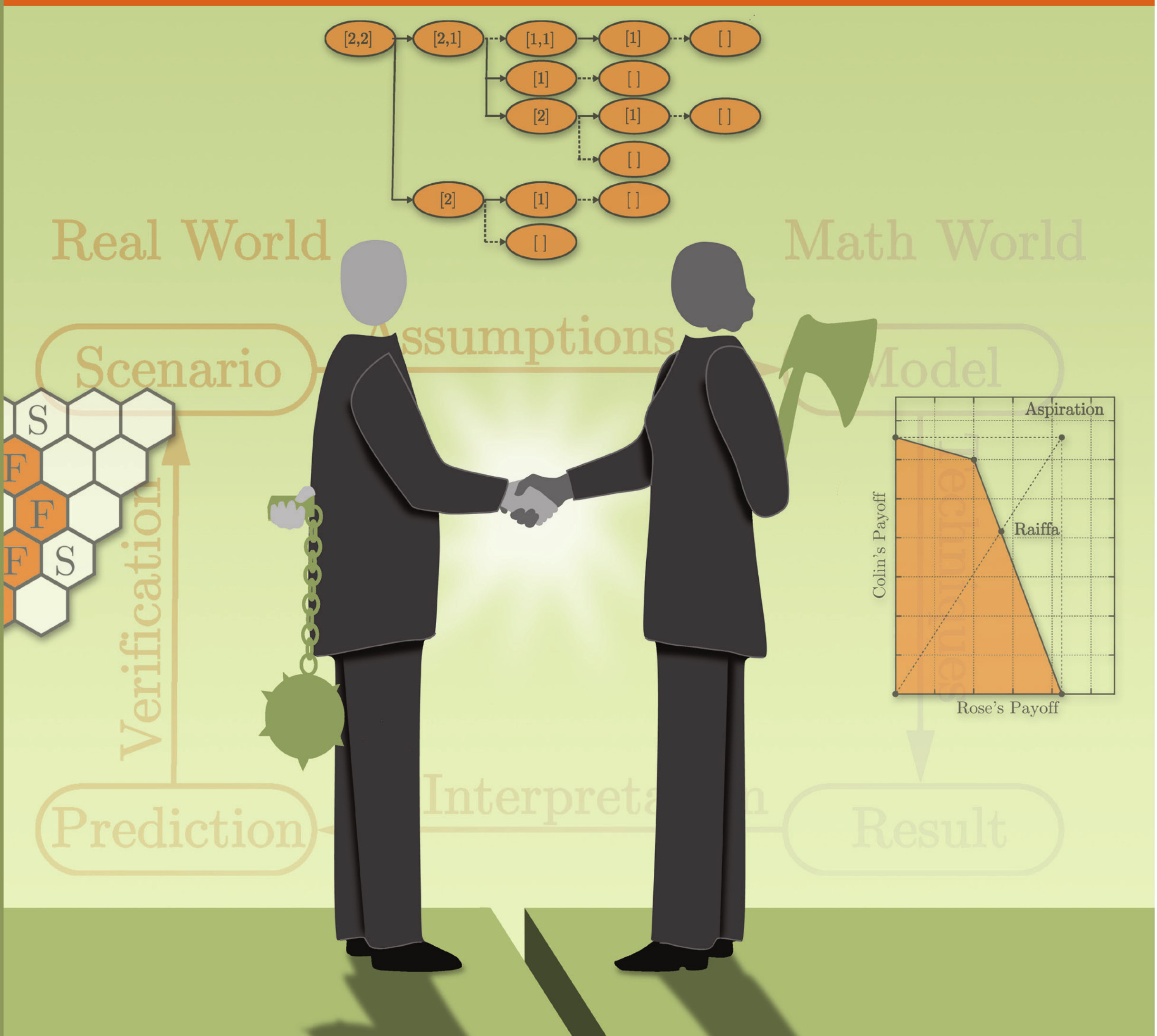
Models of Conflict and Cooperation



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Preface

Decisions. We make them every day. They are made at the personal level when resolving family tensions, deciding where to take family vacations, choosing a college, planning for retirement, or partitioning an inheritance. We make decisions at a professional level as we make business decisions about products to market, where to locate a business, when to take on a partner, or where to invest scarce resources. We make decisions on a sociopolitical level as well, as we work to resolve environmental and social problems in international politics, and make decisions about how we should be represented in government.

Almost all of these many decisions are made in situations where the resulting outcome also depends on the decisions made by other people. For example, your decision about where to go to college is influenced by the decisions of your parents, of financial aid counselors, and of your friends.

Game theory, defined in its broadest sense, is the mathematical field that studies these types of decision making problems. Specifically, it asks the question: How does a player maximize his or her return in a situation where the outcome is at least partially determined by other, independently acting players? Game theory was developed during the 1930s and 1940s in the context of the study of economic behavior, but has since expanded in scope to include many social and political contexts. In recent years, it has even been used by biologists to better understand how behaviors evolve.

The process of representing a decision, behavior, or any other real-world phenomenon in mathematical terms is called mathematical modeling. The essential features of this process are making explicit assumptions to create the model, using mathematical techniques to arrive at results, interpreting those results, and verifying their reasonableness in the original context. A primary goal of this book is to help students to learn, practice, and use this process.

To the Student

The study of mathematics is a participatory process, requiring full engagement with the material in order to learn it. READ WITH ANTICIPATION. As you read a sentence, ask yourself: What does this tell me? Where do I think that this idea leads me? How does it connect with previous material? Notice that we used the word "sentence" rather than "paragraph", "page", or "chapter" in the previous sentence. This is because reading mathematics, unlike reading literature, is a patient process. Individual sentences are meant to convey significant information and frequently need to be read two or three times for complete comprehension. Furthermore, DO THE EXERCISES. Mathematical learning requires you to pick up pencil and paper to solve problems.

As a game theory text, this book offers additional, less traditional, opportunities to engage with the material. First among these is the opportunity to PLAY THE GAMES described in almost all of the sections. In several sections, there are activities that encourage you to do this. Find some friends with whom to play the games; you will find that the games are fun and thought provoking to play. As you play a game, reflect, both personally and with the other players, on the "best" ways to play them. Identify your goals and how you would attempt to achieve these goals.

Each chapter opens with a "dialogue", an idealized conversation between two or more individuals about the topic of the chapter. Together with your friends, ACT OUT THE DIALOGUES. On the surface, these dialogues are merely a somewhat corny way to introduce the topic at hand, but at a deeper level they model the quantitatively literate conversations we hope that you will have in your life.

A Brief Description of the Chapters

Each section includes a succinct statement of the learning objectives and exercises. Most exercises are directly linked to the learning objectives but some provide extra challenge. Following the dialogue section of each chapter, there are two or three sections that contain the core ideas of the chapter. These sections carefully lay out the critical concepts, methods and algebraic tools developing these ideas. The final section of each chapter is usually an extension of the topics that could be omitted in a first reading.

Chapters 1 and 2 are key to the entire book. Chapter 1 introduces the reader to deterministic games and uses them as a means to ease the reader into quantitative thinking. Readers are introduced to the distinct concepts of heuristics and strategies in this chapter and are encouraged to use them to find ways of winning several games, including the well-known game of Nim. They are also introduced to the idea of a game tree and its application to identifying winning strategies for a game. Chapter 2 develops the critical idea of player preferences. Before an individual can make informed decisions about their actions in situations involving conflict or cooperation, he or she must be able to determine preferences among the possible outcomes for all decision making. These preferences can be quantified in a variety of ways and provide the numerical basis for much of the work in the following chapters. This quantification of preferences is our first example of the mathematical modeling process, and the diagram depicting this process is referred to throughout the rest of the book.

Chapters 3, 4, and 5 are a basic study of problems of conflict. Chapter 3 introduces the reader to strategic games and provides many opportunities for the reader to develop models of simple real-world scenarios. This chapter also introduces various solution concepts for strategic games, including the Nash equilibrium. Chapter 4 extends this work by developing the algebraic background necessary to find Nash equilibria in mixed strategies. Chapter 5 closes the theme of strategic games with a study of Prisoner's Dilemma, demonstrating the perpetual tension between rational self-interest and the collective good. This chapter investigates the repeated play solution of the dilemma, including a short study of infinite series.

Chapters 6, 7, and 8 are a basic study of problems of cooperation. Chapter 6 develops solutions to the bargaining problem, in which two players negotiate among various alternatives. As the reader progresses through the sections in this chapter, various fairness properties are introduced, leading to three solution methods. Chapter 7 explores the situation in which there are three or more players and the value of cooperation is known for each subgroup of players. Two different solutions are presented and are characterized by differing concepts of fairness. Chapter 8 continues this investigation of fairness in problems where a group of players need to partition a set of objects.

As the book moves from its beginning to its end, the general tenor and organization of the material in the chapters changes. Early chapters tend to be more conversational in tone, while later chapters are a bit more formal. This transition mirrors the way the material is developed. In early chapters readers are initially provided with experiences and contexts and are asked to generalize from them, while in the later chapters readers are asked to think about the general properties and to draw conclusions from them. This transition works since readers become increasingly more comfortable in working in a mathematical mode. By the end of the book, readers are beginning to think like mathematicians!

To the Instructor

For instructors using this book, the dialogues also provide built-in classroom activities for their students. In many sections of the book, there are activities that readers can do to develop a deeper understanding of the topic immediately at hand. Some take a longer time to complete, while others can be completed in just a few minutes. While they could easily be omitted by the reader, they represent significant opportunities to learn more by doing and can be used as homework exercises or as in-class activities by a course instructor. These are displayed in boxes for easy recognition.

Whenever they are presented, theorems are given names that reflect either their authorship or the content of the theorem. This enables the reader to readily identify the theorem and to contextualize it. We provide proofs of some of the theorems, and provide arguments based on generic examples for other theorems. Readers should practice, and learn, to work through technical material by carefully reading and, when possible, discussing these proofs and arguments with other people.

Chapter 1, in which *game* is defined, is a prerequisite to all other chapters. Chapter 2 is a prerequisite to Chapter 3 and a "soft" prerequisite for Chapters 7 and 8 (cardinal payoffs are being used but can be understood in terms of money rather than expected utilities for lotteries). Chapter 3 is a prerequisite for Chapters 4 and 5, which are independent of each other. Finally, Chapter 4 is a prerequisite for Chapter 6.

In the authors' experience, each section of each chapter requires two 50-minute periods of a course to cover satisfactorily. Thus, there is too much material in the book to be covered completely and effectively in a 3-credit course. In this setting, an instructor will need to make decisions about the material that can be omitted. The entire book can be covered in a 4-credit course, although instructors will need to identify how they want to use their time.

In either situation, there are plenty of topics omitted that individual course instructors might wish to include. There is no discussion of voting methods and Arrow's theorem, nor of the related topic of voting power. The use of linear programming methods to find Nash equilibria in strategic games and the nucleolus for coalition games is mentioned, but not covered; certainly the use of these tools to solve zero-sum games is within the reach of the general audience. Cake-cutting algorithms are not included nor are extensive discussions of multi-player games, evolutionary games, or simulation games. Any of these topics would be ideal for an instructor to use as an extension of the material, or to assign as a project for their students.

A complete set of annotated solutions can be obtained from the authors by writing either one using institutional letterhead. Please include the enrollment and a description of the course that you are teaching.

Also available from the authors is software for some of the game classes discussed in this book. The free packet includes software which can be distributed to students for installation on their personal machines. If you are interested in this software, please contact the authors for more information.

Acknowledgments

We, the authors of this book, each began teaching game theory to a general audience (nonscience, nonmathematics, noneconomics majors) more than ten years ago. We made this decision, independently, because we saw great value in enabling our students to bring mathematical tools to bear on their fundamental decision-making processes. After conversing with each other, we realized that not only did we have similar visions of what should be taking place in such a course, but also agreed that none of the resources available at the time presented the material in a way that we believed was effective for this general population of students. In particular, we realized that the basic ideas of game theory can easily be communicated using elementary mathematical tools. That is, by systematically incorporating appropriate experiences into the course, we could help students become quantitatively literate. These experiences involve using the fundamental mathematical skills of quantitative literacy: logical reasoning, basic algebra and probability skills, geometric reasoning, and problem solving. Hence, this book is explicit in its intention to teach quantitative literacy, and the layout of each chapter is designed to promote quantitative literacy.

A book of this type does not happen without inspiration or without assistance. We thank Phil Straffin, Alan Taylor, and Steven Brams for their books on game theory which provided excellent models for our own work. We thank Mary Treanor and Dan Apple for their assistance in editing the text and in improving pedagogical insights. We thank our home institutions, Valparaiso University and Goshen College, for supporting our work through internal grants, sabbatical leaves, and general encouragement. Finally, and most importantly, we thank the many students who, beginning very early in the process, patiently worked through our material as we developed ideas, examples, and text.

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CHAPTER 1

Deterministic Games

1. A Very Simple Game

In this book, you will be learning about many different games, about how to determine your preferences among the possible outcomes of a game, about how to get the best possible outcome, and about effective ways to collaborate with other players. The goal is that doing this will help you (a) recognize similar situations in the real world and (b) provide you with the methods you need to deal with them successfully. The learning objectives for each section will be enclosed in a box near the beginning of the section for easy reference. Here is our first one.

> By the end of this section, you will learn the rules for the Keypad game and will have thought about how to play the game to win.

One way that we, the authors, hope to accomplish the goals of this book is by providing you with model *dialogues* about the topic of a given chapter. While they may be a bit corny, they are meant to give you examples of conversations we hope that you will have at different points in your life. So we begin this chapter with the first of these dialogues, a conversation about a very simple game.

FATIMA: Hey, Sunil, would you like to play a game that I read about today in the *College Mathematics Journal* [**18**]?

SUNIL: Sure, why not? It's the first day of classes and I don't have any homework yet.

FATIMA: Great! It's called the **Keypad** game and here's how you play: The first player selects a number, other than zero, on the cell phone keypad. Then the second player picks a different number in the same row or column as the first. Then you find the sum of the two numbers. You continue to do this until one of the players gets a sum over 30. That player loses.

SUNIL: Huh?

FATIMA: Okay, let me show you. Here's a keypad:

1	2	3
4	5	6
7	8	9

Say I pick 7; then you can pick from 1, 4, 8, or 9.

$\boxed{1}$	2	3
$\boxed{4}$	5	6
7	$\boxed{8}$	$\boxed{9}$

SUNIL: Okay. 8.

FATIMA: Then we add. 7 plus 8 is 15. Then I have to pick from 2, 5, 7, or 9.

1	$\boxed{2}$	3
4	$\boxed{5}$	6
$\boxed{7}$	8	$\boxed{9}$

SUNIL: But you already picked 7.

FATIMA: That's okay. Numbers can be picked multiple times. All that matters is that I pick a different number in the same row or column as you did.

SUNIL: Okay.

FATIMA: I'll pick 9, so we get to 15 + 9 = 24.

SUNIL: So I've got to pick from 7, 8, 3, or 6. Let me think. Seven and 8 put the sum over 30, but 6 puts the sum right at 30, and 3 keeps it below 30. Do I lose if the sum is exactly 30?

FATIMA: No, it needs to be over 30.

SUNIL: Then I pick 6, and you'll lose with your next choice.

FATIMA: I guess you win. Play again?

SUNIL: Sure! This is easy.

FATIMA: Okay. Now we'll get serious about this. I'll start with the 9.

SUNIL: Then I pick 8, and we're up to 17 for the sum.

FATIMA: I'll pick 9 again, for a sum of 26.

SUNIL: I've got to pick the 3 to get a total of 29. But then you pick the 1 and win!

FATIMA: Now we're tied one game to one game. Play again?

SUNIL: Okay. But this time, I go first.

FATIMA: Go for it.

SUNIL: Humm. . . I'll pick 4.

FATIMA: Fine. Then I'll pick 6, and we're up to 10.

SUNIL: Humm... 3, 4, 5, or 9. I'll take the 9.

FATIMA: Okay. We're up to 19, and I'll take 8 this time, for a total of 27.

SUNIL: Wait a minute. If I take the 5, 7, or 9, I lose. If I pick the 2, then you'll pick the 1 and I still lose.

After many games, which are played very quickly, Sunil has still only won one game.

SUNIL: Fatima, you seem to know something about this game that I don't! It doesn't seem to be fair, or I would have won more often.

FATIMA: Don't blame it on the game. It could very well be fair, but you just haven't figured out how to play yet.

SUNIL: And I suppose that you've figured out how to win?

FATIMA: Of course! Remember I said that I read about the game in the *College Mathematics Journal*. The article explains how to win!

SUNIL: Let me see if I can figure this out. When you went first, you always picked 9. That was intentional, right?

FATIMA: Yes.

SUNIL: So, I would need to choose 3, 6, 7, or 8. Then if I would pick the 8...

FATIMA: I would pick the 9 again.

SUNIL: Then I'd need to pick the 3, because my other choices all put me over 30. Then you'd pick the 1, and I'd lose on my next move.

FATIMA: Picking 8 as your first move does not work very well.

SUNIL: Okay, let's start over. You opened with a 9. What if I picked the 7?

FATIMA: I would still pick the 9 again and you would still lose.

SUNIL: Because the 6, 7 and 8 are still too large, so I'd have to pick the 3 again, but this time you'd pick the 2 on your turn.

FATIMA: That's right.

SUNIL: What if I first picked the 6 instead of 7 or 8?

FATIMA: I'd respond with the 5, and after your next move, I'd pick 10 minus what ever you picked.

SUNIL: If I pick x, then you would pick $10 - x$. Hmmm. Is that always legal?

FATIMA: If you pick 2 or 8, I can pick the other since they are in the same column. Similarly, if you pick 4 or 6, I can pick the other since they are in the same row.

SUNIL: So our sum would be $9 + 6 + 5 + x + (10 - x) = 30$, no matter what I'd chosen!

FATIMA: Yes.

SUNIL: So, after your initial choice of 9, you can win if I choose 8, 7, or 6. My final possibility is to pick the 3. I suppose you can tell me how you're going to win here as well?

FATIMA: I can. After you pick the 3, I'll pick the 6, so we have a sum of 18. Now you can play one of 3, 4, 5, or 9, to which I'd respond with 9, 6, 6, or 3, respectively. Now the sum is either 28 (if you had played the 4) or 29 (if you had played the 5) or 30 (if you had played the 3 or 9). But in the first two cases, you can't pick the 1 or 2 to stay at 30 or below. So you'd lose.

SUNIL: This game isn't much fun once you know how to play is it?

FATIMA: It is if you go first and choose 9!

SUNIL: But what if I'm not playing first?

FATIMA: Then you need to hope that your opponent does not choose 9 and you need to know how to respond to your opponent's mistake.

SUNIL: I suppose that the article in the math journal tells you this, too.

FATIMA: It does. I would explain it to you, but I have a class to go to now. See you later.

Sometimes we think that it is important for the reader to engage in some activity before they continue to read. These activities will acquaint the reader with relevant games, concepts, and issues. We will enclose these in a box.

> Play the **Keypad** game with a friend. What can you find out about how to win the game?

Exercises

(1) Can Fatima lose if she starts with 9? How?
(2) Can Fatima guarantee a win if she starts with 9? How do you know?
(3) How are questions 1 and 2 different?
(4) Do you think that it really matters if you go first or second in this game?
(5) Does the game change significantly if 0 is added to the keyboard in the position indicated below?

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ & 0 & \end{array}$$

(6) Does the game change significantly if more than two players play? Note that although interesting, this is a fairly difficult and open-ended question. You may want to come back to it after working through this chapter.

(7) Play the **Keypad** game using different target sums. Do you see any patterns that help you to know how to win? Note that although interesting, this is a fairly difficult and open-ended question. You may want to come back to it after working through this chapter.

2. Rules of the Game

People play and are fascinated by all sorts of games. Some are very simple, such as Tic-Tac-Toe, or complex, such as Chess. There are board games: *Monopoly*, *Sorry!*, *Risk*, and so forth. Poker, Rummy, Euchre, and Solitaire are card games. Other games, such as football, soccer, and baseball, involve physical activity. There are role-playing games, such as *Dungeons & Dragons*, and we even play games vicariously through TV programs, such as *Jeopardy*, *Deal or No Deal*, and *The Weakest Link*.

But not all the games that people play are for entertainment. Atari founder Nolan Bushnel said, "Business is a good game—lots of competition and a minimum of rules. You keep score with money." Eighteenth-century author Percy Bysshe Shelley said, "War is the statesman's game, the priest's delight, the lawyer's jest, the hired assassin's trade." The psychiatrist Eric Berne wrote *The Games People Play* as a way of describing how people interact with each other. This is echoed in H. T. Leslie's statement that "The game of life is not so much in holding a good hand as playing a poor hand well."

Unlike Leslie, we would not say life is a game; however, we will try to model some aspects of life as games. Labor and management contract negotiations, nations deciding whether to develop and stockpile nuclear weapons, adjudicating a will, and sharing the costs of a large project among the collaborating cities are all situations that we would like to model as games. Therefore, we begin with our definition of a game.

> **Game:** A *game* consists of
> - players,
> - rules that specify what actions are available to each player,
> - outcomes that occur when the game is played, and
> - preferences each player has for the different outcomes.

Keypad is a game as defined above. There are two players (Fatima and Sunil in the dialogue). There are rules: each player is allowed to press keys in a clearly prescribed manner. There are outcomes: after a few key presses, one of the players is declared the winner and the other is declared the loser. Finally, there are preferences: each player would prefer to win. As we did with **Keypad**, names of games (as we have defined games above) will be set in boldface type.

We start with deterministic games because they are simple to describe, simple to play, and fun to play!

> By the end of this section, you should be able to identify deterministic games and know how to play **Nim**, **Hex**, and **Trickster,** change rules in order to obtain different but similar games, and enjoy playing and thinking about games.

Deterministic Game: In a *deterministic game*, players take turns in a specified manner to choose among legal moves, the only possible outcomes involve one or more players winning (with the other players losing) or a tie (no one wins or loses), there is perfect information (nothing is hidden from any player), chance is not necessary to play the game, and players most prefer winning and least prefer losing.

Keypad, **Checkers**, **Chess**, **Go**, and **Tic-tac-toe** are deterministic games. **Poker**, **Bridge**, and **Rummy** involve secrecy (each player does not know what cards are held by the other players) and chance (how the cards are dealt); therefore, they are not deterministic games. When played with money, **Poker** outcomes are not simply win/lose but also include the amount of money the player wins. **Tossing Pennies** and **Yahtzee** involve no secrecy, but they do involve chance in the tossing of the penny or dice, and so they are not deterministic games. **Matching Pennies** and **Hide-And-Seek** do involve secrecy, and so they are also not deterministic games, even though they have no inherent randomness. Note that in all of these games, even the deterministic ones, the players can choose to use chance to help them determine their moves. For example, it is okay for a player to roll a die to determine which piece to move in the deterministic game **Checkers**.

In this section, we will learn to play three types of deterministic games, and in subsequent sections, we will use simple mathematical tools to learn how to win these games. In the exercises, we will introduce you to other deterministic games.

Nim

Nim is one of the oldest and most engaging of all two-person mathematical games known today. The complete theory of the game was developed by Charles L. Bouton of Harvard University about 100 years ago [**7**]. A game of **Nim** is played with several heaps of beans (or beads or tiles). The two players, Firstus and Secondus, take turns moving, with Firstus making the first move. When it is her or his turn to move, the player removes one or more beans from a single heap. The player who removes the last bean is the winner.

The order of the heaps does not matter, so having a heap of three beans on the left and a heap of two beans on the right is the same as having a heap of two beans on the left and a heap of three beans on the right. This makes it easy for us to develop compact notation for describing **Nim**. The game with a heap of three beans and a heap of two beans can be called a $[3, 2]$ **Nim** game; we can always order the numbers from largest to smallest. For example, the game shown in Figure 2.1 has three (vertical) heaps containing three, two, and one beans, respectively. We will denote this as a $[3, 2, 1]$ **Nim** game; it is the same as a $[1, 2, 3]$ **Nim** game.

In the $[3, 2, 1]$ **Nim** game, Firstus has six possible moves: (1) remove one bean from the three-bean heap, (2) remove two beans from the three-bean heap, (3) remove three beans from the three-bean heap, (4) remove one bean from the two-bean heap, (5) remove two beans from the two-bean heap, or (6) remove one bean from the one-bean heap.

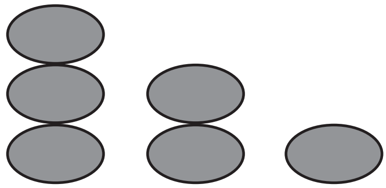


FIGURE 2.1. $[3, 2, 1]$ **Nim**

The choices for Secondus's move will depend upon Firstus's move. For example, suppose that Firstus chooses the first move described above: remove one bean from the three-bean heap. Secondus is then faced with a game of $[2, 2, 1]$ **Nim** as shown in Figure 2.2.

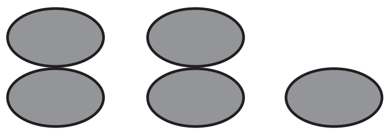


FIGURE 2.2. $[2, 2, 1]$ **Nim**

Secondus has three essentially different moves: (1) remove one bean from a two-bean heap, (2) remove two beans from a two-bean heap, or (3) remove one bean from the one-bean heap. Of course, if Secondus chooses to remove one bean from a two-bean heap, he could remove from the left or center heap. But in either case, we are left with one heap of two beans and two heaps of one bean. That is why we said that Secondus has only three essentially different moves rather than five possible moves.

The choices for Firstus's second move will depend upon Secondus's move. For example, suppose that Secondus chooses the second move described above: remove two beans from a two-bean heap. Firstus is then faced with a game of $[2, 1]$ **Nim** as shown in Figure 2.3.

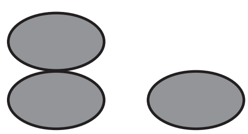


FIGURE 2.3. $[2, 1]$ **Nim**

Firstus has three possible moves: (1) remove one bean from the two-bean heap, (2) remove two beans from the two-bean heap, or (3) remove one bean from the one-bean heap. If Firstus chooses to remove two beans from the two-bean heap, Secondus will be faced with a game of $[1]$ **Nim** as shown in Figure 2.4.

Secondus has only one possible move: remove one bean from the one-bean heap. After Secondus does this move, he is declared the winner.

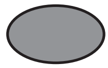


FIGURE 2.4. [1] **Nim**

If Firstus had thought ahead when she was faced with [2, 1] **Nim**, perhaps she would have realized that removing one bean from the two-bean heap was a better move. Can you see why?

> Play several games of **Nim** with several different people, using different numbers of heaps and different numbers of beans in each heap. Try to win! For each game, record the initial number of beans in each heap. Also record your moves, thoughts, observations, and insights on a piece of paper for later reference.

Hex

Hex was invented by John Nash and Piet Hein independently and marketed in the 1950s by Parker Brothers [**69**]. John Nash shared, with John Harsanyi and Reinhard Selten, the 1994 Nobel Prize in Economics for his pioneering work on game theory; this is only the first of several times that his name will appear in this book. A game of **Hex** is played on a rhombus-shaped grid of hexagons, in which two hexagons are considered adjacent if they share a side. Firstus and Secondus take turns moving. When it is her or his turn to move, the player captures one hexagon by writing her or his initial in the hexagon. They may capture any hexagon that has not already been captured, whether or not it is adjacent to a previously captured hexagon. Firstus wins if she captures a path of hexagons from top to bottom of the rhombus. Secondus wins if he captures a path of hexagons from left to right of the rhombus. The first eight moves and the winning move of a 5×5 **Hex** game are shown in Figure 2.5.

> Play several games of **Hex** with several different people using different board sizes. Remember to play to win! We have provided two 7×7 **Hex** boards in Figure 2.6. You may wish to play with photocopies of them. To play with smaller boards, simply shade the hexagons that will not be used. Record the board size, your moves, thoughts, observations, and insights on a piece of paper for later reference.

Trickster

As far as these authors can tell, **Trickster** is a new game. We tried to design a trick-taking card game similar to, but simpler than, **Bridge**, **Euchre**, or **Hearts**. In addition to simplifying the rules (e.g., there is no trump suit), elements of randomness and secrecy found in most trick-taking games were eliminated so that a

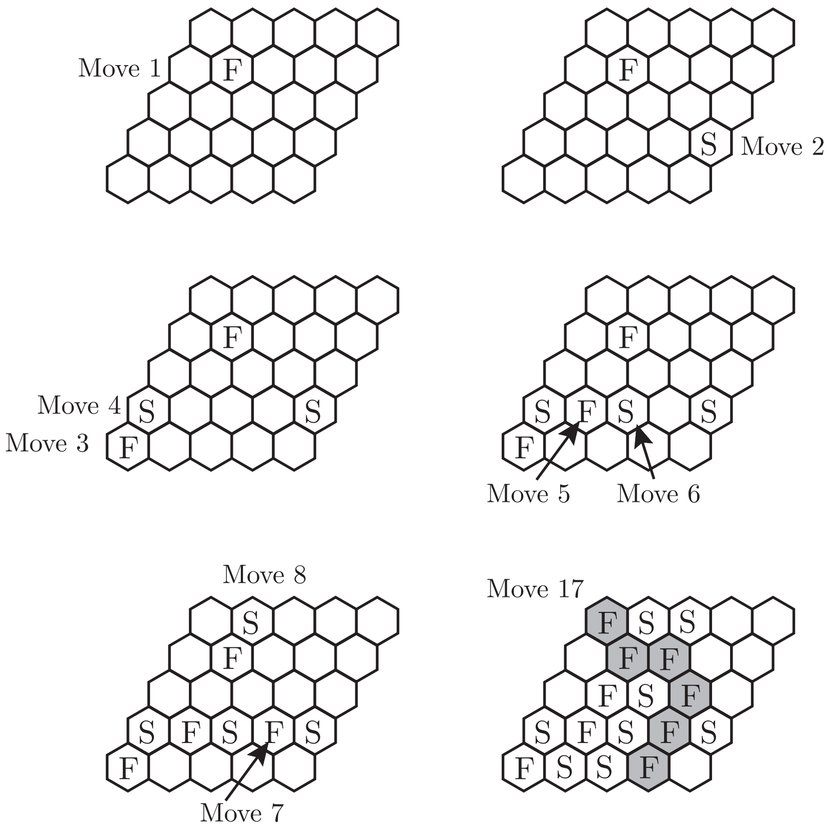


Figure 2.5. Moves in a 5×5 **Hex** game

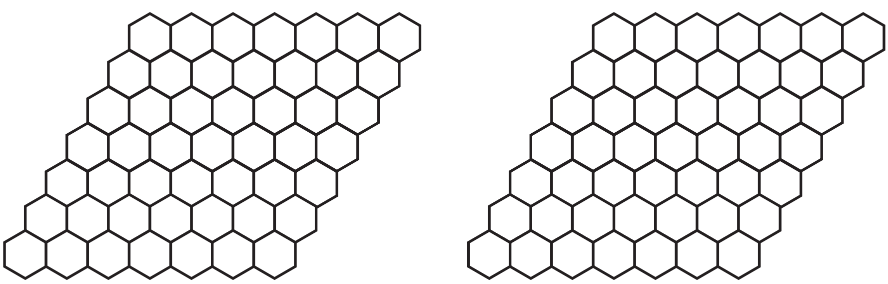


Figure 2.6. Two blank **Hex** boards

deterministic game would emerge. Even with simplification, **Trickster** has two features that neither **Nim** nor **Hex** possess: the possibility of ties and allowing more than two players.

Trickster is a game played with cards from a standard deck where there are thirteen ranks (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) and four suits (Hearts, Diamonds, Clubs, Spades). There can be two or more players. It is helpful to imagine looking down on the players sitting around a table. One player is designated the first player, and play moves clockwise from this player. In front of each player is the same number of playing cards resting face up, so that everyone can see the ranks and suits of the cards. Note that the game begins with the cards already face up on the table and does not include the deal—remember that there can be no randomness in a deterministic game. Another way to imagine this is that the players arrive at the table with the cards already in place as they take their seats.

The game is played in rounds. There is a lead player in each round. In the first round, the first player is the lead player. In subsequent rounds, the lead player is the player who won the previous round. To start a round, the lead player places any one of her cards into the center of the table. Each of the other players, playing clockwise from the lead player, will place one of his or her cards into the center of the table. Unlike the lead player who could choose any of her cards, the other players, if possible, must choose one of his or her cards that is the same suit as the card placed by the lead player. The winner of a round is the player who placed the highest ranking card of the suit of the card placed by the lead player. The winner of a round receives a trick. Any player with the largest number of tricks is a winner; ties are possible.

Figure 2.7 shows the start of a game of **Trickster** involving four players (North, East, South, and West), two cards per player, and North moving first. While the game is easily visualized with the picture below, it could also be represented with the following linear notation: (5C, 3H; 2C, 8C; 7C, 5D; 6D, 2H), which takes up less space.

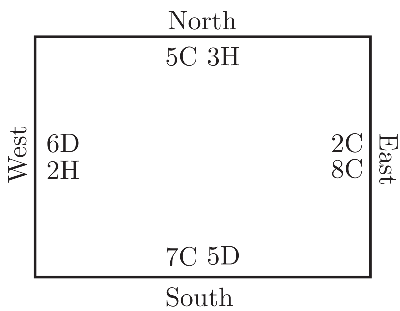


Figure 2.7. A four-player game of **Trickster**

Suppose in the first round of the game above, North placed 5C into the center. East could place either 2C or 8C into the center. Suppose East placed 2C into the center. Since players must try to match the suit of the lead card, South must place 7C into the center. Since West has no Clubs, West could place either 6D or 2H into the center. Suppose West placed 2H into the center. Figure 2.8 shows this situation.

Because she played the highest ranked Club, South is the winner of this round. In the second round, each player must place his or her remaining card in the center. Since South is the lead player, the player placing the highest ranked Diamond,

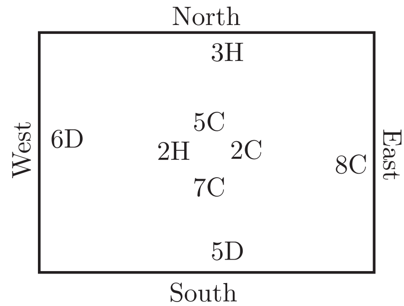


FIGURE 2.8. The first trick in **Trickster**

West, is the winner of the round. So, South and West each have one trick, and North and East each have zero tricks. South and West are the winners.

If East had played 8C, rather than 2C, in the first round, he would have won that round. He would have also won the second round because he would lead with the only remaining club. Hence, this play of the game would have resulted in East being declared the winner. Obviously, playing 2C was a very poor choice by East in the first round of the game above.

If North had played 3H, rather than 5C, in the first round, she would have won that round regardless of the other players choices, because 3H is the highest ranked Heart. North will be a winner. East will also be a winner if he kept 8C for the second round. South will be a winner if she kept the 7C for the second round and East kept 2C for the second round. Our purpose in this chapter and throughout the rest of the book is to help you to learn how to do this type of analysis in your decision making.

Play **Trickster** with one, two, or three other people. In different games, use different numbers of cards. Record the initial game, your moves, thoughts, observations, and insights on a piece of paper for later reference.

Exercises

(1) Play the three games introduced in this section several more times. It would be best if you played at least a few times with someone to whom you must explain the rules of the games. Your goal should be to understand the games well enough that you could explain to others how to play.
(2) State whether each of the following modifications of **Nim** result in a deterministic game, and why or why not.
    (a) The player who removes the last bean is the loser rather than the winner.
    (b) At the start of each player's turn, a number is picked out of a hat to determine from which heap the player must remove one or more beans.
    (c) Three, instead of two, people take turns removing beans from heaps.

(d) At the beginning of the game, both Firstus and Secondus privately write down the move each intends to do. After they are done writing, they show each other what they have written and Firstus's move is carried out. Now, if Secondus's move is still legal, it is carried out; otherwise, Secondus does nothing (e.g., Firstus may have removed all of the beans from a heap from which Secondus had planned to remove beans). Repeat until a player removes the last bean and is declared the winner.

(3) We already know that **Hex** can be played on different sized boards.
   (a) State at least one new modification of **Hex** that results in a different, but similar, deterministic game.
   (b) State at least one modification of **Hex** that results in a different, but similar, game that is not a deterministic game.

(4) There is exactly one game of **Nim** involving only one bean: [1]. There are exactly two games of **Nim** involving two beans: [2] and $[1, 1]$.
   (a) How many different games of **Nim** involve three beans? Give a list of these games.
   (b) How many different games of **Nim** involve four beans? Give a list of these games. How do you know your list is complete?
   (c) How many different games of **Nim** involve two heaps? Describe the set of these games.

(5) State whether each of the following modifications of **Trickster** results in a deterministic game, and why or why not.
   (a) Rather than starting with cards face up, first shuffle and deal the cards face up.
   (b) Rather than starting with all cards face up, start with all but one card face-up. For the first round, players can use only a face-up card. After the first round, the face-down cards are turned face-up and play continues as in the original game.
   (c) Hearts is considered *trump*. This means that for the purpose of determining a round winner, Hearts is higher than any card of a different suit.

(6) In the game **Poison**, each of two players take turns selecting one or two tiles from a single pile of tiles. The person who is forced to pick up the last tile loses. For example, if there are four tiles in the pile initially, Firstus might select one tile leaving three on the pile. Secondus may then take two tiles leaving one tile on the pile. Firstus would then be forced to take the last tile resulting in Secondus being declared the winner. Play **Poison** with some friends, and record your decisions, observations, and insights.

(7) The **Graph-Coloring** game starts with (1) a graph such as the four-pede (fewer legs than a centipede) and five-wheel examples shown in Figures 2.9 and 2.10, and (2) a set of colors such as red, green, and blue. A graph consists of open circles, called *vertices*, and lines, called *edges*, between pairs of vertices. Two vertices are said to be *adjacent* if there is an edge directly between them. Each of two players take turns selecting an uncolored vertex in the graph and color it with one of the colors. No two adjacent vertices are allowed to have the same color. The game ends when one of the players cannot make a move. Firstus wins if all of the vertices are colored; otherwise, Secondus wins. With a friend, play the **Graph-Coloring** game using either or both of the graphs below and three colors. Record your decisions, observations, and insights.

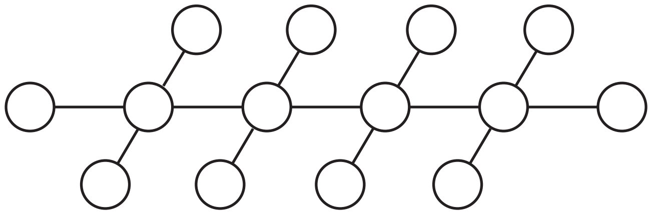


FIGURE 2.9. **Graph-Coloring** on a four-pede

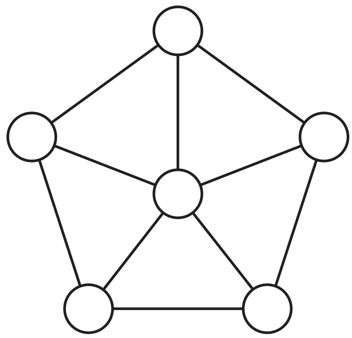


FIGURE 2.10. **Graph-Coloring** on a five-wheel

(8) The game **Chomp** [**44**] begins with a rectangular array of cookies, such as the 6×6 array shown in Figure 2.11.

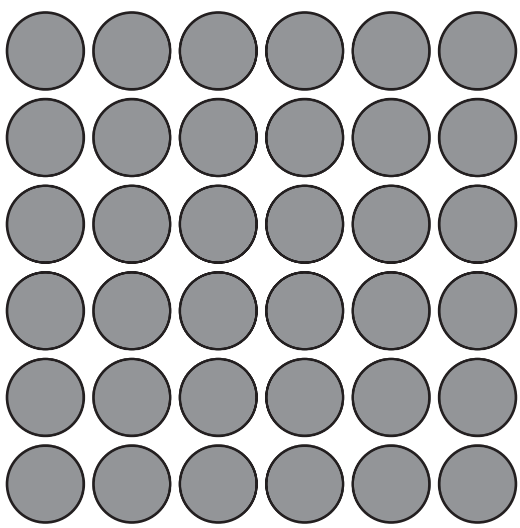


FIGURE 2.11. A 6×6 game of **Chomp**

The players take turns choosing a cookie, each time removing that cookie and all of the cookies in the rectangle of which it is the lower left corner. (They "chomp" on the array.) For example, Firstus could choose the cookie in the fourth row from the top and fifth column from the left resulting in the removal of eight cookies as shown in Figure 2.12.

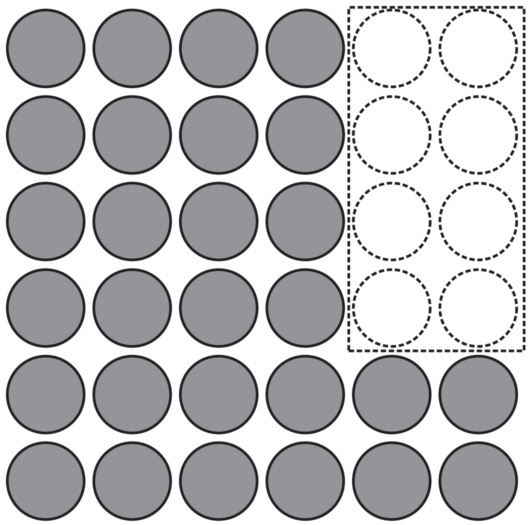


FIGURE 2.12. An initial move by Firstus

Secondus could then choose the cookie in the second row from the top and the third column from the left resulting in the removal of four cookies as shown in Figure 2.13.

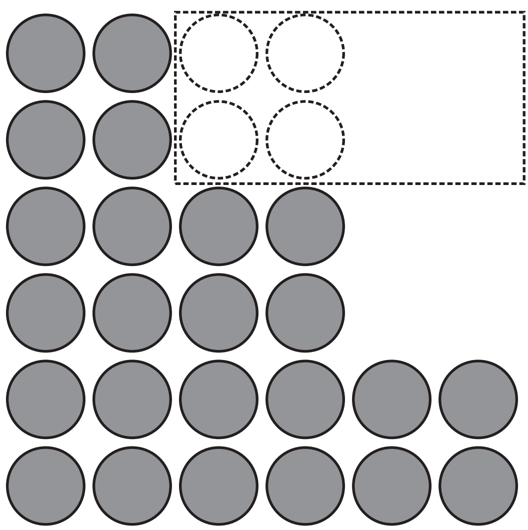


FIGURE 2.13. A response by Secondus

Firstus could then choose the cookie in the fifth row from the top and the fourth column from the left resulting in the removal of five cookies as shown in Figure 2.14.

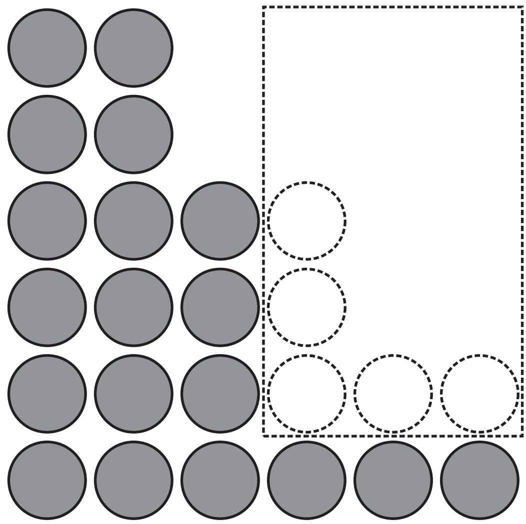


FIGURE 2.14. A response by Firstus

The loser is the player forced to take the last cookie, located in the lower left-hand corner. With a friend, play **Chomp** (start with different array sizes such as 6×6, 2×5, and 4×2); record your decisions, observations and insights.

3. Heuristics and Strategies

Since deterministic games involve no chance or secrecy, they lend themselves to mathematical analysis and frequently to complete descriptions of how to best play them. In this section, we will use deterministic games to introduce the ideas of a heuristic and a strategy.

> By the end of this section, you should be able to distinguish between heuristics and strategies, play a game using a strategy, create your own strategy for playing a game, and understand what an effective strategy is.

Heuristic: A *heuristic* is a guide for making decisions.

Strategy: A *strategy* is a complete and unambiguous description of what to do in every possible situation.

As humans, we often develop heuristics as we experience playing a game. However, a complete mathematical analysis of a game often requires us to develop strategies. The reader should be warned that colloquial use of the word "strategy" usually is in reference to a heuristic. Mathematicians need to make the distinction between a guide and a complete description. Hence, our separate definitions.

In most games, players are trying to win and are interested in knowing how to play the game to accomplish this. A good heuristic can help them win the game but cannot guarantee a win all the time because of a heuristic's imprecision. Thus, a player wants to have a strategy that will guarantee that he or she will always win the game.

Winning Strategy: A *winning strategy* is a strategy that will guarantee a win for the player using it, regardless of what strategy the opposing player is using.

Let's use the games that we introduced in the previous section to explore these ideas further.

Nim

After gaining some experience playing **Nim**, many people start adopting the following heuristic: try to even up the heaps. Faced with one heap of seven beans and one heap of three beans, which we denote as $[7, 3]$, this heuristic guides a player to remove four beans from the heap of seven. However, this heuristic does not completely specify what to do when faced with $[6, 5, 2, 2]$. Certainly, the heuristic would have us avoid removing one bean from a pile of two beans. But should three beans be removed from the heap of five so that there would be three heaps of size two? Or should one bean be removed from the heap of six so that the difference between the maximum and minimum heap sizes is minimized? Similarly, what would the heuristic tell us to do if faced with $[5, 5, 5]$? Is the heuristic telling us to take one bean from a heap of five, or five beans from a heap of five?

The point here is not whether "try to even up the heaps" is a good or bad guide for winning **Nim**. We are simply noting that "try to even up the heaps", does not always completely specify how to play the game; it is not a strategy.

Consider the following two instructions for how to play any game of **Nim**:

> **SMALL:** Remove one bean from a smallest heap.
>
> **LARGE:** Remove all beans from a largest heap.

Given any situation in a game of **Nim**, it would always be possible to determine the number of beans in each heap and so determine which heap or heaps contain the smallest (or largest) number of beans, and we agreed that all heaps of the same size are interchangeable. So, either instruction would tell us what to do. That is, they are strategies for any game of **Nim**.

Suppose Firstus and Secondus use the SMALL and LARGE strategies, respectively, when they play $[3, 3, 2, 2]$ **Nim**. According to her strategy, Firstus should remove one bean from one of the heaps of two beans. Secondus is then faced with $[3, 3, 2, 1]$. According to his strategy, Secondus should remove all of the beans from a heap of three beans. Firstus is then faced with $[3, 2, 1]$. According to her strategy, Firstus should remove the one bean from the heap of one bean. Secondus is faced with $[3, 2]$. According to his strategy, Secondus should remove all beans from the heap of three beans. Firstus is then faced with $[2]$. According to her strategy, Firstus should remove one bean from the heap of two beans. Secondus is faced with $[1]$. According to his strategy, Secondus should remove all beans from the heap of one bean. Secondus is the winner.

Some readers may say that Firstus, when faced with $[2]$, should have removed both beans from the remaining pile. It is true that by doing so Firstus would have won. We have not claimed that either strategy is effective or ineffective at winning **Nim** (although it is our eventual goal to develop strategies that are effective at winning games). We are merely illustrating that when a strategy is used, we will not violate the strategy's instruction in the middle of play. If we are asked to adopt a strategy, we are to follow the instructions regardless of consequences. As proxies, we do not make decisions; we only follow instructions.

Once strategies are developed, we do wish to evaluate their effectiveness at winning. We have already seen that Firstus's use of SMALL results in a loss to Secondus's use of LARGE in $[3, 3, 2, 2]$ **Nim**. In the exercises, you will see that Firstus's use of SMALL results in a win against Secondus's use of LARGE in $[3, 3, 2, 1]$ **Nim**. So, neither strategy is clearly better than the other. In the exercises, you will show that neither strategy is very effective at winning because there are strategies for an opponent that will always win against SMALL and LARGE in most instances of the game.

One approach for developing an effective strategy is to turn an effective heuristic into a strategy. Although "try to even up the heaps" seems to be an effective heuristic to some people, we have seen that it is not a strategy. Here is one of many ways to define a strategy that would be consistent with this heuristic:

> **EVEN WITH LARGEST:** From a largest heap, remove enough beans so that the heap's new size matches the second largest size of a heap. If all heaps are the same size, remove all the beans in one heap.

Does this new strategy improve upon LARGE and SMALL in the **Nim** games considered earlier?

Suppose Firstus uses EVEN WITH LARGEST and Secondus uses LARGE when they play $[3, 3, 2, 2]$ **Nim**.

- Faced with $[3, 3, 2, 2]$, EVEN WITH LARGEST has Firstus remove one bean from a three-bean heap.
- Faced with $[3, 2, 2, 2]$, LARGE has Secondus remove the three-bean heap.
- Faced with $[2, 2, 2]$, EVEN WITH LARGEST has Firstus remove a two-bean heap.
- Faced with $[2, 2]$, LARGE has Secondus remove a two-bean heap.
- Faced with $[2]$, EVEN WITH LARGEST has Firstus remove the remaining two-bean heap.

Firstus, using EVEN WITH LARGEST, is the winner.

Suppose Firstus uses SMALL and Secondus uses EVEN WITH LARGEST when they play $[3, 3, 2, 1]$ **Nim**.

- Faced with $[3, 3, 2, 1]$, SMALL has Firstus remove one bean from the one-bean heap.
- Faced with $[3, 3, 2]$, EVEN WITH LARGEST has Secondus remove one bean from a three-bean heap.
- Faced with $[3, 2, 2]$, SMALL has Firstus remove one bean from a two-bean heap.
- Faced with $[3, 2, 1]$, EVEN WITH LARGEST has Secondus remove one bean from the three-bean heap.
- Faced with $[2, 2, 1]$, SMALL has Firstus remove one bean from the one-bean heap.
- Faced with $[2, 2]$, EVEN WITH LARGEST has Secondus remove a two-bean heap.
- Faced with $[2]$, SMALL has Firstus remove one bean from the two-bean heap.
- Faced with $[1]$, EVEN WITH LARGEST has Secondus remove the remaining one-bean heap.

Secondus, using EVEN WITH LARGEST, is the winner.

EVEN WITH LARGEST is an improvement upon LARGE and SMALL in the two **Nim** games considered earlier. It also turns out to be quite effective on all one- and two-heap **Nim** games.

For **Nim** with a single heap, EVEN WITH LARGEST has Firstus remove all beans from the single heap, resulting in a win for Firstus. EVEN WITH LARGEST ensures a win for Firstus in a one-heap **Nim** game: regardless of what Secondus plans to do, Firstus will win.

For **Nim** with two heaps of different sizes, EVEN WITH LARGEST has Firstus remove enough beans from the larger heap to even the two heaps. Secondus is then faced with two heaps of the same size. Any legal move by Secondus will result in either (1) a single heap, or (2) two heaps of different sizes. In case (1), EVEN WITH LARGEST has Firstus remove the remaining heap, resulting in a win for Firstus. In case (2), Firstus is back to the situation that started this paragraph (although with fewer total beans). Once again EVEN WITH LARGEST will have Firstus even up the two heaps. Again any legal move by Secondus will result in either (1) a single heap, or (2) two heaps of different sizes. Since with each repetition of turns, there are fewer total beans, eventually Secondus will be forced to leave Firstus with a single heap, allowing Firstus to win. Thus, EVEN WITH LARGEST ensures a win for Firstus in **Nim** with two heaps of different sizes.

For a **Nim** game with two equal-size heaps, it is Secondus who can ensure a win by using EVEN WITH LARGEST. This is because the two players have reversed roles from the previous paragraph: Firstus is forced to make the heaps have different sizes while Secondus, using EVEN WITH LARGEST, can make both heaps even or remove the remaining single heap.

Compare the generality of these last three results with what we learned earlier. Earlier we showed that Firstus using EVEN WITH LARGEST wins against Secondus using LARGE when they play the specific game of $[3, 3, 2, 2]$ **Nim**. Now we have shown that Firstus using EVEN WITH LARGEST wins against Secondus using any strategy, not just LARGE, when they play any **Nim** game consisting of one heap or two heaps of different sizes. That is, EVEN WITH LARGEST is a winning strategy for Firstus in any **Nim** game consisting of one heap or two heaps of different sizes. Earlier we showed that Secondus using EVEN WITH LARGEST wins against Firstus using SMALL when they play the specific game of $[3, 3, 2, 1]$ **Nim**; now we have also shown that EVEN WITH LARGEST is a winning strategy for Secondus in any **Nim** game of two heaps with equal sizes.

Could there be any more effective strategy for one- and two-heap **Nim** games? For example, what is Firstus to do if faced with $[5, 5]$ **Nim**? If Secondus uses EVEN WITH LARGEST, then Firstus will lose no matter what she does. So, the best Firstus can hope for is that Secondus does not use EVEN WITH LARGEST. Similarly, the best Secondus can hope for when playing a **Nim** game with one heap or two heaps of different sizes is that Firstus does not use EVEN WITH LARGEST. If Secondus (respectively, Firstus) has a strategy that ensures a win in a particular **Nim** game, then there cannot be any strategy that will ensure a win for the other player in that particular **Nim** game. So, EVEN WITH LARGEST is as effective as any strategy could be for one- and two-heap **Nim** games: EVEN WITH LARGEST ensures a win for a player whenever there is some strategy that can ensure that player a win.

Unfortunately, EVEN WITH LARGEST is not as effective as a strategy could be for **Nim** games with three or more heaps. For example, consider $[2, 1, 1]$ **Nim**. If Firstus uses EVEN WITH LARGEST, then Firstus will remove one bean from the two-bean heap, each subsequent move involves removing a one-bean heap, and Secondus wins. However, if Firstus had initially removed the two-bean heap, then again each subsequent move involves removing a one-bean heap, but now Firstus wins. By initially removing the two-bean heap, Firstus ensures herself a win, but EVEN WITH LARGEST does not ensure Firstus a win.

In summary, we have examined a heuristic and three strategies for **Nim**, one of which is a winning strategy for **Nim** games with one or two heaps. In section four, we will describe a winning strategy for all games of **Nim**. So, the specifics of the strategies examined here are less important than the concept, use, development, and effectiveness of strategies. A last important point is to note how we have been able to do a complete analysis of one- and two-heap **Nim** but have only scratched the surface of three or more heap **Nim**. This is fairly typical for mathematicians: to solve a difficult problem, first solve simpler problems. For **Nim**, simplification occurs by considering games with few heaps or few beans.

Hex

After our lengthy discussion of **Nim**, we hope that you have the necessary tools to start an analysis of **Hex**. Notation was somewhat helpful in our discussion of **Nim**: we could write "$[3, 3, 2, 1]$" rather than write "two 3-bean heaps, one 2-bean heap, and one 1-bean heap" or draw a picture of that situation. Similarly, we can assign numbers to the cells of a **Hex** board in order to describe what cell is captured by a move. The diagram below shows two different reasonable numbering schemes.

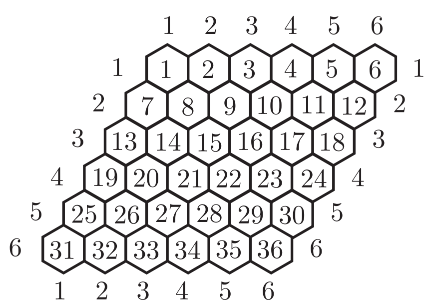


FIGURE 3.1. **Hex** numbering schemes

For example, the cell in the second row and the fourth skewed column could be described as 10 or $(2, 4)$. You may find one or the other more convenient in discussing a **Hex** board. The first numbering scheme makes it easy to describe the following strategy:

ASCENDING: Number the hexagons in reading order (left to right, top to bottom) and capture the uncaptured hexagon with the smallest number.

In the exercises, you are asked to investigate several different strategies for playing **Hex**, including strategies that you make up yourself.

Trickster

Consider the three-player game of **Trickster** in which the first player North has 4C and 4H, East has 3C and 5C, and West has 3H and 5H. If North leads with 4C, East follows with 3C, and West follows with 5H, then North will win both tricks and thus win the game as shown in Figure 3.2.

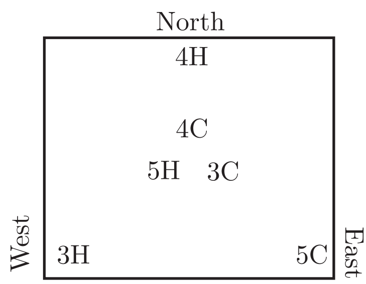


FIGURE 3.2. North leads with 4C, East follows with 3C

If North leads with 4C, East follows with 5C, and West follows with 5H, then East will win both tricks and thus win the game as shown in Figure 3.3.

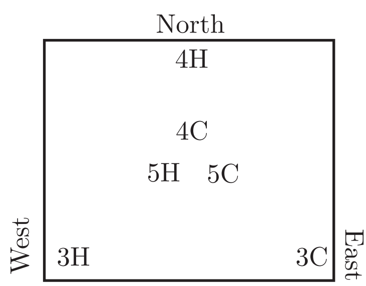


FIGURE 3.3. North leads with 4C, East follows with 5C

If North leads with 4H, East follows with 3C, and West follows with 5H, then West will win both tricks and thus win the game as shown in Figure 3.4.

So it is possible for any one of the players to win. However, none of the players can ensure themselves of a win: if North leads with 4C, North will win only if East chooses to follow with 3C, and if North leads with 4H, North will win only if West chooses to follow with 3H. East and West have a chance to win only if North leads with 4C and 4H, respectively.

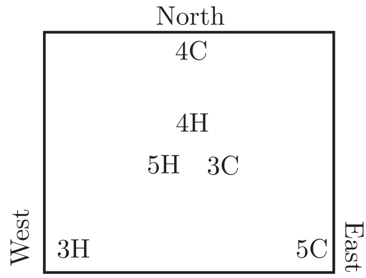


FIGURE 3.4. North leads with 4H

For some games of **Nim**, we have found strategies that ensure a win for one of the players. Apparently, as the previous example shows, that is not true for all games of **Trickster**. Are there some games of **Nim** and/or **Hex** for which neither player can ensure a win? Or is there something different about **Trickster**? We will be able to answer these questions by the end of this chapter.

Exercises

(1) For each of the following strategy choices and games, describe each move (using words or symbols) and who wins:
   (a) In $[3, 3, 2, 1]$ **Nim**, Firstus uses SMALL and Secondus uses LARGE.
   (b) In $[3, 3, 2, 2]$ **Nim**, Firstus uses SMALL and Secondus uses EVEN WITH LARGEST.
   (c) In $[7, 3, 2]$ **Nim**, both players use EVEN WITH LARGEST.
(2) Explain why "mirroring Firstus" is a strategy for Secondus in $[6, 6, 4, 4]$ **Nim**. Explain why "mirroring Firstus" is not a strategy for Secondus in $[6, 5, 4, 4]$ **Nim**. Explain why "mirroring Firstus" ensures a win for Secondus in $[6, 6, 4, 4]$ **Nim**.
(3) What happens in a game of **Nim** in which each heap has exactly one bean?
(4) Consider the following strategy for **Nim**:
   MAKE EVEN HEAPS: If there is an odd number of heaps, remove a largest heap. Otherwise, remove one bean from a largest heap.
   (a) Describe each move in $[3, 3, 2, 1]$ **Nim** if Firstus uses MAKE EVEN HEAPS and Secondus uses LARGE.
   (b) Explain why Firstus using MAKE EVEN HEAPS will win against Secondus using LARGE in every game having at least one heap with two or more beans.
   (c) Explain why Secondus using MAKE EVEN HEAPS will win against Firstus using LARGE in every game having at least two heaps with two or more beans.
(5) Consider the following strategy for **Nim**:
   MAKE EVEN BEANS: If there is an even number of beans and some heap has two or more beans, remove two beans from a largest heap. Otherwise, remove one bean from a largest heap.

(a) Describe each move in $[4, 3, 2, 1]$ **Nim** if Firstus uses MAKE EVEN BEANS and Secondus uses SMALL.
(b) Explain why Firstus using MAKE EVEN BEANS will win against Secondus using SMALL in every game having at least one heap with two or more beans.
(c) Explain why Secondus using MAKE EVEN BEANS will win against Firstus using SMALL in every game having at least two heaps and at least one heap with two or more beans.

(6) In a 3×3 **Hex** game, explain why "capture the center hexagon" does not describe a strategy for Firstus or Secondus. Describe a strategy for Firstus in a 3×3 **Hex** game that is consistent with "capture the center hexagon".

(7) In this exercise, we do an analysis of the simplest games of **Hex**.
(a) What happens in 1×1 **Hex**?
(b) Can Firstus win 2×2 **Hex**? How or why not?
(c) Can Secondus win 2×2 **Hex**? How or why not?
(d) Can Firstus ensure a win in 2×2 **Hex**? How or why not?
(e) Can Secondus ensure a win in 2×2 **Hex**? How or why not?
(f) Repeat the previous questions for 3×3 **Hex**.
(g) Repeat the previous questions for 4×4 **Hex**.

(8) In this exercise, explore playing **Hex** games with specific strategies.
(a) What happens in 4×4 **Hex** if both players use ASCENDING? What happens in 5×5 **Hex** if both players use ASCENDING?
(b) Develop a different strategy for **Hex**; we will call it YOUR STRATEGY in the following questions. What happens in 5×5 **Hex** if Firstus uses YOUR STRATEGY and Secondus uses ASCENDING?
(c) What happens in 5×5 **Hex** if Secondus uses YOUR STRATEGY and Firstus uses ASCENDING?
(d) What happens in 5×5 **Hex** if both players use YOUR STRATEGY?
(e) Does ASCENDING or YOUR STRATEGY ensure a win for one of the players in 5×5 **Hex**? Why or why not?
(f) Another heuristic often mentioned as a good way to play **Hex** is to "play near the diagonal". Develop this heuristic into a strategy. What happens when this new strategy is played against itself or against previously defined strategies?

(9) For the game of **Poison**, is the advice "always take one tile" a strategy? Why or why not? If it is a strategy, is it a winning strategy? Why or why not?

(10) For the game of **Poison**, is the advice "take two tiles if possible" a strategy? Why or why not? If it is a strategy, is it a winning strategy? Why or why not?

(11) Why is "color the leftmost available vertex with a randomly selected color from among those available" a strategy for the **Graph Coloring four-pede** game (see Figure 3.5) with three colors? Will it guarantee a win for Firstus?

(12) Consider the following purported strategy for a $2 \times n$ game of **Chomp**:
LEAVE ONE SHORTER: If the two rows have the same number of cookies, remove one cookie from the top row. Otherwise, remove enough cookies from the bottom row so that you leave the bottom row one cookie longer than the top row.

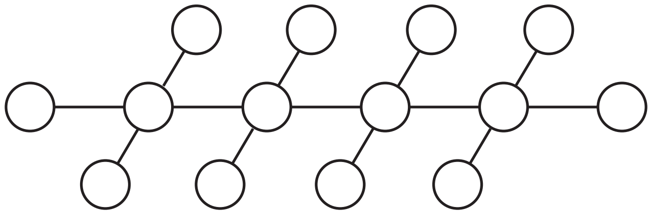


FIGURE 3.5. **Graph-Coloring** on a four-pede

(a) Is LEAVE ONE SHORTER a strategy for Firstus in a $2 \times n$ game of **Chomp**? Why or why not?
(b) Is LEAVE ONE SHORTER a strategy for Secondus in a $2 \times n$ game of **Chomp**? Why or why not?
(c) If LEAVE ONE SHORTER is a strategy for one or both players, does it ensure a win for one of the players? Why or why not?

4. Game Trees

In this section, we introduce a tool that can sometimes be effective in identifying (winning) strategies for playing a game.

By the end of this section, you should be able to create a game tree for any deterministic game, determine the number of possible ways to play a game, describe strategies for a game based on its game tree, and use backward induction to determine a strategy that ensures a win for one of the two players.

$[2, 1, 1]$ Nim

Consider $[2, 1, 1]$ **Nim**. Firstus has three legal moves: (1) remove one bean from the two-bean heap, (2) remove two beans from the two-bean heap, and (3) remove one bean from a one-bean heap. These moves can be represented by arrows going from the original game to the new games Secondus will face as shown in Figure 4.1.

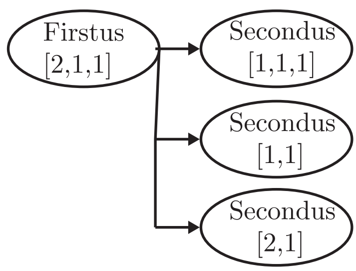


FIGURE 4.1. First moves in $[2, 1, 1]$ **Nim**

For the first two games, $[1, 1, 1]$ and $[1, 1]$, Secondus has only one legal move: remove one bean from a one-bean heap. If faced with $[2, 1]$, Secondus has three legal moves: (1) remove one bean from the two-bean heap, (2) remove two beans from the two-bean heap, and (3) remove one bean from the one-bean heap. These moves can be represented by arrows going from the game faced by Secondus to the game Firstus will face as shown in Figure 4.2.

For the first four games, Firstus has only one legal move: remove one bean from a one-bean heap. If faced with $[2]$, Firstus has two legal moves: (1) remove one bean from the two-bean heap, and (2) remove two beans from the two-bean heap. These moves can be represented by arrows going from the game faced by Firstus to the game Secondus will face as shown in Figure 4.3.

Notice that three moves result in a win for Firstus, which is indicated by the initial "F." But there are three games in which Secondus has a single legal move that results in a win: remove one bean from the one-bean heap. The complete diagram is shown in Figure 4.4.

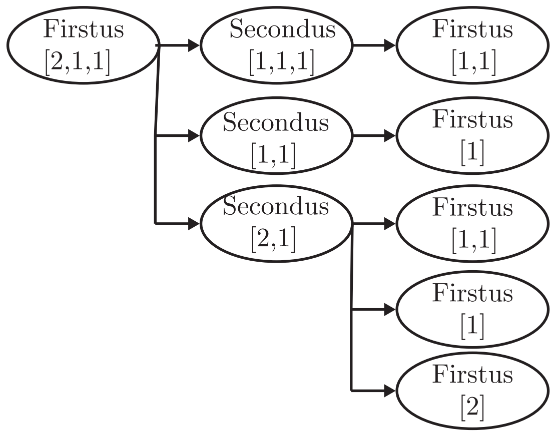


FIGURE 4.2. Second moves in $[2, 1, 1]$ **Nim**

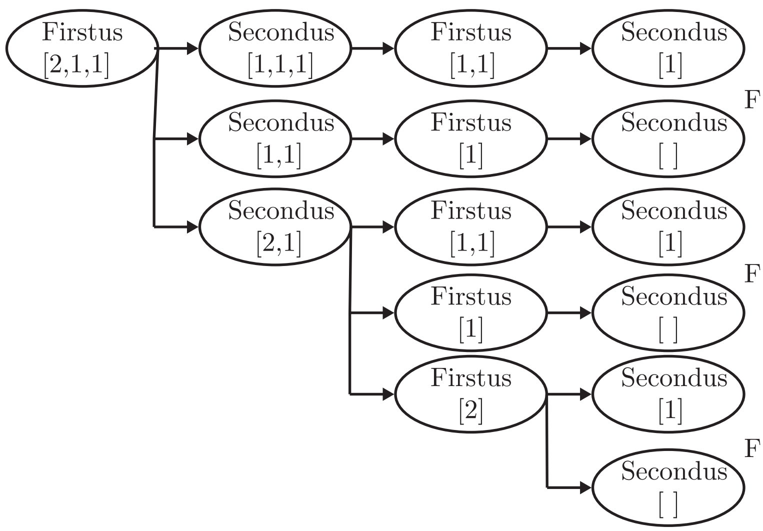


FIGURE 4.3. Third moves in $[2, 1, 1]$ **Nim**

It can be seen from the diagram that there are six distinct ways that the game could be played (corresponding to the six [ ] nodes in the diagram). Each player is the winner in three plays of the game.

Game Tree: A *game tree* is a complete visual representation of all ways a game could be played. Each node of the tree represents a game position and is labeled with a mnemonic for that position and/or the player who can move from that node. At each node in the tree, every legal move is seen as an arrow extending from that node to the node of the resulting position. Leaves (nodes with no arrows extending from them) are labeled with the outcome for that play of the game.

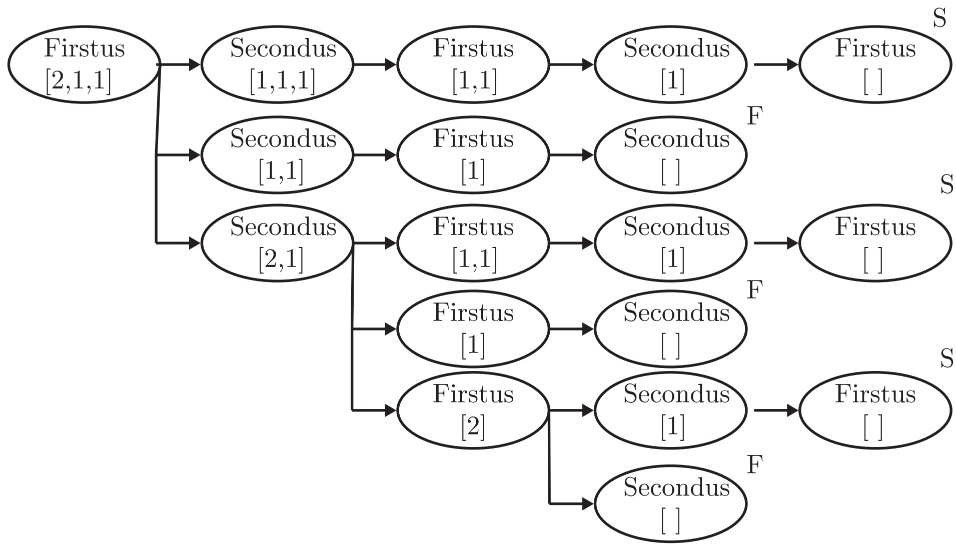


FIGURE 4.4. Fourth moves in $[2, 1, 1]$ **Nim**

Nodes tell you where you are in the game; arrows indicate the legal moves available. Each leaf must be the end of a play of the game. Therefore, the number of leaves gives us the number of ways to play the game. Sometimes it is easier to label the arrows rather than describing the positions and to only include nodes that are not leaves (see the **Trickster** examples near the end of this section).

Using the tree, we can also determine all of the strategies available to each of the players, and which of these strategies ensures a win. Recall that a strategy is a complete and unambiguous description of what to do in every possible situation. When a player has only one legal move available, there is nothing that needs to be said. However, whenever a player has more than one legal move available, a strategy must say which legal move to choose. Firstus has a choice of moves from each of the two dashed oval positions in Figure 4.5.

A strategy for Firstus specifies what she will do whenever she has a choice. In the tree shown in Figure 4.5, we see that Firstus has four strategies for $[2, 1, 1]$ **Nim**:

(1) Remove one bean from the two-bean heap.
(2) Remove two beans from the two-bean heap.
(3) Remove one bean from a one-bean heap, and if faced with [2], remove one bean.
(4) Remove one bean from a one-bean heap, and if faced with [2], remove two beans.

Firstus's first two strategies have simpler descriptions than the last two because after the initial decision, she will never again have a choice of legal moves, and when there is no choice, there is no need to talk about it. In the last two strategies, it is possible that Firstus will be faced with [2], and since there are two legal moves

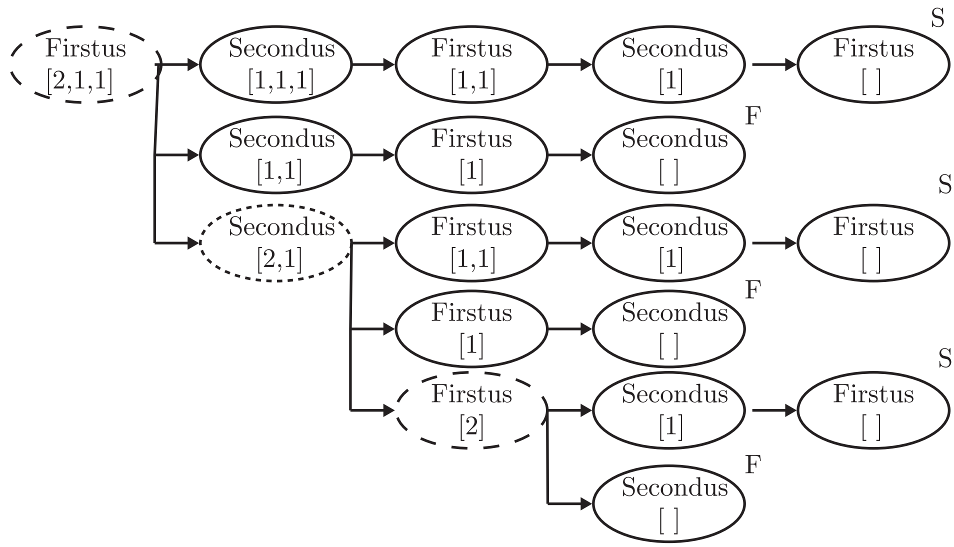


FIGURE 4.5. Decision nodes in $[2, 1, 1]$ **Nim**

available, a strategy must specify what to do. Thus strategies one and two are distinguished by the decision that Firstus makes initially, while strategies three and four are distinguished by a second decision that Firstus may need to make.

Similarly, Secondus has a decision to make only when faced with the dotted oval position $[2, 1]$. Therefore, Secondus has three strategies available for $[2, 1, 1]$ **Nim**:

(1) If faced with $[2, 1]$, remove one bean from the two-bean heap.
(2) If faced with $[2, 1]$, remove two beans from the two-bean heap.
(3) If faced with $[2, 1]$, remove one bean from the one-bean heap.

As a side note, six of the seven strategies for $[2, 1, 1]$ **Nim** described above are consistent with more general strategies we described in the previous section. Firstus's strategy 1 and Secondus's strategy 1 are EVEN WITH LARGEST. Firstus's strategy 2 and Secondus's strategy 2 are LARGE, and their third strategies are each SMALL. Only Firstus's fourth strategy is new.

Backward Induction

From the game tree, we can determine which player has a strategy that can ensure a win and what this strategy is. By reversing our perspective on the tree and working backward from the leaves, we can label each vertex with the initial of the player who can ensure a win from that node. Let us repeat that idea again because it embodies the critical step: label each vertex with the initial of the player who can ensure a win from that node. The first step backward is easy because no player has a choice of moves as shown in Figure 4.6.

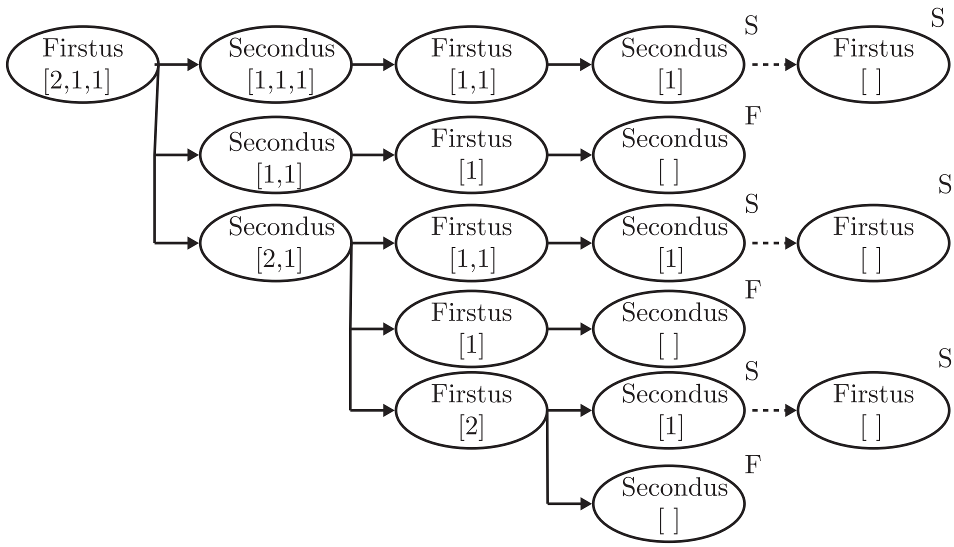


Figure 4.6. Backward Induction, first step

The next step backward has one nontrivial labeling: at [2], Firstus has two moves available. We can see that one move ensures a win for Secondus while the other move ensures a win for Firstus. Clearly, Firstus should choose the latter move, which we record by marking the arrow corresponding to that move as shown in Figure 4.7.

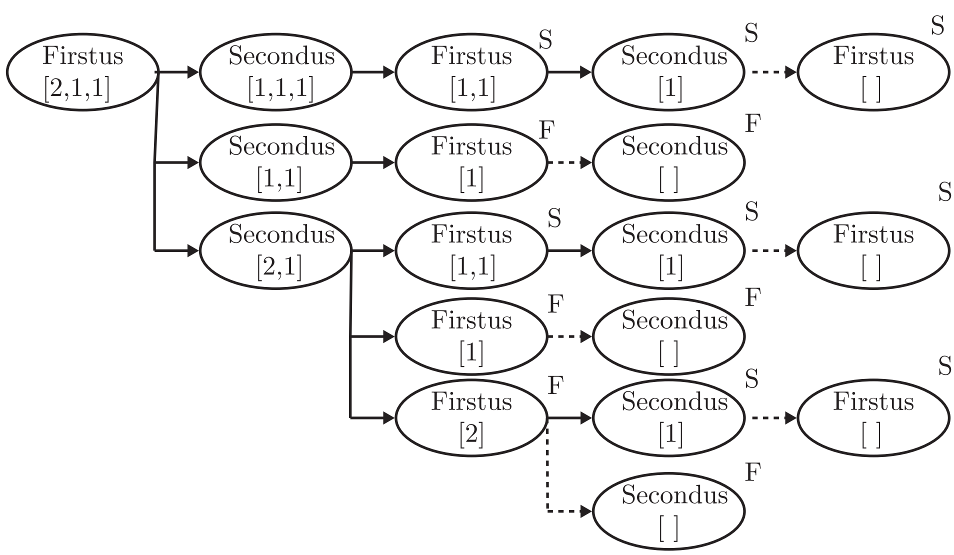


Figure 4.7. Backward Induction, second step

But Firstus can choose this move only if Secondus plays badly at position $[2, 1]$. At position $[2, 1]$, Secondus should choose the move that will ensure him a win as shown in Figure 4.8.

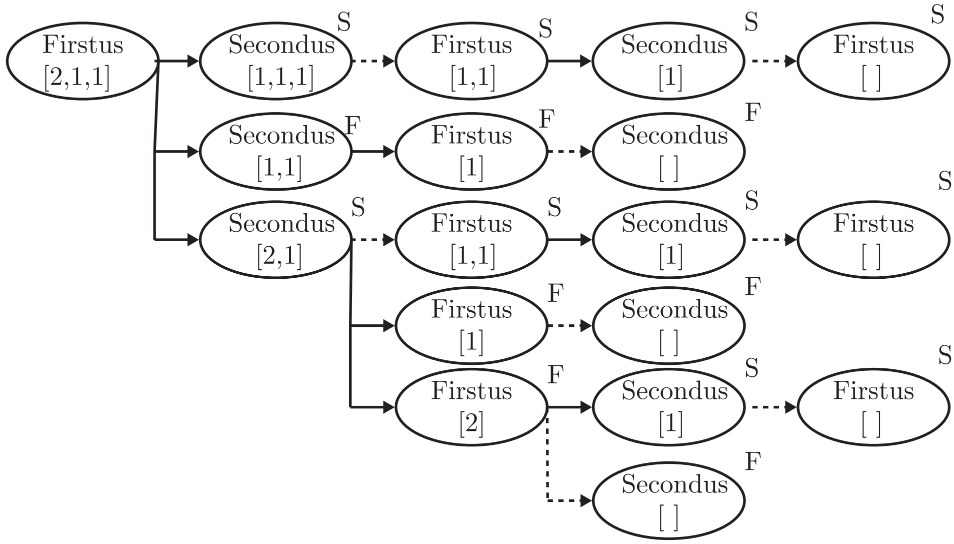


Figure 4.8. Backward Induction, third step

Finally, Firstus can choose an initial move from $[2, 1, 1]$ that ensures her a win as shown in Figure 4.9.

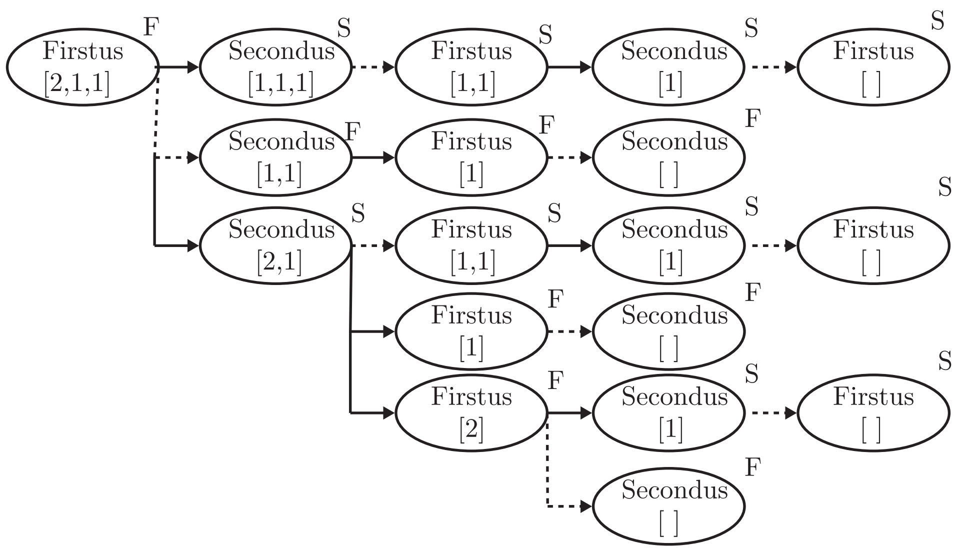


Figure 4.9. Backward Induction, fourth step

Firstus's marked arrows tell us the strategy Firstus should use to ensure her win: go from $[2,1,1]$ to $[1,1]$, which corresponds to her second listed strategy—remove two beans from the two-bean heap.

Backward Induction: *Backward induction* is the following algorithm: Label all of the leaves with the winning player. If there are unlabeled nodes, then at least one (called the parent) must have all of its branches extending to labeled nodes (called the children).

- Case 1: If any of the children are labeled with the player who can move from the parent, then label the parent with that player and mark one of the arrows from the parent to a child node that is marked with that player.
- Case 2: If none of the children are labeled the same as the player whose turn it is at the parent node, label the parent node with the other player.

Repeat this process until all nodes are labeled. The player whose name appears on the starting node has a winning strategy specified by the marked arrows this player can select.

In Figure 4.10, there are two versions of Case 1, differing by whose turn it is at the parent node.

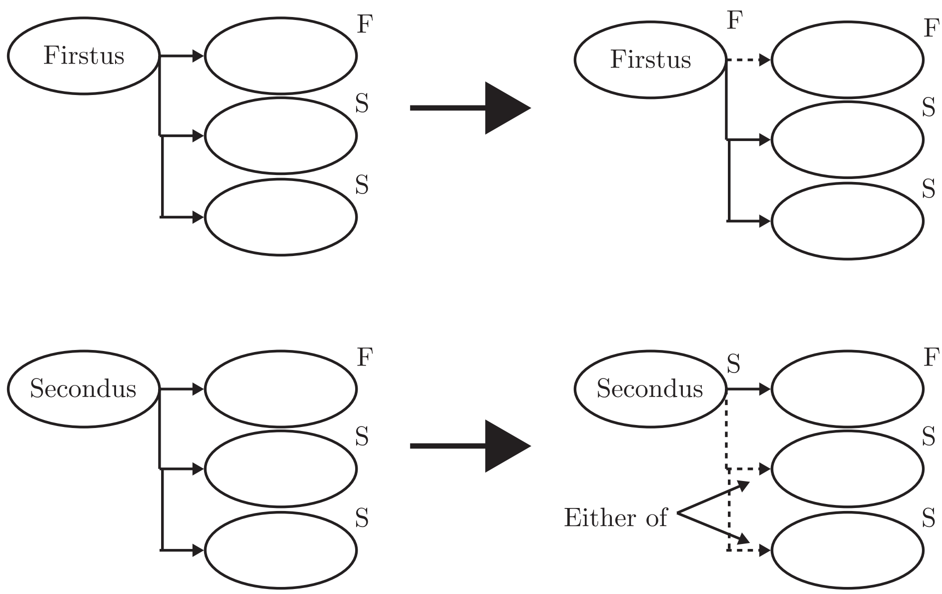


FIGURE 4.10. Case 1 of Backward Induction

Figure 4.11 shows an example of Case 2 from Secondus's perspective.

$[2,2]$ **Nim**

For another example, consider $[2,2]$ **Nim**. The game tree is shown in Figure 4.12.

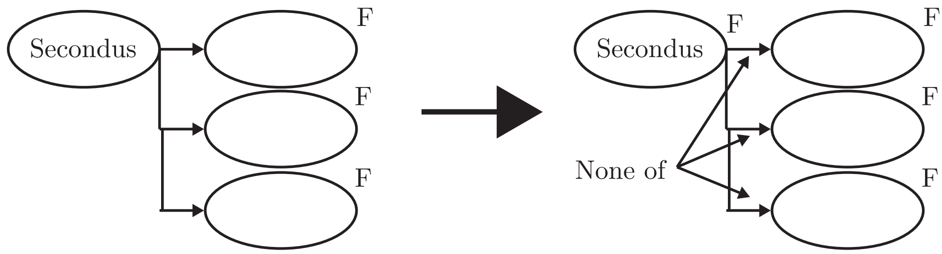


FIGURE 4.11. Case 2 of Backward Induction

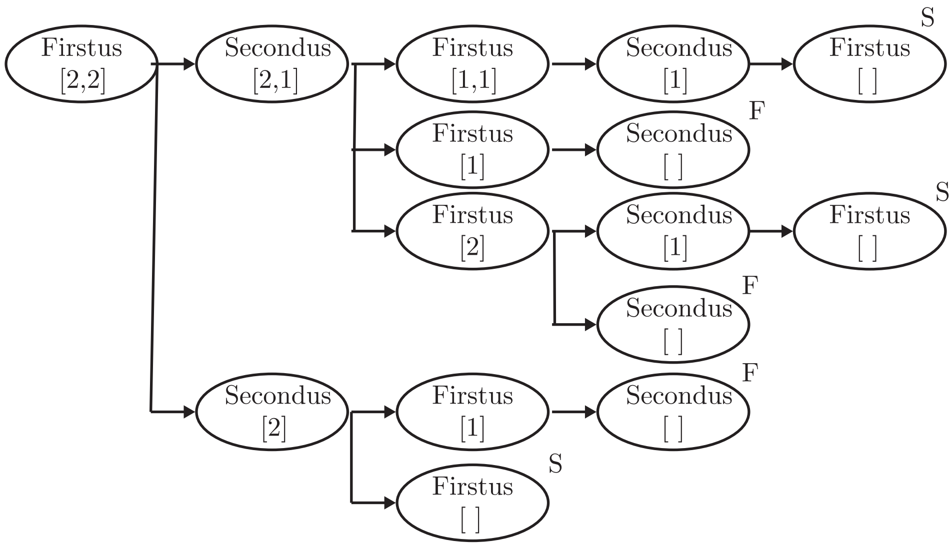


FIGURE 4.12. $[2,2]$ **Nim** game tree

There are six distinct ways that the game could be played. In Figure 4.13, the situations where Firstus or Secondus have a choice among two or more available moves have been indicated by dashes and dots, respectively.

Firstus has a choice of two moves initially, and if she chooses one of these moves, she may have another choice. This leads to the following strategies:

(1) Remove one bean from a heap of two beans, and if faced with a single heap of two beans, remove one bean.
(2) Remove one bean from a heap of two beans, and if faced with a single heap of two beans, remove two beans.
(3) Remove two beans from a heap of two beans.

At the beginning of the game, Secondus does not know whether he will be faced with $[2,1]$ or $[2]$. So, a strategy must specify what to do in either case. This leads to the following six strategies:

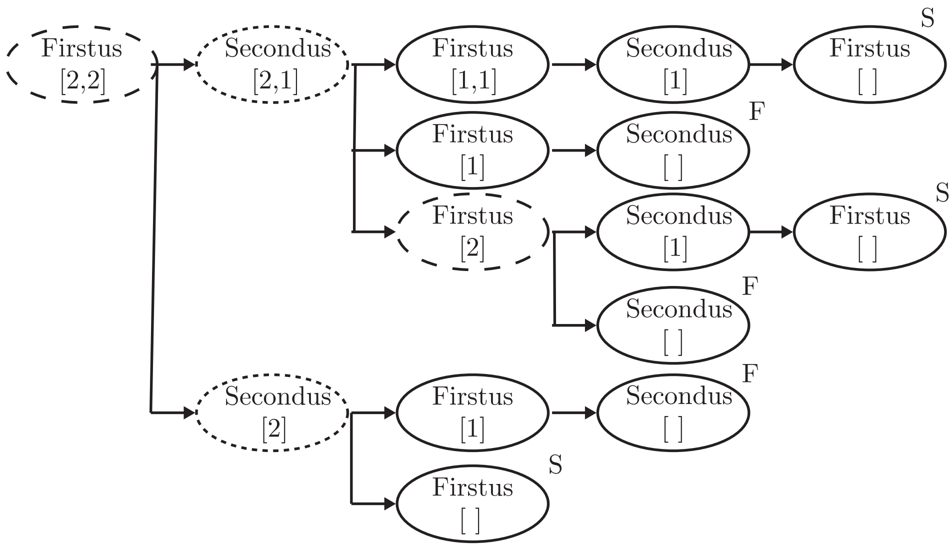


FIGURE 4.13. Decision nodes for $[2, 2]$ **Nim**

(1) If faced with $[2, 1]$, remove one bean from the two-bean heap. If faced with $[2]$, remove one bean from the two-bean heap.
(2) If faced with $[2, 1]$, remove one bean from the two-bean heap. If faced with $[2]$, remove two beans from the two-bean heap.
(3) If faced with $[2, 1]$, remove two beans from the two-bean heap. If faced with $[2]$, remove one bean from the two-bean heap.
(4) If faced with $[2, 1]$, remove two beans from the two-bean heap. If faced with $[2]$, remove two beans from the two-bean heap.
(5) If faced with $[2, 1]$, remove the one-bean heap. If faced with $[2]$, remove one bean from the two-bean heap.
(6) If faced with $[2, 1]$, remove the one-bean heap. If faced with $[2]$, remove two beans from the two-bean heap.

Backward induction leads to the labeling of the game tree as shown in Figure 4.14.

According to the labels, Secondus can ensure a win. According to the marked arrows, strategy 2 is the one that ensures a win for Secondus. It is interesting to note that Secondus's strategy 2 is a rewording of EVEN WITH LARGEST. Sometimes strategies worded very differently end up acting the same way.

Restricted Game Trees

Unfortunately, game trees can get quite big, and very quickly. In the exercises you will see that the tree for 2×2 **Hex** has 24 leaves in it. In a 3×3 game of **Hex** there are more than fifteen thousand leaves, and in a 4×4 game of **Hex**, there are over fifty million leaves! Creating the game tree appears to be impractical for all but the simplest of games. Game trees can be extremely useful in understanding

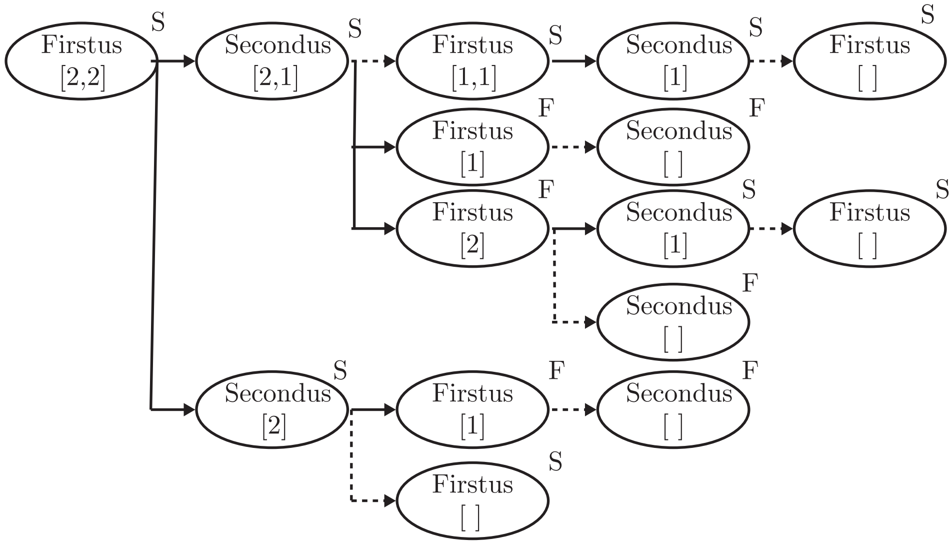


FIGURE 4.14. Backward Induction for [2, 2] **Nim**

some extremely simple games, and in the final section of this chapter, the concept of a game tree will be useful for proving a theorem about a large class of games.

Meanwhile, we can use a *restricted game tree* to evaluate whether a suggested strategy is winning. Consider the following strategy for games of **Trickster** in which each player's cards have no ties in ranks.

> **LOWEST RANK EFFECTIVE CARD:** If leading, choose the lowest ranked card. If not leading and there is a card that will win the round, choose the lowest ranked such card. Otherwise, choose the lowest ranked card that is allowable.

Suppose in (9H, 7H, 3H; 8H, 6H, 2H) **Trickster** that North uses LOWEST RANK EFFECTIVE CARD. We can build a game tree that has North using LOWEST RANK EFFECTIVE CARD and South contemplating any legal move. In a full game tree, North could initially choose any of her three cards; however, LOWEST RANK EFFECTIVE CARD forces North to choose 3H. Now South can respond by choosing any of his three cards. If he chooses 8H or 6H, South obtains a trick and will be the lead player in the next round. If South chooses 2H, North obtains a trick and will be the lead player in the next round. The start of the restricted tree, with nodes marked with the player who moves and the arrows marked with the card chosen is shown in Figure 4.15.

Again when it is North's turn, we determine his move from LOWEST RANK EFFECTIVE CARD, and when it is South's turn, we add all possible legal moves. This results in the restricted game tree as shown in Figure 4.16. Notice that labeling the arrows saves space, but leaves it to the reader to reconstruct at each node the cards that have been played and the cards that are left in each player's hand. Also

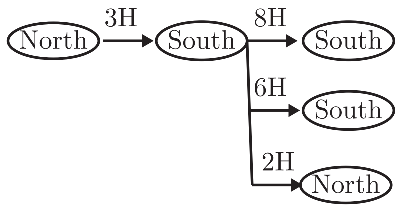


FIGURE 4.15. The beginning of a **Trickster** game tree

notice that we have saved space by not including the leaf nodes, only the outcome labels.

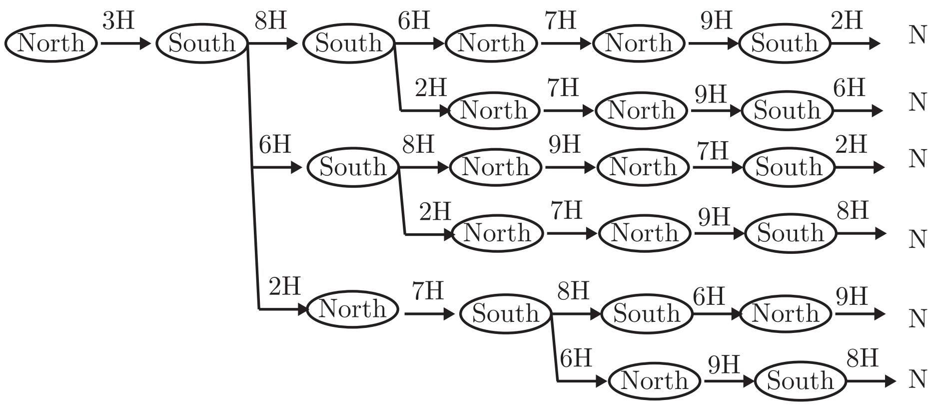


FIGURE 4.16. (9H,7H,3H;8H,6H,2H) **Trickster** restricted game tree

We see that North always wins, and so LOWEST RANK EFFECTIVE CARD is a winning strategy for North in (9H, 7H, 3H; 8H, 6H, 2H) **Trickster**.

If North uses LOWEST RANK EFFECTIVE CARD in (2C, 8D, 3S; 4C, 7D, 5H) **Trickster**, then the restricted game tree is shown in Figure 4.17.

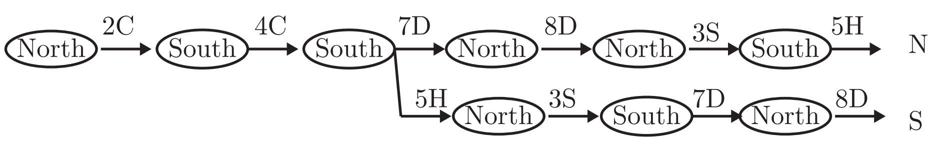


FIGURE 4.17. (2C,8D,3S;4C,7D,5H) **Trickster** restricted game tree

Notice that South has a strategy to win against North using LOWEST RANK EFFECTIVE CARD. This does not mean that South has a winning strategy for (2C, 8D, 3S; 4C, 7D, 5H) **Trickster**, only that South can win if North uses LOWEST RANK EFFECTIVE CARD. In fact, North has a winning strategy: initially choose 8D and then choose 3S in the second round.

Exercises

(1) Consider the $[2, 2, 1]$ **Nim** game.
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.
  (d) Describe one of the other player's strategies.
  (e) How many strategies does each player have?
(2) Consider the 2×2 **Hex** game.
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.
  (d) Describe one of the other player's strategies.
  (e) How many strategies does each player have?
(3) For the 3×3 **Hex** game, describe a winning strategy for Firstus without constructing a tree diagram.
(4) For the 4×4 **Hex** game, describe a winning strategy for Firstus without constructing a tree diagram.
(5) Consider the following **Trickster** game between North and South: (2H, 7H, 6C; 4H, 3C, 8C).
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.
  (d) Describe one of the other player's strategies.
  (e) How many strategies does each player have?
(6) Consider the following four-player game of **Trickster** (10C, 7D; 2H, 7C; 5C, 4S; 8D, 6C).
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Describe all strategies available to each of the four players.
  (d) Does one player have a winning strategy? If so, what is it? If not, why not?
(7) Consider the following **Trickster** game between North and South: (4H, 5C, 10C; 5H, 6C, 7C).
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Describe all strategies available to North.
  (d) Describe all strategies available to South.
  (e) Does one player have a winning strategy? If so, what is it? If not, why not?
(8) Consider the five-tile game of **Poison**.
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.
  (d) Describe one of the other player's strategies.

(e) How many strategies does each player have?

(9) Can both players have a winning strategy in some game of **Poison**? Why or why not?

(10) Consider the 2×2 game of **Chomp**.
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.
  (d) Describe one of the other player's strategies.
  (e) How many strategies does each player have?

(11) Consider **Keypad** except that the target is 7 instead of 30 and there are only four keys as shown.

1	2
3	4

To make the game tree smaller, redefine the rules in the following manner: a legal move keeps the sum under the target, and a player loses if there is no legal move available.
  (a) Construct the game tree.
  (b) How many ways can the game be played?
  (c) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.
  (d) Describe one of the other player's strategies.
  (e) How many strategies does each player have?

(12) Consider the game tree shown in Figure 4.18 (nodes are labeled with whose move it is; edges are labeled with a name for the corresponding move).

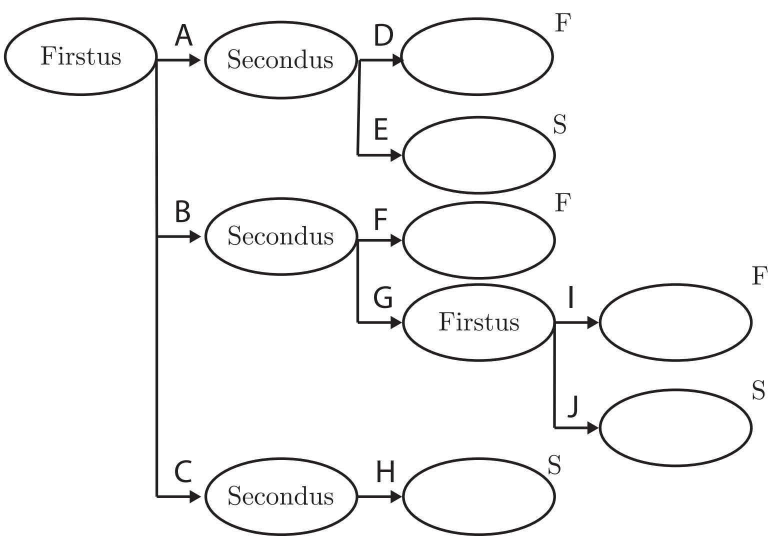


FIGURE 4.18. A generic game tree

  (a) Use backward induction to determine which player has a winning strategy. Describe the winning strategy.

(b) Describe all of Firstus's strategies.
(c) Describe all of Secondus's strategies.

(13) Use a restricted game tree to show that in (2C, 8D, 3S; 4C, 7D, 5H) **Trickster** the following is a winning strategy for North: initially choose 8D and then choose 3S in the second round.

(14) Suppose Firstus uses the following strategy in a game of **Chomp**:

LEAVE TOP ONE SHORTER: Choose the lowest cookie that will leave the top row of cookies with one fewer than the bottom row of cookies.

(a) Construct a restricted game tree to determine whether LEAVE TOP ONE SHORTER is a winning strategy for Firstus in a 2×3 game of **Chomp**.
(b) Construct a restricted game tree to determine whether LEAVE TOP ONE SHORTER is a winning strategy for Firstus in a 3×3 game of **Chomp**.
(c) Does one of the players in 2×3 **Chomp** have a winning strategy? If not, why not? If so, what is it and why does it work?
(d) Does one of the players in 3×3 **Chomp** have a winning strategy? If not, why not? If so, what is it and why does it work?
(e) Explain why LEAVE TOP ONE SHORTER is not a strategy for Secondus in 3×3 **Chomp**.

(15) Consider the **Graph-Coloring** game.

(a) Show that Secondus has a winning strategy using three colors and the five-wheel graph (see Figure 2.10).
(b) Show that Secondus has a winning strategy using three colors and the four-pede graph (see Figure 2.9).

(16) Using a restricted game tree, show that the strategy for **Keypad** described by Fatima in section 1.1 is a winning strategy.

5. A Solution for Nim

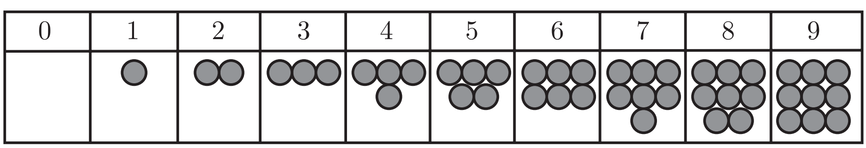
In this section, we describe for each **Nim** game who can ensure a win and the strategy that will ensure that win. We have already discovered that if there are two heaps with different numbers of beans, a winning strategy for Firstus is to even the two piles and then to mirror Secondus's moves. If the piles start out even, then Secondus can guarantee a win by using the strategy of mirroring Firstus's moves. While playing the game, you may have noticed that this idea of "evening the piles" works more generally. We want to formalize that process so that you can use it effectively. It turns out that the strategy is based on binary, instead of decimal, numbers.

By the end of this section, you will be able to relate decimal and binary representations of numbers, determine for any **Nim** game the player who can ensure a win, and use the strategy that ensures a win.

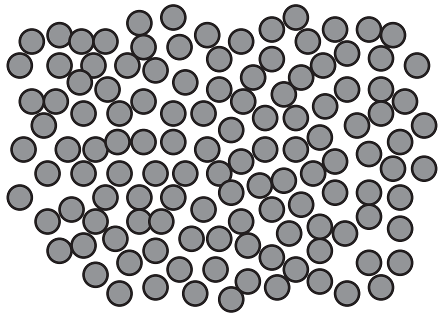
Decimal and Binary Numbers

To understand the winning strategy for **Nim**, it is helpful to understand binary numbers. To understand binary numbers, it is helpful to review decimal numbers.

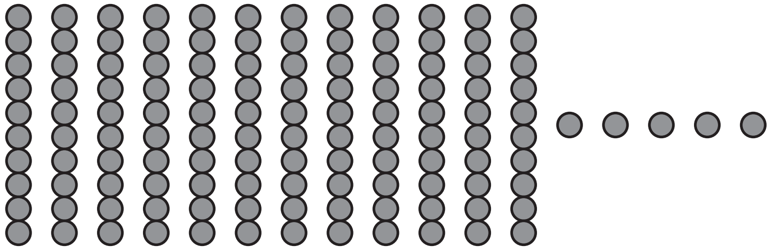
In our usual decimal representation of numbers, we use ten symbols:



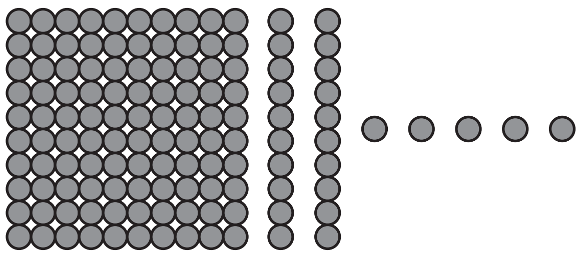
To represent a number larger than nine, we use a positional notation system. For example, say we wish to represent the number of disks shown below.



We first group the disks in piles of ten as shown next.



The remaining disks can be represented by the symbol "5". Since there are too many piles of ten disks, we group these piles in piles of ten as shown below.



The remaining piles of ten disks can be represented by the symbol "2", and the piles of "ten piles of ten disks" can be represented by the symbol "1". We concatenate these symbols in reversed order to obtain the symbol "125" to represent the number of disks. To symbolically emphasize the repetitive grouping by tens, we can write

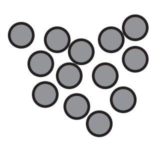
$$125 \ = \ 1 \times 10^2 \ + \ 2 \times 10^1 \ + \ 5 \times 10^0$$

to indicate that "125" means "1" pile of "ten piles of ten disks" (or "hundred disks"), "2" piles of ten disks, and "5" disks.

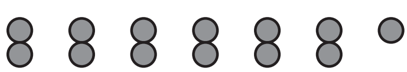
In the binary representation of numbers, we use only two symbols:

0	1

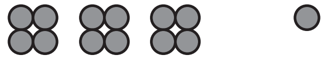
To represent a number larger than one, we again use a positional notation system. For example, say we wish to represent the number of disks shown here.



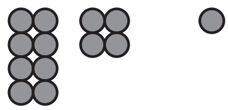
We first group the disks in piles of two as shown next.



The remaining disk can be represented by the symbol "1". Since there are too many piles of two disks, we group these piles in piles of two as shown below.



Having no remaining pile of two can be represented by the symbol "0". Since there are too many piles of "two piles of two disks" (or "four disks"), we group these piles in piles of two as shown below.



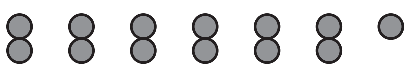
The remaining pile of four disks can be represented by the symbol "1", and the pile of "two piles of four disks" (or "eight disks") can be represented by the symbol "1". We concatenate these symbols in reverse order to obtain the symbol "1101" to represent the number of disks. To symbolically emphasize the repetitive grouping by twos, we can write

$$1101 \ = \ 1 \times 2^3 \ + \ 1 \times 2^2 \ + \ 0 \times 2^1 \ + \ 1 \times 2^0$$

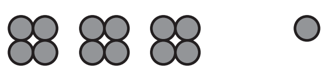
to indicate that "1101" means "one" pile of "two piles of two piles of two disks" (or "eight disks"), "one" pile of "two piles of two disks" (or "four disks"), "zero" piles of two disks, and "one" disk. Note that whereas decimal numbers use powers of ten, binary numbers use powers of two.

The repetitive process of grouping by twos is a very tactile way of determining the binary representation of a number, and therefore, it is a great approach when dealing with heaps of beans. However, we will move up one more level of abstraction and consider how to convert a decimal representation to a binary representation without the beans.

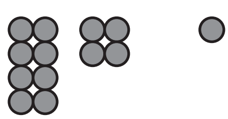
When we grouped 13 disks into piles of two,



we computed $13 \div 2 = 6$ remainder 1: the number 13 is six piles of two and one pile of one. When we grouped six piles of two into piles of two,



we computed $6 \div 2 = 3$ remainder 0: the number 6 is three piles of two and zero piles of one. When we grouped three piles of four into piles of two,



we computed $3 \div 2 = 1$ remainder 1: the number 3 is one pile of two and one pile of one. These computations can be organized as shown in Figure 5.1.

```
13 ÷ 2 = 6 remainder 1
 6 ÷ 2 = 3 remainder 0
 3 ÷ 2 = 1 remainder 1
 1 ÷ 2 = 0 remainder 1
```

FIGURE 5.1. Algorithm for decimal to binary conversion

The remainders are read from bottom to top to determine that the decimal number 13 and the binary number 1101 represent the same number. This is sometimes written: 13 (decimal) = 1101 (binary), a convention we will use.

Figure 5.2 shows one more decimal to binary conversion example. We consider 34 (decimal) and carry out the divisions by 2 (corresponding to grouping piles into piles of two). This tells us that 34 (decimal) = 100010 (binary).

```
34 ÷ 2 = 17 remainder 0
17 ÷ 2 =  8 remainder 1
 8 ÷ 2 =  4 remainder 0
 4 ÷ 2 =  2 remainder 0
 2 ÷ 2 =  1 remainder 0
 1 ÷ 2 =  0 remainder 1
```

FIGURE 5.2. An example of decimal to binary conversion

We now have two ways to obtain a binary number from a decimal number. First, produce the corresponding number of disks and then repetitively group the disks in piles of two. Second, repetitively divide decimal numbers by two. There are two analogous ways to obtain a decimal number from a binary number. First, produce the corresponding number of disks and then repetitively group the disks in piles of ten. Second, carry out the operations implied by the binary representation. Here is an example of the second approach:

$$\begin{aligned} 11010 \text{ (binary)} &= \left(1 \times 2^4\right) + \left(1 \times 2^3\right) + \left(0 \times 2^2\right) + \left(1 \times 2^1\right) + \left(0 \times 2^0\right) \\ &= (1 \times 16) + (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) \\ &= 26 \text{ (decimal)}. \end{aligned}$$

One thing that is great about mathematics is that there are often many solutions to a problem. A third way to obtain a binary number from a decimal number is to subtract powers of two from the decimal number to be converted. For example, to convert 38 (decimal) to binary, we first note that 1, 2, 4, 8, 16, and 32 are the powers of two no greater than 38. So, 38 has one 32 and $38 - 32 = 6$ left over. Now, 6 can easily be seen to have one 4 and one 2. Since 38 consists of one 32, zero 16, zero 8, one 4, one 2, and zero 1, it follows that 38 (decimal) = 100110 (binary).

Since it is essential for the winning **Nim** strategy to obtain binary representations for the number of beans in a heap, we provide a fourth method: making a conversion table for small numbers.

decimal	1	2	3	4	5	6	7	8	9	10	11	12
binary	1	10	11	100	101	110	111	1000	1001	1010	1011	1100

In addition to obtaining a binary representation for the number of beans in a heap, we will need to combine the resulting binary numbers. To do this, we will define a peculiar form of arithmetic which is NOT the equivalent of our usual decimal arithmetic. Usually 1 (binary) + 1 (binary) = 10 (binary), just as 1 (decimal) + 1 (decimal) = 2 (decimal). Instead, we will use the following *Nim-sum*.

$$\begin{array}{r} 0 \\ \oplus 0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ \oplus 1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ \oplus 0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ \oplus 1 \\ \hline 0 \end{array}$$

Notice that here $1 \oplus 1 = 0$. This is the critical difference. When adding, it eliminates any "carry" from one column to the next. Thus,

$$\begin{array}{r} 111 \\ \oplus 111 \\ \hline 000 \end{array} \qquad \begin{array}{r} 1010 \\ \oplus 0110 \\ \hline 1100 \end{array}$$

The Nim-sum Strategy

Now it is time to apply our skills with binary numbers to **Nim**. Consider the $[13, 13]$ **Nim** game. The binary representation of decimal 13 is 1101. Arranging the pile sizes vertically, the binary representations horizontally, and the Nim-sum at the bottom, we obtain the following.

$$\begin{array}{rcrrrr} 13 & = & 1 & 1 & 0 & 1 \\ 13 & = & 1 & 1 & 0 & 1 \\ \cline{3-6} & & 0 & 0 & 0 & 0 \end{array}$$

In general, when a player evens up two heaps in the game of **Nim**, the resulting Nim-sum of the number of beans in the two heaps is zero. When a player picks up the last bean on the table, the resulting Nim-sum is again zero. This suggests that zero Nim-sums is the idea of "even" that we have been looking for. Rather than trying to "even" the numbers across heaps, we will try to even the numbers in each column of the binary representations of the heap sizes.

Suppose we are playing a $[7, 6, 6, 4, 1]$ **Nim** game. Arranging the pile sizes vertically, the binary representations horizontally, and the Nim-sum at the bottom, we obtain the following.

$$\begin{array}{rcrrr} 7 & = & 1 & 1 & 1 \\ 6 & = & 1 & 1 & 0 \\ 6 & = & 1 & 1 & 0 \\ 4 & = & 1 & 0 & 0 \\ 1 & = & 0 & 0 & 1 \\ \cline{3-5} & & 0 & 1 & 0 \end{array}$$

In the sum, there is a 1 in the second column—the two's column. To "even the piles" in each column, Firstus should change one 1 in the twos column to 0. This translates into removing two beans, which can be done from the seven-bean heap or from one of the six-bean heaps. Selecting any of these three moves will result in a zero Nim-sum.

Say that Firstus removes two beans from one of the six-bean heaps. Secondus is faced with the following.

$$\begin{array}{ccccc}
7 & = & 1 & 1 & 1 \\
6 & = & 1 & 1 & 0 \\
4 & = & 1 & 0 & 0 \\
4 & = & 1 & 0 & 0 \\
1 & = & 0 & 0 & 1 \\
\hline
 & & 0 & 0 & 0
\end{array}$$

If Secondus removes beans from the seven-bean heap, then one or more of the 1s in the binary representation of the size will change to 0. This will change the Nim-sum to a nonzero quantity. If Secondus removes beans from the six-bean heap, then one or more of the 1s in the binary representation of the size will change to 0. The one 0 in the binary representation of the size may change to 1. In any case, this will change the Nim-sum to a nonzero quantity. Whatever Secondus does will change the Nim-sum to a nonzero quantity.

Say that Secondus removes all beans from the seven-bean heap. Firstus is faced with the following.

$$\begin{array}{ccccc}
6 & = & 1 & 1 & 0 \\
4 & = & 1 & 0 & 0 \\
4 & = & 1 & 0 & 0 \\
1 & = & 0 & 0 & 1 \\
\hline
 & & 1 & 1 & 1
\end{array}$$

One way to "even" the columns would be to change a 1 to a 0 in each column. This corresponds to removing $4 + 2 + 1 = 7$ beans. But there is no legal move that will remove seven beans. What should Firstus do? Well, another way to "even" a column would be to change a 0 to a 1. To obtain a legal move, all of our column changes must be done in a single row. For example, we could change the first row from 110 to 001 by removing five beans from the six-bean heap. We could also change the second or third row from 100 to 011 by removing one bean from a four-bean heap. One could also think about changing the fourth row from 001 to 110, but that would involve adding, rather than removing, beans from the one-bean heap, and so such a move would not be legal.

Suppose that Firstus removes one bean from a four-bean heap. Secondus is faced with the following.

$$\begin{array}{ccccc}
6 & = & 1 & 1 & 0 \\
4 & = & 1 & 0 & 0 \\
3 & = & 0 & 1 & 1 \\
1 & = & 0 & 0 & 1 \\
\hline
 & & 0 & 0 & 0
\end{array}$$

Again, no matter what Secondus does, the binary number in a single row will have at least one 1 changed to a 0, causing the Nim-sum to become nonzero.

Say that Secondus removes one bean from the six-bean heap. Firstus is faced with the following.

$$\begin{array}{ccccc} 5 & = & 1 & 0 & 1 \\ 4 & = & 1 & 0 & 0 \\ 3 & = & 0 & 1 & 1 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 0 & 1 & 1 \end{array}$$

To change the Nim-sum back to zero with a legal move, Firstus must change the entries in the last two columns of a single row. That would be impossible in the first row: changing 101 to 110 corresponds to adding to, rather than removing from, a heap. For the same reason, we cannot choose the second or fourth rows. Firstus is left with the third row, which could be changed legally from 011 to 000.

Suppose Firstus removes all beans from the three-bean heap. Secondus is faced with the following

$$\begin{array}{ccccc} 5 & = & 1 & 0 & 1 \\ 4 & = & 1 & 0 & 0 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 0 & 0 & 0 \end{array}$$

Again, no matter what Secondus does, the binary number in a single row will have at least one 1 changed to a 0, causing the Nim-sum to become nonzero.

Say that Secondus removes one bean from the four-bean heap. Firstus is faced with

$$\begin{array}{ccccc} 5 & = & 1 & 0 & 1 \\ 3 & = & 0 & 1 & 1 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 1 & 1 & 1 \end{array}$$

and so must change every column in a single row. This is only possible in the first row by changing 101 into 010.

So, Firstus removes three beans from the five-bean heap. Secondus is faced with the following.

$$\begin{array}{ccccc} 2 & = & 0 & 1 & 0 \\ 3 & = & 0 & 1 & 1 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 0 & 0 & 0 \end{array}$$

Again, no matter what Secondus does, the binary number in a single row will have at least one 1 changed to a 0, causing the Nim-sum to become nonzero. If Secondus removes one bean from the two-bean heap, Firstus is faced with the following.

$$\begin{array}{ccccc} 1 & = & 0 & 0 & 1 \\ 3 & = & 0 & 1 & 1 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 0 & 1 & 1 \end{array}$$

Clearly, Firstus should remove all beans from the three-bean heap. Secondus will be forced to remove one of two remaining beans, and Firstus will win on her next turn.

In general, here is the strategy that will ensure a win for the player who is faced with a game whose Nim-sum is nonzero.

> **NIM-SUM:** Remove beans from a heap so that the Nim-sum of the resulting game is zero.

If the initial game has a nonzero Nim-sum, NIM-SUM is a winning strategy for Firstus because (1) there always exists at least one move from a game with a nonzero Nim-sum to a game with a zero Nim-sum, and (2) any move from a game with a zero Nim-sum will result in a nonzero Nim-sum. This was certainly the case in the example, but how do we know that it will always be true?

If the game has a nonzero Nim-sum, there must be at least one column whose Nim-sum is 1. Consider the leftmost column whose Nim-sum is currently 1; call it column c. There must be a row, call it row r, having a 1 in column c (for otherwise, the Nim-sum of column c would have been 0). For each column with Nim-sum 1 that row r contributes to by having a 1 in it, remove sufficient beans to convert the 1 to 0. For each column with Nim-sum 1 that row r does not contribute to by having a 0 in it, add sufficient beans to convert the 0 to 1. This results in a net removal of beans because removing the beans necessary to change the 1 in column c to 0 will remove more beans than will need to be returned for those columns that a 0 needs to be changed to a 1. Hence, (1) is true.

If the game has a zero Nim-sum, then the removal of beans from a heap will change at least one 1 to a 0 in the corresponding row. This will leave an odd number of 1s in its column, and therefore a sum of 1 in that column. Hence, (2) is true.

For the same reasons, if the Nim-sum is initially zero, NIM-SUM is a winning strategy for Secondus. This completes our analysis of **Nim.** We see that for any game of **Nim**, exactly one player has a winning strategy**.** In particular, if the Nim-sum is zero, then NIM-SUM is a winning strategy for Secondus, and if the Nim-sum is nonzero, then NIM-SUM is a winning strategy for Firstus.

Exercises

(1) Write the decimal numbers 26 and 23 in binary. Convert 1110 and 10011 from binary to decimal.
(2) Find the Nim-sum of the numbers 10101 and 11110.
(3) Find two good initial moves (for Firstus) from $[7, 4, 4, 3]$ **Nim**. That is, moves that follow NIM-SUM.
(4) Show that Firstus cannot ensure a win from $[14, 12, 6, 4]$ **Nim**.
(5) There are many possible moves for a player facing $[23, 19, 13, 12, 11]$ **Nim**. Only three are consistent with NIM-SUM. What are they?
(6) Suppose a player is faced with $[5, 4, 1]$ **Nim**. For each possible move, compute the resulting Nim-sum.
(7) Explain how to visualize NIM-SUM when faced with piles of beans, rather than having to actually perform the arithmetic.
(8) For the game of **Poison**, consider the following strategy:

MULTIPLE OF THREE: If the number of tiles in the pile is a multiple of three (i.e., 3, 6, 9, 12, 15, and so forth), take two tiles; otherwise, take one tile.

(a) Show that MULTIPLE OF THREE ensures a win for Secondus whenever there are 1, 4, 7, 10, 13, and so forth, tiles in the pile.

(b) Show that MULITIPLE OF THREE ensures a win for Firstus whenever there are 2, 3, 5, 6, 8, 9, 11, 12, and so forth, tiles in the pile.

6. Theorems of Zermelo and Nash

We have examined **Nim** completely and determined when there is a winning strategy for Firstus; when there is a winning strategy, we can describe it. We also realized that if Firstus does not have a winning strategy, then Secondus does. An exercise outlines a similar analysis of the game of **Poison**, and you have worked with the game of **Hex** sufficiently that you may believe that this situation is true as well for **Hex**. That is, you may believe that one of the two players must have a winning strategy. Can this idea be generalized to other games? The answer is yes, as we will see in Zermelo's Theorem, first proven in 1912 [**74**].

> By the end of this section, you will know and be able to apply two important theorems about games, understand a fairly sophisticated mathematical argument, and be able to use similar arguments in other situations.

Zermelo's Theorem

Zermelo's Theorem: *Suppose a deterministic game has* (1) *two players,* (2) *exactly one winner for each play, and* (3) *a maximum number of moves. Then exactly one player has a winning strategy.*

Proof. We will first argue that at most one player has a winning strategy. Suppose, to the contrary, that both players have winning strategies. Have both players use their supposed winning strategies. Since there are no ties, only one of the players will win. This means that the other player's strategy did not ensure a win, contradicting our supposition. Thus, at most one player has a winning strategy.

We will now argue that at least one player has a winning strategy by constructing a game tree and carrying out backward induction. Since the game has a maximum number of moves, we can construct a game tree with a finite number of levels. Since there are no ties, we can mark each leaf with the winner of that play of the game.

We now carry out backward induction. Consider a vertex N on the next-to-last level from which player P chooses the move. For each choice available at node N, player P knows which player can then ensure a win. If player P has a choice that can ensure a win for player P, then player P can ensure herself a win from node N by making that choice (see Figure 4.10). Otherwise, since there are only two players and no ties, the other player can ensure himself a win from node N (see Figure 4.11).

This process of identifying which player can ensure a win from each node can be continued until a determination can be made from the first node. This player then has a strategy that can ensure a win. □

Each game of **Nim** has (1) two players, (2) exactly one winner for each play (the player who removes the last bean), and (3) a maximum number of moves (the total initial number of beans since every move removes at least one bean). Since **Nim** satisfies the suppositions of Zermelo's Theorem, the conclusion must be true; that is, exactly one of the two players has a strategy that will ensure a win. Of course, we already knew this based on our work in the previous section.

Some games of **Trickster** violate Zermelo's suppositions (1) and/or (2). In such cases, Zermelo's Theorem tells us nothing. It could be that multiple players have winning strategies or no player has a winning strategy. We do not know based on Zermelo's Theorem. To find out, we must go through the kind of analysis we have carried out in previous sections. Here are some examples in which only one of Zermelo's suppositions are violated.

(1) **Trickster** (5C; 4C; 3C; 2C) has four players, violating Zermelo's supposition (1), and so Zermelo's Theorem is of no help. Zermelo's conclusion actually holds because North, and only North, obviously has a winning strategy.
(2) **Trickster** (2C, 2H; 3C, 2D; 3H, 2S) has three players, again violating Zermelo's supposition (1), and so Zermelo's Theorem is of no help. Calling the players North, East, and West, this game can end in only two ways: (1) if North initially chooses 2C, East will receive both tricks and be the winner, or (2) if North initially chooses 2H, West must choose 3H and will receive both tricks and be the winner. Since North never wins, she does not have a winning strategy. Since North's initial move determines the winner, neither East nor West can ensure a win. This violates Zermelo's conclusion because no player has a winning strategy.
(3) **Trickster** (2C, 5H; 5C, 2H) always results in each player receiving one trick, and so both players are winners. This violates Zermelo's supposition (2) as well as Zermelo's conclusion that at most one player has a winning strategy.
(4) **Extended Nim** is identical to **Nim** except that a legal move consists of removing one or more beans from, or adding one or more beans to, a single heap. Consider the strategy, "if there is only one heap, remove that entire heap; otherwise, add one bean to a smallest heap". A player using this strategy cannot lose if there are at least two heaps initially. This violates Zermelo's conclusion that at least one player has a winning strategy. But this does not violate Zermelo's Theorem because even though this game satisfies Zermelo's suppositions (1) and (2), it does not satisfy Zermelo's supposition (3) because the game can keep going forever.
(5) **Larger Number** involves two players, call them Firstus and Secondus. Firstus chooses either 1 or 2. Then Secondus chooses either 1 or 2. The winner is the player that chose the larger number, and the loser is the player that chose the smaller number. If both players choose the same number, they are tied. If Firstus adopts the strategy "choose 2", then Secondus cannot win, and so Secondus must not have a winning strategy. If Secondus adopts the strategy "choose 2 no matter what Firstus chooses", then Firstus cannot win, and so Firstus must not have a winning strategy. This violates Zermelo's conclusion that at least one player has

a winning strategy. But this does not violate Zermelo's Theorem because even though this game satisfies Zermelo's suppositions (1) and (3), it does not satisfy Zermelo's supposition (2) because if both players choose 2, there is no winner.

Even though neither player has a winning strategy in **Larger Number**, we exhibited strategies for each player that ensure a tie or win. We can slightly weaken Zermelo's suppositions in order to conclude the existence of such weaker strategies.

Zermelo's Modified Theorem: *Suppose a deterministic game has* (1) *two players,* (2) *exactly one winner or a tie for each play, and* (3) *a maximum number of moves. Then exactly one player has a winning strategy or both players have a strategy that ensures at least a tie.*

An exercise gives you a chance to explain why this theorem is true. Chess is another example of a game in which ties are allowed. Since Chess satisfies the suppositions of Zermelo's Modified Theorem, either White has a winning strategy, Black has a winning strategy, or both players have strategies that ensure a tie. It is not known which of the three possibilities holds. In fact, if you think about what you've learned about playing Chess, what you know are heuristics rather than strategies. This is because of the complicated nature of the game.

Checkers is another game that satisfies the suppositions of Zermelo's Modified Theorem and so exactly one of the following holds: (1) Firstus has a winning strategy, (2) Secondus has a winning strategy, or (3) each player has a strategy that will ensure a tie. However, unlike Chess, it is now known which of the three possibilities is true. In 2007, computer scientists at the University of Alberta (Canada), after sifting through 500 billion billion checkers positions, determined a stategy for each player that ensures a tie [**40**].

Zermelo's Theorem is the generalization of the backward induction process we have been using to analyze game trees. We will be revisiting game trees and backward induction again in a later chapter.

Nash's Hex Theorem

To end this chapter, let's turn our attention back to **Hex** for a few minutes and uncover an interesting piece of information about the game. The following theorem was apparently proven by John Nash in 1949 [**35**, page 77].

Nash's Hex Theorem: *In* **Hex***, the first player has a strategy to ensure a win.*

Proof. The game of $n \times n$ **Hex** clearly has (1) two players, and (3) a maximum number of moves (the number of cells since one cell is captured on each move and the game must end by the time all cells are captured). It is clear that there is never more than one winner. But could a game of **Hex** end with no winner? No. Imagine the playing board for the game of **Hex** to be made out of paper. Whenever Firstus moves, she colors the hexagon of her choice. Whenever Secondus moves, he cuts out the hexagon of his choice. Repeat this until no one can move any more. Pick up the playing board in your hands, holding the top and bottom edges. Pull your hands apart. Either the paper stops you, in which case there must be a path of colored hexagons and so Firstus wins; or nothing stops you, in which case there are cut-out hexagons between the top and the bottom of the board that must form a path from left to right, and so Secondus wins. Clearly, one of the two must occur; and so (2) exactly one player must win.

Since Zermelo's Theorem applies, some player has a strategy to ensure a win. Suppose it is Secondus, and let WIN be the name for the strategy Secondus uses to ensure a win. Now we do not know the description for WIN, only that it ensures a win for Secondus. From our work with game trees, we know that the instructions for the WIN strategy must include an instruction for which cell Secondus should capture if faced with the position shown in Figure 6.1. We will call this cell (r, c).

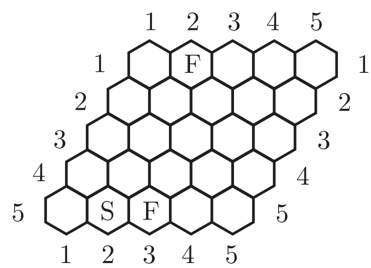


FIGURE 6.1. Three moves into **Hex**

Now define a new collection of instructions, called REFLECTED WIN, in which the WIN strategy is modified by interchanging Firstus and Secondus and interchanging the row and column numbers for every cell reference. For example, the position described by Figure 6.1 becomes the position shown in Figure 6.2, and Firstus should capture the cell (c, r).

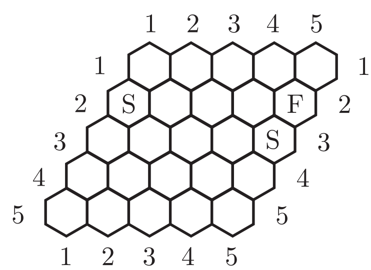


FIGURE 6.2. The **Hex** game reflected

Notice that a path from left to right for Secondus corresponds to a *reflected* path from top to bottom for Firstus.

For a moment, consider the game **Reverse Order Hex** whose rules are identical to **Hex** except that Secondus (the player who wins by forming a path from left to right) moves first instead of after Firstus (the player who wins by forming a path from top to bottom). Since WIN ensures a win in **Hex** for the player who moves second (Secondus), REFLECTED WIN must ensure a win in **Reverse Order Hex** for the player who moves second (Firstus).

Have Firstus, in **Hex**, use the REFLECTED WIN PLUS strategy: (1) on her first move, capture hexagon 1 (where the hexagons have been numbered in normal reading order), temporarily labeling hexagon 1 as the "extra" hexagon; (2) on subsequent moves, ignore the "extra" hexagon in order to use REFLECTED WIN to determine her move, unless it would instruct her to capture the "extra" hexagon, in which case capture the smallest available uncaptured hexagon, and now label this captured hexagon as the "extra" hexagon.

The "extra" hexagon only increases Firstus's chances for a win. Therefore, we have shown that if Secondus has a winning strategy, then Firstus also has a winning strategy. This absurdity means that Secondus must not have had a strategy to ensure a win. But this means Firstus must have a strategy to ensure a win. □

This fascinating result shows us that there is a strategy that will ensure a win for Firstus, but we have no clue as to what it is! In mathematics, this type of result is called an existence theorem: we show that some desired object exists. Fortunately, for problems with applications to real-world situations, most existence theorems, unlike this one, also tell us how to find the desired object.

Exercises

(1) For each of the following games, determine which of Zermelo's suppositions and conclusions are satisfied.
   (a) **Poison** with 127 tiles
   (b) **Graph Coloring** with three colors on an eleven-pede graph
   (c) 13×28 **Chomp**
   (d) **Trickster** (6C, 5H; 4C, 7H)
   (e) **Trickster** (2C, 2S; 4C, 5C; 4S, 5S)
   (f) Tic-Tac-Toe
   (g) Checkers

(2) Prove Zermelo's Modified Theorem. (Hint: Follow the proof of Zermelo's Theorem, except when doing backward induction, allow vertices in Case 2 to be marked T for "tie".)

(3) Nash argued that Secondus does not have a winning strategy by assuming Secondus has a winning strategy and finding a contradiction. Why can't we use the same argument for Firstus? That is, why can't we argue that Firstus does not have a winning strategy by assuming Firstus has a winning strategy and finding a contradiction?

(4) Nash's Theorem proved that Firstus can ensure a win in $n \times n$ **Hex**. So, we can give Secondus a better chance of winning by requiring Firstus to have longer

paths to win; that is, perhaps Secondus has a winning strategy for $(n+1) \times n$ **Hex** (this means $n+1$ rows and n slanted columns).

(a) Does Secondus have a winning strategy for 2×1 **Hex**? If so, describe it. If not, describe a winning strategy for Firstus.

(b) Does Secondus have a winning strategy for 3×2 **Hex**? If so, describe it. If not, describe a winning strategy for Firstus.

(c) Does Secondus have a winning strategy for 4×3 **Hex**? If so, describe it. If not, describe a winning strategy for Firstus.

(5) In an exercise from a previous section, it was determined that LEAVE ONE SHORTER is a strategy that ensures a win for Firstus in a $2 \times n$ game of **Chomp**. Using an argument similar to (but simpler than) Nash's argument, explain why Firstus always has a guaranteed winning strategy in $m \times n$ **Chomp**.

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CHAPTER 2

Player Preferences

1. Measurement

Most situations of conflict and/or cooperation have outcomes that are more complex than "win" or "lose." Wouldn't you have different feelings about winning $100 versus $10,000 versus $1,000,000 in a lottery? The automobile purchase price you negotiate lies along a continuum of possibilities. The brand of soda you purchase will have different effects on your pocketbook, taste buds, and consumer consciousness. Which applicant you hire will affect what work will be completed and with what quality. Whatever a regional government thinks of their relative merits, choosing among (1) a relatively inexpensive coal-fired electric power plant resulting in predictable deaths from air pollution and mining accidents, versus (2) a clean, but expensive, nuclear power plant with a small chance of a catastrophic accident, versus (3) reduced economic growth with no new power plant, it is definitely not just a question of "win" or "lose". In order to predict player choices in more complex games, we will need to measure player preferences for different outcomes.

Measurement is the assigning of a number or symbol to an attribute of something. Examples include assigning 73 inches to the height of a person, assigning 75 degrees Fahrenheit to the temperature of the water in a swimming pool, assigning 4 to the number of children in a family, assigning "marquise" to the shape of a cut ruby, and assigning "good" to the quality of service by a business. The particular manner in which numbers or other symbols are assigned is called the scale. For example, the height of a person may be measured with respect to (1) an inch scale resulting in the assignment of 73, (2) a centimeter scale resulting in the assignment of 185, or (3) a verbal scale resulting in the assignment of "tall". Different scales may be more or less informative. For example, the qualitative scale is less informative than the inch or centimeter scales when measuring the height of a person.

> By the end of this section, you will be able to differentiate among five different measurement scales and to think about how they might be used to measure preferences.

Three members of the Book Discussion Club enter their favorite local coffee shop.

ORION: So, what do you think of this book so far?

ABBY: Numbers. Mathematics is all about numbers.

NORMAN: But the first chapter spoke of games, strategies, and game trees. Surely they're not numbers!

ABBY: Ah, but they are! The number of heaps and the numbers of beans in each heap in **Nim**, the number of captured cells in **Hex**, the number of numbered cards in **Trickster**, the binary representation of numbers used in the winning strategy for **Nim**, the number of branches and paths in a game tree, and the astronomical number of strategies that come out of game trees. I reiterate: mathematics is all about numbers.

NORMAN: Surely the proofs by Zermelo and Nash were elegant, logical arguments that didn't involve numbers.

ORION: Perhaps there's more to mathematics than numbers, but even logical arguments do involve numbers. There's a clear ordering: step 1 that leads to step 2 that leads to step 3, and so forth, until the argument is complete.

ABBY: And with Orion on my side, I rest my case. Let's order some coffee.

NORMAN: I think you have failed to grasp Orion's comment that although there are numbers in mathematics, there's more than numbers to mathematics. And since I'm uncertain as to what to order, please order first.

ABBY: I'll have a number 6: a large French Roast.

ORION: Subtle, but cute, Abby. Even the drinks have numbers associated with them.

NORMAN: That French Roast smells heavenly. But that "number" 6 is not really acting like a real number 6.

ABBY: Not a real number? It certainly looks like a number to me.

NORMAN: When I say, "There are three of us", the number "3" is a numerical property associated with Abby, Orion, and Norman. There is nothing "6" about the large French Roast.

ORION: It is the sixth item on the menu board.

NORMAN: But only out of convenience. It could just as easily have been the eighth or twenty-second.

ORION: Mmmm... 22, a regular iced vanilla latte. I'll take that. How about you, Norman?

NORMAN: The iced vanilla latte sounds good, too, but I'm not sure.

At that moment, the remaining two members of the Book Discussion Club enter.

RAUL: Hello!

INEZ: Sorry we're late.

ORION: That's okay. Norman hasn't even ordered yet.

INEZ: How typical! Hey, Norman, why can't you make up your mind?

NORMAN: Perhaps I just can't quantify my life as readily as you.

RAUL: I forgot to bring some money. Inez, would you mind sharing a coffee?

INEZ: My pleasure. How about a number 8: the regular Columbian.

RAUL: With some cream?

INEZ: Sure! We're so much alike!

ABBY: Okay, Norman, you can't procrastinate any longer.

NORMAN: Sure I can.

ABBY: Look. I ordered a 6, Orion ordered a 22, and Raul and Inez ordered an 8. Taking 6 plus 22 plus 8 and dividing by the 4 of us who have ordered yields 9. So, order a 9.

NORMAN: That makes absolutely no sense. But that item seems as good as any of the others. So, I'll take a small Kenyan with a couple of ounces of skim milk to cool it off.

So, the members of the Book Discussion Club purchase their drinks and settle into a corner table.

NORMAN: So, what is the topic of discussion today?

ORION: The book authors asked that we discuss how we might measure player preferences for outcomes.

RAUL: I usually use a ruler to measure things.

INEZ: A ruler would be a good instrument to measure lengths, but I'm not so sure that it would work very well to measure player preferences.

ABBY: A ruler would not be very useful for measuring many other attributes such as wind speed, air temperature, barometric pressure, or lightning bolt voltage.

NORMAN: I like all of the weather references.

ABBY: Thank you.

INEZ: Actually a ruler would work quite well.

ABBY: How so?

INEZ: A common thermometer is just a ruler attached to a glass or plastic tube partially filled with alcohol or mercury that expands with an increase in temperature. Other thermometers have circular rulers underneath a pointer attached to coils of metal that unwind with increasing temperature. Many other measuring devices work on similar principles with a pointer and ruler to enable someone to make the measurement. That even includes that analog watch you're wearing, Abby.

ABBY: I'd never quite thought of rulers that way before, but I'll at least concede that a ruler may be an important component of many measuring devices.

RAUL: Thanks for coming to my defense, Inez.

INEZ: Anything for you.

ORION: No public displays of affection, please.

NORMAN: Rulers are not a necessary part of a measuring device. For example, I don't see the ruler in your digital watch, Inez.

ABBY: But any measuring device involves numbers. Measurement is all about numbers.

ORION: I thought it was mathematics that was all about numbers.

ABBY: So, let me modify my statement. Numbers are used in all measurements.

RAUL: I think I agree. I am going to drink about half of this regular Columbian, which is 8 ounces. Orion will probably drink all of his regular iced vanilla latte, which is 16 ounces. Even if I didn't use a ruler, I would still state a number (such as 8 or 16) when I measure volume.

NORMAN: But are those numbers really meaningful? What if I were to say that the volume you'll drink, Raul, is 237, and the volume Orion will drink is 574?

RAUL: I'd say that you're reporting volumes using milliliters rather than fluid ounces.

NORMAN: You would be correct, but it doesn't change my point that 8 and 16 are meaningless.

RAUL: I disagree. Perhaps the use of fluid ounces or milliliters is arbitrary, but once the scale is known, the numbers are quite meaningful.

NORMAN: How so?

RAUL: For example, clearly Orion will drink twice as much as I will drink.

NORMAN: I'll accept that.

RAUL: That doubling is reflected in either pair of numbers: 16 is twice 8 and 574 is twice 237.

ABBY: Cool! $\frac{16}{8} = \frac{574}{237} = 2$.

RAUL: Yes! Ratios of these measurements are meaningful.

NORMAN: A different way to say it is that the number we assign to the volume Raul will drink is perhaps arbitrary, call it R, but then the number we assign to the volume Orion will drink must be twice R, which is usually written $2R$.

RAUL: Yes! If we say I will drink 1 cup, then we must say that Orion will drink 2 cups.

ABBY: If we say Raul will drink 17 abbies, then we must say that Orion will drink 34 abbies.

ORION: Abbies?

ABBY: It is something I just made up. One abby is the volume corresponding to one of my mouthfuls.

ORION: Measuring volume with your mouth, Abby, is not a very pleasant image for me.

INEZ: I'd find it pleasant to measure in rauls.

ORION: This conversation has gone from the intellectual to the gross. Can we get back on track?

NORMAN: I think that we've made a key observation. There needs to be a standard of measurement, whether it be a fluid ounce, milliliter, cup, abby, or raul. Once we have that standard, then everything else can be measured with respect to that standard.

INEZ: So, since my regular Columbian is now about 40 degrees Fahrenheit and Abby's French Roast is about 120 degrees Fahrenheit, we can say that Abby's coffee is three times the temperature of my coffee (since $\frac{120}{40} = 3$).

ABBY: Of course! We could even use Celsius instead. Recall the conversion formula

$$\text{degrees Celsius} = \frac{5}{9}(\text{degrees Fahrenheit} - 32).$$

Inez's iced vanilla latte is about $\frac{5}{9}(40-32) = 4.4$ degrees Celsius and Abby's French Roast is about $\frac{5}{9}(120-32) = 48.9$ degrees Celsius. So, Abby's coffee is $\frac{48.9}{4.4} = 11.1$ times the temperature of Inez's coffee.

RAUL: Wait! Three times and eleven times are not the same. What's happening here?

INEZ: Temperature requires two, not one, standards. In Fahrenheit, we assign 32 to the temperature at which water freezes and 212 to the temperature at which it boils. In Celsius, we assign 0 to the temperature at which water freezes and 100 to the temperature at which it boils.

RAUL: So, while it's meaningful to say that one volume is twice another volume. . .

INEZ: It is *not* meaningful to say that one temperature is twice another temperature.

NORMAN: And the difference is that we only need one standard for volume, while we need two standards for temperature.

ABBY: In some sense, there is a second standard for volume.

NORMAN: A second standard?

ABBY: The "nothing" volume is always assigned 0 in the various units we described.

ORION: And there is no similar "nothing" temperature?

RAUL: Actually, there is. It is called "absolute zero", and it is assigned 0 in Kelvin units.

INEZ: But it turns out to be about –273 degrees Celsius and –460 degrees Fahrenheit.

NORMAN: So, perhaps I could measure volume in normans with the "nothing" volume assigned –273 normans and one cup assigned to be 0 normans.

ABBY: I don't think anyone would want to use your scale.

NORMAN: But people are willing to use a temperature scale in which it's meaningless to say that one temperature is twice another temperature.

INEZ: Although ratios are not meaningful temperature statements, intervals are.

NORMAN: What do you mean?

INEZ: Warming up Orion's coffee by 18 degrees Fahrenheit from 40 to 58 degrees Fahrenheit is twice as big as warming up Abby's coffee by 9 degrees Fahrenheit from 120 to 129 degrees Fahrenheit.

ORION: Isn't that as meaningless as saying that Abby's coffee temperature is triple my coffee temperature? Remember that when we converted temperatures into Celsius, Abby's coffee temperature suddenly seemed to be 11 times my coffee temperature.

INEZ: But if we convert to Celsius, we will be warming up Orion's coffee by 10 degrees Celsius from 4.4 to 14.4 degrees Celsius and warming up Abby's coffee by 5 degrees Celsius from 48.9 to 53.9 degrees Celsius. So, again the increase in temperature to Orion's coffee is twice as much as the increase in temperature to Abby's coffee.

ABBY: I think that Raul's ruler idea is helpful here. For both volume and temperature, different units correspond to different distances between the marks on the ruler (closer together for milliliters or Fahrenheit, farther apart for fluid ounces or

Celsius). For volume, the zero point of the ruler is fixed at the "nothing" volume, whereas for temperature, the zero point of the ruler may be placed anywhere (at the water freezing point in Celsius but below the water freezing point in Fahrenheit).

NORMAN: So, we really have two different types of scales.

RAUL: There are the ones that measure volume. They have a fixed zero and a single standard which can be assigned an arbitrary number. Since ratios of these numbers are meaningful, perhaps we should call these ratio scales.

INEZ: There are the ones that measure temperature. They have two standards which can be assigned arbitrary numbers. Since intervals between these numbers are meaningful, perhaps we should call these interval scales.

RAUL: Length, area, volume, pressure, and voltage are all attributes that we use ratio scales to measure.

INEZ: In addition to temperature, time is measured using an interval scale. It is meaningful to say that my Dad has lived twice as long as I have because the amount of time he has lived is twice the amount of time I have lived whether it is measured in years, months, days, hours, or seconds. However, there is no natural zero. In fact, track events, swimming races, and experiments like to choose zero time to occur at their start and the Gregorian, Muslim, Jewish, and Chinese calendars each assign 0 to different points in time.

RAUL: Your Dad has lived twice as long as you. So, measuring intervals on an interval scale uses a ratio scale.

INEZ: Yes! Although the two scales are different, they're closely related.

NORMAN: I hate to interrupt your self-congratulatory dialogue, but does this get us any closer to understanding how a player might measure his or her preferences for different outcomes?

RAUL: Now that we know how other things are measured, measuring preferences for outcomes should be similar.

ORION: But not everything is measured using a ratio or interval scale.

INEZ: No?

ORION: Well, consider the boldness of the coffees we have chosen to drink. I would call my iced vanilla latte "mild", Inez's and Raul's Columbian "medium", Norman's Kenyan "bold", and Abby's French Roast "extra bold".

NORMAN: Measurement without numbers!

ABBY: But couldn't we assign numbers to describe boldness?

ORION: I could assign a boldness of 1 to my latte, a 2 to the Columbian, a 3 to the Kenyan, and a 4 to the French Roast.

RAUL: Are you saying that the French Roast is four times as bold as the latte?

ORION: I don't mean to imply that.

INEZ: Are you saying that the difference in boldness between the French Roast and the Kenyan is the same as the difference in boldness between the Columbian and the latte?

NORMAN: Ridiculous!

ORION: All I wanted to convey by the number assignment was that French Roast is bolder than Kenyan which is bolder than Columbian which is bolder than my latte. I guess I could have used 8, 6, 5, and -3 instead of 4, 3, 2, and 1.

ABBY: But then aren't the numbers meaningless?

ORION: They do convey information about the order of the coffees in terms of boldness.

ABBY: But the numbers seem to imply additional relationships.

INEZ: The numbers for temperature seemed to imply that one temperature was three times another temperature, but we saw that that wasn't a meaningful relationship. When using an interval scale, we realize that while it's meaningful to consider ratios of interval sizes, it's not meaningful to consider ratios of individual measurements.

ORION: The only meaning that can be extracted from a boldness scale is which coffees are bolder than other coffees. Since only order information is meaningful in my boldness scale, perhaps we should call boldness an ordinal scale.

NORMAN: This sounds somewhat contrived. Is anything else measured using an ordinal scale?

ORION: There is the Mohs scale of mineral hardness. Diamond can scratch corundum, corundum can scratch topaz, topaz can scratch quartz, quartz can scratch feldspar, and so forth. Diamond is assigned a 10, corundum is assigned a 9, topaz is assigned an 8, quartz is assigned a 7, feldspar is assigned a 6, and so forth. The numbers are arbitrary except for their order.

NORMAN: So what happens if something has an intermediate hardness?

ORION: For example, a steel file can scratch feldspar but quartz can scratch a steel file. We can assign 6.5 to the steel file.

NORMAN: Is there a similar scratch test for measuring boldness?

ORION: Our taste buds. We can certainly taste two coffees and determine which one is bolder.

ABBY: We now have three different scale types for measuring.

RAUL: Ratio.

INEZ: Interval.

ORION: Ordinal.

NORMAN: Each new scale we have named has conveyed less information. Perhaps we could have a scale that conveys almost no information.

ABBY: What purpose would that serve?

NORMAN: I don't know. I'm just interested in the possibility. For example, the menu number for our coffees.

RAUL: The latte (menu number 22) is certainly not almost four times the French Roast (menu number 6).

INEZ: It is not even clear to me what the intervals mean let alone whether their ratios are meaningful.

ORION: And what does it mean to say that the latte is more than the French Roast?

NORMAN: That's right. The number assignment is completely arbitrary. Well, I guess it would make no sense to assign the same number to two different drinks, but otherwise, the number assignments are completely arbitrary.

ABBY: Again I ask, to what purpose?

NORMAN: I have seen such numbers on surveys used to code answers. For example, to a religious affiliation question, perhaps Catholic is assigned to 1, Lutheran is assigned to 2, Mennonite is assigned to 3, Jewish is assigned to 4, Muslim is assigned to 5, and so forth. The only meaning that can be given to the numbers are whether two people have the same or different religious affiliations.

ORION: Hey, Norman, good job in coming up with another scale type! What will you call it?

NORMAN: Since the numbers serve only as names, I would call menu items and religious affiliation nominal scales.

ABBY: Since each of you has proposed different types of measurement scales, perhaps I should join in the fun.

NORMAN: I doubt you could find a scale that conveys even less information.

ABBY: That wasn't the direction I was thinking of. Is there a scale in which all of the information implied by the numbers is meaningful?

RAUL: Don't ratio scales do that?

ABBY: I can't say the volume of coffee you will drink is 8 . . . period. The 8 must be followed by a unit name because 8 milliliters, 8 fluid ounces, and 8 gallons bring very different pictures to mind.

RAUL: Wouldn't any measurement have to have units?

ABBY: Well, couldn't I just say that the number of patrons in the coffee house right now is, let me count, 12? There are no units there.

RAUL: We could instead say that there are 1 dozen patrons. Doesn't that show that you could use different units?

ABBY: One dozen, 12, XII (Roman numeral), twelve, zwoelf (German), and 1100 binary are just different ways of expressing the same number, whereas 8 and 237 are different numbers that can both be used to describe how much coffee you have drunk, as long as each is qualified by the appropriate units.

RAUL: Okay. You can use pure numbers to measure the number of people in a room.

INEZ: What will you call your scale?

ABBY: Because they are not relative to any units, the number of people in a room is given by an absolute scale.

ORION: Would an absolute scale be used in measuring anything else?

ABBY: The fraction of milk products in a cup of coffee. Mine was 0, Inez' and Raul's was about 5%, Norman's was about 10%, and Orion's was about 30%.

NORMAN: So, each one of the five of us advocated for a different way that numbers might be assigned as measurements as shown in Table 1.1. Could those measurements be of player preferences for outcomes?

ORION: A player should certainly be able to compare two outcomes and be able to say which one would be preferred. A rank ordering of the outcomes could then be obtained. So, preferences could be measured with an ordinal scale.

INEZ: But shouldn't a player be able to ascribe an intensity of preference? Some outcomes may seem almost the same while others may be far less preferred. So, an interval scale could be used.

RAUL: Sometimes I know that one outcome is twice as good as another outcome. So, a ratio scale is needed.

ABBY: Could I honestly say that anyone could ever say that they know one outcome is worth 4 (with no associated units)? Probably not. So, perhaps an absolute scale would not be helpful to measure preferences.

NORMAN: Well, I could arbitrarily assign different numbers to different outcomes and so use a nominal scale. I'm not sure how useful it would be, but it does have

TABLE 1.1. Measurement scales

Scale (Advocate)	Description	Examples
Absolute (Abby)	Each object can have only a single unitless number assigned to it.	Count or fraction.
Ratio (Raul)	There is a natural zero point and one object is assigned an arbitrary number. Object measurements can be transformed by any positive multiple with a corresponding change in units.	Length, area, volume, velocity, or voltage.
Interval (Inez)	Two objects are assigned arbitrary numbers. Object measurements can be transformed by (1) any addend with a corresponding change in reference point, and (2) any positive multiple with a corresponding change in units.	Temperature or time.
Ordinal (Orion)	Object measurements can be transformed in any way that preserves order.	Hardness or boldness.
Nominal (Norman)	Objects are identified with an arbitrary numeral.	Religious affiliation or menu item.

the advantage that I know that it could be done. How would any of you go about assigning numbers to outcomes based on preferences?

ORION: I'd just ask players which outcome they'd prefer between the two outcomes in each pair of outcomes.

INEZ: I'd ask the player which outcomes are the most and least preferred. I'd assign 0 to the least preferred outcome and 100 to the most preferred outcome. Then for each of the other outcomes, I'd ask what percentage it is of the way from the least to most preferred outcome.

RAUL: I'd ask the player which outcome is the least preferred and assign 1 to it. Then for each of the other outcomes, I'd ask how many times more preferred it is in comparison to the least preferred outcome.

ABBY: I guess that we've found three ways to measure player preferences.

NORMAN: Maybe that's why there are three more sections to this chapter.

Exercises

(1) Students are often assigned letter grades A, B, C, D, or F in the courses that they take. These letter grades are often assigned the numbers 4, 3, 2, 1, and 0, respectively, so that an average of all grades can be computed. Describe the implications for course grades to be a nominal, ordinal, interval, or ratio scale. Conclude, with justification, what type of scale course grades are.

(2) Evaluation forms often ask people to state whether they strongly disagree, disagree, are neutral, agree, or strongly agree with a statement. People's answers are coded as the numbers 1, 2, 3, 4, and 5. The coded responses of several people are then averaged. A typical summary might be that people, on average, agreed with the statement, because the average of the coded responses was 3.9. Discuss whether this is a meaningful statement.

(3) What are the advantages and disadvantages of measuring preferences with a ratio scale? interval scale? ordinal scale?

2. Ordinal Preferences

Unlike the previous chapter, we do not always want to win at a game that we are playing. For example, when parents play games with their children, they frequently lose on purpose (sorry to disillusion all of you who think that you are smarter than your parents). Sometimes you might even try to tie because that way no one's feelings are hurt. The main point here is that outcomes (win, lose, tie) are different from our preferences among those outcomes.

Our focus in the previous chapter was finding strategies to ensure a win in games where the only possible outcomes were a win or a loss. In this chapter, we will ignore strategic aspects of games (how outcomes arise) and focus on player preferences. Instead of finding good strategies for games like **Nim** or **Hex**, we will find ways for players to describe their preferences among the possible outcomes.

> By the end of this section, you will be able to determine ordinal preferences and utilities and explain the relationship between preferences and choice.

Prizes Scenario

We call our first illustration the **Prizes** scenario. In this situation, there are five possible outcomes:

Candy: You receive a one-pound bar of Hershey's® milk chocolate with almonds.
Cards: You receive a standard deck of Bicycle Brand® plastic coated cards.
Disc: You receive a new white standard Frisbee® flying disc by Whammo®.
Money: You receive two crisp American one dollar bills.
Nothing: You receive nothing.

> Before you continue reading, we suggest that you determine and describe your preferences among the **Prizes** outcomes. If possible, share with others your descriptions and discuss the reasons for commonalities and differences in your preferences.

Coauthor David is always happy to receive things, thinks a deck of cards is worth about $2, and is trying to eat nutritious food and get some more exercise. So, when David considered **Prizes**, he came up with the following ranking from most to least preferred: (1) disc, (2) money or cards, (3) candy, and (4) nothing. By ranking both money and cards second, David means that he is indifferent between the two outcomes: neither is preferred to the other. Notice that David is able to come up with this ranking because of his clearly stated rationale.

When faced with so many outcomes, coauthor Rick looks for flexibility, and so given a choice of all five possible outcomes, Rick would choose the money. Rick was concerned about from whom the prizes were coming and so was unsure about what his second, third, or fourth choices would be. However, he knows that any of the first four outcomes would be at least four times better than receiving nothing. When pressed further, Rick said that if he were given a choice just between the money or the candy, he would choose the candy.

It is likely that your preferences are different from David's or Rick's. Were you able to rank order your preferences like David? Or did you have difficulties with ranking like Rick? Did you assign some numerical intensity to your preferences like Rick? Were you indifferent between two or more outcomes like David? Could you imagine someone who would most prefer receiving nothing?

Player preferences are the preferences of the player, not something imposed by others. This sounds almost trivial, but it is an important point. Some might say that it is human nature to desire a prize, and therefore, *any* player would least prefer the nothing outcome among the **Prizes** outcomes. However, there may be more ascetic players who would, given the choice among the five **Prizes** outcomes, happily choose to receive nothing. As game theorists, we realize that any preferences we may ascribe to a player are assumptions that should be stated explicitly.

David stated that he ranked the **Prizes** outcomes as (1) disc, (2) money or cards, (3) candy, and (4) nothing. This ranking predicts what David will choose when selecting among a subset of the five outcomes. There are a total of 26 such scenarios (one for each subset of two or more outcomes). For example, given the opportunity to choose between money and nothing, the ranking predicts that David will choose the money, which is recorded in the last row of Table 2.1. If David's choices are as shown in Table 2.1, then David will be said to have ordinal preferences.

> **Ordinal Preferences:** A player is said to have *ordinal preferences* among outcomes if these outcomes can be ranked (possibly with ties) so that whenever the player is given an opportunity to choose among a subset of the outcomes, the player chooses a top-ranked outcome among those in the subset. We will also say that the player's choices are *consistent* with the outcome ranking.

For the purposes of this book, we will generally assume that players do have ordinal preferences. This is our first example of a mathematical model. The general process of mathematical modeling is captured by Figure 2.1.

We will use this process throughout this book. There is some real-world scenario or phenomenon we wish to analyze. We try to capture the essential features of the real-world scenario in a mathematical model. The connection between the real-world scenario and the mathematical model is via explicitly stated assumptions. Mathematical techniques can then be used to analyze the model to obtain a result. The mathematical result can then be interpreted as a prediction for what will happen in the scenario. Finally, it should be possible to verify the prediction.

In **Prizes**, the scenario is how David chooses among a set of available outcomes. The mathematical model is a rank order, obtained by the assumption that David has ordinal preferences. We use the simple technique of determining the highest

TABLE 2.1. David's choices among subsets if he has ordinal preferences

Possible outcomes in the scenario					David's predicted
candy	cards	disc	money	nothing	choice
X	X	X	X	X	disc
X	X	X	X		disc
X	X	X		X	disc
X	X		X	X	cards or money
X		X	X	X	disc
	X	X	X	X	disc
X	X	X			disc
X	X		X		cards or money
X	X			X	cards
X		X	X		disc
X		X		X	disc
X			X	X	money
	X	X	X		disc
	X	X		X	disc
	X		X	X	cards or money
		X	X	X	disc
X	X				cards
X		X			disc
X			X		money
X				X	candy
	X	X			disc
	X		X		cards or money
	X			X	cards
		X	X		disc
		X		X	disc
			X	X	money

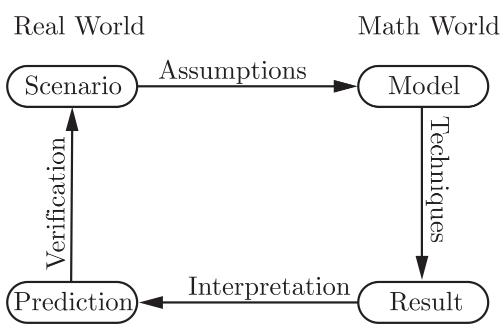

FIGURE 2.1. Mathematical Modeling Process

ranked outcome on a list to obtain a result that serves as the predicted choice for David. Finally, we can verify the prediction by directly asking David for his choice. To verify that David has ordinal preferences, we would need to verify the predictions for each of the 26 subsets of two or more outcomes.

Even though Rick did not explicitly state a rank order for the **Prizes** outcomes, perhaps we could still model his choices using a rank order. If there were such a rank order, Rick's choice of money over the other four outcomes implies that money would be ranked first. This would predict that Rick would choose money over the candy when given that choice. But this is not what he does. Thus, our initial assumption of ordinal preferences fails in the verification step. It would be inappropriate to use the ordinal preferences model to predict Rick's choices.

Is Rick being unreasonable? There is strong experimental evidence that people often do not make choices in accordance with an ordinal ranking of possible outcomes. This might be because people make mistakes and do not always think rationally. It also could be because the number, ambiguity, or complexity of the outcomes overwhelm a player's ability to make choices. Rick did indicate difficulty when choosing among many possible outcomes and noted that there was important information missing from the outcome descriptions. For a recent review of alternatives to the utility theory being developed in this chapter see *Choices, Values, and Frames* [**25**].

Acquaintance Scenario

A player's ranking can be determined by asking the player a sequence of questions. First, "If you were given the opportunity to chose among all of the outcomes, which would you choose?" The answer would be the player's first ranked outcome(s). Notice the possibility that the answer could be more than one outcome if the player would be willing to have any one of those outcomes. Next, "If you were given the opportunity to choose among all remaining outcomes, after excluding your first ranked outcome(s), which would you choose?" The answer would be the player's second ranked outcome(s). Repeat until all outcomes are ranked: "If you were given the opportunity to choose among all remaining outcomes, after excluding those already ranked, which would you choose?" The answer would be the player's next ranked outcome(s). If the player can answer all these questions (remember that Rick already had difficulty with the second question), then you will have obtained a ranking.

We illustrate this process for finding ordinal preferences in a second scenario, which we call **Acquaintance**. In this scenario, there are four possible outcomes (in which something happens to both you and your acquaintance):

80|0: You receive \$80 and your acquaintance receives \$0.
30|60: You receive \$30 and your acquaintance receives \$60.
30|30: You receive \$30 and your acquaintance receives \$30.
0|80: You receive \$0 and your acquaintance receives \$80.

Before you read how others fared in trying to find ordinal preferences among the **Acquaintance** outcomes, you may want to try it yourself. Use the approach suggested above to obtain your ranking of the **Acquaintance** outcomes. Check whether you have ordinal preferences among the **Acquaintance** outcomes. How difficult and/or time consuming was it to obtain your ranking? How difficult and/or time consuming was it to check whether you have ordinal preferences? Describe any difficulties and what caused the process to be fast or slow. If possible, share your answers with others.

For her acquaintance, Britany thought of a coworker. When she looked at all four outcomes, Britany was drawn to the symmetry of 30|30, and so ranked it first. Next considering the three unranked outcomes 80|0, 30|60, and 0|80, Britany decided she would choose 0|80. Between the remaining two outcomes 80|0 and 30|60, Britany chose 80|0. Hence, Britany obtained the only possible ranking that could model her choices: (1) 30|30, (2) 0|80, (3) 80|0, and (4) 30|60. As Britany thought about the choices she would make if faced with different subsets of outcomes, she discovered that if offered 30|30 or 80|0, she would choose 80|0 (apparently she was not drawn to symmetry so strongly when having a choice between only two outcomes). Since this choice was not consistent with her ranking, Britany's choices cannot be modeled by ordinal preferences.

For his acquaintance, Rick envisioned a friend of his who has recently come upon hard times. Being altruistic, Rick decides that he would like his friend to receive as much money as possible. This made it quite easy for Rick to answer the sequence of questions, and he obtained the ranking (1) 0|80, (2) 30|60, (3) 30|30, and (4) 80|0. Given the principle guiding how Rick would make choices, namely maximizing the money given to his friend, it is clear (without writing down the 11 subsets of two or more outcomes) that his choice among any subset will be consistent with his ranking. So, Rick has ordinal preferences among the **Acquaintance** outcomes.

For his acquaintance, David thought of his wife, with whom everything is shared. So, David is primarily concerned with maximizing the total amount given to the two players. If two outcomes provide the same total amount of money, David would prefer the one that gives him the greater share because he feels more magnanimous sharing with his wife rather than she sharing with him. This made it quite easy for David to answer the sequence of questions, and he obtained the ranking (1) 30|60, (2) 80|0, (3) 0|80, and (4) 30|30. Given the principles guiding how David would make choices, it is clear (without writing down the 11 subsets of two or more outcomes) that his choice among any subset will be consistent with his ranking. So, David has ordinal preferences among the **Acquaintance** outcomes.

For her acquaintance, Analytica thought of a close friend but not her spouse. David essentially ascribed a value of \$1.00 for each \$1.00 his wife received. Analytica, instead, ascribes a value of \$0.75 for each \$1.00 her acquaintance receives. In terms of choices, Analytica is indifferent between her receiving \$0.75 and her acquaintance receiving \$1.00. So, Analytica views 80|0 as worth \$80.00, 30|60 as worth \$30.00 + $(\frac{3}{4})$(\$60.00) = \$75.00, 30|30 as worth \$30.00 + $(\frac{3}{4})$(\$30.00) = \$52.50, and 0|80 as worth $(\frac{3}{4})(\$80.00) = \$60.00$. Thus, Analytica obtained the ranking (1) 80|0, (2) 30|60, (3) 0|80, and (4) 30|30. Given the principles guiding how Analytica would make choices, it is clear (without writing down the 11 subsets of two or more outcomes) that her choice among any subset will be consistent with her ranking. So, Analytica has ordinal preferences among the **Acquaintance** outcomes.

Your choice of acquaintance and your underlying goals may have been different from Britany's, Rick's, David's, or Analytica's, and so your ranking of the **Acquaintance** outcomes was probably different. Whether or not you had difficulty obtaining your ranking and checking whether your choices would always be consistent with that ranking may have been directly tied to how clearly you could articulate your goals. Britany did not have clearly defined motivations and found that she would not always be consistent with her ranking. Rick, David, and Analytica had clearly defined motivations that would guide any choice; thus each knew that he or she would be consistent with his or her ranking.

Ordinal Utility

Ordinal preferences are often described using a measurement scale rather than a ranking. In these scales, called utilities, the highest ranked outcome receives the largest number.

> **Ordinal Utility:** If a number $u(A)$ is assigned to each outcome A in such a way that $u(A) > u(B)$ if outcome A is ranked higher than outcome B, and $u(A) = u(B)$ if outcome A is ranked the same as outcome B, then each of these numbers is called an *ordinal utility* and u is an *ordinal utility function.*

For example, consider David's ranking of the **Prizes** outcomes: (1) disc, (2) money or cards, (3) candy, and (4) nothing. The numbers $u(\text{disc}) = 8$, $u(\text{money}) = 5$, $u(\text{cards}) = 5$, $u(\text{candy}) = 2$, and $u(\text{nothing}) = 0$ would be ordinal utilities for David. Note that the highest ranked outcome (disc) is given the largest utility (8). Since ordinal preferences do not encode intensity information, it is important to not ascribe any meaning to these numbers other than their relative magnitude; David could have just as easily use the numbers $v(\text{disc}) = 4$, $v(\text{money}) = 3$, $v(\text{cards}) = 3$, $v(\text{candy}) = 2$, and $v(\text{nothing}) = 1$ to represent the same ordinal preferences. The next two sections discuss ways of assigning numbers to outcomes that reflect intensity of preference.

In summary, we have a method to rank order a player's preferences: the first ranked outcome(s) would be those the player would choose if all outcomes were available and each successive ranked outcome(s) would be those the player would choose if all unranked outcomes were available. We could either verify directly, or simply assume, that the player has ordinal preferences with the obtained ranking. Ordinal utilities are numbers assigned to the outcomes so that higher ranked outcomes are given higher numbers.

Exercises

(1) Assign ordinal utilities to the **Acquaintance** outcomes to reflect Rick's ranking. Assign different ordinal utilities to the **Acquaintance** outcomes to again reflect Rick's ranking.
(2) What are the eleven subsets of two or more **Acquaintance** outcomes? For each subset, what would Rick's ranking predict for his choice among the outcomes in the subset?
(3) Janina is asked what prize she would choose if presented with several subsets of **Prizes**. The subsets presented and Janina's responses are given in Table 2.2. If Janina has ordinal preferences, what must be her ranking of the outcomes? Do we have sufficient information to know that Janina has ordinal preferences?

TABLE 2.2. Janina's choices

Possible outcomes in scenario					What Janina chose
candy	cards	disc	money	nothing	
X	X	X			candy
	X	X	X		cards
		X	X	X	disc
			X	X	money

(4) Charlene is planning on a get-away to the Outer Banks of North Carolina on either the first or second weekend of July. It is of primary importance to her that it be sunny rather than raining. She would prefer to spend as little as possible on airfare, although this is of only secondary importance. Unfortunately, cheap airline tickets are only available now, when it is hard to predict the weather. She could wait until early July, when she would have a better idea of the weather, to purchase expensive tickets. Finally, of tertiary importance is that an earlier get-away would be preferable to a later get-away. Each outcome in this situation can be represented as a triple of sunny or raining weather, cheap or expensive airline tickets, and first or second weekend. Of course, Charlene does not have complete control over which outcome will occur, but that is irrelevant to her preferences among the possible outcomes. Based on this scenario, describe Charlene's preferences by a rank order of the eight possible outcomes.
(5) Make a list of four to six candidates for the next United States President (or use a list provided by your instructor). Ask several people to rank the candidates in order from their most to least preferred. Record their rankings and any difficulties these people had in coming up with their ranking.
(6) Each column of Table 2.3 corresponds to a different ranking of the **Acquaintance** outcomes. For each column, describe why a player might have such a preference ranking. For example, column (a) corresponds to a player who is primarily concerned with receiving as much money as possible and secondarily concerned with her acquaintance receiving as much money as possible.
(7) Bill claims to have plans to view one of the four movies at the multiplex cinema. Bill claims that he has ordinal preferences given by the ranking (1) *Commission Possible*, (2) *Lawn of Daydreams*, (3) *Lord of the Hoops*, and (4) *Pie Rats of the Jelly Beans*. Yet, when he arrives at the multiplex cinema, Bill

TABLE 2.3. Different rankings of **Acquaintance** outcomes

Rank	a	b	c	d	e	f	g	h
1	80|0	80|0	0|80	30|60	30|30	30|30	80|0	30|60
2	30|60	30|30	30|60	80|0	30|60	0|80	30|60	0|80
3	30|30	30|60	30|30	0|80	80|0	80|0	0|80	30|30
4	0|80	0|80	80|0	30|30	0 |80	30|60	30|30	80 |0

chooses to view *Lord of the Hoops*. Give at least two scenarios in which Bill's choice would not be considered a violation of his claimed ordinal preferences.

(8) In William Styron's book *Sophie's Choice*, Sophie is forced to choose which of her children, Eva or Jan, is saved from the gas chamber in Auchwitz. If Sophie does not choose, both children will die. From her decision to allow Jan to live, can we infer that Sophie's ordinal preferences were consistent with the ranking (1) Jan lives, (2) Eva lives, and (3) neither lives?

(9) James Andreoni and John Miller [**2**] asked undergraduate students to divide tokens between themselves and a randomly chosen and anonymous other student. Tokens were converted into points at the specified rate and the points were converted into money at \$0.10 per point. For example, a student might divide 50 tokens by keeping 40 at one point each and passing 10 to the other student at three points each. The divider would receive $40 \times 1 \times \$0.10 = \$4.00$ and the other student would receive $10 \times 3 \times \$0.10 = \$3.00$.

(a) How would you make the following divisions?

(i) Divide 60 tokens: keep ____ at two points each, and pass ____ to the other student at one point each.

(ii) Divide 100 tokens: keep ____ at one point each, and pass ____ to the other student at one point each.

(iii) Divide 75 tokens: keep ____ at one point each, and pass ____ to the other student at two points each.

(iv) Divide 40 tokens: keep ____ at one point each, and pass ____ to the other student at three points each.

(b) Andreoni and Miller found that roughly half of the students maximized their own monetary payoff most of the time; that is, they kept all available tokens and passed nothing to the other student. The remaining students had one of two forms of altruism. Roughly one-sixth of the students maximized the total monetary payoff most of the time; that is, in division (i), they would keep all of the tokens, but in divisions (iii) and (iv), they would pass all of the tokens to the other student. Roughly one-third of the students gave equal monetary payoffs to themselves and the other student; that is, in division (i) they would keep 20 and pass 40 so that each would receive \$4.00, in division (ii) they would keep 50 and pass 50 so that each would receive \$5.00, in division (iii) they would keep 50 and pass 25 so that each would receive \$5.00, and in division (iv) they would keep 30 and pass 10 so that each would receive \$3.00. How did your choices compare with what was experimentally observed? How would you compare the experimental results with your *ad hoc* observations about the altruism of people?

(10) Here is an argument for why it is rational to have ordinal preferences. Suppose Firstus is money-loving, but does not have ordinal preferences among the outcomes A, B, and C. In particular, suppose Firstus prefers A to B, prefers B to C, but does not prefer A to C. Since Firstus prefers A to B and is money-loving, she should be willing to pay some amount, say \$1, to obtain A instead of B. Since Firstus prefers B to C and is money-loving, she should be willing to pay some amount, say \$1, to obtain B instead of C. Since Firstus does not prefer A to C and is money-loving, she should be willing obtain C, with any amount of money, instead of A alone. Finally, suppose Secondus has the ability to choose which of outcomes A, B, or C occurs.
   (a) If Secondus says to Firstus that outcome C will occur unless Firstus pays him \$1 to obtain outcome B, explain why Firstus will pay Secondus \$1.
   (b) If Secondus says to Firstus that outcome B will occur unless Firstus pays him \$1 to obtain outcome A, explain why Firstus will pay Secondus \$1.
   (c) If Secondus says to Firstus that outcome A will occur unless Firstus accepts \$1 and outcome C, explain why Firstus would accept the \$1.
   (d) If Secondus makes the offers suggested in parts (a), (b), and (c) in that order, explain why Firstus is \$1 poorer and expecting Secondus to select outcome C.
   (e) If Secondus repeats the process suggested in part (d), explain why Firstus can be made very poor. Further, explain why this is an absurdity.
   (f) Evaluate whether the above argument shows that it is rational to have ordinal preferences.

(11) Six games of chance (A, B, C, D, E, and F) are available. In each game of chance, a six-sided die is rolled. Depending upon which face lands face up (1, 2, 3, 4, 5, or 6), the player will receive the amount of money given in Table 2.4.

TABLE 2.4. Payoffs for dice games

		Die Face					
		1	2	3	4	5	6
Game	A	\$100	\$200	\$300	\$400	\$500	\$600
	B	\$600	\$100	\$200	\$300	\$400	\$500
	C	\$500	\$600	\$100	\$200	\$300	\$400
	D	\$400	\$500	\$600	\$100	\$200	\$300
	E	\$300	\$400	\$500	\$600	\$100	\$200
	F	\$200	\$300	\$400	\$500	\$600	\$100

Peter Fishburn, at a 1987 Operations Research Society of America meeting, stated that he would prefer A to B because in five of the six cases (all except when a 1 is rolled), he would win more money. For the same reason, he would prefer B to C, prefer C to D, prefer D to E, prefer E to F, and prefer F to A. Hence, a renowned game theorist admitted that in this situation, he did not have ordinal preferences. He stated that those were his preferences even though he knew of the "money pump" scenario described in the previous question. How does this information change or not change your beliefs about whether players must always have ordinal preferences?

3. Cardinal Preferences

Suppose that you not only want to say that some outcome is your first choice among several, but that it is your first choice way above and beyond the others. The ordinal preferences discussed in the first section do not allow you to express this. Cardinal preferences can express different intensities with numbers.

> By the end of this section, you will be able to determine cardinal preferences and utility among possible outcomes and explain the relationship between preferences and choice.

Introspection Approach

Cardinal preferences will be measured on a scale analogous to how temperatures are measured (an interval scale). With temperatures, it is possible to arbitrarily assign numbers to two temperatures, say the freezing and boiling points of water. The degree Celsius scale chooses 0 and 100, while the degree Fahrenheit scale chooses 32 and 212. Once these two numbers are assigned, the numbers assigned to all other temperatures are fixed. For example, since the normal human body temperature is 37% of the way from the freezing to boiling points for water (measured, say, by the expansion of mercury in an enclosed tube), we need to assign human body temperature to be $0+0.37\times(100-0) = 37$ degrees Celsius or $32+0.37\times(212-32) = 98.6$ degrees Fahrenheit.

Using temperature as an analogy, we will arbitrarily assign numbers to two outcomes. It is often convenient to choose the player's lowest and highest ranked outcomes and assign to them the numbers 0 and 100, respectively. Then for each of the other outcomes we try to determine the percentage of the distance that outcome is from the lowest to highest ranked outcome in terms of the player's preference (measured by introspection). The numbers assigned are called utilities, and these utilities are said to express the player's cardinal preferences. (A more formal definition will appear later in the section.)

As an illustration, consider David's ordinal ranking of the **Prizes** outcomes: (1) disc, (2) money or cards, (3) candy, and (4) nothing. David arbitrarily assigns to the nothing and disc outcomes the numbers 0 and 100. This will be denoted by $u(\text{nothing}) = 0$ and $u(\text{disc}) = 100$. Since David is trying to eat nutritious food and get some more exercise, he really prefers the disc and does not want to be tempted by the candy, although he could perhaps give the candy to his children. In any event, David believes that the candy outcome goes only a short way from the nothing outcome, and so he assigns $u(\text{candy}) = 10$. Figure 3.1 is a visual representation of David's thinking about the candy outcome being 10% of the distance from the nothing outcome to the disc outcome.

If pressed, David is willing to believe that his utility for the candy outcome could be as low as 5 and as high as 20, but it is certainly smaller than 30. David thinks that it is sometimes difficult to assign numbers to things as subjective as preferences.

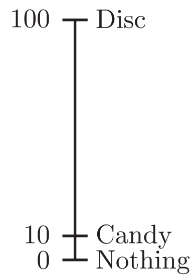

FIGURE 3.1. David's thoughts about candy

Continuing with the task at hand, David thinks that the disc might cost \$4, and there is some value in not having to go to a store to make the purchase, and thus he assigns $u(\text{money}) = 40$. Given that he ranked cards the same as money, David assigns $u(\text{cards}) = 40$. Figure 3.2 is a visual representation of David's thinking.

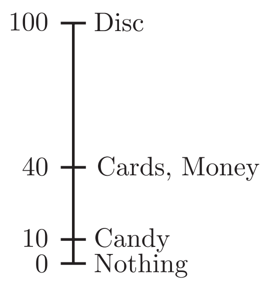

FIGURE 3.2. David's thoughts about cards and money

Table 3.1 summarizes David's rankings, ordinal utilities, and cardinal utilities for these outcomes. Notice again the reversal of number order between ranks and utilities.

TABLE 3.1. David's ordinal and cardinal utilities

Outcome	Rank	Ordinal Utility	Cardinal Utility
Candy	3	2	10
Cards	2	3	40
Disc	1	4	100
Money	2	3	40
Nothing	4	1	0

Before you continue to read, please take a few minutes to determine your own cardinal preferences for the **Prizes** outcomes by introspection. Did you find the process difficult? How so or why not?

Lottery Approach

In this section, we have been working with our preferences as if they existed as a part of ourselves; David's preference for candy over nothing is 10% as great as his preference for the disc over nothing, which is as real as David being 67 inches tall and weighing 160 pounds. Just as height and weight can be independently verified by appropriate measurements, game theorists verify David's preferences by the choices he makes. Unfortunately, David's choices among subsets of the five outcomes only implies $u(\text{disc}) > u(\text{money}) = u(\text{cards}) > u(\text{candy}) > u(\text{nothing})$. What choice could David make that would reveal that $u(\text{candy}) = 10$?

An answer is to ask David to make choices between outcomes and lotteries.

> **Lottery:** A *lottery* is an assignment of probabilities to the original outcomes or other lotteries.

For example, one lottery is "the nothing outcome occurring with probability 0.9 and the disc outcome occurring with probability 0.1." We will denote this lottery by

$$[0.9 \cdot \text{nothing} + 0.1 \cdot \text{disc}].$$

Since this is a new outcome, we need to assign it a utility.

Since David ranks the disc outcome higher than the nothing outcome, it seems that David should choose this lottery, which has some possibility of obtaining the disc, over the outcome of nothing. Similarly, it seems that David should choose the disc over this lottery. Hence, the utility assigned to the lottery should satisfy

$$u(\text{nothing}) < u([0.9 \cdot \text{nothing} + 0.1 \cdot \text{disc}]) < u(\text{disc}).$$

Consider a second lottery,

$$[0.8 \cdot \text{nothing} + 0.2 \cdot \text{disc}].$$

This lottery has the same possible outcomes as the first lottery, but there is a greater chance for the disc outcome (20% instead of 10%). Since David ranks the disc outcome higher than the nothing outcome, it would seem that David should rank this second lottery higher than the first lottery, and so the assigned utilities should satisfy

$$u([0.9 \cdot \text{nothing} + 0.1 \cdot \text{disc}]) < u([0.8 \cdot \text{nothing} + 0.2 \cdot \text{disc}]) .$$

More generally, we could consider lotteries in which the nothing outcome occurs with probability $1 - p$ and the disc outcome occurs with probability p, where p is some number between 0 and 1. We denote such a lottery by

$$[(1 - p) \cdot \text{nothing} + p \cdot \text{disc}],$$

and it must again be the case that

$$u(\text{nothing}) < u([(1 - p) \cdot \text{nothing} + p \cdot \text{disc}]) < u(\text{disc})$$

if $0 < p < 1$. The larger p is, the more probable it is for the higher ranked disc outcome to occur; therefore, $u([(1 - p)\cdot \text{ nothing } + p\cdot \text{ disc}])$ should increase as p increases. We could interpret the quantity p as the fraction of the way the lottery is from the nothing outcome to the disc outcome. This is analogous to the expansion of mercury in a tube to measure temperature.

This suggests that since the lottery [0.9· nothing + 0.1· disc] is 10% of the way from the nothing outcome to the disc outcome, then its utility should be 10% of the way from the utility of the nothing outcome (0) to the utility of the disc outcome (100), that is, $u([0.9{\cdot}\text{nothing}{+}0.1{\cdot}\text{disc}]) = 10$. Since he previously said that $u(\text{candy}) = 10$, David should be willing to choose either the candy outcome or the lottery [0.9· nothing + 0.1· disc].

We now have a way of verifying cardinal utility assignments by asking players their choice between one of the original outcomes and a lottery involving the last and first ranked outcomes. For the **Prizes** outcomes, David should verify that he is indifferent between the outcome and the lottery in each of the following three outcome-lottery pairs:

Outcome	vs.	Lottery
cards	vs.	[0.6· nothing + 0.4· disc]
money	vs.	[0.6· nothing + 0.4· disc]
candy	vs.	[0.9· nothing + 0.1· disc]

> Before you continue to read, we suggest that you determine the outcome-lottery pairs needed to verify your cardinal utility assignments for the **Prizes** outcomes. For each pair, do you agree that you would be willing to choose either the outcome or the lottery? If not, try to adjust your utility assignments so that you would be willing to choose either the outcome or the lottery in each pair.

Acquaintance Scenario

Verification is perhaps no easier than introspection; however, outcome versus lottery comparisons have the advantage of clearly linking cardinal utilities with player choices. In fact, the introspection step could be replaced by the player determining, for each outcome A, the probability p to make the player indifferent between the outcome and lottery below.

Outcome	vs.	Lottery
A	vs.	With probability $1-p$, the last ranked outcome; with probability p, the first ranked outcome.

For example, recall Rick's ranking of the **Acquaintances** outcomes: (1) 0|80, (2) 30|60, (3) 30|30, and (4) 80|0. For Rick to determine his cardinal utilities among these outcomes, he would first arbitrarily assign utilities to his last and first ranked outcomes: $u(80|0) = 0$ and $u(0|80) = 100$. Rick would then consider the following outcome-lottery pairs:

Outcome	vs.	Lottery
30|60	vs.	$[(1-q)\cdot\ 80|0 + q\cdot\ 0|80]$
30|30	vs.	$[(1-p)\cdot\ 80|0 + p\cdot\ 0|80]$

His goal would be (1) to pick q so that he would be willing to choose either the outcome 30|60 or the lottery $[(1-q)\cdot 80|0 + q\cdot\ 0|80]$, and (2) to pick p so that he would be willing to choose either the outcome 30|30 or the lottery $[(1-p)\cdot\ 80|0 + p\cdot\ 0|80]$. How would Rick do this? To pick q, he might first ask which of the following he would choose:

30|60	vs.	$[0.5\cdot\ 80|0 + 0.5\cdot\ 0|80]$

If he would choose 30|60, then we know that $0.5 < q < 1.0$, and he might ask which of the following he would choose:

30|60	vs.	$[0.2\cdot\ 80|0 + 0.8\cdot\ 0|80]$

If he would choose $[0.2\cdot\ 80|0 + 0.8\cdot\ 0|80]$, then we know that $0.5 < q < 0.8$, and he might ask which of the following he would choose:

30|60	vs.	$[0.4\cdot\ 80|0 + 0.6\cdot\ 0|80]$

If he would choose $[0.4\cdot\ 80|0 + 0.6\cdot\ 0|80]$, then we know that $0.5 < q < 0.6$, and he could continue to compare 30|60 with $[(1-q)\cdot\ 80|0 + q\cdot\ 0|80]$ for different values of q until he found himself willing to choose either. A similar approach could be used to determine p. Given his ordinal ranking, presumably Rick would choose $0 < p < q < 1$.

We can generalize our previous discussion. Suppose that u is a player's cardinal utilities, A and B are two outcomes, and p is a number between 0 and 1. The lottery in which outcome A occurs with probability $1-p$ and outcome B occurs with probability p, denoted $[(1-p)\cdot A + p\cdot B]$, is the fraction p of the distance from outcome A to outcome B. So, the player's utility for the lottery $[(1-p)\cdot A + p\cdot B]$ should be the fraction p of the distance from the utility of outcome A to the utility of outcome B. Since the distance from $u(A)$ to $u(B)$ is $u(B) - u(A)$, we obtain

$$\begin{aligned} u([(1-p)\cdot A + p\cdot B]) &= u(A) + p\cdot(u(B) - u(A)) \\ &= u(A) + p\cdot u(B) - p\cdot u(A) \\ &= 1\cdot u(A) - p\cdot u(A) + p\cdot u(B) \\ &= (1-p)\cdot u(A) + p\cdot u(B). \end{aligned}$$

This assumption about a player's preferences is called the expected utility hypothesis.

Expected Utility Hypothesis: If u is a player's cardinal utility, A and B are two original outcomes, and $0 \leq p \leq 1$, then the player's cardinal utility for the lottery $[(1-p) \cdot A + p \cdot B]$ is given by the formula

$$u([(1-p) \cdot A + p \cdot B]) = (1-p) \cdot u(A) + p \cdot u(B).$$

For David and the **Prizes** outcomes,

$$\begin{aligned} u\left([0.7 \cdot \text{candy} + 0.3 \cdot \text{money}]\right) &= 0.7 \cdot u(\text{candy}) + 0.3 \cdot u(\text{money}) \\ &= 0.7 \cdot 10 + 0.3 \cdot 40 \\ &= 19. \end{aligned}$$

Since $u(\text{candy}) = 10 < 19 < 40 = u(\text{cards})$, David should choose the lottery $[0.7 \cdot \text{candy} + 0.3 \cdot \text{money}]$ over the candy outcome and choose the cards outcome over the lottery. Also since

$$\begin{aligned} u\left(\left[\frac{2}{3} \cdot \text{candy} + \frac{1}{3} \cdot \text{disc}\right]\right) &= \frac{2}{3} \cdot u(\text{candy}) + \frac{1}{3} \cdot u(\text{disc}) \\ &= \frac{2}{3} \cdot 10 + \frac{1}{3} \cdot 100 \\ &= 40 \\ &= u(\text{cards}), \end{aligned}$$

David should be willing to choose either the lottery $\left[\frac{2}{3} \cdot \text{candy} + \frac{1}{3} \cdot \text{disc}\right]$ or the card outcome.

We have now proclaimed twice that David "should" choose a particular outcome or lottery. If he does, and he always makes choices consistent with utilities calculated from the Expected Utility Hypothesis, then David is said to have cardinal preferences.

Cardinal Preferences: A player is said to have *cardinal preferences* among outcomes if there is an assignment of numbers, called *cardinal utilities*, to lotteries so that the Expected Utility Hypothesis holds, and whenever the player is given an opportunity to choose among a subset of lotteries, the player chooses a lottery in the subset with the largest utility. When the latter condition holds, we will say that the player's choices are *consistent* with the cardinal utilities.

For an assignment of utilities to lotteries to represent a player's cardinal preferences, two conditions must be satisfied: (1) the assigned utilities must satisfy the Expected Utility Hypothesis, and (2) player's choices must be consistent with the assigned utilities. If only the latter condition holds, then the player only has ordinal preferences (see page 68). These ordinal preferences would be among all lotteries, rather than just the original outcomes, but there would be no simple way of relating the utility of different lotteries. A simple relationship between the utilities of different lotteries is provided by the Expected Utility Hypothesis.

Recall that in the Ordinal Preferences section, we modeled choices among finite sets of outcomes with a ranking and verified the model's correctness with choices among subsets of outcomes (see "Mathematical Modeling Process" diagram shown in Figure 2.1). In this section, we modeled choices, which potentially included lotteries as well as the outcomes, with utilities and the Expected Utility Hypothesis. However, to verify the consistency of the model, we would need to verify choices were consistent with claimed utilities for every subset of the outcomes and the infinite number of lotteries. This is an infinite amount of work! Since we cannot complete this verification, we could just assume that players have cardinal preferences. This is clearly a stronger assumption than the one that players have ordinal preferences. So, we should have some justification for this assumption. Our analogy with temperature provides one informal justification. Empirical tests, such as the checks performed above, could give us greater confidence or undermine our confidence in the Expected Utility Hypothesis. A more theoretical justification by John Von Neumann and Oscar Morgenstern is that if a player is "reasonably consistent" (making this more precise is beyond the scope of this book) in making pairwise choices among outcomes and lotteries, the player will have cardinal preferences given by utilities on an interval scale [**39**].

Transformation of Utilities

Let us explore the implications of cardinal utilities, just like temperatures, being an interval scale. Recall that the degree Celsius scale chooses 0 and 100 and the degree Fahrenheit scale chooses 32 and 212 for the freezing and boiling points for water. With these arbitrary assignments, we can obtain a formula for converting from degrees Fahrenheit (denoted by F) to degrees Celsius (denoted by C). The transformation must appear as a straight line on a degrees Celsius versus degrees Fahrenheit graph, as shown in Figure 3.3.

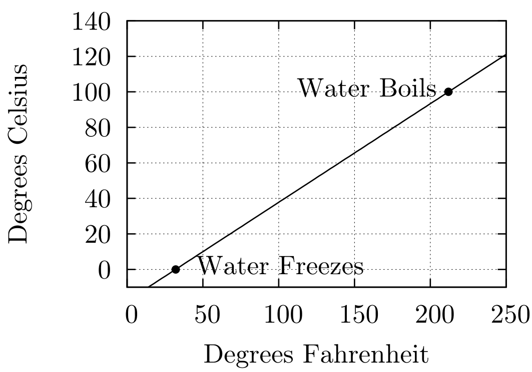

FIGURE 3.3. Temperature scale conversion

Similarly, one cardinal utility scale can be transformed to another equivalent cardinal utility scale if the utilities for both scales are known at two points. For example, Table 3.2 shows the original utilities assigned to David's preferences to the **Prizes** outcomes.

The table also shows the start of a new utility scale with which David's preferences could be described. Utilities have been arbitrarily assigned to two arbitrarily chosen

TABLE 3.2. Before new utilities are fully assigned

Outcome X	nothing	candy	cards	money	disc
Original utilities $U(X)$	0	10	40	40	100
New utilities $V(X)$	?	20	65	?	?

outcomes. With the utility assignments to the candy and cards outcomes, we can obtain a formula for converting from the original utilities (denoted by U) to the new utilities (denoted by V). The transformation must be a straight line on a new utilities versus original utilities graph, as shown in Figure 3.4.

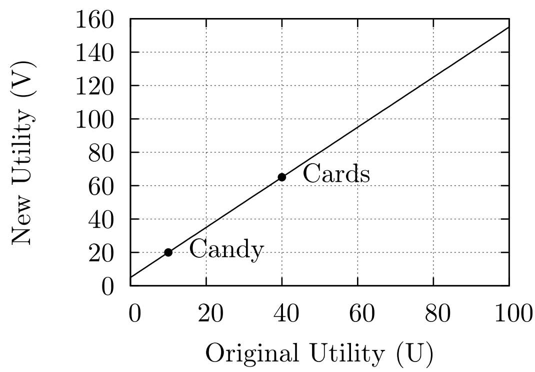

FIGURE 3.4. Utility scale conversion

Hence, the slope between $(10, 20)$ and $(40, 65)$ must equal the slope between $(10, 20)$ and (U, V). This yields the equation

$$\frac{65 - 20}{40 - 10} = \frac{V - 20}{U - 10}$$

which simplifies to

$$V = 1.5U + 5.$$

This formula can then be used to fill in the other new utilities (e.g., for the disc, $V = 1.5(100) + 5 = 155$), as shown in Table 3.3.

TABLE 3.3. After new utilities are fully assigned

Outcome X	nothing	candy	cards	money	disc
Original utilities $u(X)$	0	10	40	40	100
New utilities $v(X)$	5	20	65	65	155

The interval size-preserving nature of this scale transformation is seen vividly in Figure 3.5.

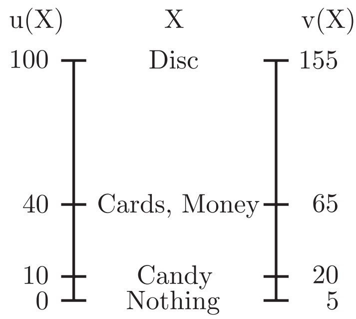

FIGURE 3.5. Visual representation of David's two utility scales

Risk

Let us consider one more example of determining cardinal utilities. The **Money** outcomes consist of receiving \$0, \$25, \$50, \$75, or \$100. When Adora, Nellie, and Loren considered these possible outcomes, they each decided that receiving \$100 was her most preferred outcome and receiving \$0 was her least preferred outcome. To determine her cardinal utilities, each woman arbitrarily assigned $u(\$0) = 0$ and $u(\$100) = 100$ and then made comparisons between lotteries of the form $[(1-p)\cdot \$0 + p \cdot \$100]$ and the \$75, \$50, and \$25 outcomes. Adora determined that she was willing to choose either the outcome or lottery in each of the following outcome-lottery pairs.

Outcome	vs.	Lottery
\$75	vs.	[0.1· \$0 + 0.9· \$100]
\$50	vs.	[0.2· \$0 + 0.8· \$100]
\$25	vs.	[0.5· \$0 + 0.5· \$100]

Analogous results were obtained by Nellie and Loren, and all of their results are summarized as utilities in Table 3.4.

TABLE 3.4. Adora, Nellie, and Loren's utilities

Player	$u(\$0)$	$u(\$25)$	$u(\$50)$	$u(\$75)$	$u(\$100)$
Adora	0	50	80	90	100
Nellie	0	25	50	75	100
Loren	0	10	30	60	100

If Adora, Nellie, and Loren have cardinal preferences, then their cardinal utilities for outcomes are given in the table and their cardinal utilities for lotteries are given by the Expected Utility Hypothesis. Of course, there would be an infinite amount of checking necessary to verify this, and so we will just assume that they have cardinal preferences and hope for the best as we consider some of the implications of the utilities given.

For a lottery between two monetary outcomes, we can calculate the expected monetary value. One interpretation of this expected value is that it is the average amount you would receive if you played the lottery many times. The lottery $[0.2 \cdot \$0 + 0.8 \cdot \$100]$ has an expected value of $(0.2)(\$0) + (0.8)(\$100) = \$80$. If you played this lottery many times, sometimes you would receive \$0 and the rest of the time you would receive \$100. In fact, you would receive \$0 about 20% of the time and receive \$100 about 80% of the time. This would result in average per-play winnings of about \$80. Of course, in the context of our scenario, you would play the lottery only once. So, while you hope for the \$100, you risk receiving only \$0.

The following are not generally the same:

(1) the expected utility of a lottery between monetary outcomes, and
(2) the utility of the lottery's expected value.

For example, the expected utility of the lottery $[0.5 \cdot \$0 + 0.5 \cdot \$100]$ is

$$u(0.5 \cdot \$0 + 0.5 \cdot \$100) = 0.5 \cdot u(\$0) + 0.5 \cdot u(\$100) = 50,$$

but the utility of the expected value of the lottery is

$$u((0.5)(\$0) + (0.5)(\$100)) = u(\$50),$$

which depends on the player. Thus, if you are like Adona and 50 is less than $u(\$50)$, you would choose a definite \$50 over a 50% chance of receiving \$100, and you would be considered risk adverse. On the other hand, if you are like Nellie and $50 = u(\$50)$, you would choose either a 50% chance of receiving \$100 or a definite \$50, and you would be considered risk neutral. And on the third hand, if you are like Loren and 50 is greater than $u(\$50)$, you would choose a 50% chance of receiving \$100 over a definite \$50, and you would be considered risk loving. In general, we have the following definition.

> **Risk Adverse, Neutral, and Loving:** A player is *risk adverse, neutral, or loving* if the expected utility of every monetary lottery is less than, equal to, or greater than, respectively, the utility of the lottery's expected value.

On its face, checking that Adora is risk adverse involves a large amount of work. For example, we should check that for each p, the expected utility of the monetary lottery

$$(1 - p) \cdot u(\$25) + p \cdot u(\$75)$$

is less than the utility of the lottery's expected value

$$u((1 - p) \cdot \$25 + p \cdot \$75),$$

expressed succinctly by

$$(1 - p) \cdot u(\$25) + p \cdot u(\$75) < u((1 - p) \cdot \$25 + p \cdot \$75)$$

for all $0 < p < 1$. Similar checks would need to be done with other pairs of monetary values substituted for \$25 and \$50. Fortunately, all of this algebraic work can be eliminated by looking at a graph of Adora's utility versus the money she received. In Figure 3.6 the triangles correspond to the five data points that we have. Thus, the right-hand side of the inequality above is represented by the vertical coordinate of a point on the dashed line in the figure. The vertical coordinates of the dotted line segment from A to B represent the left-hand side of the same inequality. Notice that the dotted line is below the dashed line; this means that the inequality above is true. If this is true for all choices of A and B, we say that Adora's utility graph is *concave down*, and that Adora is risk adverse.

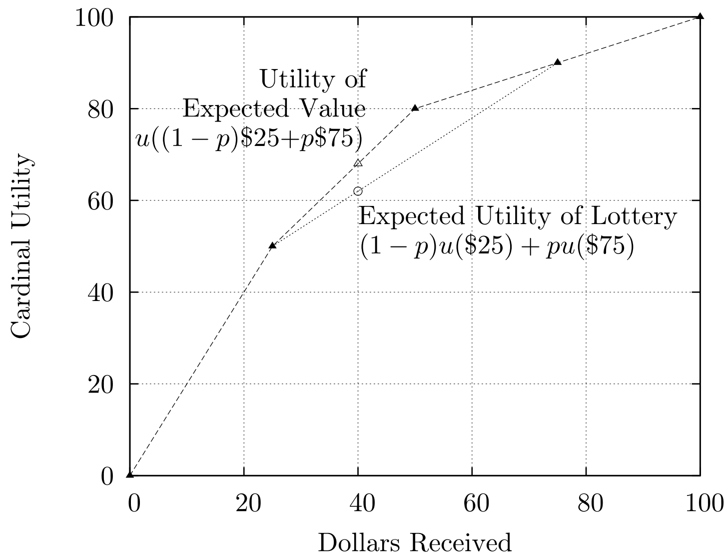

FIGURE 3.6. Adora's utility graph

So the graphical test for risk adversion is to just check whether the utility graph is concave down. Similarly, if a person's utility graph is *concave up*, then the person is risk loving. Finally, if a person's utility graph is a straight line, then that person is risk neutral. Figure 3.7 adds Nellie's and Loren's utility graphs to Adora's.

When it comes to large but unlikely winnings (the multimillion-dollar jackpot) and small but definite losses (the purchase price of a ticket), the popularity of state lotteries indicates that many people are risk loving in such situations. When it comes to large but unlikely losses (the destruction of a home or the early death of a spouse and parent) and small but definite losses (the premium payment), the popularity of home and life insurance indicates that many people are risk adverse in such situations. State governments, in the former case, and insurance companies, in the later case, are risk neutral: their priority is to maximize their expected value.

Since Loren's utility for \$50 is 30 and Adora's utility for \$50 is 80, can we say that Adora values \$50 more than Loren? Not necessarily. Remember that the utilities assigned to \$0 and \$100 were arbitrary, and unlike temperature, which can use a single device to measure the temperature of different objects, there is no single device that allows us to measure preferences of different people. Does the 0

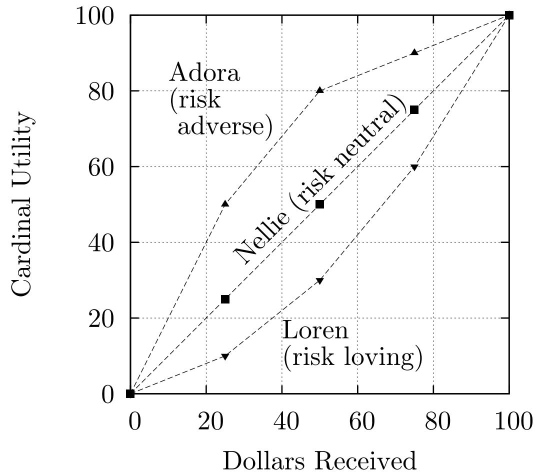

FIGURE 3.7. Comparison of risk taking

Loren assigned to \$0 mean the same thing as the 0 Adora assigned to \$0? Perhaps Loren really needs some money and would be on the brink of starvation with a \$0 outcome, while Adora is awash in cash and would barely notice the difference between receiving an extra \$100 or receiving \$0. Because of such intangibles, game theorists try to avoid such interpersonal comparisons of utility.

In summary, this section shows one way to construct a player's cardinal utilities by arbitrarily assigning the utility 0 to the last ranked outcome L and assigning the utility 100 to the first ranked outcome F, and then for each of the other outcomes A, assign the utility $100p$, where the probability p is chosen so that the player is indifferent between choosing the outcome A and the lottery $[(1-p)\cdot L+p\cdot F]$. The utility for any lottery can then be determined by the Expected Utility Hypothesis. Since cardinal utilities are an interval scale, a different equivalent scale can be obtained by linearly transforming the original scale. It is impossible to directly verify that a player has cardinal preferences based on the obtained utilities, and so we are left with assuming, when necessary, that players have cardinal preferences. When the outcomes involve only money transfers, some players can be classified as risk adverse, neutral, or loving depending upon whether the line segment formed between any two points on the utility-versus-money amount graph always lies below, on, or above, respectively, the utility graph.

Exercises

(1) Using appropriate outcome-lottery pairs, determine your cardinal utilities among the **Acquaintance** outcomes.

(2) For the **Prizes** outcomes, Samuel claims that his cardinal utilities are $u(\text{money}) = 100$, $u(\text{candy}) = 90$, $u(\text{cards}) = 60$, $u(\text{disc}) = 20$, and $u(\text{nothing}) = 0$.

(a) Would Samuel choose the cards or a lottery in which he has a 70% chance of receiving the money and a 30% chance of receiving nothing?
(b) Samuel would be indifferent between a lottery in which he has a 60% chance of receiving the money and a 40% chance of receiving nothing and what outcome?
(c) Describe a lottery for which Samuel would be indifferent to receiving the candy.
(d) Describe a lottery for which Samuel would be indifferent to receiving the disc.

(3) For the utilities you found for the **Prizes** outcomes and/or **Acquaintance** outcomes, complete a partial check of whether your choices are consistent with your utilities. More specifically, consider two new comparisons between lotteries and an outcome in a manner similar to what was done in the text on page 81. Report your comparisons and conclusions.

(4) Consider the five **Money** outcomes of receiving \$0, \$25, \$50, \$75, and \$100.
(a) Determine your cardinal utilities.
(b) Draw a graph with money received along the horizontal axis and the corresponding utility along the vertical axis. Can you estimate a curve between the five points you have plotted? Are you risk adverse, neutral, or loving?
(c) Find a lottery of the form $[(1-p)\cdot \$25 + p\cdot \$75]$ for which you would be indifferent between choosing the lottery or receiving \$50. After determining p, find $u([(1-p)\cdot\$25+p\cdot\$75])$ and compare with $u(\$50)$ as determined in part (b).
(d) Find a lottery of the form $[(1-p)\cdot \$25 + p\cdot \$100]$ for which you would be indifferent between choosing the lottery or receiving \$75. After determining p, find $u([(1-p)\cdot\$25+p\cdot\$100])$ and compare with $u(\$75)$ as determined in part (b).
(e) Based on your answers to parts (c) and (d), do you have cardinal preferences among the monetary outcomes above?

(5) Make a list of four or five animals that could be pets (or use a list provided by your instructor). Let an outcome be receiving one of these animals for free. Determine your cardinal utilities among these outcomes. Determine one other person's cardinal utilities among these four or five outcomes. Check whether the Expected Utility Hypothesis is satisfied for two lotteries; that is, show that you are indifferent between receiving one of the middle ranked animals and an appropriate lottery between your highest and lowest ranked animals. Record the cardinal utilities found and any difficulties you or others had in coming up with their cardinal utilities.

(6) Suppose Xianon has cardinal preferences and her utilities for outcomes A, B, C, and D are 0, 20, 60, and 100, respectively. Determine her utilities for the following lotteries, and then state how Xianon ranks the outcomes and lotteries.
(a) $[0.75\cdot B + 0.25\cdot D]$
(b) $[0.5\cdot B + 0.5\cdot D]$
(c) $[0.5\cdot B + 0.5\cdot C]$
(d) $[0.2\cdot A + 0.8\cdot D]$

(7) If a player has the cardinal utilities $u(A) = 90$, $u(B) = 70$, $u(C) = 20$, and $u(D) = 10$, what would be the player's cardinal utilities on a scale in which $v(D) = 0$ and $v(A) = 100$?

(8) If a player has the cardinal utilities $u(A) = 100$, $u(B) = 75$, $u(C) = 40$, $u(D) = 30$, and $u(E) = 0$, what would be the player's cardinal utilities on a scale in which $v(E) = -25$ and $v(A) = 25$?

(9) Juan plans to go on one of the four study vacations to France, Israel, Japan, or Peru being offered by his college alumni association. His first choice would be Israel. He is indifferent between choosing Japan or the lottery $[0.3 \cdot \text{France} + 0.7 \cdot \text{Peru}]$. He is also indifferent between choosing Peru or the lottery $[0.6 \cdot \text{Japan} + 0.4 \cdot \text{Israel}]$. Assuming that he has cardinal preferences, determine Juan's cardinal utilities.

(10) Investigate the following problem posed by Maurice Allais [**1**].

(a) Rank order the following four outcomes with explanation. Do this before reading about the rest of the problem.

A: You receive \$1 million with complete certainty.

B: You receive \$5 million with probability 0.10, you receive \$1 million with probability 0.89, and you receive nothing with probability 0.01.

C: You receive \$1 million with probability 0.11 and you receive nothing with probability 0.89.

D: You receive \$5 million with probability 0.10 and you receive nothing with probability 0.90.

(b) Allais found that most people ranked A over B, because \$1 million for sure is better than having a 1% chance ending up with nothing even if a bigger prize is possible. Allais also found that most people ranked D over C, because the probabilities of winning anything are almost the same but the prize in D is so much bigger than the prize in lottery C. Was your ranking consistent with Allais's empirical findings?

(c) Use the Expected Utility Hypothesis to show that a player should choose A over B if and only if the player should choose C over D.

(d) Allais argued that his empirical findings and the result of part (b) shows that most people's preferences do not follow the Expected Utility Hypothesis. Evaluate Allais's argument.

(11) Suppose you have an urn containing 30 red balls and 60 other balls that are either yellow or green. You do not know how many yellow balls or green balls there are, but that the total number of yellow and green balls is 60. The balls are well mixed so that each ball is as likely to be drawn as any other.

(a) Do this part before reading the other parts. Rank order the following four outcomes.

A: You receive \$100 if you draw a red ball; otherwise, you receive nothing.

B: You receive \$100 if you draw a yellow ball; otherwise, you receive nothing.

C: You receive \$100 if you draw a red or green ball; otherwise, you receive nothing.

D: You receive \$100 if you draw a yellow or green ball; otherwise, you receive nothing.

(b) The probability of drawing a red ball is $\frac{30}{90} = \frac{1}{3}$. Let the unknown probabilities of drawing a yellow ball and drawing a green ball be denoted y and g, respectively. Assuming that you have cardinal preferences given by the utility function u, show that $u(A) > u(B)$ if and only if $u(C) > u(D)$.

(c) Is the rank order you found in part (a) consistent with the result of part (b)?

(d) Daniel Ellsberg [**17**] reports that most people choose lottery A over lottery B and choose lottery D over lottery C. Ellsberg argues that this shows that most people do not have cardinal preferences. Evaluate his argument.

4. Ratio Scale Preferences

If utilities are a model for player choices among outcomes, then the scale can be no more informative than an interval scale. This means that two outcomes can have their utilities assigned arbitrarily, say at 0 and 100. It also means that it is not meaningful to say that one outcome is preferred twice as much as another outcome.

Yet, many people would say that one outcome is preferred twice as much as another outcome. This is because people sometimes think about preferences, not as a model for choice, but as a reality connected to, but different from, choice. In this reality, preferences are a state of mind that can be measured through introspection, and so it is reasonable to posit that the measurement can be done with a ratio scale, just like length or weight. It turns out that this way of thinking about preferences can often be helpful to individuals trying to determine their preferences among complex outcomes.

> By the end of this section, you will be able to determine ratio scale preferences among possible outcomes.

Housing Scenario

We will develop ratio scale preferences for what we call the **Housing** outcomes. Luz is trying to decide where to live next academic year. High quality and functionality, closeness to academic buildings, low cost, adequate privacy, and easy interaction with friends are all important objectives. Luz currently has the following four alternatives:

Townhouse: Three of Luz' closest friends want Luz to join them in renting a gorgeous four-bedroom and two-bathroom townhouse apartment located about a mile from academic buildings. There is a shared living room, dining room, and kitchen. This is the most expensive option, costing Luz $500 per month.

House: One of Luz' friends plans to rent a room in an eight-bedroom house located just off campus. There is another room still available in the house. The house is old and musty but has not recently been in violation of any housing ordinances. There are only three bathrooms, a small living room, and an eat-in kitchen. But this option, at $300 for Luz, is the least expensive option.

Single: With her housing lottery number, Luz could obtain a room in a four-bedroom suite in one of the newer dormatories on campus. Each pair of bedrooms shares a bathroom, there is a shared living room, and three suites share a small kitchen and lounge. Although on campus, this dormitory is almost as far from the academic buildings as the house. This option costs Luz $460 per month.

Double: A close friend and current roommate would like to share a double with Luz again. The dormitory building is adjacent to the academic buildings. The double has almost three times the area of a bedroom in the house or the single; however, twenty women share the large bathroom, lounge, and small kitchen. This option costs Luz \$340 per month.

With the five conflicting objectives, Luz finds it difficult to make a selection of her top choice, let alone rank order the four choices or provide numeric intensities. Surprisingly, Luz will be able to bring coherence to her preferences by seeking to measure her options on an even more information-intensive ratio scale! To do this, we ask Luz to first focus on the objectives and then to focus on her options one objective at a time. This is a standard problem solving approach in mathematics: divide up a complex problem into subproblems that are more easily solved and then piece together the subproblem solutions to obtain a solution to the original problem. This reductionistic approach to preference determination was first proposed by Thomas L. Saaty, who named it the Analytic Hierarchy Process [**53**].

Analytic Hierarchy Process

First, we ask Luz to focus on her objectives for a good living arrangement. Earlier we listed five objectives, but in order to minimize the arithmetic and so better illustrate the concepts, we will now assume that Luz has only three objectives: (1) low cost, (2) easy interactions with friends, and (3) high quality and functionality. We ask Luz what fraction each objective will contribute to her sense of a good living arrangement. After some thought, Luz is able to assign the numbers in Table 4.1.

TABLE 4.1. Weights on the objectives

Objective	Contribution
Low cost	0.20
Easy interactions with friends	0.40
High quality and functionality	0.40

To confirm these numbers, Luz should be able to answer "yes" to the questions (1) are "easy interactions with friends" and "high quality and functionality" equal contributors, and (2) does "easy interactions with friends" contribute twice as much as "low cost"?

Second, we ask Luz to focus on the "low cost" objective and to measure the four outcomes on the "low cost" objective. Luz knows the actual cost for each outcome, but the costs do not immediately translate into ratio scale "low cost" objective measurements. Luz needs to ask herself questions such as "How many times better is a monthly cost of \$300 than a monthly cost of \$500?" Luz decides that what is really important is the monthly savings based on the \$600 per month she had originally budgeted for lodging. For example, Luz saves \$600 − \$500 = \$100 if she selects the townhouse. The monthly cost and savings for each outcome are summarized in Table 4.2.

The normalized savings were obtained by dividing each savings by the sum of the savings, \$800. Since savings is a ratio scale, the normalized savings conveys the same information—just using different units. The conversion was chosen so that

TABLE 4.2. Preferences for the cost objective

Outcome	Cost	Savings	Normalized Savings
Townhouse	500	100	0.125
House	300	300	0.375
Single	460	140	0.175
Double	340	260	0.325

the sum of the normalized savings is one. This will be important when we later combine our work on the separate pieces.

The fact that savings is a ratio scale for Luz's "low cost" objective implies that the following statements, coming from pairwise comparisons of the outcomes, are true.

(1) The house is 3 ($\frac{300}{100} = \frac{0.375}{0.125}$) times better than the townhouse on the "low cost" objective.
(2) The single is $\frac{7}{5}$ ($\frac{140}{100} = \frac{0.175}{0.125}$) times better than the townhouse on the "low cost" objective.
(3) The double is $\frac{13}{5}$ ($\frac{260}{100} = \frac{0.325}{0.125}$) times better than the townhouse on the "low cost" objective.
(4) The house is $\frac{15}{7}$ ($\frac{300}{140} = \frac{0.375}{0.175}$) times better than the single on the "low cost" objective.
(5) The house is $\frac{15}{13}$ ($\frac{300}{260} = \frac{0.375}{0.325}$) times better than the double on the "low cost" objective.
(6) The double is $\frac{13}{7}$ ($\frac{260}{140} = \frac{0.325}{0.175}$) times better than the single on the "low cost" objective.

The pairwise comparisons above are summarized in Table 4.3 where each cell reports how many times the row outcome is better than the column outcome on the "low cost" objective. The fact that the house (row) is 3 times better than the townhouse (column) implies that the townhouse (row) is $\frac{1}{3}$ as good as the house (column). In general, the entries below the diagonal will always be the reciprocals of the corresponding entries above the diagonal.

TABLE 4.3. Pairwise comparisons on the cost objective

Low Cost	Townhouse	House	Single	Double
Townhouse	1	$\frac{1}{3}$	$\frac{5}{7}$	$\frac{5}{13}$
House	3	1	$\frac{15}{7}$	$\frac{15}{13}$
Single	$\frac{7}{5}$	$\frac{7}{15}$	1	$\frac{7}{13}$
Double	$\frac{13}{5}$	$\frac{13}{15}$	$\frac{13}{7}$	1

Notice that each column is a multiple of each other column. For example, the townhouse column numbers are 3 times the corresponding house column numbers. That is, each column conveys the same ratio scale information, but each is using different units.

Third, we ask Luz to measure the four outcomes on the "easy interactions with friends" objective (which we will shorten to just "friends"). Here it is difficult for Luz to think in terms of fractions and there is not an immediately available ratio measure. So, Luz first notes that she can readily rank order the outcomes from the best to worst on the "friends" objective: (1) townhouse, (2) double, (3) house, and (4) single. Luz can now arbitrarily assign a measure of 1 to the worst outcome, and decide how many times better each of the other outcomes is. For example, Luz thinks that the double would be 5 times as good as the single at having easy interactions with friends. All multiples are summarized in Table 4.4.

Table 4.4. Preferences based on the friends objective

Outcome	Friends	Normalized Friends
Townhouse	12	0.60
House	2	0.10
Single	1	0.05
Double	5	0.25

Again, the normalized friends was obtained by dividing the friends measure by the sum of the friends measure so that the sum of the normalized friends measurements is 1. Notice that there are six questions to which Luz should be able to answer "yes" in order for the friends measure to be accurate.

(1) Is the townhouse $\frac{12}{2} = 6$ times better than the house at having easy interactions with friends?
(2) Is the townhouse 12 times better than the single at having easy interactions with friends?
(3) Is the townhouse $\frac{12}{5} = 2.4$ times better than the double at having easy interactions with friends?
(4) Is the house 2 times better than the single at having easy interactions with friends?
(5) Is the double $\frac{5}{2} = 2.5$ times better than the house at having easy interactions with friends?
(6) Is the double 5 times better than the single at having easy interactions with friends?

If the answers are all "yes", then the pairwise comparisons can be summarized in Table 4.5 where each cell reports how many times the row outcome is better than the column outcome on the "friends" objective.

Again notice that each column is a multiple of each other column. That is, each column conveys the same ratio scale information—each using different units.

Fourth, we ask Luz to measure the four outcomes on the "high quality and functionality" objective (which we will shorten to just "quality"). Now Luz has difficulty thinking in terms of fractions (as she did for weighting the objectives), coming up with an already available ratio measure (as she had for the "low cost" objective), and even determining the rank order (as she had for the "friends" objective).

TABLE 4.5. Pairwise comparisons on the friends objective

Friends	Townhouse	House	Single	Double
Townhouse	1	6	12	$\frac{12}{5}$
House	$\frac{1}{6}$	1	2	$\frac{2}{5}$
Single	$\frac{1}{12}$	$\frac{1}{2}$	1	$\frac{1}{5}$
Double	$\frac{5}{12}$	$\frac{5}{2}$	5	1

However, we can ask Luz to make pairwise comparisons analogous to the ones we verified previously.

(1) Is the townhouse or the house better with respect to the "quality" objective? By how much?
(2) Is the townhouse or the single better with respect to the "quality" objective? By how much?
(3) Is the townhouse or the double better with respect to the "quality" objective? By how much?
(4) Is the house or the single better with respect to the "quality" objective? By how much?
(5) Is the house or the double better with respect to the "quality" objective? By how much?
(6) Is the single or the double better with respect to the "quality" objective? By how much?

If the "by how much?" question is hard to answer numerically, Thomas Saaty suggested an answer using one of the words "equally", "moderately", "strongly", "very strongly", and "extremely". These words can then be translated into the numbers 1, 3, 5, 7, and 9, respectively. With these suggestions, Luz provided the following answers.

(1) The townhouse is strongly better than the house.
(2) The townhouse is very strongly better than the single.
(3) The townhouse is very strongly better than the double.
(4) The single is moderately better than the house.
(5) The house is moderately better than the double.
(6) The double is only somewhat better than the single.

These answers are translated into the integer and reciprocal entries in Table 4.6, where each cell reports how many times the row outcome is better than the column outcome on the "quality" objective.

Notice that each column is not a multiple of each other column. That is, there are inconsistencies among Luz's answers to the questions. For example, the townhouse is 5 times better than the house and the house is 3 times better than the double, but the townhouse is 7, instead of $5 \times 3 = 15$, times better than the double. There are even some rank reversals: the answer to question 4 said that the single is better than the house, but answers to questions 5 and 6 imply that the house should be

TABLE 4.6. Pairwise comparisons on the quality objective

Quality	Townhouse	House	Single	Double
Townhouse	1	5	7	7
House	$\frac{1}{5}$	1	$\frac{1}{3}$	3
Single	$\frac{1}{7}$	3	1	$\frac{1}{2}$
Double	$\frac{1}{7}$	$\frac{1}{3}$	2	1

better than the single. These inconsistencies may be caused by our selection of the numbers used in the pairwise comparisons, or they may be caused by some unknown factor in Luz's preferences.

How do we deal with these inconsistencies? Since the four columns represent independent estimates of Luz's utilities, we will first change the units of each column to be compatible with each other, and then average the four estimates. (Just as if you had measured Luz's height with a meter stick, a yard stick, and a one-foot ruler.) One way to obtain the same units in each column is to make each column sum to 1 (achieved by dividing the numbers in the column by the sum of the numbers in the column), as seen in Table 4.7.

TABLE 4.7. Rescaling to make consistent units

Quality	Townhouse	House	Single	Double
Townhouse	0.135	0.107	0.032	0.261
House	0.096	0.321	0.097	0.043
Single	0.096	0.036	0.194	0.087
Double	0.673	0.536	0.677	0.609

Now we average the rows to smooth out the inconsistencies, as seen in Table 4.8.

TABLE 4.8. Averaged scale

Outcome	Normalized Quality
Townhouse	0.613
House	0.139
Single	0.142
Double	0.106

While a simple average of the four normalized columns gives a pretty good resolution to the inconsistencies, it does not always give an optimal solution since it may ignore information such as one column being more accurate than another. Thomas Saaty developed a computation method that obtains an optimal resolution to the inconsistencies. Using his method, the normalized qualities are 0.623 for the townhouse, 0.130 for the house, 0.139 for the single, and 0.108 for the double. However, since Saaty's method is computationally intensive, we will not include it here. If you are interested, there are software tools available to carry out the calculations.

Web-HIPRE is one free internet-based tool that can be used for this calculation as well as the entire modeling process [**34**]. For now, we will continue to use our approximation above.

Now that we have obtained a ratio scale among the objectives, and ratio scales among the outcomes for each objective as displayed in Table 4.9, we are ready to determine a composite scale for the outcomes.

TABLE 4.9. Summary of ratio scales

Weights	0.20	0.40	0.40
Outcome	Low Cost	Friends	Quality
Townhouse	0.125	0.60	0.613
House	0.375	0.10	0.139
Single	0.175	0.05	0.142
Double	0.325	0.25	0.106

The final preference measurements are obtained by matching up the objective weights with the corresponding outcome measurement, multiplying the matched numbers, and summing the products, as seen in Table 4.10.

TABLE 4.10. The final preference measurements

Outcome	Preference
Townhouse	$(0.20)(0.125) + (0.40)(0.60) + (0.40)(0.613) = 0.510$
House	$(0.20)(0.375) + (0.40)(0.10) + (0.40)(0.139) = 0.171$
Single	$(0.20)(0.175) + (0.40)(0.05) + (0.40)(0.142) = 0.112$
Double	$(0.20)(0.325) + (0.40)(0.25) + (0.40)(0.106) = 0.207$

Based on these numbers, Luz should choose the townhouse. If for some reason, the townhouse becomes unavailable, Luz should choose the double. If both the townhouse and double become unavailable, Luz should choose the house. If nothing else is available, Luz must then choose the single.

In general, measuring preferences among outcomes by a ratio scale requires that it be meaningful for the player to know how many times she prefers one outcome over another outcome. We can obtain the measurements by first measuring the objectives on a ratio scale and then measuring the outcomes on a ratio scale for each objective. Each set of measurements is normalized so that the sum of the measurements is one. The preference measurement for one outcome is then obtained by taking the product of the outcome's weight for an objective with the objective's weight and then summing over all objectives.

We saw four ways to obtain a ratio scale among objectives, or outcomes with respect to a single objective. First, the player may be able to assign weight fractions directly (as Luz did with her objectives). Second, the player may have a ratio scale already available (as Luz did for outcomes with respect to the "low cost" objective). Third, the player may determine the worst outcome and then determine how many times better each of the other outcomes is than the worst one (as Luz did for the outcomes with respect to the "friends" objective). Fourth, the player may make pairwise comparisons for each pair of outcomes and then find the best ratio scale approximation to the pairwise comparisons (as Luz did for the outcomes with respect to the "quality" objective). This last approach is very computationally intensive, and so it is best to use computer software.

To conclude this chapter, we compare the three models for preferences. Ordinal preferences (see page 68) are what a player would choose among outcomes based on a simple ranking. A player can determine his or her ranking through a small number of questions about which outcome would be chosen among a subset of the outcomes. By answering additional questions about which outcome would be chosen among other subsets of the outcomes, a player can even verify whether he or she actually has ordinal preferences. Of course, the relative ease with which ordinal preferences are determined and verified has a trade off in that ordinal preferences are the least informative model of preferences. An ordinal utility is a measurement of a player's ordinal preferences.

Cardinal preferences (see page 81) are a ranking of outcomes and lotteries among outcomes that reflect intensity. Cardinal utilities are a measure of cardinal preferences, and a player can determine his or her cardinal utilities by answering additional questions that relate the original outcomes with lotteries involving two original outcomes. It is this additional work that measures intensities that correspond to actual choices the player would make. Once the cardinal utilities of the original outcomes are determined, the Expected Utility Hypothesis provides an easy way to calculate the cardinal utility for any lottery involving two or more original outcomes. Since there is an infinity of different lotteries, there is no way to fully verify that these calculated utilities always correspond to player choices, that is, that a player truly has cardinal preferences. Thus, the more information provided by cardinal preferences is obtained at the expense of having less certainty in the accuracy of the modeled preferences.

Ratio scale preferences (see page 91) also involve assigning a number to each outcome to model an intensity of preference. However, in contrast to cardinal preferences, certain aspects of ratio scale preferences (e.g., that one outcome is preferred twice as much as another outcome) cannot be verified by any appeal to a player's choices. The ratio scale must be determined by player introspection, which may be facilitated in complicated situations by the Analytic Hierarchy Process.

Exercises

(1) Suppose in **Housing**, Luz provides the pairwise comparisons of the objectives (instead of the weights that were given), as shown in Table 4.11. Find the objectives' weights and ratio scale preferences for the outcomes.

(2) Choose a scenario in which you would have multiple objectives and several possible outcomes. Use the process described in this section to determine your ratio scale utilities for the outcomes.

TABLE 4.11. Different pairwise comparison of objectives

Objectives	Low Cost	Friends	Quality
Low Cost	1	5	10
Friends	$\frac{1}{5}$	1	4
Quality	$\frac{1}{10}$	$\frac{1}{4}$	1

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CHAPTER 3

Strategic Games

1. Tosca

One of the central themes of this book is modeling and analyzing situations involving conflict, and we begin that work in this chapter. Although "conflict" often connotes antagonism among people or nations, we take the broader perspective that there is conflict whenever two or more independent decision makers interact and might have different preferences. Over the course of this chapter and the next, you will encounter many scenarios that involve conflict of the latter kind, will learn to build mathematical models for these, will obtain solutions for the models, and will interpret these solutions back in the context of the original scenarios. Most of the scenarios described in this book will be relatively simple, selected for pedagogical purposes. More complicated scenarios can be readily found; the bibliography lists many such sources. More often, however, you will observe conflict in the world around you: in the movies you watch, in the books you read, in the news you hear, and in the conversations you have. Even exiting an interstate highway as someone is merging onto the highway at the same time is a scenario involving conflict and cooperation; both drivers' decisions about speed and direction determine to what extent each achieves his or her objective.

By the end of this section, you will have thought about a situation of conflict.

This section introduces you to problems of conflict by presenting one such conflict, extracted from Puccini's famous opera, *Tosca*. The following dialogue is a loose interpretation of the second act, taken from John Gutman's 1956 English translation of the opera [**46**].

The scene opens with Scarpia, the evil policeman, and his henchman, Spoletta, conversing in Scarpia's office about Tosca and her lover Cavaradossi. Cavaradossi is awaiting execution in another room.

SCARPIA: I'm sure she'll come out of love for her Cavaradossi. Out of love for Cavaradossi she will surrender to my will. There is no greater suffering, the suffering that love brings. Sweeter far are the raptures of a violent conquest than of willing surrender. A lover sighing and pining in the splendor of the moonlight isn't my fashion! Not for me the poetry of sunsets, of midnight serenades. I never ask a flower: "Does she love me or love me not?" Always I crave for the things that elude me. Once I've had them I can discard them. On to stronger pleasures. God created more than one wine, more than one beauty. I want to taste all I can wring from the Hand of the Maker!

SPOLETTA: Do you desire her so much that you would release Cavaradossi for her?

SCARPIA: Oh, how great would it be if I could have her and be rid of Cavaradossi! But I do desire her more than I desire his death.

SPOLETTA: But what if she does not submit to your passions.

SCARPIA: Then he will be hanged anyway! He is a revolutionary.

Spoletta looks out the window.

SPOLETTA: Tosca is at the gate.

Tosca enters the office.

SCARPIA: Ah! Tosca, my passion's desire! Spoletta, prepare Cavaradossi to meet the hangman.

Spoletta leaves.

TOSCA: Forgive him, please! Be generous.

Returning to the desk, Scarpia sits down; calmly and with a smile.

SCARPIA: I? You! You seem disheartened? Come, my lovely Signora, sit down with me. Perhaps together we can find a way to save your Cavaradossi! Sit down here, let's talk about it over a glass of wine.

Tosca sits opposite Scarpia and stares at him.

TOSCA: How much. . .

SCARPIA: How much?

TOSCA: to bribe you?

SCARPIA: Yes, I know what they say: that I can be bought. But I'm not for sale to lovely ladies for something cheap as money. No, no! No lovely lady can ever buy me with something as cheap as money. If I am asked to break the oath that I swore, I want a higher payment; I want a much higher payment. Tonight's the night I've longed for: since first I saw you, desire has consumed me, but tonight, though you hope to defy me, you can no more deny me. When you cried out despairing, passion inflamed me, and your glances almost drove beyond bearing, the lust to which you've doomed me. How your hatred enhances my resolve to possess you. I may curse or bless you, but you must be mine! Mine!

He approaches Tosca, opening his arms, and Tosca, who has listened to Scarpia horrified, rises suddenly and moves to the window.

TOSCA: Ah! I'd rather take my life first!

SCARPIA: But how about your darling, Cavaradossi?

TOSCA: How do you dare to offer me such a bargain? To yield my virtue for my lover's life!

SCARPIA: You're free to do as you please. You want to leave? You're free to, but Cavaradossi will die.

TOSCA: But without my virtue, would Cavaradossi still love me? I don't care; as long as he's alive. You foul person!

SCARPIA: How you detest me!

TOSCA: I do!

SCARPIA: And that's the way I want you!

He approaches her.

TOSCA: Do not touch me, you murd'rer! How I hate you! You coward, murd'rer!

SCARPIA: Detest me! Passionate in hating, passionate in loving.

TOSCA: Monster!

SCARPIA: Tosca!

TOSCA: I hate you!

SCARPIA: There's not much time left.

TOSCA: Love for beauty, love and compassion, they gave to my art its true inspiration. Sweet consolation I brought to those who are poor and unhappy. Always with deep emotion, my true devotion I poured out to the glory of the Lord. I brought with deep devotion flowers to adorn His house. Despairing of tomorrow, my head is bowed in sorrow. Oh why, my Lord, withdraw Your Hand from me? My worldly treasure I gladly laid on His altar; I'd never falter in singing of His greatness without measure. If love is doomed to die, Oh Lord, my Lord, tell me why? Ah. Lord, why do You withdraw Your Hand from me?

SCARPIA: Your answer?

Kneeling before Scarpia.

TOSCA: On my knees I beg for mercy. See me: like a beggar I lie before you. Hear me, hear me, I'm defeated, I implore you. Show me mercy.

SCARPIA: You're much too lovely, Tosca. Such charming graces! Well then, it seems I can't best you. You win your Cavaradossi; I, a single night's embraces!

Getting up, showing her contempt.

TOSCA: No, no! How I detest you. No, no!

Spoletta enters.

SPOLETTA: Excellency, Cavaradossi is ready for the noose. We shall proceed as you ordered.

SCARPIA: One moment. Tosca, I'm waiting...

Tosca nods "yes", then begins weeping. Scarpia turns to Spoletta.

SCARPIA: Listen,

Interrupting Scarpia.

TOSCA: You promise he'll be free before morning?

SCARPIA: That's more than I can do. I've never granted a pardon! We must make it seem as though he was really put to death. My man here will take good care of that.

TOSCA: How can I trust you?

SCARPIA: I will instruct him right now, in your presence. Spoletta, listen:

Scarpia gives Spoletta a significant glance; Spoletta indicates that he grasps Scarpia's idea.

SCARPIA: I am changing my orders: Instead of hanging Cavaradossi, we will shoot him. However, we shall proceed as we did with Palmieri!

SPOLETTA: An execution?

SCARPIA: ... without bullets! Just the same as with Palmieri! You understand me?

SPOLETTA: I understand you very well!

SCARPIA: All right, that's all. At four o'clock then.

SPOLETTA: Yes, just like Palmieri.

Spoletta leaves.

SCARPIA: You see, I've kept my promise.

TOSCA: Not completely. Give me something in writing so that we can leave the country at once.

SCARPIA: You mean you want to leave us?

TOSCA: Yes, forever!

SCARPIA: Your wishes are my orders.

Goes to his desk and writes. he looks up from his writing.

SCARPIA: Which way will you travel?

TOSCA: By the shortest!

SCARPIA: You mean by water?

TOSCA: Yes.

While Scarpia writes, Tosca walks over to the desk and lifts, with a trembling hand, the glass that Scarpia had filled; while doing so, she sees a knife on the desk; after a furtive glance at Scarpia, who is still writing, she grasps the knife and very cautiously hides it behind her back, leaning against the table and still glancing at Scarpia.

Scarpia has finished writing the safe-conduct note; he stamps it with his seal and folds it; opening his arms he goes towards Tosca to embrace her.

SCARPIA: Tosca! Now at last you're mine!

Tosca stabs him.

SCARPIA: You assassin!

TOSCA: That's the way Tosca kisses!

SCARPIA: I'm dying, help me!

TOSCA: Your own blood will choke you.

SCARPIA: Help me! Help me!

TOSCA: It's Tosca who has killed you. Now you pay for my torture!

SCARPIA: I'm dying! Help me!

Scarpia makes one last effort, then falls on his back and remains motionless.

TOSCA: He's dead. Now I forgive him.

Tosca leaves the room, and a few moments later Spoletta enters the room.

SPOLETTA: Scarpia! We've executed Cavaradossi just as we did Palmeiri. Scarpia?

Exercises

(1) Is Cavaradossi alive or dead?
(2) What choices did Scarpia and Tosca have?
(3) What outcomes would have been possible if Scarpia and Tosca had made different choices?
(4) Describe the preferences Tosca and Scarpia have over the possible outcomes.
(5) How might the preferences of a 21st-century American woman have been different from those of Tosca, an 18th-century Italian woman?

2. Fingers and Matches

In this section, we will continue our study of games by introducing strategic games. In this and following chapters, we will develop a complete theory for such games.

> **Strategic Game:** In a *strategic game*, each player has a set of strategies to choose from, outcomes determined by the strategies selected, and player preferences among the different outcomes. Players choose their strategies simultaneously and privately.

In deterministic games, each player may make many decisions as the game progresses, but there is no secrecy. In strategic games, each player makes a single decision, but that decision is made in secret and in complete ignorance of the decisions of other players. A deterministic game can be thought of as a strategic game in which each player chooses a strategy that is then carried out by a surrogate. However, the outcome of a strategic game is typically more complex than a simple win or loss.

> By the end of this section, you will be able to model simple real-world scenarios as strategic games.

Fingers

Let's consider a scenario. In **Fingers**, Rose and Colin simultaneously show one or two fingers. If each shows one finger, both players pay \$1 to the bank. If each shows two fingers, both players receive \$1 from the bank. If they show differently, then the player showing one finger receives \$4 from the bank and the player showing two fingers neither pays nor receives any money.

Since the choice of how many fingers to show is made without any knowledge of the other player's choice, this scenario can be modeled as a strategic game. The strategies can be simply described as the number of fingers to be held up. The outcomes of this game are summarized in Table 2.1.

TABLE 2.1. Summary of **Fingers** outcomes

Rose's Strategy	Colin's Strategy	Outcome
Show one finger	Show one finger	Each pays \$1
Show one finger	Show two fingers	Rose receives \$4
Show two fingers	Show one finger	Colin receives \$4
Show two fingers	Show two fingers	Each receives \$1

While the arrangement in Table 2.1 is easy to read, and can be readily adapted for more than two players, we will usually arrange the same information in the following manner:

Fingers Outcomes		Colin Show one finger	Show two fingers
Rose	Show one finger	Each pays \$1	Rose receives \$4
	Show two fingers	Colin receives \$4	Each receives \$1

Each row of this outcome matrix corresponds to a strategy available to Rose: show one finger or show two fingers. Similarly, each column of this matrix corresponds to a strategy available to Colin. Each cell is a description of the outcome when the two players choose corresponding strategies.

Arranging the relevant information in the second format clarifies what strategies are available to each player (i.e., the rows for Rose and the columns for Colin) and all of the possible combinations of choices (i.e., the cells inside the matrix). As we will see in the next section, this format also makes the determination of solutions relatively easy.

The final step in developing the strategic game model for **Fingers** is to determine player preferences. For each player, we need to assign ordinal or cardinal utilities to each outcome; in the context of games, these utilities are usually called payoffs. If we assume that each player is primarily interested in receiving individually as much money as possible, then it is easy to obtain an ordinal ranking of the outcomes. Rose would rank the outcomes (1) Rose receives \$4, (2) each receives \$1, (3) Colin receives \$4, and (4) each pays \$1. Colin would rank the outcomes (1) Colin receives \$4, (2) each receives \$1, (3) Rose receives \$4, and (4) each pays \$1. We can summarize this information in an ordinal payoff matrix:

Fingers Ordinal Payoffs (Rose, Colin)		Colin Show one finger	Show two fingers
Rose	Show one finger	$(1,1)$	$(4,2)$
	Show two fingers	$(2,4)$	$(3,3)$

The ordered pair $(4,2)$ means that Rose's ordinal payoff is 4 (the corresponding outcome is ranked first by Rose) and Colin's ordinal payoff is 2 (the corresponding outcome is ranked third by Colin). By convention, the first entry in any ordered pair is the payoff to the row player and the second entry is the payoff to the column player.

To assign cardinal payoffs to Rose, we first arbitrarily assign

$$\begin{aligned} u(\text{Each pays \$1}) &= 0 \\ u(\text{Rose receives \$4}) &= 10. \end{aligned}$$

Second, Rose determines a probability p so that she would be willing to choose either of the following outcomes

$$\boxed{\text{Colin receives \$4}} \quad \text{vs.} \quad \boxed{\begin{array}{l} \text{With probability } 1-p\text{, each pays \$1;} \\ \text{with probability } p\text{, Rose receives \$4} \end{array}}$$

and then assign

$$u(\text{Colin receives \$4}) = 10p.$$

Third, Rose determines a probability q so that she would be willing to chose either of the following outcomes

Each receives \$1	vs.	With probability $1-q$, each pays \$1; with probability q, Rose receives \$4

and then assign

$$u(\text{Each receives \$1}) \quad = \quad 10q.$$

Note that the expected monetary value to Rose of the lottery

$$[0.8 \cdot (\text{Each pays \$1}) + 0.2 \cdot (\text{Rose receives \$4})]$$

is the amount

$$(0.8)(-\$1) + (0.2)(\$4) = \$0,$$

and the expected monetary value to Rose of the lottery

$$[0.6 \cdot (\text{Each pays \$1}) + 0.4 \cdot (\text{Rose receives \$4})]$$

is the amount

$$(0.6)(-\$1) + (0.4)(\$4) = \$1.$$

So, if Rose is risk neutral, we should assign

$$\begin{aligned} u(\text{Colin receives \$4}) &= 2 \\ u(\text{Each receives \$1}) &= 4. \end{aligned}$$

If both Rose and Colin are only interested in their individual monetary gain and are risk neutral, then the cardinal payoff matrix would be the following.

Risk-Neutral Fingers Cardinal Payoffs		Colin Show one finger	 Show two fingers
Rose	Show one finger	$(0, 0)$	$(10, 2)$
	Show two fingers	$(2, 10)$	$(4, 4)$

If Rose were risk loving, the assignment would satisfy $u(\text{Colin receives \$4}) < 2$ and $u(\text{Each receives \$1}) < 4$. If Colin were risk adverse, the assignment would satisfy $u(\text{Rose receives \$4}) > 2$ and $u(\text{Each receives \$1}) > 4$. In this case, the cardinal payoff matrix might be the following.

Risk-Varying Fingers Cardinal Payoffs		Colin Show one finger	 Show two fingers
Rose	Show one finger	$(0, 0)$	$(10, 5)$
	Show two fingers	$(1, 10)$	$(3, 8)$

The above outcome matrix, together with any one of the three payoff matrices, forms a strategic game model for the **Fingers** scenario. We name these three strategic games **Fingers**, **Risk-Neutral Fingers**, and **Risk-Varying Fingers**.

Outcome and Payoff Matrices: The *outcome matrix, ordinal payoff matrix,* and *cardinal payoff matrix* for a strategic game are displays that show all of the strategies available to each player and the outcomes, ordinal payoffs, and cardinal payoffs, respectively, that result from each combination of strategies.

When there are two players, the display is typically a matrix or table with one player's strategies making up the rows and the other player's strategies making up the columns. In each cell of a payoff matrix, the payoff to the row player is written first and the payoff to the column player is written second. We will often refer to our two players as Rose and Colin to emphasize the distinction between the row and column players. This convention was popularized by Phil Straffin in his book *Game Theory and Strategy* [**64**].

We have only completed the first step of the mathematical modeling process (see the "Mathematical Modeling Process" figure on page 69): create the model by making assumptions about the real-world scenario. In section 3.3, we will explore some mathematical techniques to obtain results that can be interpreted in the real-world context. To verify those results, there needs to be experimental data for comparison. Therefore, we encourage you to experiment.

> This would be an excellent time for the reader to find a partner and play **Fingers**: decide who is the row player and who is the column player; individually decide on what strategy you should play to maximize your payoff as given in the ordinal payoff matrix; and reveal your choices to each other to determine the actual payoffs. What happened?

Matches

Consider a new scenario that we will call **Matches**. Rose and Colin just met and had a pleasant conversation at a local coffee house. Just before they depart, Rose says, "Hope to see you at the match tomorrow," and Colin responds, "It's a date!" As each heads off for "the match" the next day, each suddenly realizes that there are actually two big matches scheduled that day: tennis and soccer. From their conversation, each knows that Rose likes tennis and Colin likes soccer. Unfortunately, neither one had gotten the other's last name or telephone number, and the two games are being held at fields about one hour apart.

We can model this scenario as a strategic game since Rose and Colin will certainly make their choices without the other's knowledge; that is, we can assume that these choices are private and simultaneous. Letting Rose be the row player, we can organize the following outcome matrix.

Matches Outcomes		Colin: Go to the tennis match	Go to the soccer match
Rose	Go to the tennis match	The couple is together at tennis	Rose at tennis and Colin at soccer
	Go to the soccer match	Rose at soccer and Colin at tennis	The couple is together at soccer

Since both Rose and Colin would primarily prefer to be with each other and secondarily would like to view the sport each one liked better, the ordinal payoff matrix, with the strategies replaced with descriptive names, is as follows.

Matches Ordinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(4,3)$	$(2,2)$
	SOCCER	$(1,1)$	$(3,4)$

To assign cardinal payoffs, we need to know more about how strongly each prefers one outcome over another. Suppose Rose's primary interest in being with Colin is far greater than her secondary interest in being at the tennis match, but Colin's primary interest in being with Rose is only somewhat greater than his secondary interest in being at the soccer match. Then the cardinal payoff matrix might be as shown below.

Matches Cardinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(10,6)$	$(2,5)$
	SOCCER	$(0,0)$	$(9,10)$

More formally, we are assuming that Rose is indifferent between the following two outcomes

Rose at tennis and Colin at soccer	vs.	With probability 0.80, Rose at soccer and Colin at tennis; with probability 0.20, the couple is together at tennis

and indifferent between the following two outcomes

The couple is together at soccer	vs.	With probability 0.10, Rose at soccer and Colin at tennis; with probability 0.90, the couple is together at tennis

We are also assuming that Colin is indifferent between the following two outcomes:

Rose at tennis and Colin at soccer	vs.	With probability 0.50, Rose at soccer and Colin at tennis; with probability 0.50, the couple is together at soccer

and indifferent between the following two outcomes

The couple is together at tennis	vs.	With probability 0.40, Rose at soccer and Colin at tennis; with probability 0.60, the couple is together at soccer

These last two assumptions about Rose's and Colin's cardinal preferences could be verified by asking them.

Again, we have only completed the first step of the mathematical modeling process; we will complete the rest of the process in section 3.3. For now, we encourage you to obtain some experimental data.

This would be a good time for the reader to find a partner and play **Matches**: decide who is the row player and who is the column player; individually decide on what strategy you should play to maximize your payoff using the cardinal payoff matrix above; and reveal your choices to each other to determine the actual payoffs. What happened?

Exercises

(1) **Chicken.** On a country road closed to general traffic, two cars are racing on a collision course toward each other. The loser—the chicken—is the first to turn aside. (But is that really worse than neither one turning aside?)
 (a) Construct an outcome matrix.
 (b) Suppose a crash is worse than being a chicken. Construct an ordinal payoff matrix.
 (c) Construct a cardinal payoff matrix consistent with the ordinal matrix constructed in part (b). Describe the additional assumptions you made.

(2) **War**. Romula plans to attack Canuka. Romula has enough armaments to destroy only one of the following: a military base, school, or hospital (listed from most to least valuable for Romula to destroy). Canuka has enough resources to defend only two of the three possible attack sites. An attack by Romula on a site defended by Canuka will be unsuccessful and result in some losses to Romula. Canuka values the hospital the most and the school the least.
 (a) Construct an outcome matrix.
 (b) Construct an ordinal payoff matrix, recalling that some outcomes can be tied in rank.
 (c) Construct a cardinal payoff matrix consistent with the ordinal matrix constructed in part (b). Describe the additional assumptions you made.

(3) **Three-Player Fingers**. Rose, Colin, and Larry simultaneously show one or two fingers. If all show two fingers, each receives \$6 from the bank, and if all show one finger, each receives \$3 from the bank. If two players show two fingers each and the other player shows one finger, the two matching players each receive \$5 from the bank and the nonmatching player receives \$15. If two players show one finger each and the other player shows two fingers, the two matching players each receive \$4 from the bank and the nonmatching player receives \$15.
 (a) Construct an outcome matrix. Hint: Create a different Rose-Colin matrix for each of Larry's strategy choices, or use a single table similar to the first table we presented for **Fingers**.
 (b) Suppose the three players care only about the money they receive individually and are risk neutral. Construct a cardinal payoff matrix.
 (c) Suppose Rose and Colin, prior to playing the game, agreed to split their winnings evenly. Also suppose the three players care only about the money they receive individually, Rose and Colin are risk neutral, and Larry is risk adverse. Construct a cardinal payoff matrix.

(4) **Cigarette Tax** [**63**]. The State of Indiana is considering increasing its tax on cigarettes by \$0.50 per pack as a method to encourage smokers to stop doing so. Assume that cigarette smokers prefer smoking over not (else they would have quit already!)
   (a) Construct an outcome matrix and an ordinal payoff matrix.
   (b) Construct a cardinal payoff matrix consistent with the ordinal matrix constructed in part (a). Describe the additional assumptions you made.
   (c) How might your answers to the previous questions change if multiple tax increase levels were possible?

(5) **Two Brothers**. A family traveling on vacation has three small boys, aged 10, 9, and 4. For the evening, they rent a single hotel room with two large beds; the parents will be using one of them. The two older boys know that if they both rush to the other bed, they will be fussing about and getting in trouble all night. If only one rushes to the bed, he will share the bed with the youngest brother, while the other brother sleeps on the floor, which is much less comfortable (i.e., it is better to sleep on the bed with the youngest brother than to sleep on the floor by oneself). They both could decide to sleep on the floor, in which case they know that they can whisper and tease their younger brother all night.
   (a) Construct an outcome matrix and an ordinal payoff matrix.
   (b) Construct a cardinal payoff matrix consistent with the ordinal matrix constructed in part a. Describe the additional assumptions you made.

(6) **WWII Battle** [**75**, page 456]. Several months after the Normandy invasion in 1944, the Allied Third Army broke out of the beachhead, creating a gap between it and the Allied First Army, giving the German commander a choice between two strategies: attack the gap, or retreat toward the east. The Allies had three choices about how to deploy their reserve units: use them to reinforce the gap, send them to the east with the Third Army in an attempt to surround the German army, or hold off on the decision for 24 hours. Both, one, or neither of two possible battles will result: the Germans attacking the Allies in the gap, and the Allies attacking the Germans in the east. If the Germans attack a reinforced gap, they will lose; if the Germans attack the gap while the Allies have moved their reserves to the east, the Allies lose both battles; if the Germans attack and the Allies have held their reserves, the Allies win both battles; and finally, if the Germans retreat and the Allies have moved their reserves to the east, the Allies win the battle in the east.
   (a) Construct an outcome matrix.
   (b) Construct an cardinal payoff matrix for this game, in which the payoffs are the number of battles won minus the number of battles lost.

(7) **River Tale** [**71**, page 50]. Steve is approached by a stranger who suggests they match coins. Steve says that it's too hot for violent exercise. The stranger says, "well then, let's just lie here and speak the words 'heads' or 'tails'—and to make it interesting I'll give you \$30 when I call 'tails' and you call 'heads', and \$10 when it's the other way around. And—just to make it fair—you give me \$20 when we match."
   (a) Construct an outcome matrix for this game
   (b) If Steve and the Stranger are risk neutral, what would the cardinal payoff matrix be?

(c) If Steve is risk adverse and the Stranger is risk loving, what would be an appropriate cardinal payoff matrix?

(8) **Mating** [**64**, page 100]. A female tries to get a male to stay around and help raise a family of babies instead of going off and propagating his genes elsewhere. One possible technique for doing this is to insist on a long and arduous courtship before mating. Suppose a female can be either Coy (insisting on courtship) or Fast (willing to mate with anyone), and a male can be either Faithful (willing to go through a courtship and then help raise the babies) or Philandering (unwilling to go through a courtship and deserting any female after mating). Suppose the payoff to each parent of babies is $+15$, and the total cost of raising babies is -20, which can be split equally between both parents, or fall entirely on the female if the male deserts. Suppose the cost of a long courtship is -3 to each player.

(a) Construct an outcome matrix.
(b) Suppose payoffs are additive. Construct a cardinal payoff matrix.
(c) Why should courtship be considered to be worse than no courtship?
(d) Why should a male value having babies the same whether adopting a faithful or philandering strategy?

(9) **Bonus Points**. Each student in your class will secretly write either "1" or "2" and her or his name on a slip of paper. Pairs of paper slips will be drawn randomly. If both slips have "2", then both students will receive 10 bonus points. If both slips have "1", then both students will have 10 points deducted. If the slips have different numbers, then the student having written the "1" will receive 30 bonus points.

(a) Construct an outcome matrix for this game
(b) If you were risk loving, but the rest of the class was risk neutral, how might you construct the cardinal payoff matrix for this game?

(10) **Soccer Penalty Kicks** [**43**]. In professional soccer essentially no time passes between a penalty kicker's kick to the right or left of the goal and the goalie's decision on which corner to defend. These decisions can be assumed to be made simultaneously.

(a) Construct an outcome matrix and an ordinal payoff matrix for this game.
(b) Assume that the goalie knows that the kicker tends to kick to the right 52% of the time and that the kicker knows that the goalie can move to his right slightly better than to his left. Construct a cardinal payoff matrix for this game. Describe any additional assumptions that you are making.

(11) **Steroid Use** [**58**]. Athletes of all ages and levels are tempted to use steroids for better performance, and each player must choose whether to use steroids. Although steroids are medically unsafe in this context and an athlete would be penalized in some way if caught using steroids, using steroids provides an athlete with a significant advantage.

(a) Construct an outcome matrix in which one player is a single athlete and the other player is the "group" of other athletes.
(b) Construct a cardinal payoff matrix for your outcome matrix. Describe any additional assumptions you made.

(12) **Bush and Kerry** [**16**]. Near the end of the 2004 presidential campaign, candidates Bush and Kerry had to choose which of the swing states of Pennsylvania, Ohio, and Florida to visit. Bush needed to win two of the three

states to win the election. If they both visited the same state, there would be no impact on the outcome, but if they visited different states, the visitor would increase his chance of winning the state by 10%. At the time, Bush had a 20% chance of winning Pennsylvania, a 60% chance of winning Ohio, and an 80% chance of winning Florida.

(a) Construct an outcome matrix.

(b) Construct a cardinal payoff matrix consistent with the outcome matrix of part (a). Hint: Let the payoffs be the respective probabilities that Bush and Kerry win the election.

3. Four Solution Concepts

Now that we know how to model scenarios as strategic games, the next step is to develop techniques for obtaining results for these models; that is, how should one play a strategic game?

By the end of this section, you will be able to (1) determine prudential strategies, security levels, dominated strategies, undominated strategies, Nash equilibria, and efficient strategy pairs, and (2) interpret these solution concepts in the context of simple real-world scenarios.

We will illustrate four solution concepts with two games. **Gift Giving** is based on the following scenario: Rose and Colin are thinking about the anniversary gifts they will give to each other. To simplify matters, we will assume that each player has narrowed her or his choice down to two possibilities: EXPENSIVE (buy an expensive gift) or FRUGAL (buy an inexpensive gift). This results in the ordinal payoff matrix shown here.

Gift Giving Ordinal Payoffs		Colin EXPENSIVE	FRUGAL
Rose	EXPENSIVE	$(2, 3)$	$(4, 1)$
	FRUGAL	$(1, 2)$	$(3, 4)$

The ordinal payoffs deserve some explanation. Rose is altruistic (or some might say masochistic) in that her primary motivation is to receive as little as possible (which is why her payoffs in the FRUGAL column, which corresponds to Rose receiving little, are the highest) and her secondary motivation is to give as much as possible (which is why her payoffs in the EXPENSIVE row are larger than her corresponding payoffs in the FRUGAL row). Colin's primary motivation is equity (which is why his payoffs for the (EXPENSIVE, EXPENSIVE) and (FRUGAL, FRUGAL) strategy pairs are highest) and his secondary motivation is to receive as little as possible (which is why (FRUGAL, FRUGAL) has a higher payoff than (EXPENSIVE, EXPENSIVE), and (FRUGAL, EXPENSIVE) has a higher payoff than (EXPENSIVE, FRUGAL)).

This would be an excellent time for the reader to find a partner and play the **Gift Giving** game above. Decide who will roleplay as Rose and who will roleplay as Colin, decide individually on what strategy you should play to maximize your payoff as given in the ordinal payoff matrix, and reveal your choices to determine the actual payoffs. What happened?

For **Ad Hoc**, we do not provide a motivating scenario. Instead we simply provide the payoff matrix.

Ad Hoc		Colin			
Ordinal Payoffs		A	B	C	D
	A	$(3,6)$	$(2,8)$	$(3,2)$	$(9,5)$
Rose	B	$(6,7)$	$(8,3)$	$(1,6)$	$(5,7)$
	C	$(4,1)$	$(7,5)$	$(4,4)$	$(4,9)$

Since our focus in this section is on mathematical techniques for solving strategic game models, we do not need to know the scenario modeled by the game (or even if there was such a scenario). While we will also discuss possible interpretations for these solution concepts, these interpretations will be primarily within the mathematical world of strategic game models.

This would be an excellent time for the reader to find a partner and play the **Ad Hoc** game above. Decide who will roleplay as Rose and who will roleplay as Colin, decide individually on what strategy you should play to maximize your payoff as given in the ordinal payoff matrix, and reveal your choices to determine the actual payoffs. What happened?

Prudential Strategy

One reasonable way to play a game is to choose a strategy that gives the highest guaranteed payoff. This is a conservative approach that does not take into account what the other players will actually do, and so it involves a minimal level of strategic risk.

> **Prudential Strategy:** A *prudential strategy* for a player is a strategy, from among the player's available strategies, that will maximize the minimum payoff the player could receive when choosing that strategy.

> **Security Level:** The minimum payoff resulting from playing a prudential strategy is the player's *security level*.

To identify a prudential strategy, Rose identifies her smallest payoff for each of her strategies. She then picks the strategy that has the biggest payoff among these. In **Gift Giving** (see the "Rose Minimum" column in the ordinal payoff matrix below), Rose's smallest payoff for EXPENSIVE is 2, which is the smaller of 2 and 4; Rose's smallest payoff for FRUGAL is 1, which is the smaller of 1 and 3. Since her minimum payoffs for EXPENSIVE and FRUGAL are 2 and 1, respectively, Rose's prudential strategy is EXPENSIVE, and her security level is 2. By choosing EXPENSIVE, Rose guarantees that she will receive a payoff of at least 2. Colin follows the same procedure (see the "Colin Minimum" row in the matrix below) to identify his prudential strategy EXPENSIVE and security level 2.

Gift Giving Ordinal Payoffs		Colin EXPENSIVE	FRUGAL	Rose Minimum
Rose	EXPENSIVE	$(2,3)$	$(4,1)$	$\boxed{2}$
	FRUGAL	$(1,2)$	$(3,4)$	1
Colin	Minimum	$\boxed{2}$	1	

The same computations can be performed for **Ad Hoc** to determine that Rose's prudential strategy is ROW C and her security level is 4, while Colin's prudential strategy is COLUMN D and his security level is 5.

Ad Hoc Ordinal Payoffs		Colin A	B	C	D	Rose Minimum
	A	$(3,6)$	$(2,8)$	$(3,2)$	$(9,5)$	2
Rose	B	$(6,7)$	$(8,3)$	$(1,6)$	$(5,7)$	1
	C	$(4,1)$	$(7,5)$	$(4,4)$	$(4,9)$	$\boxed{4}$
Colin	Minimum	1	3	2	$\boxed{5}$	

If each player plays prudentially, the payoff is $(4,9)$. Notice that while Rose receives her security level, Colin actually does better than his security level.

Why should Rose settle for a payoff of 4? Suppose that she knows (we have never prohibited cheating!) that Colin is going to play his prudential strategy, COLUMN D. Then she should play her strategy ROW A for her highest payoff of 9. But if Colin knows (he can cheat too!) that Rose thinks that he is going to play his prudential strategy and that she is going to take advantage of this and choose ROW A, then he should play strategy COLUMN B and increase his payoff from 5 to 8. If Rose knew that this was Colin's thinking process and he would choose COLUMN B, she should respond by choosing ROW B, increasing her payoff from 2 to 8. But if Colin knew all of this, ...

Dominance

While choosing a prudential strategy maximizes a player's guaranteed payoff, we see that it may lead to a lot of second guessing about whether he or she could reasonably do better. We would like to arrive at a solution to a game that does not leave players guessing as to whether they could have done better. We would like stable strategy choices. For the moment, however, let's try to successively eliminate poor (definitely unstable) strategies.

Dominating Strategy: Strategy X *dominates* strategy Y if the payoff associated with playing strategy X is larger than the payoff associated with playing strategy Y in every situation. Strategy X is said to be *dominating* and strategy Y is said to be *dominated*.

In **Ad Hoc**, COLUMN D dominates COLUMN C because (with the relevant payoffs boxed in the payoff matrix):

Ad Hoc		Colin			
Ordinal Payoffs		A	B	C	D
	A	$(3,6)$	$(2,8)$	$(3,\boxed{2})$	$(9,\boxed{5})$
Rose	B	$(6,7)$	$(8,3)$	$(1,\boxed{6})$	$(5,\boxed{7})$
	C	$(4,1)$	$(7,5)$	$(4,\boxed{4})$	$(4,\boxed{9})$

- if Rose chooses ROW A, Colin is better off choosing COLUMN D with a payoff of 5 than choosing COLUMN C with a payoff of 2;
- if Rose chooses ROW B, Colin is better off choosing COLUMN D with a payoff of 7 than choosing COLUMN C with a payoff of 6; and
- if Rose chooses ROW C, Colin is better off choosing COLUMN D with a payoff of 9 than choosing COLUMN C with a payoff of 4.

Note that if Rose chooses ROW A, it is irrelevant that Colin would be even better off by choosing COLUMN B; we are only comparing COLUMN C to COLUMN D.

Again in **Ad Hoc**, we may compare Rose's strategies ROW B and ROW C (with the relevant payoffs boxed in the payoff matrix).

Ad Hoc		Colin			
Ordinal Payoffs		A	B	C	D
	A	$(3,6)$	$(2,8)$	$(3,2)$	$(9,5)$
Rose	B	$(\boxed{6},7)$	$(\boxed{8},3)$	$(\boxed{1},6)$	$(\boxed{5},7)$
	C	$(\boxed{4},1)$	$(\boxed{7},5)$	$(\boxed{4},4)$	$(\boxed{4},9)$

If Colin chooses COLUMN A, Rose is better off choosing ROW B with a payoff of 6 than choosing ROW C with a payoff of 4. So ROW C definitely does not dominate ROW B, but more checking must be done to determine whether ROW B dominates ROW C. If Colin chooses COLUMN C, Rose is better off choosing ROW C with a payoff of 4 than choosing ROW B with a payoff of 1. So ROW B does not dominate ROW C. In summary, neither ROW B nor ROW C dominates the other.

We have shown that COLUMN C is dominated. It turns out that no other strategy for Rose or Colin is dominated (this involves checking two additional pairs of Rose's strategies and five additional pairs of Colin's strategies). So we have only been able to eliminate one strategy, COLUMN C, from consideration. But if we eliminate COLUMN C from consideration, it makes sense to look at the resulting reduced game.

Reduced Ad Hoc		Colin		
Ordinal Payoffs		A	B	D
	A	$(3,6)$	$(2,8)$	$(9,5)$
Rose	B	$(6,7)$	$(8,3)$	$(5,7)$
	C	$(4,1)$	$(7,5)$	$(4,9)$

In **Reduced Ad Hoc**, ROW B dominates ROW C, and so it now seems reasonable to eliminate ROW C to obtain the following game.

Twice Reduced Ad Hoc Ordinal Payoffs		Colin A	Colin B	Colin D
Rose	A	$(3,6)$	$(2,8)$	$(9,5)$
	B	$(6,7)$	$(8,3)$	$(5,7)$

In **Twice Reduced Ad Hoc**, no strategy is dominated. So we cannot eliminate any more strategies from consideration.

> **Undominated Strategy:** An *undominated strategy* for a player is one that is not eliminated by the successive elimination of dominated strategies process.

In **Ad Hoc**, the undominated strategies are ROW A, ROW B, COLUMN A, COLUMN B, and COLUMN D. The successive elimination of dominated strategies process reduced the number of strategies for each player to consider, but it did not recommend a single strategy for each player. However, this process sometimes does result in a single strategy remaining for each player.

Consider **Gift Giving**. Neither of Colin's strategies dominates the other because he prefers to choose whatever strategy Rose has chosen. However, Rose's EXPENSIVE strategy dominates her FRUGAL strategy because the payoffs associated with her EXPENSIVE strategy (2 and 4) are larger than her corresponding payoffs associated with her FRUGAL strategy (1 and 3) no matter which strategy Colin chooses (EXPENSIVE and FRUGAL, respectively). This suggests that Rose should eliminate her FRUGAL strategy from consideration, which reduces the game matrix to the following.

Reduced Gift Giving Ordinal Payoffs		Colin EXPENSIVE	Colin FRUGAL
Rose	EXPENSIVE	$(2,3)$	$(4,1)$

In **Reduced Gift Giving**, Colin's EXPENSIVE strategy dominates his FRUGAL strategy, which leads to another reduction.

Twice Reduced Gift Giving Ordinal Payoffs		Colin EXPENSIVE
Rose	EXPENSIVE	$(2,3)$

There is a unique undominated strategy for each player: EXPENSIVE.

Nash Equilibrium

Choosing a prudential strategy may cause a player regret once the other player's strategy choice is known, in the sense that the player realizes a better payoff could have been obtained. Eliminating dominated strategies reduces the number of reasonable strategy choices, but it may not result in a unique strategy recommendation. Can we find strategy choices that neither player regrets? This "regret free" notion of stability was studied extensively by John Nash in the 1950s and was made famous in the bar scene of the movie *A Beautiful Mind*.

Nash Equilibrium: A *Nash equilibrium* is an ordered set of strategy choices, one for each player, for which no player can improve her or his own payoff by unilaterally changing her or his strategy .

In **Ad Hoc**, (ROW B, COLUMN A) is a Nash equilibrium because (1) a unilateral change by Rose from her ROW B strategy would reduce her payoff from 6 to 3 or 4 (see the boxes in the following payoff matrix),

Ad Hoc Ordinal Payoffs		Colin A	Colin B	Colin C	Colin D
	A	($\boxed{3}$, 6)	(2, 8)	(3, 2)	(9, 5)
Rose	B	($\boxed{6}$, 7)	(8, 3)	(1, 6)	(5, 7)
	C	($\boxed{4}$, 1)	(7, 5)	(4, 4)	(4, 9)

and (2) a unilateral change by Colin from his COLUMN A strategy does not increase his payoff above the 7 he is currently receiving (see the boxes in the following payoff matrix).

Ad Hoc Ordinal Payoffs		Colin A	Colin B	Colin C	Colin D
	A	(3, 6)	(2, 8)	(3, 2)	(9, 5)
Rose	B	(6, $\boxed{7}$)	(8, $\boxed{3}$)	(1, $\boxed{6}$)	(5, $\boxed{7}$)
	C	(4, 1)	(7, 5)	(4, 4)	(4, 9)

The strategy pair (ROW A, COLUMN D) is not a Nash equilibrium in **Ad Hoc**, because Colin can unilaterally change his strategy to COLUMN B and thereby obtain a payoff of 8 instead of 5. It does not matter that Rose cannot unilaterally increase her payoff. An ordered set of strategies is not a Nash equilibrium if any player can unilaterally obtain a higher payoff.

Similarly, in **Gift Giving**, the strategy pair (EXPENSIVE, EXPENSIVE) is a Nash equilibrium because (1) a unilateral change by Rose to her FRUGAL strategy reduces her payoff from 2 to 1, and (2) a unilateral change by Colin to his FRUGAL strategy reduces his payoff from 3 to 1. The strategy pair (FRUGAL, FRUGAL) is not a Nash equilibrium because a unilateral change by Rose to her EXPENSIVE strategy increases her payoff from 3 to 4.

In a Nash equilibrium, no one can unilaterally improve her or his payoff. Equivalently, each player is choosing a strategy that maximizes her or his payoff given the strategy choices of the other players. This notion of maximizing your payoff for each strategy choice of the other player can be captured visually.

Best Response Diagram: The *best response diagram* is constructed in the following manner. For each strategy choice of the other player, a player puts a box around each maximum payoff in response to that strategy. If a cell has both payoffs boxed, the corresponding strategy pair is a Nash equilibrium.

The best response diagram for **Ad Hoc** is shown in the matrix below. Notice that in response to Rose choosing ROW B, Colin has two best responses, COLUMN A and COLUMN D, each resulting in a payoff of 7.

Ad Hoc Ordinal Payoffs		Colin A	Colin B	Colin C	Colin D
	A	$(3, 6)$	$(2, \boxed{8})$	$(3, 2)$	$(\boxed{9}, 5)$
Rose	B	$(\boxed{6}, \boxed{7})$	$(\boxed{8}, 3)$	$(1, 6)$	$(5, \boxed{7})$
	C	$(4, 1)$	$(7, 5)$	$(\boxed{4}, 4)$	$(4, \boxed{9})$

Only one payoff pair has both numbers boxed, and so (ROW B, COLUMN A) is the only Nash equilibrium for **Ad Hoc**. If Rose chooses ROW B and Colin chooses COLUMN A, neither regrets her or his choice, because a unilateral change in strategy choice would not increase either player's payoff. The same thing cannot be said about any other strategy pair.

The best response diagram for **Gift Giving** is shown in the following payoff matrix.

Gift Giving Ordinal Payoffs		Colin EXPENSIVE	Colin FRUGAL
Rose	EXPENSIVE	$(\boxed{2}, \boxed{3})$	$(\boxed{4}, 1)$
	FRUGAL	$(1, 2)$	$(3, \boxed{4})$

There is a unique Nash equilibrium: (EXPENSIVE, EXPENSIVE). This result predicts that Rose and Colin will buy each other expensive gifts.

Even if we change the model of our **Gift Giving** scenario to allow preplay communication, they should both still buy expensive gifts: if Rose and Colin were to tell each other that they planned to buy an expensive gift for the other, neither would have an incentive to change her or his choice. Further, if Colin were to tell Rose that he planned to be frugal and buy her an inexpensive gift, Rose would still plan to buy Colin an expensive gift, and knowing this, Colin would want to buy Rose an expensive gift.

To change the result requires changing the players' preferences in the model. For example, if Colin valued frugality above all else and knew he could be more frugal than Rose, his ranking of the (EXPENSIVE, EXPENSIVE) and (EXPENSIVE, FRUGAL) outcomes would switch, resulting in the following payoff matrix with best response payoffs boxed.

Frugal Gift Giving Ordinal Payoffs		Colin EXPENSIVE	Colin FRUGAL
Rose	EXPENSIVE	$(\boxed{2}, 1)$	$(\boxed{4}, \boxed{3})$
	FRUGAL	$(1, 2)$	$(3, \boxed{4})$

Now the unique Nash equilibrium is (EXPENSIVE, FRUGAL), which predicts Rose will purchase an expensive gift for Colin but Colin will purchase an inexpensive gift for Rose.

When the strategic game has a unique Nash equilibrium, it is reasonable to predict that the players will identify it and choose the recommended strategies. Unfortunately, a game may have more than one Nash equilibrium. This raises the question of which Nash equilibrium should be chosen. For example, here is the best response diagram for **Matches**.

Matches Ordinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(\boxed{4}, \boxed{3})$	$(2, 2)$
	SOCCER	$(1, 1)$	$(\boxed{3}, \boxed{4})$

Clearly, (TENNIS, TENNIS) and (SOCCER, SOCCER) are the two Nash equilibria. Which one should be chosen? To this question, Nash is silent. This lack of a more precise recommendation points out the inherent tension between the two strategy pairs. An interpretation of this result back into the real-world scenario is that there is an inherent indeterminacy in the scenario. Rose and Colin may make "incorrect" choices unintentionally, and each will try to change the scenario to bring about a more strategically certain outcome (e.g., by calling someone who might be a mutual friend to obtain the other person's telephone number).

Efficient Strategy Pairs

Again, here is the best response diagram for **Gift Giving**,

Gift Giving Ordinal Payoffs		Colin EXPENSIVE	FRUGAL
Rose	EXPENSIVE	$(\boxed{2}, \boxed{3})$	$(\boxed{4}, 1)$
	FRUGAL	$(1, 2)$	$(3, \boxed{4})$

which reminds us that (EXPENSIVE, EXPENSIVE) is the unique Nash equilibrium. We have also seen that EXPENSIVE is the undominated and prudential strategy for each player. All three solution concepts recommend (EXPENSIVE, EXPENSIVE) resulting in the payoff pair $(2, 3)$.

There is a serious problem here: if Rose and Colin would only both choose FRUGAL, both would increase their payoffs! This problem suggests a new solution concept.

> **Efficient Strategy Pair:** An *efficient strategy pair* is an ordered set of strategy choices, one for each player, for which, by changing their strategies together, one player can increase his or her payoff only by decreasing another player's payoff.

(EXPENSIVE, EXPENSIVE) and (FRUGAL, EXPENSIVE) are not efficient strategy pairs because by changing their strategies together to (FRUGAL, FRUGAL), both players increase their payoff. (FRUGAL, FRUGAL) is an efficient strategy pair because making any change to their strategy choices will reduce Colin's payoff from 4. (EXPENSIVE, FRUGAL) is also an efficient strategy pair because making any change to their strategy choices will reduce Rose's payoff from 4.

An efficient strategy pair embodies an optimistic attitude: the players, even acting together, cannot further improve everyone's payoff. However, efficient strategy pairs serve more as aspirations rather than good predictors of behavior. If Rose thinks that Colin will choose FRUGAL, then Rose can choose EXPENSIVE and receive a payoff of 4 or choose FRUGAL and receive a payoff of 3. Since payoffs model choice, Rose should choose EXPENSIVE and receive a payoff of 4. To do otherwise would mean that the payoffs are not an accurate model for Rose's choices.

Since players aspire to them, when a Nash equilibrium is also an efficient strategy pair, this strengthens the players' resolve to choose Nash equilibrium strategies. For example, in **Ad Hoc**, the efficient strategy pairs are (ROW B, COLUMN A), (ROW A, COLUMN D), and (ROW C, COLUMN D). (Checking the accuracy of this statement takes a bit of work.) The first pair is the unique Nash equilibrium.

Comparison and Interpretation

All four of the solution concepts considered in this section (prudential strategies, undominated strategies, Nash equilibria, and efficient strategy pairs) depend only on the ordinal aspect of players' preferences. This is because these solution concepts all involve comparing payoffs to see which are larger than the others. This means that we may not need to find cardinal payoffs in order to complete an analysis of a strategic game.

A player's prudential strategy can be determined without knowing the payoffs for the other players. It embodies a pessimistic attitude: whatever strategy choice a player makes, others will find out and conspire to reduce his or her payoff as much as possible. Its recommendation to choose a strategy that maximizes the player's guaranteed payoff is reasonable if (1) the player does not really know the payoff possibilities of his or her opponents (in which case, we do not really have a strategic game where we assume that players do know each others' payoff possibilities); (2) the player does not think that other players will act in accordance with the assumed payoffs (in which case, the payoffs are incorrect); and/or (3) the player is adverse to strategic risk, that is, the player values a payoff with certainty more than a potentially higher payoff that depends on the actions of others.

Of course, if players initially settle on prudential strategies and do know each others' payoff possibilities, then each player should choose a strategy that will increase her or his payoff in response to the choices of the other players. This potential instability is what motivated our definitions of dominance and the Nash equilibrium.

Undominated strategies take into account the information each player has about everyone's payoffs. By successively eliminating dominated strategies, players simplify the game. This is great if the resulting game has only one strategy for each player. Unfortunately, this is rarely the case.

A Nash equilibrium is made up of undominated strategies and so is a "refinement" of successively eliminating dominated strategies. A Nash equilibrium recommends a strategy choice for each player for which there is no regret: a player can only control his or her own strategy choice, and in a Nash equilibrium, no player wants to unilaterally change his or her strategy choice because doing so will not increase his or her payoff.

An efficient strategy pair embodies an optimistic attitude: the players' payoffs are so high that they, even acting together, cannot improve anyone's payoff without harming at least one other player. Efficient strategy pairs act more as goals to achieve rather than as a prediction of what players would actually do, since it usually requires player cooperation which is not part of the private and simultaneous strategy choices that are inherent in the strategic game model.

Thus, a Nash equilibrium appears to be the best predictor of strategy choice in a strategic game. Unfortunately, some strategic games do not have a unique Nash equilibrium. When there are multiple Nash equilibria, preplay communication may help players agree on a single one. When there is no Nash equilibrium as currently defined, it becomes necessary to consider strategies involving randomization. In the next chapter we will do this and find that all strategic games have at least one Nash equilibrium.

Exercises

(1) Consider the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C
Rose	A	$(8,5)$	$(6,4)$	$(4,5)$
	B	$(5,6)$	$(2,9)$	$(9,7)$
	C	$(7,3)$	$(3,5)$	$(3,2)$

(a) Is ROW A a prudential strategy for Rose? Why or why not?
(b) Is COLUMN A a prudential strategy for Colin? Why or why not?
(c) Does ROW A dominate ROW C? Why or why not?
(d) Does COLUMN A dominate COLUMN C? Why or why not?
(e) Is (ROW A, COLUMN A) a Nash equilibrium? Why or why not?
(f) Is (ROW B, COLUMN C) a Nash equilibrium? Why or why not?
(g) Is (ROW B, COLUMN B) an efficient strategy pair? Why or why not?
(h) Is (ROW A, COLUMN A) an efficient strategy pair? Why or why not?

(2) Consider the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C	Colin D	Colin E
Rose	A	$(2,-1)$	$(2,0)$	$(-5,-1)$	$(0,0)$	$(1,-1)$
	B	$(2,2)$	$(3,3)$	$(-4,3)$	$(1,3)$	$(1,-2)$
	C	$(1,-1)$	$(-2,2)$	$(-5,4)$	$(0,4)$	$(-1,3)$

(a) What are the prudential strategies and security level for Rose?
(b) What are the prudential strategies and security level for Colin?
(c) Does ROW B dominate ROW A? Does ROW B dominate ROW C? Find all pairs of Rose's strategies in which the first strategy dominates the second strategy of the pair.
(d) Find all pairs of Colin's strategies in which the first strategy dominates the second strategy of the pair.
(e) Is (ROW A, COLUMN B) a Nash equilibrium? Why or why not?
(f) Is (ROW B, COLUMN D) a Nash equilibrium? Why or why not?

(g) What are the two efficient strategy pairs?

(3) Consider the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C
	A	$(3,0)$	$(5,2)$	$(0,4)$
Rose	B	$(2,2)$	$(1,0)$	$(3,1)$
	C	$(4,1)$	$(0,0)$	$(1,2)$

(a) What are the prudential strategies and security levels?

(b) Does the pair of prudential strategies form a Nash equilibrium? Why or why not?

(c) Does Rose have any dominated strategies? If so, eliminate them. Does Colin have any dominated strategies in the reduced game? If so, eliminate them. Repeat this process as long as either player has any dominated strategies. What strategies remain?

(4) Consider the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C	Colin D	Colin E
	A	$(5,4)$	$(3,2)$	$(4,3)$	$(9,2)$	$(3,1)$
Rose	B	$(6,6)$	$(1,5)$	$(3,4)$	$(8,3)$	$(4,4)$
	C	$(4,3)$	$(2,6)$	$(4,8)$	$(7,7)$	$(5,5)$

(a) Does Rose have any dominated strategies? If so, eliminate them. Does Colin have any dominated strategies in the reduced game? If so, eliminate them. Repeat this process as long as either player has any dominated strategies. What pair of strategies remains?

(b) What are the prudential strategies? What relationship (if any) is there between prudential strategies and the strategies that remain after successive elimination of dominated strategies? What relationship (if any) is there between prudential strategies and the eliminated strategies?

(c) Is the pair of strategies obtained in part (a) a Nash equilibrium for the original game? What relationship (if any) is there between the strategies that remain after successive elimination of dominated strategies and Nash equilibria?

(d) Is any other pair of strategies a Nash equilibrium? What relationship (if any) is there between the strategies that remain after successive elimination of dominated strategies and Nash equilibria?

(e) Is the pair of strategies obtained in part (a) an efficient strategy pair for the original game? What relationship (if any) is there between the strategies that remain after successive elimination of dominated strategies and efficient strategy pairs?

(5) Use a best response diagram to identify any Nash equilibria in the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(2,2)$	$(4,3)$
	B	$(3,4)$	$(1,1)$

(6) Use a best response diagram to identify any Nash equilibria in the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C
	A	$(3,0)$	$(5,2)$	$(0,4)$
Rose	B	$(2,2)$	$(1,1)$	$(3,3)$
	C	$(4,1)$	$(4,0)$	$(1,0)$

(7) Explain why the best response diagram identifies Nash equilibria.

(8) Consider the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(4,1)$	$(2,4)$
	B	$(3,3)$	$(1,2)$

(a) What are the prudential strategies for Rose and Colin?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you recommend that Rose and Colin choose? Why?

(9) Consider the following very simple strategic game model for the scenario described in the dialogue of section 3.1:

Tosca Ordinal Payoffs		Scarpia KILL LOVER	SPARE LOVER
Tosca	SUBMIT TO SCARPIA	$(1,4)$	$(2,3)$
	REFUSE SCARPIA	$(3,2)$	$(4,1)$

(a) Describe the motivations of Tosca and Scarpia that give rise to the stated ordinal payoffs.
(b) What are the prudential strategies?
(c) What strategies remain after successive elimination of dominated strategies?
(d) What are the Nash equilibria?
(e) What are the efficient strategy pairs?
(f) What strategies would you expect Tosca and Scarpia to choose? Why?

(10) Consider the following game.

Ordinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(4,4)$	$(1,2)$
	B	$(3,1)$	$(2,3)$

(a) What are the prudential strategies for Rose and Colin?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you recommend that Rose and Colin choose if there can be no communication between them before the game is played? Why?

(f) In what way (if any) would your recommendation change if Rose and Colin could converse before the game is played? Why?

(11) Consider the ordinal payoff matrix from **Fingers** shown here.

Fingers Ordinal Payoffs		Colin ONE	TWO
Rose	ONE	$(1, 1)$	$(4, 2)$
	TWO	$(2, 4)$	$(3, 3)$

(a) What are the prudential strategies for Rose and Colin?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you recommend that Rose and Colin choose if there can be no communication between them before the game is played? Why?
(f) In what way (if any) would your recommendation change if Rose and Colin could converse before the game is played? Why?
(g) Answer the above questions for the **Risk-Neutral Fingers** cardinal payoff matrix and the **Risk-Varying Fingers** cardinal payoff matrix found in the previous section. Compare with the results you obtained for the **Fingers** ordinal payoff matrix.

(12) Consider the following simple strategic game model for the **Steroid Use** scenario described in the section 3.2 exercises:

Steroid Use Ordinal Payoffs		Others USE	ABSTAIN
Individual	USE	$(2, 2)$	$(4, 1)$
	ABSTAIN	$(1, 4)$	$(3, 3)$

(a) What are the prudential strategies for Rose and Colin?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you expect players to choose? Why? Does it matter whether there is preplay communication?

(13) Consider the following ordinal payoff matrix from **Matches**.

Matches Ordinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(4, 3)$	$(2, 2)$
	SOCCER	$(1, 1)$	$(3, 4)$

(a) What are the prudential strategies for Rose and Colin?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you recommend that Rose and Colin choose if there can be no communication between them before the game is played? Why?

(f) In what way (if any) would your recommendation change if Rose and Colin could converse before the game is played? Why?

(14) Consider the following risk neutral cardinal payoff matrix for **River Tale**.

River Tale Cardinal Payoffs		Steve HEADS	TAILS
Stranger	HEADS	$(20, -20)$	$(-10, 10)$
	TAILS	$(-30, 30)$	$(20, -20)$

(a) What are the prudential strategies?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you recommend players choose? Why? Does it matter whether there is preplay communication?

(15) Consider the following cardinal payoff matrix for **WWII Battle** game.

WWII Battle Cardinal Payoffs		Germans ATTACK GAP	RETREAT
Allies	REINFORCE GAP	$(1, -1)$	$(0, 0)$
	HOLD RESERVES	$(2, -2)$	$(0, 0)$
	SEND RESERVES EAST	$(-2, 2)$	$(1, -1)$

(a) What are the prudential strategies for the Allies and for the Germans?
(b) What strategies remain after successive elimination of dominated strategies?
(c) What are the Nash equilibria?
(d) What are the efficient strategy pairs?
(e) What strategies would you have predicted that the Allies and the Germans used? Why?
(f) In reality, the Allies chose to hold their reserves while Hitler overruled his general's decision to retreat. Were these reasonable decisions by the Allies, by the German generals, and by Hitler? Why or why not?
(g) Suppose that the Germans, instead of the Allies, could win the battle in the east against the reserves as they retreated. How does this change the payoff matrix and your answers to parts (a) through (f)?

(16) Explain why the strategies in a Nash equilibrium must be undominated.

4. Once Again, Game Trees

In the previous two sections, all of our strategic games implicitly or explicitly required the two players to simultaneously select their strategies before playing the game. You may have also noticed that in some of the games this felt contrived; that in reality, for these games, one person would choose an action, and then the other would choose an action in response. One example of this is two cars approaching an intersection from opposite directions and each wishing to make a left-hand turn. (In much of the world, this implies that they will cross each other's paths.) A driver's decision as to whether to turn immediately or wait for the other driver frequently involves waiting to see what the other driver does. This sense of ordered play leads us to the following definition.

Sequential Game: In a *sequential game*, players take turns in a specified manner to choose among legal moves. Each play of the game results in an outcome with payoffs for each player.

Deterministic games are sequential games with additional restrictions: no secrecy, no chance, and only win, lose, or tie outcomes. Sequential games remove all of these restrictions. Fortunately, just as we found that game trees were a fruitful way to represent deterministic games, we will see here that they also represent sequential games quite well.

By the end of this section, you will be able to model simple real-world scenarios as sequential games, determine optimal strategy choices and outcomes using backward induction, and interpret these strategies and outcomes in the original scenarios.

Sequential Fingers

As a first example, Figure 4.1 is the game tree associated with **Sequential Fingers**, in which Rose holds up one or two fingers before, instead of at the same time as, Colin holds his up.

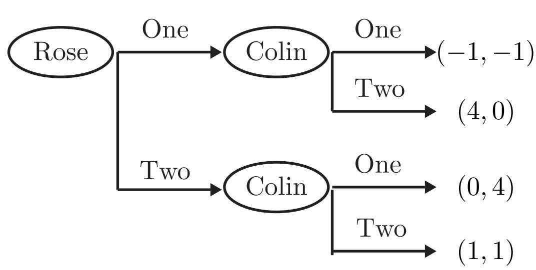

FIGURE 4.1. **Sequential Fingers** game tree

In deterministic games, we label the leaves with the winning player. For sequential games, we replace the winning player with player payoffs. So, the leaves of the **Sequential Fingers** game tree are labeled with the resulting payoffs for Rose and Colin. (In Figure 4.1, we assume that each player is risk neutral and express the payoffs as the dollars each receives.)

> This would be an excellent time for the reader to find a partner and play **Sequential Fingers**. Decide who will roleplay as Rose and who will roleplay as Colin, and then play the game. What happened?

The backward induction algorithm of Chapter 1 can again be used to find optimal play against a rational opponent in sequential games. At each node, the player whose turn it is

(1) selects the arrow that results in the best payoff from that position,
(2) marks that arrow, and
(3) labels the node with the resulting payoffs.

In **Sequential Fingers**, if Rose has already held up one finger, Colin should hold up two fingers (since a payoff of 0 is better than a payoff of -1), and if Rose has already held up two fingers, Colin should hold up one finger (since a payoff of 4 is better than a payoff of 1). This discussion has been summarized in Figure 4.2 by marking the move choices and labeling the parent nodes with the appropriate payoff pair.

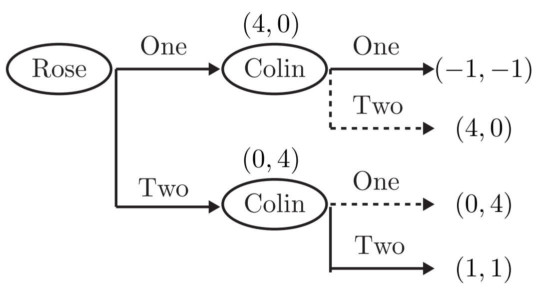

FIGURE 4.2. Colin's analysis of **Sequential Fingers**

Knowing what would be optimal for Colin to choose, Rose should hold up one finger (since a payoff of 4 is better than a payoff of 0). This is summarized in Figure 4.3.

Thus, we see that Colin's optimal strategy in **Sequential Fingers** is to do the OPPOSITE of whatever Rose does and Rose's optimal strategy is to show ONE finger. Notice that Colin's strategy must include directions for what to do regardless of what Rose chooses to do. In the actual play of the game, Rose will show one finger, and then Colin will show two fingers, resulting in the payoff pair $(4, 0)$.

We generalize this process in the following algorithm.

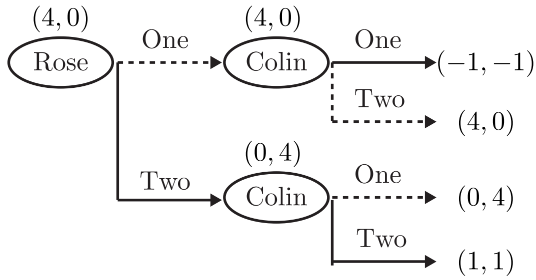

FIGURE 4.3. Rose's analysis of **Sequential Fingers**

Backward Induction: *Backward induction* is the following algorithm: Label all of the leaves with the payoff pair. If there are unlabeled nodes, then at least one (called the parent) must have all of its arrows extending to labeled nodes (called the children). Select a child with the largest payoff for the player who makes the move from the parent. Mark the arrow from the parent to the selected child and label the parent with the selected child's label. Repeat this process until all nodes are labeled. The marked arrows for each player represent a *backward induction strategy*, and the path of marked arrows from the tree root lead to a *backward induction outcome*.

In the following subsection, we will model a significant real-world scenario as a sequential game and use backward induction to predict how the players might have acted.

Cuban Missile Crisis

During 1962 Soviet Union Premier Nikita Khrushchev ordered that intermediate range nuclear missiles be secretly installed in Cuba. Upon their discovery on October 12, 1962, United States President John F. Kennedy and his national security advisors discussed possible responses including (i) a destructive air attack on the missiles, (ii) a full military invasion of Cuba, or (iii) a naval blockade of Cuba. While preparing for military options, a naval blockade was instituted and negotiations were pursued via telegrams and through intermediaries. Both President Kennedy and Premier Nikita Khrushchev threatened military retaliation if either initiated military action. Leaders of all three countries believed that war was imminent. On October 28, 1962, the crisis was resolved by a public agreement that the Soviet Union would remove its missiles from Cuba and that the United States would not invade Cuba. Privately, Kennedy gave assurances to Khrushchev that United States missiles based in Turkey would be removed.

We will model this scenario as a sequential game, called **Cuban Missile Crisis**, following the presentation in Straffin [**64**, page 39]. A good approach to modeling a situation as a game is to develop the four components of a game in order: players, rules, outcomes, and preferences.

The first step is to identify the players. Three countries, the United States, the Soviet Union, and Cuba, were involved. There were also many people from each country, not always acting consistently with each other. For example, a Soviet commander in Cuba had a United States reconnaissance jet shot down against directives from Khrushchev. In trying to model a scenario, it is best to start simple and only later add elaborations, so we will consider only Kennedy and Khrushchev to be the players.

The second step is to determine the rules, which for a sequential game involves the sequence of choices available to the players. Khrushchev will start the game by deciding whether or not to place nuclear missiles in Cuba. Historically, Khrushchev did place nuclear missiles in Cuba, but to obtain a game model, we must think about alternatives to what actually happened. If, in the game, Khrushchev places the missiles in Cuba, Kennedy will have three options: do nothing, blockade Cuba in an attempt to force the Soviets to remove them, or eliminate the missiles by a surgical air strike. Notice that, for the sake of simplicity, we have not included a military invasion of Cuba, one of the options actually discussed by Kennedy with his advisors. If Kennedy chooses one of the more aggressive actions of a blockade or an airstrike, Khrushchev may acquiesce and remove the missiles, or he may order an escalation of the confrontation, possibly leading to nuclear war. This is summarized in Figure 4.4.

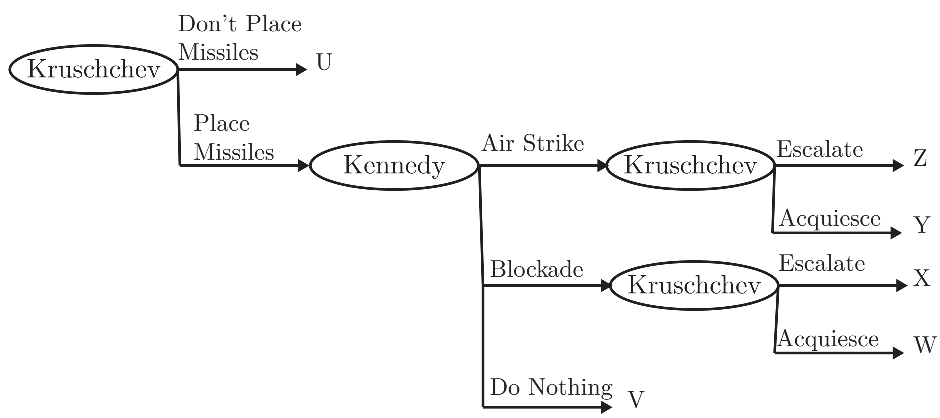

FIGURE 4.4. **Cuban Missile Crisis** game tree

The third and fourth steps are to describe the outcomes and player preferences. In Figure 4.4, the outcomes are represented by letters. Outcome X, for example, is the result of Khrushchev placing the missiles, then Kennedy blockading Cuba, and finally Khrushchev escalating the tension. While this may lead to nuclear war, Kennedy may prefer this outcome over others since he could blame the war on Khrushchev's escalation. Each of the outcomes are given analogous interpretations in Table 4.1. Ordinal payoffs were assigned assuming that each leader most desires to avoid warfare and secondarily wants to maintain a reputation of strength.

We now use backward induction on the game tree as shown in Figure 4.5 to find the predicted outcome of the crisis, as it is modeled here. Since Khrushchev prefers

TABLE 4.1. Outcome descriptions and ordinal utilities for **Cuban Missile Crisis**

Outcome	Description	Kennedy's Ordinal Payoff	Khrushchev's Ordinal Payoff
U	Status quo	4	5
V	Khrushchev obtains a military advantage	3	6
W	Kennedy shows resolve; Khrushchev loses face	6	4
X	Risk of nuclear war with Khrushchev seen as the aggressor	2	1
Y	Kennedy shows resolve but uses military resources; Khrushchev loses face and military resources	5	3
Z	Risk of nuclear war with Kennedy viewed as the aggressor	1	2

W over X, he will acquiesce if Kennedy blockades, and since he prefers Y over Z, he will acquiesce if Kennedy conducts airstrikes. Given the choices of outcome V (for doing nothing), outcome W (for blockading), and outcome Y (for airstrikes), Kennedy prefers W, so he should respond to the placement of the missiles with a blockade. To complete the backward induction, Khrushchev has a choice between outcomes W and U. Since he prefers U, according to the ordinal rankings given above, he should not place the missiles in Cuba. Thus, the backward induction strategy pair is (NOTHING, BLOCKADE).

(NOTHING, BLOCKADE) may seem strange, since the move BLOCKADE is not an option if Khrushchev makes the move NOTHING. But remember—a strategy is a complete and unambiguous description of what do in every possible situation. Before the game is played, Kennedy does not know what Khrushchev will do, so Kennedy's strategy must specify what Kennedy would do if Khrushchev chooses to place the missiles, and it is Khrushchev contemplating Kennedy choosing the BLOCKADE strategy that makes him realize that NOTHING is a better choice for him.

As it turns out, our model's prediction does not match what happened historically. This means that our game or solution concept do not accurately model the real-world scenario. It could be that the assumed preferences are incorrect (see exercise 2), or that critical potential moves and/or players have not been included (see exercise 3). Our modeling effort should not be considered a failure; it should be considered a first step toward developing a more explanatory model.

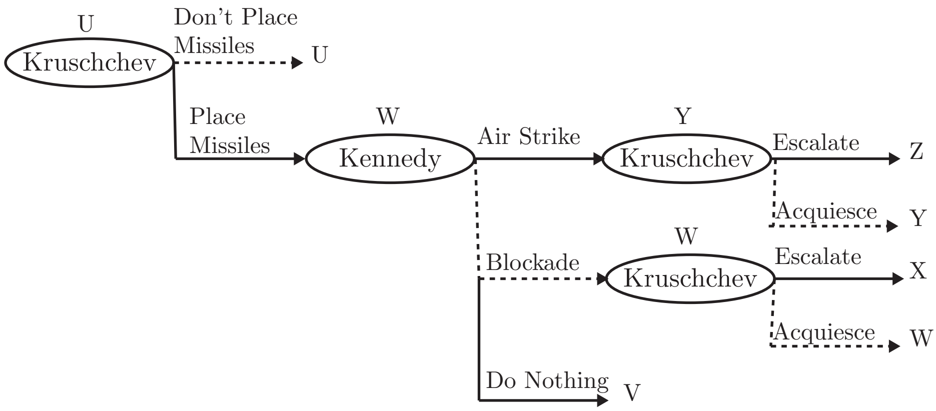

FIGURE 4.5. **Cuban Missile Crisis** backward induction solution

Exercises

(1) **Sequential Matches.** Suppose the scenario modeled by **Matches** is changed in the following manner: Colin had given Rose his cell phone number. It is now possible for Rose to send to Colin one of the following two text messages: "see you at the tennis match" or "see you at the soccer match". This changes the simultaneous move game into a sequential move game in which Rose makes her move first and Colin makes his move second, knowing Rose's move.
  (a) Draw the game tree.
  (b) Find, and describe in complete sentences, the backward induction strategies.
  (c) What happens if Rose and Colin follow their backward induction strategies?
  (d) Does the outcome in part (c) seem reasonable to you? Why or why not?

(2) Suppose the player preferences for **Cuban Missile Crisis** are changed to those Given in Table 4.2.

TABLE 4.2. Alternate preference for **Cuban Missile Crisis**

Outcome	U	V	W	X	Y	Z
Kennedy's Ordinal Payoff	5	1	6	3	4	2
Khrushchev's Ordinal Payoff	5	6	3	2	4	1

  (a) Describe in what ways these modified preferences and the implied motivations are different from the original preferences and motivations.
  (b) Use backward induction to solve the game again. What is the outcome this time?
  (c) What has to be true about the preferences for the backward induction solution to be consistent with what happened historically?

(3) How might you change the **Cuban Missile Crisis** game tree so that the backward induction strategies would be consistent with what happened historically?

(4) **Up and Down.** This is a delightfully short and mundane game. Colin says "up" or "down". Then Rose says "up" or "down". Finally, Colin says "up" or "down". Figure 4.6 shows the eight possible outcomes: each ordered pair states the number of dollars received by Rose and Colin, respectively.

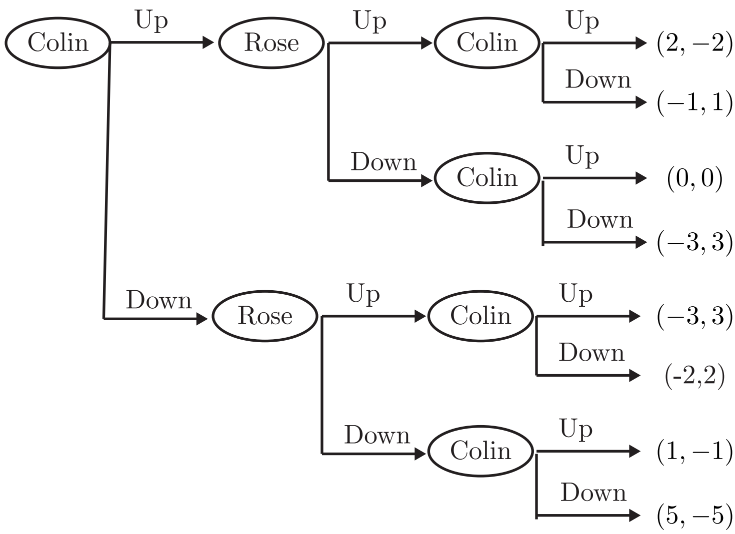

FIGURE 4.6. **Up and Down** game tree with outcomes but not preferences

(a) Suppose each player is self-interested, that is, given a choice among two or more outcomes, each player would choose the one that gives her or him the most money. Draw the game tree with ordinal payoffs consistent with self-interested players. Carry out the backward induction process. State in complete sentences the backward induction strategies for each player. State what happens if the two players use their backward induction strategies.

(b) Suppose each player is purely altruistic, that is, given a choice among two or more outcomes, each player would choose the one that gives the other player the most money. Draw the game tree with ordinal payoffs consistent with purely altruistic players. Carry out the backward induction process. State in complete sentences the backward induction strategies for each player. State what happens if the two players use their backward induction strategies.

(c) State your own assumptions about the preferences held by Rose and Colin. The only restriction is that the pair of preferences are not identical to part (a) or part (b). You could assume that Rose is self-interested and Colin is purely altruistic. Draw the game tree with ordinal payoffs consistent with your assumptions. Carry out the backward induction process. State in complete sentences the backward induction strategies for each player. State what happens if the two players use their backward induction strategies.

(5) **Adam and Eve.** In the biblical story of Adam and Eve (Genesis, Chapters 2–3), they can choose to obey God by not eating the fruit of the tree of

knowledge of good and evil, or to disobey God by eating the fruit. If they eat the fruit, God can choose to ignore their disobedience or interrogate them. If God interrogates them, Adam and Eve can choose to deny their guilt or to admit it. In either case, God can choose to ignore their transgressions, punish them, or kill them. Figure 4.7 shows the game tree for this sequential game, with one possible assignment of ordinal payoffs for Adam & Eve and God, respectively. (A simplified version of this game appears in [**9**].)

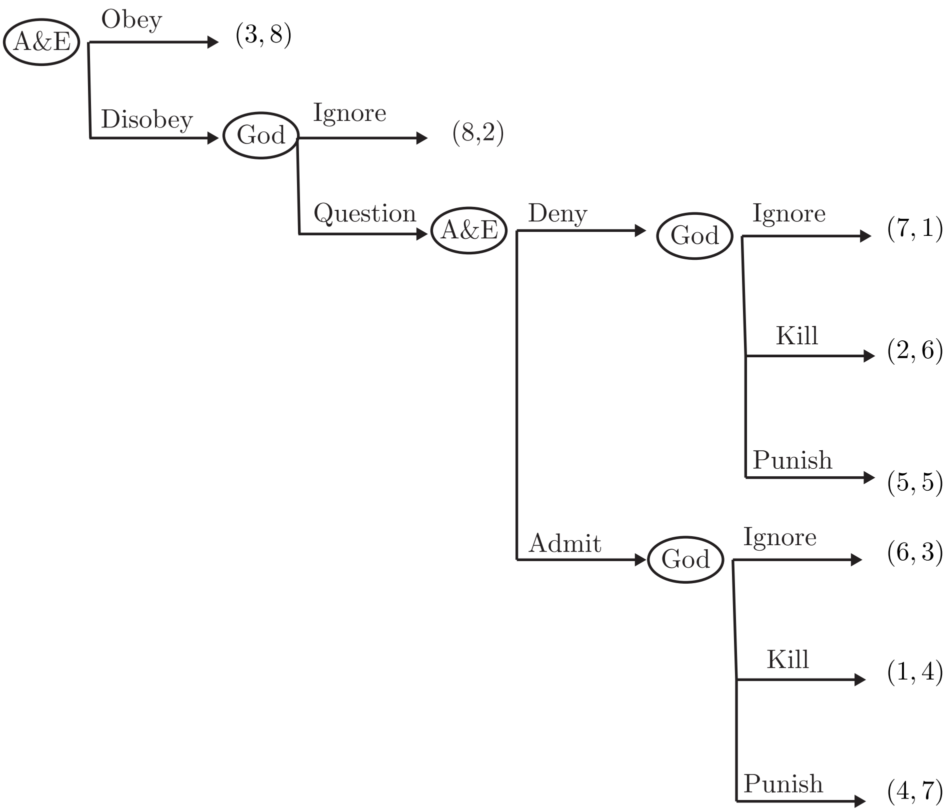

FIGURE 4.7. **Adam and Eve** game tree

(a) Describe motivations for the players that would result in these ordinal payoffs.
(b) Use backward induction to determine the best courses of action for God and for Adam and Eve.
(c) Compare part (b) with what actually happens in the biblical story.
(d) What changes could be made to the strategic game so that it would more closely model the biblical story?

(6) **Sampson and Delilah** [**9**, page 153]. In this biblical story (Judges, Chapter 16), Delilah has been paid by Samson's enemies to determine the secret to his great strength. Delilah can choose to nag him into revealing his secret, but she would prefer that he reveal it without that effort on her part. Whether

she nags him, Sampson can choose to reveal his secret or not; but if she has decided to nag him, she will persist in doing so if he refuses to tell. Model this scenario as a sequential game and determine the backward induction strategies and outcome.

(7) **Square the Diagonal** [**50**]. Castor selects a number from {1, 2, 3}, then Pollux selects a number from {1, 2}, and finally Castor again selects from a number from {1, 2, 3}, possibly the same as the first number he selected. Imagine these three numbers as the length, width, and height of a box. The number $D = l^2 + w^2 + h^2$ gives the square of the length of the diagonal of the box. (Why?) If D leaves a remainder of 0 or 1 when divided by 4, Castor wins the amount D from Pollux; otherwise, Pollux wins this amount from Castor. Model this scenario as a sequential game and determine the backward induction strategies and outcome.

(8) **Battle of Thermopylae** [**29**]**.** Around 480 BC, Xerxes, the King of Persia, decided to invade Sparta (of course, he could have decided instead not to invade). A corrupted council would not permit Leonidas, King of Sparta, to mount a defense using the full army. Leonidas could have chosen to surrender but instead chose to mount a defense with a few of his finest men. When faced with Leonidas's defense, Xerxes could have retreated or continued the invasion. Instead, Xerxes chose first to try to bribe Leonidas so that victory could be ensured without a battle. Leonidas refused the bribe and war ensued. Xerxes won the battle and killed Leonidas. Xerxes was motivated by a desire for power and territory, whereas Leonidas was motivated by honor and the protection of Sparta. Model this scenario as a sequential game, determine the backward induction strategies and outcome, and compare with what actually happened.

(9) **Vacation** [**54**]. Taylor has an opportunity to vacation with her boyfriend in Las Vegas, in Hawaii, or at home. Taylor would most prefer Las Vegas, second most prefer Hawaii, and third most prefer home. Taylor's boyfriend, Sam, would most prefer Hawaii, second most prefer home, and third most prefer Las Vegas. Taylor will first offer one of the three locations to Sam. If the offer is for home, Sam has no choice: they will vacation at home. Otherwise, Sam will either accept or reject the offer. If the offer is accepted, then the couple's plans are made. If the offer is rejected, Taylor will offer one of the remaining two locations. Again if the offer is for home, Sam has no choice but to vacation at home. Otherwise, Sam will either accept or reject the offer. If the offer is accepted, then the couple's plans are made. If the offer is rejected, the couple will vacation at home. Model this scenario as a sequential game and determine the backward induction strategies and outcome.

(10) **The Taming of the Shrew** [**20**]. Kate and Petruchio are the protagonists in Shakespeare's play *The Taming of the Shrew*. Kate is a very stubborn, loud, rebellious, and intimidating young woman. She is further enraged when her father refuses to allow anyone to court Kate's sister until Kate is married. Petruchio is stubborn, loud, intimidating, unconventional, and not very wealthy. In order to obtain the significant dowry that comes from marrying Kate, Petruchio vows to marry her and receives consent from Kate's father to woo her. From their wedding day forward, Petruchio robs Kate of every pleasure, makes her physically uncomfortable, and makes a scene reminiscent

of her own tantrums everywhere they go. The relationship suddenly changes and Kate supports everything Petruchio says, no matter how absurd. At the close of the play, the couple appear happy. Model this scenario as a sequential game, determine the backward induction strategies and outcome, and compare with what happened in the play.

(11) **Ultimatum.** This game was introduced by Werner Gûth, Rolf Schmittberger, and Bernd Schwarze [**21**]. It, and several variations, have been studied by a number of researchers [**11**, chapter 2]. The Proposer offers a division of \$3 between herself and the Responder. The offer must be in \$1 increments, that is, there are only four offers possible, \$0, \$1, \$2, or \$3 to the Responder and the remainder of the \$3 to the Proposer. After an offer is given, the Responder can either accept or reject. If the Responder accepts the offer, then each player receives in accordance with the offer. If the Responder rejects the offer, each player receives nothing.
   (a) Think about how you would play this game if you were the Proposer: how much would you offer the Responder? Think about how you would play this game if you were the Responder: what offers would you accept and which offers would you reject? If possible, play the game with another person.
   (b) Model this game as a sequential game under the assumption that each player is self-interested and risk neutral. Determine the backward induction strategies.
   (c) Change the model in part (b) by assuming that the Responder values equity over monetary gain. Determine the backward induction strategies.
   (d) How would the game change if offers could be made in increments of \$0.01 instead of \$1?

(12) **Trust**. This game was proposed by Joyce E. Berg, John Dickhaut, and Kevin McCabe [**6**] and earlier by Colin F. Camerer and Keith Weigelt [**13**]. Here is one version. The Investor is given \$2. She may keep the money, in which case the game is over. Instead, the Investor may invest \$1 or \$2. Whatever amount is invested is doubled (\$1 becomes \$2, and \$2 becomes \$4) and is given to the Trustee. The Trustee chooses how much money to keep and the rest is given back to the Investor.
   (a) Think about how you would play this game if you were the Investor: how much (if anything) would you invest? Think about how you would play this game if you were the Trustee: how much would you keep if given \$2? How much would you keep if given \$4? If possible, play the game with another person.
   (b) Model this game as a sequential game under the assumption that each player is self-interested and risk neutral. Determine the backward induction strategies.
   (c) Change the model in part (b) by assuming that the Investor's primary motivation is still her monetary gain, but she also has the Trustee's monetary gain as a secondary motivation, and the Trustee's primary motivation is now equity and his monetary gain is only a secondary motivation. Determine the backward induction strategies.
   (d) How would the game change if investments could be made in increments of \$0.01 instead of \$1?

5. Trees and Matrices

In this section, we want to build a connection between the game trees that we used to analyze sequential games and the payoff matrices that we used in sections 2 and 3 to represent (simultaneous) strategic games. In fact, we will see that every sequential game can be modeled using a payoff matrix. We will also see that, by using the concept of information sets, every strategic game with a payoff matrix can be modeled using a game tree.

> By the end of this section, you will be able to identify strategies within game trees, to construct outcome and payoff matrices from game trees, and understand corresponding solution concepts.

Trees To Matrices

We begin with the game tree associated with **Sequential Fingers**, shown in Figure 5.1.

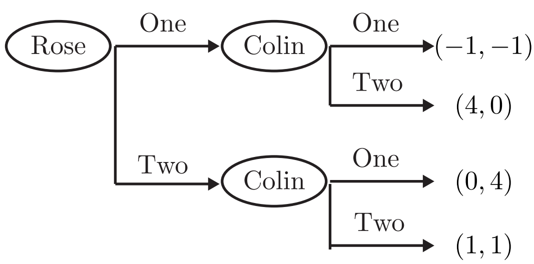

FIGURE 5.1. **Sequential Fingers** game tree

To build a payoff matrix for this game, we must first identify player strategies. Since she has only one choice of two actions, Rose has two strategies in this game as shown in Figures 5.2 and 5.3.

(1) Hold up one finger, which we abbreviate as ONE.

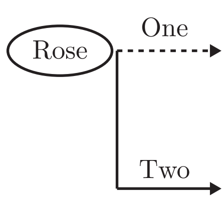

FIGURE 5.2. ONE

(2) Hold up two fingers, which we abbreviate as TWO.

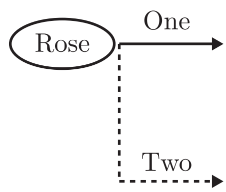

Figure 5.3. TWO

On the other hand, since he has choices available at two game tree nodes, Colin has four strategies as shown in Figures 5.4, 5.5, 5.6, and 5.7.

(1) If Rose holds up one finger, hold up one finger and if Rose holds up two fingers, hold up one finger. This can be stated more succinctly as holding up one finger regardless of what Rose does, so we abbreviate this strategy as ALWAYS ONE.

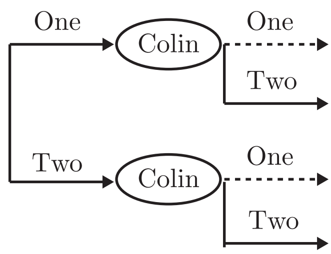

Figure 5.4. ALWAYS ONE

(2) If Rose holds up one finger, hold up two fingers and if Rose holds up two fingers, hold up two fingers. More succinctly, this strategy is to always hold up two fingers, so we abbreviate this strategy as ALWAYS TWO.

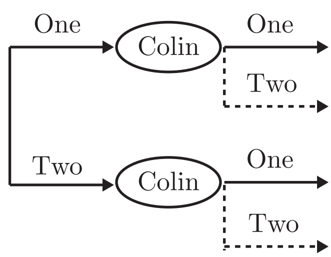

Figure 5.5. ALWAYS TWO

(3) If Rose holds up one finger, hold up one finger and if Rose holds up two fingers, hold up two fingers; that is, always do the same thing as Rose, which we abbreviate as SAME.

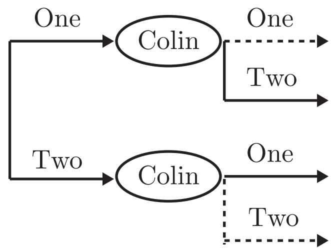

FIGURE 5.6. SAME

(4) If Rose holds up one finger, hold up two fingers and if Rose holds up two fingers, hold up one finger; that is, always do the opposite of what Rose does, which we abbreviate as OPPOSITE.

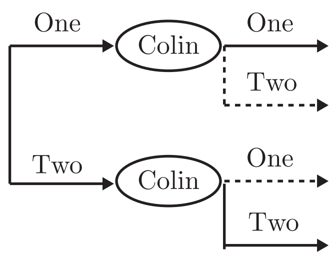

FIGURE 5.7. OPPOSITE

We can now construct a payoff matrix for this game as if Rose and Colin were playing simultaneously. The key thing to observe is that by choosing one of his four strategies, Colin predetermines his response to each of Rose's actions before the game is played. Thus, while the game may be played sequentially in real time, the strategy decisions are made simultaneously before play begins.

Sequential Fingers		Colin			
Cardinal Payoffs		ALWAYS ONE	ALWAYS TWO	SAME	OPPOSITE
Rose	ONE	$(-1,-1)$	$(4,0)$	$(-1,-1)$	$(4,0)$
	TWO	$(0,4)$	$(1,1)$	$(1,1)$	$(0,4)$

Again, to emphasize the point, the apparent paradox of a sequential game being described by simultaneous moves is resolved by the realization that Colin's choice of strategy is predetermined simultaneously with Rose's choice, but then these are played out sequentially.

Generalizing the **Sequential Fingers** example, start with any game tree. From the game tree, we can identify each player's strategies. Using the game tree again, we can then determine the payoffs for each strategy pair. Finally, we can organize this information in a payoff matrix. Thus, we have modeled a sequential game as a strategic game, and we obtain the following theorem.

Tree to Matrix Theorem: *Every sequential game, described by a game tree, can be modeled as a strategic game, described using a payoff matrix.*

Once we have a payoff matrix, it is possible to search for Nash equilibria. The best response diagram for **Sequential Fingers**, shown below, reveals three Nash equilibria: (ONE, ALWAYS TWO), (ONE, OPPOSITE), and (TWO, ALWAYS ONE).

Sequential Fingers		Colin			
Cardinal Payoffs		ALWAYS ONE	ALWAYS TWO	SAME	OPPOSITE
Rose	ONE	$(-1,-1)$	$(\boxed{4},\boxed{0})$	$(-1,-1)$	$(\boxed{4},\boxed{0})$
	TWO	$(\boxed{0},\boxed{4})$	$(1,1)$	$(\boxed{1},1)$	$(0,\boxed{4})$

Notice that pairing the backward induction strategies that we obtained for **Sequential Fingers** in section 3.4 corresponds to the Nash equilibrium (ONE, OPPOSITE). This result holds in general, so we state it as a theorem.

Backward Induction Theorem: *In a sequential game, the backward induction strategies form a Nash equilibrium for the corresponding strategic game.*

But there are two other Nash equilibria to **Sequential Fingers**. The Nash equilibrium in which Rose chooses TWO and Colin chooses ALWAYS ONE is very interesting because the player payoffs are reversed in comparison with the backward induction strategies. These Nash equilibrium strategies are marked on the game tree shown in Figure 5.8, along with the resulting payoffs.

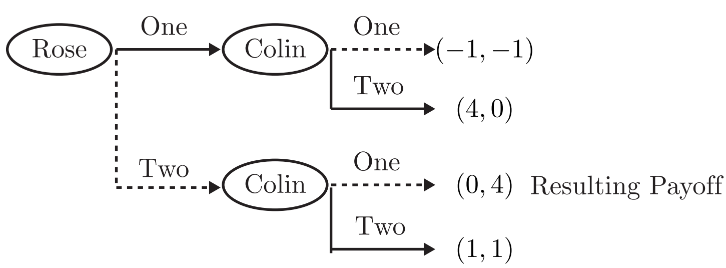

FIGURE 5.8. A **Sequential Fingers** Nash equilibrium

These payoffs can be obtained if Colin can convince Rose to choose TWO by threatening to choose ALWAYS ONE no matter what. Thus, this Nash equilibrium can only arise with preplay communication. Rose also has to believe Colin's threat that if she were to choose ONE, Colin would really harm himself by choosing ONE for a payoff of -1 instead of choosing TWO for a payoff of 0. Some would claim that such an incredible threat is not tenable; this claim suggests that sometimes the payoff matrix does not capture all of the relevant features of a game that is played out over time. Section 5.1 discusses the impact of preplay threats and promises in more detail.

Florist Scenario

The conversion of a sequential game into a strategic game can take a significant amount of work if the game tree is large or somewhat complicated. However, it is worth our time to consider another scenario because it will reinforce our understanding of what a strategy is: a complete description of how a player could play a game.

Suppose that Rose and Colin are at a florist shop to make a purchase. They agree that Rose would pick the flowers (orchids, tulips, or carnations), then Colin would pick the quantity (dozen or half dozen), and finally Rose would pick the vase (fancy or simple). Figure 5.9 illustrates this situation. The notation for the outcomes indicates that there is both a qualitative and quantitative component. For example, in outcome THF38, Rose and Colin spend \$38, and Rose gets a half dozen (H) of the tulips (T) in an fancy vase (F).

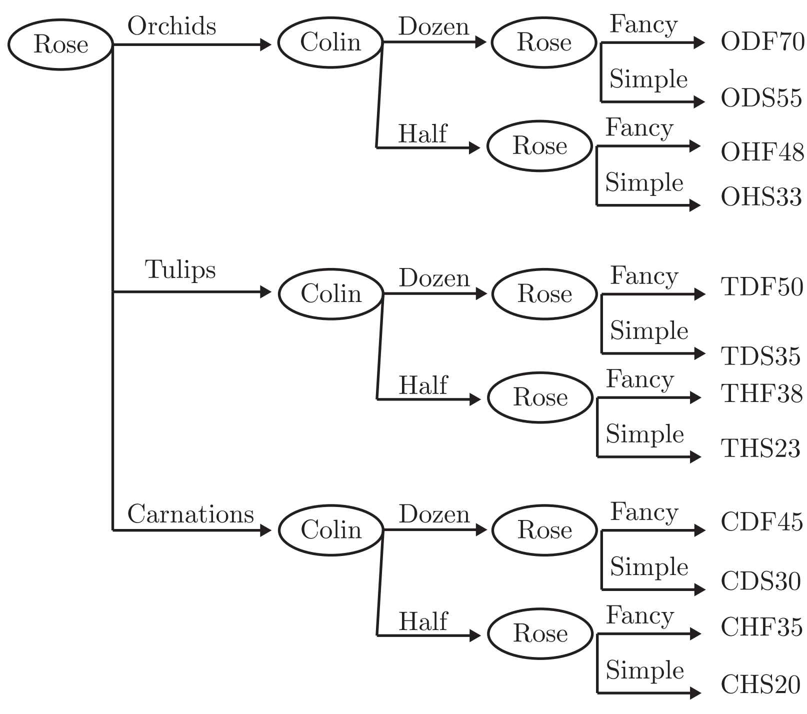

FIGURE 5.9. **Florist** game tree

Colin has eight strategies available to him. Recalling that any given strategy must respond to all three of Rose's possible initial decisions, we will denote his strategies by a triple describing how he would respond to Rose's choice of orchids, tulips and carnations in that order. Therefore, he could:

(1) (D, D, D): Always pick a dozen flowers.
(2) (D, H, H): Pick a dozen if Rose chooses orchids and a half dozen otherwise.
(3) (H, D, H): Pick a dozen if Rose chooses tulips and a half dozen otherwise.

(4) (H, H, D): Pick a dozen if Rose chooses carnations and a half dozen otherwise.
(5) (H, D, D): Pick a half dozen if Rose chooses orchids and a dozen otherwise.
(6) (D, H, D): Pick a half dozen if Rose chooses tulips and a dozen otherwise.
(7) (D, D, H): Pick a half dozen if Rose chooses carnations and a dozen otherwise.
(8) (H, H, H): Always pick a half dozen flowers.

Rose has twelve strategy options! We can abbreviate these by her choice of flowers (O, T, or C) and her selection of vase (F or S) in response to each of Colin's choices of quantity, the dozen first. We list four of them here, and encourage the reader to describe the other eight.

(1) (O, F, F): Pick orchids, then pick the fancy vase regardless of what Colin does.
(2) (O, F, S): Pick orchids, then pick the fancy vase if Colin picks the dozen and the simple vase otherwise.
(3) (O, S, F): Pick orchids, then pick the simple vase if Colin picks the dozen and the fancy vase otherwise.
(4) (O, S, S): Pick orchids, then pick the simple vase regardless of what Colin does.

These sets of strategies, and the associated outcomes come together in Table 5.1. Only a few of the outcomes are inserted here since our focus is on identifying the strategies. In the exercises, you are asked to complete the matrix.

TABLE 5.1. Partial outcome matrix for **Florist**

	(D,D,D)	(D,H,H)	(H,D,H)	(H,H,D)	(H,D,D)	(D,H,D)	(D,D,H)	(H,H,H)
(O,F,F)								
(O,F,S)	ODF70	ODF70	OHS33	OHS33	OHS33	ODF70	ODF70	OHS33
(O,S,F)								
(O,S,S)								
(T,F,F)		THF38		THF38		THF38		THF38
(T,F,S)								
(T,S,F)		THF38		THF38		THF38		THF38
(T,S,S)								
(C,F,F)								
(C,F,S)								
(C,S,F)								
(C,S,S)								

Of course, this does not complete either the sequential game or strategic game model. To do that, we would need to know Rose and Colin's preferences among the twelve outcomes.

Information Sets

To close this section, let's compare **Sequential Fingers** and **Fingers**. The essential difference between them is that in **Sequential Fingers**, Colin knows what Rose has chosen when it is his turn to make a choice. This distinction is apparent in the strategic game payoff matrices, displayed below. In **Sequential Fingers**, Colin has four strategies because he needs to account for the information that he has about Rose's choices.

Sequential Fingers Cardinal Payoffs		Colin ALWAYS ONE	ALWAYS TWO	SAME	OPPOSITE
Rose	ONE	$(-1,-1)$	$(4,0)$	$(-1,-1)$	$(4,0)$
	TWO	$(0,4)$	$(1,1)$	$(1,1)$	$(0,4)$

In **Fingers**, Colin only has two strategies since the strategies SAME and OPPOSITE do not exist when Colin will not be informed about Rose's choice.

Fingers Ordinal Payoffs		Colin ONE	TWO
Rose	ONE	$(1,1)$	$(4,2)$
	TWO	$(2,4)$	$(3,3)$

Can we capture this same distinction in a game tree? That is, can we model the strategic game **Fingers** as a sequential game? Recalling the game tree for **Sequential Fingers**, shown in Figure 5.10, we see that Colin has different infor-

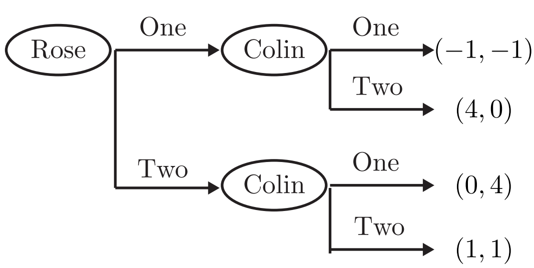

FIGURE 5.10. **Sequential Fingers** game tree

mation available at each of his two nodes: at his upper node, he knows that Rose has selected ONE, and at his lower node, he knows that Rose has selected TWO. In **Fingers,** these two nodes are not distinguishable from Colin's perspective. We can display this in the game tree by putting them in a dotted rectangle as shown in Figure 5.11.

One way of interpreting this is that Rose makes her choice and writes it on a piece of paper, but does not share her choice with Colin, who then makes his choice, after which both choices are revealed. This is clearly equivalent to simultaneous play, but conducted in a sequential manner. Notice that simultaneous play would also be equivalent to Colin making his choice and writing it on a piece of paper, but not sharing this with Rose, who then makes her choice, after which both choices are

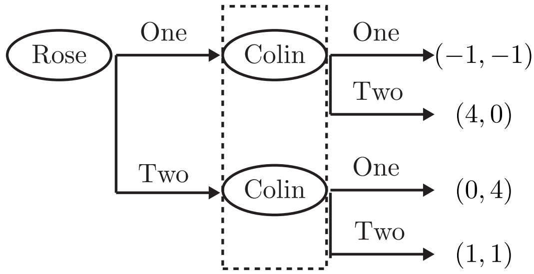

FIGURE 5.11. **Fingers** game tree, version 1

revealed. This is captured in the game tree shown in Figure 5.12. Note that even though Colin moves first, the payoffs are still ordered (Rose, Colin).

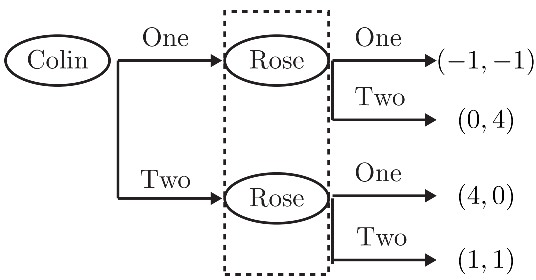

FIGURE 5.12. **Fingers** game tree, version 2

In general, the nodes in a game tree can be partitioned into information sets. When a player is at a node within an information set, he or she knows the information common to all of those nodes, but he or she cannot distinguish which specific node he or she is at. Pictorially, nodes that are in the same information set are inside a dotted rectangle. This leads us to the following theorem.

Matrix to Tree Theorem: *Every strategic game, described by a payoff matrix, can be modeled as a sequential game, described as a game tree with information sets.*

We close this section with a quick visit back to the flower shop. If Rose and Colin change their plan so that after Rose picks the type of flowers, the two of them will simultaneously choose the quantity and type of vase, then the game tree becomes the one in Figure 5.13 with the information sets as indicated. They illustrate that Rose knows which of the three initial branches she is on, but not which of the subsequent branches.

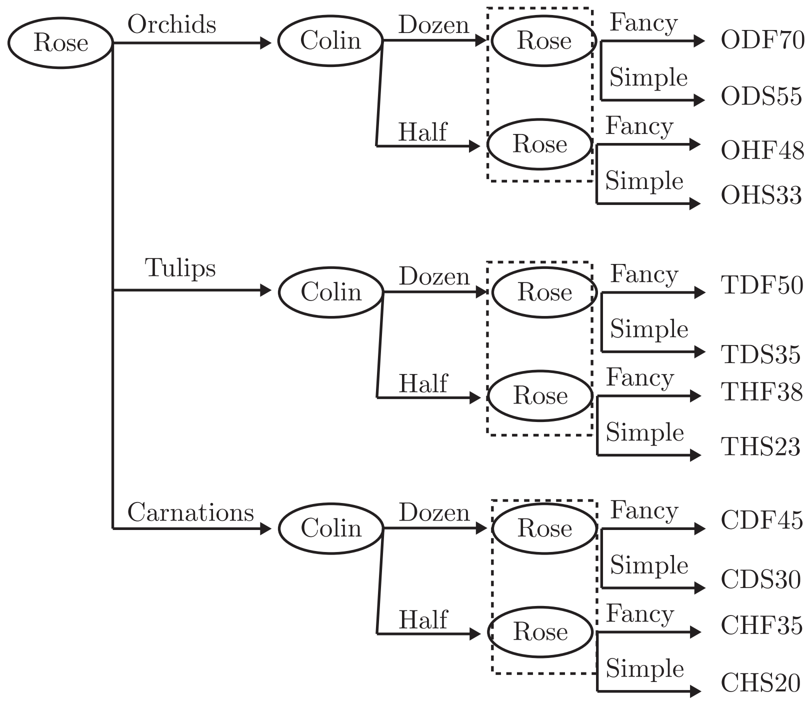

Figure 5.13. **Florist** modified game tree

Exercises

(1) Suppose the **Sampson and Delilah** scenario of the previous section is modeled as a sequential game with ordinal payoffs as shown in Figure 5.14. Identify

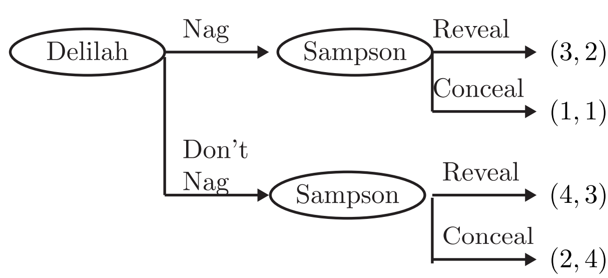

Figure 5.14. **Sampson and Delilah** game tree

Delilah's two strategies and Sampson's four strategies. Construct a payoff matrix and determine the Nash equilibria. Which one uses backward induction strategies? Is the other Nash equilibrium likely to occur? Why or why not?

(2) In the **Cuban Missile Crisis** game of the previous section, identify Krushchev's five strategies and Kennedy's three strategies. Construct a payoff matrix, and find the Nash equilibria. Which one uses backward induction strategies? Does the Nash equilibrium in which the outcome is different from the backward induction outcome help explain what actually happened historically?

(3) In the **Up and Down** game of the previous section, identify each player's strategies, construct a payoff matrix, find Nash equilibria, and compare with the backward induction strategies.

(4) Change the **Up and Down** game of the previous section in the following ways.
   (a) Have Colin make his second choice at the same time as (rather than after) Rose's only choice. Identify each player's strategies, construct a payoff matrix, and find Nash equilibria.
   (b) Have Rose make her only choice at the same time as Colin's first choice, but still have Colin make his second choice after he knows Rose's choice. Identify each player's strategies, construct a payoff matrix, and find Nash equilibria.
   (c) Have Rose make her only choice and Colin make both choices all at the same time. Identify each player's strategies, construct a payoff matrix, and find Nash equilibria.

(5) **Simple Choices**. Rose can choose "up" or "down". If Rose chooses "down", the game ends with payoffs of 3 to Rose and 10 to Colin. If Rose chooses "up", Colin can choose "up" or "down". If Colin chooses "down", the game ends with payoffs of 1 each to Rose and Colin. If Colin chooses "up", the game ends with payoffs of 10 to Rose and 3 to Colin. Construct a game tree and a payoff matrix for this game. Find the backward induction solution of the game tree and the Nash equilibria for the payoff matrix. Compare.

(6) Modify the **Adam and Eve** sequential game of the previous section by eliminating all of God's "ignore" moves. Identify each player's strategies (Adam and Eve will have three and God will have four), construct a payoff matrix, and find Nash equilibria. Compare with the backward induction strategies.

(7) Describe Rose's remaining eight strategies in the first **Florist** scenario, and complete the outcome matrix. Why can't we construct a payoff matrix?

(8) Identify all of Rose and Colin's strategies in the second **Florist** scenario (when they choose the quantity of flowers and the type of vase simultaneously.) Construct an outcome matrix.

(9) Consider the game tree form of the first **Florist** scenario. Make reasonable assumptions about the preferences Rose and Colin could have for the twelve outcomes. Use these assumptions to assign ordinal payoffs. Find the backward induction strategies and outcome for the resulting sequential game. Find the payoff matrix and Nash equilibrium for the corresponding strategic game.

(10) Consider the game tree form of the second **Florist** scenario. Make reasonable assumptions about the preferences Rose and Colin could have for the twelve outcomes. Use these assumptions to assign ordinal payoffs. Find the payoff matrix and Nash equilibrium for the corresponding strategic game.

(11) In the **Battle of Thermopylae** game of the previous section, identify each player's strategies, construct a payoff matrix, find Nash equilibria, and compare with the backward induction strategies and with what actually transpired.
(12) In the **Vacation** game of the previous section, identify each player's strategies, construct a payoff matrix, find Nash equilibria, and compare with the backward induction strategies.
(13) An especially naive game tree for **The Taming of the Shrew** is shown in Figure 5.15. Identify each player's strategies, construct a payoff matrix, find Nash equilibria, and compare with the backward induction strategies.

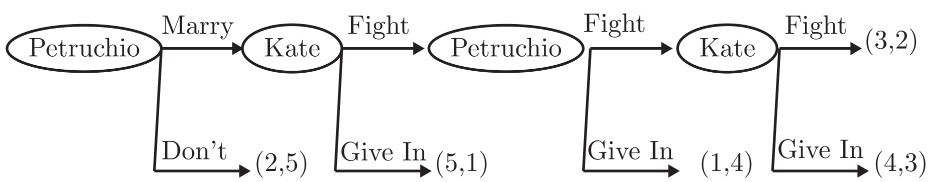

FIGURE 5.15. **The Taming of the Shrew** game tree

(14) **Stereo Equipment** [**64**, page 44]. Zeus Music is an industry leader in modern stereo equipment, a company with strong name recognition in the field. Athena Acoustics is a smaller company, well known for its high quality research and development work. Both companies have been developing a promising new sound system. It is uncertain at this point how large the market for the system will be. It might be small with potential profits of about \$24 million yearly, or it might be large with profits about \$40 million. Analysts at Zeus estimate the chances for small vs. large market at about 50-50. Zeus and Athena must each decide what type of product to produce: a high quality system to appeal to aficionados, or a slightly lower quality system with more gadgetry to appeal to a wider market. The high quality product sells better in the small market, the low quality product better in the large market. Assuming that neither Zeus nor Athena know each other's plans or have any information about the market, the game tree in Figure 5.16 illustrates this situation, with the payoffs pairs going to (Zeus, Athena).
 (a) Why are Athena's two decision nodes in the same information set?
 (b) Identify each of the player's two strategies, and construct a payoff matrix. (Hint: Use the expected value of the payoffs of the small and large markets as the payoffs.)
 (c) Eliminate any dominated strategies, and determine any Nash equilibria.
(15) Use the information from exercise 14, but now assume that Zeus must make its decision before Athena, and that Athena knows this decision before it makes its own decision.
 (a) Draw in the new information sets implied by our changed assumption.
 (b) How many strategies does each of Zeus and Athena have in this new game?
 (c) Construct the payoff matrix.
 (d) Eliminate any dominated strategies, and determine any Nash equilibria.
(16) Use the information from exercise 14, but assume that Athena must make its decision first and that Zeus will know the outcome of Athena's decision.

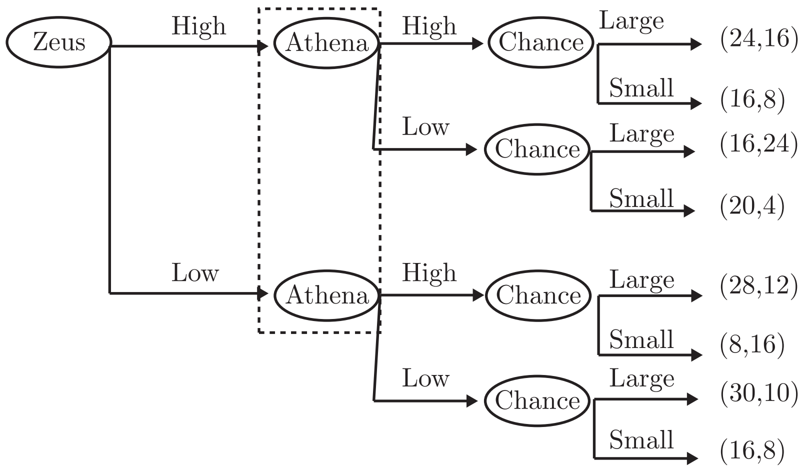

FIGURE 5.16. **Stereo Equipment** game tree

(a) Redraw the game tree showing Athena deciding first and draw in the new information sets implied by our changed assumptions.
(b) How many strategies does each of Zeus and Athena have in this new game?
(c) Construct the payoff matrix.
(d) Eliminate any dominated strategies, and determine any Nash equilibria.

(17) Use the information from exercise 14, but now assume that Zeus has also done some market research and knows whether the market will be small or large. Athena does not know this information, but can wait for Zeus's decision.
(a) Draw in the new information sets implied by our changed assumption. (Hint: Redraw the game tree, putting Chance first.)
(b) How many strategies does each of Zeus and Athena have in this new game?
(c) Construct the payoff matrix.
(d) Eliminate any dominated strategies, and determine any Nash equilibria.

(18) Russell Cooper, Douglas DeJong, Bob Forsythe, and Thomas Ross studied a game with an outside option [**14**]. Rose can either take an outside option in which both players receive a payoff of 300 or she can choose to play the strategic game shown below.

		Colin	
Cardinal payoffs		A	B
Rose	A	$(0,0)$	$(200,600)$
	B	$(600,200)$	$(0,0)$

(a) Construct the payoff matrix for the full game.
(b) Find the prudential strategies and the Nash equilibria.
(c) Discuss what is most likely to happen if two players play this game.

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CHAPTER 4

Probabilistic Strategies

1. It's Child's Play

By the end of this section, you will have considered new strategy concepts involving probabilities.

MARPS is an acronym for Monetary Asymmetric Rock-Paper-Scissors. It is based on the children's game **Rock-Paper-Scissors**. In both games, two players simultaneously shout "rock", "paper", or "scissors". The simultaneity of the shouts can be done by having both players rhythmically shout, "one, two, three" and then the chosen word. To help keep the rhythm and to add movement to the game, children often have one hand strike the open palm of their other hand with each number, and when they shout their chosen word, that hand is a fist to represent a rock, flat to represent paper, or partially closed with two fingers mimicking the two blades of scissors. Simultaneity can also be accomplished by having each player secretly write their chosen word on a piece of paper, and then the two players can show each other what they wrote. Of course, this latter approach is not as much fun as shouts and hands flying!

After the chosen words have been shouted, money may be exchanged. If both players shouted the same word, there is a tie and no money is exchanged. If the two players shout different words, one player wins and receives money from the other player. "Rock" crushes "scissors", and so the player shouting "rock" receives \$2 from the player shouting "scissors". "Scissors" cut "paper", and so the player shouting "scissors" receives \$2 from the player shouting "paper". "Paper" covers "rock", which is a somewhat wimpy victory, and so the player shouting "paper" receives only \$1 from the player shouting "rock". In contrast to the **Rock-Paper-Scissors** children's game, it is the exchange of money that adds the word "Monetary" and the differing amounts of money exchanged that adds the word "Asymmetric" to the title of the game.

Before you begin to read the dialogue below, you may want to play the game yourself. Write down the payoff matrix since it is different from that of the classical **Rock-Paper-Scissors** game, and find someone to play with. Before you play, think about how you might maximize your own payoff. Think about how you would maximize your average payoff if you played the game several times.

ROSE: This sounds like a fun game. Want to play?

COLIN: Sure! But we're talking inside of a game theory book, so we should probably first obtain the game matrices.

ROSE: Well, as long as it doesn't take too long.

COLIN: Because of the simultaneity of the shouts, a strategy can be described simply by our word choice. So, this is what I obtain for the outcome matrix:

MARPS Outcomes		Colin ROCK	Colin PAPER	Colin SCISSORS
Rose	ROCK	Nothing	Colin receives \$1 from Rose	Rose receives \$2 from Colin
Rose	PAPER	Rose receives \$1 from Colin	Nothing	Colin receives \$2 from Rose
Rose	SCISSORS	Colin receives \$2 from Rose	Rose receives \$2 from Colin	Nothing

ROSE (after comparing the matrix to the second paragraph of this section): Yes, I agree. Now we need to obtain a payoff matrix.

COLIN: What do you think each outcome is worth?

ROSE: You're a good friend, Colin, but for a game like this, I'm in it for the money.

COLIN: Me too.

ROSE: So, our cardinal payoffs should correspond directly to dollars received:

MARPS Cardinal Payoffs		Colin ROCK	Colin PAPER	Colin SCISSORS
Rose	ROCK	$(0, 0)$	$(-1, 1)$	$(2, -2)$
Rose	PAPER	$(1, -1)$	$(0, 0)$	$(-2, 2)$
Rose	SCISSORS	$(-2, 2)$	$(2, -2)$	$(0, 0)$

COLIN: That's true as long as we're also risk neutral.

ROSE: Huh?

COLIN: If you were given the choice, would you be indifferent between (1) receiving \$0 with certainty, and (2) having a 50% chance of receiving \$2 and a 50% chance of paying \$2?

ROSE: Well, if instead of receiving or paying \$2 it had been receiving or paying \$200, I'd chose the \$0 with certainty.

COLIN: So, you're risk adverse.

ROSE: Yes, for large quantities of money. But for \$2, it doesn't make much difference to me. So, it's reasonable to assume that I am risk neutral (especially if that will allow us to play the game sooner).

COLIN: I think I'd prefer the receiving or paying \$2 lottery to the \$0 with certainty.

ROSE: So, you are risk loving?

COLIN: Yes, but probably not too much. I'd be willing to accept the above payoff matrix as a good approximation.

ROSE: Let's play!

COLIN: Should we first analyze the game?

ROSE: What's there to analyze?

COLIN: There are prudential strategies...

ROSE: If I shout "paper" or "scissors", I might lose \$2.00. But if I shout "rock", the most that I can lose is \$1, so ROCK is my prudential strategy.

COLIN: Successive elimination of dominated strategies...

ROSE: No strategy dominates another strategy. So, there are no new insights here.

COLIN: ...and Nash equilibrium.

ROSE: Here's the best response diagram:

MARPS		Colin		
Cardinal Payoffs		ROCK	PAPER	SCISSORS
	ROCK	$(0,0)$	$(-1, \boxed{1})$	$(\boxed{2}, -2)$
Rose	PAPER	$(\boxed{1}, -1)$	$(0,0)$	$(-2, \boxed{2})$
	SCISSORS	$(-2, \boxed{2})$	$(\boxed{2}, -2)$	$(0,0)$

Clearly, there are no Nash equilibria.

COLIN: Wow! So, I guess that what we learned in the last chapter is not too useful.

ROSE: I guess this game theory stuff doesn't give us much guidance for how to play the game.

COLIN: If I were to shout "rock"...

ROSE: ...and I knew that you were going to do that, I should shout "paper" and win \$1 from you.

COLIN: But knowing you plan to shout “paper”, I’ll shout “scissors” instead and win $2 from you.

ROSE: I’ll be one step ahead of you, and shout “rock”.

COLIN: I’ve got you covered with “paper”.

ROSE: Very punny.

COLIN: I guess we are just seeing concretely what happens when there’s no Nash equilibrium.

ROSE: Whatever our strategy choices, at least one of us will regret our choice.

COLIN: I’d really like to win $2...

ROSE: Clever. You’re trying to make me think that you’ll shout “rock” or “scissors”, which would mean I should shout “rock” so that I either tie or win $2. Knowing this, you’ll actually shout “paper”. I won’t be fooled.

COLIN: Darn! A guy can hope.

ROSE: So, are you ready to play?

COLIN: I might as well just choose my strategy randomly.

ROSE: Will you really choose your strategy randomly? Or will you make an *ad hoc* decision?

COLIN: I was just going to arbitrarily choose a word. I guess that is really *ad hoc*. But maybe I’ll choose randomly. I have a normal 6-sided die here. I’ll roll it (out of your sight). I’ll shout “rock” if the roll is a 1 or 2, “paper” if the roll is a 3 or 4, and “scissors” if the roll is a 5 or 6.

ROSE: That’s interesting. You’ll choose each word with probability $\frac{1}{3}$.

COLIN: If I can’t fool you, I might as well.

ROSE: So, if I plan to shout “rock”, there’ll be a $\frac{1}{3}$ chance that you’ll also shout “rock” and I’ll receive $0, a $\frac{1}{3}$ chance that you’ll shout “paper” and I’ll lose $1, and a $\frac{1}{3}$ chance that you’ll shout “scissors” and I’ll receive $2.

COLIN: Yes, that’s straight from the payoff matrix.

ROSE: By shouting “rock”, my weighted average or expected payoff would be

$$(\frac{1}{3})(\$0) + (\frac{1}{3})(-\$1) + (\frac{1}{3})(\$2) = \$0.33.$$

COLIN: And my expected payoff must then be $-\$0.33$.

ROSE: Let me check my other strategies. If I shout "paper", my expected payoff would be

$$(\frac{1}{3})(\$1) + (\frac{1}{3})(\$0) + (\frac{1}{3})(-\$2) = -\$0.33.$$

If I shout "scissors", my expected payoff would be

$$(\frac{1}{3})(-\$2) + (\frac{1}{3})(\$2) + (\frac{1}{3})(\$0) = \$0.$$

So, my best approach would be to shout "rock" and win \$0.33 on average.

COLIN: This analysis is becoming a bit unpleasant.

ROSE: I'm actually starting to enjoy it.

COLIN: Well, if you plan to shout "rock", I could plan to shout "paper".

ROSE: But that just leads us back into more countermoves back and forth.

COLIN: Perhaps if I just assign more probability to "paper"... Yes, I'll shout "rock" with probability 0.3, shout "paper" with probability 0.4, and shout "scissors" with probability 0.3.

ROSE: And how do you plan to generate those probabilities?

COLIN: I have some colored beads here. I could mix three red, four purple, and three silver beads in a cup. If I draw a red bead, I shout "rock". If I draw a purple bead, I shout "paper". If I draw a silver bead, I shout "scissors".

ROSE: Okay, you've convinced me that you could carry out your plan.

COLIN: I'm sometimes blinded by my own brilliance.

ROSE: Very enlightening. Now if I shout "rock", my expected payoff would be

$$(0.3)(\$0) + (0.4)(-\$1) + (0.3)(\$2) = \$0.20.$$

If I shout "paper", my expected payoff would be

$$(0.3)(\$1) + (0.4)(\$0) + (0.3)(-\$2) = -\$0.30.$$

If I shout "scissors", my expected payoff would be

$$(0.3)(-\$2) + (0.4)(\$2) + (0.3)(\$0) = \$0.20.$$

It looks like I can still make \$0.20 on average by shouting "rock".

COLIN: That's a bit better on my wallet: \$0.20 is smaller than \$0.33. But now you have two options to earn that money.

ROSE: You're right! I could shout "scissors" instead of "rock".

COLIN: For that matter, you could even randomly choose between the two strategies.

ROSE: That's right! I could flip a coin to decide whether to shout "rock" or "scissors". Thanks for the idea, Colin!

COLIN: You're welcome. But why is it that I'm going into this game already planning to lose money?

ROSE: Perhaps because you're telling me what you plan to do, and I'm choosing the strategy that works the best for me against your plan.

COLIN: Well, I'm an "up front" and honest kind of guy.

ROSE: That's why you tried to fool me earlier.

COLIN: You caught me. Perhaps that deception idea has some merit. At this point, I have you thinking that randomly choosing between "rock" and "scissors" is a good idea.

ROSE: Only because you said that you planned to shout "rock" with probability 0.3, shout "paper" with probability 0.4, and shout "scissors" with probability 0.3.

COLIN: True, but to counter your "rock" or "scissors" choice, perhaps I should assign more probability to my choosing "rock".

ROSE: I'll just change my strategy choice to what is better.

COLIN: Fine. Suppose that I plan to shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2.

ROSE: I'll just look at my expected payoffs again. If I shout "rock", my expected payoff would be

$$(0.4)(\$0) + (0.4)(-\$1) + (0.2)(\$2) = \$0.$$

If I shout "paper", my expected payoff would be

$$(0.4)(\$1) + (0.4)(\$0) + (0.2)(-\$2) = \$0.$$

If I shout "scissors", my expected payoff would be

$$(0.4)(-\$2) + (0.4)(\$2) + (0.2)(\$0) = \$0.$$

What happened to my positive payoffs?

COLIN: Looks like they're gone.

ROSE: And it looks like there's nothing I can do to counter your plan. No matter what I do, my expected payoff is \$0.

COLIN: And if I discern what strategy you're using, I could vary my strategy to take advantage of my knowledge.

ROSE: In that case, perhaps I'll also shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2.

COLIN: I guess that would mean that whatever I tried to do, my expected payoff would be \$0, too.

ROSE: So, if both of us adopt this plan, neither one of us could do any better by adopting a different plan.

COLIN: That makes the pair of plans sound like a Nash equilibrium.

ROSE: And if just one of us adopts the plan, that person already ensures him or herself an expected payoff of at least \$0.

COLIN: That makes the plan sound like a prudential strategy.

ROSE: Of course, that would be if the plan could be considered a strategy. But we already said that there were only three strategies: rock, paper, and scissors.

COLIN: But a strategy is a complete and unambiguous description of what to do in every possible situation. Our plan tells us to shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2. That seems complete and unambiguous.

ROSE: Especially if we add a description of the bead drawing mechanism for how to generate the specified probabilities.

COLIN: Cool! I'm now ready to play **MARPS**.

ROSE: So am I!

COLIN and ROSE together: One, two, three, ...

Exercises

(1) In Chapter 1, we stated that a strategy is a complete and unambiguous description of what to do in every possible situation. Explain why Rose and Colin's use of a probabilistic strategy, which involves random choices, does not violate this definition.
(2) If you played **MARPS** several times, determine the relative frequency for each strategy choice. How do your relative frequencies compare with the probabilities that Rose and Colin adopted?
(3) If "paper" against "rock" won \$2, instead of \$1, how might this have affected Rose and Colin's discussion?
(4) If "paper" against "rock" won \$0.50, instead of \$1, how might this have affected Rose and Colin's discussion?

2. Mixed Strategy Solutions

When we introduced strategic games, our examples seemed to have a small number of strategies. Each player in **MARPS** appears to have only three strategies: shout "rock", shout "paper", or shout "scissors". But each player who is playing a strategic game has an infinite number of strategies because a player can always choose strategies in some probabilistic manner. Rose and Colin each decided to shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2. These probabilistic strategies are usually called mixed strategies, in order to distinguish them from the original strategies, which are then called pure strategies. Note that any pure strategy can be expressed as a mixed strategy: shout "rock" can be expressed as shout "rock" with probability 1.0. Formally, we have

> **Mixed Strategy:** A *mixed strategy* is an assignment of probabilities to strategies. It is usually expressed as a combination of the original, *pure*, strategies.

When we allow players to use mixed strategies, we will determine their payoffs using the Expected Utility Hypothesis. Therefore, the expected payoffs are only meaningful if cardinal, rather than ordinal, payoffs are used when we play the game.

> **Expected Payoff:** The payoff that a player receives when each is using a mixed strategy is calculated by computing the sum of the various pure strategy payoffs weighted by their probabilities. This payoff is called an *expected payoff.*

For **MARPS**, Colin finally suggested that he would shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2. We abbreviate this strategy with the notation

$$0.4 \text{ ROCK } + \ 0.4 \text{ PAPER } + \ 0.2 \text{ SCISSORS.}$$

Rose went on to calculate her expected payoff if she were to shout "rock" as

$$(0.4)(\$0) + (0.4)(-\$1) + (0.2)(\$2) = \$0.$$

There are at least three ways that we might interpret a player's expected payoff for a given mixed strategy. First, the expected payoff is simply the cardinal utility that we assign to the lottery of outcomes associated with the mixed strategy; Rose should be indifferent between (1) Rose and Colin both shouting "scissors" giving her a payoff of \$0, and (2) Rose shouting "rock" and Colin shouting "rock", "paper", or "scissors" with probability 0.4, 0.4, and 0.2, respectively, giving her an expected payoff of \$0. Second, a player's expected payoff may be interpreted as the average of his or her payoffs over many games when players consistently use the same mixed strategy; if Rose shouts "rock" and Colin shouts "rock", "paper", or "scissors" with probability 0.4, 0.4, and 0.2, respectively, Rose's average payoff after several rounds of the game should be about \$0. Third, the expected payoff may be interpreted as the average payoff from many games played by a population of players who use pure strategies in proportion to the mixed strategy probabilities; if there are ten Rose clones each shouting "rock", four Colin clones shouting "rock", four Colin clones shouting “paper”, and two Colin clones shouting “scissors”, then the average payoff for the ten Rose clones would be \$0.

The availability of mixed strategies increases the set of strategy choices that each player has. This means that there may be a mixed strategy that ensures a higher expected payoff than any of the pure strategies; that is, a mixed prudential strategy may be better than a pure prudential strategy. Also while some strategic games do not have a Nash equilibrium in pure strategies, Nash proved the following.

> **Nash Equilibrium Theorem:** *Every strategic game has at least one Nash equilibrium in pure or mixed strategies.*

In the dialogue, it was argued that each player choosing the mixed strategy of shouting “rock”, “paper”, or “scissors” with probability 0.4, 0.4, and 0.2, respectively, would form a Nash equilibrium for **MARPS**.

> By the end of this section, you will be able to calculate each player's expected payoff when players use mixed strategies, to determine whether a pair of mixed strategies is a Nash equilibrium, and to determine whether a given mixed strategy is a prudential strategy.

Calculations

Mixed strategies, mixed prudential strategies, and mixed-strategy Nash equilibria were introduced using **MARPS** in the dialog between Rose and Colin. **MARPS** is a somewhat special game since the players' interests are in complete opposition. We now develop these ideas further using a strategic game, **Impromptu**, in which the players' interests sometimes coincide. Its cardinal payoff matrix is shown here.

Impromptu Cardinal Payoffs		Colin A	Colin B	Colin C
Rose	A	$(10, 100)$	$(100, 50)$	$(60, 40)$
	B	$(0, 60)$	$(80, 70)$	$(70, 0)$
	C	$(20, 0)$	$(40, 30)$	$(50, 90)$

From the best response diagram

Impromptu Cardinal Payoffs		Colin A	Colin B	Colin C
Rose	A	$(10, \boxed{100})$	$(\boxed{100}, 50)$	$(60, 40)$
	B	$(0, 60)$	$(80, \boxed{70})$	$(\boxed{70}, 0)$
	C	$(\boxed{20}, 0)$	$(40, 30)$	$(50, \boxed{90})$

it is clear that there is no pure-strategy Nash equilibrium. Any pair of pure strategies chosen by Rose and Colin will result in at least one of the players wishing he or she had chosen differently. The only hope for stability would be to employ a probabilistic approach to choosing pure strategies; that is, employ a mixed strategy. Thus, one mixed strategy for Colin would be

$$\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C,$$

in which he selects each of his pure strategies with probability $\frac{1}{3}$. Another mixed strategy for Colin is

$$0.7A + 0.1B + 0.2C,$$

in which he selects his first strategy more often than either of the other two.

When the context is clear, we will abbreviate mixed strategies by only listing the probabilities: the two mixed strategies for Colin described in the previous sentence could be abbreviated

$$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

and

$$(0.7, 0.1, 0.2),$$

respectively. A pure strategy can be thought of as a mixed strategy in which all of the probability is concentrated on a single strategy. For example, $(0, 1, 0)$, which represents $0A + 1B + 0C$, is the same as the pure strategy COLUMN B.

Suppose Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ consisting of a 10% chance of choosing strategy ROW A, a 30% chance of choosing strategy ROW B, and a 60% chance of choosing strategy ROW C. If Colin chooses COLUMN A, then the strategy pair (ROW A, COLUMN A) occurs with probability 0.1, the strategy pair (ROW B, COLUMN A) occurs with probability 0.3, and the strategy pair (ROW C, COLUMN A) occurs with probability 0.6. A good way to visualize this is

Impromptu Cardinal Payoffs		Colin A
	$0.1A$	$(10, 100)$
Rose	$0.3B$	$(0, 60)$
	$0.6C$	$(20, 0)$

So, Rose's expected payoff is

$$(0.1)(10) + (0.3)(0) + (0.6)(20) = 13$$

(found by multiplying the probabilities times the payoffs and summing), and Colin's expected payoff is

$$(0.1)(100) + (0.3)(60) + (0.6)(0) = 28.$$

Thus, using our second interpretation of expected payoffs, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses COLUMN A in many repeated plays of **Impromptu**, then Rose's average payoff should be about 13 and Colin's average payoff should be about 28.

Similarly, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses COLUMN B, then the strategy pair (ROW A, COLUMN B) occurs with probability 0.1, the strategy pair (ROW B, COLUMN B) occurs with probability 0.3, and the strategy pair (ROW C, COLUMN B) occurs with probability 0.6. So, Rose's expected payoff is

$$(0.1)(100) + (0.3)(80) + (0.6)(40) = 58,$$

and Colin's expected payoff is

$$(0.1)(50) + (0.3)(70) + (0.6)(30) = 44.$$

Again, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses COLUMN B in many repeated plays of **Impromptu**, then Rose's average payoff should be about 58 and Colin's average payoff should be about 44.

What are the expected payoffs if both players choose mixed strategies? Suppose Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy $(0.8, 0.2, 0.0)$. Since Rose and Colin are making independent decisions, the strategy pair (ROW A, COLUMN A) occurs with probability $(0.1)(0.8) = 0.08$. Similar calculations can be performed for every other strategy pair; they are summarized, along with the payoff pairs, in the following matrix.

Probability & Payoff Pair		Colin $0.8A$	$0.2B$	$0.0C$
	$0.1A$	$(0.08)(10, 100)$	$(0.02)(100, 50)$	$(0.00)(60, 40)$
Rose	$0.3B$	$(0.24)(0, 60)$	$(0.06)(80, 70)$	$(0.00)(70, 0)$
	$0.6C$	$(0.48)(20, 0)$	$(0.12)(40, 30)$	$(0.00)(50, 90)$

So, Rose's expected payoff is

$$\begin{array}{rlclcl} & (0.08)(10) & + & (0.02)(100) & + & (0.00)(60) \\ + & (0.24)(\ 0) & + & (0.06)(\ 80) & + & (0.00)(70) \\ + & (0.48)(20) & + & (0.12)(\ 40) & + & (0.00)(50) \\ = & 22.0, & & & & \end{array}$$

and Colin's expected payoff is

$$\begin{array}{rlclcl} & (0.08)(100) & + & (0.02)(50) & + & (0.00)(40) \\ + & (0.24)(\ 60) & + & (0.06)(70) & + & (0.00)(\ 0) \\ + & (0.48)(\ \ 0) & + & (0.12)(30) & + & (0.00)(90) \\ = & 31.2. & & & & \end{array}$$

Summarizing, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy $(0.8, 0.2, 0.0)$ in many repeated plays of **Impromptu**, then Rose's average payoff should be about 22.0 and Colin's average payoff should be about 31.2.

We could have computed Rose's expected payoff in a different way. Before we do any arithmetic, Rose's expected payoff is given by the following calculation.

$$\begin{array}{rlclcl} & (0.1)(0.8)(10) & + & (0.1)(0.2)(100) & + & (0.1)(0.0)(60) \\ + & (0.3)(0.8)(\ 0) & + & (0.3)(0.2)(\ 80) & + & (0.3)(0.0)(70) \\ + & (0.6)(0.8)(20) & + & (0.6)(0.2)(\ 40) & + & (0.6)(0.0)(50) \end{array}$$

Regrouping these terms gives us the following.

$$\begin{array}{rlclcl} & (0.8)[(0.1)(\ 10) & + & (0.3)(\ 0) & + & (0.6)(20)] \\ + & (0.2)[(0.1)(100) & + & (0.3)(80) & + & (0.6)(40)] \\ + & (0.0)[(0.1)(\ 60) & + & (0.3)(70) & + & (0.6)(50)] \end{array}$$

Simplifying, we obtain

$$(0.8)(13) + (0.2)(58) + (0.0)(57) = 22.0.$$

Thus, Rose's expected payoff when Colin chooses mixed strategy $(0.8, 0.2, 0.0)$ is

$$\begin{gathered}(0.8)\times[\text{Rose's expected payoff when Colin chooses COLUMN A}]\\+\\(0.2)\times[\text{Rose's expected payoff when Colin chooses COLUMN B}]\\+\\(0.0)\times[\text{Rose's expected payoff when Colin chooses COLUMN C}].\end{gathered}$$

In general, the expected payoff associated with a mixed strategy is the weighted average of the expected payoffs associated with the pure strategies.

The conclusion of the previous paragraph implies that the expected payoff associated with a mixed strategy is no greater than the largest payoff associated with a pure strategy that is used with positive probability. In the example above, 22.0 is no greater than the larger of 13 and 58. If Rose were to choose a mixed strategy (p_A, p_B, p_C) to play against Colin's choice of $(0.8, 0.2, 0.0)$, then Rose's expected payoff would be $13p_A + 58p_B + 57p_C$, and this payoff must be no greater than 58, which is the payoff Rose would obtain by choosing the pure strategy ROW B. This shows us that there is always a pure strategy best response by a player to whatever strategy choices are made by the other players.

To illustrate these ideas from Colin's perspective, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy $(0.8, 0.2, 0.0)$, then Colin's expected payoff is

$$(0.8)(28) + (0.2)(44) + (0.0)(58) = 31.2,$$

obtained by multiplying the probability that he is in a given column times his expected payoff in that column (calculated above) and summing these products. More generally, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy (q_A, q_B, q_C), then Colin's expected payoff is

$$(q_A)(28) + (q_B)(44) + (q_C)(58),$$

which is never greater than 58. This implies that if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$, then Colin's best response is COLUMN C.

Solution Verifications

We have already seen that **Impromptu** has no Nash equilibrium using pure strategies, but the Nash Equilibrium Theorem tells us that it does have at least one Nash equilibrium in mixed strategies. Actually finding a Nash equilibrium can be a complex computational task that we will defer to the next two sections. For now we will show how we can determine whether a given pair of mixed strategies is a Nash equilibrium.

Suppose Bill claims that

$$\left(\frac{3}{8}A + 0B + \frac{5}{8}C, \frac{6}{7}A + \frac{1}{7}B + 0C\right) = \left(\frac{3}{8}A + \frac{5}{8}C, \frac{6}{7}A + \frac{1}{7}B\right)$$

is a Nash equilibrium for **Impromptu**. While not concerning ourselves with how Bill came up with this pair of strategies, we will check whether Bill's claim is correct. To do so, we need to determine (1) whether $\frac{3}{8}A + \frac{5}{8}C$ is Rose's best response to Colin choosing $\frac{6}{7}A + \frac{1}{7}B$, and (2) whether $\frac{6}{7}A + \frac{1}{7}B$ is Colin's best response to Rose choosing $\frac{3}{8}A + \frac{5}{8}C$.

We begin by checking Rose's expected payoffs using each of her pure strategies when Colin chooses $\frac{6}{7}A + \frac{1}{7}B$:

Rose's Strategy	Rose's Expected Payoff When Colin Chooses $\frac{6}{7}A + \frac{1}{7}B$
A	$(\frac{6}{7})(10) + (\frac{1}{7})(100) = \frac{160}{7}$
B	$(\frac{6}{7})(\ 0) + (\frac{1}{7})(\ 80) = \frac{80}{7}$
C	$(\frac{6}{7})(20) + (\frac{1}{7})(\ 40) = \frac{160}{7}$

Since Rose's expected payoff is highest for pure strategies A and C, they are the best pure strategy responses for Rose. Any probabilistic mixture of A and C would also give Rose an expected payoff of $\frac{160}{7}$, and so any mixed strategy involving A and C is also a best response for Rose. In particular, $\frac{3}{8}A + \frac{5}{8}C$ is a best response by Rose to Colin choosing $\frac{6}{7}A + \frac{1}{7}B$.

Now we check Colin's expected payoffs using each of his pure strategies when Rose chooses $\frac{3}{8}A + \frac{5}{8}C$:

Colin's Strategy	Colin's Expected Payoff When Rose Chooses $\frac{3}{8}A + \frac{5}{8}C$
A	$(\frac{3}{8})(100) + (\frac{5}{8})(\ 0) = \frac{300}{8}$
B	$(\frac{3}{8})(\ 50) + (\frac{5}{8})(30) = \frac{300}{8}$
C	$(\frac{3}{8})(\ 40) + (\frac{5}{8})(90) = \frac{570}{8}$

Since Colin's expected payoff is highest for pure strategy C, it is the only best response for Colin. In particular, $\frac{6}{7}A + \frac{1}{7}B$ is not a best response of Colin to Rose choosing $\frac{3}{8}A + \frac{5}{8}C$. Hence, the answer to our second question is no. Therefore, $(\frac{3}{8}A + \frac{5}{8}C, \frac{6}{7}A + \frac{1}{7}B)$ is not a Nash equilibrium.

Suppose Andrea claims that $(0.5A+0.1B+0.4C, 0.5A+0.5C)$ is a Nash equilibrium. Again, we will not concern ourselves with how Andrea came up with this pair of strategies (nor will we be concerned with why Andrea uses decimals and Bill uses fractions). We only want to know whether Andrea's claim is correct. We need to ask (1) is $0.5A+0.1B+0.4C$ a best response by Rose to Colin choosing $0.5A+0.5C$, and (2) is $0.5A+0.5C$ a best response by Colin to Rose choosing $0.5A+0.1B+0.4C$?

In order to answer the first question, we check Rose's expected payoffs using each of her pure strategies when Colin chooses $0.5A + 0.5C$:

Rose's Strategy	Rose's Expected Payoff When Colin Chooses $0.5A + 0.5C$
A	$(.5)(10) + (.5)(60) = 35$
B	$(.5)(\ 0) + (.5)(70) = 35$
C	$(.5)(20) + (.5)(50) = 35$

Since Rose's expected payoff is the same for each pure strategy, each pure strategy and any probabilistic mixture of them is a best response for Rose. In particular, $0.5A+0.1B+0.4C$ is a best response by Rose to Colin choosing $0.5A+0.5C$. Hence, the answer to the first question is yes.

In order to answer the second question, we check Colin's expected payoffs using each of his pure strategies when Rose chooses $0.5A + 0.1B + 0.4C$:

Colin's Strategy	Colin's Expected Payoff When Rose Chooses $0.5A + 0.1B + 0.4C$
A	$(.5)(100) + (.1)(60) + (.4)(\ 0) = 56$
B	$(.5)(\ 50) + (.1)(70) + (.4)(30) = 44$
C	$(.5)(\ 40) + (.1)(\ 0) + (.4)(90) = 56$

Since Colin's expected payoff is highest for pure strategies A and C, they and any probabilistic mixture of them are best responses for Colin. In particular, $0.5A+0.5C$ is a best response by Colin to Rose choosing $0.5A+0.1B+0.4C$. Hence, the answer to the second question is yes. Since both answers are yes, that is, each strategy is a best response to the other, $(0.5A+0.1B+0.4C, 0.5A+0.5C)$ is a Nash equilibrium.

We can summarize our calculations in an extended payoff matrix:

	A	B	C	$\frac{6}{7}A + \frac{1}{7}B$	$.5A + .5C$
A	$(10, \boxed{100})$	$(\boxed{100}, 50)$	$(60, 40)$	$(\boxed{\frac{160}{7}}, \frac{650}{7})$	$(\boxed{35}, 70)$
B	$(0, 60)$	$(80, \boxed{70})$	$(\boxed{70}, 0)$	$(\frac{80}{7}, \frac{430}{7})$	$(\boxed{35}, 30)$
C	$(\boxed{20}, 0)$	$(40, 30)$	$(50, \boxed{90})$	$(\boxed{\frac{160}{7}}, \frac{30}{7})$	$(\boxed{35}, 45)$
$\frac{3}{8}A + \frac{5}{8}C$	$(16.25, 37.5)$	$(62.5, 37.5)$	$(53.75, \boxed{71.25})$		
$.5A + .1B$ $+.4C$	$(13, \boxed{56})$	$(74, 44)$	$(57, \boxed{56})$		$(\boxed{35}, \boxed{56})$

The two new rows correspond to Rose using the mixed strategies $\frac{3}{8}A + \frac{5}{8}C$ and $0.5A+0.1B+0.4C$ against Colin using one of his pure strategies. We had previously computed Colin's expected payoffs; the table includes Rose's expected payoffs. The two new columns correspond to Colin using the mixed strategies $\frac{6}{7}A + \frac{1}{7}B$ and $0.5A + 0.5C$ against Rose using one of her pure strategies. We had previously computed Rose's expected payoffs; the table includes Colin's expected payoffs. One additional pair of expected payoffs is included to indicate the Nash equilibrium.

Since there are an infinite number of mixed strategies available to each player, completing the payoff table would require an infinite number of rows and columns. While we can imagine such a payoff table, we would not want to try writing it down. Furthermore, the whole table is not needed to determine whether a strategy pair is a Nash equilibrium. As we have observed, best response payoffs always occur at pure strategy payoffs. Given a mixed strategy for Rose, all of Colin's best responses are probabilistic mixtures of his pure strategy best responses. Similarly, given a mixed strategy for Colin, all of Rose's best responses are probabilistic mixtures of her pure strategy best responses. If the two strategies are best responses to each other, we have a Nash equilibrium.

It is a bit more difficult to determine whether a given mixed strategy is a prudential strategy, but it can be done. In **Impromptu**, we can show ROW C is a prudential strategy for Rose. If Rose chooses ROW C, she could receive anywhere from 20 to 50 depending on which pure or mixed strategy Colin chooses. Thus, by choosing ROW C, Rose ensures herself a payoff of at least 20. However, if Colin chooses COLUMN A, Rose will receive at most 20, depending on what pure or mixed strategy she uses. We see that Rose can guarantee herself a payoff of at least 20 by selecting ROW C, but cannot guarantee anything better since Colin may select COLUMN A. Thus, ROW C is prudential for Rose.

We claim that Colin's prudential strategy is $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$. To verify this claim, we first determine that, by using $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$, Colin receives a payoff of 48 regardless of what Rose chooses to do (see exercise 4). Further, if Rose chooses the mixed strategy $0.3A + 0.3B + 0.4C$, Colin will receive a payoff of 48 no matter what strategy he selects (see exercise 4). Thus, Rose's choice of strategy can reduce Colin's payoff to no more than 48. Since Colin can guarantee at least 48 by choosing $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$, but he cannot ensure more than this, 48 is his security level and the mixed strategy $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$ is prudential.

In the previous several paragraphs, we gave you strategies that turned out to form a Nash equilibrium and other strategies that turned out to be prudential strategies. Given a set of mixed strategies for a game, it is easy to determine whether they are a Nash equilibrium, and somewhat more difficult to determine whether either is a prudential strategy. However, it is generally a difficult task to actually find strategies that either are part of a Nash equilibrium or are prudential. In the next two sections, we will demonstrate how to do this in the two simplest cases: 2×2 and $m \times 2$ strategic games. The solution methods described in those sections illustrate the general solution method, but for larger games the algebraic manipulations become significantly more complicated.

Implementation and Interpretation

In **Impromptu**, the security levels for Rose and Colin are 20 and 48, respectively, while the payoffs from the Nash equilibrium described earlier are 35 and 56. Hence, there is no real incentive for either player to choose their prudential strategy instead of the Nash equilibrium strategies.

In the Nash equilibrium, Rose uses the strategy $0.5A + 0.1B + 0.4C$ and Colin uses the strategy $0.5A+0.5C$. How do Rose and Colin implement these mixed strategies? Colin could toss a fair coin, choose A if the coin lands heads, and choose C if the coin lands tails. Rose could place five azure, one blue, and four cyan beads in a can, shake the can, choose a bead from the can without looking, choose A if the chosen bead is azure, choose B if the chosen bead is blue, and choose C if the chosen bead is cyan. Rose and Colin could also implement their mixed strategies using dice, cards, or a computer random number generator.

But what will happen? It all depends on the random coin flip and bead choice. If the bead is blue and the coin lands heads, Rose will choose B and Colin will choose A, resulting in a payoff of 0 for Rose and 60 for Colin. Wouldn't Rose now regret her choice? Yes, she would: given that Colin chose A, Rose would have preferred to have chosen C. But her regret is *ex post* (after the bead has been chosen). When Rose was choosing her strategy, she may have thought that Colin would use the mixed strategy $0.5A + 0.5C$. Given that thought, Rose receives an expected payoff of 35 no matter what strategy she chooses (as shown earlier). So, *ex ante* (before the bead has been chosen), Rose does not regret choosing the mixed strategy $0.5A+0.1B + 0.4C$.

Since choosing a strategy randomly feels like a loss of control for most people and it is the *ex post* payoff that a player will actually receive at the end of the game, it is tempting to abandon using a mixed strategy. It is the desire to avoid the overt appearance of randomness that is the reason why humans throughout history have consulted astrologers, shamans, and counselors. But as soon as Colin decides on the pure strategy A in **Impromptu** and Rose realizes that Colin would make that decision, then Rose would choose C, whereupon Colin making this realization would switch to C, and once again there is a cycling of changing choices. If she knows that Colin will choose the mixed strategy $0.5A + 0.5C$, Rose will be willing to stick with the mixed strategy $0.5A + 0.1B + 0.4C$.

Of course, if Colin is using $0.5A + 0.5C$, Rose would receive the same expected payoff regardless of which pure or mixed strategy she uses because all of her pure strategies are best responses. So, Rose should be willing to choose any strategy, not just $0.5A+0.1B+0.4C$. The reason she is choosing the specific mixed strategy $0.5A + 0.1B + 0.4C$ is to make sure that Colin would not regret choosing $0.5A + 0.5C$.

How would Rose know that Colin is using a mixed strategy? After all, in the end, she only knows what pure strategy Colin announces. Even if Rose sees Colin flipping a coin before announcing his choice, she does not know whether Colin actually makes use of the coin flip information (maybe Colin just likes to flip coins). However, if Rose and Colin publicly announce that they will be using $0.5A + 0.1B + 0.4C$ and $0.5A + 0.5C$, then neither has an incentive do anything other than what they publicly announced. This is the inherent meaning in $(0.5A + 0.1B + 0.4C, 0.5A + 0.5C)$ being a Nash equilibrium. Of course, if Rose has an ability to predict Colin's choice (perhaps because his eyes twitch faster when he is about to choose B), she should make use of that information in order to make her choice. But if Rose cannot predict Colin's choice, Nash tells her to randomize her choices with the mixed strategy $0.5A + 0.1B + 0.4C$.

If **Impromptu** is played repeatedly, then it seems even more reasonable for Rose and Colin to choose the mixed strategies $0.5A + 0.1B + 0.4C$ and $0.5A + 0.5C$, because even though their payoffs will vary in each round, on average Rose will obtain 35 and Colin will receive 56, and neither player can obtain a higher payoff by unilaterally changing her or his strategy. The difference between the *ex ante* and *ex post* payoffs has been eliminated by the averaging that occurs in repeated play.

Instead of playing once or repeatedly involving sentient players, suppose we replace Rose and Colin with populations of mindless Rose and Colin clones that play each other. Suppose 50%, 10%, and 40% of the Rose clones are genetically programmed to use pure strategies A, B, and C, respectively. We call these Rose A clones, Rose B clones, and Rose C clones, respectively. Similarly, suppose that 50% of Colin's clones are genetically programmed to use pure strategy A, and the other 50% of Colin's clones are genetically programmed to use pure strategy C. When a Rose clone interacts with a Colin clone, progeny are produced. For example, when a Rose C clone interacts with a Colin B clone, this corresponds to the strategy choices (C, B), and the payoff pair $(40, 30)$ results in 40 new Rose C clones and 30 new Colin B clones.

Since the initial distributions of clones correspond to the Nash equilibrium, the population distributions will remain the same. That is, there will continue to be 50%, 10%, and 40% of the Rose clones genetically programmed to use pure strategies A, B, and C, and equal numbers of Colin clones genetically programmed to use pure strategies A and C. Any other non-Nash equilibrium distribution of clones would change over time. Biologists call our Nash equilibrium distributions evolutionarily stable genetic population distributions.

To experience the population interpretation for mixed strategies, play **Impromptu** in the following manner. Gather a group of at least five friends. Have each person play the role of Rose by secretly writing down on one piece of paper their name and ROW A, ROW B, or ROW C; have each person play the role of Colin by secretly writing down on one piece of paper their name and COLUMN A, COLUMN B, or COLUMN C. Collect the two sets of papers and publicly display the relative frequency distribution of Rose's strategy choices and the relative frequency distribution of Colin's strategy choices. Each player's Rose payoff is computed using their written Rose strategy versus Colin's relative frequency distribution, and similarly, each player's Colin payoff is computed using their written Colin strategy versus Rose's relative frequency distribution.

Therefore, we have at least three reasonable interpretations of the Nash equilibrium: *ex ante* regret free in a single play of the game, long term regret free in repeated play of the game, or stable population distributions. Which one is applicable depends on the original scenario that was modeled as a strategic game.

Unfortunately, difficulties arise in all three of these interpretations. When there is more than one Nash equilibrium, none of the interpretations help us select among them. Second, if the players are playing repeatedly, there could be important interactions across rounds. These two ideas are explored more in Chapter 5. Third, if populations of Rose and Colin clones do not start in a Nash equilibrium distribution, the distribution may not converge to the Nash equilibrium distribution. This final difficulty is beyond the scope of this book.

Exercises

(1) Consider the following game.

Cardinal Payoffs (Rose, Colin)		Colin A	B
Rose	A	$(2,4)$	$(1,0)$
	B	$(3,1)$	$(0,4)$

(a) What are the players' expected payoffs if Rose chooses A and Colin chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(b) What are the players' expected payoffs if Rose chooses B and Colin chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(c) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$ and Colin chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(d) What is Rose's best response(s) to Colin choosing the mixed strategy $\frac{4}{7}A+\frac{3}{7}B$?
(e) What is the lowest payoff Colin could receive if he chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(f) What are the players' expected payoffs if Rose chooses A and Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(g) What are the players' expected payoffs if Rose chooses B and Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(h) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(i) What is Rose's best response(s) to Colin choosing the mixed strategy $\frac{1}{2}A+\frac{1}{2}B$?
(j) What is the lowest payoff Colin could receive if he chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(k) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses A?
(l) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses B?
(m) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses $\frac{1}{2}A + \frac{1}{2}B$?
(n) What is (are) Colin's best response(s) to Rose choosing the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$?

(o) What is the lowest payoff Rose could receive is she chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$?

(p) Based on part (d), explain why $(\frac{3}{7}A + \frac{4}{7}B, \frac{4}{7}A + \frac{3}{7}B)$ is not a Nash equilibrium.

(q) Based on parts (i) and (n), explain why $(\frac{3}{7}A + \frac{4}{7}B, \frac{1}{2}A + \frac{1}{2}B)$ is a Nash equilibrium.

(r) Explain how Rose and Colin could implement the Nash equilibrium strategies described in part (q).

(s) If Rose and Colin use the Nash equilibrium strategies described in part (q) to play this game 100 times, roughly how many times will each outcome occur?

(t) Based on parts (e) and (j), explain why $\frac{1}{2}A + \frac{1}{2}B$ is not Colin's prudential strategy.

(u) Based on parts (e) and (k)–(m), explain why $\frac{4}{7}A + \frac{3}{7}B$ is a prudential strategy for Colin.

(v) Explain why A is prudential for Rose (even among mixed strategies).

(2) Consider the following game.

Cardinal Payoffs (Rose, Colin)		Colin A	B
	A	$(-3, 3)$	$(5, -5)$
Rose	B	$(-1, 1)$	$(3, -3)$
	C	$(2, -2)$	$(-2, 2)$
	D	$(3, -3)$	$(-6, 6)$

(a) Determine the pure prudential strategies, the dominated strategies, and the pure strategy Nash equilibria.

(b) Suppose Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$.
  (i) What is Rose's expected payoff if she chooses A?
  (ii) What is Rose's expected payoff if she chooses B?
  (iii) What is Rose's expected payoff if she chooses C?
  (iv) What is Rose's expected payoff if she chooses D?
  (v) What is Rose's best response(s) to Colin choosing $\frac{1}{2}A + \frac{1}{2}B$?

(c) What is Rose's best response(s) to Colin choosing the mixed strategy $\frac{5}{8}A + \frac{3}{8}B$?

(d) What is Colin's best response(s) to Rose choosing $\frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D$?

(e) What is Colin's best response(s) to Rose choosing $\frac{1}{2}B + \frac{1}{2}C$?

(f) Is $(\frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D, \frac{1}{2}A + \frac{1}{2}B)$ a Nash equilibrium? Why or why not?

(g) Is $(\frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D, \frac{5}{8}A + \frac{3}{8}B)$ a Nash equilibrium? Why or why not?

(h) Is $(\frac{1}{2}B + \frac{1}{2}C, \frac{1}{2}A + \frac{1}{2}B)$ a Nash equilibrium? Why or why not?

(i) Is $(\frac{1}{2}B + \frac{1}{2}C, \frac{5}{8}A + \frac{3}{8}B)$ a Nash equilibrium? Why or why not?

(j) If Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$, what is the lowest expected payoff he can obtain?

(k) If Colin chooses the mixed strategy $\frac{5}{8}A + \frac{3}{8}B$, what is the lowest expected payoff he can obtain?

(l) Based on parts (j) and (k), is $\frac{1}{2}A + \frac{1}{2}B$ a prudential strategy for Colin? Why or why not?

(m) If Rose chooses the mixed strategy $\frac{1}{2}B + \frac{1}{2}C$, what is the lowest expected payoff she can obtain?

(n) Based on parts (k) and (m), and the observation that the two players payoffs always sum to zero, explain why $\frac{1}{2}B + \frac{1}{2}C$ is a prudential strategy for Rose, and why $\frac{5}{8}A + \frac{3}{8}B$ is a prudential strategy for Colin.
(o) Is there any relationship between the prudential strategies and the Nash equilibrium strategies?
(p) How could you implement the mixed strategies $\frac{1}{2}B + \frac{1}{2}C$ and $\frac{5}{8}A + \frac{3}{8}B$?

(3) Consider the following game.

Cardinal Payoffs (Rose, Colin)		Colin A	B	C
Rose	A	$(50, 100)$	$(0, 30)$	$(30, 0)$
	B	$(80, 0)$	$(50, 20)$	$(20, 10)$
	C	$(0, 70)$	$(100, 40)$	$(50, 60)$

(a) If Rose chooses the mixed strategy $\frac{1}{2}B + \frac{1}{2}C$, what would Colin's best response(s) be?
(b) If Colin chooses the mixed strategy $\frac{3}{11}A + \frac{8}{11}C$, what would Rose's best response(s) be?
(c) Explain why the pair of strategies described in parts (a) and (b) form a Nash equilibrium, and how you would implement those strategies.
(d) If Rose chooses the mixed strategy $\frac{5}{11}B + \frac{6}{11}C$, what is Rose's minimum payoff?
(e) If Colin chooses the mixed strategy $\frac{3}{11}A + \frac{8}{11}C$, what is Rose's maximum payoff?
(f) Based on parts (d) and (e), explain why $\frac{3}{11}A + \frac{8}{11}C$ is a prudential strategy for Rose.
(g) If Colin chooses the mixed strategy B, what is Colin's minimum payoff?
(h) If Rose chooses the mixed strategy B, what is Colin's maximum payoff?
(i) Based on parts (g) and (h), explain why B is a prudential strategy for Colin.
(j) How would you expect Rose and Colin to play this game?

(4) For **Impromptu**, show that by using $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$, Colin receives a payoff of 48 regardless of what Rose chooses to do. Further, show that if Rose chooses the mixed strategy $0.3A + 0.3B + 0.4C$, Colin will receive a payoff of 48 no matter what strategy he selects. Explain why this shows that $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$ is prudential for Colin.

(5) Consider the cardinal payoff matrix obtained for **Risk-Neutral MARPS** in the dialogue, displayed again here.

Risk-Neutral MARPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS
Rose	ROCK	$(0, 0)$	$(-1, 1)$	$(2, -2)$
	PAPER	$(1, -1)$	$(0, 0)$	$(-2, 2)$
	SCISSORS	$(-2, 2)$	$(2, -2)$	$(0, 0)$

(a) Explain why this assumes self-interested and risk-neutral players.
(b) Explain why (0.4ROCK + 0.4PAPER + 0.2SCISSORS, 0.4ROCK + 0.4PAPER + 0.2SCISSORS) is a Nash equilibrium.
(c) Explain why 0.4ROCK + 0.4PAPER + 0.2SCISSORS is a prudential strategy for each player.

(d) How would you play a single round of **Risk-Neutral MARPS**? Several rounds of **Risk-Neutral MARPS**?

(6) Consider the following modified cardinal payoff matrix for **MARPS**.

Risk-Varying MARPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS
Rose	ROCK	$(0,0)$	$(-\frac{1}{2},\frac{1}{2})$	$(2,-2)$
	PAPER	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$	$(-2,2)$
	SCISSORS	$(-2,2)$	$(2,-2)$	$(0,0)$

(a) Explain why this assumes self-interested players who are risk adverse for losses (which is why people buy property insurance) and risk loving for gains (which is why people buy lottery tickets).

(b) Explain why ($\frac{4}{9}$ROCK + $\frac{4}{9}$PAPER + $\frac{1}{9}$SCISSORS, $\frac{4}{9}$ROCK + $\frac{4}{9}$PAPER + $\frac{1}{9}$SCISSORS) is a Nash equilibrium.

(c) Explain why $\frac{4}{9}$ROCK + $\frac{4}{9}$PAPER + $\frac{1}{9}$SCISSORS is a prudential strategy for each player.

(d) How would you play a single round of **Risk-Varying MARPS**? Several rounds of **Risk-Varying MARPS**?

(7) **RPS**. Consider the following modified cardinal payoff matrix for **MARPS**:

RPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS
Rose	ROCK	$(0,0)$	$(-1,1)$	$(1,-1)$
	PAPER	$(1,-1)$	$(0,0)$	$(-1,1)$
	SCISSORS	$(-1,1)$	$(1,-1)$	$(0,0)$

(a) Explain why this assumes that the players only care about a win versus a tie versus a loss.

(b) Explain why ($\frac{1}{3}$ROCK + $\frac{1}{3}$PAPER + $\frac{1}{3}$SCISSORS, $\frac{1}{3}$ROCK + $\frac{1}{3}$PAPER + $\frac{1}{3}$SCISSORS) is a Nash equilibrium.

(c) Explain why $\frac{1}{3}$ROCK + $\frac{1}{3}$PAPER + $\frac{1}{3}$SCISSORS is a prudential strategy for each player.

(d) How would you play a single round of **RPS**? Several rounds of **RPS**?

(e) Gintis [**19**, page 67] describes the side-blotched lizard *Uta stansburiana* as having three distinct male types. The orange-throats are violently aggressive, keep large harems of females, and defend large territories. The blue-throats are less aggressive, keep small harems, and defend small territories. The yellow-stripes are very docile but they look like females, so they can infiltrate another male's territory and secretly copulate with the females. So, in pairwise competitions for passing on their genes, orange-throats beat out blue throats who beat out yellow-stripes who beat out orange-throats. Explain how this scenario can be modeled by **RPS**. What should the population distribution of male *Uta stansburiana* be?

(8) **Bacteria**. Pairs of *Simplicio illustratum* often compete for food. Three behavioral patterns have been observed: AGGRESSIVE, ASSERTIVE, and COMPLIANT. An AGGRESSIVE *illustratum* will immediately attack its competitor and will fight until it wins the food or is injured. A COMPLIANT *illustratum* will only posture in hopes that this will scare off its opponent. An ASSERTIVE *illustratum* will posture first but eventually make a brief attack

(before making a hasty retreat) in hopes of scaring off its opponent. The cardinal payoff matrix assumes that the food is worth 60, posturing costs 10, a brief attack costs 20, and a long fight costs 60. So, when two AGGRESSIVE *illustrati* compete, they have a long fight and end up splitting the food, resulting in each player receiving $\frac{1}{2}60 - 60 = -30$. When an AGGRESSIVE competes with an ASSERTIVE, there is a brief attack that the AGGRESSIVE wins resulting in $60 - 20 = 40$ for the AGGRESSIVE and $0 - 20$ for the ASSERTIVE. When an AGGRESSIVE competes with a COMPLIANT, the immediate attack results in an immediate win for the AGGRESSIVE without any substantive effort on either *illustratum*'s part, resulting in 60 for the AGGRESSIVE and 0 for the COMPLIANT. When two ASSERTIVE *illustrati* compete, they have a brief attack and then split the food, resulting in each *illustratum* receiving $\frac{1}{2}60 - 20 = 10$. When an ASSERTIVE competes with a COMPLIANT, both posture and the ASSERTIVE eventually attacks and wins the food, resulting in $60 - 10 = 50$ for the ASSERTIVE and $0 - 10 = -10$ for the COMPLIANT. Finally, when two COMPLIANT *illustrati* compete, they both posture and eventually split the food, resulting in each player receiving $\frac{1}{2}60 - 10 = 20$.

Cardinal Payoffs	AGGRESSIVE	ASSERTIVE	COMPLIANT
AGGRESSIVE	$(-30, -30)$	$(40, -20)$	$(60, 0)$
ASSERTIVE	$(-20, 40)$	$(10, 10)$	$(50, -10)$
COMPLIANT	$(0, 60)$	$(-10, 50)$	$(20, 20)$

(a) Explain why each player using

$$\frac{7}{12}\text{AGGRESSIVE} + \frac{1}{12}\text{ASSERTIVE} + \frac{4}{12}\text{COMPLIANT}$$

forms a Nash equilibrium.

(b) What does the Nash equilibrium tell us about the original scenario?

(9) Construct a new strategic game in which it would be wise for Rose to play the pure strategy $(0, 1, 0)$.

(10) In this exercise, we will prove that $((0.6 - t)A + tB + 0.4C, 0.5A + 0.5C)$ for all t between 0 and 0.3 are the Nash equilibria for **Impromptu**.

(a) What is Rose's best response if Colin chooses $0.5A + 0.5C$?

(b) What is Colin's best response if Rose chooses $(0.6 - t)A + tB + 0.4C$? Your answer will depend on whether $t < 0.3$, $t = 0.3$, or $t > 0.3$.

(c) Based on parts (a) and (b), explain why $((0.6 - t)A + tB + 0.4C, 0.5A + 0.5C)$ is a Nash equilibrium for all t between 0 and 0.3.

(d) Suppose that in a Nash equilibrium, Rose uses a pure strategy. Find a contradiction to this supposition. Thus, in a Nash equilibrium, Rose cannot use a pure strategy.

(e) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1 - p)A + pB$ where $0 < p < 1$. Find a contradiction to this supposition. Thus, in a Nash equilibrium, Rose cannot use a strategy of this form.

(f) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1 - p)B + pC$ where $0 < p < 1$. Find a contradiction to this supposition. Thus, in a Nash equilibrium, Rose cannot use a strategy of this form.

(g) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1 - p)A + pC$ where $0 < p < 1$. Prove that $p = 0.4$ and Colin is using the strategy $0.5A+0.5C$, which corresponds to the Nash equilibrium described above with $t = 0$.

(h) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1 - p - q)A + pB + qC$, where $0 < p$, $0 < q$, and $p + q < 1$. Prove that $q = 0.4$, $0 < p < 0.3$, and Colin is using the strategy $0.5A + 0.5C$, which correspond to the Nash equilibria described above with $t = p$.

3. Finding Solutions in 2×2 Games

In the previous section, we claimed that prudential mixed strategies always exist and that Nash equilibria in mixed strategies always exist. We also saw how one might verify that a suggested strategy is prudential and that a pair of strategies is a Nash equilibrium. But we did not show how to find these strategies. In this section, we provide a method, with justification, for finding prudential strategies and Nash equilibria for 2×2 games.

> By the end of this section, you will be able to find Nash equilibria and prudential strategies for 2×2 games.

Nash Equilibrium in Impromptu-2

Consider the game **Impromptu-2** shown here.

Impromptu-2 Cardinal Payoffs		Colin A	B
Rose	A	$(10, 100)$	$(60, 40)$
	B	$(100, 30)$	$(40, 70)$

Recall that in a Nash equilibrium pair, each player's strategy is a best response to the other player's strategy. By looking at the best response diagram, we can see that there is no Nash equilibrium using only pure strategies in this game. Thus, if we hope to find a Nash equilibrium, we must look at mixed strategies.

Suppose Rose plays the mixed strategy $\frac{1}{4}A + \frac{3}{4}B$. Then Colin's expected payoff using COLUMN A is

$$\frac{1}{4}(100) + \frac{3}{4}(30) = 47.5$$

while his expected payoff using COLUMN B is

$$\frac{1}{4}(40) + \frac{3}{4}(70) = 62.5.$$

Clearly, Colin would prefer COLUMN B to COLUMN A. As observed in the previous section, Colin's expected payoff using a mixed strategy will be a weighted average of 47.5 and 62.5. So, Colin's only best response is the pure strategy COLUMN B.

Similarly, if Rose plays the mixed strategy $0.8A + 0.2B$, Colin's expected payoff using COLUMN A is

$$(0.8)(100) + (0.2)(30) = 86$$

and his expected payoff using COLUMN B is

$$(0.8)(40) + (0.2)(70) = 46.$$

Again, Colin has a unique best response, in this case strategy COLUMN A.

To find a Nash equilibrium for this game, both COLUMN A and COLUMN B must be best responses. Thus, Rose's goal is to identify a mixed strategy for which Colin's best response is either of his two pure strategies or, equivalently, such that Colin is indifferent among his strategy choices.

Since Rose is selecting a strategy to influence Colin's behavior, she will need to use Colin's payoffs in her calculations. To find her Nash equilibrium strategy, Rose wants to select a strategy so that Colin gets the same expected payoff regardless of which pure strategy he selects. Let's call Rose's currently unknown strategy $(1-p)A + pB$. If Colin chooses COLUMN A, his expected payoff is

$$100(1-p) + 30p$$

and if he chooses COLUMN B, then his expected payoff is

$$40(1-p) + 70p.$$

When we set these equal to each other, we can solve for a value of p that will give Colin the same expected payoff regardless of what he chooses to do. So, let's solve for p:

$$\begin{aligned}
100(1-p) + 30p &= 40(1-p) + 70p, \\
100 - 100p + 30p &= 40 - 40p + 70p, \\
100 - 70p &= 40 + 30p, \\
100 - 40 &= 70p + 30p, \\
60 &= 100p, \\
p &= 0.6.
\end{aligned}$$

Thus, if Rose uses the mixed strategy $(0.4, 0.6)$, Colin's expected payoff of 58 can be found by substituting $p = 0.6$ into either the left or right side of the first equation above, as we show here.

$$100(1-0.6) + 30(0.6) = 40(1-0.6) + 70(0.6) = 58$$

In Figure 3.1, each point on the horizontal axis represents one of Rose's mixed strategies $(1-p)A + pB$; the horizontal axis is restricted to the interval from 0 to 1 because p is a probability. We have graphed the expected payoff Colin obtains by choosing COLUMN A

$$y = 100(1-p) + 30p,$$

and the expected payoff Colin obtains by choosing COLUMN B

$$y = 40(1-p) + 70p.$$

Not simplifying these equations makes it easy to draw each line: $p = 0$ corresponds to Rose choosing ROW A and so the vertical coordinate of each of the two lines can be read from the payoff matrix, 100 and 40; $p = 1$ corresponds to Rose choosing ROW B, and so the vertical coordinate of each of the two lines can again be read from the payoff matrix, 30 and 70.

The dotted vertical lines represent Rose's two strategy choices described near the beginning of this section. The left dotted vertical line corresponds to Rose choosing the mixed strategy $(0.8, 0.2)$, and it can be seen that Colin obtains a larger expected

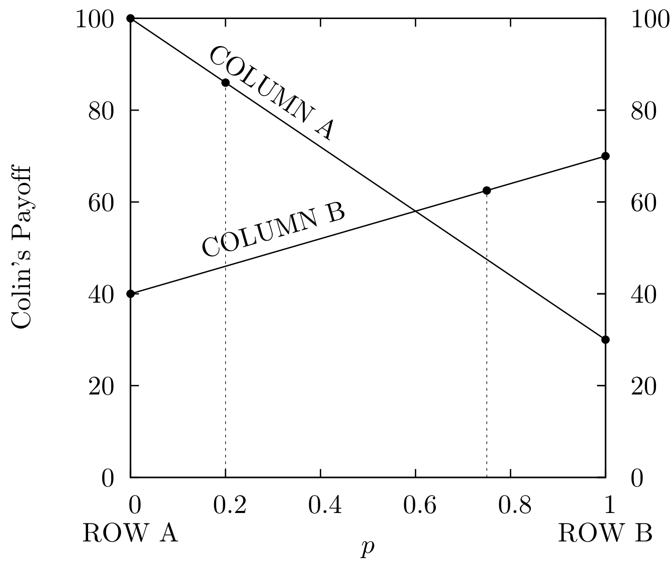

FIGURE 3.1. Rose's strategy options

payoff by using COLUMN A instead of COLUMN B. Hence, COLUMN A is Colin's best response to $(0.8, 0.2)$; the bold point is analogous to the boxed payoff in a best response diagram. Similarly, the right dotted vertical line illustrates that COLUMN B is Colin's best response to Rose's strategy $(0.25, 0.75)$.

Colin can stay on the upper expected payoff "edges" by selecting his best responses to Rose's strategies. For $p < 0.6$, Colin's best response is COLUMN A and for $p > 0.6$, Colin's best response is COLUMN B. These are bold in Figure 3.2.

Thus, there is a unique best response for all values of p except one: when $p = 0.6$, COLUMN A, COLUMN B, and any nonpure mixed strategy of COLUMN A and COLUMN B, are all best responses for Colin.

We can repeat the same process from Colin's perspective. Suppose that he plays COLUMN A the fraction $1-q$ of the time and COLUMN B the fraction q. If Rose plays ROW A, her expected payoff is $10(1-q)+60q$, and if she plays ROW B, her expected payoff is $100(1-q)+40q$. Equating these, we find

$$\begin{aligned} 10(1-q)+60q &= 100(1-q)+40q, \\ 50q+10 &= -60q+100, \\ 110q &= 90, \\ q &= \frac{9}{11}. \end{aligned}$$

Thus, if Colin uses the strategy $(\frac{2}{11}, \frac{9}{11})$, Rose's expected payoff is 50.9, as shown in Figure 3.3.

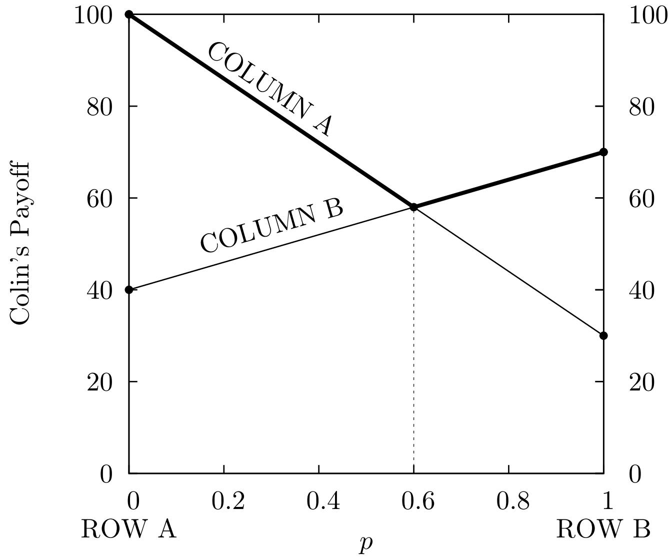

FIGURE 3.2. Rose's equalizing strategy

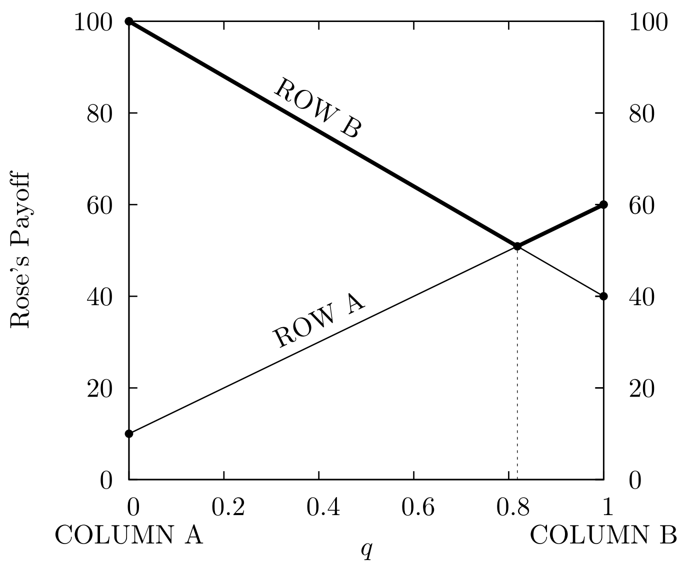

FIGURE 3.3. Colin's equalizing strategy

Once Rose and Colin are playing the mixed strategies $(\frac{2}{5}, \frac{3}{5})$ and $(\frac{2}{11}, \frac{9}{11})$, the expected payoff is $(50.9, 58)$, and, because they are each indifferent to any strategy changes that they might make independently of each other, this strategy pair is a Nash equilibrium.

Example-tu

In the following game, **Example-tu**, there are both pure strategy and nonpure mixed strategy Nash equilibria.

Example-tu Cardinal payoffs		Colin A	B
Rose	A	$(0,5)$	$(4,12)$
	B	$(2,7)$	$(3,2)$

The two pure strategy Nash equilibria are (A, B) and (B, A). To find Rose's mixed strategy $(1-p, p)$ that makes Colin indifferent between his two pure strategies, we set the two expressions

$$5(1-p)+7p$$

and

$$12(1-p)+2p$$

equal to each other and solve for p. To emphasize the point, notice that the coefficients in these equations are Colin's payoffs in the first and second columns, respectively. Solving $5(1-p)+7p = 12(1-p)+2p$ for p, we find that Rose's strategy is $(\frac{5}{12}, \frac{7}{12})$ and Colin's expected payoff is $\frac{74}{12}$.

Similarly, to find Colin's mixed strategy $(1-q, q)$ for which Rose is indifferent between her two pure strategies, we set the two expressions

$$0(1-q)+4q$$

and

$$2(1-q)+3q$$

equal to each other and solve for q. Then his strategy is $(\frac{1}{3}, \frac{2}{3})$ and Rose's expected payoff is $\frac{8}{3}$.

Thus we see that some games have Nash equilibria only in mixed strategies, others only in pure strategies, and that still others have Nash equilibria in both pure and mixed strategies. This discussion leaves us with one final question: Do Nash equilibria always exist? The answer to this is yes, and proving this result was part of the reason that John Nash won a Nobel Prize in Economics. While we present the general result here, we will develop the proof only for the 2×2 case, following the work of Webb [**70**].

Nash Equilibrium Theorem: *Every strategic game has at least one Nash equilibrium in pure or mixed strategies.*

PROOF. Consider the general 2×2 game

Cardinal Payoffs (Rose, Colin)		Colin A	B
Rose	A	(a,b)	(c,d)
	B	(e,f)	(g,h)

By considering best response diagrams, we see that if $a \geq e$ and $b \geq d$, then (A, A) is a Nash equilibrium; if $e \geq a$ and $f \geq h$, then (B, A) is a Nash equilibrium; if

$c \geq g$ and $d \geq b$, then (A, B) is a Nash equilibrium; and if $g \geq c$ and $h \geq f$, then (B, B) is a Nash equilibrium.

For this game to have no pure strategy Nash equilibrium, none of the four strategy pairs can be a Nash equilibrium. For (A, A) to not be a Nash equilibrium, we must have

$$a < e \quad \text{or} \quad b < d.$$

For (B, A) to not be a Nash equilibrium, we must have

$$e < a \quad \text{or} \quad f < h.$$

For (A, B) to not be a Nash equilibrium, we must have

$$c < g \quad \text{or} \quad d < b.$$

For (B, B) to not be a Nash equilibrium, we must have

$$g < c \quad \text{or} \quad h < f.$$

Thus, for there to be no pure strategy Nash equilibrium, at least one inequality from each of the previous four displays must hold. Suppose $a < e$ is true in the first display; since this implies that the inequality $e < a$ is not true, then $f < h$ must hold from the second display; from this, we have that in the fourth display, $g < c$ must be true; and finally, from the third display, $d < b$ must be true. Summarizing, we have

(CASE 1) $$a < e, \quad f < h, \quad g < c, \quad \text{and} \quad d < b.$$

Now suppose $b < d$ is true in the first display; this implies that $c < g$ must be true in the third display; this implies that $h < f$ must be true in the fourth display; and that $e < a$ must be true in the second display. Summarizing this case, we have

(CASE 2) $$a > e, \quad f > h, \quad g > c, \quad \text{and} \quad d > b.$$

In either of these two cases, we use the approach from this section to determine Rose's mixed strategy $(1-p, p)$ which equalizes Colin's expected payoffs for his two pure strategy choices. That is, we solve the equation

$$(1-p)b + pf = (1-p)d + ph$$

for p. Doing so results in

$$p = \frac{b-d}{(h-f)+(b-d)}.$$

In order for $(1-p, p)$ to be a mixed strategy, we need to verify that the denominator of p is not zero and p is between 0 and 1. In (CASE 1), $b-d > 0$ and $h-f > 0$. This implies that both the numerator and denominator are positive, and the numerator is smaller than the denominator. So, p is defined and is between 0 and 1. In (CASE 2), $b-d < 0$ and $h-f < 0$. This implies that both the numerator and denominator are negative, making the ratio positive, and the absolute value of the numerator is smaller than the absolute value of the denominator. So, p is defined and is between 0 and 1. An identical argument, which you are asked to write in an exercise, can be used to determine that Colin's Nash equilibrium strategy is $(1-q, q)$, where

$$q = \frac{c-g}{(c-g)+(e-a)}.$$

□

Prudential Strategies

Finding a prudential strategy in a 2×2 game is similar to the approach that we used for finding Nash equilibrium strategies, but each player uses her or his own payoffs in the calculations rather than the other player's payoffs. This is because a player is trying to maximize her or his own minimum payoff rather than to equalize the other player's payoff. To find Rose's prudential mixed strategy in the **Impromptu-2** game, we need to consider what happens as the probability p that she plays ROW B changes from 0 to 1. If Colin plays COLUMN A, her expected payoff, $10(1-p)+100p$, is graphed in Figure 3.4, as is her expected payoff, $60(1-p)+40p$, when Colin plays COLUMN B.

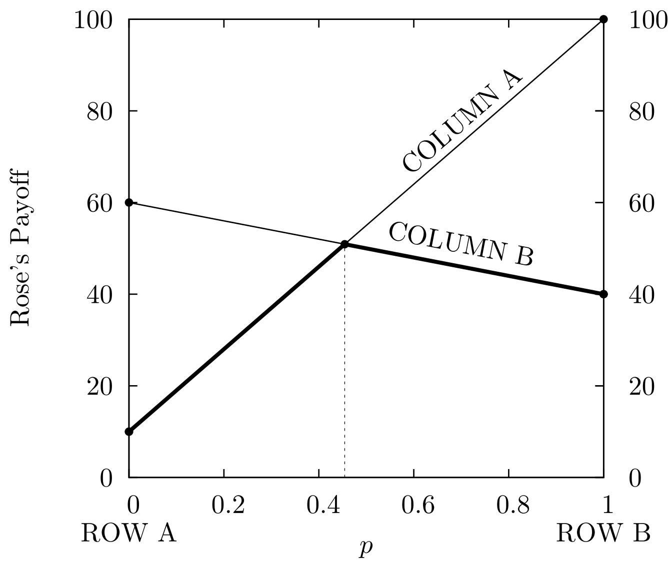

FIGURE 3.4. Rose's prudential strategy for **Impromptu-2**

Notice that if p is to the left of the intersection of the two lines, her payoff is smaller when Colin plays COLUMN A, and when p is to the right of the intersection, her payoff is smaller when Colin plays COLUMN B. The bolding in Figure 3.4 indicated Rose's smallest payoff for each strategy choice. Therefore, to maximize her minimum payoff (the goal of a prudential strategy) Rose should choose p at the point where the bolding is highest, at the intersection of the two payoff lines. We find this value of p by setting the two expressions equal to each other and solving for p:

$$\begin{aligned} 10(1-p)+100p &= 60(1-p)+40p, \\ 10+90p &= 60-20p, \\ 110p &= 50, \\ p &= \frac{5}{11}. \end{aligned}$$

Rose's minimum expected payoff is therefore

$$10(1-\frac{5}{11})+100(\frac{5}{11}) = 50.9$$

when playing the strategy $(\frac{6}{11}, \frac{5}{11})$ and her security level is 50.9.

We can find Colin's prudential strategy the same way, by setting his expected payoffs for each of Rose's two choices of pure strategies equal to each other and solving for the probability q that he is in the second column:

$$\begin{aligned} 100(1-q) + 40q &= 30(1-q) + 70q, \\ 100 - 60q &= 30 + 40q, \\ -100q &= -70, \\ q &= \frac{7}{10}. \end{aligned}$$

Playing the strategy $(\frac{3}{10}, \frac{7}{10})$ gives Colin a security level of 58.

These two prudential strategies do not form a Nash equilibrium since Colin would prefer the expected payoff of 68.2 that he gets by playing COLUMN A in response to Rose's prudential strategy. Rose would prefer the expected payoff of 58 that she gets by playing ROW B in response to Colin's prudential strategy.

Let's apply the same algebra to find Rose's prudential strategy $(1-p)A + pB$ in **Example-tu**. Set her expected payoffs for each of Colin's two choices of pure strategies equal to each other

$$0(1-p) + 2p = 4(1-p) + 3p,$$

and solving for p, we obtain

$$p = \frac{4}{3}.$$

Wait a minute! How can p, which is supposed to be a probability, be equal to a number greater than 1? Let's back up to the graph that motivates the algebra. Rose's expected payoff, $0(1-p) + 2p = 2p$, when Colin plays COLUMN A, and Rose's expected payoff, $4(1-p) + 3p = 4 - p$, when Colin plays COLUMN B are shown in Figure 3.5.

Notice that for p between 0 and 1, the first graph is always below the second graph. To maximize her minimum payoff (the goal of a prudential strategy) and since the minimum payoff always occurs on the lower graph, Rose should choose $p = 1$, where the lower graph is highest. Therefore, the prudential strategy is ROW B and it gives Rose a security level of 2.

To find Colin's prudential strategy $(1-q)A + qB$, we set his expected payoffs for each of Rose's two choices of pure strategies equal to each other

$$5(1-q) + 12q = 7(1-q) + 2q,$$

which has the solution

$$q = \frac{1}{6},$$

suggesting that Colin's prudential strategy is $(\frac{5}{6}, \frac{1}{6})$ and his security level is $\frac{37}{6}$. We could verify this by drawing the graph; however, it is sufficient to notice that Colin ensures a payoff of 5 by choosing COLUMN A and ensures a payoff of 2 by choosing COLUMN B, both of which are less than the $\frac{37}{6}$ ensured by $(\frac{5}{6}, \frac{1}{6})$.

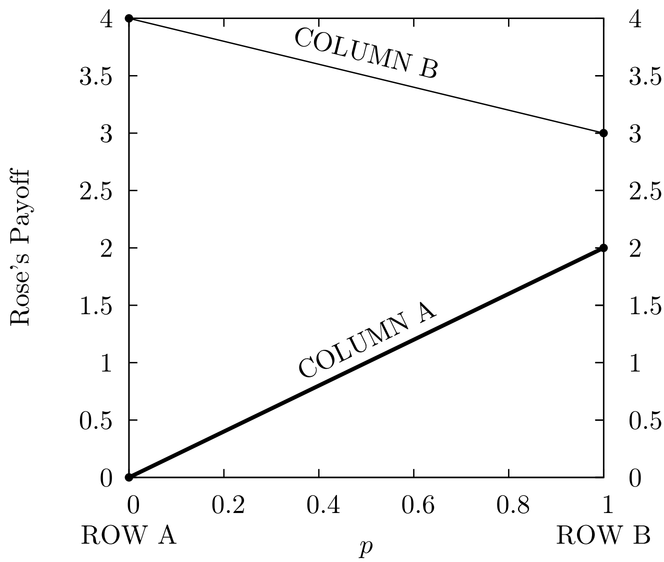

FIGURE 3.5. Rose's prudential strategy for **Example-tu**

Exercises

(1) In determining Colin's mixed Nash equilibrium strategy, why do we set Rose's expected payoffs equal to each other?

(2) **Mating**. Suppose females and males of some species of bird are genetically programmed to adopt one of two behaviors with respect to mating. The outcome matrix when a female and male interact with one or the other behavioral strategy is shown below.

Mating Outcomes		Male FAITHFUL	PHILANDERING
Female	COY	Courtship yielding a child raised by both parents.	No courtship and no child.
	FAST	A child raised by both parents	A child raised by the female alone.

If courtship translates into a loss of 3 points for each individual and a child translates into a gain of 15 points for each parent and a total loss of 20 by the parent(s) who raises the child, then the cardinal payoff matrix is as follows.

Mating Cardinal Payoffs		Male FAITHFUL	PHILANDERING
Female	COY	$(2, 2)$	$(0, 0)$
	FAST	$(5, 5)$	$(-5, 15)$

(a) Find the unique Nash equilibrium.
(b) Find the prudential strategies.
(c) Interpret your results in the context of the original scenario.

(d) If courtship translated into a gain of 1 point rather than a loss of 3 points for each individual, how would the cardinal payoff matrix, Nash equilibrium, and prudential strategies change?

(3) **Risk-Neutral Fingers**. Rose and Colin simultaneously show one or two fingers. The outcome matrix is as follows.

Risk-Neutral Fingers Outcomes		Colin ONE	TWO
Rose	ONE	Each pays \$1	Rose receives \$4
	TWO	Colin receives \$4	Each receives \$1

(a) Assume the two players are self-interested and risk neutral. Determine a cardinal payoff matrix.
(b) Find the three Nash equilibria and the corresponding payoffs.
(c) Find the prudential strategies and security levels.
(d) Based on parts (b) and (c), what should a player choose if the game is to be played once? Why? If preplay communication is possible, what effect would that have?
(e) Suppose there is a population of players genetically programmed to use one of the strategies ONE or TWO, these players randomly play each other, and the cardinal payoffs are proportional to the number of offspring that player has. Explain what should happen to the population distribution.

(4) **Risk-Varying Fingers**. The scenario is as in the previous exercise except that the cardinal payoff matrix is as follows.

Risk-Varying Fingers Outcomes		Colin ONE	TWO
Rose	ONE	$(0, 0)$	$(10, 5)$
	TWO	$(1, 10)$	$(3, 8)$

(a) State which player is risk adverse and which player is risk loving. Why?
(b) Find the three Nash equilibria and the corresponding payoffs.
(c) Find the prudential strategies and security levels.
(d) If the two players adopted their mixed strategy Nash equilibrium strategies, what would be their expected monetary winnings? How does this compare with their expected monetary winnings if they were to use their mixed strategy Nash equilibrium strategies in **Risk-Neutral Fingers**? Why do these numbers differ?

(5) **Matching Pennies**. Rose and Colin simultaneously show a penny heads up or tails up. Rose gives \$1 to Colin if the pennies match, that is, have the same side showing; otherwise, Colin gives Rose \$1. A cardinal payoff matrix could be

Matching Pennies Cardinal Payoffs		Colin Heads	Tails
Rose	Heads	$(-1, 1)$	$(1, -1)$
	Tails	$(1, -1)$	$(-1, 1)$

(a) In using the above cardinal payoff matrix to model this game, what assumption(s) are we making?
(b) Find the unique Nash equilibrium and corresponding payoff pair.

(c) Explain why the Nash equilibrium strategies are also prudential strategies.
(d) How should Rose and Colin play the game? Why? Does your answer depend on whether the game is played once or several times? Why or why not?

(6) **River Tale**. Steve and a stranger match pennies for money, but with a twist shown in the following outcome matrix.

River Tale Outcomes		Stranger Heads	Tails
Steve	Heads	Steve pays the stranger \$20.	The stranger pays Steve \$10.
	Tails	The stranger pays Steve \$30.	Steve pays the stranger \$20.

(a) Assume the two players are self-interested and risk neutral. Determine a cardinal payoff matrix.
(b) Find the unique Nash equilibrium and corresponding payoff pair.
(c) Explain why the Nash equilibrium strategies are also prudential strategies.
(d) How should Steve and the Stranger play the game? Why? Does your answer depend on whether the game is played once or several times? Why or why not?

(7) **Chicken**. Rose and Colin are driving very fast toward each other. If one turns while the other goes straight, the former player is the "loser" and the later player is the "winner". If both choose the same strategy, there is a tie, with potentially fatal consequences if both choose STRAIGHT. This is summarized in the following outcome matrix.

Chicken Outcomes		Colin STRAIGHT	TURN
Rose	STRAIGHT	Crash	Rose wins
	TURN	Colin wins	Tie

Here is a reasonable corresponding cardinal payoff matrix:

Chicken Cardinal Payoffs		Colin STRAIGHT	TURN
Rose	STRAIGHT	$(0, 0)$	$(10, 4)$
	TURN	$(4, 10)$	$(8, 8)$

(a) Describe two outcome-lottery pairs for which Rose and Colin must be indifferent for the above cardinal payoffs to be true.
(b) Find the three Nash equilibria and the corresponding payoffs.
(c) Find the prudential strategies and security levels.
(d) Based on parts (b) and (c), what should a player choose if the game is to be played once? Why? If preplay communication is possible, what effect would that have?
(e) Suppose there is a population of players genetically programmed to use one of the strategies STRAIGHT or TURN, these players randomly play each other, and the cardinal payoffs are proportional to the number of offspring that player has. This scenario usually substitutes HAWK for STRAIGHT, indicating an aggressive behavioral pattern, and substitutes DOVE for TURN, indicating a submissive behavioral pattern. Explain what should happen to the population distribution.

(8) **Soccer Penalty Kicks**. In professional soccer, essentially no time passes between a penalty kicker's kick to the right or left of the goal and the goalie's decision on which corner to defend. These decisions can be assumed to be made simultaneously. Suppose the kicker's accuracy and speed is greater when kicking to the left side of the goal. Specifically, the kicker has a 90% chance of scoring an undefended left side, an 80% chance of scoring an undefended right side, a 10% chance of scoring a defended left side, and a 0% chance of scoring a defended right side.
  (a) Determine the appropriate cardinal payoff matrix.
  (b) Find the Nash equilibrium.
  (c) Explain why the Nash equilibrium strategies are also prudential strategies.
  (d) What is somewhat surprising about the Nash equilibrium and prudential strategy for the kicker? Does it seem less surprising given the Nash equilibrium and prudential strategy for the goalie?

(9) **Matches**. Consider the scenario described in section 3.2 with the following cardinal payoff matrix.

Matches		Colin	
Cardinal Payoffs		TENNIS	SOCCER
Rose	TENNIS	$(10, 6)$	$(2, 5)$
	SOCCER	$(0, 0)$	$(9, 10)$

  (a) Find the three Nash equilibria and the corresponding payoffs.
  (b) Find the prudential strategies and security levels.
  (c) The mixed strategy Nash equilibrium may or may not make sense as a predictor of what would happen in the original scenario. Give an argument for both positions.

(10) Recall that Nash equilibrium strategies must be undominated. In the following game, successively eliminate dominated strategies. With the remaining strategies, find the Nash equilibrium. Verify that this pair of strategies is a Nash equilibrium in the original game.

Cardinal Payoffs		Colin			
(Rose, Colin)		A	B	C	D
	A	$(10, 10)$	$(9, 3)$	$(5, 0)$	$(-5, 6)$
Rose	B	$(12, -1)$	$(10, -3)$	$(6, 0)$	$(-4, -7)$
	C	$(10, 0)$	$(12, -5)$	$(10, -3)$	$(2, -5)$
	D	$(9, 10)$	$(10, 12)$	$(5, 10)$	$(-3, 5)$

(11) The game below has both pure and mixed strategy Nash equilibria. Use a best response diagram to find all pure strategy Nash equilibria. Use the best response diagram to suggest how to reduce the game to a 2×2 game. Find the mixed strategy Nash equilibrium in the reduced game. Verify that this pair of strategies is a Nash equilibrium in the original game.

Cardinal Payoffs		Colin		
(Rose, Colin)		A	B	C
	A	$(10, 10)$	$(2, 6)$	$(1, 5)$
Rose	B	$(7, 2)$	$(3, 4)$	$(2, 3)$
	C	$(4, 1)$	$(4, 7)$	$(0, 8)$

(12) Construct a 2×2 game that has only one pure strategy Nash equilibrium. Now attempt to find a nonpure mixed strategy Nash equilibrium using the method of this section. What happens?

(13) Finish the argument required to complete the proof of the Nash Equilibrium Theorem. That is, explain why

$$q = \frac{c - g}{(c - g) + (e - a)}$$

must represent a probability.

4. Nash Equilibria in $m \times 2$ Games

Obviously, not every strategic game is a 2×2 game. Unfortunately, as the game matrices get bigger, the mechanical processes for finding Nash equilibria and prudential strategies get more cumbersome. Conceptually, however, the same ideas that we saw being used graphically in the last section continue to be useful. In this section, we extend those ideas to a slightly more general class of games—those in which Rose has m strategies, but Colin still only has two. By symmetry, the method described here will also work when Rose has only two strategies and Colin has n strategies.

> By the end of this section, you will be able to find mixed strategy Nash equilibria in $m \times 2$ and $2 \times n$ games.

Let's find the Nash equilibria in the following 4×2 game, **Rectangle**.

Rectangle Cardinal Payoffs		Colin A	B
	A	$(-1, 3)$	$(5, -2)$
Rose	B	$(2, 1)$	$(4, 5)$
	C	$(4, -2)$	$(-3, 6)$
	D	$(5, 10)$	$(-4, -4)$

A best response diagram can be used to determine that (ROW D, COLUMN A) is the unique Nash equilibrium in pure strategies.

In order to determine any nonpure stategy Nash equilibria, consider Colin's perspective first since he has only two pure strategies available to him. For Colin's general mixed strategy $(1 - q, q)$, we find Rose's payoffs as we did previously.

Rectangle Cardinal Payoffs		Colin A	B	Rose's payoff
	A	$(-1, 3)$	$(5, -2)$	$-1(1-q) + 5q = 6q - 1$
Rose	B	$(2, 1)$	$(4, 5)$	$2(1-q) + 4q = 2q + 2$
	C	$(4, -2)$	$(-3, 6)$	$4(1-q) - 3q = -7q + 4$
	D	$(5, 10)$	$(-4, -4)$	$5(1-q) - 4q = -9q + 5$

We see the graphs of these four lines on the interval $0 \leq q \leq 1$ in Figure 4.1.

When Colin chooses COLUMN A, this corresponds to the left-hand side of the graph in Figure 4.1, from which it is clear that Rose's best response is ROW D. From the payoff matrix, it is clear that Colin's best response to ROW D is COLUMN A. Hence, we have reconfirmed that (ROW D, COLUMN A) is a Nash equilibrium.

When Colin chooses COLUMN B, this corresponds to the right-hand side of the graph in Figure 4.1, from which it is clear that Rose's best response is ROW A. From the payoff matrix, it is clear that Colin's best response to ROW A is COLUMN A. Hence, we have reconfirmed that (ROW A, COLUMN B) is not a Nash equilibrium.

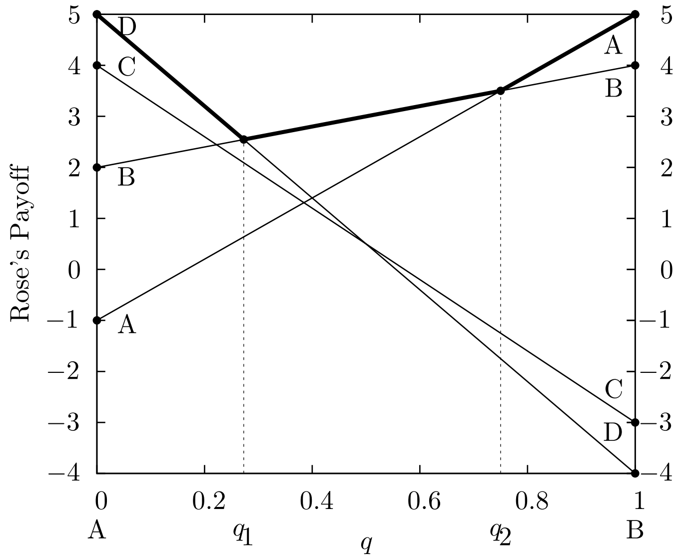

FIGURE 4.1. Colin's strategy options

Since Rose's best response payoff to a Colin strategy is always at the upper edges, to find a nonpure mixed strategy best response, we need to find the corners of this upper boundary (where her payoffs are equalized). In this example there are two such corners, (1) at the intersection of the lines for ROW B and ROW D, and (2) at the intersection of the lines for ROW A and ROW B. These are both potential Nash equilibria that we need to investigate.

To find the value of q_1 which gives the intersection of ROW B and ROW D, we solve the equation $2q_1 + 2 = -9q_1 + 5$ to obtain $q_1 = \frac{3}{11}$. Thus Colin's strategy $(\frac{8}{11}, \frac{3}{11})$ equalizes Rose's expected payoff at $2(\frac{3}{11}) + 2 = 28/11$.

With this information in hand, we turn our attention to Rose's perspective. We need to find a strategy for Rose that (1) is a best response to Colin using $(\frac{8}{11}, \frac{3}{11})$, and (2) has $(\frac{8}{11}, \frac{3}{11})$ as Colin's best response. Figure 4.1 shows that if Colin plays the strategy $(\frac{8}{11}, \frac{3}{11})$, Rose's answer to (1) is any probabilistic mixture of ROW B and ROW D; she will never play ROW A or ROW C. Therefore, we can consider the reduced game as shown below.

Reduced Rectangle Cardinal Payoffs		Colin A	B
Rose	B	$(2, 1)$	$(4, 5)$
	D	$(5, 10)$	$(-4, -4)$

To answer (2), Rose chooses her mixed strategy $(1 - p, p)$ to equalize the expected payoffs to Colin when he uses either pure strategy by solving the equation $1(1-p)+10p = 5(1-p)-4p$ to obtain $p = \frac{2}{9}$. Thus Rose's strategy $(\frac{7}{9}, \frac{2}{9})$ equalizes Colin's expected payoff at 3.

Since Rose's strategy $(\frac{7}{9}, \frac{2}{9})$ equalizes Colin's expected payoff and Colin's strategy $(\frac{8}{11}, \frac{3}{11})$ equalizes Rose's expected payoff between ROW B and ROW D,

$$\left(\left(\frac{7}{9}, \frac{2}{9}\right), \left(\frac{8}{11}, \frac{3}{11}\right)\right)$$

is a Nash equilibrium in the reduced game. Can we verify that the pair of strategies $((0, \frac{7}{9}, 0, \frac{2}{9}), (\frac{8}{11}, \frac{3}{11}))$ is a Nash equilibrium in the original game of **Rectangle**? To do so, we need to show that neither player has an incentive to change from his or her current strategy. Given the strategy that Colin is playing, Rose will never switch her strategy to include ROW A and ROW C since this would decrease her expected payoff as can be seen in the previous graph. Since she receives the same payoff whether she is in ROW B or ROW D, she has no incentive to change from what she is currently doing. Similarly, given the strategy that Rose is currently playing, Colin gets the same expected payoff in COLUMN A as in COLUMN B, so he also has no incentive to change strategy. Therefore, the strategy pair forms a Nash equilibrium.

To find the value of q_2, which gives the intersection of ROW A and ROW B, we solve the equation $6q_2 - 1 = 2q_2 + 2$ to obtain $q_2 = \frac{3}{4}$. Thus Colin's strategy $(\frac{1}{4}, \frac{3}{4})$ equalizes Rose's expected payoff at $6\frac{3}{4} - 1 = \frac{7}{2}$.

With this information in hand, we again turn our attention to Rose's perspective to find her candidate strategy. Assuming that Colin will play the strategy $(\frac{1}{4}, \frac{3}{4})$, Rose will get her best expected payoff by playing either ROW A or ROW B; she will never play ROW C or ROW D. Therefore, we can consider the following reduced game:

Reduced Rectangle		Colin	
Cardinal Payoffs		A	B
Rose	A	$(-1, 3)$	$(5, -2)$
	B	$(2, 1)$	$(4, 5)$

Rose determines her mixed strategy $(1 - p, p)$ that equalizes the expect payoffs to Colin by solving the equation $3(1 - p) + 1(p) = -2(1 - p) + 5(p)$ to obtain $p = \frac{5}{9}$. Thus, Rose's strategy $(\frac{4}{9}, \frac{5}{9})$ equalizes Colin's expected payoff at $\frac{17}{9}$ in the reduced game.

Returning to the original game, given that Colin plays $(\frac{1}{4}, \frac{3}{4})$, Rose will never switch her strategy to include ROW C or ROW D since this would decrease her expected payoff. Since she receives the same payoff whether she is in ROW A or ROW B, she has no incentive to change from what she is currently doing. Similarly, given that Rose plays $(\frac{4}{9}, \frac{5}{9}, 0, 0)$, Colin gets the same expected payoff in COLUMN A as in COLUMN B, so he also has no incentive to change strategy. Therefore, the strategy pair $((\frac{4}{9}, \frac{5}{9}, 0, 0), (\frac{1}{4}, \frac{3}{4}))$ forms a second Nash equilibrium.

The solution method that we have just described will always find a Nash equilibrium in an $m \times 2$ game, a fact that we state in the following theorem.

Nash Solution Theorem: *In an $m \times 2$ game, at least one of the intersection points or endpoints of the upper edges of the row player's expected payoff graph yields a Nash equilibrium.*

Note that this theorem does not say that every endpoint or intersection point is a Nash equilibrium; each must be checked. For example, both intersection points and the left-hand endpoint yield Nash equilibria in **Rectangle**, but the right-hand endpoint did not yield a Nash equilibrium. To demonstrate this point again, let's consider the game **Rectangle-2**, shown below, which is **Rectangle** with Colin's payoff at (ROW D, COLUMN A) changed so that this pair of strategies is not a Nash equilibrium.

Rectangle-2 Cardinal Payoffs		Colin A	B
	A	$(-1, 3)$	$(5, -2)$
Rose	B	$(2, 1)$	$(4, 5)$
	C	$(4, -2)$	$(-3, 6)$
	D	$(5, -5)$	$(-4, -4)$

Now there are no pure strategy Nash equilibria, but the graph for Rose's payoff vs Colin's strategy remains the same as the one for **Rectangle**, and since the payoff change does not effect ROW A or ROW B, $((\frac{4}{9}, \frac{5}{9}, 0, 0), (\frac{1}{4}, \frac{3}{4}))$ also remains a Nash equilibrium. So let's see what happens at the other intersection point. We still have $q_1 = \frac{3}{11}$, so Colin's candidate strategy is still $(\frac{8}{11}, \frac{3}{11})$. However, Rose is now equalizing Colin's payoffs in the reduced game below.

		Colin A	B
Rose	B	$(2, 1)$	$(4, 5)$
	D	$(5, -5)$	$(-4, -4)$

So we find her mixed strategy $(1 - p, p)$ by solving the equation $1(1 - p) - 5p = 5(1-p) - 4p$ for p, resulting in $p = \frac{4}{3}$. But this is nonsensical since p is a probability and must, therefore, be no greater than 1. The best interpretation that could be put on this is that Rose should always play ROW D, but then Colin would prefer COLUMN B over his current mixed strategy of $(\frac{8}{11}, \frac{3}{11})$, so $(\frac{8}{11}, \frac{3}{11})$ cannot be part of a Nash equilibrium. (In the exercises, you are asked to graph these two lines and explore this situation a bit more.)

We can extend this solution method to any game in which Colin has two strategies and Rose has many strategies, or conversely when Rose has two strategies and Colin has many. (See the exercises.) To solve an $m \times n$ game, in which Rose has $m > 2$ strategies and Colin has $n > 2$ strategies, we need to use the method of linear programming, which is frequently introduced in freshman-level Finite Mathematics courses. We will not be discussing this method here, beyond noting that it generalizes the idea of selecting optimal solutions along the upper edges of the graphs. There are software packages, such as GAMBIT [**31**], that can carry out these computations.

Exercises

(1) Find all of the equalizing strategies for Rose and Colin in the following game. Which strategy pairs are Nash equilibria?

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
	A	$(2, 7)$	$(7, 3)$
Rose	B	$(3, 4)$	$(4, 2)$
	C	$(6, 2)$	$(3, 6)$

(2) Find all of the equalizing strategies for Rose and Colin in the following game. Which strategy pairs are Nash equilibria?

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
	A	$(-3, 3)$	$(5, -5)$
Rose	B	$(-2, 2)$	$(3, -3)$
	C	$(2, -2)$	$(-2, 2)$
	D	$(3, -3)$	$(-6, 6)$

(3) Find all of the equalizing strategies for Rose and Colin in the following game. Which strategy pairs are Nash equilibria?

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C	Colin D	Colin E
Rose	A	$(-4, 5)$	$(3, -3)$	$(2, -5)$	$(-1, 4)$	$(1, 1)$
	B	$(3, -3)$	$(-6, 7)$	$(6, 4)$	$(0, 0)$	$(3, -2)$

(4) Find all of the equalizing strategies for Rose and Colin in the following game. Which strategy pairs are Nash equilibria? (Hint: Eliminate dominated strategies first.)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C	Colin D
	A	$(10, 1)$	$(-2, 2)$	$(-3, 3)$	$(0, 7)$
Rose	B	$(6, 0)$	$(-1, 2)$	$(2, 4)$	$(4, 3)$
	C	$(7, -3)$	$(5, 5)$	$(-2, -2)$	$(2, -1)$

(5) In the game **Rectangle** described in this section, there are three Nash equilibria: (D, A), $(\frac{7}{9}B + \frac{2}{9}D, \frac{8}{11}A + \frac{3}{11}B)$, and $(\frac{4}{9}A + \frac{5}{9}B, \frac{1}{4}A + \frac{3}{4}B)$. Since each player chooses their own strategies, what do you think happens if Rose picks her strategy from the second Nash equilibrium and Colin picks his from the third? As part of your discussion, calculate the resulting expected payoffs for each player.

(6) For the **Reduced Rectangle-2** game,

Reduced Rectangle-2 Cardinal Payoffs		Colin A	Colin B
Rose	B	$(2, 1)$	$(4, 5)$
	D	$(5, -5)$	$(-4, -4)$

graph Colin's expected payoffs $1(1-p) - 5p$ for using A and $5(1-p) - 4p$ for using B in response to Rose's strategy $(1-p)B + pD$. Explain from the graph why Colin will always prefer COLUMN B. Solving $1(1-p)-5p = 5(1-p)-4p$, we obtain $p = \frac{4}{3}$. Explain what this means graphically.

(7) Consider the following cardinal payoff matrix for the **WWII Battle** game discussed in Chapter 3.

WWII Battle Cardinal Payoffs		Germans ATTACK GAP	RETREAT
Allies	REINFORCE GAP	$(1,-1)$	$(0,0)$
	HOLD RESERVES	$(2,-2)$	$(0,0)$
	SEND RESERVES EAST	$(-2,2)$	$(1,-1)$

Find the Nash equilibrium. Given this, how unreasonable was the decision to overrule his generals decision to implement their pure prudential RETREAT strategy?

(8) **Nickel**. Barry O'Neill [**41**] and James Brown and Robert Rosenthal [**10**] ran experiments using this game. Two players simultaneously choose an ace, two, three, or jack. The outcome consists of one player winning a nickel from the other player, as shown in the outcome matrix below. Assume that players prefer winning to losing.

Nickel Winner		Colin ACE	TWO	THREE	JACK
Rose	ACE	Colin	Rose	Rose	Colin
	TWO	Rose	Colin	Rose	Colin
	THREE	Rose	Rose	Colin	Colin
	JACK	Colin	Colin	Colin	Rose

(a) Play this game at least twenty times keeping track of the strategies chosen and payoffs received.
(b) Construct a cardinal payoff matrix. Explain why the cardinal payoffs do not depend on whether the players are risk adverse, risk neutral, or risk loving.
(c) Show that both players using the mixed strategy [0.2 ACE + 0.2 TWO + 0.2 THREE + 0.4 JACK] form a Nash equilibrium.
(d) For those wanting a mathematical challenge, show that there is no Nash equilibrium other than the one described in part (c).
(e) Compare the relative frequencies of the strategies you chose with the mixed strategy used by each player in the unique Nash equilibrium.

(9) **Investment**. Amnon Rapoport and Wilfred Amaldoss [**49**] studied an investment game. Two firms have a research and development budget of \$5 million that they can invest in a patent race in \$1 million increments. Firms lose any money invested but keep the amount they do not invest. The firm that invests more wins a patent that will generate \$8 million for the firm. The firm that invests less (or both firms if they invest the same amount) does not win anything.
(a) Construct an outcome matrix. Under what conditions would the outcome matrix also be the cardinal payoff matrix?
(b) Play this game at least twenty times keeping track of the strategies chosen and payoffs received.

(c) Show that both players using the mixed strategy [0.125 INVEST \$0 + 0.125 INVEST \$1 + 0.125 INVEST \$2 + 0.125 INVEST \$3 + 0.125 INVEST \$4 + 0.375 INVEST \$5] forms a Nash equilibrium.
(d) Show that INVEST \$0 is a prudential strategy for either firm.
(e) For those wanting a mathematical challenge, show that there is no Nash equilibrium in which both firms use the same strategy other than the one described in part (c).
(f) Compare the relative frequencies of the strategies you chose with the mixed strategy used by each player in the Nash equilibrium described in part (c).

5. Zero-Sum Games

Let's consider the game **Two Finger Morra**: Each of two players shows one or two fingers, and simultaneously guesses how many fingers the other player will show. If both players guess correctly or both players guess incorrectly, there is no payoff. If just one player guesses correctly, that player wins a payoff from the other player equal to the total number of fingers shown by both players. (This is a simplified version of the game Morra, played in Italy.)

To build the payoff matrix, let the ordered pair (S, G) represent a player's selection of how many fingers to *show* and what their *guess* is about the number of fingers the other player will show. Thus, the pair (1, 2) says that a player will show one finger and guess that the other player will show two fingers. Each of the two players has the same four strategies available to him or her: (1, 1), (1, 2), (2, 1), and (2, 2). If Rose plays strategy (1, 1) and Colin plays strategy (1, 2), Rose wins two points from Colin because Rose has correctly guessed that Colin will show one finger, Colin has incorrectly guessed that Rose would show two fingers, and together they actually showed a total of two fingers. The complete payoff matrix is below.

Two Finger Morra		Colin			
Cardinal Payoffs		(1, 1)	(1, 2)	(2, 1)	(2, 2)
	(1, 1)	$(0,0)$	$(2,-2)$	$(-3,3)$	$(0,0)$
Rose	(1, 2)	$(-2,2)$	$(0,0)$	$(0,0)$	$(3,-3)$
	(2, 1)	$(3,-3)$	$(0,0)$	$(0,0)$	$(-4,4)$
	(2, 2)	$(0,0)$	$(-3,3)$	$(4,-4)$	$(0,0)$

> This is a fun game to play, so we suggest that you take some time to play it a few times now.

Is it Zero-Sum?

The feature of this game that is of significance to us is that the respective payoffs for Rose and Colin are all negatives of the other's, and thus the sum of the payoffs in each outcome is zero. Any game that has this feature is called a zero-sum game. Formally,

> **Zero-sum Game:** In a *zero-sum game,* the players have strictly opposing interests, and any gain by one player must come from another player.

> By the end of this section, you will be able to identify a zero-sum game, model simple real-world scenarios as zero-sum games, and be able to determine their Nash equilibria.

In the payoff matrix for **Two Finger Morra** above, it is easy to see that the game is zero-sum since Rose and Colin's payoffs are negatives of each other for each outcome. However, since cardinal utilities can be transformed into equivalent sets of utilities by positive linear transformations, the fact that the payoffs sum to zero is sometimes disguised in the payoff matrix. The game **Zero?**

Zero? Cardinal Payoffs		Colin A	Colin B
Rose	A	$(27, -5)$	$(17, 0)$
	B	$(19, -1)$	$(23, -3)$

does not appear to be a zero-sum game, but since cardinal utilities do not change meaning under positive linear transformations, it still may be one. To determine this, we need to find a transformation of the form $y = mx + b$ that converts Rose's payoffs x to the negatives of Colin's payoffs y as shown in Table 5.1.

TABLE 5.1. Payoff transformation

Rose's Original Payoff (x)	17	19	23	27
Rose's Transformed Payoff (y)	0	1	3	5

Since the transformation must work for all four of these payoff pairs, we can pick any two to construct the equation and then verify that the constructed equation works with the other two payoff pairs. For example, solving the system of equations

$$\begin{aligned} 0 &= m(17) + b, \\ 1 &= m(19) + b \end{aligned}$$

gives us $m = \frac{1}{2}$ and $b = -\frac{17}{2}$. We then check that

$$\begin{aligned} 3 &= \frac{1}{2}(23) - \frac{17}{2}, \\ 5 &= \frac{1}{2}(27) - \frac{17}{2}. \end{aligned}$$

In conclusion, Rose's payoff transformation $y = \frac{1}{2}x - \frac{17}{2}$ converts the game **Zero?** into the game **Transformed Zero?**, which is obviously a zero-sum game.

Transformed Zero? Cardinal Payoffs		Colin A	Colin B
Rose	A	$(5, -5)$	$(0, 0)$
	B	$(1, -1)$	$(3, -3)$

If we cannot construct an equation that maps all of Rose's payoffs to the negatives of Colin's corresponding payoffs, then the game is not a zero-sum game.

Not Zero Cardinal Payoffs		Colin A	Colin B
Rose	A	$(4, -1)$	$(2, 2)$
	B	$(0, 3)$	$(7, -4)$

For example, for the game **Not Zero** to be a zero-sum game, we would need to find a transformation of the form $y = mx + b$ that converts Rose's payoffs x to the negatives of Colin's payoffs y as shown in Table 5.2.

TABLE 5.2. Required tranformation for **Not Zero**

Rose's Original Payoff (x)	0	2	4	7
Rose's Transformed Payoff (y)	-3	-2	1	4

Since such a transformation must work for the first two of these payoff pairs,

$$\begin{aligned} -3 &= m(0) + b, \\ -2 &= m(2) + b, \end{aligned}$$

which gives us $m = \frac{1}{2}$ and $b = -3$. Now for the game to be zero-sum, the transformation $y = \frac{1}{2}x - 3$ should also satisfy

$$\begin{aligned} 1 &= \frac{1}{2}(4) - 3, \\ 4 &= \frac{1}{2}(7) - 3, \end{aligned}$$

but these are obviously not true statements. Therefore, **Not Zero** is not a zero-sum game.

Cool Properties

Now that we know what a zero-sum game is, we can discuss why they are of particular interest to us. There are two questions about Nash equilibria that you may have already asked yourself. First, what do you do when your game has multiple Nash equilibria and they have different payoffs? Second, what happens when there are multiple Nash equilibria pairs, Rose chooses a strategy for one pair, and Colin chooses a strategy from the other pair?

Matches Cardinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(10, 6)$	$(2, 5)$
	SOCCER	$(0, 0)$	$(9, 10)$

For example, recalling the cardinal payoff matrix for **Matches** which has three Nash equilibria with different payoffs, if Rose chooses TENNIS intending to reach the Nash equilibrium payoff pair $(10, 6)$, but Colin chooses SOCCER intending to reach the Nash equilibrium payoff pair $(9, 10)$, they end up with the payoff pair $(2, 5)$ instead. We would like to avoid the problems demonstrated by this example. Let's define precisely what we would like to happen.

Equivalent Nash Equilibria: Two Nash equilibrium pairs are said to be *equivalent* if they give the same payoffs to each of the players.

Interchangeable Nash Equilibria: Two Nash equilibria pairs are *interchangeable* if Rose's (Colin's) strategy in the first of the Nash equilibria can be exchanged with her (his) strategy in the second Nash equilibria pair, and the resulting strategy pairs are still Nash equilibria. For example, suppose that (A, B) and (C, D) are two Nash equilibria. If they are interchangeable, then (C, B) and (A, D) are also Nash equilibria.

The questions in the preceding paragraph can now be rephrased: What do you do when you have nonequivalent Nash equilibria? What do you do if your Nash equilibria are not interchangeable? These questions have been studied, and in general, there are no clear cut answers as to how to play the game [**11**]. However, the following theorem about zero-sum games explains why they are the nicest games with which to work.

> **Zero-Sum Games Are Cool Theorem:** *In zero-sum games, all Nash equilibria are equivalent and interchangeable. Furthermore, Nash equilibria strategies are also prudential strategies.*

Thus, in a zero-sum game, finding one Nash equilibrium tells us essentially everything about the game; there may be other Nash equilibria, but they give the players the same payoffs, and it does not matter which one an individual player chooses. To see all of this more concretely, consider the game **Zero** once again:

Zero		Colin		
Cardinal Payoffs		A	B	C
	A	$(1, -1)$	$(5, -5)$	$(1, -1)$
Rose	B	$(-2, 2)$	$(0, 0)$	$(-4, 4)$
	C	$(1, -1)$	$(2, -2)$	$(1, -1)$

Using a best response diagram, we can quickly see that the strategy pairs (A, A), (A, C), (C, A), and (C, C) are Nash equilibria. At each of them Rose and Colin get the same payoffs, demonstrating the equivalency of the equilibria. The interchangeability of equilibria is illustrated by considering the situation when Rose is intending the Nash equilibrium (A, A), and Colin is intending the Nash equilibrium (C, C); they are both content with the resulting Nash equilibrium (A, C).

Furthermore, ROW A is a prudential strategy for Rose since she can assure herself a payoff of at least 1 by selecting this strategy, and Colin can assure that Rose receives no more than 1 by selecting COLUMN A. A similar argument shows that ROW C, COLUMN A, and COLUMN C are also prudential strategies for the appropriate player.

Many children's games, like **Rock-Paper-Scissors**, are zero-sum games. When playing these games, children frequently declare whether the game is "fair" or not. From the discussion in this section, we see that this assertion about the bias of a game in favor of one player or another can be studied mathematically.

> **Fair Game:** A zero-sum game is a *fair game* if the expected payoff to each player at any Nash equilibrium is zero.

Of the games described in this section, only **Two Finger Morra** is fair. Note carefully that this does not say that each player gets a payoff of zero whenever they play the game. Rather it says that if each player is playing optimally, in the long run and over many games their average payoff will be zero.

Exercises

(1) Which of the games below is a zero-sum game?

(a)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(2, -2)$	$(0, 0)$
	B	$(1, 1)$	$(-3, 3)$

(b)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(2, 10)$	$(9, 3)$
	B	$(3, 9)$	$(12, 0)$

(c)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(10, -5)$	$(-20, 5)$
	B	$(-65, 20)$	$(60, -25)$

(d)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(10, -5)$	$(-20, 5)$
	B	$(-65, 20)$	$(70, -25)$

(2) Identify all of the Nash equilibria in the zero-sum games below. Be sure to look for Nash equilibria in pure strategies as well as in mixed strategies.

(a)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(5, -5)$	$(0, 0)$
	B	$(1, -1)$	$(3, -3)$

(b)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(1, -1)$	$(3, -3)$
	B	$(5, -5)$	$(4, -4)$

(c)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B
Rose	A	$(2, -2)$	$(-3, 3)$
	B	$(-2, 2)$	$(4, -4)$

(d)

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C
Rose	A	$(-4, 4)$	$(5, -5)$	$(-1, 1)$
	B	$(2, -2)$	$(0, 0)$	$(1, -1)$

(e)

Cardinal Payoffs (Rose, Colin)		Colin A	B
	A	$(-3,3)$	$(2,-2)$
Rose	B	$(-1,1)$	$(-1,1)$
	C	$(3,-3)$	$(-4,4)$
	D	$(0,0)$	$(-2,2)$

(3) Which of the games in exercise 2 are fair games? For those that are not fair games, which player has an advantage?

(4) **River Tale**. Steve is approached by a stranger who suggests they match coins. Steve says that it is too hot for violent exercise. The stranger says, "well then, let's just lie here and speak the words 'heads' or 'tails'—and to make it interesting I'll give you \$30 when I call 'tails' and you call 'heads,' and \$10 when it's the other way around. And—just to make it fair—you give me \$20 when we match." If the players are self-interested and risk neutral, the payoff matrix is as follows.

River Tale Cardinal Payoffs		Stranger Heads	Tails
Steve	Heads	$(-20, \boxed{20})$	$(\boxed{10}, -10)$
	Tails	$(\boxed{30}, -30)$	$(-20, \boxed{20})$

Is the game a fair game as the stranger claims?

(5) **Campaign** [**15**, page 588]. Each of two presidential candidates must decide on the best time to broadcast an hour-long television spot. Each chooses from one, two, or three days before the election. If they broadcast on the same day, there is no gain to either candidate. If they do not broadcast on the same day, the one who broadcasts closer to the election has the advantage. If there is a one-day separation, the one who broadcasts closer to the election gains 30% of the uncommitted votes. If there are two days separating them, the one who broadcasts closer gains 50% of the uncommitted votes. Write the payoff matrix for the game. Determine when each should broadcast, the resulting payoffs, and whether the game is fair.

(6) **Commuting** [**15**]. During reconstruction of Boston's Southeast Expressway, a commuter must decide on a route into the city, either the expressway, bus, or subway. The decision for each day must be made the night before. The time saved by each method is determined by whether the day is nice or rainy. The average trip takes 45 minutes. The time saved on a nice day is 10 minutes by expressway, 5 minutes by bus, and 0 minutes by subway. The time saved on a rainy day is -25 minutes by expressway, -15 minutes by bus, and 0 minutes by subway. Write the payoff matrix for the game, and determine the commuter's best strategy.

(7) **Tennis** [**72**, page 837]**.** State University is about to play Ivy College for the state tennis championship. The State team has two players (A and B), and the Ivy team has three players (X, Y, and Z). The following facts are known about the players' relative abilities: X will always beat B; Y will always beat A; A will always beat Z. In any other match, each player has a $\frac{1}{2}$ chance of winning. Each coach must determine which of their players will play the first singles match and which will play the second singles match. Assume that

each coach wants to maximize the expected number of singles matches won by the team. Find optimal strategies for each coach and the resulting expected number of singles matches won by each team.

(8) Consider the game **Zero**.
   (a) Show that the strategy $(1-p, 0, p)$ is a prudential strategy for Rose, for any value of p.
   (b) Show that $((1-p, 0, p), (1-q, 0, q))$ is a Nash equilibrium for any values of p and q.
   (c) Show that all of the Nash equilibria are of the form described in part (b).

(9) Consider the game **Two Finger Morra**, whose payoff matrix is reproduced here.

Two Finger Morra Cardinal Payoffs		Colin (1, 1)	(1, 2)	(2, 1)	(2, 2)
	(1, 1)	$(0,0)$	$(2,-2)$	$(-3,3)$	$(0,0)$
Rose	(1, 2)	$(-2,2)$	$(0,0)$	$(0,0)$	$(3,-3)$
	(2, 1)	$(3,-3)$	$(0,0)$	$(0,0)$	$(-4,4)$
	(2, 2)	$(0,0)$	$(-3,3)$	$(4,-4)$	$(0,0)$

   (a) Verify that the strategy pair $((0, 4/7, 3/7, 0), (0, 3/5, 2/5, 0))$ is a Nash equilibrium.
   (b) Verify that the strategy pair $((0, 3/5, 2/5, 0), (0, 3/5, 2/5, 0))$ is also a Nash equilibrium.
   (c) Why do parts (a) and (b) tell us that $((0, 4/7, 3/7, 0), (0, 4/7, 3/7, 0))$ must also be a Nash equilibrium?
   (d) What are the payoffs for Rose and Colin if they play the strategy in part (c)?
   (e) Is this game fair?

(10) Is $[17, 11, 8, 3, 1]$ **Nim** a zero-sum game? Is it a fair game?

(11) Is 12×12 **Hex** a zero-sum game? Is it a fair game?

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CHAPTER 5

Strategic Game Cooperation

1. Experiments

Strategic games frequently model real world interactions. Once the strategic game is specified, theorists have given us solution concepts (i.e., dominant strategies, prudential strategies, and Nash equilibria) to suggest what players are likely to do. But do real players behave as theorists predict? This is the verification step in the mathematical modeling process we have been using since Chapter 2, and illustrated again in Figure 1.1.

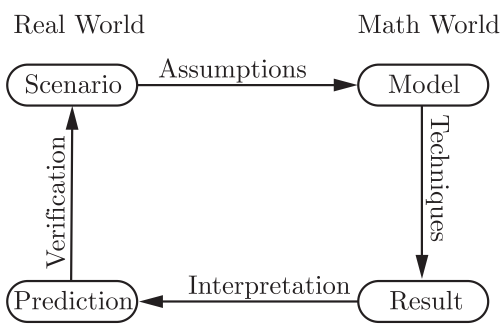

FIGURE 1.1. Mathematical Modeling Process

Appropriately designed experiments can provide an empirical check to our theories.

A comparison with the physical and biological sciences is useful. Many models are used in the sciences: Newton's laws for motion, perfectly elastic spheres for gas molecules, Mendelian genetics for breeding, and so forth. Models predict what will happen in an experiment, and the actual experimental outcome can bolster or undermine the model. Do the motions of the planets correspond to predictions from Newton's laws? Do the interactions of perfectly elastic spheres predict the behavior of gas molecules? Do the offspring in a breeding experiment have characteristics consistent with Mendelian genetics?

The direct analogy to game theory would be whether the Nash equilibrium of a strategic game model accurately predicts the actual choices of the players. Notice that such an experiment tests not just the Nash equilibrium solution concept but also the specific strategic game used. Have we included all relevant strategy choices? Are the assigned preferences accurate? It would be useful to test a solution concept independently of the game formulation; that is, we want our scenario and its model to be synonymous if possible. A growing group of researchers is trying to do just that, and one of the goals in this section is to introduce some of that work (see [**11**], [**12**]).

> By the end of this section, you will be able to describe some techniques used in game theory experiments and some of the difficulties in using the theory to predict choices.

Thirty students, including Rose, are sitting at computer terminals in a room. The furniture is arranged so that it would be very difficult for any of the persons to interact with or observe what any other person is doing. A researcher is in the front of the room.

RESEARCHER: Today you will play three strategic games. For your time, you will be given a small monetary payment. In each game, you will also be playing for chances to win more monetary rewards. At the top of your computer screen, you should see your potential monetary prize for playing the first game, called **Experiment With One Choice**.

Rose sees the following at the top of her screen:

Experiment With One Choice

Your potential monetary prize is	\$40

RESEARCHER: The reason this first game is called **Experiment With One Choice** is because the only action you will take is to choose A, B, or C with a click on the appropriate button.

Rose sees the following at the bottom of her screen:

Click your strategy choice:	A	B	C

RESEARCHER: Your name should appear as one of the players and a fake name appears for the other player. You have been randomly assigned in pairs for this first game. You will not be told either now or after the experiment the identity of your opponent. By making your opponent completely anonymous, it is hoped that you will be indifferent to your opponent's payoff and motivated to maximize your own payoff.

Rose sees the following in the middle of her screen:

Chance to Win (Rose, Colin)		Colin A	Colin B	Colin C
Rose	A	$(70, 70)$	$(0, 0)$	$(0, 100)$
Rose	B	$(0, 0)$	$(50, 50)$	$(0, 45)$
Rose	C	$(100, 0)$	$(45, 0)$	$(10, 10)$

RESEARCHER: Again, the only action you will take in this first game is to choose one of the three strategies available to you: A, B, or C. After both you and your opponent have made your choices privately, those choices will be revealed and the appropriate cell in the table will tell you both your chances and your opponent's chances of winning a monetary prize. For example, if you (Rose) choose C and your opponent (Colin) chooses B, then you (Rose) have a 45% chance of winning a monetary prize and your opponent (Colin) has a 0% chance of winning a monetary prize. Any questions?

ROSE (raising her hand): Is my opponent's monetary prize the same as mine?

RESEARCHER: Not necessarily. Monetary prizes can be anywhere between \$1 and \$90. The monetary prizes for each person in this room were selected randomly. Again, one reason for the secrecy and difference is to give you motivation to be interested only in maximizing your own chance of winning your monetary prize.

A few more questions are asked and answered.

RESEARCHER: The strategy choice buttons will become active in five minutes and will remain active for two more minutes. This should give you sufficient time to think about your choice and then make it. Once you click on one of the strategy choice buttons, you will not be able to change your choice. Remember you will only be playing this game one time. You may commence thinking!

> Before you read Rose's thoughts about this game, you should spend a few minutes thinking about what strategy you would choose.

Rose starts to think about what strategy she should choose. Those thoughts come to her as if three people were having a conversation in her mind.

OLGA: Wow! 100 is possible if Rose chooses C!

PRADEEP: 0 is possible if Rose chooses A or B.

OLGA: Then C it is.

PRADEEP: How likely is it to actually obtain 100? Colin would need to choose A.

OLGA: Can't we hope?

MARK: On what basis?

PRADEEP: Colin is probably thinking like Olga, and plans to choose C in hopes of obtaining 100 for himself.

OLGA: Oh, my! If Colin chooses C, then Rose would only obtain 10 by choosing C.

PRADEEP: And 10 is the best Rose could achieve.

MARK: Looks like (C, C) is a Nash equilibrium.

PRADEEP: C is also Rose's prudential strategy: Rose will receive 10 or more by choosing C, but Rose may receive 0 if she chooses A or B.

OLGA: What a depressing thought. Why can't Colin see that we'd both be far better off if we were to both choose A: 70 each instead of 10 each.

PRADEEP: That's exactly what Colin hopes we'll think. If we think that way, Rose will choose A while Colin chooses C: Rose receives 0 and Colin receives 100.

OLGA: Oh, my!

MARK: Why should either Colin or Rose choose A? Strategy A is dominated by strategy C: whatever Colin chooses to do, Rose will receive more by choosing C than by choosing A.

PRADEEP: So, it's impossible for us to each obtain 70.

OLGA: But couldn't we both obtain 50 by both choosing B?

MARK: Perhaps. The strategy pair (B, B) is another Nash equilibrium, and so neither player would have an incentive to switch strategies as was the case with (A, A).

OLGA: Colin, realizing that there are two Nash equilibria...

MARK: Actually, there is a third Nash equilibrium in which each player uses the mixed strategy $\frac{2}{3}$B $+\frac{1}{3}$C, resulting in a payoff of $33\frac{1}{3}$ to each player.

PRADEEP: How would Colin realize that Rose is using a mixed strategy? How would Rose realize that Colin is using a mixed strategy?

MARK: Each player could use a mixed strategy whether or not the other player realizes it.

OLGA: It really doesn't matter. Rose and Colin, realizing that there are three Nash equilibria, will naturally choose the one yielding the highest payoffs, especially since there's one Nash equilibrium that yields the highest payoffs for each player: (B, B).

MARK: That seems like a reasonable argument.

PRADEEP: But what if Olga is wrong?

MARK: How could that be? Colin has the same amount of time as Rose to think about this game. Surely he'll realize, just as Rose has, that there are three Nash equilibria, and the common best one is (B, B). Clearly, Colin will conclude, just as we have concluded, that he should choose B.

PRADEEP: Very eloquent. But earlier we considered choosing A and C with reasonable arguments...

OLGA: By choosing A, both players may obtain 70.

MARK: By choosing C, Rose assures herself of 10 or more. Also C dominates A, and (C, C) is a Nash equilibrium.

PRADEEP: So, there's a reasonable chance that Colin will choose A or C instead of B.

MARK: Perhaps.

OLGA: I guess so.

PRADEEP: And so if we choose B, then there's also a reasonable chance that Rose will receive 0.

OLGA: But Colin will most likely choose B, and so most likely Rose will obtain 50.

PRADEEP: Suppose there is a 90% chance Colin will choose B and only a 5% chance Colin will choose A and a 5% chance Colin will choose C.

MARK: If Rose chooses B, Rose's expected payoff would be $(0.05)(0)+(0.90)(50)+(0.05)(0) = 45$, but if Rose chooses C, Rose's expected payoff would be $(0.05)(100)+(0.90)(45)+(0.05)(10) = 46$.

PRADEEP: Even if there's a good chance that Colin will choose B, it's still to Rose's advantage to choose C.

MARK: That was interesting. Notice how the idea that Colin would most likely choose one pure strategy but could mistakenly choose a different pure strategy could be thought of as Colin using a mixed strategy.

OLGA: But thought of in that way, wouldn't there be a very small probability of Colin choosing A? We've already noted that A is dominated by C, and there's no Nash equilibrium involving A. So, even if Colin had only an 80% chance of choosing B and a 20% chance of choosing C, ...

MARK: If Rose chooses B, Rose's payoff would be $(0.80)(50) + (0.20)(0) = 40$, and if Rose chooses C, Rose's payoff would be $(0.80)(45) + (0.20)(10) = 38$.

OLGA: So, now it'd be to Rose's advantage to choose B.

PRADEEP: I think that it's far more likely that Colin will choose C. Say he chooses B and C each with a 50% chance.

MARK: If Rose chooses B, Rose's payoff would be $(0.50)(50) + (0.50)(0) = 25$, and if Rose chooses C, Rose's payoff would be $(0.50)(45) + (0.50)(10) = 27.5$.

PRADEEP: So, again it's to Rose's advantage to choose C.

MARK: Perhaps two-thirds of the people in this room are optimistic like Olga and decide to choose B, while one-third of the people in this room are pessimistic like Pradeep and decide to choose C. Then Rose would obtain $33\frac{1}{3}$ whether she chooses B or C.

PRADEEP: It's safer to choose C.

OLGA: It's more likely that we'll receive a higher payoff if B is chosen.

MARK: I'm not sure which argument I find more persuasive. I do think it's interesting that we can think of a mixed strategy as a statement about the population of players from which our opponent has been selected rather than as a device used by our opponent to select a strategy.

RESEARCHER: You have one more minute left to make your strategy choice.

Rose makes a choice and then feels elated, despondent, or angry depending upon Colin's strategy choice. The Researcher has a student pull a random ball out of a container having balls numbered from 0 to 99. If the number drawn is less than a player's game payoff, then that player wins the monetary prize offered. Otherwise, the player wins nothing.

RESEARCHER: Let's continue to our second game, called **Experiment With Proposals**. You've been randomly matched with a new opponent, who, as with the first game, will remain anonymous. You'll play the same strategic game as before; however, you'll be given an opportunity to communicate with your opponent.

Rose sees the following information on her computer screen:

Experiment With Proposals

Your potential monetary prize is	\$22

Chance to Win (Rose, Colin)		Colin		
		A	B	C
Rose	A	$(70, 70)$	$(0, 0)$	$(0, 100)$
	B	$(0, 0)$	$(50, 50)$	$(0, 45)$
	C	$(100, 0)$	$(45, 0)$	$(10, 10)$

Compose and Send a Proposal	
1. Rose proposes that	
Rose choose	○A ○B ○C
2. Rose proposes that	
Colin choose	○A ○B ○C
3. Rose is ready to play.	○Yes ○No
4. Send the proposal to Colin.	

Messages Sent and Received

RESEARCHER: You can propose to your opponent what strategy you'll choose (click on the appropriate radio button for item 1 to choose A, B, or C), what strategy your opponent should choose (click on the appropriate radio button for item 2 to choose A, B, or C for your opponent), and whether or not you are ready to play (click on the appropriate radio button for item 3). The proposal is sent when you click on the button item 4. You may send identical or different proposals as many times as you like. Each time you or your opponent make a proposal, it will appear in the "Messages Sent and Received" text area. When both players signal their readiness to play, the ability to communicate will be lost, strategy choice buttons

Click your strategy choice:	A	B	C

will appear, and the game must actually be played by each player clicking on a strategy choice button. Any questions?

ROSE: What if I say that I'm ready to play but later change my mind?

RESEARCHER: If your opponent's most recent proposal has said that he's ready to play and you've just sent a proposal in which you've stated your readiness to play, then the ability to communicate has already been lost. If, however, your opponent has not yet made a proposal or his most recent proposal stated that he was not ready to play, then you should promptly send the same or another proposal stating that you're now not ready to play.

ANOTHER STUDENT: If I propose to choose B, must I then actually choose B when it comes time to play the game?

RESEARCHER: No. Your strategy choice is completely independent of your proposals and your opponent's proposals. Of course, the communication process may help you to decide what strategy to choose.

RESEARCHER (after a pause): Any more questions? Okay, you may begin proposing and playing.

> Before you read Rose's thoughts about this game and the proposals she sends and receives, you should spend a few minutes thinking about how you would use your ability to communicate and the strategy you would eventually choose after the communication ends.

Those three voices continue to converse inside Rose's mind.

MARK: Rose should suggest the "best" Nash equilibrium (B, B), and in that way both Rose and Colin would receive 50.

PRADEEP: Seems reasonable. Without communication in the first game, we couldn't be sure that Colin understood the advantages of both choosing B. Now Rose can be assured of Colin's understanding.

MARK: Should we say that we're ready to play?

PRADEEP: I'd like to see Colin respond first.

MARK: Okay.

Rose selects the two B radio buttons and the No radio button, but before she presses the "4. Send the proposal to Colin" button, a message appears.

Colin proposes that Rose choose A. Colin proposes that Colin choose A. Colin is ready to play.

OLGA: Colin has done you both one better. Why should the two players settle for 50 each when each could obtain 70?

PRADEEP: I'm more cynical. I think Colin is trying to make Rose think like Olga so that she'll choose A. But Colin will actually choose C instead of A. Rose will obtain 0 and Colin will obtain 100.

Before Rose takes any more action, another message appears:

Colin proposes that Rose choose A. Colin proposes that Colin choose A. Colin is ready to play.

OLGA: See? Colin is making Rose a promise to choose A if she chooses A.

PRADEEP: I think his persistence is a ruse intended to deceive Rose so that he can obtain 100.

OLGA: I wish our communication wasn't so limited.

PRADEEP: Would you ask Colin his intentions? I'm sure he'd insist upon the value for both of us choosing A and so promise to choose A. All the while, he'd be planning his double-cross.

MARK: Shall we respond similarly? We could also propose that each of us choose A, but then we could choose C instead.

OLGA: So, perhaps Rose will receive 100!

PRADEEP: At least Rose would not receive less than 10.

OLGA: So why don't we go back to our original idea and propose that we both choose B.

MARK: That's a promise we both can keep. Surely Colin will understand that.

Rose sends the message:

Rose proposes that Rose choose B. Rose proposes that Colin choose B. Rose is not ready to play.

Almost immediately, another message appears:

Colin proposes that Rose choose A. Colin proposes that Colin choose A. Colin is ready to play.

PRADEEP: Does Colin think Rose is a fool?

OLGA: Isn't it wonderful how hard Colin is trying to get Rose and him to agree to cooperate to obtain even higher payoffs?

PRADEEP: But his promise is not credible. It's in Colin's interest to convince Rose to choose A as much as it's in his interest to choose C.

MARK: Perhaps we should be as insistent on the best Nash equilibrium.

Rose again sends the message:

Rose proposes that Rose choose B. Rose proposes that Colin choose B. Rose is not ready to play.

and almost immediately, another message appears:

Colin proposes that Rose choose A. Colin proposes that Colin choose A. Colin is ready to play.

MARK: I think we have two options. First, we could agree with Colin's proposal and then actually choose C. Second we could reiterate our best Nash equilibrium proposal, but also say Rose is ready to play.

PRADEEP: The second option would show Colin our seriousness of choosing B, and so he'll be forced to rethink his choice.

OLGA: And he'd then have to choose B?

PRADEEP: I suppose he could be obstinate or stupid.

MARK: And so we risk obtaining 0 by choosing B. Perhaps it's best to agree with Colin's proposal and then double-cross him by choosing C. Maybe we'll receive 100, but we'd be assured of at least 10.

PRADEEP: How has communication helped us?

OLGA: It could have if we believed Colin's promise or if Colin saw the reasonableness of Rose's offer.

RESEARCHER: You only have one minute left to end your communication and play the game.

Once again, Rose makes a strategy choice, and then she feels elated, despondent, or angry depending upon Colin's strategy choice. The Researcher has another student pull a random ball out of a container having balls numbered from 0 to 99. If the number drawn is less than a player's game payoff, then that player wins the monetary prize offered. Otherwise, the player wins nothing.

RESEARCHER: Let's continue to our third and final game, called **Experiment With Ability to Reject**. You have been randomly matched with a new opponent, who, as with the previous games, will remain anonymous. You will again be given an opportunity to communicate with your opponent, and you will play the same strategic game as before with one change: After each player has chosen A, B, or C

Click your strategy choice:	A	B	C

by clicking the appropriate strategy choice, the choices will be announced to both players,

Rose chose		
Colin chose		
Click on one:	Accept	Reject

and each player can either accept or reject the outcome. If both players accept, then the payoffs are as before. If either player rejects, then neither player can win the monetary prize.

Rose sees the following information on her computer screen:

Experiment With Ability to Reject

Your potential monetary prize is	$73

Chance to Win (Rose, Colin)		Colin A	Colin B	Colin C
Rose	A	$(70, 70)$	$(0, 0)$	$(0, 100)$
Rose	B	$(0, 0)$	$(50, 50)$	$(0, 45)$
Rose	C	$(100, 0)$	$(45, 0)$	$(10, 10)$

Compose and Send a Proposal	
1. Rose proposes that	
Rose choose	○A ○B ○C
If Colin chooses A, Rose will	○Accept ○Reject
If Colin chooses B, Rose will	○Accept ○Reject
If Colin chooses C, Rose will	○Accept ○Reject
2. Rose proposes that	
Colin choose	○A ○B ○C
If Rose chooses A, Colin will	○Accept ○Reject
If Rose chooses B, Colin will	○Accept ○Reject
If Rose chooses C, Colin will	○Accept ○Reject
3. Rose is ready to play.	○Yes ○No
4. Send the proposal to Colin.	

Messages Sent and Received

RESEARCHER: As before, you can make proposals to your opponent. However, now there is more to a strategy. In addition to the initial choice of A, B, or C, each player should state whether to accept or reject a proposal dependent upon what initial choice of A, B, or C the opponent has made. Again, you may, but need not, follow the last strategy proposed by you or your opponent.

> Before you read Rose's thoughts about this game, you should spend a few minutes thinking about how you would use your ability to communicate and the strategy you would eventually choose after the communication ends.

Rose's three voices continue their debate.

MARK: What does this extra stage do for us?

PRADEEP: If it looks as though we'll obtain a halfway decent payoff, then Colin will probably reject it.

MARK: Not if Colin thinks that the proposal is fair.

OLGA: Could we convince Colin that it would be desirable for each of us to obtain a payoff of 70?

MARK: We would need to propose that each player choose A.

PRADEEP: Which is dominated by C.

OLGA: Yes, but what if we said that we'd accept only if Colin chooses A?

PRADEEP: In other words, we should make a threat to reject if Colin chooses B or C?

OLGA: Well, the word "threat" sounds so negative, but I guess that would be a different way to describe my proposal.

MARK: To make it sound fair, shouldn't both players do the same thing?

OLGA: Okay, we should both choose A, accept if the other chooses A, and reject if the other chooses B or C.

MARK: If Rose and Colin both use that strategy, then they'll both choose A and both receive a payoff of 70.

PRADEEP: Of course, Colin will screw things up by choosing C instead and receive 100.

MARK: But if Colin chose C, then Rose would reject and Colin would receive only 0.

OLGA: Colin has a choice of 70 by choosing A and 0 by choosing C. Hence, Colin would not want to choose C instead of A.

PRADEEP: What happened to C dominating A?

MARK: This two stage game is a new game. The instruction "Choose C" by itself is not a strategy. A strategy must include instructions for what to do in response to each of Colin's choices of A, B, or C.

PRADEEP: So, the three-by-three matrix is not really the payoff matrix for this game?

MARK: That's right. To specify a strategy, a player must choose one of A, B, or C as the initial move and must choose accept or reject for each of the three possible initial moves by the other player. Thus, there are $3 \times 2 \times 2 \times 2 = 24$ strategies.

PRADEEP: So, the real payoff matrix for this game is 24-by-24?

MARK: Yes.

PRADEEP: That's too big for me to think about.

OLGA: So, don't think about it. I've already proposed a Nash equilibrium that gives each player 70!

PRADEEP: Is it really a Nash equilibrium?

MARK: If either player unilaterally changes her or his strategy, then rather than 70, she or he will obtain 0. There is no incentive for either player to change strategy.

Rose sends the message:

Rose proposes that Rose choose A and then accept only if Colin chooses A. Rose proposes that Colin choose A and then accept only if Rose chooses A. Rose is not ready to play.

OLGA: Colin is thinking about our proposal. I'm sure he will agree.

MARK: It's hard to imagine him not going along with this reasonable proposal.

Rose receives the message:

Colin proposes that Rose choose A and then accept only if Colin chooses C. Colin proposes that Colin choose C and then accept only if Rose chooses A. Colin is ready to play.

OLGA: What is this?

PRADEEP: Colin wants Rose to choose A and for him to choose C so that Rose receives 0 and Colin receives 100. I knew Colin would screw things up!

OLGA: There must be a way for Rose to thwart this ridiculous proposal!

MARK: Humm. This is strange. With the strategy Colin proposes to use, Rose will obtain 0 no matter what she does.

OLGA: Surely, Rose can do something.

PRADEEP: Of course, Rose can choose any of the twenty-four available strategies, But for each one, either Rose chooses A, Colin chooses C, both accept, and so Rose receives 0, or one of the players will reject and Rose receives 0.

MARK: So Rose has no incentive to change her strategy, and Colin cannot do better than 100. Thus, Colin has proposed a second Nash equilibrium.

OLGA: Rose cannot change her own payoff, but she could choose C. If both choose C, would Colin really reject in order to receive 0 instead of 10?

MARK: It certainly would seem unreasonable for Colin to carry out his threat to reject especially since Rose's personal payoff is unaffected by a change in strategy.

OLGA: Rose has made a more credible threat. She has proposed to reject only when she would already be receiving 0.

RESEARCHER: Once again, your time is nearly up. Please send your last communications and play the game.

No easy answers come to Rose as she and Colin exchange their final proposals. Rose and Colin state that they are ready to play, and each makes a choice among A, B, or C. After learning what was chosen by the other player, each player decides whether to accept or reject. The game is over, and prizes are awarded as previously.

RESEARCHER: Thank you for taking time to help with my dissertation!

Exercises

(1) In **Experiment With One Choice**, is it possible to make a promise or a threat? If so, when is the promise or threat credible? If not, why not? Answer the same questions for **Experiment With Proposals** and **Experiment With Ability to Reject**.

(2) In **Experiment With One Choice**, what is the best choice to make? Why? Does your answer depend on whether your opponent will be chosen from a group of friends, a group of business owners, a group of people from a different race or ethnic group, or some other specific group of people? Does your answer depend on whether the prize you might win is worth $10 or $1,000,000? Answer the same questions for **Experiment With Proposals** and **Experiment With Ability to Reject**.

(3) Consider the **Matches** game.

Matches Cardinal Payoffs		Colin Tennis	Soccer
Rose	Tennis	$(10,6)$	$(2,5)$
	Soccer	$(0,0)$	$(9,10)$

Do the best choices depend on whether the players can communicate before strategy choices are made? Why or why not?

(4) Consider the **Coordination** game.

Coordination Cardinal Payoffs		Colin A	B
Rose	A	$(10,10)$	$(0,6)$
	B	$(6,0)$	$(8,8)$

Do the best choices depend on whether the players can communicate before strategy choices are made? Why or why not?

2. The Prisoners' Dilemma

In the games we have considered so far, players do not explicitly cooperate. In some games, like **Nim**, **Hex**, and **Matching Pennies**, player interests are in perfect opposition (if each wants to win), and so there is no incentive to cooperate. Nonetheless, in most of the games discussed in Chapters 3 and 4, certain outcomes are preferred by both players to other outcomes, and so there is a potential benefit to the players acting together. But just because there is a potential benefit to cooperation does not mean that it will occur. This chapter explores what benefits can be derived from cooperation and under what circumstances cooperation might actually occur.

> By the end of this section, you will be able to describe and identify prisoners' dilemma scenarios.

Consider the following scenario, often called the **Prisoners' Dilemma**. The police have arrested two suspected accomplices to a crime. The police have enough hard evidence to convict each suspect of a misdemeanor, but a felony conviction would require a confession. The suspects are put into separate interrogation rooms. Each suspect may either confess to the police or remain quiet. If neither confesses, each will get a light sentence for the misdemeanor. If exactly one of the two confesses, the prosecutor will make a deal with the confessor to receive probation only, while the other will receive a heavy sentence. If both confess, they each get a moderate sentence.

We model this scenario as a strategic game. The two suspects, each confined in a separate interrogation room, are the players and they each have two strategies available to them, which we abbreviate as QUIET and CONFESS. The outcome matrix is as follows.

Prisoners' Dilemma Outcomes		Second Suspect QUIET	CONFESS
First Suspect	QUIET	light sentence for each	heavy sentence for first suspect; probation for second suspect
	CONFESS	probation for first suspect; heavy sentence for second suspect	moderate sentence for each

If each suspect is primarily concerned about his or her own sentence, the cardinal payoff matrix for this game could be the following.

Prisoners' Dilemma Cardinal Payoffs		Second Suspect QUIET	CONFESS
First	QUIET	$(6, 6)$	$(0, 10)$
Suspect	CONFESS	$(10, 0)$	$(4, 4)$

From this we see that confessing to the police is the prudential and dominating strategy for each of the suspects and (CONFESS, CONFESS) is the unique Nash equilibrium, leading to payoffs $(4, 4)$. This congruence of three solution concepts suggests that players will choose CONFESS. But the payoffs $(6, 6)$, which are obtained when both suspects choose QUIET, are better for both players. There is a tension between (1) choosing the dominating strategy, which will always be better for you regardless of the other player's choice, and (2) knowing that a better outcome might be possible if both players choose their dominated strategy. This tension is what puts the dilemma into the **Prisoners' Dilemma**.

Before you continue to read, play the **Prisoners' Dilemma** game with someone else. To simulate the conditions as much as possible, (1) the players should not communicate before making their choices, (2) choices should be made secretly and written on a piece of paper before being simultaneously revealed, and (3) each player should be trying to maximize his or her own payoff. To create even more anonymity, have several players write both their strategy choice and name on a piece of paper and then randomly pair the pieces of paper. How many players chose QUIET? How many players chose CONFESS? Which players received higher payoffs? Would allowing preplay communication have changed the results?

Could the two suspects be friends and care about what happens to the other? Perhaps so! But suppose their friendship is not totally altruistic, that is, neither wants to be a patsy. Given this context, the outcome descriptions should be extended:

Coordination Outcomes		Second Suspect QUIET	CONFESS
First Suspect	QUIET	light sentence for each and equity maintained	heavy sentence for the first suspect; probation for second suspect but friendship violated
	CONFESS	probation for first suspect but friendship violated; heavy sentence for second suspect	moderate sentence for each and equity maintained

If each suspect is concerned primarily about not being a patsy, secondarily about maintaining the friendship, and with all else equal about his or her own sentence, the cardinal payoff matrix for this game could be the following.

Coordination Cardinal Payoffs		Second Suspect QUIET	CONFESS
First	QUIET	$(10, 10)$	$(0, 6)$
Suspect	CONFESS	$(6, 0)$	$(8, 8)$

This game is no longer a **Prisoners' Dilemma** game. Instead, we shall call it the **Coordination** game. There are two pure strategy Nash equilibria and one mixed strategy Nash equilibrium: (QUIET, QUIET), (CONFESS, CONFESS), and ($\frac{2}{3}$QUIET + $\frac{1}{3}$CONFESS, $\frac{2}{3}$QUIET + $\frac{1}{3}$CONFESS). Although the (QUIET, QUIET) Nash equilibrium leads to the outcome most preferred by each player, there must be a twinge of doubt about their partner's choice because (CONFESS, CONFESS) is also a Nash equilibrium, and the CONFESS strategy choice ensures a payoff of at least 6, while the QUIET strategy choice can only ensure a payoff of 0. That is, CONFESS is the pure prudential strategy. A player can ensure an even greater expected payoff by using the prudential strategy $\frac{1}{6}$QUIET + $\frac{5}{6}$CONFESS, which ensures the security level of 6.67.

Before you continue to read, play the **Coordination** game with someone else in the same way you played **Prisoners' Dilemma** above. How do the results of playing the **Coordination** game compare with the results of playing the **Prisoners' Dilemma** game? Would allowing preplay communication have changed the results?

In the **Prisoners' Dilemma** game, a player's best response does not depend on the opponent's choice: CONFESS always leads to a higher payoff. Although the outcome obtained from the strategy choices (QUIET, QUIET) is better for each player than the outcome obtained from the dominant strategy choices (CONFESS, CONFESS), each player would regret not obtaining an even higher payoff by deviating from the (QUIET, QUIET) choices. In the **Coordination** game, a player's best response depends on the opponent's choice: QUIET leads to a higher expected payoff if there is at least a 2/3 chance that the opponent chooses QUIET. While choosing QUIET is riskier than choosing CONFESS, neither player has any regrets if the strategy choices are (QUIET, QUIET).

Prisoners' Dilemma Scenario: A scenario is said to be a *prisoners' dilemma scenario* if it can be modeled as a strategic game in which there is a single strategy for each player that dominates all of that player's other strategies (a *dominant* strategy), but all players would receive a higher payoff if they would together choose a specific dominated, rather than the dominant, strategy. Since the mutual benefit result requires all players to choose the dominated strategy, it is often called the *COOPERATE* strategy, and since there is always an incentive for any individual player to switch her or his choice to the dominant strategy, the dominant strategy is often called the *DEFECT* strategy.

Since payoffs are supposed to reflect player choices, one can argue that players in a prisoners' dilemma scenario must choose the dominant, or DEFECT, strategy, for to do otherwise would be to choose an outcome with a lower payoff than is available, a contradiction to the direct connection between payoffs and choice.

If the prisoners' dilemma scenario involves exactly two players and each player has exactly two strategies, then its payoff matrix must look like the following:

General Prisoners' Dilemma Cardinal Payoffs		Column Player COOPERATE	Column Player DEFECT
Row Player	COOPERATE	(R, R)	(S, T)
	DEFECT	(T, S)	(P, P)

where the "sucker" payoff is S, "punishment" payoff is P, "reward" payoff is R, and "temptation" payoff is T, and they satisfy $S < P < R < T$. The reader can verify that DEFECT is the prudential and the dominant strategy for each player, (DEFECT, DEFECT) is the unique Nash equilibrium, and each player would be better off if the strategy pair (COOPERATE, COOPERATE) were chosen instead of the strategy pair (DEFECT, DEFECT). The restriction $S + T < 2R$ is often added to ensure that the two players could not achieve even greater payoffs by publicly flipping a coin and choosing (COOPERATE, DEFECT) if heads and (DEFECT, COOPERATE) if tails. More will be said about such "correlated" strategies in section 6.2.

There are many real world prisoners' dilemma scenarios. A prisoners' dilemma scenario may, and frequently does, involve more than two players. We describe three here and ask you to consider some others in the exercises.

If all of the gas stations in a town charge the same prices for gasoline, then they will each sell roughly the same amount of gasoline and obtain a certain profit. If the "Moxil" gas station sets its price slightly lower than the other gas stations, then Moxil will attract customers away from the other gas stations. Although Moxil will obtain slightly less revenue from each customer (because of its lower prices), it will attract so many additional customers (again because of its lower prices) that its overall profit will increase. Of course, once Moxil lowers its prices, the other gas stations lose customers resulting in less profit. In order to increase their profits, the other gas stations need to lower their prices. So, the dominant strategy for a gas station is to decrease its prices. However, if every gas station chooses to lower its prices, then no one gains market share and each receives a smaller profit. If all the gas stations could together choose a high price, then each will make a larger profit with the same market share. Of course, with its competitors choosing a high price, any single gas station can make even more profit by choosing a lower price. The tension between wanting every gas station to charge high prices (COOPERATE) but individually wanting to charge lower prices (DEFECT) is the dilemma in this prisoners' dilemma scenario.

A second example of a real-world prisoners' dilemma scenario involves student contributions to a group assignment. Suppose Amy, Jose, Ibrahim, and Xianon have to complete a group assignment and each is primarily interested in obtaining a high grade with as little effort as possible. If three of the four students put in maximal effort, then an A can be achieved. But this means that one student can put in minimal effort and still receive the group's A grade. If all members of the group put in minimal effort, then the grade is more likely to be a D. But with the other three members putting in minimal effort, then one student putting in maximal effort is not likely to change the grade: extra work for little or no grade benefit. So the dominant strategy for each student is to put in minimal effort. But everyone would be better off if all put in maximal effort. Of course, with everyone putting in maximal effort, any single student can be made even better off by choosing to put in minimal effort. The tension between wanting everyone to put in maximal effort (COOPERATE) but individually wanting to put in minimal effort (DEFECT) is the dilemma in this prisoners' dilemma scenario.

A third example is the **Hand** game, played with two or more players. Each player simultaneously shows an open or closed hand. Your payoff is determined by the number of other players' open hands that you see as specified by the payoff matrix below.

Hand		Open Hands Seen Among Other Players					
Your payoff		0	1	2	3	4	n
You	OPEN	0	5	10	15	20	$5n$
	CLOSED	10	15	20	25	30	$5n + 10$

For example, if Amy, Jose, and Ibrahim show open hands and Xianon shows a closed hand, then Amy, Jose, and Ibrahim each see two open hands among the other players and so receive a payoff of 10, and Xianon sees three open hands among the other players and so receives a payoff of 25.

> Before you continue to read, play the **Hand** game with at least two other players. When you played the game, how many players chose to open their hands? How many chose to close their hands? Who received the greater payoff? Why is this game a prisoners' dilemma scenario?

We have described four prisoners' dilemma scenarios in this section. To make the parallel structure clear, Table 2.1 summarizes the two strategies in each of the four scenarios.

TABLE 2.1. Comparison of Prisoners' Dilemma Scenarios

Scenario	Prisoners' Dilemma	Gas Stations	Student Group Project	Hand Game
COOPERATE	QUIET	Higher prices	Maximal effort	OPEN
DEFECT	CONFESS	Lower prices	Minimal effort	CLOSED

Sometimes what may appear to be a prisoners' dilemma scenario could be a different game. For example, the original scenario described in this section, instead of being a **Prisoners' Dilemma** game, could really be a **Coordination** game if the players are motivated by friendship or loyalty to each other. If all gas stations seem to be charging the same high prices, could it be that loyalty toward other gas stations is as important a motivation as profits? If students put in maximal effort on a group project, could it be that loyalty toward other students is as important a motivation as grades and minimizing effort? That is, was the prisoners' dilemma real or only apparent? And if the dilemma is real, is there some explanation for why players sometimes forgo their dominant strategy? We address these questions in the next section.

Exercises

(1) Which of the many games described in Chapter 3 could be considered a prisoners' dilemma scenario?

(2) Determine which of the following are prisoners' dilemma scenarios. To do so, you will need to answer the following questions: Who are the players? What are the COOPERATE and DEFECT strategies? Is an individual player better off choosing the DEFECT strategy regardless of what other players do? Why or why not? Are all players better off if they all choose the COOPERATE strategy than if they all choose the DEFECT strategy? Why or why not? (Based on [**64**, page 144].)
   (a) Nations deciding whether to develop and stockpile nuclear weapons.
   (b) Couples in a poor country deciding whether to have another child.
   (c) Automobile drivers deciding to drive on the right or left side of the road.
   (d) Deciding whether or not to turn on your air conditioner during an extremely hot and humid summer day.
   (e) Standing or sitting on your seat at a rock concert.
   (f) Students deciding whether to attend a class.
   (g) Whether to pay to ride the subway or to jump over the turnstile and not pay.
   (h) Whether prey flee from or try to fight a predator.
   (i) Municipalities surrounding a lake deciding whether to build sewage treatment plants.

(3) From your own experiences, describe a prisoners' dilemma scenario. Write the story. Identify the players, strategies, and payoffs. Identify the dominant strategy, and explain why it is dominant. Identify a dominated strategy that will then be mutually beneficial if every player chooses it, and explain why it is dominated but would be mutually beneficial if every player chooses it.

(4) From your own experiences, describe a scenario that could be modeled as a strategic game similar to the **Coordination** game. Write the story. Identify the players, strategies, and payoffs. Identify the two pure strategy Nash equilibria. Explain why it might or might not be difficult for the players to pick the same Nash equilibrium.

(5) **Commons** [**64**, page 143]. Six farmers live around a common which can support six cows of value \$1000 each. For each additional cow beyond six which is grazed on the common, the value of every cow grazed on the common decreases by \$100. Each farmer has two strategies: graze one or two cows on the common.

(a) Fill in the table

Commons Increased value of the Farmer's cows		Number of other farmers choosing to graze only one cow 0	1	2	3	4	5
Farmer's Strategy	Graze one cow						
	Graze two cows						

(b) Explain why this game is a prisoners' dilemma scenario.
(c) What is the smallest size coalition of farmers that could benefit by having all of its members choose to graze only one cow? Assume that the farmers not in the coalition would continue to pursue their individual self-interest by choosing to graze two cows. If such a coalition formed, would you prefer to be in it or not in it?

(6) **Continental Divide.** Consider the following five-person cooperation-defection game, based on [**64**, page 143].

Continental Divide Cardinal Payoffs		Number of others choosing C 0	1	2	3	4
Player Chooses	C	-3	-1	1	3	5
	D	-1	0	1	2	3

(a) Explain why this is not a prisoners' dilemma scenario.
(b) If all players play C, would a player think it beneficial to have chosen D?
(c) Suppose four players choose C and one player chooses D (which we denote CCCCD). Would a C player regret his or her decision? Would the D player regret his or her decision?
(d) Repeat the analysis for plays of CCCDD, CCDDD, CDDDD, and DDDDD.
(e) What would you expect to happen in this game?
(f) How is the expectation that you described in (e) similar to the behavior of rainfall near a continental divide?
(g) Can you think of any real life scenarios like this?

(7) **Beauty Contest.** First presented by Herve Moulin [**33**], in this game each of two or more players chooses a number (which need not be an integer) between 0 and 100 inclusive. Calculate the average of the numbers chosen and multiply by 0.7 to obtain the target number. Whichever player chose the number closest to the target number wins a prize (which can be divided equally among two or more players if there is a tie).
(a) Gather a small group of people and play **Beauty Contest** two or more times. Describe what happened.
(b) Explain why someone might choose 35.
(c) Explain why someone might choose 24.5.
(d) Explain why someone might choose 17.15.
(e) Explain why someone might choose 0.
(f) Is this game a prisoners' dilemma scenario? Why or why not?

3. Resolving the Prisoners' Dilemma

Two players are faced with a prisoners' dilemma scenario. Specifically, we consider the strategic game introduced in the previous section and reproduced here.

Prisoners' Dilemma Cardinal Payoffs		Colin COOPERATE	DEFECT
Rose	COOPERATE	$(6,6)$	$(0,10)$
	DEFECT	$(10,0)$	$(4,4)$

DEFECT is the dominant strategy for each player resulting in a payoff of 4 for each player. It would be mutually beneficial for each player to choose COOPERATE, resulting in a payoff of 6 for each player. Both in experiments like those described in section 5.1 and in real-world scenarios like those described in section 5.2, players sometimes choose a COOPERATE strategy. This phenomenon, of some players not choosing the dominant strategy, has puzzled mathematicians, economists, psychologists, and biologists. Social planners and activists would like to find ways to encourage cooperation, so this section explores possible reasons for cooperation in a prisoners' dilemma scenario.

By the end of this section, you will be able to calculate payoffs in a repeated strategic game and explain how repetition is one way to resolve the dilemma in prisoners' dilemma scenarios.

One way to obtain cooperation is by fiat: both players are forced to cooperate. Consider people either giving or not giving money to establish a public park. If people are both money loving and enjoy having a public park, then they would most enjoy keeping their own money and establishing a park with others' money. Typically, no one person has sufficient money to single handedly establish a park. Since a single person giving money makes little difference in establishing a park, not giving money is a dominant strategy. But if all players give no money, then the public park would not be established. Yet, if each player gave money, then the benefits derived from the resulting public park would more than offset the small amount paid by each player. If each player's choice is voluntary, no public park is established. However, if a government imposes taxes on the players, then every player will be better off. So, one purpose for government is to impose mutually beneficial outcomes upon players who would otherwise choose strategies leading to mutually worse outcomes.

There are three problems with fiat as an explanation for cooperation in a prisoners' dilemma scenario. First, no authority can always dictate player actions. Second, from the game theory point of view, fiat eliminates strategic choice, so there is no game left to analyze. Finally, without fiat, players often cooperate, and so we need a better explanation.

Cooperation may be obtained through moral suasion: players are convinced that choosing COOPERATE is the only morally correct choice. In particular, the Golden Rule asks us to place the preferences of others ahead of our own. Applying this to the suspects example, each suspect should want to minimize the amount of jail time the other suspect receives. In terms of the payoff matrix above, the payoffs should be interchanged, resulting in a different game, **Moral Suasion**, with the following payoff matrix:

Moral Suasion Cardinal Payoffs		Colin COOPERATE	DEFECT
Rose	COOPERATE	$(6,6)$	$(10,0)$
	DEFECT	$(0,10)$	$(4,4)$

Each player now chooses COOPERATE because it has become the dominant strategy. Other forms of moral suasion might be to value equity and friendship, which would change the payoffs to form the **Coordination** game discussed in the previous section.

Generally, moral suasion assigns preferences as they "should be" rather than what they are. This explanation for cooperation is that the player who chooses COOPERATE is not actually playing a game in which DEFECT is dominant but is instead playing a game in which COOPERATE is dominant. Of course, there is no dilemma for such a player. However, there are players who actually see the dilemma but still choose to cooperate. Moral suasion does not explain their behavior.

Perhaps communication between the players before they make their final strategy choices can lead to cooperation. For example, communication is helpful in the **Coordination** game, shown here.

Coordination Cardinal Payoffs		Second Suspect COOPERATE	DEFECT
Rose	COOPERATE	$(10,10)$	$(0,6)$
	DEFECT	$(6,0)$	$(8,8)$

Without communication, there is uncertainty about the Nash equilibrium each player will choose. However, if communication is allowed before strategy choices are finalized, the two players could each commit to COOPERATE because the resulting outcome is best possible for each. When the players finalize their choices, each player has no reason to doubt the commitment, and so each will actually choose COOPERATE.

Unfortunately, communication before choices are finalized will not guarantee cooperation in a prisoners' dilemma scenario. Again, the two players could each commit to COOPERATE because the resulting outcome is better than if they each choose DEFECT; however, there is no reason for either player to honor their commitment when it comes time to choose privately. In some scenarios, communication may not even be possible. For example, in the gas station scenario described in section 5.2, such communication would be called collusion, and any choices agreed upon could result in criminal charges of price fixing.

The last explanation for cooperation we will consider is repeated play. Gas stations can change their prices daily. Students work with each other in a variety of contexts. Can repeated play encourage more cooperative choices? We explore this possibility with the **Repeated Prisoners' Dilemma** game. This involves playing

Prisoners' Dilemma, shown here again, in one or more rounds.

Prisoners' Dilemma Cardinal Payoffs		Colin COOPERATE	DEFECT
Rose	COOPERATE	$(6,6)$	$(0,10)$
	DEFECT	$(10,0)$	$(4,4)$

In each round, players simultaneously choose their strategies, after which they reveal their strategy choices, and finally payoffs are allocated in accordance with the table above. After playing a round, a six-sided die is rolled. The game is over if the die shows a six; otherwise, another round is played. The overall payoff is the sum of the round payoffs.

Before you continue to read, play the **Repeated Prisoners' Dilemma** game with someone else. What strategies seem to result in higher total payoffs? What is the effect of communication before and/or during play? How did play in this repeated game compare with the single round **Prisoners' Dilemma** game played in the previous section? Would a successful strategy be different if you knew that there would be exactly three rounds?

In the 1980's Robert Axelrod sponsored a tournament in which contestants could submit computer programs designed to play **Prisoners' Dilemma** repeatedly. Rather than the number of rounds being determined randomly, as in **Repeated Prisoners' Dilemma**, Axelrod had a large fixed number of rounds [**5**]. Each program was the contestant's strategy for playing Axelrod's game. The winner of the tournament would be the contestant whose program had the highest score after playing thousands of games in a round robin fashion. Fourteen programs were submitted to the contest, and the winner, submitted by Anatol Rapport, was

TIT FOR TAT: Choose COOPERATE in the first round. In subsequent rounds, choose the strategy your opponent chose in the previous round.

Axelrod reported the results of his tournament and called for contestants in a second tournament. Sixty-two people submitted programs for the second tournament, with several designed specifically to defeat TIT FOR TAT. Surprisingly, TIT FOR TAT won again! Although it lost rounds to the programs designed to beat it, when these played each other as part of the round robin competition, they lowered each other's scores sufficiently for TIT FOR TAT to have the highest score in the end.

To obtain a theoretical understanding of Robert Axelrod's empirical results, we will model repeated play of the **Prisoners' Dilemma** game as the single game **Repeated Prisoners' Dilemma**. Since a strategy is a complete description of what to do in every possible scenario and it is possible for there to be any number of rounds, a **Repeated Prisoners' Dilemma** game strategy must describe what to do in round n for each positive integer n. A strategy could be fairly complex, such as the following:

STRANGE: In odd numbered rounds choose COOPERATE with probability 0.5 and choose DEFECT with probability 0.5, while in even numbered rounds choose the action that has been used less by the other player in previous rounds.

A strategy could be very simple, such as one of the following:

COOPERATE ALWAYS: Choose COOPERATE in each round, regardless of what the other player has chosen in previous rounds.

DEFECT ALWAYS: Choose DEFECT in each round, regardless of what the other player has chosen in previous rounds.

Since two possibilities for round n are COOPERATE or DEFECT and the choices in each round can be different, there are an infinite number of strategies. We do not have sufficient pages in this book to exhibit a payoff matrix with an infinite number of rows and columns. So, we will have to be content with analyzing just a few of the possibilities.

But before we can analyze strategies, we must clearly define payoffs. At the end of a single play of the **Repeated Prisoners' Dilemma** game, a player's payoff will, of course, depend on the strategies chosen but also on the number of rounds actually played. Because of the indeterminate number of rounds that actually occur, the expected payoff is calculated in the following manner.

$$\begin{aligned}
\text{expected payoff} \quad = \quad & (\text{payoff in round 1})\,(\text{probability round 1 occurs}) \\
& + (\text{payoff in round 2})\,(\text{probability round 2 occurs}) \\
& + (\text{payoff in round 3})\,(\text{probability round 3 occurs}) \\
& + \cdots \\
& + (\text{payoff in round } n)\,(\text{probability round } n \text{ occurs}) \\
& + \cdots
\end{aligned}$$

Clearly, round 1 will be played with probability 1. Round 2 will be played if 1 through 5 appears when the six-sided die is rolled the first time, which occurs with probability $\frac{5}{6}$. Round 3 will be played if 1 through 5 appears when the six-sided die is rolled the first time and if 1 through 5 appears when the six-sided die is rolled the second time, which occurs with probability $\frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^2$. Round 4 will be played if 1 through 5 appears when the six-sided die is rolled on three separate occasions, which occurs with probability $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^3$. In general, round n will be played if 1 through 5 appears when the six-sided die is rolled on $n-1$ separate occasions, which occurs with probability $\left(\frac{5}{6}\right)^{n-1}$. Hence, a player's expected payoff can be rewritten as follows:

$$\begin{aligned}\text{expected payoff} &= (\text{payoff in round 1})\\ &\quad + (\text{payoff in round 2})\left(\frac{5}{6}\right)\\ &\quad + (\text{payoff in round 3})\left(\frac{5}{6}\right)^2\\ &\quad + \cdots\\ &\quad + (\text{payoff in round } n)\left(\frac{5}{6}\right)^{n-1}\\ &\quad + \cdots\end{aligned}$$

How do we sum an infinite number of numbers? Sometimes it is possible to obtain a simple formula. For example, consider the following sum.

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \cdots$$

We can first examine sums of the first few terms:

$$\begin{aligned}1 + \frac{1}{2} &= \frac{3}{2},\\ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 &= \frac{7}{4},\\ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 &= \frac{15}{8},\\ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 &= \frac{31}{16}.\end{aligned}$$

After staring at these numbers for a short time, you might notice that the denominators are increasing by powers of 2. In fact, the denominator of the nth sum is 2^n. After staring at these numbers for a while longer, you might notice that each numerator is 1 less than the next denominator. This would mean that the denominator of the nth sum is $2^{n+1} - 1$. These observations for $n \leq 4$ suggest the following conjecture about the general form of these sums:

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n = \frac{2^{n+1} - 1}{2^n}.$$

It may not be immediately clear how to prove that this conjecture is true for all values of n. The search for a proof is often difficult. That is why we celebrate when someone makes the creative leap that conveys understanding. In this case, one creative leap is to repeat the last term and sum from the right. So for $n = 3$,

$$\begin{aligned}1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\\ &= 1 + \frac{1}{2} + \frac{1}{2}\\ &= 1 + 1\\ &= 2.\end{aligned}$$

Similarly, for $n = 4$,

$$\begin{aligned} 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^4 &= 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^3 \\ &= 2. \end{aligned}$$

And in general,

$$\begin{aligned} 1+\frac{1}{2} &+ \left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^n \\ &= 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{2}\right)^{n-1} \\ &= 2. \end{aligned}$$

So

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^n = 2-\left(\frac{1}{2}\right)^n,$$

which is exactly the formula we obtained earlier in a slightly different form. Notice that as the number of terms increases, $\left(\frac{1}{2}\right)^n$ decreases toward 0. We conclude that the infinite sum

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^n+\cdots = 2.$$

Another clever insight is that

$$\begin{aligned} & (1-p)(1+p+p^2+\cdots+p^{n-1}) \\ = & (1+p+p^2+\cdots+p^{n-1}) - p(1+p+p^2+\cdots+p^{n-1}) \\ = & (1+p+p^2+\cdots+p^{n-1}) - (p+p^2+p^3\cdots+p^n) \\ = & 1-p^n. \end{aligned}$$

So

(Finite Geometric Sum) $$1+p+p^2+\cdots+p^{n-1} = \frac{1-p^n}{1-p}.$$

If $0 \leq p < 1$, then as n increases, p^n decreases toward 0. We conclude that the infinite sum

(Infinite Geometric Sum) $$1+p+p^2+\cdots+p^n+\cdots = \frac{1}{1-p}.$$

These formulas will be helpful in the calculations to follow.

Suppose that Rose and Colin both choose TIT FOR TAT. In round 1, Rose and Colin will choose COOPERATE. Since Rose chose COOPERATE in the first round, TIT FOR TAT will have Colin choose COOPERATE in the second round. Likewise, since Colin chose COOPERATE in the first round, TIT FOR TAT will have Rose choose COOPERATE in the second round. Therefore, by following TIT FOR TAT, Rose and Colin will choose COOPERATE in every round. Since each player's payoff is 6 in each round that they play, each player's expected payoff is

$$6+6\left(\frac{5}{6}\right)+6\left(\frac{5}{6}\right)^2+\cdots+6\left(\frac{5}{6}\right)^{n-1}+\cdots.$$

Factoring out the 6, we obtain

$$6\left(1+\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)^2+\cdots+\left(\frac{5}{6}\right)^{n-1}+\cdots\right).$$

Inside the parentheses is an infinite geometric sum with $p=\frac{5}{6}$. By the (Infinite Geometric Sum) formula, each player's expected payoff is

$$6\frac{1}{1-\frac{5}{6}}=36.$$

Now suppose that Rose chose TIT FOR TAT and that Colin chose DEFECT ALWAYS. In round 1, Rose would choose COOPERATE while Colin would choose DEFECT, resulting in payoffs of 0 and 10, respectively. In subsequent rounds, Colin will continue to choose DEFECT, and since Rose copies what Colin has done in previous rounds, she will also choose DEFECT in every subsequent round. Each player will receive a payoff of 4 in these subsequent rounds. So, the payoff for Rose is

$$\begin{aligned}
& 0+4\left(\frac{5}{6}\right)+4\left(\frac{5}{6}\right)^2+\cdots+4\left(\frac{5}{6}\right)^{n-1}+\cdots \\
= \quad & 4\left(\frac{5}{6}\right)\left(1+\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)^2+\cdots+\left(\frac{5}{6}\right)^{n-2}+\cdots\right) \\
= \quad & 4\left(\frac{5}{6}\right)\frac{1}{1-\frac{5}{6}}=20,
\end{aligned}$$

and the payoff for Colin is

$$\begin{aligned}
& 10+4\left(\frac{5}{6}\right)+4\left(\frac{5}{6}\right)^2+\cdots+4\left(\frac{5}{6}\right)^{n-1}+\cdots \\
= \quad & 10+4\left(\frac{5}{6}\right)\left(1+\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)^2+\cdots+\left(\frac{5}{6}\right)^{n-2}+\cdots\right) \\
= \quad & 10+4\left(\frac{5}{6}\right)\frac{1}{1-\frac{5}{6}}=30.
\end{aligned}$$

Notice that if Rose uses TIT FOR TAT, Colin obtains 36 by choosing TIT FOR TAT and only 30 by choosing DEFECT ALWAYS. Thus, the strategy TIT FOR TAT is a better response than DEFECT ALWAYS to a player using TIT FOR TAT. But perhaps Colin could find an even better response to Rose—after all, there are an infinite number of strategies to chose from. The next theorem shows that Colin cannot find a better response to Rose than TIT FOR TAT.

Tit For Tat Theorem: *The strategy pair* (TIT FOR TAT, TIT FOR TAT) *is a Nash equilibrium for the* **Repeated Prisoners' Dilemma** *game.*

Proof. Suppose that Rose has chosen TIT FOR TAT and that Colin is searching for his best response. We already know that if Colin chooses COOPERATE in each round, his expected payoff will be 36, and if Colin chooses DEFECT in each round, his expected payoff will be 30. Since Colin's best response should give him an expected payoff of at least 36, his best response must involve him choosing COOPERATE in some round.

Suppose that Colin chooses DEFECT in the first $n \geq 1$ rounds and then chooses COOPERATE. Colin's expected payoff from just the first $n+1$ rounds would be

$$\begin{aligned}
P_D &= 10 + 4\left(\frac{5}{6}\right) + 4\left(\frac{5}{6}\right)^2 + \cdots + 4\left(\frac{5}{6}\right)^{n-1} + 0\left(\frac{5}{6}\right)^n \\
&= 10 + 4\left(\frac{5}{6}\right)\left(1 + \left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 + \cdots + \left(\frac{5}{6}\right)^{n-2}\right) \\
&= 10 + 4\left(\frac{5}{6}\right)\frac{1-(\frac{5}{6})^{n-1}}{1-\frac{5}{6}} \\
&= 30 - 20\left(\frac{5}{6}\right)^{n-1}.
\end{aligned}$$

Had Colin, instead, chosen COOPERATE in the first $n+1$ rounds, then Colin's expected payoff from just the first $n+1$ rounds would have been

$$\begin{aligned}
P_C &= 6 + 6\left(\frac{5}{6}\right) + 6\left(\frac{5}{6}\right)^2 + \cdots + 6\left(\frac{5}{6}\right)^{n-1} + 6\left(\frac{5}{6}\right)^n \\
&= 6\left(1 + \left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 + \cdots + \left(\frac{5}{6}\right)^{n-1} + \left(\frac{5}{6}\right)^n\right) \\
&= 6\frac{1-(\frac{5}{6})^{n+1}}{1-\frac{5}{6}} \\
&= 36 - 36\left(\frac{5}{6}\right)^{n+1} \\
&= 36 - 25\left(\frac{5}{6}\right)^{n-1}.
\end{aligned}$$

But for each $n \geq 1$,

$$\begin{aligned}
6 &> 5\left(\frac{5}{6}\right)^{n-1}, \\
36 - 30 &> 25\left(\frac{5}{6}\right)^{n-1} - 20\left(\frac{5}{6}\right)^{n-1}, \\
36 - 25\left(\frac{5}{6}\right)^{n-1} &> 30 - 20\left(\frac{5}{6}\right)^{n-1}, \\
P_C &> P_D.
\end{aligned}$$

Notice also that changing his choices in the first n rounds will not change any of Rose's choices in round $n+2$ or after. This means that if Colin uses a strategy that has him DEFECT in the first n rounds and then COOPERATE in the $n+1$st round, he would obtain a larger payoff by changing all DEFECT choices in the first n rounds to COOPERATE choices. So, Colin's best response has him COOPERATE in round 1.

Suppose now that Colin chooses COOPERATE in the first $m \geq 1$ rounds and chooses DEFECT in round $m+1$. Colin's expected payoff in the entire game can be written as the sum of Colin's expected payoff in the first m rounds, and $\left(\frac{5}{6}\right)^m$ times the expected payoff in rounds $m+1$ onward, thought of as round 1 onward of a new game. But the previous arguments can be repeated to show that Colin's best response has him COOPERATE in round $m+1$. This shows that Colin's best response is COOPERATE in every round. Since TIT FOR TAT will result in Colin choosing COOPERATE in each round, TIT FOR TAT is a best response to TIT FOR TAT. □

The Tit for Tat Theorem shows that even when players are selfish and would benefit from not cooperating with other players, they may still cooperate if multiple games will be played over time with the same players. A student who might prefer to slack off on a group assignment knows that others may slack off on him or her in future assignments. Gas stations which try to capture greater profits by lowering their prices today know that the other gas stations may retaliate tomorrow by lowering their prices. Even among self-interested individuals, cooperation can become a best response when there may be future games and players are willing to reward cooperative behavior and punish noncooperative behavior.

Exercises

(1) Work with a small group of friends to simulate Axelrod's tournament. That is, explain to your friends how to play **Repeated Prisoners' Dilemma** and hold a small tournament with some small "prize" for the person with the best score. Save TIT FOR TAT for yourself. Write about your friend's strategies and the outcome of the tournament.

(2) If the other player is using COOPERATE ALWAYS, show that DEFECT ALWAYS is a better response than COOPERATE ALWAYS. Recalling that a best response to TIT FOR TAT is TIT FOR TAT, explain why willingness to punish a player who chooses to DEFECT is important to ensuring that players actually do COOPERATE.

(3) Unfortunately, (TIT FOR TAT, TIT FOR TAT) is not the only Nash equilibrium for the **Repeated Prisoners' Dilemma** game. Show that the (DEFECT ALWAYS, DEFECT ALWAYS) strategy pair is also a Nash equilibrium.

(4) **Repeated Matches.** Suppose Rose and Colin play a repeated version of the **Matches** game with a 90% chance of playing a subsequent round. The cardinal payoffs for a single round are given in the matrix shown here.

Matches Cardinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(10, 8)$	$(4, 0)$
	SOCCER	$(0, 4)$	$(8, 10)$

Suppose Rose and Colin both use the strategy TENNIS in odd number rounds and SOCCER in even number rounds. What are the player's expected payoffs? Show that the pair of strategies is a Nash equilibrium for **Repeated Matches**.

(5) **Repeated Hand.** This is a repeated version of the **Hand** game, with a 90% chance of playing a subsequent round. Suppose that there are at least four players. Show that the following strategy played by each player is a Nash equilibrium: show an open hand in all rounds until a closed hand is seen, after which always show a closed hand. More colloquially, COOPERATE as long as others COOPERATE, but punish forever by always choosing to DEFECT once another player has chosen to DEFECT.

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CHAPTER 6

Negotiation and Arbitration

1. A Simple Negotiation

In a strategic game, we assume that players choose their strategies simultaneously and independently. In Chapter 5, we saw that preplay communication and repeated play can sometimes foster implicit collaboration resulting in improved payoffs for the players. In this chapter, we examine what might happen in a strategic game if, during preplay communication, players could make binding agreements with each other about how to play the game. Rather than just suggesting how each other might play, players can agree upon how they must play with an unbreakable contract. The preplay communication becomes a negotiation with players making, rejecting, modifying, and accepting proposals. A sense of fairness often guides players to acceptable proposals. If negotiations break down, an arbitrator may be needed to determine a fair resolution.

> By the end of this section, you will be able to describe some notions of fairness in the context of a cooperatively played strategic game.

PROFESSOR: You have been paired to play the **Chances** game. The matrix below states your chance to win a prize.

Chances		Colin	
(Rose, Colin)		A	B
Rose	A	$(50, 20)$	$(70, 80)$
	B	$(0, 100)$	$(100, 0)$

If each of you is indifferent to your opponent's chance of winning a prize, then these chances are your cardinal payoffs. The matrix makes it look like a strategic game, but there are two changes to the rules of play. First, rather than choosing a strategy privately and simultaneously, your goal is to agree upon a strategy pair. For example, you might agree upon the strategy pair (0.5A + 0.5B,A) which would result in the payoff pair $(0.5)(50, 20)+(0.5)(0, 100) = (25, 60)$. Any questions about this first rule?

ROSE: Could we agree upon the strategy pair (A,B) and also agree to reallocate the resulting payoff pair $(70, 80)$ to be the even split $(75, 75)$?

PROFESSOR: No, you may only agree upon strategies.

COLIN: Do we have to come to an agreement?

PROFESSOR: No. The second rule is that if you do not come to an agreement before time is called, then you will receive the payoff pair $(50, 50)$.

COLIN: Why that pair?

PROFESSOR: Those are your security levels obtained with the prudential strategies A for Rose and 0.5A + 0.5B for Colin.

ROSE: And how do we come to an agreement?

PROFESSOR: Persuasion. Negotiation.

COLIN (speaking to Rose): I'll treat you to pizza after class if you agree to (B, A).

PROFESSOR (speaking to Colin): Sorry, that is not permissible. You may not promise or threaten anything unrelated to this game.

COLIN: It was worth a try.

PROFESSOR: You now have twenty minutes to agree upon a strategy pair, write it on a piece of paper (your contract), and hand it to me (who will enforce the contract). If I have not received your contract within twenty minutes, then you will receive the payoff pair $(50, 50)$. Begin.

> Before you read about how Rose and Colin fared, it would be fun and instructive for you to play **Chances** with a partner. Keep track of how the game was played.

COLIN: Since I could obtain 50 without an agreement, I'd never agree to (A, A) or (B, B).

ROSE: And since I could obtain 50 without an agreement, I'd never agree to (B, A).

COLIN: That only leaves (A,B) for which we would both benefit from an agreement.

ROSE: By using mixed strategies, perhaps we could both receive even larger payoffs.

COLIN: The only pure outcome in which you obtain more than 70 is when we both choose B.

ROSE: So, if you choose B and I choose the mixed strategy $(1-p)$A + pB, then I will obtain $70(1-p) + 100p = 70 + 30p$, which is more than 70 as long as $p > 0$.

COLIN: But then I'll obtain $80(1-p) + 0p = 80 - 80p$, which is less than 80 as long as $p > 0$. Thus, you cannot receive more than 70 unless I receive less than 80.

ROSE: True, but I won't agree to the $(70, 80)$ payoff pair. It is not fair that you receive more than I do.

COLIN: What would you suggest?

ROSE: I think we should choose our strategy pair so that we obtain the same payoffs. That means solving $70 + 30p = 80 - 80p$ to obtain $p = \frac{1}{11}$. Hence, if we agree to the strategy pair $(\frac{10}{11}\text{A} + \frac{1}{11}\text{B}, \text{B})$, then we both obtain 72.7.

COLIN: It's not clear to me that equal payoffs is fair in this game.

ROSE: Why not?

COLIN: We don't have the same opportunities. For example, $(70, 80)$ is a feasible payoff pair but $(80, 70)$ is not. This suggests that I should receive a larger payoff than you.

ROSE: Well, $(50, 20)$ is feasible while $(20, 50)$ is not feasible. Shouldn't that suggest that I receive a larger share?

COLIN: Not really, since neither one of us would agree to a payoff less than 50. Here's a game in which I would agree that it would be fair for us to receive equal payoffs.

Chances Symmetrized Cardinal Payoffs		Colin A	B	C
Rose	A	$(50, 20)$	$(70, 80)$	$(50, 100)$
	B	$(0, 100)$	$(100, 0)$	$(100, 50)$

ROSE: Why should we get the same payoffs in this game?

COLIN: Because in this game we have the same opportunities.

ROSE: What do you mean?

COLIN: Notice that the sums of payoff pairs in the matrix are never greater than 150, and both the payoff pairs in column C sum to 150. In fact, if we agreed to a strategy pair of the form $((1-p)\text{A} + p\text{B}, \text{C})$, the sum of our payoffs, $50 + 50p$ to you and $100 - 50p$ to me, is 150.

ROSE: Yes, but why does this mean that we have the same opportunities?

COLIN: If we were to agree to any strategy pair, then our combined payoff is no greater than 150, but by agreeing to a strategy pair of the form $((1-p)\text{A} + p\text{B}, \text{C})$, your worst payoff 50 is paired with my best payoff 100, your best payoff 100 is paired with my worst payoff 50, and we could receive any payoff pair $(50+50p, 100-50p)$ between these two extremes with the correct choice of p.

ROSE: So if we do this, then we each receive at least the 50 we would receive without an agreement and have the same opportunity to receive as much as 100.

COLIN: Yes, and since our payoffs would sum to 150, we are not wasting any potential payoffs.

ROSE: So, in some sense, this makes the game symmetric.

COLIN: Yes, and in this game we should each receive the same payoff, which would be the average of our worst and best expectations, 75, by agreeing to the "middle" strategy pair (0.5A + 0.5B, C).

ROSE: I agree with your analysis of **Chances Symmetrized**, but what does this have to do with **Chances**?

COLIN: Nothing, and that's my point. Here's the game that's intimately related to **Chances**.

Chances Extended Cardinal Payoffs		Colin A	B	C
Rose	A	$(50, 20)$	$(70, 80)$	$(50, 110)$
	B	$(0, 100)$	$(100, 0)$	$(90, 50)$

ROSE: Both **Chances Symmetrized** and **Chances Extended** add a column to **Chances**. The only difference I see is the payoff pairs in the new column. I don't see why **Chances Extended** should be more related to **Chances** than **Chances Symmetrized** is.

COLIN: That will take a bit of explanation. First notice another similarity between **Chances Extended** and **Chances Symmetrized**: the payoffs in column C extend from (1) the worst you can expect and the best I can expect, $(50, 110)$ in **Chances Extended** and $(50, 100)$ in **Chances Symmetrized**, to (2) the best you can expect and the worst I can expect, $(90, 50)$ in **Chances Extended** and $(100, 50)$ in **Chances Symmetrized**.

ROSE: Why is 90 the best I can expect in **Chances Extended**?

COLIN: The only way you could receive more is if we agreed to a strategy pair $((1-p)$A + pB, B) with a relatively large p, but then I'd receive less than 50, and so there would be no agreement.

ROSE: Okay. Then we should each receive the average of our worst and best expectations, $(70, 80)$, by agreeing to the "middle" strategy pair (0.5A + 0.5B, C).

COLIN: Yes! It is clear for **Chances Symmetrized** and **Chances Extended** that the most fair agreement would be the strategy pair (0.5A + 0.5B, C) resulting in the payoff pairs $(75, 75)$ and $(70, 80)$, respectively.

ROSE: Yes, I agree.

COLIN: And would you also agree that there are more possibilities in **Chances Symmetrized** and **Chances Extended** than in **Chances**?

ROSE: Yes, of course: the new games add a strategy to **Chances**. But I still don't see the relevance to **Chances**.

COLIN: Let me explain with an analogy. The basketball concession stand usually offers hot dogs, hamburgers, and chicken tenders. Which would you order?

ROSE: Mmmm. . . chicken tenders.

COLIN: Okay. Now suppose that when they're ready to take your order, they're out of chicken tenders.

ROSE: I'd have to decide between a hot dog or a hamburger. I'd choose the hot dog.

COLIN: Your choice among a hot dog, hamburger, or chicken tenders is analogous to our choice among the possibilities in **Choices Symmetrized**. Your first choice, chicken tenders, is analogous to our choice of the most fair payoff pair, $(75, 75)$. Unfortunately, when it came time for you to order, chicken tenders were no longer available, and when it comes time for us to play **Choices Symmetrized**, we no longer will have $(75, 75)$ as a possibility because we will actually play **Choices**.

ROSE: I had to rethink my order at the concession stand. Similarly, knowing what to do in **Choices Symmetrized** doesn't help us with our choice in **Choices**.

COLIN: And that's why I said that **Choices Symmetrized** has nothing to do with **Choices**.

ROSE: Okay.

COLIN: Now suppose when the person at the concession stand is ready to take your order, they've run out of hot dogs instead of chicken tenders.

ROSE: I would still order the chicken tenders.

COLIN: Why?

ROSE: Don't be dense! If chicken tenders is the best among the three possibilities, then it would still be the best between the two remaining possibilities.

COLIN: And that is the same principle I want us to use here. Your choice among a hot dog, hamburger, or chicken tenders is analogous to our choice among the possibilities in **Choices Extended**. Your first choice, chicken tenders, is analogous to our choice of the most fair payoff pair, $(70, 80)$. Fortunately, when it came time for you to order, chicken tenders were still available, and when it comes time for us to play **Choices**, instead of **Choices Extended**, we still have $(70, 80)$ as a possibility. As you so eloquently said, "Don't be dense." The payoff pair that was most fair for **Choices Extended** must still be the most fair for **Choices**.

ROSE: I follow your argument, but I don't like the conclusion. So there must be something wrong with your argument.

COLIN: You're just a sore loser!

ROSE: No. Why couldn't we use the same argument with a third extended game and obtain a different "most fair" payoff pair?

COLIN: I don't think that would be possible, but you're welcome to try to come up with one.

PROFESSOR: You have one minute to hand me an agreement paper.

ROSE: I guess I don't have time to try.

COLIN: So, do you agree to choose the strategy pair (A, B) and receive the payoff pair $(70, 80)$?

ROSE: I would have preferred the strategy pair ($\frac{10}{11}$A + $\frac{1}{11}$B, B) resulting in the payoff pair $(72.7, 72.7)$.

COLIN: And I would have preferred the strategy pair (B, A) resulting in the payoff pair $(0, 100)$.

ROSE: But that is blatantly unfair.

COLIN: And I have provided an argument for why $(70, 80)$ is fair.

ROSE: I guess I agree.

COLIN: Professor, here's our contract!

Exercises

(1) List the proposals made by Rose and Colin. Which one seems the most fair? Why? Would you make a different proposal? Why?
(2) List the fairness properties suggested by Rose and Colin. Which properties do you agree are important aspects of fairness? Why? Are there other fairness properties that you might suggest?
(3) If no agreement was made, the players were told that they would receive their security levels. Why does that seem reasonable? Are there other payoff pairs that would be more reasonable? Why or why not?
(4) At the beginning the Professor states, "If each of you is indifferent to your opponent's chance of winning a prize, then these chances are your cardinal payoffs."
    (a) Explain why this is true.
    (b) If a player were not indifferent to the chances received by his or her opponent, how would that change his or her cardinal payoffs?

2. Bargaining Games

In this chapter, we examine scenarios in which players can make binding agreements with each other. In order for players to agree with a proposal, they need to know all possible proposals, what will happen if no agreement is found, and be convinced of the fairness of a given proposal. This leads to the following definition.

Bargaining Game: In a *bargaining game*, there are two players, a set of feasible payoff pairs, a disagreement payoff pair, and a time limit. Players may propose payoff pairs at any time. The game ends when both players agree upon a payoff pair, which becomes the outcome, or the time limit is reached resulting in the disagreement payoff pair becoming the outcome. Each player most prefers to maximize her or his own payoff.

> By the end of this section, you will be able to geometrically describe all payoff possibilities in a two player bargaining scenario and how to obtain a desired payoff possibility.

By allowing players to make binding agreements in strategic games, we create bargaining games. So our first example is the **Chances** game described in the previous section, and displayed again here.

Chances Cardinal Payoffs		Colin A	B
Rose	A	$(50, 20)$	$(70, 80)$
	B	$(0, 100)$	$(100, 0)$

The unique Nash equilibrium is $(\frac{5}{8}\text{A} + \frac{3}{8}\text{B}, \frac{3}{8}\text{A} + \frac{5}{8}\text{B})$, which yields the payoff pair $(62.5, 50)$. Clearly, both Rose and Colin would be better off if they were to choose (A, B), which yields the payoff pair $(70, 80)$, but then Rose would have an incentive to change her strategy to B, and then Colin would have an incentive to change his strategy to A, and we get into a never ending cycle of best response reactions to the other player's best response reactions. But when viewed as a bargaining game, we can prevent the cycling by having Rose and Colin sign a binding agreement committing them to implement their agreed upon proposal. Since the two players have agreed to cooperate, the essential question is what to do jointly. The first step in answering that question is to identify what payoffs are possible.

Feasible Payoff Pair: A *feasible payoff pair* is a payoff for each player that can be obtained simultaneously.

For example, $(50, 20)$ is a feasible payoff pair in **Chances** because Rose will receive a payoff of 50 and Colin will receive a payoff of 20 if Rose and Colin both choose the pure strategy A. Another feasible payoff pair in **Chances** is $(62.5, 50)$ because Rose will receive a payoff of 62.5 and Colin will receive a payoff of 50 if Rose chooses the mixed strategy $\frac{5}{8}\text{A} + \frac{3}{8}\text{B}$ and Colin chooses the mixed strategy $\frac{3}{8}\text{A} + \frac{5}{8}\text{B}$. The payoff pair $(100, 100)$ is not a feasible payoff pair in **Chances** because it is impossible for Rose and Colin to obtain payoffs of 100 simultaneously. It does not matter that Rose could obtain 100 by the players choosing the strategy pair (B, B) and that Colin could obtain 100 by the player choosing the strategy pair (B, A).

The feasible payoff pairs can be displayed graphically. The strategy pair (A, A) yields the payoff pair $(50, 20)$, and so we plot that pair and label it (A, A). This is repeated for the three other pure strategy payoff pairs in Figure 2.1.

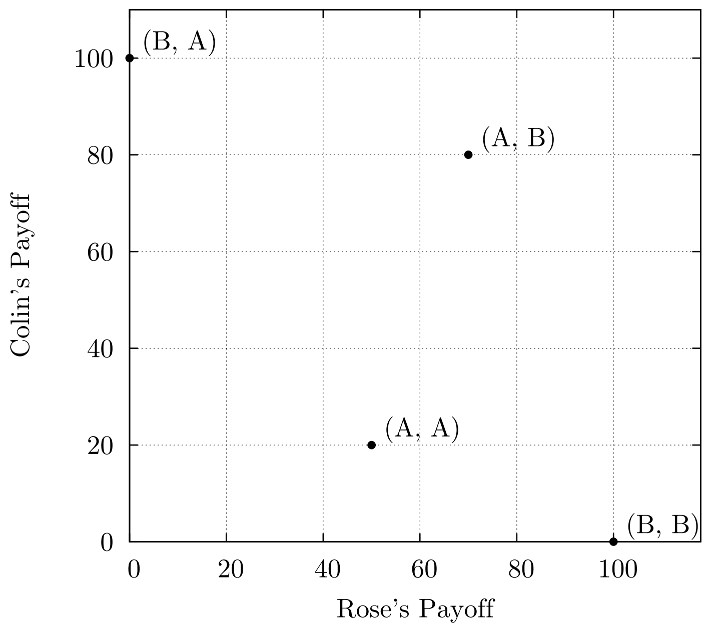

FIGURE 2.1. **Chances** pure payoff pairs

Of course, there are more feasible payoff pairs that come from players choosing mixed strategies. For example, if Rose chooses the mixed strategy 0.5A + 0.5B and Colin chooses the pure strategy A, then the payoff pair,

$$\begin{aligned}(0.5)(50, 20) + (0.5)(0, 100) &= (0.5 \times 50 + 0.5 \times 0, 0.5 \times 20 + 0.5 \times 100) \\ &= (25, 60),\end{aligned}$$

lies midway between the payoff pairs arising from the strategy pairs (A, A) and (B, A). Similarly, if the players choose the strategy pair (0.8A + 0.2B, A), then the payoff pair,

$$\begin{aligned}(0.8)(50, 20) + (0.2)(0, 100) &= (0.8 \times 50 + 0.2 \times 0, 0.8 \times 20 + 0.2 \times 100) \\ &= (40, 36),\end{aligned}$$

lies 20% of the way from the payoff pair arising from the strategy pair (A, A) to the payoff pair arising from the strategy pair (B, A). Figure 2.2 shows both of these feasible strategy pairs.

Notice that the two new feasible payoff pairs lie on the line $y = 100 - 1.6x$ determined by the feasible payoff pairs $(50, 20)$ and $(0, 100)$. This is not an accident. If

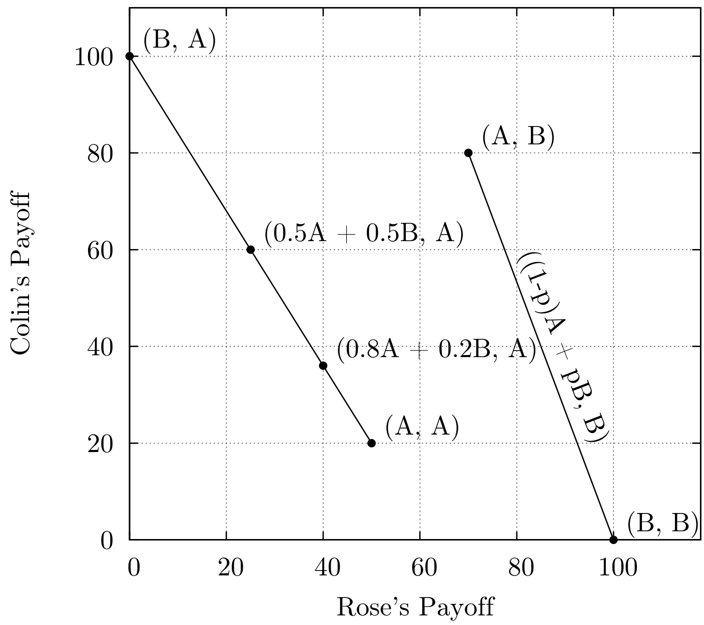

FIGURE 2.2. **Chances** payoff pairs when Rose uses mixed and Colin uses pure strategies

the players choose the strategy pair $((1-p)\mathrm{A} + p\mathrm{B}, \mathrm{A})$, then the corresponding payoff pair,

$$(1-p)(50, 20) + (p)(0, 100) = (50 - 50p, 20 + 80p),$$

lies on this line because

$$20 + 80p = 100 - 1.6(50 - 50p).$$

It is p of the distance from the payoff pair labeled (A, A) to the payoff pair labeled (B, A). Further, since p ranges from 0 to 1, every point on the line segment from $(50, 20)$ to $(0, 100)$ is a feasible payoff pair. Similarly, every point on the line segment connecting the payoff pairs labeled (A, B) and (B, B) are feasible payoff pairs.

To obtain payoff pairs on the line segment connecting the points labeled (A, A) and (B, B), Rose and Colin need to coordinate their actions. This coordination is accomplished using a correlated strategy.

Correlated Strategy: A *correlated strategy* is a lottery among strategy pairs. They are usually expressed as a linear combination of pure strategy pairs.

When Rose and Colin use the correlated strategy

$$\frac{1}{2}(\mathrm{A},\ \mathrm{A}) + \frac{1}{2}(\mathrm{B},\ \mathrm{B}).$$

they receive the expected payoff pair

$$\frac{1}{2}(50,20) + \frac{1}{2}(100,0) = (75,10).$$

The players implement this correlated strategy by publicly flipping a fair coin. If the coin lands heads, Rose and Colin will both choose A, and if the coin lands tails, Rose and Colin will both choose B. For any p between 0 and 1, $(1-p)$(A, A) + p(B, B) is a correlated strategy, and the resulting payoff pairs lie on the line segment connecting the (A, A) and (B, B) payoff pairs.

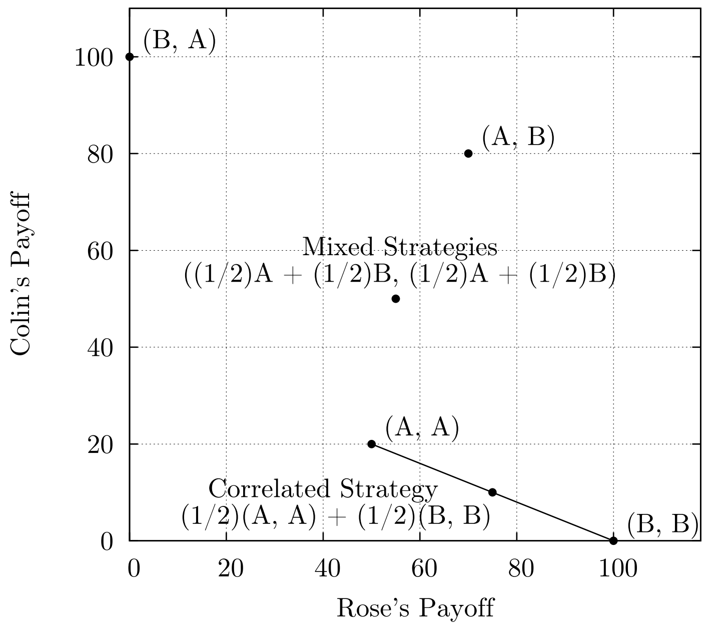

FIGURE 2.3. **Chances** correlated versus mixed strategies

Notice that the correlated strategy $\frac{1}{2}$(A, A) + $\frac{1}{2}$(B, B) resulting in the payoff pair $(75,10)$ is very different from the mixed strategy pair ($\frac{1}{2}$A + $\frac{1}{2}$B, $\frac{1}{2}$A + $\frac{1}{2}$B), with the resulting payoff pair

$$\frac{1}{2}\frac{1}{2}(50,20) + \frac{1}{2}\frac{1}{2}(70,80) + \frac{1}{2}\frac{1}{2}(0,100) + \frac{1}{2}\frac{1}{2}(100,0) = (55,50).$$

In the correlated strategy, the two players coordinate their choice of pure strategy on a single public coin toss, while for the mixed strategies, the two players each make separate and independent coin tosses to make their strategy choices. Figure 2.3 compares these two payoff pairs.

Similarly, any payoff pair on the line segment connecting payoff pairs labeled (A, B) and (B, A) can also be obtained by using correlated strategies.

The players could even use a correlated strategy over all four pure strategy pairs

$$a(\text{A, A}) + b(\text{A, B}) + c(\text{B, A}) + d(\text{B, B})$$

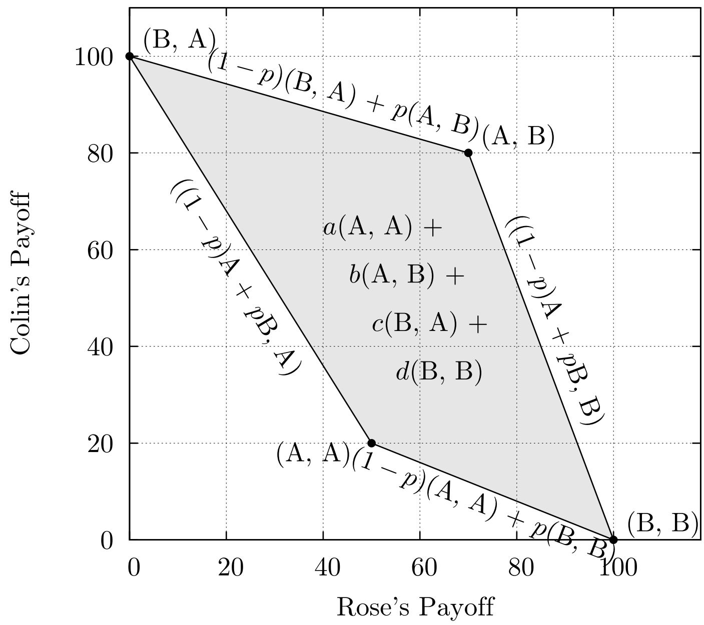

FIGURE 2.4. **Chances** feasible payoff pairs

by choosing nonnegative a, b, c, and d that sum to 1. This results in the expected payoff pair

$$(x, y) = a(50, 20) + b(70, 80) + c(0, 100) + d(100, 0),$$

which is a weighted average of the four pure payoff pairs. Just as correlated strategies of the form $(1 - p)$(A, A) + p(B, B) lie between the payoff pairs labeled (A, A) and (B, B), the payoff pairs (x, y) lie in the shaded quadrilateral in Figure 2.4. Thus, the quadrilateral is the set of feasible payoff pairs.

To complete our description of **Chances** as a bargaining game, we need to identify the disagreement payoff pair. In the first section, the pair of security payoffs $(50, 50)$ served as the **Chances** disagreement payoff pair (although it was not given that name then). Some might argue that if **Chances** were played without an agreement, the unique Nash equilibrium would be played, resulting in the payoff pair $(62.5, 50)$. Colin might argue that he would rather choose his prudential strategy, which ensures that he will receive a payoff of 50, than choose his Nash equilibrium strategy, for which there is a danger of him receiving less than 50. In response, Rose may decide to use her prudential strategy. If both players use their prudential strategies, the resulting payoff pair is $(60, 50)$.

Since the choice of the disagreement payoff pair is a modeling decision, any of the three is mathematically valid. This decision is significant since, as we see in **Chances**, the three alternatives give Rose different bargaining positions. We will use $(60, 50)$ as the disagreement payoff pair throughout this chapter, and you will explore the other two possibilities in the exercises. Thus, the **Chances** bargaining game can be summarized in Figure 2.5, where DPP is an abbreviation for "disagreement payoff pair".

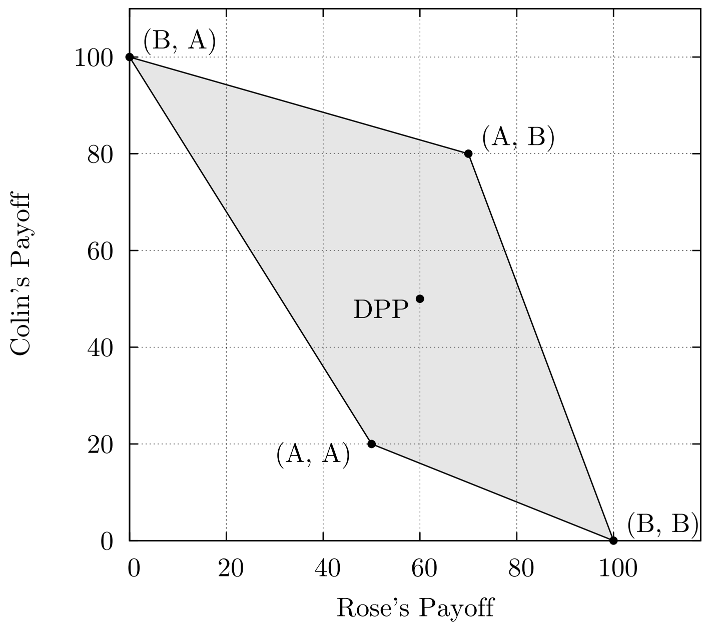

FIGURE 2.5. **Chances** bargaining game

Let us summarize what we have accomplished. We started with the scenario in which Rose and Colin were to play the **Chances** *strategic* game with one major change in the rules: Rose and Colin are permitted in enter into binding agreements about their strategy choices. We modeled this scenario as the **Chances** *bargaining* game consisting of all payoff pairs they could obtain and a disagreement payoff pair corresponding to each using her or his prudential strategy. The bargaining game itself consists of the two players proposing and potentially agreeing to payoff pairs. If, for example, Rose and Colin agree to the payoff pair $(40, 70)$, then the bargaining game is over. This result in the bargaining game can be interpreted in the original strategic game scenario by determining the correlated strategy that Rose and Colin should agree to use to obtain this payoff pair.

The payoff pair $(40, 70)$ is clearly within the filled quadrilateral in Figure 2.5. Therefore there must be nonnegative a, b, c, and d that sum to 1 and satisfying the equation

$$(40, 70) = a(50, 20) + b(70, 80) + c(0, 100) + d(100, 0).$$

Further, since $(40, 70)$ is within the filled triangle whose vertices are $(50, 20)$, $(70, 80)$, and $(0, 100)$, we can let $d = 0$. So,

$$(40, 70) = a(50, 20) + b(70, 80) + (1 - a - b)(0, 100),$$

where we used the fact that $a+b+c+d=1$ and $d=0$ to eliminate c. Separating this equation into the two individual payoffs, we obtain

$$\begin{aligned} 40 &= 50a+70b+0(1-a-b), \\ 70 &= 20a+80b+100(1-a-b), \end{aligned}$$

which simplifies to

$$\begin{aligned} 50a+70b &= 40, \\ 80a+20b &= 30 \end{aligned}$$

and has the solution $a=\frac{13}{46}$ and $b=\frac{17}{46}$. Thus, if the players use the correlated strategy $\frac{13}{46}$(A, A) + $\frac{17}{46}$(A, B) + $\frac{16}{46}$(B, A), then they will obtain the payoff pair

$$\frac{13}{46}(50,20)+\frac{17}{46}(70,80)+\frac{16}{46}(0,100)=(40,70),$$

which is what they had agreed upon.

Acme Industrial

We close this section with a scenario that is not easily modeled as a strategic game but is easily modeled as a bargaining game. The labor contract at **Acme Industrial** is up for renewal. (This example is patterned on an problem originally described by E. Allen Layman in [**28**] and then redescribed by P. Straffin in [**64**].) There are three issues over which the two sides, Labor and Management, disagree:

(1) Wages. Labor wants a 3% annual increase and Management wants no annual increase.
(2) Benefits. Labor wants to maintain all current benefits and Management wants employees to pay for one-half of their medical insurance.
(3) Security. Labor wants assurances that no jobs will be eliminated during the contract period and Management wants the capability to eliminate up to six hundred union jobs during the contract period.

If the two sides are to obtain a new contract, there must be an agreement about which side wins or what compromise is reached on each issue.

In order to model this scenario as a bargaining game, we need to determine the preferences of Labor and Management over the possible resolutions of the issues. In order to model this scenario as a (sequential) strategic game, we would need, in addition, to specify the process of negotiation (what moves are available to each player and in what sequence). Here we will avoid this additional work by modeling the scenario as a bargaining game.

To keep our model reasonably simple, we assume that the issues are independent and their associated payoffs are additive. Since two of the outcomes can be assigned arbitrary cardinal payoffs while determining preferences, we set the payoffs to be 0 when a player loses on all three issues and 100 when a player wins on all three of the issues. Now, based on our additive assumption, we can have each player decide what percentage of the whole each issue is worth. Table 2.1 shows what the players decided.

TABLE 2.1. Worth of issues to Labor and Management

Issue	Labor Worth	Management Worth
Wages	50	10
Benefits	30	30
Security	20	60
Total	100	100

Using these preferences, if Labor were to win the wages and benefits issues and Management were to win the security issue, then Labor's payoff would be $50+30 = 80$ and Management's payoff would be 60, and, as usual, we combine these payoffs into a payoff pair $(80, 60)$. Table 2.2 lists the eight "pure" outcomes in which each issue is won by one side or the other. The outcome names are the initials of the issues Labor won followed, after the comma, by the initials of the issues Management won.

TABLE 2.2. Outcomes and worth to Labor and Management

Outcome	Issue Winner			Labor	Management
Name	Wages	Benefits	Security	Payoff	Payoff
(WBS, ∅)	L	L	L	100	0
(WB, S)	L	L	M	80	60
(WS, B)	L	M	L	70	30
(W, BS)	L	M	M	50	90
(BS, W)	M	L	L	50	10
(B, WS)	M	L	M	30	70
(S, WB)	M	M	L	20	40
(∅, WBS)	M	M	M	0	100

We assume that a compromise on any single issue can be treated as a lottery. For example, if Labor were to win the wages and benefits issues and there were a 40% Labor and 60% Management compromise on the security issue, then this assumption means that we view this outcome as the lottery 0.4(WBS, ∅) + 0.6(WB, S) and the corresponding payoff pair is $(0.4)(100, 0) + (0.6)(80, 60) = (88, 36)$. What such a compromise might mean in practice would have to be negotiated. In our simplified description of the security issue as the desire by Management to be able to eliminate up to 600 union jobs, a 60% win by Management could mean giving Management the right to eliminate up to $(0.6)(600) = 360$ union jobs.

What will happen if the contract negotiations fail? A strike? A lockout? A continuance of the current contract? An imposition of an agreement by an arbitrator? These are all possibilities, and in a real situation, the likelihood of each would need to be evaluated by the negotiators or arbitrator. We will assume that if an agreement is not reached, the result will be far more costly to Labor than to Management. Specifically, we will assume that $(0, 50)$ is the disagreement payoff pair.

With the above information, the **Acme Industrial** bargaining game can be summarized in Figure 2.6.

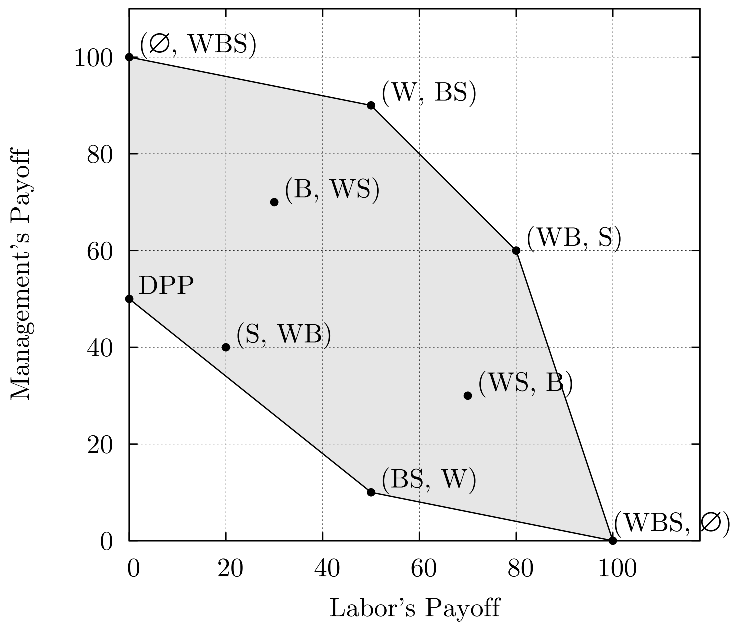

FIGURE 2.6. **Acme Industrial** bargaining game

Notice that in contrast to **Chances**, some of the pure payoff pairs for **Acme Industrial** are not vertices of the polygon enclosing all feasible payoff possibilities. Recall that any lottery of the pure outcomes is a possible outcome, and so feasible payoff pairs extend to the "outermost" pure payoff pairs. A good way to visualize the boundary of the polygon of feasible payoff pairs is to plot all of the pure payoff pairs and then imagine the shape of a taut rubber band surrounding all the pure payoff pairs.

If the negotiating teams in **Acme Industrial** agree upon the payoff pair $(20, 96)$, then how should Labor and Management interpret this result? Luckily, $(20, 96)$ lies on the line segment between $(0, 100)$ and $(50, 90)$, labeled ($\emptyset$, WBS) and (W, BS), respectively, on the graph. This means that we find the appropriate lottery by solving

$$(1-t)(0,100) + t(50,90) = (20,96),$$

which is equivalent to

$$\begin{aligned} 50t &= 20, \\ 100 - 10t &= 96, \end{aligned}$$

which has the solution

$$t = 0.4.$$

This means that $(20, 96)$ will be obtained from the lottery

$$(0.6)(\emptyset,\ \text{WBS}) + (0.4)(\text{W, BS}).$$

One way of interpreting this is that Management wins all the issues with probability 0.6 and Labor wins on the wage issue only with probability 0.4. However, these negotiators are unlikely to report to their respective constituents that they pulled colored beads from a hat to settle the contract. Thus a more appropriate interpretation of the lottery corresponds to Management winning on the benefits and security issues and a compromise is reached on the wage issue that provides a 60% win for Management and a 40% win for Labor. A reasonable compromise could be a $0.4 \times 3\% = 1.2\%$ wage increase.

Before you continue to the next section, it would be fun and instructive to play the **Acme Industrial** bargaining game. Gather some people into Management and Labor negotiating teams. Negotiate a contract within a specified amount of time. If no agreement is reached by the end of the time period, the teams receive the payoffs specified in the disagreement payoff pair. Each team should be trying to maximize their own payoff. Record how the negotiations proceeded.

Different people playing the **Acme Industrial** bargaining game will arrive at different agreements, and some may seem better than others. Some of the processes used may be more collaborative than others. What would a good collaborative solution look like? How would you know when you had one? These are the questions to which we provide some answers in the following sections.

Exercises

(1) Consider the following **Asymmetric Prisoner's Dilemma** strategic game.

Asymmetric Prisoner's Dilemma		Colin	
Cardinal Payoffs		COOPERATE	DEFECT
Rose	COOPERATE	$(90, 60)$	$(0, 100)$
	DEFECT	$(100, 0)$	$(10, 40)$

(a) What assumption must we make for this strategic game to be modeled as a bargaining game?

(b) Graph the pure strategy payoff pairs, the payoff pair corresponding to the strategy pair

$$(\text{DEFECT},\ 0.4\text{COOPERATE} + 0.6\text{DEFECT}),$$

and the correlated strategy

$$0.5(\text{COOPERATE},\ \text{COOPERATE}) + 0.5(\text{DEFECT},\ \text{DEFECT}).$$

(c) Graph the feasible payoff pairs and disagreement payoff pair $(10, 40)$.

(d) Explain why $(10, 40)$ is a reasonable disagreement payoff pair for this game. (Hint: Consider the four solution concepts for strategic games.)

(e) If possible, determine the pure, mixed, or correlated strategies that will result in the following payoff pairs. If not possible, explain why not.

(i) $(5, 70)$

(ii) $(9, 96)$
(iii) $(70, 30)$
(iv) $(100, 10)$
(v) $(72, 18)$

(2) Consider the **Matches** strategic game described in section 3.2:

Matches Cardinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(10, 6)$	$(2, 5)$
	SOCCER	$(0, 0)$	$(9, 10)$

(a) In the original description of the **Matches** strategic game, Rose and Colin could not communicate with each other before making their strategy choices. In what way(s) must the situation change in order for the bargaining game to be an appropriate model?

(b) Graph the pure strategy payoff pairs. Also graph the payoff pair corresponding to the strategy pair

$$(0.5\text{TENNIS} + 0.5\text{SOCCER}, 0.5\text{TENNIS} + 0.5\text{SOCCER}),$$

and the correlated strategy

$$0.5(\text{TENNIS, TENNIS}) + 0.5(\text{SOCCER, SOCCER}).$$

(c) Graph the feasible payoff pairs and disagreement payoff pair $(2, 5)$.

(d) Discuss whether the security levels $(2, 5)$ are appropriate for the disagreement payoffs.

(e) If possible, determine the pure, mixed, or correlated strategies that will result in the following payoff pairs. If not possible, explain why not.
(i) $(2.5, 1.5)$
(ii) $(3.6, 5.2)$
(iii) $(9.5, 8.0)$
(iv) $(10.0, 10.0)$
(v) $(6.0, 6.0)$

(3) Steven Brams and Alan Taylor [**8**] describe a hypothetical **Divorce Settlement** between Tom and Mary. Issues to be settled are custody of their son John, alimony payments from Tom to Mary between 0 and 50% of Tom's current salary, and ownership of the house. The judge who will make the decision asked each person to assign a percentage to each of the three issues with the results shown in Table 2.3.

TABLE 2.3. Issue and worth to Tom and Mary

Issue	Tom Worth	Mary Worth
Custody	25	65
Alimony	60	25
House	15	10
Total	100	100

(a) Graph the feasible payoff pairs and $(50, 50)$ as the disagreement payoff pair.

(b) Discuss whether $(50, 50)$ is appropriate as the disagreement payoff pair.

(c) For each of the following payoff pairs (Tom, Mary), determine the resolution of the issues that result in the payoff pair.
   (i) $(85, 10)$
   (ii) $(87.5, 32.5)$
   (iii) $(12, 95)$

(4) Howard Raiffa [**48**] reports that a consulting firm interviewed members of the United States negotiating team after the second of the **Panama Canal** treaty negotiations with the Panamanian government. Ten issues had been identified during negotiations, and the consulting firm estimated the worth of each issue to each side as summarized in Table 2.4.

TABLE 2.4. Issues and worth to US and Panama

Issue	United States Worth	Panama Worth
Expansion rights	14	3
U.S. defense rights	22	9
Use rights	22	15
Expansion routes	6	5
Land and water	15	15
Duration	11	15
Compensation	4	11
Jurisdiction	2	7
U.S. military rights	2	7
Defense role of Panama	2	13
Total	100	100

(a) With three issues for the **Acme Industrial** bargaining game, we obtained eight payoff pairs to plot based on the possible agreements involving no compromise. With ten issues for the **Panama Canal** bargaining game, how many payoff pairs are there to plot based on the possible agreements involving no compromise?

(b) Figure 2.7 shows the region of feasible payoff pairs and the assumed disagreement payoff pair $(0, 0)$.
For each of the following resolutions of the issues, identify the corresponding payoff pair on the graph.
   (i) Panama wins on the last issue (defense role of Panama), and the United States wins on all other issues.
   (ii) Panama wins on the last three issues (jurisdiction, U.S. military rights, and defense role of Panama), and the United States wins on all other issues.
   (iii) Panama wins on the "duration" issue, and the United States wins on all other issues.
   (iv) Panama wins on the last four issues (compensation, jurisdiction, U.S. military rights, and defense role of Panama), the two countries split the "duration" issue 50-50, and the United States wins on all other issues.

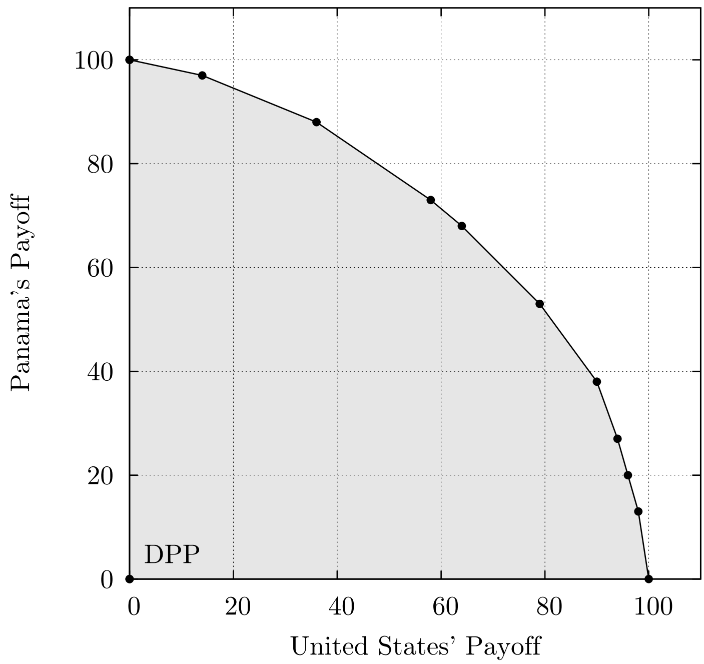

FIGURE 2.7. **Panama Canal** bargaining game

(c) For each of the following payoff pairs, identify the corresponding point on the graph and determine the resolution of the issues that result in that pair.
   (i) $(0, 100)$
   (ii) $(58, 73)$
   (iii) $(37, 70)$
   (iv) $(71.5, 60.5)$

(d) Discuss whether $(0, 0)$ is appropriate as the disagreement payoff pair.

(5) Consider the **Risk-Neutral Fingers** strategic game described in section 3.2:

Risk-Neutral Fingers Cardinal Payoffs		Colin ONE	TWO
Rose	ONE	$(0, 0)$	$(10, 2)$
	TWO	$(2, 10)$	$(4, 4)$

(a) Graph the feasible payoff pairs and disagreement payoff pair $(2.5, 2.5)$.

(b) Discuss whether the unique Nash equilibrium payoffs $(2.5, 2.5)$ are appropriate for the disagreement payoffs.

(c) If possible, determine the pure, mixed, or correlated strategies that will result in the following payoff pairs. If not possible, explain why not.
   (i) $(7, 7)$
   (ii) $(6, 6)$
   (iii) $(5, 5)$
   (iv) $(4, 4)$
   (v) $(1, 1)$

(6) Consider the **River Tale** strategic game described in a section 3.2 exercise:

River Tale Cardinal Payoffs		Stranger Heads	Tails
Steve	Heads	$(-20, 20)$	$(30, -30)$
	Tails	$(10, -10)$	$(-20, 20)$

(a) Graph the feasible payoff pairs and disagreement payoff pair $(-1.25, 1.25)$.
(b) Verify that $(-1.25, 1.25)$ is the unique Nash equilibrium payoff pair, and comment on the appropriateness of using this as the disagreement payoff pair.
(c) What is special about the possible payoff pairs for this bargaining game? Why does this happen?

(7) Explain why it is not meaningful to graph the expected payoff possibilities for a game for which only ordinal payoffs are known.

(8) Complete the following steps to prove that $(75, 10)$ cannot be obtained by players choosing mixed strategies in **Chances**.
(a) Explain why if $(75, 10)$ could be obtained by players choosing mixed strategies in **Chances**, then there must be a p and q satisfying

$$(1-p)(1-q)(50, 20) + (1-p)q(70, 80) + p(1-q)(0, 100) + pq(100, 0) = (75, 10),$$
$$0 \leq p \leq 1,$$
$$0 \leq q \leq 1.$$

(b) Show that the payoff pair equation in part (a) can be simplified to the system of equations

$$\begin{aligned} 20q - 50p + 80pq &= -40, \\ 80p + 60q - 160pq &= 55. \end{aligned}$$

(c) By adding 2 times the first equation in part (b) to the second equation in part (b) and then simplifying, we can obtain

$$p = 5q + 1.25.$$

(d) Explain why the equation in part (c) and the inequalities in part (a) cannot have a solution.
(e) Explain why the proof is now complete.

3. The Egalitarian Method

A bargaining game specifies possible agreements (the feasible payoff pairs) and the consequence for no agreement (the disagreement payoff pair). This is sufficient information for the players to negotiate without telling them how to negotiate. This is also sufficient information for an arbitrator to propose or impose an agreement between the players. Whether the game will be negotiated or arbitrated, fairness will be central. Typically an arbitrator is charged with making a fair proposal. If the players are negotiating, a player is unlikely to accept a proposal that seems unfair. This section and the following two sections examine three different methods for obtaining a proposal and discuss in what ways each method is fair or unfair.

> By the end of this section, you will be able to use the egalitarian method and to describe its fairness properties.

Three Fundamental Fairness Properties

Before we describe the egalitarian method, we describe three fundamental fairness properties: rational, efficient, and unbiased. We will use the **Chances** bargaining game as an illustration. Figure 3.1 gives its graphical description.

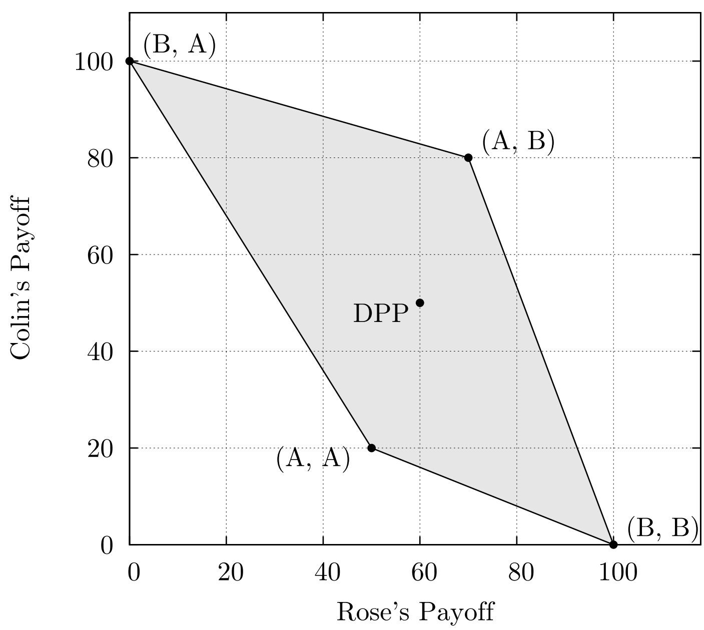

FIGURE 3.1. **Chances** bargaining game

The disagreement payoff pair in **Chances** is $(60, 50)$; that is, without an agreement, Rose will receive 60 and Colin will receive 50. Since cardinal payoffs model choice, Rose should be unwilling to agree to any proposal that gives her less than 60, and Colin should be unwilling to agree to any proposal that gives him less than 50. The rational fairness property formalizes this idea.

> **Rational Property:** A feasible payoff pair is *rational* if each player obtains at least as much as the player would obtain from the disagreement payoff pair. A method is *rational* if it determines a rational payoff pair for each bargaining game.

In **Chances**, $(65, 60)$ and $(60, 80)$ are rational while $(50, 20)$ and $(0, 100)$ are not rational. Graphically, the rational payoff pairs are the points upward and rightward (to the "northeast") of the disagreement payoff pair. For **Chances**, the rational payoff pairs are the points in the filled black quadrilateral in Figure 3.2.

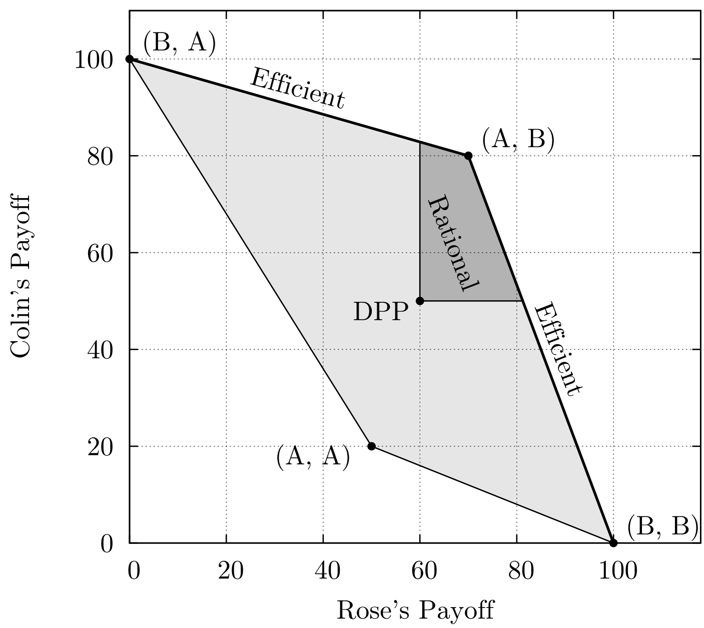

FIGURE 3.2. **Chances** rational and efficient payoff pairs

In **Chances**, it seems unreasonable for Rose and Colin to agree to $(65, 60)$ when each could do better by agreeing to $(70, 80)$. The efficient fairness property formalizes this idea.

> **Efficient Property:** A feasible payoff pair is *efficient* if it is impossible to increase the payoff of one player without decreasing the payoff to the other player. A method is *efficient* if it produces an efficient payoff pair for each bargaining game.

For example, $(65, 60)$ and $(60, 80)$ are inefficient in **Chances** because of the $(70, 80)$ feasible payoff pair. Graphically, in comparison to the points $(65, 60)$ and $(70, 75)$, the point $(70, 80)$ lies rightward ($65 < 70$ and $70 \leq 70$) and upward ($20 < 80$ and

$75 < 80$). The feasible payoff pair $(70, 80)$ is efficient in **Chances** because (1) increasing Rose's payoff would correspond to moving to the right on the graph and all feasible payoff pairs to the right of $(70, 80)$ are below $(70, 80)$, which corresponds to a decrease in Colin's payoff, and (2) increasing Colin's payoff would correspond to moving upward on the graph and all feasible payoff pairs above $(70, 80)$ are to the left of $(70, 80)$, which corresponds to a decrease in Rose's payoff. In general, the efficient payoff pairs have no feasible payoff pairs rightward ("east") and upward ("north"). That is, the efficient payoff pairs lie on the northeast boundary of the feasible payoff pairs region. For **Chances**, the efficient payoff pairs are the two thick black line segments in Figure 3.2.

If the players have agreed to cooperate, then it seems reasonable to believe that they will choose a rational and efficient payoff pair. Therefore, we focus our attention on these payoff pairs by zooming in on the rational payoff pairs, which we have done in Figure 3.3.

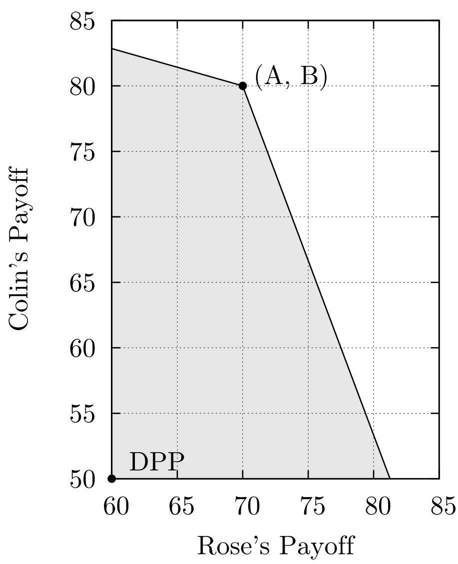

FIGURE 3.3. **Chances** rational payoff pairs

Shortly, it will be useful to have an algebraic description for the payoff pairs that are both rational and efficient. The point-slope equation for the line going through the payoff pairs $(0, 100)$ and $(70, 80)$, corresponding to the strategy pairs (B, A) and (A, B), respectively, is

$$y - 80 = \frac{100 - 80}{0 - 70}(x - 70),$$

and

$$60 \leq x \leq 70$$

provides the restriction to the payoff pairs that are rational ($x \geq 60$) and are to the left of $(70, 80)$. Simplifying, we obtain that the **Chances** "upper" efficient and rational line segment are the payoff pairs (x, y) satisfying

$$\text{(CHANCES UPPER)} \qquad y = -\frac{2}{7}x + 100, \quad 60 \leq x \leq 70.$$

Similarly, the **Chances** "lower" efficient and rational line segment is determined by the payoff pairs $(70, 80)$ and $(100, 0)$, corresponding to the strategy pairs (A, B) and (B, B), respectively, and so consists of the payoff pairs (x, y) satisfying

$$\text{(CHANCES LOWER)} \qquad y = -\frac{8}{3}x + \frac{800}{3}, \quad 70 \leq x \leq 81.25.$$

The upper bound on x was obtained by finding the value of x in (CHANCES LOWER) when $y = 50$, Colin's disagreement payoff.

The rational and efficient fairness properties restrict what payoff pairs are thought to be fair in any bargaining game. The third fairness property applies only to very special bargaining games, those in which each player has the same opportunity for gain from his or her respective disagreement payoff. Geometrically, this occurs when the region of rational payoff pairs is symmetric across the line with slope one through the DPP. In such a game, a fair proposal would give each player the same gain on their disagreement payoff. The unbiased fairness property formalizes this idea.

Unbiased Property: A bargaining game is *symmetric* if the region of rational payoff pairs is symmetric about the 45° line through the disagreement payoff pair. A method is *unbiased* if, for symmetric games, the method gives each player the same additional payoff above their disagreement payoffs.

As an example of a symmetric game, consider the **Kite** bargaining game, shown in Figure 3.4, in which the disagreement payoff vector is $(10, 40)$ and the rational payoff pairs consist of the filled quadrilateral with vertices $(10, 40)$, $(45, 40)$, $(40, 70)$, and $(10, 75)$.

Kite is a symmetric game because dividing the region of rational payoff pairs along the 45° line through DPP results in two regions which are mirror images of each other. Therefore, an unbiased method will produce a payoff pair of the form $(10+t, 40+t)$, which is somewhere on the 45° line through DPP. If, in addition, the solution is efficient, the method must produce the payoff pair at the intersection of the 45° line through DPP and the efficient line segments: $(40, 70)$.

In **Chances**, the region of rational payoff pairs is not symmetric about the 45° line through DPP, and so whether a method is unbiased or not tells us nothing directly about the payoff pair produced by that method.

The Egalitarian Method

We have already noted that a rational and efficient method will produce a payoff pair along the northeast boundary of the rational payoff pairs. There are no further restrictions for an unbiased method applied to **Chances** because the rational payoff pairs are not symmetric about the 45° line through the disagreement payoff pair.

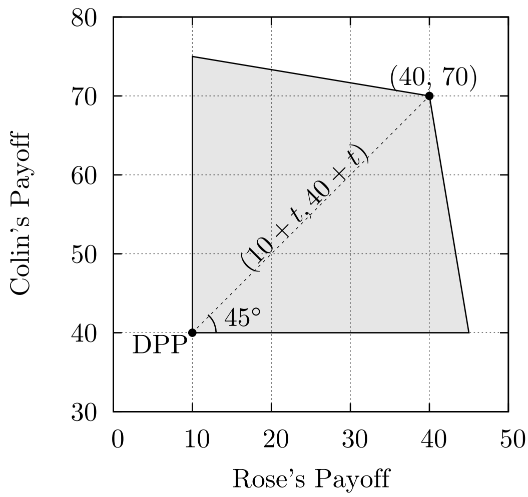

FIGURE 3.4. **Kite** bargaining game

However, in the dialogue in section 6.1, Rose initially ignored the fact that **Chances** is not symmetric and suggested that each player should receive the same additional amount above her or his disagreement payoff. Rose's idea is formalized as the egalitarian method.

Egalitarian Method: First give the players their disagreement payoff pair (r, c). Then give the same additional amount, t, to each player, with the value t chosen as large as possible so that the resulting *egalitarian payoff pair* $(r + t, c + t)$ remains feasible.

To find the egalitarian payoff pair for the **Chances** bargaining game, we start at the disagreement payoff pair $(60, 50)$ and add the same additional amount t to each player's payoff to obtain the payoff pairs $(60 + t, 50 + t)$ that lie on the dashed line in Figure 3.5.

The amount added is made as large as possible while keeping $(60+t, 50+t)$ feasible. Looking at the graph, we can see that the egalitarian payoff pair is roughly $(75, 65)$.

To obtain the egalitarian payoff pair precisely, we need to do some algebra. From Figure 3.5 it is clear that the egalitarian payoff pair lies on the lower efficient and rational line segment, and so we can find the solution by substituting the payoff pair $(60 + t, 50 + t)$ into the (CHANCES LOWER) equation to obtain

$$50 + t = -\frac{8}{3}(60 + t) + \frac{800}{3}.$$

Solving for t, we obtain

$$t = \frac{170}{11}.$$

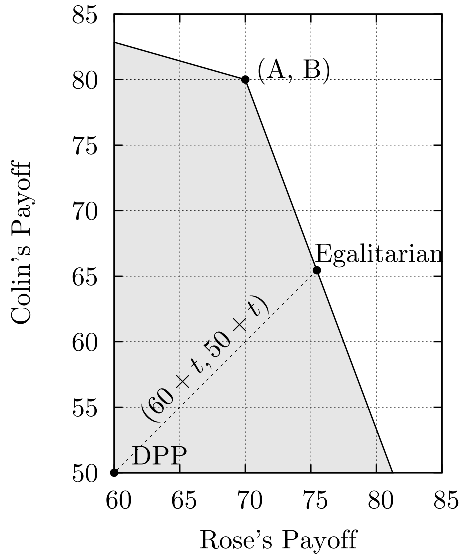

FIGURE 3.5. **Chances** egalitarian payoff pair

Finally substituting the value of t back into the payoff pair, we obtain the egalitarian payoff pair

$$(60+t, 50+t) = \left(60 + \frac{170}{11}, 50 + \frac{170}{11}\right) \approx (75.5, 65.5).$$

This completes our analysis of the **Chances** bargaining game; Rose and Colin agree that Rose gets 75.5 and that Colin gets 65.5. But the original scenario had Rose and Colin negotiating how to play a strategic game in which each needed to select a strategy. In that scenario, the egalitarian payoff pair can be obtained by the correlated strategy $(1-p)$(A, B) + p(B, B) in which p satisfies $(1-p)(70, 80) + p(100, 0) \approx (75.5, 65.5)$. In particular, the second coordinate equation is $80(1-p) \approx 65.5$, which has the solution $p \approx 0.18$. Notice that $p \approx 0.18$ also solves the first coordinate equation. Hence, the correlated strategy 0.82(A, B) + 0.18(B, B) will result in the egalitarian payoff pair. Since Colin always chooses B, we can also describe this correlated strategy as Rose agreeing to use the mixed strategy 0.82A + 0.18B and Colin agreeing to use the pure strategy B.

We now turn our attention to **Acme Industrial**. Figure 3.6 zooms in on the rational payoff pairs.

The efficient and rational payoff pairs (x, y) consist of three line segments that can be described algebraically by

(ACME UPPER) $$y = -0.2x + 100, \quad 0 \le x \le 50,$$

(ACME MIDDLE) $$y = -x + 140, \quad 50 \le x \le 80,$$

and

(ACME LOWER) $$y = -3x + 300, \quad 80 \le x \le 83.33.$$

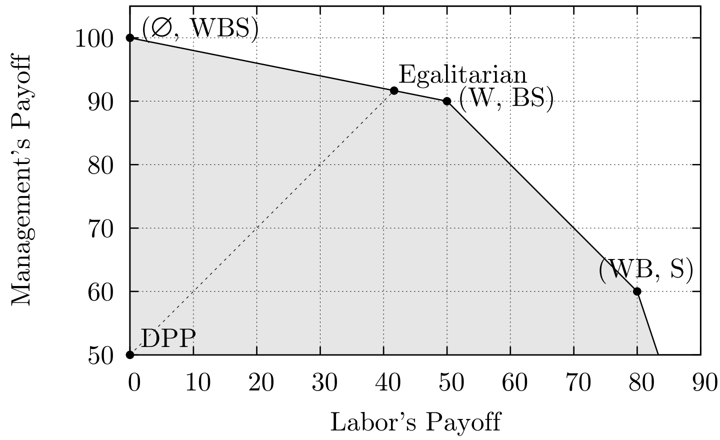

FIGURE 3.6. **Acme Industrial** egalitarian payoff pair

The egalitarian payoff pair $(x, y) = (0 + t, 50 + t)$ clearly lies on the upper efficient and rational line segment and so satisfies

$$50 + t = -0.2(0 + t) + 100,$$

which has the solution $t = 125/3 \approx 41.7$. Hence, the egalitarian payoff pair is $(125/3, 275/3) \approx (41.7, 91.7)$.

The egalitarian payoff pair can be obtained with the lottery $(1 - p)(\emptyset,\ \mathrm{WBS}) + p(\mathrm{W},\ \mathrm{BS})$ in which p satisfies

$$(1 - p)(0, 100) + p(50, 90) \approx (41.7, 91.7).$$

In particular, the first coordinate equation is $50p \approx 41.7$, which has the solution $p \approx 0.83$. Hence, the players could use the lottery

$$0.17(\emptyset,\ \mathrm{WBS}) + 0.83(\mathrm{W},\ \mathrm{BS}).$$

Thus, Labor should win 83% of the wage issue and Management should win 17% of the wage issue and all of the benefits and security issues. This agreement might be interpreted as a 2.5% annual increase in wages, employees paying for one-half of their medical insurance, and Management having the right to eliminate up to 600 union jobs.

Strongly Monotone Property

The egalitarian method is rational because for each bargaining game, we start at the disagreement payoff pair and then increase both players' payoffs. The egalitarian method is unbiased because it always gives each player the same additional payoff above their disagreement payoffs whether or not the game is symmetric. Surprisingly, the egalitarian method does not always produce an efficient payoff pair (see exercise 2).

The egalitarian method also satisfies another fairness property, which we motivate with an analogy. According to the first section dialogue, if offered a hot dog, hamburger, or chicken tenders, Rose would choose the chicken tenders. What would happen if Rose were offered a hot dog, hamburger, chicken tenders, or barbecue? Presumably, Rose will choose the chicken tenders or barbecue, whichever she likes more. Certainly, it would not make sense for Rose to choose something she likes less when there are more options available. In other words, introducing more possibilities should not make Rose worse off. This idea is formalized by the strongly monotone property.

Strongly Monotone Property: A method is *strongly monotone* if, whenever a new bargaining game is obtained by adding more feasible payoff pairs (without eliminating any feasible payoff pair or changing the disagreement payoff pair), the method does not give less to either player in the new game than what they had received in the original game.

Ehud Kalai [**26**] proved the following theorem.

Egalitarian Characterization Theorem: *If a method is efficient, unbiased, and strongly monotone, then that method must be the egalitarian method.*

Proof. We will illustrate Kalai's proof with the **Chances** bargaining game with the disagreement payoff pair $(60, 50)$. Suppose some unknown method, called the X method, is efficient, unbiased and strongly monotone. We need to show that the X method produces the same payoff pair for **Chances** as the egalitarian method produces: $(75.5, 65.5)$.

Since **Chances** itself is not symmetric, we need to define a symmetric game that is somehow related to **Chances**. To use the strongly monotone property, we need the rational payoff pairs of this symmetric game to be contained in the rational payoff pairs for **Chances**. A first attempt at defining this symmetric game, having the same DPP as **Chances**, is **KiteA** whose rational payoff pairs are shown in dark grey in Figure 3.7.

Since **KiteA** is a symmetric game, an unbiased and efficient method will produce the upper vertex $(70, 60)$. **Chances** has more rational payoff pairs than the kite shaped game, and so a strongly monotone and efficient method must produce a payoff pair among those that are marked in bold. We can narrow the possibilities further by making the kite shaped game bigger. In order to retain a symmetric game, the larger kite shaped game must also have its upper vertex on the dotted $45°$ line. Thus we extend the kite along the $45°$ line until it touches the boundary for **Chances** at the payoff pair $(75.5, 65.5)$; we name this new kite **KiteC** as illustrated in Figure 3.8.

KiteC is symmetric, and so the X method, which is efficient and unbiased, will produce the upper vertex $(75.5, 65.5)$. The **Chances** bargaining game has the same disagreement payoff pair as **KiteC** and has more rational payoff pairs than **KiteC**, and so the X method, which is strongly monotone, must produce a payoff pair for

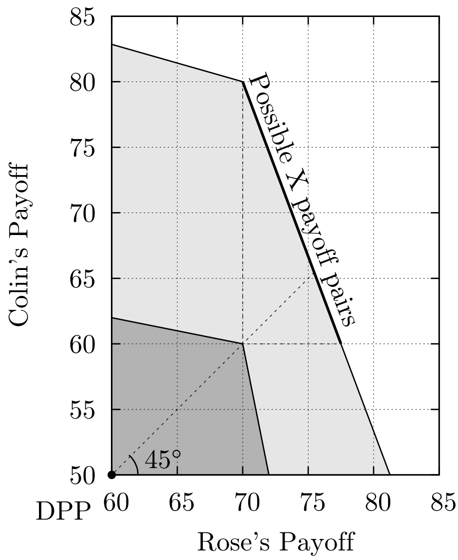

FIGURE 3.7. Payoff pairs for **KiteA** and **Chances** using the X method

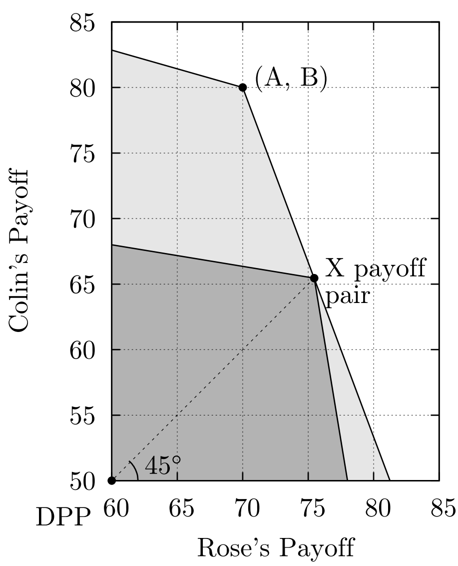

FIGURE 3.8. Payoff pair for both **KiteC** and **Chances** using the X method

Chances at or to the northeast of $(75.5, 65.5)$. Since there are no feasible payoff pairs to the northeast of $(75.5, 65.5)$, the X method must produce $(75.5, 65.5)$ for **Chances**. $\square$

This theorem tells us that we should use the egalitarian method if we believe that a fair method should (1) not give payoffs for which at least one player can be improved without harming the other player, (2) treat players who have equivalent options equally, and (3) not punish either player if more options become available. Since these three properties seem to be reasonable descriptions of some aspects of fairness, it may seem like our work is done; we have found a fair method. Unfortunately, the egalitarian method does not satisfy properties that some people believe are more descriptive of fairness than the strongly monotone property.

We motivated the definition of the strong monotone property with an analogy about Rose selecting food at a concession stand. It seemed rather compelling that if Rose was offered more choices she should not be less happy. One difference between this analogy and bargaining games is that this analogy involved only one person while bargaining games involve two players. Let's consider a different analogy. Suppose Rose and Colin have enough money to share a one scoop ice cream cone. Suppose further that coffee, vanilla, and chocolate are the available flavors, Rose ranks coffee over vanilla over chocolate, and Colin ranks chocolate over vanilla over coffee. The fair compromise seems to be vanilla. Alternatively, suppose that seven flavors are now available, Rose ranks strawberry over peach over raspberry over maple over coffee over vanilla over chocolate, and Colin ranks those flavors in reverse order. A strongly monotone method would require that neither Rose nor Colin be harmed by the four added flavor possibilities, meaning that vanilla would again be chosen. To many people, this does not seem fair. Since Rose and Colin's flavor rankings are in complete opposition, to them, it would seem more fair that the compromise should be the middle ranked flavor for both: maple. While the addition of flavor possibilities harms Colin, these people would contend that since all of the new flavors are better for Rose and worse for Coln, there is no reason to expect Colin to be unharmed by these new options.

This new analogy suggests that the strongly monotone property is not as good of a formalization of our intuition about fairness as we first had thought. As we continue, we seek other properties that might better formalize our intuitions about fairness.

Scale Invariant Property

The next fairness property we will describe arises from our usual assumption that players have cardinal preferences described by interval scales. Just as reporting outdoor temperatures in degrees Fahrenheit instead of degrees Celsius does not change how warm it is outside, rescaling a player's cardinal payoffs should not have a real effect on the proposed method.

> **Scale Invariant Property:** If a new bargaining game is obtained by a positive linear rescaling of a player's payoffs, then the payoffs produced by a *scale invariant* method for the new game should be the same positive linear rescaling of the payoffs produced for the original bargaining game.

For example, divide Colin's payoffs in **Chances** by two to obtain the **Rescaled Chances** bargaining game. The disagreement payoff pair in **Chances**, $(60, 50)$, will become the disagreement payoff pair in **Rescaled Chances**, $(60, 25)$. The other three vertices of the rational payoff pairs in **Chances**, $(60, 82.86)$, $(70, 80)$, and $(81.25, 50)$, will become the rational payoff pairs in **Rescaled Chances**, $(60, 41.43)$, $(70, 40)$, and $(81.25, 25)$, respectively, and the egalitarian payoff pair in **Chances**, $(75.46, 65.46)$, will be rescaled to $(75.46, 32.73)$. In Figure 3.9, the two sets of rational payoff pairs are graphed side by side.

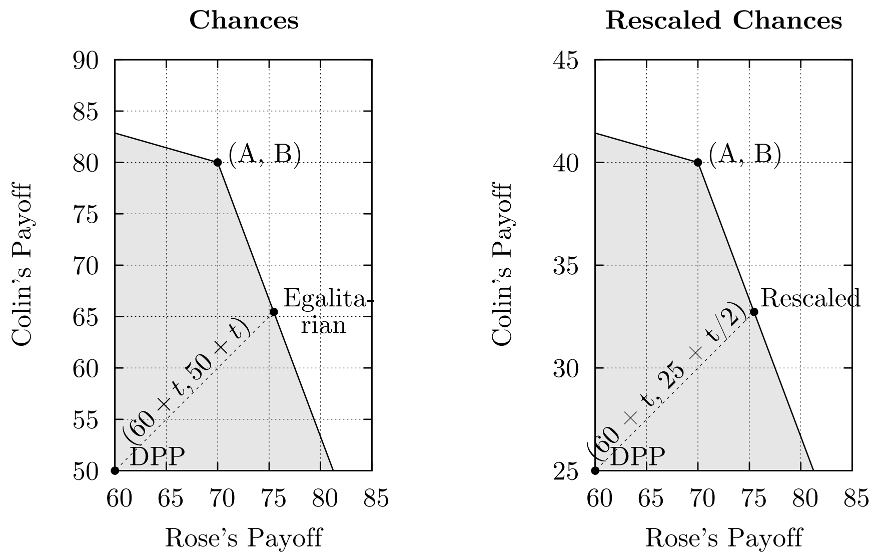

FIGURE 3.9. Rational payoff pairs for **Chances** and **Rescaled Chances**

Observe that the two graphs look identical except for (1) the "Colin's Payoff" axis scales, and (2) the dotted line labeling: for both, Colin's payoffs in **Rescaled Chances** are one-half of the corresponding payoffs in **Chances**. Because the dotted line in the right graph above is not the 45° line, the rescaled egalitarian payoff pair $(75.5, 32.7)$ from **Chances** is not the egalitarian payoff pair in **Rescaled Chances,** which is $(72.1, 37.1)$. The two payoff pairs are compared in Figure 3.10.

Further, the egalitarian payoff pair in **Chances** is implemented by the strategy pair (0.82A + 0.18B, B), while the egalitarian payoff pair in **Rescaled Chances** is implemented by the strategy pair (0.93A + 0.07B, B). The choice of scales makes real changes in the outcomes when the egalitarian method is used!

The example above shows that the egalitarian method is not scale invariant. Since the egalitarian method is the only efficient, unbiased, and strongly monotone method, it follows that there is no method that is efficient, unbiased, strongly monotone, and scale invariant. Thus, we cannot expect any method to have all of the fairness properties that we have defined or might define. Ultimately, we will need to decide which fairness properties are of most relevance to the real world situation that is being modeled.

Since the egalitarian method is not scale invariant, it can be appropriate only when it makes sense to compare the two players' cardinal preferences via some external

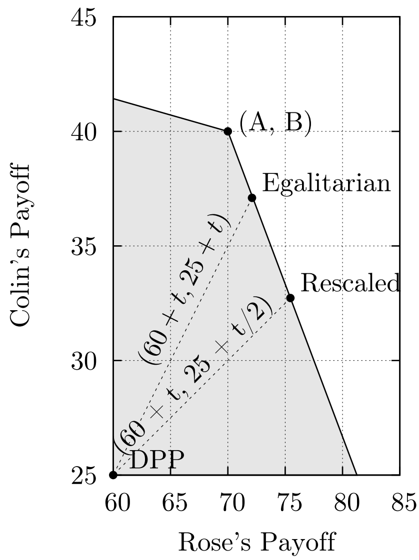

FIGURE 3.10. Egalitarian payoff pair for **Rescaled Chances**

standard. For example, if the outcomes involve money transfers and the amount of money received by each player is deemed to be more important with respect to fairness than the happiness experienced by each player from this money, then "amount of money" acts as an external standard through which player preferences are compared. If you believe that "chance of winning a prize" should act as an external standard with which to compare the two players' cardinal preferences in **Chances**, then use of the egalitarian method seems reasonable. But if unknown differences in economic well-being or in the worth of the prizes suggest that it is meaningless to make comparisons between the two players' preferences, then the egalitarian method should not be used.

If it is not meaningful to compare the cardinal preferences of the players, which is our usual assumption when we model a scenario as a strategic game, then the egalitarian method is fundamentally unfair because it is not scale invariant. The next two sections describe methods that are scale invariant.

Exercises

(1) Consider the following **Asymmetric Prisoner's Dilemma** strategic game considered in section 6.2, exercise 1:

Asymmetric Prisoner's Dilemma		Colin	
Cardinal Payoffs		COOPERATE	DEFECT
Rose	COOPERATE	$(90, 60)$	$(0, 100)$
	DEFECT	$(100, 0)$	$(10, 40)$

(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the egalitarian payoff pair is located.

(b) Describe algebraically the payoff pairs (x, y) that are both efficient and rational.
(c) Find the egalitarian payoff pair.
(d) Describe how to implement the egalitarian payoff pair using a correlated strategy in the strategic game.
(e) Following the argument in the Egalitarian Characterization Theorem, explain why an efficient, unbiased, and strongly monotone method must produce the egalitarian payoff pair in this game.

(2) Consider the **Matches** strategic game described in section 3.2 and the corresponding bargaining game considered in section 6.2, exercise 2:

Matches Cardinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(10, 6)$	$(2, 5)$
	SOCCER	$(0, 0)$	$(9, 10)$

(a) Graph the rational payoff pairs and identify the efficient payoff pairs.
(b) Describe algebraically the payoff pairs (x, y) that are both efficient and rational.
(c) Find the egalitarian payoff pair.
(d) Explain why the egalitarian payoff pair is not efficient.

(3) Consider the **Divorce Settlement** bargaining game described in section 6.2, exercise 3.
(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the egalitarian payoff pair is located.
(b) Describe algebraically the payoff pairs (x, y) that are both efficient and rational.
(c) Find the egalitarian payoff pair.
(d) Describe how to implement the egalitarian payoff pair.
(e) Following the argument in the Egalitarian Characterization Theorem, explain why an efficient, unbiased, and strongly monotone method must produce the egalitarian payoff pair in this game.

(4) Consider the **Panama Canal** bargaining game described in section 6.2, exercise 4.
(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the egalitarian payoff pair is located.
(b) Find the egalitarian payoff pair.
(c) Describe how to implement the egalitarian payoff pair.

(5) Consider the **Risk-Neutral Fingers** bargaining game described in section 6.2, exercise 5.
(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the egalitarian payoff pair is located.
(b) Find the egalitarian payoff pair.
(c) Describe how to implement the egalitarian payoff pair.
(d) What two properties are sufficient to determine a unique payoff pair in this bargaining game?

(6) Consider the **River Tale** bargaining game described in section 6.2, exercise 6.
(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the egalitarian payoff pair is located.

(b) Find the egalitarian payoff pair.
(c) Describe how to implement the egalitarian payoff pair.
(d) What single property is sufficient to determine a unique payoff pair in this bargaining game?

(7) Consider the **Chances** bargaining game with disagreement payoff pair $(60, 50)$ discussed in the text.
(a) Find the egalitarian payoff pair if the disagreement payoff pair is changed to the security levels $(50, 50)$.
(b) Find the egalitarian payoff pair if the disagreement payoff pair is changed to the unique Nash equilibrium payoff pair $(62.5, 50)$.
(c) Compare the egalitarian payoff pairs for the three different disagreement payoff pairs.

4. The Raiffa Method

The egalitarian method is a seemingly fair approach to a bargaining game: start with the disagreement payoff pair and give each player the same additional utility. The problem with this approach is that by adding the same utility to both players, we are assuming that personal utilities are comparable. In this section, we will discuss a similar method that does not make interpersonal utility comparisons.

> By the end of this section, you will be able to use the Raiffa method and to describe its fairness properties.

Since using cardinal preferences allows two of the outcome payoffs to be set arbitrarily, Howard Raiffa [**47**] suggested that a payoff pair for a bargaining game should be on a line between two outcomes: the disagreement payoff pair and an "aspiration" payoff pair.

> **Raiffa Method:** Find the *aspiration payoff pair* formed by the maximum individual payoffs among the rational payoff pairs. The *Raiffa payoff pair* is the efficient payoff pair on the line segment formed between the disagreement and aspiration payoff pairs.

Recall the graphical description of **Chances** rational payoff pairs, as shown in Figure 4.1.

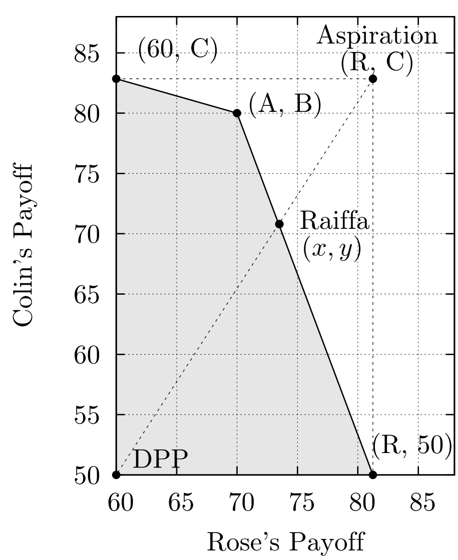

FIGURE 4.1. The Raiffa payoff pair for **Chances**

Rose's maximum possible payoff R occurs at the payoff pair (R, 50) on the lower efficient line segment

$$\text{(CHANCES LOWER)} \qquad y = -\frac{8}{3}x + \frac{800}{3},$$

which implies R $= \frac{325}{4} = 81.25$. Colin's maximum possible payoff C occurs at the payoff pair (60, C) on the upper efficient line segment

$$\text{(CHANCES UPPER)} \qquad y = -\frac{2}{7}x + 100,$$

which implies C $= \frac{580}{7} \approx 82.857$. The line determined by the disagreement payoff pair $(60, 50)$ and the aspiration payoff pair (R, C) $= (81.25, 82.857)$ has the point slope form

$$y - 50 = \frac{82.857 - 50}{81.25 - 60}(x - 60),$$

which simplifies to

$$y = 1.546x - 42.773.$$

Solving this equation simultaneously with (CHANCES LOWER) yields $x = 73.5$ and $y = 70.8$ for the Raiffa payoff pair.

The Raiffa payoff pair can be obtained by a correlated strategy $(1 - p)$(A, B) + p(B, B) in which p satisfies $(1 - p)(70, 80) + p(100, 0) = (73.5, 70.8)$. In particular, the second coordinate equation is $80(1-p) \approx 70.8$, which has the solution $p \approx 0.11$. Since this solution also satisfies the first coordinate equation, the correlated strategy 0.89(A, B) + 0.11(B, B) implements the Raiffa payoff pair. Since Colin always chooses B, we can also describe this correlated strategy as Rose choosing the mixed strategy 0.89A + 0.11B and Colin choosing the pure strategy B.

Acme Industrial

Figure 4.2 shows the graphical description of **Acme Industrial** rational payoff pairs with the aspiration payoff pair $(83.33, 100)$ included.

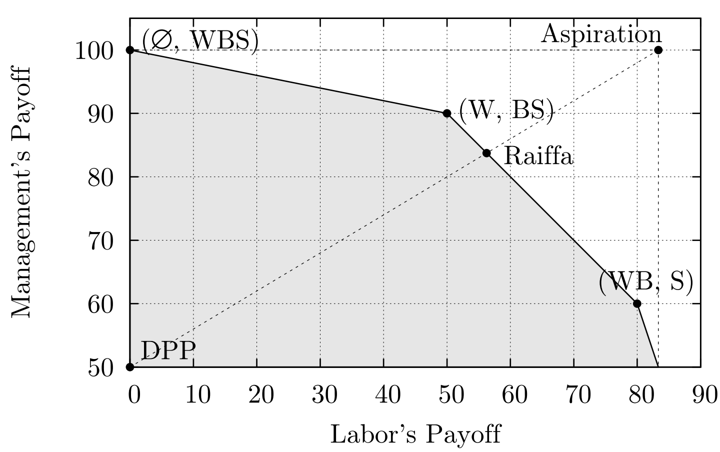

FIGURE 4.2. The Raiffa payoff pair for **Acme Industrial**

The Raiffa payoff pair (x, y) is the intersection of the middle efficient line segment

(Acme middle) $$y = -x + 140$$

and disagreement to aspiration line segment

$$y - 50 = \frac{100 - 50}{83.33 - 0}(x - 0),$$

which has the solution $(56.25, 83.75)$.

The Raiffa payoff pair can be obtained with the lottery $(1 - p)$(W, BS) + p(WB, S) in which p satisfies $(1 - p)(50, 90) + p(80, 60) \approx (56.25, 83.75)$. The solution to both coordinate equations is $p \approx 0.21$, and hence, the lottery 0.79(W, BS)+ 0.21(WB, S) implements the Raiffa payoff pair. This can be interpreted as Labor winning the wage issue and 21% of the benefits issue, and Management winning 79% of the benefits issue and all of the security issue. Specifically, this suggests that there should be a 3% annual increase in wages, employees will pay for about 40% of their medical insurance, and Management has the right to eliminate up to 600 union jobs.

Raiffa Method Properties

The Raiffa method is scale invariant, because any rescaling moves both the disagreement and aspiration payoff pairs. Here is an illustration. Recall that we divided Colin's payoffs in **Chances** by two to obtain the **Rescaled Chances** bargaining game. Notice that, in Figure 4.3, except for the scale along the vertical axis, the two graphs are identical.

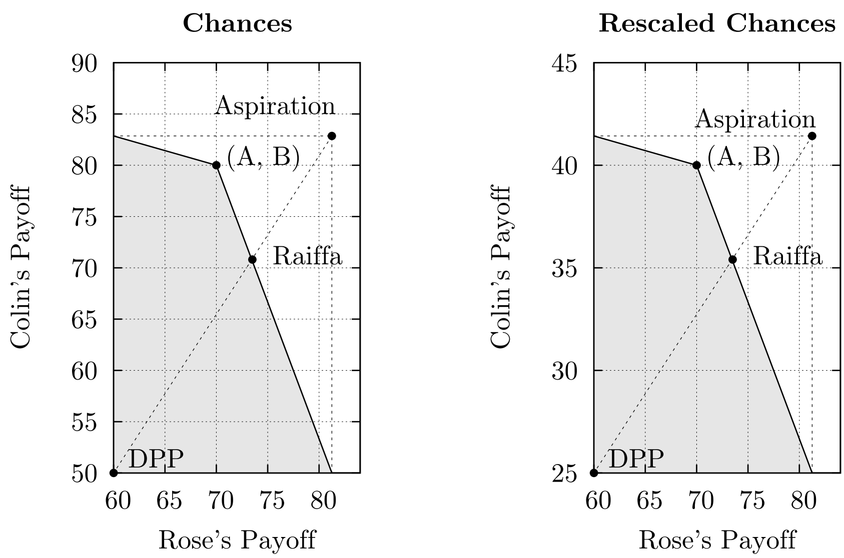

Figure 4.3. Rational payoff pairs for **Chances** and **Rescaled Chances**

By definition, the Raiffa method is efficient. Further, the Raiffa method is unbiased: if the region of rational payoff pairs is symmetric about the 45° line through the disagreement payoff pair, then the aspiration payoff pair must be on this line, and so the Raiffa method gives each player the same additional payoff above their disagreement payoffs.

By the Egalitarian Characterization Theorem, the egalitarian method is the only efficient, unbiased, and strongly monotone method. Since, the Raiffa method is efficient and unbiased, it cannot be strongly monotone. This can be illustrated with the **KiteC** bargaining game, whose feasible payoff pairs are marked in dark grey in Figure 4.4, and the **Chances** bargaining game, whose feasible payoff pairs are marked in light and dark grey in the same figure. The disagreement payoff pairs are the same for both bargaining games.

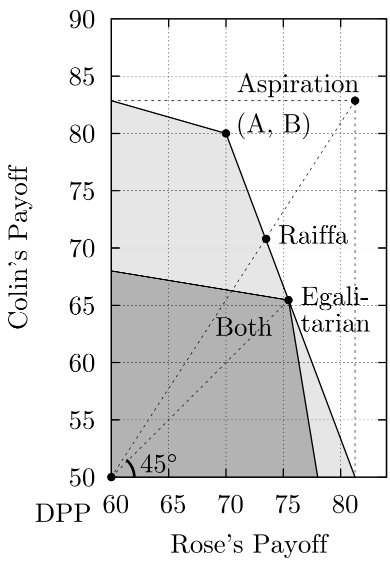

FIGURE 4.4. Raiffa and egalitarian payoff pairs for **Chances**

As we argued in the previous section, any efficient and unbiased method applied to **KiteC** must produce $(75.5, 65.5)$, marked "Both" in Figure 4.4. If the method is also strongly monotone, as is the egalitarian method, then $(75.5, 65.5)$ must also be produced for **Chances**. However, we can see that the Raiffa payoff pair is $(73.4, 70.8)$, which is clearly detrimental to Rose. Even though there are more feasible payoff pairs in **Chances** than there are in **KiteC**, the Raiffa method gives Rose less in **Chances** than it gives Rose in **KiteC** in violation of the strongly monotone property.

On the other hand, if you ignore the scales in the diagram and only consider the shapes of the two rational payoff pair regions, we see that the additional payoff pairs available in **Chances**, in comparison with **KiteC**, are heavily in favor of Colin. This makes it seem more reasonable that Colin might benefit at the expense of Rose in any negotiation. The Raiffa method incorporates this reasoning. An extreme example is the **Colin Weighted** bargaining game in which all of the payoff pairs added to **KiteC** favor Colin, as seen in Figure 4.5.

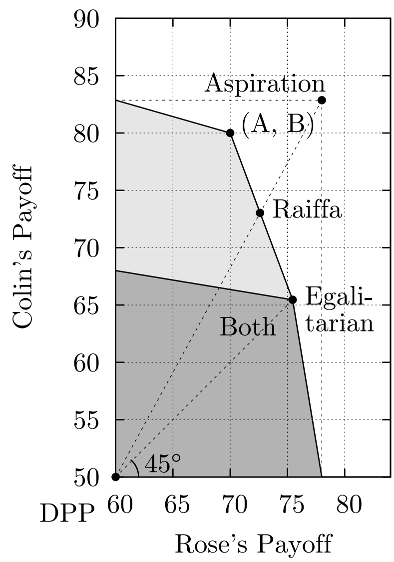

FIGURE 4.5. Raiffa and egalitarian payoff pairs for **Colin Weighted**

Despite the extra feasible payoff pairs favoring Colin, the egalitarian payoff pair is the same in **Colin Weighted** and **KiteC**. It is clear that Colin's payoff in **Colin Weighted** should not decrease, but it is less clear that Rose's payoff must not decrease. These considerations suggest the following weakening of the strongly monotone property.

Individually Monotone Property: If a new bargaining game is obtained by adding more feasible payoff pairs without changing one player's aspiration payoff, then an *individually monotone* method does not give less to the other player in the new game.

For example, in going from **KiteC** to **Colin Weighted**, Rose's aspiration payoff does not change and so an individually monotone method should give Colin at least as large a payoff in **Colin Weighted** as in **KiteC**. However, Colin's aspiration payoff increased, so the individually monotone property tells us nothing about how Rose's payoffs in **Chances** and **KiteC** payoff pairs should compare. Since the Raiffa method is determined by the aspiration payoffs, the Raiffa method is individually monotone. Ehud Kalai and Meir Smorodinsky [**27**] proved the following.

Raiffa Characterization Theorem: *The Raiffa method is the only efficient, unbiased, scale invariant, and individually monotone method.*

PROOF. We again illustrate the proof using **Chances**. Suppose some unknown method, called the X method, is efficient, unbiased, scale invariant, and individually monotone. We need to show that the X method produces the same payoff pair for **Chances** as the Raiffa method produces: $(73.4, 70.8)$.

The first step is to rescale player payoffs (x, y) in **Chances** to payoffs (x', y') in a new game we will call **CT** (short for **Chances Transformed**) so that the disagreement payoff pair is $(0, 0)$ and the aspiration payoff pair is $(100, 100)$. The appropriate rescaling is

$$\begin{aligned} x' &= \frac{100}{81.25 - 60}(x - 60), \\ y' &= \frac{100}{82.86 - 50}(y - 50). \end{aligned}$$

Note that the (A, B) payoff pair is transformed from $(70, 80)$ to $(47.1, 91.3)$, and the Raiffa payoff pair is transformed from $(73.5, 70.8)$ to $(63.3, 63.3)$. The second step is to consider the game that we will call **CK** (short for **Chances Kite**), whose disagreement payoff pair is $(0, 0)$, and whose feasible payoff pairs are in the kite having vertices $(0, 0)$, $(100, 0)$, $(63.3, 63.3)$, and $(0, 100)$. Figure 4.6 exhibits the **CK** feasible payoff pairs in dark grey and the **CT** feasible payoff pairs in both light and dark grey.

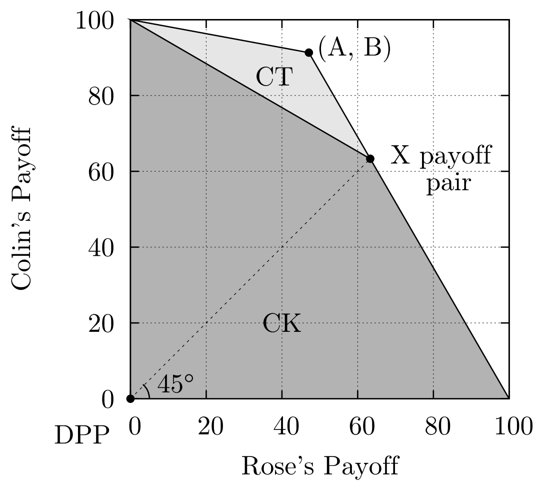

FIGURE 4.6. Rational and method X payoff pairs for **Chances Transformed** and **Chances Kite**

CK is a symmetric game, and therefore the X method, which is efficient and unbiased, produces the efficient payoff pair $(63.3, 63.3)$. Since **CT** adds feasible payoff pairs without changing either aspiration payoff, the X method, which is individually monotone, produces a payoff pair in which each player receives at least as much in **CT** as he or she received in **CK**. But there is only one feasible payoff pair satisfying that constraint: $(63.3, 63.3)$. Finally, since the X method is scale invariant, it produces a payoff pair for **Chances** that is the appropriate rescaling of $(63.3, 63.3)$, which is $(73.5, 70.8)$. $\square$

The egalitarian and Raiffa methods are both efficient, unbiased, and individually monotone (the egalitarian method is individually monotone because it is strongly monotone, which implies that it is individually monotone). In addition, the egalitarian method is strongly monotone and the Raiffa method is scale invariant. By the characterization theorems, the egalitarian method is not scale invariant and the Raiffa method is not strongly monotone. Therefore, the difference between the two methods is clear. If, in your view, strongly monotone is a more important fairness property (which makes most sense if the payoffs are measured against a mutually agreeable external standard), then use the egalitarian method. If on the other hand, in your view, scale invariance is a more important fairness property (which makes most sense if interpersonal comparison of payoffs is not possible), then use the Raiffa method.

Exercises

(1) Consider the following **Asymmetric Prisoner's Dilemma** strategic game considered in exercise 1 of sections 6.2–6.3:

Asymmetric Prisoners' Dilemma Cardinal Payoffs		Colin COOPERATE	DEFECT
Rose	COOPERATE	$(90, 60)$	$(0, 100)$
	DEFECT	$(100, 0)$	$(10, 40)$

(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Raiffa payoff pair is located.
(b) Find the Raiffa payoff pair.
(c) Describe how to implement the Raiffa payoff pair using a correlated strategy in the strategic game.
(d) Following the argument in the Raiffa Characterization Theorem, explain why an efficient, unbiased, scale invariant, and individually monotone method must produce the Raiffa payoff pair in this game.

(2) Consider the **Matches** strategic game described in section 3.2 and the corresponding bargaining game considered in exercise 2 of sections 6.2–6.3.

Matches Cardinal Payoffs		Colin TENNIS	SOCCER
Rose	TENNIS	$(10, 6)$	$(2, 5)$
	SOCCER	$(0, 0)$	$(9, 10)$

(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Raiffa payoff pair is located.
(b) Find the Raiffa payoff pair.
(c) Describe how to implement the Raiffa payoff pair using a correlated strategy in the strategic game.

(3) Consider the **Divorce Settlement** bargaining game described in section 6.2, exercise 3.
(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Raiffa payoff pair is located.
(b) Find the Raiffa payoff pair.
(c) Describe how to implement the Raiffa payoff pair using a correlated strategy in the strategic game.

(d) Following the argument in the Raiffa Characterization Theorem, explain why an efficient, unbiased, scale invariant, and individually monotone method must produce the Raiffa payoff pair in this game.

(4) Consider the **Panama Canal** bargaining game described in section 6.2, exercise 4.
  (a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Raiffa payoff pair is located.
  (b) Find the Raiffa payoff pair.
  (c) Describe how to implement the Raiffa payoff pair.

(5) Consider the **Risk-Neutral Fingers** bargaining game described in section 6.2, exercise 5.
  (a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Raiffa payoff pair is located.
  (b) Find the Raiffa payoff pair.
  (c) Describe how to implement the Raiffa payoff pair.
  (d) What two properties are sufficient to determine a unique payoff pair in this bargaining game?

(6) Consider the **River Tale** bargaining game described in section 6.2, exercise 6.
  (a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Raiffa payoff pair is located.
  (b) Find the Raiffa payoff pair.
  (c) Describe how Steve and the stranger could obtain the Raiffa payoff pair.
  (d) What single property is sufficient to determine a unique payoff pair in this bargaining game?

(7) Consider the **Chances** bargaining game with disagreement payoff pair $(60, 50)$ discussed in the text.
  (a) Find the Raiffa payoff pair if the disagreement payoff pair is changed to the security levels $(50, 50)$.
  (b) Find the Raiffa payoff pair if the disagreement payoff pair is changed to the unique Nash equilibrium payoff pair $(62.5, 50)$.
  (c) Compare the Raiffa payoff pairs for the three different disagreement payoff pairs.

(8) In section 6.3, an ice cream flavor choice analogy was used to suggest that strongly monotone was too restrictive a property. Specifically, suppose Rose and Colin have enough money to share a one scoop ice cream cone. In the first scenario, coffee, vanilla, and chocolate are the available flavors, Rose ranks coffee over vanilla over chocolate, and Colin ranks chocolate over vanilla over coffee. The fair compromise seems to be vanilla. In the second scenario seven flavors are available, Rose ranks strawberry over peach over raspberry over maple over coffee over vanilla over chocolate, and Colin ranks those flavors in reverse order. A strongly monotone method would require that neither Rose nor Colin be harmed by the four added flavor possibilities, meaning that vanilla would again be chosen. To many people, this does not seem to be a fair outcome for the second scenario.
  (a) What would an individually monotone method that produces vanilla in the first scenario need to produce in the second scenario?

(b) Consider a third scenario in which there are six flavors available: fudge, caramel, cherry, coffee, vanilla, and chocolate. If an individually monotone method produces vanilla in the first scenario and then must produce vanilla in the third scenario, what would have to be true about the players' preference orders over the six flavors?

(9) The last paragraph of the text in this section states that the Raiffa method is not strongly monotone and that any strongly monotone method must be individually monotone. This exercise explores why.

(a) Explain why the Egalitarian Characterization Theorem and the Raiffa Characterization Theorem imply that the Raiffa method is not strongly monotone.

(b) With the Raiffa payoff pairs for **Colin Weighted** and **KiteC**, explain why the Raiffa method is not strongly monotone.

(c) Explain why a method that is strongly monotone must also be individually monotone.

5. The Nash Method

The egalitarian method has been recommended as a fair approach when player payoffs are measured by comparable scales such as money or probability of winning a prize. The Raiffa method has been recommended as a fair approach when, as is typically assumed, player payoffs cannot be directly compared. Both solutions are efficient and unbiased. The egalitarian method is then characterized by adding strongly monotone to the list of desired fairness properties. Alternatively, the Raiffa method is characterized by adding scale invariant and individually monotone to the list of desired fairness properties. In this section, we introduce a third monotonicity property and compare it to the previous two.

> By the end of this section, you will be able to use the Nash method and to describe its fairness properties.

Returning to a previously described analogy, suppose that Rose is getting hungry at the basketball game. In the past, the concession stand has offered hot dogs, hamburgers, and chicken tenders. Suppose that chicken tenders is Rose's most preferred option. When Rose reaches the concession stand, she finds out that pulled pork sandwiches have been added to the menu. Rose may still order the chicken tenders, but she may order the pulled pork sandwich if she prefers it to the chicken tenders. The addition of more options has left Rose at least as well off as before (and perhaps better). This is the gist of the strongly and individually monotone properties for bargaining game solutions: the addition of feasible payoff pairs should leave both players or one player, respectively, at least as well off as before.

When John Nash [**36**] introduced bargaining games in 1950, he had a different view of monotonicity. Suppose when Rose reaches the concession stand, she finds out that only hamburgers and chicken tenders are available. Since Rose had planned to choose chicken tenders when she thought hot dogs, hamburgers, and chicken tenders would be offered, she should certainly still choose chicken tenders when hot dogs are eliminated from consideration. In a sense, the hot dog option was irrelevant to Rose's decision between chicken tenders and hamburgers. He formalized this notion of monotonicity with the following definition.

Independent of Irrelevant Payoff Pairs ($\mathbf{I^2P^2}$) Property: Given a bargaining game, an *independent of irrelevant payoff pairs* method produces the same payoff pair (X, Y) in any new game as it produced in the original game, as long as the new game was obtained by eliminating some feasible payoff pairs from the original game other than the payoff pair (X, Y) produced for the original game.

For example, consider **More**, whose rational payoff pairs consist of both the light and dark grey regions in Figure 5.1, and **Less**, whose rational payoff pairs consist of only the dark grey region. Notice that **Less** can be obtained from **More** by eliminating the light grey regions of rational payoff pairs.

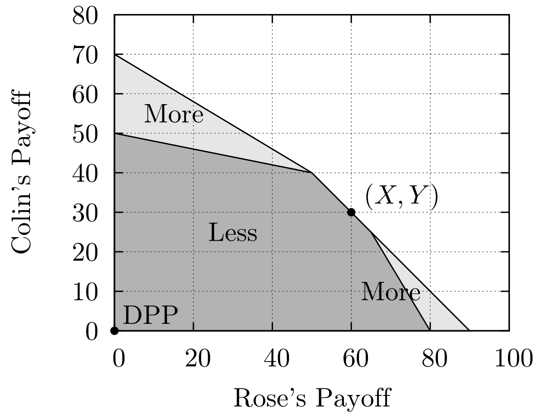

FIGURE 5.1. Rational payoff pairs for **More** and **Less**

If a method is $\mathrm{I}^2\mathrm{P}^2$ and produces (X, Y) for **More,** then that method must also produce (X, Y) for **Less**.

Suppose that the X method is efficient, unbiased, scale invariant, and $\mathrm{I}^2\mathrm{P}^2$. Then, on the game **Isoceles Triangle**, illustrated in Figure 5.2, which is symmetric, the X method must produce the payoff pair $(50, 50)$.

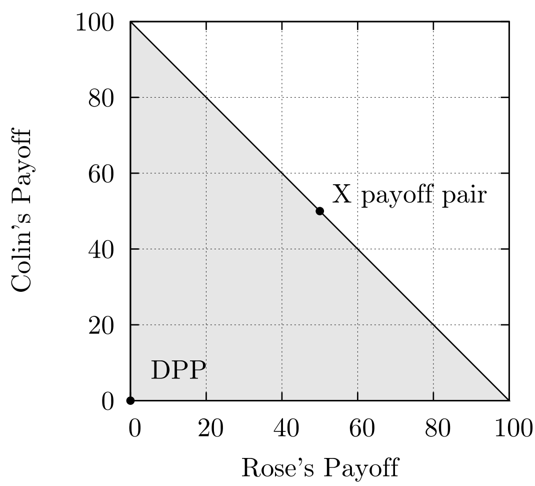

FIGURE 5.2. Rational and method X payoff pairs for **Isoceles Triangle**

We can now use the rescaling

$$\begin{aligned} x' &= 60 + \frac{81.25 - 60}{100}x, \\ y' &= 50 + \frac{106.67 - 50}{100}x \end{aligned}$$

to transform pairs (x, y) in **Isoceles Triangle** into pairs (x', y') in the game **Right Triangle**, shown in Figure 5.3.

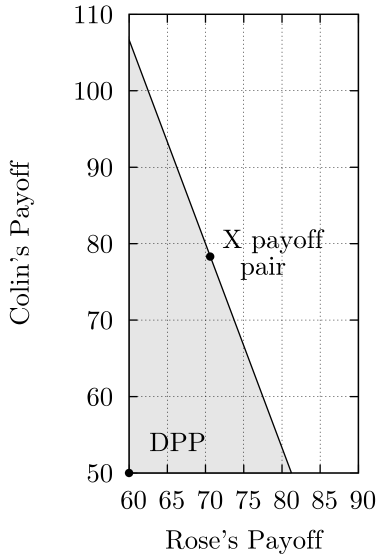

FIGURE 5.3. Rational and method X payoff pairs for **Right Triangle**

The three **Isoceles Triangle** vertices $(0, 0)$, $(100, 0)$, and $(0, 100)$ are transformed into the **Right Triangle** vertices $(60, 50)$, $(81.25, 50)$, and $(60, 106.67)$, respectively. Further, the **Isoceles Triangle** payoff pair $(50, 50)$ is transformed into the **Right Triangle** payoff pair $(70.6, 78.3)$. This is true in general: midpoints of line segments are transformed to midpoints of line segments under linear transformations. Further, since the X method is scale invariant and produces $(50, 50)$ for **Isoceles Triangle**, it must also produce $(70.6, 78.3)$ as the payoff pair when applied to **Right Triangle**. In general, the X method must always produce the midpoint of the hypotenuse when the region of rational payoff pairs is a right triangle.

Observe, coincidentally, that **Chances** can be obtained from **Right Triangle** by eliminating the feasible payoff pairs in the region shaded light grey, as shown in Figure 5.4.

The X method is I^2P^2 and produces $(70.6, 78.3)$ as a payoff pair for **Right Triangle.** Since $(70.6, 78.3)$ is feasible in **Chances**, the X method must produce $(70.6, 78.3)$ as a payoff pair for **Chances** as well. Thus, any efficient, unbiased, scale invariant, and I^2P^2 method applied to **Chances** must produce $(70.6, 78.3)$.

In summary, if the region of rational payoff pairs is a right triangle, an efficient, unbiased, and scale invariant method must produce the midpoint of the hypotenuse. If the region of rational payoff pairs is not a right triangle but we are able to extend the region to a right triangle whose hypotenuse midpoint is still in the original region of rational payoff pairs, then an I^2P^2 method will produce this midpoint. John Nash created an algorithm to find such a triangle [**36**].

Nash Method: Extend one of the efficient and rational line segments to the disagreement payoff lines forming the hypotenuse to a right triangle. If

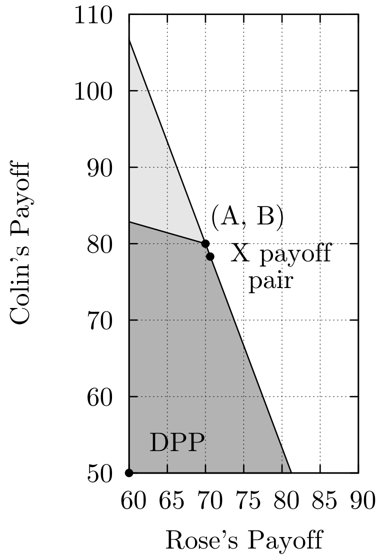

FIGURE 5.4. Rational and method X payoff pairs for **Right Triangle** and **Chances**

the midpoint of the hypotenuse is rational, then it is the *Nash payoff pair*. If the midpoint is to the left or right of the rational payoff pairs on the hypotenuse, then select the next efficient and rational line segment in the corresponding direction and repeat these steps from the beginning. If at some stage you are supposed to reconsider a line segment, then the payoff pair shared by the two most recently considered efficient and rational line segments is the *Nash payoff pair*.

Let's use the Nash method on **Chances.** Suppose we start with the upper efficient and rational line segment, obtaining Figure 5.5.

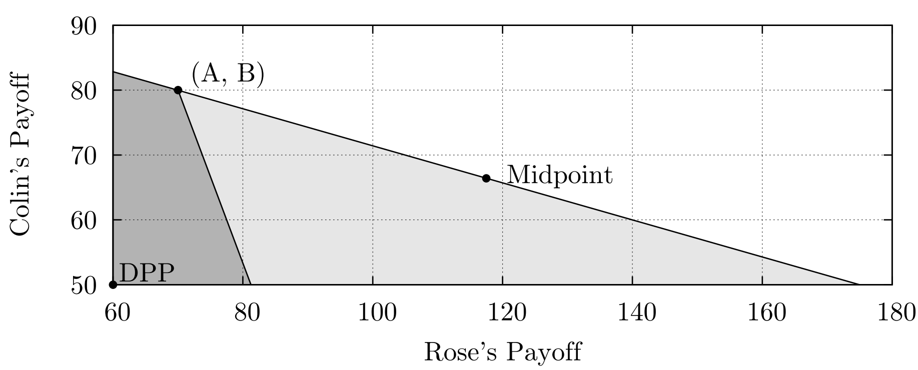

FIGURE 5.5. Step 1 of the Nash method for **Chances**

The algebraic description of the payoff pairs (x, y) on the upper line is

(CHANCES UPPER) $$y = -\frac{2}{7}x + 100.$$

This line intersects Rose's disagreement line $x = 60$ at $(60, 82.86)$ and intersects Colin's disagreement line $y = 50$ at $(175, 50)$. Thus, the midpoint of the hypotenuse is

$$\frac{1}{2}(60, 82.86) + \frac{1}{2}(175, 50) = (117.5, 66.4).$$

Since this midpoint is not a rational payoff pair and is to the right of the rational payoff pairs on the hypotenuse, the next step is to repeat these steps using the lower efficient and rational line segment, resulting in Figure 5.6.

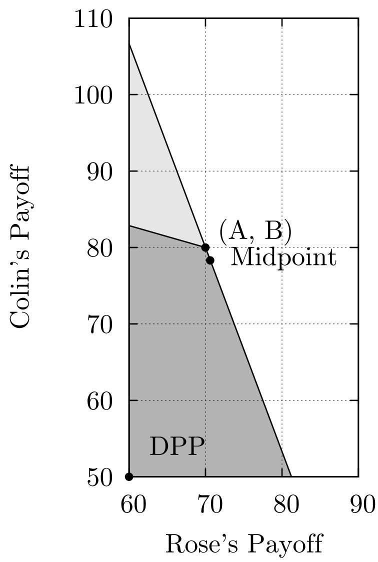

FIGURE 5.6. Step 2 of the Nash method for **Chances**

The algebraic description of the payoff pairs (x, y) on the lower line is

(CHANCES LOWER) $$y = -\frac{8}{3}x + \frac{800}{3}.$$

This line intersects Rose's disagreement line $x = 60$ at $(60, 106.67)$ and intersects Colin's disagreement line $y = 50$ at $(81.25, 50)$. Thus, the midpoint of the hypotenuse is

$$\frac{1}{2}(60, 106.67) + \frac{1}{2}(81.25, 50) = (70.6, 78.3).$$

Now the midpoint is rational, and so $(70.6, 78.3)$ is the Nash payoff pair. This is the same payoff pair that we found earlier by using the X method. This observation illustrates the proof of the following theorem, originally proven by John Nash [**36**].

> **Nash Characterization Theorem:** *The Nash method is the only efficient, unbiased, scale invariant, and independent of irrelevant payoff pairs method.*

Acme Industrial

We now illustrate the Nash method with **Acme Industrial**. For our first step, we use the right-most efficient and rational line (ACME LOWER) $y = -3x + 300$. From Figure 5.7, it is clear that the hypotenuse midpoint is to the left of the feasible payoff pairs on the line. So for our second step, we use (ACME MIDDLE) $y = -x + 140$, which has intersections with the disagreement payoff lines $x = 0$ and $y = 50$ at $(0, 140)$ and $(90, 50)$, making the midpoint of the hypotenuse $\frac{1}{2}(0, 140) + \frac{1}{2}(90, 50) = (45, 95)$. We see again that this point is to the left of the rational payoff pairs on the line.

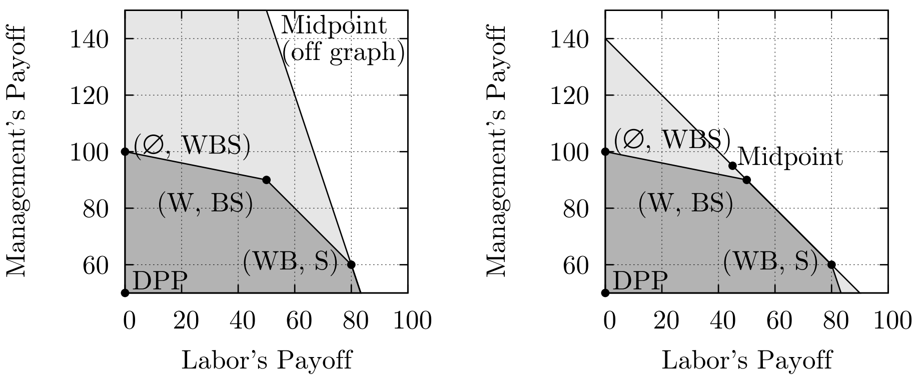

FIGURE 5.7. Steps 1 and 2 of the Nash method for **Acme Industrial**

So for our third step, we use (ACME UPPER) $y = -0.2x + 100$. From Figure 5.8, it is clear that the midpoint of the hypotenuse is to the right of the rational payoff pairs on the line. With this switch in sides, the Nash payoff pair is identified as the payoff pair shared by the lines in steps 3 and 2: $(50, 90)$. This payoff pair can be implemented by having Labor win on the wage issue and having Management win on the benefits and security issues.

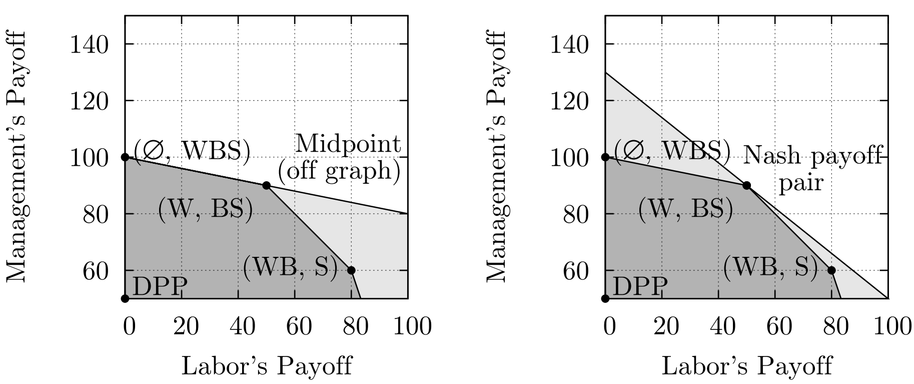

FIGURE 5.8. Steps 3 and 4 of the Nash method for **Acme Industrial**

Although unnecessary for the completion of the Nash method, step 4 as shown in Figure 5.8 includes the right triangle that justifies our selection of $(50, 90)$ for the

Nash payoff pair. That is, the disagreement payoff lines intersection points $(0, 130)$ and $(100, 50)$ were chosen so that $(50, 90)$ would be their midpoint.

Comparison of Methods

Tables 5.1 and 5.2 summarize the payoff pairs produced by each of the methods described in this chapter for the two games used as illustrations.

TABLE 5.1. Comparison of solutions for **Chances**

Method	Payoff Pair	Mixed Strategy Implementation
Egalitarian	$(75.5, 65.5)$	(0.82A + 0.18B, B)
Raiffa	$(73.5, 70.8)$	(0.89A + 0.11B, B)
Nash	$(70.6, 78.3)$	(0.98A + 0.02B, B)

TABLE 5.2. Comparison of solutions for **Acme Industrial**

Method	Payoff Pair	Outcome Lottery Implementation
Egalitarian	$(41.7, 91.7)$	0.17(∅, WBS) + 0.83(W, BS)
Raiffa	$(56.2, 83.7)$	0.79(W, BS) + 0.21(WB, S)
Nash	$(50.0, 90.0)$	(W, BS)

The payoff pairs and implementations produced by the three methods are sufficiently different from each other that players could argue for a long time about the relative fairness of each option.

One goal of this chapter has been to convince you that, rather than arguments over specific payoffs, the arguments should be over which general fairness properties are relevant to the situation. Table 5.3 summarizes the fairness properties that each method satisfies.

TABLE 5.3. Properties satisfied by each method

Property	Method		
	Egalitarian	Raiffa	Nash
Efficient	Yes	Yes	Yes
Unbiased	Yes	Yes	Yes
Rational	Yes	Yes	Yes
Scale Invariant	No	Yes	Yes
Strongly Monotone	Yes	No	No
Individually Monotone	Yes	Yes	No
Independent of Irrelevant Payoff Pairs	Yes	No	Yes

We have used the fairness properties to characterize the three methods. Depending upon which properties best define fairness in a specific scenario, the appropriate method can be chosen and used. In particular, if the player payoffs are measured with respect to a comparable external standard, then the properties suggest use of the egalitarian method. If player payoffs are not comparable (our usual assumption for strategic games), then the Raiffa or Nash methods are more appropriate. Which one of these two is chosen depends on which monotonicity property is thought to be most descriptive of fairness. If potential changes in the rationality of payoff pairs are likely to affect only one of the two players, then anomalies will be avoided by choosing the individually monotone Raiffa method. If potential changes in the rationality of payoff pairs are likely to affect both players, then anomalies will be avoided by choosing the I^2P^2 Nash method.

Exercises

(1) Consider the following **Asymmetric Prisoner's Dilemma** strategic game considered in exercise 1 of sections 6.2–6.4:

Asymmetric Prisoners' Dilemma		Colin	
Cardinal Payoffs		COOPERATE	DEFECT
Rose	COOPERATE	$(90, 60)$	$(0, 100)$
	DEFECT	$(100, 0)$	$(10, 40)$

(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Nash payoff pair is located.
(b) Find the Nash payoff pair.
(c) Describe how to implement the Nash payoff pair.
(d) In the context of this game, explain why an efficient, unbiased, scale invariant, and I^2P^2 method must produce the Nash payoff pair.

(2) Consider the **Matches** strategic game described in section 3.2 and the corresponding bargaining game considered in exercise 2 of sections 6.2–6.4.

Matches		Colin	
Cardinal Payoffs		TENNIS	SOCCER
Rose	TENNIS	$(10, 6)$	$(2, 5)$
	SOCCER	$(0, 0)$	$(9, 10)$

(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Nash payoff pair is located.
(b) Find the Nash payoff pair.
(c) Describe how to implement the Nash payoff pair.
(d) In the context of this game, explain why an efficient, unbiased, scale invariant, and I^2P^2 method must produce the Nash payoff pair.

(3) Consider the **Divorce Settlement** bargaining game described in section 6.2, exercise 3.
(a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Nash payoff pair is located.
(b) Find the Nash payoff pair.
(c) Describe how to implement the Nash payoff pair.
(d) In the context of this game, explain why an efficient, unbiased, scale invariant, and I^2P^2 method must produce the Nash payoff pair.

(4) Consider the **Panama Canal** bargaining game described in section 6.2, exercise 4.
   (a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Nash payoff pair is located.
   (b) Find the Nash payoff pair.
   (c) Describe how to implement the Nash payoff pair.
(5) Consider the **Risk-Neutral Fingers** bargaining game described in section 6.2, exercise 5.
   (a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Nash payoff pair is located.
   (b) Find the Nash payoff pair.
   (c) Describe how to implement the Nash payoff pair.
   (d) What two properties are sufficient to determine a unique payoff pair in this bargaining game?
(6) Consider the **River Tale** bargaining game described in section 6.2, exercise 6.
   (a) Graph the rational payoff pairs, identify the efficient payoff pairs, and sketch where the Nash payoff pair is located.
   (b) Find the Nash payoff pair.
   (c) Describe how to implement the Nash payoff pair.
   (d) What single property is sufficient to determine a unique payoff pair in this bargaining game?
(7) Consider the **Chances** bargaining game with disagreement payoff pair $(60, 50)$ discussed in the text.
   (a) Find the Nash payoff pair if the disagreement payoff pair is changed to the security levels $(50, 50)$.
   (b) Find the Nash payoff pair if the disagreement payoff pair is changed to the unique Nash equilibrium payoff pair $(62.5, 50)$.
   (c) Compare the Nash payoff pairs for the three different disagreement payoff pairs.
(8) Provide an explanation for why the following statements from the final table in this section are true.
   (a) The Nash method is not individually or strongly monotone.
   (b) The egalitarian method is independent of irrelevant payoff pairs.
   (c) The Raiffa method is not independent of irrelevant payoff pairs.

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CHAPTER 7

Coalition Games

1. A Savings Allocation Problem

Cooperation hopefully results in outcomes more beneficial to all cooperating players. If we know the gains obtainable from cooperation, all that remains is to determine how those gains are allocated among the cooperating players.

> By the end of this section, you should understand that subgroups of players will have an impact on the allocation of savings obtainable by a group, and you will have thought about what might constitute a fair method of allocation.

The Environmental Protection Agency (EPA) has mandated improvements in the sewage treatment facilities in the cities of Avon, Barport, Claron, and Delmont. Each city could work separately, but \$140 million would be saved by all four working together. If one of the cities were unwilling to cooperate, some triples of cities could also save money: without Delmont's cooperation, Avon, Barport, and Claron could save \$108 million; without Claron's cooperation, Avon, Barport, and Delmont could save \$96 million; without Belmont's cooperation, Avon, Claron, and Delmont could save \$84 million; and without Claron's or Delmont's cooperation, Avon and Barport could save \$24 million. No other subset of the cities could save money over completing the projects individually. In particular, Barport, Claron, and Delmont cannot save any money without the assistance of Avon. This information is summarized in Table 1.1.

Table 1.1. Savings for coalitions in EPA scenario

Coalition	ABCD	ABC	ABD	ACD	AB	any other
Savings	140	108	96	84	24	0

For cooperation to occur and the savings to be obtained, there must be a signed agreement among the cities stating how the savings is to be allocated among the cities. With no agreement, each city saves nothing (\$0 million).

Angelina, Bob, Celine, and Diego are the negotiators representing Avon, Barport, Claron, and Delmont, respectively.

Before you read about their negotiation, we encourage you to gather together four people to act as negotiators. Come to your own settlement. Then see how your ideas compare with the ideas expressed here.

ANGELINA: Welcome to our negotiation session, being held in my fair city of Avon.

BOB: Barport looks forward to a mutually beneficial resolution from the negotiation.

CELINE: As Claron's representative, I find it curious that the first letters of our names match the first letters of the cities we represent.

DIEGO: Delmont takes delight in your observation. But what would you expect from a story written by a pair of mathematicians?

BOB: We should focus on the purpose of this meeting.

DIEGO: Since \$140 million in savings is obtained through all four of our cities cooperating, I propose that we allocate that savings equally: \$35 million for each city.

CELINE: While your suggestion may seem reasonable to a neophyte negotiator, I must point out that there is no incentive for the rest of us to agree to your proposal. Avon, Barport, and Claron could choose to leave out Delmont and allocate the \$108 million saved equally. Thereby, Avon, Barport, and Claron would each save \$36 million, instead of the \$35 million you proposed. And Delmont would need to do its improvements alone, saving nothing. It seems only fair that no coalition of cities should be allocated less than what they could obtain on their own.

DIEGO: Forgive my naiveté. I should have taken into account the greater contributions to savings provided by the rest of you. I certainly do not want to be left out of a joint agreement. With \$140 million in savings available to the four of us, why should you settle for saving only \$108 million? I now propose that Avon, Barport, and Claron each receive \$37 million in savings leaving poor Delmont with only \$29 million in savings.

BOB: Poor Delmont? You have suggested that the three of us should split whatever savings we can muster on our own (\$108 million) and then you should take the additional savings brought about when you join with us (\$140 − \$108 = \$32 million) less a paltry \$1 million for each of us for our unwillingness to expose your deception. I offer Barport to play the role of "victim" instead. Avon, Claron, and Delmont can split the \$84 million in savings they can obtain on their own, and Barport will be content with the remaining \$140 − \$84 = \$56 million less \$2 million for each of you. Hence, I propose that Avon, Claron, and Delmont each receive \$30 million (\$84 / 3 + \$2) in savings and Belmont receive \$50 million in savings.

ANGELINA: Bob, you must be joking! By the same argument, but with Avon the "victim" city, Barport, Claron, and Delmont split the \$0 million in savings they can achieve without Avon. Then Avon should receive the entire \$140 million less a token amount for each of the other cities.

DIEGO: Please accept my apologies for not possessing the experience and wisdom of my negotiating colleagues. It seemed reasonable to me that since your three cities on their own could save only \$108 million and with Delmont we four could save \$140 million, then Delmont deserves most of the \$32 million difference.

CELINE: Since your three cities on their own could save only \$96 million and with Claron we four could save \$140 million, then Claron deserves most of the \$44 million difference.

BOB: Since your three cities on their own could save only \$84 million and with Barport we four could save \$140 million, then Barport deserves most of the \$56 million difference.

ANGELINA: Since your three cities on their own could save only \$0 million and with Avon we four could save \$140 million, then Avon deserves most of the \$140 million difference.

CELINE: Avon deserves \$140 million, Barport deserves \$56 million, Claron deserves \$44 million, and Delmont deserves \$32 million. Since that sums to \$272 million and there is only \$140 million to share, we have a problem.

BOB: Well, no one said that their city deserves the full difference. How about each of us takes only half of the difference?

CELINE: That still would not have us allocate exactly \$140 million. To turn the \$272 million sum into a \$140 million sum, we should allocate 140/272 of the differences. Then

- Avon receives $140 \times (\frac{140}{272}) = \$72.1$ million,
- Barport receives $56 \times (\frac{140}{272}) = \$28.8$ million,
- Claron receives $44 \times (\frac{140}{272}) = \$22.6$ million, and
- Delmont receives $32 \times (\frac{140}{272}) = \$16.5$ million.

DIEGO: I'm not sure that I'll keep my job as Delmont's negotiator if I agree to such a small amount of savings.

CELINE: The key is to explain the method we developed. Each city should receive the same fraction of the savings as that city's cooperation contributed to the overall savings.

- Avon's contribution to overall savings is $140 - 0 = \$140$ million,
- Barport's contribution to overall savings is $140 - 84 = \$56$ million,
- Claron's contribution to overall savings is $140 - 96 = \$44$ million, and
- Delmont's contribution to overall savings is $140 - 108 = \$32$ million.

When we give $\frac{140}{272}$, or a little more than 51%, of the contributions to each city, the \$140 million in overall savings is distributed fairly.

ANGELINA: I agree that the method sounds fair.

DIEGO: I'm not so sure. I still like the idea of an equal split, perhaps applied more broadly.

ANGELINA: We have already said that an equal split would not be agreeable to Avon, Barport, and Claron.

BOB: It is true that an equal split gives Avon, Barport, and Claron only \$105 million versus the \$108 million that we could obtain on our own. However, the equal split proposal would give Barport \$35 million rather than the \$28.8 million that Celine's proposal gives my city. I, for one, would like to hear Diego's idea.

DIEGO: I propose that equal split be applied to each coalition, not just to the coalition of all of us. I have done this in Table 1.2.

TABLE 1.2. Equal splits for all coalitions in the EPA scenario

Coalition	Avon	Barport	Claron	Delmont
ABCD	35	35	35	35
ABC	36	36	36	0
ABD	32	32	0	32
ACD	28	0	28	28
AB	12	12	0	0
any other	0	0	0	0
Sum	143	115	99	95

BOB: Instead of focusing on contributions to overall savings, you're suggesting that we focus on the sum of equal splits of the savings for each coalition of cities. Sounds like an interesting alternative.

DIEGO: Yes. Unfortunately, the total of these sums is \$452 million. But, if we multiply them by 140/452, then each city would receive the same fraction of the sum of equal splits of coalition savings.

- Avon receives $143 \times (\frac{140}{452}) = \$44.3$ million,
- Barport receives $115 \times (\frac{140}{452}) = \$35.6$ million,
- Claron receives $99 \times (\frac{140}{452}) = \$30.7$ million, and
- Delmont receives $95 \times (\frac{140}{452}) = \$29.4$ million.

Avon, Barport, and Claron receive a total of $44.3 + 35.6 + 30.7 = \$110.6$, which is more than the \$108 million you could obtain without Delmont's cooperation.

ANGELINA: Why distribute equal fractions? Why not subtract an equal amount from each of the sums? We would need to subtract $(452 - 140)/4 = \$78$ million from each sum. Then

- Avon receives $143 - 78 = \$65$ million,
- Barport receives $115 - 78 = \$37$ million,

- Claron receives $99 - 78 = \$21$ million, and
- Delmont receives $95 - 78 = \$17$ million.

CELINE: Why *not* distribute equal fractions? Why subtract an equal amount from each of the sums?

ANGELINA: We are summing the savings obtained by a city in each coalition. Since that turns out to be more than can be allocated, we should do the corresponding operation, subtraction, not multiplication or division, to obtain the right amounts.

BOB: I don't buy this equal adjustment idea. If we apply the same idea to the contributions to overall savings, then

- Avon receives $140 - 33 = \$107$ million,
- Barport receives $56 - 33 = \$23$ million,
- Claron receives $44 - 33 = \$7$ million, and
- Delmont receives $32 - 33 = -\$1$ million.

DIEGO: My city would pay $1 million to cooperate?

BOB: Yes, if we make an equal adjustment to the contributions to overall savings.

ANGELINA: But I was suggesting an equal adjustment to the sum of equal splits of coalition savings.

BOB: Hold on a minute! Before we continue, let me try to summarize how far we have come. I think that we have three somewhat reasonable proposals that I've shown in Table 1.3.

TABLE 1.3. Comparison of proposals

Proposal	Avon	Barport	Claron	Delmont
Celine: each city should receive the same fraction of the savings as is generated by that city's contribution to the overall savings.	72.1	28.8	22.6	16.5
Angelina: each city should receive an equal adjustment to the sum of the equal splits of coalition savings.	65.0	37.0	21.0	17.0
Diego: each city should receive the same fraction of the sum of the equal splits of coalition savings.	44.3	35.6	30.7	29.4

BOB: I have not included (1) an equal split because Avon, Barport, and Claron could do better without Delmont, nor (2) an equal adjustment from contribution to the overall savings because Delmont could do better without the other cities.

DIEGO: I say that we put it to a vote.

CELINE: It's clear that Diego's proposal would receive the plurality of votes.

ANGELINA: Perhaps I'll choose to not cooperate. Then none of you will obtain any savings.

BOB: Of course, but you'll receive no savings either.

ANGELINA: But it is clearly unfair for Avon to receive only 32% of the savings when without Avon there can be no savings to anyone.

BOB: Fair? What is fair?

Exercises

(1) Determine the allocation of savings that you believe to be the most fair. Why do you believe your allocation is fair? How does your allocation compare with the various proposals made during the above negotiation?
(2) During the negotiation, Celine argues that fairness implies that no coalition should be allocated less than what they could obtain on their own. What else should be true for an allocation to be considered fair?

2. Two Properties and Five Methods

In this chapter we ignore strategy and assume that players want to cooperate with each other. Rather than a payoff to each player for any particular outcome, we assume a total payoff that can be allocated among the players in any way that they together choose. Rather than asking what strategy to choose to maximize my payoff, we ask what is the fair way to allocate among the players the total payoff available. We will suggest some allocation methods, formalize fairness notions as properties of allocation methods, and determine which methods satisfy which properties. This is the same approach we took with bargaining games in Chapter 6.

> By the end of this section, you should be able to model cooperative scenarios as coalition games, identify efficient and rational allocations, and use some simple allocation methods.

Coalition Game: A *coalition game* consists of players and a gain associated with each *coalition* (nonempty set of players). The players in a coalition can allocate their gain in any manner to which each in the coalition agrees; when each player in a coalition agrees to a proposed allocation, we say that each player has completed an agreement. Each player can complete an agreement with only one coalition. The game ends when all players have completed agreements or a preset time limit has expired, whereupon each player who has not completed an agreement is considered to have completed an agreement with the coalition containing only that player for the associated gain. The collection of agreed upon payoffs, called an *allocation*, constitute the outcome for the game. Each player receives the payoff to which she or he agreed, and each player most prefers to maximize her or his own payoff.

An example of a coalition game is the savings allocation problem of the first section, that we will call the **EPA** game. The players are the four cities, abbreviated here by the letters A, B, C, and D. The gain of a coalition is the savings obtained when that coalition cooperates to fulfill the EPA mandate. The coalition gains are summarized in Table 2.1. It is reasonable to assume that each city's primary concern is the savings it receives.

TABLE 2.1. **EPA** coalition game

Coalition	Gain
ABCD	140
ABC	108
ABD	96
ACD	84
AB	24
other	0

Here is one way the game might be played: Player D may propose that the coalition ABCD form and each player receives 35, which can be denoted $(35, 35, 35, 35)$. Player B may then propose that the coalition ABC form and each player in the coalition receives 36, which can be denoted $(36, 36, 36, 0)$. Player C may then voice agreement to $(36, 36, 36, 0)$. Player A may then propose that the coalition ABD form, A receives 46, B receives 40, and D receives 10, which can be denoted $(46, 40, 0, 10)$. Players B and D may then agree to $(46, 40, 0, 10)$, completing the agreement. If time ran out now, then the allocation would be $(46, 40, 0, 10)$ because A, B, and D had agreed to $(46, 40, 0, 10)$ and the gain associated with C is 0.

A second way the game might be played would be as in the previous paragraph except that more time was available. With the additional time, player C may propose $(50, 40, 30, 20)$ for the coalition ABCD. Players A, B, and D may then agree, whereupon the game ends (even if time has not run out), and the allocation is $(50, 40, 30, 20)$.

Coalition games are similar to bargaining games in that there are benefits to cooperation and players make payoff proposals and attempt to obtain agreement from other players. There are two significant differences between coalition games and bargaining games. Coalition games are more general than bargaining games in that coalition games may involve more than two players, and so there may be many opportunities for subgroups to cooperate by forming coalitions. Surprisingly, coalition games are also more specialized than bargaining games in the sense that each coalition has only a single number, its gain, associated with it, while in a bargaining game the coalition of the two players has a whole set of feasible payoff pairs associated with it. The gain in a two player coalition game corresponds to the set of all efficient payoff pairs in a **Triangle** like bargaining game. It is possible to associate a set of payoff vectors with each of the coalitions instead of a single number; however, these “nontransferable utility” or “NTU” games are beyond the scope of this book.

In our two example plays of **EPA** above, the allocations were $(46, 40, 0, 10)$ and $(50, 40, 30, 20)$. The later allocation is better than the former because players A, C, and D prefer the later allocation over the former allocation and player B is indifferent between the two allocations. We formalize this informal idea about what is a good allocation in the following definition.

> **Efficient Property:** An allocation is *efficient* if it is impossible to increase the payoff of one player without decreasing the payoff to another player.

The allocation $(46, 40, 0, 10)$ is not efficient because $(50, 40, 30, 20)$ is another allocation in which at least one player has an increased payoff without decreasing the payoffs of any players. The allocation $(50, 40, 30, 20)$ is efficient because the sum of the payoffs is $50 + 40 + 30 + 20 = 140$, which is the most the four players' payoffs can sum, and so to increase one player's payoff would force a decrease in some other player's payoff. We generalize these specific results in the following theorem.

> **Efficient Coalition Game Allocation Theorem:** *Suppose that it is always beneficial for larger coalitions to form. In particular, the gains from*

nonoverlapping coalitions sum to no more than the overall gain*, that is, the gain obtained from the entire group of players cooperating. Then an allocation is efficient if and only if the sum of the payoffs equals the overall gain.*

It does not seem reasonable for players to agree to an allocation that is not efficient: if there is another allocation with some additional benefit and in which no one is harmed, we should not settle for the former allocation. Nonetheless, not all efficient allocations seem fair. For example, $(50, 20, 20, 50)$ is an efficient allocation for **EPA**, but few would argue that it is fair. Specifically, players A, B, and C together receive $50 + 20 + 20 = 90$, but the coalition ABC could obtain a greater amount, 108, on their own. Why should players A, B, and C agree to $(50, 20, 20, 50)$ when they could agree to $(58, 25, 25, 0)$ instead? Of course, player D would object to $(58, 25, 25, 0)$, but ABC can obtain $58+25+25 = 108$ on their own, so player D's objections could be ignored. Of course, $(58, 25, 25, 0)$ is not efficient, and so the efficient allocation $(65, 30, 30, 15)$ would be even better. This discussion suggests that a fair allocation should not only be efficient, but it should also satisfy the following property.

Rational Property: An allocation is *rational* if each coalition receives at least as much as its gain.

The allocation $(65, 30, 30, 15)$ is both efficient and rational, as is shown in Table 2.2. The first row verifies efficiency and all the rows together verify rationality.

TABLE 2.2. Demonstration that $(65, 30, 30, 15)$ is efficient and rational

Coalition	Payoff Received											Gain
ABCD	65	+	30	+	30	+	15	=	140	=	140
ABC	65	+	30	+	30			=	125	≥	108
ABD	65	+	30			+	15	=	110	≥	96
ACD	65			+	30	+	15	=	110	≥	84
BCD			30	+	30	+	15	=	75	≥	0
AB	65	+	30					=	95	≥	24
AC	65			+	30			=	95	≥	0
AD	65					+	15	=	80	≥	0
BC			30	+	30			=	60	≥	0
BD			30			+	15	=	45	≥	0
CD					30	+	15	=	45	≥	0
A	65							=	65	≥	0
B			30					=	30	≥	0
C					30			=	30	≥	0
D							15	=	15	≥	0

Clearly, a rational allocation must be efficient. However, the allocation $(40, 30, 30, 40)$ is efficient but not rational because coalition ABC receives $40 + 30 + 30 = 100$ but their gain is 108. Further, the allocation $(20, 20, 20, 20)$ is neither efficient nor rational.

Although many would agree that being efficient and rational are minimal requirements for an allocation to be fair, there are many efficient and rational allocations for **EPA**. If we do not want to settle for many fair possibilities but would instead like to recommend a single most fair allocation, we must find additional fairness criteria. One criterion would be that allocations are selected by a method that is applicable to any coalition game. As captured in the previous section's conversation, people who have engaged in the **EPA** game as negotiators eventually suggest allocation methods that work for all games rather than for just the **EPA** game.

Five Allocation Methods

We will now describe five allocation methods people often suggest when presented with a coalition game such as **EPA**. We do not recommend the use of any of these methods; however, the ideas are interesting to contemplate.

> **Equal Split Method:** The *equal split allocation* gives each player an equal split of the overall gain.

For **EPA**, the equal split allocation is $(35, 35, 35, 35)$. More importantly, equal split could be used to select the allocation for any other coalition game. In order to emphasize that allocation methods select allocations for every coalition game, it would be useful to have a second coalition game example. So, let us describe another scenario involving cooperation.

Adonis, Beautex, and Celestron are three companies planning to collaborate on a new product. Adonis estimates that it could develop the product on its own and achieve a profit of $2 million, but neither Beautex nor Celestron thinks that either could develop the product independently and turn a profit. Together the three companies estimate that they could develop the product and obtain a profit of $50 million. Adonis and Beautex could obtain a profit of $38 million without Celestron. Adonis and Celestron could obtain a profit of $38 million without Beautex. Beautex and Celestron could obtain a profit of $18 million without Adonis. How should the three companies allocate the profits from the new product?

We can model this situation as a coalition game, which we will call the **New Product** game, shown in Table 2.3.

TABLE 2.3. **New Product** coalition gains

Coalition	Gain
ABC	50
AB	38
AC	38
BC	18
A	2
B	0
C	0

The players are the three companies, which are represented by the first letters of their names. The gain of a coalition is synonymous with the profit generated by that coalition. The equal split allocation is $(\frac{50}{3}, \frac{50}{3}, \frac{50}{3})$, which is approximately $(16.7, 16.7, 16.7)$.

Notice that while equal split always selects efficient allocations, equal split does not yield rational allocations for either **EPA** ($35 + 35 + 35 = 105$ is less than ABC can obtain on its own) or **New Product** ($\frac{50}{3} + \frac{50}{3} = \frac{100}{3}$ is less than AB or AC can obtain on their own). At a more fundamental level, equal split ignores information about the gains of all coalitions except for the coalition of all players. It would seem that a fair method should take into account as much relevant information as possible, as will be done with the remaining methods described in this section.

> **Adjusted Marginals Method:** Each player first receives the player's *marginal contribution* to the overall gain: the difference between the overall gain and the gain to the coalition of all players except the one player under consideration. At this point, the sum of what the players have received may be more or less than the overall gain. Find the difference (which may be positive, negative, or zero) between the overall gain and the sum of the marginal contributions. The *adjusted marginals allocation* gives each player an equal split of this difference.

For **EPA**, we first give each player their marginal contribution to the overall gain (see the first two columns of Table 2.4). The sum of these marginal contributions is 272, which is more than the overall gains, 140. So, each player is given an equal split of the difference between the overall gain and the sum of the marginal contributions, $140 - 272$.

TABLE 2.4. Adjusted marginal payoffs for **EPA**

EPA Player	Marginal Contribution to the Overall Gain	Adjusted Marginals Payoff
A	$140 - 0 = 140$	$140 + \frac{1}{4}(140 - 272) = 107$
B	$140 - 84 = 56$	$56 + \frac{1}{4}(140 - 272) = 23$
C	$140 - 96 = 44$	$44 + \frac{1}{4}(140 - 272) = 11$
D	$140 - 108 = 32$	$32 + \frac{1}{4}(140 - 272) = -1$
Sum	272	140

For **New Product**, we follow the same procedure in Table 2.5.

By definition (the adjustment is chosen so that the sum of payoffs equals the overall gain), the adjusted marginals method produces efficient allocations. While the

TABLE 2.5. Adjusted marginal payoffs for **New Product**

New Product Player	Marginal Contribution to the Overall Gain	Adjusted Marginals Payoff
A	$50 - 18 = 32$	$32 + \frac{1}{3}(50 - 56) = 30$
B	$50 - 38 = 12$	$12 + \frac{1}{3}(50 - 56) = 10$
C	$50 - 38 = 12$	$12 + \frac{1}{3}(50 - 56) = 10$
Sum	56	50

New Product allocation is rational, the **EPA** allocation is not rational because D receives -1 while D could obtain 0 on its own.

> **Proportional-to-Marginals Method:** Calculate each player's *marginal contribution* to the overall gain: the difference between the overall gain and the gain to the coalition of all players except the one player under consideration. The *proportional-to-marginals allocation* gives to each player the player's marginal contribution multiplied by the ratio of the overall gain to the sum of the marginal contributions.

For **EPA**, we first calculate marginal contributions (as shown in Table 2.6). Since the overall gain is 140 and the sum of the marginal contributions is 272, each marginal contribution is multiplied by the ratio $\frac{140}{272}$ in order to obtain the payoffs. The allocation $(72.1, 28.8, 22.6, 16.5)$ is efficient and rational.

TABLE 2.6. Proportional-to-marginal payoffs for **EPA**

EPA Player	Marginal Contribution to the Overall Gain	Proportional-to-Marginals Payoff
A	$140 - 0 = 140$	$140 \times \frac{140}{272} \approx 72.1$
B	$140 - 84 = 56$	$56 \times \frac{140}{272} \approx 28.8$
C	$140 - 96 = 44$	$44 \times \frac{140}{272} \approx 22.6$
D	$140 - 108 = 32$	$32 \times \frac{140}{272} \approx 16.5$
Sum	272	140

For **New Product**, we have similar results in Table 2.7. The allocation $(28.6, 10.7, 10.7)$ is efficient and rational.

TABLE 2.7. Proportional-to-marginals payoffs for **New Product**

New Product Player	Marginal Contribution to the Overall Gain	Proportional-to-Marginals Payoff
A	$50 - 18 = 32$	$32 \times \dfrac{50}{56} \approx 28.6$
B	$50 - 38 = 12$	$12 \times \dfrac{50}{56} \approx 10.7$
C	$50 - 38 = 12$	$12 \times \dfrac{50}{56} \approx 10.7$
Sum	56	50

By definition (the constant of proportionality is chosen to make the allocation efficient), the proportional-to-marginals allocation method always produces efficient allocations. Based on the **EPA** and **New Product** games, it might seem reasonable to conjecture that the proportional-to-marginals allocation should always be rational. That turns out to not be the case, as will be seen in the exercises.

Adjusted Equal Splits Method: Each player first receives an equal split from the gain to each coalition to which the player belongs. At this point, the sum of what the players have received will be more than the overall gain. Find the difference between the overall gain and the sum of the equal splits of gains. The *adjusted equal splits* method gives each player an equal split of this difference.

For **EPA**, we first determine the equal splits of gains for each coalition that has a positive gain (reflected in the first several columns of Table 2.8). At this point, the players have received 452 in total, which is more than the overall gain of 140. So, each player is then given an equal split of the deficit, $140 - 452$.

TABLE 2.8. Adjusted equal splits payoffs for **EPA**

EPA	Equal Splits of Gains						Adjusted Equal Splits
Player	ABCD	ABC	ABD	ACD	AB	Sum	Payoff
A	35	36	32	28	12	143	$143 + \frac{1}{4}(140 - 452) = 65$
B	35	36	32	0	12	115	$115 + \frac{1}{4}(140 - 452) = 37$
C	35	36	0	28	0	99	$99 + \frac{1}{4}(140 - 452) = 21$
D	35	0	32	28	0	95	$95 + \frac{1}{4}(140 - 452) = 17$
Sum	140	108	96	84	24	452	140

For **New Product**, we have Table 2.9.

TABLE 2.9. Adjusted equal splits payoffs for **New Product**

New Product Player	Equal Splits of Gains						Adjusted Equal Splits Payoff
	ABC	AB	AC	BC	A	Sum	
A	$\frac{50}{3}$	19	19	0	2	$\frac{170}{3}$	$\frac{170}{3} + \frac{1}{3}(50 - 146) \approx 24.6$
B	$\frac{50}{3}$	19	0	9	0	$\frac{134}{3}$	$\frac{134}{3} + \frac{1}{3}(50 - 146) \approx 12.7$
C	$\frac{50}{3}$	0	19	9	0	$\frac{134}{3}$	$\frac{134}{3} + \frac{1}{3}(50 - 146) \approx 12.7$
Sum	50	38	38	18	2	146	50

By definition, adjusted equal splits selects efficient allocations. While the **EPA** allocation is rational, the **New Product** allocation is not rational because AB receives 37.3 while AB could obtain 38 on its own.

Proportional-to-Equal Splits Method: Calculate each player's sum of equal splits of gains from each coalition containing the player. The *proportional-to-equal splits allocation* gives to each player the player's sum of equal splits of gains multiplied by the ratio of the overall gain to the sum of the equal splits of gains.

For **EPA**, we have the same computation of equal splits of gains. These equal splits of gains are then multiplied by the ratio of the overall gain, 140, to the sum of the equal splits of gains, 452, as shown in Table 2.10.

TABLE 2.10. Proportional-to-equal splits payoffs for **EPA**

EPA Player	Equal Splits of Gains						Proportional-to-Equal Splits Payoff
	ABCD	ABC	ABD	ACD	AB	Sum	
A	35	36	32	28	12	143	$143 \times \frac{140}{452} \approx 44.3$
B	35	36	32	0	12	115	$115 \times \frac{140}{452} \approx 35.6$
C	35	36	0	28	0	99	$99 \times \frac{140}{452} \approx 30.7$
D	35	0	32	28	0	95	$95 \times \frac{140}{452} \approx 29.4$
Sum	140	108	96	84	24	452	140

For **New Product**, we have similar results in Table 2.11.

TABLE 2.11. Proportional-to-equal splits payoffs for **New Product**

New Product Player	Equal Splits of Gains						Proportional-to-Equal Splits Payoff
	ABC	AB	AC	BC	A	Sum	
A	$\frac{50}{3}$	19	19	0	2	$\frac{170}{3}$	$\frac{170}{3} \times \frac{50}{146} \approx 19.4$
B	$\frac{50}{3}$	19	0	9	0	$\frac{143}{3}$	$\frac{143}{3} \times \frac{50}{146} \approx 15.3$
C	$\frac{50}{3}$	0	19	9	0	$\frac{143}{3}$	$\frac{143}{3} \times \frac{50}{146} \approx 15.3$
Sum	50	38	38	18	2	146	50

By definition, proportional-to-equal splits selects efficient allocations. While the **EPA** allocation is rational, the **New Product** allocation is not rational because AB receives 34.7 while AB could obtain 38 on its own.

Method Properties

We have defined two properties that allocations may possess: efficient and rational. We can extend these definitions so that they apply to allocation methods as well as to specific allocations.

> **Efficient Property:** An allocation method is *efficient* if the method always selects efficient allocations.

Clearly, all five allocation methods described in this section are efficient.

We cannot similarly extend the definition of rational, requiring that an allocation method *always* selects a rational allocation, because some coalition games have *no* rational allocations! We can only expect an allocation method to select a rational allocation when there are rational allocations available. Thus, we use the following definition.

> **Rational Property:** An allocation method is *rational* if it selects a rational allocation whenever a rational allocation exists.

As an example of a game without a rational allocation, consider a committee consisting of three voting members operating under the rule that motions pass if at least two of the three members vote in favor. We assign a gain of 1 to coalitions that can pass a measure and 0 to coalitions that cannot pass a measure. This results in the **Simple Majority** game shown in Table 2.12.

Suppose that (a, b, c) is a rational allocation. Since (a, b, c) is an allocation,

$$a + b + c \leq 1.$$

Table 2.12. **Simple Majority** coalition game

Coalition	Gain
ABC	1
AB	1
AC	1
BC	1
A	0
B	0
C	0

Because (a, b, c) is rational and the gain of ABC is 1,

$$a + b + c \geq 1.$$

Combining the previous two inequalities, we obtain

$$a + b + c = 1.$$

Since (a, b, c) is rational and the gain of AB is 1, $a + b \geq 1$, and by the previous equality, $a + b = 1 - c$, and so $1 - c \geq 1$, or

$$c \leq 0.$$

Since (a, b, c) is rational and the gain of C is 0,

$$c \geq 0.$$

Combining the previous two inequalities, we obtain

$$c = 0.$$

A similar argument using the gains of AC and B leads to the conclusion

$$b = 0,$$

and a similar argument using the gains of BC and A leads to the conclusion

$$a = 0.$$

But $(0, 0, 0)$ does not satisfy $a+b+c = 1$. Thus, **Simple Majority** has no rational allocation, and thus we cannot expect any allocation method to select a rational allocation for **Simple Majority**.

We have seen that both **EPA** and **New Product** have rational allocations. We have also seen that the equal split, adjusted marginals, proportional-to-equal splits, and adjusted equal splits methods each selects an allocation that is not rational in at least one of **EPA** or **New Product**. This shows that four of the methods introduced in this section are not rational. It turns out that even the fifth method, the proportional-to-marginals method, is also not rational (see exercise 4). How disappointing! The rational property was discussed almost immediately in the previous section's negotiation, and negotiators and arbitrators in many situations find it important that allocations be rational because there will be a coalition who will try to block any proposed allocation that is not rational.

The rest of the chapter will describe two more allocation methods, each of which can be considered more fair than any of the five methods described above. We will also consider a number of other properties which people have thought that a fair allocation method should have.

Exercises

(1) **Apartment Sharing**. Abeje, Belva, and Corrine are considering whether to obtain separate apartments or to share an apartment. If all three share an apartment, they will save \$120 weekly. If only Abeje and Belva share an apartment, the pair will save \$90 weekly. If only Abeje and Corrine share an apartment, the pair will save \$60 weekly. If only Belva and Corrine share an apartment, the pair will save \$30 weekly.
   (a) Model this scenario as a coalition game.
   (b) Find the equal split allocation.
   (c) Find the proportional-to-marginals allocation.
   (d) Find the adjusted marginals allocation.
   (e) Find the proportional-to-equal splits allocation.
   (f) Find the adjusted equal splits allocation.
   (g) Which of the above allocations are rational?
   (h) What would *you* say is the fair way to allocate the savings?

(2) **Veto Power**. Aamir, Brice, and Cayman have been given the opportunity to split \$90. However, the person providing the money is also willing to give the money to the pair Aamir and Brice or to the pair Aamir and Cayman. All that is necessary is for some pair containing Aamir or all three men to come to an agreement as to how to allocate the money. Since no money will be received without Aamir's consent, he is said to have veto power.
   (a) Model this scenario as a coalition game.
   (b) Find the equal split allocation.
   (c) Find the proportional-to-marginals allocation.
   (d) Find the adjusted marginals allocation.
   (e) Find the proportional-to-equal splits allocation.
   (f) Find the adjusted equal splits allocation.
   (g) Which of the above allocations are rational?
   (h) Verify that there is a unique efficient and rational allocation.
   (i) What would *you* say is the fair way to allocate the money?

(3) **Loner Power**. Ron and Stephen have been given the opportunity to split \$180 whether or not Tabitha cooperates. No other individual or pair can make an agreement to obtain any money. Since what Tabitha decides to do has no real effect on the money available, she could be called a loner.
   (a) Model this scenario as a coalition game.
   (b) Find the equal split allocation.
   (c) Find the proportional-to-marginals allocation.
   (d) Find the adjusted marginals allocation.
   (e) Find the proportional-to-equal splits allocation.
   (f) Find the adjusted equal splits allocation.
   (g) Which of the above allocations are rational?
   (h) Verify that Tabitha receives a payoff of 0 in any rational allocation.
   (i) What would *you* say is the fair way to allocate the money?

(4) **Not Finding Rational**. Consider the four player coalition game with the players A, B, C, and D, and the gains specified in Table 2.13 below.

TABLE 2.13. **Not Finding Rational** coalition game

Coalition	Gain
ABCD	432
ABC	384
ABD	384
ACD	96
BCD	96
CD	84
other	0

(a) Verify that the allocation $(170, 170, 46, 46)$ is efficient and rational.
(b) Find the equal split allocation and verify that it is not rational.
(c) Find the proportional-to-marginals allocation and verify that it is not rational.
(d) Find the adjusted marginals allocation and verify that it is not rational.
(e) Find the proportional-to-equal splits allocation and verify that it is not rational.
(f) Find the adjusted equal splits allocation and verify that it is not rational.
(g) The earlier parts of this problem show that none of the five methods considered in this section is rational. What is your conclusion?

3. The Shapley Method

The Shapley method can be considered a generalization of the adjusted marginals method, taking into account contributions of all coalitions rather than simply the coalition of all the players. More importantly, the Shapley method satisfies and is characterized by four fairness properties. So, if an arbitrator believes that a fair allocation method must satisfy these four properties, then the Shapley method should be used.

> By the end of this section, you should be able to calculate the Shapley allocation in two different ways and to describe why the Shapley method can be considered fair.

Shapley Method: The *Shapley allocation* gives each player that player's marginal contribution to all coalitions, averaged over all player orders. Given an ordering of the players, a player i's *marginal contribution* to a coalition is the difference between (1) the gain obtained by the coalition containing i and all players before i in the ordering, and (2) the gain obtained by the coalition containing all players before i in the ordering.

Let us explain more concretely in the context of **EPA**, displayed again in Table 3.1. A player order can be considered the order in which the players decide that

TABLE 3.1. **EPA** coalition game

Coalition	Gain
ABCD	140
ABC	108
ABD	96
ACD	84
AB	24
other	0

they want to cooperate. For example, the player order BADC represents player B choosing to cooperate first, player A choosing to cooperate second, player D choosing to cooperate third, and player C choosing to cooperate last. A player's marginal contribution is the additional gain that player generates by choosing to cooperate. For the player order BADC, player B alone has a gain of 0, and so generates 0 additional gain. The pair BA has a gain of 24, and so player A generates $24 - 0 = 24$ additional gain. The triple BAD has a gain of 96, and so player D generates $96 - 24 = 72$ additional gain. Finally, BADC has a gain of 140, and so player C generates $140 - 96 = 44$ additional gain. These marginal contributions are recorded in Table 3.2 in the row for the BADC order.

TABLE 3.2. Shapley payoffs for **EPA**

Order	Marginal Contribution			
	A	B	C	D
ABCD	0	$24-0=24$	$108-24=84$	$140-108=32$
ABDC	0	$24-0=24$	$140-96=44$	$96-24=72$
ACBD	0	$108-0=108$	$0-0=0$	$140-108=32$
ACDB	0	$140-84=56$	$0-0=0$	$84-0=84$
ADBC	0	$96-0=96$	$140-96=44$	$0-0=0$
ADCB	0	$140-84=56$	$84-0=84$	$0-0=0$
BACD	$24-0=24$	0	$108-24=84$	$140-108=32$
BADC	$24-0=24$	0	$140-96=44$	$96-24=72$
BCAD	$108-0=108$	0	$0-0=0$	$140-108=32$
BCDA	$140-0=140$	0	$0-0=0$	$0-0=0$
BDAC	$96-0=96$	0	$140-96=44$	$0-0=0$
BDCA	$140-0=140$	0	$0-0=0$	$0-0=0$
CABD	$0-0=0$	$108-0=108$	0	$140-108=32$
CADB	$0-0=0$	$140-84=56$	0	$84-0=84$
CBAD	$108-0=108$	$0-0=0$	0	$140-108=32$
CBDA	$140-0=140$	$0-0=0$	0	$0-0=0$
CDAB	$84-0=84$	$140-84=56$	0	$0-0=0$
CDBA	$140-0=140$	$0-0=0$	0	$0-0=0$
DABC	$0-0=0$	$96-0=96$	$140-96=44$	0
DACB	$0-0=0$	$140-84=56$	$84-0=84$	0
DBAC	$96-0=96$	$0-0=0$	$140-96=44$	0
DBCA	$140-0=140$	$0-0=0$	$0-0=0$	0
DCAB	$84-0=84$	$140-84=56$	$0-0=0$	0
DCBA	$140-0=140$	$0-0=0$	$0-0=0$	0
Average	$1464/24=61$	$792/24=33$	$600/24=25$	$504/24=21$

Table 3.2 includes all 24 ways to order the four players. The marginal contributions for each player have been summed and then averaged.

A marginal contribution formalizes the notion of how much a player adds by joining a coalition. It would not seem fair to specify a particular ordering of the players, but by averaging over all possible orders, each player is being treated in the same manner. The differences in player payoffs occur because of the real differences in how much each player contributes.

While the argument in the preceding paragraph may convince some that the Shapley method is fair, similar *ad hoc* arguments could be written for each of the five methods described in the previous section. We need to be more precise in our description of fairness and then determine whether the Shapley or any other method satisfies our description of fairness.

Shapley Properties

So far in this chapter, we have described six allocation methods, each with some *ad hoc* and informal arguments justifying their fairness. We now take a different approach. Rather than describing a method, we will instead describe fairness properties and then use those fairness properties to determine the method used.

We want to investigate properties that most people would agree describe aspects of fairness. In order to do this, we will not focus on a complex game like **EPA**, for which there is plenty of disagreement. Instead, we will look at fairly simple games for which there is a greater chance for agreement. A very simple coalition game is one in which gain can occur only when all players cooperate, as shown in Table 3.3.

TABLE 3.3. **Very Simple** coalition game

Coalition	Gain
ABCD	100
other	0

This is a scenario in which few disagree: each player should receive 25, which we denote by $(25, 25, 25, 25)$. But can we articulate why we consider this to be the uniquely fair allocation? One way to say it is that (1) we have allocated all of the 100 that is available to distribute, and (2) since there is nothing to distinguish the players, each should receive the same amount. Item (1) is the conclusion of the Efficient Coalition Game Allocation Theorem. Item (2) we formalize as the unbiased property.

> **Unbiased Property:** An allocation is *unbiased* if players who are distinguishable only by their names are allocated the same payoff. An allocation method is *unbiased* if the method always selects unbiased allocations.

The efficient and unbiased properties seem to be minimal requirements for fairness. It would be unfair to not allocate all available gain or to favor one player over another simply because of their names. Let's now make sure that these properties alone determine the allocation for **Very Simple**.

Suppose that (a, b, c, d) is the allocation for the **Very Simple** coalition game produced by an efficient and unbiased allocation method. Efficient implies $a+b+c+d = 100$. Unbiased implies $a = b = c = d$. Substituting the unbiased equalities into the efficient equation, we obtain

$$\begin{aligned} d + d + d + d &= 100, \\ 4d &= 100, \\ d &= 25. \end{aligned}$$

Hence, the allocation should be $(25, 25, 25, 25)$. Our formal argument replicates our informal reasoning.

Next consider the only slightly more complicated coalition game shown in Table 3.4. One way to describe this game is that gain can occur only when the players A,

TABLE 3.4. **Somewhat Simple** coalition game

Coalition	Gain
ABCD	150
ABC	150
other	0

B, and C are in the coalition, and player D never contributes to or detracts from gains. Again, most agree that the uniquely fair allocation is $(50, 50, 50, 0)$: since D never contributes to or detracts from gains, he should receive 0, and since the other three players are indistinguishable except for their names, they should split the 150 available equally. More formally, we add a third allocation method fairness property.

Subsidy Free Property: An allocation is *subsidy free* if players that never contribute to or detract from gains are allocated zero. An allocation method is *subsidy free* if the method always selects subsidy free allocations.

Suppose (a, b, c, d) is the allocation for the **Somewhat Simple** coalition game produced by an efficient, unbiased, and subsidy free allocation method. Efficient implies $a+b+c+d = 150$. Unbiased implies $a = b = c$. Subsidy free implies $d = 0$. Substituting the unbiased and subsidy free equalities into the efficient equation, we obtain

$$\begin{aligned} c+c+c+0 &= 150, \\ 3c &= 150, \\ c &= 50. \end{aligned}$$

Hence, the allocation should be $(50, 50, 50, 0)$. Again, our formal argument replicates our informal reasoning.

Consider now the even more complicated coalition game shown in Table 3.5.

TABLE 3.5. **Not So Simple** coalition game

Coalition	Gain
ABCD	250
ABC	150
other	0

Suppose that (a, b, c, d) is the allocation for the **Not So Simple** coalition game produced by an efficient, unbiased, and subsidy free allocation method. Efficient implies $a + b + c + d = 250$. Unbiased implies $a = b = c$. Unfortunately, player D contributes to one coalition: the gain of ABCD is more than ABC. So, we cannot make use of the subsidy free property.

After careful thought, some people will have the crucial insight that the **Not So Simple** coalition game is a sum of the two earlier games, as shown in Table 3.6.

TABLE 3.6. **Not So Simple** as a sum of two other games

Coalition	**Very Simple** Gain		**Somewhat Simple** Gain		**Not So Simple** Gain
ABCD	100	+	150	=	250
ABC	0	+	150	=	150
other	0	+	0	=	0

Why should this matter? If players plan to play multiple coalition games, it may be useful to combine them into one coalition game. For example, four cities may plan to cooperate on sewage treatment improvements, road repair, library book acquisition, and law enforcement. The cities, for accounting purposes, may wish to keep these games separate. But they may also wish to combine them for the purpose of arriving at a fair allocation of the savings. The savings allocation obtained for the separate games should sum to the savings allocation for the combined game. Otherwise, the cities will also have to negotiate over whether to combine or separate projects on paper as well as negotiate the allocations for each of the games. The accounting procedure used should not have an effect on what allocation is considered fair. This discussion requires that we add a fourth allocation method fairness property:

> **Additive Property:** An allocation method for coalition games is *additive* if whenever a coalition game is the sum or difference of other coalition games, the allocation for the original coalition game is the corresponding sum or difference of the allocations of the other coalition games.

Suppose our allocation method is efficient, unbiased, subsidy free, and additive. As we have already seen, the first three properties select the allocations for **Very Simple** and **Somewhat Simple**. Since **Not So Simple** is the sum of **Very Simple** and **Somewhat Simple**, additive implies that the allocation for **Not So Simple** is the sum of the allocations of **Very Simple** and **Somewhat Simple**. The computation is summarized in Table 3.7.

TABLE 3.7. Payoffs for **Not So Simple**

Coalition	**Very Simple** Gain		**Somewhat Simple** Gain		**Not So Simple** Gain
ABCD	100	+	150	=	250
ABC	0	+	150	=	150
other	0	+	0	=	0
Player	**Very Simple** Allocation		**Somewhat Simple** Allocation		**Not So Simple** Allocation
A	25	+	50	=	75
B	25	+	50	=	75
C	25	+	50	=	75
D	25	+	0	=	25

We have argued, with fairly simple and compelling examples, that a fair allocation method should be efficient, unbiased, subsidy free, and additive. In 1953, Lloyd Shapley proved the following theorem [**60**].

> **Shapley Characterization Theorem:** *The Shapley method is the only allocation method that is efficient, unbiased, subsidy free, and additive.*

PROOF. We illustrate Shapley's proof with the **EPA** game. Suppose some method, which we name the X method, is efficient, unbiased, subsidy free, and additive. We will show that the X method produces the same allocation as the Shapley method in **EPA**.

The idea is to break apart the game into a sum and difference of games whose allocations can be selected by the efficient, unbiased, and subsidy free properties. Games that work well are ones similar to **Very Simple** and **Somewhat Simple**, in which there is a set of players that generates all gain and the remaining players (if any) never contribute to or detract from gain. The latter players are allocated zero and the former players equally split the overall savings. It is best to start with the smallest coalition that produces a positive gain (AB in **EPA**) and move toward larger coalitions. So, Table 3.8 was built from left to right. In the following paragraphs, we explain how we chose these games.

TABLE 3.8. Shapley payoffs for **EPA**

Coalition	**G1**		**G2**		**G3**		**G4**		**G5**		**EPA**
ABCD	24	+	84	+	72	+	84	−	124	=	140
ABC	24	+	0	+	0	+	84	−	0	=	108
ABD	24	+	0	+	72	+	0	−	0	=	96
ACD	0	+	84	+	0	+	0	−	0	=	84
AB	24	+	0	+	0	+	0	−	0	=	24
anything else	0	+	0	+	0	+	0	−	0	=	0
Player	A1		A2		A3		A4		A5		**EPA**
A	12	+	28	+	24	+	28	−	31	=	61
B	12	+	0	+	24	+	28	−	31	=	33
C	0	+	28	+	0	+	28	−	31	=	25
D	0	+	28	+	24	+	0	−	31	=	21

The first game (labeled **G1**) was chosen to make sure that the gain of the pair AB would be correct in the sum: so the gain to AB was set to 24 and AB was chosen to be the set of players that generates all gain. In order to have C and D neither contribute to nor detract from the gain of any coalition, all coalitions containing A and B must have the same gain (so the gains to ABC, ABD, and ABCD were set to 24) and all other coalitions should have a zero gain. With the game defined, the allocation (labeled A1) could be determined. By the subsidy free property, C and D each receive 0. By the efficient and unbiased properties, A and B split the 24 available.

Game **G2** was chosen to make sure that the gain of the triple ACD would be correct in the sum: so the gain to ACD was set to 84. In order to have the remaining player, B, neither contribute to nor detract from the gain of any coalition, the gain of ABCD was also set to 84 and the gains to all other coalitions were set to zero. With the game defined, the allocation (labeled A2) could be determined. By the subsidy free property, B receives 0. By the efficient and unbiased properties, A, C, and D split the 84 available.

Game **G3** was chosen to make sure that the gain of the triple ABD would be correct in the sum: since game **G1** already has a gain of 24 for ABD, the gain to ABD in **G3** was set to $96 - 24 = 72$. In order to have C neither contribute to nor detract from the gain of any coalition, the gain of ABCD was also set to 72 and the gains to all other coalitions were set to zero. With the game defined, the allocation (labeled A3) could be determined. By the subsidy free property, C receives 0. By the efficient and unbiased properties, A, B, and D split the 72 available.

Game **G4** was chosen to make sure that the gain of the triple ABC would be correct in the sum: since game **G1** already has a gain of 24 for ABC, the gain to ABC in **G4** was set to $108 - 24 = 84$. In order to have D neither contribute to nor detract from the gain of any coalition, the gain of ABCD was also set to 84 and the gains to all other coalitions were set to zero. With the game defined, the allocation (labeled A4) could be determined. By the subsidy free property, D receives 0. By the efficient and unbiased properties, A, B, and C split the 84 available.

At this point the gain of ABCD from the first four games summed to $24 + 84 + 72 + 84 = 264$. So, game **G5** was chosen so that the five games would sum to the original game: so the gain to ABCD was set to $264 - 140 = 124$ and the gains to all other coalitions were set to zero. With the game defined, the allocation (labeled A5) could be determined. By the efficient and unbiased properties, the four players split what is available.

Finally, the allocation to the original game was obtained by carrying out the indicated sums and difference (since the X method is additive). The resulting allocation $(61, 33, 25, 21)$ is the same allocation as produced by the Shapley method. □

The four negotiators now have an argument to take back to their respective government officials and citizens. A fair allocation method should (1) always distribute all of the available gain, (2) depend only on coalition gains and not the names of the players, (3) neither reward nor punish a player who neither contributes to nor detracts from the gains of coalitions, and (4) not depend on the accounting procedure used to represent allocation problems. And so the Shapley Characterization Theorem says that the Shapley method must be used.

New Product

Let's apply these ideas to the **New Product** coalition game with the gains given in Table 3.9.

If the companies decide to allocate to use the Shapley method, then the allocation can be computed from Table 3.10.

TABLE 3.9. **New Product** coalition game

Coalition	Gain
ABC	50
AB	38
AC	38
BC	18
A	2
B	0
C	0

TABLE 3.10. Marginal contributions for **New Product**

Order	Marginal Contribution		
	A	B	C
ABC	2	36	12
ACB	2	12	36
BAC	38	0	12
BCA	32	0	18
CAB	38	12	0
CBA	32	18	0
Average	24	13	13

If, instead, the companies decide that the allocation method used should be efficient, unbiased, subsidy free, and additive, then the allocation can be obtained directly from the calculations given in Table 3.11.

TABLE 3.11. **New Product** payoffs directly from the properties

Coalition	Gains										
ABC	2	+	18	+	36	+	36	−	42	=	50
AB	2	+	0	+	0	+	36	−	0	=	38
AC	2	+	0	+	36	+	0	−	0	=	38
BC	0	+	18	+	0	+	0	−	0	=	18
A	2	+	0	+	0	+	0	−	0	=	2
anything else	0	+	0	+	0	+	0	−	0	=	0
Player	Allocations										
A	2	+	0	+	18	+	18	−	14	=	24
B	0	+	9	+	0	+	18	−	14	=	13
C	0	+	9	+	18	+	0	−	14	=	13

Notice that the two computational approaches lead to the same allocation: \$24 million to Adonis and \$13 million each to Beautex and Celestron. Of course, the Shapley Characterization Theorem told us that this would be the case.

Observe that Adonis and Beautex together are allocated \$37 million, but they could have obtained \$38 million on their own. Also Adonis and Celestron together are allocated \$37 million, but they could have obtained \$38 million on their own. So, the Shapley allocation is not rational. Had we allocated \$26 million to Adonis and \$12 million each to Beautex and Celestron, then every coalition would have been allocated at least as much as they could have on their own. So, there is at least one rational allocation. Hence, the Shapley method is not rational.

In conclusion, the Shapley method has a fair sounding description as the average marginal contribution over all player orders. The Shapley method satisfies four properties indicative of fairness. If your notion of fairness includes the efficient, unbiased, subsidy free, and additive properties, then you must use the Shapley method. Unfortunately, the Shapley method is not rational: some coalition may receive less than they would be able to obtain on their own even though some allocation is available for which every coalition would receive at least what they are able to obtain on their own.

Exercises

(1) **Apartment Sharing**. Consider the game described in exercise 1 of section 7.2 and summarized in Table 3.12.

TABLE 3.12. **Apartment Sharing** coalition game

Coalition	Gain
ABC	120
AB	90
AC	60
BC	30
A	0
B	0
C	0

(a) Use the Shapley method to produce an allocation.
(b) Use the computational approach suggested by the proof of the Shapley Characterization Theorem to produce an allocation.
(c) What does the Shapley Characterization Theorem tell you about your answers to parts (a) and (b)?
(d) Do you agree that a fair allocation has been found?

(2) **Veto Power**. Consider the game described in exercise 2 of section 7.2 and summarized in Table 3.13.
(a) Use the Shapley method to produce an allocation.
(b) Use the computational approach suggested by the proof of the Shapley Characterization Theorem to produce an allocation.
(c) What does the Shapley Characterization Theorem tell you about your answers to parts (a) and (b)?
(d) Do you agree that a fair allocation has been found?

TABLE 3.13. **Veto Power** coalition game

Coalition	Gain
ABC	90
AB	90
AC	90
BC	0
A	0
B	0
C	0

(3) **Loner Power.** Consider the game described in exercise 3 of section 7.2 and summarized in Table 3.14.

TABLE 3.14. **Loner Power** coalition game

Coalition	Gain
RST	180
RS	180
RT	0
ST	0
R	0
S	0
T	0

(a) Use the Shapley method to produce an allocation.
(b) Use the computational approach suggested by the proof of the Shapley Characterization Theorem to produce an allocation.
(c) What does the Shapley Characterization Theorem tell you about your answers to parts (a) and (b)?
(d) Do you agree that a fair allocation has been found?

(4) **Not Finding Rational**. Consider the game described in exercise 4 of section 7.2 and summarized in Table 3.15.

TABLE 3.15. **Not Finding Rational** coalition game

Coalition	Gain
ABCD	432
ABC	384
ABD	384
ACD	96
BCD	96
CD	84
other	0

(a) Use the Shapley method to produce an allocation.
(b) Use the computational approach suggested by the proof of the Shapley Characterization Theorem to produce an allocation.

(c) What does the Shapley Characterization Theorem tell you about your answers to parts (a) and (b)?
(d) Do you agree that a fair allocation has been found?

(5) **TVA**. The Tennessee Valley Authority (TVA) plans to build a dam in order to provide electricity, flood control, and recreation facilities. By planning the dam for all three purposes, there will be a savings of $120 million over what it would have cost to provide for each of the three purposes separately. If the TVA had left out recreation, the savings would have been only $100 million. If the TVA had left out flood control, the savings would have been only $80 million. If the TVA had left out electricity, the savings would have been only $60 million.
(a) Model this situation as a coalition game.
(b) Use either computational approach for the Shapley method to produce an allocation.
(c) Do you agree that a fair allocation has been found?

(6) **EPA After Cost Overrun**. Suppose Avon, Barport, Claron, and Delmont agree to allocate savings in accordance with the Shapley method. Also suppose that there is a cost overrun that changes the savings generated by the four cities from $140 million to $120 million. Suppose that there would not have been a cost overrun for any other coalition.
(a) Model this situation as a coalition game.
(b) Use either computational approach for the Shapley method to produce an allocation.
(c) Do you agree that a fair allocation has been found?

(7) In the Shapley Characterization Theorem, we argued that if a method is efficient, unbiased, subsidy free, and additive, then that method must be the Shapley method. Prove the converse; that is, prove that the Shapley method is efficient, unbiased, subsidy free, and additive.

4. The Nucleolus Method

Two serious objections can be made to the fairness of the Shapley method. First, the Shapley method is not rational: sometimes a coalition may receive less from the Shapley allocation than they could obtain on their own. This was illustrated with **New Product** and some of the exercises in the previous section. Second, while the additive property seems reasonable, the games used in conjunction with the additive property seem somewhat artificial. Perhaps it would be more reasonable to compare the original game allocation with allocations obtained in games more naturally related to the original game. Our goal in this section is to describe another allocation method, called the nucleolus, that responds to our two objections.

> By the end of this section, you should be able to calculate the nucleolus, verify the consistency of the nucleolus, and to describe why the nucleolus can be considered fair.

Excess as Happiness

Recall the **EPA Game**, whose coalition gains are repeated in Table 4.1. The Shapley allocation was determined to be $(61, 33, 25, 21)$.

TABLE 4.1. **EPA** coalition game

Coalition	Gain
ABCD	140
ABC	108
ABD	96
ACD	84
AB	24
other	0

While the Shapley method has an economic interpretation (marginal contributions of players), the nucleolus has a political interpretation (actual coalition gain). Suppose that the Shapley allocation $(61, 33, 25, 21)$ was proposed. Coalition ABC could obtain 108 on its own but actually receives $61 + 33 + 25 = 119$. Certainly, coalition ABC should be happy about this. One way to quantify coalition ABC's happiness is the additional amount received over what ABC could obtain on its own: $119 - 108 = 11$. In general, a coalition's happiness with a proposed allocation will be modeled with the difference between what the coalition receives and what the coalition could obtain on its own. We will call this difference the coalition's *excess* with the proposed allocation (because it is the gain the coalition receives in excess of what it could obtain on its own). Table 4.2 shows the calculation of coalition excesses with the Shapley allocation $(61, 33, 25, 21)$. Notice that the last four excesses duplicate the allocations to the individual players.

All coalitions have a positive excess, which means that all coalitions are happy. However, coalition ABC has the smallest excess, and so it is the coalition that is

TABLE 4.2. Excesses for the Shapley allocation of **EPA**

Coalition	Allocation to A		B		C		D	−	Gain	=	Excess
ABC	61	+	33	+	25			−	108	=	11
ABD	61	+	33			+	21	−	96	=	19
ACD	61			+	25	+	21	−	84	=	23
BCD			33	+	25	+	21	−	0	=	79
AB	61	+	33					−	24	=	70
AC	61			+	25			−	0	=	86
AD	61					+	21	−	0	=	82
BC			33	+	25			−	0	=	58
BD			33			+	21	−	0	=	54
CD					25	+	21	−	0	=	46
A	61							−	0	=	61
B			33					−	0	=	33
C					25			−	0	=	25
D							21	−	0	=	21

least happy with the Shapley allocation. Our goal will be to make the least happy coalition as happy as possible. This could be considered a fairness idea: help those who are least well off.

This also has a political interpretation: suppose that the game is being played by voters for a political office. Then the office holder, through his or her decisions and votes, allocates gains to the voters: tax breaks for some, economic benefits for others, and consumer or environmental protection legislation for yet others. Each coalition could be considered a special interest group, and the coalitions with the smallest excess are likely to make the most noise in the media against the office holder. Thus the office holder may want to make decisions and vote in a manner that will make the least happy coalition as happy as possible.

Table 4.3 shows the coalition excesses with six different proposed allocations (the allocations can be recovered by looking at the last four rows of excesses). In the following paragraphs, we explain how we came to choose these six allocations.

In column E1, we arbitrarily started with the Shapley allocation $(61, 33, 25, 21)$. Coalition ABC has the smallest excess. To increase ABC's excess (to make ABC happier), we need to increase the payoffs to ABC. The only possible way to increase the payoffs to ABC is to reduce the payoff to D. As a first experiment, we will shift 3 from D to ABC.

To construct column E2, we shifted 3 from D equally to A, B, and C to obtain the second proposed allocation $(61, 33, 25, 21) + (1, 1, 1, -3) = (62, 34, 26, 18)$. Notice that coalition ABC still has the smallest excess, but the smallest excess has increased from 11 to 14; we are moving in the right direction. At the same time the excesses for coalitions ABD and D have both decreased to 18; we don't like making some coalitions less happy, but it is most important to make the least happy coalition as happy as possible.

TABLE 4.3. Re-allocations for **EPA**

Coalition	Gain	E1	E2	E3	E4	E5	E6
ABC	108	11	14	17	16	16	16
ABD	96	19	18	17	18	22	22
ACD	84	23	22	21	21	21	28
BCD	0	79	78	77	77	73	66
AB	24	70	72	74	74	78	78
AC	0	86	88	90	89	89	96
AD	0	82	80	78	79	83	90
BC	0	58	60	62	61	57	50
BD	0	54	52	50	51	51	44
CD	0	46	44	42	42	38	38
A	0	61	62	63	63	67	74
B	0	33	34	35	35	35	28
C	0	25	26	27	26	22	22
D	0	21	18	15	16	16	16

In column E3, we again increase the excess of ABC by shifting 3 more from D to A, B, and C to obtain the third allocation $(62, 34, 26, 18) + (1, 1, 1, -3) = (63, 35, 27, 15)$. Coalition ABC's excess increased from 14 to 17, but D's excess decreased from 18 to 15. Coalition D now has the smallest excess, and so it is necessary to shift some payoff back to D. How much? Let's make the ABC and D excesses the same, which can be accomplished by shifting 1 to D.

In column E4, we shifted 1 to D from the arbitrarily chosen C. We obtain the allocation $(63, 35, 27, 15)+(0, 0, -1, 1) = (63, 35, 26, 16)$. The smallest excess is now 16. It is not possible to increase the smallest excess any further because to increase ABC's excess above 16 would require a transfer from D to the players in ABC, to increase D's excess above 16 would require a transfer to D from the players in ABC, and it is impossible to both transfer from and to D simultaneously. Although it is impossible to make the smallest excess greater than 16, notice that the third smallest excess with $(63, 35, 26, 16)$, held by coalition ABD, is 18 (the two 16s are considered the first and second smallest excesses). It would be possible to increase ABD's excess, without changing ABC's excess or D's excess, by shifting from C to A and/or B. Again, we don't want C's excess to become smaller than ABD's excess. So, we just shift half of the current difference in excesses $(26 - 18)/2 = 4$ from C.

In column E5, because of the low happiness of ACD, we will transfer 4 from C to A to obtain the fifth allocation $(63, 35, 26, 16) + (4, 0, -4, 0) = (67, 35, 22, 16)$. Notice that it is now impossible to increase both ABD's excess and C's excess any further because that would require that C receive both more and less than 22. Unfortunately, the third smallest excess for $(67, 35, 22, 16)$, as can be seen from E5, is now 21 associated with coalition ACD, not the 22 associated with coalitions ABD and C. Fortunately, ACD's excess can be reduced without changing ABC's, D's, ABD's, and C's excesses, by shifting from B to A. Again, we do not want B's excess to become smaller than ACD's excess. So, we just shift half of the current difference in excesses $(35 - 21)/2 = 7$ from B to A.

Column E6 displays the final allocation $(67, 35, 22, 16)+(7, -7, 0, 0)=(74, 28, 22, 16)$. Because D cannot both give and receive, the smallest excesses, held by ABC and D, cannot be increased. Because C cannot both give and receive, the next smallest excesses, held by ABD and C, cannot be increased. Because B cannot both give and receive, the next smallest excesses, held by ACD and B, cannot be increased. We have now argued that to successively maximize the smallest coalition excesses, the payoffs to D, C, and B must be 16, 22, and 28, respectively. Since we want to allocate 140, player A must receive $140 - (16 + 22 + 28) = 74$.

The efficient allocation $(74, 28, 22, 16)$ successively maximizes the smallest excesses: the least happy coalition is as happy as possible; given that, the second least happy coalition is as happy as possible; given that, the third least happy coalition is as happy as possible; and so forth. This allocation method was first described by David Schmeidler in 1969 [**57**].

> **Nucleolus Method:** The *nucleolus* is the efficient allocation that successively maximizes the smallest excesses.

Notice that our definition of the nucleolus is not in terms of an algorithm (a collection of rules to be carried out step by step). Instead we provide a goal to achieve. For **EPA**, we made a sequence of informed guesses and provided a reasoned argument to verify that the nucleolus had been found. Note that if we had miraculously picked $(74, 28, 22, 16)$, rather than the Shapley allocation, to start with, we would have been done after one paragraph of verification. Different people will make different sequences of informed guesses but should eventually end in the same place. There are formal algorithms for computing the nucleolus that do not involve guessing; however, they involve solving a sequence of linear programs, which is beyond the scope of this book.

New Product

We use our heuristic of informed guessing to find the nucleolus for **New Product**, whose gains are repeated in Table 4.4.

TABLE 4.4. **New Product** coalition game

Coalition	Gain
ABC	50
AB	38
AC	38
BC	18
A	2
B	0
C	0

As can be seen in Table 4.5, we first examine the Shapley allocation $(24, 13, 13)$, and find that coalitions AB and AC have the smallest excesses. A negative excess indicates that the corresponding coalition receives less than it could obtain on its own. In order to increase both excesses, the payoff to A must be increased at an expense to B and C. Since B and C are indistinguishable except by their name, we keep the payoffs to B and C the same.

TABLE 4.5. Adjustments to the excesses for **New Product**

Coalition	Gain	Excess With		
		$(24, 13, 13)$	$(26, 12, 12)$	$(30, 10, 10)$
AB	38	-1	0	2
AC	38	-1	0	2
BC	18	8	6	2
A	2	22	24	28
B	0	13	12	10
C	0	13	12	10

The first change is to take 1 from each of B and C and give 2 to A to obtain $(24, 13, 13) + (2, -1, -1) = (26, 12, 12)$. This increases the first and second smallest excesses from -1 to 0 and lowers the third smallest excess from 8 to 6. Extrapolating these changes, we can see that by shifting 2 more from each of B and C to A, the three smallest excesses should become equal with $(26, 12, 12) + (4, -2, -2) = (30, 10, 10)$.

It is not as clear with this example whether there is a way to increase the smallest excess further; however, this can be handled algebraically. Suppose (a, b, c) is an efficient allocation whose minimum excess is at least 2. In particular, the AB, AC, and BC excesses must be each at least 2, which can be written

$$\begin{aligned} a + b - 38 &\geq 2, \\ a + c - 38 &\geq 2, \\ b + c - 18 &\geq 2. \end{aligned}$$

Summing these three inequalities and using efficiency, we obtain

$$\begin{aligned} 2a + 2b + 2c - 94 &\geq 6, \\ 2(a + b + c) &\geq 100, \\ 2(50) &\geq 100, \\ 100 &\geq 100. \end{aligned}$$

Since 100 is not strictly greater than 100, it must be the case that all previous inequalities must hold with equality; that is,

$$\begin{aligned} a + b - 38 &= 2, \\ a + c - 38 &= 2, \\ b + c - 18 &= 2. \end{aligned}$$

This system of three equations in the three unknowns a, b, and c, has a unique solution, $(a, b, c) = (30, 10, 10)$. This verifies that $(30, 10, 10)$ is the nucleolus.

Two-Player Games

Table 4.6 is the **Arbitrary Two-Person** coalition game. The suggested allocation follows the principle that each player should receive what it can obtain on its own (A receives a and B receives b), and any surplus $(c-a-b)$ should be split equally between the two players. The nucleolus method produces the suggested allocation because the excesses for coalitions A and B are both $\frac{1}{2}(c-a-b)$, and it is impossible to simultaneously increase both excesses.

TABLE 4.6. **Arbitrary Two-Person** coalition game

Coalition	Gain
AB	c
A	a
B	b
Player	**Allocation**
A	$a+\frac{1}{2}(c-a-b)$
B	$b+\frac{1}{2}(c-a-b)$

Nucleolus Properties

The nucleolus method, like the Shapley method, is efficient, unbiased, and subsidy free. Unlike the Shapley method, which is not rational, we have the following theorem.

Nucleolus Rationality Theorem: *The nucleolus method is rational.*

PROOF. If there is a rational allocation, then for that rational allocation, each coalition receives at least as much as it could obtain on their own, and so all excesses (being the difference between what each coalition receives and what it could obtain on its own) are nonnegative. But the nucleolus maximizes the smallest excess, and so all excesses for the nucleolus must also be nonnegative, and thus the nucleolus is a rational allocation. □

Our first objection to the Shapley method was that it is not rational; the previous theorem shows that we cannot raise the same objection to the nucleolus. Our second objection to the Shapley method was that the games used in conjunction with the additive property seemed somewhat artificial and unrelated to the original game. We will now introduce a new property for which the related games arise naturally from the original game.

Suppose that the Shapley allocation $(61, 33, 25, 21)$ has been proposed for **EPA**. Also suppose that everyone agrees that Claron and Delmont should receive their \$25 million and \$21 million in savings, respectively, but Avon and Barport do not agree on how to share the remaining \$94 million. If the Shapley method is our definition of fair, then the Shapley method applied to the reduced game involving only Avon and Barport should again yield the split: \$61 million for Avon and \$33 million for Barport. What reduced game? Clearly, cooperation has led to $\$61 + \$33 = \$94$ million in savings. Certainly, each city can obtain a savings of 0 without any cooperation. But Avon has another option: cooperation with Claron and Delmont leads to a savings of \$84 million, Claron is satisfied with \$25 million, Delmont is satisfied with \$21 million, and so Avon could save $\$84 - \$25 - \$21 = \$38$ million without the cooperation of Barport. Barport cannot obtain a similar (greater than zero) savings without the help of Avon. Thus, the subgame involving Avon and Barport, along with the Shapley allocation computation is given in Table 4.7.

TABLE 4.7. Shapley allocation for the reduced **EPA** coalition game

Coalition	Gains				
AB	38	+	56	=	94
A	38	+	0	=	38
B	0	+	0	=	0
Player	**Allocations**				
A	38	+	28	=	66
B	0	+	28	=	28

The Shapley allocation of the reduced game (\$68 million for Avon and \$28 for Barport) is not the same as their Shapley allocations in the original game (\$61 and \$33 millions). We say that the Shapley method is not consistent on reduced games, a notion that we formalize in the following definition.

Consistent Property: An allocation method is *consistent* if the allocation for reduced games is the same as the allocation for the original game. The reduced game on a coalition S is played among the players in S, the gain of S is the sum of the amounts allocated in the original game to the players in S, and the gain of each subcoalition R is the maximum obtainable by having R join with zero or more players outside of S and giving those outside players what they were allocated in the original game.

Let's illustrate this property by showing that the nucleolus method is consistent for **New Product**. We previously showed that $(30, 10, 10)$ is the nucleolus. Consider the reduced game on $S = \text{AB}$. This corresponds to everyone being satisfied with C's payoff, but A and B being dissatisfied with their own; that is, A and B wish to renegotiate their 30/10 split. In this game, the gain to the coalition AB is $30 + 10 = 40$. If A does not form a coalition with B, certainly A could stay by itself and obtain the 2 A would have obtained in the original game. But A could instead form a coalition with C and obtain 28, which is the difference between what AC obtains in the original game, 38, and the 10 that is considered satisfactory by C. Since A can choose either approach, the gain for coalition A is $\max\{2, 38-10\} = 28$. Similarly, the gain for coalition B is $\max\{0, 18 - 10\} = 8$. The resulting game is shown in Table 4.8.

TABLE 4.8. The reduced **New Product** coalition game on AB

Coalition	Gain
AB	$30 + 10 = 40$
A	$\max\{2, 38 - 10\} = 28$
B	$\max\{0, 18 - 10\} = 8$

So the payoffs to A and B are initially 28 and 8, respectively, and the surplus of $40 - 28 - 8 = 4$ is then split evenly, resulting in the payoffs of 30 and 10 to A and B, respectively. These payoffs match these players' payoffs in the original game.

The verification for AC is the same with B replaced by C. The reduced game on BC is shown in Table 4.9.

TABLE 4.9. The reduced **New Product** coalition game on BC

Coalition	Gain
BC	$10 + 10 = 20$
B	$\max\{0, 38 - 30\} = 8$
C	$\max\{0, 38 - 30\} = 8$

So the payoffs to B and C are both initially 8, and the surplus of $20 - 8 - 8 = 4$ is then split evenly resulting in the payoffs of 10 to each of B and C. These payoffs match these players' payoffs in the original game. This shows that the nucleolus is consistent on **New Product**.

The nucleolus method, unlike the Shapley method, is consistent. Since the nucleolus method is efficient, unbiased, subsidy free, rational, and consistent, perhaps the nucleolus method, just like the Shapley method, may be characterized with fairness properties. A. I. Sobolev provided such a characterization theorem in 1975 [**61**], which was strengthened by Orshan in 1993 [**42**]:

Nucleolus Characterization Theorem: *The nucleolus method is the only allocation method that is efficient, unbiased, scale invariant, and consistent.*

Before illustrating the proof, note that Sobolev and Orshan used one property we have not previously defined. This property for coalition games is similar to the property we defined for bargaining games.

Scale Invariant Property: An allocation method is *scale invariant* if (1) multiplying all gains by the same number M results in the method multiplying all payoffs by M, and (2) if changing all gains containing a player i by the same amount B results in the method changing player i's payoff by B.

PROOF. We illustrate Sobolev's proof with the **EPA** game. Suppose some method, which we name the X method, is efficient, unbiased, scale invariant, and consistent. We will show that the X method produces the same allocation as the nucleolus method in **EPA**.

Suppose the X method produces the allocation (a, b, c, d) for **EPA**. The reduced game on AB and the corresponding allocation is given in Table 4.10.

TABLE 4.10. The X Method allocation on the reduced **EPA** coalition game on AB

Coalition	Gain
AB	$a+b$
A	$\max\{0, 0-c, 0-d, 84-c-d\} = 84-c-d$
B	$\max\{0, 0-c, 0-d, 0-c-d\} = 0$
Player	**Allocations**
A	$(84-c-d) + \frac{1}{2}(a+b-(84-c-d)-0)$
B	$0 + \frac{1}{2}(a+b-(84-c-d)-0)$

The allocation to player B can be simplified to

$$\begin{aligned} \frac{1}{2}(a+b-84+c+d) &= \frac{1}{2}(a+b+c+d-84) \\ &= \frac{1}{2}(140-84) \\ &= 28, \end{aligned}$$

where the second equality follows from the efficient property. For the allocation method to be consistent, it must be that this allocation to player B in the reduced game is the same as the allocation to player B in the original game; that is,

$$b = 28.$$

The reduced game on AC and the corresponding allocation is given in Table 4.11.

TABLE 4.11. The X Method allocation on the reduced **EPA** coalition game on AC

Coalition	Gain
AC	$a+c$
A	$96-b-d$
C	0
Player	**Allocations**
A	$(96-b-d) + \frac{1}{2}(a+c-(96-b-d)-0)$
C	$0 + \frac{1}{2}(a+c-(96-b-d)-0)$

With the efficient property, the allocation to player C can be simplified (in a manner similar to the previous game) to 22. For the allocation method to be consistent, it must be that this allocation to player C in the reduced game is the same as the allocation to player C in the original game; that is,

$$c = 22.$$

The reduced game on AD and the corresponding allocation is given in Table 4.12.

TABLE 4.12. The X Method allocation on the reduced **EPA** coalition game on AD

Coalition	Gain
AD	$a + d$
A	$108 - b - c$
D	0
Player	**Allocations**
A	$(108 - b - c) + \frac{1}{2}(a + d - (108 - b - c) - 0)$
D	$0 + \frac{1}{2}(a + d - (108 - b - c) - 0)$

With the efficient property, the allocation to player D can be simplified to 16. For the allocation method to be consistent, it must be that this allocation to player D in the reduced game is the same as the allocation to player D in the original game:

$$d = 16.$$

We now know the allocation to three of the four players, and so the allocation to player A can be obtained from the efficient property:

$$a = 140 - b - c - d = 74.$$

Thus, the X method produces the allocation $(74, 28, 22, 16)$, which is also produced by the nucleolus method. □

To conclude this section, there are four reasons to use the nucleolus method. First, the nucleolus has a fair sounding description as the efficient allocation that makes the least happy coalitions as happy as possible (at least if happiness can be modeled by the excess). Second, the nucleolus is rational: no coalition receives less than they would be able to obtain on their own (at least if any such allocation is available). Third, the nucleolus is consistent: there is no incentive for coalitions to reallocate amongst themselves. Finally, if your notion of fairness includes the efficient, unbiased, scale invariant, and consistent properties, then the Nucleolus Characterization Theorem says that you should use the nucleolus.

Exercises

(1) **Apartment Sharing**. Consider the game described in exercise 1 of section 7.2 and summarized again in Table 4.13.

TABLE 4.13. **Apartment Sharing** coalition game

Coalition	Gain
ABC	120
AB	90
AC	60
BC	30
A	0
B	0
C	0

(a) Find the nucleolus.
(b) Find the reduced game on AB using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(c) Find the reduced game on AC using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(d) Find the reduced game on BC using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(e) Compare the Shapley allocation obtained in exercise 1 of section 7.3 with the nucleolus. Which one would you consider more fair? Why?

(2) **Veto Power**. Consider the game described in exercise 2 of section 7.2 and summarized again in Table 4.14.

TABLE 4.14. **Veto Power** coalition game

Coalition	Gain
ABC	90
AB	90
AC	90
BC	0
A	0
B	0
C	0

(a) Find the nucleolus.
(b) Find the reduced game on AB using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(c) Find the reduced game on AC using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(d) Find the reduced game on BC using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(e) Compare the Shapley allocation obtained in exercise 2 of section 7.3 with the nucleolus. Which one would you consider more fair? Why?

(3) **Loner Power.** Consider the game described in exercise 3 of section 7.2 and summarized again in Table 4.15.

TABLE 4.15. **Loner Power** coalition game

Coalition	Gain
RST	180
RS	180
RT	0
ST	0
R	0
S	0
T	0

(a) Find the nucleolus.
(b) Find the reduced game on RS using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(c) Find the reduced game on RT using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(d) Find the reduced game on ST using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(e) Compare the Shapley allocation obtained in exercise 3 of section 7.3 with the nucleolus. Which one would you consider more fair? Why?

(4) **Not Finding Rational**. Consider the game described in exercise 4 of section 7.2 and summarized again in Table 4.16.

TABLE 4.16. **Not Finding Rational** coalition game

Coalition	Gain
ABCD	432
ABC	384
ABD	384
ACD	96
BCD	96
CD	84
other	0

(a) Find the nucleolus.
(b) Find the reduced game on AC using the nucleolus. Show that the nucleolus method is consistent on this reduced game.
(c) Compare the Shapley allocation obtained in exercise 4 of section 7.3 with the nucleolus. Which one would you consider more fair? Why?

(5) **TVA**. Consider the game described in exercise 5 of section 7.3.
(a) Find the nucleolus.
(b) Compare the Shapley allocation obtained in exercise 5 of section 7.3 with the nucleolus. Which one would you consider more fair? Why?

(6) **EPA After Cost Overrun**. Consider the game described in exercise 6 of section 7.3.

(a) Find the nucleolus.
(b) Compare the Shapley allocation obtained in exercise 6 of section 7.3 with the nucleolus. Which one would you consider more fair? Why?

5. You Can't Always Get What You Want

The Shapley method is efficient, unbiased, subsidy free, scale invariant, and additive. The nucleolus method is efficient, unbiased, subsidy free, scale invariant, consistent, and rational. Unfortunately, the Shapley method is neither consistent nor rational, and the nucleolus method is not additive.

Is there a method that satisfies all of these fairness properties? We have already seen that the answer is "no". The Shapley method is the only allocation method that is efficient, unbiased, subsidy free, and additive. So, if any method were to satisfy all of the previously listed fairness properties, then the Shapley value would have to be that method. But we know that the Shapley method is neither consistent nor rational.

Negotiators or arbitrators must decide what properties of fairness are most important to keep and which ones may be safely ignored. "What is fair?" is a question that has not been answered. However, we have raised the level of discussion from *ad hoc* arguments about allocations (Is $(61, 33, 25, 21)$ or $(74, 28, 22, 16)$ more fair?) or methods (Is it more fair to focus on marginal contributions or coalition excesses?) to arguments about principles (Does fairness require allocation methods to be additive or consistent?) The mathematical theorems show us that we must think carefully about what we mean by fairness, and they also help to delineate possible fairness properties.

> By the end of this section, you will know one last fairness property, be able to summarize the fairness properties of the Shapley and nucleolus methods, and to know how easy it is to ask for fairness to mean too much.

We reexamine the situation that opened this chapter. Avon, Barport, Claron, and Delmont finally agree upon either the Shapley or nucleolus method. As often happens in the real world, there is a cost overrun of \$20 million. If we assume that there would not have been any cost overruns had other coalitions of players completed the work themselves, then the coalition game is the same as **EPA** except that the gain of coalition ABCD will change from 140 to 120, as shown in Table 5.1.

TABLE 5.1. **EPA After Cost Overrun** coalition game

Coalition	ABCD	ABC	ABD	ACD	AB	other
Gain	120	108	96	84	24	0

It is interesting to compare the allocations from before and after the cost overrun (the calculations for the **EPA After Cost Overrun** game were performed in previous section exercises). These are shown in Table 5.2.

With the Shapley method, each city shares in the cost overrun equally: each city's payoff is reduced by \$5 million. Unfortunately, it now would have been better for

TABLE 5.2. Comparison of allocations

Method and Game	A	B	C	D
Shapley for **EPA**	61	33	25	21
Shapley for **EPA After Cost Overrun**	56	28	20	16
Nucleolus for **EPA**	74	28	22	16
Nucleolus for **EPA After Cost Overrun**	90	18	12	0

ABC to have done the project on their own with a savings of \$108 million rather than the $56 + 28 + 20 =$ \$104 savings allocated. Once again we are reminded that the Shapley method is not rational.

Nonetheless, it is the nucleolus method that seems far stranger. The \$20 million cost overrun has resulted in Avon receiving greater savings! If the cities planned to use the nucleolus method, there was incentive for Avon to purposely damage the project in some way so as to cause a cost overrun. This is enormously perverse! Changes in the coalition game gains should result in reasonable changes in the allocations. To avoid this perversity, a fair allocation method should have the following property.

Coalition Monotone Property: An allocation method is *coalition monotone* if increases (decreases) in a single coalition's gain do not result in a decrease (increase) in the payoff for any player who is in the coalition.

Of course, we already know that no allocation method could possess all of the properties we have considered so far. What is surprising is how few properties are needed to obtain mutual incompatibility.

Coalition Game Impossibility Theorem: *There is no allocation method for coalition games with four or more players that is efficient, rational, and coalition monotone.*

This impossibility was shown for five or more players by H. Peyton Young [**73**]. David Housman and Lori Clark [**23**] proved the impossibility for four or more players and showed that there are many efficient, rational, and monotone allocation methods (the nucleolus method being one) when restricted to three-player games.

PROOF. The key to the proof is to analyze the five four-player coalition games defined in Table 5.3 (momentarily ignore the allocation information). Suppose that method X is efficient, rational, and coalition monotone, and suppose that the allocation produced by the X method for the **AA** coalition game is (a, b, c, d). Since (a, b, c, d) is efficient and the gain for the coalition containing all players is 2,

$$a + b + c + d = 2.$$

Note that $(0, 0, 1, 1)$ is a rational allocation for **AA**, and so the X method must produce a rational allocation, that is, (a, b, c, d) must be rational. Since (a, b, c, d) is rational and the gain for coalition ACD is 2,

$$a + c + d \geq 2,$$

TABLE 5.3. Five four-player coalition games

Coalition	**AA** Gain	**BB** Gain	**CC** Gain	**DD** Gain	**ZZ** Gain
ABCD	2	2	2	2	2
ACD	2	1	1	1	1
BCD	1	2	1	1	1
ABC	1	1	2	1	1
ABD	1	1	1	2	1
BD	1	1	1	1	1
AC	1	1	1	1	1
AD	1	1	1	1	1
BC	1	1	1	1	1
other	0	0	0	0	0
Player	Allocations				
A	≤ 0				≤ 0
B		≤ 0			≤ 0
C			≤ 0		≤ 0
D				≤ 0	≤ 0

and since the gain for coalition B is 0,

$$b \geq 0.$$

Using the efficient equality, rewritten as $a+c+d = 2-b$, in the inequality $a+c+d \geq 2$, we obtain $2 - b \geq 2$, which implies

$$b \leq 0.$$

Combining $b \geq 0$ and $b \leq 0$, we obtain

$$b = 0.$$

Since (a, b, c, d) is rational, $b + d \geq 1$ and $b + c \geq 1$. Substituting $b = 0$ into these inequalities, we obtain the following.

$$\begin{aligned} d &\geq 1, \\ c &\geq 1. \end{aligned}$$

Rewriting the efficiency equation and using previous results, we next obtain the following.

$$\begin{aligned} a &= 2 - b - c - d, \\ a &\leq 2 - 0 - 1 - 1, \\ a &\leq 0. \end{aligned}$$

This result is included in the allocation information for **AA** in the table.

Now notice that the only difference between **AA** and **ZZ** is that the gain for ACD is greater in **AA** than it is in **ZZ**. Since the X method is coalition monotone, the payoff to A in **AA** must be larger than the payoff to A in **ZZ**. But since the payoff to A in **AA** is no greater than zero, the payoff to A in **ZZ** must be no greater than zero. This result is included in the allocation information for **ZZ** in the table.

Analogous arguments can be used to show that player B's payoff in **BB** and **ZZ** must be no greater than zero, player C's payoff in **CC** and **ZZ** must be no greater than zero, and player D's payoff in **DD** and **ZZ** must be no greater than zero. These results are included in the allocation information for the appropriate games in the table. We can then conclude that the sum of the player payoffs in **ZZ** is no greater than zero. By the efficient property, the sum of the player payoffs in **ZZ** equals 2. Since a sum cannot be both no greater than zero and equal to 2, this contradiction shows that the X method could not have been efficient, rational, and coalition monotone. □

Early in the chapter, it was argued that fair allocation methods should be efficient and rational. In this section, we argued that fair allocation methods should also be coalition monotone. The Coalition Game Impossibility Theorem tells us that there is no allocation method that satisfies all three properties. Therefore, we need to decide which properties are most important for the situation in which the method will be used. If players make and break agreements freely, then violations of the rational property would be disastrous for collaborative agreements. However, if cooperation will be enforced by government or moral decree, then violations of the rational property (when they occur) may not be too objectionable. If the gains change often or if players can unilaterally sabotage projects, then violations of the coalition monotone property could be psychologically detrimental to the players. However, if the gains are clear and impossible to manipulate, then the coalition monotone property may be less relevant.

If the situation limits the possible games that can arise, sometimes it is possible to have methods that satisfy more properties on that restricted class of games. For example, we have already noted that the nucleolus method is efficient, rational, and coalition monotone for three-player games. It is also known that if the games under consideration have gains that increase rapidly with the size of the coalition, then the Shapley method is efficient, rational, and coalition monotone.

In this chapter, we have seen that whenever fairness is desired in allocating the gains obtained through collaboration, it is important to clearly describe what is meant by fairness. Once fairness has been clearly defined, mathematics can then be used to determine whether any fair method exists, and if so, how to find allocations with such a method. Mathematics can facilitate a conversation on what fairness means, but in the end, mathematics cannot decide what fairness means, people must make that decision.

Exercises

(1) Under what circumstances would you recommend that the Shapley method be used for allocating payoffs for a coalition game?
(2) Under what circumstances would you recommend that the nucleolus be used for allocating payoffs for a coalition game?
(3) Verify in the proof of the Coalition Game Impossibility Theorem that player B's payoff in **BB** and **ZZ** must be no greater than zero, that player C's payoff

in **CC** and **ZZ** must be no greater than zero, and that player D's payoff in **DD** and **ZZ** must be no greater than zero.

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CHAPTER 8

Fair Division

1. An Inheritance Problem

Bob, Carol, and Doug have inherited equal shares in their mother's estate, which for simplicity, consists of a cabin, a car, and silverware. The three siblings are friendly with each other, but it is important that they each believe that the division of property is fair. So, their first step is to try to determine the monetary value of each item in the estate.

By the end of this section, you will be able to describe why individual values of items are important data for determining a fair division, and you will have thought about ways to carry out a fair division.

CAROL: The cabin is a small building in somewhat poor condition on an acre of land next to a national forest. For tax purposes, the property is assessed at $25,000. We asked two realtors what price they could sell the property for. One proposed a listing price of $61,000 and the other proposed a listing price of $50,000.

DOUG: But both said that real estate sales in the area near the cabin were fairly slow and that accepted offers have been between 80% and 105% of the listing price. If a realtor were to handle the listing and sale of the property, he or she would charge 6% of the sale price, and there would be an additional $1,000 in closing costs (primarily local government transfer tax).

BOB: Of course, that's if we were to sell the cabin. I've often stayed there and hunted in the national forest. If we were to sell the cabin, I might want to purchase a similar property and so would have to pay perhaps $61,000 plus buyer closing costs. To avoid the hassle of a new purchase, the cabin is probably worth about $66,000 to me.

CAROL: If I were to acquire the property, I would sell. Since I have some expertise at selling property, I believe I could obtain the higher $61,000 proposed listing price and then have to pay only the seller closing costs. So, the cabin is worth about $60,000 to me.

DOUG: I've never been as optimistic as you, Carol. I think that the actual selling price would be closer to $45,000. Since I would need to use a real estate agent, in the end I would probably see only $42,000 in cash.

CAROL: I respectfully disagree with your assessment, Doug.

BOB: Even with your more optimistic assessment of the property value, Carol, shouldn't you subtract something for the time you would need to devote to the process of selling the property? I added to my value of the property because I didn't want to go through the hassle of buying a property.

CAROL: I actually enjoy selling property, so my time is free.

BOB: Since I would like to keep the cabin, wouldn't it make sense for my portion of the estate to include the cabin? In which case, we would just need to agree on how much it is worth to determine whether I have received a one-third share of the estate.

CAROL: That's not completely clear to me. As I said, I like to sell property, and so perhaps I should receive the pleasure of selling it.

DOUG: I'm a little strapped for cash right now and wouldn't mind having the cabin sold so that I could receive my one-third share in money.

BOB: If I were to receive the cabin and, because of that, was receiving more than one-third of the estate, I would be willing to give each of you money to even up our shares of the estate.

CAROL: I would also be willing to give money to either or both of you in order to even out our shares of the estate.

DOUG: I'm glad to hear this. So, all we really need to do is to determine the value of each item in the estate, decide who will receive the items, and then transfer money accordingly. For example, if we say that the cabin is worth \$66,000 (the amount Bob said he valued it) and Bob is given the cabin, then he should give Carol and me each \$22,000. That way, each of us will have received one-third of the cabin.

CAROL: I agree with your idea, but I think that the cabin is only worth \$60,000.

BOB: Be honest, Doug. You believe that the cabin is worth only \$42,000. So, shouldn't I have to pay you and Carol only \$14,000 each?

DOUG: I guess that I do believe the cabin is worth only \$42,000 to me personally. But you honestly believe the cabin is worth \$66,000.

BOB: True.

DOUG: And so that should somehow be taken into account.

BOB: Well, it would be in the sense that since I value the cabin the most, I should be the one to receive the cabin.

DOUG: And that would happen if we were to auction off the cabin to the three of us. I would bid \$42,000, Carol would bid \$60,000, and you would bid \$66,000. You'd win the cabin and have to pay $66,000. That money could then be divided among us equally. You would have the cabin and have spent $44,000 while Carol and I would each have received $22,000. We would clearly have each received one-third of the cabin.

CAROL: An auction is an interesting idea. But in most auctions that I've experienced, people make multiple ascending bids. In that case, Bob is not going to have to bid much more than $60,000 in order to win the cabin because neither Doug nor I would want to bid over $60,000.

BOB: So, perhaps I should give $20,000 to each of you in order to obtain the cabin.

CAROL: Well, it's an idea. It's not clear that we can agree on a single number as the value of the cabin.

DOUG: Perhaps the cabin was not the right item to begin discussing. Perhaps if we discuss the other items now, we will know better what to do with the cabin afterward.

BOB: Okay. The car should be easy. It is a 2003 Oldsmobile Alero GX four-door sedan in excellent condition and with 12,000 miles on the odometer. We should be able to obtain a value from Kelly Blue Book™.

CAROL: Let me check out the web site (www.kbb.com). Hmmm. The web site provides three different values!

BOB: What?

CAROL: Yes, let me click through these questions. It says here that we could expect to be offered $7,875 by a dealer if we were trading in this car, to be offered $10,075 if selling this car to another consumer, and to be asked to pay $12,380 if purchasing the car from a dealer.

DOUG: That's a wide variation in values. Does it have something to do with nationwide averages?

CAROL: Actually, the site asked for a zip code and I put in Mom's old zip code. If I put in your zip code, Doug, it appears that the values are somewhat lower.

DOUG: So, they're using average selling prices within different locales.

CAROL: Yes.

BOB: Well, I'd probably try to sell the car, and so I'd place its value at about $10,000.

DOUG: I wouldn't want to take the time trying to sell it on the open market. I'd just take it to a dealer in my hometown, and so it would only be worth about $7,000 to me.

CAROL: Oh, I really like that car and am hoping to purchase one. So, I'd be willing to pay $12,000 for this one and not have to look for another like it.

BOB: It doesn't seem that we're able to agree upon a single value for the car either. However, I think that it should be somewhat easier to obtain a valuation for the silverware.

DOUG: Why?

BOB: Because stores still sell the same pattern. It would cost $22,000 to purchase all of the same pieces as in Mom's sterling silverware set. Since her silverware has been used, I would say that the set is worth about $17,000.

CAROL: Used? They're antiques! Heirlooms! They've been in the family for three generations! They're definitely worth more than a new set, perhaps $27,000.

DOUG: Well, perhaps they're antiques to you, Carol, but personally, I would rather just obtain some money. How much could we sell the silverware for? I doubt that anyone would buy our silverware for more than an identical new set. In fact, I doubt that anyone would pay more than half, or $11,000.

BOB: It turns out that we are spread apart even farther on how much to value the silverware!

DOUG: So, it appears that there's really no single objective value that can be placed on any item.

BOB: I guess not. Perhaps that is why some people are buyers and some are sellers: buyers value an item more than the sellers, and so an intermediate price makes both happy.

CAROL: I summarized how each of us values each of the items we have inherited in Table 1.1.

Table 1.1. Personal valuations (in thousands of dollars) of the items by the siblings

Sibling/Item	Cabin	Car	Silverware	Total
Bob	66	10	17	93
Carol	60	12	27	99
Doug	42	7	11	60

DOUG: I'd agree with the numbers in the table.

BOB: And so would I.

CAROL: Seeing everyone elses' values doesn't change your values?

DOUG: Well, it would be wonderful if we were to receive $66,000 for the cabin so that I could receive $22,000 for my one-third share. However, I honestly believe that we would only net about $42,000 if we were to sell the cabin. So, I respectfully disagree with both of your values.

BOB: And I disagree with both of your values.

CAROL: I would say the same.

DOUG: So, a fair division of the estate should somehow take into account our different values of the items in the estate.

BOB: Since these are firmly held opinions, yes, I believe that they should somehow be taken into account.

CAROL: And remember that we're willing to exchange some money in order to even things out.

DOUG: And so, what would be a fair division?

> This would be a good time to join with two other people and have each of you act as Bob, Carol, or Doug in a negotiation to achieve a fair division based on the values given in the previous table. Alternatively, you could act as an arbitrator and suggest a fair division.

Exercises

(1) Bob, Carol, and Doug are unable to agree on the value of any item. Are they acting out of ignorance? Are they being obstinate? Or is this reasonable human behavior?
(2) What items in an estate might a group of inheritors all value equally?
(3) Carol said that the silverware was worth $27,000 to her. What does she mean by that? Might she have different monetary valuations for the silverware dependent upon who receives ownership (Bob, Carol, Doug, or a nonfamily member)?
(4) What would you consider to be a fair division of the items and transfers of money?

2. Fair Division Games and Methods

In this section, we model scenarios similar to the three-sibling inheritance scenario of the previous section as fair division games and suggest four methods for solving these games.

> By the end of this section, you should be able to model a variety of scenarios as fair division games, to select divisions in accordance with four methods, and to compute each player's value of what he or she received in monetary terms and as a share of what was to be divided.

Fair Division Game: In a *fair division game*, there are items and each player has a monetary value for each item. A *division* assigns each item to a player and transfers money among players. Players may propose divisions at any time. The game ends when all players agree upon a division, which becomes the outcome. The *value* of a division to a player is the sum of the player's monetary values for the items she or he is assigned and the net money received. Each player most prefers to maximize her or his value.

The inheritance scenario described in the previous section can be modeled as the **Inheritance** fair division game in Table 2.1. The columns correspond to the three players (Bob, Doug, and Carol); the first three rows correspond to the three items (cabin, car, and silverware); and the cells report the monetary values each player has assigned to each item. The final row reports each player's assessment of the value of the estate, which is simply the sum of the monetary values for the individual items.

Table 2.1. The **Inheritance** fair division game

	Bob	Carol	Doug
Cabin	$66,000	$60,000	$42,000
Car	$10,000	$12,000	$7,000
Silverware	$17,000	$27,000	$11,000
Total Estate	$93,000	$99,000	$60,000

One of our modeling assumptions is that players can place monetary values on each item. For example, Bob placed a monetary value of $66,000 on the cabin. By this, we mean that if Bob owned the cabin, he would be willing to sell it for $66,000, and if Bob did not own the cabin, he would be willing to purchase it for $66,000. Another way of stating this is that if Bob was presented with the choice of receiving the cabin or $66,000, he would be willing to choose either. This determination of monetary values is similar to determining cardinal utilities for outcomes. So, it is not surprising that many players find it difficult to determine monetary values, especially when there may be emotional attachments to the items.

One example of a division would be to (1) assign ownership of the cabin to Bob, the car to Carol, and the silverware to Doug, and (2) transfer $20,000 from Bob, $10,000 to Carol and $10,000 to Doug. We name this division *first-try* because it was chosen merely to illustrate what a division is. We are not recommending first-try, but will use it to illustrate some ideas. In fact, in the next section we will show that first-try is not fair for several reasons.

Since Bob has received the cabin, which he values at $66,000, and spends $20,000, Bob places a

$$\$66,000 - \$20,000 = \$46,000$$

value on this division. Since Bob values the estate at $93,000, he has received

$$\frac{\$46,000}{\$93,000} = 49.5\%$$

of the estate from his perspective. We say that Bob received a 49.5% *share* of the estate.

> **Share:** Given a division, the *share* for a player is the ratio of that player's value of the division to that player's value for the estate.

Similar calculations can be done for Carol and Doug; the division and each player's values and shares, are displayed in Table 2.2.

TABLE 2.2. A first-try division for **Inheritance**

Player	Bob	Carol	Doug
Received	cabin − $20,000	car + $10,000	silver + $10,000
Value	$66,000 − $20,000 = $46,000	$12,000 + $10,000 = $22,000	$11,000 + $10,000 = $21,000
Share	$46,000/$93,000 = 49.5%	$22,000/$99,000 = 22.2%	$21,000/$60,000 = 35.0%

The division tells us physically what happens: Bob obtains ownership of the cabin and gives away $20,000; Carol obtains ownership of the car and receives $10,000; and Doug obtains ownership of the silverware and receives $10,000. A value tells us for what dollar amount that player would be willing to purchase or sell his or her portion of the estate: Bob would be willing to sell the cabin and his $20,000 debt for $46,000; Carol would be willing to sell the car and $10,000 in cash for $22,000; and Doug would be willing to sell the silverware and $10,000 in cash for $21,000. A share tells us what percentage of the estate that player thinks he or she has received: Bob thinks he has received almost half of the estate; Carol thinks she has received about 22% of the estate; and Doug thinks that he has received 35% of the estate.

You may wonder why the share percentages would add up to 49.5% + 22.2% + 35.0% = 106.7% rather than 100%. How could the three siblings receive more than 100% of the estate? It is important to remember that a player's share is from that player's perspective. Bob thinks that he has obtained 49.5% of the estate because he values what he obtained at $46,000 and he values the estate at $93,000. Carol has a very different view of what Bob has received. Since Carol values the cabin at \$60,000, she values what Bob has received at \$60,000 − \$20,000 = \$40,000. Since Carol values the estate at \$99,000, she thinks that Bob has received only $\frac{\$40{,}000}{\$99{,}000} =$ 40.4% of the estate.

Therefore, it is not clear exactly what it means to add the percentage of the estate that Bob thinks he has received (49.5%) to the percentage of the estate which Carol thinks she has received (22.2%) to the percentage of the estate that Doug thinks he has received (35.0%). Just because we have a set of numbers does not mean that it is meaningful to add them! For example, although one author biked 66 miles in 10 hours on August 17, that does not mean that $66 + 10 + 17 = 93$ has any meaning. Similarly, adding up percentages of different things does not have any clear meaning.

A sum that would make sense is to add the percentage of the estate Bob thinks he has received

$$\frac{(\$66,000 - \$20,000)}{\$93,000} = 49.5\%$$

to the percentage of the estate Bob thinks Carol has received

$$\frac{(\$10,000 + \$10,000)}{\$93,000} = 21.5\%$$

to the percentage of the estate Bob thinks Doug has received

$$\frac{(\$17,000 + \$10,000)}{\$93,000} = 29.0\%.$$

These percentages sum to 100% because from Bob's perspective, the value of the estate has been divided up among the three siblings. In Table 2.3, we summarize these calculations for Bob and similar calculations for Carol and Doug.

TABLE 2.3. First-try values and shares from each player's perspective

	Bob	Carol	Doug	Sum
Received	cabin − \$20,000	car + \$10,000	silver + \$10,000	cabin + car + silver + \$0
Bob's Perspective	\$46,000 (49.5%)	\$20,000 (21.5%)	\$27,000 (29.0%)	\$93,000 (100.0%)
Carol's Perspective	\$40,000 (40.4%)	\$22,000 (22.2%)	\$37,000 (37.4%)	\$99,000 (100.0%)
Doug's Perspective	\$22,000 (36.7%)	\$17,000 (28.3%)	\$21,000 (35.0%)	\$60,000 (100.0%)

Summing the numbers in a row makes sense because they are from a single perspective. The percentages we were trying to sum from the previous table are the diagonal elements from this table, and it is not clear if any meaning can be attached to this sum.

This would be an appropriate time for the reader to stop and think for a few minutes about why you might consider first-try to be unfair and what a fair division might be.

Four Methods

We will now describe four methods of division that seem reasonable and have been recommended by various scholars. The first method is based on the idea that players' monetary values for an item could be thought of as auction bids that the players submit in writing in sealed envelopes. After the envelopes are submitted, they are opened, the highest bid wins the item being auctioned, and the winner pays the amount he or she bid.

> **First-Price Auction Method:** Assign ownership of each item to a player who values it the most. Have each player place into a pot money equal to their monetary values of the items received. Finally, split the money in the pot equally among the players.

Let's use the first-price auction method on **Inheritance**. Since Bob values the cabin the most ($66,000 versus $60,000 and $42,000), Bob receives the cabin and places $66,000 into the pot. Similarly, since Carol values the car the most ($12,000 versus $10,000 and $7,000), Carol receives the car and places $12,000 into the pot. Finally, since Carol also values the silverware the most ($27,000 versus $17,000 and $11,000), she receives the silverware and places $27,000 into the pot, which now contains $105,000, as shown in Step 1 of Table 2.4.

TABLE 2.4. First price auction steps for **Inheritance**

Player	Bob	Carol	Doug	Pot
Step 1	cabin − $66,000	car − $12,000 silver − $27,000		$66,000 $12,000 $27,000 $105,000
Step 2	+ $105,000/3	+ $105,000/3	+ $105,000/3	− $105,000
Net	cabin− $31,000	silver + car − $4,000	$35,000	$0

Since the pot contains $105,000, each player receives $\frac{\$105,000}{3} = \$35,000$, as shown in Step 2 of Table 2.4. Because Bob put $66,000 into the pot and received $35,000 from the pot, Bob has effectively spent $31,000. Similarly, because Carol put $39,000 into the pot and received $35,000 from the pot, Carol has effectively spent $4,000. Finally, Doug put nothing into the pot and received $35,000, as shown in the "Net" row of Table 2.4. This division is summarized in Table 2.5.

Our next division method is again based on using players' monetary values as the basis for auctioning off each item. However, the format of the auction involves

TABLE 2.5. First-price auction division for **Inheritance**

Player	Bob	Carol	Doug
Received	cabin − $31,000	car + silver − $4,000	$35,000
Value	$35,000	$35,000	$35,000
Share	37.6%	35.4%	58.3%

publicly stated bids announced as an auctioneer calls for higher and higher bids. The bidder with the highest value for an item will still win the item; however, the bidding will stop at an amount only slightly larger than the second highest value, because no one should bid higher than their value for the item, and so the bidder with the highest value will be able to stop raising his or her bid shortly after the other players stop entering bids.

> **Second-Price Auction Method:** Assign ownership of each item to a player who values it the most. Have each player place into a pot money equal to the highest monetary value other players have for the items received. Finally, split the money in the pot equally among the players.

Let's use the second-price auction method on **Inheritance**. Since Bob values the cabin the most ($66,000 versus $60,000 and $42,000), Bob receives the cabin and places the second highest value of $60,000 into the pot. Similarly, since Carol values the car the most ($12,000 versus $10,000 and $7,000), Carol receives the car and places the second highest value of $10,000 into the pot. Finally, since Carol also values the silverware the most ($27,000 versus $17,000 and $11,000), she receives the silverware and places the second highest value of $17,000 into the pot, which now contains $87,000, as shown in Step 1 of Table 2.6.

TABLE 2.6. Second-price auction division steps for **Inheritance**

Player	Bob	Carol	Doug	Pot
Step 1	cabin − $60,000	car − $10,000 silver − $17,000		$60,000 $10,000 $17,000 $87,000
Step 2	+ $87,000/3	+ $87,000/3	+ $87,000/3	− $87,000
Net	cabin − $31,000	silver + car + $2,000	$29,000	$0

Since the pot contains $87,000, each player receives $\frac{\$87{,}000}{3} = \$29{,}000$, as shown in Step 2 of Table 2.6. Because Bob put $60,000 into the pot and received $29,000 from the pot, Bob has effectively spent $31,000. Similarly, because Carol put $27,000 into the pot and received $29,000 from the pot, Carol has effectively received $2,000. Finally, Doug put nothing into the pot and received $29,000, as shown in the "Net" row of Table 2.6. This division is summarized in Table 2.7.

It is important to emphasize that a player's value is based on that player's values! Bob values the cabin at $66,000, not at $60,000, and so Bob's value is $66,000 − $31,000 = $35,000.

TABLE 2.7. Second-price auction division for **Inheritance**

Player	Bob	Carol	Doug
Received	cabin − $31,000	car + silver + $2,000	$29,000
Value	$35,000	$41,000	$29,000
Share	37.6%	41.4%	48.3%

In order to motivate our third method, since each player has equal ownership in the estate, perhaps each player should receive the same share of the estate. The next method is based on this interpretation of ownership.

> **Equal Shares Method:** Assign ownership of each item to a player who values it the most. Have each player give or receive money so that every player's share is the same.

Let's use the equal shares method on **Inheritance**. For the same reasons as before, Bob will receive the cabin and Carol will receive the car and silverware. Presumably, Carol and Doug will receive some money from Bob. Since we do not know what those amounts are, we use the variables Y and Z to denote them, as shown in the "Received" row of Table 2.8.

TABLE 2.8. Equal shares set up for **Inheritance**

Player	Bob	Carol	Doug	Sum
Received	cabin − $\$Y - \$Z$	silver + car + $\$Y$	$\$Z$	cabin + car + silver
Value based on Division	$\$66,000 - \$Y - \$Z$	$\$12,000 + \$27,000 + \$Y$	$\$Z$	$105,000
Share	S	S	S	
Value based on Share	$\$93,000S$	$\$99,000S$	$\$60,000S$	$\$252,000S$

Since Bob values the cabin at $66,000 and Carol values the car and silverware at $39,000, there is $105,000 in value available to divide among the three players (see the "Value based on Division" row). Each player is to receive the same share of the estate, but since we do not know what that share is, we let S be the variable representing the share of the estate that each player will receive (see the "Share" row). Since Bob values all items at $93,000, his value for the division should be $\$93,000S$. Similarly, Carol's and Doug's values for the division should be $\$99,000S$ and $\$60,000S$, respectively (see the "Value based on Share" row). The sum of the values received by the three players should equal the total value available, leading to the equations,

$$
\begin{aligned}
252,000S &= 105,000, \\
S &= \frac{105,000}{252,000}, \\
S &= \frac{5}{12} \approx 41.7\%.
\end{aligned}
$$

With this value for S, Doug's value is

$$Z = 60,000 \cdot \frac{5}{12} = 25,000,$$

and Carol's value is

$$\$12,000 + \$27,000 + \$Y = \$99,000 \cdot \frac{5}{12},$$

which implies that $Y = 2,250$, and so Bob must give $Y + Z = 27,250$. The resulting equal shares division is summarized in Table 2.9.

TABLE 2.9. Equal shares division for **Inheritance**

Player	Bob	Carol	Doug
Received	cabin $-$ \$27,250	car + silver + \$2,250	\$25,000
Value	\$38,750	\$41,250	\$25,000
Share	41.7%	41.7%	41.7%

Notice that the money transfers sum to zero, as they must for any division. If the money transfers had not summed to zero, it would mean that we had made a mistake somewhere in our calculations.

Our fourth method was suggested by Bronislaw Knaster in 1945 (as attributed by [**62**]) and was the first of the four that was studied in the context of fair division games.

> **Knaster's Method:** Given a fair division game with n players, assign ownership of each item to a player who values it the most. Have each player give or receive money so that every player receives the same additional value above $\frac{1}{n}$ of their value for the estate.

Let's use Knaster's method on **Inheritance**. For the same reasons as before, Bob will receive the cabin, Carol will receive the car and silverware, and money will again be transferred among players (see "Received" row in Table 2.10). Player values can be calculated easily (see "Value based on Division" row in Table 2.10).

TABLE 2.10. Set up of Knaster's method for **Inheritance**

Player	Bob	Carol	Doug	Sum
Received	cabin $-$ \$$Y$ $-$ \$$Z$	silver + car + \$$Y$	\$$Z$	cabin + car + silver
Value based on Division	\$66,000 $-$ \$$Y$ $-$ \$$Z$	\$12,000 +\$27,000 + \$$Y$	\$$Z$	\$105,000
Value based on Knaster's	$\frac{1}{3}\$93,000 + \$A$	$\frac{1}{3}\$99,000 + \$A$	$\frac{1}{3}\$60,000 + \$A$	\$84,000 + 3\$$A$

Let A be the additional value each player receives above $\frac{1}{3}$ of their individual values for the estate. Since he values the estate at \$93,000, Bob's value for the division should be $\frac{1}{3}\$93,000 + \$A$. Similarly, since she values the estate at \$99,000, Carol's value for the division should be $\frac{1}{3}\$99,000 + \$A$. Finally, since he values the estate at $\$60,000$, Doug's value for the division should be $\frac{1}{3}\$60,000+\$A$ (see the "Value based on Knaster's" row). The sum of the values received by the three players should equal the total value available, leading to the equations,

$$\begin{aligned}\left(\frac{1}{3}\$93,000+\$A\right)+\left(\frac{1}{3}\$99,000+\$A\right)+\left(\frac{1}{3}\$60,000+\$A\right) &= \$105,000,\\ 84,000+3A &= 105,000,\\ 3A &= 21,000,\\ A &= 7,000.\end{aligned}$$

With $A=7,000$, Bob should value what he receives at $\frac{1}{3}\$93,000+\$7,000=\$38,000$, and since Bob will receive the cabin, he needs to pay $\$Y+\$Z=\$66,000-\$38,000=\$28,000$. Similarly, Carol should value what she receives at $\frac{1}{3}\$99,000+\$7,000=\$40,000$, and since Carol will receive the car and silverware, Carol needs to be paid $\$Y=\$40,000-\$39,000=\$1,000$. Finally, Doug should value what he receives at $\frac{1}{3}\$60,000+\$7,000=\$27,000$, and since he receives none of the items, Doug needs to be paid $\$Z=\$27,000$. In summary, the Knaster division is shown in Table 2.11.

TABLE 2.11. Knaster division for **Inheritance**

Player	Bob	Carol	Doug
Received	cabin − $28,000	car + silver + $1,000	$27,000
Value	$38,000	$40,000	$27,000
Share	40.9%	40.4%	45.0%

Knaster's method is a compromise between the equal shares and first-price auction methods because a portion of the estate is allocated by shares (each player receives $\frac{1}{n}$ of their value for the estate to start with) and the remainder of the estate is allocated by monetary value (each player receives the same additional amount A). If the disagreement payoff tuple for an n-player estate bargaining game is each player receiving $\frac{1}{n}$ of their value for the estate, then the egalitarian, Raiffa, and Nash methods would each produce the payoff tuples made up of the values obtained for the Knaster's division.

Painting Game

For practice, let's apply these four methods to another scenario. Three sisters had been given a painting as a gift. When it came time to leave their parents' home, they asked who would be the one to take the painting with her. In order for the sisters to use one of the division methods suggested above, each must be able to place a monetary value on the painting and be willing to exchange money. The market value of the painting is not clear to any of the sisters, and besides, there is too much sentimental value for any of the sisters to imagine selling the painting to a stranger. Yet, each sister thinks that she can place a monetary value on being able to keep the painting in her home. Sarah decides the painting is worth $900, Elizabeth decides the painting is worth $810, and Nada decides the painting is worth $90. The **Painting** fair division game is given in Table 2.12.

TABLE 2.12. **Painting** fair division game

Player	Sarah	Elizabeth	Nada
Painting	\$900	\$810	\$90

Since there is only one item in the estate, there is no need for a separate row for the total estate values.

In each of the four division methods previously described, each item goes to the player who values that item the most. So, each method will assign the painting to Sarah.

With the first-price auction method, Sarah will place \$900 into the pot and each player will then receive \$300 from the pot, as shown in Table 2.13.

TABLE 2.13. First-price auction division for **Painting**

Player	Sarah	Elizabeth	Nada
Receives	Painting − \$600	\$300	\$300
Value	\$300	\$300	\$300
Share	33.3%	37.0%	333.3%

With the second-price auction method, Sarah will place \$810 into the pot and each player will then receive \$270 from the pot as shown in Table 2.14.

TABLE 2.14. Second-price auction division for **Painting**

Player	Sarah	Elizabeth	Nada
Receives	Painting − \$540	\$270	\$270
Value	\$360	\$270	\$270
Share	40.0%	33.3%	300.0%

With the equal shares method, Sarah, Elizabeth, and Nada will receive $900S$, $810S$, and $90S$ dollars in value, where S is the share of the estate each player will receive and the sum of these values should be equal to the total available value of \$900. This leads to the equations

$$\begin{aligned} 900S + 810S + 90S &= 900, \\ 1800S &= 900, \\ S &= \frac{900}{1800} = \frac{1}{2}, \end{aligned}$$

which means that Nada will receive $\$90(\frac{1}{2}) = \$45$, Elizabeth will receive $\$810(\frac{1}{2}) = \$405$, and Sarah will spend $\$900 - \$900(\frac{1}{2}) = \$450$, as shown in Table 2.15.

With Knaster's method, Sarah, Elizabeth, and Nada will receive $\frac{1}{3}900 + A$, $\frac{1}{3}810 + A$, and $\frac{1}{3}90 + A$ dollars in value, where A is the additional amount each player will receive, and the sum of these values should be equal to the total available value of

TABLE 2.15. Equal shares division for **Painting**

Player	Sarah	Elizabeth	Nada
Receives	Painting − \$450	\$405	\$45
Value	\$450	\$405	\$45
Share	50.0%	50.0%	50.0%

\$900. This leads to the equations

$$\begin{aligned} \left(\frac{1}{3}900 + A\right) + \left(\frac{1}{3}810 + A\right) + \left(\frac{1}{3}90 + A\right) &= 900, \\ 600 + 3A &= 900, \\ A &= 100, \end{aligned}$$

which means that Nada will receive $\frac{1}{3}\$90 + \$100 = \$130$, Elizabeth will receive $\frac{1}{3}\$810 + \$100 = \$370$, and Sarah will spend $\$900 - \left(\frac{1}{3}\$900 + \$100\right) = \$500$, as shown in Table 2.16.

TABLE 2.16. Knaster's division for **Painting**

Player	Sarah	Elizabeth	Nada
Receives	Painting − \$500	\$370	\$130
Value	\$400	\$370	\$130
Share	44.4%	45.7%	43.3%

Which of these divisions would you describe as fair? Is there a different division you would find to be more fair? These questions will be addressed in the next section. The exercises for this section ask you to model some scenarios as fair division games and to select divisions in accordance with the four methods described above. Some of the exercises will challenge you to extend some of the ideas developed here.

Exercises

(1) **Three Brothers** Jesse, Kaleab, and Ulises have equal shares in the inheritance of their parents' musical estate. The items in the estate and each brother's monetary value for each item is given in Table 2.17.

TABLE 2.17. **Three Brothers** fair division game

Value	Jesse	Kaleab	Ulises
Piano	\$4,000	\$3,500	\$3,100
Trumpet	\$90	\$130	\$100
Cello	\$1,000	\$1,400	\$1,800
Flute	\$70	\$100	\$70
Total Estate	\$5,160	\$5,130	\$5,070

(a) What do the monetary values mean? In order to use one of the division methods described in this section, what must the brothers be willing to do?
(b) Consider the first-try division in which the trumpet and \$1,740 is given to Jesse; the piano and flute are given to and \$1,770 is given away by Kaleab; and the cello and \$30 is given to Ulises. Determine each brother's value of what is received and each brother's share of the estate.
(c) Determine the first-price auction division, each brother's value, and each brother's share.
(d) Determine the second-price auction division, each brother's value, and each brother's share.
(e) Determine the equal shares division, each brother's value, and each brother's share.
(f) Determine Knaster's division, each brother's value, and each brother's share.
(g) Which division (among the five above or one of your own devising) do you believe is the most fair? Why?

(2) **Four Siblings.** Four siblings have been given equal shares in a prize of a diamond pendant. Kate, Genevieve, Stephen, and Laura think the pendant is worth \$600, \$560, \$560, and \$520, respectively.
(a) What do the values mean? In order to use one of the division methods described in this section, what must the siblings be willing to do?
(b) Consider the first-try division in which Stephen receives the diamond pendant and gives \$150 to Kate, \$140 to Genevieve, and \$130 to Laura. Determine each sibling's value of what is received and each sibling's share of the prize.
(c) Determine the first-price auction division, each sibling's value, and each sibling's share.
(d) Determine the second-price auction division, each sibling's value, and each sibling's share.
(e) Determine the equal shares division, each sibling's value, and each sibling's share.
(f) Determine Knaster's division, each sibling's value, and each sibling's share.
(g) Which division (among the five above or one of your own devising) do you believe is the most fair? Why?

(3) **Divorce.** After a short marriage, Paul and Mary have filed for divorce. There are only three joint assets that must be divided between them. Those joint assets and the monetary value each person has for each item is given in Table 2.18.

TABLE 2.18. **Divorce** fair division game

Value	Paul	Mary
Media Center	\$1,000	\$1,200
Bedroom Furniture	\$1,200	\$1,800
Living Room Furniture	\$1,600	\$1,000
Total Estate	\$3,800	\$4,000

(a) Consider the first-try division in which Paul receives the media center (MC) and living room furniture (LRF) and Mary receives the bedroom furniture (BF). No money is exchanged. Determine each person's value, and each person's share of the joint assets.
(b) Determine the first-price auction division, each person's value, and each person's share.
(c) Determine the second-price auction division, each person's value, and each person's share.
(d) Determine the equal shares division, each person's value, and each person's share.
(e) Determine Knaster's division, each person's value, and each person's share.
(f) If the marriage had lasted longer, there may have been issues of child custody, child support, and alimony. Could such issues be items in a fair division game between the two people? If so, how? If not, why not?

(4) **Candy Bar.** Suppose three friends have been given a gift of a large and exquisite chocolate candy bar. Of course, they could simply divide the candy bar into thirds. However, they have been reading this book and each is willing to give the others some money for the privilege of having the entire candy bar. Leslie is willing to give \$6.00, Michael is willing to give \$4.20, and Owen is willing to give \$3.00.
(a) Consider the first-try division in which Leslie receives the candy bar and gives \$1.50 each to Michael and Owen. Determine each person's value and share.
(b) Determine the first-price auction division, each person's value, and each person's share.
(c) Determine the second-price auction division, each person's value, and each person's share.
(d) Determine the equal shares division, each person's value, and each person's share.
(e) Determine Knaster's division, each person's value, and each person's share.
(f) Which division (among the five above or one of your own devising) do you believe is the most fair? Why?

(5) **Uneven Inheritance.** Suppose that in **Inheritance**, the three siblings were not given equal shares of the estate. Suppose instead that Bob, Carol, and Doug were suppose to receive 1/6, 1/3, and 1/2 of the estate, respectively.
(a) Modify the first-price auction method to be able to handle players owning unequal shares of an estate. Apply your modification to select an appropriate division, each sibling's value, and each sibling's share.
(b) Modify the second-price auction method to be able to handle players owning unequal shares of an estate. Apply your modification to select an appropriate division, each sibling's value, and each sibling's share.
(c) Modify the equal shares method to be able to handle players owning unequal shares of an estate. Apply your modification to select an appropriate division, each sibling's value, and each sibling's share.
(d) Modify Knaster's method to be able to handle players owning unequal shares of an estate. Apply your modification to select an appropriate division, each sibling's value, and each sibling's share.

(6) **Ailing Parent.** Barbara, John, and Marc have a father who can no longer live by himself in his home. Each would be willing to lodge and provide daily care for their father and/or provide financial support for the sibling who will provide the care. Each sibling determines how much it would cost her or him to make the necessary home changes and to provide the care: \$12,000, \$15,000, and \$30,000 for Barbara, John, and Marc, respectively. (In order to model this scenario as a fair division game, it is helpful to think of the costs as negative values.)
  (a) Determine the first-price auction division, each sibling's cost, and each sibling's share.
  (b) Determine the second-price auction division, each sibling's cost, and each sibling's share.
  (c) Determine the equal shares division, each sibling's cost, and each sibling's share.
  (d) Determine Knaster's division, each sibling's cost, and each sibling's share.

(7) **Committee Work.** An academic department needs to assign two of its members to be representatives to the Judicial and Governance Committees. The representatives can expect a fairly high work load, and none of the department members is especially interested in taking on either job. In fact, many members of the department realize that they would be willing to give up some of their salary rather than take on either job, and some in the department would be more willing to take on one of the jobs if their salary were higher. Table 2.19 records how much each department member values each committee position (negative numbers because each member views each item as a "bad" instead of a "good").

TABLE 2.19. **Committee Work** fair division game

Value	Ian	Calista	Renee	Kaelem	Urbana
Judiciary	$-\$1,000$	$-\$1,500$	$-\$2,500$	$-\$2,500$	$-\$2,500$
Governance	$-\$1,500$	$-\$1,000$	$-\$2,000$	$-\$2,000$	$-\$2,000$
Total Estate	$-\$2,500$	$-\$2,500$	$-\$4,500$	$-\$4,500$	$-\$4,500$

  (a) Determine the first-price auction division, each department member's cost, and each department member's share.
  (b) Determine the second-price auction division, each department member's cost, and each department member's share.
  (c) Determine the equal shares division, each department member's cost, and each department member's share.
  (d) Determine Knaster's division, each department member's cost, and each department member's share.
  (e) University rules do not allow one person to be assigned to both positions. Discuss how you would modify the four methods if Ian had valued the Governance position at $-\$800$ instead of $-\$1,500$.

(8) How might a labor contract negotiation be modeled as a fair division game? What would be the most difficult part to model?

(9) Which of the five suggested divisions for **Inheritance** would you consider to be fair? Why? Is there a different division that you would find to be more fair? Why?

(10) Which of the four suggested divisions for **Painting** would you consider to be fair? Why? Is there a different division you would find to be more fair? Why?

3. Fairness Properties

What division is fair? Certainly the use of a method provides a sense of fairness because of its consistency across different specific games. An even better approach would be to define fairness as precisely as possible so that the fairness of a division or a method could be measured against the definition.

> By the end of this section, you will be able to describe several fairness properties and determine whether a division satisfies each property.

The **Inheritance** fair division game and the five divisions described in the previous section are summarized in Table 3.1.

TABLE 3.1. **Inheritance** fair division game and five divisions

	Bob	Carol	Doug
Cabin	$\$66,000$	$\$60,000$	$\$42,000$
Car	$\$10,000$	$\$12,000$	$\$7,000$
Silver	$\$17,000$	$\$27,000$	$\$11,000$
Total Estate	$\$93,000$	$\$99,000$	$\$60,000$
First-try value (share)	cabin $-$ $\$20,000$ $\$46,000$ (49.5%)	car + $\$10,000$ $\$22,000$ (22.2%)	silver + $\$10,000$ $\$21,000$ (35.0%)
First-price value (share)	cabin $-$ $\$31,000$ $\$35,000$ (37.6%)	car + silver $-$ $\$4,000$ $\$35,000$ (35.4%)	$\$35,000$ $\$35,000$ (58.3%)
Second-price value (share)	cabin $-$ $\$31,000$ $\$35,000$ (37.6%)	car + silver + $\$2,000$ $\$41,000$ (41.4%)	$\$29,000$ $\$29,000$ (48.3%)
Equal shares value (share)	cabin $-$ $\$27,250$ $\$38,750$ (41.7%)	car + silver + $\$2,250$ $\$41,250$ (41.7%)	$\$25,000$ $\$25,000$ (41.7%)
Knaster's value (share)	cabin $-$ $\$28,000$ $\$38,000$ (40.9%)	car + silver + $\$1,000$ $\$40,000$ (40.4%)	$\$27,000$ $\$27,000$ (45.0%)

Notice that from her perspective, Carol receives only 22.2% of the estate in the first-try division. Since she is supposed to have $1/3 \approx 33.3\%$ ownership of the estate, the first-try division seems extremely unfair to Carol. We formalize this concept of fairness by defining the following property.

Proportionate Property: A division is *proportionate* if each player's share is at least $1/n$, where n is the number of players.

A scan of the **Inheritance** table reveals that all shares are at least $\frac{1}{3} = 33.3\%$ except for Carol's share in the first-try division, and so while the first-try division is not proportionate, the other four divisions are proportionate. The proportionate property for fair division game divisions is similar, in spirit, to the rational property for bargaining game payoff pairs and coalition game allocations: all three specify minimum acceptable payoffs for each player. However, the rational properties are based on what is possible for each player without cooperation with the others. It is not clear what an individual player can do without an agreement in a fair division game, and so the minimum is based on ownership rights.

Observe that the sum of each row of the **Inheritance** share table is greater than 100%. For example, the sum of the first-price auction shares is $37.6\% + 35.4\% + 58.3\% = 131.3\%$. What does this mean? How could the players receive more than 100% of the estate? In section 2 we briefly discussed this in the context of the first-try division, but it is a point worth repeating and expanding upon. Remember that each share is computed from one player's perspective. For the first-price auction, Bob thinks that he has received 37.6% of the estate, Carol thinks that she has received 35.4% of the estate, and Doug thinks that he has received 58.3% of the estate. Since these numbers are from three different perspectives, there is no clear meaning attached to their sum.

If we instead focus on the perspective of one player, then a meaningful sum can be found. Again for the first-price auction, Bob thinks that he has received 37.6% of the estate. Continuing to take Bob's perspective, he thinks that the car and silverware Carol has received is valued at $\$10,000 + \$17,000 = \$27,000$. Since Carol also gave away $\$4,000$, Bob thinks that the value of the items Carol received and her money transfer is $\$27,000 - \$4,000 = \$23,000$. Since he values the estate at $\$93,000$, Bob thinks that Carol has received $\frac{\$23{,}000}{\$93{,}000} \approx 24.7\%$ of the estate. Similarly, Doug received $\$35,000$, which from Doug's perspective is $\frac{\$35{,}000}{\$60{,}000} \approx 58.3\%$ of the estate; however, from Bob's perspective, Doug received $\frac{\$35{,}000}{\$93{,}000} \approx 37.6\%$ of the estate. So, from Bob's perspective, he himself received 37.6% of the estate, Carol received 24.7% of the estate, and Doug received 37.6% of the estate. The sum of these percentages (up to round-off error) is 100%. This means that from Bob's perspective, the entire estate was distributed among the three players.

If the players all agreed on the value of each item, the best we could do would be to give each player exactly one-third of the estate. What makes it possible for each player to think that he or she has obtained more than one-third of the estate is that the players value each item differently, and the suggested divisions give the items to players who value the items more. So, the disagreements over what value to give to each item, which led to arguments and exasperation among the siblings, is exactly what makes it possible for each sibling to think that he or she has received more than the one-third share to which he or she is entitled!

Remembering whose perspective is being considered is important when viewing the **Inheritance** table. Otherwise, you may incorrectly argue that equal shares is not fair to Doug because he receives only $\$25,000$ while Carol receives $\$41,250$. But these are the values from Doug and Carol's perspectives, respectively. From Doug's perspective, what Carol receives (car + silverware + $\$2,250$) is worth $\$7,000 + \$11,000 + \$2,250 = \$20,250$, and so Doug thinks that the $\$25,000$ in cash that he receives is far better than what Carol receives. Of course, from Carol's perspective, what she receives is worth \$41,250, which is better than the \$25,000 in cash that Doug receives. So, in the equal shares division, both Carol and Doug are happy with what they have each received in comparison with what the other has received.

This is not the case for the first-try division. There Carol values what Doug receives, the silverware and \$10,000, at $\$27,000 + \$10,000 = \$37,000$, which is more than \$22,000, her value for what she receives. Carol would be happy to trade with Doug (although Doug would not be happy to trade with Carol). In other words, Carol would prefer to have what Doug receives instead of what she receives. We say that Carol is envious of Doug. It would be good for a division to avoid envy.

> **Envy Free Property:** A division is *envy free* if each player values the items and money received or spent at least as much as that player would value the items and money received or spent by any other player.

By the previous discussion, the first-try division for **Inheritance** is not envy free. In order to show that a division is envy free, we must consider the value of what each player receives from the perspective of each player. We have done that for the equal shares division in Table 3.2.

TABLE 3.2. An envy free analysis of the equal shares division for **Inheritance**

Player	Bob	Carol	Doug
Received	cabin − \$27,250	car + silver + \$2,250	\$25,000
Bob's perspective	$\$66,000 - \$27,250$ $= \$38,750$	$\$10,000 + \$17,000 + \$2,250$ $= \$29,250$	\$25,000
Carol's perspective	$\$60,000 - \$27,250$ $= \$32,750$	$\$12,000 + \$27,000 + \$2,250$ $= \$41,250$	\$25,000
Doug's perspective	$\$42,000 - \$27,250$ $= \$14,750$	$\$7,000 + \$11,000 + \$2,250$ $= \$20,250$	\$25,000

The \$20,250 value of Carol's items and money from Doug's perspective was calculated in a previous paragraph. From Carol's perspective, what Bob receives (cabin − \$27,250) is worth $\$60,000 - \$27,250 = \$32,750$. The other cells of the table are calculated in a similar manner. Now it is important to notice that

- Bob values what he receives (\$38,750) more than what the other players receive (\$29,250 and \$25,000),
- Carol values what she receives (\$41,250) more than what the other players receive (\$32,750 and \$25,000), and
- Doug values what he receives (\$25,000) more than what the other players receive (\$14,750 and \$20,250).

Thus, the equal shares division for **Inheritance** is envy free. In the exercises, you are asked to verify that the first-price auction, second-price auction, and Knaster's divisions for **Inheritance** are envy free, too.

We have seen already that the first-try division is not fair because it is neither proportionate nor envy free. A third reason why first-try is not fair is that it is possible to make someone better off without making anyone worse off, as we now show. By having Carol and Doug trade $11,000 for the silverware, the Better division is obtained, as shown in Table 3.3.

TABLE 3.3. A better division for **Inheritance**

	Bob	Carol	Doug
First-try value	cabin − $20,000 $46,000	car + $10,000 $22,000	silver + $10,000 $21,000
Better value	cabin − $20,000 $46,000	car + silver − $1,000 $28,000	$21,000 $21,000

Notice that Bob and Doug are indifferent between first-try and better. However, Carol values what she receives in better at $\$12,000 + \$27,000 - \$1,000 = \$28,000$, which is bigger than Carol's value for what she receives in first-try, $22,000. The better division makes Carol better off (i.e., increases her value) without hurting either of the other players (i.e., does not decrease their values). Bob, Carol, and Doug should not settle for first-try when they can have better.

> **Efficient Property:** A division is *efficient* if any other division that increases the value of one or more players decreases the value of one or more players.

This is the same efficiency concept used for strategic game payoff pairs, bargaining game payoff pairs, and coalition game allocations.

By the previous discussion, the first-try division is not efficient: in comparison with the first-try division, the better division increases Carol's value without decreasing Bob's and Doug's values.

Is the first-price auction division efficient? In comparison with the first-price auction division, the second-price auction division increases Carol's value but decreases Doug's value. In comparison with the first-price auction division, the division in which Bob receives the cabin but gives $30,000, Carol receives the car and silverware but gives $7,000, and Doug receives $37,000, increases Bob's and Doug's values but decreases Carol's value. In comparison with the first-price auction division, the division car to Bob, cabin to Carol, and silver to Doug increases Carol's value but decreases Bob's and Doug's values. We have compared the first-price auction division with three other divisions and have, so far, not found a division that increases the value of one or more players without decreasing the value one or more of the other players. Have we checked enough? No! There are many more (in fact, an infinity) of other divisions we should check. It seems almost impossible to determine whether the first-price auction division (or any other division) is efficient.

It turns out that divisions that are efficient can be described fairly easily. Notice how we showed that first-try is not efficient. We "sold" an item from a player at that player's value to a player who valued the item more. This meant that only the player receiving the item had a value change, and that value would have to increase because the receiving player valued the item more than the player giving up the item. We can always do this kind of trade if an item is owned by a player who does not value it the most. On the other hand, if items are owned by players who value them the most, then there are no opportunities to increase the overall value of the items. So, any further trades that would benefit one or more players would have to harm one or more different players. Thus, we have just proven the following theorem.

Efficient Division Theorem: *A division is efficient if and only if each item is owned by a player who values the item the most.*

In the first step of each of the four methods, each item is assigned to a player who values it the most. By the Efficient Division Theorem, this is enough to guarantee that the division is efficient. Thus, the first-price auction, second-price auction, equal shares, and Knaster's **Inheritance** divisions are efficient. Notice the power of the Efficient Division Theorem: a check of an infinite number of other divisions (impossible to do in our finite lifetimes) was replaced with a single, easily completed check!

Since each player has equal ownership of the items, a fair division should treat all players equally. But what is equal treatment? Two possibilities are that the players receive equal values or equal shares.

Value Equitable Property: A division is *value equitable* if the players' values are equal.

Share Equitable Property: A division is *share equitable* if the players' shares are equal.

Looking again at the **Inheritance** summary table presented at the beginning of this section, we see that the three players' values are equal only for the first-price auction division. Similarly, we see that the three players' shares are equal only for the equal shares division.

We can summarize our findings so far in Table 3.4.

TABLE 3.4. Comparison of methods by properties for **Inheritance**

	Proportionate	Envy free	Efficient	Value equitable	Share equitable
First-try	no	no	no	no	no
First-price	yes	yes	yes	yes	no
Second-price	yes	yes	yes	no	no
Equal shares	yes	yes	yes	no	yes
Knaster's	yes	yes	yes	no	no

Clearly, the first-try division is unacceptable since it satisfies none of the five fairness properties. The divisions selected by the four methods are much better since each satisfies three or four of the five fairness properties. Do these results change for different fair division games?

Consider the **Painting** fair division game, as summarized in Table 3.5.

TABLE 3.5. A summary of divisions for **Painting**

	Sarah	Elizabeth	Nada
Painting	\$900	\$810	\$90
First-price auction division value (share)	painting − \$600 \$300 (33.3%)	\$300 \$300 (37.0%)	\$300 \$300 (333.3%)
Second-price auction division value (share) division	painting − \$540 \$360 (40.0%)	\$270 \$270 (33.3%)	\$270 \$270 (300.0%)
Equal shares division value (share)	painting − \$450 \$450 (50.0%)	\$405 \$405 (50.0%)	\$45 \$45 (50.0%)
Knaster's division value (share)	painting − \$500 \$400 (44.4%)	\$370 \$370 (45.7%)	\$130 \$130 (43.3%)

It is easy to verify four of the properties for each of the divisions. Since each player's share for each division is at least $\frac{1}{3} = 33.3\%$, all four divisions are proportionate. All four divisions give the painting to the player who values it the most, so all four divisions are efficient. Since the three players' values are equal only for the first-price auction division, only the first-price auction division is value equitable. The three players' shares are equal only for the equal shares division, so only the equal shares division is share equitable.

Verifying the envy-free property for each division requires more thought. For the first-price auction division, Sarah values each portion at \$300, Elizabeth values her and Nada's portions at \$300 and Sarah's portion at only \$210, and Nada values her and Elizabeth's portions at \$300 and Sarah's portion at a negative amount. So, the first-price auction division is envy free. For the second-price auction division, Sarah values her portion at \$360 and Elizabeth's and Nada's portion at only \$270, Elizabeth values each portion at \$270, and Nada values her and Elizabeth's portions at \$270 and Sarah's portion at a negative amount. So, the second-price auction division is envy free. The equal shares division is not envy free because Nada would prefer to have the \$405 received by Elizabeth rather than the \$45 that she received. Finally, Knaster's division is not envy free because Nada would prefer to have the \$370 received by Elizabeth rather than the \$130 that she received.

We summarize our analysis in Table 3.6.

TABLE 3.6. Comparison of methods by properties for **Painting**

	Proportionate	Envy free	Efficient	Value equitable	Share equitable
First-price	yes	yes	yes	yes	no
Second-price	yes	yes	yes	no	no
Equal shares	yes	no	yes	no	yes
Knaster's	yes	no	yes	no	no

The four method divisions for **Inheritance** and **Painting** have the same properties except that the equal shares and Knaster's divisions for **Inheritance** are envy free while the equal shares and Knaster's divisions for **Painting** are not envy free. What properties would the divisions recommended by our four methods satisfy for other fair division games? We answer that question in the next section.

Exercises

(1) Consider the **Three Brothers** game described in section 8.2, exercise 1.
  (a) Which of the five divisions is proportionate? Explain your answers.
  (b) Which of the five divisions is envy free? Explain your answers.
  (c) Which of the five divisions is efficient? Explain your answers.
  (d) Which of the five divisions is value equitable? Explain your answers.
  (e) Which of the five divisions is share equitable? Explain your answers.

(2) Consider the **Four Siblings** game described in section 8.2, exercise 2.
  (a) Which of the five divisions is proportionate? Explain your answers.
  (b) Which of the five divisions is envy free? Explain your answers.
  (c) Which of the five divisions is efficient? Explain your answers.
  (d) Which of the five divisions is value equitable? Explain your answers.
  (e) Which of the five divisions is share equitable? Explain your answers.

(3) Consider the **Divorce** game described in section 8.2, exercise 3.
  (a) Which of the five divisions is proportionate? Explain your answers.
  (b) Which of the five divisions is envy free? Explain your answers.
  (c) Which of the five divisions is efficient? Explain your answers.
  (d) Which of the five divisions is value equitable? Explain your answers.
  (e) Which of the five divisions is share equitable? Explain your answers.

(4) Consider the **Candy Bar** game described in section 8.2, exercise 4.
  (a) Which of the five divisions is proportionate? Explain your answers.
  (b) Which of the five divisions is envy free? Explain your answers.
  (c) Which of the five divisions is efficient? Explain your answers.
  (d) Which of the five divisions is value equitable? Explain your answers.
  (e) Which of the five divisions is share equitable? Explain your answers.

(5) Verify that the first-price auction, second-price auction, and Knaster's divisions for **Inheritance** are envy free.

(6) Which of the properties described in this section do you believe to be most descriptive of fairness? Does your answer depend on the context of the fair division game? Are there other properties that you believe to be descriptive of fairness?

(7) Sometimes players do not have equal ownership in the estate. For example, see **Uneven Inheritance** described in the section 8.2 exercises. How would you redefine (if at all) proportionate, envy free, efficient, value equitable, and share equitable if each player has a specified ownership fraction of the estate?

(8) Sometimes rather than dividing items that players want ("goods"), the problem is to divide items that the players do not want ("bads"). For example, see **Ailing Parent** and **Committee Work** described in the section 8.2 exercises. How would you redefine (if at all) proportionate, envy free, efficient, value equitable, and share equitable if the items are "bads" rather than "goods"?

4. Choosing a Fair Method

If we define fairness by a number of properties, then we should determine which methods satisfy which properties. We have done that for four methods (first-price auction, second-price auction, equal shares, and Knaster's) on a couple of specific games (**Inheritance** and **Painting**). Do the divisions generated for other games by the four methods always satisfy the properties that the divisions for **Inheritance** and **Painting** had? More generally, are there other methods that satisfy more fairness properties?

> By the end of this section, you will be able to make some general arguments about when certain methods satisfy certain properties and to determine an appropriate method to use depending upon which fairness properties are considered important.

We answer the first question with the following theorem. This single theorem consists of twenty subtheorems, one for each cell in the table. Therefore, the entire proof is rather long, but it can be easily digested one paragraph or subproof at a time.

Fair Division Properties Theorem: *Table 4.1 describes which of the four division methods satisfy which of the five properties. By "always," we mean that the division always satisfies the property. By "rarely," we mean that the division satisfies the property only under very specific conditions, and by "sometimes," we mean there are many games for which the property is satisfied and many for which it is not.*

TABLE 4.1. A comparison of methods by property

	Proportionate	Envy free	Efficient	Value equitable	Share equitable
First-price	always	always	always	always	rarely
Second-price	always	always	always	rarely	rarely
Equal shares	always	sometimes	always	rarely	always
Knaster's	always	sometimes	always	rarely	rarely

PROOF. This will be a long proof, but only because we are really proving twenty separate statements (one for each "always/sometimes/rarely" cell in Table 4.1). We will treat at most one column at a time. The first sentence of each paragraph states what will be proved in the rest of the paragraph.

Equal shares and Knaster's divisions are always proportionate. Recall our earlier discussion about summing the shares: because the sum of the largest value for each item is at least as great as any individual player's value for the estate, the sum of the shares will always be at least 100% and definitely more if there are some differences among the values players place on the items. This makes it always possible for each player to receive at least $\frac{1}{n}$ of the estate from his or her perspective. Since equal shares gives each player the same share of the estate, it must be that each player's share is at least $\frac{1}{n}$. Thus, equal shares divisions are always proportionate. In Knaster's, each player is given a $\frac{1}{n}$ share. Since these amounts sum to no more than what is available, the additional amount A is nonnegative, and so each player receives at least a $\frac{1}{n}$ share. Thus, Knaster's divisions are always proportionate.

First-price auction divisions are always proportionate. When the first-price auction method is applied to a single item, a player who has the highest value for the item, call it u, is assigned the item and pays u into the pot. Call the player assigned the item the winner and call the other players losers. The money in the pot is then divided equally among the players, and so each player receives $\frac{u}{n}$ in money. The winner values what he receives at $u - u + \frac{u}{n} = \frac{u}{n}$ and has a $\frac{(\frac{u}{n})}{u} = \frac{1}{n}$ share of the item. A loser values the item at $v \leq u$ and receives $\frac{u}{n}$ in money, and so has a $\frac{(\frac{u}{n})}{v}$ share of the item. Since $\frac{(\frac{u}{n})}{v} \geq \frac{(\frac{v}{n})}{v} = \frac{1}{n}$, each player has at least a $\frac{1}{n}$ share of the item. The first-price auction division for multiple items can be obtained by combining the first-price auction divisions for each item taken individually. Since each player obtains at least $\frac{1}{n}$ share of each item, each player obtains at least $1/n$ share of the estate. Thus, first-price auction divisions are always proportionate.

Second-price auction divisions are always proportionate by a similar argument (see the exercises).

First-price auction divisions are always envy free. Let u again represent the highest value of an item. The winner of the item is assigned the item, pays $u - \frac{u}{n}$, and values the combination at $u - u + \frac{u}{n} = \frac{u}{n}$. Each loser receives $\frac{u}{n}$ in money. So, the winner does not envy the losers. A loser values the item at $v \leq u$, and so values what the winner received at $v - u + \frac{u}{n} \leq \frac{u}{n}$, which is the amount of money received by the loser. Since no player envies another over a first-price auction division of a single item and the first-price auction division for all items simultaneously can be obtained by combining the first-price auction division for each item taken individually, the first-price auction divisions are always envy free.

Second-price auction divisions are always envy free by a similar argument (see the exercises).

Equal shares and Knaster's divisions are sometimes envy free. We have already seen that the equal shares and Knaster's divisions can be either envy free (in **Inheritance**) or not envy free (in **Painting**). A closer look at Table 3.2 reveals that the closest anyone gets to being envious is Doug, who finds what Carol received to be worth only \$4,750 less than what he received. We can obtain many different games by making changes in the values that the players assign to the items; if the differences in the resulting values do not change by more than \$4,750, then the equal share division will remain envy free. Similarly, fairly large changes in the original **Painting** values would not change that Nada is envious of Elizabeth. Thus, there are many games in which the equal shares division is envy free and many games in which the equal shares division is not envy free.

First-price auction, second-price auction, equal shares, and Knaster's divisions are always efficient. In the first step of each of the four methods, each item is assigned to a player who values it most. By the Efficient Division Theorem, this is enough to guarantee that the division is efficient. Thus, the divisions selected by each of the four methods are always efficient.

First-price auction divisions are always value equitable. As in earlier arguments, let u be the value of the item to the player who receives it. As shown previously, the player who is assigned the item values the item and money paid out at $u - u + \frac{u}{n} = \frac{u}{n}$. The remaining players receive $\frac{u}{n}$ in money, which they each value at $\frac{u}{n}$. Clearly, each player receives the same value $\frac{u}{n}$.

Second-price auction, equal shares, and Knaster's divisions are rarely value equitable. Suppose s is the share received by each player in an equal shares division. If two players value the estate at u and v, respectively, then they value what they receive at su and sv, respectively. To be value equitable, $su = sv$, or $u = v$. That is, an equal shares division is value equitable if and only if the players have the same value for the estate. Of course, it is possible that all players would have the same value for the estate, but it would not be likely. A similar argument can be used for Knaster's divisions (see the exercises). The argument for second-price auction divisions is a bit trickier; the key idea is that for a division to be value equitable requires that the n values of the division must satisfy a particular set of n linear equations, and that would be an extremely rare occurrence.

Equal shares divisions are always share equitable. The second step of the method is to "[h]ave each player give or receive money so that every player receives the same share of the estate." So, by definition, equal shares divisions are always share equitable.

First-price auction, second-price auction, and Knaster's divisions are rarely share equitable. We have already seen that a first-price auction division is value equitable, that is, each player has the same value, call it r, for what they have received in a first-price auction division. If two players value the estate at u and v, respectively, then their share of the estate is $\frac{r}{u}$ and $\frac{r}{v}$, respectively. But $\frac{r}{u} = \frac{r}{v}$ if and only if $u = v$. So, a first-price auction division is share equitable if and only if the players have the same value for the estate, something that would occur only rarely. A similar argument can be used for Knaster's divisions (see the exercises). The argument for second-price auction divisions is again a bit trickier; the key idea again is that for a division to be share equitable requires that the n values of the division must satisfy a particular set of n linear equations, and that would be an extremely rare occurrence. $\square$

So, what does this theorem tell us? For a division to be fair, it seems that it should be efficient (there should not be another division which is better for some players without being worse for any player), proportionate (each player should receive at least a $\frac{1}{n}$ share of the estate from their own perspective), envy free (no player should prefer what a different player received), value equitable (the players should receive equal monetary values), and share equitable (the players should receive equal shares). Unfortunately, none of the four methods we have described always satisfies these five properties. Is there a different method that would satisfy all five properties? The answer is "no".

Characterization Theorems

We have defined five properties that capture elements of fairness; it is surprising that a method can be completely characterized by only two of the five properties. Together, the following two theorems demonstrate that there is no method that will always produce efficient, value equitable, and share equitable divisions. Each theorem tells us what method to choose if we want to be guaranteed value equitable or guaranteed share equitable divisions.

> **Value-Equitable Division Theorem:** *A division is efficient and value equitable if and only if it is the first-price auction division.*

PROOF. By the Fair Division Properties Theorem, the first-price auction divisions are always efficient and value equitable. Conversely, suppose that a division is efficient and value equitable. We will show that this division is the first-price auction division. By the Efficient Division Theorem, our supposed efficient division must assign the items to players who value the items the most; this is the first step of the first-price auction method. Let u_j be the value the jth player has for the items assigned to the jth player, and let d_j be the dollars the jth player will receive (and so will be negative if the jth player gives money away). Since money is being transferred from player to player, the sum of the dollars received should be zero: $d_1 + d_2 + \cdots + d_n = 0$. Because our supposed division is value equitable, $u_1 + d_1 = u_2 + d_2 = \cdots = u_n + d_n$. By temporarily naming the common value v, the previous equalities can be written as follows.

$$\begin{aligned} u_1 + d_1 &= v \\ u_2 + d_2 &= v \\ &\vdots \\ u_n + d_n &= v. \end{aligned}$$

Summing these equations, we obtain

$$u_1 + u_2 + \cdots + u_n + d_1 + d_2 + \cdots + d_n = nv.$$

Recalling that $d_1 + d_2 + \cdots + d_n = 0$ and dividing by n, we obtain

$$v = \frac{1}{n}(u_1 + u_2 + \cdots + u_n).$$

Substituting this value of v back into the earlier equalities, we obtain the following.

$$\begin{aligned} d_1 &= -u_1 + \frac{1}{n}(u_1 + u_2 + \cdots + u_n) \\ d_2 &= -u_2 + \frac{1}{n}(u_1 + u_2 + \cdots + u_n) \\ &\vdots \\ d_n &= -u_n + \frac{1}{n}(u_1 + u_2 + \cdots + u_n). \end{aligned}$$

In words, each player pays an amount equal to the value of the items received and then receives an equal share of the amount paid by each player. This is exactly the rest of the description of the first-price auction method. □

Since the first-price auction division is rarely share equitable, the previous theorem tells us that there is no method that will always select efficient, value equitable, and share equitable divisions.

> **Share Equitable Division Theorem:** *A division is efficient and share equitable if and only if it is the equal shares division.*

PROOF. By the Fair Division Properties Theorem, equal shares divisions are always efficient and share equitable. Conversely, suppose that a division is efficient and share equitable. We will show that this division is the equal shares division. By the Efficient Division Theorem, our supposed efficient division must assign the items to players who value the items the most; this is the first step of the equal shares method. Now our division is also supposed to be share equitable. But this is just the second step of the equal shares method. □

Since the equal shares division is rarely value equitable, the Share Equitable Division Theorem tells us, again, that there is no method that will always select efficient, value equitable, and share equitable divisions. Since the equal shares division is only sometimes envy free, this theorem also tells us that there is no method that will always select efficient, envy free, and share equitable divisions.

Even without efficiency, it is rare that a single division can be both value equitable and share equitable. For example, when there are three players and only one item, two of the players will not receive the item; to be value equitable, these two players must receive the same amount of money; and to be share equitable in addition, these two players must have the same value for the item.

Which of value equitable or share equitable is more descriptive of fairness? It depends on the context of the fair division game. In an actual inheritance, the legal requirement is usually that each person receives an appropriate proportion of the estate in terms of monetary values. Our difficulty is in determining the monetary value for each item. The individual differences of opinion about the monetary value for the items helps us to estimate a single value for each item and create a division that takes into account players' preferences. One could argue that while this technique of having players provide individual monetary valuations has therefore been useful, it should not change the legal requirement of making the division value equitable.

In other circumstances, one could argue that players' values for the items should have an even greater weight in the creation of a division. For example, if one player thinks that the estate is worthless, that is, the player places a monetary value of $0 on each item, then why should that player receive anything? It appears that the others are subsidizing this player's presence. This would, perhaps, lead one to argue that a division should be share equitable.

If you favor the idea that fairness involves the players receiving equal monetary values, then you should use the first-price auction division. As a bonus, the division will be proportionate and envy free.

If you favor the idea that fairness involves players receiving equal shares of the estate, then you should use the equal shares division. As a bonus, the division will be proportionate but, unfortunately, the division may not be envy free.

Efficient, Envy Free, and Close to Share Equitable

What if you definitely want an efficient and envy free division but favor equal shares over equal values as a guiding fairness principle? The equal shares division works sometimes, as in the **Inheritance** game. In games where the equal shares division is not envy free, such as in **Painting**, the goal would be to choose an efficient and envy free division that is as close to giving players equal shares as is possible. For a **Painting** division to be efficient, the painting must be given to Sarah. Let d_e and d_n be the amount of money Sarah gives to Elizabeth and Nada, respectively. For the division to be envy free, the following inequalities must hold:

$$\begin{array}{ll} 900 - d_e - d_n \geq d_e & \text{Sarah does not envy Elizabeth,} \\ d_e \geq 810 - d_e - d_n & \text{Elizabeth does not envy Sarah,} \\ 900 - d_e - d_n \geq d_n & \text{Sarah does not envy Nada,} \\ d_n \geq 90 - d_e - d_n & \text{Nada does not envy Sarah,} \\ d_e \geq d_n & \text{Elizabeth does not envy Nada,} \\ d_n \geq d_e & \text{Nada does not envy Elizabeth.} \end{array}$$

The last two inequalities tell us that $d_n = d_e$. Using the symbol d to denote the amount both Elizabeth and Nada receive, we can simplify the first four inequalities to

$$270 \leq d \leq 300.$$

So, all efficient and envy free divisions are of the form shown in Table 4.2 with $270 \leq d \leq 300$.

TABLE 4.2. Template for efficient and envy free divisions for **Painting**

Player	Sarah	Elizabeth	Nada
Division	Painting $-$ \$2d	\$$d$	\$$d$
Value	\$900 $-$ 2d	\$$d$	\$$d$
Share	$\frac{900-2d}{900}100\%$	$\frac{d}{810}100\%$	$\frac{d}{90}100\%$

When $d = 270$, we obtain the second-price auction division. When $d = 300$, we obtain the first-price auction division. So, the two auction divisions are at the "endpoints" of the envy free divisions. We are interested in making player shares as equal as possible, so let's look at the possible player shares graphically, as shown in Figure 4.1.

We can see that Nada will have a relatively large 300% to 333% share of the estate, but Sarah's and Elizabeth's shares of the estate are somewhat comparable. Nada's share and Elizabeth's share both increase with increasing d because they

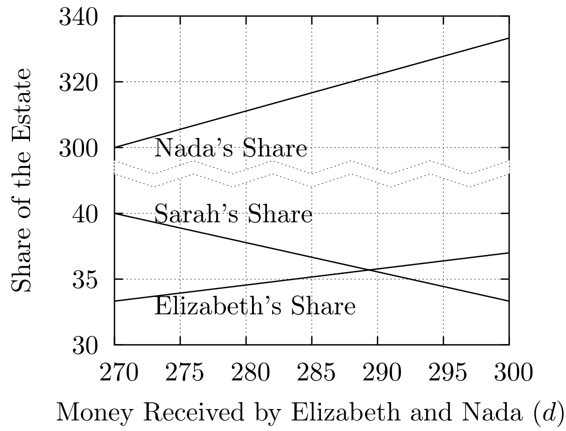

FIGURE 4.1. Efficient and envy free **Painting** divisions

are receiving more money. Although Nada's share and Elizabeth's share seem to be increasing in parallel, Nada's share is actually increasing more rapidly because $\frac{1}{90} > \frac{1}{810}$. Sarah's share decreases with increasing d because that results in her giving more money to the other women.

Now it is important to decide what you mean by being "close" to giving players equal shares. For example, suppose that you want the smallest share to be as big as possible. On the graph, we need to find where the lowest of the three graphs is as high as possible. Notice that this will occur when Sarah's and Elizabeth's shares of the estate are equal:

$$\begin{aligned}\frac{900-2d}{900} &= \frac{d}{810},\\ 810(900-2d) &= 900d,\\ 900\times 810 &= (900+2\times 810)d,\\ d &= \frac{900\times 810}{900+2\times 810} = \frac{2025}{7} \approx 289.\end{aligned}$$

Suppose, instead, that you want the spread of shares to be as small as possible. From the graph, it can be seen that this will not occur for $d > 289$ because the difference between Nada's largest share and Sarah's smallest share is increasing when $d \geq 289$. Hence, we only need to check for $270 \leq d \leq 289$. In that interval, Nada's largest share increases at the rate $\frac{1}{90}$, while Elizabeth's share increases at the rate $\frac{1}{810}$, and thus the spread is increasing (at the rate $\frac{1}{90} - \frac{1}{810} > 0$). Therefore, the minimum spread occurs when $d = 270$, which corresponds to the second-price auction division.

We can summarize what we have learned in Table 4.3.

Notice that the smallest share in the Maximize the smallest share division is 35.7%, which is larger (as should be the case) than the smallest share 33.3% in either of the other two divisions. Similarly, the second-price auction division spread of shares, $300.0 - 33.3 = 266.7$, is smaller than the first-price auction division spread of shares,

TABLE 4.3. The efficient and envy free divisions for **Painting**

	Sarah	Elizabeth	Nada
First-price auction	painting − \$600	\$300	\$300
value	\$300	\$300	\$300
(share)	(33.3%)	(37.0%)	(333.3%)
Maximize the smallest share	painting − \$578	\$289	\$289
value	\$322	\$289	\$289
(share)	(35.7%)	(35.7%)	(321.1%)
Second-price auction Minimize the spread of shares	painting − \$540	\$270	\$270
value	\$360	\$270	\$270
(share)	(40.0%)	(33.3%)	(300.0%)

$333.3 - 33.3 = 300$, or the Maximize the smallest share division spread of shares, $321.1 - 35.7 = 285.4$.

Is the second-price auction division always the efficient and envy free division that minimizes the spread of shares? The answer is "no" (see the exercises).

So, our grappling with fairness properties has suggested two additional methods. If you definitely want an efficient and envy free division and would like to equalize shares as much as possible, then you can either choose to maximize the smallest share or to minimize the spread of shares among efficient and envy free divisions. As we did above, it is possible to calculate either of these divisions by hand if there is only one item. Unfortunately, if there is more than one item and the equal shares division does not turn out to be envy free, then finding either one of these divisions involves solving a linear program. Since linear programming is studied in other mathematics courses, perhaps this book has motivated you to want to take one of them!

Exercises

(1) Explain why second-price auction divisions are always proportionate.
(2) Explain why second-price auction divisions are always envy free.
(3) Explain why a Knaster's division is value equitable if and only if the players have the same value for the estate.
(4) Explain why a Knaster's division is share equitable if and only if the players have the same value for the estate.
(5) Explain why an envy free division is also proportionate.
(6) **Candy Bar.** Recall this fair division game first described in section 8.2, exercise 4. Suppose three friends have been given a gift of a large and exquisite chocolate candy bar. Of course, they could simply divide the candy bar into thirds. However, they have been reading this book, and each is willing to give the others some money for the privilege of having the entire candy bar. Leslie is willing to give \$6.00, Michael is willing to give \$4.20, and Owen is willing to give \$3.00.

(a) Determine the efficient and envy free division that maximizes the smallest share and each friend's value and share for what is received.
(b) Determine the efficient and envy free division that minimizes the spread of shares and each friend's value and share for what is received.
(c) Compare the two divisions obtained here and the four divisions obtained in the previous section. From your perspective, which division is the most fair? Why?

(7) Which of the properties described in this section do you believe to be most descriptive of fairness? Does it depend on the context of the fair division game? Are there other properties that you believe to be descriptive of fairness?

(8) Name three game theory solution concepts described in this book that involve maximizing a minimum.

(9) **Apartment.** The following scenario cannot be modeled as a fair division game (the way we have defined one); however, we can use many of the same ideas. Diana, Stacey, and Tien-Yee plan to share a three bedroom townhouse apartment. They must still decide which bedroom each will use and how much each should contribute to the \$600 per month rent. There are two bedrooms on the second floor: the largest one and the intermediate size one with the most picturesque view. The third bedroom is the smallest and on the first floor. The women individually determined how much they thought each room was worth; these values are summarized in Table 4.4. Note that each woman needs to assign values to the three rooms so that the sum equals the rent.

TABLE 4.4. Values for **Apartment**

	Diana	Stacey	Tien-Yee
Large	\$300	\$200	\$275
Medium	\$200	\$250	\$225
Small	\$100	\$150	\$100
Apartment	\$600	\$600	\$600

(a) A reasonable approach would be to assign a room to the person who values it most. However, even though Stacey places the highest value on two of the rooms, it does not make sense to give her both rooms. We could give Diana the large bedroom, Stacey the medium bedroom, and Tien-Yee the small bedroom. Given what the women are willing to pay for their bedrooms, this assignment would mean the women would be willing to pay $\$300 + \$250 + \$100 = \$650$, and since the rent is only \$600, they can each pay less than they were willing, saving \$50. Consider each of the six possible ways to assign bedrooms among the women and compute the sums of what the women are willing to pay. Which bedroom assignment maximizes the amount they save?
(b) To obtain an efficient division, the bedroom assignment should be one that maximizes the sum of what the women are willing to pay. An auction-like division would have each woman first pay the amount they value the bedroom assigned. The sum of these payments will be greater than the \$600 needed for the rent. One-third of the amount over \$600 would be returned to each woman. Find this division and show that one woman envies another.

(c) A different approach would be to have each woman first pay the highest value, which may be different from their own value, for the bedroom they have been assigned. The sum of these payments will be greater than the \$600 needed for the rent. One-third of the amount over \$600 would be returned to each woman. Find this division and show that no woman envies any other woman.

(d) What division would you propose? Why?

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Epilogue

It is very possible that you are quite exhilarated by the things that you've learned in this book. At a minimum, you've had a chance to improve your algebra skills and mathematical reasoning skills. Even better, you might appreciate the work that mathematicians do more than you had previously. But better still, you may have gained some insight into how to arrive at decisions in problems involving more than one person, each of whose actions are important and critical to the outcome.

From Chapter 1, you learned the value of developing action plans (strategies) completely, so that you will be able to respond effectively to any situation that arises. The work in that chapter also helped to improve your ability to develop these strategies by understanding the different outcomes that may result from a particular course of action. Backward analysis provides you with a method for selecting an optimal strategy when the game tree is available.

Following this, in Chapter 2, you realized the value of being able to prioritize the possible outcomes of a situation, and understand that other players may not have the same priorities as yourself. Although it may not be easy to verify that a player ranks or assigns intensities to outcomes in a specified way, the attempt to do so helps us to recognize explicitly the assumptions we are making in modeling a person, company, or nation as a player and provides useful insights into why players act the way they do. Finally, the Analytic Hierarchy Process provided you with a method for making difficult decisions in your own life.

Chapters 3 and 4 pull together the ideas of the previous chapters to investigate situations in which the several players are interacting, but prioritizing their own interests. To be successful in these situations, you learned that it helps to truly understand yourself: are you risk adverse, risk loving, or risk neutral? It also helps to understand the other players: are they cautious, or do they have "all or nothing" attitudes? With these important traits clearly identified, the methods in these chapters lead you to take effective action to reach the highest possible payoff. Your study of game theory experiments and the **Prisoners' Dilemma** game in Chapter 5 taught you about credibility, promises, threats, trust, short versus long term gains, and altruism.

In Chapters 6, 7, and 8 you investigated situations in which players are attempting to cooperate, although they have differing goals and values. In all of these games, you observed that the key to selecting the most effective method and achieving the optimal outcome is to come to an agreement on what fairness means.

In sum, this study of game theory taught you that when making decisions, it is absolutely critical to know and understand the framework of the situation: Who are the players? What courses of action are available to each? What are the possible consequences of our collective actions? How do each of us rank or value the different possible outcomes? Are there opportunities for collaboration or negotiation? Can any agreements made be enforced?

On the other hand, it is also possible that you are frustrated and very confused after reading this book. It may seem that mathematical modeling hasn't contributed anything to your decision-making process. In Chapter 1, you learned many deterministic games, but you learned how to win only one of them. Zermelo's Theorem told you that under a very particular set of conditions, one of the players always has a winning strategy, and Nash's Hex Theorem told you that in the game **Hex**, this is always the first player. But neither theorem tells you what this winning strategy is. For games like Chess, you only know that optimal strategies exist, without knowing what they are. In Chapters 3, 4, and 5, you discovered how to play various strategic games and that the Nash equilibrium is the best solution. But from studying the **Prisoners' Dilemma** game in Chapter 5, you know that this sometimes gives less than optimal payoffs. Further, in many games there is more than one Nash equilibria, and so you don't know which one to select. You could decide to use a prudential strategy, but then you might find yourself regretting your decision.

When you studied bargaining games, coalition games, and fair division games in Chapters 6, 7, and 8, respectively, you discovered that each type of game has multiple, mutually incompatible, solutions. Getting a group of players to agree to a solution requires that the group agrees on their assumptions about fairness; for example, the egalitarian solution to the bargaining game assumes that players should get equal benefit from cooperation, while the Raiffa solution assumes that it is important that each player gets as close as possible to their individual goals. Your study of coalition games and fair division games also revealed that sometimes there is no method of solving the problems that satisfies all of the fairness conditions that you would like.

However, these unresolved questions are the life blood of mathematics. Both mathematical and social progress is made as individuals struggle to answer the questions left unresolved in this book. The answers that people have provided in the past and will provide in the future are widely recognized for contributing to our understanding of society and human behavior.

Nobel Prizes

We've mentioned several times that John Nash won a Nobel Prize for his work in game theory. He shared his 1994 prize with John Harsanyi and Reinhard Selton for their "pioneering analysis of equilibria in the theory of non-cooperative games". Not only did Nash develop the concept of what is now known as the Nash equilibrium, which we studied in Chapters 3, 4, and 6, but he was also the first to clearly distinguish between cooperative and noncooperative games. Nash was also one of the first people to propose the interpretations of the mathematical results that we saw in this book: mixed strategies can be viewed as either the choice of players making completely rational decisions or as a statistical average formed by many different players playing the game [**37**][**38**].

Since there are frequently more than one Nash equilibria, Reinhard Selton investigated the stronger conditions necessary to eliminate some Nash equilibria, or to at least avoid selecting them [**51**]. He further refined Nash's equilibrium concept for application to dynamic interactions: situations in which there is a small probability of error [**59**].

A key assumption made throughout this book is that players knew each other's preferences. Nash's second coprize winner, John Harsanyi, investigated situations in which the players are faced with incomplete information about each other's preferences. Harsanyi was able to show that under reasonable assumptions, every game with incomplete information could be transformed into a game with complete information and handled with standard methods [**22**].

More recently, in 2005, Robert Aumann and Thomas Schelling also shared a Nobel Prize in Economics for their work in game theory. Aumann's prize was awarded for developing a theory for understanding the long-term behavior of repeated play games. We introduced this topic with our work in Chapter 5, as we tried to resolve the **Prisoners' Dilemma** game. Aumann also introduced the concept of a correlated strategy, which we discussed in Chapter 6 [**3**][**4**].

Aumann's coprize winner, Thomas Schelling, was recognized for demonstrating that a player can strengthen his or her position in a game by worsening his or her options, by having a capacity to retaliate, and maintain uncertainty [**55**]. Among other contributions, Schelling was recognized for his application of game theoretic notions to many problems in the social sciences [**56**].

The Minimax Theorem

The Nobel Prize winning work described above was presented in a more mathematically abstract, general, and rigorous way than has been presented in this book. Mathematicians want to know that their results are precise, cover as many cases as possible, and are true beyond any doubt. To give you some sense of that mathematical abstraction, generality and rigor, we present a proof of the Minimax Theorem, which is essentially what we called the Zero-Sum Games Are Cool Theorem in section 4.5. The first proof of this result was given by John Von Neumann in 1928. We follow here a proof by Carsten Thomassen published in 2000 [**67**].

> **Minimax Theorem:** *Consider a zero-sum game in which Rose has m strategies, Colin has n strategies, and the payoff to Rose is a_{ij} and the payoff to Colin is $-a_{ij}$ when Rose uses strategy i* and *Colin uses strategy j. There exists at least one mixed prudential strategy for each player. Further, if $x = (x_1, x_2, \ldots, x_m)$ is a mixed prudential strategy and α is the security level for Rose and $y = (y_1, y_2, \ldots, y_n)$ is a mixed prudential strategy and β is the security level for Colin, then $\alpha = -\beta$ and (x, y) is a Nash equilibrium.*

PROOF. We will not prove the existence of prudential strategies here.

We first prove that Rose's security level is no greater than the negative of Colin's security level; symbolically, $\alpha \leq -\beta$. If Rose uses the mixed strategy x and Colin uses the pure strategy j, then Rose's payoff is $x_1a_{1j} + x_2a_{2j} + \cdots + x_ma_{mj}$. Since x is a prudential strategy and α is the security level for Rose, α must equal the smallest such payoff, which implies that $\alpha \leq x_1a_{1j} + x_2a_{2j} + \cdots + x_ma_{mj}$ for each pure strategy j. Since $y_1 + y_2 + \cdots + y_n = 1$ and $y_j \geq 0$ for each pure strategy j, it follows that

$$\begin{aligned}
\alpha &= (y_1 + y_2 + \cdots + y_n)\alpha \\
&= y_1\alpha + y_2\alpha + \cdots + y_n\alpha \\
&\leq y_1(x_1a_{11} + x_2a_{21} + \cdots + x_ma_{m1}) \quad + \\
&\quad\; y_2(x_1a_{12} + x_2a_{22} + \cdots + x_ma_{m2}) \quad + \\
&\quad\; \cdots \quad + \\
&\quad\; y_n(x_1a_{1n} + x_2a_{2n} + \cdots + x_ma_{mn})
\end{aligned}$$

This expression can be rearranged in the following manner.

$$\begin{aligned}
\alpha &\leq y_1x_1a_{11} + y_1x_2a_{21} + \cdots + y_1x_ma_{m1} \quad + \\
&\quad\; y_2x_1a_{12} + y_2x_2a_{22} + \cdots + y_2x_ma_{m2} \quad + \\
&\quad\; \cdots \quad + \\
&\quad\; y_nx_1a_{1n} + y_nx_2a_{2n} + \cdots + y_nx_ma_{mn} \\
&= x_1y_1a_{11} + x_1y_2a_{12} + \cdots + x_1y_na_{1n} \quad + \\
&\quad\; x_2y_1a_{21} + x_2y_2a_{22} + \cdots + x_2y_na_{2n} \quad + \\
&\quad\; \cdots \quad + \\
&\quad\; x_my_1a_{m1} + x_my_2a_{m2} + \cdots + x_my_na_{mn} \\
&= x_1(y_1a_{11} + y_2a_{12} + \cdots + y_na_{1n}) \quad + \\
&\quad\; x_2(y_1a_{21} + y_2a_{22} + \cdots + y_na_{2n}) \quad + \\
&\quad\; \cdots \quad + \\
&\quad\; x_m(y_1a_{m1} + y_2a_{m2} + \cdots + y_na_{mn})
\end{aligned}$$

Since y is a prudential strategy and β is the security level for Colin, $\beta \leq y_1(-a_{i1}) + y_2(-a_{i2}) + \cdots + y_n(-a_{in})$ for each pure strategy i. Since $x_1 + x_2 + \cdots + x_m = 1$ and $x_i \geq 0$ for each pure strategy i, it follows from the above expression that

$$\alpha \leq x_1(-\beta) + x_2(-\beta) + \cdots + x_m(-\beta) = -\beta.$$

Thus, $\alpha \leq -\beta$.

We now prove, by induction on $m+n$, that Rose's security level equals the negative of Colin's security level; symbolically, $\alpha = -\beta$. If $m = n = 1$, then each player must use her or his only strategy 1, Rose's payoff and security level is a_{11}, Colin's payoff and security level is $-a_{11}$, and thus the statement is true for $m + n = 2$.

Now assume that $m + n > 2$ and that the statement is true when there are fewer than $m + n$ strategies. Renaming Colin's strategies, if necessary, we may assume that when Rose uses her prudential strategy x, she receives her security level α

when Colin uses any of his first k pure strategies and she receives more than α when Colin uses any of his remaining $n - k$ pure strategies.

Assume first that $k < n$, and consider the smaller game in which Colin can use only his first k pure strategies, and let α' and β' be the security levels for Rose and Colin, respectively, in this smaller game. By the induction hypothesis, $\alpha' = -\beta'$. Since Colin has fewer strategies in the smaller game, $\alpha \leq \alpha'$ and $\beta' \leq \beta$. If $\alpha = \alpha'$, then by the previous two sentences, $\alpha \geq -\beta$, and thus by the second paragraph, $\alpha = -\beta$. If $\alpha < \alpha'$ instead, then there is a strategy x' for Rose for which she receives a payoff of at least α' against any of Colin's first k pure strategies. Imagine Rose using the strategy $z = ((1-t)x_1 + tx'_1, (1-t)x_2 + tx'_2, \ldots, (1-t)x_m + tx'_m)$ where t is a very small positive number. When Colin uses any of his first k pure strategies, Rose receives at least α when she uses x and receives at least α' when she uses x', and thus Rose receives more than α when she uses z. When Colin uses any of his last $n - k$ pure strategies, Rose receives more than α when she uses x, and thus Rose receives more than α when she uses z as long as t is small enough. But then z would ensure Rose a payoff greater than α, a contradiction to α being Rose's security level. Since this contradiction was derived from our assumption that $k < n$, we can conclude that $k = n$; that is, Rose receives her security payoff of α when she uses x and Colin uses any strategy.

Interchanging the roles of Rose and Colin, we can similarly argue that either the security level statement is true or that Colin receives his security payoff of β when he uses y and Rose uses any strategy. In particular, if Rose uses x and Colin uses y, then each receives her or his security level. Since this is a zero-sum game, the sum of the security levels must be 0, and so the security level statement follows.

Finally, consider the strategy pair (x, y). Since these are prudential strategies for each player, each player must receive at least her or his security level. Since the sum of the security levels is zero and this is a zero-sum game, each player must receive exactly her or his security level. Since y is prudential for Colin, any strategy change on the part of Rose will not decrease Colin's payoff and so not increase Rose's payoff since Rose's payoff is the negative of Colin's payoff. An analogous argument can be made for Colin. Thus, (x, y) is a Nash equilibrium. □

Further Reading

If you are interested in more rigorous treatments of the topics discussed in this book, we recommend that you read *Game Theory and Strategy* [**64**], *Introduction to Game Theory* [**68**], *Game Theory: Decisions, Interaction and Evolution* [**70**], or *An Introduction to Game-Theoretic Modeling* [**32**]. These are just a few of the many formal treatments of game theory that are available.

Finally, if you are interested in the topic of game theory, but would prefer to see different applications of the material rather than a rigorous treatment, we suggest that you read *Social Choice and the Mathematics of Manipulation* [**65**], *Mathematics and Politics* [**66**], or *Decisions and Elections* [**52**] to explore applications to politics, elections, and voting theory, which is a very popular topic at the moment.

Prisoner's Dilemma [**45**] is an excellent book about the interpretation of this game in many different contexts. *Strategy in Poker, Business, and War* [**30**] was the first popular book written on game theory and is still a fun read. There is also currently a lot of interest in experimental game theory, much like what we modeled in the dialogue for Chapter 5, which connects the theoretical game theory models to the actions that individuals actually take when confronted with these situations. Several, somewhat difficult, sources that you might read are *Behavioral Game Theory* [**11**] and *The Handbook of Experimental Economics* [**24**].

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Answers to Selected Exercises

Deterministic Games

A Very Simple Game

1. Yes, Fatima loses if Fatima chooses 9, Sunil chooses 7, Fatima chooses 8, Sunil chooses 5, and Fatima makes any legal move.

4. It really matters if you go first or second in this game because, as explained by Fatima, the first player can guarantee a win.

5. Fatima can still ensure a win using the strategy described in the dialogue with one addition. If the choices are 9, 6, and 5, then Sunil could choose 0. In response, Fatima should choose 5, resulting in a sum of 25. Sunil must now choose 8, 6, 4, or 2. The choices or 8 or 6 result in a win for Fatima. In response to 4 or 2, Fatima should choose 1 and 3, respectively, resulting in a sum of 30 when Sunil next moves.

Rules of the Game

2. (a) Deterministic. (b) Not deterministic. (c) Deterministic. (d) Not deterministic.

4. (a) The **Nim** games involving three beans: $[3]$, $[2, 1]$, $[1, 1, 1]$. (b) The **Nim** games involving four beans: $[4]$, $[3, 1]$, $[2, 2]$, $[2, 1, 1]$, $[1, 1, 1, 1]$. (d) The **Nim** games involving two heaps is infinite: $[1, 1]$, $[2, 1]$, $[3, 1]$, $[2, 2]$, $[4, 1]$, $[3, 2]$, and so forth.

Heuristics and Strategies

2. For $[6, 6, 4, 4]$, if Firstus takes x beans from a pile of n beans, then there is still a pile of n beans from which Secondus can take x beans, and this will remain true as the game progresses. Since Firstus cannot remove two piles in one move, mirroring always allows Secondus to remove the last pile when starting with $[6, 6, 4, 4]$.

4. (b) Firstus eventually removes the last heap since she is always faced with an odd number of heaps. (c) LARGE has Firstus remove one heap. Since the game starts with at least two heaps with two or more beans, Secondus is now faced with at least one heap with two or more beans. The game will now continue to be played like the previous one with the roles of Firstus and Secondus interchanged.

7. (a) Firstus wins 1×1 **Hex** with the only legal move available. (b) Yes, by capturing cell 1 and then cell 3 or 4. (c) Yes, if Firstus captures 1, Secondus captures 3, Firstus captures 2, and Secondus captures 4. (d) Yes, by capturing cell 1 and then cell 3 or 4. (e) No, because Firstus can ensure a win. (f) Firstus can ensure a win in 3×3 **Hex** by capturing cell 5 on her first move. If Secondus captures cell 1 or 2, then Firstus should capture the other cell. If Secondus captures cell 8 or 9, then Firstus should capture the other cell. If Secondus captures a different cell, then Firstus should capture cell 1 if it is not already captured and capture cell 9 otherwise.

12. (a) LEAVE ONE SHORTER is a strategy for Firstus because she certainly can follow the instruction on her first move, and once the upper row is one shorter than the lower row, Secondus must either make the upper row two or more shorter than the lower row or make the two rows even, and this makes it possible for Firstus to follow the LEAVE ONE SHORTER instruction. (b) LEAVE ONE SHORTER is not a strategy for Secondus, because Firstus on her first move could leave the upper row one shorter than the lower row making it impossible for Secondus to follow the instruction. (c) LEAVE ONE SHORTER ensures a win for Firstus.

Game Trees

1. (b) 16 ways. (c) A winning strategy for Firstus is "Remove one bean from the heap of one. If faced with $[2,1]$, remove one bean from the heap of two. If faced with $[2]$, remove both beans from the only heap". (d) One strategy for Secondus would be "If faced with $[2,1,1]$, remove the heap of two. If faced with $[2,1]$, remove one from the heap of two. If faced with $[2,2]$, remove one from a heap of two. If faced with $[2]$, remove two from the heap of two". (e) Firstus has $3+2+3\times2=11$ strategies. Secondus has $(1+1+2)(3)(2+1)=36$ strategies.

3. Initially capture cell 5. If Secondus captures 1 or 2, capture the other; if Secondus captures 8 or 9, capture the other; otherwise, take 1. On the third move, at least one of 1, 2, 8, and 9 are available and would give Firstus the win if captured; capture a cell that will cause a win.

6. (b) 5 ways. (c) North has two strategies: (1) open with 10C, and (2) open with 7D. East also has two strategies: (1) If North opens with 7D, place 2H; and (2) if North opens with 7D, place 7C. South has four strategies: (1) Place 5C in the first round; (2) if North opens with 7D, place 4S in the first round; (3) if East places 2H or 7C in the first round, place 5C or 4S in the first round, respectively; and (4) if East places 2H or 7C in the first round, place 4S or 5C in the first round, respectively. West has only one strategy: (1) play legally. (d) No matter what strategy each player adopts, the outcome is always the same: North and West tie.

8. (b) 8 ways. (c) The winning strategy for Firstus: Take one tile initially. Next, do the opposite of Secondus. (d) A strategy for Secondus: If Firstus takes one tile initially, take one tile. If Firstus takes two initially, take two tiles. If Firstus again takes one tile, take one tile, otherwise take two. (e) Secondus has six strategies. Firstus has six strategies.

12. (a) Winning strategy: Choose B; if G, choose L. (b) Firstus's strategies: (1) Choose A. (2) Choose B; if G, choose L. (3) Choose B; if G, choose M. (4) Choose C. (c) Secondus's strategies: (1) If A, choose D; if B, choose F. (2) If A, choose D; if B, choose G. (3) If A, choose E; if B, choose F. (4) If A, choose E; if B, choose G.

14. (a) The partial game tree shows that LEAVE TOP ONE SHORTER is a winning strategy for Firstus. (b) Secondus can win against Firstus using LEAVE TOP ONE SHORTER. (c) Part (a) shows that LEAVE TOP ONE SHORTER is a winning strategy for Firstus in 2×3 **Chomp**. (d) Firstus has a winning strategy in 3×3 **Chomp**: Initially choose the middle cookie and then "mirror" Secondus around the resulting "L" shape. (e) If, on her first move, Firstus takes the cookie in the second row from the bottom and the rightmost column, then it would be impossible for Secondus to carry out the instruction.

A Solution for Nim

1. 26 (decimal) = 11010 (binary); 23 (decimal) = 10111 (binary); 1110 (binary) = 14 (decimal); 10011 (binary) = 19 (decimal).

2. $10101 \oplus 11110 = 01011$.

3. The two good moves available are take four from the heap of seven or take four from a heap of four.

8. (a) If Firstus is faced with $3n + 1$ tiles, she leaves Secondus with either $3n$ tiles (if $n = 0$, Secondus has already won) or $3n - 1$ tiles. By MULTIPLE OF THREE, Secondus will take two tiles in the former case and one tile in the later case. In either case, Firstus will now be faced with $3n-2 = 3(n-1)+1$ tiles. So, eventually, Firstus will be faced with $3(0) + 1 = 1$ tile and will lose.

Theorems of Zermelo and Nash

1. (b) Each **Graph Coloring** game satisfies all suppositions and so satisfies the conclusion. (d) **Trickster** (2C, 2S; 4C, 5C; 4S, 5S) satisfies suppositions (2) and (3) but does not satisfy supposition (1). The conclusion is not satisfied because North controls who will win.

3. If we assume that Firstus has a winning strategy, there is no clear way of turning it into a strategy for Secondus. Of course, since Nash has proved that Firstus *has* a winning strategy, if we *assume* Firstus has a winning strategy, we cannot come to any contradictions.

5. If Secondus had a winning strategy, Firstus could have employed this strategy beginning on the first move by using Secondus' response to Firstus chomping on the top right cookie.

Player Preferences

Measurement

1. If course grades were a nominal scale, then it would be meaningless to say that a student assigned an A had performed better than a student assigned a B. If course grades were an ordinal scale, then it would be meaningful to say that one student performed better than another student, but it would be meaningless to compute average grades. If course grades were an interval scale, then it would be meaningful to compute average grades, but it must be the case that the difference between an A and a B is the same as the difference between a D and an F. If course grades were a ratio scale, then it would be meaningful to say that a student receiving an A performed twice as well as a student receiving a C and infinitely better than a student receiving an F.

Ordinal Preferences

3. Janina must rank candy > cards > disc > money > nothing. No, the only way to know that for sure that Janina has ordinal preferences would be to ask her the 22 other "choose from a subset of prizes" questions.

4. If we describe each of the eight possible outcomes by a triple of characteristics, then the ranking from first to last is (sunny, cheap, first), (sunny, cheap, second), (sunny, expensive, first), (sunny, expensive, second), (raining, cheap, first), (raining, cheap, second), (raining, expensive, first), (raining, expensive, second).

8. No, the bottom two could have been reversed. Or the top two could have been tied and Sophie randomly chose between the two alternatives.

Cardinal Preferences

6. (a) $u([0.75B + 0.25D]) = 0.75u(B) + 0.25u(D) = 0.75(20) + 0.25(100) = 40$.

7. $v(A) = 100$, $v(B) = 75$, $v(C) = 12.5$, and $v(D) = 0$

9. One possible answer: $u(\text{Israel}) = 100$, $u(\text{Peru}) = 69.0$, $u(\text{Japan}) = 48.3$, and $u(\text{France}) = 0$.

11. (b) Arbitrarily assign $u(\$0) = 0$ and $u(\$100) = 1$. Then $u(A) = 1/3$, $u(B) = y$, $u(C) = 1/3 + g$, and $u(D) = y + g$. Thus, $u(A) > u(B)$ is equivalent to $1/3 > y$ is equivalent to $1/3 + g > y + g$ is equivalent to $u(C) > u(D)$.

Ratio Scale Preferences

1. The approximation is

$$\begin{bmatrix} 1.000 & 5.000 & 10.000 \\ 0.200 & 1.000 & 4.000 \\ 0.100 & 0.250 & 1.000 \end{bmatrix} \overset{\text{normalize}}{\Rightarrow} \begin{bmatrix} 0.769 & 0.800 & 0.667 \\ 0.154 & 0.160 & 0.267 \\ 0.077 & 0.040 & 0.067 \end{bmatrix} \overset{\text{average}}{\Rightarrow} \begin{bmatrix} 0.710 \\ 0.231 \\ 0.060 \end{bmatrix}.$$

The revised preferences are below.

Outcome	Preference
Townhouse	$(0.710)(0.125) + (0.231)(0.60) + (0.060)(0.623) = 0.264$
House	$(0.710)(0.375) + (0.231)(0.10) + (0.060)(0.130) = 0.297$
Single	$(0.710)(0.175) + (0.231)(0.05) + (0.060)(0.139) = 0.144$
Double	$(0.710)(0.325) + (0.231)(0.25) + (0.060)(0.108) = 0.295$

Notice now the first choice is the house, second choice is the double, third choice is the townhouse, and fourth choice is the single.

Strategic Games

TOSCA

2. At minimum, Scarpia can either execute or not execute Cavaradossi. At minimum, Tosca can either refuse or accept Scarpia's sexual advances, and then after acceptance, either follow through or double-cross.

3. Here is one possible answer:

Tosca Outcomes		Scarpia EXECUTE	DON'T EXECUTE
Tosca	ACQUIESCE	Tosca loses honor and lover	Tosca loses honor but lover is alive
	FIGHT	Tosca is honorable but her lover dies	Tosca is honorable and her lover is alive

Fingers and Matches

The exercises are modeling exercises and so a variety of answers could be acceptable, depending upon how comprehensive the assumptions are in the situation description. The answers we provide are typically the most simple model consistent with the situation described.

1. (a)

Chicken Outcomes		Colin STRAIGHT	TURN
Rose	STRAIGHT	Both dead	Colin coward
	TURN	Rose coward	No loss of face

(b)

Chicken Ordinal Payoffs		Colin STRAIGHT	TURN
Rose	STRAIGHT	(1, 1)	(4, 2)
	TURN	(2, 4)	(3, 3)

(c) The critical issue is the relative importance of death versus reputation. In the table below, we have assumed that (1) each driver is indifferent between "no loss of face" versus "a 20% chance of death and an 80% chance of surviving", and (2) each driver is indifferent between "being labeled a coward" versus "an 80% chance of death and an 20% chance of surviving".

Chicken Cardinal Payoffs		Colin STRAIGHT	TURN
Rose	STRAIGHT	(0, 0)	(100, 20)
	TURN	(20, 100)	(80, 80)

4. (a) The outcome table assumes that even if Indiana smokers were to stop smoking, the state would continue to obtain some cigarette tax revenue from nonstate residents buying cigarettes in Indiana.

Cigarette Tax Outcomes		State of Indiana Increase Taxes	Don't increase taxes
Smokers	Keep smoking	State fails, but has more revenue	Nothing changes
	Stop smoking	State achieves its goal, and loses revenue	State achieves its goal, but loses the most revenue

Assume that smokers' primary and secondary motivations are to continue smoking and low cost, and the state's primary and secondary motivations are stop smoking and have more revenue.

Cigarette Tax Ordinal Payoffs		State of Indiana Increase Taxes	Don't increase taxes
Smokers	Keep smoking	(3, 2)	(4, 1)
	Stop smoking	(1, 3)	(2, 4)

(b) The critical issue for the smokers is the trade-off between smoking and cost. The table below assumes that a change in tax rate is small in comparison with smokers' desire to continue smoking. The critical issue for the state is the amount of revenue generated in each outcome. The table below assumes that a tax rate increase results in a significant increase in revenue if the smokers keep smoking but only a small increase in revenue if the smokers stop.

Cigarette Tax Cardinal Payoffs		State of Indiana Increase Taxes	Don't increase taxes
Smokers	Keep smoking	(90, 20)	(100, 0)
	Stop smoking	(0, 99)	(10, 100)

(c) If multiple tax increase levels were possible, then the number of strategies available to the state would increase, and that would increase the number of columns in each table.

10. "Right" and "left" are from the perspective of the goalie. (a) In the outcome matrix, we assume that the Goalie is 100% effective if he moves to the correct side and 0% effective if he moves to the wrong side. This assumption is not necessary for the ordinal payoffs. It is sufficient for "goal" to mean "relatively high probability

of a goal" and "no goal" to mean "relatively low probability of a goal". We have definitely assumed a symmetry between the two sides.

Soccer Penalty Kicks Outcomes		Goalie Left	Right
Kicker	Left	No goal	Goal
	Right	Goal	No goal

Soccer Penalty Kicks Ordinal Payoffs		Goalie Left	Right
Kicker	Left	$(-1, 1)$	$(1, -1)$
	Right	$(1, -1)$	$(-1, 1)$

(b) That "the goalie can move to his right slightly better than to his left" implies that the goalie is more effective at stopping goal attempts on his right. That "the kicker tends to kick to the right 52% of the time" could mean either (1) the kicker is slightly better at scoring to his right, or (2) the kicker has chosen right more often even though he is no more effective in that direction. Using the first interpretation, we obtain outcomes similar to these:

Soccer Penalty Kicks Outcomes		Goalie Left	Right
Kicker	Left	10% chance of goal	95% chance of goal
	Right	96% chance of goal	5% chance of goal

The chance of a goal could be used for the kicker's cardinal payoffs and 100% minus the chance of a goal could be used for the goalie's cardinal payoffs.

Soccer Penalty Kicks Ordinal Payoffs		Goalie Left	Right
Kicker	Left	$(10, 90)$	$(95, 5)$
	Right	$(96, 4)$	$(5, 95)$

12. (a) The probability of Bush winning the election is the probability that he wins at least two of the three states. More specifically, it is the sum of the probabilities that Bush (1) wins all three states, (2) wins Pennsylvania and Ohio, (3) wins Pennsylvania and Florida, and (4) wins Ohio and Florida. If they visit the same state, this probability is $(.2)(.6)(.8) + (.2)(.6)(.2) + (.2)(.4)(.8) + (.8)(.6)(.8) = 0.57$. The other entries are computed similarly.

Bush and Kerry Chance Bush wins		Kerry Pennsylvania	Ohio	Florida
Bush	Pennsylvania	57%	55%	56%
	Ohio	60%	57%	57%
	Florida	58%	55%	57%

(b) The chance of winning the election can be used as the cardinal payoffs.

Bush and Kerry Cardinal Payoffs		Kerry Pennsylvania	Ohio	Florida
Bush	Pennsylvania	$(57, 43)$	$(55, 45)$	$(56, 44)$
	Ohio	$(60, 40)$	$(57, 43)$	$(57, 43)$
	Florida	$(58, 42)$	$(55, 45)$	$(57, 43)$

Four Solution Concepts

1. (a) Yes. (b) No. (c) Yes. (d) No. (e) Yes. (f) No. (g) Yes. (h) No.

4. (a) Rose has no dominated strategies, but we can eliminate COLUMN D (dominated by COLUMN C) and COLUMN E (dominated by COLUMN B). Then eliminate ROW C (now dominated by ROW A). Now eliminate COLUMN B and COLUMN C (now dominated by COLUMN A). Finally, eliminate ROW A (now dominated by ROW B). Thus, (ROW B, COLUMN A) is the strategy pair that remains. (b) ROW A, COLUMN A, and COLUMN C are the prudential strategies. Strategies that remain after elimination may (COLUMN A) or may not (ROW B) be prudential. Eliminated strategies may (ROW A and COLUMN C) or may not (ROW C, COLUMN B, COLUMN D, and COLUMN E) be prudential. (c) (ROW B, COLUMN A) is a Nash equilibrium. In general, if a pair of strategies is obtained through successive elimination of dominated strategies, then that pair is a Nash equilibrium. (d) The only Nash equilibrium is the pair of undominated strategies. In general, if a pair of strategies is obtained through successive elimination of dominated strategies, then that pair is the only Nash equilibrium. (e) (ROW B, COLUMN A) is not an efficient strategy pair because (ROW C, COLUMN D) leads to an increased payoff for each player. There is no general relationship between undominated strategy pairs and efficient strategy pairs.

6. (B, C) and (C, A) are Nash equilibria.

7. The boxed payoff is the best payoff in each row or column, respectively. If both payoffs are boxed, it is simultaneously the best payoff in the row and in the column. So neither player can unilaterally improve his or her payoff.

11. (a) TWO is the prudential strategy for each player. (b) All strategies remain because no strategy is dominated. (c) (ONE, TWO) and (TWO, ONE) are the Nash equilibria. (d) Every strategy pair except (ONE, ONE) is an efficient strategy pair. (e) There is a tension between "playing it safe" by choosing the prudential strategy TWO and "taking advantage of the other player who is hopefully playing it safe" by choosing ONE. (f) Preplay communication may ameliorate the tension if the players can verbally agree to one of the Nash equilibria. An agreed upon Nash equilibrium is self-enforcing. However, the two players clearly prefer different Nash equilibria, and so preplay communication may degenerate into posturing or a show of resolve that may result in both players choosing ONE, leading to both players' least preferred outcome. (g) The answers do not change with different cardinal payoffs.

15. (a) The prudential strategies for the Allies are REINFORCE GAP and HOLD RESERVES, and the security level is 0. The prudential strategy for the Germans is RETREAT, and the security level is -1. (b) All strategies are undominated. (c) There is no Nash equilibrium in pure strategies. (d) All strategy pairs are efficient. (e) Different answers are possible, but HOLD RESERVES and RETREAT seem to have the most game theoretic backing. (f) The Allies' choice and the German generals' decision are consistent with the prudential solution concept. Hitler's choice seems less reasonable especially since it seems unlikely that the Allies would choose SEND RESERVES EAST. (g) This changes the payoff matrix and best response

payoffs as follows:

WWII Battle Cardinal Payoffs		Germans ATTACK GAP	RETREAT
Allies	REINFORCE GAP	$(1, -1)$	$(\boxed{0}, \boxed{0})$
	HOLD RESERVES	$(\boxed{2}, -2)$	$(\boxed{0}, \boxed{0})$
	SEND RESERVES EAST	$(-2, \boxed{2})$	$(\boxed{0}, 0)$

The prudential strategies stay the same but the German security level increases to 0. All strategies are still undominated. There are now two pure strategy Nash equilibria: (REINFORCE GAP, RETREAT) and (HOLD RESERVES, RETREAT). All strategy pairs remain efficient. Different answers are possible, but HOLD RESERVES and RETREAT seem to have even more game theoretic backing. The Allies' choice and the German generals' decision are consistent with the prudential and Nash solution concepts. Hitler's choice seems even less reasonable especially since it seems unlikely that the Allies would choose SEND RESERVES EAST.

Once Again, Game Trees

1. (b) The backward induction strategies are "tennis" for Rose and "whatever Rose chooses" for Colin. (c) Rose and Colin will both choose to go to the tennis match. (d) This certainly seems reasonable because Colin's payoffs indicate that he will choose whatever Rose chooses, and knowing this, Rose's payoffs indicate that she will choose to go to the tennis match.

2. (a) Kennedy has reversed his preferences for outcomes Y and U: the reputation benefits of Khrushchev acquiescing to an air strike are outweighed by the military and diplomatic costs associated with carrying out the air strikes. Kennedy dropped outcome V below outcomes X and Z: the reputation benefits of challenging Khrushchev outweigh the potentially high costs associated with the challenge. Khrushchev interchanged W and Y and interchanged Z and X: Khrushchev is less embarrassed to acquiesce in the face of more aggressive United States action (air strike vs. blockade) and is more willing to be aggressive if the United States is less aggressive (blockade vs. air strike). (b) The backward induction strategies and the outcome are the same as before: Krushchev does nothing. (c) Kennedy must prefer W to V and W to whichever of Y and Z that Khrushchev prefers; this is easy to fulfill since we have already been assuming that Kennedy ranks W highest. Khrushchev must prefer W to X and W to U. The former has already been satisfied under the assumption that Khrushchev primarily wants to avoid war. The later means that placing missiles and then removing them in response to a blockade by the United States needs to be preferable to never placing the missiles. This seems hard to believe unless Khrushchev obtained something in return for removing the missiles; this could have been the United States withdrawal of missiles from Turkey.

5. (a) Adam and Eve's motivations (in decreasing order of importance) are life, acquisition of knowledge, avoidance of punishment, and avoidance of accountability. God's motivations (in decreasing order of importance) are Adam and Eve's obedience, justice, and life for His creation. (b) Adam and Eve should eat the fruit and admit if interrogated. God should interrogate if Adam and Eve eat the

fruit, kill if they deny, and punish if they admit. (c) The biblical story follows the backward induction path: Adam and Eve eat the fruit, God interrogates them, they admit their disobedience, and God punishes them. The story does not tell us what might have happened if the players had chosen differently. (d) There may have been many more options available to each player. After Adam and Eve ate the fruit, God could have immediately punished or killed them. During the interrogation, Adam and Eve could have not only admitted their disobedience but have also accepted responsibility for their actions rather than trying to shift the blame. God could have had the choice of punishments of different severities.

11. (b) There are two backward induction equilibria: (offer \$1, accept if offered at least \$1 and reject otherwise) and (offer \$0, always accept). (c) The backward induction equilibrium is now (offer \$2, accept an offer of \$2 and reject all other offers). (d) There will be many more branches out of the initial node. In the part (b) version of the game, the backward induction equilibria will involve offers of \$0.01 or \$0.00 and acceptance. In the part (c) version of the game, the backward induction equilibrium will probably be an offer of \$1.50 and acceptance only if the offer is at least (or exactly equal to) \$1.50.

Trees and Matrices

1. Delilah's strategies are nag and don't nag. Sampson's strategies are (R, R): reveal regardless; (R, Q): reveal if nagged, stay quiet if not nagged; (Q, R): stay quiet if nagged, reveal if not nagged; and (Q, Q): stay quiet regardless. There are two Nash equilibria: (nag, (R, Q)) corresponds to the backward induction strategies and (don't nag, (Q, Q)) involves Sampson making the noncredible threat to remain quiet if Delilah nags.

2. Here is the outcome matrix:

	Do Nothing	Blockade	Air Strike
Don't place missiles	U	U	U
Place, then always acquiesce	V	W	Y
Place, then always escalate	V	X	Z
Place, then escalate only on blockade	V	X	Y
Place, then, escalate only on airstrike	V	W	Z

Here is the ordinal payoff matrix with the best response payoffs boxed:

	Do Nothing	Blockade	Air Strike
Don't place missiles	$(5, \boxed{4})$	$(\boxed{5}, \boxed{4})$	$(\boxed{5}, \boxed{4})$
Place, then always acquiesce	$(\boxed{6}, 3)$	$(4, \boxed{6})$	$(3, 5)$
Place, then always escalate	$(\boxed{6}, \boxed{3})$	$(1, 2)$	$(2, 1)$
Place, then escalate only on blockade	$(\boxed{6}, 3)$	$(1, 2)$	$(3, \boxed{5})$
Place, then, escalate only on airstrike	$(\boxed{6}, 3)$	$(4, \boxed{6})$	$(2, 1)$

There are three Nash equilibria: (Don't place missiles, Blockade) corresponds to the backward induction strategies, (Don't place missiles, Air Strike) results in the same outcome U, and (Place, then always escalate, Do Nothing) which involves a noncredible threat on the part of Khrushchev in order to keep Kennedy doing nothing. Historically, both leaders threatened the other with retaliation. Historically, it was also the case that the missiles were placed in Cuba secretly and the Soviet Union continuously denied their existence to the United States until there was incontrovertible evidence. This shows that Khrushchev understood that his threats would not be credible.

5. The backward induction strategies are (UP, UP) with resulting payoff pair (10, 3). In addition to the backward induction strategies, there is a second Nash equilibrium (DOWN, DOWN) with resulting payoff pair (3, 10) that involves Colin making a noncredible threat.

11. The strategies, payoff matrix, and best response payoffs are exhibited below. There is a unique Nash equilibrium that corresponds to the backward induction strategies and the actual outcome. Note also that the backward induction strategies are dominant for each player.

Battle of Thermopylae		Leonidas		
Ordinal Payoffs		Surrender	Fight Refuse	Fight Surrender
Xerxes	Invade/Bribe/Attack	$(\boxed{6}, 1)$	$(\boxed{4}, \boxed{3})$	$(\boxed{5}, 0)$
	Invade/Bribe/Retreat	$(\boxed{6}, 1)$	$(0, \boxed{6})$	$(\boxed{5}, 0)$
	Invade/Attack	$(\boxed{6}, 1)$	$(3, \boxed{2})$	$(3, \boxed{2})$
	Invade/Retreat	$(\boxed{6}, 1)$	$(1, \boxed{5})$	$(1, \boxed{5})$
	Don't Invade	$(2, \boxed{4})$	$(2, \boxed{4})$	$(2, \boxed{4})$

15. (b) Zeus has two strategies; Athena has four strategies. (c) The payoff matrix and best response diagram:

		Athena ALWAYS LO	ALWAYS HI	SAME	OPPOSITE
Zeus	LO	$(\boxed{23}, 9)$	$(18, \boxed{14})$	$(\boxed{23}, 9)$	$(\boxed{18}, \boxed{14})$
	HI	$(18, \boxed{14})$	$(\boxed{20}, 12)$	$(20, 12)$	$(\boxed{18}, \boxed{14})$

(d) OPPOSITE OF ZEUS is a dominant strategy for Athena. Neither of Zeus's strategies are dominated by the other. There are two Nash equilibria: (LO, OPPOSITE OF ZEUS) and (HI, OPPOSITE OF ZEUS). Both have payoffs $(18, 14)$.

18. (a)

BOS with Outside Option		Colin	
Outcome in Tickets		Left	Right
Rose	Top	0,0	200,600
	Bottom	600,200	0,0
	Outside	300,300	300,300

(b) Rose's prudential strategy is the Outside option. Colin's pure prudential strategy is either Left or Right. Colin's mixed prudential strategy is 0.75 Left + 0.25 Right, which provides a security level of 150. There are two pure strategy Nash equilibria: (Outside, Right) and (Bottom, Left). There are many mixed strategy Nash equilibria: (Outside, $(1 - q)$ Left + q Right), where $1/2 \leq q \leq 1$. (c) In the cited experiment, 20% of Roses chose the Outside option, and 90% of those who did not chose the Outside option ended up in the better (from Rose's perspective) Nash equilibrium.

Probabilistic Strategies

It's Child's Play

1. A probabilistic strategy, which involves random choices, is still a complete and unambiguous description of what to do.

Mixed Strategy Solutions

1. (a) $(\frac{11}{7}, \frac{16}{7})$. (b)$(\frac{12}{7}, \frac{16}{7})$. (c)$(\frac{23}{14}, \frac{16}{7})$. (d) B. (e)$\frac{16}{7}$. (f) $(1.5, 2)$. (g)$(1.5, 2.5)$. (h)$(1.5, \frac{16}{7})$. (i)A, B, or any probabilistic mixture of A and B. (j) 2. (k) $(\frac{18}{7}, \frac{16}{7})$. (l) $(\frac{3}{7}, \frac{16}{7})$. (m) $(\frac{3}{2}, \frac{16}{7})$. (n) A, B, or any probabilistic mixture of A and B. (o) $\frac{3}{7}$. (p)According to part (d), B is the only best response by Rose to Colin choosing $\frac{4}{7}A + \frac{3}{7}B$. Since $\frac{3}{7}A + \frac{4}{7}B$ is not a best response by Rose to Colin choosing $\frac{4}{7}A + \frac{3}{7}B$, the pair of strategies is not a Nash equilibrium. (q) According to part (i), $\frac{3}{7}A + \frac{4}{7}B$ is a best response by Rose to Colin choosing $\frac{1}{2}A + \frac{1}{2}B$. According to part (n), $\frac{1}{2}A + \frac{1}{2}B$ is a best response by Colin to Rose choosing $\frac{3}{7}A + \frac{4}{7}B$. Since each strategy is a best response to the other, the pair of strategies is a Nash equilibrium. (r) Rose could place 3 azure and 4 blue beads in a can, shake the can, select one bead without looking, choose A if the bead is azure, and choose B if one bead is blue. Colin could flip a coin, choose A if the coin lands heads, and choose B if the coin lands tails. (t) According to part (j), Colin could receive 2 if he chooses $\frac{1}{2}A + \frac{1}{2}B$. According to part (e), Colin will not receive below $\frac{16}{7}$ if he chooses $\frac{4}{7}A + \frac{3}{7}B$. Since Colin can guarantee himself a higher payoff by choosing $\frac{4}{7}A + \frac{3}{7}B$ than by choosing $\frac{1}{2}A + \frac{1}{2}B$, the later strategy must not be prudential. (u) According to parts (k)-(m), if Rose chooses $\frac{3}{7}A + \frac{4}{7}B$, then Colin's payoff will be $\frac{16}{7}$ no matter what strategy he chooses. This means Colin cannot guarantee himself more than $\frac{16}{7}$. According to part (e), Colin will not receive below $\frac{16}{7}$ if he chooses $\frac{4}{7}A + \frac{3}{7}B$. Since Colin cannot guarantee himself more, $\frac{4}{7}A + \frac{3}{7}B$ must be prudential.

5. (a) In **MARPS**, a player can win \$2, win \$1, win \$0, lose \$1, or lose \$2. A self-interested player will rank order the five possible outcomes in the order given. Suppose we arbitrarily assign $u(\text{win } \$2) = 2$ and $u(\text{lose } \$2) = -2$. Since $\$0 = \frac{1}{2}(\$2) + \frac{1}{2}(-\$2)$, a risk neutral player is indifferent between the outcome win \$0 and the lottery $\frac{1}{2}(\text{win } \$2) + \frac{1}{2}(\text{lose } \$2)$, which implies that $u(\text{win } \$0) = u(\frac{1}{2}(\text{win } \$2) + \frac{1}{2}(\text{lose } \$2)) = \frac{1}{2}u(\text{win } \$2) + \frac{1}{2}u(\text{lose } \$2) = \frac{1}{2}(2) + \frac{1}{2}(-2) = 0$. Since $\$1 = \frac{3}{4}(\$2) + \frac{1}{4}(-\$2)$, a risk neutral player is indifferent between the outcome win \$1 and the lottery $\frac{3}{4}(\text{win } \$2) + \frac{1}{4}(\text{lose } \$2)$, which implies that $u(\text{win } \$1) = u(\frac{3}{4}(\text{win } \$2) + \frac{1}{4}(\text{lose } \$2)) = \frac{3}{4}u(\text{win } \$2) + \frac{1}{4}u(\text{lose } \$2) = \frac{3}{4}(2) + \frac{1}{4}(-2) = 1$. Similarly, $u(\text{lose } \$1) = -1$ for a risk neutral player.

(b) As can be seen in the matrix below, if Colin uses the strategy $0.4R + 0.4P + 0.2S$, then any strategy, including $0.4R + 0.4P + 0.2S$, is a best response for Rose.

By the symmetry of the payoffs, $0.4R + 0.4P + 0.2S$ is a best response for Colin to Rose using $0.4R + 0.4P + 0.2S$.

Risk Neutral MARPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS	$0.4R + 0.4P + 0.2S$
	ROCK	$(0, 0)$	$(-1, 1)$	$(2, -2)$	$(0, 0)$
Rose	PAPER	$(1, -1)$	$(0, 0)$	$(-2, 2)$	$(0, 0)$
	SCISSORS	$(-2, 2)$	$(2, -2)$	$(0, 0)$	$(0, 0)$

(c) As can be seen in the matrix above, if Colin uses the strategy $0.4R + 0.4P + 0.2S$, then Colin is assured of obtaining an expected payoff of 0. If Rose uses the strategy $0.4R + 0.4P + 0.2S$, then Colin will be unable to guarantee more. So, $0.4R + 0.4P + 0.2S$ must be Colin's (and by symmetry, Rose's) prudential strategy.

8. (a) As can be seen in the matrix below, $\frac{7}{12}$AGGRESSIVE $+\frac{1}{12}$ASSERTIVE $+ \frac{4}{12}$COMPLIANT is a best response to itself.

Payoffs	AG	AS	CO	$\frac{7}{12}$AG $+\frac{1}{12}$AS $+\frac{4}{12}$CO
AG	$(-30, -30)$	$(40, -20)$	$(60, 0)$	$\frac{7}{12}(-30) + \frac{1}{12}(40) + \frac{4}{12}(60) = \frac{35}{6}$
AS	$(-20, 40)$	$(10, 10)$	$(50, -10)$	$\frac{7}{12}(-20) + \frac{1}{12}(10) + \frac{4}{12}(50) = \frac{35}{6}$
CO	$(0, 60)$	$(-10, 50)$	$(20, 20)$	$\frac{7}{12}(0) + \frac{1}{12}(-10) + \frac{4}{12}(20) = \frac{35}{6}$

(b) The population distribution will remain stable if $\frac{7}{12}$ are AGGRESSIVE, $\frac{1}{12}$ are ASSERTIVE, and $\frac{4}{12}$ are COMPLIANT.

Finding Solutions in 2×2 Games

1. In determining Colin's mixed Nash equilibrium strategy, we set Rose's expected payoffs equal to each other so that she would have a mixed strategy best response to Colin's strategy.

2. (a) From a best response diagram, it is easy to see that there is no pure strategy Nash equilibrium. The Nash equilibrium is ($\frac{5}{6}$COY $+$ $\frac{1}{6}$FAST, $\frac{5}{8}$FAITHFUL $+$ $\frac{3}{8}$PHILANDERING). (b) COY is prudential for the female. FAITHFUL is prudential for the male. (c) A biological interpretation of the Nash equilibrium is that a population of 5/6 coy and 1/6 fast females and 5/8 faithful and 3/8 philandering males would be evolutionarily stable. Evolution could lead to populations of individuals with different innate behaviors. The prudential strategies do not tell us anything about this scenario because if there were a population of mostly COY females and FAITHFUL males, then there is a competitive advantage to FAST females who would produce a disproportionate number of children. With more FAST females, there would be a competitive advantage for PHILANDERING males. So, it seems that the population distribution will move toward the Nash equilibrium. (d) The payoff pair for (COY, FAITHFUL) would increase by 4 points for each player, and there is a pure strategy Nash equilibrium of (COY, FAITHFUL). These payoff increases only make COY and FAITHFUL even more strongly prudential. In this case, all females will eventually be COY and all males will eventually be FAITHFUL.

4. (a) The lottery $[0.8(-\$1) + 0.2(\$4)]$ has expected value of \$0 and an expected payoff of 2 for each player, and the lottery $[0.6(-\$1) + 0.4(\$4)]$ has expected value of \$1 and an expected payoff of 4 for each player. Since Rose's payoff for receiving \$0 with certainty is $1 < 2$ and for receiving \$1 with certainty is $3 < 4$, Rose is risk loving. Since Colin's payoff for receiving \$0 with certainty is $5 > 2$ and for receiving \$1 with certainty is $8 > 4$, Colin is risk adverse. (b) (ONE, TWO) and (TWO, ONE) are two Nash equilibria with payoffs $(10, 5)$ and $(1, 10)$. ($\frac{2}{7}$ONE $+\frac{5}{7}$TWO, $\frac{7}{8}$ONE $+\frac{1}{8}$TWO) is a Nash equilibrium and the corresponding payoff pair is $(\frac{10}{8}, \frac{50}{7})$. (c) Rose's prudential strategy is TWO with a security level of 1, which corresponds to receiving nothing. Colin's prudential strategy is TWO with a security level of 5, which corresponds to receiving nothing. (d) If the players use ($\frac{2}{7}$ONE $+\frac{5}{7}$TWO, $\frac{7}{8}$ONE $+\frac{1}{8}$TWO), then their expected monetary winnings are $(-\frac{1}{56}, \frac{131}{56})$. If the players use ($\frac{3}{4}$ONE $+\frac{1}{4}$TWO, $\frac{3}{4}$ONE $+\frac{1}{4}$TWO), then their expected monetary winnings are $(\frac{1}{4}, \frac{1}{4})$. In **Risk-Varying Fingers**, Rose's risk-loving nature results in an expected monetary loss!

7. (a) Rose and Colin must be (1) indifferent between a tie and the lottery [0.8Win + 0.2Crash] and (2) indifferent between a loss and the lottery [0.4Win + 0.6Crash]. (b) From the best response diagram, it is clear that (TURN, STRAIGHT) and (STRAIGHT, TURN) are two pure strategy Nash equilibria with corresponding payoffs $(4, 10)$ and $(10, 4)$, respectively. ($\frac{1}{3}$STRAIGHT + $\frac{2}{3}$TURN, $\frac{1}{3}$STRAIGHT + $\frac{2}{3}$TURN) is a Nash equilibrium with expected payoffs $(\frac{20}{3}, \frac{20}{3})$. (c) Rose's prudential strategy is TURN with a security level of 4, which corresponds to losing. By the symmetry of the payoff matrix, Colin's prudential strategy is also TURN with a security level of 4. (d) With three Nash equilibria and different prudential strategies, it is difficult to give a recommendation. If a player is strategically risk adverse, then she or he should turn. If a player perceives her or his opponent to be strategically risk adverse, she or he should go straight. With preplay communication, it would be important to convince your opponent that you are planning to go straight no matter what your opponent decides to do. (e) The mixed strategy Nash equilibrium would be a good prediction for the population distribution. That is, there should eventually be $\frac{1}{3}$ STRAIGHT players and $\frac{2}{3}$ TURN players.

9. (a) From the best response diagram, it is clear that (TENNIS, SOCCER) and (SOCCER, TENNIS) are two pure strategy Nash equilibria with corresponding payoffs $(10, 6)$ and $(6, 10)$, respectively. ($\frac{10}{11}$TENNIS + $\frac{1}{11}$SOCCER, $\frac{7}{17}$TENNIS + $\frac{10}{17}$SOCCER) is a Nash equilibrium with expected payoffs $(\frac{90}{17}, \frac{60}{11})$. (b) Rose's prudential strategy is $\frac{9}{17}$TENNIS + $\frac{8}{17}$SOCCER, and her security level is $\frac{90}{17}$. Colin's prudential strategy is $\frac{5}{11}$TENNIS + $\frac{6}{11}$SOCCER, and his security level is $\frac{60}{11}$. (c) It seems reasonable to believe that the other player, because of the difficulty of the situation, may be randomizing (implicitly if not explicitly). This suggests that the mixed strategy Nash equilibrium makes sense as a predictor of what would happen. On the other hand, if a player if going to choose a mixed strategy, the prudential strategy guarantees an expected payoff equal to the expected payoff obtained using the Nash equilibrium strategies. So, there is little incentive to choose the more strategically risky Nash equilibrium strategies rather than the prudential strategies.

11. From the best response diagram, (A, A) is a Nash equilibrium, and $(\frac{1}{2}B + \frac{1}{2}C, \frac{2}{3}B + \frac{1}{3}C)$ is a Nash equilibrium with expected payoffs $(\frac{8}{3}, 5.5)$.

12. When the equation for p or q is solved, it turns out that the solution is either negative or greater than 1; that is, a valid probability is not found.

Nash Equilibria in $m \times 2$ Games

1. The intersection of the ROW A and ROW C payoff lines is at $q = \frac{1}{2}$. The equalizing strategy $(1-p)A + pC$ for Rose is obtained by solving $7(1-p) + 2p = 3(1-p) + 6p$, which implies $p = \frac{1}{2}$. The pair $(\frac{1}{2}A + \frac{1}{2}C, \frac{1}{2}A + \frac{1}{2}B)$ is a Nash equilibrium.

7. The unique Nash equilibrium is ($\frac{3}{5}$HOLD RESERVES + $\frac{2}{5}$SEND RESERVES EAST, $\frac{1}{5}$ATTACK GAP + $\frac{4}{5}$RETREAT). If the Allies had adopted their Nash equilibrium strategy, then the German's expected payoff would have been the same regardless of their strategy choice. So, the choice really is about how much strategic risk to take on: ATTACK GAP is strategically more risky than RETREAT.

9. (a) The outcomes are in millions of dollars. These would correspond to cardinal payoffs if the firms are self-interested and risk neutral.

Investment Outcome	Invest \$0	Invest \$1	Invest \$2	Invest \$3	Invest \$4	Invest \$5
Invest \$0	(5,5)	(5,12)	(5,11)	(5,10)	(5,9)	(5,8)
Invest \$1	(12,5)	(4,4)	(4,11)	(4,10)	(4,9)	(4,8)
Invest \$2	(11,5)	(11,4)	(3,3)	(3,10)	(3,9)	(3,8)
Invest \$3	(10,5)	(10,4)	(10,3)	(2,2)	(2,9)	(2,8)
Invest \$4	(9,5)	(9,4)	(9,3)	(9,2)	(1,1)	(1,8)
Invest \$5	(8,5)	(8,4)	(8,3)	(8,2)	(8,1)	(0,0)

(c) Verify that (1) when Firm 2 uses

$$(0.125, 0.125, 0.125, 0.125, 0.125, 0.375),$$

Firm 1 receives 5 no matter which strategy it chooses, and (2) when Firm 1 uses

$$(0.125, 0.125, 0.125, 0.125, 0.125, 0.375),$$

Firm 2 receives 5 no matter which strategy it chooses. (d) INVEST \$0 ensures a payoff of exactly 5 for Firm 1. If Firm 1 uses any other strategy, then against Firm 2 using INVEST \$5, Firm 1 will obtain a payoff of no more than 5.

Zero-Sum Games

1. (a) No. (b) Yes. (c) No. (d) Yes.

2. (a) From the best response diagram, it is clear there is no pure strategy Nash equilibrium. $(\frac{2}{7}A+\frac{5}{7}B, \frac{3}{7}A+\frac{4}{7}B)$ is the Nash equilibrium with payoff pair $(\frac{15}{7}, -\frac{15}{7})$. (b) From the best response diagram, it is clear that (B, B) is the one pure strategy equilibrium. (c) From the best response diagram, it is clear there is no pure strategy Nash equilibrium. $(\frac{6}{11}A+\frac{5}{11}B, \frac{7}{11}A+\frac{4}{11}B)$ is the Nash equilibrium with payoff pair

$(-\frac{2}{11}, \frac{2}{11})$. (d) From the best response diagram, it is clear there is no pure strategy Nash equilibrium. The pair $(\frac{1}{7}A + \frac{6}{7}B, \frac{2}{7}B + \frac{5}{7}C)$ is a Nash equilibrium with payoff pair $(\frac{5}{7}, -\frac{5}{7})$. (e) The pair $(\frac{7}{12}A + \frac{5}{12}C, \frac{1}{2}A + \frac{1}{2}B)$ is a Nash equilibrium with the payoff pair $(-\frac{1}{2}, \frac{1}{2})$.

3. None of the games in exercise 2 is fair. Rose has the advantage in parts (a), (b), and (d). Colin has the advantage in parts (c) and (e).

5. Using the percentage of the electoral gains as the payoff (divided by 10), we obtain the following payoff matrix.

Cardinal Payoffs (Rose, Colin)		Colin 1	2	3
	1	$(\boxed{0}, \boxed{0})$	$(\boxed{3}, -3)$	$(\boxed{5}, -5)$
Rose	2	$(-3, \boxed{3})$	$(0, 0)$	$(3, -3)$
	3	$(-5, \boxed{5})$	$(-3, 3)$	$(0, 0)$

To advertise one day before the election is the dominant strategy, and $(1, 1)$ is the unique Nash equilibrium. With the payoff pair $(0, 0)$, this game is fair.

7. A strategy for each team consists of which players will play first and second singles. The payoff is the expected number of singles matches won by the team.

Cardinal Payoffs (State, Ivy)		Ivy XY	XZ	YX	YZ	ZX	ZY
State	AB	$(1, 1)$	$(1, 1)$	$(0, 2)$	$(\frac{1}{2}, \frac{3}{2})$	$(1, 1)$	$(\frac{3}{2}, \frac{1}{2})$
	BA	$(0, 2)$	$(1, 1)$	$(1, 1)$	$(\frac{3}{2}, \frac{1}{2})$	$(1, 1)$	$(\frac{1}{2}, \frac{3}{2})$

The unique Nash equilibrium is $(\frac{1}{2}\text{AB} + \frac{1}{2}\text{BA}, \frac{1}{2}\text{XY} + \frac{1}{2}\text{YX})$ resulting in 0.5 expected singles wins for State and 1.5 expected wins for Ivy.

Strategic Game Cooperation

Experiments

1. Since there is no communication and only one simultaneously made choice, there can be no promises or threats made in **Experiment With One Choice**. The ability to communicate strategy proposal in **Experiment With Proposals** and **Experiment With Ability to Reject** makes it possible to make promises and threats. Since choices are made twice and players learn of each other's first choice before each makes his or her second choice in **Experiment With Ability to Reject**, there is some ability to link actions with communication. The credibility of any promises or threats was discussed in the dialogue.

The Prisoner's Dilemma

1. **Tosca**, **Two Brothers**, and **Steroid Use** are prisoners' dilemma scenarios.

2. (a) This could be a prisoners' dilemma scenario with each nation as a player. COOPERATE: do not develop and stockpile nuclear weapons. DEFECT: develop

and stockpile nuclear weapons. If the power and safety obtained by defecting outweighs the financial and moral expense, then DEFECT is a dominate strategy. If equal power and safety is obtained with nuclear parity, then it would be better for all nations to avoid the expense of defecting in comparison to cooperating. (c) This is a coordination game with each automobile driver being a player. Everyone driving on the right (such as in the United States and the cities of Bolivia) is a Nash equilibrium and everyone driving on the left (such as in England and the mountain roads in Bolivia) is a second Nash equilibrium. No strategy is dominant. (f) This is unlikely to be a prisoners' dilemma scenario.

6. (a) Neither strategy is dominant. (b) No, because he would drop from a payoff of 5 to a payoff of 3 assuming the other players continued to choose C. (c) The C players would like to stick with C (3 versus 2) and the D player would prefer to switch to C (5 versus 3). (d) For CCCDD, the C players are indifferent (1 versus 1) and the D players would prefer to switch (3 versus 2). For CCDDD, the C players would prefer to switch to D (0 versus -1) and the D players are indifferent (1 versus 1). For CDDDD, the C player would prefer to switch to D (-1 versus -3) and the D players would prefer to stick with D. For DDDDD, the players would prefer to stick with D. (e) There appear to be two Nash equilibria: DDDDD and CCCCC. (g) When competing standards are introduced. For example, HD and Blue Ray were introduced as competing standards for high definition DVDs. Eventually consumers all chose Blue Ray, but it could have gone in the other direction. As soon as a sufficient mass of consumers favors one over the other, it is to everyone else's advantage to adopt that standard.

Resolving the Prisoner's Dilemma

3. Suppose both Rose and Colin use the DEFECT ALWAYS strategy. Then both players defect in each round and then each player receives a payoff of 4 in each round. If Rose were to change her strategy, Colin will continue to defect in every round, but Rose may cooperate in some rounds. In the rounds Rose cooperates, she receives a payoff of 0 instead of 4. So, Rose's expected payoff using the new strategy cannot be any more than her expected payoff using the DEFECT ALWAYS strategy, and so the DEFECT ALWAYS strategy is a best response to the DEFECT ALWAYS strategy.

4. Rose's expected payoff is

$$\begin{aligned} 10 + 8p + 10p^2 + 8p^3 + \cdots &= 10(1 + p^2 + p^4 + \cdots) + 8p(1 + p^2 + p^4 + \cdots) \\ &= \frac{10 + 8p}{1 - p^2}. \end{aligned}$$

Colin's expected payoff is

$$\begin{aligned} 8 + 10p + 8p^2 + 10p^3 + \cdots &= 8(1 + p^2 + p^4 + \cdots) + 10p(1 + p^2 + p^4 + \cdots) \\ &= \frac{8 + 10p}{1 - p^2}. \end{aligned}$$

In each round, each player is choosing the unique best response to the other player. So, no unilateral change of strategy would improve either player's payoff.

Negotiation and Arbitration

A Simple Negotiation

1. If Rose is indifferent to Colin's chances of winning a prize, then there are only two outcomes from her perspective: she wins a prize or she does not win a prize. If we arbitrarily assign utilities of 100 and 0, respectively, to the two possible outcomes, then by the Expected Utility Hypothesis, a $100p$ chance of winning the prize should have utility $100p$.

4. Various answers are possible, but here is one possibility. If Rose were not indifferent to Colin's chances to receiving a prize, then Rose's actual payoffs may be a linear combination of her and his chances:

Chances		Colin	
Rose's Modified Payoffs		A	B
Rose	A	$50s + 20t$	$70s + 80t$
	B	$0s + 100t$	$100s + 0t$

where s and t indicate how important Rose's and Colin's chances, respectively, are to Rose. For example, if $t = 0$, then Rose is indifferent to Colin's chances. If $s = 0$ and $t = 1$, then Rose is indifferent to receiving a prize but would like to see Colin win a prize. If $s = 1$ and $t = -1$, then Rose's desire to win a prize is equal to her desire for Colin to not win a prize.

Bargaining Games

2. (a) Rose and Colin must be able to communicate with each other and make a binding agreement. (b) The mixed strategy pair $(.5T + .5S, .5T + .5S)$ yields the payoff pair $(5.25, 5.25)$, and the correlated strategy $.5(T, T) + .5(S, S)$ yields the payoff pair $(9.5, 8)$. Notice that the two payoff pairs are not equal. (d) The security levels seem appropriate for the disagreement payoffs. If each player uses her or his prudential strategy, the payoffs are the security levels. No strategy is dominated, and there are three Nash equilibria, and so it is not clear that any of them could reasonably be adopted as what would happen without cooperation. (e) (i) Strategy $(0.25)T + (0.75)S$ and Colin could choose the pure strategy T. (ii) $(3.6, 5.2) = (0.2)(10, 6) + (0.8)(2, 5)$, and so Rose could choose the pure strategy T and Colin could choose the mixed strategy $(0.2)T + (0.8)S$. (iii) $(9.5, 8.0) = (0.5)(10, 6) + (0.5)(9, 10)$, and so Rose and Colin could choose the correlated strategy $(0.5)(T, T) + (0.5)(S, S)$. (iv) $(10.0, 10.0)$ cannot be achieved. Graphically, this point lies outside of the feasible payoff pair region. (v) $(6.0, 6.0) = \frac{13}{33}(10, 6) + \frac{16}{33}(2, 5) + \frac{4}{33}(9, 10)$ so Rose and Colin could choose the correlated strategy $\frac{13}{33}(T, T) + \frac{16}{33}(T, S) + \frac{4}{33}(S, S)$.

4. (a) There are $2^{10} = 1024$ possibilities. (b) The four resolutions yield the payoff pairs $(98, 13)$, $(94, 27)$, $(89, 15)$, and $(84.5, 45.5)$. (c) $(0, 100)$ corresponds to Panama winning on all issues. $(58, 73)$ corresponds to the United States winning on the first three issues: expansion rights, U.S. defense rights, and use rights. $(37, 70)$ corresponds to the United States winning on use rights and land and water. $(71.5, 60.5)$ corresponds to the United States winning on the first four issues (expansion rights, U.S. defense rights, use rights, and expansion routes) and compromising 50-50 on

the fifth issue (land and water). (d) It is not clear from the given information what the disagreement payoffs would be. The implication of $(0,0)$ is that without an agreement, each side loses on all issues.

7. The axes of a graph are an interval scale.

The Egalitarian Method

2. (b) $y - 6 = \frac{10-6}{9-10}(x - 10)$ or $y = 46 - 4x$ satisfying $9 \le x \le 10$. (c) $(2, 5)$. (d) $(9, 10)$ is a rational payoff pair which is better for both players.

4. (b) $(66, 66)$. (c) The United States should win on the first four issues (expansion rights, U.S. defense rights, use rights, and expansion routes), Panama should win on the last five issues (duration, compensation, jurisdiction, U.S. military rights, and defense role of Panama), and there should be a compromise on the remaining issue (land and water). More precisely, the United States should win $\frac{2}{15}$ and Panama should win $\frac{13}{15}$ of the land and water issue.

5. (b) $(6, 6)$. (c) $(6, 6) = \frac{1}{2}(10, 2) + \frac{1}{2}(2, 10)$ and so the players should use the correlated strategy $\frac{1}{2}$(ONE, TWO) + $\frac{1}{2}$(TWO, ONE). (d) Efficient and unbiased (since the set of rational payoff pairs is symmetric about the 45-degree line from the disagreement payoff pair).

7. **Chances** with different disagreement payoff pairs:

DPP	Egalitarian PP
$(50, 50)$	$(72.7, 72.7)$
$(60, 50)$	$(75.5, 65.5)$
$(62.5, 50)$	$(76.1, 63.6)$

The Raiffa Method

2. (b) $(\frac{338}{37}, \frac{350}{37}) \approx (9.14, 9.46)$. (c) From the parametric description of the efficient payoff pairs, the Raiffa payoff pair satisfies

$$\frac{338}{37} = 9(1 - t) + 10t,$$

which has the solution $t = \frac{5}{37} \approx 0.135$. Hence, Rose and Colin would choose the correlated strategy $\frac{32}{37}(S, S) + \frac{5}{37}(T, T)$. With an 86% chance, Rose and Colin should both go to the soccer match, and with a 14% chance, Rose and Colin should both go to the tennis match.

4. For the **Panama Canal** bargaining game the answers here (for the Raiffa method) are identical to the answers given in exercise 4 of the previous section (for the egalitarian method).

7. **Chances** with different disagreement payoff pairs:

DPP	Aspiration PP	Raiffa PP
$(50, 50)$	$(81.25, 85.714)$	$(68.2, 80.5)$
$(60, 50)$	$(81.25, 82.857)$	$(73.5, 70.8)$
$(62.5, 50)$	$(81.25, 82.143)$	$(74.5, 68.1)$

8. (a) Since Colin's aspiration payoff (chocolate) has not changed, Rose must not be harmed. Since Rose's aspiration payoff has changed (from coffee to strawberry), there is no restriction on Colin being harmed. So, an individually monotone method could produce any of strawberry, peach, raspberry, maple, coffee, or vanilla. (b) For no change to be forced in the third scenario, the aspiration payoffs must be unchanged; that is, both Rose and Colin must rank fudge, caramel, and cherry between coffee and chocolate.

The Nash Method

2. (b) $(9, 10)$. (c) The strategy pair (S, S) should be used. (d) Suppose we are using a method X that is efficient, unbiased, scale invariant, and $\mathrm{I}^2\mathrm{P}^2$. As seen in the text, the efficient and unbiased method X will produce $(50, 50)$ for **Isoceles Triangle**. A payoff pair (x, y) in **Isoceles Triangle** can be transformed to a payoff pair (x', y') in **Extended Matches**, the light and dark grey triangle in the part (a) diagram by the positive linear transformations

$$\begin{aligned} x' &= 2 + \frac{16-2}{100}x, \\ y' &= 5 + \frac{15-5}{100}x. \end{aligned}$$

Since the X method is scale invariant, the X method will produce the payoff pair $(x', y') = (9, 10)$ in **Extended Matches** because it produced $(x, y) = (50, 50)$ in **Triangle**. Now **Matches** is just **Extended Matches** with some payoff pairs, those in light grey, no longer feasible. Since the X method is $\mathrm{I}^2\mathrm{P}^2$, it will also produce $(9, 10)$, which is the Nash payoff pair, for **Matches**.

4. (b) $(66, 66)$. (c) The United States should win on the first four issues (expansion rights, U.S. defense rights, use rights, and expansion routes), Panama should win on the last five issues (duration, compensation, jurisdiction, U.S. military rights, and defense role of Panama), and there should be a compromise on the remaining issue (land and water). More precisely, the United States should win $\frac{2}{15}$ and Panama should win $\frac{13}{15}$ of the land and water issue.

7. **Chances** with different disagreement payoff pairs:

DPP	Nash PP
$(50, 50)$	$(70.0, 80.0)$
$(60, 50)$	$(70.6, 78.3)$
$(62.5, 50)$	$(71.9, 75.0)$

8. (a) The Nash method is not individually or strongly monotone. One argument is to note that the Nash method is efficient, unbiased, and scale invariant. By

the Raiffa Characterization Theorem, if the Nash method was also individually monotone, then the Nash method would have to be the same as the Raiffa method. Since the two methods select different payoff pairs for at least one bargaining game, we have a contradiction. (b) The egalitarian method is $\mathrm{I}^2\mathrm{P}^2$. Removing feasible payoff pairs other than the egalitarian payoff pair (x, y) will not change that (x, y) is the efficient payoff pair on the 45-degree line from the disagreement payoff pair. (c) The Raiffa method is not $\mathrm{I}^2\mathrm{P}^2$. A direct argument is to note that the Raiffa payoff pair for **Triangle** is $(50, 50)$, and the Raiffa payoff pair for **Triangle** after removal of the rational payoff pairs (x, y) satisfying $x \geq 80$ is $(44.4, 55.6)$, which is not the same as $(50, 50)$.

Coalition Games

A Savings Allocation Problem

Two Properties and Five Methods

1. (a) Let Abeje, Belva, and Corrine be abbreviated by their first initials. Have the gains made synonymous with the savings.

Coalition	ABC	AB	AC	BC	A	B	C
Gain	120	90	60	30	0	0	0

(b) $(40, 40, 40)$. (c) $(60, 40, 20)$. (d) $(70, 40, 10)$. (e) $(46, 40, 34)$. (f) $(55, 40, 25)$. (g) Equal split and proportional to equal splits are not rational. The others are rational.

3. (a) Let Ron, Stephen, and Tabitha be abbreviated by their first initials. Have the gains made synonymous with the money.

Coalition	RST	RS	RT	ST	R	S	T
Gain	180	180	0	0	0	0	0

(b) $(60, 60, 60)$. (c) $(90, 90, 0)$. (d) $(120, 120, -60)$. (e) $(75, 75, 30)$. (f) $(90, 90, 0)$. (g) Equal split, equal adjustment to marginals, proportional to equal splits are not rational. The other two methods determine rational allocations. (h) Suppose (r, s, t) is a rational allocation. Allocation implies $r + s + t \leq 180$. Rational implies $r + s \geq 180$ and $t \geq 0$. Combining these inequalities, we obtain $180 = 180 + 0 \leq r + s + t \leq 180$. So, the inequalities must hold with equality. In particular, $t = 0$.

The Shapley Method

1. (a)

Order	Marginal Contribution		
	A	B	C
ABC	0	90	30
ACB	0	60	60
BAC	90	0	30
BCA	90	0	30
CAB	60	60	0
CBA	90	30	0
Average	55	40	25

(b)

Coalition	Gains								
ABC	30	+	60	+	90	−	60	=	120
AB	0	+	0	+	90	−	0	=	90
AC	0	+	60	+	0	−	0	=	60
BC	30	+	0	+	0	−	0	=	30
anything else	0	+	0	+	0	−	0	=	0
Player	**Allocations**								
A	0	+	30	+	45	−	20	=	55
B	15	+	0	+	45	−	20	=	40
C	15	+	30	+	0	−	20	=	25

3. (a)

Order	Marginal Contribution		
	R	S	T
RST	0	180	0
RTS	0	180	0
SRT	180	0	0
STR	180	0	0
TRS	180	180	0
TSR	180	0	0
Average	90	90	0

(b)

Coalition	Gains
RST	180
RS	180
anything else	0
Player	**Allocations**
R	90
S	90
T	0

6. (a)

Coalition	ABCD	ABC	ABD	ACD	AB	other
Gain	120	108	96	84	24	0

(b) $(56, 28, 20, 16)$.

The Nucleolus Method

1. (a) Starting from the Shapley allocation, shift from C to A so as to even the complaints of AB and C. Next shift from B to A to even the complaints of AC and BC.

Coaliton	Gain	Excess For		
		$(55, 40, 25)$	$(65, 40, 15)$	$(67.5, 37.5, 15)$
AB	90	5	15	15
AC	60	20	20	22.5
BC	30	35	25	22.5
A	0	55	65	67.5
B	0	40	40	37.5
C	0	25	15	15

It is impossible to simultaneously increase the AB and C excesses because C cannot both receive more and less. Now that C's payoff is fixed, it is impossible to simultaneously increase the AC and BC excesses because A cannot both receive more and less.
(b) The reduced game on AB is

Coalition	Gain
AB	$67.5 + 37.5 = 105$
A	$\max\{0, 60 - 15\} = 45$
B	$\max\{0, 30 - 15\} = 15$

and so the payoffs to A and B are initially 45 and 15, respectively, and the surplus of $105 - 45 - 15 = 45$ is then split evenly resulting in the payoffs of 67.5 and 37.5 to A and B, respectively. These payoffs match these players payoffs in the original game.
(c) The reduced game on AC is

Coalition	Gain
AC	$67.5 + 15 = 82.5$
A	$\max\{0, 90 - 37.5\} = 52.5$
C	$\max\{0, 30 - 37.5\} = 0$

and so the payoffs to A and C are initially 52.5 and 0, respectively, and the surplus of $82.5 - 52.5 - 0 = 30$ is then split evenly resulting in the payoffs of 67.5 and 15 to A and C, respectively. These payoffs match these players payoffs in the original game.
(d) The reduced game on BC is

Coalition	Gain
BC	$37.5 + 15 = 52.5$
B	$\max\{0, 90 - 67.5\} = 22.5$
C	$\max\{0, 60 - 67.5\} = 0$

and so the payoffs to B and C are initially 22.5 and 0, respectively, and the surplus of $52.5 - 22.5 - 0 = 30$ is then split evenly resulting in the payoffs of 37.5 and 15 to B and C, respectively. These payoffs match these players' payoffs in the original game.

3. (a) Starting from the Shapley allocation, the table below shows the solution immediately.

Coaliton	Gain	Excesses For $(90, 90, 0)$
RS	180	0
RT	0	-90
ST	0	-90
R	0	-90
S	0	-90
T	0	0

It is impossible to simultaneously increase the RS and T excesses because T cannot both receive more and less. It is now impossible to simultaneously increase the R and S excesses because S cannot both receive more and less.
(b) The reduced game on RS is

Coalition	Gain
RS	$90 + 90 = 180$
R	$\max\{0, 0 - 90\} = 0$
S	$\max\{0, 0 - 90\} = 0$

and so the payoffs to R and S are 90 and 90, respectively. These payoffs match these players' payoffs in the original game.
(c) The reduced game on RT is

Coalition	Gain
RT	$90 + 0 = 90$
R	$\max\{0, 180 - 90\} = 90$
T	$\max\{0, 0 - 0\} = 0$

and so the payoffs to R and T are 90 and 0, respectively. These payoffs match these players' payoffs in the original game.
(d) The reduced game on ST is

Coalition	Gain
ST	$90 + 0 = 90$
S	$\max\{0, 180 - 90\} = 90$
T	$\max\{0, 0 - 0\} = 0$

and so the payoffs to S and T are 90 and 0, respectively. These payoffs match these players' payoffs in the original game.

6. (a)

Coaliton	Gain	Excesses For		
		$(56, 28, 20, 16)$	$(66, 28, 20, 6)$	$(84, 18, 12, 6)$
ABC	108	4	−6	−6
ABD	96	−4	−4	−12
ACD	84	−8	−8	−18
BCD	0	−64	−54	−36
AB	24	−60	−70	−78
AC	0	−76	−86	−96
AD	0	−72	−72	−90
BC	0	−48	−48	−30
BD	0	−44	−34	−24
CD	0	−36	−26	−18
A	0	−56	−66	−84
B	0	−28	−28	−18
C	0	−20	−20	−12
D	0	−16	−06	−6

You Can't Always Get What You Want

3. For **BB**, use the argument for **AA** with A and B interchanged. For **CC**, use the argument for **AA** with A and C interchanged and with B and D interchanged. For **DD**, use the argument for **AA** with A and D interchanged and with B and C interchanged.

Fair Division

An Inheritance Problem

1. We believe this to be normal human behavior. Doug sums it up with the statement, "there is really no single objective value that can be placed on any item", and Bob continues with "Perhaps that is why some people are buyers and some are sellers; buyers value an item more than the sellers, and so an intermediate price makes both happy."

2. Cash, certificates of deposit, stocks, or other financial instruments would likely to be valued the same by each inheritor.

Fair Division Games and Methods

1. (a) A monetary value means the price at which that player would be willing to buy or sell that item. The brothers must be willing to exchange money for any of the division methods to be used.

(b)–(f) are summarized in the table.

Three Brothers: Divisions, Values, and Shares			
Value	Jesse	Kaleab	Ulises
Piano	$4,000	$3,500	$3,100
Trumpet	$90	$130	$100
Cello	$1,000	$1,400	$1,800
Flute	$70	$100	$70
Estate	$5,160	$5,130	$5,070
Ad Hoc	Trumpet + $1,740	Piano + Flute − $1,770	Cello + $30
	$1,830 (35.5%)	$1,830 (35.7%)	$1,830 (36.1%)
First-Price	Piano − $1,990 $2,010 (39.0%)	Trumpet + Flute + $1,780 $2,010 (39.2%)	Cello + $210 $2,010 (39.6%)
Second-Price	Piano − $1,810 $2,190 (42.4%)	Trumpet + Flute + $1,520 $1,750 (34.1%)	Cello + $290 $2,090 (41.2%)
Equal Shares	Piano − $1,974 $2,026 (39.3%)	Trumpet + Flute + $1,784 $2,014 (39.3%)	Cello + $190 $1,990 (39.3%)
Knaster's	Piano − $1,976 $2,024 (39.2%)	Trumpet + Flute + $1,783 $2,013 (39.2%)	Cello + $193 $1,993 (39.3%)

2. (b)–(f) are summarized in the table.

Four Siblings: Divisions, Values, and Shares				
Value	Kate	Genevieve	Stephen	Laura
Pendant	$600	$560	$560	$520
First-Try	$150	$140	Pendant − $420	$130
	$150 (25.0%)	$140 (25.0%)	$140 (25.0%)	$130 (25.0%)
First-Price	Pendant − $450 $150 (25.0%)	$150 $150 (26.8%)	$150 $150 (26.8%)	$150 $150 (28.8%)
Second-Price	Pendant − $420 $180 (30.0%)	$140 $140 (25.0%)	$140 $140 (25.0%)	$140 $140 (26.9%)
Equal Shares	Pendant − $439 $161 (26.8%)	$150 $150 (26.8%)	$150 $150 (26.8%)	$139 $139 (26.8%)
Knaster's	Pendant − $440 $160 (26.7%)	$150 $150 (26.8%)	$150 $150 (26.8%)	$140 $140 (26.9%)

6. (a)–(d) are summarized in the table.

Ailing Parent: Divisions, Values, and Shares			
Value	Barbara	John	Marc
Care for Father	$-\$12,000$	$-\$15,000$	$-\$30,000$
First-Price	Father + \$8,000 −\$4,000 (33.3%)	−\$4,000 −\$4,000 (26.7%)	−\$4,000 −\$4,000 (13.3%)
Second-Price	Father + \$10,000 −\$2,000 (16.7%)	−\$5,000 −\$5,000 (33.3%)	−\$5,000 −\$5,000 (16.7%)
Equal Shares	Father + \$9,474 −\$2,526 (21.1%)	−\$3,158 −\$3,158 (21.1%)	−\$6,316 −\$6,316 (21.1%)
Knaster's	Father + \$10,333 −\$1,667 (13.9%)	−\$2,667 −\$2,667 (17.8%)	−\$7,667 −\$7,667 (25.6%)

Fairness Properties

1. The table summarizes the answers.

Three Brothers	Proportionate	Envy free	Efficient	Value equitable	Share equitable
Ad Hoc	yes	no	no	yes	no
First-Price	yes	yes	yes	yes	no
Second-Price	yes	yes	yes	no	no
Equal Shares	yes	yes	yes	no	yes
Knaster's	yes	yes	yes	no	no

2. (a) Since each player's share for each division is at least 1/4 = 25.0%, all five divisions are proportionate. (b) For the ad hoc division, the following table shows that Kate is envious of Stephen, Genevieve is envious of Kate, Stephen is envious of Kate, and Laura is envious of Kate and Genevieve.

Division	Kate's Portion \$150	Genevieve's Portion \$140	Stephen's Portion Pendant −\$420	Laura's Portion \$130
Value from Kates's perspective	\$150	\$140	\$600 – \$420 = \$180	\$130
Value from Genevieve's perspective	\$150	\$140	\$560 – \$420 = \$140	\$130
Value from Stephen's perspective	\$150	\$140	\$560 – \$420 = \$140	\$130
Value from Laura's perspective	\$150	\$140	\$520 – \$420 = \$100	\$130

For the first-price auction division, the following table shows that no player is envious of another player (the diagonal entry is the largest in each row):

Division	Kate's Portion Pendant −\$450	Genevieve's Portion \$150	Stephen's Portion \$150	Laura's Portion \$150
Value from Kates's perspective	\$600 – \$450 = \$150	\$150	\$150	\$150
Value from Genevieve's perspective	\$560 – \$450 = \$110	\$150	\$150	\$150
Value from Stephen's perspective	\$560 – \$450 = \$90	\$150	\$150	\$150
Value from Laura's perspective	\$520 – \$450 = \$70	\$150	\$150	\$150

For the second-price auction division, the following table shows that no player is envious of another player (the diagonal entry is the largest in each row):

Division	Kate's Portion Pendant −\$450	Genevieve's Portion \$140	Stephen's Portion \$140	Laura's Portion \$140
Value from Kates's perspective	\$600 – \$420 = \$180	\$140	\$140	\$140
Value from Genevieve's perspective	\$560 – \$420 = \$140	\$140	\$140	\$140
Value from Stephen's perspective	\$560 – \$420 = \$120	\$140	\$140	\$140
Value from Laura's perspective	\$520 – \$420 = \$100	\$140	\$140	\$140

For the equal shares division, the following table shows that Laura is envious of Genevieve and Stephen.

Division	Kate's Portion Pendant −$439	Genevieve's Portion $150	Stephen's Portion $150	Laura's Portion $139
Value from Kates's perspective	$600 − $439 = $161	$150	$150	$139
Value from Genevieve's perspective	$560 − $439 = $121	$150	$150	$139
Value from Stephen's perspective	$560 − $439 = $121	$150	$150	$139
Value from Laura's perspective	$520 − $439 = $81	$150	$150	$139

For Knaster's division, the following table shows that Laura is envious of Genevieve and Stephen.

Division	Kate's Portion Pendant −$440	Genevieve's Portion $150	Stephen's Portion $150	Laura's Portion $140
Value from Kates's perspective	$600 − $440 = $161	$150	$150	$140
Value from Genevieve's perspective	$560 − $440 = $121	$150	$150	$140
Value from Stephen's perspective	$560 − $440 = $121	$150	$150	$140
Value from Laura's perspective	$520 − $440 = $81	$150	$150	$140

(c) Since the ad hoc division gives the pendant to Stephen instead of Kate (who values it the most) by the Efficient Division Theorem the ad hoc division is not efficient. This can also be seen directly by noting that in comparison with the ad hoc division, the equal shares division increases each player's value. Since the other four divisions give the pendant to the player who values it the most (Kate), by the Efficient Division Theorem, the other four divisions are efficient. (d) Since the four players' values are equal only for the first-price auction division, only the first-price auction division is value equitable. (e) Since the four players' shares are equal only for the ad hoc and equal shares divisions, only these divisions are share equitable.

Choosing a Fair Method

1. When the second-price auction method is applied to a single item, the player who has the highest value for the item, call it u, is assigned the item and pays the highest value for the item among the other players, call it v, into the pot. The money in the pot is then divided equally among the players, and so each player receives v/n in money. The player who is assigned the item values what he received at $u-v+v/n$ and has a $(u-v+v/n)/u$ share of the item. Since the highest value is at least as great as the highest remaining value,

$$\begin{aligned}
u &\geq v, \\
(n-1)u &\geq (n-1)v, \\
nu-u &\geq nv-v, \\
nu-nv+v &\geq u, \\
u-v+v/n &\geq u/n, \\
(u-v+v/n)/u &\geq 1/n.
\end{aligned}$$

So, the player who is assigned the item has at least a $1/n$ share of the item. Now any other player values the item at $w \leq v$ and receives v/n in money, and so has a $(v/n)/w$ share of the item. Since $(v/n)/w \geq (v/n)/v = 1/n$, any other player has at least a $1/n$ share of the item. The second-price auction division for all items simultaneously can be obtained by summing the second-price auction division for each item taken individually. Since each player obtains at least $1/n$ share of each item, each player obtains at least $1/n$ share of the estate. Thus, second-price auction divisions are always proportionate.

3. If two players value the estate at u and v, respectively, then they will value what they receive in a Knaster's division at $\frac{1}{n}u+a$ and $\frac{1}{n}v+a$, respectively, where a is the additional amount each player receives. But $u=v$ if and only if $\frac{1}{n}u+a=\frac{1}{n}v+a$. So, the players have the same value for the estate if and only if the Knaster's division is value equitable.

6. Here is a summary table for all six methods.

Candy Bar: Divisions, Values, and Shares			
Item/Method	Leslie	Michael	Owen
Candy Bar	\$6.00	\$4,20	\$3.00
Equal Shares	Candy − \$3.27 \$2.73 (45.5%)	\$1.91 \$1.91 (45.5%)	\$1.36 \$1.36 (45.5%)
Knaster's	Candy − \$3.47 \$2.53 (42.2%)	\$1.93 \$1.93 (46.0%)	\$1.53 \$1.53 (51.0%)
First-Price	Candy − \$4.00 \$2.00 (33.3%)	\$2.00 \$2.00 (47.6%)	\$2.00 \$2.00 (66.7%)
Maximize the Smallest Share	Candy − \$3.50 \$2.50 (41.7%)	\$1.75 \$1.75 (41.7%)	\$1.75 \$1.75 (58.3%)
Minimize the Spread of Shares	Candy − \$3.00 \$3.00 (50%)	\$1.50 \$1.50 (35.7%)	\$1.50 \$1.50 (50%)
Second-Price	Candy − \$2.80 \$3.40 (56.7%)	\$1.40 \$1.40 (33.3%)	\$1.40 \$1.40 (46.7%)

(a) For a **Candy Bar** division to be efficient, the candy bar must be given to Leslie. To minimize the share spread, we should choose d so that Leslie's share and Owen's shares are equal:

$$\begin{aligned} \frac{6.00 - 2d}{6.00} &= \frac{d}{3.00}, \\ 3(6 - 2d) &= 6d, \\ d &= \frac{3 \times 6}{6 + 3 \times 2} = 1.50. \end{aligned}$$

8. A prudential strategy maximizes the minimum payoff to a player. The nucleolus allocation maximizes the minimum excess. The maximize the smallest share division is a third solution concept.

9. (a) The bedroom assignment that maximizes the sum of what the women are willing to pay gives Diana the large bedroom, Stacey the small bedroom, and Tien-Yee the medium bedroom.

Diana	Stacey	Tien-Yee	Amount Willing To Pay
Large	Medium	Small	$\$300 + \$250 + \$100 = \$650$
Large	Small	Medium	$\$300 + \$150 + \$225 = \$675$
Medium	Large	Small	$\$200 + \$200 + \$100 = \$500$
Medium	Small	Large	$\$200 + \$150 + \$275 = \$625$
Small	Large	Medium	$\$100 + \$200 + \$225 = \$525$
Small	Medium	Large	$\$100 + \$250 + \$275 = \$625$

(b) Diana should pay $\$300 - \$25 = \$275$ for the large bedroom; Stacey should pay $\$150 - \$25 = \$125$ for the small bedroom; and Tien-Yee should pay $\$225 - \$25 = \$200$ for the medium bedroom. Stacey is currently paying \$25 less than she would have been willing to pay for the bedroom she is in. However, if Stacey were to have the medium bedroom for what Tien-Yee is paying, Stacey would be paying \$50 less than she would have been willing to pay for the bedroom. So, Stacey envies Tien-Yee.

(c) Diana should pay $\$300 - \$33\frac{1}{3} = \$266\frac{2}{3}$ for the large bedroom; Stacey should pay $\$150 - \$33\frac{1}{3} = \$116\frac{2}{3}$ for the small bedroom; and Tien-Yee should pay $\$250 - \$33\frac{1}{3} = \$216\frac{2}{3}$ for the medium bedroom. Note in the following table that the diagonal entries are maximal in their column; this means that no woman would prefer another woman's bedroom and payment over her current bedroom and payment.

Division Values from Each Player's Perspective			
Division/Perspective	Diana	Stacey	Tien-Yee
Diana: Large $-\$266\frac{2}{3}$	$\$33\frac{1}{3}$	$-\$66\frac{2}{3}$	$\$8\frac{1}{3}$
Stacey: Small $-\$116\frac{2}{3}$	$-\$16\frac{2}{3}$	$\$33\frac{1}{3}$	$-\$16\frac{2}{3}$
Tien-Yee: Medium $-\$216\frac{2}{3}$	$-\$16\frac{2}{3}$	$\$33\frac{1}{3}$	$\$8\frac{1}{3}$

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