known properties of S it follows that

$$\liminf_{n\to\infty} \frac{S_*(n)}{S(n)} = 0, \quad \limsup_{n\to\infty} \frac{S_*(n)}{S(n)} = 1. \tag{20}$$

For odd values n, we clearly have $S_*(n) = 1$.

Proposition 5. For $n \geq 3$ one has

$$S_*(n! + 2) = 2 \tag{21}$$

and more generally, if p is a prime, then for $n \geq p$ we have

$$S_*(n! + (p-1)!) = p - 1. \tag{22}$$

Proof. (21) is true, since $2|(n! + 2)$ and if one assumes that $k!|(n! + 2)$ with $k \geq 3$, then $3|(n! + 2)$, impossible, since for $n \geq 3$, $3|n!$. So $k \leq 2$, and remains $k = 2$.

For the general case, let us remark that if $n \geq k+1$, then, since $k|(n!+k!)$, we have $S_*(n!+k!) \geq k$.

On the other hand, if for some $s \geq k+1$ we have $s!|(n!+k!)$, by $k+1 \leq n$ we get $(k+1)|(n!+k!)$ yielding $(k+1)|k!$, since $(k+1)|n!$. So, if $(k+1)|k!$ is not true, then we have

$$S_*(n!+k!) = k. \tag{23}$$

Particularly, for $k = p-1$ (p prime) we have $p \nmid (p-1)!$.

Corollary. For infinitely many m one has $S_*(m) = p-1$, where p is a given prime.

Proposition 6. For all $k, m \geq 1$ we have

$$S_*(k!m) \geq k \tag{24}$$

and for all $a, b \geq 1$,

$$S_*(ab) \geq \max\{S_*(a), S_*(b)\}. \tag{25}$$

Proof. (24) trivially follows from $k!|(k!m)$, while (25) is a consequence of $(S_*(a))!|a \Rightarrow (S_*(a))!|(ab)$ so $S_*(ab) \geq S_*(a)$. This is true if a is replaced by b, so (25) follows.

Proposition 7. $S_*[x(x-1)\dots(x-a+1)] \geq a$ for all $x \geq a$ (x positive integer).(26)

Proof. This is a consequence of the known fact that the product of $\underline{a}$ consecutive integers is divisible by $a!$.

We now investigate certain properties of $S_*(a!b!)$. By (24) or (25) we have $S_*(a!b!) \geq \max\{a, b\}$. If the equation

$$a!b! = c! \tag{27}$$

is solvable, then clearly $S_*(a!b!) = c$. For example, since $3! \cdot 5! = 6!$, we have $S_*(3! \cdot 5!) = 6$. The equation (27) has a trivial solution $c = k!$, $a = k!-1$, $b = k$. Thus $S_*(k!(k!-1)!) = k$.

In general, the nontrivial solutions of (27) are not known (see e.g. [3], [1]).

We now prove:

Proposition 8. $S_*((2k)!(2k+2)!) = 2k+2$, if $2k+3$ is a prime; (28)

$S_*((2k)!(2k+2)!) \geq 2k+4$, if $2k+3$ is not a prime. (29)

Proof. If $2k+3=p$ is a prime, (28) is obvious, since $(2k+2)!|(2k)!(2k+2)!$, but $(2k+3)!\nmid(2k)!(2k+2)!$. We shall prove first that if $2k+3$ is not prime, then

$$(2k+3)|(1\cdot 2\ldots(2k)) \qquad (*)$$

Indeed, let $2k+3=ab$, with $a,b\geq 3$ odd numbers. If $a<b$, then $a<k$, and from $2k+3\geq 3b$ we have $b\leq \frac{2}{3}k+1<k$. So $(2k)!$ is divisible by ab, since a,b are distinct numbers between 1 and k. If $a=b$, i.e. $2k+3=a^2$, then $(*)$ is equivalent with $a^2|(1\cdot 2\ldots a)(a+1)\ldots(a^2-3)$. We show that there is a positive integer k such that $a+1<ka\leq a^2-3$ or. Indeed, $a(a-3)=a^2-3a<a^2-3$ for $a>3$ and $a(a-3)>a+1$ by $a^2>4a+1$, valid for $a\geq 5$. For $a=3$ we can verifiy $(*)$ directly. Now $(*)$ gives

$$(2k+3)!|(2k)!(2k+2)!,\ \text{if } 2k+3\neq \text{prime} \qquad (**)$$

implying inequality (29).

For consecutive odd numbers, the product of factorials gives for certain values

$$S_*(3!\cdot 5!)=6,\quad S_*(5!\cdot 7!)=8,\quad S_*(7!\cdot 9!)=10,$$

$$S_*(9!\cdot 11!)=12,\quad S_*(11!\cdot 13!)=16,\quad S_*(13!\cdot 15!)=16,\quad S_*(15!\cdot 17!)=18,$$

$$S_*(17!\cdot 19!)=22,\quad S_*(19!\cdot 21!)=22,\quad S_*(21!\cdot 23!)=28.$$

The following conjecture arises:

Conjecture. $S_*((2k-1)!(2k+1)!)=q_k-1$, where q_k is the first prime following $2k+1$.

Corollary. From $(q_k-1)!|(2k-1)!(2k+1)!$ it follows that $q_k>2k+1$. On the other hand, by $(2k-1)!(2k+1)!|(4k)!$, we get $q_k\leq 4k-3$. Thus between $2k+1$ and $4k+2$ there is at least a prime q_k. This means that the above conjecture, if true, is stronger than Bertrand's postulate (Chebyshev's theorem [1], [8]).

6. Finally, we make some remarks on the functions defined by (4), (5), other functions of this type, and certain other generalizations and analogous functions for further study, related to the Smarandache function.

First, consider the function F_φ of (4), defined by

$$F_\varphi=\min\{k\in\mathbb{N}^*:\ n|\varphi(k)\}.$$

First observe that if $n+1 = \text{prime}$, then $n = \varphi(n+1)$, so $F_\varphi(n) = n+1$. Thus

$$n+1 = \text{prime} \;\Rightarrow\; F_\varphi(n) = n+1. \tag{30}$$

This is somewhat converse to the φ-function property

$$n+1 = \text{prime} \;\Rightarrow\; \varphi(n+1) = n.$$

Proposition 9. Let ϕ_n be the nth cyclotomic polynomial. Then for each $a \geq 2$ (integer) one has

$$F_\varphi(n) \leq \phi_n(a) \text{ for all } n. \tag{31}$$

Proof. The cyclotomic polynomial is the irreducible polynomial of grade $\varphi(n)$ with integer coefficients with the primitive roots of order n as zeros. It is known (see [2]) the following property:

$$n|\varphi(\phi_n(a)) \text{ for all } n \geq 1, \text{ all } a \geq 2. \tag{32}$$

The definition of F_φ gives immediately inequality (31).

Remark. We note that there exist in the literature a number of congruence properties of the function φ. E.g. it is known that $n|\varphi(a^n-1)$ for all $n \geq 1$, $a \geq 2$. But this is a consequence of (32), since $\phi_n(a)|a^n-1$, and $u|v \;\Rightarrow\; \varphi(u)|\varphi(v)$ implies (known property of φ) what we have stated.

The most famous congruence property of φ is the following

Conjecture. (D.H. Lehmer (see [4])) If $\varphi(n)|(n-1)$, then $n = \text{prime}$.

Another congruence property of φ is contained in Euler's theorem: $m|(a^{\varphi(m)}-1)$ for $(a,m) = 1$. In fact this implies

$$S_*[a^{\varphi(m!)}-1] \geq m \text{ for } (a,m!) = 1 \tag{33}$$

and by the same procedure,

$$S_*(\varphi(a^{n!}-1)] \geq n \text{ for all } n. \tag{34}$$

As a corollary of (34) we can state that

$$\limsup_{k\to\infty} S_*[\varphi(k)] = +\infty. \tag{35}$$

(It is sufficient to take $k = a^{n!} - 1 \to \infty$ as $n \to \infty$).

7. In a completely similar way one can define $F_d(n) = \min\{k : \ n|d(k)\}$, where $d(k)$ is the number of distinct divisors of k. Since $d(2^{n-1}) = n$, one has

$$F_d(n) \leq 2^{n-1}. \tag{36}$$

Let now $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be the canonical factorization of the number n. Then Smarandache ([9]) proved that $S(n) = \max\{S(p_1^{\alpha_1}), \dots, S(p_r^{\alpha_r})\}$.

In the analogous way, we may define the functions $S_\varphi(n) = \max\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S_\sigma(n) = \max\{\sigma(p_1^{\alpha_1}), \dots, \sigma(p_r^{\alpha_r})\}$, etc.

But we can define $S_\varphi^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, etc. For an arithmetical function f one can define

$$\Delta_f(n) = l.c.m.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}$$

and

$$\delta_f(n) = g.c.d.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}.$$

For the function $\Delta_\varphi(n)$ the following divisibility property is known (see [8], p.140, Problem 6).

If $(a, n) = 1$, then

$$n|[a^{\Delta_\varphi(n)} - 1]. \tag{37}$$

These functions and many related others may be studied in the near (or further) future.

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Note added in proof: In a recent paper, F. Luca [On a divisibility property involving factorials, C.R. Acad. Bulg. Sci. **53**(2000), no.6, 35-38] has proved affirmatively the conjecture of this paper.

9 On an inequality for the Smarandache function

1. In paper [2] the author proved among others the inequality $S(ab) \leq aS(b)$ for all a, b positive integers. This was refined to

$$S(ab) \leq S(a) + S(b) \tag{1}$$

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

$$S(a_1 a_2 \ldots a_n) \leq S(a_1) + S(a_2) + \ldots + S(a_n) \tag{2}$$

for all integers $a_i \geq 1$ $(i = 1, \ldots, n)$. When $a_1 = \ldots = a_n = n$ we obtain

$$S(a^n) \leq nS(a). \tag{3}$$

For three applications of this inequality, remark that

$$S((m!)^n) \leq nS(m!) = nm \tag{4}$$

since $S(m!) = m$. This is inequality 3) part 1. from [3]. By the same way, $S((n!)^{(n-1)!}) \leq (n-1)!S(n!) = (n-1)!n = n!$, i.e.

$$S((n!)^{(n-1)!}) \leq n! \tag{5}$$

Inequality (5) has been obtained in [3] by other arguments (see 4) part 1.).

Finally, by $S(n^2) \leq 2S(n) \leq n$ for n even (see [3], inequality 1), $n > 4$, we have obtained a refinement of $S(n^2) \leq n$:

$$S(n^2) \leq 2S(n) \leq n \tag{6}$$

for $n > 4$, even.

3. Let m be a divisor of n, i.e. $n = km$. Then (1) gives $S(n) = S(km) \leq S(m) + S(k)$, so we obtain:

If $m|n$, then

$$S(n) - S(m) \leq S\left(\frac{n}{m}\right). \tag{7}$$

As an application of (7), let $d(n)$ be the number of divisors of n. Since $\prod_{k|n} k = n^{d(n)/2}$, and $\prod_{k\leq n} k = n!$ (see [3]), and by $\prod_{k|n} k | \prod_{k\leq n} k$, from (7) we can deduce that

$$S(n^{d(n)/2}) + S(n!/n^{d(n)/2}) \geq n. \tag{8}$$

This improves our relation (10) from [3].

4. Let $S(a) = u$, $S(b) = v$. Then $b|v!$ and $u!|x(x-1)\dots(x-u+1)$ for all integers $x \geq u$. But from $a|u!$ we have $a|x(x-1)\dots(x-u+1)$ for all $x \geq u$. Let $x = u+v+k$ $(k \geq 1)$. Then, clearly $ab(v+1)\dots(v+k)|(u+v+k)!$, so we have $S[ab(v+1)\dots(v+k)] \leq u+v+k$. Here $v = S(b)$, so we have obtained that

$$S[ab(S(b)+1)\dots(S(b)+k)] \leq S(a) + S(b) + k. \tag{9}$$

For example, for $k = 1$ one has

$$S[ab(S(b)+1)] \leq S(a) + S(b) + 1. \tag{10}$$

This is not a consequence of (2) for $n = 3$, since $S[S(b)+1]$ may be much larger than 1.

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10 The Smarandache function of a set

1. In paper [1] we have introduced certain generalizations and duals of Smarandache function. We have considered also the following extension. Let A be a set of positive integers (i.e. $A \subset N^*$), $A \neq \emptyset$, having the following property: for each $n \geq 1$ there exists at least a $k \in A$ such that $n|k!$. Let

$$S_A(n) = \min\{k \in A : \ n|k!\} \tag{1}$$

For example, if $A = N^*$, $S_N(n) = S(n)$ the Smarandache function. When $A = P =$ set of prime numbers, we obtain a new arithmetic function, denoted by us by $P(n) =$ least prime p such that $n|p!$. This function was not studied by Erdös (as is mentioned in [3]), and coincides for squarefree n with the usual function of the greatest prime factor of n (called by Sabin Tabârcă as "Erdös function").

In paper [1] we have considered certain properties of this function, e.g.

$$2p + 1 \leq P(p^2) \leq 3p - 1 \tag{2}$$

$$mp + 1 \leq P(p^m) \leq (m+1)p - 1, \quad p \geq p_0 \ (p \text{ prime}, \ m \geq 1) \tag{3}$$

$$S(n) \leq P(n) \leq 2S(n) - 1 \tag{4}$$

and

$$P(m) \leq 2[P(n) + P(m)] - 1, \quad n, m = 1, 2, \ldots \tag{5}$$

The aim of this note is to introduce other functions of this type, i.e. other particular cases of (1).

2. First let $A = \{k^2 : \ k \in N^*\} =$ sequence of squares. Let us denote the function obtained from (1) by

$$Q(n) = \min\{m^2 : \ m|(m^2)!\} \tag{6}$$

e.g. $Q(1) = 1$, $Q(2) = 4$, $Q(5) = 9$, $Q(6) = 4$, $Q(11) = 16$, $Q(12) = 4$, etc. We have the followinf remarks.

Proposition 1. *Let p be prime such that $m^2 < p < (m+1)^2$. Then $Q(p) = (m+1)^2$. Indeed, $p|[(m+1)^2]!$, but $p \nmid (n^2)!$ for any $n \leq m$.*

Remark. It is a difficult open problem the existence of a prime between two consecutive squares (for all $m \geq 1$), see [2]. By using the "integer part" function (which is only a notation, and is not a simple function!), certain values of Q can be expressed. E.g.

Proposition 2.

$$Q(p) = ([\sqrt{p}] + 1)^2 \tag{7}$$

$$Q(p^2) = ([\sqrt{2p}] + 1)^2 \text{ for } p \geq 3 \tag{8}$$

$$Q(p^3) = ([\sqrt{3p} + 1)^2 \text{ for } p \geq 5 \tag{9}$$

$$Q(p^k) = ([\sqrt{kp} + 1)^2 \text{ for } p > k \tag{10}$$

Proof. We give the proof of (9), the other are similar.

$p^3 | 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2p \cdot \ldots \cdot m^2$ is true with the least m if $3p \leq m^2$, $p > 3$. Then $m \geq \sqrt{3p}$ and the least integer with this property, is clearly $[\sqrt{3p}] + 1$.

Proposition 3. *If $p < q$ are primes, then*

$$Q(pq) = ([\sqrt{q}] + 1)^2 \tag{11}$$

Proof. $pq | 1 \cdot 2 \cdot 3 \cdot \ldots \cdot p \cdot \ldots \cdot q \cdot \ldots \cdot m^2$, so $m^2 \geq q$, giving $m \geq [\sqrt{q}] + 1$.

Proposition 4. *If $p < q < 2p$, then*

$$Q(p^2q) = ([\sqrt{2p}] + 1)^2. \tag{12}$$

If $q > 2p$, then

$$Q(p^2q) = ([\sqrt{q}] + 1)^2 \tag{13}$$

Proof. Indeed, $1 \cdot 2 \cdot 3 \cdot \ldots \cdot p \cdot \ldots \cdot 2p \cdot \ldots \cdot q \cdot \ldots \cdot m^2$ or $1 \cdot 2 \cdot 3 \cdot \ldots \cdot p \cdot \ldots \cdot q \cdot \ldots \cdot 2p \cdot \ldots \cdot m^2$ are the possible cases, and proceed as above, For pq^2 $(p < q)$ the things are simpler:

Proposition 5. *If $p < q$ are primes, then*

$$Q(p^2q) = ([\sqrt{2q}] + 1)^2 \tag{14}$$

Indeed, $1 \cdot 2 \cdot 3 \cdot \ldots \cdot p \cdot \ldots \cdot q \cdot \ldots \cdot 2p \cdot \ldots \cdot m^2$ is divisible by pq^2 and the least m is $[\sqrt{2q}] + 1$. The following result gives a connection with the Smarandache function S.

Proposition 6. *If $S(n)$ is a square, then $Q(n) = S(n)$. If $S(n)$ is not a square, then*

$$Q(n) = ([\sqrt{S(n)}] + 1)^2$$

Proof. Clearly $n|1 \cdot 2 \cdot \ldots \cdot S(n)$ with the least $S(n)$, so if $S(n) = m^2$, then $Q(n) = m^2$. Otherwise, $Q(n)$ is the least square $> S(n)$, which is given by (15).

3. Let now $A =$ set of squarefree numbers. A number m is called squarefree if doesn't have square divisors. Therefore $m > 1$ must be prime or product of distinct primes. Let Q_1 be the Smarandache function of the squarefree numbers, i.e.

$$Q_1(n) = \text{least squarefree } m \text{ such that } n|m! \tag{15}$$

Clearly $Q_1(p) = p$ for prime p, $Q_1(n) = P(n)$ for $n =$ squarefree. More precisely,

Proposition 7. *Let $n = p_1 p_2 \ldots p_r$ with $p_1 < p_2 < \ldots < p_r$. Then $Q_1(n) = p_r$.*

Proposition 8. *$Q_1(p^2) = 2p$ for $p > 2$, $Q_1(p^3) = 3p$ for $p > 3$, $Q_1(p^k) = kp$ for $p > k$, $Q_1(p^2q) = q$ for $2p < q$, $Q_1(p^2q) = q$ for $q < 2p$, $Q_1(p^2q^2) = 2q$ in all cases $(p < q)$; $Q_1(pq^2) = 2q$ for $p < q$.*

Proof. We prove only $Q_1(p^2q^2) = 2q$. The number $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2p \cdot \ldots \cdot q \cdot \ldots \cdot 2q$ is divisible by p^2q^2, and the same is true for $1 \cdot 2 \cdot 3 \cdot \ldots \cdot p \cdot \ldots \cdot q \cdot \ldots \cdot 2p \cdot \ldots \cdot 2q$.

Proposition 9. *If $S(n)$ is squarefree, then $Q_1(n) = S(n)$, otherwise $Q_1(n)$ is the least squarefree numbers which is greatest than $S(n)$.*

Proof. $S(n)$ is least with $n|1 \cdot 2 \cdot 3 \cdot \ldots \cdot S(n)$ and the result follows.

Proposition 10. *$Q_1(n) \geq S(n)$ for all n, and if $n \geq 2$, then $Q_1(n) \leq 2S(n)$.*

Proof. $Q_1(n) \geq S(n)$ follows from Proposition 9. Now, if $n \geq 2$, then $S(n) \geq 2$, and it is known by Chebysheff theorem the existence of a prime p between $S(n)$ and $2S(n)$. But p is squarefree, so the result follows.

Corollary. $\sqrt[n]{Q_1(n)} \to 1$ *as* $n \to \infty$.

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11 On the Pseudo-Smarandache function

Kashihara [2] defined the Pseudo-Smarandache function Z by

$$Z(n) = \min\left\{m \geq 1 : \ n|\frac{m(m+1)}{2}\right\}.$$

Properties of this function have been studied in [1], [2] etc.

1. By answering a question by C.Ashbacher, Maohua Le ([6]) proved that $S(Z(n)) - Z(S(n))$ changes signs infinitely often. Put

$$\Delta_{S,Z}(n) = |S(Z(n)) - Z(S(s))|.$$

We will prove first that

$$\liminf_{n\to\infty} \Delta_{S,Z}(n) \leq 1 \tag{1}$$

and

$$\limsup_{n\to\infty} \Delta_{S,Z}(n) = +\infty. \tag{2}$$

Indeed, let $n = \frac{p(p+1)}{2}$, where p is an odd prime. Then it is not difficult to see that $S(n) = p$ and $Z(n) = p$. Therefore,

$$|S(Z(n)) - Z(S(n))| = |S(p) - Z(p)| = |p - (p-1)| = 1$$

implying (1). We note that if the equation $S(Z(n)) = Z(S(n))$ has infinitely many solutions, then clearly the lim inf in (1) is 0, otherwise is 1, since

$$|S(Z(n)) - Z(S(n))| \geq 1,$$

$S(Z(n)) - Z(S(n))$ being an integer.

Now let $n = p$ be an odd prime. Then, since $Z(p) = p - 1$, $S(p) = p$ and $S(p-1) \leq \frac{p-1}{2}$ (see [4]) we get

$$\Delta_{S,Z}(p) = |S(p-1) - (p-1)| = p - 1 - S(p-1) \geq \frac{p-1}{2} \to \infty \text{ as } p \to \infty$$

proving (2). Functions of type $\Delta_{f,g}$ have been studied recently by the author [5] (see also [3]).

2. Since $n|\dfrac{(2n-1)2n}{2}$, clearly $Z(n) \leq 2n-1$ for all n.

This inequality is best possible for even n, since $Z(2^k) = 2^{k+1}-1$. We note that for odd n, we have $Z(n) \leq n-1$, and this is the best possible for odd n, since $Z(p) = p-1$ for prime p. By $\dfrac{Z(n)}{n} \leq 2 - \dfrac{1}{n}$ and $\dfrac{Z(2^k)}{2^k} = 2 - \dfrac{1}{2^k}$ we get

$$\limsup_{n\to\infty} \frac{Z(n)}{n} = 2. \tag{3}$$

Since $Z\left(\dfrac{p(p+1)}{2}\right) = p$, and $\dfrac{p}{p(p+1)/2} \to 0$ $(p \to \infty)$, it follows

$$\liminf_{n\to\infty} \frac{Z(n)}{n} = 0. \tag{4}$$

For $Z(Z(n))$, the following can be proved. By $Z\left(Z\left(\dfrac{p(p+1)}{2}\right)\right) = p-1$, clearly

$$\liminf_{n\to\infty} \frac{Z(Z(n))}{n} = 0. \tag{5}$$

On the other hand, by $Z(Z(n)) \leq 2Z(n)-1$ and (3), we have

$$\limsup_{n\to\infty} \frac{Z(Z(n))}{n} \leq 4. \tag{6}$$

3. We now prove

$$\liminf_{n\to\infty} |Z(2n) - Z(n)| = 0 \tag{7}$$

and

$$\limsup_{n\to\infty} |Z(2n) - Z(n)| = +\infty. \tag{8}$$

Indeed, in [1] it was proved that $Z(2p) = p-1$ for a prime $p \equiv 1 \pmod 4$. Since $Z(p) = p-1$, this proves relation (7).

On the other hand, let $n = 2^k$. Since $Z(2^k) = 2^{k+1}-1$ and $Z(2^{k+1}) = 2^{k+2}-1$, clearly $Z(2^{k+1}) - Z(2^k) = 2^{k+1} \to \infty$ as $k \to \infty$.

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12 On certain inequalities for $Z(n)$

We have to determine the best $a, b > 0$, such that: ([1])

$$a\left(\sqrt{2n+\frac{1}{4}}-\frac{1}{2}\right) \leq Z(n) \leq b\left(\sqrt{2n-\frac{1}{4}}+\frac{1}{2}\right) \tag{1}$$

where

$$Z(n) = \min\left\{k:\ k \in N,\ n|\frac{k(k+1)}{2}\right\} \tag{2}$$

(see [3]).

We will prove that $a = 1$ is the best constant, while doesn't exist. Clearly $n|\frac{Z(n)(Z(n)+1)}{2}$, so $\frac{Z(n)(Z(n)+1)}{2} \geq n$. Resolving this inequality, easily follows:

$$Z(n) \geq -\frac{1}{2}+\sqrt{2n+\frac{1}{4}} \tag{3}$$

This inequality cannot be improved, since for infinitely many n there is equality. Indeed, put $n = \frac{p(p+1)}{2}$, where p is a prime. Then $Z(n) = p$ and

$$-\frac{1}{2}+\sqrt{\frac{2p(p+1)}{2}+\frac{1}{4}} = -\frac{1}{2}+\sqrt{\left(p+\frac{1}{2}\right)^2} = -\frac{1}{2}+p+\frac{1}{2} = p.$$

On the other hand, we have recently proved ([2]) that:

$$\limsup_{n\to\infty}\frac{Z(n)}{n} = 2 \tag{4}$$

Therefore $\limsup_{n\to\infty}\frac{Z(n)}{\sqrt{n}} = +\infty$, so the right side of (1) cannot be true for sufficiently large n. In fact, we note that an upper inequalities for $Z(n)$ is (which is best possible).

$$Z(n) \leq \begin{cases} n, & \text{for } n \text{ odd} \\ 2n-1, & \text{for } n \text{ even} \end{cases} \tag{5}$$

Indeed, for n odd $n|\frac{n(n+1)}{2}$ and for all n, $n|\frac{(2n-1)2n}{2}$, while $Z(p) = p$ for a prime p,

$$Z(2^k) = 2^{k+1}-1 = 2\cdot 2^k - 1 \quad (k \in N).$$

Therefore for $n =$ prime and $n = 2^k$ there is equality in (5).

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13 On a dual of the Pseudo-Smarandache function

Introduction

In paper [3] we have defined certain generalizations and extensions of the Smarandache function. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetic function with the following property: for each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$. Let

$$F_f : \mathbb{N}^* \to \mathbb{N}^* \text{ defined by } F_f(n) = \min\{k \in \mathbb{N}^* : \ n|f(k)\}. \tag{1}$$

This function generalizes many particular functions. For $f(k) = k!$ one gets the Smarandache function, while for $f(k) = \dfrac{k(k+1)}{2}$ one has the Pseudo-Smarandache function Z (see [1], [4-5]). In the above paper [3] we have defined also dual arithmetic functions as follows: Let $g : \mathbb{N}^* \to \mathbb{N}^*$ be a function having the property that for each $n \geq 1$ there exists at least a $k \geq 1$ such that $g(k)|n$.

Let

$$G_g(n) = \max\{k \in \mathbb{N}^* : \ g(k)|n\}. \tag{2}$$

For $g(k) = k!$ we obtain a dual of the Smarandache function. This particular function, denoted by us as S_* has been studied in the above paper. By putting $g(k) = \dfrac{k(k+1)}{2}$ one obtains a dual of the Pseudo-Smarandache function. Let us denote this function, by analogy by Z_*. Our aim is to study certain elementary properties of this arithmetic function.

The dual of yhe Pseudo-Smarandache function

Let

$$Z_*(n) = \max\left\{m \in \mathbb{N}^* : \ \frac{m(m+1)}{2}|n\right\}. \tag{3}$$

Recall that

$$Z(n) = \min\left\{k \in \mathbb{N}^* : \ n|\frac{k(k+1)}{2}\right\}. \tag{4}$$

First remark that

$$Z_*(1) = 1 \quad \text{and} \quad Z_*(p) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{5}$$

where p is an arbitrary prime. Indeed, $\frac{2\cdot 3}{2} = 3|3$ but $\frac{m(m+1)}{2}|p$ for $p \neq 3$ is possible only for $m = 1$. More generally, let $s \geq 1$ be an integer, and p a prime. Then:

Proposition 1.

$$Z_*(p^s) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{6}$$

Proof. Let $\frac{m(m+1)}{2}|p^s$. If $m = 2M$ then $M(2M+1)|p^s$ is impossible for $M > 1$ since M and $2M+1$ are relatively prime. For $M = 1$ one has $m = 2$ and $3|p^s$ only if $p = 3$. For $m = 2M-1$ we get $(2M-1)M|p^k$, where for $M > 1$ we have $(M, 2M-1) = 1$ as above, while for $M = 1$ we have $m = 1$.

The function Z_* can take large values too, since remark that for e.g. $n \equiv 0(\mathrm{mod}6)$ we have $\frac{3\cdot 4}{2} = 6|n$, so $Z_*(n) \geq 3$. More generally, let a be a given positive integer and n selected such that $n \equiv 0(\mathrm{mod}a(2a+1))$. Then

$$Z_*(n) \geq 2a. \tag{7}$$

Indeed, $\frac{2a(2a+1)}{2} = a(2a+1)|n$ implies $Z_*(n) \geq 2a$.

A similar situation is in

Proposition 2. Let q be a prime such that $p = 2q-1$ is a prime, too. Then

$$Z_*(pq) = p. \tag{8}$$

Proof. $\frac{p(p+1)}{2} = pq$ so clearly $Z_*(pq) = p$.

Remark. Examples are $Z_*(5\cdot 3) = 5$, $Z_*(13\cdot 7) = 13$, etc. It is a difficult open problem that for infinitely many q, the number p is prime, too (see e.g. [2]).

Proposition 3. For all $n \geq 1$ one has

$$1 \leq Z_*(n) \leq Z(n). \tag{9}$$

Proof. By (3) and (4) we can write $\frac{m(m+1)}{2}|n|\frac{k(k+1)}{2}$, therefore $m(m+1)|k(k+1)$. If $m > k$ then clearly $m(m+1) > k(k+1)$, a contradiction.

Corollary. One has the following limits:

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 1. \tag{10}$$

Proof. Put $n = p$ (prime) in the first relation. The first result follows by (6) for $s = 1$ and the well-known fact that $Z(p) = p$. Then put $n = \dfrac{a(a+1)}{2}$, when $\dfrac{Z_*(n)}{Z(n)} = 1$ and let $a \to \infty$.

As we have seen,

$$Z\left(\frac{a(a+1)}{2}\right) = Z_*\left(\frac{a(a+1)}{2}\right) = a.$$

Indeed, $\dfrac{a(a+1)}{2}\Big|\dfrac{k(k+1)}{2}$ is true for $k = a$ and is not true for any $k < a$. In the same manner, $\dfrac{m(m+1)}{2}\Big|\dfrac{a(a+1)}{2}$ is valied for $m = a$ but not for any $m > a$. The following problem arises: What are the solutions of the equation $Z(n) = Z_*(n)$?

Proposition 4. All solutions of equation $Z(n) = Z_*(n)$ can be written in the form $n = \dfrac{r(r+1)}{2}$ $(r \in \mathbb{N}^*)$.

Proof. Let $Z_*(n) = Z(n) = t$. Then $n|\dfrac{t(t+1)}{2}|n$ so $\dfrac{t(t+1)}{2} = n$. This gives $t^2 + t - 2n = 0$ or $(2t+1)^2 = 8n+1$, implying $t = \dfrac{\sqrt{8n+1}-1}{2}$, where $8n + 1 = m^2$. Here m must be odd, let $m = 2r + 1$, so $n = \dfrac{(m-1)(m+1)}{8}$ and $t = \dfrac{m-1}{2}$. Then $m - 1 = 2r$, $m + 1 = 2(r+1)$ and $n = \dfrac{r(r+1)}{2}$.

Proposition 5. One has the following limits:

$$\lim_{n\to\infty} \sqrt[n]{Z_*(n)} = \lim_{n\to\infty} \sqrt[n]{Z(n)} = 1. \tag{11}$$

Proof. It is known that $Z(n) \leq 2n - 1$ with equality only for $n = 2^k$ (see e.g. [5]). Therefore, from (9) we have

$$1 \leq \sqrt[n]{Z_*(n)} \leq \sqrt[n]{Z(n)} \leq \sqrt[n]{2n-1},$$

and by taking $n \to \infty$ since $\sqrt[n]{2n-1} \to 1$, the above simple result follows.

As we have seen in (9), upper bounds for $Z(n)$ give also upper bounds for $Z_*(n)$. E.g. for $n =$ odd, since $Z(n) \leq n - 1$, we get also $Z_*(n) \leq n - 1$. However, this upper bound is too large. The optimal one is given by:

Proposition 6.

$$Z_*(n) \leq \frac{\sqrt{8n+1}-1}{2} \text{ for all } n. \tag{12}$$

Proof. The definition (3) implies with $Z_*(n) = m$ that $\frac{m(m+1)}{2}|n$, so $\frac{m(m+1)}{2} \leq n$, i.e. $m^2 + m - 2n \leq 0$. Resolving this inequality in the unknown m, easily follows (12). Inequality (12) cannot be improved since for $n = \frac{p(p+1)}{2}$ (thus for infinitely many n) we have equality. Indeed,

$$\left(\sqrt{\frac{8(p+1)p}{2}+1}-1\right)/2 = \left(\sqrt{4p(p+1)+1}-1\right)/2 = [(2p+1)-1]/2 = p.$$

Corollary.

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = \sqrt{2}. \tag{13}$$

Proof. While the first limit is trivial (e.g. for n = prime), the second one is a consequence of (12). Indeed, (12) implies $Z_*(n)/\sqrt{n} \leq \sqrt{2}\left(\sqrt{1+\frac{1}{8n}} - \sqrt{\frac{1}{8n}}\right)$, i.e. $\overline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} \leq \sqrt{2}$. But this upper limit is exact for $n = \frac{p(p+1)}{2}$ $(p \to \infty)$.

Similar and other relations on the functions S and Z can be found in [4-5].

An inequality connecting $S_*(ab)$ with $S_*(a)$ and $S_*(b)$ appears in [3]. A similar result holds for the functions Z and Z_*.

Proposition 7. For all $a, b \geq 1$ one has

$$Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}, \tag{14}$$

$$Z(ab) \geq \max\{Z(a), Z(b)\} \geq \max\{Z_*(a), Z_*(b)\}. \tag{15}$$

Proof. If $m = Z_*(a)$, then $\frac{m(m+1)}{2}|a$. Since $a|ab$ for all $b \geq 1$, clearly $\frac{m(m+1)}{2}|ab$, implying $Z_*(ab) \geq m = Z_*(a)$. In the same manner, $Z_*(ab) \geq Z_*(b)$, giving (14).

Let now $k = Z(ab)$. Then, by (4) we can write $ab|\frac{k(k+1)}{2}$. By $a|ab$ it results $a|\frac{k(k+1)}{2}$, implying $Z(a) \leq k = Z(ab)$. Analogously, $Z(b) \leq Z(ab)$, which via (9) gives (15).

Corollary. $Z_*(3^s \cdot p) \geq 2$ for any integer $s \geq 1$ and any prime p. (16)

Indeed, by (14), $Z_*(3^s \cdot p) \geq \max\{Z_*(3^s), Z(p)\} = \max\{2, 1\} = 2$, by (6).

We now consider two irrational series.

Proposition 8. The series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{n!}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} Z_*(n)}{n!}$ are irrational.

Proof. For the first series we apply the following irrationality criterion ([6]). Let (v_n) be a sequence of nonnegative integers such that

(i) $v_n < n$ for all large n;

(ii) $v_n < n-1$ for infinitely many n;

(iii) $v_n > 0$ for infinitely many n.

Then $\sum_{n=1}^{\infty} \frac{v_n}{n!}$ is irrational.

Let $v_n = Z_*(n)$. Then, by (12) $Z_*(n) < n-1$ follows from $\frac{\sqrt{8n+1}-1}{2} < n-1$, i.e. (after some elementary fact, which we omit here) $n > 3$. Since $Z_*(n) \geq 1$, conditions (i)-(iii) are trivially satisfied.

For the second series we will apply a criterion from [7]:

Let $(a_k), (b_k)$ be sequences of positive integers such that

(i) $k | a_1 a_2 \dots a_k$;

(ii) $\frac{b_{k+1}}{a_{k+1}} < b_k < a_k$ $(k \geq k_0)$. Then $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{b_k}{a_1 a_2 \dots a_k}$ is irrational.

Let $a_k = k$, $b_k = Z_*(k)$. Then (i) is trivial, while (ii) is $\frac{Z_*(k+1)}{k+1} < Z_*(k) < k$. Here $Z_*(k) < k$ for $k \geq 2$. Further $Z_*(k+1) < (k+1)Z_*(k)$ follows by $1 \leq Z_*(k)$ and $Z_*(k+1) < k+1$.

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14 On Certain Arithmetic Functions

In the recent book [1] there appear certain arithmetic functions which are similar to the Smarandache function. In a recent paper [2] we have considered certain generalization or duals of the Smarandache function $S(n)$. In this note we wish to point out that the arithmetic functions introduced in [1] all are particular cases of our function F_f, defined in the following manner (see [2] or [3]).

Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetical function which satisfies the following property:

(P_1) For each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbb{N}^* \to \mathbb{N}*$ defined by

$$F_f(n) = \min\{k \in \mathbb{N}^* : n|f(k)\} \tag{1}$$

In Problem 6 of [1] it is defined the "ceil function of t-th order" by $S_t(n) = \min\{k : n|k^t\}$. Clearly here one can select $f(m) = m^t$ $(m = 1, 2, \ldots)$, where $t \geq 1$ is fixed. Property (P_1) is satisfied with $k = n^t$. For $f(m) = \dfrac{m(m+1)}{2}$, one obtains the "Pseudo-Smarandache" function of Problem 7. The Smarandache "double-factorial" function

$$SDF(n) = \min\{k : n|k!!\}$$

where

$$k!! = \begin{cases} 1 \cdot 3 \cdot 5 \ldots k & \text{if } k \text{ is odd} \\ 2 \cdot 2 \cdot 6 \ldots k & \text{if } k \text{ is even} \end{cases}$$

of Problem 9 [1] is the particular case $f(m) = m!!$. The "power function" of Definition 24, i.e. $SP(n) = \min\{k : n|k^k\}$ is the case of $f(k) = k^k$. We note that the Definitions 39 and 40 give the particular case of S_t for $t = 2$ and $t = 3$.

In our paper we have introduced also the following "dual" of F_f. Let $g : \mathbb{N}^* \to \mathbb{N}*$ be a given arithmetical function, which satisfies the following assumption:

(P_3) For each $n \geq 1$ there exists $k \geq 1$ such that $g(k)|n$.

Let $G_g : \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$G_g(n) = \max\{k \in \mathbb{N}^* : g(k)|n\}. \tag{2}$$

Since $k^t|n$, $k!!|n$, $k^k|n$, $\frac{k(k+1)}{2}|n$ all are verified for $k=1$, property (P_3) is satisfied, so we can define the following duals of the above considered functions:

$$S_t^*(n) = \max\{k : k^t|n\};$$

$$SDF^*(n) = \max\{k : k!!|n\};$$

$$SP^*(n) = \max\{k : k^k|n\};$$

$$Z^*(n) = \max\left\{k : \frac{k(k+1)}{2}|n\right\}.$$

These functions are particular cases of (2), and they could deserve a further study, as well.

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15 On a new Smarandache type function

Let $C_n^k = \binom{n}{k}$ denote a binomial coefficient, i.e.

$$C_n^k = \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\dots k} = \frac{n!}{k!(n-k)!} \text{ for } 1 \le k \le n.$$

Clearly, $n|C_n^1$ and $n|C_n^{n-1} = C_n^1$. Let us define the following arithmetic function:

$$C(n) = \max\{k:\ 1 \le k < n-1,\ n|C_n^k\} \tag{1}$$

Clearly, this function is well-defined and $C(n) \ge 1$. We have supposed $k < n-1$, otherwise on the basis of $C_n^{n-1} = C_n^1 = n$, clearly we would have $C(n) = n-1$.

By a well-known result on primes, $p|C_p^k$ for all primes p and $1 \le k \le p-1$.

Thus we get:

$$C(p) = p-2 \text{ for primes } p \ge 3. \tag{2}$$

Obviously, $C(2) = 1$ and $C(1) = 1$. We note that the above result on primes is usually used in the inductive proof of Fermat's "little" theorem.

This result can be extended as follows:

Lemma. *For $(k,n) = 1$, one has $n|C_n^k$.*

Proof. Let us remark that

$$C_n^k = \frac{n}{k}\cdot\frac{(n-1)\dots(n-k+1)}{(k-1)!} = \frac{n}{k}\cdot C_{n-1}^{k-1} \tag{3}$$

thus, the following identity is valid:

$$kC_n^k = nC_{n-1}^{k-1} \tag{4}$$

This gives $n|kC_n^k$, and as $(n,k) = 1$, the result follows.

Theorem. *$C(n)$ is the greatest totient of n which is less than or equal to $n-2$.*

Proof. A totient of n is a number k such that $(k,n) = 1$. From the lemma and the definition of $C(n)$, the result follows.

Remarks. 1) Since $(n-2,n) = (2,n) = 1$ for odd n, the theorem implies that $C(n) = n-2$ for $n \ge 3$ and odd. Thus the real difficulty in calculating $C(n)$ is for n an even number.

2) The above lemma and Newton's binomial theorem give an extension of Fermat's divisibility theorem $p|(a^p - a)$ for primes p ([3]).

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16 On an additive analogue of the function S

The function S, and its dual S_* are defined by

$$S(n) = \min\{m \in \mathbb{N} : \ n|m!\};$$

$$S_*(n) = \max\{m \in \mathbb{N} : \ m!|n\} \quad \text{(see e.g. [1])}$$

We now define the following "additive analogue", which is defined on a subset of real numbers.

Let

$$S(x) = \min\{m \in \mathbb{N} : \ x \leq m!\}, \quad x \in (1, \infty) \tag{1}$$

as well as, its dual

$$S_*(x) = \max\{m \in \mathbb{N} : \ m! \leq x\}, \quad x \in [1, \infty). \tag{2}$$

Clearly, $S(x) = m$ if $x \in ((m-1)!, m!]$ for $m \geq 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

In the same manner, $S_*(x) = m$ if $x \in [m!, (m+1)!)$ for $m \geq 1$, i.e. $S_* : [1, \infty) \to \mathbb{N}$ (while $S : (1, \infty) \to \mathbb{N}$).

It is immediate that

$$S(x) = \begin{cases} S_*(x) + 1, & \text{if} \quad x \in (k!, (k+1)!) \quad (k \geq 1) \\ S_*(x), & \text{if} \quad x = (k+1)! \qquad (k \geq 1) \end{cases} \tag{3}$$

Therefore, $S_*(x) + 1 \geq S(x) \geq S_*(x)$, and it will be sufficient to study the function $S_*(x)$.

The following simple properties of S_* are immediate:

1° S_* is surjective and an increasing function

2° S_* is continuous for all $x \in [1, \infty) \setminus A$, where $A = \{k!, \ k \geq 2\}$, and since $\lim\limits_{x \nearrow k!} S_*(x) = k - 1$, $\lim\limits_{x \searrow k!} S_*(x) = k$ $(k \geq 2)$, S_* is continuous from the right in $x = k!$ $(k \geq 2)$, but it is not continuous from the left.

3° S_* is differentiable on $(1, \infty) \setminus A$, and since $\lim\limits_{x \searrow k!} \dfrac{S_*(x) - S_*(k!)}{x - k!} = 0$, it has a right-derivative in $A \cup \{1\}$.

4° S_* is Riemann integrable in $[a,b] \subset \mathbb{R}$ for all $a < b$.

a) If $[a,b] \subset [k!,(k+1)!)$ $(k \geq 1)$, then clearly

$$\int_a^b S_*(x)dx = k(b-a) \tag{4}$$

b) On the other hand, since

$$\int_{k!}^{l!} = \int_{k!}^{(k+1)!} + \int_{(k+1)!}^{(k+2)!} + \ldots + \int_{(k+l-k-1)!}^{(k+l-k)!}$$

(where $l > k$ are positive integers), and by

$$\int_{k!}^{(k+1)!} S_*(x)dx = k[(k+1)! - k!] = k^2 \cdot k!, \tag{5}$$

we get

$$\int_{k!}^{l!} S_*(x)dx = k^2 \cdot k! + (k+1)^2(k+1)! + \ldots + [k+(l-k-1)]^2[k+(l-k-1)!] \tag{6}$$

c) Now, if $a \in [k!,(k+1)!]$, $b \in [l!,(l+1)!)$, by

$$\int_a^b = \int_a^{(k+1)!} + \int_{(k+1)!}^{l!} + \int_{l!}^{k}$$

and (4), (5), (6), we get:

$$\int_a^b S_*(x)dx = k[(k+1)! - a] + (k+1)^2(k+1)! + \ldots +$$

$$+[k+1+(l-k-2)]^2[k+1+(l-k-2)!] + l(b-l!) \tag{7}$$

We now prove the following

Theorem 1.

$$S_*(x) \sim \frac{\log x}{\log\log x} \quad (x \to \infty) \tag{8}$$

Proof. We need the following

Lemma. *Let* $x_n > 0$, $y_n > 0$, $\dfrac{x_n}{y_n} \to a > 0$ *(finite) as* $n \to \infty$, *where* $x_n, y_n \to \infty$ $(n \to \infty)$. *Then*

$$\frac{\log x_n}{\log y_n} \to 1 \quad (n \to \infty). \tag{9}$$

Proof. $\log \dfrac{x_n}{y_n} \to \log a$, i.e. $\log x_n - \log y = \log a + \varepsilon(n)$, with $\varepsilon(n) \to 0$ $(n \to \infty)$. So

$$\frac{\log x_n}{\log y_n} - 1 = \frac{\log a}{\log y_n} + \frac{\varepsilon(n)}{\log y_n} \to 0 + 0 \cdot 0 = 0.$$

Lemma 2. a) $\dfrac{n \log\log n!}{\log n!} \to 1$;

b) $\dfrac{\log n!}{\log(n+1)!} \to 1$;

c) $\dfrac{\log\log n!}{\log\log(n+1)!} \to 1$ as $n \to \infty$ (10)

Proof. a) Since $n! \sim Ce^{-n}n^{n+1/2}$ (Stirling's formula), clearly $\log n! \sim n \log n$, so b) follows by $\dfrac{\log n}{\log(n+1)} \sim 1$ ((9), since $\dfrac{n}{n+1} \sim 1$). Now c) is a consequence of b) by the Lemma. Again by the Lemma, and $\log n! \sim n \log n$ we get

$$\log\log n! \sim \log(n \log n) = \log n + \log\log n \sim \log n$$

and a) follows.

Now, from the proof of (8), remark that

$$\frac{n \log\log n!}{\log(n+1)!} < \frac{S_*(x) \log\log x}{\log x} < \frac{n \log\log(n+1)!}{\log n!}$$

and the result follows by (10).

Theorem 2. *The series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^{\alpha}}$ *is convergent for* $\alpha > 1$ *and divergent for* $a \leq 1$.

Proof. By Theorem 1,

$$A\frac{\log n}{\log\log n} < S_*(n) < B\frac{\log n}{\log\log n}$$

$(A, B > 0)$ for $n \geq n_0 > 1$, therefore it will be sufficient to study the convergence of $\displaystyle\sum_{n \geq n_0}^{\infty} \frac{(\log\log n)^{\alpha}}{n(\log n)^{\alpha}}$.

The function $f(x) = (\log\log x)^{\alpha}/x(\log x)^{\alpha}$ has a derivative given by

$$x^2(\log x)^{2\alpha} f'(x) = (\log\log x)^{\alpha-1}(\log x)^{\alpha-1}[1 - (\log\log x)(\log x + \alpha)]$$

implying that $f'(x) < 0$ for all sufficiently large x and all $\alpha \in \mathbb{R}$. Thus f is strictly decreasing for $x \geq x_0$. By the Cauchy condensation criterion ([2]) we know that $\sum a_n \leftrightarrow$

$\sum 2^n a_{2^n}$ (where $\leftrightarrow$ means that the two series have the same type of convergence) for (a_n) strictly decreasing, $a_n > 0$. Now, with $a_n = (\log\log n)^\alpha/n(\log n)^\alpha$ we have to study $\sum \frac{2^n(\log\log 2^n)^\alpha}{2^n(\log 2^n)^\alpha} \leftrightarrow \sum \left(\frac{\log n + a}{n+b}\right)^\alpha$, where a, b are constants ($a = \log\log 2$, $b = \log 2$). Arguing as above, (b_n) defined by $b_n = \left(\frac{\log n + a}{n+b}\right)^\alpha$ is a strictly positive, strictly decreasing sequence, so again by Cauchy's criterion

$$\sum_{n\geq m_0} b_n \leftrightarrow \sum_{n\geq m_0} \frac{2^n(\log 2^n + a)^\alpha}{(2^n+b)^\alpha} = \sum_{n\geq m_0} \frac{2^n(nb+a)^\alpha}{(2^n+b)^\alpha} = \sum_{n\geq m_0} c_n.$$

Now, $\lim\limits_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{1}{2^{\alpha-1}}$, by an easy computation, so D'Alembert's criterion proves the theorem for $\alpha \neq 1$. But for $\alpha = 1$ we get the series $\sum \frac{2^n(nb+a)}{2^n+b}$, which is clearly divergent.

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17 On the difference of alternate compositions of arithmetic functions

Let $\Delta_{f,g}(n) = |f(g(n)) - g(f(n))|(n = 1, 2, \ldots))$, where $f, g : N \to R$ are certain arithmetic functions. In OQ.350, OQ.351, OQ.352 ([1]) the irrationality of series of type $\sum_{n=1}^{\infty} \frac{1}{\Delta_{f,g}(n)}$ is announced. Before asking the irrationality, a more modest problem is the convergence of this series. For example, when $\Delta_{f,g}(n) = 0$, the above series is not defined; but even when $\Delta_{f,g}(n) \neq 0$, a problem arises: Is $\lim_{n\to\infty} |\Delta_{f,g}(n)| = +\infty$? This simpler question is even difficult in certain cases. For example, let φ be the Euler totient, and S be the Smarandache function. We will prove that:

$$\limsup_{n\to\infty} |\Delta_{f,g}(n)| = +\infty \tag{1}$$

$$\liminf_{n\to\infty} |\Delta_{f,g}(n)| \leq 1 \tag{2}$$

thus, the series in OQ.352 is not convergent.

Indeed, let $p \geq 3$ be a prime. Then

$$\Delta_{S,g}(p) = |S(p-1) - (p-1)| = p - 1 - S(p-1),$$

since $p - 1$ being even,

$$S(p-1) \leq \frac{p-1}{2}$$

(see [7]). This implies

$$p - 1 - S(p-1) \geq p - 1 - \frac{p-1}{2} \to \infty \text{ as } p \to \infty.$$

Now let $n = p^2$. By $S(p^2) = 2p$, $\varphi(2p) = p - 1$, $\varphi(p^2) = p(p-1)$, $S(p(p-1)) = \max\{S(p), S(p-1)\}$ (since p, $p - 1 = 1$) we get

$$S(p(p-1)) = p,$$

by $S(p-1) \leq \frac{p-1}{2} < p = S(p)$. Therefore $\Delta_{S,g}(p^2) = |p - (p-1)| = 1$. This implies (2). But here the following open problem arises:

What are the solutions of the equation $S(\varphi(n)) = \varphi(S(n))$?

2. Now we prove that:

$$\liminf_{n\to\infty} \Delta_{d,\varphi}(n) = 0 \tag{3}$$

and

$$\limsup_{n\to\infty} \Delta_{d,\varphi}(n) = +\infty \tag{4}$$

where $d(n)$ denotes the number of distinct divisors of n.

Let $p \geq 3$ be a prime and put $n = 2^{p-1}$. Then

$$\Delta_{d,\varphi}(n) = |d(2^{p-2}) - \varphi(p)| = |p - 1 - p - 1| = 0.$$

In fact $\inf \Delta_{d,\varphi}(n) = 0$. Now put $n = 2^{p-1} \cdot 3^{q-1}$, with $p, q \geq 3$ distinct primes. Then

$$\varphi(n) = 2^{p-1} \cdot 3^{q-2}, \quad d(\varphi(n)) = p(q-1), \quad \varphi(d(n)) = \varphi(pq) = (p-1)(q-1).$$

Thus

$$\Delta_{d,\varphi}(n) = p(q-1) - (p-1)(q-1) = q - 1 \to \infty \text{ as } q \to \infty.$$

This proves (4).

In fact (4) follows from the stronger result by Prachar [5] that

$$d(p-1) > \exp\{c \log p / \log\log p\} \quad (c > 0)$$

for infinitely primes p. As we have seen, $d(\varphi(n)) = \varphi(d(n))$ for $n = 2^{p-1}$ ($p \geq 3$ prime). Clearly, $n = 1$ is also solution. What are the general solutions of this equation?

3. Let $\sigma(n)$ be the sum of divisors of n. Let us assume the conjecture that for infinitely many primes p, $2p - 1$ is prime, too, is valid. Let $q = p - 1$. Then

$$\Delta_{d,\sigma}(q) = |d(q+1) - 3| = |d(2p) - 3| = 1.$$

Thus:

$$\liminf_{n\to\infty} |\Delta_{d,\sigma}(n)| \leq 1 \tag{5}$$

Let now $n = 2^{p-1}$ ($p \geq 3$ prime), and assume that the conjecture on the infinitude of Mersenne primes $2^p - 1$ is true. Then

$$\sigma(d(n)) = p + 1, \quad d(\sigma(n)) = d(2^p - 1) = 2$$

for infinitely many p. Thus:

$$\limsup_{n\to\infty} |\Delta_{d,\sigma}(n)| = +\infty \tag{6}$$

Probably, (5) and (6) can be proved without any assumption. The following problem seems difficult: What are the solutions of the equation $d(\sigma(n)) = \sigma(d(n))$?

4. For an odd prime p,

$$\Delta_{\varphi,\sigma}(p) = |\varphi(p+1) - \sigma(p-1)| = \sigma(p-1) - \varphi(p+1)$$

as

$$\sigma(p-1) > p > \frac{p+1}{2} \geq \varphi(p+1).$$

This implies also

$$\sigma(p-1) - \varphi(p+1) > p - \frac{p+1}{2} = \frac{p-1}{2} \to \infty \text{ as } p \to \infty.$$

Thus:

$$\limsup_{n\to\infty} \Delta_{\varphi,\sigma}(n) = +\infty \tag{7}$$

Another sequence for which the limit is $+\infty$ is $n = 2^{p-1}$ ($p \geq 3$ prime). By a result of Bojanić [2] we have:

$$\frac{\varphi(2^p-1)}{2^p-1} \to 1 \text{ as } p \to \infty \tag{8}$$

Now,

$$\Delta_{\varphi,\sigma}(2^{p-1}) = |\varphi(2^p-1) - (2^{p-1}-1| = \left|2^{p-1}\left[2\cdot\frac{\varphi(2^p-1)}{2^p-1}\right] - \frac{\varphi(2^p-1)}{2^p-1} + 1\right| \to \infty$$

by (8) and $2^{p-1} \to \infty$ as $p \to \infty$. Probably:

$$\liminf_{n\to\infty} \Delta_{\varphi,\sigma}(n) = 0 \tag{9}$$

Golomb [3] remarks that if for a prime p, $\dfrac{3^p-1}{2}$ is prime too (e.g. $p = 2, 7, 13, 71, 103$), then $n = 3^{p-1}$ is a solution of $\Delta_{\varphi,\sigma}(n) = 0$. What are the most general solutions of $\varphi(\sigma(n)) = \sigma(\varphi(n))$? Other problems on the composition of arithmetical functions can be found e.g. in [4], [6].

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18 On multiplicatively deficient and abundant numbers

Definition 28 of [1] introduces the so-called "impotent numbers" n whose proper divisors product is less than n. It is mentioned there that the sequence of these numbers contains terms with the forms p and p^2, where p is a prime.

Let $T(n)$ denote the product of all divisors of n. Then $T(n) = n^2$ iff n is a multiplicatively-perfect (or shortly m-perfect) number. In a recent paper [2] we have studied these numbers or, for example, numbers satisfying equations of type $T(T(n)) = n^2$ (called m-superperfect numbers). Clearly, the above impotent numbers satisfy the inequality

$$T(n) < n^2 \tag{1}$$

i.e. they are multiplicatively deficient (or "m-deficient") numbers. Therefore it is not necessary to introduce a new terminology in this case.

First remark, that all m-deficient numbers can be written in the forms $1, p, p^2, pq, p^2q$, where p, q are distinct primes. Indeed, if $d_1, d_2, \ldots, d_s$ are all divisors of n, then

$$\{d_1, \ldots, d_s\} = \left\{\frac{n}{d_1}, \ldots, \frac{n}{d_s}\right\},$$

implying that

$$d_1 d_2 \ldots d_s = \frac{n}{d_1} \cdot \frac{n}{d_2} \cdots \frac{n}{d_s},$$

i.e.

$$T(n) = n^{s/2} \tag{2}$$

where $s = d(n)$ denotes the number of distinct divisors of n. Therefore inequality (1) is satisfied only when $d(n) < 4$, implying $n \in \{1, p, p^2, pq, p^2q\}$. Clearly, n is m-abundant when

$$T(n) > n^2 \tag{3}$$

implying $d(n) > 4$. Since for $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ one has $d(n) = (\alpha_1 + 1) \ldots (\alpha_r + 1)$, in the case $r = 1$, (3) is true only for $\alpha_1 > 3$; when $r = 2$ for $\alpha_1 = 1$ we must have $\alpha_2 \geq 2$,

while for $\alpha_1 \geq 2$, $\alpha_2 \geq 2$ this is always valid; for $r \geq 3$, (3) always holds true. Therefore, all m-abundant numbers are of the forms $n = p^{\alpha}$ $(\alpha \geq 4)$; pq^{β} $(\beta \geq 2)$, $p^{\alpha}q^{\beta}$ $(\alpha, \beta \geq 2)$; $w(n) \geq 3$ (where p, q are distinct primes and $w(n)$ denotes the number of distinct prime divisors of n).

On the other hand, let us remark that for $n \geq 2$ one has $d(n) \geq 2$, so

$$T(n) \geq n \tag{4}$$

with equality, only for $n =$ prime. If $n \neq$ prime, then $d(n) \geq 3$ gives

$$T(n) \geq n^{3/2} \quad (n \neq \text{prime}). \tag{5}$$

Now, relations (4) and (5) give together

$$T(T(n)) \geq n^{9/4} \quad \text{for} \quad n \neq \text{prime} \tag{6}$$

Since $9/4 > 2$, we have obtained that for all composite numbers we have $T(T(n)) > n^2$, i.e. all composite numbers are m-super abundant. Since $T(T(p)) = p < p^2$, all prime numbers are m-super deficient. Therefore we can state the following "primality criterion".

Theorem 1. *The number $n > 1$ is prime if and only if it is m-super deficient.*

In fact, by iteration from (6) we can obtain

$$\underbrace{T(T(\ldots T(n)\ldots))}_{k} \geq n^{3^k/2^k}, \quad n \neq \text{prime}.$$

Since $3^k > 2^k \cdot k$ for all $k \geq 1$, we have the following generalization.

Theorem 2. *The number $n > 1$ is prime if and only if it is m-k-super deficient.*

(n is m-k-super deficient if $\underbrace{T(T(\ldots T(n)\ldots))}_{k} < n^k$).

For related results see [2].

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19 On values of arithmetical functions at factorials I

1. The Smarandache function is a characterization of factorials, since $S(k!) = k$, and is connected to values of other arithmetical functions at factorials. Indeed, the equation

$$S(x) = k \quad (k \geq 1 \text{ given}) \tag{1}$$

has $d(k!) - d((k-1)!)$ solutions, where $d(n)$ denotes the number of divisors of n. This follows from $\{x : \ S(x) = k\} = \{x : \ x|k!, \ x \nmid (k-1)!\}$. Thus, equation (1) always has at least a solution, if $d(k!) > d((k-1)!)$ for $k \geq 2$. In what follows, we shall prove this inequality, and in fact we will consider the arithmetical functions $\varphi, \sigma, d, \omega, \Omega$ at factorials. Here $\varphi(n)$ = Euler's arithmetical function, $\sigma(n)$ = sum of divisors of n, $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of total divisors of n. As it is well known, we have $\varphi(1) = d(1) = 1$, while $\omega(1) = \Omega(1) = 0$, and for $1 < \prod_{i=1}^{r} p_i^{a_i}$ ($a_i \geq 1$, p_i distinct primes) one has

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right),$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

$$\omega(n) = r,$$

$$\Omega(n) = \sum_{i=1}^{r} a_i,$$

$$d(n) = \prod_{i=1}^{r} (a_i + 1). \tag{2}$$

The functions φ, σ, d are multiplicative, ω is additive, while Ω is totally additive, i.e. φ, σ, d satisfy the functional equation $f(mn) = f(m)f(n)$ for $(m, n) = 1$, while ω, Ω satisfy the equation $g(mn) = g(m) + g(n)$ for $(m, n) = 1$ in case of ω, and for all m, n is case of Ω (see [1]).

2. Let $m = \prod_{i=1}^{r} p_i^{\alpha_i}$, $n = \prod_{i=1}^{r} p_i^{\beta_i}$ ($\alpha_i, \beta_i \geq 0$) be the canonical factorizations of m and n. (Here some α_i or β_i can take the values 0, too). Then

$$d(mn) = \prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \geq \prod_{i=1}^{r} (\beta_i + 1)$$

with equality only if $\alpha_i = 0$ for all i. Thus:

$$d(mn) \geq d(n) \tag{3}$$

for all m, n, with equality only for $m = 1$.

Since $\prod_{i=1}^{r}(\alpha_i + \beta_i + 1) \leq \prod_{i=1}^{r}(\alpha_i + 1)\prod_{i=1}^{r}(\beta_i + 1)$, we get the relation

$$d(mn) \leq d(m)d(n) \tag{4}$$

with equality only for $(n, m) = 1$.

Let now $m = k$, $n = (k-1)!$ for $k \geq 2$. Then relation (3) gives

$$d(k!) > d((k-1)!) \text{ for all } k \geq 2, \tag{5}$$

thus proving the assertion that equation (1) always has at least a solution (for $k = 1$ one can take $x = 1$).

With the same substitutions, relation (4) yields

$$d(k!) \leq d((k-1)!)d(k) \text{ for } k \geq 2 \tag{6}$$

Let $k = p$ (prime) in (6). Since $((p-1)!, p) = 1$, we have equality in (6):

$$\frac{d(p!)}{d((p-1)!)} = 2, \quad p \text{ prime.} \tag{7}$$

3. Since $S(k!)/k! \to 0$, $\dfrac{S(k!)}{S((k-1)!)} = \dfrac{k}{k-1} \to 1$ as $k \to \infty$, one may ask the similar problems for such limits for other arithmetical functions.

It is well known that

$$\frac{\sigma(n!)}{n!} \to \infty \text{ as } n \to \infty. \tag{8}$$

In fact, this follows from $\sigma(k) = \sum_{d|k} d = \sum_{d|k} \dfrac{k}{d}$, so

$$\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n,$$

as it is known.

From the known inequality ([1]) $\varphi(n)\sigma(n) \leq n^2$ it follows

$$\frac{n}{\varphi(n)} \geq \frac{\sigma(n)}{n},$$

so $\dfrac{n!}{\varphi(n!)} \to \infty$, implying

$$\frac{\varphi(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{9}$$

Since $\varphi(n) > d(n)$ for $n > 30$ (see [2]), we have $\varphi(n!) > d(n!)$ for $n! > 30$ (i.e. $n \geq 5$), so, by (9)

$$\frac{d(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{10}$$

In fact, much stronger relation is true, since $\dfrac{d(n)}{n^\varepsilon} \to 0$ for each $\varepsilon > 0$ ($n \to \infty$) (see [1]). From $\dfrac{d(n!)}{n!} < \dfrac{\varphi(n!)}{n!}$ and the above remark on $\sigma(n!) > n! \log n$, it follows that

$$\limsup_{n\to\infty} \frac{d(n!)}{n!} \log n \leq 1. \tag{11}$$

These relations are obtained by very elementary arguments. From the inequality $\varphi(n)(\omega(n) + 1) \geq n$ (see [2]) we get

$$\omega(n!) \to \infty \text{ as } n \to \infty \tag{12}$$

and, since $\Omega(s) \geq \omega(s)$, we have

$$\Omega(n!) \to \infty \text{ as } n \to \infty. \tag{13}$$

From the inequality $nd(n) \geq \varphi(n) + \sigma(n)$ (see [2]), and (8), (9) we have

$$d(n!) \to \infty \text{ as } n \to \infty. \tag{14}$$

This follows also from the known inequality $\varphi(n)d(n) \geq n$ and (9), by replacing n with $n!$. From $\sigma(mn) \geq m\sigma(n)$ (see [3]) with $n = (k-1)!$, $m = k$ we get

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \geq k \quad (k \geq 2) \tag{15}$$

and, since $\sigma(mn) \leq \sigma(m)\sigma(n)$, by the same argument

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \leq \sigma(k) \quad (k \geq 2). \tag{16}$$

Clearly, relation (15) implies

$$\lim_{k\to\infty}\frac{\sigma(k!)}{\sigma((k-1)!)}=+\infty. \tag{17}$$

From $\varphi(m)\varphi(n)\leq\varphi(mn)\leq m\varphi(n)$, we get, by the above remarks, that

$$\varphi(k)\leq\frac{\varphi(k!)}{\varphi((k-1)!)}\leq k,\quad (k\geq 2) \tag{18}$$

implying, by $\varphi(k)\to\infty$ as $k\to\infty$ (e.g. from $\varphi(k)>\sqrt{k}$ for $k>6$) that

$$\lim_{k\to\infty}\frac{\varphi(k!)}{\varphi((k-1)!}=+\infty. \tag{19}$$

By writing $\sigma(k!)-\sigma((k-1)!)=\sigma((k-1)!)\left[\frac{\sigma(k!)}{\sigma((k-1)!)}-1\right]$, from (17) and $\sigma((k-1)!)\to\infty$ as $k\to\infty$, we trivially have:

$$\lim_{k\to\infty}[\sigma(k!)-\sigma((k-1)!)]=+\infty. \tag{20}$$

In completely analogous way, we can write:

$$\lim_{k\to\infty}[\varphi(k!)-\varphi((k-1)!)]=+\infty. \tag{21}$$

4. Let us remark that for $k=p$ (prime), clearly $((k-1)!,k)=1$, while for $k=$ composite, all prime factors of k are also prime factors of $(k-1)!$. Thus

$$\omega(k!)=\begin{cases}\omega((k-1)!k)=\omega((k-1)!)+\omega(k) & \text{if } k \text{ is prime}\\ \omega((k-1)!) & \text{if } k \text{ is composite } (k\geq 2).\end{cases}$$

Thus

$$\omega(k!)-\omega((k-1)!)=\begin{cases}1, & \text{for } k=\text{prime}\\ 0, & \text{for } k=\text{composite}\end{cases} \tag{22}$$

Thus we have

$$\begin{aligned}&\limsup_{k\to\infty}[\omega(k!)-\omega((k-1)!)]=1\\ &\liminf_{k\to\infty}[\omega(k!)-\omega((k-1)!)]=0\end{aligned} \tag{23}$$

Let p_n be the nth prime number. From (22) we get

$$\frac{\omega(k!)}{\omega((k-1)!)}-1=\begin{cases}\frac{1}{n-1}, \text{if } k=p_n\\ 0, \text{if } k=\text{composite.}\end{cases}$$

Thus, we get

$$\lim_{k\to\infty} \frac{\omega(k!)}{\omega((k-1)!)} = 1. \tag{24}$$

The function Ω is totally additive, so

$$\Omega(k!) = \Omega((k-1)!k) = \Omega((k-1)!) + \Omega(k),$$

giving

$$\Omega(k!) - \Omega((k-1)!) = \Omega(k). \tag{25}$$

This implies

$$\limsup_{k\to\infty} [\Omega(k!) - \Omega((k-1)!)] = +\infty \tag{26}$$

(take e.g. $k = 2^m$ and let $m \to \infty$), and

$$\liminf_{k\to\infty} [\Omega(k!) - \Omega((k-1)!)] = 2$$

(take $k =$ prime).

For $\Omega(k!)/\Omega((k-1)!)$ we must evaluate

$$\frac{\Omega(k)}{\Omega((k-1)!)} = \frac{\Omega(k)}{\Omega(1) + \Omega(2) + \ldots + \Omega(k-1)}.$$

Since $\Omega(k) \leq \dfrac{\log k}{\log 2}$ and by the theorem of Hardy and Ramanujan (see [1]) we have

$$\sum_{n\leq x} \Omega(n) \sim x \log\log x \quad (x \to \infty)$$

so, since $\dfrac{\log k}{(k-1)\log\log(k-1)} \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{\Omega(k!)}{\Omega((k-1)!)} = 1. \tag{27}$$

5. Inequality (18) applied for $k = p$ (prime) implies

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\varphi(p!)}{\varphi((p-1)!)} = 1. \tag{28}$$

This follows by $\varphi(p) = p - 1$. On the other hand, let $k > 4$ be composite. Then, it is known (see [1]) that $k|(k-1)!$. So $\varphi(k!) = \varphi((k-1)!k) = k\varphi((k-1)!)$, since $\varphi(mn) = m\varphi(n)$ if $m|n$. In view of (28), we can write

$$\lim_{k\to\infty} \frac{1}{k} \cdot \frac{\varphi(k!)}{\varphi((k-1)!)} = 1. \tag{29}$$

For the function σ, by (15) and (16), we have for $k = p$ (prime) that $p \leq \dfrac{\sigma(p!)}{\sigma((p-1)!)} \leq \sigma(p) = p + 1$, yielding

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\sigma(p!)}{\sigma((p-1)!)} = 1. \tag{30}$$

In fact, in view of (15) this implies that

$$\liminf_{k\to\infty} \frac{1}{k} \cdot \frac{\sigma(k!)}{\sigma((k-1)!)} = 1. \tag{31}$$

By (6) and (7) we easily obtain

$$\limsup_{k\to\infty} \frac{d(k!)}{d(k)d((k-1)!)} = 1. \tag{32}$$

In fact, inequality (6) can be improved, if we remark that for $k = p$ (prime) we have $d(k!) = d((k-1)!) \cdot 2$, while for $k =$ composite, $k > 4$, it is known that $k|(k-1)!$. We apply the following

Lemma. *If $n|m$, then*

$$\frac{d(mn)}{d(m)} \leq \frac{d(n^2)}{d(n)}. \tag{33}$$

Proof. Let $m = \prod p^{\alpha} \prod q^{\beta}$, $n = \prod p^{\alpha'}$ $(\alpha' \leq \alpha)$ be the prime factorizations of m and n, where $n|m$. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod(\alpha + \alpha' + 1)\prod(\beta + 1)}{\prod(\alpha + 1)\prod(\beta + 1)} = \prod\left(\frac{\alpha + \alpha' + 1}{\alpha + 1}\right).$$

Now $\dfrac{\alpha + \alpha' + 1}{\alpha + 1} \leq \dfrac{2\alpha' + 1}{\alpha' + 1} \Leftrightarrow \alpha' \leq \alpha$ as an easy calculations verifies. This immediately implies relation (33).

By selecting now $n = k$, $m = (k-1)!$, $k > 4$ composite we can deduce from (33):

$$\frac{d(k!)}{d((k-1)!)} \leq \frac{d(k^2)}{d(k)}. \tag{34}$$

By (4) we can write $d(k^2) < (d(k))^2$, so (34) represents indeed, a refinement of relation (6).

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20 On certain inequalities for σ_k

1. Let k be a real number and $n \geq 1$ a positive integer. Let $d_1, d_2, \ldots, d_s$ be all distinct divisors of n. Put

$$\sigma_k(n) = \sum_{i=1}^{s} d_i^k \tag{1}$$

for the sum of kth powers of all divisors of n. For $k = 1$ one obtains the sum of divisors $\sigma_1(n) = \sigma(n)$, while for $k = 0$ we have $\sigma_0(n) = d(n) = s$ - the number of divisors of n. For $k = -1$ one has

$$\sigma_{-1}(n) = \frac{1}{d_1} + \ldots + \frac{1}{d_s} = \frac{1}{n}\left(\frac{n}{d_1} + \ldots + \frac{n}{d_s}\right) = \frac{1}{n}(d_1 + \ldots + d_s) = \frac{\sigma(n)}{n};$$

since $\{d_1, \ldots, d_s\} = \left\{\frac{n}{d_1}, \ldots, \frac{n}{d_s}\right\}$.

Remark that the arithmetic mean of divisors of n is

$$A(d_1, \ldots, d_s) = \frac{d_1 + \ldots + d_s}{s} = \frac{\sigma(n)}{d(n)};$$

the harmonic mean is

$$H(d_1, \ldots, d_s) = \frac{s}{\sigma_{-1}(n)} = \frac{nd(n)}{\sigma(n)},$$

while the geometric mean is

$$G(d_1, d_2, \ldots, d_s) = \sqrt[s]{d_1 d_2 \ldots d_s} = \sqrt{n}.$$

Indeed, $d_1 d_2 \ldots d_s = \frac{n}{d_1} \cdot \frac{n}{d_2} \ldots \frac{n}{d_s}$, so $d_1 d_2 \ldots d_s = n^{s/2}$ (see [1]). Now, applying the classical inequalities $A \geq G \geq H$, one can deduce

$$\frac{\sigma(n)}{d(n)} \geq \sqrt{n} \tag{2}$$

and in more general form

$$\frac{\sigma_k(n)}{d(n)} \geq n^{k/2} \tag{3}$$

due to R. Sivaramakrishnan and C.S. Venkataraman [2].

S. Philipp [4] deduced the complementary inequality $\frac{nd(n)}{\sigma(n)} \geq \frac{4}{3}$, i.e.

$$\frac{\sigma(n)}{d(n)} \leq \frac{3}{4}n, \tag{4}$$

while E.S. Langford [2] improved this to

$$\frac{\sigma(n)}{d(n)} \leq \frac{n+1}{2}. \tag{5}$$

We will obtain strong generalized refinements of these relations.

2. Let the numbers (a_k) $(k = \overline{1,s})$ satisfy $0 < m \leq a_k \leq M$. Then the following inequality is due to P. Schweitzer [6]:

$$\left(\sum_{k=1}^{s} a_k\right)\left(\sum_{k=1}^{s} \frac{1}{a_k}\right) \leq \frac{s^2(M+m)^2}{4mM} \tag{6}$$

G. Pólya and G. Szegö [5] have generalized (5) as follows:

If $0 < a \leq a_k \leq A$, $0 < b \leq b_k \leq B$, then

$$\frac{(a_1^2 + \ldots + a_s^2)(b_1^2 + \ldots + b_s^2)}{(a + 1b_1 + \ldots + a_s b_s)^2} \leq \frac{(AB+ab)^2}{4ABab} \tag{7}$$

Remark that if $d_1, \ldots, d_s$ are the divisors of n, then

$$\sum_{i=1}^{s} d_i^k = \sum_{i=1}^{s} \left(\frac{n}{d_i}\right)^k,$$

so

$$\sum_{i=1}^{s} d_i^{-k} = n^{-k} \sum_{i=1}^{s} d_i^k \tag{8}$$

Let $k, t > 0$ be real numbers, and apply (6) to $a_i = d_i^{k/2}$, $b_i = d_i^{-t/2}$ $(i = \overline{1,s})$. Then $a = 1$, $A = n^{k/2}$, $b = n^{-t/2}$, $B = 1$ (suppose $n \geq 2$). After substitution in (6), by taking into account of (5) one obtains (by taking squaroots)

$$\frac{(\sigma_k(n)\sigma_t(n))^{1/2}}{\sigma_{(k-t)/2}(n)} \leq n^{-(k-t)/4} \frac{n^{(k+t)/2}+1}{2} \tag{9}$$

For $k = t$, a generalization of (4) is obtained:

$$\frac{\sigma_k(n)}{d(n)} \leq \frac{n^k+1}{2} \tag{10}$$

For $t = 0$ (8) gives the curious inequality

$$\frac{(\sigma_k(n))^{1/2}}{\sigma_{k/2}(n)} \leq n^{-k/4} \frac{n^{k/2}+1}{2} \tag{11}$$

3. A form of the Chebysheff inequalities [3] such that if $a_1 \leq a_2 \leq \ldots \leq a_s$ and $b_1 \leq b_2 \leq \ldots \leq b_s$ are real numbers, then

$$\left(\sum_{i=1}^{s} a_i\right)\left(\sum_{i=1}^{s} b_i\right) \leq s\sum_{i=1}^{s} a_i b_i \tag{12}$$

Let $a_i = d_i^k$, $b_i = d_i^t$ $(i = \overline{1,s})$, where $k, t > 0$. Then

$$\sigma_k(n)\sigma_t(n) \leq d(n)\sigma_{k+t}(n) \tag{13}$$

For an application of (12) we note that by (2) this gives

$$\frac{\sigma_{k+t}(n)}{\sigma_t(n)} \geq \frac{\sigma_k(n)}{d(n)} \geq n^{k/2} \tag{14}$$

We note that a simple proof of (2) follows by the application of the inequality

$$(a_1 + a_2 + \ldots + a_s)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_s}\right) \geq s^2$$

for $a_i = d_i^k$, by taking into account of (7).

We now write the following extension of the Chebysheff inequality ([3]): If $0 \leq a_1^1 \leq \ldots \leq a_n^1$, $0 \leq a_1^2 \leq \ldots \leq a_n^2, \ldots$, $0 \leq a_2^m \leq \ldots \leq a_n^m$ (where the superscripts are not powers!) then

$$\sum_{i=1}^{s} a_i^1 \sum_{i=1}^{s} a_i^2 \ldots \sum_{i=1}^{s} a_i^m \leq s^{m-1} \sum_{i=1}^{s} a_i^1 a_i^2 \ldots a_i^m \tag{15}$$

Let now in (14) $a_i^1 = d_i^{k_1}, \ldots, a_i^m = d_i^{k_m}$, where d_i $(i = \overline{1,s})$ are the divisors of n, while $k_1, \ldots, k_m > 0$ are given real numbers. One obtains

$$\sigma_{k_1}(n)\sigma_{k_2}(n)\ldots\sigma_{k_m}(n) \leq (d(n))^{m-1}\sigma_{k_1+\ldots k_2+\ldots+k_m}(n) \tag{16}$$

This is an extension of (12). Applying (2) this yields:

$$\frac{\sigma^{(n)}_{k_1+k_2+\ldots+k_m}}{\sigma_{k_i}(n)} \geq n^{\frac{1}{2}\sum\limits_{j=1}^{m-1} k_j}, \quad j \neq i \tag{17}$$

For $m = 2$ one reobtains the weaker form of (13). Other relations can be found in [7].

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21 Between totients and sum of divisors: the arithmetical function ψ

Introduction

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ (p_i primes, $\alpha_i \geq 1$, $i = \overline{1,r}$) be the canonical representation of the natural number $n > 1$. Then it is well-known that the Euler function φ has the representation

$$\varphi(n) = n \prod_{i=1}^{r} (1 - p_i^{-1}), \quad \varphi(1) = 1.$$

This arithmetical function plays an important role in many problems of number theory, algebra, geometry, etc. Another important function is the sum-of-divisors function σ, defined by

$$\sigma(n) = \sum_{d|n} d,$$

where the sum runs through all divisors d of n. We have

$$\sigma(n) = \prod_{i=1}^{r} (p_i^{\alpha_i+1} - 1)/(p_i - 1)$$

(see [1], [3], [7], [9], [22]). These two functions have been studied extensively and many interesting and important results have been proved. On the other hand there are some open problems which are very difficult to study at the present state of the science (e.g. the study of the equation $\sigma(n) = 2n$ for odd n (odd perfect numbers), problems on $\sigma(\sigma(n))$, $\sigma(\varphi(n))$, etc.).

Dedekind's arithmetical function ψ is defined by

$$\psi(n) = n \prod_{i=1}^{r} (1 + p_i^{-1}), \quad \psi(1) = 1.$$

This function proved to be useful in some problems of number theory ([2], [13], [14], [18], [24], [25]) and has interesting connections with the above mentioned and other arithmetical functions. There are known some generalizations ([4], [18], [19]), one of which will be used by us, namely the function

$$\psi_k(n) = n^k \prod_{i=1}^{r} (1 + p_i^{-k}), \quad k \geq 1.$$

Dedekind's function satisfies $\varphi(n) \leq \psi(n) \leq \sigma(n)$ and sometimes it is easier to study ψ than φ or σ. In our opinion a more penetrating study of this function may be important in elucidating some properties of φ and σ. The aim of this paper is to examine some less known and new properties of Dedekind's arithmetical function. Our results generalize or improve upon known theorems, have several interesting applications and lead to a new approach of old problems.

In what follows it will be convenient to use the well-known Jordan generalization of Euler's function,

$$\varphi_k(n) = n^k \prod_{i=1}^{r}(1 - p_i^{-k}),$$

and a generalization of σ:

$$\sigma_k(n) = \sum_{d|n} d^k = \prod_{i=1}^{r}(p_i^{k(\alpha_i+1)} - 1)/(p_i^k - 1)$$

([7], [9]). $\omega(n)$ and $\Omega(n)$ will denote the distinct and total number - of prime factors of n, respectively, i.e.

$$\omega(n) = r, \quad \Omega(n) = \sum_{i=1}^{r} \alpha_i;$$

$d(n)$ will be the number of all divisors of n, in expression

$$d(n) = \prod_{i=1}^{r}(\alpha_i + 1)$$

(see [1], [3], [7], [9], [22]). A divisor δ of n is called "unitary divisor", in notation $\delta \| n$, iff $\delta | n$ and $(\delta, n/\delta) = 1$. Define $\sigma_k^*(n) = \sum_{\delta \| n} \delta^k =$ the sum of kth powers of unitary divisors of n ([6], [17], [23]). Then it is not difficult to show that

$$\sigma_k^*(n) = \prod_{i=1}^{r}(p_i^{k\alpha_i} + 1).$$

Finally, the "core of n" is defined by

$$\gamma(n) = \prod_{i=1}^{r} p_i,$$

i.e. the greatest squarefree divisor of n.

Some inequalities

1. As we claimed, the following inequality is true:

$$\varphi_k(n) \leq \psi_k(n) \leq \sigma_k(n), \quad n \geq 1,\ k \geq 1 \tag{1}$$

The involved functions being multiplicative (i.e. they all satisfy the functional equation $f(ab) = f(a)f(b)$ for $(a, b) = 1$), it is sufficient to consider prime powers $n = p^\alpha$ (p prime, $\alpha \geq 1$). Then (1) becomes

$$p^{k\alpha} - 1 \leq p^{k\alpha} + 1 \leq (p^{k(\alpha+1)} - 1)/(p^k - 1),$$

which is obvious.

2. By considering the fraction

$$\varphi_k(n)/\psi_k(n) = \prod_{p|n}(1 - p^{-k})/(1 + p^{-k})$$

and using the decreasing function

$$f(x) = (1 - x)/(1 + x), \quad x \in (0, 1),$$

with $1/p \leq 1/2$ for n even and $1/p \leq 1/3$ for n odd, we immediately can deduce:

$$\psi_k(n) \leq \left(\frac{2^k + 1}{2^k - 1}\right)^{\omega(n)} \varphi_k(n),\ \text{for } n \text{ even};\ \leq \left(\frac{3^k + 1}{3^k - 1}\right)^{\omega(n)} \cdot \varphi_k(n),\ \text{for } n \text{ odd} \tag{2}$$

where $r = \omega(n)$ denotes the number of distinct prime factors of n. For $k = 1$ this means that

$$\psi(n) \leq 3^{\omega(n)}\varphi(n),\ n \text{ even};\ \leq 2^{\omega(n)}\varphi(n),\ n \text{ odd} \tag{3}$$

with equality only when $n = 2^m$ and $n = 3^m$, respectively. In the same way, the equality

$$\varphi_k(n)\psi_k(n)/n^{2k} = \prod_{p|n}(1 - p^{-2k})$$

combined with Euler's formula

$$\zeta(s) = \prod_{p\,prime}(1 - p^{-s})^{-1},$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

represents Riemann's zeta function ([1], [3], [7], [21]), yields the relations

$$(\zeta(2k))^{-1} < \varphi_k(n)\psi_k(n)/n^{2k} < 1. \tag{4}$$

3. We now prove two inequalities for the sum and difference, of φ and ψ, respectively. Firstly,

$$\varphi_k(n) + \psi_k(n) \geq 2n^k \tag{5}$$

This trivially follows from the algebraic inequality

$$\prod_{i=1}^{r}(1+x_i) + \prod_{i=1}^{r}(1-x_i) \geq 2$$

for $0 < x_i < 1$, $i = \overline{1,r}$ (selecting $x_i = 1/p_i$). We omit the (simple) proof of this inequality, which easily can be done e.g. by induction. A similar algebraic inequality is

$$\prod_{i=1}^{r}(y_i^k + 1) - \prod_{i=1}^{r}(y_i - 1)^k \geq 2^{kr}$$

of $y_i \geq 2$ are real numbers ($i = \overline{1,r}$) and $k, r \geq 1$ are natural numbers. This can be proved e.g. by induction with respect to r. Applying it for $y_i = p_i$, we get

$$\psi_k(n) - (\varphi(n))^k \geq \left(\frac{n}{\gamma(n)} \cdot 2^{\omega(n)}\right)^k \geq 2^{k\omega(n)} \tag{6}$$

where $\gamma(n) = \prod_{i=1}^{r} p_i \leq n$ denotes the core of n. As a consequence, we have:

$$\psi(n) \geq \varphi(n) + 2^{\omega(n)} \tag{7}$$

with equality for $n = p^\alpha$ (prime power).

Lastly we quote the inequality ([11])

$$\psi_k(n) \geq 2^{\omega(n)} \cdot n^{k/2} \tag{8}$$

which can be proved by an application of the arithmetic mean-geometric mean inequality. In section 8 we shall obtain a stronger form of this inequality.

Some limits

1. By (1) we have $\psi(n)/\varphi(n) \geq 1$. If p is a prime number,

$$\psi(p)/\varphi(p) = (p+1)/(p-1) \to 1$$

as $p \to \infty$. On the other hand, if p_i denotes the ith prime number,

$$\psi(p_1 \dots p_r)/\varphi(p_1 \dots p_r) = \prod_{i=1}^{r}(1+p_i^{-1})/(1-p_i^{-1}) > \prod_{i=1}^{r}(1+p_i^{-1}) \to \infty$$

if $r \to \infty$, as it is well-known ([1], [7]). Thus,

$$\liminf \psi(n)/\varphi(n) = 1, \quad \limsup \psi(n)/\varphi(n) = \infty \tag{9}$$

An analogous argument shows that

$$\liminf \psi(n)/n = 1, \quad \limsup \psi(n)/n = \infty \tag{10}$$

Using the same idea, by $\psi(p)\varphi(p)/p^2 = (p-1)(p+1)/p^2 \to 1$ and

$$\psi(p_1 \dots p_r)\varphi(p_1 \dots p_r)/(p_1 \dots p_r)^2 = \prod_{i=1}^{r}(1-p_i^{-2}) \to \prod_{i=1}^{\infty}(1-p_i^{-2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

in view of (4) we obtain

$$\liminf \psi(n)\varphi(n)/n^2 = \frac{6}{\pi^2}, \quad \limsup \psi(n)\varphi(n)/n^2 = 1 \tag{11}$$

2. Clearly, $\psi(n) \geq n+1$ (with equality for $n =$ prime). On the other hand if

$$n = \prod_{i=1}^{r} p_i^{\alpha_i},$$

then

$$\psi(n) = n\prod_{i=1}^{r}(1+p_i^{-1}),$$

so by $p_i \geq i+1$ $(i = \overline{1,r})$, we can write:

$$\psi(n) \leq n\left(1+\frac{1}{2}\right)\dots\left(1+\frac{1}{r+1}\right) = n(r+2)/2 = n(\omega(n)+2/2$$

Remarking that for $n = p_i p_{i+1}$ (p_i denotes the ith prime) we have

$$\frac{\psi(n)}{n\omega(n)} = (p_i + 1)(p_{i+1} + 1)/2p_i p_{i+1} \to 1/2$$

as $i \to \infty$, the above inequality leads to

$$\limsup \frac{\psi(n)}{n\omega(n)} = \frac{1}{2} \tag{12}$$

For the lim inf we note that by Mertens' formula ([1], [7])

$$\prod_{i=1}^{r}(1 + p_i^{-1}) \sim C \cdot \log\log n \quad (r \to \infty),$$

where $n = p_1 \dots p_r$ and p_i is the ith prime number (C a positive constant), so we immediately obtain

$$\liminf \frac{\psi(n)}{n\omega(n)} = 0 \tag{13}$$

Let $d(n)$ denote the number of all divisors of n. Then taking the sequence (2^m) and using the relation $\omega(n) \leq d(n)$, and (12) imply

$$\liminf \frac{\psi(n)}{nd(n)} = 0, \quad \limsup \frac{\psi(n)}{nd(n)} = \frac{1}{2} \tag{14}$$

3. In connection with the difference $\psi(n+1) - \psi(n)$, we can prove:

$$\liminf(\psi(n+1) - \psi(n)) = -\infty, \quad \limsup(\psi(n+1) - \psi(n)) = \infty \tag{15}$$

More generally, we shall prove that for each $A > 0$ there exists n such that $\psi(n-1) - \psi(n) > A$ and $\psi(n+1) - \psi(n) > A$. Indeed, let $n = 2^k m$, ($k \geq 1$, m odd) be an even number. The multiplicativity of ψ implies

$$\psi(n) = \psi(m)\psi(2^k) \geq m \cdot 2^k \cdot \frac{3}{2} = \frac{3}{2}n,$$

thus $\psi(n) \geq \frac{3}{2}n$ for n even. Now let p be an odd prime. Then $\psi(p) = p + 1$, so

$$\psi(p-1) - \psi(p) \geq \frac{3}{2}(p-1) - (p+1) = \frac{p-5}{2}$$

and

$$\psi(p+1) - \psi(p) \geq \frac{3}{2}(p+1) - (p+1) = \frac{p+1}{2}.$$

Selecting e.g. $p > A$, the above assertion is proved. Of course, (15) follows also from the equations

$$\liminf \frac{\psi(n+1)}{\psi(n)} = 0, \quad \limsup \frac{\psi(n+1)}{\psi(n)} = \infty \tag{16}$$

which are more difficult to prove. In order to obtain a more general result, we quote a theorem of A. Schinzel and W. Sierpinski [15]: For each $n \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n > 1$, such that $\varphi(n)/\varphi(n-1) > m$ and $\varphi(n)/\varphi(n+1) > m$. By applying this result, for $m = 8M$, the inequality (4) ($k = 1$ and $6/\pi^2 > 1/2$) one gets:

$$\psi(n-1)/\psi(n) > ((n-1)^2/2n^2)\varphi(n)/\varphi(n-1) > 1/8\varphi(n)/\varphi(n-1) > M$$

and

$$\psi(n+1)/\psi(n) > ((n+1)^2/2n^2)\varphi(n)/\varphi(n+1) > 4M.$$

Thus we have proved that for each M there exists $n > 1$ with:

$$\frac{\psi(n-1)}{\psi(n)} > M, \quad \frac{\psi(n+1)}{\psi(n)} > M \tag{17}$$

This clearly yields (16).

Density problems

We shall study the density of the sequences $(\psi(n)/n)$ and $(\psi(n)/\varphi(n))$.

$$\text{The sequence } (\psi(n)/n) \text{ is everywhere dense in } (1, \infty) \tag{18}$$

$$\text{The sequence } (\psi(n)/\varphi(n)) \text{ is everywhere dense in } (1, \infty) \tag{19}$$

1. In order to prove (18), let $1 < a < b$ be given real numbers and let p_i be the ith prime number. Select k in such a way that $1 + \dfrac{1}{p_k} < \dfrac{b}{a}$ and let l be chosen such that:

$$(1 + 1/p_k) \dots (1 + 1/p_{k+l}) \geq b, \quad (1 + 1/p_k) \dots (1 + 1/p_{k+l-1}) < b$$

(A such l exists because of $(1 + 1/p_k) \dots (1 + 1/p_{k+m}) \to 0$ as $m \to \infty$). Define

$$\alpha = (1 + 1/p_k) \dots (1 + 1/p_{k+l-1}).$$

Then $\alpha(1+1/p_{k+l}) \geq b$, hence

$$\alpha \geq b/(1+1/p_{k+l}) > b/(1+1/p_k) > a.$$

Thus $a < \alpha < b$. On the other hand,

$$\psi(p_k p_{k+1} \dots p_{k+l-1})/p_k p_{k+1} \dots p_{k+l-1} = (1+1/p_k) \dots (1+1/p_{k+l-1}) = \alpha$$

finishing the proof of (18).

2. For (19) we shall use a method of B.S.K.R. Somayajulu [16], who proved the following assertion: Let (a_n) be a positive, strictly decreasing sequence with $a_n \to 0$ $(n \to \infty)$ (in notation $0 < a_n \downarrow 0$) and suppose that the series $\sum a_n$ is divergent. Then for each $s > 0$ there exists an infinite subseries of $\sum a_n$, which converges to s.

Let p_n be the nth prime and choose u_n with $1 + u_n = (p_n + 1)/(p_n - 1)$. Then clearly $0 < u_n \downarrow 0$ and

$$\prod_{n=1}^{\infty}(1+u_n) = \infty$$

(see the proof of (9)). Select now in the above lemma $a_n = \log(1+u_n)$ in order to get the proposition: For each $\delta > 1$ there exists $(u_{n_r})_r$ such that

$$\prod_{r=1}^{\infty}(1+u_{n_r}) = \delta$$

Take $m_r = p_{n_1} \dots p_{n_r}$, when

$$\psi(m_r)/\varphi(m_r) = \prod_{k=1}^{r}(1+u_{n_k}) \to \delta$$

as $r \to \infty$. This completes the proof of (19).

Divisibility properties

1. We start with a simple but useful property:

$$a|b \;\Rightarrow\; \psi(a)|\psi(b) \tag{20}$$

Indeed, let

$$a = \prod p^{\alpha}, \quad b = \prod p^{\alpha+\alpha'} \prod q^{\beta},$$

be the prime factorizations of a and b. (For simplicity we do not use indices), p, q being primes, $(p, q) = 1$. Then

$$\psi(b)/\psi(a) = \prod p^{\alpha'} \prod (q^{\beta} + q^{\beta-1}) \in \mathbb{N}$$

thus $\psi(a)$ divides $\psi(b)$.

2. Next, we prove that:

$$a|b \;\Rightarrow\; \frac{\psi(a)}{a} \leq \frac{\psi(b)}{b} \tag{21}$$

We prove a more general relation, namely

$$\psi(A)\psi(B) \geq \psi(AB) \geq A\psi(B) \tag{22}$$

Let

$$S = \prod p^{\alpha} \prod q^{\beta}, \quad B = \prod p^{\alpha'} \prod t^{\gamma}$$

with $(p, q) = (p, t) = (q, t) = 1$, be the prime factorizations of A and B. Then

$$\psi(AB)/\psi(B) = \prod p^{\alpha} \prod q^{\beta} \prod (1 + 1/q) \geq A$$

with equality if all $\beta = 0$, i.e. if for each prime $p|A$ we have also $p|B$. The left-side inequality follows by the same lines.

Write now $b = qa$, $q \geq 1$ and apply (22):

$$\psi(b) = \psi(qa) \geq q\psi(a) = \frac{b}{a}\psi(a)$$

yielding (21).

3. We note that the following inequalities also hold true:

$$\sigma(A)\sigma(B) \geq \sigma(AB) \geq A\sigma(B) \tag{23}$$

$$\varphi(A)\varphi(B) \leq \varphi(AB) \leq A\varphi(B) \tag{24}$$

having the consequences

$$a|b \;\Rightarrow\; \frac{\sigma(a)}{a} \leq \frac{\sigma(b)}{b} \tag{25}$$

$$a|b \;\Rightarrow\; \frac{\varphi(a)}{a} \geq \frac{\varphi(b)}{b} \tag{26}$$

As an interesting consequence of (20), we can state:

$$\psi((m,n))|(\psi(m),\psi(n)) \tag{27}$$

$$[\psi(m),\psi(n)]|\psi([m,n]) \tag{28}$$

where (u,v) and $[u,v]$ are the g.c.d. and l.c.m., respectively, of a,b.

Power and composite functions

1. For the function $\sigma(n)^{\varphi(n)}$ we have proved [12] that $\sigma(n)^{\varphi(n)} < n^n$ for all $n \geq 2$, which in the light of (1) yields

$$\psi(n)^{\varphi(n)} < n^n, \quad n \geq 2 \tag{29}$$

On the other direction, one can prove: If all prime factors of n are ≥ 5:

$$\varphi(n)^{\psi(n)} > n^n \tag{30}$$

First we prove (30) for prime powers $n = p^\alpha$. Relations (22) and (24) imply at once

$$p^{\alpha-1}(p+1) \leq \psi(p^\alpha) \leq (p+1)^\alpha$$

and

$$(p-1)^\alpha \leq \varphi(p^\alpha) \leq p^{\alpha-1}(p-1).$$

Therefore

$$\varphi(p^\alpha)^{\psi(p^\alpha)} \geq (p-1)^{\alpha p^{\alpha-1}(p+1)} > p^{\alpha p^\alpha}$$

iff $(p-1)^{p+1} > p^p$. This is true for all $p \geq 5$. Indeed, we have to show that

$$p-1 > (1+1/(p-1))^{p-1}(1+1/(p-1)).$$

Here $(1+1/(p-1))^{p-1} < e < 3$ and $3p/(p-1) < p-1$ iff $p^2+1 > 5p$, which is true for $p \geq 5$.

Next we remark that if (30) is valid for $n = a$ and $n = b$ with $(a,b) = 1$, then it is valid also for $n = ab$:

$$\varphi(ab)^{\psi(ab)} = ((\varphi(a)^{\psi(a)})^{\psi(b)}(\varphi(b)^{\psi(b)})^{\psi(a)}) > a^{a\psi(b)}b^{b\psi(a)} > a^{ab} \cdot b^{ab} = (ab)^{ab}.$$

This remark (and the above proof for $n = p^\alpha$) concludes the proof of (30).

2. The composite function $\varphi(\psi(n))$ behaves very irregularity, but we are able to prove that

$$\varphi\left(n\left[\frac{\psi(n)}{n}\right]\right) \leq n \tag{31}$$

where $[a]$ denotes the integer part of real number a. More generally,

$$\varphi_k\left(n\left[\frac{(\psi_k(n))^{1/k}}{n}\right]\right) \leq n^k \tag{32}$$

By (4) and (24) we have

$$\varphi_k\left(n\left[\frac{(\psi_k(n))^{1/k}}{n}\right]\right) \leq \frac{\psi_k(n)}{n^k}\varphi_k(n) \leq n^k,$$

proving (32).

We can use a unified method to prove that

$$\liminf \frac{\varphi(\psi(n))}{n} = 0 \tag{33}$$

$$\limsup \frac{\psi(\varphi(n))}{n} = \infty \tag{34}$$

$$\limsup \frac{\psi(\psi(n))}{\psi(n)} = \infty \tag{35}$$

Dirichlet's theorem on arithmetical progressions ([1], [3], [7]) asserts that if $(a, b) = 1$, then there are infinitely many primes of the form $ak + b$. Let p_i be the ith prime and consider a prime p of the form $p \equiv -1 \pmod{p_1 \dots p_r}$. Since $p_1 \dots p_r | (p+1)$, by (26) we can write:

$$\varphi(p_1 \dots p_r)/p_1 \dots p_r \geq \varphi(p+1)/(p+1) = \varphi(\psi(p))/p + 1,$$

thus

$$\varphi(\psi(p))/p \leq \frac{p+1}{p}\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_r}\right) \to 0$$

if $r \to \infty$, proving (33). By (21), we deduce in the same manner:

$$\frac{\psi(\psi(p))}{\psi(p)} \geq \left(1 + \frac{1}{p_1}\right) \dots \left(1 + \frac{1}{p_r}\right) \to \infty$$

if $r \to \infty$, getting (35).

Let q be a prime of the form $q \equiv 1 \pmod{p_1 \dots p_r}$. Then (21) shows that

$$\psi(p_1 \dots p_r)/p_1 \dots p_r \leq \psi(p-1)/(p-1) = \psi(\varphi(p))/(p-1),$$

yielding (34).

3. We note that A. Makowski and A. Schinzel [10] have proved that

$$\liminf \sigma(\sigma(n))/n = 1, \quad \limsup \varphi(\sigma(n))/n = \infty,$$

which in base of (1) trivially imply

$$\limsup \frac{\psi(\sigma(n))}{n} = \infty, \quad \liminf \frac{\psi(\sigma(n))}{n} \leq 1 \tag{36}$$

4. In connection with the function $\psi(\psi(n))/n$, (35) shows that

$$\limsup \psi(\psi(n))/n = \infty.$$

We now prove that

$$\liminf \frac{\psi(\psi(n))}{n} = \frac{3}{2} \tag{37}$$

if we assume the existence of an infinity of Mersenne primes ([1], [7]), i.e. primes of the form $2^m - 1$. First remark that for $n \geq 3$, $\psi(n)$ is always even so (20) and (21) give us

$$\psi(\psi(n)) \geq \frac{3}{2}\psi(n) \tag{38}$$

Using (10), we get the $\liminf \geq 3/2$. Let n be a prime of the form $2^m - 1$. Then

$$\psi(n) = 2^m$$

and

$$\psi(\psi(n))/n = 3/2 \cdot 2^m/(2^m - 1) \to 3/2$$

if $m \to \infty$, proving (37).

5. For the function $\psi(\varphi(n))$ we can prove an interesting result, for which we need some preliminaries. In certain cases the inequality (22) can be improved. First let us introduce the following notation. Let $a \wedge b$ denote the property: there exists at least a prime $q|a$, with $q \nmid b$. We claim

$$\psi(ab) \geq (a+1)\psi(b), \text{ if } a \wedge b \tag{39}$$

Let

$$a = \prod p^{\alpha} \prod q^{\beta}, \quad b = \prod p^{\alpha'} \prod t^{\gamma}, \quad ((p,q) = (p,t) = (q,t) = 1)$$

be the prime factorizations of a and b, where $\alpha, \alpha', \gamma \geq 0$; $\beta \geq 1$ (i.e. $a \wedge b$). Then, by the definition of ψ, we get

$$\psi(ab)/\psi(b) = a \prod (1 + 1/q).$$

Here

$$\prod (1 + 1/q) \geq 1 + 1/\left(\prod q\right) \geq 1 + 1/\left(\prod q^{\beta}\right) \geq 1 + 1/\left(\prod q^{\beta} \prod p^{\alpha}\right) = 1 + 1/a.$$

Therefore we obtain (39). The second lemma we need below says that: If n is an even number and $\psi(\varphi(m)) \geq m$, where m denotes the greatest odd divisor of n, then

$$\psi(\varphi(n)) \geq \frac{n}{2} \tag{40}$$

To prove this, let $n = 2^k \cdot m$. Then $\varphi(n) = 2^{k-1}\varphi(m)$ and by taking into account (22) and the assumed condition, we find that

$$\psi(\varphi(n)) \geq 2^{k-1}m = n/2.$$

We are now in a position to formulate the above mentioned result: Let S denote the set of odd numbers m whose prime factors $p_1, \ldots, p_r$ satisfy the following conditions:

$$(p_2 - 1) \wedge (p_1 - 1),\ (p_3 - 1) \wedge (p_1 - 1)(p_2 - 1), \ldots, (p_r - 1) \wedge (p_1 - 1) \ldots (p_{r-1} - 1).$$

Then $\psi(\varphi(m)) \geq m$ for $m \in S$. If n is even number and $m \in S$ (where m denotes the greatest odd divisor of n), then

$$\psi(\varphi(n)) \geq \frac{n}{2} \tag{41}$$

This can be proved immediately, using (39) (the first part by induction) and (40).

We conjecture that $\psi(\varphi(m)) \geq m$ for all off m, but this seems to be very difficult (see 10.).

Dirichlet series

1. Denote

$$D(f,s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

the Dirichlet series of the arithmetical function f. It is well-known that

$$D(f*g,s)=D(f,s)D(g,s),$$

where

$$(f*g)(n)=\sum_{d|n}f(d)g\left(\frac{n}{d}\right)$$

is the Dirichlet product of f and g. (Suppose that the considered series are absolute convergent, see [20]). Denoting $E_k(n)=n^k$, $I(n)=1$, one has

$$\sigma_k(n)=\sum_{i|n}E_k(i)I\left(\frac{n}{i}\right).$$

Since

$$D(E_k,s)=\sum_{n=1}^{\infty}\frac{n^k}{n^s}=\zeta(s-k)\quad(Re\,s>k+1)$$

and

$$D(I,s)=\zeta(s),$$

the zeta function of Riemann, one gets

$$D(\sigma_k,s)=\zeta(s-k)\zeta(s) \tag{42}$$

For $k=0$ this gives

$$D(d,s)=\zeta^2(s) \tag{43}$$

Similarly, from $\varphi_k(n)=n^k\sum_{i|n}\mu(i)/i^k$ (μ denotes the Möbius function [1], [3], [7], [22]) we obtain

$$D(\varphi_k,s)=\frac{\zeta(s-k)}{\zeta(s)}\quad(Re\,s>k) \tag{44}$$

2. These results follow also from the Euler formula

$$\zeta(s)=\prod_p(1-p^{-s})^{-1}$$

and a theorem on multiplicative arithmetical functions f ([1], [7]): If f is multiplicative, then

$$D(f,s) = \prod_p \{1 + f(p)p^{-s} + f(p^2)p^{-2s} + \ldots\} \tag{45}$$

Let $\sigma_k^*(n)$ be the sum of kth powers of unitary divisors of n (see 1.). Then by (45),

$$D(\sigma_k^*, s) = \prod_p \{1 + \sigma_k^*(p)p^{-s} + \sigma_k^*(p^2)p^{-2s} + \ldots\} =$$

$$= \prod_p (1 + (1+p^k)p^{-s} + (1+p^{2k})p^{-2s} + \ldots = \prod_p (1 - p^{-(2s-k)})/(1-p^{-s})(1-p^{-(s-k)} =$$

$$= \zeta(s)\zeta(s-k)/\zeta(2s-k),$$

by Euler's formula. Thus

$$D(\sigma_k^*, s) = \frac{\zeta(s)\zeta(s-k)}{\zeta(2s-k)}, \quad Re\, s > k \tag{46}$$

A similar argument shows that

$$D(\psi_k, s) = \prod_p \{1 + (1+p^k)p^{-s} + (p^k + p^{2k})p^{-2s} + \ldots\} =$$

$$= \prod_p (1 - p^{-2(s-k+1)})/(1-p^{-(s-k)})(1-p^{-(s-k+1)},$$

so

$$D(\psi_k, s) = \frac{\zeta(s-k+1)\zeta(s-k)}{\zeta(2(s-k+1))}, \quad Re\, s > k \tag{47}$$

3. Dirichlet series can be used well to obtain interesting relations. Let us consider first the identity

$$\frac{\zeta(s-k)}{\zeta(s)} \cdot \zeta^2(s) = \zeta(s-k)\zeta(s)$$

which in view of (42), (43), (44) has the form

$$D(\varphi_k * d, s) = D(\sigma_k, s).$$

It is well-known - by the uniqueness theorem of Dirichlet series ([20]) - that this implies $\varphi_k * d = \sigma_k$, i.e.

$$\sum_{i|n} \varphi_k(i) d\left(\frac{n}{i}\right) = \sigma_k(n) \tag{48}$$

Writing the identity

$$\frac{\zeta(s-k)}{\zeta(s)}\zeta(s-k)\zeta(s)=\zeta^2(s-k),$$

we obtain

$$\sum_{i|n}\varphi_k(i)\sigma_k\left(\frac{n}{i}\right)=n^k d(n).$$

By

$$\frac{\zeta^2(s)}{\zeta(2s)}\cdot\frac{\zeta(s-1)}{\zeta(s)}=\frac{\zeta(s)\zeta(s-1)}{\zeta(2s)},$$

taking into account of $D(2^\omega,s)=\zeta^2(s)/\zeta(2s)$ (see [21]), via (44), (47) we get

$$\psi(n)=\sum_{i|n}2^{\omega(i)}\varphi\left(\frac{n}{i}\right) \tag{49}$$

Using $D(\sigma d,s)=\zeta^2(s)\zeta^2(s-10/\zeta(2s-1)$ ([21]), we get the identity:

$$\sum_{i|n}\sigma(i)\sigma^*\left(\frac{n}{i}\right)=\sigma(n)d(n) \tag{50}$$

For an application of (49) notice that $i|n\ \Rightarrow\ 2^{\omega(i)}\le 2^{\omega(n)}$. For a such S we obtain

$$\psi(n)\le\sum_{i|n,i\in S}\varphi(i)+2^{\omega(n)}\left(n-\sum_{i|n,i\in S}\varphi(i)\right) \tag{51}$$

Let $S=\{n\}$. Then we have:

$$\psi(n)\le\varphi(n)+2^{\omega(n)}(n-\varphi(n)) \tag{52}$$

with equality only if n is a prime (compare with (7)).

Asymptotic results

1. Relation (49) informs us on a connection between ω,ψ,φ. However it is more convenient to use the following expression:

$$\psi(n)=\sum_{d^2k=n}\mu(d)\sigma(k) \tag{53}$$

where, as usual, μ denotes the Möbius function. This can be proved easily if we observe that the sum on the right side is multiplicative, and a simple computation for $n = p^\alpha$ (prime powers) yields the equality.

We now are interested for an asymptotic formula with remainder term for $\sum_{n\le x}\psi(n)$. On base of (53) this sum may be written as

$$\sum_{d^2k\le x}\mu(d)\sigma(k) = \sum_{d\le x^{1/2}}\mu(d)\sum_{k\le x/d^2}\sigma(k).$$

It is well-known that ([1], [7])

$$\sum_{k\le m}\sigma(k) = \frac{\pi^2}{12}m^2 + O(m\log m),$$

thus

$$\sum_{n\le x}\psi(n) = \frac{\pi^2x^2}{12}\left(\sum_{d=1}^{\infty}\mu(d)d^{-4} + O\left(\sum_{d>x^{1/2}}d^{-4}\right)\right) + O(x\log x) =$$

$$= \frac{\pi^2x^2}{12}\cdot\frac{1}{\zeta(4)} + O(x^{1/2}) + O(x\log x) = \frac{15}{2\pi^2}x^2 + O(x\log x)$$

(because of $\zeta(4) = \pi^4/90$ and the known relation $D(\mu, s) = 1/\zeta(s)$). Thus:

$$\sum_{n\le x}\psi(n) = \frac{15}{2\pi^2}x^2 + O(x\log x) \tag{54}$$

Remark. Using the Weyl exponential sums ([26]) it can be shown that

$$\sum_{n\le m}\sigma(n) = \frac{\pi^2}{12}m^2 + O(m\log^{2/3}m),$$

implying the refinement

$$\sum_{n\le x}\psi(n) = \frac{15}{2\pi^2}x^2 + O(x\log^{2/3}x) \tag{54$'$}$$

For $k > 1$ we can prove a similar formula, but with a weaker remainder term. Namely, we have

$$\sum_{n\le x}\psi_k(n) = x^{k+1}\frac{\zeta(k+1)}{(k+1)\zeta(2k+2)} + O(x^k), \quad k > 1 \tag{55}$$

The proof runs on the same lines, based on

$$\varphi_k(n) = \sum_{d^2l=n}\mu(d)\sigma_k(l)$$

and

$$\sum_{n\leq x}\sigma_k(n)=\frac{\zeta(k+1)}{k+1}x^{k+1}+O(x^k)\text{ for }k>1$$

([7], [9]). As a corollary,

$$\sum_{n\leq x}\psi_k(n)\sim\frac{\zeta(k+1)}{(k+1)\zeta(2k+2)}\quad(x\to\infty),\ k\geq 1 \tag{56}$$

2. Recall the definition of "normal order of magnitude" ([7]). Let P be a property in the set of natural numbers and set $a_p(n)=1$ if n has the property P; =0, contrary. Let

$$A_p(x)=\sum_{n\leq x}a_p(n).$$

If $a_p(x)\sim x$ for $x\to\infty$ we say that the property P holds for almost all natural numbers. We say that the normal order of magnitude of the arithmetical function $f(n)$ is the function $g(n)$, if for each $\varepsilon>0$, the inequality $|f(n)-g(n)|<\varepsilon g(n)$ holds true for almost all natural numbers n.

Our aim is to prove the following theorem: The nomal order of magnitude of $\log\psi_k(n)d(n)/\sigma_k(n)$ is

$$\log 2\log\log n\quad(k\geq 1) \tag{57}$$

First we shall prove the double inequality:

$$2^{\omega(n)}\leq\psi_k(n)d(n)/\sigma_k(n)\leq 2^{\Omega(n)} \tag{58}$$

where $\omega(n)$ and $\Omega(n)$ denote the distinct and total number - of prime factors of n, respectively.

The right side of (58) is quite obvious from (1) and $d(n)\leq 2^{\Omega(n)}$ (this follows from $\alpha+1\leq 2^\alpha$). Apply now the Cauchy mean-value theorem for $f(x)=x^{\alpha+1}$, $g(x)=x^2$ on $x\in[1,a]$:

$$\frac{a^{\alpha+1}-1}{a^2-1}=\frac{\alpha+1}{2}\xi^{\alpha-1}\leq\frac{\alpha+1}{2}a^{\alpha-1}$$

for $\alpha,a\geq 1$. Take $a=p^k$, which transforms the above inequality into

$$2\frac{p^{k(\alpha+1)}-1}{(p^k-1)(\alpha+1)}\leq p^{k\alpha}\left(1+\frac{1}{p^k}\right),$$

i.e. the left side of (58) for $n = p^\alpha$. This finishes the proof of this inequality for all n, because the involved arithmetical functions all are multiplicative.

Remark. It is known ([9]) that $\sigma_k(n) \geq n^{k/2}d(n)$, so the obtained inequality implies (with a new proof) relation (8). Combining (58) with (3) we get the interesting inequality $\sigma(n) \leq \varphi(n)d(n)$, for n odd.

We now recall a result of G.H. Hardy and S. Ramanujan ([1], [7], [9]) which says that the normal order of magnitudes of $\omega(n)$ and $\Omega(n)$ is $\log\log n$. Therefore, in the light of (58) one has:

$$(1-\varepsilon)\log\log n < \omega(n) \leq \frac{1}{\log 2}\log \psi_k(n)d(n)/\sigma_k(n) \leq \Omega(n) < (1+\varepsilon)\log\log n$$

for almost all n. This proves the validity of (57) (see also [13] for a stronger result).

3. We touch here also a problem related to the maximal order of magnitude ([1], [7]) of ψ. More precisely, we prove that:

There exists a constant $C_1 > 0$ such that

$$\psi(n) < C_1 n\log\log n \text{ for all } n \geq 2; \tag{59}$$

there exists a constant $C_2 > 0$ such that for infinitely many n one has

$$\psi(n) > C_2 n\log\log n. \tag{60}$$

(59) is a consequence of (1) and of Gronwall's theorem ([7], [9]):

$$\sigma(n) < C_1 n\log\log n.$$

For (60) we use a theorem of Mertens ([7], [9]):

$$\prod_{p\leq m}(1+1/p) \sim C\log m \quad (m\to\infty),$$

where $C > 0$ is a constant and p runs through all primes below m. Let $n = p_1 \dots p_r$, with p_i the ith prime. The prime number theorem (of Hadamard and de la Vallée Poussin) ([7], [9]) asserts that $n \sim e^r$ $(r\to\infty)$, i.e. $\log n \sim r$. By $p_r \sim r\log r$ we have $\log p_r \sim \log\log n$, so

$$\prod_{i\leq r}(1+1/p_i) \sim C\log\log n.$$

For sufficiently large r we obtain $\psi(n)/n > C_2 \log\log n$ with suitable positive constant C_2.

Remark. A. Ivić [8] has proved that for $n \geq 31$,

$$\prod(1 + 1/p) < \frac{28}{15} \log\log n,$$

which in our terminology means that

$$\psi(n) < \frac{28}{15} n \log\log n \text{ for } n \geq 31.$$

4. Lastly we deal with the representation of ψ by Ramanujan sums ([1], [7], [9]). Remark that $\psi(n) = \varphi_2(n)/\varphi(n)$, where $\varphi_2(n)$ is Jordan's arithmetical function ($k = 2$). Now

$$\varphi_2(n) = \sum_{t|n} \mu(t)/t^2,$$

thus

$$\frac{\psi(n)\varphi(n)}{n^2} = \sum_{t|n} \mu(t)/t^2 = \sum_{t=1}^{\infty} (t)/t^3 \sum_{m|t} c_m(n)$$

with

$$c_m(n) = \sum_{(k,m)=1} \exp(2\pi i n k/m)$$

the Ramanujan arithmetical function. Thus

$$\psi(n)\varphi(n)/n^2 = \sum_{m=1}^{\infty} c_m(n)/m^3 \sum_{d=1}^{\infty} \mu(dm)/d^3 =$$

$$= \sum_{m=1}^{\infty} c_m(n)\mu(m)/m^{k+1} \sum_{(d,m)=1} \mu(d)/d^3.$$

The function $f : d \to \mu(d)$ for $(d, m) = 1$; 0 in other cases,, is evidently multiplicative, so we can apply (45):

$$\sum_{(d,m)=1} \mu(d)/d^3 = \prod_p \left(\sum_{i=0}^{\infty} f(p^i)/p^{3i} \right) = \prod_{p \nmid m} (1 - p^{-3}) = \frac{1}{\zeta(3)} \prod_{p|m} (1 - p^{-k-1})^{-1}.$$

Therefore

$$\psi(n) = \frac{1}{\zeta(3)} \cdot \frac{n^2}{\varphi(n)} \sum_{m=1}^{\infty} \frac{c_m(n)\mu(m)}{\varphi_3(m)} \tag{61}$$

(where $\varphi_3(m) = m^3 \prod_{p|m} (1 - p^{-3})$ is the 3-order Jordan function) which is the desired representation.

Some diophantine equations

1. The numbers n satisfying the equation $\sigma(n) = kn$ are called k-fold perfect numbers ($k \geq 2$). It is not known ([6]) if there exist odd k-fold perfect numbers. However we can study the equation $\psi(n) = kn$. We shall use a notion and method of Ch. Wall [23]. We say that n is ω-multiple of m if $m|n$ and the set of prime factors of m and n are identical. We need a simple result on $\psi(n)/n$: If m and n are squarefree numbers and $\psi(n)/n = \psi(m)/m$, then

$$m = n \tag{62}$$

Without loss of generality we may suppose $(m, n) = 1$ and $m, n > 1$. Let $m \neq n$, $n = p_1 \dots p_k$ ($p_1 < \dots < p_k$), $m = q_1 \dots q_j$ ($q_1 < \dots < q_j$). Then the assumed equality has the form

$$n(1 + q_1) \dots (1 + q_j) = m(1 + p_1) \dots (1 + p_k).$$

Since $p_k|m$, the relation $p_k|(1 + p_1) \dots (1 + p_{k-1})(1 + p_k)$ implies $p_k|(1 + p_i)$ for some i ($i = \overline{1, k}$). Here $1+p_1 < \dots < 1+p_{k-1} < 1+p_k$, so we must have $p_k|(1+p_{k-1})$. This happens only when $k = 2$, $p_1 = 2$, $p_2 = 3$; $j = 2$, $q_1 = 2$, $q_2 = 3$. In this case $(n, m) = 6 \neq 1$, a contradiction. Thus $k = j$ and $p_k = q_j$, etc.

Let us suppose now that the least solution n_k of $\psi(n) = kn$ is a squarefree number. We shall prove that

$$\text{all solutions of our equation are the } \omega\text{-multiples of } n_k. \tag{63}$$

Indeed, if n is ω-multiple of n_k, then clearly

$$\psi(n)/n = \psi(n_k)/n_k = k,$$

by the definition of ψ. Conversely, if n if a solution, set $m = \gamma(n)$ - the greatest squarefree divisor of n. Then

$$\psi(n)/n = \psi(m)/m = k = \psi(n_k)/n_k$$

By (62) $m = n_k$, i.e. n is ω-multiple of n_k.

Remark. As an application we obtain that all solutions of the equation $\psi(n) = \frac{9}{5}n$ have the form $2^a 5^b$ $(a, b \geq 1)$. (Choose $n_k = 10$, $k = 9/5$). For $k = 2$ we get the solutions

$$\{2^a 3^b : \ a, b \geq 1\}. \tag{64}$$

2. An interesting simple equation is $\psi(\psi(n)) = \frac{3}{2}\psi(n)$. As we have proved in (38) one has $\psi(\psi(n)) \geq \frac{3}{2}\psi(n)$ for all n. The proof given there shows that equality one has only when $\psi(n)$ has not other prime factors than 2, i.e. $\psi(n) = 2^m$. We prove that all solutions of this equation may be written as $n = p_1 \dots p_k$, with p_i $(i = \overline{1,k})$ distinct Mersenne primes. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

and write

$$\psi(n) = \prod_{i=1}^{r} p_i^{\alpha_i - 1}(p_i + 1).$$

Then $\psi(n) = 2^m$ iff $\alpha_i - 1 = 0$ (i.e. n is squarefree) and $p + i + 1 = 2^a$, thus $p_i = 2^a - 1 =$ Mersenne prime $(i = \overline{1,k})$.

Remarks. There are many other equations (solved or open problems, see 10.). As a sample, we give the equation $\psi(\psi(n)) = 2n$ which has the only solution $n = 3$ (see [14]). In [14] we have studied among others the equations

$$\psi(\sigma(n)) = 2n, \quad \sigma(\psi(n)) = 2n, \quad \psi(\sigma^*(n)) = 2n \pm 1, \text{ etc.}$$

Some open problems

1. (37) says that $\liminf \psi(\psi(n))/n = 3/2$. In the proof of this result we have accepted the existence of an infinity of Mersenne primes. However this conjecture about Mersenne prime numbers is not proved and is one of the most notorious open problems in Number Theory ([6]).

Could we prove (37) without this assumptions? (65)

2. In (36) we have obtained $\liminf \psi(\sigma(n))/n \leq 1$.

What is the value of this $\liminf$? (66)

3. The theorem in (40) is based on the inequality $\psi(\varphi(m)) \geq m$, where m is the greatest odd divisor of n.

Is it true the inequality $\psi(\varphi(m)) \geq m$ for all odd m? (67)

In view of (1), this problem is stronger than the conjecture of Makowski and Schinzel [10]: $\sigma(\varphi(m)) \geq m$ for all odd m. For results connected with this conjecture see also [2], [12].

We prove that

$$\sigma(mn) \geq \sigma(m)\psi(m) \text{ if } n \wedge m \tag{68}$$

(see (39)) i.e. there exists a prime factor $t|n$, $t \nmid m$.

Let

$$m = \prod p^{\alpha} \prod q^{\beta}, \quad n = \prod p^{\alpha'} \prod t^{\gamma}, \quad (\gamma \geq 1).$$

Then

$$\sigma(mn)/\sigma(m) = \prod_{p^{\alpha+\alpha'+1}-1} /(p^{\alpha+1}-1) \prod (t^{\gamma+1}-1)/(t-1).$$

The elementary inequalities

$$(p^{\alpha+\alpha'+1}-1)/(p^{\alpha+1}-1) \geq p^{\alpha'} \quad (\alpha, \alpha' \geq 0);$$

$$(t^{\gamma+1}-1)/(t-1) \geq t^{\gamma}(1+1/t) \quad (\gamma \geq 1)$$

imply at once (68).

Using (68) and the method of (41)

how could we attack the conjecture of Makowski and Schinzel? (69)

4. In [14] we have found all even solutions of the equations $\sigma(\psi(n)) = 2n$ and $\psi(\sigma(n)) = 2n$.

What are the odd solutions of these equations? (70)

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22 A note on certain arithmetic functions

In what follows, we will consider arithmetic functions $f : N \to N$, where $N = \{1, 2, \ldots\}$. Then f is called a multiplicative function (shortly: **m**-function) if $(a, b) = 1 \Rightarrow f(ab) = f(a)f(b)$. In the recent note [1] there are considered certain **m**-functions with the property

$$(a, b) = 1 \Rightarrow (f(a), f(b)) = 1. \tag{1}$$

We shall call such function as "relative-primality preserving function" (shortly: **r.p.**-function). In [1] an **m**-function, which is also **r.p.**, is called 1-multiplicative. However, this name is not justified, as there are many nations of higher-order multiplicative functions in the literature (see e.g. [2]). Let (x, y) denote the g.c.d. of x and y. A function f with the property

$$(f(a), f(b)) = f((a, b)) \text{ for all } a, b \tag{2}$$

will be called as "greatest common divisor preserving function" (shortly: **g.p.**-function). If

$$a|b \Rightarrow f(a)|f(b) \text{ for all } a, b \tag{3}$$

then we shall say that f is a "divisibility preserving function" (shortly **d.p.**-function).

Finally, we note that f is called totaly-multiplicative (or: **t.m.**-function) if $f(ab) = f(a)f(b)$ holds for all $a, b \in N$. The structure of the above classes of functions is not so simple, as is expected in [1]. For example, it is questioned if a function which is **m** and **r.p.**, must be **t.m.** function? That this is not true, follows from the following examples. Let

$$f(n) = \begin{cases} 1, & n = 1 \\ p_1 \ldots p_r, & n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} \geq 2 \end{cases} \tag{4}$$

where p_i $(i = \overline{1, r})$ are distinct primes, $\alpha_i \geq 1$. Then f is **m**-function, since if $m = q_1^{\beta_1} \ldots q_s^{\beta_s}$, then

$$f(nm) = (p_1 \ldots p_r)(q_1 \ldots q_s) = f(n)f(m)$$

if $(n, m) = 1$. Clearly, f is **r.p.**-function, since if $(n, m) = 1$, clearly $q_i \neq p_j$, so $(p_1 \ldots p_r, q_1 \ldots q_s) = 1$, But f is not **t.m.**-function, since e.g. $f(p^2) = p$, but $f(p)f(p) =$

$pp = p^2 \neq p$. Thus $f(p^2) \neq (f(p))^2$, implying that f is not totally multiplicative. Let now

$$f(n) = \begin{cases} 1, & n = 1 \\ (p_1 - 1)^{\alpha_1} \dots (p_r - 1)^{\alpha_r}, & n = p_1^{\alpha_1} \dots p_r^{\alpha_r} \geq 2. \end{cases} \tag{5}$$

Then $f(p) = p-1$, so $f(p)^\alpha = (p-1)^\alpha = (f(p))^\alpha$ for each prime p and $\alpha \geq 0$. Similarly

$$f(p^\alpha q^\beta) = (p-1)^\alpha (q-1)^\beta = (f(p))^\alpha (f(q))^\beta = f(p)^\alpha f(q)^\beta,$$

(even for $p = q$). This implies immediately (by induction) that f is a **t.m.**-function. But f is not a relative-primality preserving function! Indeed, let p and q be odd primes, $p \neq q$. Then $f(p) = p - 1$, $f(q) = q - 1$ which are both divisible by 2. Thus $(p, q) = 1 \not\Rightarrow (f(p), f(q)) = 1$.

The above considered classes of functions do not coincide, as can be seen by various examples. A **g.p.**-function with $f(1) = 1$ is also an r.p.-function. Indeed, let $(a, b) = 1$ in (2). Then $(f(a), f(b)) = f(1) = 1$. But conversely, it is not true, an example is $F_n = 2^{2^n} + 1$ (Fermat numbers). It is well known that $(F_n, F_m) = 1$ for all $n \neq m$. Therefore, (1) is true, but (2) not. Also (2) is not true for (F_n). Euler's function φ has property (3), but (1) is not true, since $\varphi(n)$ is always even for $n \geq 3$. Relation (2) is true for $f(a) = 2^a - 1$ and $f(a) = F(a) =$ Fibonacci number of order a (see Applications of Theorem 2). Clearly (2) $\Rightarrow$ (3), since if $a|b$, one has $(a, b) = a$ and $(f(a), f(b)) = f(a)$ implies $f(a)|f(b)$. But (3) $\not\Rightarrow$ (2); take e.g. $f(n) = \varphi(n)$. The following general results are true:

Theorem 1. *Let $f : \mathbb{N} \to \mathbb{N}$ be a* **t.m.***-function. Then f is* **r.p.** *if and only if is* **g.p.**

Proof. If f is **t.m.**, then $f(1) = 1$, since $f(1 \cdot 1) = f(1)f(1)$ and $f(1) \neq 0$. If f is **g.p.**, then as we seen in the above remarks, f is **r.p.**, too. Reciprocally, let us suppose, that f is **r.p**. Let $a = da_1$, $b = db_1$, with $(a_1, b_1) = 1$. Then $f(a) = f(d)f(a_1)$, $f(b) = f(d)f(b_1)$, so $(f(a), f(b)) = f(d)(f(a_1), f(b_1)) = f(d)$, since $(a_1, b_1) = 1 \Rightarrow (f(a_1), f(b_1)) = 1$. This finishes the proof of Theorem 1.

Theorem 2. *Let f be a* **d.p.***-function which satisfies the following property: For all $x, y \in \mathbb{N}$, $a, b \in \mathbb{N}$ with $ax > by$ there exist $A, B, C \in \mathbb{Z}$ such that $(C, f(b)) = 1$ or $(C, f(a)) = 1$ and*

$$Cf(ax - by) = Af(a) + Bf(b). \tag{6}$$

Then f is a **g.p.**-*function.*

Proof. Let $a = da_1$, $b = db_1$, where $(a_1, b_1) = 1$. Thus $d = (a, b)$. Then f being a **d.p.**-function, one has $f(d)|f(a)$, $f(d)|f(b)$, implying

$$f(d)|(f(a), f(b)). \tag{7}$$

Now, we shall prove that conversely

$$(f(a), f(b))|f(d) \tag{8}$$

is true, which together with (7) implies that f is **g.p.**-function. It is well-known that there exist $x, y \in \mathbb{N}$, such that $(a, b) = d = ax - by$. Therefore, by (6) one can write $Cf(d) = Af(a) + Bf(b)$. Now, let k be any common divisor of $f(a)$ and $f(b)$. Then, by the above relation k divides also $Cf(d)$. Since $(C, f(b)) = 1$, and k divides $f(b)$, clearly $(C, k) = 1$. Therefore $k|f(d)$. This implies (8).

Application 1. Let $f(a) = u^a - v^a$, where $u > v$, $(u, v) = 1$. Then f is **g.p.**-function.

Proof. Clearly $(u^a - v^a)|(u^b - v^b)$ for $a|b$; which follows by an algebraic identity. Remark now that

$$(uv)^{by}(u^{ax-by} - v^{ax-by}) = v^{by}(u^{ax} - v^{ax}) - v^{ax}(u^{by} - v^{by}). \tag{9}$$

Since $u^{ax} - v^{ax} = (u^a - v^a)[(u^a)^{xy} + \ldots + (v^a)^{xy}]$, etc. one can immediately remark, that (6) holds true with

$$C = (uv)^{by}, \quad A = (u^a)^{x-1} + \ldots + (v^a)^{x-1}, \quad B = (u^b)^{y-1} + \ldots + (v^b)^{y-1}.$$

Application 2. Let $f(a) = F(a) =$ Fibonacci number of order a. These numbers are defined by $F(1) = F(2) = 1$, $F(n+1) = F(n) + F(n-1)$, $(n \geq 2)$. The following identities are valid (e.g. by induction)

$$F(n+1)F(n-1) - (F(n))^2 = (-1)^n, \quad (n \geq 2) \tag{10}$$

$$F(n+m) = F(n-1)F(m) + F(n)F(m+1), \quad (n \geq 2), \quad m \geq 1 \tag{11}$$

Now, (10) implies $(F(n), F(n+1)) = 1$. By letting $m = (k-1)n$, $(k \geq 2)$ in (11) we get

$$F(kn) = F(n-1)F((k-1)n) + F(n)F((k-1)n+1).$$

For $k = 2$ one has $F(n)|F(2n)$. By admitting $F(n)|F((k-1)n)$, we get $F(n)|F(kn)$. Therefore F is a **d.p.**-function (indeed, $a|b$ means $b = ka$). Now, let $n = ax - by$, $m = by$ in (11). One gets

$$F(xa) = F(n-1)F(by) + F(ax - by)F(by + 1). \tag{12}$$

Here $F(ax)$ is a multiple of $F(a)$ and $F(by)$ a multiple of $F(b)$. Therefore, (6) is satisfied with $C = F(by + 1)$, which is relatively prime to $F(by)$, so to $F(b)$, too. Thus F is indeed a **g.p.**-function by Theorem 2. Finally, we state:

Theorem 3. *Let f be an* **m***-function, which is an* **r.p.***-function, too.*

Then $g : \mathbb{N} \to \mathbb{N}$ defined by $g(a) = f^n(a)$ has these two properties, too (for any $n \geq 1$). Here $f^n(a) = f(\underbrace{f \ldots (f(a)) \ldots}_{n-times})$ is the composition of f n-times at a.

Proof. Let $(a, b) = 1$. Then $f(ab) = f(a)f(b)$. But $(f(a), f(b)) = 1$, so $f[f(ab)] = f[f(a)f(b)] = f[f(a)]f[f(b)]$. Thus $f \circ f$ is an **m**-function. But it is an **r.p.**-function too, since $(f[f(a)], f(f(b))) = 1$ by $(f(a), f(b)) = 1$ for $(a, b) = 1$. Theorem 3 immediately follows, fro all n, by induction.

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23 A generalized Pillai function

Arithmetic functions involving sums on (k,n) where $k \leq n$ and (k,n) denotes the g.c.d. of k and n were considered first by Pillai. Particularly, the function

$$P(n) = \sum_{k \leq n} (k,n) \tag{1}$$

is named as "Pillai's arithmetic function", first studied in 1933 [2]. Various extensions or analogues have been studied along the years (see [1], [4], [8], [3]). Some of these functions were rediscovered by Tabîrcă and Tabîrcă [5], [6].

Let now $f : N \to N$ be an arbitrary arithmetic function and denote

$$P_f(n) = \sum_{k \leq n} f((k,n)) \tag{2}$$

When $f(n) = n^{-\alpha}$, $(\alpha \in R)$ we get

$$P_f(n) = \frac{1}{n^\alpha} \sum_{k \leq n} \left(\frac{n}{(k,n)} \right)^\alpha,$$

where the function

$$\Psi_\alpha(n) = \sum_{k \leq n} \left(\frac{n}{(k,n)} \right)^\alpha$$

has been introduced and studied in [3]. For $\alpha = -1$, this clearly contains also Pillai's function P, since

$$\Psi_{-1}(n) = \frac{1}{n} \sum_{k \leq n} (k,n).$$

Therefore, (2) is a common generalization of Ψ_α and P.

Lemma 1. *Let P_f given by (2). Then*

$$P_f(n) = \sum_{d|n} \varphi(d) f\left(\frac{n}{d}\right) \tag{3}$$

Proof. Let $A = \{1, 2, \ldots, n\}$, $A_d = \{i \in A : (i,n) = d\}$. Then $(i,n) = d \Leftrightarrow \left(\frac{i}{d}, \frac{n}{d}\right) = 1$ and $1 \leq \frac{i}{d} \leq \frac{n}{d}$. Therefore, A_d has $\varphi\left(\frac{n}{d}\right)$ elements. Since $A = \bigcup_{d|n} A_d$ and $card A_d = \varphi\left(\frac{n}{d}\right)$, we have

$$P_f(n) = \sum_{d \in A} f(d) = \sum_{d|n} card A_d \cdot f(d) = \sum_{d|n} f(d) \varphi\left(\frac{n}{d}\right).$$

Lemma 2. *When f is multiplicative, P_f is multiplicative too.*

Proof. By $P_f = f * \varphi$ - the Dirichlet convolution of f and φ; since $f * \varphi$ is known to be multiplicative, P_f is a multiplicative function.

Theorem. *When f is multiplicative, we have the following relation:*

$$P_f(n) = \prod_{i=1}^{r} \sum_{k=0}^{\alpha_i} f(p_i^k)(p_i^{\alpha_i - k} - p_i^{\alpha_i - k - 1}) \tag{4}$$

where $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the prime factorization of $n > 1$ (and $p_i^{-1} := 0$).

Proof. $P_f(n) = \prod_{i=1}^{r} P_f(p_i^{\alpha_i})$, since P_f is multiplicative. By (3) one has

$$P_f(p_i^{\alpha_i}) = \sum_{d | P_i^{\alpha_i}} \varphi(d) f(p_i^{\alpha_i}/d) = \sum_{k=0}^{\alpha_i} (p_i^k - p_i^{k-1}) f(p_i^{\alpha_i - k}),$$

which is the same as (4).

Remark. Let $f(n) = n^{\alpha}$. Then we get

$$P_f(n) = \prod_{i=1}^{r} \sum_{k=0}^{\alpha_i} p_i^{k\alpha}(p_i^{\alpha_i - k} - p_i^{\alpha_i - k - 1}) \tag{5}$$

Various other relations for particular f can be obtained. For asymptotic relations and open problems, see [1], [3], [7], [8].

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24 A note on semigroup valued multiplicative functions

Let $(G, *)$ be a commutative semigroup with unit element, where $* : G \times G \to G$ denotes the semigroup operation in G. We say that a function $f : N^* \to G$ is $(G, *)$-multiplicative (or "G-multiplicative"), if

$$f(ab) = f(a) * f(b) \text{ for all } (a, b) = 1 \tag{1}$$

where (a, b) denotes the g.c.d. of the natural numbers a and b.

The aim of this note is to prove the following result.

Theorem. *Let $f : N^* \to G$ be a G-multiplicative function and suppose that $f(k)$ has an inverse element $f^{-1}(k)$ in G. Then the function $F_k : N^* \to G$, defined by*

$$F_k(n) = f(nk) * f^{-1}(k), \quad n \in N^* \tag{2}$$

is G-multiplicative, too.

Proof. A proof can be obtained by generalizing the idea from [2], but here we shall adopt another method. For $n \in N^*$ and p prime denote $a_p(n) = t \in N$ if $p^t | n$, $p^{t+1} \nmid n$, i.e. the greatest power of p which divides n. Then clearly,

$$n = \prod_{p|k} p^{a_p(n)} \cdot \prod_{p|n, p\nmid k} p^{a_p(n)}; \quad k = \prod_{p|n} p^{a_p(k)} \cdot \prod_{p|k, k\nmid n} p^{a_p(k)} \tag{3}$$

by the prime factorization of n and k. Let us denote by $\prod_{p\in P}^{*} x_p$ a product (with respect to the operation $*$) of elements $x_p \in G$, when p runs over a set P. By using (1) and (3), a simple calculation gives

$$F_k(n) = \prod_{p|n}^{*} [f(p^{a_p(n)+a_p(k)}) * f^{-1}(p^{a_p(k)})], \tag{4}$$

where we used the properties of G and the obvious fact that $a * b$ has an inverse $(a * b)^{-1}$ in G if and only if a and b have inverses. Of course, $(a * b)^{-1} = a^{-1} * b^{-1}$. Now the G-multiplicativity of F_k follows at once from (4) by remarking that $a_p(xy) = a_p(x)$ for $p|x$, $p|y$. Finally, we note that this proof is based on the explicit expression of (2).

Remarks. 1) The notion of G-multiplicative functions is a common generalization of the classical multiplicative and additive functions. Let $(G, *) = (R, \cdot)$ the set of real numbers, endowed with the usual product operation. Then each $a \in R$, $a \neq 0$, has an inverse, so we obtain (with a new proof) the result from [2]: If f is a multiplicative function and $f(k) \neq 0$, then $F_k(n) = f(kn)/f(k)$ is multiplicative, too. We note that this theorem extends a result of [1].

2) Let $(G, *) = (R, +)$. Then we get the theorem: If f is an additive function (i.e. satisfying $f(xy) = f(x) + f(y)$ for $(x, y) = 1$, then $F_k(n) = f(kn) - f(k)$ is additive, too. As a consequence we can remark that if f and g are additive functions, then the function h defined by $h(n) = g(k)f(kn) - f(k)g(kn)$ is additive, too. This follows from the equality $h(n) = g(k)[f(kn) - f(k)] - f(k)[g(kn) - g(k)]$. Another corollary says that if f is additive, then for $(k, r) = 1$, the function s defined by $s(n) = f(krn) - f(kn) - f(rn)$ is additive, too.

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Chapter 4. Divisibility properties of numbers and functions

"... one of the attractions of the subject-matter is that, whilst it is rich in ideas, it requires little initiation..."

(H. Halberstam and K.F. Roth, Sequences, Springer, 1983)

1 On a divisibility property

Let $p > 3$ be an odd prime (i.e. $p \geq 5$). Then Wolstenholme's theorem asserts that the numerator of the fraction in $1 + \frac{1}{2} + \ldots + \frac{1}{p-1}$ is divisible by p^2, in other words, if

$$\frac{a}{b} = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{p-1},$$

then

$$p^2|a \quad ([2]). \tag{1}$$

By using properties of the field Z_p, in [2] it is proved that:

1) If $p \geq 5$ and

$$\frac{a}{b} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(p-1)^2},$$

then

$$p|a \tag{2}$$

and

2) If $p \geq 3$ and if

$$\frac{a}{b} = \frac{1}{1^3} + \frac{1}{2^3} + \ldots + \frac{1}{(p-1)^3},$$

then

$$p|a. \tag{3}$$

We shall determine all positive integers m and k such that:

3) If

$$\frac{a}{b} = \frac{1}{1^m} + \frac{1}{2^m} + \ldots + \frac{1}{(p-1)^m},$$

then

$$p^k|a. \tag{4}$$

Clearly, (1) is not true for $p = 3$; and $p^3|a$ is not generally true (take e.g. $p = 5$). Therefore, for $m = 1$ we must have $k = 2$ in (4). 2) is true for $p \geq 5$, but $p^2 \nmid a$ generally. Therefore for $m = 2$ one has $k = 1$. For 3) we will prove that $p^2|a$, but $p^3 \nmid a$ generally. Let us suppose first that m is odd. We will prove that for sufficiently large p, one has $p^2|a$.

If m is even, then for sufficiently large p one has $p|a$. Therefore, one can say that for m odd we have $k = 2$, while for m even, $k = 1$.

In the proof the method shown in the Niven-Zuckerman book ([1]) will be used. Let $m = 2n - 1$ $(n \geq 1)$. Then

$$a = \frac{((p-1)!)^{2n-1}}{1^{2n-1}} + \frac{((p-1)!)^{2n-1}}{2^{2n-1}} + \ldots + \frac{((p-1)!)^{2n-1}}{(p-1)^{2n-1}}.$$

To each member of a corresponds an element $\widetilde{x} \in Z_p$. In fact

$$x \equiv \frac{((p-1)!)^{2n-1}}{j^{2n-1}} \pmod{p},$$

with $1 \leq i \leq p-1$ satisfying

$$xi^{2n-1} \equiv -1 \pmod{p}.$$

Therefore $\widetilde{x} = (-\widetilde{i^{2n-1}})^{-1} \in Z_p$, and since the inverse is unique, $-\widetilde{x}$ is one of the 2^{2n-1} powers of $\widetilde{1}, \widetilde{2}, \ldots, \widetilde{p-1}$. Therefore,

5) $a \equiv -(1 + 2^{2n-1} + \ldots + (p-1)^{2n-1}) \pmod{p}$

But the expression in the right-hand side can be evaluated with the Bernoulli numbers:

$$1^{2n-1} + 2^{2n-1} + \ldots + (p-1)^{2n-1} = \sum_{i=0}^{2n-1} \frac{1}{2n-i} C_{2n-1}^{i} B_i,$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_{2n+1} = 0$ are the Bernoulli numbers. This gives

$$2n(1^{2n-1} + 2^{2n-1} + \ldots + (p-1)^{2n-1}) = \sum_{i=0}^{2n-1} \frac{2n}{2n-i} C_{2k-1}^{i} p^{2n-i} B_i.$$

The right-hand side is divisible by p^2 by $B_{2n-1} = 0$, and hence $p^2|2n(1^{2n-1} + \ldots + (p-1)^{2n-1})$. Let now $p - 1 \geq 2n$. Then $p^2|a$, so the assertion is proved. When m is even, $m = 2n$ $(n \geq 1)$, the proof is very similar, and we omit the details.

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2 On a non-divisibility property

Let $n > 1$ be a positive integer. An old problem states that $2^n - 1$ cannot be divisible by n. We will prove that $2^{n-\varphi(n)} - 1$ is not divisible by n, too (where φ is Euler's totient); and in fact a more general result will be obtained. Our method is based on the following

Lemma. *Let $a > b$, $(a, b) = 1$. Then*

$$(a^n - b^n, a^m - b^m) = a^{(n,m)} - b^{(n,m)}, \tag{1}$$

where (u, v) denotes the g.c.d. of u and v.

For a proof of (1), see [1]; and in more general contexts, see [2] (which shows that $f(n) = a^n - b^n$ is a g.c.d. preserving function).

Theorem. *Let $a > b$, $(a, b) = 1$. Suppose that $(a, n) = (b, n) = (a - b, n) = 1$. Then*

1) $n \nmid (a^n - b^n)$

2) $n \nmid (a^{n-\varphi(n)} - b^{n-\varphi(n)})$.

Proof. Let us suppose on the contrary that there exists $n > 1$ such that 1) holds. By Euler's divisibility theorem, for $(a, n) = 1$ one has $n|(a^{\varphi(n)} - 1)$. Now, since $a^{(\varphi(n)} - b^{\varphi(n)} = (a^{\varphi(n)} - 1) - (b^{\varphi(n)} - 1)$, for $(a, n) = (b, n) = 1$, n divides $a^{\varphi(n)} - b^{\varphi(n)}$, so by the Lemma n divides also $a^d - b^d$, where $d = (n, \varphi(n))$. For $d = 1$ we would have $n|(a - b)$, which is impossible by $(a - b, n) = 1$. Therefore $d > 1$. On the other hand, $(a, d) = (b, d) = (a - b, d) = 1$ (d being a divisor of n) and $d < n$ by $\varphi(n) < n$ for $n > 1$. Then continuing indefinitely, one obtains an infinite sequence (d_k) of positive integers which is strictly decreasing. This is impossible (i.e. we have applied the well-known Fermat descent method). This proves 1).

Ramarking that $(n - \varphi(n), \varphi(n)) = (n, \varphi(n))$, by the Lemma

$$(a^{n-\varphi(n)} - b^{n-\varphi(n)}, a^{\varphi(n)} - b^{\varphi(n)}) = (a^n - b^n, a^{\varphi(n)} - b^{\varphi(n)})$$

so 2) follows at once.

Remarks. Let $b = 1$. Then $n \nmid (a^n - 1)$ if $(a, n) = (a - 1, n) = 1$. This is true for $a = 2$. In this case $n \nmid (2^{n-\varphi(n)} - 1)$ is also true.

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3 On two properties of Euler's totient

Let φ be Euler's totient. The following two properties of φ are well-known:

(1) If $(a, n) = 1$, then $n|[a^{\varphi(n)} - 1]$ (Euler's theorem); and

(2) If $a > 1$ then $n|\varphi(a^n - 1)$

which as far as we know was first discovered by U. Scarpis [1] (see also Guderson [3], Rotkiewicz [2]). The aim of this note is the characterization of functions $f : N^* \to N^*$ such that:

(3) For $(a, n) = 1$ we have $n|[a^{f(n)} - 1]$ and

(4) For all $a > 1$, $n|f(a^n - 1)$.

We shall prove that (3) and (4) are valid if and only if

(5) $\varphi(n)|f(n)$ for all n.

First, let us suppose that (5) holds. Then since $(a^k - 1)|(a^m - 1)$ for $k|m$, from (1) and (5) it follows that (3) is true. Now, by (2) and (5) we have $n|\varphi(a^n - 1)|f(a^n - 1)$, so (4) is also true. Reciprocally, let us suppose that (3) is true. Let us consider the multiplicative group Z_n^* of reduced classed $\pmod n$, i.e. $\widehat{a} \in Z_n^*$ if $(a, n) = 1$. This is a finite group of order $\varphi(n)$. Now it is well-known that in a finite group if $a^k = e$ (e = unity element), then k is multiple of order of the group. In our case (3) means that $\widehat{a}^{f(n)} = 1$ in Z_n^*, therefore $\varphi(n)|f(n)$. By (2) we get $m|\varphi(a^m - 1)|f(a^m - 1)$ (here $n = a^m - 1$), so (4) is a consequence of (3). In fact, (1) is a consequence of the known fact that in a finite group G of order t, one has $x^t = e$ ($x \in G$). For a proof of (2) let us consider the group $Z^*_{a^n-1}$. Then $(a, a^n - 1) = 1$ and $a^n \equiv 1 \pmod{a^n - 1}$, but $a^s \not\equiv 1 \pmod{a^n - 1}$ for $s < n$. Therefore the order of $\widehat{a}$ is n. This must divide the order of the group, as it is well-known. (This follows also from Lagrange's theorem, which says that the order of a subgroup divides the order of the group - here one considers cyclic subgroups). Other similar properties of φ are included in [4].

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4 On a conjecture of Smarandache on prime numbers

Let p_n denote the n-th prime number. One of Smarandache conjectures in [3] is the following inequality:

$$p_{n+1}/p_n \leq 5/3, \text{ with equality for } n = 2. \tag{1}$$

Clearly, for $n = 1, 2, 3, 4$ this is true and for $n = 2$ there is equality. Let $n > 4$. Then we prove that (1) holds true with strict inequality. Indeed, by a result of Dressler, Pigno and Young (see [1] or [2]) we have

$$p_{n+1}^2 \leq 2p_n^2. \tag{2}$$

Thus $p_{n+1}/p_n \leq \sqrt{2} \leq 5/3$, since $3\sqrt{3} < 5$ (i.e. $18 < 25$). This finishes the proof of (1).

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5 On consecutive primes

Let p_n denote the nth prime ($p_1 = 2,\ p_2 = 3, \ldots$). In paper [2] it is conjectured that:

$$\frac{p_{n+1}}{p_n} \leq \frac{5}{3} \tag{1}$$

with equality only for $n = 2$. In what follows we shall prove the stronger relation:

$$\frac{p_{n+1}}{p_n} \leq \frac{3}{2}, \quad (n \neq 2, 4) \tag{2}$$

with equality for $n = 1$. Since $\frac{3}{2} < \frac{5}{3}$, clearly (2) improves relation (1) for $n \neq 2, 4$. We have

$$\frac{p_2}{p_1} = \frac{3}{2}, \quad \frac{p_3}{p_2} = \frac{5}{3} > \frac{3}{2}, \quad \frac{p_4}{p_3} = \frac{7}{5} < \frac{3}{2}, \quad \frac{p_5}{p_4} = \frac{11}{7} > \frac{3}{2}.$$

Thus (2) is not valid for $n = 2, 4$.

Let now $n > 4$. By a result of Dressler, Pigno and Young [1] one has:

$$p_{n+1}^2 \leq 2p_n^2. \tag{3}$$

Thus, by (3) we can write $\frac{p_{n+1}}{p_n} \leq \sqrt{2} < \frac{3}{2}$ for $n > 4$.

Clearly, relation (1) holds true for all n, with equality only for $n = 2$.

In paper [2] it is conjectured also that:

$$\frac{1}{p_n} - \frac{1}{p_{n+1}} \leq \frac{1}{6}. \tag{4}$$

By (2) one has:

$$\frac{1}{p_n} - \frac{1}{p_{n+1}} \leq \frac{3}{2} \cdot \frac{1}{p_{n+1}} - \frac{1}{p_{n+1}} = \frac{1}{2} \cdot \frac{1}{p_{n+1}} \text{ for } n \neq 2, 4.$$

By $\frac{1}{p_{n+1}} < \frac{1}{3}$, this yields relation (4), which holds true also for $n = 1, 3$ (direct verification), with equality only for $n = 1$.

As an application of (2) and (1) note that an improvement of the well-known relation:

$$p_n + p_{n+1} \geq p_{n+2} \tag{5}$$

can be deduced. Indeed,

$$p_n + p_{n+1} \geq \frac{2}{3}p_{n+1} + p_{n+1} = \frac{5}{3}p_{n+1} \geq p_{n+2}$$

by (2) and (1). Thus, one has:

$$p_n + p_{n+1} \geq \frac{5}{3} p_{n+1} \geq p_{n+1} \quad (n \neq 2, 4) \tag{6}$$

which sharpens relation (5).

2. A similar relation to (5) and (6) follows by the above proved inequality:

$$\frac{1}{p_n} - \frac{1}{p_{n+1}} \leq \frac{1}{2} \frac{1}{p_{n+1}}$$

and the known relation $p_{n+2} < 2p_{n+1}$ (which is a consequence of the Chebyshev Theorem, that there exists a prime between a and $2a$ for $a \geq 2$, see [5]).

We can deduce:

$$\frac{1}{p_n} \leq \frac{3}{2} \cdot \frac{1}{p_{n+1}} < \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}}. \tag{7}$$

3. The Open Question 171 by Mihály Bencze (Octogon Mathematical Magazine, vol.6(1998), No.2, pp.219) asks for the inequality

$$\sqrt{\frac{p_{n^2+1}}{p_n}} - \sqrt{p_n} < 1,34. \tag{8}$$

In fact, we shall prove that $\sqrt{\frac{p_{n^2+1}}{p_n}} - \sqrt{p_n} < 0$ for $n \geq 5$.

This is equivalent to:

$$p_{n^2+1} < p_n^2, \quad n \geq 5. \tag{9}$$

We note that by the prime number theorem, $p_n \sim n \log n$ $(n \to \infty)$ (see e.g. [5]) it follows easily $\frac{p_{n^2+1}}{p_n} \to 0$ as $n \to \infty$, so $\frac{p_{n^2+1}}{p_n} < 1$ for $n \geq n_0$, but this "n_0" cannot be determined by this way. Thus inequality (9) can be much improved for greater values of n. However, we shall apply here a general simple method based on the Rosser-Schoenfeld inequalities ([3]):

$$p_m < m \log m + m \log \log m \quad (m \geq 6) \tag{10}$$

$$p_m > m \log m \quad (m \geq 2)$$

Now, since

$$\log(a+1) - \log a < \frac{1}{\sqrt{a(a+1)}}$$

(see e.g. [4]) and

$$\log x \leq x - 1 \quad (x > 0),$$

we have

$$\log(n^2+1) < \log n^2 + \frac{1}{\sqrt{n^2(n^2+1)}} = 2\log n + \frac{1}{n\sqrt{n^2+1}}.$$

By using the first part of (10) with $m = n^2 + 1$ ($n \geq 3$), and the second part of (10) with $m = n$, it is immediate that to prove (9) it is sufficient to deduce an inequality

$$4(n^2+1)\log n < n^2(\log n)^2.$$

Putting $n^2 = t$ this becomes equivalent to

$$\log t > 8\left(1 + \frac{1}{t}\right). \tag{11}$$

With the increasing function $f(t) = \log t - 8\left(1 + \frac{1}{t}\right)$ it can be proved that (11) holds true e.g. for $t \geq e^9$. Thus (9) is proved for $n \geq \sqrt{e^9} = e^{4,5}$; for $5 \leq n < e^{4,5}$ a direct computation can be done, and this finishes the proof of (9).

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6 On Bonse-type inequalities

1. Let p_n be the nth prime. Euclid's proof of the infinitude of primes implies $p_{n+1} \leq p_1 p_2 \dots p_n - 1$ (indeed, the least prime divisor of the right side is distinct from each of $p_1, p_2, \dots, p_n$, so is at least p_{n+1}).

In what follows, by similar simple considerations we shall deduce some results, which particularly contain the Bonse inequalities ([1]). G. Pólya applied the method for Fermat numbers $F_n = 2^{2^n+1}$ and deduced $p_{n+2} \leq 2^{2n} + 1$ ([2]).

P. Erdös has proved the following (see [3]): If $1 < a_1 < a_2 < \dots < a_k \leq x$ is a sequence of integers such that none of them divides the product of others, then $k \leq \pi(x)$, where $\pi(x)$ denotes the number of primes $\leq x$.

This implies $p_k \leq a_k$. Indeed, if $a_k < p_k$, then $\pi(a_k) \leq \pi(p_k) = k$ and applying Erdös' theorem for $x = a_k$ one can write $k \leq \pi(a_k) \leq k$, i.e. $\pi(a_k) = k$, impossible.

2. Let us consider now the sequence of general term $a_k = kp_1 p_2 \dots p_{n-1} - p_n$ ($n \geq 2$ fixed). Clearly $a_1 < a_2 < \dots < a_s$ with $s = p_{n-1}$ and Euclid's theorem implies $a_1 \geq 1$. On the other hand, $(a_i, a_j) = 1$ for $1 \leq i, j \leq s$; since if there would be a prime p dividing a_i and a_j, then p would divide also $a_j - a_i = (j-i)p_1 p_2 \dots p_{n-1}$, and since $j - i < p_{n-1}$, by $p \leq p_{n-1}$ the term a_i would not be divisible by p. By Erdös' theorem $p_{n-1} \leq \pi(x)$ if $x \geq a_{p_{n-1}} = p_1 p_2 \dots p_{n-1}^2 - p_n$, so

$$p_1 p_2 \dots p_{n-2} p_{n-1}^2 - p_n \geq p_{p_{n-1}} \quad (n \geq 3) \tag{1}$$

By taking $s = p_n - 1$, then $j - i \leq p_n - 2$ and the above method gives

$$p_1 p_2 \dots p_n \geq p_1 p_2 \dots p_{n-1} + p_n + p_{p_{n-1}} \quad (n \geq 3) \tag{2}$$

We note that this implies the famous Bonse inequality

$$p_1 p_2 \dots p_n > p_{n+1}^2 \quad (n \geq 4) \tag{3}$$

Indeed, for $n \leq 8$, a direct computation can be done. For $n \geq 9$ by induction it follows $p_n \geq 2n + 5$. It is immediate that

$$p_1 p_2 \dots p_n > (p_1 p_2 \dots p_{[\frac{n}{2}]})^2,$$

so applying (1) to $\left[\frac{n}{2}\right]$ in place of n, we get

$$(p_1p_2\dots p_{\left[\frac{n}{2}\right]}) > p^2_{p_{\left[\frac{n}{2}\right]-1}}$$

and

$$p_{\left[\frac{n}{2}\right]-1} \geq 2\left(\left[\frac{n}{2}\right]-1\right)+5 > n+1,$$

so $p_1p_2\dots p_n > p^2_{n+1}$.

3. Let now $b_k = kp_1p_2\dots p_{n-1} - p_n^2$ $(n \geq 4)$. By Bonse's inequality one has $1 \leq b_1$. On the other hand, $b_1 < b_2 < \dots < b_k$, and $(b_i, b_j) = 1$ (which can be proved as above). Therefore we get

$$p_1p_2\dots p_n \geq p_1p_2\dots p_{n+1} + p_n^2 + p_{p_n-1} \quad (n \geq 4) \tag{4}$$

Applying (1) for $n-1$, (2) for $n-2$, we obtain

$$p_1p_2\dots p_n > p_{n-1} + p_{n-1}^2 + p_{p_n-1} + p_{p_{n+1}-1} + p_n^2 \quad (n \geq 6) \tag{5}$$

Now, remark that $p_1p_2\dots p_n > (p_1p_2\dots p_{\left[\frac{n}{3}\right]})^3$. It is easy to see that for $k \geq 17$ one has $p_k > 3k+7$ (indeed, $p_{15} = 53$, $3\cdot 15+7 = 52$). Then $p_{\left[\frac{n}{3}\right]-1} \geq 3\left(\left[\frac{n}{3}\right]-1\right)+7 > n+1$. Bonse's second inequality

$$p_1p_2\dots p_n > p_{n+1}^3 \text{ is true for } n \geq 5. \tag{6}$$

4. If $c_k = kp_1p_2\dots p_{n-1} - p_n^3$, all method can be maded once again, and it follows

$$p_1p_2\dots p_n \geq p_1p_2\dots p_{n-1} + p_n^3 + p_{p_{n-1}} \quad (n \geq 5) \tag{7}$$

and applying it once again:

$$p_1p_2\dots p_n \geq p_1p_2\dots p_{n-2} + p_{n-1}^3 + p_{p_{n-1}-1} > 2p_{n-1}^3 + p_{p_n-1} - 1 \tag{8}$$

At last, notice that Erdös' theorem has a simple proof: By decomposing in prime powers all of the k numbers, then each number a_i has a prime p_{k_i} which has a higher power than all the others. Since each a_i has a such prime divisor, clearly $k \leq \pi(x)$.

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7 On certain inequalities for primes

Let p_n denote the n-th prime. The following Open Problem proposed by M. Bencze has five parts, namely:

i. Determine all $\alpha > 0$ such that $p_{n+2}^{\alpha} \leq p_{n+1}^{\alpha} + p_n^{\alpha}$

ii. Determine all $\beta > 0$ such that $p_{n+2}^{\beta} \leq p_1^{\beta} + p_2^{\beta} + \ldots + p_n^{\beta}$

iii. Determine all $\gamma > 0$ with $p_{n+2}^{-\gamma} \leq p_{n_1}^{-\gamma} + p_n^{-\gamma}$

iv. Determine all $\delta > 0$ so that $p_{n+2}^{\delta} \leq p_{n+1}^{\delta} p_n^{\delta}$

v. Determine all $\varepsilon > 0$ so that $p_{n+2}^{\varepsilon} \leq \frac{2p_n^{\varepsilon} p_{n+1}^{\varepsilon}}{p_n^{\varepsilon} + p_{n+1}^{\varepsilon}}$.

First remark that v. is false for all $\varepsilon > 0$, since written equivalently:

$$\frac{p_{n+2}^{\varepsilon}}{p_{n+1}^{\varepsilon}} \leq \frac{2p_n^{\varepsilon}}{p_n^{\varepsilon} + p_{n+1}^{\varepsilon}} < 1$$

since $p_n^{\varepsilon} < p_{n+1}^{\varepsilon}$. On the other hand $\dfrac{p_{n+2}^{\varepsilon}}{p_{n+1}^{\varepsilon}} > 1$. Relation iii. is trivial for all $\gamma > 0$ since as can be easily seen, it can be written equivalently as

$$p_n^{\gamma} p_{n+1}^{\gamma} \leq p_{n+2}^{\gamma}(p_n^{\gamma} + p_{n+1}^{\gamma}),$$

and this is trivial since

$$p_{n+1}^{\gamma} < p_{n+2}^{\gamma}, \quad p_n^{\gamma} < p_n^{\gamma} + p_{n+1}^{\gamma}.$$

To obtain a strong inequality of this type, remark that

$$\frac{1}{p_n} < \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}}$$

(see [3]). Now, by the following Lemma, for all $0 < \gamma \leq 1$ one has

$$\frac{1}{p_n^{\gamma}} < \left(\frac{1}{p_{n+1}} + \frac{1}{p_{n+2}}\right)^{\gamma} \leq \frac{1}{p_{n+1}^{\gamma}} + \frac{1}{p_{n+2}^{\gamma}},$$

therefore:

$$\frac{1}{p_n^{\gamma}} < \frac{1}{p_{n+1}^{\gamma}} + \frac{1}{p_{n+2}^{\gamma}},$$

for all $0 < \gamma \leq 1$. Relation iv. is not true, since it is well known by a result of Erdös and Turán [2] that $p_{n+2}^2 > p_{n+1}p_n$ for infinitely many n. Inequalities i. and ii. are true for all $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, and this is based on the following:

Lemma. *Let* $a, b > 0$, $0 < \alpha \leq 1$. *Then* $(a+b)^\alpha \leq a^\alpha + b^\alpha$. *For* $\alpha > 1$, *we have* $(a+b)^\alpha > a^\alpha + b^\alpha$.

Proof. This Lemma is well-known, so we give only a hint for proof. By putting $\frac{a}{a+b} = x$, the first inequality states that

$$f(x) = x^\alpha + (1-x)^\alpha - 1 \geq 0 \quad (\text{for } 0 < \alpha \leq 1).$$

By extending the definition of f to $[0,1]$ and using a little differential calculus, the result follows. Now, in i. we must have $0 < \alpha \leq 1$, since for $n = 1$ we must have $5^\alpha \leq 2^\alpha + 3^\alpha$ and by Lemma this is true only for $0 < \alpha \leq 1$. Now, by $p_{n+2} \leq p_{n+1} + p_n$ one can write

$$p_{n+2}^\alpha \leq (p_{n+1} + p_n)^\alpha \leq p_{n+1}^\alpha + p_n^\alpha.$$

Therefore i. is valid for $0 < \alpha \leq 1$. The same is true for ii. Indeed, for $n = 1$ we must have $5^\beta \leq 2^\beta + 3^\beta$ and this yields $0 < \beta \leq 1$. We now prove by induction that ii. is true for all n. By $p_{n+2} \leq p_{n+1} + p_n$ one has

$$p_{n+2}^\beta \leq (p_{n+1}p_n)^\beta \leq p_{n+1}^\beta + p_n^\beta$$

(see Lemma)

$$\leq p_1^\beta + p_2^\beta + \ldots + p_{n-1}^\beta + p_n^\beta.$$

So, if the property is true for $n-1$, it is also true for n.

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8 On certain new conjectures in prime number theory

The OQ.413 [1], M. Bencze states the conjecture that if p_n denotes the n-th prime number, then with $p_1, p_2, \ldots p_{n^2}$ it is not possible to construct a magic square.

Let d be the magic constant of such a square. Then clearly the sum of all rows is nd, i.e.

$$\sum_{k \leq n^2} p_k = nd.$$

This particularly gives the divisibility property

$$n|(p_1 + p_2 + \ldots + p_{n^2}) \tag{1}$$

Ley n be even. Then $p_1 + p_2 + \ldots + p_{n^2}$ contains an odd number of odd primes, so it is odd. Since $p_1 = 2$, $p_1 + \ldots + p_{n^2}$ is odd. Therefore (1) cannot be true for $n =$ even. This shows that the above conjecture is true for n even. For $n = 3$, however, (1) is true. For $n = 5$, it is not true. In the book [2], by Rouse Ball one can find that J.N. Muncey in 1913 constructed a magic square of 12th order which involve the first 143 odd primes and 1, namely 1, 3, 5, 7, 11,. . . , 827 (and this cannot be true for squares of order $n < 12$). Now, this result for general n raises the problem

$$n|(p_1 + p_2 + \ldots + p_{n^2-1} - 1) \tag{2}$$

which is true for $n = 12$. Since $p_1 + p_2 + \ldots + p_{n^2} = (1 + 3 + \ldots + p_{n^2-1}) + (p_{n^2} + 1)$, if

$$n|(p_{n^2} + 1) \tag{3}$$

then if (2) is true, clearly (1) cannot be true, and vice-versa. The curious divisibility property (3) is true for $n = 1, 2, 3, 8, 12, 37, 72$ etc. I conjecture here that (3) holds true for infinitely many n. A similar conjecture would be

$$n|(p_{n^2} - 1) \tag{4}$$

This is true for $n = 1, 2, 4, 6, 10, 11$, etc. Probably, there are infinitely many. I cannot decide for what odd numbers n, relation (1) holds true. The same for numbers n with the property (2). I conjecture that (1), as well as (2), hold true for infinitely many n.

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9 On certain conjectures by Russo

In a recent note [1] F. Russo published ten conjectures on prime numbers. Here we prove three of them. (For solutions of other conjectures for large n, see [2]).

Conjecture 3 is the following:

$$e^{\sqrt{\frac{n+1}{p_{n+1}}}}/e^{\sqrt{\frac{p_n}{n}}} < e^{\sqrt{\frac{3}{5}}}/e^{\sqrt{\frac{3}{2}}} \tag{1}$$

Written equivalently as

$$e^{\sqrt{\frac{n+1}{p_{n+1}}}+\sqrt{\frac{3}{2}}} < e^{\sqrt{\frac{p_n}{n}}+\sqrt{\frac{3}{5}}},$$

we have to prove that

$$\sqrt{\frac{n+1}{p_{n+1}}} + \sqrt{\frac{3}{2}} < \sqrt{\frac{p_n}{n}} + \sqrt{\frac{3}{5}}. \tag{2}$$

For $n \leq 16$, (2) can be verified by calculations. Now, let $n \geq 17$. Then $p_n > 3n$. Indeed, $p_{17} = 53 > 3 \cdot 17 = 51$. Assuming this inequality to be valid for n, one has $p_{n+1} \geq p_n + 2 > 3n + 2$ so $p_{n+1} \geq 3n + 3 = 3(n + 1)$. But $3(n + 1)$ is divisible by 3, so $p_{n+1} > 3(n + 1)$. Since $\dfrac{n+1}{p_{n+1}} \leq \dfrac{1}{3}$, it is sufficient to prove that

$$\sqrt{3} + \sqrt{\frac{3}{5}} > \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{3}},$$

i.e. $3 + \dfrac{3}{\sqrt{5}} > \dfrac{3}{\sqrt{2}} + 1$ or $2 > 3\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{5}}\right)$, i.e. $2\sqrt{10} > 3\left(\sqrt{5} - \sqrt{2}\right)$. This is easily seen to be true. Therefore (2), i.e. (1) is proved.

Remark. The proof shows that (2) is valid whenever a sequence (p_n) of positive integers satisfies $p_n > 3n$.

Conjecture 5 is

$$\log d_n - \log \sqrt{d_n} < \frac{1}{2} n^{3/10}, \quad \text{where} \quad d_n = p_{n+1} - p_n. \tag{3}$$

By $\log \sqrt{d_n} = \dfrac{1}{2} \log d_n$, (3) can be written as

$$\log d_n < n^{3/10}. \tag{4}$$

It is immediate that (4) holds for sufficiently large n since $d_n < p_n$ and $\log p_n \sim \log n$ $(n \to \infty)$ while $\log n < n^{3/10}$ for sufficiently large n. Such arguments appear in [2].

Now we completely prove the left side of **conjecture 8**. We will prove a stronger relation, namely

$$\frac{\sqrt{p_{n+1}} - \log p_{n+1}}{\sqrt{p_n} - \log p_n} > 1 \quad (n \geq 3) \tag{5}$$

Since $\dfrac{\sqrt{3} - \log 3}{\sqrt{2} - \log 2} < 1$, (5) will be an improvement. The logarithmic mean of two positive numbers is

$$L(a, b) = \frac{b - a}{\log b - \log a}.$$

It is well-known that $L(a, b) > \sqrt{ab}$ for $a \neq b$. Now let $a = p_{n+1}$, $b = p_n$. Then $\sqrt{ab} > \sqrt{a} + \sqrt{b}$ is equivalent to $\sqrt{p_{n+1}}\left(\sqrt{p_n} - 1\right) > \sqrt{p_n}$. If $\sqrt{p_n} - 1 \geq 1$, i.e. $p_n \geq 4$ $(n \geq 3)$, this is true. Now,

$$\frac{p_{n+1} - p_n}{\log p_{n+1} - \log p_n} > \sqrt{p_n p_{n+1}} > \sqrt{p_n} + \sqrt{p_{n+1}}$$

gives

$$\frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} > \log p_{n+1} - \log p_n,$$

i.e.

$$\sqrt{p_{n+1}} - \log p_{n+1} > \sqrt{p_n} - \log p_n.$$

This is exactly inequality (5). We can remark that (5) holds true for any strictly increasing positive sequence such that $p_n \geq 4$.

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10 On certain limits related to prime numbers

1. Let p_n denote the nth prime number. The famous prime number theorem states (in equivalent form) that

$$\frac{p_n}{n\log n} \to 1 \text{ as } n \to \infty. \tag{1}$$

(In what follows, for simplicity we will note $x_n \to a$ when $\lim\limits_{n\to\infty} x_n = a$). There are some immediate consequences of this relation, for example:

$$\frac{p_{n+1}}{p_n} \to 1; \tag{2}$$

$$\frac{\log p_n}{\log n} \to 1. \tag{3}$$

Without logarithms, (1) and (3) have the form

$$n^{n/p_n} \to e; \tag{4}$$

$$p_n^{1/\log n} \to e. \tag{5}$$

From (2) easily follows

$$\sqrt[n]{p_n} \to 1; \tag{6}$$

while (1) and (2) imply

$$\frac{p_{n+1} - p_n}{n\log n} \to 0. \tag{7}$$

In paper [1] there were stated a number of 106 Conjectures on certain inequalities related to (p_n). The above limits, combined with Stolz-Cesaro's theorem, Stirling's theorem on $n!$, simple inequalities imply the following relations (see [7], [8]):

$$\frac{\log n}{\dfrac{1}{p_1} + \ldots + \dfrac{1}{p_n}} \to \infty; \tag{8}$$

$$\frac{p_1 + p_2 + \ldots + p_n}{\dfrac{n(n+1)}{2}\log n} \to 1; \tag{9}$$

$$\frac{p_{[\log n]}}{\log p_n} \to \infty; \tag{10}$$

$$\sqrt[p_n]{p_{n+1}p_{n+2}} \to 1; \tag{11}$$

$$\frac{\sqrt[n]{p_1 p_2 \dots p_n}}{n!} \to 0; \tag{12}$$

$$\frac{p_{(n+1)!} - p_{n!}}{n p_n} \to \infty; \tag{13}$$

$$\frac{p_{n!}}{(p_n)!} \to 0; \tag{14}$$

$$\frac{p_{n!}}{p_1 p_2 \dots p_n} \to 0; \tag{15}$$

$$\frac{p_{(n+1)!} - p_{n!}}{(p_{n+1} - p_n)!} \to \infty; \tag{16}$$

$$\frac{p_1 + p_2 + \ldots + p_n}{p_{1!} + p_{2!} + \ldots + p_{n!}} \to 0; \tag{17}$$

$$\frac{\log\log p_{n+1} - \log\log p_n}{\log p_{n+1} - \log p_n} \to 0; \tag{18}$$

$$\frac{1}{p_n} \log \frac{e^{p_{n+1}} - e^{p_n}}{p_{n+1} - p_n} \to 1; \tag{19}$$

$$\limsup \left(\frac{p_{p_{n+1}} - p_{p_n}}{p_{n+1} - p_n} \right) = +\infty; \tag{20}$$

$$\liminf \left(\sqrt[3]{p_{n+1} p_{n+2}} - \sqrt[3]{p_n p_{n+1}} \right) = 0; \tag{21}$$

$$\liminf \left(p_{\left[\sqrt{n+1}\right]} - p_{\left[\sqrt{n}\right]} \right) = 0; \tag{22}$$

$$\limsup \left(p_{\left[\sqrt{n+1}\right]} - p_{\left[\sqrt{n}\right]} \right) = \infty; \tag{23}$$

$$\liminf p_n^{\lambda} \left(\sqrt{p_{n+1}} - \sqrt{p_n} \right) = 0 \quad \left(\lambda \in \left(0, \frac{1}{2} \right) \right); \tag{24}$$

$$\limsup p_n^{1-\frac{1}{k}} \left(\sqrt[k]{p_{n+1}} - \sqrt[k]{p_n} \right) = +\infty \quad (k \geq 2,\ k \in \mathbb{N}), \quad \text{etc.} \tag{25}$$

With the use of these limits, a number of conjectures were shown to be false or trivial. On the other hand, a couple of conjectures are very difficult at present. Clearly, (24) implies

$$\liminf \frac{p_{n+1} - p_n}{\sqrt{p_n}} = 0. \tag{26}$$

A famous unproved conjecture of Cramér [3] states that

$$\liminf \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1. \tag{27}$$

If this is true, clearly one can deduce that

$$\limsup \frac{p_{n+1} - p_n}{(\log p_n)^2} \leq 1. \tag{28}$$

Even

$$\limsup \frac{p_{n+1}-p_n}{(\log p_n)^2} < \infty \tag{29}$$

seems very difficult. A conjecture of Schinzel [2] states that between x and $x+(\log x)^2$ there is always a prime. This would imply $p_n < p_{n+1} < p_n + (\log p_n)^2$, so

$$\frac{p_{n+1}-p_n}{\sqrt{p_n}} \to 0. \tag{30}$$

Probably, this is true. A result of Huxley [4] says that with the notation $d_n = p_{n+1}-p_n$ one has $d_n < p_n^{\frac{7}{12}+\varepsilon}$ ($\varepsilon > 0$), and the Riemann hypothesis would imply $d_n < p_n^{\frac{1}{2}+\varepsilon}$. Even these statements wouldn't imply (30). Erdös and Turán [5] have proved that $\frac{d_{n+1}}{d_n} > 1$ for infinitely many n, while $\frac{d_{m+1}}{d_m} < 1$ for infinitely many m; probably

$$\limsup \frac{d_{n+1}}{d_n} = +\infty \tag{31}$$

is true.

2. In [12] it is shown that

$$\log p_n - \frac{p_n}{n} \to 1. \tag{32}$$

Therefore

$$\log p_{n+1} - \frac{p_{n+1}}{n+1} - \log p_n + \frac{p_n}{n} \to 0,$$

so by putting $x_n = \frac{p_{n+1}}{n+1} - \frac{p_n}{n}$, by $\log p_{n+1} - \log p_n \to 0$, we get

$$x_n \to 0. \tag{33}$$

Thus

$$|x_n| \to 0, \tag{34}$$

implying $|x_n| \le 1/2$ for sufficiently large n. This settles essentially conjecture 81 of [1] (and clearly, improves it, for large n). Now, by a result of Erdös and Prachar [6] one has

$$c_1 \log^2 p_n < \sum_{m=1}^{n} |x_m| < c_2 \log^2 p_n$$

($c_1, c_2 > 0$ constants), so we obtain

$$\limsup \left(\frac{|x_1| + \ldots + |x_n|}{\log^2 p_n} \right) < \infty; \tag{35}$$

$$\liminf \left(\frac{|x_1| + \ldots + |x_n|}{\log^2 p_n} \right) > 0; \tag{36}$$

it would be interesting to obtain more precise results. By applying the arithmetic-geometric inequality, one obtains

$$\limsup \frac{n}{\log^2 p_n} |x_1 x_2 \ldots x_n|^{1/n} < \infty. \tag{37}$$

What can be said on lim inf of this expression?

3. In paper [11] there are stated ten conjectures on prime numbers. By the following limits we can state that the inequalities stated there are true for all sufficiently large values of n. By Huxley's result (for certain improvements, see [2]),

$$\frac{n^\alpha d_n}{p_{n+1} p_n} < \frac{n^\alpha}{n^{5/12-\varepsilon} (\log n)^{5/12-\varepsilon}} \to 0,$$

so if $\alpha < 5/12 - \varepsilon$, we have

$$\frac{p_{n+1} - p_n}{p_{n+1} + p_n} < n^{-\alpha} \tag{38}$$

for sufficiently large n. This is related to conjecture 2 of [11].

We now prove that

$$\frac{n^{\log p_{n+1}}}{(n+1)^{\log p_n}} \to 1, \tag{39}$$

this settles conjecture 7 for all large n, since $\dfrac{1}{2^{\log 2}} < 1$ and $\dfrac{30^{\log 127}}{31^{\log 113}} > 1$. In order to prove (39), remark that the expression can be written as $\left(\dfrac{n}{n+1} \right)^{\log p_n} \cdot \left(n^{\log p_{n+1} - \log p_n} \right)$. Now,

$$\left(\frac{n+1}{n} \right)^{\log p_n} = \left[\left(\frac{n+1}{n} \right)^{\log n} \right]^{\log p_n / \log n} \to 1^1 = 1,$$

since

$$\left(\frac{n+1}{n} \right)^{\log n} = \left[\left(1 + \frac{1}{n} \right)^n \right]^{\frac{\log n}{n}} \to e^0 = 1$$

and apply relation (3). Therefore, it is sufficient to prove

$$n^{\log p_{n+1} - \log p_n} \to 1. \tag{40}$$

By Lagrange's mean value theorm applied to the function $t \mapsto \log t$ on $t \in [p_n, p_{n+1}]$ we easily can deduce

$$\frac{p_{n+1} - p_n}{p_{n+1}} < \log p_{n+1} - \log p_n < \frac{p_{n+1} - p_n}{p_n}.$$

Therefore, it is sufficient to prove

$$n^{(p_{n+1}-p_n)/p_n} \to 1; \tag{41}$$

$$n^{(p_{n+1}-p_n)/p_{n+1}} \to 1. \tag{42}$$

By (2), (42) follows from (41). Now, for (41) it is enough to prove (by taking logarithms) that $\dfrac{p_{n+1}-p_n}{p_n}\log n \to 0$, or, by using (1); that

$$\frac{p_{n+1}-p_n}{n} \to 0. \tag{43}$$

This is stronger than (7), but it is true, and follows clearly e.g. by $d_n < n^{7/12+\varepsilon}$. This finishes the proof of (39).

Conjectures (8) and (10) of [11] are clearly valid for sufficiently large n, since

$$\frac{\sqrt{p_{n+1}}-\log p_{n+1}}{\sqrt{p_n}-\log p_n} \to 1 \tag{44}$$

and

$$\frac{\sqrt{p_n}-\log p_{n+1}}{\sqrt{p_{n+1}}-\log p_n} \to 1. \tag{45}$$

Indeed,

$$\frac{\sqrt{p_{n+1}}\left(1-\log p_{n+1}/\sqrt{p_{n+1}}\right)}{\sqrt{p_n}\left(1-\log p_n/\sqrt{p_n}\right)} \to 1\cdot\left(\frac{1-0}{1-0}\right)=1, \quad \text{etc.}$$

Now, conjecture (9) is true for large n, if one could prove that

$$\frac{(\log p_{n+1})^{\sqrt{p_n}}}{(\log p_n)^{\sqrt{p_n+1}}} \to 1. \tag{46}$$

Since this expression can be written as $\left(\dfrac{\log p_{n+1}}{\log p_n}\right)^{\sqrt{p_n}}(\log p_n)^{\sqrt{p_n+1}-\sqrt{p_n}}$, we will prove first that

$$(\log p_n)^{\sqrt{p_{n+1}}-\sqrt{p_n}} \to 1. \tag{47}$$

By logarithmation,

$$(\sqrt{p_{n+1}}-\sqrt{p_n})\log\log p_n = \frac{d_n}{\sqrt{p_n}+\sqrt{p_{n+1}}}\log\log p_n < \frac{p_n^{7/12+\varepsilon}}{2\sqrt{p}_n}\log\log p_n \to 0,$$

so indeed (47) follows.

Now, the limit

$$\left(\frac{\log p_{n+1}}{\log p_n}\right)^{\sqrt{p_n}} \to 1 \tag{48}$$

seems difficult. By taking logarithms, $\sqrt{p_n}\log\left(\frac{\log p_{n+1}}{\log p_n}\right) \to 0$ will follow, if we suppose that

$$\log\left(\frac{\log p_{n+1}}{\log p_n}\right) < \frac{1}{n} \tag{49}$$

is true for sufficiently large n. This is exactly conjecture 6 of [11]. Now, by (49) we get (48), since clearly $\frac{\sqrt{p_n}}{n} \to 0$ (e.g. by (1)). Therefore one can say that conjecture 6 implies conjecture 9 in [11] (for large values of n).

4. I can prove that Conjecture 6 holds true for infinitely many n, in fact a slightly stronger result is obtainable. The logarithmic mean $L(a,b)$ of two positive numbers a, b is defined by

$$L(a,b) = \frac{b-a}{\log b - \log a}.$$

It is well-known that (see e.g. [13])

$$\sqrt{ab} < L(a,b) < \frac{a+b}{2}.$$

Thus

$$\log\left(\frac{\log p_{n+1}}{\log p_n}\right) = \log(\log p_{n+1}) - \log(\log p_n) < \frac{\log p_{n+1} - \log p_n}{\sqrt{\log p_n \log p_{n+1}}} <$$

$$< \frac{p_{n+1} - p_n}{\sqrt{p_n p_{n+1} \log p_n \log p_{n+1}}} < \frac{p_{n+1} - p_n}{\log p_n} \cdot \frac{1}{p_n} = \frac{b_n}{p_n}.$$

Now, if

$$b_n < \frac{p_n}{n}, \tag{50}$$

then Conjecture 6 is proved. The sequence (b_n) has a long history. It is known (due to Erdös) that $b_n < 1$ for infinitely many n. Since $\frac{p_n}{n} > 1$, clearly (50) holds for infinitely many n. It is not known that

$$\liminf b_n = 0, \tag{51}$$

but we know that

$$\limsup b_n = +\infty. \tag{52}$$

The relation

$$\frac{b_1 + b_2 + \ldots + b_n}{n} \to 1 \tag{53}$$

is due to L. Panaitopol, many other results are quoted in [9].

Remarks. 1) Conjecture 5, i.e. $\log d_n < n^{3/10}$ is true for large n by Huxley's result.

2) Conjectures 3 and 8 (left side) are completely settled by other methods ([10]).

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11 On the least common multiple of the first n positive integers

1. A. Murthy [1] and F. Russo [2] recently have considered the sequence $(a(n))$, where $a(n) = [1, 2, \ldots, n]$ denotes the l.c.m. of the positive integers $1, 2, \ldots, n$.

We note that $a(n)$ has a long-standing and well known connection with the famous "prime-number theorem". Indeed, let Λ be the Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \ (p \text{ prime}) \\ 0, & \text{otherwise} \end{cases}$$

Put $\psi(x) = \sum_{n \le x} \Lambda(n)$, known as one of the Chebysheff's function. Now

$$\sum_{m \le n} \Lambda(m) = \sum_{p^k \le n} \log p = \log \prod_{p^k \le n} p.$$

Let k_p be the largest positive integer with $p^{k_p} \le n$. Then

$$\log \prod_{p \le n} p^{k_p} = \log a(n)$$

on the base of the known calculation of l.c.m. Therefore

$$a(n) = e^{\psi(n)} \tag{1}$$

where $e^x = \exp(x)$. By the equivalent formulation of the prime number-theorem one has $\dfrac{\psi(n)}{n} \to 1$ as $n \to \infty$, giving by (1):

$$\lim_{n \to \infty} \sqrt[n]{a(n)} = e. \tag{2}$$

Now, by Cauchy's test of convergence of series of positive terms, this gives immediately that

$$\sum_{n \ge 1} \frac{1}{a(n)} \quad \text{and} \quad \sum_{n \ge 1} \frac{a(n)}{n!} \tag{3}$$

are convergent series; the first one appears also as a problem in Niven-Zuckerman [3]. Problem 21.3.2 of [4] states that this series is irrational. A similar method shows that the second series is irrational, too.

2. Relation (2) has many interesting applications. For example, this is an important tool in the Apéry proof of the irrationality of $\zeta(3)$ (where ζ is the Riemann zeta function). For same methods see e.g. Alladi [7]. See also [8]. From known estimates for the function ψ, clearly one can deduce relations for $a(n)$. For example, Rosser and Schoenfeld [5] have shown that $\dfrac{\psi(x)}{x}$ takes its maximum at $x = 113$ and $\dfrac{\psi(x)}{x} < 1.03883$ for $x > 0$. Therefore $\left(\sqrt[n]{a(n)}\right)$ takes its greatest value for $n = 113$, and

$$\sqrt[n]{a(n)} < e^{1.03883} \quad \text{for all} \quad n \geq 1. \tag{4}$$

Costa Pereira [6] proved that $\dfrac{530}{531} < \dfrac{\psi(x)}{x}$ for $x \geq 70841$ and $\dfrac{\psi(x)}{x} < \dfrac{532}{531}$ for $x \geq 60299$; giving

$$e^{530/531} < \sqrt[n]{a(n)} < e^{532/531} \quad \text{for} \quad n \geq 70841. \tag{5}$$

A. Perelli [9] proved that if $N^{\theta+\varepsilon} < H \leq N$, then $\psi(x+H) - \psi(x) \sim H$ for almost all x ($\theta \in (0,1)$ is given), yielding:

$$\log \frac{a(n+H)}{a(n)} \sim H \quad \text{for almost all} \quad n, \tag{6}$$

for $N^{\theta+\varepsilon} < H \leq N$.

M. Nair [10] has shown by a new method that $\sum\limits_{n \leq x} \psi(n) \geq \alpha x^2$ for all $x \geq x_0$, where $\alpha = 0.49517\ldots$; thus:

$$\sum_{m \leq n} \log a(m) \geq \alpha n^2 \quad \text{for} \quad n \geq n_0. \tag{7}$$

Let $\Delta(x) = \psi(x) - x$. Assuming the Riemann hypothesis, it can be proved that $\Delta(x) = O\left(\sqrt{x}\log^2 x\right)$; i.e.

$$\log a(n) - n = O\left(\sqrt{n}\log^2 n\right). \tag{8}$$

This is due to von Koch [11]. Let

$$D(x) = \frac{1}{x}\int_1^x |\Delta(t)|dt.$$

By the Riemann hypothesis, Cramér [12] proved that $D(x) = O\left(\sqrt{x}\right)$ and S. Knaponski [13] showed that

$$D(x) > \sqrt{x}\exp\left(-c\frac{\log x}{\log\log x}\cdot\log\log\log x\right).$$

Without any hypothesis, J. Pintz [14] proved that

$$D(x) > \frac{\sqrt{x}}{2200} \quad \text{for} \quad x > 2. \tag{9}$$

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Chapter 5. Some irrationality results

"... it is often possible to attain by elementary methods results which are inaccessible to the powerful analytic methods which operate so successfully in other cases..."

(A.O. Gelfond and Yu.V. Linnik, Elementary methods in the analytic theory of numbers, Pergamon Press, 1966)

1 An irrationality criterion

There are many results on the irrationality of infinite series. These results are proved sometimes by Euler-type arguments or by some special tools from diophantine approximation theory (see e.g. [1-3]).

In what follows, we will obtain by very simple arguments a general theorem on the irrationality of some series and - as we shall see - with non-trivial applications. See also [4].

Theorem. *Let $(p_n), (q_n)$ be sequences of natural numbers and*

$$u_n = \sum_{k=1}^{n} p_k/q_k, \quad n = 1, 2, 3, \ldots$$

Suppose that the following conditions are satisfied

1) $q_n \geq 1,\ n = 1, 2, \ldots;\ p_n > 0$ for infinitely many numbers n;

2) there exists a sequence (a_n) of real numbers such that $|u_m - u_n| \leq a_n$, for all $m, n = 1, 2, \ldots,\ m > n$;

3) $\lim\limits_{n\to\infty} a_n[q_1, \ldots, q_n] = 0$, where $[q_1, \ldots, q_n]$ denotes the least common multiple of the numbers $q_1, \ldots, q_n$.

Then the series $\sum\limits_{k=1}^{\infty} p_k/q_k$ is convergent and has an irrational value.

Proof. Condition (3) implies $a_n \to 0$ $(n \to \infty)$, so by (2), the sequence (u_n) is fundamental. It is well-known that such a sequence must be convergent, let θ be its limit. Let us assume now that θ would be rational, i.e. $\theta = a/b$ with $a, b \in Z,\ b > 0$. Letting $m \to \infty$ in (2) we get

$$\left|\frac{a}{b} - u_n\right| \leq a_n,$$

i.e.

$$|a[q_1, \ldots, q_n] - bh_n| \leq ba_n[q_1, \ldots, q_n], \qquad (*)$$

where h_n denotes the numerator of the fraction in $u_n = \sum\limits_{k=1}^{n} p_k/q_k$, after we have effectuated the common denominator. Using again (3), we find that the right side of $(*)$ is inferior of 1 for sufficiently large n $(n \geq n_0)$. On the other hand, the left side of $(*)$ is an integer number, so evidently $a[q_1, \ldots, q_n] - bh_n = 0,\ n \geq n_0$.

Thus $u_n = h_n/[q_1, \ldots, q_n] = p/b = \text{constant}$ for $n \geq n_0$. Since $u_{n+1} = u_n + p_n/q_n$ and $p_n < 0$ for infinitely many a, on base of (1), this condition is impossible. In other words, we have obtained a contradiction which finishes the proof of the irrationality of θ and thus of the theorem.

Corollary. *Let (a_n) be a sequence of real numbers with the property $n!a_n \to 0$ $(n \to \infty)$ and let (p_n) be a sequence of natural numbers, satisfying*

i) $p_n > 0$ for infinitely many numbers n;

ii) $p_{n+1} < 2p_n$, for all sufficiently large n;

iii) $p_{n+1}/(n+1) \leq n!a_n/2$, for all sufficiently large n.

Then $\sum_{n=1}^{\infty} p_n/n!$ is irrational.

Proof. Apply the theorem with $q_n = n!$. Then (1), (3) hold trivially, so we have to prove relation (2). We have

$$|u_m - u_n| = \frac{1}{(n+1)!}\left[p_{n+1} + \frac{p_{n+2}}{n+2} + \ldots + \frac{p_m}{(n+2)(n+3)\ldots(n+m-n)}\right] <$$

$$< \frac{p_{n+1}}{(n+1)!}\left[1 + \frac{2}{n+2} + \left(\frac{2}{n+2}\right)^2 + \ldots + \left(\frac{2}{n+2}\right)^{m+n-1}\right]$$

since (ii) implies $p_{n+k} < 2^{k-1}p_{n+1}$ and $1/(n+2)(n+3) < (n+2)^{-2}, \ldots, 1/(n+2)\ldots(n+m-1) < (n+2)^{-m+n}$ are immediate, for all sufficiently large n. Now,

$$|u_m - u_n| < \frac{p_{n+1}}{(n+1)!}(1 - (2/(n+2))^{m-n})/(1 - 2/(n+2)) < a_n,$$

by (iii). Thus the theorem may be applied to this particular case.

Application. $\sum_{n=1}^{\infty} \frac{[\ln n]}{n!} \notin Q$ (where $[x]$ is the integer part of x).

Proof. One has $[\ln(n+1)] \leq \ln(n+1) < 2(\ln n - 1)$ by $n^2 > e^2(n+1)$, true for large n. Since $2(\ln n - 1) < 2[\ln n]$, (ii) holds true. Let $a_n = \ln(n+1)/(n!\sqrt{n})$. Then, clearly, $n!a_n \to 0$ $(n \to \infty)$ and

$$p_{n+1}/(n+1) \leq \ln(n+1)/(n+1) \leq n!\ln(n+1)/(2n!\sqrt{n})$$

since $2\sqrt{n} \leq n+1$. So, condition (iii) is also verified; therefore the corollary gives the irrationality of the above series.

Remarks.

1) The proof shows that one can assume (p_n), (q_n) integers, $q_n \geq 1$, $p_n \geq 0$ for infinitely many n.

2) For a recent application of the Theorem of this note, we quote [5].

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2 On the irrationality of certain alternative Smarandache series

1. Let S be the Smarandache function. In paper [1] it is proved the irrationality of $\sum_{n=1}^{\infty} \frac{S(n)}{n!}$. We note here that this result is contained in the following more general theorem (see e.g. [2]).

Theorem 1. *Let (x_n) be a sequence of natural numbers with the properties:*

(1) there exists $n_0 \in N^*$ such that $x_n \leq n$ for all $n \geq n_0$;

(2) $x_n < n-1$ for an infinity of n;

(3) $x_m > 0$ for infinitely many m.

Then the series $\sum_{m=1}^{\infty} \frac{x_n}{n!}$ is irrational.

By letting $x_n = S(n)$, it is well known that $S(n) \leq n$ for $n \geq n_0 \equiv 1$, and $S(n) \leq \frac{2}{3}n$ for $n > 4$, composite. Clearly, $\frac{2}{3}n < n-1$ for $n > 3$. Thus the irrationality of the second constant of Smarandache ([1]) is contained in the above result.

2. We now prove a result on the irrationality of the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{S(n)}{n!}$.

We can formulate our result more generally, as follows:

Theorem 2. *Let $(a_n), (b_n)$ be two sequences of positive integers having the following properties:*

(1) $n | a_1 a_2 \dots a_n$ for all $n \geq n_0$ ($n_0 \in N^*$);

(2) $\frac{b_{n+1}}{a_{n+1}} < b_n \leq a_n$ for $n \geq n_0$;

(3) $b_m < a_m$, where $m \geq n_0$ is composite. Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{b_n}{a_1 a_2 \dots a_n}$ is convergent and has an irrational value.

Proof. It is sufficient to consider the series $\sum_{n=n_0}^{\infty}(-1)^{n-1}\frac{b_n}{a_1 a_2 \dots a_n}$. The proof is very similar (in some aspect) to Theorem 2 in our paper [3]. Let $x_n = \frac{b_n}{a_1 a_2 \dots a_n}$ ($n \geq n_0$). Then $x_n \leq \frac{1}{a_1 \dots a_{n-1}} \to 0$ since (1) gives $a_1 \dots a_k \geq k \to \infty$ (as $k \to \infty$). On the other hand, $x_{n+1} < x_n$ by the first part of (2). Thus the Leibniz criteria assures the convergence of the series. Let us now assume, on the contrary, that the series has a rational value,

say $\frac{a}{k}$. First we note that we can choose k in such a manner that $k+1$ is composite, and $k > n_0$. Indeed, if $k+1 = p$ (prime), then $\frac{a}{p-1} = \frac{ca}{c(p-1)}$. Let $c = 2ar^2 + 2r$, where r is arbitrary. Then $2a(2ar^2+2r)+1 = (2ar+1)^2$, which is composite. Since r is arbitrary, we can assume $k > n_0$. By multiplying the sum with $a_1 a_2 \dots a_k$, we can write:

$$a\frac{a_1 \dots a_k}{k} = \sum_{n=n_0}^{k} (-1)^{n-1} \frac{a_1 \dots a_k}{a_1 \dots a_n} b_n + (-1)^k \left(\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1}a_{k+2}} + \dots \right).$$

The alternating series on the right side is convergent and must have an integer value. But it is well known its value lies between $\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1}a_{k+2}}$ and $\frac{b_{k+1}}{a_{k+1}}$. Here $\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1}a_{k+2}} > 0$ on base of (3). On the other hand $\frac{b_{k+1}}{a_{k+1}} < 1$, since $k+1$ is a composite number. Since an integer number has a value between 0 and 1, we have obtained a contradiction, finishing the proof of the theorem.

Corollary. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{S(n)}{n!}$ *is irrational.*

Proof. Let $a_n = n$. Then condition (1) of Theorem 2 is obvious for all n; (2) is valid with $n_0 = 2$, since $S(n) \le n$ and $S(n+1) \le n+1 = (n+1) \cdot 1 < (n+1)S(n)$ for $n \ge 2$. For composiye m we have $S(m) \le \frac{2}{3}m < m$, thus condition (3) is verified, too.

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3 On the Irrationality of Certain Constants Related to the Smarandache Function

1. Let $S(n)$ be the Smarandache function. Recently I. Cojocaru and S. Cojocaru [2] have proved the irrationality of $\sum_{n=1}^{n} \frac{S(n)}{n!}$.

The author of this note [5] showed that this is a consequence of an old irrationality criteria (which will be used here once again), and proved a result implying the irrationality of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{S(n)}{n!}$.

E. Burton [1] has studied series of type $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$, which has a value $\in \left(e-\frac{5}{2}, \frac{1}{2}\right)$. He showed that the series $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$ is convergent for all $r \in \mathbb{N}$. I. Cojocaru and S. Cojocaru [3] have introduced the "third constant of Smarandache" namely $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$, which has a value between $\frac{71}{100}$ and $\frac{97}{100}$. Our aim in the following is to prove that the constants introduced by Burton and Cojocaru-Cojocaru are all irrational.

2. The first result is in fact a refinement of an old irraionality criteria (see [4] p.5):

Theorem 1. *Let (x_n) be a sequence of nonnegative integers having the properties:*

(1) there exists $n_0 \in \mathbb{N}^*$ such that $x_n \leq n$ for all $n \geq n_0$;

(2) $x_n < n-1$ for infinitely many n;

(3) $x_m > 0$ for an infinity of m.

Then the series $\sum_{n=1}^{\infty} \frac{x_n}{n!}$ is irrational.

Let now $x_n = S(n-1)$. Then

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!} = \sum_{n=3}^{\infty} \frac{x_n}{n!}.$$

Here $S(n-1) \leq n-1 < n$ for all $n \geq 2$; $S(m-1) < m-2$ for $m > 3$ composite, since by $S(m-1) < \frac{2}{3}(m-1) < m-2$ for $m > 4$ this holds true. (For the inequality $S(k) < \frac{2}{3}k$ for $k > 3$ composite, see [6]). Finally, $S(m-1) > 0$ for all $m \geq 1$. This proves the irrationality of $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$.

Analogously, write

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} = \sum_{m=r+2}^{\infty} \frac{S(m-r)}{m!}.$$

Put $x_m = S(m-r)$. Here $S(m-r) \leq m-r < m$, $S(m-r) \leq m-r < m-1$ for $r \geq 2$, and $S(m-r) > 0$ for $m \geq r+2$. Thus, the above series is irrational for $r \geq 2$, too.

3. The third constant of Smarandache will be studied with the following irrationality criterion (see [4], p.8):

Theorem 2. *Let $(a_n), (b_n)$ be two sequences of nonnegative integers satisfying the following conditions:*

(1) $a_n > 0$ for an infinity of n;

(2) $b_n \geq 2$, $0 \leq a_n \leq b_n - 1$ for all $n \geq 1$;

(3) there exists an increasing sequence (i_n) of positive integers such that

$$\lim_{n\to\infty} b_{i_n} = +\infty, \quad \lim_{n\to\infty} a_{i_n}/b_{i_n} = 0.$$

Then the series $\sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \dots b_n}$ is irrational.

Corollary. *For $b_n \geq 2$, (b_n positive integers), (b_n) unbounded the series $\sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \dots b_n}$ is irrational.*

Proof. Let $a_n \equiv 1$. Since $\limsup_{n\to\infty} b_n = +\infty$, there exists a sequence (i_n) such that $b_{i_n} \to \infty$. Then $\frac{1}{b_{i_n}} \to 0$, and the three conditions of Theorem 2 are verified.

By selecting $b_n \equiv S(n)$, we have $b_p = S(p) = p \to \infty$ for p a prime, so by the above Corollary, the series $\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)\dots S(n)}$ is irrational.

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4 On the irrationality of e^t $(t \in \mathbb{Q})$

Let t be a nonzero rational number. The irrationality of e^t was demonstrated by Lambert and Legendre ([3]) by using continuous fractions, then by Hermite ([2]) with the application of integral calculus. A general criterion which particularly implies this irrationality has been found by Froda ([1]).

In what follows we shall obtain a new proof, suggested by an inequality for the exponential function ([4]).

Let $F(x) = 1 + \dfrac{x}{1!} + \ldots + \dfrac{x^n}{n!}$. By the infinite series of e^x one gets

$$\frac{e^x - F(x)}{x^{n+1}} = \frac{1}{(n+1)!}\left[1 + \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} + \ldots\right] \tag{1}$$

By taking into account of the differentiability of power series, we may take the derivative by n-times in (1). Applying the Leibniz formula for

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

for $f(x) = e^x - F(x)$, $g(x) = x^{-n-1}$, we immediately obtain

$$\left(\frac{e^x - F(x)}{x^{n+1}}\right)^{(n)} = \sum \binom{n}{k}\left[e^x - \left(1 + \frac{x}{1!} + \ldots + \frac{x^k}{k!}\right)\right](-1)^k \frac{(n+k)\ldots(n+1)}{x^{n+k+1}} =$$

$$= \sum \binom{n}{k} \frac{e^x(-1)^k(n+k)\ldots(n+1)}{x^{n+k+1}} -$$

$$- \sum \binom{n}{k}\left(1 + \frac{x}{1!} + \ldots + \frac{x^k}{k!}\right)(-1)^k(n+k)\ldots(n+1)/x^{n+k+1} =$$

$$= \frac{e^x G(x) - H(x)}{x^{2n+1}},$$

where $G(x)$ and $H(x)$ are polynomials with integer coefficients. By taking the derivative of the right side of (1), one obtains the equality

$$\frac{e^x G(x) - H(x)}{x^{2n+1}} = \frac{1}{(n+1)!} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)\ldots(m+n)}{(n+2)(n+3)\ldots(2n+m+1)} x^m \tag{2}$$

Let now x be a positive integer and suppose that there exist positive integers a, b such that $e^x = a/b$. From (2) we get

$$aG(x) - bH(x) = bx^{n+1} \sum_{n=0}^{\infty} \frac{(m+n)!}{m!(2n+m+1)!} x^m \tag{3}$$

On the left side of (3) there is an integer number. On the other hand, the obvious inequality $\dfrac{(n+m)!}{(2n+m+1)!} < \dfrac{n!}{(2n+1)!}$ implies that the right-hand side of (3) is less than $\dfrac{bx^{2n+1}n!}{(2n+1)!}e^x$. Since

$$\lim_{n\to\infty} \frac{x^{2n+1}n!}{(2n+1)!} = 0,$$

for suficiently large n, the above expression is < 1. The obtained contradiction proves the irrationality of e^x ($x \in \mathbb{N}^*$). The case $x = \dfrac{p}{q}$ ($p > 0$, $q \in \mathbb{Z}$) reduces to this case, since $e^{p/q} = a/b$ would imply $e^p = (a/b)^q = A/B \in Q$, contradiction.

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5 A transcendental series

Let $G_k(n)$ denote the number of digits in the decimal expansion of k^n, where $k \in \{2, 3, \ldots, 9\}$. Let us consider the sum

$$S_k = \sum_{n=1}^{\infty} \frac{G_k(n)}{k^n}. \tag{1}$$

We shall prove that S_k is **transcendental** for each k. Let n_m denote the least positive integer such that $G_k(n_m) = m$. Then $10^{m-1} \leq k^{n_m} < 10^m$, which by logarithmation gives

$$(m-1)\log 10 \leq n_m \log k \leq m \log 10,$$

i.e.

$$n_m < \frac{m \log 10}{\log k} \quad \text{and} \quad n_m \geq \frac{(m-1)\log 10}{\log k},$$

implying

$$n_m = 1 + \left[(m-1)\frac{\log 10}{\log k} \right], \quad (m \geq 1),$$

where $[x]$ denotes the greatest integer $\leq x$. On the other hand,

$$S_k = \sum_{m=1}^{\infty} \left(\sum_{n=n_m}^{\infty} k^{-n} \right) = \sum_{m=1}^{\infty} k^{1-n_m}.$$

Let

$$S(\lambda) = \sum_{m=1}^{\infty} k^{-[m\lambda]}, \quad \lambda \in R.$$

It is a consequence of the elaborated theorem by Roth that for irrational λ, $S(\lambda)$ is transcendental (for $k \geq 2$ integer). For such result, we quote the paper [1]. Now, since

$$S_k = 1 + S\left(\frac{\log 10}{\log k} \right),$$

it is sufficient to prove that $\dfrac{\log 10}{\log 2} = a$ is irrational. Let us suppose, on the contrary that $a = \dfrac{\log 10}{\log 2} = \dfrac{u}{v}$ $(u, v > 0$, integers). Then $10^v = 2^u$, which is a contradiction, since 2^u is not divisible by 5.

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6 Certain classes of irrational numbers

The theory of irrational numbers is a creation of modern mathematics, with its first results due to K. Weierstrass. Further, G. Cantor and R. Dedekind made the appearance of the conception of the whole construction of the system of real numbers, including the irrationals too.

However, this construction has not clarified a wide part a of the mystery of irrationality, and these secrets still remain not completely elucidated even today. It is sufficient to give some classical examples, as the open problems of the irrationality of $e + \pi$ or of Euler's constant

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right).$$

Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $(s > 1)$ be the classical Riemann ζ-function. In 1978, R. Apéry [1] proved the irrationality of $\zeta(3)$. (For new proofs, see F. Beukers [2] and M. Prévost [8]).

If $s = 2k$ is an even integer, then it is well-known that $\zeta(s)$ is irrational, and even transcendental. For odd values of $s \geq 5$, however, there remains in doubt the irrationality of $\zeta(s)$. Recently, in this direction we note a result by T. Rivoal [9] who proved that $\zeta(s)$ must be irrational for infinitely many odd values of s (but no one particular case is known for $s \geq 5$!).

1. The irrationality of many numbers may be easily established by using a well known property of polynomials with integer coefficients.

Proposition 1. *If the polynomial*

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \in \mathbb{Z}[x], \quad a_0 \neq 0$$

admits as a root an irreducible fraction $\frac{a}{b} \neq 0$*, then* a *divides* a_n *and* b *divides* a_0*.*

Corollary 1. *If* $k > 1$ *is an integer which is not an* n*-th power, then* $\sqrt[n]{k}$ *is irrational.*

Proof. Let us consider $P(x) = x^n - k$. By the made assumption, P has no integer roots. Let $x_0 = \frac{a}{b}$ be a rational root (where $(a, b) = 1$). Then by Proposition 1 one gets b divides 1, so $b = \pm 1$, impossible, since then $x_0 = \pm a =$ integer.

Therefore, all of $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$, etc. are irrationals.

The next corollary gives an information on certain irrational values of a trigonometric function.

Corollary 2. *For all integers $m \geq 5$, $\tan \dfrac{\pi}{m}$ is irrational.*

Let us first consider $m =$ odd. Remark that $\tan(m\alpha)$ can be represented as

$$\tan(m\alpha) = \frac{\pm x^m + P_m(x)}{1 + Q_m(x)}, \text{ where } x = \text{tg } \alpha,$$

and $P_m(x), Q_m(x)$ are certain polynomials of integer coefficients, of order at most $m-1$. This assertion follows at once by induction. Indeed, it is obvious for $m = 1$. Now, let us suppose that it is true for m, and consider

$$\tan(m+2)\alpha = \tan(m\alpha + 2\alpha) = \frac{\tan m\alpha + \tan 2\alpha}{1 - (\text{tg } m\alpha)(\tan 2\alpha)} =$$

$$= \left(\frac{\pm x^m + P_m(x)}{1 + Q_m(x)} + \frac{2x}{1-x^2}\right) \Big/ \left[1 - 2x\left(\frac{\pm x^m + P_m(x)}{(1-x^2)(1+Q_m(x)}\right)\right] =$$

$$= \frac{\pm x^{m+2} \pm x^m + (1-x^2)P_m(x) + 2x(1+Q_m(x))}{1 + Q_m(x) - x^2(1+Q_m(x)) - 2x(\pm x^m + P_m(x))}$$

which clearly imply that the representation is valid also for $m+2$.

Let now $\alpha = \dfrac{\pi}{m}$. From the above it follows that $\pm x_0^m + P_m(x_0) = 0$, thus Proposition 1 implies that $x_0 = \text{tg } \alpha$ must be an integer. On the other hand, for $m \geq 5$, one has $0 < \text{tg } \alpha < 1$, giving a contradiction.

Let now m be even, i.e. $m = 2^k n$, with $n =$ odd. By

$$\tan(2\alpha) = 2\tan\alpha/(1 - \tan^2\alpha)$$

and supposing $\tan \pi/2^k n \in \mathbb{Q}$ we obtain successively that

$$\tan(\pi/2^{k-1}n), \tan(\pi/2^{k-2}n), \ldots, \tan(\pi/2^0 n) = \tan \pi/n$$

are all rational, in contradiction with the above proved result.

2. By generalizing the Euler series, one obtains:

Proposition 2. *Let $1 \leq v_1 < v_2 < \ldots < v_n < \ldots$ be a sequence of integers. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(v_n)!}$ is irrational.*

Proof. Clearly $v_n \geq n$, so the Euler series of e implies the convergence. Let us suppose now that there exist integers $p, q \geq 1$ so that

$$\frac{1}{v_1!} + \frac{1}{v_2!} + \ldots + \frac{1}{v_n!} + \ldots = \frac{p}{q} \tag{1}$$

Since $v_n \to \infty$ $(n \to \infty)$, there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $v_n \geq q$. This gives

$$q | (v_n)! \tag{2}$$

From relation (1) one obtains

$$v_n! \left(\frac{1}{v_1!} + \ldots + \frac{1}{v_n!} \right) + \frac{v_n!}{v_{n+1}!} + \ldots = \frac{p}{q} v_n!$$

(where $n = N$). On base of (2), the expression

$$t = \frac{1}{(v_n + 1) \ldots v_{n+1}} + \ldots + \frac{1}{(v_n + 1) \ldots v_{n+k}} \tag{3}$$

must be an integer number. But, remark that, since $v_{n+m} - v_n \geq m$ (which follows at once from $v_{n+1} - v_n \geq 1$), we can write

$$0 < t < \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(n+1) \ldots (n+k)} + \ldots <$$

$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots = \frac{1}{n} < 1,$$

giving a contradiction to the integrality of t.

The following result has a similar proof:

Proposition 3. *Let us suppose that the sequence (v_n) of nonnegative integers satisfies the following conditions:*

(i) there exists $n_0 \in \mathbb{N}$ such that $v_n < n$ for all $n \geq n_0$;

(ii) $v_n < n - 1$ for infinitely many n;

(iii) $v_m > 0$ for infinitely many m.

Then the series $\sum_{n=1}^{\infty} \frac{v_n}{n!}$ is irrational.

Proof. We proceed as above, with the difference that we select n such that $n - 1 > q$, $v_n < n - 1$, $n \geq n_0$. Then multiplying both sides of the similar relation with (1) with $(n-1)!$, we get that

$$v = \frac{v_n}{n} + \frac{v_{n+1}}{n(n+1)} + \ldots$$

is an integer. On the other hand, by (iii) we have $v > 0$. By (ii) and the selection of n, one has

$$v < \frac{n-2}{n} + \frac{1}{n} + \frac{1}{n^2} + \ldots = \frac{n-2}{n} + \frac{1}{n-1} < 1,$$

so $0 < v < 1$, giving the desired contradiction.

Application. ([10])

1) $\sum_{n \text{ composite}} \varphi(n)/n! \notin \mathbb{Q}$

2) $\sum_{n=1}^{\infty} \varphi(n)/n! \notin \mathbb{Q}$,

3) $\sum_{p \text{ prime}} 1/p! \notin \mathbb{Q}$ (where φ denotes Euler's totient).

Proof. 1) Let $v_n = 0$, if n is prime; $= \varphi(n)$, if n is composite or $n = 1$. Now (i), (iii) are obvious, we prove (ii). Let n be even. Then it is well known that $\varphi(n) \leq \frac{n}{2} < n - 1$.

2) Select $v_n = \varphi(n)$.

3) Let $v_n = 1$ if n is prime; $= 0$, if n is composite or $n = 1$.

The above result cannot be applied when $v_n > n$ eventually. The following theorem extends the applicability of irrationality of certain series.

Proposition 4. (P. Erdös [5]) *Let (v_n) be a sequence of positive integers such that*

(i) $v_n/n^2 \to 0$ $(n \to \infty)$

(ii) $\{v_n/n\} \nrightarrow 1$ $(n \to \infty)$, where $\{x\}$ denotes the fractional part of x.

Then the series $\sum_{n=1}^{\infty} \frac{v_n}{n!}$ is irrational.

Proof. Clearly, the series is convergent, let S be its sum. Let us suppose that $S = \frac{p}{q}$, and let $k > q$. Then

$$\frac{v_k}{k} + \frac{v_{k+1}}{k(k+1)} + \frac{v_{k+2}}{k(k+1)(k+2)} + \ldots = (k-1)!\frac{p}{q}$$

is an integer; therefore

$$\frac{v_k}{k} - \left[\frac{v_k}{k}\right] + \frac{v_{k+1}}{k(k+1)} + \ldots \geq 1,$$

where $\left[\frac{v_k}{k}\right]$ is the integer part of $\frac{v_k}{k}$. Since $v_k/k - [v_k/k] = \{v_k/k\}$ doesn't tend to 1 as $n \to \infty$, there exist infinitely many k such that $\{v_k/k\} \leq 1/2$, and for these values of k

one gets

$$\frac{v_{k+1}}{k(k+1)}+\frac{v_{k+2}}{k(k+1)(k+2)}+\ldots\geq\frac{1}{2}.$$

But, for sufficiently large k, this cannot be true by condition (i) (which is an easy exercise).

Application. Let p_k be the k-th prime. Then $\sum_{k=1}^{\infty}\frac{p_k}{k!}\notin\mathbb{Q}$.

Proof. By a well-known theorem, the sequence $(\{a_n\})$ is everywhere dense in $(0,1)$, if the following two conditions are valid:

1) $a_n\to\infty$

2) $a_{n+1}-a_n<o(1)$ (as $n\to\infty$) (see G. Pólya - G. Szegö [7]). Now the inequality $\frac{p_{n+1}}{n+1}-\frac{p_n}{n}<\frac{p_{n+1}-p_n}{n}$ and $p_{n+1}-p_n<o(n)$ imply that $\left(\left\{\frac{p_n}{n}\right\}\right)$ is dense in $(0,1)$, which is much stronger than the fact that $\left\{\frac{p_n}{n}\right\}$ doesn't have 1 as limit for $n\to\infty$.

Remark. It would be interesting to obtain an elementary proof of the fact that $\left\{\frac{p_n}{n}\right\}\nrightarrow 1$ as $n\to\infty$.

3. The sequence $(n!)$ is a particular case of $(b_1b_2\ldots b_n)$. In what follows we shall study the arithmetic character of the series $\sum\frac{a_n}{b_1b_2\ldots b_n}$, by admitting certain restrictions.

Proposition 5. *If $a_n\geq 0$, $b_n\geq 1$ are integers ($n\in\mathbb{N}$), and*

(i) $a_n>0$ for infinitely many n;

(ii) $b_n\geq 2$, $0\leq a_n\leq b_n-1$ for all $n\geq 1$;

(iii) there exists a subsequence (i_n) of positive integers such that $b_i\to\infty$, $\frac{a_i}{b_i}\to 0$ $(n\to\infty)$, where $i=i_n$, then the above series is irrational.

Proof. Let

$$t_i=\frac{a_i}{b_i}+\frac{a_{i+1}}{b_ib_{i+1}}+\ldots$$

and $t=t_1$. Then one has $b_it_i=a_i+t_{i+1}$. Thus, if $t=p/q$, then $t_i=p_i/q$ $(p_i\in\mathbb{N})$, thus all t_i have the same denominator q. Reciprocally, if there exists $j\in\mathbb{N}$ with $p_j\in\mathbb{N}^*$, such that $t_j=p_j/q$, then there exists $p\in\mathbb{N}^*$ with $t=p/q$.

By remarking that between p/q and $(p+1)/q$ there is no other rational number of denominator q, we get the affirmation: "If for all $q\in\mathbb{N}^*$ there exist $i\in\mathbb{N}^*$, $p_i\in\mathbb{N}^*$ such that $\frac{p_i}{q}<t_i<\frac{p_{i+1}}{q}$, then t is irrational (here $i=i(q)$)."

Now, condition (iii) implies $a_i < b_i - 1$ for infinitely many $i = i_n$. We have $0 \leq a_i/b_i < t_i < a_i/b_i + 1/b_i$ since $0 \leq t_1 \leq 1$ and $a_i/b_i \leq t_i \leq (a_1 + 1)/b_i$ (without equality since (i), (ii) do hold). Let $q \in \mathbb{N}^*$ be given. Then there is an $n = n_0$ with $0/q < t_i < 1/q$ $(i = i_n)$. On the base of the above affirmation, t must be irrational.

Application. Let $b_n \geq 2$ $(n = 1, 2, \ldots)$ be a sequence of integers such that $\limsup_{n\to\infty} b_n = +\infty$ (i.e. (b_n) is unbounded). Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \ldots b_n}$ is irrational.

Proof. Let $a_i = 1$ an $b_{i_n} \to \infty$ $(n \to \infty)$ in Proposition 5.

4. A general, simple irrationality criterion can be deduced from the following lemma.

Lemma 1. *Let $f : \mathbb{N}^* \to \mathbb{R}_+$ be a function with the property $nf(n) \to 0$ $(n \to \infty)$. Let us suppose that there exist infinitely many distinct rational numbers p/q $(p \in \mathbb{Z},\ q \in \mathbb{N}*)$ such that $\left|a - \dfrac{p}{q}\right| \leq f(q)$, where a is a real number. Then a is irrational.*

Proof. Let (p_k/q_k) be the sequence of rational numbers such that

$$|a - p_k/q_k| \leq f(q_k) \ (k = 1, 2, \ldots).$$

Thus

$$-q_k f(q_k) + q_k a \leq p_k \leq q_k f(q_k) + q_k a.$$

This implies that for each number q_k (fixed) there are a finit number of numbers p_k. If the sequence (q_k) would be bounded, then the sequence (p_k/q_k) would have a finite number of terms, in contradiction with the made assumption.

Therefore, the sequence (q_k) is unbounded. Let (k_i) be a strictly increasing subsequence such that $q_k \to \infty$ $(k = k_i)$ as $i \to \infty$, and select j such that $a = \dfrac{r}{s} = p_k/q_k$ (where $k = k_j$). We then have

$$\left|\frac{r}{s} - \frac{p_k}{q_k}\right| = \frac{|rq_k - sq_k|}{sq_k} \geq \frac{1}{sq_k}.$$

The equality $\lim_{i\to\infty} q_{k_i} f(q_{k_i}) = 0$ implies the existence of an i_0 such that $q_{k_i} f(q_{k_i}) < \dfrac{1}{s}$, $i \geq i_0$. If we select $j \geq i_0$, then

$$\left|\frac{r}{s} - \frac{p_k}{q_k}\right| > f(q_k)$$

(where $k = k_j$), a contradiction.

Proposition 6. ([11]) *Let $(a_n), (b_n)$ be two sequences of positive integers such that with $u_n = a_n/b_n$, the series $\sum_{n=1}^{\infty} u_n$ to be convergent. Let $f : \mathbb{N}^* \to \mathbb{R}_+$ be a function with the property $\lim_{n\to\infty} nf(n) = 0$. If the following conditions*

(i) $u_{k+p} \leq u_{k+1}^p$ $(k, p = 1, 2, \ldots)$.

(ii) $u_{k+1} < \dfrac{f(b_1 b_2 \ldots b_k)}{1 + f(b_1 b_2 \ldots b_k)}$ $(k = 1, 2, \ldots)$

are satisfied, then the above series has an irrational sum.

Proof. Let us select

$$\frac{p_k}{q_k} = \frac{a_1}{b_1} + \ldots + \frac{a_k}{b_k},$$

where p_k is the numerator of the right-hand expression, while $q_k = b_1 \ldots b_k$. Let t be the sum of the series. Then

$$\left| t - \frac{p_k}{q_k} \right| = \frac{a_{k+1}}{b_{k+1}} + \ldots \leq \frac{a_{k+1}}{b_{k+1}} + \left(\frac{a_{k+1}}{b_{k+1}} \right)^2 + \ldots = \frac{a_{k+1}}{b_{k+1} - a_{k+1}} < f(b_1 b_2 \ldots b_k),$$

on base of (i) and (ii).

All conditions of Lemma 1 are valid, so t must be irrational.

Application. $\sum_{k=1}^{\infty} 2^{-3^k} \cdot 3^{-4^k} \notin \mathbb{Q}$.

Proof. Let $b_k = 2^{3^k} \cdot 3^{4^k}$, $a_k = 1$, $f(q) = 1/q^2$.

5. The irrationality of certain series of type $\sum_{n=1}^{\infty} \frac{v_n}{p^n}$, where $v_n \leq p - 1$ can be deduced from the theory of p-adic fractions (see e.g. [6]), which states that the series in question is rational if and only if the sequence (v_n) is periodic. For example, by use of this theorem the following is true:

Application. $\sum_{n=1}^{\infty} \frac{1}{2^{p_n}} \notin \mathbb{Q}$, where p_n is the nth prime.

Proof. Let $p = 2$, $v_n = 1$ if n is prime; $= 0$, if n is composite, or $n = 1$. If the sequence (v_n) would be periodic, then we could find $a \in \mathbb{N}^*$, $N \in \mathbb{N}^*$ such that $v_{n+a} = v_n$ $(n \geq N)$. Then $v_{kn+a} = v_n$ for all $n \geq N$. Let N be a prime number. Then $v_{kN+a} = v_N = 1$, so the sequence (x_k) defined by $x_k = kN + a$ is an infinite arithmetical progression which contains only primes. But this is impossible, since if $a > 1$, one has $x_a = aN + a =$ composite, and if $a = 1$, then $x_{N+2} = (N + 2)N + 1 = (N + 1)^2 =$ composite.

The following theorem extends the applicability for other possible cases of the sequence (v_n) ([12]).

Proposition 7. *Let (v_n) be a sequence of nonnegative integers such that:*

(i) $\limsup\limits_{n\to\infty} \sqrt[n]{v_n} < 2$;

(ii) $v_n > 0$ *for infinitely many* n;

(iii) $\liminf\limits_{n\to\infty} \dfrac{v_1 + v_2 + \ldots + v_n}{n} = 0$.

Then, for all $p \geq 2$, *the series* $\displaystyle\sum_{n=1}^{\infty} \frac{v_n}{p^n}$ *has an irrational value.*

Proof. Let us suppose that the value of the series is $\dfrac{a}{b}$ $(a, b \geq 1)$. We can consider $b = 1$ since in place of the sequence (v_n) we may work with the sequence (bv_n), when conditions (i)-(iii) are valid, too.

By (i) there exist certain r, k_0 with $r < 1$ and

$$v_k < (pr)^k \text{ if } k \geq k_0 \quad (p \geq 2) \tag{4}$$

Let $s \geq 1$ be a positive integer such that

$$pr^s < 1 \text{ and } pr^s < \left(\frac{1-r}{r} \cdot \frac{p-1}{p}\right)^{1/k_0} \tag{5}$$

Further, let $n \geq sk_0$ and

$$m = \left[\frac{n}{s}\right] \quad (\text{so } m \geq k_0) \tag{6}$$

For an arbitrary $h \in \mathbb{N}^*$, put

$$A_h = \sum_{h=k+1}^{\infty} v_k/p^{k-h}.$$

By (iii) one has $A_h \in \mathbb{N}$, while (ii) implies $A_h > 0$, i.e. $A_h \geq 1$. Let $h \leq m$, when clearly $h < n$ and

$$A_h = \sum_{k=h+1}^{n} v_k/p^{k-h} + \sum_{k=h+1}^{\infty} v_k/p^{k-h} \tag{7}$$

On base of (4), (5), (6) one can write

$$\sum_{k=n+1}^{\infty} v_k/p^{k-h} < p^h \sum_{k=h+1}^{\infty} r^k \leq p^m r^n \frac{r}{1-r} \leq (pr^s)^m \frac{r}{1-r} < \frac{p-1}{p},$$

where $p \geq 2$.

On the other hand, $A_h \geq 1$, so

$$\sum_{k=h+1}^{\infty} v_k/p^{k-h} > 1 - \frac{p-1}{p} = \frac{1}{p}$$

for all $h \leq m$. By summing up these inequalities for $h = \overline{1,m}$, one obtains

$$\frac{m}{p} < \sum_{h=1}^{m} \sum_{k=h+1}^{n} v_k/p^{k-h} < \sum_{k=2}^{n} v_k \sum_{h=1}^{k-1} 1/p^{k-h} \leq$$

$$\leq \sum_{k=2}^{n} v_k \sum_{h=1}^{k-1} 1/2^{k-h} < \sum_{k=1}^{n} v_k.$$

Thus

$$\left(\sum_{k=1}^{n} v_k \right) /n > m/pn = \left[\frac{n}{s} \right] /pn,$$

implying

$$\liminf_{n\to\infty} \left(\sum_{k=1}^{n} v_k \right) /n \geq 1/ps > 0,$$

a contradiction to condition (iii). This proves Proposition 7.

Application. ([12]) Let $p \geq 2$ integer, (a_n) a sequence of positive integers such that $\lim_{k\to\infty} (a_{k+1} - a_k) = +\infty$. Then the series $\sum_{k=1}^{\infty} 2^{a_k}/p^{2^{a_k}}$ is irrational.

Proof. Let $v_n = n$, if there is $k \in \mathbb{N}$ with $n = 2^{a_k}$; $= 0$, in the other cases. Then

$$\limsup_{n\to\infty} \sqrt[n]{v_n} = \lim_{k\to\infty} \sqrt[2^{a_k}]{2^{a_k}} = 1 < 2,$$

so (i) is satisfies. Let now $n_k = 2^{a_k} - 1$. Then

$$\frac{v_1 + \ldots + v_{n_k}}{n_k} = \frac{2^{a_1} + \ldots + 2^{a_{k-1}}}{2^{a_k} - 1} < \frac{(k-1)2^{a_{k-1}}}{2^{a_k} - 1} \to 0$$

by $a_k - a_{k-1} \to +\infty$.

6. A generalization of the p-adic expansion is contained in the so-called Cantor expansion ([3]). Let (b_n) be a sequence ("base sequence") of positive integers, $b_n \geq 2$ for all n. Let α be a real number. Put $a_0 = [\alpha]$ and select $0 \leq \gamma_1 < b_1$ such that $\alpha = a_0 + \gamma_1/b_1$. With $a_1 = [\gamma_1]$ one has $\gamma_1 = a_1 + \frac{\gamma_2}{b_2}$, where $0 \leq \gamma_2 < b_2$. By continuing one obtains a sequence (γ_n) such that

(a) $\gamma_n = a_n + \frac{\gamma_{n+1}}{b_{n+1}}$ $(n = 0, 1, 2, \ldots)$;

(b) $0 \leq \gamma_{n+1} < b_{n+1}$.

From (a) it is immediate that

$$\alpha = a_0 + \sum_{n=1}^{k} \frac{a_n}{b_1 b_2 \ldots b_n} + \frac{\gamma_{k+1}}{b_1 b_2 \ldots b_{k+1}},$$

while from (b) one has

$$0 \leq \frac{\gamma_{k+1}}{b_1 b_2 \dots b_{k+1}} < \frac{1}{b_1 b_2 \dots b_k} \leq \frac{1}{2^k},$$

so α can be written as

$$\alpha = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \dots b_n} \tag{8}$$

where $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{Z}$, $a_n \geq 0$, $a_n \leq b_n - 1$ for $n > 1$.

This is the Cantor-expansion of α. It is immediate that this representation of α is unique. On the other hand, for infinitely many n one has $a_n \leq b_n - 2$. Indeed, if $a_n = b_n - 1$ for all $n \geq N$, then

$$\gamma_n = b_n - 1 + \frac{\gamma_{n+1}}{b_{n+1}},$$

so

$$b_n - \gamma_n = \frac{b_{n+1} - \gamma_{n+1}}{b_{n+1}} = \frac{b_{n+2} - \gamma_{n+2}}{b_{n+1} b_{n+2}} = \dots = \frac{b_{n+r+1} - \gamma_{n+r+1}}{b_{n+1} b_{n+2} \dots b_{n+r+1}} \leq \frac{1}{b_{n+1} \dots b_{n+r}} \leq \frac{1}{2^r},$$

so for $r \to \infty$ one gets $b_n \leq \gamma_n$, in contradiction to (b).

Remark that for $a_n = v_n$, $b_n = n$ one obtains the series studied at paragraph 2, while for $a_n = v_n$, $b_n = p$ the one from paragraph 5.

From relation (8) the following important formula can be deduced:

$$a_n = [b_n b_{n-1} \dots b_2 b_1 \alpha] - b_n [b_{n-1} \dots b_2 b_1 \alpha] \quad (n \geq 1) \tag{9}$$

Let us give for example, a new proof of Proposition 3. By (9) one has

$$v_n = [n!\alpha] - n[(n-1)!\alpha].$$

If $\alpha = \frac{p}{q}$, then select $n > \max\{q+1, n_0\}$ such that $v_n > 0$ (this is possible by (iii)). Then $q | n!$ and $q | (n-1)!$ since $n - 1 > q$, so $v_n = n!\alpha - n(n-1)!\alpha = 0$, giving $0 > 0$, a contradiction.

We now prove (using an idea by S. Drobot [13])

Proposition 8. *Let p_n be the nth prime and let (α_n) be an arbitrary sequence of strictly positive integers. Then the series $\sum_{n=1}^{\infty} \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}$ is irrational.*

Proof. If t denotes the sum of the series, suppose $t = \dfrac{r}{q}$ with $(r, q) = 1$. Select n so that

$$q \leq p_{n-1} \tag{10}$$

By (9) one can write

$$1 = \left[p_n^{\alpha_n} p_{n-1}^{\alpha_{n-1}} \dots p_1^{\alpha_1} \frac{r}{q}\right] - p_n^{\alpha_n} \left[p_{n-1}^{\alpha_{n-1}} \dots p_1^{\alpha_1} \frac{r}{q}\right] \tag{11}$$

There are two possibilities.

Case a): All prime factors in the prime factorization of q are at the power 1. Then, since $\alpha_n \geq 1$ for all n, by (10), relation (11) yields $1 = 0$, which is impossible.

Case b): q has also prime factors having exponent > 1. Then it may happen that for the common prime factors in $p_{n-1}^{\alpha_{n-1}} \dots p_2^{\alpha_2} p_1^{\alpha_1}$ and q the powers α_i are greater or equal than the corresponding prime exponents of p_i in q. Then one obtains again $1 = 0$.

The other case is when there exists at least a prime divisor p_i of q having an exponent $> \alpha_i$. Then after simplification one arrives at a number s such that

$$2s \leq q \tag{12}$$

Indeed, $2s = p_1 s \leq p_{k-1} s \leq q$. After division one obtains the equalities

$$\begin{aligned} p_n^{\alpha_n} p_{n-1}^{\alpha_{n-1}} \dots p_2^{\alpha_2} p_1^{\alpha_1} \frac{r}{q} &= A + \frac{a}{s} \quad (\text{where } 1 \leq a < s) \\ p_{n-1}^{\alpha_{n-1}} \dots p_2^{\alpha_2} p_1^{\alpha_1} \frac{r}{q} &= B + \frac{b}{s} \quad (1 \leq b < s) \end{aligned} \tag{13}$$

so by (11) one can write

$$1 = A - p_n^{\alpha_n} B \tag{14}$$

On the other hand,

$$A + \frac{a}{s} = p_n^{\alpha_n} \left(B + \frac{b}{s}\right),$$

and by taking into account of (14),

$$p_n^{\alpha_n} = \frac{a + s}{b} < 2s \leq 2q,$$

by (12). But $p_n \le p_n^{\alpha_n} \le q$ implies $p_n \le q$. Relation (10) on the other hand says that $q \le p_{n-1}$, so $p_n \le p_{n-1}$, a trivial contradiction.

Remark. Proposition 8 can be extended as follows:

Proposition 9. *Let (x_n) and (α_n) be sequences of positive integers such that*

$$x_n \le [2p_n^{\alpha_n}/p_{n-1}] - 1 \text{ for all } n \ge n_0.$$

Then the series $\displaystyle\sum_{n=1}^{\infty} x_n/p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$ is irrational.

Proof. The same idea applies. Now, in place of (11) one writes

$$x_n = \left[p_n^{\alpha_n}\dots p_1^{\alpha_1}\frac{r}{q}\right] - p_n^{\alpha_n}\left[p_{n-1}^{\alpha_{n-1}}\dots p_1^{\alpha_1}\frac{r}{q}\right].$$

Let $t = \dfrac{r}{q}$ where $(r,q) = 1$ and let $q \le p_{n-1}$ if $n \ge n_0$. The same conclusion as above holds, i.e. $2s \le q$, and (13) holds again, but now this implies

$$p_n^{\alpha_n}\left(B + \frac{b}{s}\right) = A + \frac{a}{s},$$

i.e.

$$p_n^{\alpha_n}\frac{b}{s} = x_n + \frac{a}{s},$$

giving

$$p_n^{\alpha_n} = \frac{sx_n + a}{b} < \frac{s(x_n+1)}{b} \le s(x_n+1) \le \frac{q}{2}(x_n+1) \le \frac{p_{n-1}}{2}(x_n+1).$$

Since

$$x_n + 1 \le \left[\frac{2p_n^{\alpha_n}}{p_{n-1}}\right] \le \frac{2p_n^{\alpha_n}}{p_{n-1}},$$

we get

$$\frac{p_{n-1}}{2}(x_n+1) \le p_n^{\alpha_n}.$$

Finally one can deduce that $p_n^{\alpha_n} < p_n^{\alpha_n}$, i.e. a contradiction.

Remark. Let $\alpha_n \ge 2$. Then $p_n^{\alpha_n}/p_{n-1} \ge p_n^2/p_{n-1} > p_n$ so, if $x_n \le 2p_n - 1$, then the condition of Proposition 9 is satisfied.

So for $\alpha_n \ge 2$ and

$$x_n \le 2p_n - 1 \quad (n = 1, 2, \dots) \tag{15}$$

the series considered above is irrational.

Application. 1) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(p_1p_2\dots p_n)^{2k+1}}\notin\mathbb{Q}$ for all $k\geq 0$, integer; (For $k=0$ see S. Drobot [13])

2) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{p_n(p_1p_2\dots p_{n-1})^{\alpha}}\notin\mathbb{Q}$ for all $\alpha\in\mathbb{N}$.

7. We now consider series $\displaystyle\sum_{n=1}^{\infty}\frac{v_n}{n!}$, where $v(n)$ may take relatively large values.

Proposition 10. *Let us suppose that*

(i) there exists $N\in\mathbb{N}$ such that $n\leq v_n\leq\dfrac{n(n+1)}{2}$ for all $n\geq N$;

(ii) for infinitely many n one has $v_n\leq n+1$.

Then $\displaystyle\sum_{n=1}^{\infty}\frac{v_n}{n!}$ is irrational.

Proof. The series is convergent, since

$$\sum_{n=1}^{\infty}\frac{u_n}{n!}=\sum_{n=1}^{N-1}\frac{u_n}{n!}+\sum_{n=N}^{\infty}\frac{v_n}{n!}\leq\sum_{n=1}^{N-1}\frac{v_n}{n!}+\frac{1}{2}\sum_{n=N}^{\infty}\frac{n}{(n-1)!}<\infty$$

by $\dfrac{n}{(n-1)!}<\dfrac{n}{2^{n-1}}$ and $\sum\dfrac{n}{2^{n-1}}<\infty$.

Let us now suppose that $\displaystyle\sum_{n=1}^{\infty}\frac{v_n}{n!}=\frac{p}{q}\in\mathbb{Q}$ and let $m>\max(q,N)=n_0$ such that $v_m\leq m+1$ (possible, by (ii)). Since $(m-1)!\dfrac{p}{q}$ is an integer, the same is true for

$$R=\sum_{i=m}^{\infty}\frac{v_i}{i!}(m-1)!.$$

On the other hand,

$$R=\frac{v_m}{m}+\frac{v_{m+1}}{m(m+1)}+\ldots=\frac{m+1}{m}+\frac{v_{m+1}}{m(m+1)}+\ldots$$

Since $v_{m+i}\leq\dfrac{(m+i)(m+i+1)}{2}$ and by the obvious inequality

$$m(m+i+1)<(m+2)(m+i-1)$$

one can write

$$\frac{v_{m+i}}{m(m+1)\dots(m+i)}<\frac{m+2}{2m^i}\quad(i\geq 1),$$

thus

$$R\leq 1+\frac{1}{m}+\sum_{i=1}^{\infty}\frac{m+2}{2m^i}=1+\frac{1}{m}+\frac{m+2}{2(m-1)}<1+\frac{1}{m}+\frac{m-1}{m}=2.$$

The inequality $R > 1$ is trivially satifies by (i), implying $1 < R < 2$, contradicting the fact that R is an integer.

Remark. The proof shows that the following variant of the above proposition holds true also:

Proposition 11. *If*

(i) $v_n \leq \dfrac{n(n+1)}{2}$ *for all* $n \geq N$*;*

(ii) $v_n = n+1$ *for infinitely many* n*;*

then $\displaystyle\sum_{n=1}^{\infty} \frac{v_n}{n!}$ *is irrational.*

Applications. 1) $\displaystyle\sum_{p\ prime} \frac{1}{(p-1)!} + \sum_{p\ prime} \frac{1}{p!} \notin \mathbb{Q}$;

2) $\displaystyle\sum_{n\ odd} \frac{n+1}{n!} + \frac{1}{2} \sum_{n\ even} \frac{n^2+n}{n!} \notin \mathbb{Q}$;

3) $\displaystyle\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!} \notin \mathbb{Q}$ (where $\sigma(n)$ denotes the sum of all divisors of n) ([4]).

Proof. 1) $v_n = n+1$ if $n =$ prime; $= 0$, $n \neq$ prime in Proposition 11;

2) $v_n = n+1$ if $n =$ odd; $= \dfrac{n(n+1)}{2}$, if $n =$ even;

3) $v_n = \sigma(n)$.

Then $v_p = p+1$ for all primes p.

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Note added in proof. Recently, W. Zudilin [One of the numbers $\zeta(2), \zeta(7), \zeta(9), \zeta(11)$ is irrational, Russian Math. Surveys, **56**(2001), No.4, 774-776] proved the result in the title of his paper.

7 On the irrationality of $\cos 2\pi s$ $(s \in \mathbb{Q})$

Let $\phi_n(x) = \prod_{(r,n)=1} (x - e^{2\pi ir/n})$ be the n-th cyclotomic polynomial (where $e^{2\pi ir/n}$ for $(r,n) = 1$ are the n-th primitive roots of unity). It is well-known that $\phi_n(x)$ is a polynomial with integer coefficients, and that this is an irreducible polynomial over $\mathbb{Z}[x]$. Let $n \geq 7$ and put $\alpha = 2\pi ir/n$. By another property of $\phi_n(x)$ one has $x^{\varphi(n)}\phi_n\left(\frac{1}{x}\right) = \phi_n(x)$ (i.e. the coefficients from the extrems are equal), so the equation $K_n(\alpha) = 0$ takes the form

$$\alpha^{\varphi(n)/2} + \alpha^{-\varphi(n)/2} + a_1\left(\alpha^{\frac{\varphi(n)}{2}-1} + \alpha^{-\frac{\varphi(n)}{2}+1}\right) + \ldots + a_{\frac{\varphi(n)}{2}-1}(\alpha + \alpha^{-1}) + a_{\frac{\varphi(n)}{2}} = 0 \quad (1)$$

Since $2\cos 2\pi r/n = \alpha + \alpha^{-1}$, $(2\cos 2\pi r/n)^2 = \alpha^2 + \alpha^{-2} + 2, \ldots$ the above equation (1) can be written also as

$$\left(2\cos\frac{2\pi r}{n}\right)^{\frac{\varphi(n)}{2}} + b_1\left(2\cos\frac{2\pi r}{n}\right)^{\frac{\varphi(n)}{2}-1} + \ldots + b_{\frac{\varphi(n)}{2}} = 0 \quad (2)$$

where $n \geq 7$, and $b_1, \ldots, b_{\varphi(n)/2}$ are some integers.

This gives a polynomial in $2\cos\frac{2\pi r}{n}$, which cannot be reducible over $\mathbb{Z}[x]$, since then the one from (1) would be reducible, too (impossible, by the irreducibility of the cyclotomic polynomial K_n).

Since the leader coefficient in (2) is 1, the number $2\cos\frac{2\pi r}{n}$ is an algebraic integer number, of order $\frac{\varphi(n)}{2} > 1$ (by $n \geq 7$). It is a well-known result that an algebraic integer can be rational iff has order 1, so this clearly implies that for $n \geq 7$, $(r,n) = 1$, the number $\cos\frac{2\pi r}{n}$ is irrational. An easy examination for $n \leq 6$ yields the following

Proposition. ([3]) *Let $(r,n) = 1$, $r \leq n$. Then $\cos\frac{2\pi r}{n}$ is irrational if and only if $n \notin \{1,2,3,4,6\}$.*

Remark. Many other properties of cyclotomic polynomials appear in [1]. See also article 3.8. For applications of irrationality in geometry, we quote [2].

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Author Index

This book contains short notes or articles, as well as studies on several topics of Geometry and Number theory. The material is divided into five chapters: Geometric theorems; Diophantine equations; Arithmetic functions; Divisibility properties of numbers and functions; and Some irrationality results. Chapter 1 deals essentially with geometric inequalities for the remarkable elements of triangles or tetrahedrons. Other themes have an arithmetic character (as 9-12) on number theoretic problems in Geometry. Chapter 2 includes various Diophantine equations, some of which are treatable by elementary methods; others are partial solutions of certain unsolved problems. An important method is based on the famous Euler-Bell-Kalmár lemma, with many applications. Article 20 may be considered also as an introduction to Chapter 3 on Arithmetic functions. Here many papers study the famous Smarandache function, the source of inspiration of so many mathematicians or scientists working in other fields.

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