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GRAVITATIONAL STABILITY OF SCALAR MATTER

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The question of the stability of matter against gravitational collapse is of general interest to astrophysics. In this work we investigate the stability against small radial oscillations of equilibrium configurations of cold, gravitationally bound states of complex scalar fields, known as boson stars. These equilibrium configurations exhibit a mass profile against central density which is very similar to that of ordinary neutron stars, with a pronounced maximum mass at $M_c = 0.633 M_{Pl}^2/m$, where M_{Pl} is the Planck mass, for a certain value of the central density $\sigma_c(0)$. We give analytical and numerical proof that configurations with central densities greater than $\sigma_c(0)$ are unstable against radial perturbations by studying the behavior of the eigenfrequencies of the perturbations for different values of $\sigma(0)$.

1. Introduction

The general problem of the stability of relativistic matter, be it via Coulomb or gravitational interactions is of fundamental importance in physics. A great deal of progress has been made in the past years by studying the boundedness of the quantum mechanical hamiltonian operator of a system of particles with the above interactions [1]. Using these methods Lieb and Yau [2] were able to reproduce the results of the semi-classical Chandrasekhar analysis to less than 0.01% accuracy. This general framework gives good results for fermionic bound systems, despite the exclusion of general relativistic effects. However, results on the stability of bosonic bound systems with the inclusion of strong gravitational effects are still lacking.

It is our intention in this work to continue the previous efforts of one of us [3] to study the stability of gravitational bound states of complex scale fields (known as boson stars) taking into account the full effects of general relativity. As in ref. [3], we only consider stability of the lowest mass configurations, i.e., with nodeless scalar fields in the first spherical shell. The motivation for studying such objects is twofold; there is a growing interest in cosmology in the use of scalar fields to explain several problems of the so-called standard model, such as the horizon and flatness problems and the generation of large-scale structure [4]. Although so far no mechanism for the formation of boson stars has been successfully proposed [5], it is important to study their general properties as a first step towards understanding their possible role in cosmology. In fact, recent work on boson stars was generally

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motivated by the axion field [6], proposed in connection with a possible solution to the strong *CP* problem in QCD, and also by the interesting relation between these objects and non-topological solitons, as stressed in the thorough analysis of Lee and his collaborators [7].

The second reason for studying these objects comes from relativistic astrophysics and has to do with the unusual properties of bosonic condensates at zero temperature. As shown in the analysis of Ruffini and Bonazzola [8], boson stars naturally exhibit fractional anisotropy; the radial and tangential components of the pressure are different, precluding a macroscopic description of these objects in terms of an effective perfect fluid energy–momentum tensor. In the mid-seventies, there were some efforts to understand how the inclusion of fractional anisotropy could influence general properties of spherically symmetric and static distributions of matter such as critical masses and surface redshifts [9]. The motivation for that work came from the fact that deviations from the perfect fluid assumption for nuclear matter are expected in the presence of strong gravitational fields. Nevertheless, in the works of ref. [9] local anisotropy is unavoidably included in a very ad hoc fashion. Boson stars provide a unique way for studying the effects of local anisotropy from first principles [3], since for any given model we start with an explicit knowledge of the interactions involved. In particular, it was shown in ref. [3] how the inclusion of a $\lambda|\phi|^4$ self-interaction for the scalar field provides a way of varying the fractional anisotropy for the equilibrium configurations, the greater the value of λ the smaller the fractional anisotropy for a given central density [10]. As can be seen from the results of ref. [3], configurations with higher fractional anisotropy were stable up to higher values of the central density, the limiting case being of course for $\lambda = 0$.

However, the results of ref. [3] were based on the construction of a variational principle for the linearized perturbations that allows one to compute upper bounds for the values of the eigenfrequencies that describe their time evolution, as in the work of Chandrasekhar concerning the dynamical instability of gaseous masses against baryon number conserving, infinitesimal radial perturbations [11]. As the analysis involves a variational principle, the estimates for the eigenfrequencies depend on the choice of trial functions used. (Of course, this is also the advantage of the variational formulation, to obtain an estimate for the eigenfrequencies without explicitly solving the linearized perturbation equations, which in general is not extremely sensitive to the choice of trial functions, so long as the trial functions obey the correct boundary conditions at the origin and, in the boson star case, at infinity.) Contrary to the case studied by Chandrasekhar, it is not possible to reduce the problem to one linearized equation; as shown in ref. [3] (see also ref. [12]), one obtains two coupled second order differential equations for the two components of the scalar field, making the choice of trial functions a non-trivial problem. Also, due to the complexity of the Einstein–Klein–Gordon system, the method used in ref. [3] to obtain the two linearized equations is not the most natural to the problem, as we hope to show in the present work.

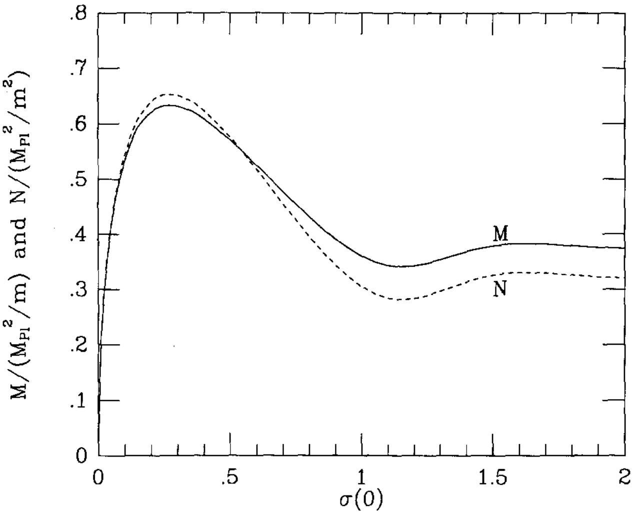

Fig. 1. Boson star mass in units of $M_{\rm Pl}^2/m$ (continuous line) and particle number in units of $M_{\rm Pl}^2/m^2$ (dotted line) as a function of the central density. Note the location of the critical points where $\mathrm{d}N/\mathrm{d}\sigma(0)=0$. The fundamental mode goes unstable at $\sigma(0)=0.271$, the first such point. Each successive critical point has a higher mode going unstable.

Thus, we can improve dramatically the results of ref. [3] (and, for similar reasons, those of ref. [12]), so as to have instability for configurations with considerably smaller central densities than those quoted in ref. [3]. In fact, we show that boson stars behave qualitatively in a similar way as neutron stars [13]; configurations with central densities higher than the critical value (the critical value being the one corresponding to the maximum mass, see fig. 1) are all unstable against small radial perturbations.

The paper is organized as follows. In sect. 2 we obtain the equilibrium configurations and briefly describe their properties. More details can be found in refs. [6–8, 10]. In sect. 3 we obtain the linearized perturbation equations and discuss the role of charge conservation in studying the stability problem. Sect. 4 gives a general argument as to why the critical points of fig. 1 correspond to changes in stability; at each successive critical point another mode becomes negative. In sect. 5 we give the results of the numerical integration of the pulsation equations and show how the eigenvalues change with central density. The analysis confirms the results of sect. 4. We summarize our results in sect. 6.

2. Equilibrium configurations

The discussion of the equilibrium configurations that follows is essentially the same as that of ref. [3]. In the present analysis we do not include the scalar self-interaction term. The results obtained here can be trivially extended to the self-interacting case without any qualitative change. (As shown in ref. [10], although the inclusion of the self-interaction changes the value of the maximum mass and the corresponding critical central density, the qualitative features are essentially as in fig. 1).

The starting point for the calculation is the action

$$\mathscr{S}=\int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi G}+g^{\mu\nu}\phi^*_{;\mu}\phi_{;\nu}-m^2|\phi|^2\right). \tag{1}$$

This action is invariant under a global phase transformation, $\phi \to e^{i\theta}\phi$, that implies the conservation of its generator N, the total particle number. By varying the action with respect to $g^{\mu\nu}$ and ϕ (or equivalently ϕ^*), we obtain respectively Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi GT_{\mu\nu}, \tag{2}$$

with

$$T_{\mu\nu}=\phi^*_{,\mu}\phi_{,\nu}+\phi^*_{,\nu}\phi_{,\mu}-g_{\mu\nu}\left[g^{\alpha\beta}\phi^*_{,\alpha}\phi_{,\beta}-m^2|\phi|^2\right], \tag{3}$$

and the scalar field equation

$$g^{\alpha\beta}\phi_{;\alpha\beta}+m^2\phi=0. \tag{4}$$

As we are considering a spherically symmetric system with motions only in the radial direction, we take as the spacetime metric

$$\mathrm{d}s^2=\mathrm{e}^{\nu}\mathrm{d}t^2-\mathrm{e}^{\lambda}\mathrm{d}r^2-r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{5}$$

where ν and λ are functions of r and t only.

It proves convenient to write the scalar field as

$$\phi(r,t)=\left[\phi_1(r,t)+i\phi_2(r,t)\right]\mathrm{e}^{-i\omega t}, \tag{6}$$

where $\phi_1(r,t)$ and $\phi_2(r,t)$ are real functions. With the metric (5) Einstein equations are

$$R^0_0-\tfrac{1}{2}R=-8\pi GT^0_0\equiv-8\pi G\rho=\frac{1}{r^2}\left(r\mathrm{e}^{-\lambda}\right)'-\frac{1}{r^2}, \tag{7}$$

$$R^1_1-\tfrac{1}{2}R=-8\pi GT^1_1\equiv8\pi Gp_{\mathrm{r}}=\mathrm{e}^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \tag{8}$$

$$\begin{aligned}R^2_2-\tfrac{1}{2}R=R^3_3-\tfrac{1}{2}R&=-8\pi GT^2_2=-8\pi GT^3_3\equiv8\pi Gp_{\perp}\\&=\mathrm{e}^{-\lambda}\left(\tfrac{1}{2}\nu''-\tfrac{1}{4}\nu'\lambda'+\tfrac{1}{4}\nu'^2+\frac{1}{2r}(\nu'-\lambda')\right)\\&\quad-\mathrm{e}^{-\nu}\left[-\tfrac{1}{4}\dot{\lambda}\dot{\nu}+\tfrac{1}{2}\ddot{\lambda}+\tfrac{1}{4}(\dot{\lambda})^2\right],\end{aligned} \tag{9}$$

and

$$R^1_0 = -8\pi G T^1_0 = \frac{e^{-\lambda}}{r}\dot{\lambda}, \tag{10}$$

where the prime and the dot denote differentiation with respect to r and t respectively. For the reader's convenience, the usual definitions of the energy density ρ, the radial pressure p_r, and the tangential pressure $p_\perp$ were introduced. Combining eqs. (7) and (8) we obtain the useful relation

$$\frac{e^{-\lambda}}{r}(\lambda' + \nu') = -8\pi G\left(T^1_1 - T^0_0\right). \tag{11}$$

As is well known, eqs. (7)–(10) are not all independent due to the Bianchi identities. As a consequence, the conservation of energy–momentum $T^{\nu}_{\mu;\nu} = 0$ leads, in the present framework, to

$$\dot{T}^0_0 + T^{1\prime}_0 + \tfrac{1}{2}\left(T^0_0 - T^1_1\right)\dot{\lambda} + T^1_0\left(\tfrac{1}{2}(\lambda' + \nu') + \frac{2}{r}\right) = 0, \tag{12}$$

and

$$\dot{T}^0_1 + T^{1\prime}_1 + \tfrac{1}{2}T^0_1(\dot{\lambda} + \dot{\nu}) + \tfrac{1}{2}\left(T^1_1 - T^0_0\right)\nu' + \frac{2}{r}\left(T^1_1 - T^2_2\right) = 0. \tag{13}$$

The equations for ϕ_1 and ϕ_2 can be obtained by using eqs. (5) and (6) into eq. (4) and its complex conjugate giving

$$\phi_1'' + \left(\frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2}\right)\phi_1' + e^{\lambda}\left(\omega^2 e^{-\nu} - m^2\right)\phi_1$$
$$- e^{\lambda-\nu}\ddot{\phi}_1 + \tfrac{1}{2}e^{\lambda-\nu}(\dot{\nu} - \dot{\lambda})\dot{\phi}_1 \pm \tfrac{1}{2}e^{\lambda-\nu}(\dot{\nu} - \dot{\lambda})\omega\phi_2 \mp 2e^{\lambda-\nu}\omega\dot{\phi}_2 = 0, \tag{14}$$

where the equation for ϕ_2 is obtained by interchanging the subscripts $1 \leftrightarrow 2$ and by taking the lower signs.

Finally, as mentioned before, the global invariance of the action (1) leads to the continuity equation

$$J^{\mu}_{;\mu} = 0, \tag{15}$$

where the current four-vector J^{μ} is given by

$$J^{\mu} = i g^{\mu\nu}\left(\phi_{,\nu}\phi^* - \phi^*_{,\nu}\phi\right). \tag{16}$$

The conserved charge is then

$$N = \int d^3x \sqrt{-g}\, J^0. \tag{17}$$

For the equilibrium configurations the metric functions are time independent. Also, the scalar field components can be written as $\phi_1(r, t) = \phi_0(r)$ and $\phi_2(r, t) = 0$. Thus, there are only three unknown functions of r to be determined; ν_0, λ_0 and ϕ_0, where the subscript 0 is used to characterize the equilibrium quantities. There must be only three independent equations. First note that for the static solutions, eqs. (10), (12) and (15) are trivially satisfied. Also, eq. (13) (the hydrostatic equilibrium equation) is identical to eq. (14) which can be obtained from a combination of eqs. (7)–(9). It is then a matter of choice which set of equations is taken to be independent; one can use Einstein eqs. (7)–(9) or, say, eqs. (7), (8) and (13). We will follow the current practice [6–8, 10] and take the latter combination. We are left with the three equations

$$\left(r\,\mathrm{e}^{-\lambda_0}\right)' = 1 - 8\pi G r^2\left[\left(m^2 + \mathrm{e}^{-\nu_0}\omega^2\right)\phi_0^2 + \mathrm{e}^{-\lambda_0}\phi_0'^2\right], \tag{18}$$

$$\frac{\mathrm{e}^{-\lambda_0}}{r}\nu_0' - \frac{1}{r^2}\left(1 - \mathrm{e}^{-\lambda_0}\right) = -8\pi G\left[\left(m^2 - \mathrm{e}^{-\nu_0}\omega^2\right)\phi_0^2 - \mathrm{e}^{-\lambda_0}\phi_0'^2\right], \tag{19}$$

$$\phi_0'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\phi_0' - \mathrm{e}^{\lambda_0}\left(m^2 - \mathrm{e}^{-\nu_0}\omega^2\right)\phi_0 = 0. \tag{20}$$

Eq. (11) becomes

$$\lambda_0' + \nu_0' = 16\pi G r\left(\mathrm{e}^{\lambda_0 - \nu_0}\omega^2\phi_0^2 + \phi_0'^2\right). \tag{21}$$

These equations can be integrated numerically once we introduce the dimensionless variables $x \equiv rm$, $\sigma(r) \equiv (8\pi G)^{1/2}\phi_0(r)$, with the factor ω^2/m^2 being absorbed into the definition of the metric function e^{ν_0}. The boundary conditions are $\lambda_0(r=0) = 0$, $\sigma(r=0) = \sigma(0)$, $\sigma'(r=0) = 0$ and $\sigma(\infty) = 0$. Note that the equilibrium configurations can be parametrized by the value of the scalar field at the origin $\sigma(0)$. For each value of $\sigma(0)$ it is possible to calculate all relevant quantities of the equilibrium configurations such as the mass M and the charge N. For details see refs. [6–8]. In fig. 1 we plot the total mass M and particle number N against $\sigma(0)$. The total mass M is defined by

$$M = 4\pi\int_0^{\infty}\rho r^2\,\mathrm{d}r. \tag{22}$$

Note the existence of a critical mass and particle number and how the extrema of the two curves overlap. Note also that for a finite value of the central density the binding energy of the configurations $E_b \equiv M - Nm$ becomes positive, signaling the existence of configurations with excess energy. This property is also shared by neutron stars and expresses a global instability of the equilibrium configurations against dispersion of the particles to infinity. The excess energy is translated into kinetic energy of the free particles at infinity.

3. Radial perturbations

We now wish to study the behavior of small, radial perturbations about the equilibrium configurations described above. We write the perturbed fields as

$$\lambda = \lambda_0 + \delta\lambda, \qquad \nu = \nu_0 + \delta\nu, \qquad \phi_1 = \phi_0(1 + \delta\phi_1), \qquad \phi_2 = \phi_0\,\delta\phi_2 \tag{23}$$

where all the quantities are taken to be functions of r and t only, consistent with radial perturbations. Plugging these fields into eqs. (7)–(10) and keeping only the first order terms in the perturbations we find

$$\left(r\,e^{-\lambda_0}\,\delta\lambda\right)' = 8\pi G r^2\,\delta T_0^0, \tag{24}$$

with

$$\delta T_0^0 = \delta\rho = -\delta\nu\,\omega^2\phi_0^2\,e^{-\nu_0} - \delta\lambda\,\phi_0'^2\,e^{-\lambda_0} + 2\,e^{-\nu_0}\,\omega^2\phi_0^2\,\delta\phi_1 - 2\,e^{-\nu_0}\,\omega\phi_0^2\,\delta\dot{\phi}_2$$
$$+ 2\,e^{-\lambda_0}\,\phi_0'^2\,\delta\phi_1 + 2\,e^{-\lambda_0}\,\phi_0\phi_0'\,\delta\phi_1' + 2m^2\phi_0^2\,\delta\phi_1, \tag{25}$$

$$\frac{1}{r}\left(\delta\nu' - \nu_0'\,\delta\lambda\right)e^{-\lambda_0} = \frac{e^{-\lambda_0}}{r^2}\,\delta\lambda - 8\pi G\,\delta T_1^1, \tag{26}$$

with

$$\delta T_1^1 = -\delta p_r = -\delta T_0^0 + 4m^2\phi_0^2\,\delta\phi_1, \tag{27}$$

$$e^{-\lambda_0}\left(\tfrac{1}{2}\,\delta\nu'' - \tfrac{1}{4}\nu_0'\,\delta\lambda' - \tfrac{1}{4}\lambda_0'\,\delta\nu' + \tfrac{1}{2}\nu_0'\,\delta\nu' + \frac{1}{2r}\left(\delta\nu' - \delta\lambda'\right)\right) - \tfrac{1}{2}\,e^{-\nu_0}\,\delta\ddot{\lambda}$$
$$-\delta\lambda\left(\tfrac{1}{2}\nu_0'' - \tfrac{1}{4}\nu_0'\lambda_0' + \tfrac{1}{4}\nu_0'^2 + \frac{1}{2r}\left(\nu_0' - \lambda_0'\right)\right)e^{-\lambda_0} = -8\pi G\,\delta T_2^2 \tag{28}$$

with

$$\delta T_2^2 = -\delta p_\perp = \delta\nu\,\omega^2\phi_0^2\,e^{-\nu_0} - \delta\lambda\,e^{-\lambda_0}\,\phi_0'^2 - 2\,e^{-\nu_0}\,\omega^2\phi_0^2\,\delta\phi_1 + 2\,e^{-\nu_0}\,\omega\phi_0^2\,\delta\dot{\phi}_2$$
$$+ 2\,e^{-\lambda_0}\,\phi_0'^2\,\delta\phi_1 + 2\,e^{-\lambda_0}\,\phi_0\phi_0'\,\delta\phi_1' + 2m^2\phi_0^2\,\delta\phi_1, \tag{29}$$

and, finally,

$$\delta\dot{\lambda} = 16\pi G r\phi_0\left[\phi_0'\,\delta\dot{\phi}_1 - \omega\phi_0\,\delta\phi_2'\right]. \tag{30}$$

Similarly, using eq. (14) we obtain

$$\delta\phi_1'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2} + 2\frac{\phi_0'}{\phi_0}\right)\delta\phi_1' + \frac{1}{2}\frac{\phi_0'}{\phi_0}\left(\delta\nu' - \delta\lambda'\right) - e^{\lambda_0-\nu_0}\left(2\omega\,\delta\dot{\phi}_2 + \delta\ddot{\phi}_1\right)$$
$$+ e^{\lambda_0-\nu_0}\,\omega^2\left(\delta\lambda - \delta\nu\right) - e^{\lambda_0}\,m^2\,\delta\lambda = 0, \tag{31}$$

and

$$\delta\phi_2'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2} + 2\frac{\phi_0'}{\phi_0}\right)\delta\phi_2' - \tfrac{1}{2}\,\mathrm{e}^{\lambda_0-\nu_0}\,\omega(\delta\dot{\nu} - \delta\dot{\lambda}) - \mathrm{e}^{\lambda_0-\nu_0}\left(\delta\ddot{\phi}_2 - 2\omega\,\delta\dot{\phi}_1\right) = 0\,. \tag{32}$$

Instead of taking eqs. (24) and (26) independently, it proves more convenient to take as independent equations, eq. (24) and the difference between eqs. (26) and (24),

$$\delta\nu' - \delta\lambda' = \left(\nu_0' - \lambda_0' + \frac{2}{r}\right)\delta\lambda - 32\pi G r\,\mathrm{e}^{\lambda_0} m^2\phi_0^2\,\delta\phi_1\,. \tag{33}$$

Note that we have six equations and only four unknown functions. As for the equilibrium configurations, it is easy to show that the equations for the components of the scalar field perturbations (the linearized Klein–Gordon equations), eqs. (31) and (32), are a consequence of energy–momentum conservation. Thus, they can both be obtained from appropriate combinations of the linearized Einstein equations, eqs. (24)–(30). It is thus a matter of choice what set of four equations forms the independent set to be solved, the goal being to eventually obtain eigenvalue equations for the perturbations. After some frustrated attempts, it became clear that the most appropriate independent set of equations for this purpose consists of eqs. (24), (28), (31) and (33). (Note that the present approach differs from that of ref. [3]. As we will see later, the results of ref. [3] are not the most general possible.)

This system of four coupled differential equations is still quite complicated. It turns out, however, that these equations can be reduced to two second order differential equations by eliminating $\delta\phi_2$ and $\delta\nu$. Eq. (24) can be solved for $\delta\dot{\phi}_2$. Plugging this into eq. (31), we obtain an equation involving $\delta\nu$ only through its first derivative, $\delta\nu'$. This can then be eliminated using eq. (33), and we end up with a second order equation for $\delta\phi_1$:

$$\begin{aligned}
&\delta\phi_1'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\delta\phi_1' + \frac{\delta\lambda'}{8\pi G\phi_0^2 r} - \mathrm{e}^{\lambda_0-\nu_0}\,\delta\ddot{\phi}_1 \\
&\quad + \left[\left(\frac{\phi_0'}{\phi_0}\right)^2 + \mathrm{e}^{\lambda_0}\left(\frac{1 - r\lambda_0'}{8\pi G r^2\phi_0^2}\right) + \mathrm{e}^{\lambda_0-\nu_0}\,\omega^2 - \mathrm{e}^{\lambda_0} m^2 + \left(\frac{\phi_0'}{\phi_0}\right)\left(\frac{\nu_0' - \lambda_0'}{2} + \frac{1}{r}\right)\right]\delta\lambda \\
&\quad - 2\,\mathrm{e}^{\lambda_0}\left[\mathrm{e}^{-\nu_0}\,\omega^2 + \mathrm{e}^{-\lambda_0}\left(\frac{\phi_0'}{\phi_0}\right)^2 + m^2 + 8\pi G r m^2\phi_0'\phi_0\right]\delta\phi_1 = 0\,.
\end{aligned} \tag{34}$$

To get our second equation, we first add eq. (24) to eq. (28). This gives us an equation involving $\delta\nu$ only through $\delta\nu'$ and $\delta\nu''$. These terms can be eliminated by using eq. (33) and its derivative. The resulting equation is

$$\delta\lambda'' + \tfrac{3}{2}(\nu_0' - \lambda_0')\,\delta\lambda' + 32\pi G\left(2\phi_0\phi_0' - rm^2 e^{\lambda_0}\phi_0^2\right)\delta\phi_1' - e^{\lambda_0-\nu_0}\,\delta\ddot{\lambda}$$
$$-\left(32\pi G\phi_0'^2 + \lambda_0'' + \frac{2}{r^2} - \frac{(\nu_0'-\lambda_0')^2}{2} - \frac{2\nu_0'+\lambda_0'}{r}\right)\delta\lambda$$
$$+32\pi G\left[2\phi_0'^2 - e^{\lambda_0} rm^2\phi_0^2\left(2\frac{\phi_0'}{\phi_0} + \frac{2\nu_0'+\lambda_0'}{2}\right)\right]\delta\phi_1 = 0\,. \qquad (35)$$

The change in the total charge of the star due to the perturbations can be obtained from eq. (17) and is given to first order in the perturbations by

$$\delta N = 8\pi\omega\int_0^\infty \mathrm{d}r\,r^2 e^{(\lambda_0-\nu_0)/2}\,\phi_0^2\left[2\,\delta\phi_1 + \tfrac{1}{2}(\delta\lambda - \delta\nu) - \frac{1}{\omega}\,\delta\dot{\phi}_2\right]. \qquad (36)$$

Using eqs. (24) and (25) we can eliminate $\delta\nu$ and $\delta\dot{\phi}_2$.

In terms of $\delta\phi_1$ and $\delta\lambda$ we have

$$\delta N = \frac{8\pi}{\omega}\int_0^\infty \mathrm{d}r\,r^2 e^{-(\lambda_0-\nu_0)/2}\,\phi_0^2$$
$$\times\left\{\frac{\delta\lambda'}{16\pi Gr\phi_0^2} + \frac{1}{2}\left[e^{(\lambda_0-\nu_0)}\omega^2 + \left(\frac{\phi_0'}{\phi_0}\right)^2 + \frac{(1-r\lambda_0')}{8\pi Gr^2\phi_0^2}\right]\delta\lambda\right.$$
$$\left. - \left[-e^{(\lambda_0-\nu_0)}\omega^2 + \left(\frac{\phi_0'}{\phi_0}\right)^2 + e^{\lambda_0}m^2\right]\delta\phi_1\right\}. \qquad (37)$$

As in the case with the conventional stability analysis (see e.g. refs. [11,13]) it is necessary to impose that the radial perturbations conserve charge. In Chandrasekhar's analysis, charge conservation guarantees that the perturbation in the fluid's pressure can be expressed in terms of the lagrangian displacement. Here, it will guarantee the proper behavior of the perturbations at infinity and will be a fundamental tool in looking for the solutions of eqs. (34) and (35).

Suppose we now assume a harmonic time dependence for the perturbations with frequency χ. The system of two coupled equations, along with the condition $\delta N = 0$, defines a characteristic value problem for χ^2. Furthermore, it is easy to show that the system is self-adjoint so that the values of χ^2 must be real. The question of stability is thus reduced to a study of the possible values of χ^2 [13]; if any of the values of χ^2 are found to be negative, then the perturbations will grow and the boson star will be unstable against radial oscillations. Of course, as the eigenvalues form an infinite discrete ordered sequence, if the fundamental radial mode of the star is stable ($\chi_0^2 > 0$), then all higher modes are stable. In other words, examining how χ_0^2 changes for different values of the central density should be enough to find instability since it will be the first mode to go unstable.

4. Zero frequency perturbations

We are interested in finding the values of $\sigma(0)$ for which the equilibrium configurations are stable. (Recall that the central density can be parametrized by the value of the scalar field at the origin, $\sigma(0)$.) As mentioned above, by studying the behavior of χ_0^2 with $\sigma(0)$ we can establish the boundaries between stable and unstable configurations; it is clear that such boundaries will be defined whenever $\chi_0^2 = 0$, i.e., where the lowest frequency perturbations are static. Thus a study of static perturbations will tell us for which values of the central density the stars are stable.

The analysis described in sect. 3 simplifies considerably in the case of static perturbations. In particular, it is easy to show that in the case of static perturbations the perturbed quantities defined in eq. (23) ϕ, λ and ν satisfy the same equations as the equilibrium solutions ϕ_0, λ_0, and ν_0. Thus if we begin with an equilibrium configuration with $\sigma(0)$, the perturbed fields will describe another equilibrium configuration with $\sigma(0) + \delta\sigma(0)$, for some infinitesimal $\delta\sigma(0)$. However, as we stated above, perturbed configurations must have the same charge as the equilibrium configuration. Thus zero frequency perturbations will exist if and only if there exist two neighboring equilibrium solutions with the same charge; i.e. when $\mathrm{d}N/\mathrm{d}\sigma(0) = 0$. Note that the presence of a zero frequency perturbation need not signal a change in stability. If the lowest χ_0^2 perturbation is already negative, this could be a higher frequency perturbation going negative, in which case the stability will be unchanged. Stability is established by examining for which values of $\sigma(0)$ the fundamental mode crosses through zero.

Examination of fig. 1 reveals that the condition $\mathrm{d}N/\mathrm{d}\sigma(0) = 0$ is satisfied only at a discrete set of points. Thus transitions from stability to instability or vice versa can occur at these points only. This is very similar to the usual stability analysis of compact objects [13].

5. Numerical study of perturbations

In this section we will use numerical integration to determine the characteristic values for the frequency χ^2. The idea is to choose a value of χ^2 and then integrate the perturbation equations (eqs. (34) and (35)) from the origin out to infinity. If the resulting perturbation satisfies $\delta N = 0$, then the chosen χ^2 is a characteristic value. If not, then another value is chosen and the process is repeated.

In general, this is a very tedious procedure. However, we recall that our interest lies in finding points where the lowest characteristic value is 0, and we know that these points occur only when $\mathrm{d}N/\mathrm{d}\sigma(0) = 0$. Thus our problem is reduced to finding these zero frequency perturbations and determining whether they are the lowest frequency mode. This involves counting the number of nodes of the perturbations. (Remember that the eigenfunction corresponding to the fundamental mode χ_0^2 has no nodes from $0 < r < \infty$ and that the eigenfunction corresponding to the nth mode has n nodes.)

A further check is required to insure that the characteristic value actually crosses zero and in which direction. To determine this we consider points close to where $\mathrm{d}N/\mathrm{d}\sigma(0) = 0$ and determine the sign of χ_0^2. Since χ_0^2 will deviate from zero only slightly around the critical point, finding its value at these points is relatively simple.

For simplicity, we use the same dimensionless variables and rescalings introduced for numerical work in sect. 2. Before we can integrate eqs. (34) and (35) we must discuss the boundary conditions at the origin. A study of the equations for small x

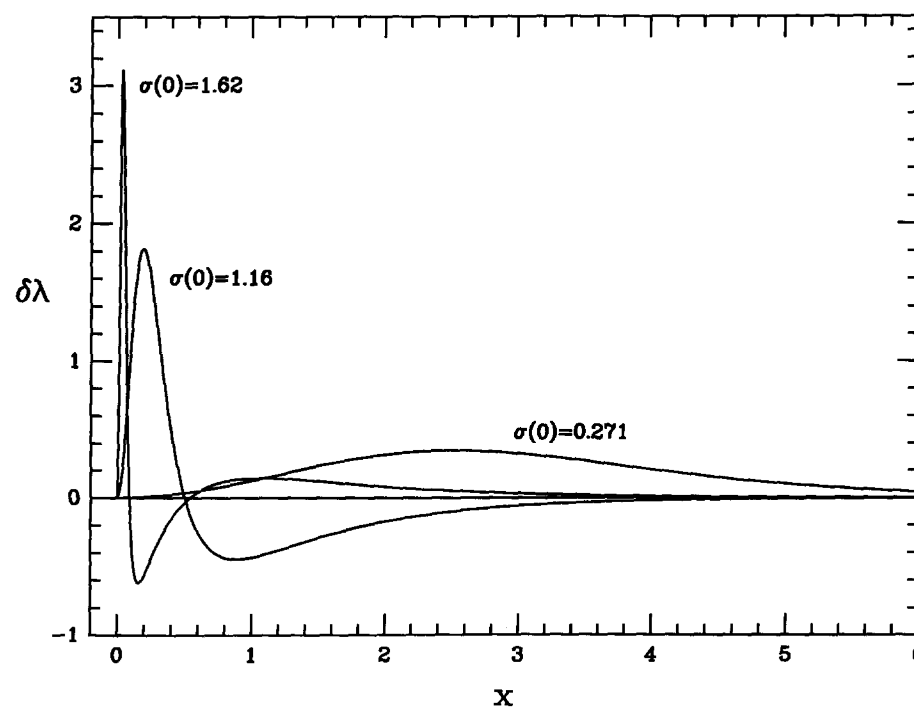

Fig. 2. The perturbation in the metric function $\delta\lambda$ is shown as a function of the radial coordinate for the first three modes. The number of nodes is related in a trivial way with the discrete sequence of eigenfrequencies.

shows that the only regular solution for $\delta\phi_1$ and $\delta\lambda$ has the form

$$\delta\phi_1 = 1 + \frac{1}{3}\left(\frac{1-\chi^2}{b} + 1 - \frac{\gamma}{\sigma(0)^2}\right)x^2 + O(x^4),$$

$$\delta\lambda = \gamma x^2 + O(x^4), \tag{38}$$

where $b = e^{\nu_0}(x=0)$, γ is an undetermined constant, and we have used the linearity of eqs. (34) and (35) to scale $\delta\phi_1(x=0)$ to one.

The presence of an undetermined constant in eq. (38) complicates somewhat our procedure (a similar problem arises in the case of polytropes [13]). For each value of χ_0^2 we must search the entire range of γ in order to see if there exists a solution with $\delta N = 0$. Again, this is greatly simplified by our only looking for characteristic values near zero at points where we know they must exist.

In fig. 2 we plot $\delta\lambda$ for the first three local extrema. By counting nodes, it is clear that the lowest characteristic value is zero at the first local extrema, the next highest at the second, etc. By looking at points on either side of the local extrema we determined that the characteristic values were in fact crossing from positive to negative. In fig. 3 we present a schematic drawing of how the characteristic values vary as we change $\sigma(0)$ in a small neighborhood of the zero modes.

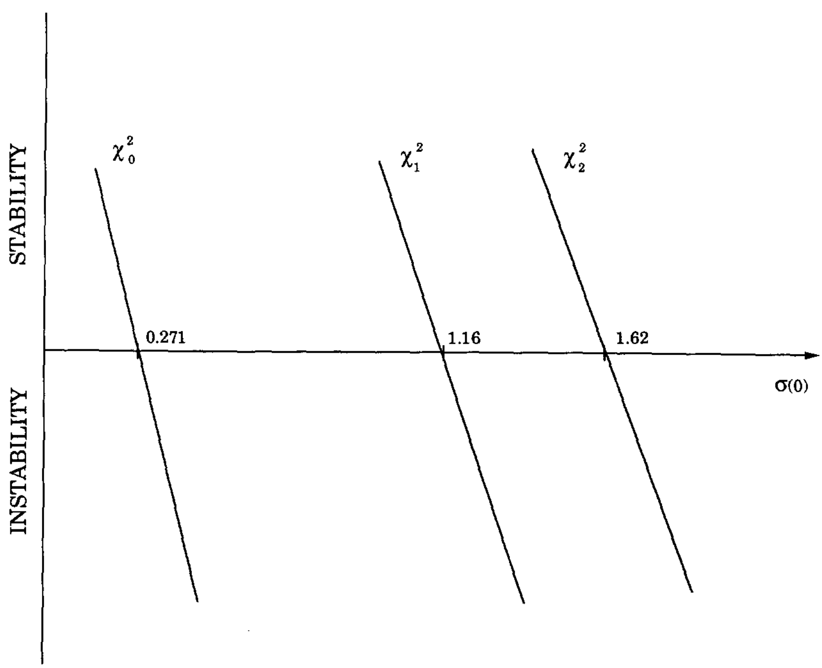

Fig. 3. Schematic behavior of the first three eigenfrequencies as a function of the central density. Each related mode of vibration goes unstable at the crossing point where the frequencies are zero.

6. Conclusion

We have examined the stability against small radial oscillations of spherically symmetric gravitational bound states of complex scalar fields known as boson stars. We gave analytical and numerical arguments to prove that the transition from stability to instability occurs at the critical points of the profile of mass (or charge) against central density. In particular, as the fundamental mode is the first to go unstable, we found that the critical value for stability is given by $M_c = 0.633 M_{Pl}^2/m$ for a central value of the scalar field given by $\phi_0(0) = 0.271(8\pi G)^{-1/2}$. Using a definition for the radius of the configuration as suggested by Lee and his collaborators [7], $R = 4\pi M^{-1}\int_0^\infty \rho r^3\, dr$, the density of the critical equilibrium configuration is

$$\rho_c = 0.633 \frac{M_{Pl}^2}{m}\left(\frac{m}{3.11}\right)^3 \simeq 5.0 \times 10^{53}\left(\frac{m}{\text{GeV}}\right)^2 \text{ g cm}^{-3}.$$

Note that the inclusion of general relativistic effects decreases the upper bound for stability found by analysing the relativistic hamiltonian [14], $N < 1.273 M_{Pl}^2/m^2$, as expected.

The fact that such objects have radii very close to their Schwarzschild limit produces very high surface redshifts. If they exist in appreciable numbers, their astrophysical effects can be very interesting and deserve further study.

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Upon completion of this work it came to our attention that similar results have been very recently obtained in a thorough analysis by Lee and Pang [15]. They have also shown that higher spherical configurations (with $l = 0$) for the scalar field are unstable. Thus, the limit quoted above is the absolute limit for the stability of boson stars.

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